1. Solution:

All of them are Markov chains!

X is a Markov chain if and only if

$$P\left(X_{n+1}=k \middle| X_n=i_n,\cdots,X_0=i_0\right)=P\left(X_{n+1}=k \middle| X_n=i_n\right)$$

(a)

We want to check if

$$P\left(X_{m+r}=k \middle| X_{m+r-1}=i_{m+r-1},\cdots,X_m=i_m\right)=P\left(X_{m+r}=k \middle| X_{m+r-1}=i_{m+r-1}\right).$$

It is true based on the Markov property of X.

(b)

Let $\{\text{even}\}=\{x_{2r}=i_{2r},0\le r\le m\}$ and $\{\text{odd}\}=\{x_{2r+1}=i_{2r+1},0\le r\le m-1\}$

$$P\left(X_{2m+2}=k \middle| \text{even}\right)=\frac{P\left(X_{2m+2}=k,\text{even}\right)}{P(\text{even})}$$

$$=\frac{\sum P\left(X_{2m+2}=k,X_{2m+1}=i_{2m+1},\text{even},\text{odd}\right)}{P(\text{even})}$$

(All the possible values of odds form a partition, so we can employ the Law of Total Probability. The sum is taken over all the possible values of odds)

$$=\frac{\sum P\left(X_{2m+2}=k,X_{2m+1}=i_{2m+1} \middle| \text{even},\text{odd}\right)P(\text{even},\text{odd})}{P(\text{even})}$$

$$=\frac{\sum P\left(X_{2m+2}=k,X_{2m+1}=i_{2m+1} \middle| X_{2m}=i_{2m}\right)P(\text{even},\text{odd})}{P(\text{even})}$$

(by the Markov property of X)

$$=\frac{P\left(X_{2m+2}=k \middle| X_{2m}=i_{2m}\right)P(\text{even})}{P(\text{even})}=P\left(X_{2m+2}=k \middle| X_{2m}=i_{2m}\right)$$

(note that the sum is taken over all the possible values of odds, we can use the inversion of Law of Total Probability)

Eventually, we have $P\left(X_{2m+2}=k \middle| \text{even}\right)=P\left(X_{2m+2}=k \middle| X_{2m}=i_{2m}\right)$, which means $X_{2m},m\ge 0$ is Markov.

(c)

$$P\left((X_{n+1},X_{n+2})=(k,l) \middle| (X_n,X_{n+1})=(i_n,k),\cdots,(X_0,X_1)=(i_0,i_1)\right)$$

$$=P\left((X_{n+1},X_{n+2})=(k,l) \middle| X_{n+1}=k,X_n=i_n,\cdots,X_0=i_0\right)$$

$$= \frac{P(X_{n+2} = l, X_{n+1} = k, X_n = i_n, \dots)}{P(X_{n+1} = k, X_n = i_n, \dots)} \quad \text{(by definition)}$$

$$= \frac{P(X_{n+2} = l, X_{n+1} = k, | X_n = i_n, \dots) P(X_n = i_n, \dots)}{P(X_{n+1} = k \mid X_n = i_n, \dots) P(X_n = i_n, \dots)}$$

$$= \frac{P(X_{n+2} = l, X_{n+1} = k, | X_n = i_n)}{P(X_{n+1} = k \mid X_n = i_n)} \quad \text{(by the Markov property of X)}$$

$$= \frac{P(X_{n+2} = l, X_{n+1} = k, X_n = i_n)}{P(X_{n+1} = k, X_n = i_n)}$$

$$= P\left((X_{n+1}, X_{n+2}) = (k, l) \middle| X_{n+1} = k, X_n = i_n\right)$$

$$= P\left((X_{n+1}, X_{n+2}) = (k, l) \middle| (X_n, X_{n+1}) = (i_n, k)\right)$$

So, (X_n, X_{n+1}) is a Markov chain.

2. Solution:

Again, all are Markovian.

(a)

Let Y_n be the outcome of the n-th row, then according to the definition $X_n = \max(Y_1, \cdots, Y_n)$, we have $X_n = \max(X_{n-1}, Y_n)$. So it is Markovian. The one-step transition matrix is

$$P_{i,j} = \begin{cases} 0, & j < i \\ i/6, & j = i \\ 1/6, & j > i \end{cases}.$$

Further, the n-step transition matrix is

$$P_{i,j}(n) = \begin{cases} 0, & j < i \\ (i/6)^n, & j = i \\ (j/6)^n - ((j-1)/6)^n, & j > i \end{cases}$$

where for $j > i$, $P_{i,j}(n) = P(X_n = j)$, here we define $X_n = \max(Y_1, \cdots, Y_n)$. So,

$$
\begin{aligned}
P(X_n = j) &= P(X_n \le j) - P(X_n \le j-1) \\
&= P(Y_1 \le j)\cdots P(Y_n \le j) - P(Y_1 \le j-1)\cdots P(Y_n \le j-1) \\
&= \left(\frac{j}{6}\right)^n - \left(\frac{j-1}{6}\right)^n
\end{aligned}
$$

where the second equality holds because the maximum is smaller than j if and only if each $Y_m, m = 1, \dots, n$ is smaller than j, and the third is from that $P(Y_m \le j) = j/6$.

(b)

According to the definition, it is not difficult to understand that $N_{n+1} - N_n$ is independent of $N_1, N_2, \cdots, N_n$, so that N_n is Markov. The one-step transition matrix is $P_{i,j} = \begin{cases} 1/6, & j = i+1 \\ 5/6, & j = i \\ 0, & \text{otherwise} \end{cases}$

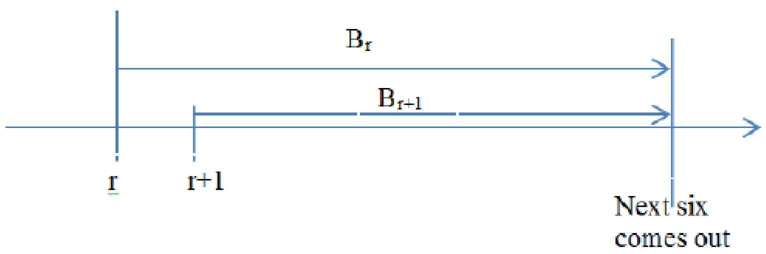
(c)

As C_r is the time since the most recent six, the relationship between C_r and C_{r+1} can be described as (at the (*r*+1)th step)

$$
C_{r+1} = \begin{cases} 0, & \text{if 6 shows } (p = 1/6) \\ C_r + 1, & \text{otherwise} \end{cases}
$$

The one-step transition matrix is $P_{i,j} = \begin{cases} 1/6, & j = 0 \\ 5/6, & j = i+1 \\ 0, & \text{otherwise} \end{cases}$

(d)



See the figure, at time r, B_r is the time until the next six comes out. B_{r+1} means that from time $r+1$, the time until the next six comes out. We have rolled the dice once between r and $r+1$, so it is easy to find that $B_r = B_{r+1} + 1$, if $B_r > 0$.

If $B_r = 0$ and $B_{r+1} = j$, it means the six comes out at time r and the next $j - 1$ rolls are all not six. Thus, we can get

$$P_{i,j} = P\left(B_r = i, B_{r+1} = j\right) = \begin{cases} 1, & j = i-1, i > 0 \\ \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{j-1}, & j \geq 1, i = 0 \end{cases}$$

3. Solution:

For any sequence $i_0, i_1, \cdots, i_k, \cdots$,

$$P\left(Y_{k+1} = i_{k+1} \middle| Y_r = i_r, \text{ for any } 0 \leq r \leq k\right)$$

$$= P\left(X_{n_{k+1}} = i_{k+1} \middle| X_{n_r} = i_r, \text{ for any } 0 \leq r \leq k\right)$$

$$= \frac{P\left(X_{n_r} = i_r, \text{ for any } 0 \leq r \leq k+1\right)}{P\left(X_{n_r} = i_r, \text{ for any } 0 \leq r \leq k\right)}$$

$= \dfrac{\prod_{r=0}^{k} P_{i_r, i_{r+1}}\left(n_{r+1} - n_r\right)}{\prod_{r=0}^{k-1} P_{i_r, i_{r+1}}\left(n_{r+1} - n_r\right)}$, where $P_{i_r, i_{r+1}}\left(n_{r+1} - n_r\right)$ is the probability of

$$\left\{\text{from } X_{nr} = i_r \text{ to } X_{n(r+1)} = i_{r+1}, \text{ through } \left(n_{r+1} - n_r\right) \text{ steps}\right\}$$

$$= P_{i_k, i_{k+1}}\left(n_{k+1} - n_k\right) = P\left(Y_{k+1} = i_{k+1} \middle| Y_k = i_k\right)$$

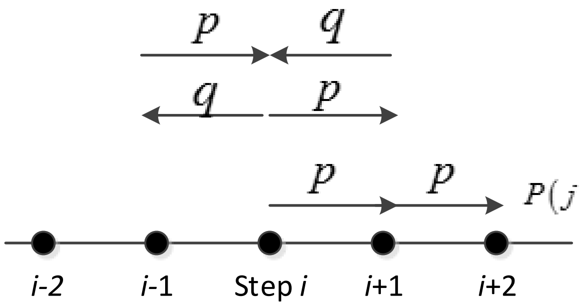
So, Y_r is a Markov chain.

When X is a simple random walk, the transition matrix of X_{2r} is:

$$P_{i,j} = \begin{cases} p^2, & j = i+2 \\ 2pq, & j = i \\ q^2, & j = i-2 \end{cases}$$

It is not difficult if we consider the following figure.

$$P(j=i)=P(i\to i+1, i+1\to i)+P(i\to i-1, i-1\to i)=2pq$$



$$P(j=i+2)=p^2$$

4. Solution:

Firstly, let's consider a simply case where $N = 2$, i.e. there are two states: 1 and 2, with $P_{1,2} = \alpha$, $P_{2,1} = \beta$. Denote the transition matrix as P and $P^{(n+1)} = P^{(n)} \cdot P$

So, $P_{1,1}^{(n+1)} = P_{1,2}^{(n)} P_{2,1} + P_{1,1}^{(n)} P_{1,1}$

$$= \left(1 - P_{1,1}^{(n)}\right)\beta + P_{1,1}^{(n)}(1-\alpha)$$

$$= (1-\alpha-\beta) P_{1,1}^{(n)} + \beta \quad \text{(with } P_{1,1}^{0} = 1\text{)} \qquad (1)$$

Now we want to find the value of $P_{1,1}^{(n)}$.

We have reason to believe that $P_{1,1}^{(n)}$ has the form as following:

$$P_{1,1}^{(n)} = c_1 (1-\alpha-\beta)^n + c_2 \qquad (2)$$

where c_1, c_2 are unknown constants which need to be determined.

From Eq. (1) and (2), we can establish:

$$c_1(1-\alpha-\beta)^{n+1} + c_2 = (1-\alpha-\beta)\left(c_1(1-\alpha-\beta)^n + c_2\right) + \beta$$

$$\Rightarrow c_1(1-\alpha-\beta)^{n+1} + c_2 = c_1(1-\alpha-\beta)^{n+1} + (1-\alpha-\beta)c_2 + \beta$$

$$\Rightarrow c_2 = (1-\alpha-\beta)c_2 + \beta$$

$$\Rightarrow c_2 = \frac{\beta}{\alpha+\beta}$$

Besides, from $1 = P_{1,1}^{0} = c_1 + c_2$, we know that $c_1 = \dfrac{\alpha}{\alpha+\beta}$.

Then, we can get

$$P_{1,1}^{(n)} = \begin{cases} \dfrac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n + \dfrac{\beta}{\alpha+\beta}, & \alpha+\beta > 0 \\ 1, & \alpha+\beta = 0 \end{cases}$$

Secondly, for the general case $N \geq 2$, the following is straightforward, if we notice $\beta = \dfrac{\alpha}{N-1}$. The final result is $\dfrac{1}{N} + \left(1 - \dfrac{1}{N}\right)\left(1 - \dfrac{\alpha N}{N-1}\right)^n$

5. Solution:

It is easy to get the following distribution:

Y_{2k} \ Y_{2k+2}	1	-1	
1	1/4	1/4	1/2
-1	1/4	1/4	1/2
	1/2	1/2	

For example, if we want calculate $P(Y_{2k+2}=1, Y_{2k}=1)$, then according to $Y_{2k} = Y_{2k-1}Y_{2k+1}$ and $Y_{2k+2} = Y_{2k+1}Y_{2k+3}$, there are two cases, i.e.

$$Y_{2k-1} = -1, Y_{2k+1} = -1, Y_{2k+3} = -1$$

$$Y_{2k-1} = 1, Y_{2k+1} = 1, Y_{2k+3} = 1$$

which satisfy the event $Y_{2k+2} = 1, Y_{2k} = 1$. The probabilities of these two cases are both 1/8, and then $P(Y_{2k+2}=1, Y_{2k}=1)$ =1/4.

With this distribution, it is not difficult to verify that the sequence $Y_2, Y_4, \cdots$ are i.i.d. random variables.

Furthermore, we also can obtain the joint distribution of (Y_{2k}, Y_{2k+1})

Y_{2k} \ Y_{2k+1}	1	-1	
1	1/4	1/4	1/2
-1	1/4	1/4	1/2

	1/2	1/2	

We can find that

$$P\left(Y_{2k}=i, Y_{2k+1}=j\right)=P\left(Y_{2k}=i\right)P\left(Y_{2k+1}=j\right),\ i=\pm 1,\ j=\pm 1$$

which means the sequence is pairwise independent.

However, $Y_{2k-1}, Y_{2k}, Y_{2k+1}$ are not independent because, for example,

$P\left(Y_{2k-1}=1, Y_{2k}=-1, Y_{2k+1}=1\right)=0$, but $P\left(Y_{2k-1}=1\right)P\left(Y_{2k}=-1\right)P\left(Y_{2k+1}=1\right)=\frac{1}{8}$.

Moreover,

$$P\left(Y_{2k+1}=1 \middle| Y_{2k}=-1\right)=\frac{1}{2}$$

but $P\left(Y_{2k+1}=1 \middle| Y_{2k}=-1, Y_{2k-1}=1\right)=0$.

So $Y_1, Y_2, Y_3, \cdots$ is not a Markov chain.

(Note that $Z_n=\left(Y_n, Y_{n+1}\right)$ in state space $\{-1,+1\}^2$ is a Markov chain. In fact, for example,

$P\left(Z_{n+1}=(1,1) \middle| Z_n=(1,1)\right)=\begin{cases}1/2, & n \text{ even} \\ 1, & n \text{ odd}\end{cases}$)