1. Solution:

For n samples we have

$$f(\underline{x}, c) = c^{4n}(x_1 \cdots x_n)^{3n} e^{-c(x_1 + \cdots + x_n)} \quad [3E]$$

$$\frac{\partial f(\underline{x}, c)}{\partial c} = 4n \cdot c^{4n-1} \cdot (x_1 \cdots x_n)^{3n} e^{-c(x_1 + \cdots + x_n)} \quad [2E]$$

$$- (x_1 + \cdots + x_n)\, c^{4n} (x_1 \cdots x_n)^{3n} e^{-c(x_1 + \cdots + x_n)}$$

$$= \left[\frac{4n}{c} - (x_1 + \cdots + x_n)\right] f(\underline{x}, c) \quad [2E]$$

$$= 0$$

$$c = \frac{4n}{x_1 + \cdots + x_n} \quad [2E]$$

In this problem, $n = 5$

$$c = \frac{4 \times 5}{30} = \frac{4}{6} = \frac{2}{3} \quad [2E]$$

2. Solution:

Let $\bar{x}$ denote the average.

The joint density

$$f(X, c) = c^n e^{-cn(\bar{x} - x_0)}$$

has maximum if

$$\frac{\partial f(X, c)}{\partial c} = 0 \quad \Rightarrow \hat{c} = \frac{1}{\bar{x} - x_0}$$

obviously, $\bar{x} = 9$ in this problem. So

$$\hat{c} = \frac{1}{9 - 5} = \frac{1}{4}$$

3. Solution:

Note that the tranfer function is

$$H(z) = \frac{1}{1-\alpha z^{-1}} = \sum_{n=0}^{\infty} \alpha^n z^{-n} \qquad [2B]$$

So

$$h(n) = \alpha^n \qquad n \geq 0 \qquad [2B]$$

Therefore,

$$\begin{aligned} R_y(n) &= R_x(n) \circledast h(-n) \otimes h(n) \\ &= h(-n) \otimes h(n) \end{aligned} \qquad [2B]$$

Since $R_x(n) = \delta(n)$.

$$R_y(n) = \begin{cases} \sum_{k=0}^{\infty} \alpha^{-(n-k)} \alpha^k & n < 0 \\ \sum_{k=n}^{\infty} \alpha^{-(n-k)} \alpha^k & n > 0 \end{cases} \qquad [2B]$$

$$= \begin{cases} \alpha^{-n} \sum_{k=0}^{\infty} \alpha^{2k} & n < 0 \\ \alpha^{n} \sum_{k=0}^{\infty} \alpha^{2k} & n > 0 \end{cases} \qquad [1B]$$

$$= \begin{cases} \alpha^{-n} \frac{1}{1-\alpha^2} & n < 0 \\ \alpha^{n} \frac{1}{1-\alpha^2} & n > 0 \end{cases} \qquad [1B]$$

$$= \alpha^{|n|} \frac{1}{1-\alpha^2} \qquad [1B]$$

ii) The Wiener-Hopf equation reads

$$\begin{pmatrix} R_y(0) & R_y(1) & \cdots & R_y(n-1) \\ R_y(1) & R_y(0) & \cdots & R_y(n-2) \\ & \vdots & & \\ R_y(n-1) & \cdots & & R_y(0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \\ c_n \end{pmatrix} = \begin{pmatrix} R_y(n) \\ R_y(n-1) \\ \\ R_y(1) \end{pmatrix}$$

that is

$$\begin{pmatrix} 1 & \alpha & \cdots & \alpha^{n-1} \\ \alpha & 1 & \cdots & \alpha^{n-2} \\ & & \vdots & \\ \alpha^{n-1} & & \cdots & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \alpha^n \\ \alpha^{n-1} \\ \vdots \\ \alpha \end{pmatrix}$$

whose solution is

$$c_n = \alpha, \qquad c_i = 0 \qquad i < n$$

Consequently, the MMSE prediction is

$$y(n+1) = \alpha y(n)$$

The mean-square error is given by

$$\begin{aligned} MSE &= E[y(n+1) - \alpha y(n)]^2 \\ &= E[y^2(n+1) - 2\alpha y(n+1) y(n) + \alpha^2 y^2(n)] \\ &= R_y(0) - 2\alpha R_y(1) + \alpha^2 R_y(0) \\ &= \frac{1 - 2\alpha^2 + \alpha^2}{1 - \alpha^2} \\ &= 1 \end{aligned}$$