Machine Learning
Reinforcement Learning

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Asian Data Science and Artificial Intelligence Master's Program

Co-funded by the Erasmus+ Programme of the European Union

Readings

Readings for these lecture notes:

- Ng, A. (2017), *Reinforcement Learning and Control*. Lecture note set 12 for CS229, Stanford University.
- Sutton, R.S. and Barto, A.G. (2018), *Reinforcement Learning: An Introduction*, 2nd ed., MIT Press.

These notes contain material © Ng (2017) and Sutton and Barto (2018).

Outline

1. Introduction
2. Markov decision processes
3. Value iteration and policy iteration
4. Learning a model for a MDP
5. Temporal-Difference Methods
6. Continuous state MDPs
7. Finite-horizon MDPs
8. Linear quadratic regulation (LQR)
9. LQR for nonlinear dynamics
10. Partially observable Markov decision processes
11. Linear quadratic Gaussian (LQG) regulation

Introduction

Now we consider the problem of reinforcement learning.

In the supervised learning setting, we give the machine examples of what it should do in particular circumstances.

Sometimes it is not easy or even possible to come up with such a training set. Examples include robot motion control, game playing systems, stock trading policies, dynamic allocation of resources to a Web application, ...

In some of these domains, rather than providing a definite answer, we provide the machine with a reward function indicating when it is doing well and when it is doing poorly.

Introduction

Example reward functions:

- Robot control: give positive rewards for moving forward and negative rewards for falling over.
- Game playing: give positive rewards for winning, negative rewards for losing.
- Robotic exploration: give positive rewards for expanding the "frontier," negative rewards for going over the same old ground.
- Robotic vacuum cleaning: give positive rewards for cleaning dirty areas, negative rewards for spending time in clean areas.
- Web application resource allocation: positive rewards for serving requests quickly, negative rewards for using excessive resources.

Usually, reinforcement learning problems are posed as Markov decision processes.

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Markov decision processes

Formal definition

A Markov decision process is a tuple $(S, A, \{P_{sa}\}, \gamma, R)$, where

- S is a set of states.
- A is a set of actions.
- P_{sa} are the state transition probabilities. For each state $s \in S$ and action $a \in A$ P_{sa} is a distribution over S. $P_{s_i a_j}(s_k)$ gives the probability that we transition to state s_k when we perform action a_j in state s_i.
- $\gamma \in [0, 1)$ is called the discount factor.
- $R : S \times A \mapsto \mathbb{R}$ is the reward function. Sometimes we use $R : S \mapsto \mathbb{R}$.

Markov decision processes

Payoff

A MDP proceeds as follows:

- Start in some state s_0.
- Choose action $a_0 \in A$.
- Sample $s_1 \sim P_{s_0 a_0}$.
- Choose action a_1.
- Sample $s_2 \sim P_{s_1 a_1}$.
- Repeat...

We can represent the progress of the process as

$$s_0 \xrightarrow{a_0} s_1 \xrightarrow{a_1} s_2 \xrightarrow{a_2} s_3 \xrightarrow{a_3} \cdots$$

Given such a sequence, we can compute the total payoff as

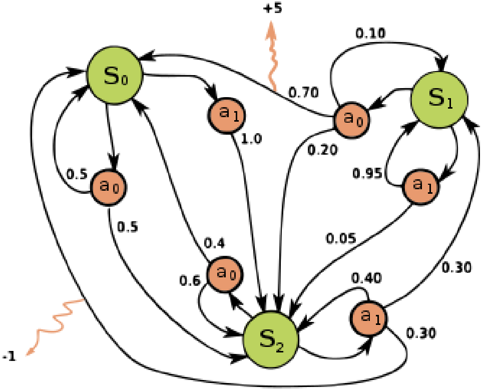
$$R(s_0, a_0) + \gamma R(s_1, a_1) + \gamma^2 R(s_2, a_2) + \cdots$$

or

$$R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots .$$

Markov decision processes

Simple MDP example



https://en.wikipedia.org/wiki/Markov_decision_process

Markov decision processes

Discount

The reward at timestep t is discounted by a factor γ^t.

In some applications, time might not matter and we would use $\gamma = 1$.

In many other applications, we want to accrue positive rewards as quickly as possible, so $\gamma < 1$.

When $R(\cdot)$ is the amount of money we make, γ would represent the effect of interest (money accrued today is more valuable in the long term than money accrued tomorrow).

How should our learning agent behave?

A policy is a function $\pi : S \mapsto A$ mapping from states to actions.

Executing policy π means when we are in state s, we take action $a = \pi(s)$.

The value function for policy π is

$$V^{\pi}(s) = \mathbb{E}[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots \mid s_0 = s; \pi],$$

i.e., the expected sum of discounted rewards upon starting in state s and taking actions according to π.

Markov decision processes

Bellman equations

Given a fixed policy π, the value function V^{π} satisfies the Bellman equations

$$V^{\pi}(s) = R(s) + \gamma \sum_{s' \in S} P_{s\pi(s)}(s') V^{\pi}(s').$$

$R(s)$ is called the immediate reward we get for starting in state s.

The second term (the summation) can be rewritten $\mathbb{E}_{s' \sim P_{s\pi(s)}}[V^{\pi}(s')]$, i.e., the expected sum of discounted rewards for starting in state s', where s' is distributed according to $P_{s\pi(s)}$, the distribution over states resulting from action $\pi(s)$ in state s.

The Bellman equations give us a simple way to determine V^π for finite-state MDPs ($|S| < \infty$).

Simply write down the equation $V^\pi(s)$ for each $s \in S$.

When R, P_{sa}, and π are fixed, we have $|S|$ linear equations in $|S|$ unknowns.

[Exercise: find V^π for a small MDP such as a robot maze escape.]

Markov decision processes

Optimal value function and policy

We define the optimal value function as

$$V^*(s) = \max_\pi V^\pi(s).$$

We can write the Bellman equations for the optimal value function as

$$V^*(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s') V^*(s').$$

We have the immediate reward plus the maximum over all a of the expected future sum of discounted rewards we'll get from a.

We define the optimal policy $\pi^* : S \mapsto A$ as

$$\pi^*(s) = \operatorname*{argmax}_{a \in A} \sum_{s' \in S} P_{sa}(s') V^*(s') \tag{1}$$

Markov decision processes

Optimal policy

Some facts:

$$\forall s, V^*(s) = V^{\pi^*}(s).$$

$$\forall s, \pi, V^{\pi^*}(s) \geq V^{\pi}(s).$$

i.e., π^* as defined in Equation (1) is the optimal policy.

Note that π^* is optimal for *all* states s. So there is no need to modify π^* according to the start state.

So the question, then, is how can we find π^* for a given MDP?

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Value iteration and policy iteration

Value iteration

For the time being, we assume our MDP has a finite state space and action space ($|S| < \infty$, $|A| < \infty$).

Algorithm VALUE-ITERATION

```
For each state $s$
    $V(s) \leftarrow 0$
Repeat until convergence
    For each state $s$
        $V(s) \leftarrow R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s') V(s')$
```

The inner loop can be synchronous (compute all values before overwriting) or asynchronous (compute and update values one by one in some order).

Theorem: under synchronous or asynchronous value iteration, V converges to V^*.

Value iteration and policy iteration

Policy iteration

After running value iteration, we use Equation (1) to find π^*.

Policy iteration computes π^* somewhat more directly:

Algorithm Policy-Iteration

```
Initialize $\pi$ randomly
Repeat until convergence
    $V \leftarrow V^\pi$
    For each state $s$
        $\pi(s) \leftarrow \text{argmax}_{a \in A} \sum_{s' \in S} P_{sa}(s') V(s')$
```

Theorem: after a finite number of iterations, V will converge to V^* and π will converge to π^*.

Value iteration and policy iteration

Which to use?

Policy iteration converges quickly for small MDPs.

However, computing V^{π} requires solving a prohibitively large linear system for large problems.

Value iteration is therefore more often used in practice.

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Learning a model for a MDP

Assumptions thus far

So we see that reinforcement learning is straightforward when

- $|S| < \infty$
- $|A| < \infty$
- P_{sa} is given
- $R(s)$ is given
- The outcome s' obtained from executing action a in state s is observable.
- The reward $R(s)$ obtained from state s is immediately observable.
- The horizon is infinite (the total payoff calculation is an infinite sum).

We would want to relax each of these assumptions.

Relaxing assumptions 3 and 4 require estimating P_{sa} and/or $R(s)$ from data.

[Think about what is known in example real-world problems.]

Learning a model for a MDP

Estimating P_{sa}

When P_{sa} is unknown, we run the MDP multiple times in a simulator or the real world and observe the behavior of the system:

$$s_0^{(1)} \xrightarrow{a_0^{(1)}} s_1^{(1)} \xrightarrow{a_1^{(1)}} s_2^{(1)} \xrightarrow{a_2^{(1)}} s_3^{(1)} \xrightarrow{a_3^{(1)}} \cdots$$

$$s_0^{(2)} \xrightarrow{a_0^{(2)}} s_1^{(2)} \xrightarrow{a_1^{(2)}} s_2^{(2)} \xrightarrow{a_2^{(2)}} s_3^{(2)} \xrightarrow{a_3^{(2)}} \cdots$$

Examples:

- For an inverted pendulum, we run until the pendulum falls over.
- For a game, we try a large but finite number of moves.

Learning a model for a MDP

Estimating P_{sa}

With enough experience, the maximum likelihood estimate of the state transition probabilities are

$$P_{sa}(s') = \frac{\text{\# of times we took action } a \text{ in state } s \text{ and got to } s'}{\text{\# of times we took action } a \text{ in state } s}.$$

(When we obtain $0/0$ we can assume $P_{sa}(s') = 1/|S|$.)

To do this efficiently, we just need to record the counts for the numerator and denominator over time and increment the appropriate counts each time we take an action and observe the result.

Estimating R is similar. Just keep track of the number of times we entered state s and what reward we received in order to calculate the expected reward.

Learning a model for a MDP

Estimating P_{sa}

Putting value iteration together with model learning for P_{sa}, we obtain

Algorithm VALUE-ITERATION-WITH-TRANSITION-LEARNING

```
Initialize $\pi$ randomly
Repeat
    Execute $\pi$ in the MDP for some number of trials
    Use accumulated experience to update $P_{sa}$
    Use value iteration to estimate $V$
    Update $\pi$ to be the greedy policy for $V$
```

Value iteration will converge more quickly if we begin with the V obtained in the previous step.

Note that there is a related, more stochastic algorithm called Q-learning that learns the expected value of taking action a in state s.

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TD learning combines ideas from Monte Carlo methods with ideas from dynamic programming.[1]

TD methods update estimates of V or Q iteratively based on other learned estimates.

[1] "Dynamic programming" refers to any computational algorithm that solves a complex problem by breaking it down into smaller problems recursively.

Temporal-Difference Methods

TD prediction

Our first TD method is TD prediction, which is the TD version of policy evaluation, aiming to estimate V^{π}.

Monte Carlo methods wait until the real return following a visit is known:

$$V(S_t) \leftarrow V(s_t) + \alpha \left[G_t - V(s_t) \right],$$

where G_t is the actual return obtained in state s_t following π to the end of the episode.

TD methods don't wait until G_t is known. Instead, the simplest TD method, TD(0), makes the update

$$V(s_t) \leftarrow V(s_t) + \alpha \left[R_{t+1} + \gamma V(s_{t+1}) - V(s_t) \right]$$

right after transitioning to s_{t+1} and receiving R_{t+1}.

Temporal-Difference Methods

TD prediction

Take note of the difference: MC uses a target of G_t, whereas TD(0) uses a target of $R_{t+1} + \gamma V(s_{t+1})$.

Algorithm TABULAR TD(0) FOR V^π

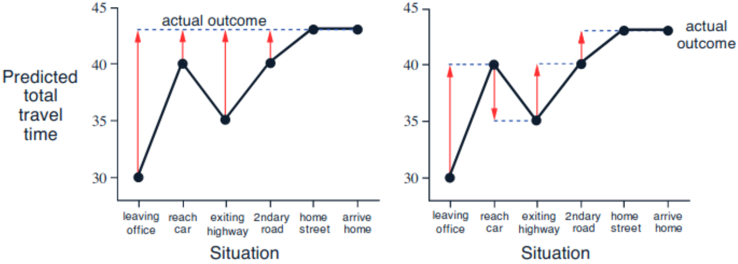
```
Input: policy $\pi$, learning rate $\alpha$
Initialize $V(s)$ for all $s \in S$ arbitrarily except $V(\mathrm{terminal}) = 0$
For each episode:
    Initialize $s$
    For each step in episode:
        $a \leftarrow \pi(s)$
        $s' \sim P_{sa}$
        $V(s) \leftarrow V(s) + \alpha\,[R(s') + \gamma V(s') - V(s)]$
```

Temporal-Difference Methods

TD prediction

There's a nice illustration of the difference between MC and TD policy evaluation in Sutton and Barto (p. 122): driving home from work every day and continually updating an estimate of how long it will take to get home.

Reward is the elapsed time on each leg of your journey (when we switch to control, we would have to use negative reward).



Sutton and Barto (2018), Figure 6.1: MC (left) vs. TD (right).

Temporal-Difference Methods

TD prediction

Both the MC and TD methods are guaranteed to converge to V^π if α is small enough and decreases over time.

MC is fine for short episodes, but TD is preferred when episodes are long or infinite.

Empirical studies usually find that TD methods converge faster than MC.

Temporal-Difference Methods

TD control: SARSA

Now we consider the control problem, the task of finding π^*.

SARSA is an on-policy TD policy iteration method.

We introduce the action value function $q_\pi(s, a)$, which is the discounted return expected after executing action a in state s then following π.

The SARSA update is

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha \left[R(s_{t+1}) + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t) \right].$$

The method is called SARSA due to the quintuple $(s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1})$.

Temporal-Difference Methods

TD control: SARSA

Algorithm SARSA ON-POLICY TD CONTROL

```
Input: learning rate $\alpha$, exploration probability $\varepsilon$
$\forall s, a$, initialize $Q(s, a)$ arbitrarily except $Q(\mathrm{terminal}, \cdot) = 0$
For each episode:
    Initialize $s$, choose $a$ using policy derived from $Q$ (e.g., $\varepsilon$-greedy)
    For each step in episode:
        Take action $a$, observe $r$, $s'$
        Choose $a'$ from $s'$ using policy derived from $Q$ (e.g., $\varepsilon$-greedy)
        $Q(s, a) \leftarrow Q(s, a) + \alpha \left[r + \gamma Q(s', a') - Q(s, a)\right]$
        $s \leftarrow s'; a \leftarrow a'$
```

TD control: SARSA

There's a nice example of SARSA for the "Windy Gridworld" example in Sutton and Barto on p. 130.

Exercise: reproduce the SARSA result shown in the graph on p. 130, then do Exercise 6.9 on p. 131.

Q-learning (Watkins, 1989) was one of the early breakthroughs in RL.

It is also fundamental to the Mnih et al. (2013) breakthrough with DQN.

The Q-learning update is

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha \left[R(s_{t+1}) + \gamma \max_a Q(s_{t+1}, a) - Q(s_t, a_t) \right].$$

Compare with SARSA.

Actions are taken using ε-greedy, but the updates are based on the action that would be taken with the optimal policy, without ε-greedy.

Because of this difference, Q-learning is an off-policy TD method.

Temporal-Difference Methods

TD control: Q-learning

Algorithm Q-LEARNING OFF-POLICY TD CONTROL

```
Input: learning rate $\alpha$, exploration probability $\varepsilon$
$\forall s, a$, initialize $Q(s, a)$ arbitrarily except $Q(\mathrm{terminal}, \cdot) = 0$
For each episode:
    Initialize $s$
    For each step in episode:
        Choose $a$ using policy derived from $Q$ (e.g., $\varepsilon$-greedy)
        Take action $a$, observe $r$, $s'$
        $Q(s, a) \leftarrow Q(s, a) + \alpha \left[ r + \gamma \max_{a'} Q(s', a') - Q(s, a) \right]$
        $s \leftarrow s'$
```

Note that Q-learning is a little simpler than SARSA, and note precisely why it is off-policy.

There is a nice demonstration of the differences between SARSA and Q-learning, the "Cliff Walking" example in Sutton and Barto on p. 132.

Exercise: implement SARSA and Q-learning for the cliff walking example to demonstrate that Q-learning learns an optimal policy that is different from (and better than) the ε-greedy policy learned by SARSA.

The TD control algorithms involve maximization during policy construction.

Q-learning's target policy is the greedy policy, which involves taking action a maximizing $Q(s, a)$.

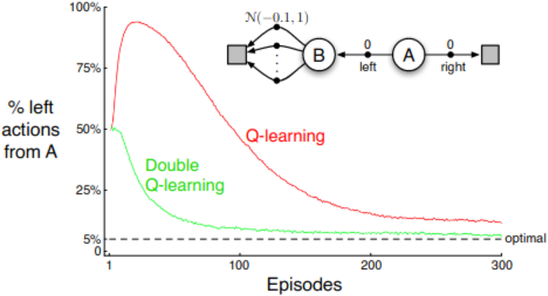
SARSA's policy is, for example, ε-greedy, also involving maximization.

The max operator tends to introduce bias into the value estimates.

Consider a state s for which all actions a have 0 value. With uncertain but unbiased estimates of $Q(s, a)$, some will be positive and some will be negative.

$\max_a Q(s, a)$, on the other hand, will almost always be positive, a positive bias called maximization bias.

Sutton and Barto give an example MDP in which the optimal action is always to move “right” from state A, but due to maximization bias and the high variance of the distribution over rewards, Q-learning will overestimate the value of a “left” move.



Sutton and Barto (2018), Fig. 6.5

Idea to learn without maximization bias: divide the episodes into two sets and use them to learn two separate estimates Q_1 and Q_2 of $Q(a)$.

We use Q_1 for maximization:

$$a^* = \underset{a}{\operatorname{argmax}}\, Q_1(a),$$

and we use Q_2 as the estimate of the value:

$$Q_2(a^*) = Q_2(\underset{a}{\operatorname{argmax}}\, Q_1(a)).$$

The expectation

$$\mathbb{E}\left[Q_2(a^*)\right] = q(a^*)$$

is unbiased.

We also use Q_1 as an unbiased estimate of the value of actions selected using Q_2:

$$Q_1(a^*) = Q_1(\operatorname*{argmax}_a Q_2(a)).$$

This is called double learning.

We learn two estimates, but only one is updated on each episode.

Double Q-learning uses this idea to get more accurate, less biased estimates of values and can learn an optimal policy for the left/right environment above.

Temporal-Difference Methods

Maximization bias and double learning

Algorithm DOUBLE Q-LEARNING for $Q_1 \approx Q_2 \approx q$

```
Input: learning rate $\alpha$, exploration probability $\varepsilon$
$\forall s, a$, initialize $Q_1(s, a)$ and $Q_2(s, a)$ arbitrarily except $Q(\text{terminal}, \cdot) = 0$
For each episode:
    Initialize $s$
    For each step in episode:
        Choose $a$ using $\varepsilon$-greedy with $Q_1 + Q_2$
        Take action $a$, observe $r$, $s'$
        With probability $\frac{1}{2}$:
            $Q_1(s, a) \leftarrow Q_1(s, a) +$
                $\alpha \left[r + \gamma Q_2(s', \operatorname{argmax}_{a'} Q_1(s', a')) - Q_1(s, a)\right]$
        else:
            $Q_2(s, a) \leftarrow Q_2(s, a) +$
                $\alpha \left[r + \gamma Q_1(s', \operatorname{argmax}_{a'} Q_2(s', a')) - Q_2(s, a)\right]$
        $s \leftarrow s'$
```

That's it for TD methods for now.

We'll return to TD methods when we consider Deep RL.

First we describe some of the methods for dealing with continuous state MDPs.

Then we'll come to the modern use of deep neural networks to approximate value functions in both continuous state spaces and large discrete state spaces.

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Continuous state MDPs

Examples

Now that we can estimate models for P_{sa} and R, we have the following remaining assumptions:

- $|S| < \infty$
- $|A| < \infty$
- The outcome s' obtained from executing action a in state s is observable.
- The reward $R(s)$ obtained from state s is immediately observable.

What about relaxing the finite state assumption?

In a car or ground robot, we might have a state $(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$. $S = \mathbb{R}^6$.

In the inverted pendulum problem, we have $(x, \theta, \dot{x}, \dot{\theta})$. $S = \mathbb{R}^4$.

For an aircraft, we have $(x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, \dot{\phi}, \dot{\theta}, \dot{\psi})$. $S = \mathbb{R}^{12}$.

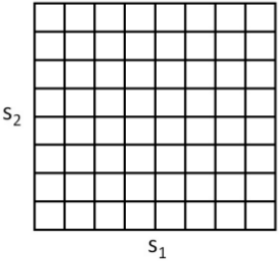
Continuous state MDPs

Discretization

Depending on the situation, we may be able to discretize the state space.

If so, we can simply run value iteration or policy iteration on the discrete approximation to the state space.

For example, with $S = \mathbb{R}^2$, we use a grid. Rather than $\mathrm{s} = (s_1, s_2) \in \mathbb{R}^2$ we break up the range of s_1 into k bins and s_2 into l bins and use $\overline{S} = \{1, ..., k\} \times \{1, ..., l\}$.

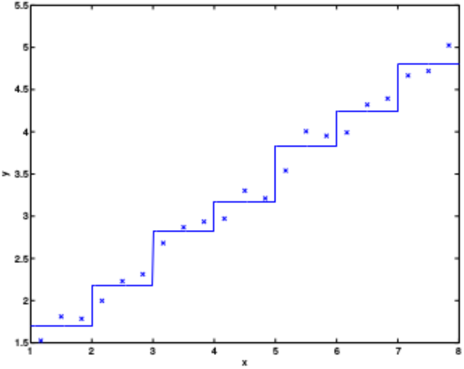


Ng (2017), CS 229 lecture notes, set 12

Continuous state MDPs

Discretization

Problem 1 with discretization: inaccuracy. Suppose the value $V(s)$ was really linear in s. We might obtain something like this:



Ng (2017), CS 229 lecture notes, set 12

Problem 2 with discretization: the curse of dimensionality.

The number of parameters we have to estimate is exponential in the number of variables in the state.

Ng:

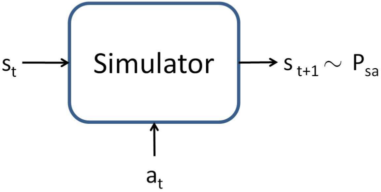
- Discretization is good for 1D and 2D problems.
- Discretization sometimes works with clever representations for 3D-6D problems.
- It never works for higher dimensional problems!

Continuous state MDPs

Value function approximation

The next technique in arsenal for dealing with continuous state spaces is value function approximation.

Value function approximation works best when we have a simulator for the MDP at hand or when running trials in the real world is fast and cheap.



Ng (2017), CS 229 lecture notes, set 12

[Simulating an inverted pendulum is straightforward if all parameters like the pendulum mass are known. Think about simulators for other tasks.]

Continuous state MDPs

Value function approximation

We first need to know P_{sa}. One possible form is

$$\mathtt{s}_{t+1} = \mathtt{A}\mathtt{s}_t + \mathtt{B}\mathtt{a}_t + \boldsymbol{\epsilon}_t,$$

where $\epsilon_t \sim \mathcal{N}(0, \Sigma)$.

$\mathtt{A}$ and $\mathtt{B}$ might be given, or we could estimate them.

To estimate $\mathtt{A}$ and $\mathtt{B}$, first we collect data from the simulation or real world MDP:

$$s_0^{(1)} \xrightarrow{a_0^{(1)}} s_1^{(1)} \xrightarrow{a_1^{(1)}} s_2^{(1)} \xrightarrow{a_2^{(1)}} \dots \xrightarrow{a_{T-1}^{(1)}} s_T^{(1)}$$

$$s_0^{(2)} \xrightarrow{a_0^{(2)}} s_1^{(2)} \xrightarrow{a_1^{(2)}} s_2^{(2)} \xrightarrow{a_2^{(2)}} \dots \xrightarrow{a_{T-1}^{(2)}} s_T^{(1)}$$

The maximum likelihood estimator, if $\epsilon_t \sim \mathcal{N}(0, \sigma^2 \mathtt{I})$, would be

$$\mathtt{A}^*, \mathtt{B}^* = \underset{\mathtt{A},\mathtt{B}}{\operatorname{argmin}} \sum_{i=1}^{m} \sum_{t=0}^{T-1} \left\| \mathrm{s}_{t+1}^{(i)} - \left(\mathtt{A}\mathrm{s}_t^{(i)} + \mathtt{B}\mathrm{a}_t^{(i)} \right) \right\|^2 .$$

(If Σ is unknown, we could also estimate it.)

Continuous state MDPs

Value function approximation

Once we have $\mathtt{A}$, $\mathtt{B}$, and possibly Σ, we can either build a deterministic model in which we assume

$$\mathtt{s}_{t+1} = \mathtt{A}\mathtt{s}_t + \mathtt{B}\mathtt{a}_t$$

precisely, or a stochastic model in which we treat $\mathtt{s}_{t+1}$ as random.

Other alternatives: add nonlinear mappings ϕ_s and ϕ_a to obtain

$$\mathtt{s}_{t+1} = \mathtt{A}\phi_s(\mathtt{s}_t) + \mathtt{B}\phi_a(\mathtt{a}_t),$$

or use locally weighted linear regression to estimate $\mathtt{s}_{t+1}$ from $\mathtt{s}_t$ and $\mathtt{a}_t$.

Continuous state MDPs

Value function approximation

Modeling the state transition is just the first step!

Next, we perform fitted value iteration to approximate the value function.

For simplicity, we'll assume $S = \mathbb{R}^n$ but $A = \{1, \ldots, k\}$ for a small number k of possible discrete actions.

(Even if A is really continuous, discretizing A is often feasible due to lower dimensionality than S.)

We will essentially, then, approximate $V(\mathrm{s})$ as a linear or nonlinear function of the state:

$$V(\mathrm{s}) = \boldsymbol{\theta}^\top \phi(\mathrm{s})$$

where ϕ is an appropriate feature mapping.

For each state in each sampled state transition sequence, we will compute a training signal $y^{(i)}$ that approximates $R(\mathrm{s}) + \gamma \max_a \mathbb{E}_{\mathrm{s}' \sim P_{\mathrm{s}a}}[V(\mathrm{s}')]$, i.e., the value function.

Continuous state MDPs

Value function approximation

Here is the complete algorithm:

```
**Algorithm** FITTED-VALUE-ITERATION
    Randomly sample $m$ states $s^{(1)}, s^{(2)}, \ldots, s^{(m)} \in S$
    $\boldsymbol{\theta} \leftarrow 0$
    Repeat
        For $i \in 1..m$
            For each action $a \in A$
                Sample $s'_1, \ldots, s'_k \sim P_{s^{(i)}a}$ (using a model of the MDP)
                // Set $q(a)$ to an estimate of $R(s^{(i)}) + \gamma \mathbb{E}_{s' \sim P_{s^{(i)}a}}[V(s')]$
                $q(a) \leftarrow \frac{1}{k} \sum_{j=1}^{k} R(s^{(i)}) + \gamma V(s'_j)$
            // Set $y^{(i)}$ to an estimate of $R(s^{(i)}) + \gamma \max_a \mathbb{E}_{s' \sim P_{s^{(i)}a}}[V(s')]$
            $y^{(i)} \leftarrow \max_a q(a)$
        $\boldsymbol{\theta} \leftarrow \operatorname{argmin}_{\boldsymbol{\theta}} \frac{1}{2} \sum_{i=1}^{m} \left(\boldsymbol{\theta}^\top \phi(s^{(i)}) - y^{(i)}\right)^2$
```

Continuous state MDPs

Value function approximation

Rather than linear regression, we can use any appropriate regression algorithm, such as locally weighted linear regression.

Unfortunately, fitted value iteration is not guaranteed to converge like discrete-space value iteration.

In practice, it does converge for many problems.

If we have a deterministic model of the MDP, then $k = 1$ is sufficient.

The approximately optimal policy is determined by $V(\mathrm{s})$, which is an approximation to V^*:

$$\pi(\mathrm{s}) = \underset{a}{\operatorname{argmax}}\, \mathbb{E}_{\mathrm{s}' \sim P_{\mathrm{s}a}}[V(\mathrm{s}')].$$

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Finite-horizon MDPs

Up to now, we always assumed an infinite horizon (our learning agent's total payoff is a infinite sum).

However, in the real world, we would often want to consider a finite horizon MDP $(\mathcal{S}, \mathcal{A}, P_{sa}, T, R)$ with $T > 0$ placing a limit on the time the agent has to accrue its payoff:

$$R(s_0, a_0) + \gamma R(s_1, a_1) + \cdots + R(s_T, a_T)$$

This leads to some important consequences, especially that the optimal policy becomes non-stationary (different depending on the start state).

With some other generalizations (P_{sa} and R are dynamic, i.e., changing over time), value iteration turns into a dynamic programming algorithm for solving the Bellman equations.

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Linear quadratic regulation (LQR)

Suppose $\mathcal{S} = \mathbb{R}^n$ and $\mathcal{A} = \mathbb{R}^d$.

Suppose also linear transitions

$$\mathtt{s}_{t+1} = \mathtt{A}\mathtt{s}_t + \mathtt{B}\mathtt{a}_t + \mathtt{w}_t$$

with $\mathtt{w}_t \sim \mathcal{N}(0, \Sigma_t)$.

Finally, suppose we have quadratic rewards

$$R^{(t)}(\mathtt{s}_t, a_t) = -\mathtt{s}_t^\top \mathtt{U}_t \mathtt{s}_t - \mathtt{a}^\top \mathtt{W} \mathtt{a}_t,$$

where $\mathtt{U}_t \in \mathbb{R}^{n \times n}$ and $\mathtt{W}_t \in \mathbb{R}^{d \times d}$ are positive definite, forcing the reward to always be negative.

Linear quadratic regulation (LQR)

The quadratic rewards mean that we want to take small, smooth actions that keep the system as close to $\mathsf{s}_t = 0$ as possible.

Think about deviation of the inverted pendulum, deviation of our car from the center of the lane, etc.

The LQR algorithm first estimates $\mathtt{A}_t$, $\mathtt{B}_t$, and Σ_t, then uses dynamic programming to find the optimal policy.

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LQR for nonlinear dynamics

Now we suppose that the dynamics, rather than being linear

$$\mathsf{s}_{t+1} = \mathtt{A}\mathsf{s}_t + \mathtt{B}\mathsf{a}_t + \mathsf{w}_t,$$

we instead have a nonlinear relationship:

$$\mathsf{s}_{t+1} = f(\mathsf{s}_t, \mathsf{a}_t) + \mathsf{w}_t.$$

An example is the inverted pendulum, in which the elements of the next position vector depend on the cosine or sine of the current angle and so on.

In this situation, we can obtain a Taylor expansion of $f(\cdot,\cdot)$ around s_t and a_t to linearize the function, then perform LQR as before.

There is a further generalization called differential dynamic programming for the case in which the reward function is more complicated than quadratic rewards (for example, approximating a gold standard trajectory for a rocket).

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Partially observable Markov decision processes (POMDPs)

Sometimes we are not able to fully observe s_t.

Instead, we may only be able to take an observation related to s_t, such taking a picture with a camera or a distance scan with a laser or a GPS reading from a GPS device.

This leads to the idea of the partially observable Markov decision process (POMDP).

We assume that at each timestep t we obtain an observation $\mathrm{z}_t \in \mathcal{O}$ related to the state s_t by a distribution $P_{zs}(\mathrm{z} \mid \mathrm{s})$.

A POMDP with finite horizon is thus given by a tuple $(\mathcal{S}, \mathcal{O}, \mathcal{A}, P_{sa}, P_{zs}, T, R)$.

In the POMDP setting, we will maintain a belief over the states $p(\mathrm{s}_t \mid \mathrm{z}_1, \ldots, \mathrm{z}_t)$, and the policy will map beliefs to actions rather than crisp states to actions.

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Linear quadratic Gaussian (LQG) regulation

In the special case that the dynamics and observations are both linear with Gaussian noise, i.e.,

$$\begin{aligned} \mathtt{s}_{t+1} &= \mathtt{A}\mathtt{s}_t + \mathtt{B}\mathtt{a}_t + \mathtt{w}_t \\ \mathtt{z}_t &= \mathtt{C}\mathtt{s}_t + \mathtt{v}_t \end{aligned}$$

with $\mathtt{w}_t \sim \mathcal{N}(0, \Sigma_s)$ and $\mathtt{v}_t \sim \mathcal{N}(0, \Sigma_z)$, we obtain the Kalman filter for maintaining our belief $p(\mathtt{s}_t \mid \mathtt{z}_1, \ldots, \mathtt{z}_t)$.

The Kalman filter gives us a mean and covariance for the belief at each time t.

Based on mean at time t, we can use the regular LQR for deterimining the optimal action.