The formal demography of kinship: Two-sex models

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Why kinship?

The kinship model

The kin of Focal as a population

$$
\begin{aligned}
\mathbf{k}(x+1) &= \mathbf{U}\mathbf{k}(x) + \boldsymbol{\beta}(x) \\
\mathbf{k}(0) &= \mathbf{k}_0
\end{aligned}
$$

where

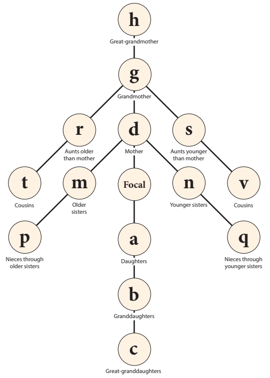
$$
\begin{aligned}
\mathbf{k}(x) &= \text{age distribution of a kin at age } x \text{ of Focal}^1 \\
\boldsymbol{\beta}(x) &= \text{recruitment `subsidy' at age } x \text{ of Focal} \\
&= \begin{cases} \mathbf{0} & \text{no subsidy} \\ \mathbf{F}\,\mathbf{k}^*(x) & \text{subsidy from } \mathbf{k}^* \end{cases} \\
\mathbf{k}_0 &= \text{initial condition}
\end{aligned}
$$

[1]To be more precise, the expectation of the age distribution.

Dynamics of kin of Focal at age x

Symbol	Kin	i.c. $\mathbf{k}_0$	$\beta(x)$
a	daughters	$\mathbf{0}$	$\mathbf{Fe}_x$
b	granddaughters	$\mathbf{0}$	$\mathbf{Fa}(x)$
c	great-granddaughters	$\mathbf{0}$	$\mathbf{Fb}(x)$
d	mothers	π	$\mathbf{0}$
g	grandmothers	$\sum_i \pi_i \mathbf{d}(i)$	$\mathbf{0}$
h	great-grandmothers	$\sum_i \pi_i \mathbf{g}(i)$	$\mathbf{0}$
m	older sisters	$\sum_i \pi_i \mathbf{a}(i)$	$\mathbf{0}$
n	younger sisters	$\mathbf{0}$	$\mathbf{Fd}(x)$
p	nieces via older sisters	$\sum_i \pi_i \mathbf{b}(i)$	$\mathbf{Fm}(x)$
q	nieces via younger sisters	$\mathbf{0}$	$\mathbf{Fn}(x)$
r	aunts older than mother	$\sum_i \pi_i \mathbf{m}(i)$	$\mathbf{0}$
s	aunts younger than mother	$\sum_i \pi_i \mathbf{n}(i)$	$\mathbf{Fg}(x)$
t	cousins from aunts older than mother	$\sum_i \pi_i \mathbf{p}(i)$	$\mathbf{Fr}(x)$
v	cousins from aunts younger than mother	$\sum_i \pi_i \mathbf{q}(i)$	$\mathbf{Fs}(x)$

Female kin through female lines of descent



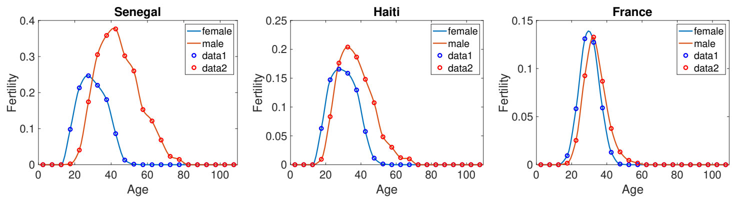
We want both sexes through all lines of descent

Two-sex kinship: demographic components

$$
\begin{aligned}
\mathbf{U}_f, \mathbf{U}_m &= \text{female and male survival} \\
\mathbf{F}_f, \mathbf{F}_m &= \text{female and male fertility} \\
\boldsymbol{\pi}_f, \boldsymbol{\pi}_m &= \text{female and male ages at birth} \\
\alpha &= \text{proportion male births} \\
\bar{\alpha} &= 1 - \alpha
\end{aligned}
$$

Female and male rates[2]

- mortality differences universal and well known
- fertility differences can be large
  - in timing
  - in amount



[2] Schoumaker, B. 2019. Male Fertility Around the World and Over Time: How Different is it from Female Fertility? Population and Development Review 45(3): 459-487

Female and male fertility

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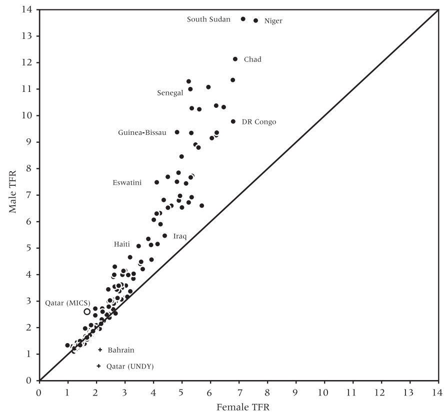
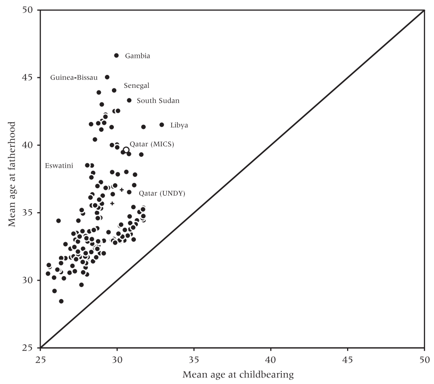
FIGURE 2 Comparison of male and female total fertility rates, circa 2011 in 163 countries



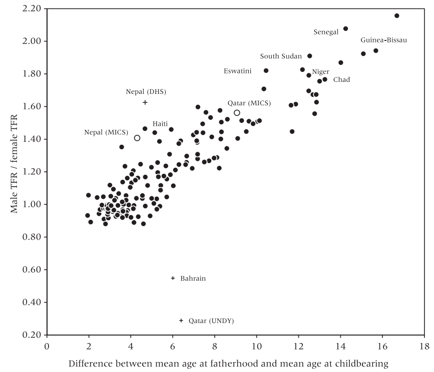
Female and male fertility

FIGURE 3 Comparison of mean age at childbearing (women) and mean age at fatherhood (men), circa 2011 in 163 countries



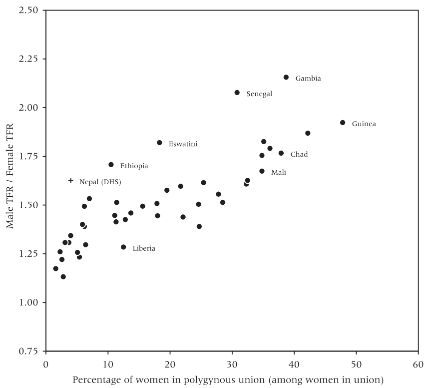
Female and male fertility

FIGURE 5 Relationship between the ratio of male TFR to female TFR and the difference between mean age at fatherhood and mean age at childbearing, circa 2011 in 163 countries

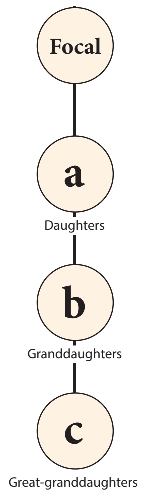


Female and male fertility

FIGURE 6 Relationship between male–female fertility ratio and polygyny, circa 2011 in 43 countries

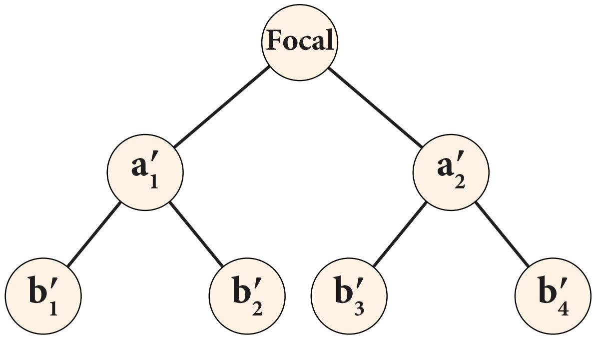


Descendants: one-sex lineage



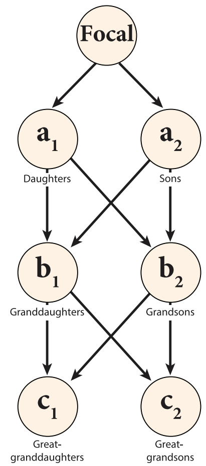
Descendants of both sexes, both lineages

female and male lineages

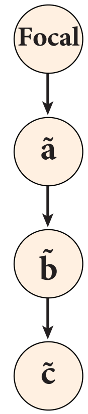


Descendants of both sexes

rates independent of lineage



block-structured vectors



Two-sex models

Block-structured population vector

$$\widetilde{\mathbf{k}} = \left(\begin{array}{c} \mathbf{k}_\mathrm{f} \\ \hline \mathbf{k}_\mathrm{m} \end{array} \right)$$

Two-sex models

Block-structured matrices

$$\widetilde{\mathbf{U}} = \left(\begin{array}{c|c} \mathbf{U}_\mathrm{f} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{U}_\mathrm{m} \end{array} \right)$$

$$\widetilde{\mathbf{F}} = \left(\begin{array}{c|c} \bar{\alpha}\mathbf{F}_f & \bar{\alpha}\mathbf{F}_\mathrm{m} \\ \hline \alpha\mathbf{F}_f & \alpha\mathbf{F}_\mathrm{m} \end{array} \right)$$

$$\widetilde{\mathbf{F}}^* = \left(\begin{array}{c|c} \bar{\alpha}\mathbf{F}_\mathrm{f} & \mathbf{0} \\ \hline \alpha\mathbf{F}_\mathrm{f} & \mathbf{0} \end{array} \right)$$

where

$$\begin{aligned} \alpha &= \text{sex ratio (proportion male births)} \\ \bar{\alpha} &= 1 - \alpha \end{aligned}$$

Age at maternity and paternity

$$\widetilde{\pi} = \begin{pmatrix} \pi_{\mathrm{f}} \\ \hline \pi_{\mathrm{m}} \end{pmatrix}$$

where

$$\|\pi_{\mathrm{f}}\| = \|\pi_{\mathrm{m}}\| = 1$$

The dynamics of $\widetilde{\mathbf{k}}(x)$

$$\begin{aligned} \begin{pmatrix} \mathbf{k}_{\mathrm{f}} \\ \hline \mathbf{k}_{\mathrm{m}} \end{pmatrix}(x+1) &= \left(\begin{array}{c|c} \mathbf{U}_{\mathrm{f}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{U}_{\mathrm{m}} \end{array}\right) \begin{pmatrix} \mathbf{k}_{\mathrm{f}} \\ \hline \mathbf{k}_{\mathrm{m}} \end{pmatrix}(x) + \begin{pmatrix} \beta_{\mathrm{f}} \\ \hline \beta_{\mathrm{m}} \end{pmatrix}(x) \\ \widetilde{\mathbf{k}}(x+1) &= \widetilde{\mathbf{U}}\widetilde{\mathbf{k}}(x) + \widetilde{\beta}(x). \end{aligned}$$

Children, grandchildren, ...

Children

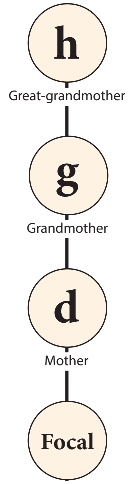
$$\widetilde{\mathbf{a}}(x+1) = \widetilde{\mathbf{U}}\widetilde{\mathbf{a}}(x) + \widetilde{\mathbf{F}}\widetilde{\phi}(x)$$

Grandchildren

$$\widetilde{\mathbf{b}}(x+1) = \widetilde{\mathbf{U}}\widetilde{\mathbf{b}}(x) + \widetilde{\mathbf{F}}\widetilde{\mathbf{a}}(x)$$

etc.

One-sex lineage of ancestry



Parents, grandparents, ...

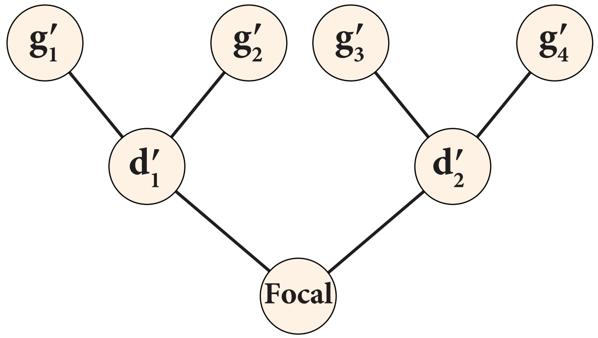
Parents

$$\begin{aligned}\tilde{\mathbf{d}}(x+1) &= \tilde{\mathbf{U}}\tilde{\mathbf{d}}(x) + \mathbf{0}\\ \tilde{\mathbf{d}}(0) &= \tilde{\pi}\end{aligned}$$

Grandparents

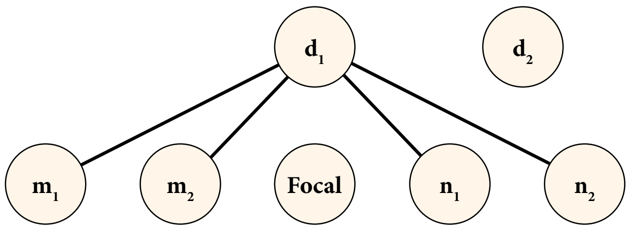
$$\begin{aligned}\tilde{\mathbf{g}}(x+1) &= \tilde{\mathbf{U}}\tilde{\mathbf{g}}(x) + \mathbf{0}\\ \tilde{\mathbf{g}}(0) &= \sum_i \left[\pi_{\mathrm{f}}(i) + \pi_{\mathrm{m}}(i)\right] \tilde{\mathbf{d}}(i)\end{aligned}$$

Counting ancestors of both sexes



(and then combine to get block-structured vectors, as with descendants)

Ancestors don't reproduce independently



$$\begin{pmatrix}\mathbf{n}_{\mathrm{f}}\\ \mathbf{n}_{\mathrm{m}}\end{pmatrix}(x+1) = \tilde{\mathbf{U}}\begin{pmatrix}\mathbf{n}_{\mathrm{f}}\\ \mathbf{n}_{\mathrm{m}}\end{pmatrix}(x) + \left(\begin{array}{c|c}\bar{\alpha}\mathbf{F}_{\mathrm{f}} & \mathbf{0}\\ \alpha\mathbf{F}_{\mathrm{f}} & \mathbf{0}\end{array}\right)\begin{pmatrix}\mathbf{d}_{\mathrm{f}}\\ \mathbf{d}_{\mathrm{m}}\end{pmatrix}(x)$$

$$\tilde{\mathbf{n}}(x+1) = \tilde{\mathbf{U}}\,\tilde{\mathbf{n}}(x) + \tilde{\mathbf{F}}^{*}\,\tilde{\mathbf{d}}(x)$$

Recruitment

1. by a direct ancestor of Focal

$$\begin{pmatrix}\beta_{\mathrm{f}}\\ \beta_{\mathrm{m}}\end{pmatrix}(x) = \left(\begin{array}{c|c}\bar{\alpha}\mathbf{F}_{\mathrm{f}} & \mathbf{0}\\ \alpha\mathbf{F}_{\mathrm{f}} & \mathbf{0}\end{array}\right)\begin{pmatrix}\mathbf{k}^{*}{}_{\mathrm{f}}\\ \mathbf{k}^{*}{}_{\mathrm{m}}\end{pmatrix}(x)$$

$$\tilde{\beta}(x) = \tilde{\mathbf{F}}^{*}\tilde{\mathbf{k}}(x)$$

2. by any other kin type

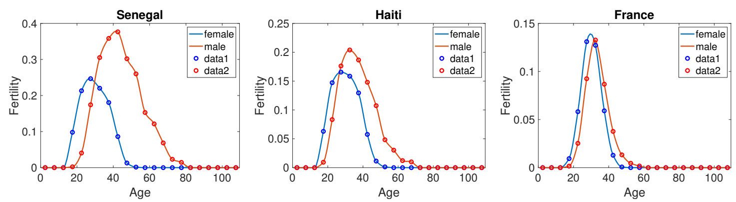
$$\begin{pmatrix}\beta_{\mathrm{f}}\\ \beta_{\mathrm{m}}\end{pmatrix}(x) = \left(\begin{array}{c|c}\bar{\alpha}\mathbf{F}_{\mathrm{f}} & \bar{\alpha}\mathbf{F}_{\mathrm{m}}\\ \alpha\mathbf{F}_{\mathrm{f}} & \bar{\alpha}\mathbf{F}_{\mathrm{m}}\end{array}\right)\begin{pmatrix}\mathbf{k}^{*}{}_{\mathrm{f}}\\ \mathbf{k}^{*}{}_{\mathrm{m}}\end{pmatrix}(x)$$

$$\tilde{\beta}(x) = \tilde{\mathbf{F}}\tilde{\mathbf{k}}(x)$$

3. no recruitment subsidy, then $\tilde{\beta} = \mathbf{0}$.

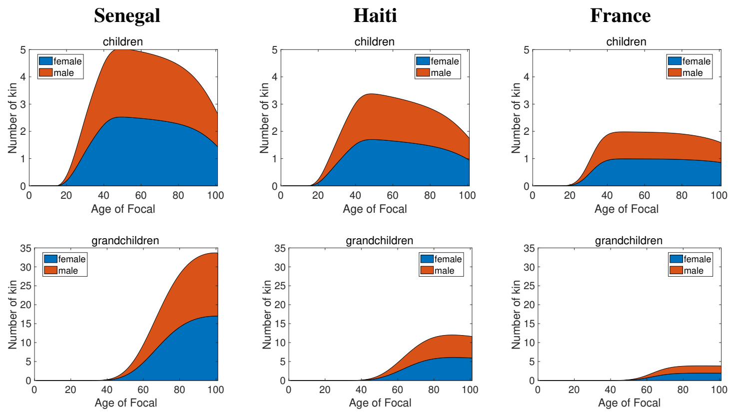
The complete two-sex model

Symbol	Kin	initial condition $\mathbf{k}_0$	Subsidy $\beta(x)$
$\tilde{\mathbf{a}}$	children	$\mathbf{0}$	$\tilde{\mathbf{F}}\phi(x)$
$\tilde{\mathbf{b}}$	grandchildren	$\mathbf{0}$	$\tilde{\mathbf{F}}\tilde{\mathbf{a}}(x)$
$\tilde{\mathbf{c}}$	great-grandchildren	$\mathbf{0}$	$\tilde{\mathbf{F}}\tilde{\mathbf{b}}(x)$
$\tilde{\mathbf{d}}$	parents	$\tilde{\boldsymbol{\pi}}$	$\mathbf{0}$
$\tilde{\mathbf{g}}$	grandparents	$\sum_i \pi_{\mathrm{fm}}(i)\,\tilde{\mathbf{d}}(i)$	$\mathbf{0}$
	great-grandparents	$\sum_i \pi_{\mathrm{fm}}(i)\,\tilde{\mathbf{g}}(i)$	$\mathbf{0}$
$\tilde{\mathbf{m}}$	older siblings	$\sum_i \pi_{\mathrm{f}}(i)\,\tilde{\mathbf{a}}(i)$	$\mathbf{0}$
$\tilde{\mathbf{n}}$	younger siblings	$\mathbf{0}$	$\tilde{\mathbf{F}}^*\,\tilde{\mathbf{d}}(x)$
$\tilde{\mathbf{p}}$	nieces/nephews via older siblings	$\sum_i \pi_{\mathrm{f}}(i)\,\tilde{\mathbf{b}}(i)$	$\tilde{\mathbf{F}}\tilde{\mathbf{m}}(x)$
$\tilde{\mathbf{q}}$	nieces/nephews via younger siblings	$\mathbf{0}$	$\tilde{\mathbf{F}}\tilde{\mathbf{n}}(x)$
$\tilde{\mathbf{r}}$	aunts/uncles older than mother	$\sum_i \pi_{\mathrm{fm}}(i)\,\tilde{\mathbf{m}}(i)$	$\mathbf{0}$
$\tilde{\mathbf{s}}$	aunts/uncles younger than mother	$\sum_i \pi_{\mathrm{fm}}(i)\,\mathbf{n}(i)$	$\tilde{\mathbf{F}}^*\,\tilde{\mathbf{g}}(x)$
$\tilde{\mathbf{t}}$	cousins from aunts/uncles older than mother	$\sum_i \pi_{\mathrm{fm}}(i)\,\tilde{\mathbf{p}}(i)$	$\tilde{\mathbf{F}}\tilde{\mathbf{r}}(x)$
$\tilde{\mathbf{v}}$	cousins from aunts/uncles younger than mother	$\sum_i \pi_{\mathrm{fm}}(i)\,\tilde{\mathbf{q}}(i)$	$\tilde{\mathbf{F}}\tilde{\mathbf{s}}(x)$

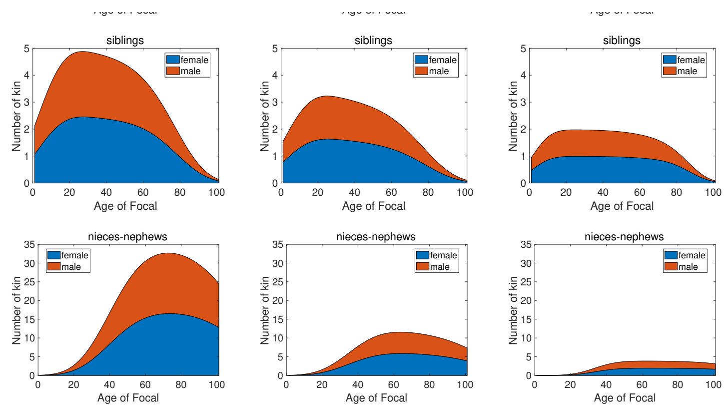


	Senegal	Haiti	France
TFR female	5.3	3.7	2.0
TFR male	11.0	5.1	2.0
age at maternity	29.8	30.0	30.1
age at paternity	44.1	37.0	33.5

Female and male kin, through all lineages



Female and male kin, through all lineages



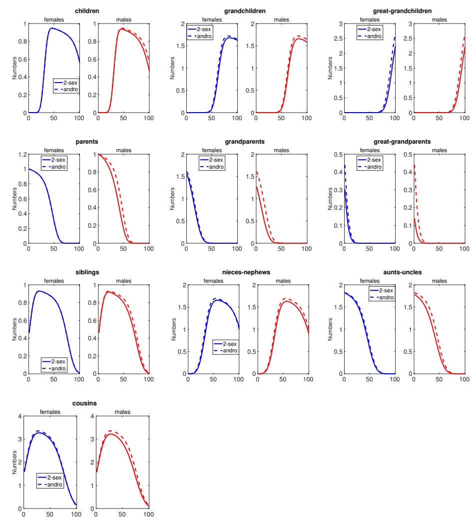
Approximations to a two-sex model

$$\tilde{\mathbf{U}} = \left(\begin{array}{c|c} \mathbf{U}_\mathrm{f} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{U}_\mathrm{m} \end{array} \right) \qquad \tilde{\mathbf{F}} = \left(\begin{array}{c|c} \bar{\alpha}\mathbf{F}_f & \bar{\alpha}\mathbf{F}_\mathrm{m} \\ \hline \alpha\mathbf{F}_f & \alpha\mathbf{F}_\mathrm{m} \end{array} \right)$$

- model 1: full two-sex model
- model 2: use female fertility for both sexes
- model 3: use female mortality for both sex
- model 4: males and females identical (the ‘androgynous approximation’)

Put appropriate survival and fertility matrices into $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{F}}$

The androgynous approximation[3]



[3] for Senegal

Androgynous approximation: the GKP factors[4]

$$\Big(\text{1-sex } \mathbf{k}(x)\Big) \times \text{GKP factor} = \Big(\text{androgynous 2-sex } \mathbf{k}(x)\Big)$$

[4] Originally suggested by Goodman, Keyfitz, Pullum 1974

Androgynous approximation: the GKP factors