Matrix Population Models 1

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"Matrices unify and simplify demography. ... For many of us, including myself, multidimensionality is esoteric ... But we can be sure that what is strange and difficult to us will be natural and easy to our children. What is a narrow and almost closed sect within the profession today is going to become standard and obvious demogaphic technique in the next decade. A table of working life made the way Dublin et al (1949) made theirs, or a table of marriage similarly calculated, will be regarded as quaint; it will compare with the matrix method discussed in this issue as a cumbersome and inflexible desk calculator of the 1940s would compare with a hand-programmable calculator of the late 1970s."

Nathan Keyfitz (1984)

Starting simple

Age classes 1,2,3

$$\begin{aligned} n_1(t+1) &= F_1 n_1(t) + F_2 n_2(t) + F_3 n_3(t) \\ n_2(t+1) &= P_1 n_1(t) \\ n_3(t+1) &= P_2 n_2(t) \end{aligned}$$

or

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}(t+1) = \begin{pmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}(t)$$

or

$$\mathbf{n}(t+1) = \mathbf{A}\,\mathbf{n}(t)$$

So, what is this doing?[1]

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}(t+1) = \begin{pmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}(t)$$

[1] transformation, distribution, projection, age, time, . . .

P. H. Leslie (1900–1972)[2]

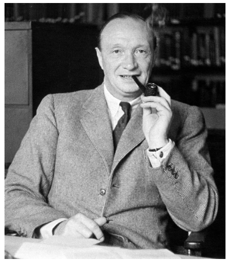
VOLUME XXXIII, PART III NOVEMBER 1945

ON THE USE OF MATRICES IN CERTAIN POPULATION MATHEMATICS

BY P. H. LESLIE, *Bureau of Animal Population, Oxford University*

CONTENTS

$$A = \begin{bmatrix} F_0 & F_1 & F_2 & F_3 & \dots & F_{k-1} & F_k \\ P_0 & . & . & . & \dots & . & . \\ . & P_1 & . & . & \dots & . & . \\ . & . & P_2 & . & \dots & . & . \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ . & . & . & . & \dots & P_{k-1} & . \end{bmatrix}$$



[2]1945, Biometrika

Nathan Keyfitz (1913–2010)[3]

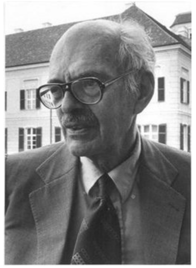
THE POPULATION PROJECTION AS A MATRIX OPERATOR

N. KEYFITZ
University of Chicago

RESUMEN

El método de "componentes" para proyecciones de población, uno de los métodos demográficos más valiosos, ha sido tratado casi enteramente como método empírico, desprovisto de una teoría básica o una estructura matemática. Este artículo considera la proyección como una matriz, con

$$M = \begin{pmatrix} 0 & 0 & .0570 & .2316 & .3338 & .2622 & .1565 & .0692 & .0158 \\ .99108 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .99781 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .99546 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .99329 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .99140 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .98954 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .98664 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .98237 & 0 \end{pmatrix}.$$



[3]Demography (1964). Volume 1. Issue 1!

Leonard Lefkovitch (1929–2010)[4]

THE STUDY OF POPULATION GROWTH IN ORGANISMS GROUPED BY STAGES

L. P. LEFKOVITCH
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SUMMARY

In this extension to the use of matrices in population mathematics (Lewis [1942] and Leslie [1945]), the division of a population into equal age groups is replaced by one of unequal stage groups, no assumptions being made about the variation of the duration of the stage that different individuals may show. This extension has application in ecological studies where the age of an individual is rarely known. The model is briefly applied to three experimental situations.

$$\mathbf{M}_{\mathrm{I}'} = \begin{bmatrix} \cdot & 1.978 & \cdot & \cdot \\ 1.203 & 0.998 & 27.776 & 15.309 \\ 0.362 & 0.191 & \cdot & \cdot \\ \cdot & 0.835 & \cdot & \cdot \end{bmatrix}$$

[4]Biometrics 21:1–18, 1965

Why did it take 20 years?

State variables for individuals and populations

individual state (i-state):

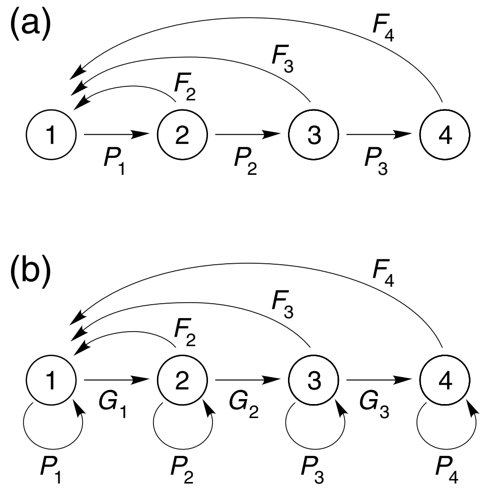
- information needed to determine the probabilities of survival, reproduction, and transition from t to $t+1$
- state + environment at time t determines state at time $+1$
- demograpers' favorite i-state variable is age

population state (p-state):

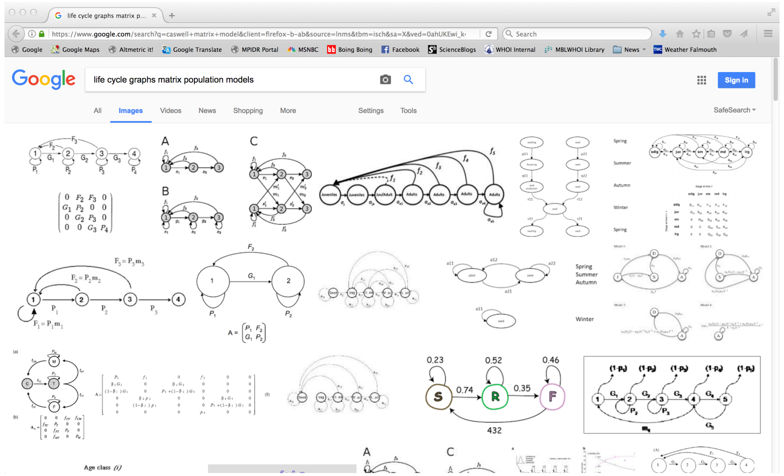
- distribution of individuals over i-state categories[5]
  - age $\longrightarrow$ age distribution
  - size $\longrightarrow$ size distribution
  - health status $\longrightarrow$ health status distribution
  - you get the idea

[5] Subject to some conditions

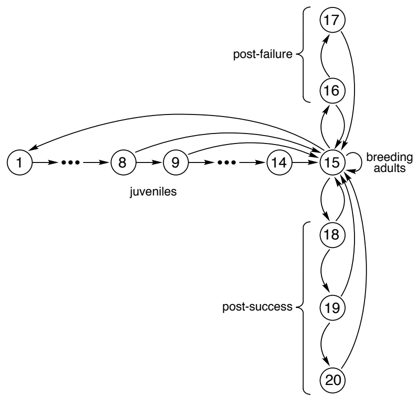
i-state variables and life cycle graphs



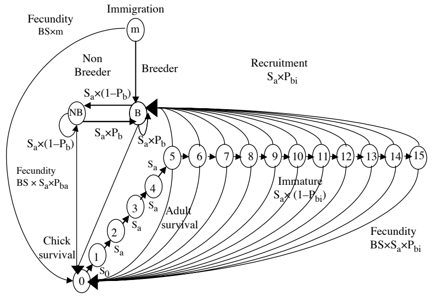
Life cycle graphs



Wandering albatross

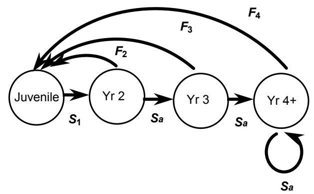


Snow petrel



mean	=	0.98
SD	=	3.8
CV	=	3.9
skew	=	5.8
Crow's $\mathcal{I}$	=	15.2

Greater snow goose



mean	=	2.1
SD	=	4.8
CV	=	2.3
skew	=	3.5
Crow's $\mathcal{I}$	=	5.3

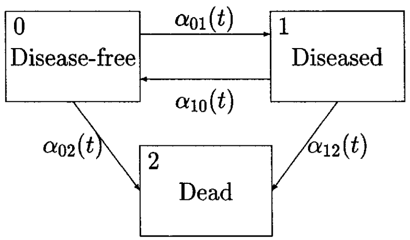
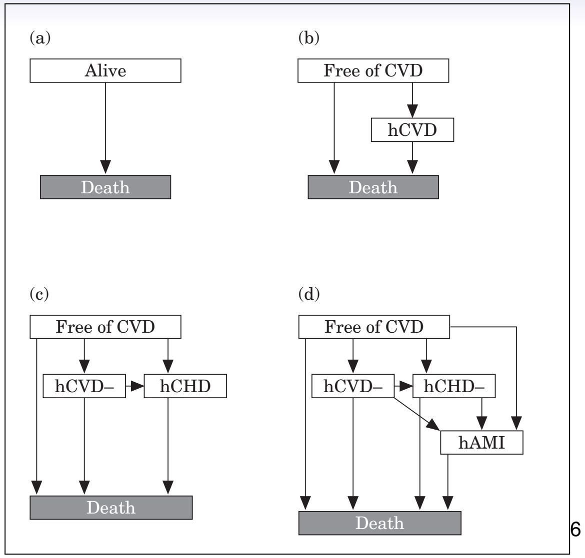


Figure 3 The illness–death or disability model.



[6]

[6] Peeters et al. 2002

process. from two different sets of starting values.

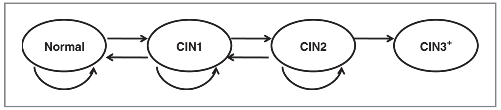


Figure 2. Underlying four states model for the progression of cervical cancer. Bidirectional transitions could only appear between normal and CIN1, CIN1 and CIN2, rather than directly between normal and CIN2. CIN3+ is considered an absorbing state with no further transitions to other states.

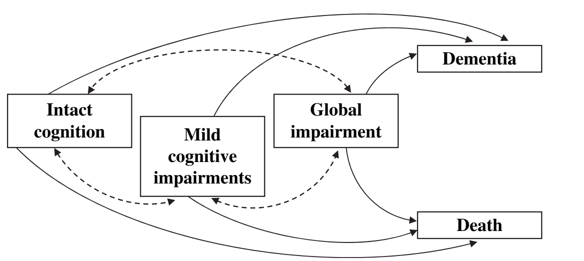


FIGURE 1. Possible cognitive transitions between transient states (intact cognition, mild cognitive impairments, global impairment) and absorbing states (dementia, death). [7]

[7] Tyas et al. 2007

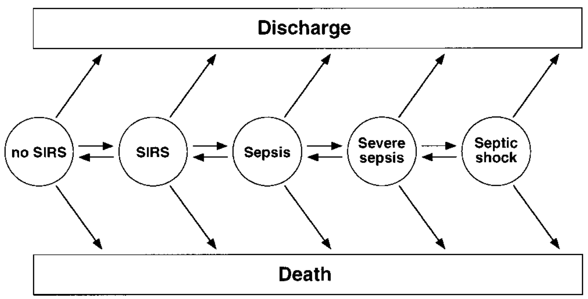
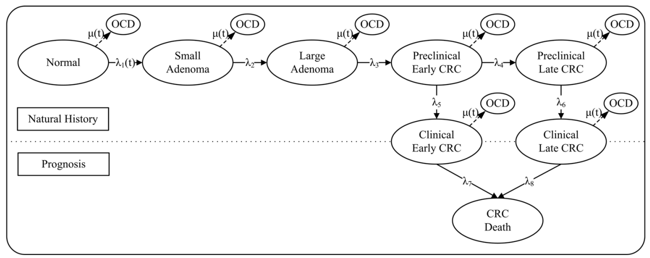


Figure 1. Transitional probability of the sepsis cascade. The open circles indicate the starting stages (no systemic inflammatory response syndrome [SIRS], SIRS, sepsis, severe sepsis, and septic shock). The arrows represent forward or backward progression from one stage of the sepsis cascade to another or from one stage to an absorbing state (rectangles). The probabilities of advancing to a more advanced stage [8]

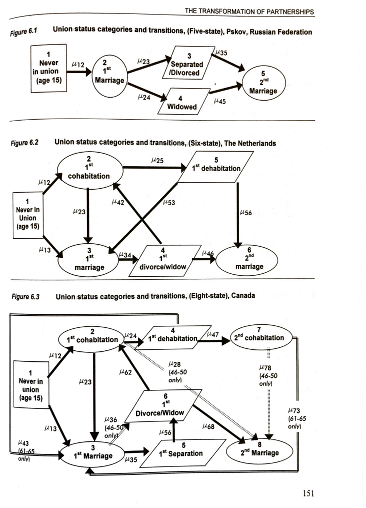
[8] Ragnel-Frausto et al. 1998

Colorectal cancer

BMC Cancer 2006, **6**:136 http://www.biomedcentral.com/1471-2407/6/136

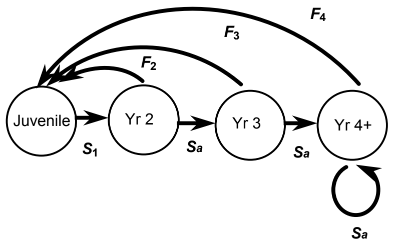


Union status[9]



[9] Mills 2000

From the life cycle graph to the matrix



$$\mathbf{A} = \begin{pmatrix} 0 & F_2 & F_3 & F_4 \\ S_1 & 0 & 0 & 0 \\ 0 & S_a & 0 & 0 \\ 0 & 0 & S_a & S_a \end{pmatrix}$$

Important

$a_{i,j}$ = coefficient **from** j **to** i

A useful decomposition

$$\mathbf{A} = \mathbf{U} + \mathbf{F}$$

$$\mathbf{A} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ S_1 & 0 & 0 & 0 \\ 0 & S_a & 0 & 0 \\ 0 & 0 & S_a & S_a \end{pmatrix}}_{\mathbf{U}} + \underbrace{\begin{pmatrix} 0 & F_2 & F_3 & F_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\mathbf{F}}$$

Population projection

$$\mathbf{n}(t+1) = \mathbf{A}\mathbf{n}(t)$$

projection initial population $\mathbf{n}(0)$

$$\begin{aligned} \mathbf{n}(1) &= \mathbf{A}\mathbf{n}(0) \\ \mathbf{n}(2) &= \mathbf{A}\mathbf{n}(1) \\ &= \mathbf{A}^2\mathbf{n}(0) \\ \mathbf{n}(3) &= \mathbf{A}\mathbf{n}(2) \\ &= \mathbf{A}^3\mathbf{n}(0) \\ &\vdots \\ \mathbf{n}(t) &= \mathbf{A}^t\mathbf{n}(0) \end{aligned}$$

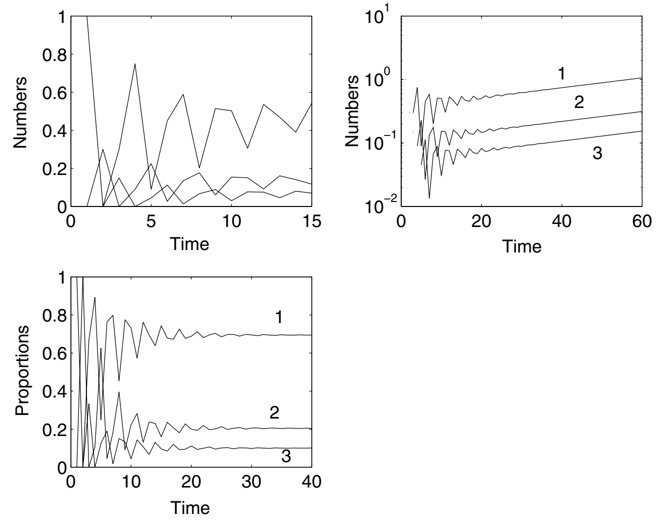
Transient and asymptotic dynamics

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 5 \\ .3 & 0 & 0 \\ 0 & .5 & 0 \end{pmatrix} \qquad \mathbf{n}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

projecting

$$\mathbf{n}(1) = \begin{pmatrix} 0 \\ 0.3 \\ 0 \end{pmatrix} \quad \mathbf{n}(2) = \begin{pmatrix} 0.3 \\ 0 \\ 0.15 \end{pmatrix} \quad \ldots$$

Transient and asymptotic dynamics



Some types of matrix models

- linear, time-invariant

$$\mathbf{n}(t+1) = \mathbf{A}\mathbf{n}(t)$$

- what else?

Next

- solve the projection equation
- explain transient dynamics
- characterize asymptotic dynamics