A formal approach to child survival from the perspective of mothers

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Abstract

Background

The expected number of descendants for mothers that reach a certain age is an important relation in kinship theory. Child survival (and child death) affect mothers in a variety of ways, yet it is not clear how changes in mortality affect the number of surviving children for mothers.

Objective

Existing formal demographic tools are used within a stable population framework to characterize child survival from the point of view of a mother, considering both absolute and relative changes in mortality patterns.

Results

The expected change in child survival given an absolute change in mortality depends on the difference between maternal age and the age structure of her surviving children. The effects of absolute changes in mortality are explained using a fertility-weighted measure of entropy.

Contribution

The paper proposes new expresions to characterize the effects of changes in mortality on child survival. The different meassures it proposes can be applied to populations at different stages of the demographic transition.

1. Relationship

CS_a is the expected number of surviving children to a woman aged a in a stable female population with fertility rates m_x and force of mortality μ_x:

$$CS_a = \int_{\alpha}^{a} m_x l_{a-x} dx. \tag{1}$$

We now consider the consequences of a change δ in mortality in the range $[0, a-\alpha]$, where α is the start age of fertility risk (Goodman, Keyfitz, and Pullum (1974)) and δ can be absolute or relative (Wrycza and Baudisch (2012)).

A discrete approximate of the effect on CS_a of an absolute change in mortality $m_{x,\delta} = m_x + \delta$ is:

$$\frac{\Delta CS_a}{CS_a} \approx -(a - \overline{x}_{(CS)})\Delta\delta. \tag{2}$$

In this equation, the expected change in child survival is inversely proportional to the difference between maternal age a and the mean age of the mother at the birth of her surviving daughters $\overline{x}_{(CS)}$. In other words, the magnitude of the change depends negatively on the age distribution of the surviving offspring (as younger children experience longer periods of exposure to risk).

The relationship is less clear for proportional changes in mortality $m_{x,\delta} = m_x(1+\delta)$ given the interaction of birth and mortality rates (Keyfitz and Caswell (2005)). Neverthless we can say that, in discrete terms:

$$\frac{\Delta CS_a}{CS_a} \approx -H_{CS_a}\Delta\delta. \tag{3}$$

Where H_{CS_a} can be interpreted as a temporary entropy constant until age $a-\alpha$ but weighted by the fertility pattern between. In this sense, depending the way in that the survival childs are accumulated by age $C_{CS_a}(x) = \frac{m_x l_{a-x}}{CS_a}$ is the way that is taken each age contribution to the disparity measure $H_{CS_a} = \int_\alpha^a C_{CS_a}(x)H(x)$. The effect in the child survival experienced by the mother will be bigger when the mortality is the same in all ages, no matter where are concentrated the newborns.

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#> [1] "graph"
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2. Proofs

2.1. Absolute change

Considering that $m_{x,\delta} = m_x + \delta$ and $l_{a-x} = e^{-\int_0^{a-x}(\mu_t+\delta)}$:

$$CS_{(a)}^{\delta} = \int_\alpha^a m_x l_{a-x} e^{-\delta(a-x)}dx. \tag{4}$$

To find the effects on daughter survival of adding δ to the age-specific death rates of daughters, we get the derivative of $dCS_{(a)}^{\delta}/d\delta$, evaluated near zero (Keyfitz & Caswell, 2005, section 4.3):

$$= -a\int_\alpha^a m_x l_{a-x}dx + \int_\alpha^a m_x l_{a-x}x dx. \tag{5}$$

Since $CS_a = \int_\alpha^a m_x l_{a-x}dx$, we can rewrite equation (5) as

$$\frac{dCS^{\delta}}{d\delta} = -aCS_a + \int_\alpha^a x m_x l_{a-x}dx. \tag{6}$$

Dividing both sides by CS_a and multiplying by $d\delta$, we get

$$\frac{dCS_a}{CS_a} = -(a - \overline{x}_{(CS)})d\delta \tag{7}$$

where $\overline{x}_{(CS)}$ is the mean age of women at the birth of their surviving daughters in a stationary population.

2.2. Relative change

Considering that $m_{x,\delta} = m_x(1+\delta)$, $l_{a-x,\delta} = e^{-\int_0^{a-x}\mu_t(1+\delta)dt} = l_{a-x}^{(1+\delta)}$, and:

$$CS_a^{\delta} = \int_\alpha^a m_x l_{a-x}^{(1+\delta)}dx. \tag{8}$$

Using the derivative of $dCS_{(a)}^{\delta}/d\delta = log(l_{a-x})l_{a-x}^{(1+\delta)}$ and evaluating δ near zero we get the final expression:

$$\frac{dCS_a}{CS_a} = \frac{\int_\alpha^a m_x l_{a-x}\log(l_{a-x})dx}{\int_\alpha^a m_x l_{a-x}}d\delta \tag{9}$$

Using that $C_{CS_a}(x) = \frac{m_x l_{a-x}}{CS_a}$ is a weighting function and making an analogy with the classical entropy fucntion H where the weights are just the survival function, we can define this inforamtion function $H_{CS_a} = \int_\alpha^a C_{CS_a}H(x)$, and get:

$$\frac{dCS_a}{CS_a} = -H_{CS_a} d\delta. \tag{10}$$

3. History

Kinship theory has a long pedigree in mathematical demography. Lotka (1931) started to model the orphanhood in a theorical population, comparing the effect in different demographic regimes. Equation 1 was originally proposed by Brass (1953) to estimate child mortality. The notion of estimating the expected number of living daughters in a stable population was extended by Goodman, Keyfitz, and Pullum (1974) to other kin relations (granddaughters, cousins, etc.). The "counting method" approach, being the net reproduction rate R_0 limited to a, or the (not complete) ratio between generations (multiplying and dividing by the mother generation B), was further popularized in Keyfitz and Caswell (2005). Bongaarts (1987) used a similar approach to estimate descendants in his 'Family Status Model'. More recently, Wrycza and Baudisch (2012) has looked at the formal effect of different kind of changes in mortality by age in a paper published in this series.

4. Related meassures

4.1. Time with children lost

Some additional measures are proposed to measure the survival experience by a mother aged a. One is an absolute measure of the *Mean Time Spent with Lost*, which can be expressed in terms of a temporary expected lost years measure $e^\dagger$:

$$MTSL_a = \int_\alpha^a m_x \int_0^{a-x} d_t e_{t,a-x} dt dx = \int_\alpha^a m_x e^\dagger_{o,a-x} dx. \tag{11}$$

But maybe a mother would relate these lost years with the expected ones, feeling the *intensity* of her losts (*Intensity Time Lost*), and make possible the comparison between fertility regimes:

$$ITL_a = \frac{\int_\alpha^a m_x e^\dagger_{o,a-x} dx}{\int_\alpha^a m_x e_{o,a-x} dx} \tag{12}$$

4.2. Mean Age at Child Loss

Finally, different populations could have different mean age at lost of mothers with death childs:

$$MAL_a = K_a + \int_\alpha^a f_{x,a} MAD_{a-x} \tag{13}$$

K_a is the mean age at birth for womens aged a, MAD_{a-x} refers to the mean age at death for those newborns who dies before $a-x$, TFR_a is the cumulated fertility for a women aged a, and $f_{x,a}$ is the cumulative distribution of fertility until age a.

Equation 13 can be derived by starting with the integral of the age $x+t$ at each death child weighted by fertility and survival function

$$
\begin{aligned}
MAL_a &= \frac{\int_\alpha^a m_x \int_0^{a-x} \frac{d_t(x+t)dt}{\int_0^{a-x} d_t dt}}{\int_\alpha^a m_x} \\
MAL_a &= \frac{\int_\alpha^a m_x x}{TFR_a} + \frac{\int_\alpha^a m_x MAD_{a-x}}{TFR_a} \\
MAL_a &= K_a + \frac{\int_\alpha^a m_x MAD_{a-x}}{TFR_a} \\
MAL_a &= K_a + \int_\alpha^a f_{x,a} MAD_{a-x}
\end{aligned} \tag{14}
$$

5. Applications

The proposed meassures are now applied to two different demographic regimes.

[PENDING]

A computation approach is used to evaluate the relative contribution of each component given a relative change in δ.

[PENDING]

5. Bibliography

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