AD Hub Swaption Pricing and Analysis

EN.553.628 Stochastic Processes and Applications to Finance II

Final Project

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Contents

1 Introduction

1.1 Introduction to Swaptions

Swaptions, or options to enter into a swap agreement at a future date, represent a cornerstone of financial derivatives used in risk management and speculative strategies. These options are highly customizable and can be tailored to suit a vast array of financial needs, accommodating various underlying assets including interest rates, currencies, and commodities.

The flexibility offered by swaptions is invaluable in financial markets characterized by significant uncertainty and fluctuation. The academic and practical approaches to understanding and utilizing swaptions have been shaped by an extensive range of research focusing on both theoretical valuation models and empirical market behaviors.

Pioneering work by Hull and White introduced frameworks for pricing interest rate derivatives, which have been adapted and extended to other types of swaptions [5]. Similarly, the Monte Carlo methods popularized by Clewlow and Strickland for energy derivatives provide crucial techniques for dealing with the complex, stochastic nature of energy prices [1].

Recent studies have further refined our understanding of these instruments. The insights provided by Galeeva and Thomas on parameterized calendar correlations illuminate the intricate dynamics of commodity markets [2]. Recent studies, such as those by Galeeva exploring volatility smiles in oil futures in 2020 [4], and research into measuring correlation risk in energy derivatives [3], add depth to our understanding of these complex financial instruments.

1.2 Swaptions in Commodity Markets

Commodity swaptions serve a critical role in the financial strategies of companies involved in the production, trading, or consumption of physical commodities. These options enable firms to hedge against future price uncertainties, thereby ensuring financial stability in environments prone to abrupt price changes due to geopolitical, environmental, or economic developments.

In-depth studies and continuous research in the field offer advanced insights and methodologies for valuing these options accurately. The integration of models that consider market volatilities, correlation factors, and even the nuanced 'Samuelson effect', which observes that the volatility of futures prices increases as the maturity date approaches, are essential for sophisticated pricing models in today's financial markets.

These contributions have not only enhanced our theoretical understanding but have also significantly improved the practical applications of swaptions in commodity markets, providing market participants with better tools for financial planning and risk management.

2 Research Objectives

2.1 Derivative Product Description

Our study concentrates on AD Hub Electricity Swaptions, which are specifically designed financial instruments for the electricity market in the AD Hub region. The key features of these derivative products are:

- **Underlying Assets:** The derivatives are based on baskets of 12 futures for each month of the calendar years 2025, 2026, and 2027.
- **Evaluation Date:** The base date for our analysis is March 5, 2024.
- **Payment Schedule:** Payments for each future within the swap occur on the 20th of the month following the contract month. For instance, payments for January 2025 futures are exchanged on February 20, 2025.
- **Strike Price:** The strike price K is set at-the-money (ATM), equivalent to the swap price on the evaluation date.
- **Volume:** Each month's futures within the swap have a uniform volume of 1000 MWh.
- **Expiration Dates:** The swaptions expire on December 20 of each year for 2024, 2025, and 2026.

The payoff for a commodity swaption, which hinges on the future prices of the underlying commodity, can be modeled with the following equation:

$$\text{Payoff} = \max\left(\frac{\sum_i df_i \cdot N_i \cdot F_i(T)}{\sum_i df_i \cdot N_i} - K, 0\right) \tag{1}$$

This formula encapsulates the essence of swaptions by focusing on the expected future prices $F_i(T)$, the discount factors df_i, the volumes N_i, and the strike price K. It illustrates how the value of the option is derived from the weighted average of the future prices adjusted by their respective discount factors and volumes, subtracting the predetermined strike price.

2.2 Research Directions

Our research will utilize and assess various methodologies for accurately pricing AD Hub Electricity Swaptions. The methodologies and analytical approaches include:

- **Pricing Methods:** Utilization of **Monte Carlo** simulations and **Moment Matching** techniques to handle the pricing complexities of swaptions.
- **Calculation of Greeks:** Computing the Greeks to measure the sensitivity of the swaption prices to changes in market variables. Specifically, we calculate the delta for each month, reflecting how swaption prices are sensitive to movements in the underlying asset prices.
- **Market Effects:**
  - Consideration of the "Samuelson effect," which predicts increasing volatility as futures contracts approach maturity.
  - Calibration of volatility to reflect actual market conditions, ensuring that our models are both realistic and relevant.
  - Incorporation of seasonal impacts, crucial for accurately pricing electricity derivatives due to variable demand across different times of the year.

Through these methodologies, our research aims to develop robust models for pricing and analyzing the risks associated with electricity swaptions, providing insights into the dynamics of electricity derivatives markets.

3 Dataset Description

In our research, we utilize a comprehensive dataset that includes several key metrics vital for the pricing and analysis of AD Hub Electricity Swaptions. Below is a detailed description of the data currently available:

- **Electricity Futures Prices:**
  - Our dataset contains the On Peak electricity futures prices for AD Hub, spanning from April 2024 through December 2027.
  - These prices serve as the basis for our swaption pricing models.
- **Implied Volatility:**

- We have data on the at-the-money (ATM) implied volatility for futures from April 2024 to March 2026.
- Implied volatility is crucial for understanding market expectations of future price fluctuations and is a vital input for our derivative pricing models. This data will also be instrumental in analyzing the Samuelson effect, which predicts an increase in volatility as maturity approaches.

Note on Data Limitations: Although our dataset includes implied volatility data up to March 2026, we require this data through December 2027 to complete our analysis, including a thorough examination of the Samuelson effect over the entire study period. In the subsequent section, we will discuss our approach to extrapolating the missing volatility data to ensure our models remain robust and comprehensive.

4 Mapping Market Implied Volatilities to Earlier Expirations using Samuelson

4.1 Introduction to the Samuelson Effect

The Samuelson effect, named after economist Paul Samuelson, elucidates the observable increase in volatility of commodity prices as their futures contracts approach maturity. This phenomenon is a crucial aspect of commodity markets, reflecting the inherent uncertainties that decrease as the time to delivery shortens.

In practical terms, the Samuelson effect influences the correlation dynamics among commodity prices over time. By modeling these dynamics, analysts can effectively adapt correlation structures for any designated time interval. This adaptability proves particularly valuable in the pricing of commodity swaptions, enabling a more accurate reflection of market behaviors and risk assessments.

The effectiveness of utilizing the Samuelson effect in financial models for commodity derivatives is well-documented. It enhances the precision of pricing strategies by aligning the model's output closer to actual market conditions, thereby providing a robust framework for financial analysis and decision-making in volatile markets.

4.2 Derivation of Relevant Variable

Objective: To derive the initial volatility parameter ($\sigma_{0,i}$) and the adjusted volatility for a specific interval ($\sigma_{i,[t_1,t_s]}$) using known parameters: volatility decay factor (B), start time (t_1), swaption expiry time (t_s), futures expiration time (T_i), and implied volatility (σ_{imp}).

Step 1: Calibration of $\sigma_{0,i}$

To convert the implied volatility, σ_i, which is realized over T_i, to the shorter time period t_s, we use the Samuelson form of instantaneous volatility:

$$\sigma_{i,\text{inst}}(t) = \sigma_{0,i} e^{-B(T_i - t)} \tag{2}$$

First, we need to calibrate $\sigma_{0,i}$ to match the market-implied volatility. We calculate the total realized variance from today, t_1, to the expiry of the option (contract delivery), T_i:

$$\text{Var}_{0,T} = \int_{t_1}^{T_i} \left(\sigma_{0,i} e^{-B(T_i - s)}\right)^2 ds = \sigma_{\text{imp}}^2 (T_i - t_1) \tag{3}$$

Integral Simplification:

$$\sigma_{0,i}^2 \int_{t_1}^{T_i} e^{-2B(T_i - s)}\, ds$$

Using the integral of exponential:

$$\begin{aligned} \text{Var}_{0,T} &= \sigma_{0,i}^2 \int_{t_1}^{T_i} e^{-2B(T_i - s)}\, ds \\ &= \sigma_{0,i}^2 \left[\frac{1}{2B} e^{-2B(T_i - s)}\right]_{t_1}^{T_i} \\ &= \sigma_{0,i}^2 \frac{1}{2B} (1 - e^{-2B(T_i - t_1)}) \\ &= \sigma_{\text{imp}}^2 (T_i - t_1) \end{aligned} \tag{4}$$

Solving for $\sigma_{0,i}$:

$$\sigma_{0,i} = \sigma_{imp} \sqrt{\frac{2B(T_i - t_1)}{1 - e^{-2B(T_i - t_1)}}} \tag{5}$$

This setup effectively calibrates the initial volatility parameter $\sigma_{0,i}$ according to the Samuelson effect and aligns it with market data.

Step 2: Calculating Variance for Swaptions ($\sigma_{i,[t_1,t_s]}$)

After calibrating $\sigma_{0,i}$ for each contract on the swap underlying the swaption, we can calculate the variance realized between today and the swaption expiration:

$$\mathrm{Var}_i^{[t_1,t_s]} = \int_{t_1}^{t_s} (\sigma_i^{\mathrm{inst}}(s))^2 \, ds \tag{6}$$

and the volatility to be used in Monte Carlo (MC) simulations and Moment Matching Calculation:

$$\sigma_i^{[t_1,t_s]} = \sqrt{\frac{\mathrm{Var}_i^{[t_1,t_s]}}{t_s - t_1}} \tag{7}$$

Calculating for interval $[t_1, t_s]$:

$$\mathrm{Var}_i^{[t_1,t_s]} = \int_{t_1}^{t_s} \left(\sigma_{0,i} e^{-B(T_i - s)}\right)^2 ds \tag{8}$$

Simplify the integral:

$$\mathrm{Var}_i^{[t_1,t_s]} = \sigma_{0,i}^2 \frac{1}{2B} \left(e^{-2B(T_i - t_s)} - e^{-2B(T_i - t_1)}\right) \tag{9}$$

Extracting the volatility:

$$\sigma_{i,[t_1,t_s]} = \sigma_{0,i} \sqrt{\frac{e^{-2B(T_i - t_s)} - e^{-2B(T_i - t_1)}}{2B(t_s - t_1)}} \tag{10}$$

This comprehensive approach allows for the accurate mapping of market implied volatilities to earlier expirations, critical for pricing AD Hub Electricity Swaptions under varying market conditions and timeframes.

4.3 Expand Incomplete Value

Our implied volatility data is available from April 2024 to March 2026. However, based on the above calculations, we need implied volatility for all months from 2025 to 2027 for calibration. Therefore, we need to find a method to simulate the missing data.

4.3.1 Assumption of Seasonal Characteristics

Considering the seasonal characteristics of electricity options, we assume that each month has a σ_0 and that the σ_0 for the same month across different years is identical. Thus, for the months with missing data, we can use the same σ_0 as the corresponding month with available data, and then proceed with the calculation of variance for swaptions.

To support our assumption, we can use the complete data to calculate σ_0 for each future using different values of B through Equation (5). Our goal is to find the best B that minimizes the difference in σ_0 for the same month across different years.

4.3.2 Calculation of σ_0 for Different Values of B and Selection of the Best B

The complete data contains two full years of futures data. We calculate σ_0 for $B = 0.1, 0.15, 0.2, \ldots, 0.9, 0.95, 1$, and compute the MSE of σ_0 for the same month across different years. The results are shown in the figure below.

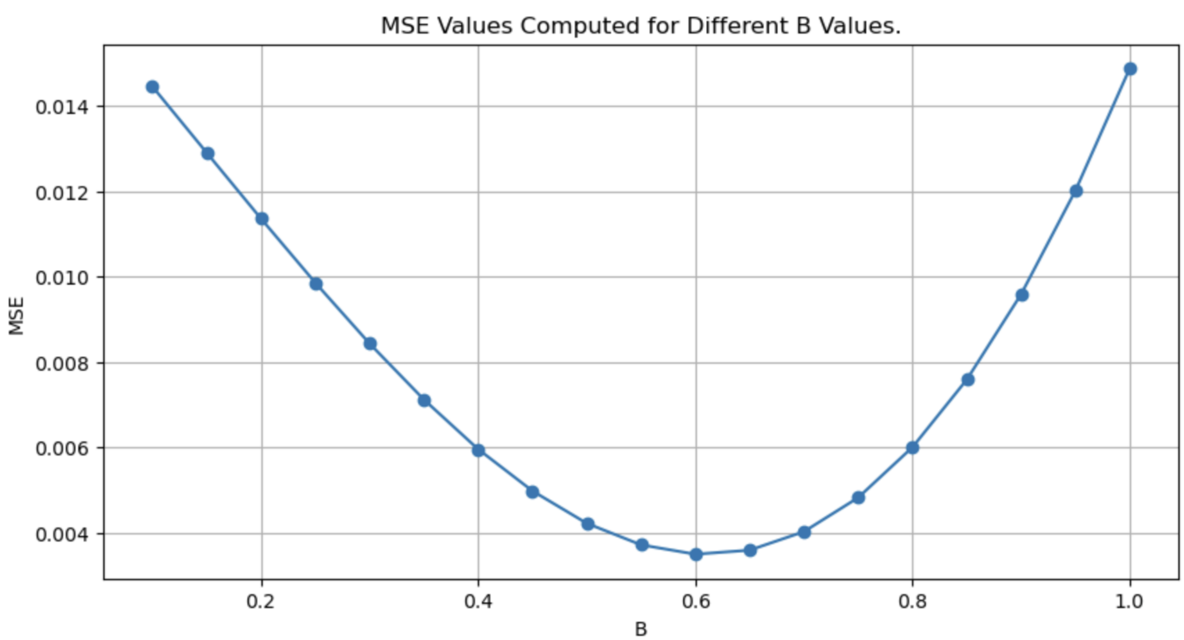

Figure 1: MSE of σ_0 for the same month across different years for various values of B

When $B = 0.6$, the MSE is minimized, so we choose the best B as 0.6.

4.3.3 Extending Incomplete Values

For the missing data, we extend σ_0 to match the corresponding σ_0 calculated from the complete data for the same month, and then calculate the variance for swaptions ($\sigma_{i,[t_1,t_s]}$) using σ_0 according to Equation (10).

4.4 Displaying Mapped Implied Volatility

Based on previous calculations, we have determined the Mapped Implied Volatility for each month from 2025 to 2027 for AD Hub electricity futures, also known as the variance for swaptions ($\sigma_{i,[t_1,t_s]}$). The figure below shows the Original Implied Volatility and the Mapped Implied Volatility for each month.

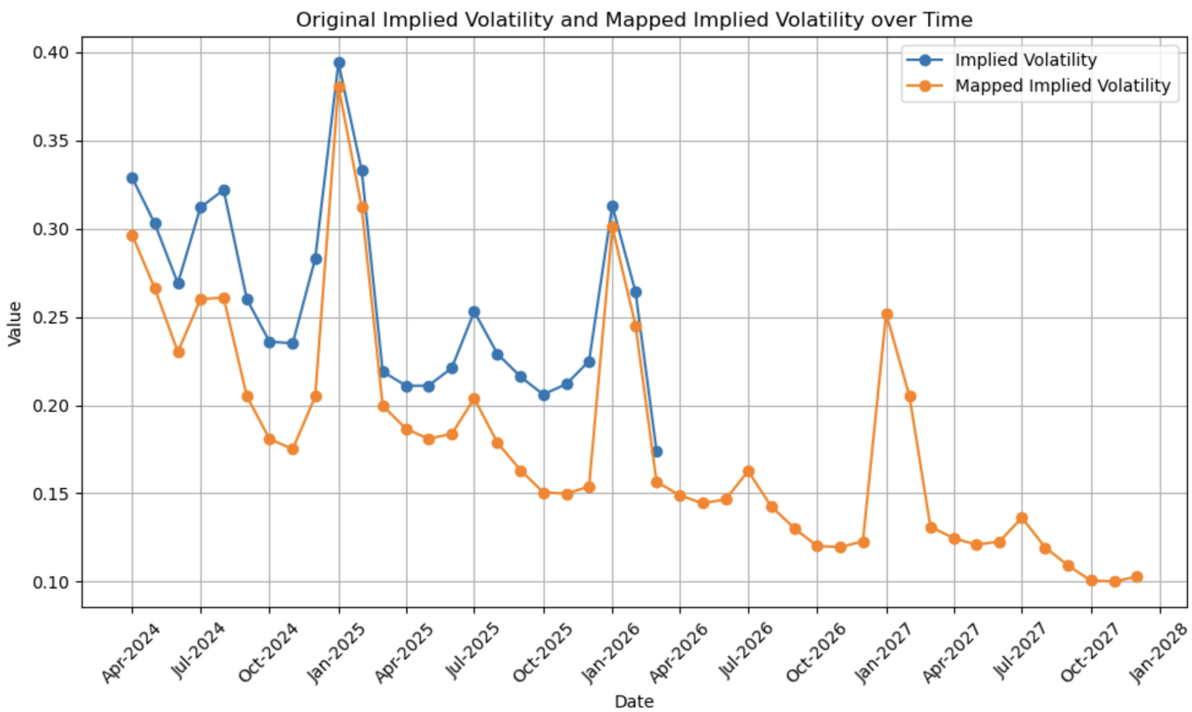

Figure 2: Original Implied Volatility and Mapped Implied Volatility for each month

As shown in the figure, the Implied Volatility of electricity futures exhibits clear seasonal characteristics. The Implied Volatility tends to increase during the winter months due to higher electricity demand, leading to increased uncertainty and volatility in the market. This seasonal effect is consistent across the years and can be clearly seen in the figure.

It can also be observed that the Mapped Implied Volatility is lower than the Original Implied

Volatility.

5 Swaption Pricing

5.1 Set Up for MM & MC Calculation

5.1.1 Correlation Matrix

The correlation matrix, 12 by 12, could come from a parameterization, which needs to reflect accordingly the properties of calendar correlation. Here, we assume that the correlation decay follows an Angular Decay Form, which means that each element of the correlation matrix is calculated according to the following formula[2]:

$$C(\beta, T_i, T_j) = \sin\left(\frac{\pi}{2}\exp^{-\beta(T_i - T_j)}\right), \beta > 0 \tag{11}$$

For our calculation, we set $\beta = 0.5$.

5.1.2 Interest Rate

As of March 5, 2024, we are pricing three swaptions with futures maturing in 2025, 2026, and 2027. The interest rates used for pricing these swaptions are the Treasury rates for different maturities as of March 5, 2024:

Expiry Year	Interest Rate	Description
2025	0.0505	1 Year Treasury Rate (I:1YTCMR)
2026	0.0472	2 Year Treasury Rate (I:2YTCMR)
2027	0.0451	3 Year Treasury Rate (I:3YTCMR)

Table 1: Interest Rates as of 2024-03-05 from `ycharts.com/indicators`

The above information is obtained from YCharts.

5.2 Valuing Swaption using Monte Carlo

5.2.1 Monte Carlo Framework

A Monte Carlo simulation method provides a versatile approach for valuing derivatives, especially when analytical solutions are not feasible. This method is particularly useful when multiple random factors are present, as is often the case with energy derivatives. A correlation matrix is required to account for the relationships between these random factors.

A typical swaption might be an option to enter a swap for the next calendar year, involving a basket of 12 underlying contracts, which could have equal or different weights.

The correlation matrix needs to accurately reflect the properties of calendar correlation, as calculated in the previous subsection 5.1.1.

The volatilities of the underlying futures are determined by mapping the market implied volatilities to the swaption expiration, as calculated in chapter 4.

Here is how to simulating correlated commodity futures:

- To simulate correlated commodity futures, we start with the description of Monte Carlo simulation for one underlying future:

$$F(t_s, T_i) = F(t_1, T_i) \exp\left(-\frac{1}{2}\left(\sigma_i^{[t_1,t_s]}\right)^2 (t_s - t_1) + \sigma_i^{[t_1,t_s]}\sqrt{t_s - t_1}\, x_i\right) \tag{12}$$

  where $x_i \in N(0,1)$ is a random variable from the standard normal distribution.

- To create correlated random variables, we use the standard Cholesky decomposition for a correlation matrix C:

$$B^T B = C \tag{13}$$

  - Generate *NumSim* samples of independent random variables y_i^k, where $k = 1, \ldots, NumSim$ and $i = 1, \ldots, 12$.
  - Use the lower Cholesky decomposition and create a set of correlated standard normal variables x_i^k:

$$\vec{X} = B\vec{Y} \tag{14}$$

5.2.2 Model Test

To verify the accuracy of the Monte Carlo (MC) method, we provide model tests of the evaluation. By setting the strike price and interest rate to zero, the simulated swaption price should theoretically equal the current swap price. The following table shows the results:

Year	Simulated Swaption Price	Current Swap Price	Error (%)
2025	50.1389	50.1333	0.0110%
2026	53.2629	53.2625	0.0009%
2027	53.3204	53.4083	0.1648%

Table 2: Comparison of Simulated Swaption Prices and Current Swap Prices (K = 0; r = 0)

As shown in the table, the swaption prices are very close to the current swap prices, indicating that the model test is successful.

5.2.3 Simulation Results

We conducted 10,000 Monte Carlo simulations to calculate the swaption value V_{swpn}. This process was repeated 60 times to estimate the error of the Monte Carlo value. The following histograms display the frequency distribution of the swaption price for 2025, 2026, and 2027, respectively, over the 60 repetitions of 10,000 simulations each.

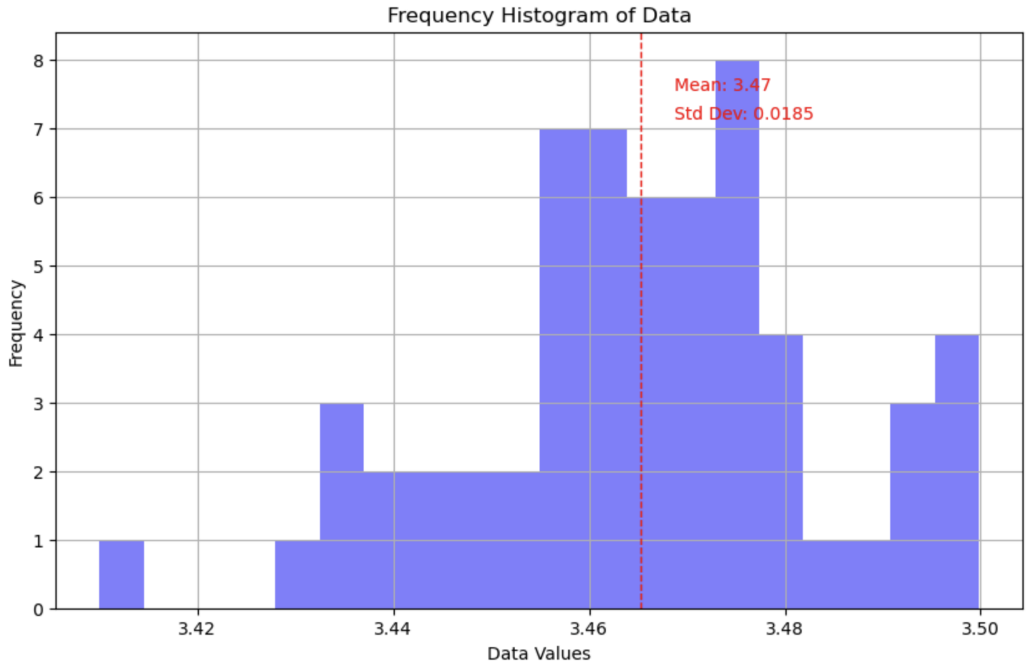

Figure 3: Histogram of 2025 Swaption Prices (60 repetitions of 10,000 simulations)

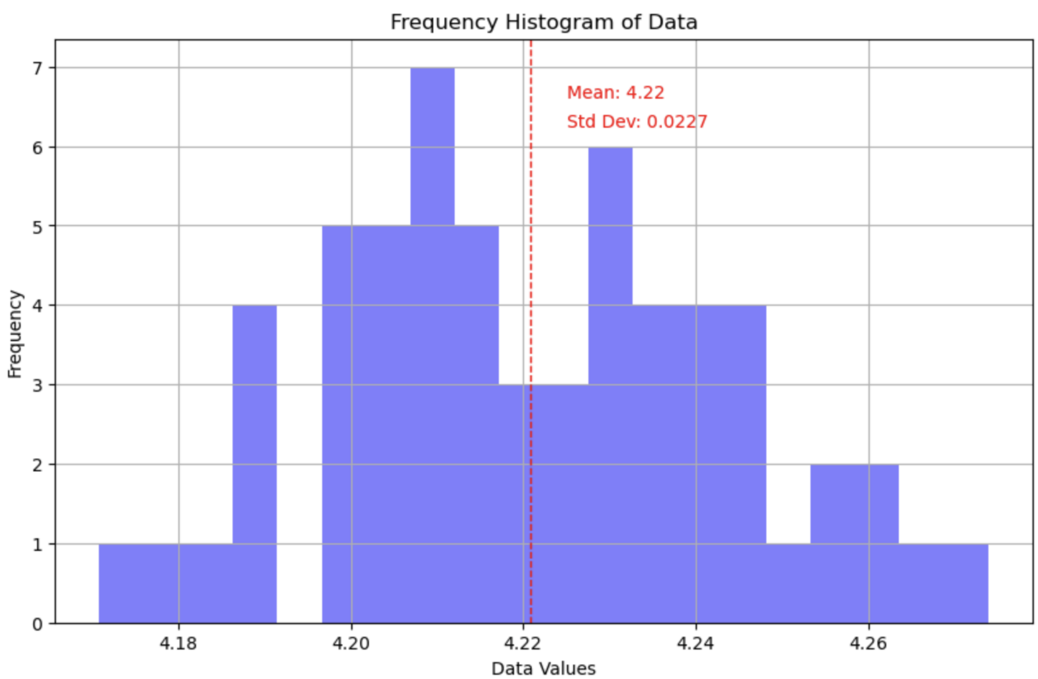

Figure 4: Histogram of 2026 Swaption Prices (60 repetitions of 10,000 simulations)

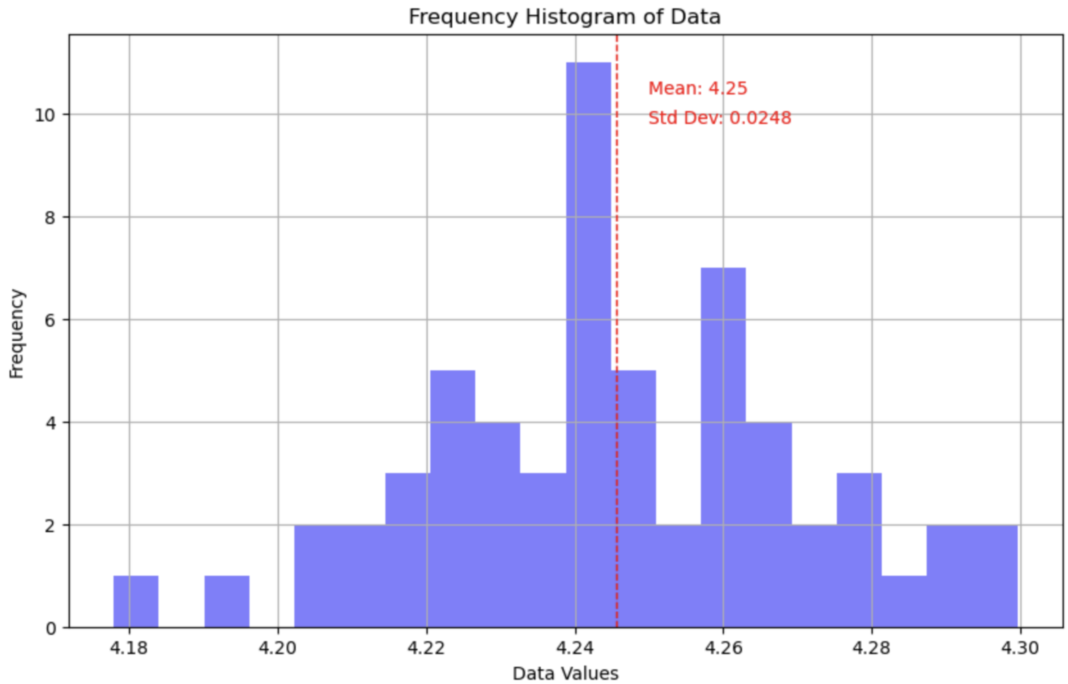

Figure 5: Histogram of 2027 Swaption Prices (60 repetitions of 10,000 simulations)

The results show stability in the swaption prices across the different years, indicating the reliability of the Monte Carlo simulation method for valuing swaptions.

5.3 Valuing Swaption using Moment Matching

5.3.1 Moment Matching Framework

Moment matching techniques provide a straightforward approach to approximating the value of swaptions, which is effective in most practical scenarios. For simplicity, we consider the case of 12 futures. The idea is to approximate the sum of 12 lognormal variables by another lognormal variable that matches two moments.

This method can be generalized to the case of N futures in the swap. Based on the monthly volumes and the discounts, we can calculate the weights ω_i as follows:

$$\omega_i = \frac{N_i DF(t, T_i)}{\sum_{j=1}^{N} N_j DF(t, T_j)} \tag{15}$$

where N_i are the monthly volumes and $DF(t, T_i^{\text{pay}})$ are the discounts.

The steps involved in the evaluation are as follows:

1. Calculate the expectation (the first moment M_1) of the sum of 12 futures:

$$M_1 = \sum_{i=1}^{12} \omega_i F_i \tag{16}$$

2. Calculate the second moment of the sum of the 12 lognormal prices:

$$M_2 = \sum_{i=1}^{12} \sum_{j=1}^{12} \omega_i \omega_j F_i F_j \left(e^{\rho_{ij} \sigma_i^* \sigma_j^* (t_s - t_1)} - 1 \right) \tag{17}$$

Where ρ_{ij} is the ij-th element of the correlation matrix, which is calculated in the previous subsection 5.1.1

σ_i^* are determined by mapping the market implied volatilities to the swaption expiration, as calculated in chapter 4.

3. Calculate the volatility:

$$\sigma^s = \sqrt{\frac{\log\left(\frac{M_2}{M_1^2} + 1\right)}{t_s - t_1}} \tag{18}$$

4. Calculate the option price with price M_1, volatility σ^s, and time to expiration $t_s - t_1$ using BS formula.

The Black-Scholes formula for calculating the price of a European call option is given by:

$$C_{price} = (M_1 \Phi(d_1) - K\Phi(d_2)) e^{-r(t_s - t_1)} \tag{19}$$

where

$$d_1 = \frac{\log\left(\frac{M_1}{K}\right) + \left(\frac{(\sigma^s)^2}{2}\right)(t_s - t_1)}{\sigma^s \sqrt{t_s - t_1}} \tag{20}$$

$$d_2 = d_1 - \sigma^s \sqrt{t_s - t_1} \tag{21}$$

Here, C_{price} is the call option price, M_1 is the expected price, K is the strike price, r is the risk-free interest rate, σ^s is the volatility, $t_s - t_1$ is the time to expiration, and Φ is the cumulative distribution function of the standard normal distribution.

5.3.2 Calculation Results

The following table presents the calculation results for the swaption prices and volatilities (σ^s) for the years 2025, 2026, and 2027 using the moment matching (MM) technique.

Year	Swaption Price (MM)	σ^s (MM)
2025	3.48	0.2033
2026	4.24	0.1624
2027	4.27	0.1361

Table 3: Calculation Results for Swaption Prices and Volatilities using Moment Matching

5.4 Comparison of Calculation Results (Monte Carlo vs. Moment Matching)

The following table presents the comparison of swaption prices calculated using the Monte Carlo and Moment Matching techniques for the years 2025, 2026, and 2027, along with the calculation of the error in percentage explicitly.

Year	Price (MC)	Price (MM)	Error (%)
2025	3.47	3.48	0.29%
2026	4.22	4.24	0.47%
2027	4.25	4.27	0.47%

Table 4: Comparison of Swaption Prices using Monte Carlo and Moment Matching Techniques

As shown in the table, the difference in swaption prices obtained by the Monte Carlo and Moment Matching techniques is very small, highlighting the effectiveness and reliability of both methods in practical applications.

6 Greek Analysis

6.1 Calculating Delta with Numerical Method

Delta (Δ) measures the sensitivity of the swaption's value to changes in the forward prices of the underlying assets. The numerical "bump and reval method" is a straightforward technique to compute the Delta by slightly altering the forward prices and recalculating the swaption's value to observe the change. This method provides a practical approach to estimate how small changes in market conditions affect the price of the derivative.

The Delta with respect to the forward price of the i-th future (F_i) can be calculated using the following formula:

$$\Delta_i = \frac{\partial V_{\text{swpn}}}{\partial F_i} \approx \frac{V_{\text{swpn}}(F_i + \epsilon) - V_{\text{swpn}}(F_i - \epsilon)}{2\epsilon} \tag{22}$$

where:

- $V_{\text{swpn}}(F_i + \epsilon)$ is the value of the swaption when the forward price F_i is increased by a small amount ϵ.
- $V_{\text{swpn}}(F_i - \epsilon)$ is the value of the swaption when the forward price F_i is decreased by ϵ.
- ϵ is a small increment chosen to minimize the approximation error without causing numerical instability.

This method is particularly useful when an analytical expression for the derivative's sensitivity is difficult to derive or when the model does not easily lend itself to sensitivity analysis. By using this approach, traders and risk managers can gain insights into how changes in forward prices impact the financial derivative's market value, enabling more informed hedging and trading strategies.

6.2 Calculating Delta with Analytical Method

We can also calculate the Delta (Δ) values from the formula of swaption value we got from Moment Matching method.

From the Moment Matching method, we have:

$$M_1 = \sum_{i=1}^{12} \omega_i F_i,$$

$$M_2 = \sum_{k=1}^{12}\sum_{j=1}^{12}\omega_k\omega_j F_k F_j B_{kj} = \sum_{k=1}^{12}\omega_k^2 F_k^2 B_{kk} + \sum_{k=1}^{12}\sum_{j=1,j\neq k}^{12}\omega_k\omega_j F_k F_j B_{kj},$$

where: $B_{kj} = e^{\rho_{kj}\sigma_k^*\sigma_j^*(t_s-t_1)} - 1$. Therefore, we could get following partial derivatives to the i-th future F_i of M_1 and M_2:

$$\frac{\partial M_1}{\partial F_i} = \omega_i, \tag{23}$$

$$\begin{aligned}\frac{\partial M_2}{\partial F_i} &= 2\omega_i^2 F_i B_{ii} + \sum_{k=1,k\neq i}^{12}\omega_k\omega_i F_k B_{ki} + \sum_{j=1,j\neq i}^{12}\omega_j\omega_i F_j B_{ij}\\ &= 2\omega_i^2 F_i B_{ii} + \sum_{j=1,j\neq i}^{12}\omega_j\omega_i F_j (B_{ij}+B_{ji})\\ &= \sum_{j=1}^{12}\omega_j\omega_i F_j (B_{ij}+B_{ji}).\end{aligned} \tag{24}$$

When calculating the $sigma_s$ which will be used in the BS model, we use the following formula:

$$\sigma_s = \sqrt{\frac{log(\frac{M_2}{M_1^2}+1)}{t_s-t_1}}.$$

From the equation above, we could get

$$\begin{aligned}\frac{\partial\sigma_s}{\partial M_1} &= \frac{1}{2}\left(\frac{log(\frac{M_2}{M_1^2}+1)}{t_s-t_1}\right)^{-\frac{1}{2}}\frac{1}{t_s-t_1}\frac{M_1^2}{M_2+M_1^2}\frac{-2M_2}{M_1^3}\\ &= -\frac{1}{\sigma_s(t_s-t_1)}\frac{1}{M_1(1+\frac{M_1^2}{M_2})},\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\partial\sigma_s}{\partial M_2} &= \frac{1}{2}\left(\frac{log(\frac{M_2}{M_1^2}+1)}{t_s-t_1}\right)^{-\frac{1}{2}}\frac{1}{t_s-t_1}\frac{M_1^2}{M_2+M_1^2}\frac{1}{M_1^2}\\ &= \frac{1}{2\sigma_s(t_s-t_1)}\frac{1}{M_2(1+\frac{M_1^2}{M_2})},\end{aligned} \tag{26}$$

Using chain rule, we could easily get

$$\begin{aligned}\frac{\partial\sigma_s}{\partial F_i} &= \frac{\partial\sigma_s}{\partial M_1}\frac{\partial M_1}{\partial F_i} + \frac{\partial\sigma_s}{\partial M_2}\frac{\partial M_2}{\partial F_i}\\ &= \frac{1}{2\sigma_s(t_s-t_1)}\frac{-2\frac{\partial M_1}{\partial F_i}M_2 + \frac{\partial M_2}{\partial F_i}M_1}{M_1M_2(1+\frac{M_1^2}{M_2})},\end{aligned} \tag{27}$$

When calculating the value of swaption, we use the Black-Scholes formula:

$$V_{\text{swpn}} = e^{-r(t_s-t_1)}[M_1N(d_1) - KN(d_2)], \tag{28}$$

where N represents the culmulative probability function, $d_1 = \frac{log(\frac{M_1}{K})+\frac{1}{2}\sigma_s^2(t_s-t_1)}{\sigma_s\sqrt{t_s-t_1}}$, and $d_2 = d_1 - \sigma_s\sqrt{t_s-t_1}$. Then we could get following derivatives:

$$\frac{\partial d_1}{\partial M_1} = \frac{\partial d_2}{\partial M_1} = \frac{1}{M_1\sigma_s\sqrt{t_s-t_1}}, \tag{29}$$

$$\begin{aligned}\frac{\partial d_1}{\partial \sigma_s} &= \frac{\sigma_s^2(t_s-t_1)^{\frac{3}{2}} - \sqrt{t_s-t_1}[log(\frac{M_1}{K}) + \frac{1}{2}\sigma_s^2(t_s-t_1)]}{\sigma_s^2(t_s-t_1)} \\ &= -\frac{log(\frac{M_1}{K})}{\sigma_s^2\sqrt{(t_s-t_1)}} + \frac{1}{2}\sqrt{t_s-t_1},\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{\partial d_2}{\partial \sigma_s} &= \frac{-\sigma_s^2(t_s-t_1)^{\frac{3}{2}} - \sqrt{t_s-t_1}[log(\frac{M_1}{K}) + \frac{1}{2}\sigma_s^2(t_s-t_1)]}{\sigma_s^2(t_s-t_1)} \\ &= -\frac{log(\frac{M_1}{K})}{\sigma_s^2\sqrt{(t_s-t_1)}} - \frac{1}{2}\sqrt{t_s-t_1}.\end{aligned} \tag{31}$$

Using the chain rule and incorporating Eq.(27)(29)(30)(31), we get

$$\frac{\partial d_1}{\partial F_i} = \frac{\partial d_1}{\partial M_1}\frac{\partial M_1}{\partial F_i} + \frac{\partial d_1}{\partial \sigma_s}\frac{\partial \sigma_s}{\partial F_i}, \tag{32}$$

$$\frac{\partial d_2}{\partial F_i} = \frac{\partial d_2}{\partial M_1}\frac{\partial M_1}{\partial F_i} + \frac{\partial d_2}{\partial \sigma_s}\frac{\partial \sigma_s}{\partial F_i}. \tag{33}$$

From the BS formula Eq.(28), we could find

$$\frac{\partial V_{\text{swpn}}}{\partial M_1} = N(d_1)e^{-r(t_s-t_1)}, \tag{34}$$

$$\frac{\partial V_{\text{swpn}}}{\partial d_1} = M_1 p(d_1)e^{-r(t_s-t_1)}, \tag{35}$$

$$\frac{\partial V_{\text{swpn}}}{\partial d_2} = -Kp(d_2)e^{-r(t_s-t_1)}. \tag{36}$$

The Delta with respect to the forward price of the i-th future (F_i) can be calculated using the chain rule:

$$\begin{aligned}\Delta_i &= \frac{\partial V_{\text{swpn}}}{\partial F_i} \\ &= \frac{\partial V_{\text{swpn}}}{\partial M_1}\frac{\partial M_1}{\partial F_i} + \frac{\partial V_{\text{swpn}}}{\partial d_1}\frac{\partial d_1}{\partial F_i} + \frac{\partial V_{\text{swpn}}}{\partial d_2}\frac{\partial d_2}{\partial F_i}.\end{aligned} \tag{37}$$

We could find all the intermediate variable in the Delta equation (Eq.(37)) from Eq.(23) (36) to calculate Δ_i

6.3 Comparative Analysis of Delta using Numerical and Analytical Methods

We compare the Delta values for swaptions across different months for the years 2025, 2026, and 2027 calculated using both numerical and analytical methods. The results help us understand the precision and effectiveness of each method under varying market conditions.

Delta Comparison for 2025

Date	Delta Analytical	Delta Numerical	Error (%)
Jan-25	0.048812	0.048442	0.76%
Feb-25	0.046733	0.046456	0.59%
Mar-25	0.043387	0.043164	0.51%
Apr-25	0.042860	0.042638	0.52%
May-25	0.042538	0.042319	0.51%
Jun-25	0.042431	0.042214	0.51%
Jul-25	0.042807	0.042595	0.50%
Aug-25	0.041888	0.041677	0.50%
Sep-25	0.041239	0.041027	0.51%
Oct-25	0.040684	0.040470	0.53%
Nov-25	0.040434	0.040220	0.53%
Dec-25	0.040309	0.040096	0.53%

Table 5: Comparison of Delta values for 2025 using Analytical and Numerical Methods

Delta Comparison for 2026

Date	Delta Analytical	Delta Numerical	Error (%)
Jan-26	0.048020	0.047669	0.73%
Feb-26	0.045618	0.045399	0.48%
Mar-26	0.041863	0.041713	0.36%
Apr-26	0.041418	0.041270	0.36%
May-26	0.041093	0.040946	0.36%
Jun-26	0.041013	0.040872	0.34%
Jul-26	0.041485	0.041351	0.32%
Aug-26	0.040476	0.040340	0.34%
Sep-26	0.039777	0.039637	0.35%
Oct-26	0.039184	0.039039	0.37%
Nov-26	0.038940	0.038796	0.37%
Dec-26	0.038837	0.038697	0.36%

Table 6: Comparison of Delta values for 2026 using Analytical and Numerical Methods

Delta Comparison for 2027

Date	Delta Analytical	Delta Numerical	Error (%)
Jan-27	0.046426	0.045983	0.95%
Feb-27	0.044026	0.043709	0.72%
Mar-27	0.040273	0.040002	0.67%
Apr-27	0.039841	0.039567	0.69%
May-27	0.039527	0.039251	0.70%
Jun-27	0.039461	0.039184	0.70%
Jul-27	0.039946	0.039674	0.68%
Aug-27	0.038948	0.038666	0.72%
Sep-27	0.038261	0.037966	0.77%
Oct-27	0.037680	0.037373	0.81%
Nov-27	0.037448	0.037139	0.83%
Dec-27	0.037357	0.037047	0.83%

Table 7: Comparison of Delta values for 2027 using Analytical and Numerical Methods

6.4 Analysis of Delta

The Delta values in the tables show a high level of accuracy between the numerical and analytical methods, with error percentages generally below 1%. This close alignment suggests that both methods are reliable for practical use.

In practice, Delta is used to hedge risk. A Delta close to 0.05, as seen in the calculations for various months, indicates that for a one-unit change in the underlying asset's price, the swaption's value will change by approximately 0.05 units. This information is crucial for traders when constructing a hedging strategy to mitigate potential losses from adverse price movements.

For example, a trader holding a portfolio with a Delta of 0.05 can hedge against price changes by taking an offsetting position in the underlying asset. If the price of the underlying asset increases by one unit, the swaption's price will increase by 0.05 units, offsetting the loss from the underlying position.

Understanding Delta's practical significance and the accuracy of different calculation methods is crucial for effective risk management and informed decision-making in financial markets. The minimal error percentages confirm the robustness of the numerical method, making it a valuable tool when analytical solutions are challenging or infeasible.

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