Physics 121 and 122

Statistics:

$$\langle f(j)\rangle = \sum_{j=0}^{\infty} f(j)\,P(j)$$

$$\langle x\rangle = \int_{-\infty}^{\infty} x\rho(x)\,\mathrm{d}x$$

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} f(x)\rho(x)\,\mathrm{d}x$$

$$\langle \hat{Q}\rangle = \int_{-\infty}^{\infty} \psi^*\hat{Q}\psi\,\mathrm{d}x = \langle\psi|\hat{Q}|\psi\rangle$$

$$\sigma^2 = \langle(\Delta j)^2\rangle$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2}$$

$$P_{ab} = \int_a^b \rho(x)\,\mathrm{d}x$$

$$\int_{-\infty}^{\infty} |\Psi(x)|^2\,\mathrm{d}x = \langle\psi|\psi\rangle = 1 \text{ (if normalized)}$$

Quantum Mechanics:

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} = m\frac{\mathrm{d}\langle\hat{x}\rangle}{\mathrm{d}t}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) = \frac{\hat{p}^2}{2m} + V$$

$$\left[\hat{x}\,,\hat{L}_z\right] = -i\hbar\hat{y} \text{ (cyclic permutations)}$$

$$\frac{\mathrm{d}\langle\hat{Q}\rangle}{\mathrm{d}t} = \frac{i}{\hbar}\left\langle\left[\hat{H}\,,\hat{Q}\right]\right\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$$

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$$\left[\hat{A}\,,\hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$[\hat{x}\,,\hat{p}] = i\hbar$$

$$\sigma_A^2\sigma_B^2 \geq \left|\frac{1}{2i}\left\langle\left[\hat{A}\,,\hat{B}\right]\right\rangle\right|^2$$

$$\sigma_x^2\sigma_p^2 \geq \frac{\hbar}{2}$$

$$\left[\hat{A}\,, f(\hat{B})\right] = \left[\hat{A}\,,\hat{B}\right]f'(\hat{B})$$

$$\left[\hat{A}\hat{B}\,,\hat{C}\right] = \hat{A}\left[\hat{B}\,,\hat{C}\right] + \left[\hat{A}\,,\hat{C}\right]\hat{B}$$

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} \text{ (TISE sol.)}$$

$$c_n = \int_{-\infty}^{\infty}\psi_n^*(x)f(x)\,\mathrm{d}x$$

Infinite Square Well:

$$V(x) = \begin{cases} 0, & 0 \leq x \leq a, \\ \infty, & \text{otherwise}\end{cases}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \qquad n = 1,2,3...$$

Quantum Harmonic Oscillator:

$$V = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2\,, \qquad \omega \equiv \sqrt{\frac{k}{m}}$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\hat{a}_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega\hat{x})\,,\ [\hat{a}_-\,,\hat{a}_+] = 1$$

$$\psi_n(x) = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n\,\psi_0(x), \quad n = 0,1,2,3...$$

$$\hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}\,, \quad \hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

$$\hat{H}\psi = \hbar\omega\left(\hat{a}_\pm\hat{a}_\mp \pm \frac{1}{2}\right)\psi = E\psi$$

Quantum Free Particle:

$$V = 0; \quad \hat{H} = E\psi \Rightarrow \frac{\partial^2\psi}{\partial x^2} = -\left(\frac{\sqrt{2mE}}{\hbar}\right)^2\psi$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$\Psi(x,t) = Ae^{ik\left(x - \frac{\hbar k}{2m}t\right)} + Be^{-ik\left(x + \frac{\hbar k}{2m}t\right)}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{ik\left(x - \frac{\hbar k}{2m}t\right)}\,\mathrm{d}k$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)e^{-ikx}\,\mathrm{d}x$$

$$T = \frac{|\text{coeff out}|^2}{|\text{coeff in}|^2}\,, \qquad R = \frac{|\text{coeff back}|^2}{|\text{coeff in}|^2}$$

$$R + T = 1$$

Bound State ($E < 0$): $E < V(x)$, $x \to \infty$

Scattering State ($E > 0$): $E > V(x)$, $x \to \infty$

Plancherel's Theorem:

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\Psi(x,t)e^{-ipx/\hbar}\,\mathrm{d}x$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\Phi(p,t)e^{ipx/\hbar}\,\mathrm{d}p$$

3D Quantum Mechanics:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \left[V + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu$$

$$\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)Y_\ell^m(\theta,\phi)$$

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}e^{im\phi}P_\ell^m(\cos\theta)$$

$$P_\ell^m(x) \equiv (-1)^m(1-x)^{\frac{m}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m P_\ell(x)$$

$$P_\ell(x) \equiv \frac{1}{2^\ell\ell!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell(x^2-1)^\ell$$

The Hydrogen Atom:

$$V(r) = -\frac{e^2}{4\pi\varepsilon_0 r}$$

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx 1/137$$

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{m_ee^2} = \frac{\hbar}{\alpha m_ec} \approx 5.29177\times10^{-11}\ \text{m}$$

$$-E_1 = \frac{\hbar^2}{2m_ea_0^2} = \frac{\alpha^2m_ec^2}{2} \approx 13.6057\ \text{eV}$$

$$E_n = \frac{E_1}{n^2}, \qquad n = 1,2,3...$$

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}$$

$$\psi_{n\ell m}(r,\theta,\phi) = \sqrt{\left(\frac{2}{na_0}\right)^3\frac{(n-\ell-1)!}{2n(n+\ell)!}}e^{-r/na_0}$$

$$\times\left(\frac{2r}{na_0}\right)^\ell\left[L_{n-\ell-1}^{2\ell+1}\left(\frac{2r}{na_0}\right)\right]Y_\ell^m(\theta,\phi)$$

$$L_q^p(x) \equiv (-1)^p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^p L_{p+q}(x)$$

$$L_q(x) \equiv \frac{e^x}{q!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^q(e^{-x}x^q)$$

$$R_{n\ell}(r) = \frac{1}{r}\rho^{\ell+1}e^{-\rho}v(\rho)$$

$$v(\rho) = \sum_{j=0}^{\infty}c_j\rho^j, \quad c_{j+1} = \frac{2(j+\ell+1-n)}{(j+1)(j+2\ell+2)}c_j$$