Kinship: Multistate models and parity

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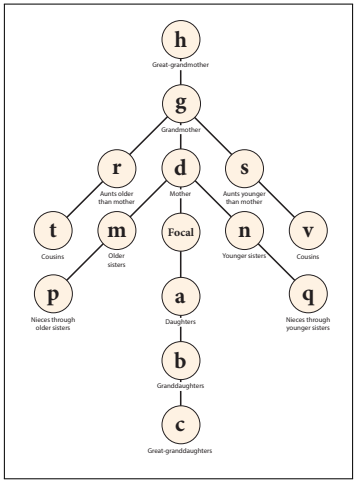
IBED 128 May 2023

The GKP kinship network[1]



[1] Keyfitz and Caswell 2005

Multistate kinship models: age and something else[2]

$$
\begin{aligned}
s &= \text{number of 'stages'} \\
\omega &= \text{number of age classes}
\end{aligned}
$$

Joint age×stage structure of kin of type **k**:

$$
\widetilde{\mathbf{k}}(x) = \begin{pmatrix} k_{11} \\ \vdots \\ k_{s1} \\ \hline \vdots \\ \hline k_{1\omega} \\ \vdots \\ k_{s\omega} \end{pmatrix}
$$

[2] Caswell 2020, Demographic Research

Multistate model: vec-permutation construction[3]

$$
\begin{aligned}
\mathbf{U}_i &= \text{transition matrix for age class } i \qquad i = 1, \ldots, \omega \\
\mathbf{D}_j &= \text{age advancement matrix for stage } j \qquad j = 1, \ldots, s \\
\mathbf{F}_i &= \text{fertility matrix for age class } i \qquad i = 1, \ldots, \omega \\
\mathbf{H}_j &= \text{offspring assignment matrix for stage } j \qquad j = 1, \ldots, s
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathbf{U}} &= \mathbf{K}_{s,\omega}^{\mathsf{T}} \, \mathbb{D}\, \mathbf{K}_{s,\omega} \, \mathbb{U} \\
\widetilde{\mathbf{F}} &= \mathbf{K}_{s,\omega}^{\mathsf{T}} \, \mathbb{H}\, \mathbf{K}_{s,\omega} \, \mathbb{F}
\end{aligned}
$$

Dynamics:

$$
\begin{aligned}
\widetilde{\mathbf{k}}(x+1) &= \widetilde{\mathbf{U}}\widetilde{\mathbf{k}}(x) + \widetilde{\boldsymbol{\beta}}(x) \\
\widetilde{\mathbf{k}}(0) &= \widetilde{\mathbf{k}}_0
\end{aligned}
$$

[3] Caswell 2018, Ecological Monographs

The joint age×stage distribution of mothers in the stable population is

$$\widetilde{\pi} = \frac{\left(\mathbf{1}_{s\omega}^{\mathsf{T}}\widetilde{\mathbf{F}}\right)^{\mathsf{T}} \circ \widetilde{\mathbf{w}}}{\left\|\left(\mathbf{1}_{s\omega}^{\mathsf{T}}\widetilde{\mathbf{F}}\right)^{\mathsf{T}} \circ \widetilde{\mathbf{w}}\right\|} \qquad s\omega \times 1. \tag{1}$$

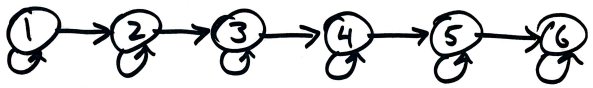
The marginal age distribution of mothers

$$\pi^{\text{age}} = \left(\mathbf{I}_{\omega} \otimes \mathbf{1}_{s}^{\mathsf{T}}\right) \widetilde{\pi} \qquad s \times 1.$$

Symbol	Kin	initial condition $\mathbf{k}_0$	Subsidy $\beta(x)$
$\widetilde{\phi}$	Focal	$\widetilde{\phi}_0$	$\mathbf{0}$
$\widetilde{\mathbf{a}}$	daughters	$\mathbf{0}$	$\widetilde{\mathbf{F}}\widetilde{\phi}(x)$
$\widetilde{\mathbf{b}}$	granddaughters	$\mathbf{0}$	$\widetilde{\mathbf{F}}\widetilde{\mathbf{a}}(x)$
$\widetilde{\mathbf{c}}$	great-granddaughters	$\mathbf{0}$	$\widetilde{\mathbf{F}}\widetilde{\mathbf{b}}(x)$
$\widetilde{\mathbf{d}}$	mothers	$\widetilde{\pi}$	$\mathbf{0}$
$\widetilde{\mathbf{g}}$	grandmothers	$\sum_i \pi_i^{\text{age}}\widetilde{\mathbf{d}}(i)$	$\mathbf{0}$
$\widetilde{\mathbf{h}}$	great-grandmothers	$\sum_i \pi_i^{\text{age}}\widetilde{\mathbf{g}}(i)$	$\mathbf{0}$
$\widetilde{\mathbf{m}}$	older sisters	$\sum_i \pi_i^{\text{age}}\widetilde{\mathbf{a}}(i)$	$\mathbf{0}$
$\widetilde{\mathbf{n}}$	younger sisters	$\mathbf{0}$	$\widetilde{\mathbf{F}}\widetilde{\mathbf{d}}(x)$
$\widetilde{\mathbf{p}}$	nieces via older sisters	$\sum_i \pi_i^{\text{age}}\widetilde{\mathbf{b}}(i)$	$\widetilde{\mathbf{F}}\widetilde{\mathbf{m}}(x)$
$\widetilde{\mathbf{q}}$	nieces via younger sisters	$\mathbf{0}$	$\widetilde{\mathbf{F}}\widetilde{\mathbf{n}}(x)$
$\widetilde{\mathbf{r}}$	aunts older than mother	$\sum_i \pi_i^{\text{age}}\widetilde{\mathbf{m}}(i)$	$\mathbf{0}$
$\widetilde{\mathbf{s}}$	aunts younger than mother	$\sum_i \pi_i^{\text{age}}\widetilde{\mathbf{n}}(i)$	$\widetilde{\mathbf{F}}\widetilde{\mathbf{g}}(x)$
$\widetilde{\mathbf{t}}$	cousins: aunts older than mother	$\sum_i \pi_i^{\text{age}}\widetilde{\mathbf{p}}(i)$	$\widetilde{\mathbf{F}}\widetilde{\mathbf{r}}(x)$
$\widetilde{\mathbf{v}}$	cousins: aunts younger than mother	$\sum_i \pi_i^{\text{age}}\widetilde{\mathbf{q}}(i)$	$\widetilde{\mathbf{F}}\widetilde{\mathbf{s}}(x)$

Multistate age×parity

Stage (parity) transition structure: parity 0 to parity 5+



$$\mathbf{U}_i = \begin{pmatrix} 1-f_1 & 0 & 0 & 0 & 0 & 0 \\ f_1 & 1-f_2 & 0 & 0 & 0 & 0 \\ 0 & f_2 & 1-f_3 & 0 & 0 & 0 \\ 0 & 0 & f_3 & 1-f_4 & 0 & 0 \\ 0 & 0 & 0 & f_4 & 1-f_5 & 0 \\ 0 & 0 & 0 & 0 & f_5 & 1 \end{pmatrix} \qquad i = 1, \ldots, \omega$$

where

p_j = probability of reproduction in parity class j

Multistate age×parity

Notice that column sums of **U** equal 1, so mortality will be included in the age advancement matrix **D**

$$\mathbf{D}_j = \begin{pmatrix} 0 & 0 & 0 \\ p_1 & 0 & 0 \\ 0 & p_2 & [p_3] \end{pmatrix} \qquad j = 1, \ldots, s$$

where p_i is $1 - q_i$ in parity stage j

An example

Slovakia 1960 – 2014

	1960	2014	
TFR	3.6	1.5	−63%
Life expectancy	62	80	+29%

Marginal distributions

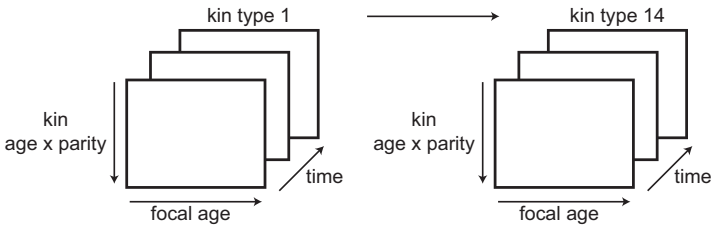
Joint age×stage structure of kin of type **k**:

$$\widetilde{\mathbf{k}}(x) = \begin{pmatrix} k_{11} \\ \vdots \\ k_{s1} \\ \hline \vdots \\ \hline k_{1\omega} \\ \vdots \\ k_{s\omega} \end{pmatrix}$$

marginal age and parity vectors

$$\begin{aligned} \mathbf{k}^{\text{age}}(x) &= \left(\mathbf{I}_\omega \otimes \mathbf{1}_s^{\mathsf{T}}\right) \widetilde{\mathbf{k}}(x) \qquad \omega \times 1 \\ \mathbf{k}^{\text{stage}}(x) &= \left(\mathbf{1}_\omega^{\mathsf{T}} \otimes \mathbf{I}_s\right) \widetilde{\mathbf{k}}(x) \qquad s \times 1. \end{aligned}$$

Output data structure



Sisters (aunts to Focal's daughters)

older sisters

$$\begin{aligned}\widetilde{\mathbf{m}}(x+1) &= \widetilde{\mathbf{U}}\widetilde{\mathbf{m}}(x)+\mathbf{0}\\ \widetilde{\mathbf{m}}_0 &= \sum_i \pi_i^{\text{age}}\,\widetilde{\mathbf{a}}(i).\end{aligned}$$

younger sisters

$$\begin{aligned}\widetilde{\mathbf{n}}(x+1) &= \widetilde{\mathbf{U}}\widetilde{\mathbf{n}}(x)+\widetilde{\mathbf{F}}\widetilde{\mathbf{d}}(x) &(2)\\ \widetilde{\mathbf{n}}_0 &= \mathbf{0}. &(3)\end{aligned}$$

Aunts (aunts to Focal)

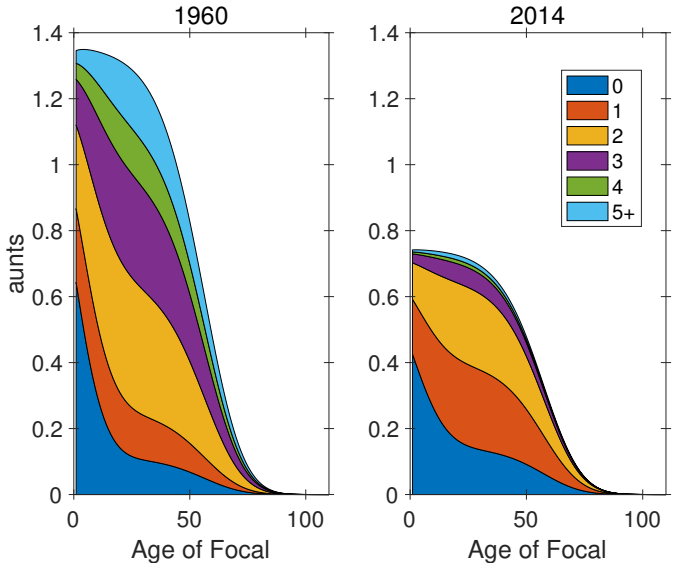
aunts older than Mother

$$\begin{aligned}\widetilde{\mathbf{r}}(x+1) &= \widetilde{\mathbf{U}}\widetilde{\mathbf{r}}(x)+\mathbf{0}\\ \widetilde{\mathbf{r}}_0 &= \sum_i \pi_i^{\text{age}}\,\widetilde{\mathbf{m}}(i).\end{aligned}$$

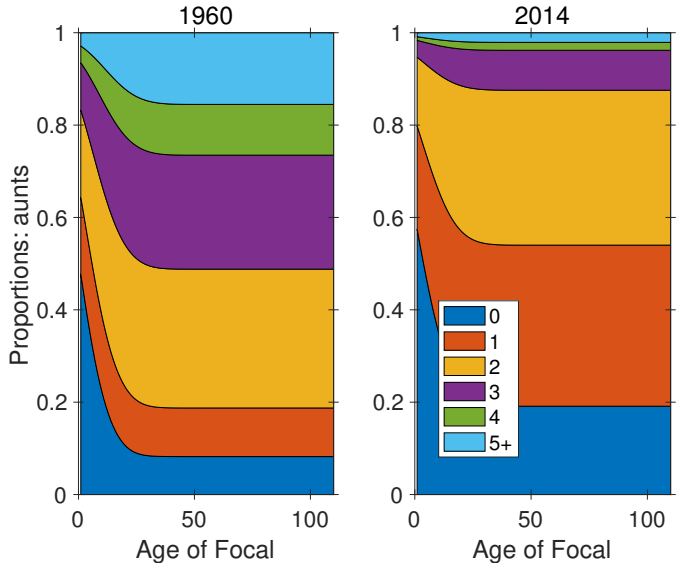
aunts younger than Mother

$$\begin{aligned}\widetilde{\mathbf{s}}(x+1) &= \widetilde{\mathbf{U}}\widetilde{\mathbf{s}}(x)+\widetilde{\mathbf{F}}\widetilde{\mathbf{g}}(x)\\ \widetilde{\mathbf{s}}_0 &= \sum_i \pi_i^{\text{age}}\widetilde{\mathbf{n}}(i).\end{aligned}$$

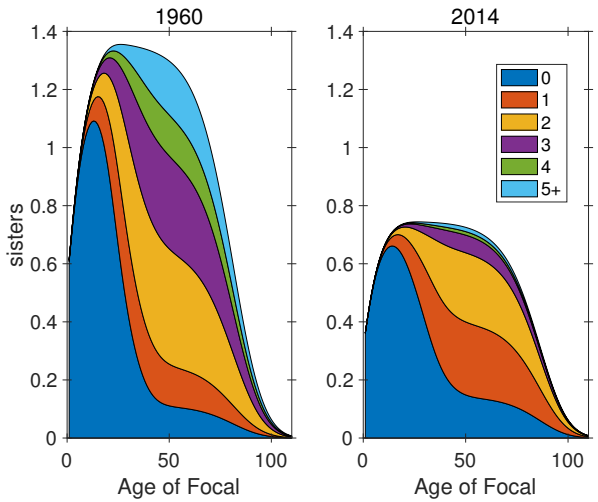
Slovakia: marginal parity structure of aunts



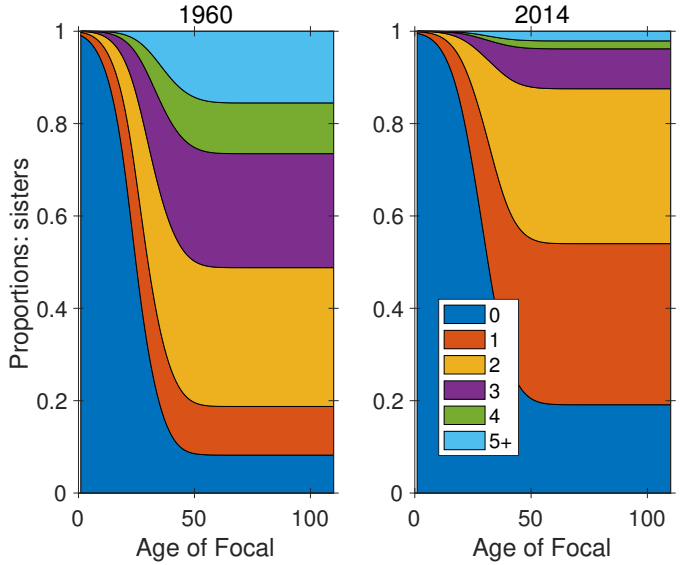
Slovakia: proportional parity structure of aunts



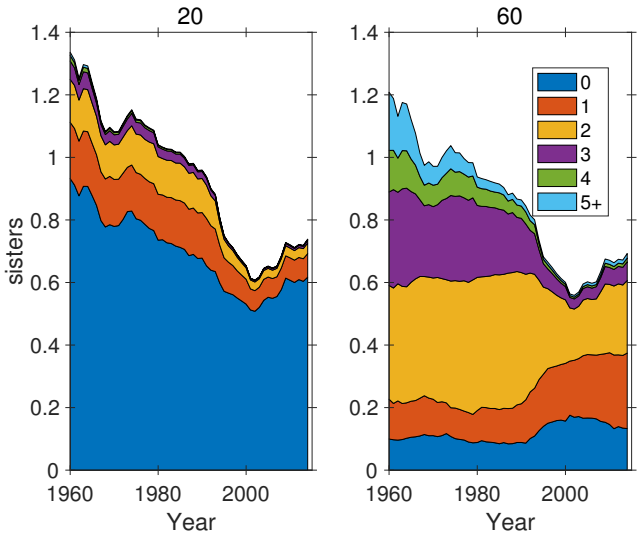
Marginal parity structure of sisters



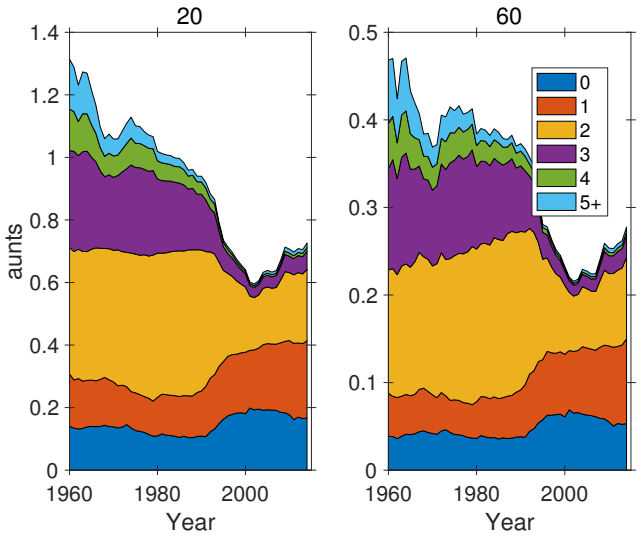
Slovakia: proportional parity structure of sisters



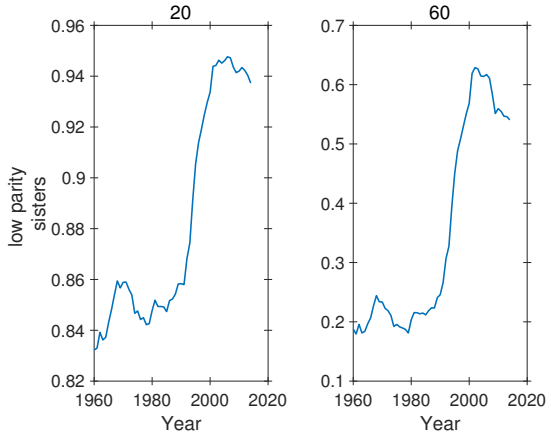
Parity structure of sisters over time



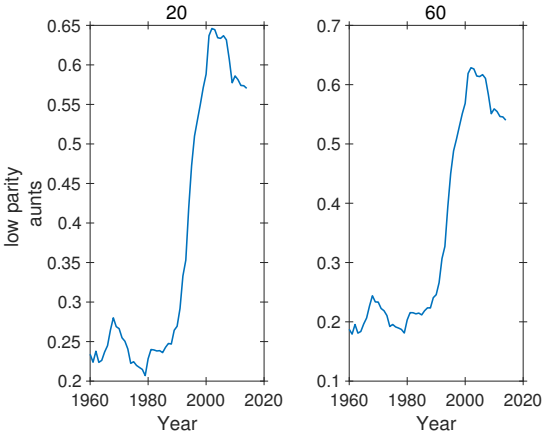
Parity structure of aunts over time



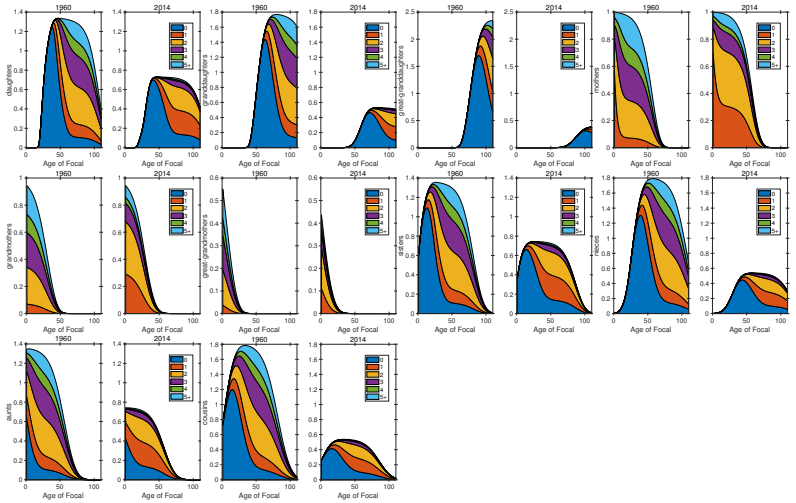
Proportion low parity (0 and 1) sisters



Proportion low parity (0 and 1) aunts

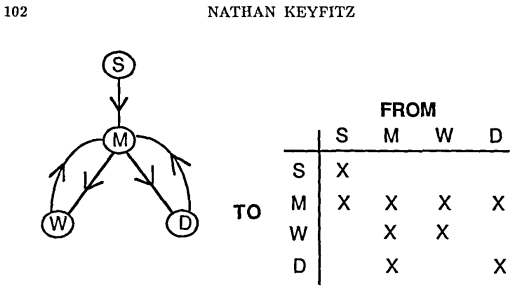


Lots of kin



Other age-stage possibilities?

Marital status[4]



[4] Keyfitz, N. 1988. A Markov chain for calculating the durability of marriage. Math. Pop. Studies 1:101-121.

Other age-stage possibilities?

Health, employment[5]

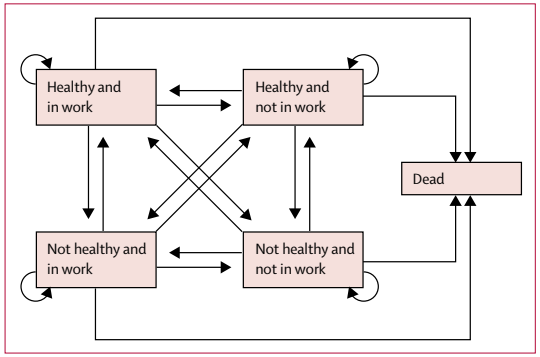


Figure 1: **Multi-state model of healthy working life expectancy**

[5]Parker et al. 2020. Lancet Public Health 5:e395-403.

Other age-stage possibilities?

- location
- SES variables (education,. . .)
- ?

What is needed to incorporate these into the kinship model?