EE4-65/EE9-SO27 Wireless Communications Problem Sheets

Bruno Clerckx
Communications and Signal Processing Group
Department of Electrical and Electronic Engineering
Imperial College London
email: b.clerckx@imperial.ac.uk

January 7, 2019

Note: For references to equations and figures, please refer to the MIMO Wireless Networks reference book.

Here are problems for you to practice during the lab sessions and at home. Solving those problems will help you grasp the content of the lectures and prepare for the courseworks. Questions follow the flow of the lectures. Do not hesitate to ask questions to the GTA.

Using Matlab, ...

1. [**PART 1: channel**] Generate of a log-normal shadowing with 8dB standard deviation and observe the distribution of this shadowing.

2. [**PART 1: channel**] Generate a circularly symmetric complex random Gaussian variable $h \sim \mathcal{CN}(0,1)$ and observe the distribution of its magnitude $|h|$ and phase. Observe the distribution of the squared magnitude $|h|^2$.
   N.B: Function randn may be helpful.

3. [**PART 1: channel**] Generate the channel $h = \sqrt{\frac{K}{1+K}}\bar{h} + \sqrt{\frac{1}{1+K}}\tilde{h}$ with $\bar{h} = e^{j\alpha}$ and $\tilde{h}$ a circularly symmetric complex random Gaussian variable with real and imaginary parts having variance equal to $1/2$. Observe the distribution of its magnitude $|h|$ as a function of K.

4. [**PART 1: channel**] Generate n independent and identically distributed circularly symmetric complex random Gaussian variable $\mathcal{CN}(0,1)$ and observe the distribution of the sum of the squared magnitude.

5. [**PART 1: SISO/SIMO/MISO**] Evaluate the bit error rate vs SNR performance of

   - BPSK/QPSK over an AWGN channel.
   - BPSK/QPSK over a SISO Rayleigh fading channel.

- BPSK/QPSK over a SIMO i.i.d. Rayleigh fading channel with MRC combining and two receive antennas.
- BPSK/QPSK over a MISO i.i.d. Rayleigh fading channel with transmit MRC (Matched Beamforming) and two transmit antennas.
- Alamouti scheme with BPSK/QPSK over a MISO i.i.d. Rayleigh fading channel two transmit antennas.

Explain the observed results and the achieved diversity and array gains. Compare with Fig. 1.4.

6. **[PART 2: MIMO i.i.d.]** Evaluate the bit error rate vs SNR performance of

    - QPSK over a 2×2 MIMO i.i.d. Rayleigh fading channel with dominant eigenmode transmission.
    - Alamouti scheme with QPSK over a 2×2 MIMO i.i.d. Rayleigh fading channel.

    Explain the observed results and the achieved diversity and array gains. Compare with Fig. 1.5.

7. **[PART 2: MIMO correlated]** Consider 2×2 MIMO spatially correlated Rayleigh fading channel. Correlation is modeled using the Kronecker model with $r = 0$ (no receive correlation) and t. Plot the distribution of the two eigenvalues of the channel matrix for $t = 0$ and $t = 0.9$. Explain the general behavior.

    - QPSK over a 2×2 MIMO spatially correlated Rayleigh fading channel with dominant eigenmode transmission.
    - Alamouti scheme with QPSK over a 2×2 MIMO spatially correlated Rayleigh fading channel.

8. **[PART 2: MIMO correlated]** Using the Kronecker model, evaluate the bit error rate performance of

    - QPSK over a 2×2 MIMO spatially correlated Rayleigh fading channel with dominant eigenmode transmission.
    - Alamouti scheme with QPSK over a 2×2 MIMO spatially correlated Rayleigh fading channel.

    Vary the transmit and receive correlation coefficient and observe the evolution of the error rate.

9. **[PART 2: MIMO Capacity]** For SNR ranging from -10 dB till 20dB, compare the ergodic capacity of 2×2, 4×2 and 2×4 i.i.d. Rayleigh fading channels with full (CSIT) and partial (CDIT) knowledge at the transmitter. Compare with Figure 5.3.

10. **[PART 2: Code design]** Consider two space-time codes, namely the Golden Code and the Tilted QAM Code discussed in Section 6.5.7. Using the rank-determinant criterion, predict which code is the best for several QAM constellation sizes.

11. **[PART 2: MIMO Tx and Rx i.i.d.]** Evaluate the bit error rate performance of

    - 16-QAM over a 1×2 SIMO i.i.d. Rayleigh fading channel with MRC combining.
    - Spatial Multiplexing with ML receiver and QPSK constellation over a 2×2 MIMO i.i.d. Rayleigh fading channel.
    - Spatial Multiplexing with ZF receiver and QPSK constellation over a 2×2 MIMO i.i.d. Rayleigh fading channel.
    - Spatial Multiplexing with MMSE receiver and QPSK constellation over a 2×2 MIMO i.i.d. Rayleigh fading channel.
    - Spatial Multiplexing with unordered ZF SIC receiver and QPSK constellation over a 2×2 MIMO i.i.d. Rayleigh fading channel.
    - Spatial Multiplexing with ordered ZF SIC receiver and QPSK constellation over a 2×2 MIMO i.i.d. Rayleigh fading channel.
    - Alamouti scheme with 16-QAM over a 2×2 MIMO i.i.d. Rayleigh fading channel.

    Explain the observed results and the achieved diversity and array gains. Compare with Fig. 6.2 and Fig. 6.4.

12. **[PART 2: MIMO Tx and Rx correlated]** Evaluate the bit error rate performance of

    - 16-QAM over a 1×2 SIMO spatially correlated Rayleigh fading channel with MRC combining. Investigate the performance as a function of the receive correlation coefficient.
    - Spatial Multiplexing with ML receiver and QPSK constellation over a 2×2 MIMO spatially correlated Rayleigh fading channel. The spatial correlation is modeled using the Kronecker model. Investigate the performance as a function of the transmit and receive correlation coefficient (magnitude and phase).
    - Alamouti scheme with 16-QAM over a 2×2 MIMO spatially correlated Rayleigh fading channel. The spatial correlation is modeled using the Kronecker model. Investigate the performance as a function of the transmit and receive correlation coefficient.

    Explain the observed results.

13. **[PART 2: Quantized Precoding]** Using the codebook in Table 10.1, evaluate the SER performance of quantized dominant eigenmode transmission using BPSK in 1×3 i.i.d. Rayleigh MISO channels. Compare with perfect dominant eigenmode transmission. Compare with Figure 10.8.

14. **[PART 3: PF scheduling]** Evaluate the sum-rate of a SISO BC with PF scheduling at SNR=0 dB as a function of the number of users K, the scheduling time scale t_c and the channel model

    $$h_k = \epsilon h_{k-1} + \sqrt{1-\epsilon^2} n_k$$

    with ϵ the channel time correlation coefficient. Compare with Fig. 12.7.

On a sheet of paper, ...

1. **[PART 1]** Verify the expression (1.22) and the high SNR approximation (1.23).

2. **[PART 1]** Verify the expression (1.47) and the high SNR approximation (1.48).

3. **[PART 1]** Assuming the channel is normalized such that $\mathcal{E}\{\|\mathbf{h}\|^2\} = 1$, derive an estimate of the error probability of MRC over i.i.d. Rayleigh fading channels. Take the limit for a very large number of receive antennas. What do you get? Interpret the result.

   Hint: Recall that (1.47) was found assuming $2\sigma^2 = 1$, i.e. $\mathcal{E}\{\|\mathbf{h}\|^2\} = n_r$.

4. **[PART 1]** Assume a transmission of a signal c from a single antenna transmitter to a multi-antenna receiver through a SIMO channel $\mathbf{h}$. The transmission is subject to the interference from another transmitter sending signal x through the interfering SIMO channel $\mathbf{h}_i$. The received signal model writes as

   $$\mathbf{y} = \mathbf{h}c + \mathbf{h}_i x + \mathbf{n}$$

   where $\mathbf{n}$ is the zero mean complex additive white Gaussian noise (AWGN) vector with $\mathcal{E}\{\mathbf{n}\mathbf{n}^H\} = \sigma_n^2 \mathbf{I}_{n_r}$.

   We apply a combiner $\mathbf{g}$ at the receiver to obtain the observation $z = \mathbf{g}\mathbf{y}$. Derive the expression of the MMSE combiner and the SINR at the output of the combiner.

5. **[PART 1]** Verify the high SNR assumption (1.56).

6. **[PART 1]** Verify the high SNR assumption (1.62). Explain the difference with (1.56).

7. **[PART 1/3]** A system is made of one single-antenna transmitter and K single-antenna receivers. The transmitter decides to send data in a TDMA manner, i.e. one user a a time. The user selected to receive data is the one with the largest instantaneous channel magnitude. Assuming all K users experience independent and identically distributed Rayleigh fading channels, derive the expression of the average SNR after user selection. How does this average SNR change as a function of K?

   Hint: View a user as an antenna and reuse the derivations made for receive diversity via selection combining.

8. **[PART 1]** Explain what happens if Alamouti is applied to a MISO channel that is varying over two consecutive symbol durations?

9. **[PART 2]** Show that the optimum (in the sense of SNR maximization) transmit precoder and combiner in dominant eigenmode transmission is given by the dominant right and left singular vector of the channel matrix, respectively.

10. **[PART 2]** Verify (1.80) and the high SNR assumption (1.81). What is the difference with the output SNR and error rate achieved in a SIMO channel with 4 receive antennas?

11. **[PART 2]** Assume a MISO system with two transmit antennas. The channel gains are identically distributed circularly symmetric complex Gaussian (with real and imaginary parts having a variance 1/2) but can be correlated and are denoted as h_1 and h_2. Write the expression of the transmit correlation matrix $\mathbf{R}_t$ and derive the eigenvalues and eigenvectors of $\mathbf{R}_t$ as a function of the transmit correlation coefficient t.

12. **[PART 2]** Assume a dominant eigenmode transmission in a MIMO channel $\mathbf{H} = \mathbf{H}_w \mathbf{R}_t^{1/2}$ with $\mathbf{H}_w$ a random fading matrix made of i.i.d. circularly symmetric complex Gaussian entries (with real and imaginary parts having a variance 1/2) and $\mathbf{R}_t$ the transmit correlation matrix. Considering two transmit antennas, discuss how the array gain behaves as a function of the number of receive antennas n_r and the transmit correlation coefficient. For simplicity, look at the case where $n_r = 1$ and $n_r \to \infty$.

13. **[PART 2]** Is the rate achievable in a MIMO channel with multiple eigenmode transmission and uniform power allocation across modes always larger than that achievable with dominant eigenmode transmission?

14. **[PART 2]** Using Jensen inequality, upper bound the ergodic capacity of Rayleigh fast fading MIMO channels with transmit correlation only ($\mathbf{R}_r = \mathbf{I}_{n_r}$) and derive a power allocation strategy that maximizes this upper bound. Interpret the results.

15. **[PART 2]** Discuss the validity of the statement "Spatial correlations degrade the achievable rate of MIMO channels." Give examples and/or counter-examples.

16. **[PART 2]** Derive the asymptotic (infinite SNR) multiplexing-diversity trade-off of a SIMO i.i.d. Rayleigh fading channel.

    Hint: For u that is χ^2_{2n} distributed, $P(u \leq \epsilon) \approx \epsilon^n$.

17. **[PART 2]** Discuss the validity of the statement "The achievable rate of a $n_r \times n_t$ MIMO channel is increased by transmitting a larger number of streams." Give examples and/or counter-examples.

18. **[PART 2]** Discuss the validity of the statement "In a point to point SISO system where only the receiver has knowledge of the channel and the codewords span many coherence times, fading is detrimental to channel capacity."

19. **[PART 2]** Compare the capacity with perfect CSIT of a deterministic $n \times m$ MIMO channel $\mathbf{H}$ with that of the $m \times n$ MIMO channel $\mathbf{H}^T$ under a fixed total transmit power constraint. What do you observe? Would this observation also be true for the ergodic capacities (with CDIT) of i.i.d. $n \times m$ and $m \times n$ MIMO Rayleigh fading channels?

20. **[PART 2]** Assume $n_t = n_r$ and full rank transmit/receive correlation matrices, verify the expression (5.87).

21. **[PART 2]** Derive the asymptotic (at infinite SNR) diversity-multiplexing tradeoff of a $1 \times n_t$ MISO i.i.d. Rayleigh slow fading channel. Assume that the input covariance matrix that minimizes the outage probability at high

SNR is the identity matrix $\mathbf{Q} = \frac{1}{n_t}\mathbf{I}_{n_t}$. Explain your reasoning and the meaning of the result.
Hint: Note that for a χ^2_{2n} random variable u, $P[u \leq \delta] \approx \delta^n$ for small δ.

22. **[PART 2]** Consider the transmission $\mathbf{y} = \mathbf{H}\mathbf{c}' + \mathbf{n}$ with perfect CSIT over a deterministic point to point MIMO channel whose matrix is given by

$$\mathbf{H} = \begin{bmatrix} a & 0 & a & 0 \\ 0 & b & 0 & b \end{bmatrix}$$

where a and b are complex scalars with $|a| \geq |b|$. The input covariance matrix is given by $\mathbf{Q} = \mathcal{E}\left\{\mathbf{c}'\mathbf{c}'^H\right\}$ and is subject to the transmit power constraint $\mathrm{Tr}\left\{\mathbf{Q}\right\} \leq P$.

- Compute the capacity with perfect CSIT of that deterministic channel. Particularize to the case $a = b$. Explain your reasoning.
- Explain how to achieve that capacity.
- In which deployment scenario, could such channel matrix structure be encountered?

23. **[PART 2]** A common expression for the ZF applied to Spatial Multiplexing is given by $\mathbf{G}_{ZF} = \mathbf{H}^\dagger$ (see (6.91) where we simply omit additional scaling factor) where $\mathbf{H}$ is the MIMO channel matrix. Write an alternative expression for the ZF filter/combiner illustrating that ZF receiver actually maximizes the SNR under the constraint that the interferences from all other layers are nulled out. Specifically, focus on a given layer q and write the expression of the combiner such that this layer is detected through a projection of the output vector $\mathbf{y}$ onto the direction closest to $\mathbf{H}(:,q)$ within the subspace orthogonal to the one spanned by the set of vectors $\mathbf{H}(:,p)$, $p \neq q$.

24. **[PART 2]** Show that at low SNR, the pairwise error probability $P(\mathbf{C} \rightarrow \mathbf{E})$ in slow fading i.i.d. channel is mainly dominated by the trace of error matrix $\mathrm{Tr}\{\tilde{\mathbf{E}}\}$. Derive a space-time code design criterion for low SNR. Explain the meaning and relate to SISO AWGN channel.

25. **[PART 2]** Relying on the rank-determinant criterion, show that delay diversity achieves full diversity. Assume for simplicity two transmit antennas. Recall that delay diversity is characterized by the fact that the sequence of symbols transmitted on the first antenna also appears on the second antenna with a delay of one symbol duration. Generalize to n_t transmit antennas. Discuss the pros and cons of such scheme versus O-STBC.

26. **[PART 2]** Let us assume that c_1, c_2, c_3 and c_4 are constellation symbols taken from a unit average energy QAM constellation. A narrowband transmission using a Linear Space-Time Block Code, characterized by codewords

$$\mathbf{C} = \frac{1}{2}\begin{bmatrix} c_1 + c_3 & c_2 + c_4 \\ c_2 - c_4 & c_1 - c_3 \end{bmatrix},$$

is considered for low velocity deployments where the transmit/receive antennas are widely spaced and the transmitter/receiver are surrounded by a large number of scatterers. You are asked to provide a recommendation on the suitability of that code in slow fading scenarios. Is that a good space-time code? Justify your answer.

27. **[PART 3]** What is the capacity region of a Three-User SISO Multiple Access Channel?

28. **[PART 3]** Explain why in a two-user SISO BC, the channels have to be ordered in order to achieve the capacity region with superposition coding with SIC.

29. **[PART 3]** Show that the capacity region of a two-user SISO BC is a triangle if the normalized channel gains are equal, i.e. $|\mathsf{h}_1|^2 = |\mathsf{h}_2|^2 = |\mathsf{h}|^2$.

30. **[PART 3]** Show that the sum-rate capacity of a two-user SISO BC is achieved by allocating the transmit power to the strongest user.

31. **[PART 3]** Consider a MU-MIMO transmission with K streams transmitted to K users. Each receiver is equipped with multiple antennas and a single stream is intended to each user. The transmitter precodes user data information using a general precoder $\mathbf{P}$ (size $n_t \times K$). Assume that each receiver is equipped with a MMSE receiver. Write the expression of the MMSE receiver of user 1.

32. **[PART 3]** Consider a multi-cell network where each transmitter is equipped with 4 antennas and a user in the centre cell is equipped with 2 receive antennas. Assume two streams are transmitted to the user. Write a system model, the expression of the MMSE receiver for that user and the expression of the rate achievable by that user. Do the same for a ZF, ZF-SIC and MMSE-SIC receiver.