1D finite difference
1st order upwind HD solver

Ivo Cools

2 november 2018

This report is about the development of a finite difference 1st order upwind scheme, specifically to solve hydrodynamical equations. To do this, some guidelines were made available, for which step one is the good ol' handywork-calculations. During this paper, bold-faced characters are indicative of vectors while overlined, bold characters are matrices/tensors.

The original, 1D Euler-equation that is to be solved has the form:

$$\partial_t \mathbf{U} + \nabla_x \mathbf{F}(\mathbf{U}) = \mathbf{0} \tag{1}$$

which in it's full glory becomes:

$$\partial_t \begin{pmatrix} \rho \\ \rho v \\ \frac{P}{\gamma-1} + \frac{1}{2}\rho v^2 \end{pmatrix} + \nabla_x \begin{pmatrix} \rho v \\ \rho v^2 + \mathtt{P} \\ (\frac{P}{\gamma-1} + \frac{1}{2}\rho v^2 + P)v \end{pmatrix} = \mathbf{0} \tag{2}$$

Step one is to write the flux term $\mathbf{F}$ in function of the original $\mathbf{U}$ variables $u_1 = \rho$; $u_2 = \rho v$ and $u_3 = \frac{P}{\gamma-1} + \frac{1}{2}\rho v^2$ where γ is the "ratio of specific heats". It is defined as $\gamma = \frac{\alpha+2}{\alpha}$ where α the total number of degrees of freedom (3 in this case). [Euler . pdf]

This gives a flux term, in function of these "conserved"variables:

$$\mathbf{F} = \begin{pmatrix} u_2 \\ (\gamma-1)u_3 + \frac{1}{2}(3-\gamma)\frac{u_2^2}{u_1} \\ \gamma\frac{u_3 u_2}{u_1} - \frac{1}{2}(\gamma-1))\frac{u_2^3}{u_1^2} \end{pmatrix} \tag{3}$$

Naturally, another way to write equation 1 is:

$$\partial_t \mathbf{U} + \overline{\mathbf{J}}(\mathbf{U})\nabla_x \mathbf{U} = 0 \tag{4}$$

Where $\overline{\mathbf{J}}(\mathbf{U})$ is defined as $\frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}}$

This gives a Jacobian matrix of the flux-term in the form of :

$$\overline{\mathbf{J}}_F = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2}(3-\gamma)\frac{u_2^2}{u_1^2} & (3-\gamma)\frac{u_2}{u_1} & \gamma-1 \\ -\gamma\frac{u_3 u_2}{u_1^2} + \frac{1}{2}(\gamma-1))\frac{u_2^3}{u_1^2} & \gamma\frac{u_3}{u_1} + \frac{3}{2}(\gamma-1)\frac{u_2^2}{u_1} & \gamma\frac{u_2}{u_1} \end{pmatrix} \tag{5}$$

Now, we should calculate the eigenvalues and vectors to more easily solve the Euler-equations. By pure brilliance and by using my infinite mathematical knowledge I realised (i) matrixes which are similar have the same eigenvalues and (ii) matrixes which are obtained by a change of variables (such as conservative to primitive variables) are similar. [el Ilyan]

Restarting from step 1 and reworking with "primitive"variables ρ, v and P [Euler(1).pdf], we note that we can work in a similar way as above. We start by defining the primitive variable vector $\mathbf{K}$:

$$\mathbf{K} = \begin{pmatrix} \rho \\ v \\ P \end{pmatrix} . \tag{6}$$

This vector can then be used to rewrite $\mathbf{U}$ in function of $\mathbf{K}$:

$$\begin{aligned}
\partial_i \mathbf{U} &= \partial \begin{pmatrix} \rho \\ \rho v \\ \frac{P}{\gamma-1} + \frac{1}{2}\rho v^2 \end{pmatrix} \\
&= \begin{pmatrix} \rho' \\ \rho \mathbf{v}' + \rho' v \\ \frac{P'}{\gamma-1} + \frac{1}{2}\rho' v^2 + \rho v \mathbf{v}' \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ v & \rho & 0 \\ \frac{1}{2}v^2 & \rho v & \frac{1}{\gamma-1} \end{pmatrix} \begin{pmatrix} \rho' \\ v' \\ P' \end{pmatrix} \\
&= \overline{\mathbf{M}} \partial_i \mathbf{K} \quad ,
\end{aligned} \tag{7}$$

which in turn can be used to rewrite equation 4 as follows:

$$\overline{\mathbf{M}} \partial_t \mathbf{K} + \mathbf{J}(\mathbf{K}) \overline{\mathbf{M}} \nabla_x \mathbf{K} \quad . \tag{8}$$

Multiplying eq. 8 with $\overline{\mathbf{M}}^{-1}$ on the left gives:

$$\partial_t \mathbf{K} + \widetilde{\mathbf{J}(\mathbf{K})} \nabla_x \mathbf{K} \tag{9}$$

defining $\widetilde{\mathbf{J}(\mathbf{K})}$ as :

$$\widetilde{\mathbf{J}(\mathbf{K})} = \overline{\mathbf{M}}^{-1} \overline{\mathbf{J}}(\mathbf{K}) \overline{\mathbf{M}} \quad . \tag{10}$$

Using eq. 10 and a lot of manual calculations (which were obviously crosschecked with online sources), we get:

$$\widetilde{\mathbf{J}(\mathbf{K})} = \begin{pmatrix} v & \rho & 0 \\ 0 & v & \frac{1}{\rho} \\ 0 & \gamma \mathrm{P} & v \end{pmatrix} \tag{11}$$

Obviously, this matrix is similar to $\mathbf{J}(\mathbf{K})$, and as such has the same eigenvalues, but *not* the same eigenvectors! Using the miracle that is basic arithmetic (combined with MatLab), we are able to calculate all the eigenvalues, *with* corresponding eigenvectors!

$$\begin{cases} v & \texttt{with eigenvector} & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ v + \sqrt{\gamma \frac{\mathtt{P}}{\rho}} & \texttt{with eigenvector} & \begin{pmatrix} \frac{\rho}{\gamma \mathtt{P}} \\ -\sqrt{\frac{1}{\gamma \mathtt{P} \rho}} \\ 1 \end{pmatrix} \\ v - \sqrt{\gamma \frac{\mathtt{P}}{\rho}} & \texttt{with eigenvector} & \begin{pmatrix} \frac{\rho}{\gamma \mathtt{P}} \\ \sqrt{\frac{1}{\gamma \mathtt{P} \rho}} \\ 1 \end{pmatrix} \end{cases} \tag{12}$$

So using this we can finally diagonalise $\widetilde{\mathbf{J}(\mathbf{K})}$:

$$\begin{aligned} \widetilde{\mathbf{J}(\mathbf{K})} &= \begin{pmatrix} 1 & \frac{\rho}{\gamma \mathtt{P}} & \frac{\rho}{\gamma \mathtt{P}} \\ 0 & -\sqrt{\frac{1}{\gamma \mathtt{P} \rho}} & \sqrt{\frac{1}{\gamma \mathtt{P} \rho}} \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} v & 0 & 0 \\ 0 & v + \sqrt{\gamma \frac{\mathtt{P}}{\rho}} & 0 \\ 0 & 0 & v - \sqrt{\gamma \frac{\mathtt{P}}{\rho}} \end{pmatrix} \begin{pmatrix} 1 & \frac{\rho}{\gamma \mathtt{P}} & \frac{\rho}{\gamma \mathtt{P}} \\ 0 & -\sqrt{\frac{1}{\gamma \mathtt{P} \rho}} & \sqrt{\frac{1}{\gamma \mathtt{P} \rho}} \\ 0 & 1 & 1 \end{pmatrix}^{-1} \\ &=: \overline{\mathbf{A}}(\mathbf{K}) \cdot \overline{\mathbf{D}}(\mathbf{K}) \cdot \overline{\mathbf{A}}(\mathbf{K})^{-1} \end{aligned}$$

for later, it is easier to multiply equation 9 with $\overline{\mathbf{A}}^{-1}$ on the left:

$$\begin{aligned} 0 &= \overline{\mathbf{A}}(\mathbf{K})^{-1} \partial_t \mathbf{K} + \overline{\mathbf{D}}(\mathbf{K}) \overline{\mathbf{A}}(\mathbf{K})^{-1} \nabla_x \mathbf{K} \\ &= \partial_t [\overline{\mathbf{A}}(\mathbf{K})^{-1} \mathbf{K}] + \overline{\mathbf{D}}(\mathbf{K}) \nabla_x [\overline{\mathbf{A}}(\mathbf{K})^{-1} \mathbf{K}] \\ &= \partial_t [\overline{\mathbf{A}}(\mathbf{K})^{-1} \mathbf{K}] + \begin{pmatrix} v & 0 & 0 \\ 0 & v + \sqrt{\gamma \frac{\mathtt{P}}{\rho}} & 0 \\ 0 & 0 & v - \sqrt{\gamma \frac{\mathtt{P}}{\rho}} \end{pmatrix} \nabla_x [\overline{\mathbf{A}}(\mathbf{K})^{-1} \mathbf{K}] \\ &=: \partial_t \mathbf{W} + \begin{pmatrix} v & 0 & 0 \\ 0 & v + \sqrt{\gamma \frac{\mathtt{P}}{\rho}} & 0 \\ 0 & 0 & v - \sqrt{\gamma \frac{\mathtt{P}}{\rho}} \end{pmatrix} \nabla_x \mathbf{W} \end{aligned} \tag{13}$$