

THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. a) This is the joint pdf of two independent Gaussian RVs with zero mean and variance 2. Hence $P(X \leq 0.5 \cap Y \leq 0.7) = P(X \leq 0.5)P(Y \leq 0.7)$. After standardizing the two random variables, we find $P(X \leq 0.5) = P(Z_1 \leq 0.5/\sqrt{2}) \approx 0.63$ and $P(Y \leq 0.7) = P(Z_2 \leq 0.7/\sqrt{2}) \approx 0.69$ such that $P(X \leq 0.5 \cap Y \leq 0.7) \approx 0.43$. [2 - E]

b) $f_X(x) = \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}}$. [2 - E]

c) $E(X) = 0$, [2 - E]

$\text{Var}(X) = 2$, [2 - E]

We can find these results by directly computing the integrals but it would be simpler to note from the marginal PDF that $X \sim N(0, 2)$.

d) $f_Y(y) = \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}}$. [2 - E]

e) $E(Y) = 0$, [2 - E]

$\text{Var}(Y) = 2$ [2 - E]

f) $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$. [1 - E]
 $\text{Corr}(X, Y) = 0$ [1 - E]

- g) X and Y are uncorrelated since $\text{Corr}(X, Y) = 0$. [1 - E]
 They are also independent since the joint pdf is written as the product of marginals. [1 - E]

- h) We can first compute the Jacobian and write

$$\begin{vmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{vmatrix} = \begin{vmatrix} \cos V & -U \sin V \\ \sin V & U \cos V \end{vmatrix} = U$$

[2 - B]

We then write

$$f_{U,V}(u, v) = \frac{u}{4\pi} e^{-\frac{u^2}{4}}, \quad u > 0, -\pi \leq v \leq \pi.$$

[2 - B]

- i) The marginal are obtained by integration of the joint pdf as follows

$$f_U(u) = \int_{-\pi}^{\pi} f_{U,V}(u,v)dv = \frac{u}{2}e^{-\frac{u^2}{4}}, \quad u > 0$$

$$f_V(v) = \int_0^{\infty} f_{U,V}(u,v)du = \frac{1}{2\pi}, \quad -\pi \leq v \leq \pi$$

U is Rayleigh distributed and V is uniformly distributed over $[-\pi, \pi]$.

[2 - B]

- j) Since $f_{U,V}(u,v) = f_U(u)f_V(v)$, U and V are two independent random variables.

[2 - A]

- k) The conditional pdf $f_{U|V}(u|v)$ is given as

$$f_{U|V}(u|v) = f_U(u) = \frac{u}{2}e^{-\frac{u^2}{4}}, \quad u > 0$$

[2 - A]

- l) $E(U|V) = E(U) = \sqrt{\pi}$.

[2 - A]

2. a) The pdf is valid since $f_X(x) \geq 0$ and $\int_{-\infty}^{\infty} f_X(x)dx = 1$. This can be easily verified by noting that $X \sim \text{EXPO}(1/2)$. [4 - A]

b) The CDF is given by $F_X(x) = \int_{-\infty}^x f_X(x)dx$ which leads to

$$F_X(x) = 1 - e^{-\frac{x}{2}}, \quad x \geq 0$$

[4 - A]

c) $E(X) = \int_{-\infty}^{\infty} x f_X(x)dx$. We get $E(X) = 2$.

[2 - A]

$$\text{Var}(X) = E(X^2) - E(X)^2 = 4.$$

[2 - A]

d) We write $m_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$.

[1 - A]

By integration,

$$m_X(t) = \int_0^{\infty} e^{tx} \frac{1}{2} e^{-\frac{x}{2}} dx = \frac{1}{2} \int_0^{\infty} e^{(t-1/2)x} dx = \frac{1}{1-2t}$$

[1 - A]

We can compute $E(X) = m'_X(0)$ and $E(X^2) = m''_X(0)$.

We get $m'_X(0) = \frac{2}{(1-2t)^2} \Big|_{t=0} = 2$.

[1 - A]

Similarly $m''_X(0) = 8$ such that $\text{Var}(X) = 8 - (2)^2 = 4$.

[1 - A]

e) The exact value can be computed as follows

$$\begin{aligned} P\left(\left|X - \frac{1}{3}\right| \geq \frac{1}{4}\right) &= 1 - P\left(\left|X - \frac{1}{3}\right| \leq \frac{1}{4}\right) \\ &= 1 - P\left(-\frac{1}{4} \leq X - \frac{1}{3} \leq \frac{1}{4}\right) \\ &= 1 - P\left(\frac{1}{12} \leq X \leq \frac{7}{12}\right) \\ &= 1 - F_X\left(\frac{7}{12}\right) + F_X\left(\frac{1}{12}\right) \\ &= 1 - (1 - e^{-7/24}) + (1 - e^{-1/24}) \\ &= 0.7062 \end{aligned}$$

[4 - A]