1(a) solution:

If all the possible values of a random variable are x_1, x_2, …, x_n, respectively, and the corresponding probabilities are $p(x_i), i = 1, \dots, n$, then the expectation of X is defined as:

$$E(x) = \sum_{i=1}^{n} x_i p(x_i)$$

As for this problem, Y=1 and Y=0 are all the possible values and their corresponding probabilities are both equal to ½. So, $E(Y) = 1 * P(Y = 1) + 0 * P(Y = 0) = 1 * \frac{1}{2} + 0 * \frac{1}{2} = 1/2$.
Y is called a Bernoulli random variable.

1(b) solution:

The probability of having j successes knowing that you have flipped the coin for n times is
the probability of j successes in any n independent experiments. If we denote the X as the number of successes, then random variable X obeys a binomial distribution:

$$P(X = j) = \binom{n}{j} p^j (1-p)^{(n-j)}$$

$\binom{n}{j}$ (read as "n choose j") is the binomial coefficient, the number of j-combinations from a given set of n elements is also denoted by a variation such as C_n^j.

1(c) solution:

If random variable X represents the number of flips, the x=k mean that the "success" comes at the k-th flips and all the k-1 flips before are "failed", so the probability of P(x=k) is

$$p(1-p)^{k-1}, k = 1,2, \dots,$$

and it a Geometric distribution.

2(a) solution:

There are 36 equally outcomes, just 10 of them, i.e.

(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)

contain exactly one six. The answer is

$$\frac{10}{36} = \frac{5}{18}$$

2(b) solution:

P(a die shows an odd)=1/2, then by independence,
P(both odd)=P(first die shows an odd)*P(second die shows an odd)=1/2*1/2=1/4.

2(c) solution:

The sum of the scores is 4. Obviously, there are {1, 3}, {2, 2} and {3, 1}, so the probability is 3/36.

2(d) solution:

The sum of the scores is divisible by 3. Denote the sum as S, then
P(S divisible by 3) = P(S=3) + P(S=6) + P(S=9) + P(S=12) = 2/36 + 5/36 + 4/36 + 1/36 = 12/36.
where S=3 => {1, 2}, {2, 1}
S=6 => {1, 5}, {2, 4}, {3, 3}, {4, 2}, {5, 1}
S=9 => {3, 6}, {4, 5}, {5, 4}, {6, 3}
S=12 => {6, 6}

3 a fair coin is thrown repeatedly. What is the probability that on the n-th throw?
(a) Solution:

A head happens for the first time. The same as the problem 1(c), it is a geometric distribution with p=1/2, so P(x=n) = $\frac{1}{2} * \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n}$.

(b) Solution:

If n is odd, so the probability is zero.
If n is even, It is equal to the fact that n/2 times "successes" in *n* independent Bernoulli experiments. So it is an binomial distribution

$$P(x=\frac{n}{2})=\binom{n}{n/2}\left(\frac{1}{2}\right)^{n/2}\left(\frac{1}{2}\right)^{n/2}=\binom{n}{n/2}\left(\frac{1}{2}\right)^{n}$$

(c) Solution:

If exactly two heads have appeared, then he probability is

$$P(x=2)=\binom{n}{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{n-2}=\binom{n}{2}\left(\frac{1}{2}\right)^{n}$$

(d) Solution:

At least two heads have appeared. You can solve it directly, say the event {at least two heads} = {two heads} + {three heads} + … + {n heads} = …

Or, you can consider this problem from the opposite side:

P{at least two heads} = 1 – P{no heads} – P{only one head}= $1-\left(\frac{1}{2}\right)^n-\binom{n}{1}\left(\frac{1}{2}\right)^n$

4. Solution:

The sample space S={0,1,…9} and we can simply obtain:

A={1,3,5,7,9}, B={3,6,9}, C={0,1,2,3,4}

so P(A)=5/10, P(B)=3/10, P(C)= 5/10.
the probability of P(AUBUC) =

$$
\begin{aligned}
&P(A)+P(B)+P(C)-P(AB)-P(AC)-P(BC)+P(ABC) \\
&=\frac{5}{10}+\frac{3}{10}+\frac{5}{10}-\frac{2}{10}-\frac{2}{10}-\frac{1}{10}+\frac{1}{10} \\
&=\frac{9}{10}
\end{aligned}
$$

5. Solution:

The decoder takes a majority vote to decide on what transmitted bit was. It means if there is only one error in the 3 transmitted bits, then the decoder will give a correct decision. If there are two or more errors, then the decoder will give an incorrect decision.
So, if we denote X as the numbers of errors at the decoder, then

P(incorrect decision) = P(X is greater than or equals to 2)

each transmission is a Bernoulli trial in which a “success” means an error, so the corresponding probability is p=0.001.

P(incorrect decision) = $P(X\geq 2)=\binom{3}{2}(0.001)^2(0.999)+\binom{3}{3}(0.001)^3\approx 3\times 10^{-6}$

6. Solution:

“pick two numbers x and y uniformly at random between 0 and 1” means we get a two-dimensional sample space: unit square.
If we draw the regions corresponding to the events, we can get the probabilities according to the area of regions.
P(A) = area of A / area of the sample space = 1/2.

Similarly, P(B) =1/2, P(C) = 1/2.

P(D) = 0.

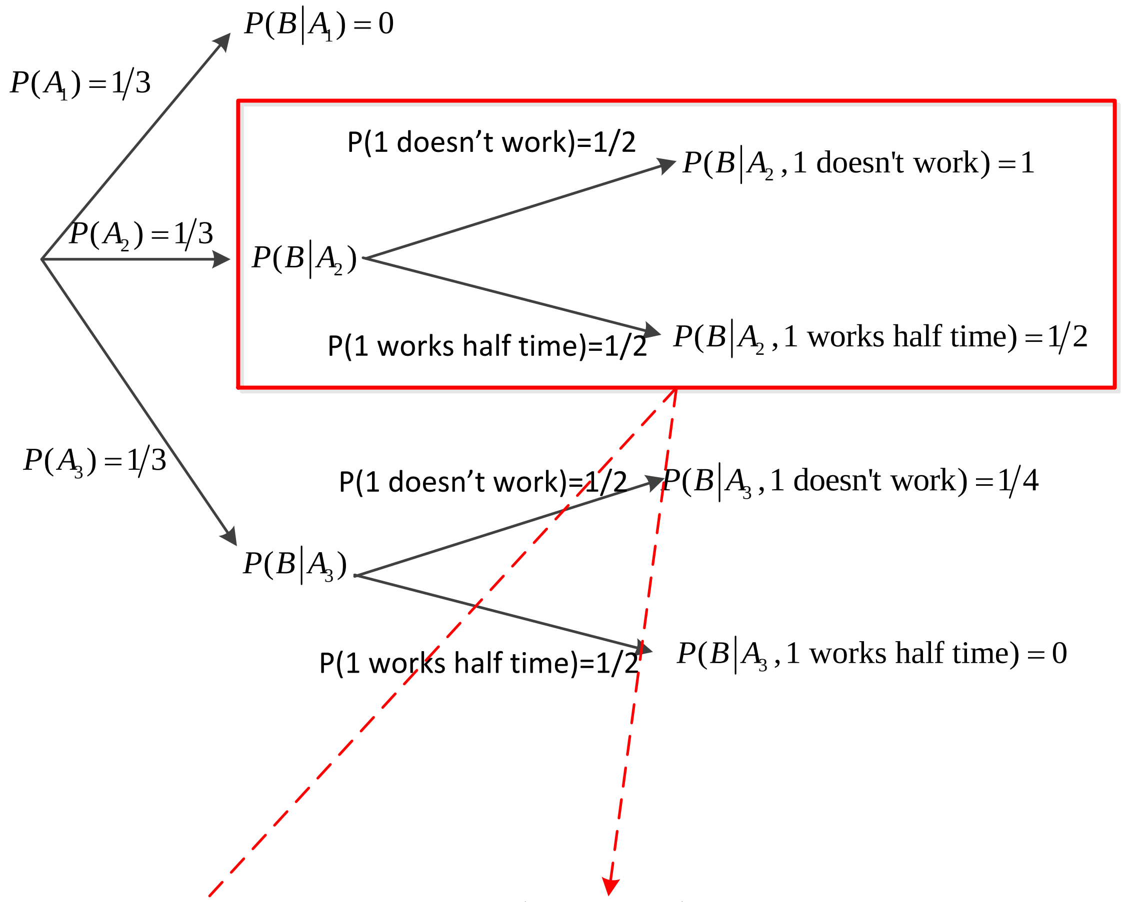
7. Solution:

Firstly, we list the relevant events:
A1={the first phone is reliable},
A2={the second phone is reliable},

A3={the third phone is reliable}

B={try one phone and it doesn't work and try another twice in succession and it works both times}

According to the problem, we want to know P(A2|B). To solve this kind of problems more clearly, we can use the tree diagram.



$$P(A_2|B)=\frac{P(B|A_2)P(A_2)}{P(B)}=\frac{\left(1\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\right)\times\frac{1}{3}}{P(B|A_1)P(A_1)+P(B|A_2)P(A_2)+P(B|A_3)P(A_3)}$$

$$=\frac{\left(1\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\right)\times\frac{1}{3}}{0\times\frac{1}{3}+\left(1\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\right)\times\frac{1}{3}+\left(\frac{1}{4}\times\frac{1}{2}+\frac{1}{2}\times 0\right)\times\frac{1}{3}}=\frac{6}{7}$$