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Relations and Functions (continued)

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Sets, relations, functions

- Powerset is the type constructor for sets of elements
- Cartesian product is the type constructor for pairs of elements
- A relation is a set of pairs
  - Domain and range of a relation
  - Relational image
  - Restriction and subtraction
- A function is a special case of a relation
  - Many-to-one: each domain element mapped to a unique range element
  - Partial function, function application
  - Function override
  - Total functions
- Relational inverse
- Relational composition
- Relation operators apply to functions – with caution!

Total Functions

A total function is a special kind of partial function. To declare f as a total function:

$$\boxed{f \;\in\; X \rightarrow Y}$$

This means that f is well-defined for every element in X, i.e., $f \;\in\; X \rightarrow Y$ is shorthand for

$$f \;\in\; X \nrightarrow Y \quad \wedge \quad dom(f) = X$$

Birthday Book Example

Birthday book relates people to their birthday.

Each person has one birthday.

People can share birthdays.

sets $\mathit{PERSON} \quad \mathit{DATE}$

variables $\mathit{birthday}$

invariants $\mathit{birthday} \in \mathit{PERSON} \nrightarrow \mathit{DATE}$

initialisation $\mathit{birthday} := \{\}$

Modelling with Total functions

We can re-write the invariant for the birthday book to use total functions:

variables $birthday, person$
invariants

$$person \subseteq PERSON$$
$$birthday \in person \to DATE$$

Using the total function arrow means that we don't need to explicitly specify that $dom(birthday) = person$.

We can use $person$ as a guard instead of $dom(birthday)$:

$$Check \;\hat{=}\; \textbf{any } p, result \textbf{ where } \quad p \in person \quad result = birthday(p) \quad \textbf{end}$$

AddEntry needs to be modified

Add an entry to the directory:

$$
\begin{array}{lcl}
AddEntry & \triangleq & \textbf{any } p, d \textbf{ where} \\
& & \quad p \in PERSON \\
& & \quad p \notin person \\
& & \quad d \in DATE \\
& & \textbf{then} \\
& & \quad birthday := birthday \cup \{p \mapsto d\} \\
& & \quad person := person \cup \{p\} \\
& & \textbf{end}
\end{array}
$$

Relational Inverse

Given $R \in S \leftrightarrow T$, the relational inverse of R is written $\boxed{R^{-1}}$

Predicate	Definition
$y \mapsto x \in R^{-1}$	$x \mapsto y \in R$

Example:

$$directory = \{\ mary \mapsto 287573,\ mary \mapsto 398620,\ jim \mapsto 398620\ \}$$

$$directory^{-1} = \{\ 287573 \mapsto mary,\ 398620 \mapsto mary,\ 398620 \mapsto jim\ \}$$

$$directory^{-1}[\ \{398620\}\ ] = \{\ mary,\ jim\ \}$$

Inverse Queries

Return all the people associated with a number in the directory:

$$
\begin{array}{lll}
GetNames & \widehat{=} & \textbf{any } n, result \textbf{ where} \\
& & \quad n \in PhoneNum \\
& & \quad result = dir^{-1}[\ \{n\}\ ] \\
& & \textbf{end}
\end{array}
$$

Return all the people associated with a set of numbers:

$$
\begin{array}{lll}
GetMultiNames & \widehat{=} & \textbf{any } ns, result \textbf{ where} \\
& & \quad ns \subseteq PhoneNum \\
& & \quad result = dir^{-1}[\ ns\ ] \\
& & \textbf{end}
\end{array}
$$

Function inverse

Check birthdays on a particular date:

$$
\begin{array}{lll}
\mathit{Who} & \hat{=} & \textbf{any } d, \mathit{result} \textbf{ where} \\
 & & \quad d \in \mathit{Date} \\
 & & \quad \mathit{result} = \mathit{birthday}^{-1}(d) \\
 & & \textbf{end}
\end{array}
$$

- Is this mathematically valid?

Function inverse

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$$
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Who & \triangleq & \textbf{any } d, result \textbf{ where} \\
 & & \quad d \in Date \\
 & & \quad result = birthday^{-1}(d) \\
 & & \textbf{end}
\end{array}
$$

- Is this mathematically valid?
- No: $birthday^{-1}$ might not be a function.

Function inverse

$birthday^{-1}$ is a relation:

$$birthday^{-1} \quad \in \quad Date \leftrightarrow Person$$

Check birthdays on a particular date:

$$\begin{aligned} Who \quad \triangleq \quad & \textbf{any } d, result \textbf{ where} \\ & \quad d \in Date \\ & \quad result = birthday^{-1}[\{d\}] \\ & \textbf{end} \end{aligned}$$

Alternative:

$$\begin{aligned} Who \quad \triangleq \quad & \textbf{any } d, result \textbf{ where} \\ & \quad d \in Date \\ & \quad result = dom(birthday \triangleright \{d\}) \\ & \textbf{end} \end{aligned}$$

Relational Composition

Given $Q \in S \leftrightarrow T$ and $R \in T \leftrightarrow U$,
the relational composition of Q and R is written $\boxed{Q \; ; R}$

We have that $Q \; ; R \in S \leftrightarrow U$

Predicate	Definition
$x \mapsto z \in (Q \; ; R)$	$\exists y \cdot x \mapsto y \in Q \wedge y \mapsto z \in R$

Example:

$$
\begin{aligned}
M &= \{ a \mapsto l, b \mapsto m, c \mapsto n \} \\
N &= \{ l \mapsto 4, n \mapsto 6, p \mapsto 8 \} \\
M \; ; N &= \; ?
\end{aligned}
$$

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Composition and Image

Given $Q \in S \leftrightarrow T$ and $R \in T \leftrightarrow U$ and $A \subseteq S$

$$(Q \,;\, R)[\, A \,] \;=\; R[\, Q[\, A \,]\,]$$

Example:

$$\begin{aligned} M &= \{\, a \mapsto l, b \mapsto m, c \mapsto n \,\} \\ N &= \{\, l \mapsto 4, n \mapsto 6, p \mapsto 8 \,\} \end{aligned}$$

$$(M \,;\, N)\ [\ \{a, b\}\ ] \;=\; N[\ M[\ \{a, b\}\ ]\ ]\ \ ?$$

Composition and Image

Given $\quad Q \in S \leftrightarrow T \quad$ and $\quad R \in T \leftrightarrow U \quad$ and $\quad A \subseteq S$

$$(Q \,;R)[\, A\,] \;=\; R[\, Q[\, A\,]\,]$$

Example:

$$\begin{aligned} M &= \{\, a \mapsto l, b \mapsto m, c \mapsto n \,\} \\ N &= \{\, l \mapsto 4, n \mapsto 6, p \mapsto 8 \,\} \end{aligned}$$

$$(M \,; N)\, [\, \{a, b\}\,] \;=\; (\{\, a \mapsto 4, c \mapsto 6 \,\})\, [\, \{a, b\}\,] \;=\; \{4\}$$

$$N[\, M[\, \{a, b\}\,]\,] \;=\; N[\{l, m\}] \;=\; \{4\}$$

Extend directory with friends

variables $dir, friend$

invariants

$$
\begin{array}{lcl}
friend & \in & Person \leftrightarrow Person \\
dir & \in & Person \leftrightarrow PhoneNum
\end{array}
$$

Return the telephone numbers of all friends of p:

$$
\begin{array}{l}
GetFriendNumbers \;\; \hat{=} \\
\quad \textbf{any } p, result \textbf{ where} \\
\qquad p \in Person \\
\qquad result = (friend;dir)[\ \{p\}\ ] \\
\quad \textbf{end}
\end{array}
$$

Function Operators

All the relational operators can be used on functions (restriction, subtraction, image, composition, etc).

Be careful with some operators!

Assume that f and g are functions.

- Set Union: $f \cup g$ is a function provided

$$x \in \mathrm{dom}(f) \ \wedge \ x \in \mathrm{dom}(g) \quad \Longrightarrow \quad f(x) = g(x)$$

  Why?

- Inverse: f^{-1} is not always a function as we have seen.
- Composition? Is $f ; g$ also a function?