Computer Vision
Projective Geometry

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Readings

Readings for these lecture notes:

- Hartley, R., and Zisserman, A. *Multiple View Geometry in Computer Vision*, Cambridge University Press, 2004, Chapters 1–3.
- Szeliski, R. *Computer Vision: Algorithms and Applications*, Springer, 2021.

These notes contain material © Hartley and Zisserman (2004) and Szeliski (2021).

Outline

We begin with 2D projective geometry because it's simple, then we'll generalize to 3D.

In particular, we consider what happens when we perform projective transformations of the plane.

Projective transformations model the distortions introduced by projective cameras (more on cameras later).

In projective cameras, funny things happen. Although straight lines stay straight, parallel lines are no longer parallel.

Projective geometry gives us the mathematics for these kinds of transformations.

A point in the plane can be represented as a pair (x, y) in $\mathbb{R}^2$.

We consider $\mathbb{R}^2$ as a vector space and we write the point (x, y) as a vector.

This makes it possible to write transformations of points as matrices.

Generally, we write transformations on the left and points on the right, so we need to write points as column vectors, i.e. $(x, y)^T$.

We will typically write column vectors using bold upright symbols, e.g., $\mathbf{x} = (x, y)^T$.

A line in the plane is normally represented by an equation like $ax + by + c = 0$. The parameters a, b, and c give us different lines.

This means we can write a line in $\mathbb{R}^2$ as the vector $(a, b, c)^T$.

$(a, b, c)^T$ represents the same line as $k(a, b, c)^T$ for any non-zero constant k.

A homogeneous vector is an equivalence class of vectors defined by scaling.

The set of homogeneous equivalence classes of vectors in $\mathbb{R}^3 - (0, 0, 0)^T$ is called the projective space $\mathbb{P}^2$.

Since a point $\mathrm{x} = (x, y)^T$ lies on a line $\mathrm{l} = (a, b, c)^T$ iff $ax + by + c = 0$, we can equivalently write the inner product $(x, y, 1)(a, b, c)^T = (x, y, 1)\mathrm{l} = 0$.

If $(x, y, 1)\mathrm{l} = 0$, it is also true that $(kx, ky, k)\mathrm{l} = 0$.

This makes it convenient to represent a point $\mathrm{x} = (x, y)^T$ in $\mathbb{R}^2$ with the homogeneous vector $(x, y, 1)^T$.

This means the arbitrary point $\mathrm{x} = (x_1, x_2, x_3)^T$ in $\mathbb{P}^2$ can be used to represent the point $(x_1/x_3, x_2/x_3)$ in $\mathbb{R}^2$.

This is nice! Why? Because now we can say that a homogeneous point $\mathrm{x} = (x_1, x_2, x_3)^T$ lies on line l iff $\mathrm{x}^T\mathrm{l} = 0$.

Another nice property: the intersection of two lines l and l' is the point $\mathrm{x} = \mathrm{l} \times \mathrm{l}'$.

Reminder: the cross product of two vectors $\mathrm{x}_1 = (x_1, x_2, x_3)^T$ and $\mathrm{x}' = (x_1', x_2', x_3')^T$ is defined as

$$\mathrm{x} \times \mathrm{x}' = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \end{vmatrix}$$

Another reminder: if $\mathrm{x} = \mathrm{l} \times \mathrm{l}'$, x is the vector normal to l and l' with magnitude equal to the area of the parallelogram formed by l and l'.

Proof: Since x is orthogonal to both l and l' we know $\mathrm{l}^T\mathrm{x} = 0$ and $\mathrm{l}'^T\mathrm{x} = 0$, meaning x lies on both l and l'!

Similarly, the line l joining two points is just $\mathrm{l} = \mathrm{x} \times \mathrm{x}'$.

Where do the parallel lines $(a, b, c)^T$ and $(a, b, c')^T$ intersect?

The cross product turns out to be $(c' - c)(b, -a, 0)^T = (b, -a, 0)^T$ in $\mathbb{P}^2$ but this point has no inhomogeneous representation. (What is $(b/0, -a/0)^T$?)

We call such a point $(x_1, x_2, 0)^T$ in $\mathbb{P}^2$ an ideal point or a point at infinity along the direction $(x_1, x_2)^T$.

All points at infinity lie on the line at infinity $\mathbf{l}_\infty = (0, 0, 1)^T$ (remember that lines in $\mathbb{P}^2$ correspond to planes in $\mathbb{R}^3$, points at infinity lie on the plane $x_3 = 0$, so we represent the plane $x_3 = 0$ by its normal vector $(0, 0, 1)^T$).

In $\mathbb{P}^2$, then, we can say that any two lines intersect, even if they are parallel.

Here's how to think of $\mathbb{P}^2$: it is a set of rays through the origin in $\mathbb{R}^3$.

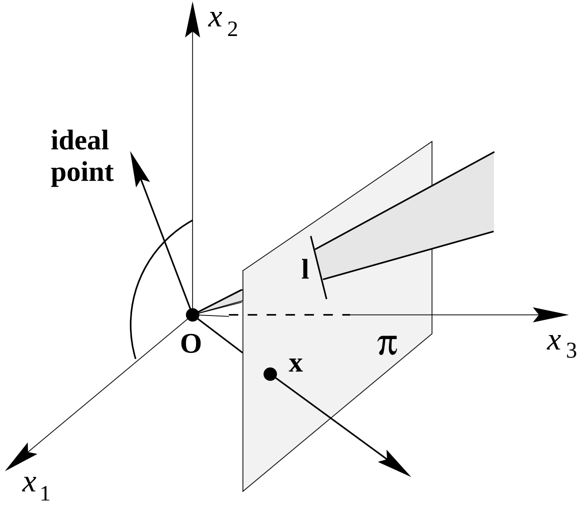
If we consider all vectors $k(x_1, x_2, x_3)^T$ as k varies, we obtain a ray through the origin. Each such ray is a single point in $\mathbb{P}^2$.

A line between any two different points in $\mathbb{P}^2$ forms a plane through the origin.

Inhomogeneous representations of points and lines can be obtained by finding the intersection of their rays and planes with the plane $x_3 = 1$.

2D projective geometry

A model for the projective plane



Hartley and Zisserman (2004), Fig. 2.1

In $\mathbb{R}^3$, lines through the origin that lie in the $x_1 x_2$ plane represent ideal points in $\mathbb{P}^2$.

All other lines through the origin represent points in $\mathbb{P}^2$.

Planes through the origin in $\mathbb{R}^3$ represent lines in $\mathbb{P}^2$.

The vector $(a, b, c)^T$ representing a line in the Euclidean plane, when interpreted as a vector in $\mathbb{R}^3$, is orthogonal to the $\mathbb{R}^3$ plane representing the line in $\mathbb{P}^2$.

You should try to prove that this must be so.

Conic sections or just conics are curves in the plane described by 2nd-degree equations.

In Euclidean geometry, conics can be parabolas, hyperbolas, and ellipses, or "degenerate" conics (two lines or a single line).

In inhomogeneous coordinates, we write conics as

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

We homogenize the equation by replacing x by x_1/x_3 and y by x_2/x_3. This gives us

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

With homogeneous points, similar to the point-line equation, conics can be written in matrix form:

$$\mathtt{x}^T \mathtt{C} \mathtt{x} = 0$$

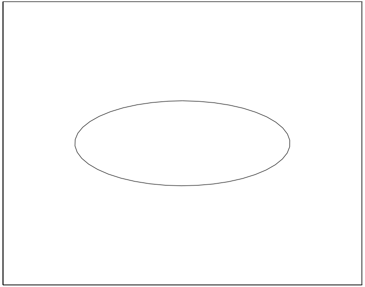
Where the (symmetric) conic coefficient matrix $\mathtt{C}$ is given by

$$\mathtt{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

$\mathtt{C}$ is homogeneous because scaling by a non-zero constant does not change the conic.

$\mathtt{C}$ has five degrees of freedom, and each point on the conic gives us one equation linear in the conic's parameters, so we need five points to uniquely define a conic up to scale.

If we find the points x satisfying $\mathrm{x}^T \mathtt{C}\mathrm{x} = 0$, we obtain the plane curve described by $\mathtt{C}$:



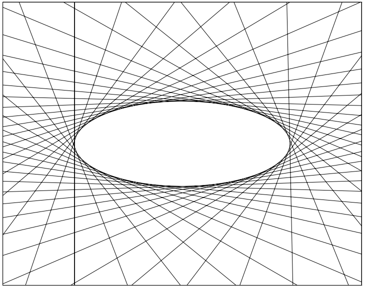
Solution of $\mathrm{x}^T \mathtt{C}\mathrm{x} = 0$
Hartley and Zisserman (2004), Fig. 2.2(a)

Exercise: write the circle about the origin with radius r in conic matrix form, and verify that the points on the circle satisfy $\mathrm{x}^T \mathtt{C}\mathrm{x} = 0$.

What happens if we take an arbitrary point $\mathbf{x}$ on the conic and compute $\mathtt{C}\mathbf{x}$, treating $\mathtt{C}$ as a linear operator?

We obtain a 3-vector whose inner product with $\mathbf{x}$ is 0. This means that $\mathbf{l} = \mathtt{C}\mathbf{x}$ necessarily represents a line passing through $\mathbf{x}$.

In fact, the resulting line is special: it is the tangent to the conic at $\mathbf{x}$.



The line $\mathbf{l} = \mathtt{C}\mathbf{x}$ is the tangent to $\mathtt{C}$ through $\mathbf{x}$.

Hartley and Zisserman (2004), Fig. 2.2(b)

There is an important class of transformations on $\mathbb{P}^2$ called projectivities or homographies or collineations.

A projectivity (homography) is a transform that can be represented by a 3×3 non-singular matrix $\mathtt{H}$.

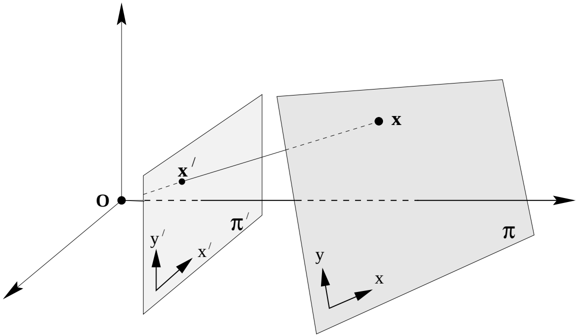
Homographies are linear mappings of homogeneous coordinates:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or simply $\mathrm{x}' = \mathtt{H}\mathrm{x}$.

Since scaling $\mathtt{H}$ doesn't change the result, we say $\mathtt{H}$ is a homogeneous matrix with 8 degrees of freedom.

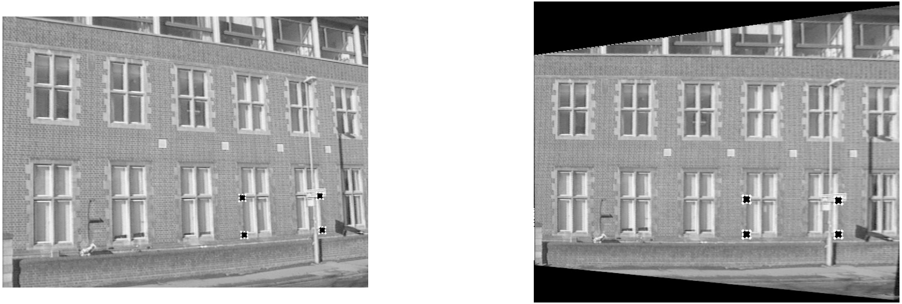
Homographies can also be thought of as central projections mapping one plane to another:



Hartley and Zisserman (2004), Fig. 2.3

[Actually, central projections between rectilinear coordinate systems as shown here are called perspectivities and only have 6 degrees of freedom.]

One application of homographies is rectification. As an example, suppose we want to remove projective distortion from a perspective image of a plane:



(a) (b)

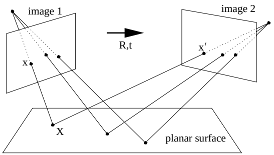
Hartley and Zisserman (2004), Fig. 2.4

By picking any four points in an original image and the desired corresponding points in the new image, we obtain 8 linear equations in the 9 unknowns of `H`, allowing us to compute the parameters of `H`:

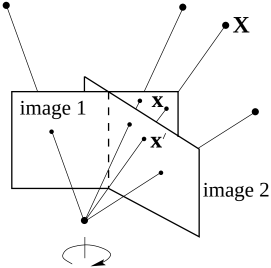
$$x' = \frac{x_1'}{x_3'} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}, \qquad y' = \frac{x_2'}{x_3'} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}.$$

See the example `compute_homography.m` that implements this idea.

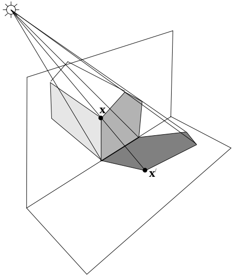
Here are a few of the most important examples of homographies (Hartley and Zisserman, 2004, Fig. 2.5):



Images of a plane from two cameras related by a rotation and a translation.



Images of arbitrary objects from two cameras related by a rotation.



Images of shadows of planar objects.

Note that images of arbitrary objects from two cameras related by a rotation and translation are not related by homographies.

Given a point homography $\mathrm{x}' = \mathtt{H}\mathrm{x}$:

- The corresponding line homography is $\mathrm{l}' = \mathtt{H}^{-T}\mathrm{l}$
- The corresponding conic homography is $\mathtt{C}' = \mathtt{H}^{-T}\mathtt{C}\mathtt{H}^{-1}$

We can create a hierarchy of transformations based on the restrictions that we put on a linear transformation $\mathtt{H}$.

The real linear group $GL(3)$ consists of all invertible real 3×3 matrices.[1]

When we place all members of $GL(3)$ related by scale in an equivalence class, we obtain the projective linear group $PL(3)$.

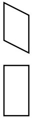
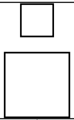
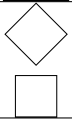
There are three important subgroups of $PL(3)$:

- The affine group in which the bottom row is constrained to $(0, 0, 1)$;
- The similarity group in which the rows and columns of the upper-left 2×2 submatrix are orthogonal;
- The Euclidean or isometry group in which the upper-left 2×2 submatrix is orthonormal.

[1] Recall that a group is a set paired with an operation that has an inverse, associativity, an identity, and closure.

2D projective geometry

A hierarchy of transforms

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, order of contact, tangent discontinuities and cusps, cross ratios
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines, linear combinations of vectors, the line at infinity $\mathbf{l}_\infty$
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Hartley and Zisserman (2004), Table 2.1

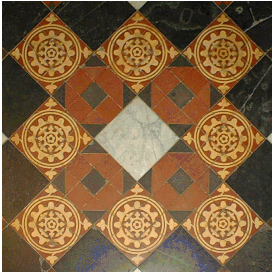
What happens when we apply a homography $\mathtt{H}$ to a point at infinity?

Affinities map ideal points to ideal points:

$$\begin{bmatrix} \mathtt{A} & \mathtt{t} \\ \mathtt{0}^T & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathtt{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

but general projectivities can map ideal points to finite points:

$$\begin{bmatrix} \mathtt{A} & \mathtt{t} \\ \mathtt{v}^T & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathtt{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$



Similarity: circularity is invariant

Affinity: parallelism is invariant

Projectivity: the line at infinity becomes finite (parallel lines on the plane intersect at finite points on l_∞)

Hartley and Zisserman (2004), Fig. 2.6

Outline

Now we move to projective 3-space, or $\mathbb{P}^3$.

Things will mostly generalize from $\mathbb{P}^2$.

We'll use homogeneous coordinates in $\mathbb{R}^4$ to represent points and planes in $\mathbb{P}^3$.

We'll see that parallel lines and parallel planes intersect on the plane at infinity π_∞.

3D projective geometry

Points in $\mathbb{P}^3$

We represent a 3D point $(X, Y, Z)^T$ in homogeneous coordinates $\mathrm{X} = (X_1, X_2, X_3, X_4)^T$ with $X_4 \neq 0$ and

$$X = X_1/X_4, Y = X_2/X_4, Z = X_3/X_4.$$

Homogeneous coordinates with $X_4 = 0$ represent points at infinity.

A projective transformation on $\mathbb{P}^3$ is a linear transformation on homogeneous 4-vectors, represented by a non-singular 4×4 matrix: $\mathrm{X}' = \mathtt{H}\mathrm{X}$.

$\mathtt{H}$ is homogeneous and in general has 15 degrees of freedom.

Whereas points and lines are dual in $\mathbb{P}^2$, points and planes are dual in $\mathbb{P}^3$.

A plane in 3-space is written

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0.$$

The equation is unaffected by scalar multiplication, so we can represent a plane as the homogeneous vector $(\pi_1, \pi_2, \pi_3, \pi_4)^T$.

Homogenizing the point X, we get

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$

or simply $\boldsymbol{\pi}^T \mathrm{X} = 0$ to express that the point X lies on the plane $\boldsymbol{\pi}$.

Suppose we have 3 points in general position. We can find the plane they lie in by constructing the equation

$$\mathtt{X}\boldsymbol{\pi} = \begin{bmatrix} \mathrm{X}_1^T \\ \mathrm{X}_2^T \\ \mathrm{X}_3^T \end{bmatrix} \boldsymbol{\pi} = 0$$

As long as the 3×4 matrix $\mathtt{X}$ has rank[2] 3, the solution $\boldsymbol{\pi}$ is the right null space[3] of $\mathtt{X}$.

If $\mathtt{X}$ is rank 2, we have collinear points and the equation defines a pencil of planes[4] with the line of collinear points as its axis.

[2] The rank of matrix $\mathtt{A}$ is the number of linearly independent rows or columns in $\mathtt{A}$.
[3] The right null space of matrix $\mathtt{A}$, written $\mathcal{N}(\mathtt{A})$, is the set of all solutions of $\mathtt{A}\mathrm{x} = 0$.
[4] Just a fancy term for the set of planes intersecting at a given line.

Analogous to the definition of the line through two points in $\mathbb{P}^2$, the plane through three points in $\mathbb{P}^3$ can be written in terms of determinants and minors.

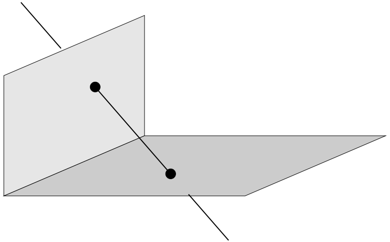
We finally obtain (see text for proof)

$$\boldsymbol{\pi} = \begin{pmatrix} (\tilde{\mathtt{X}}_1 - \tilde{\mathtt{X}}_3) \times (\tilde{\mathtt{X}}_2 - \tilde{\mathtt{X}}_3) \\ -\tilde{\mathtt{X}}_3^T (\tilde{\mathtt{X}}_1 \times \tilde{\mathtt{X}}_2) \end{pmatrix}$$

where $\tilde{\mathtt{X}}$ is the inhomogeneous representation of point $\mathtt{X}$.

So we can find the plane through three points by finding $\mathcal{N}(\mathtt{X})$ or by using some vector algebra.

Lines in 3-space can be defined by their intersection with two given planes:



Hartley and Zisserman (2004) Fig. 3.1

This means a line in 3-space has 4 degrees of freedom.

Wouldn't it be natural to represent lines as 5-element homogeneous vectors?

Unfortunately this doesn't work well with 4-vectors for lines and planes.

Several alternative representations for lines have been proposed.

In the homogeneous point representation of a line, given two homogeneous points A and B on the line, we write

$$\mathtt{W} = \begin{bmatrix} \mathsf{A}^T \\ \mathsf{B}^T \end{bmatrix},$$

in which case:

- The span[5] of $\mathtt{W}^T$ is the pencil of points[6] $\lambda \mathsf{A} + \mu \mathsf{B}$ on the line.[7]
- The span of the 2-dimensional right null-space of $\mathtt{W}$ is the pencil of planes with the line as the axis.

[5] The set of all linear combinations of the columns of a matrix.

[6] Just a fancy term for the set of points on a line.

[7] To convince yourself of this, consider the inhomogeneous parametric representation of the line $\tilde{\mathsf{A}} + t(\tilde{\mathsf{B}} - \tilde{\mathsf{A}})$ with $t = \frac{\mu}{\lambda+\mu}$.

There is a dual representation of a line as the intersection of two planes P and Q:

$$\mathtt{W}^* = \begin{bmatrix} \mathrm{P}^T \\ \mathrm{Q}^T \end{bmatrix}$$

in this representation,

- The span of ${\mathtt{W}^*}^T$ is the pencil of planes $\lambda' \mathrm{P} + \mu' \mathrm{Q}$ with the line as an axis.
- The span of the 2-dimensional null-space of $\mathtt{W}^*$ is the pencil of points on the line.

Example: find $\mathtt{W}$ and $\mathtt{W}^*$ for the points $(1,0,0,1)^T$ and $(2,0,0,1)^T$ and verify that $\mathtt{W}^*\mathtt{W}^T = \mathtt{W}{\mathtt{W}^*}^T = 0$.

We can also represent a line through A and B by a 4×4 skew-symmetric[8] homogeneous matrix $\mathtt{L}$ defined by

$$l_{ij} = A_i B_j - B_i A_j$$

or equivalently,

$$\mathtt{L} = \mathrm{A}\mathrm{B}^T - \mathrm{B}\mathrm{A}^T.$$

$\mathtt{L}$ is the Plücker matrix representation of a line.

[8] Recall that a skew-symmetric matrix is a matrix $\mathtt{A}$ such that $\mathtt{A}^T = -\mathtt{A}$.

Some nice properties of $\mathtt{L}$:

- The rank of $\mathtt{L}$ is 2 and its 2-dimensional null-space is spanned by the pencil of planes with the line as an axis ($\mathtt{L}\mathtt{W}^{*T} = \mathtt{0}$).
- $\mathtt{L}$ has 4 degrees of freedom, the same as a line, since it is skew-symmetric (having only 6 independent non-zero elements), homogeneous, and constrained so that $\det \mathtt{L} = 0$ (its rank is 2).
- The relation $\mathtt{L} = \mathrm{A}\mathrm{B}^T - \mathrm{B}\mathrm{A}^T$ is the generalization to $\mathbb{P}^3$ of the relation $\mathrm{l} = \mathrm{x} \times \mathrm{y}$ of $\mathbb{P}^2$.
- $\mathtt{L}$ is independent of the points A and B.
- Under the point transformation $\mathrm{X}' = \mathtt{H}\mathrm{X}$, the matrix $\mathtt{L}$ transforms as $\mathtt{L}' = \mathtt{H}\mathtt{L}\mathtt{H}^T$.

Example: write the line through $(1, 0, 0, 1)$ and $(2, 0, 0, 1)$ as a Plücker matrix.

A dual Plücker representation can be obtained using the intersection of two planes P and Q:

$$\mathtt{L}^* = \mathrm{P}\mathrm{Q}^T - \mathrm{Q}\mathrm{P}^T$$

$\mathtt{L}^*$ can be derived directly from $\mathtt{L}$ by a simple rewrite rule (see text).

The advantage of the Plücker matrix and dual Plücker matrix representations is that joins and indicence properties are easily represented:

- The plane defined by the join of a point X and a line $\mathtt{L}$ is $\boldsymbol{\pi} = \mathtt{L}^*\mathrm{X}$, and $\mathtt{L}^*\mathrm{X} = 0$ iff X is on $\mathtt{L}$.
- The point defined by the intersection of a line $\mathtt{L}$ and a plane $\boldsymbol{\pi}$ is $\mathrm{X} = \mathtt{L}\boldsymbol{\pi}$, and $\mathtt{L}\boldsymbol{\pi} = 0$ iff $\mathtt{L}$ is on $\boldsymbol{\pi}$.

Example: find the intersection of the line through $(1, 0, 0, 1)$ and $(2, 0, 0, 1)$ with the plane $X = 1$ (note that the plane $X = 1$ is represented as $(1, 0, 0, -1)^T$).

A line can also be represented by the 6 independent non-zero elements of the skew-symmetric Plücker matrix $\mathtt{L}$.

This representation is called the Plücker line coordinate representation of a line.

See the text for the properties of this representation.

A quadric is the 3D analog of a conic, defined by

$$\mathrm{X}^T \mathtt{Q} \mathrm{X} = 0$$

where $\mathtt{Q}$ is a symmetric 4×4 matrix.

Here are some properties of quadrics:

- 9 degrees of freedom (10 independent elements, one lost to scale)
- Defined by 9 points in general position
- If $\mathtt{Q}$ is singular, the quadric is degenerate and can be described by fewer points
- $\boldsymbol{\pi} = \mathtt{Q}\mathrm{X}$ is the polar plane of X with respect to $\mathtt{Q}$. If X is outside $\mathtt{Q}$ then $\boldsymbol{\pi}$ is defined by the points of contact of the rays through X tangent to $\mathtt{Q}$; if X is on $\mathtt{Q}$ then $\boldsymbol{\pi}$ is the tangent plane to $\mathtt{Q}$ at X.
- The intersection of a plane $\boldsymbol{\pi}$ and a quadric $\mathtt{Q}$ is always a conic $\mathtt{C}$.
- Given a 3D homography $\mathrm{X}' = \mathtt{H}\mathrm{X}$, the quadric $\mathtt{Q}$ transforms as

$$\mathtt{Q}' = \mathtt{H}^{-T} \mathtt{Q} \mathtt{H}^{-1}$$

The dual of a point quadric is an equation on planes: the tangent planes $\boldsymbol{\pi}$ to point quadric $\mathtt{Q}$ satisfying $\pi^T \mathtt{Q}^* \pi = 0$.

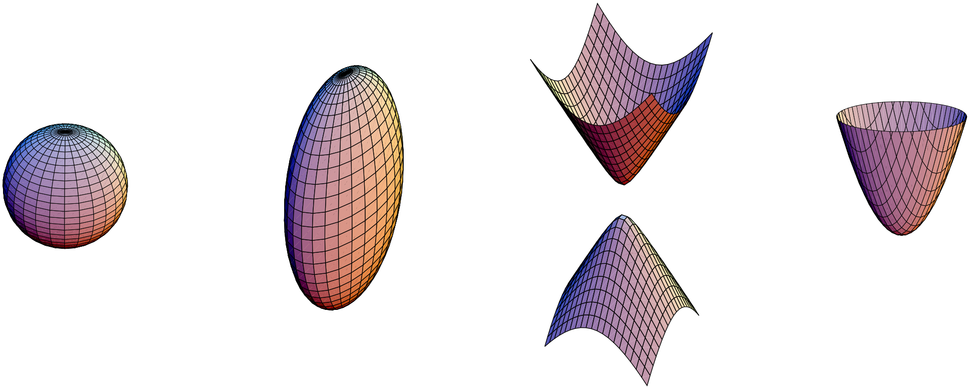
Here $\mathtt{Q}^* = \mathtt{Q}^{-1}$ if $\mathtt{Q}$ is invertible, or $\mathtt{Q}^* = \mathrm{adjoint}\ \mathtt{Q}$ otherwise.[9]

Dual quadrics transform as

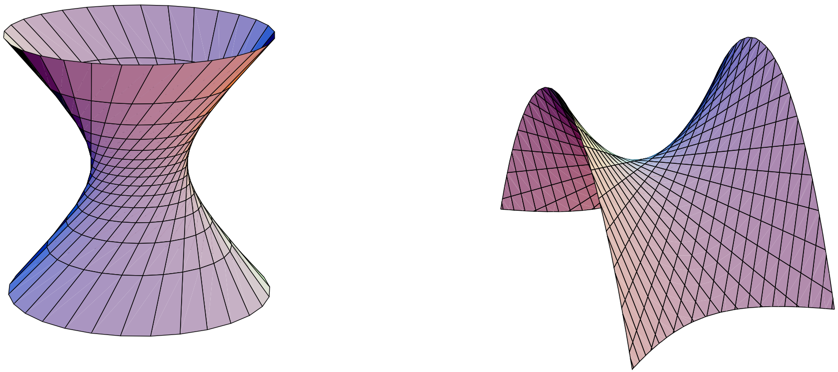
$$\mathtt{Q}^{*\prime} = \mathtt{H}\mathtt{Q}^*\mathtt{H}^T.$$

[9]For the curious, the adjoint of matrix $\mathtt{M}$ is defined as the transpose of the matrix where element ij is $(-1)^{i+j} \det \hat{\mathtt{M}}_{ij}$ where $\hat{\mathtt{M}}_{ij}$ is obtained from $\mathtt{M}$ by striking out the i-th row and j-th column.

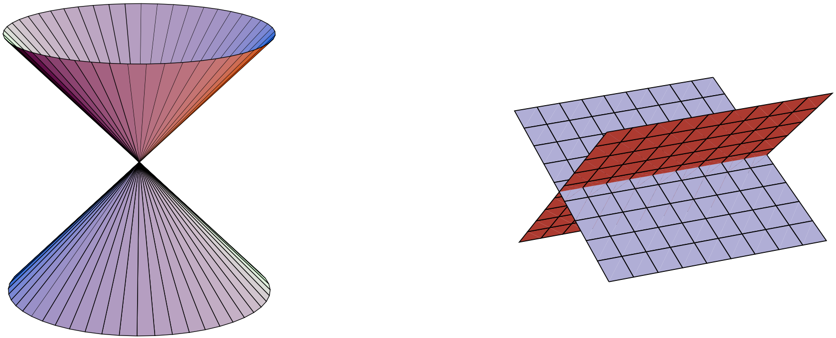
Depending on the rank and signs of the singular values of Q, we obtain a variety of different topologies:



Quadrics projectively equivalent to a spheres (ellipsoid, hyperboloid of two sheets,



Hyperboloids of one sheet, also called ruled quadrics, Hartley and Zisserman (2004), Fig. 3.3



Degenerate quadrics, Hartley and Zisserman (2004), Fig. 3.4

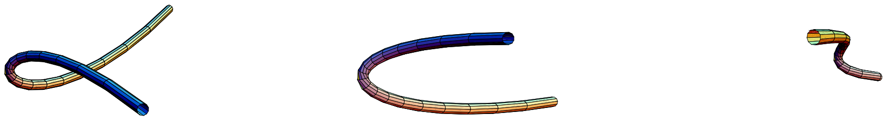
Conics in 2D can be expressed parametrically in $\mathbb{P}^2$ as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathtt{A} \begin{pmatrix} 1 \\ \theta \\ \theta^2 \end{pmatrix} \tag{1}$$

By analogy, various kinds of space curves called twisted cubics can be epressed parametrically in $\mathbb{P}^3$ as

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \mathtt{A} \begin{pmatrix} 1 \\ \theta \\ \theta^2 \\ \theta^3 \end{pmatrix} \tag{2}$$

Some views of a twisted cubic:

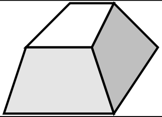
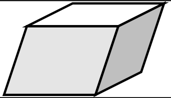
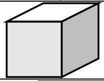
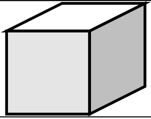


Hartley and Zisserman (2004), Fig. 3.5

3D projective geometry

Hierarchy of transforms

As we saw in 2D, 3D homographies form a hierarchy from most general to most constrained:

Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^T & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at inifinty π_∞
Similarity 7 dof	$\begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$		The absolute conic Ω_∞.
Euclidean 6 dof	$\begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$		Volume

Hartley and Zisserman (2004), Table 3.1

Recall that in planar projective geometry the line at infinity l_∞ contained the intersections of parallel lines.

In the projective geometry of 3-space, the corresponding object is the plane at infinity $\boldsymbol{\pi}_\infty$.

The canonical position of the plane at infinity is $\boldsymbol{\pi}_\infty = (0, 0, 0, 1)^T$ in affine 3-space.

Properties of the plane at infinity:

- Two planes are parallel iff their line of intersection is on $\boldsymbol{\pi}_\infty$.
- Two lines are parallel iff their point of intersection is on $\boldsymbol{\pi}_\infty$.

Why do we care about π_∞?

- Just like l_∞ in $\mathbb{P}^2$, π_∞ in $\mathbb{P}^3$ is fixed under affine transformations but moved under projective transformations.
- When we work on 3D reconstruction from multiple views, we'll get a projective reconstruction then we'll need to find a transformation $\mathtt{H}$ giving us a Euclidean reconstruction.
- One way to do this is to first transform from a projective frame to an affine frame, then from affine to Euclidean.
- In planar geometry, if we could find l_∞ in the image then apply $\mathtt{H}_p$ mapping l_∞ to $(0, 0, 1)$, we could also use $\mathtt{H}_p$ to transform our projective reconstruction into a 2D affine frame.
- Similarly, in 3-space, if we can find π_∞ (e.g. using parallel lines and planes in an image), we could find a transform $\mathtt{H}_p$ mapping the observed π_∞ to $(0, 0, 0, 1)$. $\mathtt{H}_p$ would map our reconstruction into a 3D affine frame.

The absolute conic Ω_∞ is a conic on π_∞.

As π_∞ is fixed under affine transformations, the absolute conic is fixed under similarity transforms.

In a metric frame, $\Omega_\infty = \mathtt{I}$.

Informally we can say that Ω_∞ records non-metric distortions we've applied in 3-space.

In principle, if we could find Ω_∞ in an affine frame (by e.g. comparing observed to known angles), we could apply an affine transform $\mathtt{H}_a$ mapping Ω_∞ to $\mathtt{I}$. This would also undo the affine distortions to the scene.

The dual of the absolute conic is a degenerate quadric called the dual absolute quadric $\mathtt{Q}^*_\infty$.

The dual absolute conic is fixed under similarity transforms and can be identified directly in a projective frame by observing angles between planes.

Outline

Rigid or Euclidean transformations involve only rotations and translations.

We usally think of rigid transformations as transforms between different coordinate systems in $\mathbb{R}^3$.

The vector representation of a point $\mathrm{X} = (X, Y, Z)^T$ should be understood as $\mathrm{X} = X\mathrm{i} + Y\mathrm{j} + Z\mathrm{k}$ where i, j, k are an orthonormal right-handed basis for $\mathbb{R}^3$ with respect to some origin O.

Suppose we have two coordinate systems or frames $A = (\mathrm{O}_A, \mathrm{i}_A, \mathrm{j}_A, \mathrm{k}_A)$ and $B = (\mathrm{O}_B, \mathrm{i}_B, \mathrm{j}_B, \mathrm{k}_B)$.

Problem: given a point X in frame A, what is that same point represented frame B?

Any rigid transform can be decomposed into a rotation and translation:

$$\mathrm{X}' = \mathtt{R}_{A/B}\mathrm{X} + \mathrm{O}_{A/B}$$

where $\mathtt{R}_{A/B}$ is a rotation matrix rotating points from frame A to frame B and $\mathrm{O}_{A/B}$ is the representation of the origin of frame A in frame B.

If X and X' are represented in homogeneous coordinates, we write

$$\mathrm{X}' = \mathtt{H}\mathrm{X} = \begin{bmatrix} \mathtt{R}_{A/B} & \mathrm{O}_{A/B} \\ 0^T & 1 \end{bmatrix} \mathrm{X}.$$

The rotation $\mathtt{R}_{A/B}$ of a point from frame $A = (\mathrm{O}_A, \mathrm{i}_A, \mathrm{j}_A, \mathrm{k}_A)$ to frame $B = (\mathrm{O}_B, \mathrm{i}_B, \mathrm{j}_B, \mathrm{k}_B)$ can be written

$$\mathtt{R}_{A/B} = \begin{bmatrix} \mathrm{i}_A \cdot \mathrm{i}_B & \mathrm{j}_A \cdot \mathrm{i}_B & \mathrm{k}_A \cdot \mathrm{i}_B \\ \mathrm{i}_A \cdot \mathrm{j}_B & \mathrm{j}_A \cdot \mathrm{j}_B & \mathrm{k}_A \cdot \mathrm{j}_B \\ \mathrm{i}_A \cdot \mathrm{k}_B & \mathrm{j}_A \cdot \mathrm{k}_B & \mathrm{k}_A \cdot \mathrm{k}_B \end{bmatrix} \tag{3}$$

The columns of $\mathtt{R}$ are the projections of the basis vectors for A onto the basis vectors for B.

Some important properties of rotation matrices:

- A rotation matrix is orthogonal.
- Any rotation matrix can be decomposed into the product of 3 simple rotations, i.e., around i, j, and k.
- The inverse of a rotation matrix is its transpose.
- The determinant of a rotation matrix is 1.
- The columns of a rotation matrix form a right-handed coordinate system.
- The rows of a rotation matrix form a right-handed coordinate system.

Rigid (Euclidean) transformations

Simple rotation matrices

Here are the simple rotation matrices for points:

Rotation of α around i:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

Rotation of β around j:

$$\begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

Rotation of γ around k:

$$\begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In robotics, it is more common to express rotations as being applied to the coordinate frame rather than points. A frame rotation by angle α is equivalent to a point rotation by angle $-\alpha$:

Rotation of α around i:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

Rotation of β around j:

$$\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

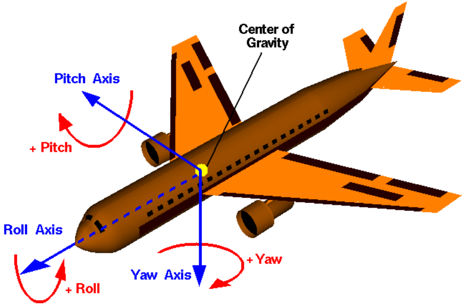
Rotation of γ around k:

$$\begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that in aerial robotics, particular conventions are used for the local coordinate system.

The X axis is the forward direction of the aircraft, Y is right, and Z is down.

Rotations are specified in terms of roll ϕ then pitch θ then yaw ψ.



http://en.wikipedia.org/wiki/File:Rollpitchyawplain.png

Rigid (Euclidean) transformations

Simple rotation matrices

Roll ϕ is a rotation around X, pitch θ is a rotation around Y, and yaw ψ is a rotation around Z:

$$
\begin{aligned}
\mathtt{R} &= \mathtt{R}_Z(\psi)\mathtt{R}_Y(\theta)\mathtt{R}_X(\phi) \\
&= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \\
&= \begin{bmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\phi - \sin\psi\cos\theta & \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\theta & \cos\psi\sin\theta\cos\phi - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}
\end{aligned}
$$

Source: Sturm et al., TUMx: AUTONAVx Autonomous Navigation for Flying Robots, `http://courses.edx.org`. Note error in the video version.

Another representation of rotations is the axis/angle a.k.a. twist coordinate representation.

We specify a rotation axis n and a rotation angle θ.

The 4 parameter version is not minimal.

For a minimal 3 parameter version: use the length of the vector as the rotation angle.

Rodriguez's formulae:

$$
\begin{aligned}
\mathtt{R}(\mathrm{n}, \theta) &= \mathtt{I} + \sin\theta \, [\mathrm{n}]_\times + (1 - \cos\theta) \, [\mathrm{n}]_\times^2 \\
\theta &= \cos^{-1}\left(\frac{\operatorname{trace}\mathtt{R} - 1}{2}\right) \\
\mathrm{n} &= \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}
\end{aligned}
$$

Let $A = ((0,0,0)^T, (1,0,0)^T, (0,1,0)^T, (0,0,1)^T)$ (i.e., the world coordinate frame), and let $B = ((2,-1,1)^T, (0,1,0)^T, (-1,0,0)^T, (0,0,1)^T)$.

Try to write the rotation from frame A to frame B and the origin of frame A, represented in frame B.

Then use that to transform the point $(1, \frac{1}{2}, \frac{1}{2})^T$ from frame A to frame B.

Rigid (Euclidean) transformations
Example

Step 1: compute the rotation matrix from frame A to frame B:

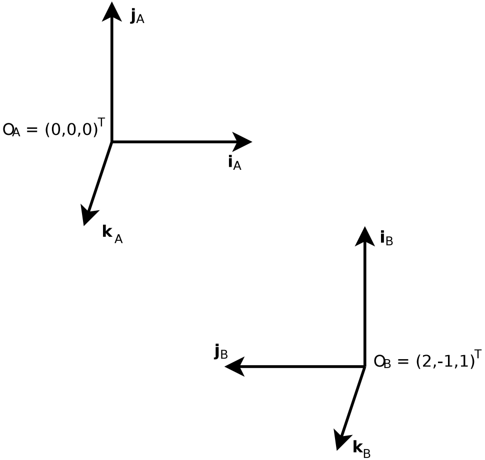
$$\mathrm{R} = \begin{bmatrix} \mathsf{i}_A \cdot \mathsf{i}_B & \mathsf{j}_A \cdot \mathsf{i}_B & \mathsf{k}_A \cdot \mathsf{i}_B \\ \mathsf{i}_A \cdot \mathsf{j}_B & \mathsf{j}_A \cdot \mathsf{j}_B & \mathsf{k}_A \cdot \mathsf{j}_B \\ \mathsf{i}_A \cdot \mathsf{k}_B & \mathsf{j}_A \cdot \mathsf{k}_B & \mathsf{k}_A \cdot \mathsf{k}_B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2: compute the origin of frame A in frame B. We compute the difference between the two frames' origins in world coordinates then project the result onto frame B's basis:

$$\begin{bmatrix} \mathrm{i}_B & \mathrm{j}_B & \mathrm{k}_B \end{bmatrix}^T (O_A - O_B) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Now we can easily transform from frame A to frame B.

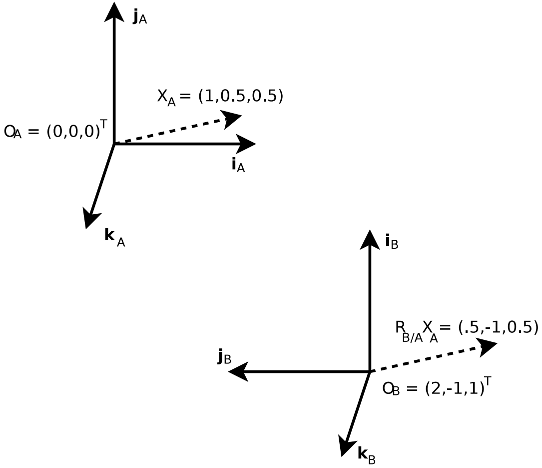
Now let's transform the point $(1, \frac{1}{2}, \frac{1}{2})^T$ in frame A:



Rigid (Euclidean) transformations

Example

First we rotate the point into frame B:



Then we translate by $O_{A/B}$ (the origin of frame A represented in frame B):