

Wavelets, Sparsity and their Applications

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Session Five: Multiresolution Analysis and Splines

Multi-Resolution Analysis

Definition By a multi-resolution analysis we mean a sequence of embedded closed subspaces

$$\dots V_2 \subset V_1 \subset V_0 \subset V_{-1} \dots$$

such that

1. Upward Completeness: $\lim_{m \rightarrow -\infty} V_m = \bar{\bigcup_{m \in \mathbb{Z}} V_m} = L_2(\mathbb{R})$. (finite energy)
2. Downward Completeness: $\lim_{m \rightarrow \infty} V_m = \bigcap_{m \in \mathbb{Z}} V_m = \{0\}$.
3. Scale Invariance: $f(t) \in V_m \leftrightarrow f(2^m t) \in V_0$.
4. Shift Invariance: $f(t) \in V_0 \rightarrow f(t - n) \in V_0$ for all $n \in \mathbb{Z}$.
5. Existence of a Basis. There exists $\varphi(t) \in V_0$, such that $\{\varphi(t - n)\}_{n \in \mathbb{Z}}$ is an orthonormal basis for V_0 .

Consequences of the Multi-Resolution Analysis

First notice that:¹

$$\langle \varphi(t - n), \varphi(t - m) \rangle = \delta_{m,n} \iff \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2k\pi)|^2 = 1.$$

Since V_1 is included in V_0 , if $\varphi(t/2)$ belongs to V_1 , it belongs to V_0 as well. Thus:

$$\varphi(x/2) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(x - n)$$

or

$$\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t - n).$$

This is the **two-scale relation**.

¹see Appendix

$$\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{+\infty} g_0[n] \varphi(2t-n)$$

↪ FT

Consequences of the Multi-Resolution Analysis

$$\hat{\varphi}(\omega) = \sqrt{2} \sum_{n=-\infty}^{+\infty} g_0[n] \int_{-\infty}^{+\infty} \varphi(2t-n) e^{-j\omega t} dt$$

$$\stackrel{\substack{x=2t-n \\ t=\frac{x+n}{2}}}{=} \frac{\sqrt{2}}{2} \sum_{n=-\infty}^{+\infty} g_0[n] \int_{-\infty}^{+\infty} \varphi(x) e^{-j\omega \frac{x}{2}} e^{-j\omega \frac{n}{2}} dx$$

By taking the Fourier transform of both sides of the two-scale relation, we obtain

$$= \frac{\sqrt{2}}{2} \sum_{n=-\infty}^{+\infty} g_0[n] e^{j\omega \frac{n}{2}} \int_{-\infty}^{+\infty} \varphi(x) e^{j\omega \frac{x}{2}} dx$$

$$= \frac{\sqrt{2}}{2} G_0(e^{j\omega/2}) \hat{\varphi}(\omega/2)$$

$$\hat{\varphi}(\omega) = \frac{1}{\sqrt{2}} G_0(e^{j\omega/2}) \hat{\varphi}(\omega/2)$$

where

$$\therefore \sum_{k=-\infty}^{+\infty} |\hat{\varphi}(2\omega + 2k\pi)|^2 = 1$$

$$G_0(e^{j\omega}) = \sum_n g_0[n] e^{-j\omega n}. \quad G_0(z) G_0(z^{-1}) + G_0(-z) G_0(-z^{-1}) = 2$$

and because of the orthogonality of $\varphi(t)$, we obtain

$$\therefore \frac{1}{2} \sum_k |G_0(e^{j(\omega+k\pi)})|^2 |\hat{\varphi}(\omega+k\pi)|^2 = 1$$

$$|G_0(e^{j\omega})|^2 + |G_0(e^{j(\omega+\pi)})|^2 = 2.$$

$$\therefore \frac{1}{2} \left(\sum_k |G_0(e^{j(\omega+2k\pi)})|^2 |\hat{\varphi}(\omega+2k\pi)|^2 + \sum_k |G_0(e^{j(\omega+(2k+1)\pi)})|^2 |\hat{\varphi}(\omega+(2k+1)\pi)|^2 \right) = 1$$

$$\therefore \underbrace{\frac{1}{2} |G_0(e^{j\omega})|^2 \sum_k |\hat{\varphi}(\omega+2k\pi)|^2}_{=1} + \underbrace{\frac{1}{2} |G_0(e^{j(\omega+\pi)})|^2 \sum_k |\hat{\varphi}(\omega+(2k+1)\pi)|^2}_{=1} = 1$$

$$\therefore |G_0(e^{j\omega})|^2 + |G_0(e^{j(\omega+\pi)})|^2 = 2$$

Consequences of the Multi-Resolution Analysis

Theorem 1. Let $\{V_n\}$, $n \in \mathbb{Z}$ be a multiresolution analysis with the scaling function $\varphi(t)$. There exists an orthonormal basis for $L_2(\mathbb{R})$:

$$\psi_{m,n}(t) = 2^{-m/2} \psi(2^{-m}t - n) \quad m, n \in \mathbb{Z}$$

φ : scaling function
(basis of V_0)

and

$$\psi(t) = \sum_{n=-\infty}^{\infty} (-1)^n g_0[1-n] \varphi(2t-n)$$

such that $\{\psi_{m,n}\}$, $n \in \mathbb{Z}$ is an orthonormal basis for W_m , where W_m is the orthogonal complement of V_m in V_{m+1} .

Construct $\psi(t)$ by $\begin{cases} \textcircled{+} & (\text{should satisfy 3 conditions}) \\ g_0[n] & (\text{choose } g_0[n] = (-1)^n g_0[1-n] \text{ for orthogonality of } \varphi(t-n) \\ & \text{that span } W_0 \perp V_0 \end{cases}$

$$V_1 \subset V_0$$

$$\textcircled{+} \quad \textcircled{+}$$

$$W_1 \quad W_0 \leftarrow \psi(t) = \sum_n g_0[n] \varphi(t-n)$$

$$\swarrow \quad \psi\left(\frac{t}{2}\right) = \sum_n g_1[n] \varphi(t-n)$$

Scaling Function and Splines

Central to multiresolution analysis is the design of a proper scaling function. It is possible to show that $\varphi(t)$ is an admissible scaling function of $L_2(\mathbb{R})$ if and only if it satisfies the three following conditions:

existence of a (biorthogonal) basis:

$\varphi(t) \in V_0$. $\{\varphi(t-n)\}_{n \in \mathbb{Z}}$ is a (orthogonal) basis for V_0 .

1. Riesz basis criterion: There exists two constants $A > 0$ and $B < +\infty$ such that

$$A \leq \sum_{k \in \mathbb{Z}} |\hat{\varphi}(\omega + 2\pi k)|^2 \leq B$$

orthogonal basis
 $\langle \varphi(t), \varphi(t-n) \rangle = \delta_n$
 $\Downarrow \Uparrow$
 $\sum_{k \in \mathbb{Z}} |\varphi(t+2\pi k)|^2 = 1$
 (1)

2. Two scale relation \Leftrightarrow
 - scale invariance: $f(t) \in V_m \Leftrightarrow f(2^m t) \in V_0$.
 - shift invariance: $f(t) \in V_0 \Leftrightarrow f(t-n) \in V_0$ for all $n \in \mathbb{Z}$.
$$\varphi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} g_0[k] \varphi(2t - k)$$
 (2)

3. Partition of unity \Leftrightarrow
 - upward completeness $\lim_{m \rightarrow \infty} V_m = \bigcap_{m \in \mathbb{Z}} V_m = \{0\}$.
 - downward completeness $\lim_{m \rightarrow -\infty} V_m = \bigcup_{m \in \mathbb{Z}} V_m = L_2(\mathbb{R})$.
$$\sum_{k \in \mathbb{Z}} \varphi(t - k) = 1.$$
 (3)

spline: have a certain number of continuous derivatives.

$\beta_N(t)$: $(N-1)$ -order derivatives is continuous.

N -order derivatives is not continuous.

Scaling Function and Splines

$$\hat{\beta}_0^+(w) = \frac{1-e^{-jw}}{jw} \quad \hat{\beta}_N^+(w) = \left(\frac{1-e^{-jw}}{jw}\right)^{N+1}$$

$\beta_0(t)$: box function 

$$\hat{\beta}_0(w) = \text{sinc} \frac{w}{2} \quad \hat{\beta}_N(w) = \left(\text{sinc} \frac{w}{2}\right)^{N+1}$$

$\beta_1(t)$: triangular function 

A remarkable example of scaling functions is given by the family of B-splines. A B-spline $\beta_N(t)$ of order N is obtained from the $(N+1)$ -fold convolution of the box function $\beta_0(t)$ or

All B-splines $\beta_i(t)$ are valid scaling function.

$$\beta_N(t) = \underbrace{\beta_0(t) * \beta_0(t) * \dots * \beta_0(t)}_{N+1 \text{ times}}$$

$$\text{with } \hat{\beta}_0(w) = \frac{1 - e^{-jw}}{jw}$$

• Riesz basis criterion
 $A \leq \sum_{k \in \mathbb{Z}} |\hat{f}(w+2\pi k)|^2 \leq B$

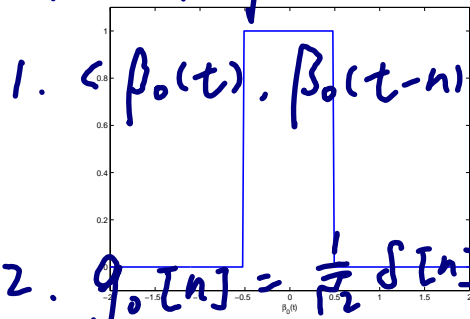
• Two scale relation

$$f(t) = \sum_{k \in \mathbb{Z}} g_0(k) f(2t-k)$$

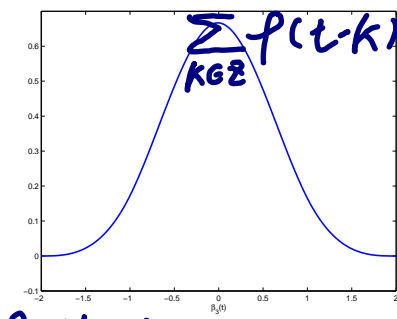
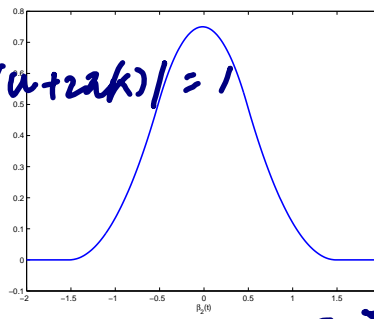
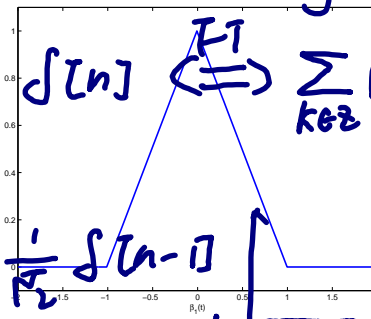
• Partition of unity

where $\hat{\beta}(w)$ is the Fourier transform of $\beta(t)$.

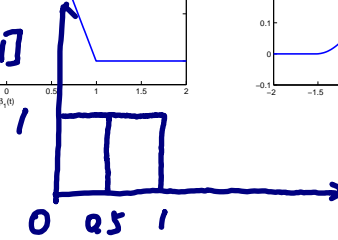
$\beta_0(t)$ box function is a valid scaling function:



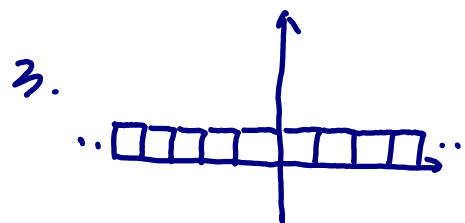
$$1. \langle \beta_0(t), \beta_0(t-n) \rangle = \delta[n] \Leftrightarrow \sum_{k \in \mathbb{Z}} |\hat{\beta}(w+2\pi k)| = 1$$



$$2. g_0[n] = \frac{1}{\sqrt{2}} \delta[n] + \frac{1}{\sqrt{2}} \delta[n-1]$$



$$f[n] = \sum_{k \in \mathbb{Z}} g_0[k] f[2n-k]$$

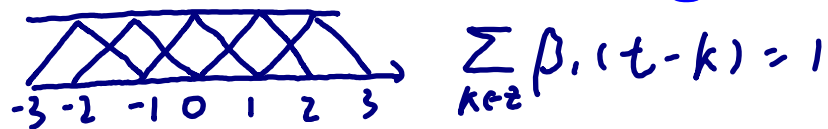


$$\Rightarrow \sum_{k \in \mathbb{Z}} \beta(t-k) = 1$$

$N=0$: orthogonal
 $N \geq 1$: biorthogonal

$\beta_1(t)$ (linear spline \wedge) is a valid scaling function.

3. Scaling Function and Splines



Theorem 2. Given two valid biorthogonal scaling functions $\varphi(t)$ and $\tilde{\varphi}(t)$ satisfying the following two scale relations

2. $\beta_1(\frac{t}{2}) = \frac{1}{2}\beta_1(t+1) + \beta_1(t) + \frac{1}{2}\beta_1(t-1)$ $P(z) = \frac{1}{16\alpha} (1+z)^2(1+z^{-1})^2(1-\alpha z)(1-\alpha z^{-1})$

$$\varphi(t) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} g_0[k] \varphi(2t - k)$$

$$g_0[-1] = \frac{1}{2\sqrt{2}}, g_0[0] = \frac{1}{\sqrt{2}}, g_0[1] = \frac{1}{2\sqrt{2}}$$

$$G_0(z) = \frac{1}{2\sqrt{2}} (1+z)(1+z^{-1})$$

$$\tilde{\varphi}(t) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} h_0[k] \tilde{\varphi}(2t - k)$$

$$H_0(z) = \frac{\sqrt{2}}{8\alpha} (1+z)(1+z^{-1})(1-\alpha z)(1-\alpha z^{-1})$$

There exist two biorthogonal wavelets ψ and $\tilde{\psi}$ such that

1. ^{finite} compact support: check in time then FT.

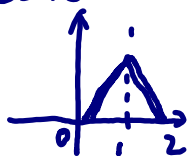
$$\psi(t) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} (-1)^{k-1} h_0[1-k] \varphi(2t - k)$$

$$A(e^{j\omega}) = \sum_n |f(\omega + 2\pi n)|^2$$

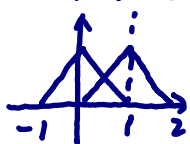
$$\{a[n] = \langle \varphi(t), \varphi(t-n) \rangle\} \xLeftrightarrow{FT} \sum_n |f(\omega + 2\pi n)|^2$$

$$\therefore \frac{1}{3} \leq \sum_n |f(\omega + 2\pi n)|^2 \leq 1$$

$$\tilde{\psi}(t) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} (-1)^{k-1} g_0[1-k] \tilde{\varphi}(2t - k)$$



$$a[0] = 2 \int_0^1 t^2 dt = \frac{2}{3}$$



$$a[-1] = a[1] = \int_0^1 t(1-t) dt = \frac{1}{6}$$

$$\therefore A(e^{j\omega}) = \frac{2}{3} + \frac{1}{6}(e^{-j\omega} + e^{j\omega})$$

$$= \frac{2}{3} + \frac{1}{3} \cos \omega$$

(linear spline: biorthogonal)

$$\psi(t) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} (-1)^{n-1} h_0[1-n] \phi_{1/2}(t-n)$$

Scaling Function and Splines

$$h_0[-n] \Leftrightarrow H_0(z^{-1}) = \frac{\sqrt{2}}{8} (-z^{-2} + 2z^{-1} + 6 + 2z - z^2)$$

$$h_0[1-n] \Leftrightarrow \frac{\sqrt{2}}{8} (-z^{-3} + 2z^{-2} + 6z^{-1} + 2 - z)$$

$$(-1)^{n-1} h_0[1-n] \Leftrightarrow \frac{\sqrt{2}}{8} (-z^{-3} - 2z^{-2} + 6z^{-1} - 2 - z)$$

Example

Assume that $\varphi(t)$ is a linear spline. The two scale equation is satisfied when $G_0(z) = (\frac{1}{2}z^{-1} + 1 + \frac{1}{2}z)/\sqrt{2}$.

The biorthogonality relation says that

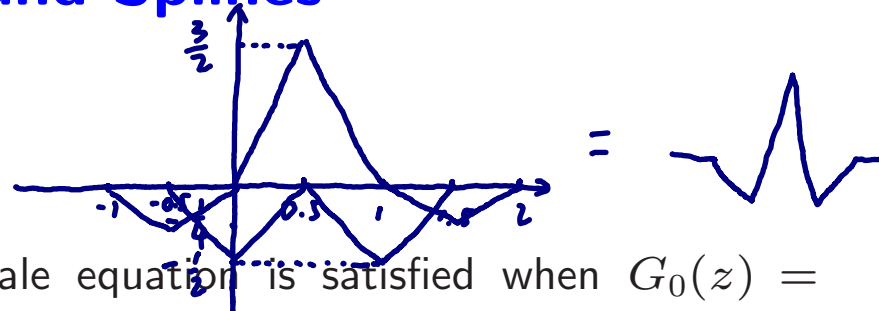
$$\langle \tilde{\varphi}(t), \varphi(t-n) \rangle = \delta_n.$$

Since both $\varphi(t)$ and $\tilde{\varphi}(t)$ satisfy a two scale relation, it follows that

$$\langle \tilde{\varphi}(t), \varphi(t-n) \rangle = \langle h_0[k], g_0[k-2n] \rangle = \delta_n.$$

The above relation is equivalent to the condition $P(z) + P(-z) = 2$ with $P(z) = H_0(z^{-1})G_0(z)$. Here $G_0(z)$ is known and has two zeros at $\omega = \pi$, the shortest $H_0(z)$ with the same number of zeros at π is then

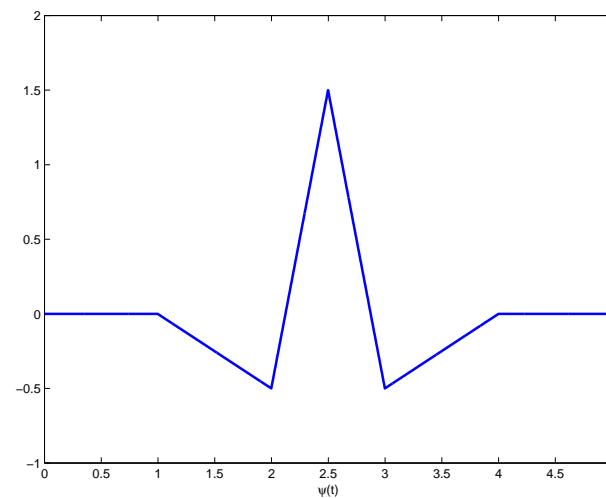
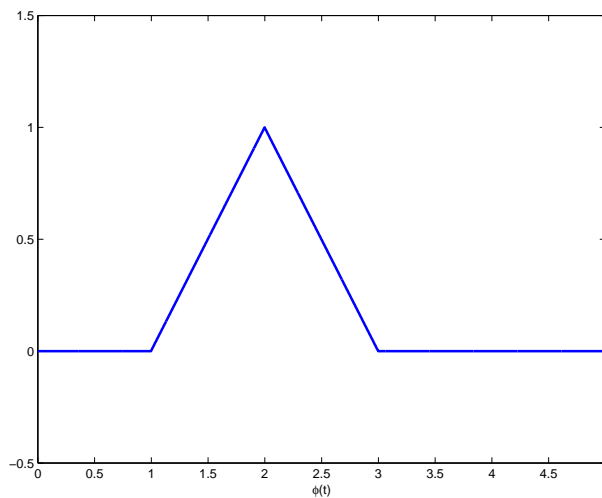
$$H_0(z) = \frac{\sqrt{2}}{8} (1+z)(1+z^{-1})(-z+4-z^{-1}) = \frac{\sqrt{2}}{8} (z^{-1}+2+z)(-z+4-z^{-1}).$$



$$f(t) = \sum_n c_{j,n} \phi_{j,n}(t) + \sum_{m=-\infty}^I \sum_n d_{n,m} \psi_{m,n}(t)$$



Given $H_0(z)$ the construction of the wavelet $\psi(t)$ is then straightforward. The scaling function $\phi(t)$ and wavelet $\psi(t)$ for this example are shown below.



$$A[n] = \langle f(t), f(t-n) \rangle = \int_n$$

$$A(\tau) = \langle f(t), f(t-\tau) \rangle = \int_{-\infty}^{+\infty} f^*(t) f(t-\tau) dt$$

Appendix

↓ FT

$$A(\omega) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f^*(t) f(t-\tau) dt e^{-j\omega\tau} d\tau$$

Claim: $\underline{\tau = t-x}$ $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f^*(t) e^{-j\omega t} dt f(x) e^{j\omega x} dx = \left[\int_{-\infty}^{+\infty} f(t) e^{j\omega t} dt \right]^* \left[\int_{-\infty}^{+\infty} f(x) e^{j\omega x} dx \right]$

$$\langle \varphi(t-n), \varphi(t-m) \rangle = \delta_{m,n} \iff \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2k\pi)|^2 = 1.$$

$$= \hat{\varphi}^*(-\omega) \hat{\varphi}(-\omega) = |\hat{\varphi}(-\omega)|^2$$

Proof:

Define $p(\tau) = \langle \varphi(t), \varphi(t-\tau) \rangle$. Then $\langle \varphi(t), \varphi(t-m) \rangle$ is obtained by sampling $p(\tau)$ with sampling period $T = 1$. The Fourier transform of $p(\tau)$ is given by: $\therefore \text{ans} = \hat{\varphi}^*(\omega) \hat{\varphi}(\omega) = |\hat{\varphi}(\omega)|^2$.

$$\int_{-\infty}^{\infty} p(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} \langle \varphi(t), \varphi(t-\tau) \rangle e^{-j\omega\tau} d\tau = |\hat{\varphi}(\omega)|^2.$$

Applying the rule that sampling in time corresponds to replica in frequency leads to the desired equality.