Задание №11. Функции нескольких переменных. Непрерывность, частные производных

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№3202.

$$f(x,y)=\begin{cases}\frac{2xy}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

(*) Заметим, что $f(x,y)$ - симметрична отн-но x и y.

В (3183.1) мы доказали, что $f(x,y)$ разрывна по совокупности $(x,y)\to(0,0)$.
Осталось доказать, что f непрерывна по одной из переменных:

1) $y\neq 0, x\in\mathbb{R}$: $\lim\limits_{x\to x_0} f(x,y)=\frac{2x_0y}{x_0^2+y^2}=f(x_0,y)$

2) $y=0 \Rightarrow \forall x_0\neq 0$ $\lim\limits_{x\to x_0} f(x,0)=0=\lim\limits_{x\to x_0} f(x_0,0)$

3) $y=0$ и $x=0 \Rightarrow \lim\limits_{x\to 0} f(x,0)=f(0,0)=0$

Для каждого y f непр. по x

Из (*) $\Rightarrow$ для каждого x f непр. по y.

~~что~~

№3203

$$f(x,y)=\begin{cases}\frac{x^2y}{x^4+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

непрерывна вдоль каждого луча $\begin{cases}x=t\cos a\\ y=t\sin a\end{cases}$ $(0\leq t<+\infty)$, т.е.

$\lim\limits_{t\to 0} f(t\cos a, t\sin a)=f(0,0)$, но не явл. непрерывной в точке $(0,0)$.

Рассмотрим 2 случая:

1) $\alpha=\frac{\pi k}{2}, k\in\mathbb{Z} \Rightarrow f(t\cos\alpha, t\sin\alpha)=\lim\limits_{t\to 0} f(t\cos\alpha, t\sin\alpha)=f(0,0)=0$

2) $\alpha\neq\frac{\pi k}{2}, k\in\mathbb{Z}$ $\lim\limits_{t\to 0} f(t\cos\alpha, t\sin\alpha)=\lim\limits_{t\to 0}\frac{t\cos^2\alpha\sin\alpha}{\underbrace{(t^2\cos^4\alpha+\sin^2\alpha)}_{\to\sin^2\alpha}}=0=f(0,0)\Rightarrow$

$\Rightarrow f$ непрерывна в $O(0,0)$

~~Но из предыдущих номеров (3202(а,б))~~

Но при $x_n = \frac{1}{n},\ y_n = \frac{1}{n^2},\ (x_n, y_n) \xrightarrow[n\to\infty]{} (0,0)$

$$\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^4}}{\frac{1}{n^4} + \frac{1}{n^4}} = \frac{1}{2} \neq f(0,0)$$

~~что~~

№3204. $f(x,y) = x \sin \frac{1}{y}$ $\quad y \neq 0$ и $f(x,0) = 0$

Показать, что мн-во точек разрыва не является замкнутым.

$y_n = \left(\frac{\pi}{2} + 2\pi k\right)^{-1}$

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} \frac{n x_0}{n+1} \sin\left(\frac{\pi}{2} + 2\pi k\right) = x_0 \neq$$

$x_n = \frac{n}{n+1} x_0 \xrightarrow[n\to\infty]{} x_0$

$\neq f(x_0, 0) = 0,\ x_0 \neq 0 \Rightarrow (x_0, 0)$ - точка разрыва $\quad x_0 \neq 0.$

Т.к. $|f(x,y)| < |x| < \Rightarrow$ $f(x,y)$ непр. в точке $(0,0)$

$< \delta_1 = \varepsilon \Rightarrow$

$\Rightarrow$ Ось Ox — множ-во точек разрыва $\setminus \{0,0\}$

Но $\{0,0\}$ - предельная точка этого мн-ва $\Rightarrow$ f - не замкнута

№3211. $f'_x(x,b) = \frac{d}{dx}\left[\overset{g(x)}{f(x,b)}\right]$

По определению: $f'_x(x,b) = \lim\limits_{\Delta x\to 0} \frac{f(x+\Delta x, b) - f(x,b)}{\Delta x} = \lim\limits_{\Delta x} \frac{g(x+\Delta x) - g(x)}{\Delta x} =$

$$= g'(x) = \frac{d}{dx}\left[f(x,b)\right]$$

№3212 $f(x,y) = x + (y-1)\arcsin\sqrt{\frac{x}{y}}$ $\quad$ $f'_x(x,1)$?

$$f'_x(x,1) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x, 1) - f(x,1)}{\Delta x} = \lim_{x\to 0} \frac{x + \Delta x - x}{\Delta x} = 1$$

Ответ: 1

№3223

$$u = \operatorname{arctg}\left(\frac{x+y}{1-xy}\right)$$

1) $$\frac{\partial u}{\partial x} = \left(\operatorname{arctg}\left(\frac{x+y}{1-xy}\right)\right)' = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{1+y^2}{(1-xy)^2} =$$

$$= \frac{1+y^2}{1-2xy+x^2y^2+x^2+2xy+y^2} = \frac{1}{1+x^2}$$

2) $$\frac{\partial u}{\partial y} = \boxed{\frac{1}{1+y^2}}$$ (так как аргумент arctg симметричен отн-но xy)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x}\left(\frac{1}{1+x^2}\right) = -\frac{2x}{(1+x^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{2y}{(1+y^2)^2}$$ (анал-но)

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{1+x^2}\right) = \boxed{0}$$

№3224

$u = \arcsin\frac{x}{\sqrt{x^2+y^2}}$

$$\frac{\partial u}{\partial x} = \left(\arcsin\frac{x}{\sqrt{x^2+y^2}}\right)' = \frac{1}{\sqrt{\left(1-\frac{x^2}{x^2+y^2}\right)}} \cdot \frac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} =$$

$$= \frac{1}{|y| \cdot \sqrt{x^2+y^2}} \cdot \frac{y^2}{\sqrt{x^2+y^2}} = \boxed{\frac{|y|}{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = \left(\arcsin\frac{x}{\sqrt{x^2+y^2}}\right)' = -\frac{1 \cdot x \cdot 1 \cdot y}{\sqrt{1-\frac{x^2}{x^2+y^2}} \cdot (x^2+y^2) \cdot \sqrt{x^2+y^2}} = \frac{-xy}{|y|(x^2+y^2)} = -\frac{(\operatorname{sgn} y)x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2x|y|}{(x^2+y^2)^2};\quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{xy}{|y|(x^2+y^2)}\right) = \frac{2x|y|}{(x^2+y^2)^2}$$

Задание № 12.

№3227 $u = x^{\frac{y}{z}}$

1) $\frac{\partial u}{\partial x} = \frac{y}{z} x^{\frac{y}{x}-1} = \frac{y \cdot u}{z \cdot x}$

2) $\frac{\partial u}{\partial y} = (\ln x) \cdot x^{\frac{y}{z}} \cdot \frac{1}{z} = u \frac{\ln x}{z}$

3) $\frac{\partial u}{\partial z} = \ln x \cdot x^{\frac{y}{z}} \cdot \left(\frac{y}{z}\right)' = -\frac{u \cdot \ln x}{z^2} \cdot y$

4) $\frac{\partial^2 u}{\partial x^2} = \left(\frac{y \cdot u}{z x}\right)' = \frac{y \cancel{xz} \frac{\partial u}{\partial x} - yuz}{z^2 x^2} = \frac{y^2 u - yuz}{z^2 x^2} = \frac{uy(y-z)}{x^2 z^2}$

5) $\frac{\partial^2 u}{\partial y^2} = \left(u \frac{\ln x}{z}\right)' = \frac{u^2 \ln^2 x}{z^2}$

6) $\frac{\partial^2 u}{\partial z^2} = \left(-\frac{uy(\ln x)}{z^2}\right)' = -y \ln x \left(\frac{z^2 u_z' - 2uz}{z^4}\right) = \frac{-y \ln x (2z + y \ln x)}{z^4}$

7) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{yu}{zx}\right) = \frac{(uy)'}{zx} = \frac{1}{xz}\left(u + y \frac{\partial u}{\partial y}\right) = \frac{1}{xz}\left(u + y \cdot u \frac{\ln x}{z}\right) =$

$= \frac{u}{xz^2}(z + y \ln x)$

8) $\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial z}\left(u \frac{\ln x}{z}\right) = \ln x \left(\frac{1}{z} \cdot \frac{\partial u}{\partial z} - \frac{u}{z^2}\right) = \ln x \left(\frac{-1}{z^3} \cdot u \ln x y - \frac{u}{z^2}\right) =$

$= -\frac{\ln x \cdot u}{z^3}(z + y \ln x)$

9) $\frac{\partial^2 u}{\partial z \partial x} = -\frac{y}{z^2}\left(\ln x \, u' + \frac{u}{x}\right) = -\frac{y}{z^2}\left(\frac{\ln x \cdot y \cdot u}{z x} + \frac{u}{x}\right) = \frac{-yu(z + y \ln x)}{x \cdot z^3}$

№3229

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

а) $u = x^2 - 2xy - 3y^2$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(2x - 2y) = -2$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(\cancel{-2xy} - 2x - 6y) = -2$$

б) $u = x^{y^2}$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(y^2 \cdot x^{y^2-1}\right) = \cancel{2} \cdot 2y \cdot x^{y^2-1} + y^2 \cdot \ln x \cdot x^{y^2-1} \cdot 2y$$

$$\cancel{\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\ln x \cdot x^{y^2} \cdot 2x\right) = \frac{\partial}{\partial x}\left(2 \ln x \cdot x^{y^2+1}\right) = \frac{2}{x} \cdot x^{y^2+1} + 2\ln x \cdot x^{y^2} \cdot x = (y^2+1)\, x^{y^2} \cdot \ln x =}$$

$$\cancel{= 2x^{y^2} + y^2 \ln x \cdot x^{y^2} + x^{y^2} \ln x =}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\ln x \cdot x^{y^2} \cdot 2y\right) = 2y \cdot \left(x^{y^2-1} + \ln x \cdot y^2 \cdot x^{y^2-1}\right)$$

~~чтд~~

в) $u = \arccos\sqrt{\frac{x}{y}}$, если $\frac{x}{y} \geq 0$:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{1}{\sqrt{1-\frac{x}{y}}} \cdot \frac{1}{2\sqrt{x \cdot y}}\right) = \frac{\partial}{\partial y}\left(-\frac{1}{2\sqrt{x(y-x)}}\right) =$$

$$= \frac{1}{4\sqrt{x}\,(y-x)^{3/2}}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{\sqrt{1-\frac{x}{y}}}\left(-\frac{\sqrt{x}}{2y^{3/2}}\right)\right) = \frac{\partial}{\partial x}\left(\frac{\sqrt{x}}{2\sqrt{y^2(y-x)}}\right) = \frac{1}{4\sqrt{x(y^2(y-x))}} + \frac{\sqrt{x}}{4y(y-x)^{3/2}} =$$

$$= \frac{1}{4\sqrt{x}\,(y-x)^{3/2}}$$

для $\begin{cases} x \geq y \\ x < 0 \end{cases}$ проверяется все аналогично, только везде знаки „–"

№3245(а)

а) $1{,}002 \cdot 2{,}003^2 \cdot 3{,}004^3$

$\Delta f(x_0, y_0, z_0) \approx df(x_0, y_0, z_0)$

$f(x,y,z) = xy^2z^3$

$x_0 = 1 \quad \Delta x = 0{,}002$

$y_0 = 2 \quad \Delta y = 0{,}003$

$z_0 = 3 \quad \Delta z = 0{,}004$

$f = (x_0 + \Delta x) y_0^2 z_0^3 + x_0 z_0^3 (y_0 + \Delta y)^2 + x_0 y_0^2 (z_0 + \Delta z^3) =$

$= xy^2z^3 \left(1 + \frac{\Delta x}{x}\right)\left(1 + \frac{\Delta y}{y}\right)^2\left(1 + \frac{\Delta z}{z}\right)^3 =$

$= \underline{108{,}972}$

Через дифференциал:

$f(x_0 + \Delta x, y_0 + \Delta y, z + \Delta z_0) \approx f(x_0, y_0, z_0) +$

$+ \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z = 108 + \Delta x y^2 z^3 + 2yxz^3 \Delta y + \Delta z \, 3z^2 x y^2 =$

$= 108 + 0{,}972 = 108{,}972$

№3245(в) $\sqrt{1{,}02^3 + 1{,}97^3}$

$x_0 = 1 \quad \Delta x = 0{,}02$

$y_0 = 2 \quad \Delta y = -0{,}03$

$f(x,y) = \sqrt{x^3 + y^3}$

$f(x + \Delta x, y + \Delta y) \approx 3 + \frac{\Delta x \cdot 3x^2}{2\sqrt{x^3+y^3}} + \frac{\Delta y \cdot 3y^2}{2\sqrt{x^3+y^3}} =$

$= 3 + \frac{1}{2} \cdot 3 \cdot \frac{1}{\sqrt{x^3+y^3}} \left(\Delta x x^2 - \Delta y y^2\right) = 3 - \frac{1}{20} = 2{,}95$

№3248 Д-ть, что отн-ая погр-ть произв-ия $\approx$ сумме отн-ых погрешностей сомножителей

Пусть $g(x,y) = xy \qquad dg = x\,dy + y\,dx$

$\frac{dg}{xy} = \frac{dy}{y} + \frac{dx}{x} \Rightarrow \frac{dg}{g} = \frac{dx}{x} + \frac{dy}{y}$

$\left|\frac{dg}{g}\right| \leq \left|\frac{dx}{x}\right| + \left|\frac{dy}{y}\right|$ (нер-во треугольника)

$\sqrt{x}$

$4y(y-x)^{3/2}$

№3267

$u = f(xyz)$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = F(t)$$

$t = xyz$ и найти $F(t)$

$$\frac{\partial u}{\partial x} = u'_x = f'(xyz)(xyz)' = f'(xyz)\, yz$$

$$\frac{\partial u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(f'(xyz)\, yz\right) = yz f''(xyz) \cdot xz + z f'(xyz) = xyz^2 f''(t) + z f'(t)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial z}\left(xyz^2 f''(t) + z f'(t)\right) = x^2y^2z^2 f'''(t) + 2xyz f''(t) +$$

$$+ f'(t) + xyz f''(t) = x^2y^2z^2 f^{(3)}(t) + 3xyz f''(t) + f'(t) =$$

$$= t^2 f^{(3)}(t) + 3t f''(t) + f'(t) = F(t)$$

№3283

$u = f(x^2+y^2+z^2)$

$$\frac{\partial u}{\partial x} = 2x \cdot f'(x^2+y^2+z^2) \qquad \frac{\partial^2 u}{\partial x^2} = 2f'(x^2+y^2+z^2) + 4x^2 f''(x^2+y^2+z^2)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(2x f'(x^2+y^2+z^2)\right) = 4xy f''(x^2+y^2+z^2)$$

№3270 $u = \sin(x^2+y^2)$ d^3u - ?

$du = 2x\cos(x^2+y^2)dx + 2y\cos(x^2+y^2)dy = 2\cos(x^2+y^2)(x\,dx+y\,dy)$.

$d^2u = -4\sin(x^2+y^2)(x\,dx+y\,dy)^2 + 2\cos(x^2+y^2)(dx^2+dy^2)$

$d^3u = -8\cos(x^2+y^2)(x\,dx+y\,dy)^3 - 8\sin(x^2+y^2)(x\,dx+y\,dy)(dx^2+dy^2) -$
$- 4\sin(x^2+y^2)(x\,dx+y\,dy)(dx^2+dy^2) = -8(x\,dx+y\,dy)^3\cos(x^2+y^2) -$
$-12(x\,dx+y\,dy)(dx^2+dy^2)\sin(x^2+y^2)$

№3284: $u = f(x, \frac{x}{y})$

$u'_y = f'_1(x, x/y)(x)'_y + f'_2(x, x/y)(x/y)'_y$, $(x)'_y = 0$

$\approx 2{,}35$

f_i - произв. по i перем

$$\frac{\partial u}{\partial x} = f'_1(x; \frac{x}{y}) + f'_2(x, \frac{x}{y}) \cdot \frac{1}{y}; \quad \frac{\partial u}{\partial y} = -\frac{x}{y^2} f'_2(x, \frac{x}{y})$$

$$\frac{\partial^2 u}{\partial x^2} = f''_{11}(x, \frac{x}{y}) + \frac{2}{y} f''_{12}(x, \frac{x}{y}) + \frac{1}{y^2} f''_{22}(x, \frac{x}{y})$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{x}{y^2} f''_{12}(x, \frac{x}{y}) - \frac{1}{y^2} f'_{12}(x, \frac{x}{y}) - \frac{x}{y^3} f''_{22}(x, \frac{x}{y})$$

$y^2 + z$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2x}{y^3} f'_{2}(x, \frac{x}{y}) + \frac{x^2}{y^4} f''_{22}(x, \frac{x}{y})$$