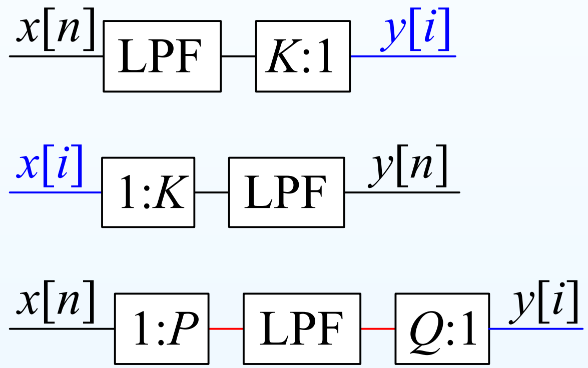
# 13: Resampling Filters

# Resampling

Suppose we want to change the sample rate while preserving information: e.g. Audio $44.1$ kHz$\leftrightarrow$$48$ kHz$\leftrightarrow$$96$ kHz




Downsample:

LPF to new Nyquist bandwidth: $\omega_0 = \frac{\pi}{K}$

Upsample:

LPF to old Nyquist bandwidth: $\omega_0 = \frac{\pi}{K}$

Rational ratio: $f_s \times \frac{P}{Q}$

LPF to lower of old and new Nyquist bandwidths: $\boxed{\omega_0 = \frac{\pi}{\max(P,Q)}}$

- Polyphase decomposition reduces computation by $\underline{K = \max(P,Q)}$.
- The transition band centre should be at the Nyquist frequency, $\omega_0 = \frac{\pi}{K}$
- Filter order $\boxed{M \approx \frac{d}{3.5\Delta\omega}}$ where $d$ is stopband attenuation in dB and $\Delta\omega$ is the transition bandwidth (Remez-exchange estimate).
- Fractional semi-Transition bandwidth, $\alpha = \frac{\Delta\omega}{2\omega_0}$, is typically fixed.

  e.g. $\alpha = 0.05 \quad \Rightarrow \quad M \approx \frac{dK}{7\pi\alpha} = 0.9dK$ (where $\omega_0 = \frac{\pi}{K}$)

# Halfband Filters

If $K = 2$ then the new Nyquist frequency is $\omega_0 = \frac{\pi}{2}$.

We multiply ideal response $\frac{\sin \omega_0 n}{\pi n}$ by a Kaiser window. All even numbered points are zero except $h[0] = 0.5$.

If $4 \mid M$ ! and we make the filter causal ($\times z^{-\frac{M}{2}}$),

$$H(z) = 0.5z^{-\frac{M}{2}} + z^{-1}\sum_{r=0}^{\frac{M}{2}-1} h_1[r]z^{-2r}$$

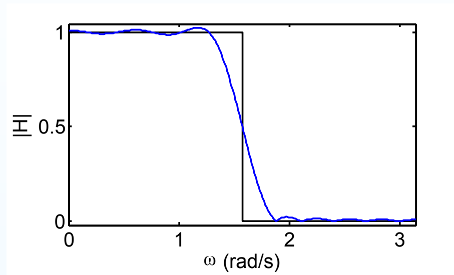
where $h_1[r] = h[2r+1-\frac{M}{2}]$

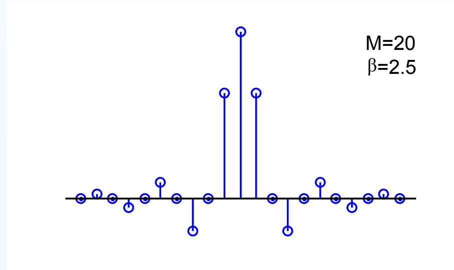
Half-band upsampler:

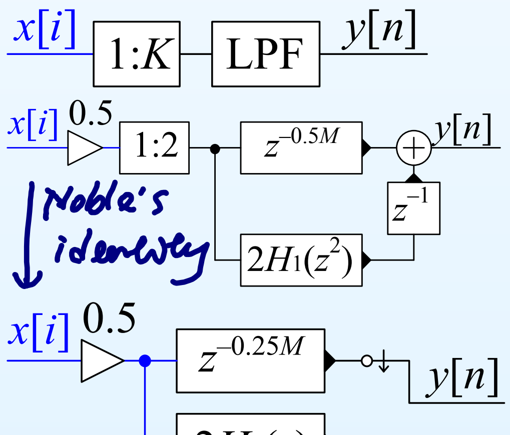
We interchange the filters with the 1:2 block and use the commutator notation.

$H_1(z)$ is symmetrical with $\frac{M}{2}$ coefficients (non-zero) so we need $\frac{M}{4}$ multipliers in total (input gain of $0.5$ can usually be absorbed elsewhere).

Computation: $\frac{M}{4}$ multiplies per input sample



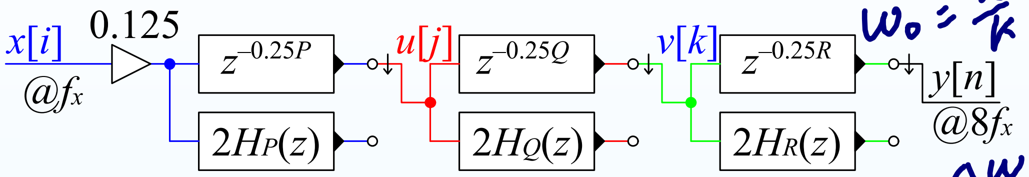



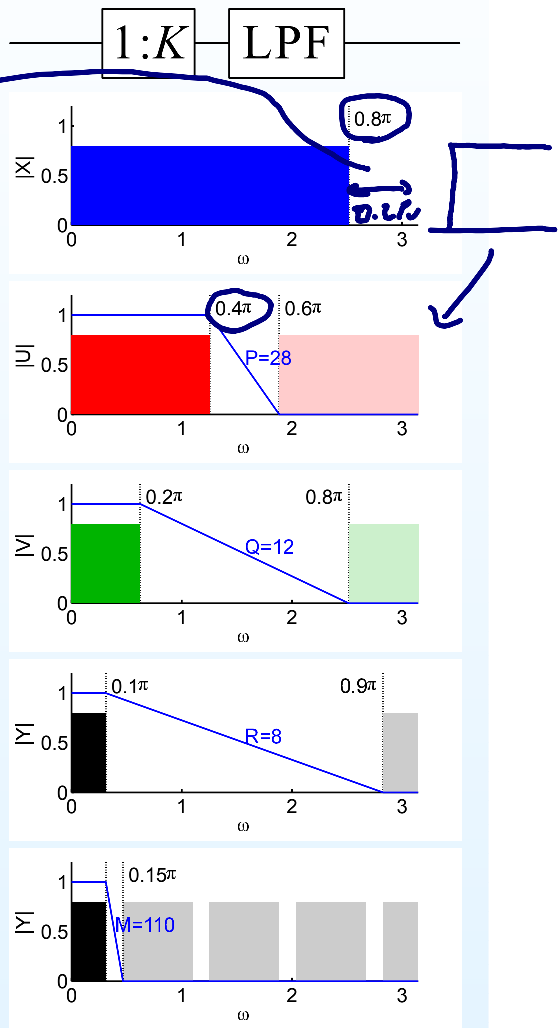

Noble's identity

filter at lower freq.

# Dyadic 1:8 Upsampler




$W_0 = \frac{\pi}{K} = \frac{\pi}{2}$




Suppose $X(z)$: BW = $0.8\pi \Leftrightarrow \alpha = 0.2$ $\quad \alpha = \frac{\Delta W}{2W_0} = 0.2$

Upsample 1:2 $\rightarrow U(z)$: $\quad P \approx \frac{d}{3.5\Delta W}$

Filter $H_P(z)$ must remove image: $\Delta\omega = 0.2\pi$

For attenuation = 60 dB, $P \approx \frac{60}{3.5\Delta\omega} = 27.3$

Round up to a multiple of 4: $P = 28$

Upsample 1:2 $\rightarrow V(z)$: $\Delta\omega = 0.6\pi \Rightarrow Q = 12$

Upsample 1:2 $\rightarrow Y(z)$: $\Delta\omega = 0.8\pi \Rightarrow R = 8$

[diminishing returns + higher sample rate]

Multiplication Count:

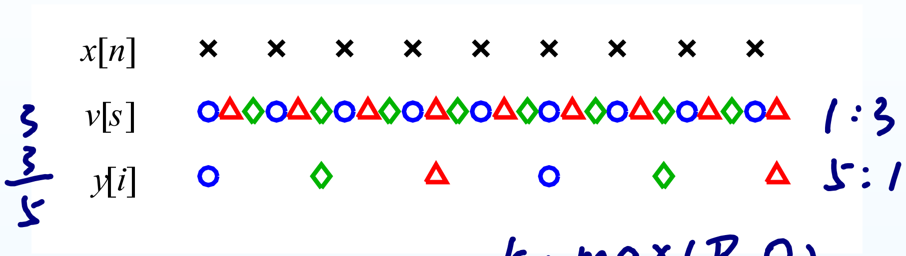
$$\left(1 + \frac{P}{4}\right) \times f_x + \frac{Q}{4} \times 2f_x + \frac{R}{4} \times 4f_x = 22f_x$$

Alternative approach using direct 1:8 upsampling:

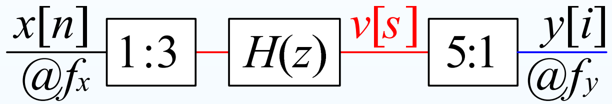
$\Delta\omega = 0.05\pi \Rightarrow M = 110 \Rightarrow 111f_x$ multiplications (using polyphase)

# Rational Resampling






To resample by $\frac{P}{Q}$ do 1:$P$ then LPF, then $Q$:1.




$k = \max(P, Q)$

Resample by $\frac{P}{Q} \Rightarrow \omega_0 = \frac{\pi}{\max(P,Q)}$ → steepness

$\Delta\omega \triangleq 2\alpha\omega_0 = \frac{2\alpha\pi}{\max(P,Q)}$ $\alpha = \frac{\Delta\omega}{2\omega_0}$

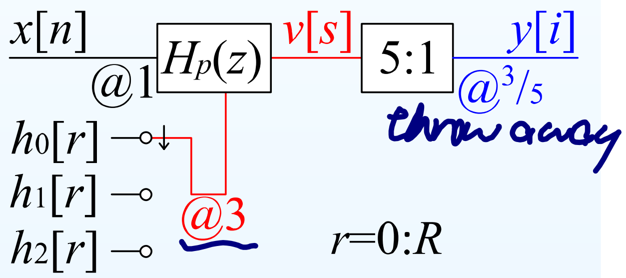
Polyphase: $H(z) = \sum_{p=0}^{P-1} z^{-p} H_p(z^P)$

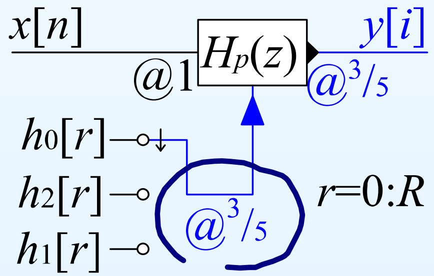
Commutate coefficients:

$v[s]$ uses $H_p(z)$ with $p = s \bmod P$

Keep only every $Q^{\text{th}}$ output:

$y[i]$ uses $H_p(z)$ with $p = Qi \bmod P$




Multiplication Count:

$H(z)$: $M + 1 \approx \frac{60\text{ [dB]}}{3.5\Delta\omega} = \frac{2.7\max(P,Q)}{\alpha}$

$H_p(z)$: $R + 1 = \frac{M+1}{P} = \frac{2.7}{\alpha}\max\left(1, \frac{Q}{P}\right)$ $M + 1$ coeficients in all

Multiplication rate: $\boxed{\frac{2.7}{\alpha}\max\left(1, \frac{Q}{P}\right) \times f_y = \frac{2.7}{\alpha}\max\left(f_y, f_x\right)}$

# Arbitrary Resampling



(adjust a little)

Sometimes need very large $P$ and $Q$:

e.g. $\frac{44.1\text{ kHz}}{48\text{ kHz}} = \frac{147}{160}$

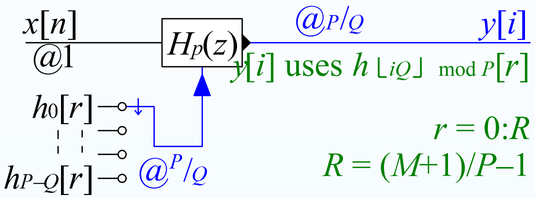
Multiplication rate OK: $\frac{2.7\max(f_y, f_x)}{\alpha}$ × P.Q

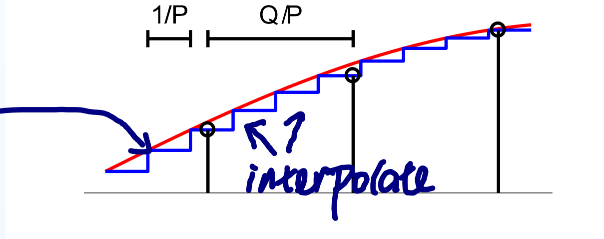
However # coefficients: $\frac{2.7\max(P, Q)}{\alpha}$

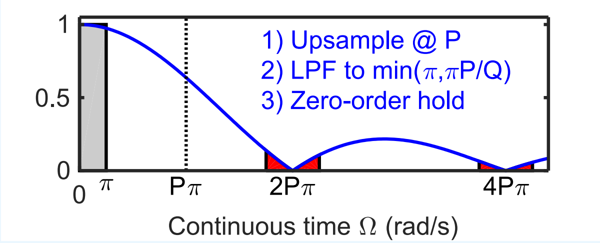
Alternatively, use any large integer $P$ and round down to the nearest sample:

E.g. for $y[i]$ at time $i\frac{Q}{P}$ use $h_p[r]$

where $p = (\lfloor iQ \rfloor)_{\text{mod } P}$




Equivalent to converting to analog with zero-order hold and resampling at $f_y = \frac{P}{Q}$.

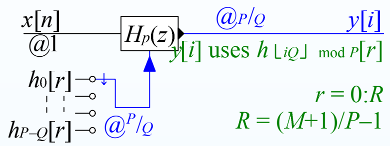
Zero-order hold convolves with rectangular $\frac{1}{P}$-wide window ⇒ multiplies periodic spectrum by $\frac{\sin\frac{\Omega}{2P}}{\frac{\Omega}{2P}}$. Resampling aliases $\Omega$ to $\Omega_{\text{mod }\frac{2P\pi}{Q}}$.

Unit power component at $\Omega_1$ gives alias components with total power:

$$\sin^2\frac{\Omega_1}{2P}\sum_{n=1}^{\infty}\left(\frac{2P}{2nP\pi+\Omega_1}\right)^2+\left(\frac{2P}{2nP\pi-\Omega_1}\right)^2 \approx \frac{\omega_1^2}{4P^2}\frac{2\pi^2}{6\pi^2} = \frac{\Omega_1^2}{12P^2}$$

For worst case, $\Omega_1 = \pi$, need $P = 906$ to get $-60$ dB ☹

# Polynomial Approximation




Suppose $P = 50$ and $H(z)$ has order $M = 249$

$H(z)$ is lowpass filter with $\omega_0 \approx \frac{\pi}{50}$

Split into $50$ filters of length $R + 1 = \frac{M+1}{P} = 5$:

$h_p[0]$ is the first $P$ samples of $h[m]$

$h_p[1]$ is the next $P$ samples, etc.

$h_p[r] = h[p + rP]$

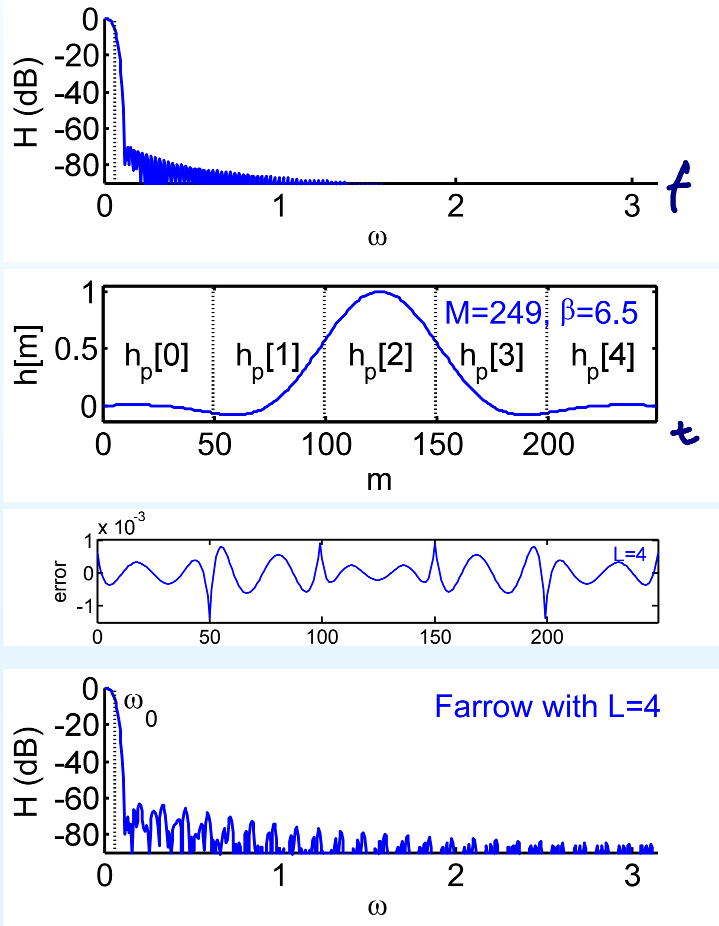
Use a polynomial of order $L$ to approximate each segment:

$h_p[r] \approx f_r\left(\frac{p}{P}\right)$ with $0 \le \frac{p}{P} < 1$

$h[m]$ is smooth, so errors are low.

E.g. error $< 10^{-3}$ for $L = 4$

- Resultant filter almost as good
- Instead of $M + 1 = 250$ coefficients we only need $(R + 1)(L + 1) = 25$ where $R + 1 = \frac{2.7}{\alpha} \max\left(1, \frac{Q}{P}\right)$



# Farrow Filter

+

Filter coefficients depend on fractional part of $i\frac{Q}{P}$:

$\Delta[i] = i\frac{Q}{P} - n$ where $n = \left\lfloor i\frac{Q}{P} \right\rfloor$

calculate on the fly

$y[i] = \sum_{r=0}^{R} f_r(\Delta[i])x[n-r]$

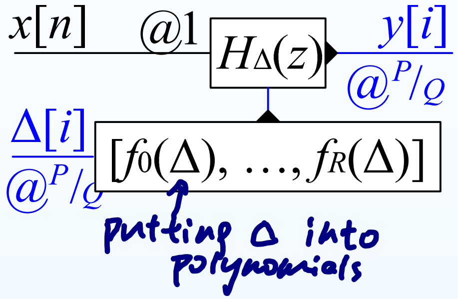
where $f_r(\Delta) = \sum_{l=0}^{L} b_l[r]\Delta^l$

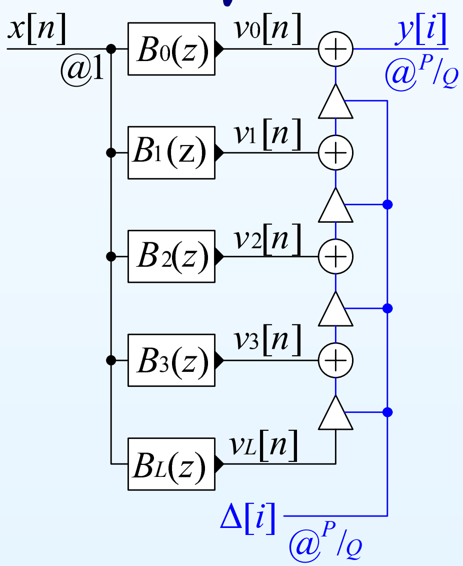
$$y[i] = \sum_{r=0}^{R}\sum_{l=0}^{L} b_l[r]\Delta[i]^l x[n-r]$$
$$= \sum_{l=0}^{L}\Delta[i]^l \underbrace{\sum_{r=0}^{R} b_l[r]x[n-r]}_{\text{filter coef} \cdot \text{data}}$$
$$= \sum_{l=0}^{L}\Delta[i]^l v_l[n]$$

where $v_l[n] = b_l[n] * x[n]$

[like a Taylor series expansion]

$R+1 = \frac{M+1}{P} = 5$




Horner's Rule:

$y[i] = v_0[n] + \Delta\left(v_1[n] + \Delta\left(v_2[n] + \Delta\left(\cdots\right)\right)\right)$

Multiplication Rate:

Each $B_l(z)$ needs $R+1$ per input sample

Horner needs $L$ per output sample

Total: $(L+1)(R+1)f_x + Lf_y = \frac{2.7(L+1)}{\alpha}\max\left(1, \frac{f_x}{f_y}\right)f_x + Lf_y$

$R+1 \approx \frac{2.7}{\alpha}\max\left(1, \frac{Q}{P}\right)$

# Summary

- Transition band centre at $\omega_0$
  - $\omega_0$ = the lower of the old and new Nyquist frequencies
  - Transition width = $\Delta\omega = 2\alpha\omega_0$, typically $\alpha \approx 0.1$
- Factorizing resampling ratio can reduce computation
  - halfband filters very efficient (half the coefficients are zero)
- Rational resampling $\times \frac{P}{Q}$
  - # multiplies per second: $\frac{2.7}{\alpha} \max(f_y, f_x)$
  - # coefficients: $\frac{2.7}{\alpha} \max(P, Q)$
- Farrow Filter
  - approximate filter impulse response with polynomial segments
  - arbitrary, time-varying, resampling ratios
  - # multiplies per second: $\frac{2.7(L+1)}{\alpha} \max(f_y, f_x) \times \frac{f_x}{f_y} + Lf_y$
    - $\approx (L+1)\frac{f_x}{f_y}$ times rational resampling case
  - # coefficients: $\frac{2.7}{\alpha} \max(P, Q) \times \frac{L+1}{P}$
  - coefficients are independent of $f_y$ when upsampling

For further details see Mitra: 13 and Harris: 7, 8.

# MATLAB routines

| | |
|---|---|
| gcd(p,q) | Find $\alpha p + \beta q = 1$ for coprime $p$, $q$ |
| polyfit | Fit a polynomial to data |
| polyval | Evaluate a polynomial |
| upfirdn | Perform polyphase filtering |
| resample | Perform polyphase resampling |