Markov chains with rewards

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IBED

Individual lives

- trajectories through the life cycle
- probabilities given by transition matrix

$$\mathbf{P} = \left(\begin{array}{c|c} \mathbf{U} & \mathbf{0} \\ \mathbf{M} & \mathbf{I} \end{array} \right)$$

- trajectories (lives) are stochastic

Lifetime accumulation

- imagine accumulating something over lifetime
- call it a “reward”

(much like kaiten sushi)

Examples of rewards

reward	lifetime accumulation
a year of life	longevity
a year of healthy life	healthy longevity
an offspring	lifetime reproductive output
income	lifetime income

Analysis

transition matrix

$$\mathbf{P} = \left(\begin{array}{c|c} \mathbf{U} & \mathbf{0} \\ \hline \mathbf{M} & \mathbf{I} \end{array}\right)$$

define a reward matrix

$$\mathbf{R}(i,j) = \text{reward for transition } j \to i$$

but rewards are random, so define the moments of reward

$$\mathbf{R}_m(i,j) = m\text{th moment of reward} \qquad m = 1, 2, 3, \ldots$$

assume no rewards for the dead

$$\mathbf{R}_m(i,j) = 0 \qquad \text{if } j \text{ an absorbing state}$$

Analysis[1]

$$\tilde{\rho}_i = \text{vector of } i\text{th moments of lifetime rewards}$$

$$\begin{aligned}
\tilde{\rho}_1 &= \mathbf{N}^{\mathsf{T}}\mathbf{Z}(\mathbf{P} \circ \mathbf{R}_1)^{\mathsf{T}}\mathbf{1}_{s+1} \\
\tilde{\rho}_2 &= \mathbf{N}^{\mathsf{T}}\left[\mathbf{Z}(\mathbf{P} \circ \mathbf{R}_2)^{\mathsf{T}}\mathbf{1}_{s+1} + 2(\mathbf{U} \circ \tilde{\mathbf{R}}_1)^{\mathsf{T}}\tilde{\rho}_1\right] \\
\tilde{\rho}_3 &= \mathbf{N}^{\mathsf{T}}\left[\mathbf{Z}(\mathbf{P} \circ \mathbf{R}_3)^{\mathsf{T}}\mathbf{1}_{s+1} + 3(\mathbf{U} \circ \tilde{\mathbf{R}}_2)^{\mathsf{T}}\tilde{\rho}_1 + 3(\mathbf{U} \circ \tilde{\mathbf{R}}_1)^{\mathsf{T}}\tilde{\rho}_2\right]
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{N} &= (\mathbf{I} - \mathbf{U})^{-1} \\
\mathbf{Z} &= \left(\begin{array}{c|c} \mathbf{I}_{\tau\times\tau} & \mathbf{0}_{\tau\times\alpha} \end{array}\right) \\
\tilde{\mathbf{R}}_m &= \mathbf{R}_m(1:\tau, 1:\tau)
\end{aligned}$$

[1]van Daalen and Caswell 2017, Theoretical Ecology, Thm. 1.

Analysis

$$\begin{aligned}
E(\tilde{\rho}) &= \tilde{\rho}_1 \\
V(\tilde{\rho}) &= \tilde{\rho}_2 - \tilde{\rho}_1 \circ \tilde{\rho}_1 \\
SD(\tilde{\rho}) &= \sqrt{V(\tilde{\rho})}
\end{aligned}$$

(and skewness, kurtosis, etc.)

Healthy longevity: prevalence[2]

- prevalence is probability of having condition of interest
- prevalence measured from cross-sectional data
- health expectancy (HE), via Sullivan method
- Sullivan method for health expectancy

$$\begin{aligned}
\ell(x) &= P[\text{survival to age } x] \\
v(x) &= \text{prevalence of health at age } x \\
HE(a) &= \frac{1}{\ell(a)}\int_a^\infty \ell(x)v(x)dx \\
&= E\,(\text{healthy years remaining at age } a)
\end{aligned}$$

- treat health condition as a reward

[2]Caswell and Zarulli 2018, Population Health Metrics

Reward matrices for prevalence

$$v_j = \text{prevalence in age class } j$$

then

$$\text{reward } j \to i = \begin{cases} 1 & \text{with probability } v_j \\ 0 & \text{with probability } 1 - v_j \end{cases}$$

so

$$\mathbf{R}_1 = \left(\begin{array}{ccc|c} v_1 & \cdots & v_\omega & 0 \\ \vdots & & \vdots & \vdots \\ v_1 & \cdots & v_\omega & 0 \\ \hline v_1/2 & \cdots & v_\omega/2 & 0 \end{array} \right)$$

and

$$\mathbf{R}_2 = \mathbf{R}_3 = \cdots = \mathbf{R}_1$$

Stochasticity in healthy longevity: the missing ingredient

- stochasticity in trajectories
- stochasticity in health outcomes (prevalence)
- variance among individuals in lifetime experience

Types of health rewards

- binary (e.g., disability-free)
  - Bernoulli distribution
- counts (e.g., number of hospitalizations)
  - maybe Poisson distribution
- quantitative measures (e.g., grip strength)
  - empirical measurement of moments
- amount of life lost (e.g., in DALY)
  - moments calculated from Markov chain model

SHARE survey data

- Survey of Health, Ageing and Retirement in Europe
- wave 4, 2011, data for Germany, Sweden, France, Denmark, Switzerland, Belgium, Czech Republic, Portugal, and Estonia
- prevalence of disability (binary): moments calculated from Bernoulli distribution
- grip strength[3] (continuous): moments calculated from empirical individual-level measures
- measurements only reliable to age 90

[3] Affects mortality, myocardial infarction, stroke, cardiovascular mortality

An example: healthy longevity from SHARE data[4]

v_j = prevalence of disability-freedom in age class j

Rewards

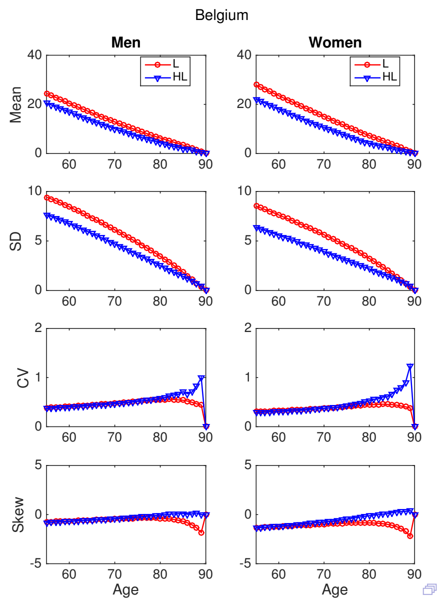
$$r_{ij} = \begin{cases} 1 & \text{with probability } v_j \\ 0 & \text{with probability } 1 - v_j \end{cases}$$

[4]Caswell and Zarulli 2018, Population Health Metrics

Disability-free longevity (Belgium)

		Men		Women	
		55	75	55	75
mean	L	24.5	10.1	27.9	11.8
	HL	20.6	7.6	22.2	7.9
SD	L	9.4	5.1	8.5	5.1
	HL	7.6	3.8	6.4	3.2
CV	L	0.38	0.50	0.31	0.40
	HL	0.37	0.50	0.29	0.41
Sk	L	-0.75	-0.38	-1.34	-0.89
	HL	-0.80	-0.28	-1.35	-0.51

Disability-free longevity by age



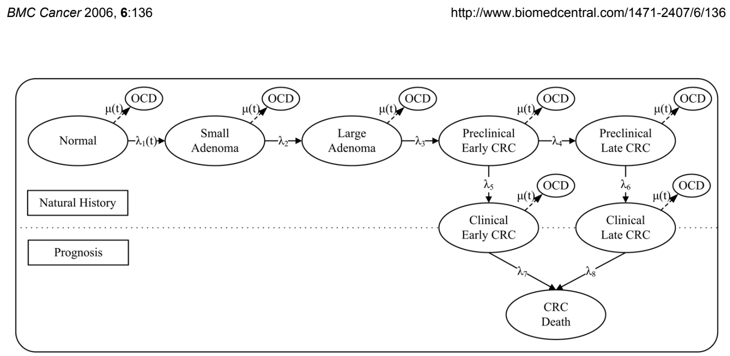
See the paper for more results

Healthy longevity: incidence models[5]

- incidence refers to transitions among health states
- requires longitudinal data
- age-specific health transition matrices . . .
- lead to age-stage multistate Markov chains
- health is part of the i-state, not just a prevalence
- basic questions
  - occupancy
  - longevity (total, healthy)
  - eventual fate
  - accumulation of rewards
  - life lost due to causes
  - sensitivity analysis

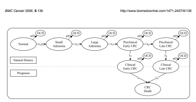
[5] Caswell and van Daalen 2021, Demographic Research

An example: a model for colorectal cancer[6]



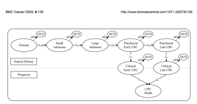
[6] Wu et al. 2006, BMC Cancer

Multistate model: ages and health stages



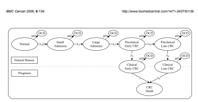
$x = 1$

$\searrow$



$x = 2$

$\ddots$



$x = \omega$

How to measure “healthy longevity”

1. **Occupancy of health states.** How much life spent in a specified health condition, including combinations of ages and health stages?
2. Transitions among health states.
3. The “value” of occupancy of health states.
4. The “value” of transitions among states.

Constructing age × stage matrix models[7]

$$
\begin{aligned}
\mathbf{U}_i &= \text{age-specific state transitions} \\
\mathbb{U} &= \text{block diagonal matrix} \\
\mathbf{D}_j &= \text{stage-specific age transitions} \\
\mathbb{D} &= \text{block diagonal matrix} \\
\\
\mathbf{K} &= \text{vec-permutation matrix}
\end{aligned}
$$

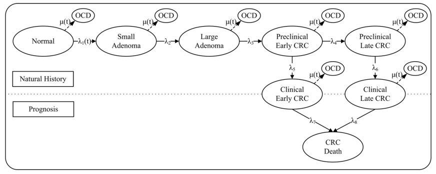
Then

$$
\begin{aligned}
\tilde{\mathbf{U}} &= \mathbf{K}^{\mathsf{T}}\, \mathbb{D}\, \mathbf{K}\, \mathbb{U} \\
\tilde{\mathbf{P}} &= \left(\begin{array}{c|c} \tilde{\mathbf{U}} & \mathbf{0} \\ \hline \mathbf{M} & \mathbf{I} \end{array} \right)
\end{aligned}
$$

[7]Caswell et al. 2018. *Ecological Monographs*; Caswell 2019, *Demographic Research Monographs*.

Define "healthy" longevity in the CRC model

BMC Cancer 2006, **6**:136 http://www.biomedcentral.com/1471-2407/6/136



For example:

1. **Cancer-free longevity.** Time spent in any of the cancer-free stages (1–3) over a lifetime.
2. **Old age clinical cancer.** The time spent in any of the clinical cancer stages (6–7) after a specified age (65 years)

Longevity by age and stage: the health matrix

Create an array **H** to specify age-stage combinations of interest

	Ages					
Stages	1	2	3	4	...	ω
healthy						
small adenoma						
large adenoma						
preclinical early CRC						
preclinical late CRC						
clinical early CRC						
clinical late CRC						

e.g., Cancer-free longevity

Put a 1 in cells that count as "healthy"

	Ages					
Stage	1	2	3	4	...	ω
normal cells	1	1	1	1	1	1
small adenoma	1	1	1	1	1	1
large adenoma	1	1	1	1	1	1
preclinical early CRC						
preclinical late CRC						
clinical early CRC						
clinical late CRC						

(zeros elsewhere)

Old age clinical cancer

Stages	Ages 1	2	3	4	...	ω
healthy						
small adenoma						
large adenoma						
preclinical early CRC						
preclinical late CRC						
clinical early CRC				1	...	1
clinical late CRC				1	...	1

(zeros elsewhere)

How many definitions of health are there?

How many definitions of health are there?

$$\begin{aligned} \text{7 cancer stages} \times \text{50 ages} &= \text{350 combinations} \\ \text{Number of definitions of health} &= 2^{350} \approx 10^{105} \end{aligned}$$

Compare this to 10^{24} stars in the visible universe

Reward matrices

Including partial occupancy when transitions into or out of states occur

Define

$$\mathbf{r} = \operatorname{vec} \mathbf{X}$$

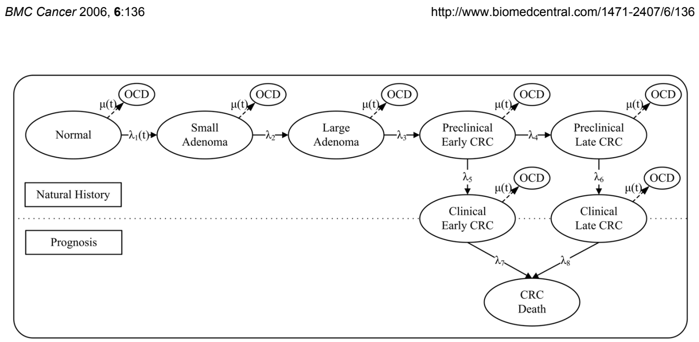
Then

$$\widetilde{\mathbf{R}}_1 = \mathbf{r}\mathbf{r}^{\mathsf{T}} + 0.5\,(\sim \mathbf{r})\,(\mathbf{r}^{\mathsf{T}}) + 0.5\,(\mathbf{r})\,(\sim \mathbf{r}^{\mathsf{T}})$$

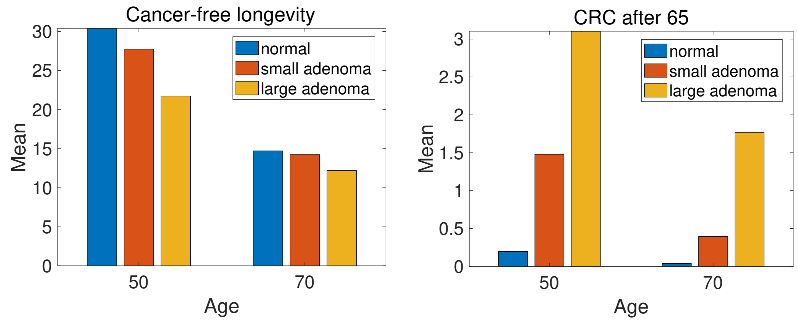
and

$$\begin{aligned} \widetilde{\mathbf{R}}_2 &= \widetilde{\mathbf{R}}_1 \circ \widetilde{\mathbf{R}}_1 \\ \widetilde{\mathbf{R}}_3 &= \widetilde{\mathbf{R}}_1 \circ \widetilde{\mathbf{R}}_1 \circ \widetilde{\mathbf{R}}_1 \end{aligned}$$

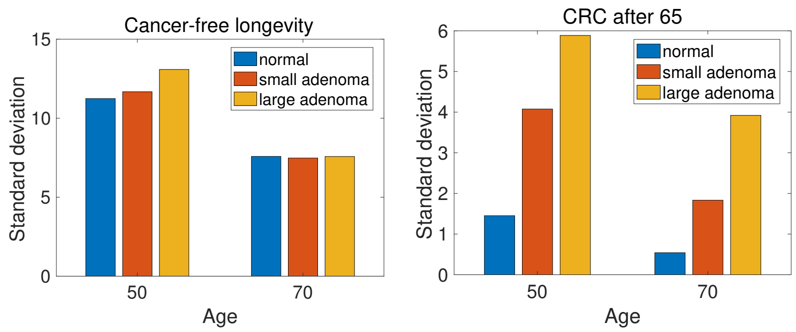
Colorectal cancer



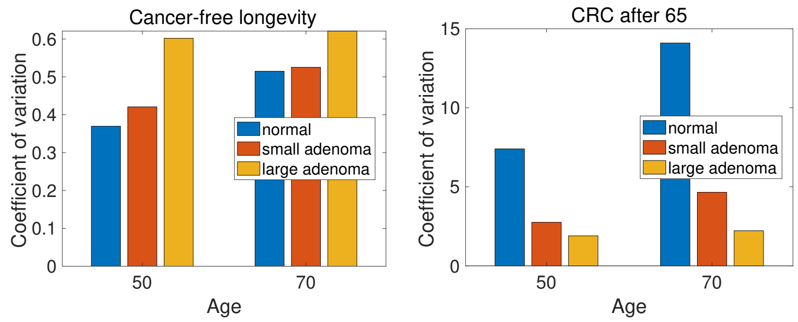
Mean healthy longevity



Standard deviation healthy longevity



Coefficient of variation of healthy longevity



Concluding thoughts: MCWR

- very powerful tool for analyzing variation in lifetime experiences
- a fixation on expectancies neglects important aspects related to risk

- Markov chains with rewards give all the moments of lifetime accumulation
- in the case of health, provides limitless defintion of "healthy" longevity: any combination of health stages and ages

- extensions $\longrightarrow$ value or cost of occupancy, transitions into or out of stages, components of variance, decomposition of differences