

# Changes in mortality and child survival for mothers

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## Abstract

### Background

The expected number of descendants for mothers is an important relation in kinship theory. Child survival (and child death) affect mothers in a variety of ways, yet it is not clear how changes in mortality schedules affect the number of surviving children from the standpoint of mothers.

### Objective

Existing formal demographic tools are used within a stable population framework to characterize child survival from the point of view of a mother, considering both absolute and relative changes in mortality patterns.

### Results

The expected change in child survival given an absolute change in mortality depends on the difference between the age of a mother and the age structure of her surviving children. The effects of absolute changes in mortality are explained using a fertility-weighted measure of entropy.

### Contribution

The paper proposes new expressions to characterize the effects of changes in mortality on child survival. The different measures it introduces can be applied to populations at different stages of the demographic transition.

## 1. Relationship

$CS_a$  is the expected number of surviving children to a woman aged  $a$  in a stable female population with fertility rates  $m_x$  and force of mortality  $\mu_x$ :

$$CS_a = \int_{\alpha}^a m_x l_{a-x} dx. \quad (1)$$

We now consider the consequences of a change  $\delta$  in mortality in the range  $[0, a - \alpha]$ , where  $\alpha$  is the start age of fertility risk, as proposed by Goodman, Keyfitz, and Pullum (1974), and  $\delta$  can be absolute or relative.

A discrete approximate of the effect on  $CS_a$  given an absolute change in mortality  $m_{x,\delta} = m_x + \delta$  is:

$$\frac{\Delta CS_a}{CS_a} \approx -(a - \bar{x}_{(CS)}) \Delta \delta. \quad (2)$$

In this equation, the expected change in child survival is inversely proportional to the difference between maternal age  $a$  and the mean age of the mother at the birth of her surviving daughters  $\bar{x}_{(CS)}$ . In other words, the magnitude of the change depends negatively on the age distribution of the surviving offspring (as younger children experience longer periods of exposure to risk).

The relationship is less clear for proportional changes in mortality  $m_{x,\delta} = m_x(1 + \delta)$  given the interaction of birth and mortality rates (Keyfitz and Caswell (2005)). Nevertheless we can say that, in discrete terms:

$$\frac{\Delta CS_a}{CS_a} \approx -H_{CS_a} \Delta \delta. \quad (3)$$

Where  $H_{CS_a}$  can be interpreted as a temporary entropy constant until age  $a - \alpha$  but weighted by the fertility pattern between [falta algo aqui?]. In this sense, depending the way in that the survival child is accumulated by age  $CS_a(x) = \frac{m_x l_{a-x}}{CS_a}$  is the way that is taken each age contribution to the disparity measure  $H_{CS_a} = \int_{\alpha}^a CS_a(x) H(x)$ . The effect in the child survival experienced by the mother will be bigger when the mortality is the same in all ages, no matter where the newborns are concentrated.

#> [1] "graph"

## 2. Proofs

### 2.1. Absolute change

Considering that  $m_{x,\delta} = m_x + \delta$  and  $l_{a-x} = e^{-\int_0^{a-x} (\mu_t + \delta) dt}$ :

$$CS_{(a)}^{\delta} = \int_{\alpha}^a m_x l_{a-x} e^{-\delta(a-x)} dx. \quad (4)$$

To find the effects on daughter survival of adding  $\delta$  to the age-specific death rates of daughters, we get the derivative of  $dCS_{(a)}^{\delta}/d\delta$ , evaluated near zero (Keyfitz & Caswell, 2005, section 4.3):

$$= -a \int_{\alpha}^a m_x l_{a-x} dx + \int_{\alpha}^a m_x l_{a-x} x dx. \quad (5)$$

Since  $CS_a = \int_{\alpha}^a m_x l_{a-x} dx$ , we can rewrite equation (5) as

$$\frac{dCS_a^{\delta}}{d\delta} = -aCS_a + \int_{\alpha}^a x m_x l_{a-x} dx. \quad (6)$$

Dividing both sides by  $CS_a$  and multiplying by  $d\delta$ , we get

$$\frac{dCS_a}{CS_a} = -(a - \bar{x}_{(CS)}) d\delta \quad (7)$$

where  $\bar{x}_{(CS)}$  is the mean age of women at the birth of their surviving daughters in a stationary population.

### 2.2. Relative change

Considering that  $m_{x,\delta} = m_x(1 + \delta)$  and  $l_{a-x,\delta} = e^{-\int_0^{a-x} \mu_t(1+\delta) dt} = l_{a-x}^{(1+\delta)}$ :

$$CS_a^{\delta} = \int_{\alpha}^a m_x l_{a-x}^{(1+\delta)} dx. \quad (8)$$

Using the derivative of  $dCS_{(a)}^{\delta}/d\delta = \log(l_{a-x}) l_{a-x}^{(1+\delta)}$  and evaluating  $\delta$  near zero we get the final expression:

$$\frac{dCS_a}{CS_a} = \frac{\int_{\alpha}^a m_x l_{a-x} \log(l_{a-x}) dx}{\int_{\alpha}^a m_x l_{a-x}} d\delta \quad (9)$$

Using that  $CS_a(x) = \frac{m_x l_{a-x}}{CS_a}$  is a weighting function and making an analogy with the classical entropy function  $H$  where the weights are just the survival function, we can define this information function  $H_{CS_a} = \int_{\alpha}^a CS_a H(x)$ , and get:

$$\frac{dCS_a}{CS_a} = -H_{CS_a} d\delta. \quad (10)$$

### 3. History

Kin count estimation has a long pedigree in mathematical demography. Lotka (1931) proposed to model the orphanhood in a theoretical population, comparing the effect across demographic regimes. Equation 1 was originally proposed by Brass (1953) to estimate child mortality. The notion of estimating the expected number of living daughters in a stable population was generalized by Goodman, Keyfitz, and Pullum (1974) for other kin relations (granddaughters, cousins, etc.). The “counting method” approach, being the net reproduction rate  $R_0$  limited to  $a$ , or the (not complete) ratio between generations (multiplying and dividing by the mother generation  $B$ ), was further popularized by Keyfitz and Caswell (2005). Bongaarts (1987) used a similar approach to estimate descendants in his ‘Family Status Model’. More recently, Wrycza and Baudisch (2012) has looked at the formal effect of different kind of changes in mortality by age in a paper published in this series.

### 4. Related measures

#### 4.1. Time with children lost

This section proposes two additional measures of child survival for a mother aged  $a$ .  $MTSL$  is an absolute measure of the *Mean Time Spent with Lost*, which can be expressed in terms of a temporary expected lost years measure  $e^\dagger$ :

$$MTSL_a = \int_{\alpha}^a m_x \int_0^{a-x} d_t e_{t,a-x} dt dx = \int_{\alpha}^a m_x e_{o,a-x}^\dagger dx. \quad (11)$$

It is reasonable to expect a mother to relate these lost years with the expected ones, feeling the *intensity* of her losses (*Intensity Time Lost*). This makes it possible to compare between fertility regimes:

$$ITL_a = \frac{\int_{\alpha}^a m_x e_{o,a-x}^\dagger dx}{\int_{\alpha}^a m_x e_{o,a-x} dx} \quad (12)$$

#### 4.2. Mean Age at Child Loss

Different populations could have different mean age at lost of mothers with death child:

$$MAL_a = K_a + \int_{\alpha}^a f_{x,a} MAD_{a-x} \quad (13)$$

$K_a$  is the mean age at birth for womens aged  $a$ ,  $MAD_{a-x}$  refers to the mean age at death for those newborns who dies before  $a - x$ ,  $TFR_a$  is the cumulated fertility for a women aged  $a$ , and  $f_{x,a}$  is the cumulative distribution of fertility until age  $a$ .

Equation 13 can be derived by starting with the integral of the age  $x + t$  at each death child weighted by fertility and survival function:

$$\begin{aligned}
MAL_a &= \frac{\int_{\alpha}^a m_x \int_0^{a-x} \frac{d_t(x+t)dt}{\int_0^{a-x} d_t dt}}{\int_{\alpha}^a m_x} \\
MAL_a &= \frac{\int_{\alpha}^a m_x x}{TFR_a} + \frac{\int_{\alpha}^a m_x MAD_{a-x}}{TFR_a} \\
MAL_a &= K_a + \frac{\int_{\alpha}^a m_x MAD_{a-x}}{TFR_a} \\
MAL_a &= K_a + \int_{\alpha}^a f_{x,a} MAD_{a-x}
\end{aligned} \tag{14}$$

## 5. Applications

The proposed measures are applied to two different demographic regimes.

[PENDING]

A computation approach is used to evaluate the relative contribution of each component given a relative change in  $\delta$ .

[PENDING]

## 5. Bibliography

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