Mortality change and child survival from the point of view of a prospective mother

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Abstract

Individuals are poor arbiters of macro-level demographic change. For example, it would be difficult for a given woman to infer the existence of a population-level decline in infant mortality from her own experience of child survival alone. Here, we establish the relationship between population-level demographic change and child survival for mothers in a stable population framework. We build on existing formal demographic work, mainly the Goodman, Keyfitz and Pullum kinship equations, to characterize the effect of a change in all-age mortality or mortality at first age on child survival. After establishing the nature of the relationship, we introduce two related measures to characterize numerically the experience of maternal bereavement. Finally, we use empirical data from Latin America to exemplify our findings and to show that our proposed method yields acceptable approximations.

Introduction

The question of child survival for mothers sits at the very center of demographic theory. Offspring survival is usually studied in the context of the demographic transition, but it also matters for post-transitional societies. Lower fertility and higher life expectancy means that fewer children are expected to provide key emotional, social, and financial transfers to aging parents for longer periods of time. With increasing periods of generational overlap, individuals find themselves 'sandwiched' between aging parents and young children requiring their simultaneous attention and care (Daatland, Veenstra, and Lima 2010). In the context of global population aging, elderly parents without access to formal social security and pension systems are particularly reliant on these transfers (Smith-Greenaway and Trinitapoli 2020). The demographic processes that shape parental bereavement remain an important but unexplored field of enquiry.

How do changes in mortality affect the availability of offspring over age from the point of view of a prospective mother? In this brief paper, we aim to formalize the relationship between population-level changes in mortality rates (additive or multiplicative) and changes in the lived experience of child survival in a stable framework. This is key for understanding the impact of age-specific excess mortality on the resilience or otherwise of kinship networks for the older population. We also introduce measures to describe the intensity of bereavement and the mean age at child loss. The last section of the paper provides numerical Applications using data from Latin America in the second half of the twentieth century. Our work contributes to a long and fruitfully tradition in the mathematical demography of kinship modelling, starting with the fundamental work of Lotka (1931) on orphanhood in theoretical populations, Brass (1953) on child survival, Goodman, Keyfitz, and Pullum (1974) on kin survival in stable populations, and recently Caswell (2019) in matrix framework.

Relationships

Child Survival

Let CS_a be the expected number of surviving children to a mother alive at aged a in a female stable population with fertility rates m_x, mortality hazard μ_x and survival function $l_x = e^{-\int_0^x \mu_t\, dt}$ (with unit radix $l_0 = 1$), as proposed by Goodman, Keyfitz, and Pullum (1974):

$$CS_a = \int_0^a m_x l_{a-x} dx$$

An approximation

As an initial step, we seek an intuitive understanding of the previous relation. Building on work by Keyfitz and Caswell (2005) for the probability of a living mother, we can show which features explains the expected number of children. For this we use an approximate of l_x using Taylor´s theorem until second order around the mean age of childbearing κ:

$$\begin{aligned} CS_a \approx & l_{a-\kappa} \int_0^a m_x dx + (l_{a-\kappa})' \int_0^a (x-\kappa) m_x dx + (l_{a-\kappa})'' \int_0^a \frac{(x-\kappa)^2}{2} m_x dx \\ & \approx F_a l_{a-\kappa} + \frac{\sigma^2}{2} F_a (l_{a-\kappa})'' \\ & \approx F_a\, l_{a-\kappa} [1 + \frac{\sigma^2}{2} \frac{(l_{a-\kappa})''}{l_{a-\kappa}}] \end{aligned}$$

Here the fertility pattern by age is concentrated around κ and the accumulated fertility (or gross reproduction rate in our female-dominant scenario) is $F_a = \int_\alpha^a m_x dx$. The second Taylor´s term is null because $\int_\alpha^a x\, m_x dx = \kappa\, F_a$. We find that, seen from the perspective of a mother, child survival mainly depends on the cumulative fertility function and the survival of daughters from birth to $a - \kappa$. The approximation is affected negatively by the dispersion of fertility over age (variance σ^2) and the (negative) curvature of the survival curve in an age range where l_x is typically very flat in transitioned populations (range 20-40 years old)[1].

Mortality Changes

We now consider the consequences of an absolute change δ in mortality in the range $[0, a-\alpha]$, where α is the start age of fertility risk. Let us consider $m_{x,\delta} = m_x + \delta$ (Wrycza and Baudisch (2012)) and $l^\delta_{a-x} = e^{-\int_0^{a-x} (\mu_t + \delta) dt}$:

$$CS^\delta_a = \int_\alpha^a m_x l_{a-x} e^{-\delta(a-x)} dx$$

We get the derivative of $dCS^\delta_{(a)}/d\delta$ evaluated near zero (Keyfitz and Caswell (2005)) to find the effects of adding δ to hazard rates:

$$\begin{aligned} \frac{dCS^\delta}{d\delta} &= -a \int_\alpha^a m_x l_{a-x} dx + \int_\alpha^a x m_x l_{a-x} dx \\ &= -a\, CS_a + \int_\alpha^a x\, m_x l_{a-x} dx \end{aligned}$$

Dividing both sides by CS_a in a discrete approximation we get:

[1] This approximation is useful also for get an idea of impact because of changes in the fertility average age κ, keeping constant the level: taking logs in $F_a\, l_{a-\kappa}$ and deriving we get $\frac{\Delta CS_a}{CS_a} \approx \mu_{a-\kappa} \Delta\kappa$, being daughters exposed to less time because of the delay in pattern.

$$\frac{\Delta CS_a}{CS_a} \approx -(a-k_a)\Delta\delta$$

The expected change in descendants survival is inversely proportional to the difference between maternal age a and the mean age of the mother at the birth of her surviving daughters k_a (different than κ_a for total fertility until age a). The magnitude of the change depends negatively on the age distribution of the surviving offspring (the younger the fertility bigger the impact), which is intuitive considering that older descendants experiences longer periods of exposure to risk. This change applies only to surviving offspring, like an excess mortality effect.

However, it is unlikely for mortality to change at the same rate at all ages. In the course of the demographic transition, for example, we would expect larger changes in young-age mortality at first. Consider a change δ_0 in infant mortality in the age range $[0;1)$. We can inspect the effect in child survival $CS_a^{\delta_0}$ by splitting the integral between those daughters who were exposed to the mortality change all the year and those who were not (now generalizing for α positive and $a > 1$ for simplicity):

$$\begin{aligned} CS_a^{\delta_0} &= \int_0^{a-1} m_x e^{-\int_1^{a-x}\mu_t\,dt - \int_0^1(\mu_t+\delta)\,dt}dx + \int_{a-1}^{a} m_x e^{-\int_0^{a-x}(\mu_t+\delta)\,dt}dx \\ &= \int_0^{a-1} m_x l_{a-x} e^{-\delta}dx + \int_{a-1}^{a} m_x l_{a-x} e^{-\delta(a-x)}\,dx \end{aligned}$$

Deriving by δ and valuating near 0, we get:

$$\frac{dCS^{\delta_0}}{d\delta_0} = -\int_0^{a-1} m_x l_{a-x}dx - a\int_{a-1}^{a} m_x l_{a-x}dx + \int_{a-1}^{a} x\, m_x l_{a-x}dx$$

Defining the expected living daughters born in last year $CS_{a-1,a} = \int_{a-1}^{a} m_x l_{a-x}dx$, we can express the first term in the right as $CS_a - CS_{a-1,a}$. The righ-most term could also be expressed as $k_{a-1,a}CS_{a-1,a}$, being $k_{a-1,a}$ mean age at childbearing that last year, so that:

$$\frac{dCS^{\delta_0}}{d\delta_0} = -[CS_a - CS_{a-1,a}] - aCS_{a-1,a} + k_{a-1,a}CS_{a-1,a}$$

$$\frac{dCS^{\delta_0}}{d\delta_0} = -CS_a + CS_{a-1,a}(1 - a + k_{a-1,a})$$

The factor $1-(a-k_{a-1,a})$ lies between 0 and 1, and means the average portion of time spent with living daughters during age *a-1*. Assuming that all births happen at the middle point of each age, discretizing the change and dividing by CS_a we get the final expression for an absolute change in infant mortality:

$$\frac{\Delta CS_a^{\delta_0}}{CS_a} \approx -\left[1 - \frac{CS_{a-1,a}}{CS_a}\frac{1}{2}\right]\Delta\delta_0$$

We find that the effect of a decline in infant hazard rates is proportional at all ages, except for "half" the portion $\frac{CS_{a-1,a}}{CS_a}$ the mother's last fertility year. The degree to which this will matter for a given woman will be a function of the relative contribution of this last year to the number o living offspring of a woman. If a is near β or older (no additional daughters in last year) then $\frac{\Delta CS_a^{\delta_0}}{CS_a} \approx -\Delta\delta_0$.

Given the known pattern of mortality risk by age, could be more useful evaluate changes in relative terms. We now consider the consequences of a proportional change in mortality on child survival. Given that $\mu_{x,\delta} = \mu_x(1+\delta)$ (Wrycza and Baudisch, 2012), then $l_{a-x}^{\delta} = e^{-\int_0^{a-x}\mu_t(1+\delta)dt} = (l_{a-x})^{(1+\delta)}$. As a result:

$$CS_a^\delta = \int_0^a m_x l_{a-x}^{(1+\delta)} dx.$$

Using the derivative $\frac{dl_{a-x}^{(1+\delta)}}{d\delta} = log(l_{a-x})l_{a-x}^{(1+\delta)}$, and in the third row reversing integrals between t and x gets:

$$\begin{aligned}\frac{dCS_a^\delta}{d\delta} &= \int_0^a m_x l_{a-x} \log(l_{a-x}) dx \\ &= -\int_0^a m_x l_{a-x} \int_0^{a-x} \mu_t \, dt \, dx \\ &= -\int_0^a \mu_t \int_0^{a-t} m_x l_{a-t-x} \, dx \, dt\end{aligned}$$

Considering that the last integral is equal to CS_{a-t}, we divide by CS_a and multiply by a discrete small change in δ to obtain:

$$\frac{\Delta CS_a}{CS_a} \approx -\left[\int_0^a \mu_x \frac{CS_{a-x}}{CS_a} dx\right] \Delta\delta$$

This is a change in the negative of cumulative hazard $H_a = \int_0^a \mu_x \, dx$ but considering a positive factor $\frac{CS_{a-x}}{CS_a}$ that takes in account the relative amount of surviving descendants that would be lost at each child age at risk. This gives more weight to first ages because children of all parities were exposed at that age.

Grasping this relationship intuitively is more difficult given the interaction of birth and mortality rates (Keyfitz and Caswell (2005)). The factor $\frac{CS_{a-x}}{CS_a}$ has a S-shape because of the l_x curvature and fertility accumulation. To better understand that, we can analyze the difference between consecutive ages $a-1$ and a. We have $CS_{a-1} = \int_0^{a-1} m_x l_{x-a+1} \, dx$ and $CS_a = \int_0^{a-1} m_x l_{a-x} \, dx + CS_{a-1,a}$. The difference is explained by adding a second term in CS_{a-1} (children born after $a-1$) and accounting for offspring survival via $\frac{l_{a-x}}{l_{a-1-x}}$. In advanced ages when fertility is complete, the difference is only explained by the surviving part, that is why this factor is not a strictly increasing function (see Figure 1).

Again, if the relative change only affects daughters in their first year of life, and for simplicity a is β or older, we have:

$$\begin{aligned}CS_a^{\delta_0} &= \int_0^{a-1} m_x e^{-\int_1^{a-x} \mu_t \, dt - \int_0^1 \mu_t (1+\delta) \, dt} dx \\ &= \int_0^{a-1} m_x l_{a-x} \, l_1^\delta dx\end{aligned}$$

Note that the limit of the integral goes to $a-1$ instead of a, being the same if $m_x = 0$ in that last age, as the assumption was made. Applying logs and deriving by δ finally we get a discrete approximation on how this modify the expected survival daughter to a mother aged a a:

$$\frac{\Delta CS_a}{CS_a} \approx -H_1 \Delta\delta$$

In the next section we provide a numerical evaluation of both analytical expressions (absolute and relative) in the Applications section.

Related Measures

Burden and timing of maternal bereavement

The *Mean Time Spent in Bereavement* (*MTSB*) is a measure of the daughter person-years 'lost' to a mother because of the death of her daughter. It is the expected time that a mother aged a could have spend with

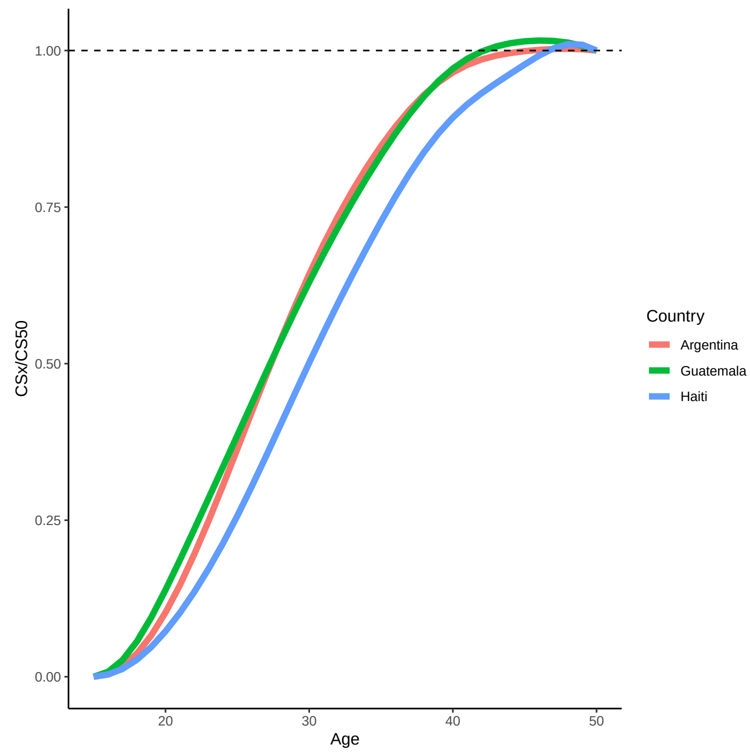

Figure 1: $\frac{CS_x}{CS_{50}}$ for Argentina, Guatemala and Haiti in 1950-1955. Weight factor for cumulative hazard in relative change case.

a living daughter if the daughter would have survived at each age. This can be expressed in terms of a temporary expected lost years index, in line with $e^{\dagger}$ (Vaupel (1986)):

$$MTSB_a = \int_0^a m_x \int_0^{a-x} d_t e_{0|a-x-t} \, dt \, dx = \int_0^a m_x e^{\dagger}_{0|a-x} \, dx$$

Here, d_t is the death distribution from birth, $e_{0|a-x-t}$ is the life expectancy at birth until age $a-x-t$ and $e^{\dagger}_{0|a-x}$ the temporary dispersion measure. A more informative measure compares these 'lost' years with the years that a mother should expect to live (and share) with their daughters. We call this the *Intensity Time in Bereavement* (ITB): a ratio between the expected bereavement time (in daughter person-years lost, as introduced above) and the expected daughters-time with living ancestors. This measure allows to make comparisons between population regimes.

$$ITL_a = \frac{\int_0^a m_x e^{\dagger}_{0|a-x} dx}{\int_0^a m_x e_{0|a-x} dx}$$

This is a ratio between child-years in two radically different states. It is similar to the transcendental entropy measure $\overline{H}$ (Keyfitz (2005)) but it considers all the cohorts born during the mother´s life, weighted by their relative size m_x. If we add the additional assumption of fertility concentration in the mean age like in An approximation, we can express this measure as $ITL_a \approx \overline{H}_{a-\kappa}$, the entropy measure but truncated to $a-\kappa$: more loss intensity she would experience when more life span inequality are between her daughters.

Another important factor for the child survival experience of mothers, is the mean age at child loss, called here MAL. This can be derived by starting with the mother age $x+t$ at each death child age t, weighted by the fertility and survival function. In it, κ is the mean age at childbirth for women aged a, MAD_{a-x} refers to the mean age at death for newborns that die before $a-x$, and F_a is the accumulated fertility for a women aged a:

$$
\begin{aligned}
MAL_a =& \frac{\int_0^a m_x \frac{\int_0^{a-x} l_t\, \mu_t (x+t) dt}{\int_0^{a-x} l_t\, \mu_t dt} dx}{\int_0^a m_x dx} \\
MAL_a =& \frac{\int_0^a m_x \left[x + \frac{\int_0^{a-x} l_t\, \mu_t\, t\, dt}{\int_0^{a-x} l_t\, \mu_t dt} \right] dx}{\int_0^a m_x} \\
MAL_a =& \frac{\int_0^a m_x\, x\, dx}{F_a} + \frac{\int_0^a m_x MAD_{a-x} dx}{F_a} \\
MAL_a =& \kappa + \frac{\int_0^a m_x MAD_{a-x} dx}{F_a}
\end{aligned}
$$

Following this, in populations with high infant mortality $MAD_{a-x} < 1$ for all x, so $MAL_a \approx \kappa_a + f_0$, being f_0 the average time spent for those child that dies in their first year of life. A numerical approximation of both measures is done in Applications section.

Applications

Data

We motivate this paper with an empirical example using fertility and mortality rates from the 2019 Revision of the UN World Population Prospects (UN WPP) for the Latin American Region (Nations (2019)). We smoothed female l_x in quinquennial ages, using cubic-splines constrained to monotonic decrease, taking L_0 and T_{100} from raw life tables as inputs for year-person calculations. For splitting fertility five groups was used quadratic optimization approach by Michalski and Gorlishchev (2018), with an desirable property for our purpose which is a good fitting in parity. Also was assumed an unique female percentage of newborns of 0.49 for all period-country cases. Calculations were done in a discrete way assuming that the m_x live births are born at exact mother´s age x. An already known log relation between infant mortality and TFR is shown in Latin American experience (Figure 2).

Numerical Results

Consider a Latin American woman standing before us on her 50^{th} birthday. If this were 1950-1955, she could expect to have 2.2 surviving daughters from a 2.9 parity, $\frac{CS_{50}}{F_{50}}$=.79. In 2015-2020 a woman in the same age would only have .98 living daughters from 1, with $\frac{CS_{50}}{F_{50}}$=.97. The difference is explained by reduced fertility and improved mortality in the region (a decrease in q_0 from .11 to .01, especially because a rapid demographic transition in populated countries like Brazil, that affects average).

We start by considering the approximation of child survival proposed $CS_a \approx F_a\, l_{a-\kappa}$, which is really precise, especially for recent periods. For all the region the relative error $\frac{aprox-empirical}{empirical}$ for age 30 goes from .8% to .005% in time range, and for age 50 goes from -.8% to -.08%. An example on this is the Guatemala case (red dots in figure 3), where this improve over time. This can be explained mostly by the rectangularization of l_x, a lower mortality in youth ages causing second derivatives close to zero in that range (graph D), than because a less dispersion in age fertility pattern (graph E).

The main goal of this work was to evaluate an analytical formulation for mortality change in child survival experience, for small changes under a stable population regime. We considered a range for δ from .0001 to .1. Figure 4 shows that goodness of fit decrease with change size, for absolute and relative change. Taking again the entire region for women aged 50, in the case of absolute change, an small increase of .0001 in all rates produces a not really meaningful change in CS_{50} of -.01 daughters, with an error of -3 percent, very similar for all period. In the case of $\delta = .01$ (it leads to $\frac{l_{50}}{e^{\delta\, 50}} = \frac{l_{50}}{1.65}$, a big change), there would be a change of -.4

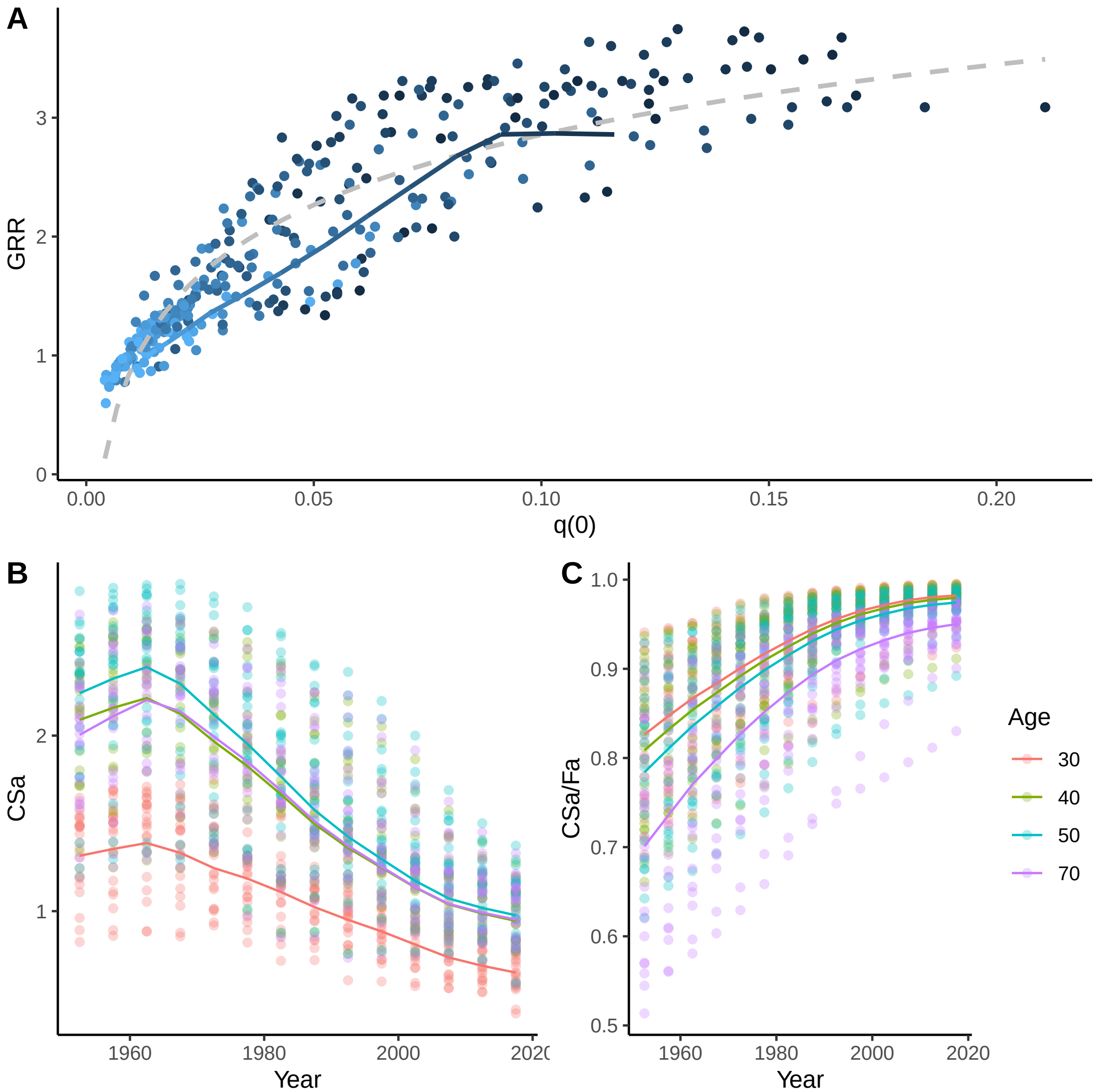

Figure 2: A) Female probability of death at birth by gross reproduction rate. Latin America countries in period 1950-2015 (color scale from dark to lighter with time). B) Child Survival for different age of mothers. C) Portion of alive parity by age of mother.

daughters in CS_{50} for the period 1950-1955 with an 8 percent of error, being this change of -.2 daughters in 2015-2020, with a similar error.

For the relative case calculation was assumed a constant rate within each interval $\mu_{x+t} = \mu_x$ for t between 0 and 1, $\int_x^{x+1} \mu_t dt = \log(l_{x+1}) - \log(l_x)$, and was used this approximation on a discrete way on the empirical data: $\frac{\Delta CS_a}{CS_a} \approx -\left[\sum_0^{a-1} [\log(l_{x+1}) - \log(l_x)] \frac{CS_{a-1-x}}{CS_a}\right] \Delta\delta$, following Mortality Changes section. The conclusion is the same but with better results than the absolute case, because the relative impact of δ on $l_x^{(1+\delta)}$ is lower and limited in 1, in comparison with the absolute case of $e^{\delta(a-x)}$ not limited.

Lastly, we consider ITB, the related measure introduced in Related Measures. Figure 5 shows that the time that a mother spend, on average, in a state of bereavement out of the total daughter person-years that she expect to enjoy were it not for offspring bereavement.As an extreme case in Latin America experience, women aged 30 in Haiti would have experienced an intensity of 4.8% in 1950-1955 and of 1.4% in 2010-2015. The

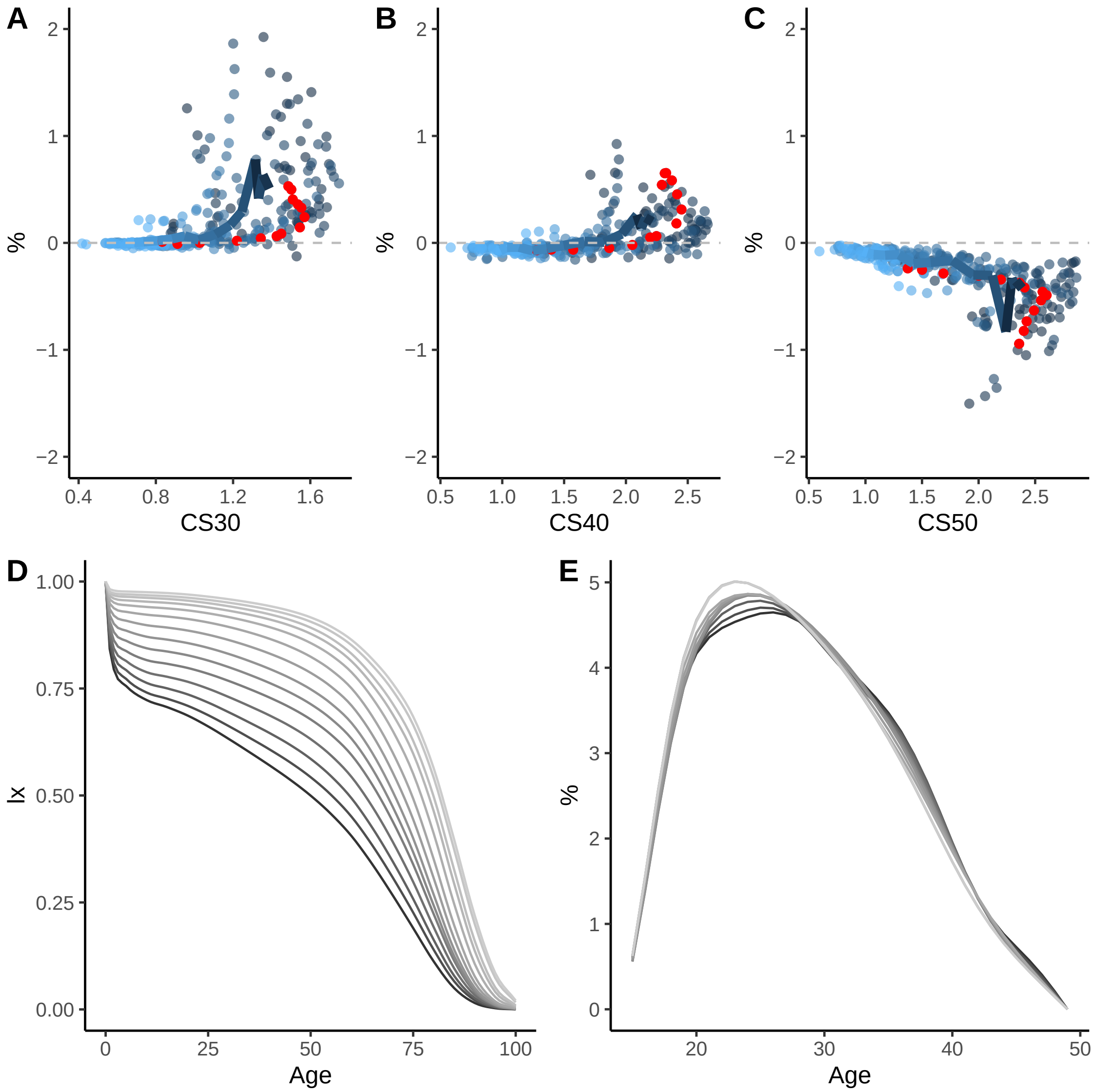

Figure 3: Relative error in approximation of CS for ages 30 (A), 40 (B) and 50 (C), from years 1950-1955 (darker) to 2015-2020 (lighter) for all Latin American countries (blue) and Guatemala (red). D) Guatemala: l_x, same period range. E) Fertility distribution by age in Guatemala, same period range.

equivalent values for women in Costa Rica would be 2.3% and 0.2%. The average in the region was 2.7% and .3% for the most recent period. The survival experience at higher maternal ages depends less on infant mortality, that is why values are lower than 2% for women aged 50, eventually converging to 0 with time.

The average age at child loss MAL for a bereaved Latin American woman aged 50 who bore three daughters was round 30 at half century XX. In 2015-2020 with a third part of fertility, the age for those women that lost a child is near 27 age old, which responds to a lower mean age at birth (also less dispersion) and a more concentration of daughters death in the first year of life.

Conclusion

How do changes in mortality affect the availability of offspring over age from the point of view of a prospective mother? We proposed analytical expressions to evaluate the effect of absolute and relative changes in mortality

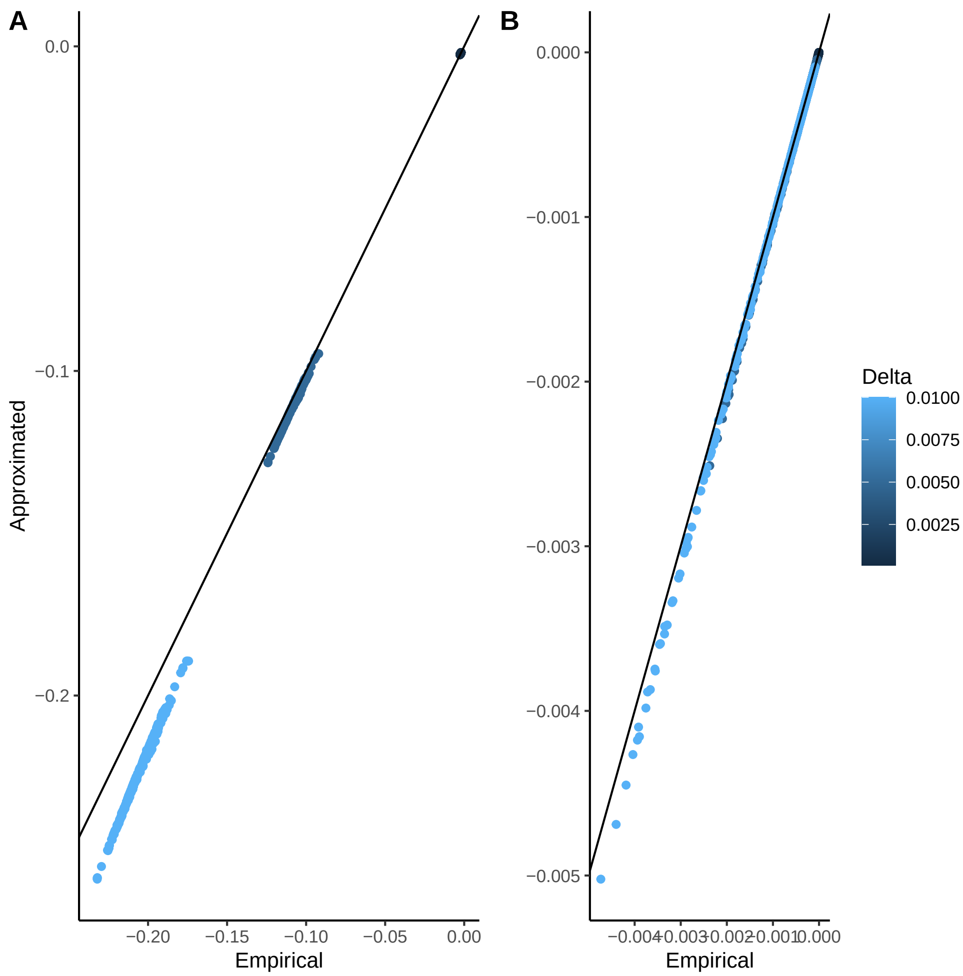

Figure 4: A) Empirical and analytical approximation on change in CS_a for an absolute change in μ_x for different change sizes (δ). Effect of absolute and relative change in mortality on child survival by maternal age of women in Latin America for the 1950-2015 period. Each point is a combination of country and period. B) Same but for a relative change.

on child survival. Some limitations must be remarked. The most obvious is that that results are valid in a stable and homogeneous population, where fertility and mortality are independent. We also assume that mortality change is permanent. This does not apply to mortality shocks, like the ongoing COVID-19 pandemic. This work contributes to the formal understanding of the effects of mortality change on maternal bereavement (or otherwise). Hopefully, it can inspired further development of indirect estimations methods that rely on widely available data sources.

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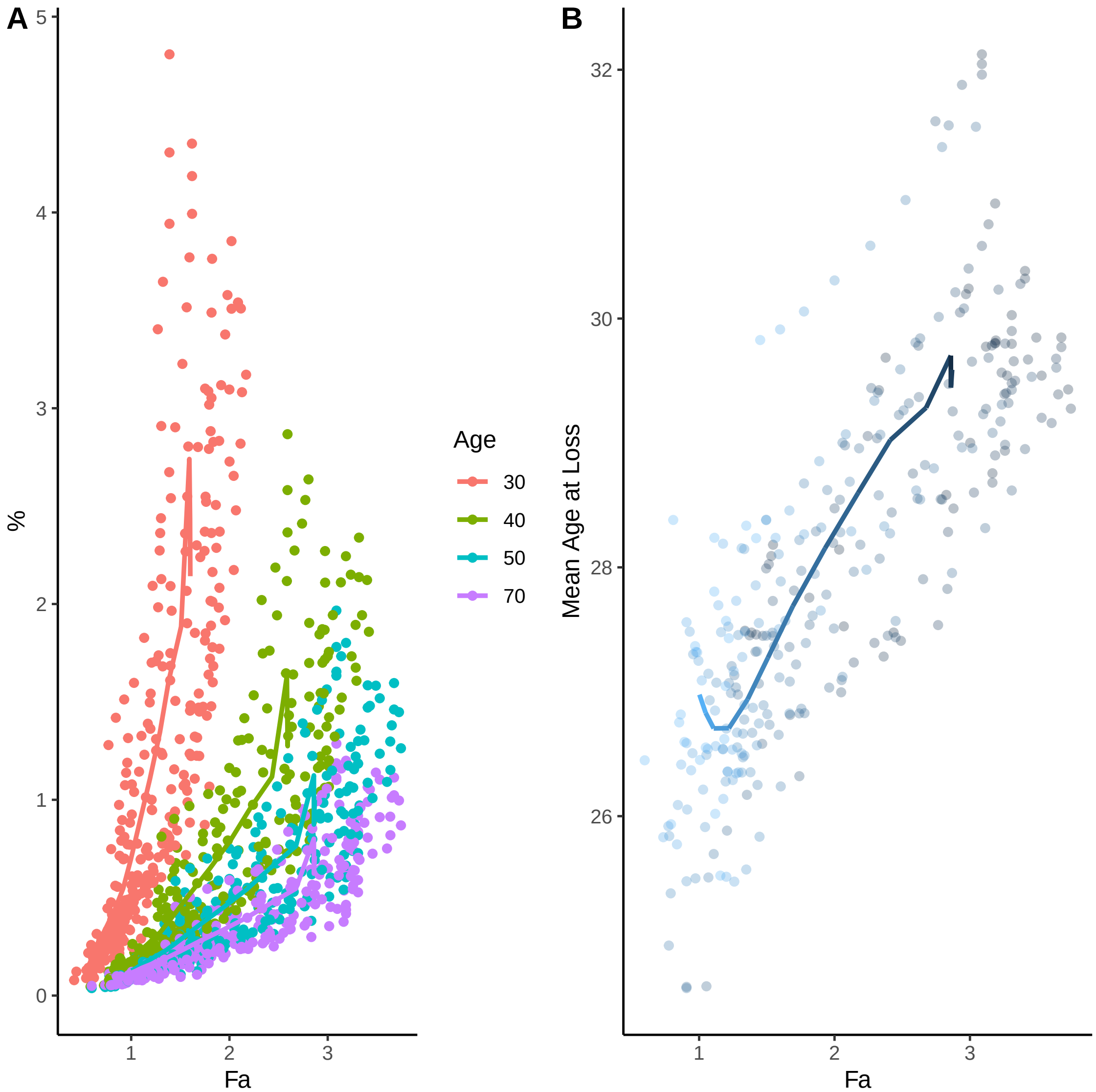

Figure 5: A) Intensity Time Lost of women aged 30, 40 and 50 for all period-country combination. B) Mean age at Loss for women aged 40. Years 1950 (darker) to 2015 (lighter). Latin American countries in period 1950-2015

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