Child survival for mothers: mortality change and related measures

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Introduction

The question of child survival for mothers sits at the very center of demographic theory. Offspring survival is usually studied alongside fertility decline in the context of the demographic transition, but it also matters for post-transitional societies. Lower fertility and higer life expectancy means that fewer children are expected to provide key emotional, social, and financial transfers to aging parents for longer periods of time. With increasing periods of generational overlap, individuals find themselves 'sandwiched' between aging parents and young children requiring their simultaneous attention and care (Daatland, Veenstra, and Lima 2010). In the context of global population aging, elderly parents without access to formal social security and pension systems are particularly reliant on these transfers (Smith-Greenaway and Trinitapoli 2020). Given the known psychological, health, and social consequences of child loss for bereaved parents and families (Hendrickson 2009; Lee et al. 2014), it is surprising that the demographic processes that shape parental bereavement remain very poorly understood.

The Covid-19 pandemic has underscored the need to understand how sudden changes in mortality affect the availability of kin. The elderly depend on kin support, especially during periods of lockdown and in the absence of governmental support mechanisms. Elderly mothers may be at a higher risk of losing the key support provided by their adult children given the known age-gradient in the Covid-19 case fatality rate. How do changes in mortality affect the availability of offspring over age from the point of view of a prospective mother? How do these processes play out in a heterogeneous population?

In this paper, we aim to formalize the relationship between population-level changes in mortality rates (additive or multiplicative) and changes in the lived experience of kin death. This is key for understanding the impact of age-specific excess mortality on the resilience or otherwise of kinship networks for the vulnerable elderly population. This paper uncovers formal relationships and provides numerical applications using data from Latin America in the second half of the twentieth century using period measures in a stable context.[1] We continue a long tradition in mathematical demography, starting with the work of Lotka (1931) on modelling orphanhood in theoretical populations, Brass (1953) on child survival, Goodman, Keyfitz, and Pullum (1974) on kin survival in stable populations, and recently, Caswell (2019) kinship matrices.

Relationship (and proofs)

Child Survival

Let CS_a be the expected number of surviving children to a mother[2] alive at aged a in a female stable population with fertility rates m_x, mortality hazard μ_x and survival function $l_x = e^{-\int_0^x \mu_t \, dt}$ (with unit radix $l_0 = 1$), as proposed by Goodman, Keyfitz, and Pullum (1974):

$$CS_a = \int_0^a m_x l_{a-x} dx$$

[1] See the Applications section for more details on the source, calculation procedures and main numerical results.

[2] We use woman and mother interchangeably in this paper, assuming that all women are exposed to the same fertility rates.

An approximation

As an initial step, we seek an intuitive understanding of the previous relation. Building on work by Keyfitz and Caswell (2005) for the probability of a living mother, we can show which features of the child survival function explain the expected number of children. For this we use an approximate of l_x using Taylor´s theorem until second order around the mean age of childbearing κ:

$$
\begin{aligned}
CS_a \approx & l_{a-\kappa} \int_0^a m_x dx + (l_{a-\kappa})' \int_0^a (x-\kappa) m_x dx + (l_{a-\kappa})'' \int_0^a \frac{(x-\kappa)^2}{2} m_x dx \\
&\approx F_a l_{a-\kappa} + \frac{\sigma^2}{2} F_a (l_{a-\kappa})'' \\
&\approx F_a \, l_{a-\kappa} [1 + \frac{\sigma^2}{2} \frac{(l_{a-\kappa})''}{l_{a-\kappa}}]
\end{aligned}
$$

where the fertility pattern by age is concentrated around κ and the accumulated fertility (or gross reproduction rate in our female-dominant scenario) is $F_a = \int_\alpha^a m_x dx$. The second Taylor´s term is null because $\int_\alpha^a x \, m_x dx = \kappa \, F_a$.

We find that, seen from the perspective of a mother, child survival mainly depends on the cummulative fertility function and the survival of daughters from birth to $a - \kappa$. The approximation is affected negatively by how disperse is the fertility by age (variance σ^2) and the survival curvature around (with negative sign), in an age range where l_x is tipycally very flat in transitioned populations (range 20-40 years old)[3].

Mortality Changes

We now consider the consequences of an absolute change δ in mortality in the range $[0, a - \alpha]$, where α is the start age of fertility risk. For now, let us consider $m_{x,\delta} = m_x + \delta$ (Wrycza and Baudisch (2012)) and $l^\delta_{a-x} = e^{-\int_0^{a-x} (\mu_t + \delta) dt}$:

$$
CS^\delta_a = \int_\alpha^a m_x l_{a-x} e^{-\delta(a-x)} dx
$$

We get the derivative of $dCS^\delta_{(a)}/d\delta$ evaluated near zero (Keyfitz and Caswell (2005)) to find the effects of adding δ to hazard rates:

$$
\begin{aligned}
\frac{dCS^\delta}{d\delta} &= -a \int_\alpha^a m_x l_{a-x} dx + \int_\alpha^a x m_x l_{a-x} dx \\
&= -a \, CS_a + \int_\alpha^a x \, m_x l_{a-x} dx
\end{aligned}
$$

Dividing both sides by CS_a in an discrete approximation we get:

$$
\frac{\Delta CS_a}{CS_a} \approx -(a - k_a) \Delta\delta
$$

The expected change in descendants survival is inversely proportional to the difference between maternal age a and the mean age of the mother at the birth of her surviving daughters k_a. The magnitude of the change depends negatively on the age distribution of the surviving offspring (the younger the fertility bigger the impact), which is intuitive considering that older descendants experiences longer periods of exposure to risk.

[3] This approximation is useful also for get an idea of impact because of changes in the fertility average age κ, keeping constant the level: taking logs in $F_a \, l_{a-\kappa}$ and deriving we get $\frac{\Delta CS_a}{CS_a} \approx \mu_{a-\kappa} \Delta\kappa$, being daughters exposed to less time beacuse of the delay in pattern

However, it is unlikely for mortality to change at the same rate at all ages. In the course of the demographic transition, for example, we would expect larger changes in young-age mortality. Consider a change in infant mortality in the age range $[0;1)$. We can inspect the effect in child survival $CS_a^{\delta_0}$ by splitting the integral between those daughters who were exposed to the mortality change and those who were not (now generalizing for α positive and $a > 1$ for simplicity):

$$
\begin{aligned}
CS_a^{\delta_0} = \int_0^{a-1} m_x e^{-\int_1^{a-x} \mu_t \, dt - \int_0^1 (\mu_t + \delta) \, dt} dx + \int_{a-1}^{a} m_x e^{-\int_0^{a-x} (\mu_t + \delta) \, dt} dx \\
= \int_0^{a-1} m_x l_{a-x} e^{-\delta} dx + \int_{a-1}^{a} m_x l_{a-x} e^{-\delta(a-x)} \, dx
\end{aligned}
$$

Deriving by δ and valuating near 0, we get:

$$
\frac{dCS^{\delta_0}}{d\delta_0} = -\int_0^{a-1} m_x l_{a-x} dx - a \int_{a-1}^{a} m_x l_{a-x} dx + \int_{a-1}^{a} x \, m_x l_{a-x} dx
$$

Given that $CS_{a-1,a} = \int_{a-1}^{a} m_x l_{a-x} dx$, we can express the first term in the right as $CS_a - CS_{a-1,a}$. The righ-most term could also be expressed as $\kappa_{a-1,a} CS_{a-1,a}$, so that:

$$
\frac{dCS^{\delta_0}}{d\delta_0} = -CS_a + CS_{a-1,a}(1 - a + \kappa_{a-1,a})
$$

The factor $1 - (a - \kappa_{a-1,a})$, which lies between 0 and 1, is equal to the average portion of time spent with living daughters durings age a. Assuming a uniform distribution of those births, discretizing the change and dividing by CS_a we get the final expression for change in infant mortality:

$$
\frac{\Delta CS_a^{\delta_0}}{CS_a} \approx -\left[1 - \frac{CS_{a-1,a}}{CS_a} \frac{1}{2}\right] \Delta\delta
$$

This means that an absolute change in infant hazard rate affects proportionally for all the age range extension (cohorts aged more than 1), except for "half" the portion $\frac{CS_{a-1,a}}{CS_a}$ on how important was this las age in total "successful" experience (in terms of alive descendants). If a is near β then $\frac{\Delta CS_a^{\delta_0}}{CS_a} \approx -\Delta\delta$.

We now consider the consequences of a proportional change in mortality on child survival. Given a proportional change in mortality $\mu_{x,\delta} = \mu_x(1+\delta)$ so that $l_{a-x}^{\delta} = e^{-\int_0^{a-x} \mu_t (1+\delta) dt} = (l_{a-x})^{(1+\delta)}$, it follows that:

$$
CS_a^{\delta} = \int_{\alpha}^{a} m_x l_{a-x}^{(1+\delta)} dx.
$$

Using the derivative $\frac{dl_{a-x}^{(1+\delta)}}{d\delta} = log(l_{a-x}) l_{a-x}^{(1+\delta)}$, and in the third row reversing integrals between t and x:

$$
\begin{aligned}
\frac{dCS_a^{\delta}}{d\delta} &= \int_0^a m_x l_{a-x} \log(l_{a-x}) dx \\
&= -\int_0^a m_x l_{a-x} \int_0^{a-x} \mu_t \, dt \, dx \\
&= -\int_0^a \mu_t \int_0^{a-t} m_x l_{a-t-x} \, dx \, dt
\end{aligned}
$$

Considering that the last integral is equal to CS_{a-t}, we divide by CS_a and multiply by δ to obtain:

$$
\frac{\Delta CS_a}{CS_a} \approx -\left[\int_0^a \mu_x \frac{CS_{a-x}}{CS_a} \, dx\right] \Delta\delta
$$

This is the negative of cumulative hazard H_a but considering a positive factor $\frac{CS_{a-t}}{CS_a}$ that takes in account the relative amount of surviving descendants that would be lost at each child age at risk, that gives more weight to first ages because all the parity was exposed in that age.

Grasping this relationship intuitively is more difficult given the interaction of birth and mortality rates (Keyfitz and Caswell (2005)). The factor $\frac{CS_{a-t}}{CS_a}$ has a S-shape because l_x curvature and fertility accumulation. Taking the difference between consecutive ages a and $a+1$, we have $CS_{a-1} = \int_0^{a-1} m_x l_{a-1}\,dx$, $CS_a = \int_0^{a-1} m_x l_a\,dx + CS_{a-1,a}$. In the last one, a second term is added because of additional born daughters after $a-1$, and in the first term the existing one are survived at on $\frac{l_{a-x}}{l_{a-1-x}}$. In advanced ages when fertility is complete, the difference is only explained by the surviving part, that is why this factor is not a strictly increasing function but for mother ages close to β (last theoretical fecundity age). We provide a numerical evaluation of both analytical expressions (absolute an relative) in the Applications section.

Related Measures

Burden and timing of maternal bereavement

We call *Mean Time Spent in Bereavement* (*MTSB*) the absolute measure of the expected total lifetime with a death daughter for a mother aged a, which can be expressed in terms of a temporary expected lost years index, in line with $e^\dagger$ (Vaupel (1986)):

$$MTSB_a = \int_0^a m_x \int_0^{a-x} d_t e_{0|a-x-t}\,dt\,dx = \int_\alpha^a m_x e^\dagger_{0|a-x}\,dx$$

.

Where d_t is the death distribution from birth, $e_{0|a-x-t}$ is the life expectancy at birth until age $a-x-t$ and $e^\dagger_{0|a-x}$ the temporary dispersion measure. But most interesting would be to compare these years with the time that mothers would expect to live with their daughters. We call this the *Intensity Time in Bereavement* (*ITB*): a ratio between expected time with a "lost" and expected time with a "life", that allows to make comparisons between population regimes.

$$ITL_a = \frac{\int_0^a m_x e^\dagger_{0|a-x} dx}{\int_0^a m_x e_{0|a-x} dx}$$

This is a ratio between child-years in two radically different states. Looks similar to the transcendental entropy measure H (Keyfitz and Caswell (2005)) but considering all the cohorts born during the mother´s life, weighted by their relative size m_x. In the figure @ref(fig:plot_ITL_MAL) is shown that *ITB* is bigger for young women because of the weight of infant mortality in their mother experience, also with more dispersion for same levels of parity.

Another important factor for the child survival experience of mothers, is the mean age at child loss, called here *MAL*. This relation can be derived by starting with the mother age $x+t$ at each death child age t at death, weighted by the fertility and survival function. In it, κ is the mean age at childbirth for women aged a, MAD_{a-x} refers to the mean age at death for newborns that die before $a-x$ and F_a is the accumulated fertility for a women aged a:

$$
\begin{aligned}
MAL_a =& \frac{\int_0^a m_x \frac{\int_0^{a-x} l_t\, \mu_t (x+t) dt}{\int_0^{a-x} l_t\, \mu_t dt} dx}{\int_0^a m_x dx} \\
MAL_a =& \frac{\int_0^a m_x \left[x + \frac{\int_0^{a-x} l_t\, \mu_t\, t\, dt}{\int_0^{a-x} l_t\, \mu_t dt} \right] dx}{\int_0^a m_x} \\
MAL_a =& \frac{\int_0^a m_x\, x\, dx}{F_a} + \frac{\int_0^a m_x MAD_{a-x} dx}{F_a} \\
MAL_a =& \kappa + \frac{\int_0^a m_x MAD_{a-x} dx}{F_a}
\end{aligned}
$$

Following this, in populations with high infant mortality $MAD_{a-x} < 1$ for all x, so $MAL_a \approx \kappa_a + f_0$, being f_0 the average time spent for those child that dies in their first year of life. A numerical approximation of both measures are done in Applications section.

Heterogeneity (first ideas)

In a model with heterogeneity, determined at birth by a multiplicative effect, CS_a could be interpreted as a conditional expectation with random variables K for fertility heterogeneity and Z for mortality frailty.

$$
CS_a(k, z) = \int_0^\infty m(k)_x\, l(z)_{a-x}\, dx
$$

Following Coresh and Goldman (1988) we can express the fertility part in a multiplicative way with variability in the level but not in the shape, as $m(k)_x = F_\beta\, k\, r_x$, where F_β is the baseline cumulated fertility until upper limit in all age range until β, k is a random variable with mean 1 that allows variability between groups and r_x is the fertility structure by age ($\int_0^\beta r_x = 1$). In the other side, frailty part can be thought in a cohort effect way, also in the multiplicative assumption as Vaupel and Missov (2014), with $l(z)_x = e^{-H_x z} = l_x^z$, with baseline hazard μ. Considering the joint distribution f_{kz}, the intuition could be expressed as $f_{zk} = f_{k|z} f_z$: groups with higher descendant´s mortality would adjust their fertility level, with positive correlation. Replacing both variables, the unconditional mean $\overline{CS}_a$ would be[4]:

$$
\begin{aligned}
\overline{CS}_a &= \int_0^\infty \int_0^\infty \left[\int_0^a m(k)_x l(z)_{a-x}\, dx \right] f_{kz}\, dz\, dk \\
&= F_\beta \int_0^{a-x} r_x \int_0^\infty \int_0^\infty (l_{a-x})^z\, k\, f_{kz}\, dk\, dz\, dx
\end{aligned}
$$

Isolating for an age x we can inspect the part related to z and k: $(l_{a-x})^z\, k\, f_{kz}$. Following Gupta (2018), one way to create a correlated bivariate distribution is assume for example that $Z = Y_0 + Y_1$ and $K = e^{(Y_0+Y_2)}$, given the fact that the historical relation is not linear between both components (see figure @ref(fig:plot_tfr_q0), a funtion f like $F_\beta = f(ln(q_{0,5}))$. The auxiliar variables Y_0, Y_1 and Y_2 are independet and Gamma distribuited. Then we can express, using Laplace transform properties:

$$
\begin{gathered}
\int_0^\infty \int_0^\infty (l_{a-x})^z\, k\, f_{kz}\, dk\, dz \\
\int_0^\infty \int_0^\infty e^{-H_x (Y_0+Y_1)} e^{Y_0+Y_2} \\
L_{Y_0}[1 - H_x] L_{Y_1}[-H_x] L_{Y_2}\, dY_0\, dY_1\, dY_2
\end{gathered}
$$

[4]Note that given k and z values, fertility and child survival are independent.

To continue…

Applications

Data

We motivate this paper with an empirical example using fertility and mortality rates from the 2019 Revision of the UN World Population Prospects (UN WPP) for the Latin American Region (Nations (2019)). We smoothed female l_x in quinquennial ages, using cubic-splines constrained to monotonic decrease, taking L_0 and T_{100} from raw life tables as inputs for year-person calculations. For splitting fertility five groups was used quadratic optimization approach by Michalski and Gorlishchev (2018), with an desirable property for our purpose which is a good fitting in parity. Also was assumed an unique female percentage of newborns of 0.49 for all period-country cases. Calculations were done in a discrete way assuming that the m_x live births are born at exact mother´s age x. An already known log relation between infant mortality and TFR is shown in Latinoamerican experience (@ref(fig:plot_tfr_q0)).

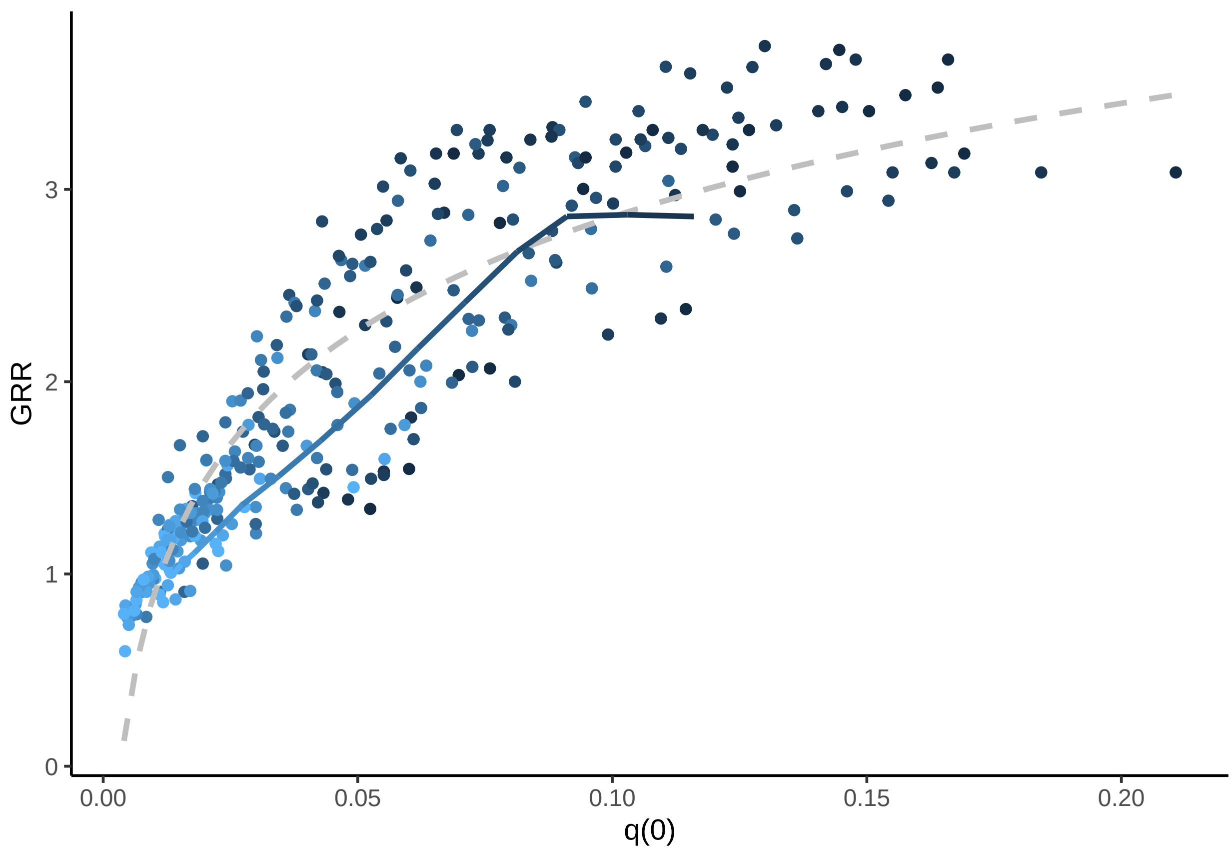

Figure 1: Female probability of death at birth by gross reproduction rate. Latin America countries in period 1950-2015

Numerical Results

Consider a Latin American woman standing before us. If this were 1950-1955, she could reasonable expect to have 2.23 surviving daughters on her 50^{th} birthday. In 2010-2015 a woman the same age would only have 1.1 living children. The difference of 1.13 children is explained by reduced fertility and improved mortality in the region [give stats]. We now remove the effect of changing fertility by considering the number of daughters surviving up to maternal age a as a proportion of the daughters ever born to a woman that age, $\frac{CS_a}{F_a}$. Given Eq. **??**, the increase from 0.78 in 1950-1955 to 0.97 in 2010-2015 must be explained by a change in mortality.

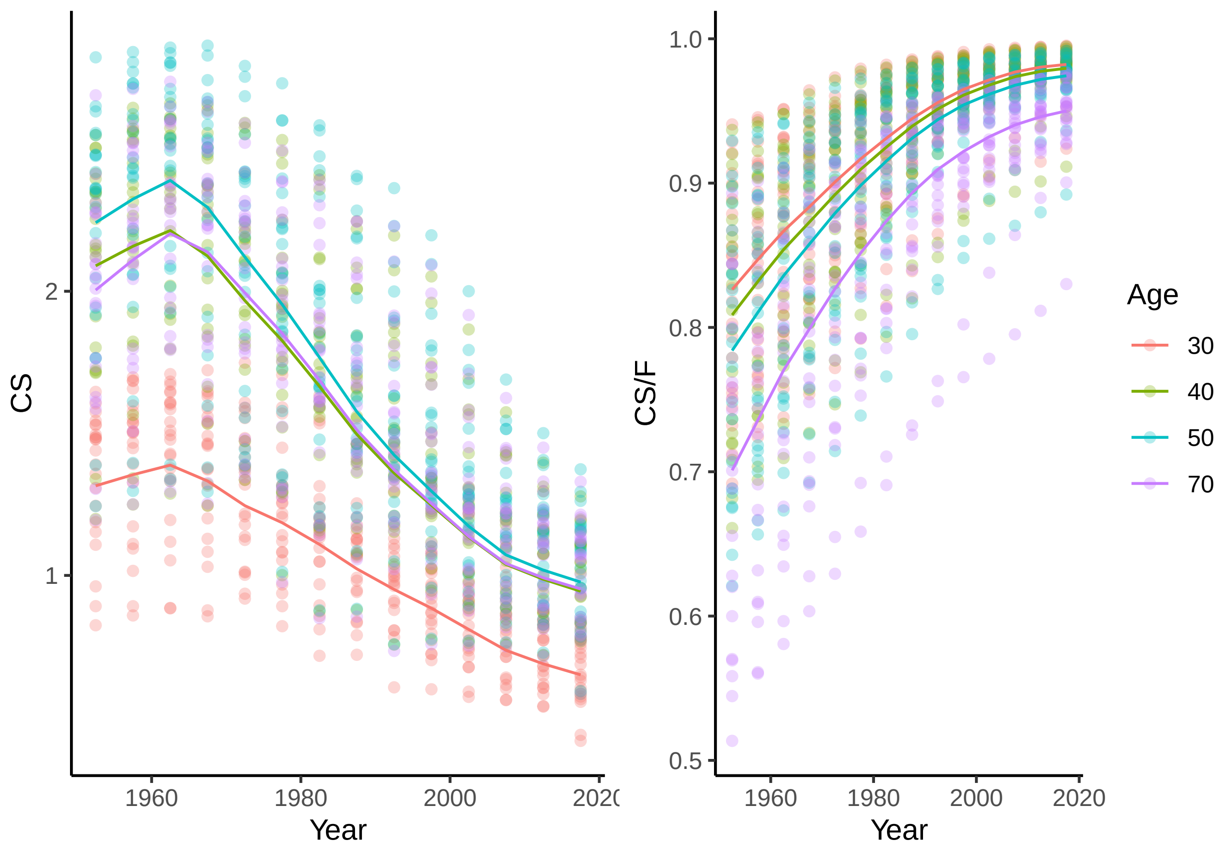

Figure 2: Child Survival and Child survival as a share of cummulative fertility by age, for women aged 30, 40 and 50. Estimates using UN WPP data for Latin American countries in the period 1950-2015 period

The approximation proposed for CS_a is realy precise, especially for recent periods. For example, Guatemala improved the approximation with years due to rectangularization process in l_x (figure @ref(fig:plot_CS_aprox)).

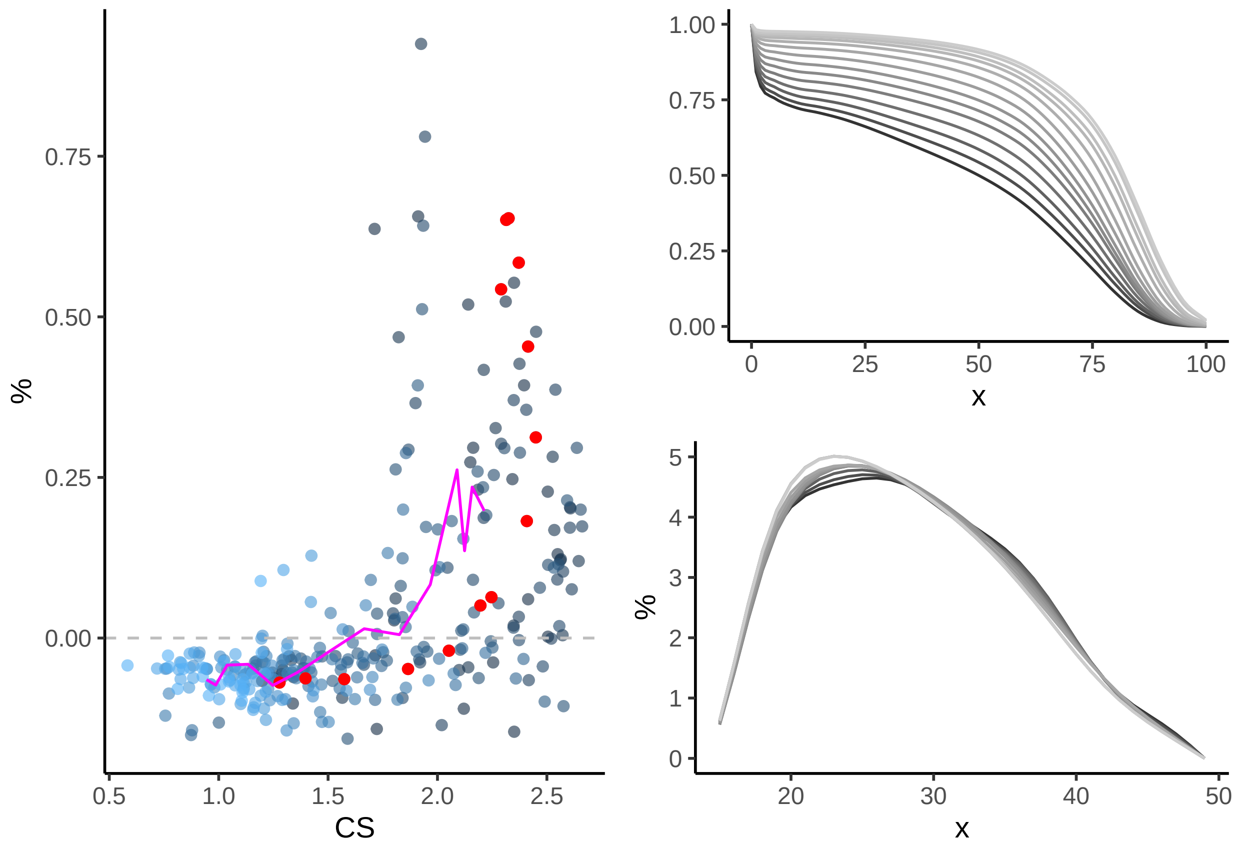

Figure 3: Left: Error in approximation for a=40, from years 1950 (darker) to 2015 (lighter) for all Latin American countries (blue) and Guatemala (red). Right: Change in survival and fertility by age in Guatemala.

The figure @ref(fig:CS_abs_app) shows that goodness of fit is decreasing with the change size, given that δ is assumed near zero. To ilustrate that considering $\delta = 0.01$ means at age 50, it means for the absolute change case that $\frac{l_{50}}{e^{\delta\,50}} = l_{50}/$ 1.6, a big change). For the relative case was assumed a constant within each interval $\mu_{x+t} = \mu_x$ for t between 0 and 1, $\int_x^{x+1} \mu_t dt = \log(l_{x+1}) - \log(l_x)$, and was used this approximation on the empirical data $\frac{\Delta CS_a}{CS_a} \approx -\left[\sum_0^{a-1} \left[\log(l_{x+1}) - \log(l_x)\right] \frac{CS_{a-1-x}}{CS_a}\right] \Delta\delta$.

More develop here showing the factor and ggplot type...

Assuming constant period rates the expected time in years that a mother aged 30 passed with a death son was around 4% in some countries at middle XX Century. When increasing age, the survival experience depends less on infant mortality, and the distribution is around less than 2% for women aged 50, converging to 0 on time. As extreme cases, in 1950-1955 Haiti women aged 30 would have experienced an intensity of 4.8%, and in 2010-2015 1.4%, while the women of Costa Rica 2.3%, and in 2010-2015 0.2%. For those Latin American mothers aged 50 with 3 daughters born who suffered a lost, they experienced that at age 30 in average (figure @ref(fig:plot_ITL_MAL)).

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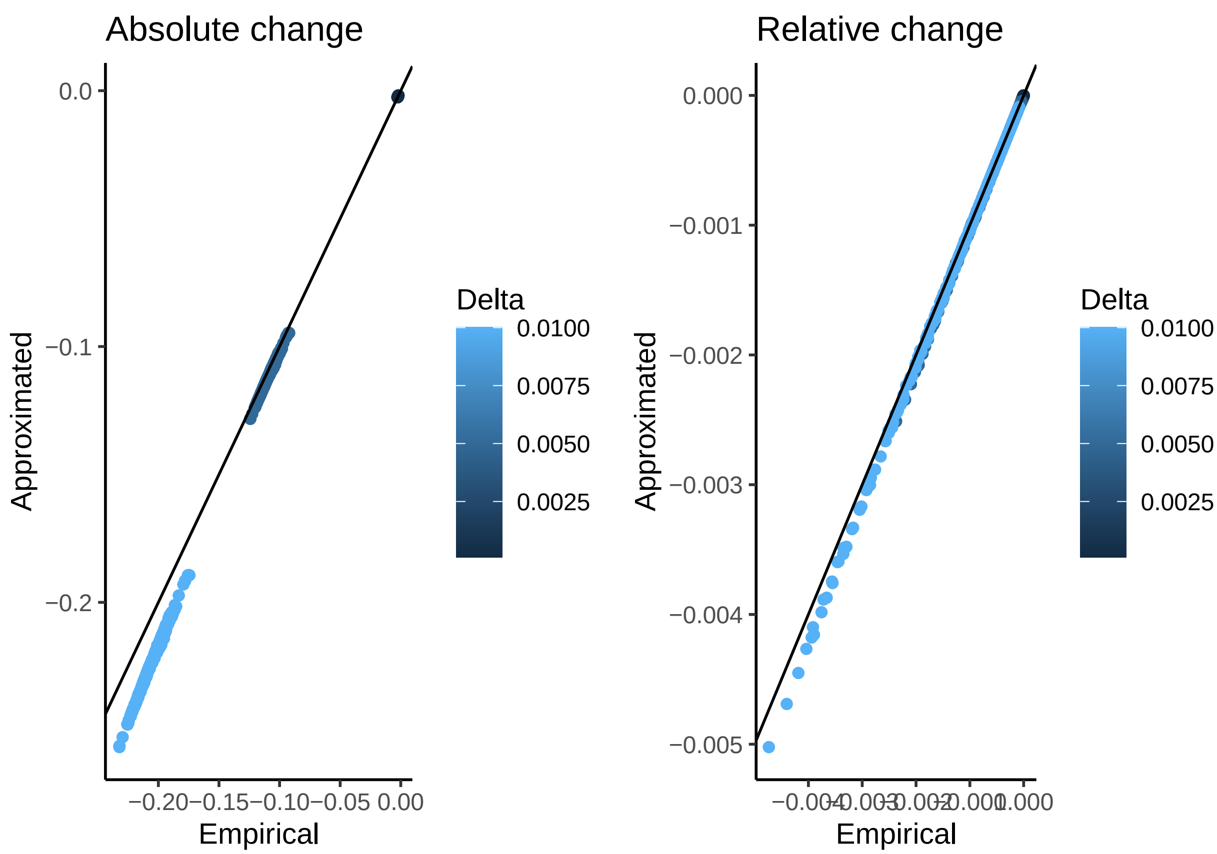

Figure 4: Effect of absolute and relative change in mortality on child survival by maternal age of women in Latin America for the 1950-2015 period. Goodness of fit of approximation compared to direct estimation

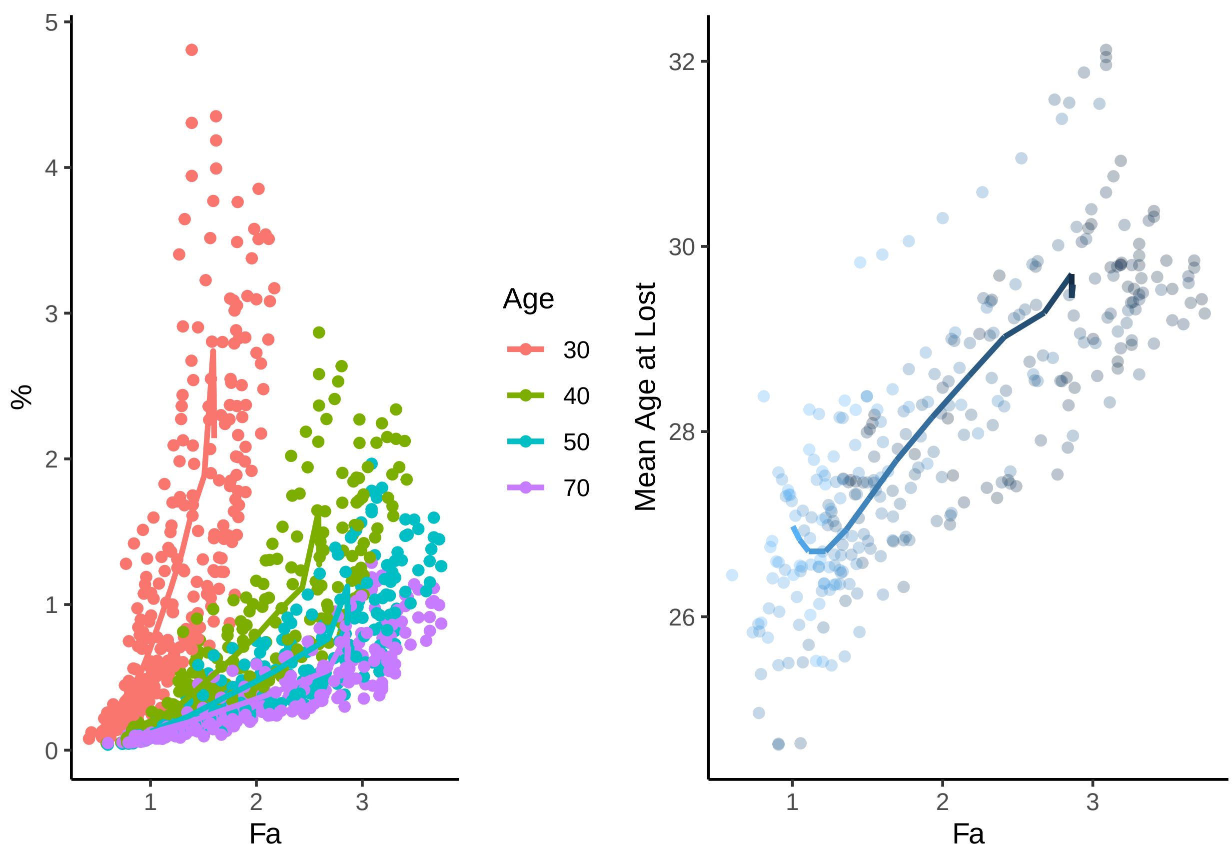

Figure 5: a) Intensity Tome Lost of women aged 30, 40 and 50. b) Mean age at Lost for women aged 40. Years 1950 (darker) to 2015 (lighter). Latinamerican countries in period 1950-2015

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