The formal demography of kinship: The basic model

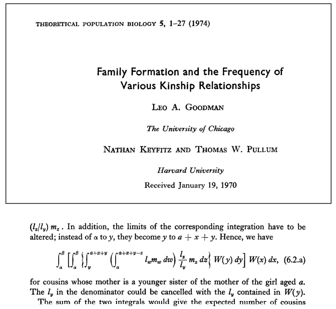
Hal Caswell
University of Amsterdam

IDEM 128 May 2023

The conditions that present themselves in an actual population are always excessively complicated. Whoever has failed to grasp clearly the necessary relations among the characteristics of theoretical population, subject to simple hypotheses, will certainly be unable to manage in the much more complicated relations that exist in a real population.

A.J. Lotka (1939)

Kinship: Goodman, Keyfitz, and Pullum 1974



THEORETICAL POPULATION BIOLOGY 5, 1–27 (1974)

Family Formation and the Frequency of Various Kinship Relationships

LEO A. GOODMAN

The University of Chicago

NATHAN KEYFITZ AND THOMAS W. PULLUM

Harvard University

Received January 19, 1970

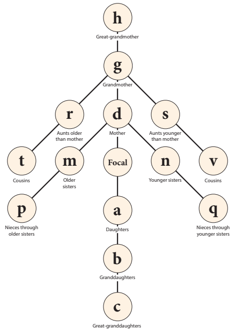
$(l_z/l_y)\, m_z$. In addition, the limits of the corresponding integration have to be altered; instead of α to y, they become y to $a + x + y$. Hence, we have

$$\int_\alpha^\beta \left[\int_\alpha^\beta \left\{ \int_y^{a+x+y} \left(\int_\alpha^{a+x+y-z} l_w m_w \, dw \right) \frac{l_z}{l_y} m_z \, dz \right\} W(y)\, dy \right] W(x)\, dx, \quad (6.2.a)$$

for cousins whose mother is a younger sister of the mother of the girl aged a. The l_y in the denominator could be cancelled with the l_y contained in $W(y)$. The sum of the two integrals would give the expected number of cousins

---

The kinship network[1]



[1] Keyfitz and Caswell 2005

Meet Focal

- female (or male) of specified age
- specified mortality and fertility
- distribution π of ages of mothers at birth

Let's suppose:

- uniformity
- time-invariance
- female model

The kin of Focal **are a population**

. . . so we might as well model them as one

A matrix formulation of kinship demography

mortality schedule ↘
kin at any age of Focal
fertility schedule ↗

- age distribution of kin
- and things that can be calculated from age distributions
- extendable beyond age-classification
- extendable beyond time-invariance
- extendable to two sexes
- easy to compute; no simulations required

The pieces

- survival matrix $\mathbf{U}$
- fertility matrix $\mathbf{F}$
- recruitment subsidy β
- initial condition $\mathbf{k}(0)$
- distribution of age at maternity π

The kinship model

The kin of Focal as a population

$\mathbf{k}$ is a generic kin population vector

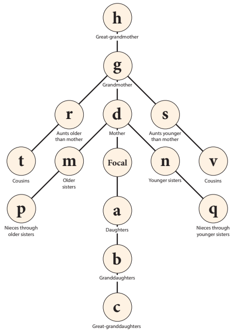
$$\begin{aligned} \mathbf{k}(x+1) &= \mathbf{U}\mathbf{k}(x) + \beta(x) \\ \mathbf{k}(0) &= \mathbf{k}_0 \end{aligned}$$

where

$$\begin{aligned} \mathbf{k}(x) &= \text{age distribution of a kin at age } x \text{ of Focal}^{2} \\ \beta(x) &= \text{recruitment `subsidy' at age } x \text{ of Focal} \\ &= \begin{cases} \mathbf{0} & \text{no subsidy} \\ \mathbf{F}\,\mathbf{k}^*(x) & \text{subsidy from } \mathbf{k}^* \end{cases} \\ \mathbf{k}_0 &= \text{initial condition} \end{aligned}$$

[2] To be more precise, the expectation of the age distribution.

The kinship network[3]



[3] Keyfitz and Caswell 2005

What about dynamics of Focal?

- Kin network for a living Focal individual of age x

$$\phi(x) = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \end{pmatrix}^{\mathsf{T}}$$

- Kin network x years after birth of Focal

$$\phi(x+1) = \mathbf{U}\phi(x)$$

- may want to think about the stage distribution of Focal at age x

Daughters

$\mathbf{a}(x)$ = daughters of Focal

- initial condition? Focal has no daughters at birth.

$$\mathbf{a}_0 = \mathbf{0}$$

- recruitment? New daughters are the result of reproduction by Focal.

$$\beta(x) = \mathbf{F}\phi(x)$$

so

$$\mathbf{a}(x+1) = \mathbf{U}\mathbf{a}(x) + \mathbf{F}\phi(x)$$

Granddaughters

$$\mathbf{b}(x) = \text{granddaughters of Focal}$$

- initial condition

$$\mathbf{b}(0) = \mathbf{0}$$

- recruitment

$$\beta(x) = \mathbf{F}\,\mathbf{a}(x)$$

Great-granddaughters

$$\mathbf{c}(x) = \text{great-granddaughters of Focal}$$

- initial condition

$$\mathbf{c}(0) = \mathbf{0}$$

- recruitment

$$\beta(x) = \mathbf{F}\,\mathbf{b}(x)$$

- dynamics

$$\mathbf{c}(x+1) = \mathbf{U}\mathbf{c}(x) + \mathbf{F}\mathbf{b}(x)$$

Mothers

$\mathbf{d}(x)$ = mothers of Focal

- initial condition? Focal has one mother at birth; age unknown but distributed as $\boldsymbol{\pi}$

$$\mathbf{d}_0 = \sum_i \pi_i \mathbf{e}_i \;=\; \boldsymbol{\pi}$$

- recruitment? No new mothers obtained after birth of Focal

$$\beta(x) = \mathbf{0}$$

so

$$\mathbf{d}(x+1) = \mathbf{U}\mathbf{d}(x) + \mathbf{0}$$

Grandmothers

$$\mathbf{g}(x) = \text{grandmothers of Focal}$$

- initial condition: Grandmothers of Focal are the mothers of Focal's mom. Focal's mom's age is unknown but distributed as $\boldsymbol{\pi}$, so

$$\mathbf{g}(0) = \sum_i \pi_i \mathbf{d}(i)$$

- recruitment

$$\beta(x) = \mathbf{0}$$

- dynamics

$$\mathbf{g}(x+1) = \mathbf{U}\mathbf{g}(x) + \mathbf{0}$$

Great-grandmothers

$\mathbf{h}(x)$ = great-grandmothers of Focal.

$$\begin{aligned} \mathbf{h}(x+1) &= \mathbf{U}\mathbf{h}(x) + \mathbf{0} \\ \mathbf{h}_0 &= \sum_i \pi_i \mathbf{g}(i) \end{aligned}$$

ancestors of order n

$$\begin{aligned} \mathbf{k}_{n+1}(x+1) &= \mathbf{U}\mathbf{k}_{n+1}(x) + \mathbf{0} \\ \mathbf{k}_{n+1}(0) &= \sum_i \pi_i \mathbf{k}_n(i) \end{aligned}$$

Older sisters

$\mathbf{m}(x)$ = older sisters of Focal

- initial condition? Older sisters of Focal at birth are the children of Focal's mother at the birth of Focal

$$\mathbf{m}_0 = \sum_i \pi_i \mathbf{a}(i)$$

- recruitment? Focal acquires no new older sisters after she is born

$$\beta(x) = \mathbf{0}$$

so

$$\mathbf{m}(x+1) = \mathbf{U}\mathbf{m}(x) + \mathbf{0}$$

Younger sisters

$\mathbf{n}(x)$ = younger sisters of Focal

- initial condition?
$$\mathbf{n}_0 = \mathbf{0}$$
- recruitment?
$$\beta(x) = \mathbf{Fd}(x)$$

so
$$\mathbf{n}(x+1) = \mathbf{Un}(x) + \mathbf{Fd}(x)$$

Kinship[4] for multiple births[5]

[4] Princess Josephine and Prince Vincent (older by 26 minutes) of Denmark.
[5] Including animals

Older ($\mathbf{m}$) and younger ($\mathbf{n}$) sisters of Focal

$$\begin{aligned}\mathbf{m}(x+1) &= \mathbf{Um}(x) + \mathbf{0} \\ \mathbf{n}(x+1) &= \mathbf{Un}(x) + \mathbf{Fd}(x)\end{aligned}$$

Initial condition, single birth

$$\begin{aligned}\mathbf{m}_0 &= \sum_{i=1}^{\omega} \pi_i^{\text{age}} \mathbf{a}(i) \\ \mathbf{n}_0 &= \mathbf{0}\end{aligned}$$

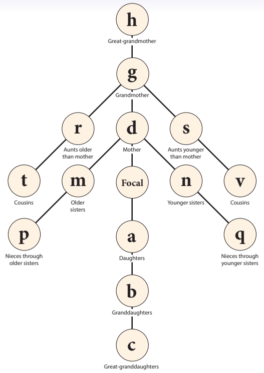
Older ($\mathbf{m}$) and younger ($\mathbf{n}$) sisters of Focal

Initial condition, multiple births

$$\begin{aligned}E(L) &= \text{mean litter size} \\ E(L|\ell \geq 1) &= \frac{E(L)}{P(L>0)}\end{aligned}$$

On average, half of Focal's littermates are older and half younger, so

$$\begin{aligned}\mathbf{m}_0 &= \sum_{i=1}^{\omega} \pi_i^{\text{age}} \mathbf{a}(i) + \frac{1}{2}\left[E(L|\ell > 1) - 1\right] \\ \mathbf{n}_0 &= \frac{1}{2}\left[E(L|\ell > 1) - 1\right]\end{aligned}$$



Nieces of Focal at age x

$\mathbf{p}(x)$ = nieces through older sisters of Focal.

$$\begin{aligned}\mathbf{p}(x+1) &= \mathbf{Up}(x) + \mathbf{Fm}(x) \\ \mathbf{n}_0 &= \sum_i \pi_i \mathbf{b}(i)\end{aligned}$$

$\mathbf{q}(x)$ = nieces through younger sisters of Focal.

$$\begin{aligned}\mathbf{q}(x+1) &= \mathbf{Uq}(x) + \mathbf{Fn}(x) \\ \mathbf{q}_0 &= \mathbf{0}\end{aligned}$$

Aunts of Focal at age x

$\mathbf{r}(x)$ = aunts older than mother of Focal.

$$\begin{aligned}\mathbf{r}(x+1) &= \mathbf{Ur}(x) + \mathbf{0} \\ \mathbf{r}_0 &= \sum_i \pi_i \mathbf{m}(i)\end{aligned}$$

$\mathbf{s}(x)$ = aunts younger than mother of Focal.

$$\begin{aligned}\mathbf{s}(x+1) &= \mathbf{Us}(x) + \mathbf{Fg}(x) \\ \mathbf{s}_0 &= \sum_i \pi_i \mathbf{n}(i)\end{aligned}$$

Cousins of Focal at age x

$\mathbf{t}(x)$ = cousins from aunts older than mother of Focal.

$$\begin{aligned}\mathbf{t}(x+1) &= \mathbf{Ut}(x) + \mathbf{Fr}(x) \\ \mathbf{t}_0 &= \sum_i \pi_i \mathbf{p}(i)\end{aligned}$$

$\mathbf{v}(x)$ = cousins from aunts younger than mother of Focal.

$$\begin{aligned}\mathbf{v}(x+1) &= \mathbf{Uv}(x) + \mathbf{Fs}(x) \\ \mathbf{v}_0 &= \sum_i \pi_i \mathbf{q}(i)\end{aligned}$$

Dynamics of kin of Focal at age x

Symbol	Kin	i.c. $\mathbf{k}_0$	$\beta(x)$
a	daughters	**0**	$\mathbf{Fe}_x$
b	granddaughters	**0**	$\mathbf{Fa}(x)$
c	great-granddaughters	**0**	$\mathbf{Fb}(x)$
d	mothers	π	**0**
g	grandmothers	$\sum_i \pi_i \mathbf{d}(i)$	**0**
h	great-grandmothers	$\sum_i \pi_i \mathbf{g}(i)$	**0**
m	older sisters	$\sum_i \pi_i \mathbf{a}(i)$	**0**
n	younger sisters	**0**	$\mathbf{Fd}(x)$
p	nieces via older sisters	$\sum_i \pi_i \mathbf{b}(i)$	$\mathbf{Fm}(x)$
q	nieces via younger sisters	**0**	$\mathbf{Fn}(x)$
r	aunts older than mother	$\sum_i \pi_i \mathbf{m}(i)$	**0**
s	aunts younger than mother	$\sum_i \pi_i \mathbf{n}(i)$	$\mathbf{Fg}(x)$
t	cousins from aunts older than mother	$\sum_i \pi_i \mathbf{p}(i)$	$\mathbf{Fr}(x)$
v	cousins from aunts younger than mother	$\sum_i \pi_i \mathbf{q}(i)$	$\mathbf{Fs}(x)$

The distribution of age at maternity π

- could be measured, or
- could be calculated from an age distribution $\mathbf{w}$

$$\pi = \frac{\mathbf{F}(1,:)^{\mathsf{T}} \circ \mathbf{w}}{\|\mathbf{F}(1,:)^{\mathsf{T}} \circ \mathbf{w}\|}$$

where
- $\mathbf{w}$ a measured age distribution, or
- $\mathbf{w}$ the stable age distribution implied by $\mathbf{U}$ and $\mathbf{F}$

What have we got?

For every type of kin $\mathbf{k}$, at every age x of Focal:

$$\mathbf{k}(x) = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_\omega \end{pmatrix}(x)$$

Analyses: Functions of the age distribution

- numbers of kin
$$\mathbf{1}^{\mathsf{T}}\mathbf{k}$$
- weighted numbers of kin (prevalence of disease, relatedness, coresidence, employment, ...)
$$\mathbf{c}^{\mathsf{T}}\mathbf{k}$$
- ratios of weighted numbers of kin (e.g., dependency ratios)
$$\frac{\mathbf{c}_1^{\mathsf{T}}\mathbf{k}}{\mathbf{c}_2^{\mathsf{T}}\mathbf{k}}$$
- mean and variance of age of kin

Analyses

- mean and variance of age of kin

$$\text{mean age} = \frac{1}{\|\mathbf{k}\|}\begin{pmatrix} 0.5 & \cdots & \omega - 0.5 \end{pmatrix}\begin{pmatrix} k_1 \\ \vdots \\ k_\omega \end{pmatrix}$$

- fraction of population experiencing a kin (Poisson approximation). Let K be a kin number:

$$P[K > 0] = 1 - \exp(-K)$$

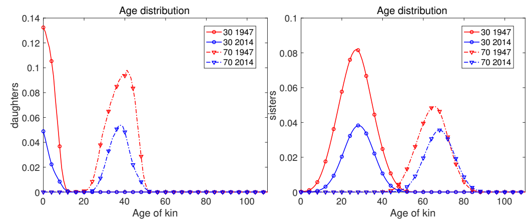
- combining types of kin (e.g., by degree of kinship)

Example: Japan 1947 and 2014[6]

	1947	2014	%
life exp	54	87	+61%
TFR	4.6	1.4	−70%
R_0	1.7	0.7	−59%

[6] Caswell 2019 Demographic Research

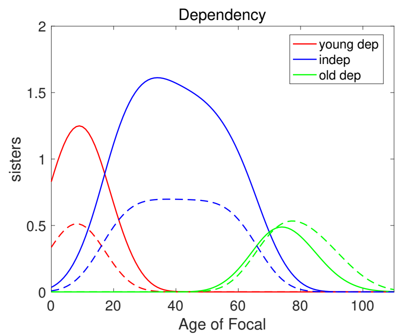
Age distributions: daughters and sisters



Dependent kin

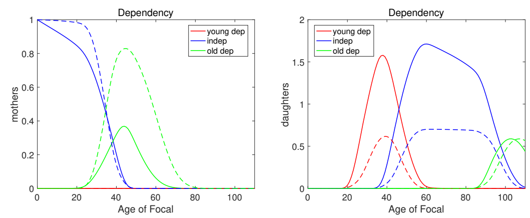
- young dependent
- independent
- old dependent

Dependent kin: sisters



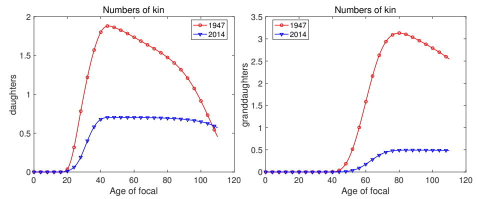
Dependent kin: daughters and mothers

Sandwich

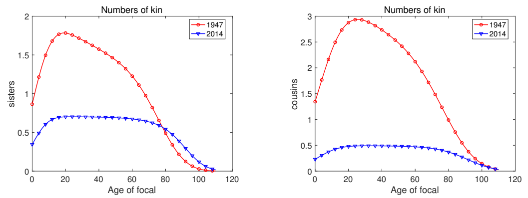


Numbers of kin

daughters and granddaughters

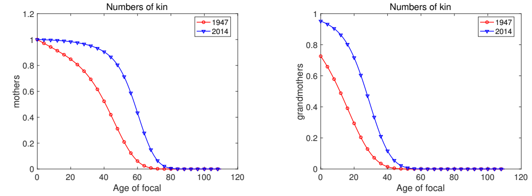


Numbers of kin: sisters and cousins



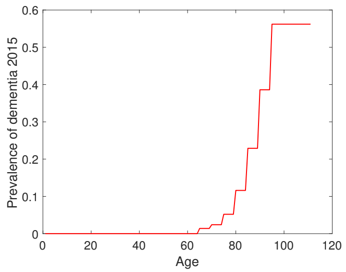
Numbers of kin

mothers and grandmothers



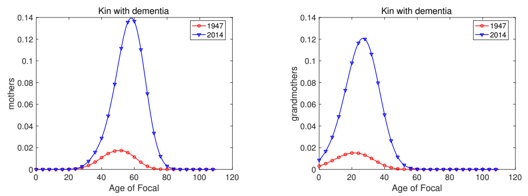
Kin numbers weighted by prevalence

Prevalence of dementia 2015[7]

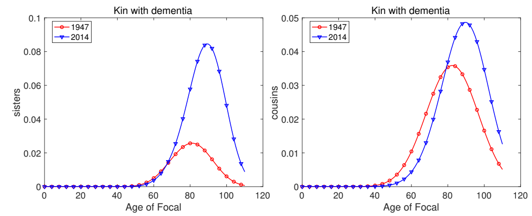


[7] Fukawa 2018, Health and Primary Care 2:1–6

Kin with dementia: mothers and grandmothers



Kin with dementia: sisters and cousins



The death of kin[8]

$$\tilde{\mathbf{k}} = \left(\begin{array}{c}\mathbf{k}_{\text{living}}\\ \hline \mathbf{k}_{\text{dead}}\end{array}\right)$$

$$\tilde{\mathbf{F}} = \left(\begin{array}{c|c}\mathbf{F} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{0}\end{array}\right)$$

$$\tilde{\mathbf{U}} = \begin{cases} \left(\begin{array}{c|c}\mathbf{U} & \mathbf{0}\\ \hline \mathbf{M} & \mathbf{0}\end{array}\right) & \textbf{deaths experienced} \text{ at age } x \\ \left(\begin{array}{c|c}\mathbf{U} & \mathbf{0}\\ \hline \mathbf{M} & \mathbf{I}\end{array}\right) & \textbf{cumulative deaths} \text{ to age } x \end{cases}$$

[8] See Alburez-Gutierrez et al. (2021)for some of the effects of bereavement due to loss of kin.
`https://DOI10.1215/00703370-9420770`

The death of kin

mortality matrix[9]

$$\mathbf{M} = \begin{pmatrix} q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & q_\omega \end{pmatrix}$$

initial condition

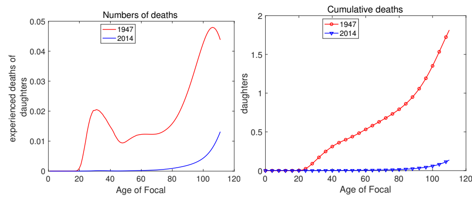
$$\tilde{\mathbf{k}}_0 = \left(\begin{array}{c}\mathbf{k}_0\\ \hline \mathbf{0}\end{array}\right)$$

So finally

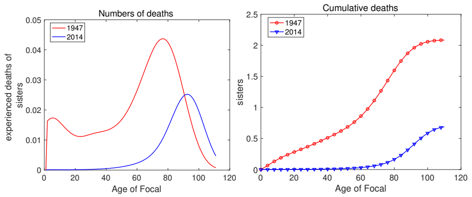
$$\tilde{\mathbf{k}}(x+1) = \tilde{\mathbf{U}}\tilde{\mathbf{k}}(x) + \tilde{\beta}(x)$$

[9] can include age and cause of death; see Caswell et al. (2023) SocArXiv
`https://doi.org/10.31235/osf.io/mk64p`

Deaths of kin: daughters



Deaths of kin: sisters



OK, time to try it out