Exam 2016 solutions

B—bookwork, E—new example, T—new theory

1.

Students did well.

a)

$$Z = X + Y > 0, \qquad W = X - Y > 0$$
$$x_1 = \frac{z+w}{2}, \qquad y_1 = \frac{z-w}{2}$$

is the only solution. Moreover

[2E]

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

so that

[2E]

$$f_{ZW}(z,w) = \frac{1}{|J|} f_{XY}(x_1, y_1) = \frac{1}{2} e^{-(z+w)/2}, \quad 0 < w < z < \infty$$

$$F_Z(z) = \int_0^z f_{ZW}(z,w)\, dw = \tfrac{1}{2} e^{-z/2} \left. \frac{e^{-w/2}}{(-1/2)} \right|_0^z$$ [3E]

$$= \tfrac{1}{2} e^{-z/2} \left. \frac{e^{-w/2}}{(-1/2)} \right|_0^z = e^{-z/2}(1 - e^{-z/2}), \quad z > 0$$

b) We denote

$$I_k = \begin{cases} 1, & \text{the k-th post is rotten} \\ 0, & \text{otherwise} \end{cases}$$

$k+1, k+2, \ldots, k+7$ are 7 consecutive posts (if $k+i > 17$, then $k+i \Leftarrow k+i-17$)

$R_k = \sum_{i=1}^{7} I_{k+i}$ is the number of rotten posts. [3B]

So, $E(R_k) = E\left(\sum_{i=1}^{7} I_{k+i}\right) = \sum_{i=1}^{7} E(I_{k+i}) = 7E(I_k) = \frac{35}{17} > 2$

where $E(I_k) = \frac{5}{17}$ (5 of 17 posts are rotten)

[3B]

Since R_k should be integer, it must be the case that $P(R_k \geq 3) > 0$, for some k, i.e. there necessarily exists a run of 7 consecutive posts at least 3 of which are rotten.

This is the probabilistic approach given in the tutorial, which most students followed. Yet, a combinatorial approach is also possible (i.e., without using any probability). Marks are given as long as your proof is logical.

c) Part c is a bit tricky. Students did less well.

Proof is not unique. Marks are given as long as your proof makes sense.

i)

$$\mathbb{E}(X) = \sum_{m=0}^{\infty} m\mathbb{P}(X=m) = \sum_{m=0}^{\infty}\sum_{n=0}^{m-1} \mathbb{P}(X=m) = \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} \mathbb{P}(X=m) = \sum_{n=0}^{\infty} \mathbb{P}(X>n).$$ [6T]

ii) Denote by r and b the numbers of red and blue balls, respectively. Let N be the number of balls drawn. From i), we have

$$\mathbb{E}(N) = \sum_{n=0}^{r} \mathbb{P}(N>n) = \sum_{n=0}^{r} \mathbb{P}(\text{first } n \text{ balls are red})$$

$$= \sum_{n=0}^{r} \frac{r}{b+r}\frac{r-1}{b+r-1}\cdots\frac{r-n+1}{b+r-n+1} = \sum_{n=0}^{r} \frac{r!}{(b+r)!}\frac{(b+r-n)!}{(r-n)!}$$ [2T]

$$= \frac{r!\,b!}{(b+r)!}\sum_{n=0}^{r}\binom{n+b}{b} = \frac{b+r+1}{b+1},$$ [2T]

Substituting in r =10 and b = 10, we obtain

$$E(N) = \frac{21}{11} \approx 2$$ [2T]

2. A routine question. Students did well in general.

a) The joint distribution of the samples is given by

$$f_X(x_1, \dots, x_n; \lambda) = \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} = e^{-n\lambda} \frac{\lambda^{\sum x_i}}{\prod x_i!}$$

Then the log-likelihood function is [3E]

$$\ln f_X(x_1, \dots, x_n; \lambda) = -n\lambda + \sum x_i \ln\lambda - \ln \prod x_i!$$

Now take the derivatives:

$$\frac{\partial}{\partial\lambda} \ln f_X(x_1, \dots, x_n; \lambda) = -n + \frac{\sum x_i}{\lambda}$$

[3E]

$$\frac{\partial^2}{\partial\lambda^2} \ln f_X(x_1, \dots, x_n; \lambda) = -\frac{\sum x_i}{\lambda^2}$$

Then the Fisher information is given by

$$I = -E\left[-\frac{\sum x_i}{\lambda^2}\right] = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$$ [2E]

So the Cramer-Rao bound is

$$\mathrm{Var}[\hat{\lambda}] = \frac{1}{I(\lambda)} = \frac{\lambda}{n} = \frac{1}{10} = 0.1$$ [2E]

Similar to a question in coursework. Did well in general.

b) Recall the Wiender-Hopf equation

$$\mathbf{c} = \mathbf{R}^{-1}\mathbf{r}$$

$$\sigma^2 = r_0 - \mathbf{r}^T\mathbf{R}^{-1}\mathbf{r}$$ [2E]

i) If n=1, the Wiener-Hopf equation trivially reads

$$R_Y(0)c_1 = R_Y(1)$$

Therefore,

$$c_1 = \frac{R_Y(1)}{R_Y(0)} = \frac{J_0(2\pi f_d)}{J_0(0)} = J_0(2\pi f_d)$$ [2E]

because $J_0(0) = 1$. Therefore,

$$c_1 = J_0(2\pi \times 0.3) = 0.291$$ [2E]

$$\sigma^2 = 1 - r_1^2 = 1 - 0.291^2 = 0.915$$

ii) When n = 2, we have

$$\mathbf{R} = \begin{bmatrix} 1 & 0.291 \\ 0.291 & 1 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} -0.402 \\ 0.291 \end{bmatrix}$$ [3E]

Thus the coefficient vector

[3E]

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{R}^{-1}\mathbf{r} = \begin{bmatrix} 1 & 0.291 \\ 0.291 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -0.402 \\ 0.291 \end{bmatrix} = \begin{bmatrix} -0.532 \\ 0.446 \end{bmatrix}$$

Mean-square error

$$\sigma^2 = 1 - \begin{bmatrix} -0.402 & 0.291 \end{bmatrix} \begin{bmatrix} 1 & 0.291 \\ 0.291 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -0.402 \\ 0.291 \end{bmatrix} = 0.657$$ [3E]

Did well in general. Surprisingly, a non-negaligible number of students stopped here (i.e., didn't simplify to cos(m\lambda), apparently couldn't use trigonometrical formula. Then, they concluded it is not stationary, which lost mark.

3.

a)

i) It's clear that $E[X(n)] = 0$ and $\mathrm{Var}[X(n)] = 1$. The autocorrelation function [2E]

$$R_X(m, m+n) = \cos(n\lambda)\cos\big((n+m)\lambda\big) + \sin(n\lambda)\sin\big((n+m)\lambda\big) = \cos(m\lambda)$$ [2E]

ii) Clearly it is wide-sense stationary. [2E]

iii) In general, this random process is not strict-sense stationary. To see this, let's consider an example where $\lambda = \frac{\pi}{2}$. Then

[2E]

$$X(n) = A\cos(n\frac{\pi}{2}) + B\sin(n\frac{\pi}{2})$$

Obviously, the samples at different times

$$X(1) = B$$
$$X(2) = -A$$ [2E]

have different distributions.

Again, approach is not unique for iii). Another approach, as demonstrated in lecture and used by many students, is to compute the third moment, i.e., E[X(n1)X(n2)X(n3)], and show it is not shift invariant.

b)

i) The autocorrelation function is

$$R_X(k) = \begin{cases} 1+\alpha^2 & if\ k = 0 \\ \alpha & if\,|k| = 1 \\ 0 & \text{otherwise} \end{cases}$$ [3E]

Hence the power spectral density is

$$S_X(\omega) = R_X(0) + R_X(1)e^{-j\omega} + R_X(-1)e^{j\omega} = 1 + \alpha^2 + 2\alpha\cos(\omega), \quad |\omega| < \pi$$ [3E]

Did well in general

ii) In the general case, the autocorrelation function is

[3E]

$$R_X(k) = \sum_{i=0}^{r} \alpha_i \alpha_{k+i}$$

where the convention that $\alpha_s = 0$ if $s < 0$ or $s > r$. Thus,

[3E]

$$S_X(\omega) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-jk\omega} = \sum_{k=-\infty}^{\infty} \sum_{i=0}^{r} \alpha_i \alpha_{k+i}\, e^{-jk\omega}$$

$$= \textstyle\sum_{k=-\infty}^{\infty} \alpha_{k+i} e^{-j(k+i)\omega} \sum_{i=0}^{r} \alpha_i e^{ji\omega}$$

$$= \sum_{k'=0}^{r} \alpha_{k'} e^{-j(k')\omega} \sum_{i=0}^{r} \alpha_i e^{ji\omega} = |A(e^{j\omega})|^2, \quad |\omega| < \pi \qquad [3E]$$

where $A(Z) = \sum_{i=0}^{r} \alpha_i Z^i$ (the last step holds because that $\alpha_s = 0$ if $s < 0$ or $s > r$).

4. Standard question for Markov chains.

a) From the equation $\pi = \pi P$, we obtain

$\pi_1 = \frac{3}{4}\pi_1 + \frac{1}{4}\pi_2 \qquad \Rightarrow \pi_1 = \pi_2$

$\pi_2 = \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 \qquad \Rightarrow \pi_2 = \pi_3$ [3E]

…

$\pi_M = \frac{1}{4}\pi_{M-1} + \frac{3}{4}\pi_M \qquad \Rightarrow \pi_{M-1} = \pi_M$

So all the π_i's are equal, and

$$\pi_i = \frac{1}{M} \quad \text{for all } i.$$ [2E]

b) Part b is bookwork.

i) $\{S_n\}$ is a martingale. This is because

$$E[S_{n+1}|X_1,\dots,X_n] = E\left[\sum_{i=1}^{n+1} X_i \middle| X_1,\dots,X_n\right] = S_n + E[X_{n+1}] = S_n$$ [3B]

ii) $\{T_n\}$ is not a martingale. This is because

$E[T_{n+1}|X_1,\dots,X_n] = E[S_n^2 + 2S_nX_{n+1} + X_{n+1}^2|X_1,\dots,X_n] = T_n + E[X_{n+1}^2] \geq T_n$ [4B]

Thus, $\{T_n\}$ is not a martingale.

iii) But it's a submartingale. [3B]

c)

i) Of the $2n$ steps, suppose the chain goes upward for *i* steps, leftward for *j* steps, and inward for *k* steps. It returns to the origin if and only it also goes downward for *i* steps, rightward for *j* steps, and outward for *k* steps. Here we must have $i + j + k = n$.

[2E]

Therefore,

$$P\{X_{2n} = (0,0,0)\} = \left(\frac{1}{6}\right)^{2n} \sum_{i+j+k=n} \frac{(2n)!}{(i!\,j!\,k!)^2}$$ [2E]

ii) The above formula may be rewritten as

$$P\{X_{2n} = (0,0,0)\} = \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \sum_{i+j+k=n} \frac{(n)!}{3^n(i!\,j!\,k!)} \frac{(n)!}{3^n(i!\,j!\,k!)}$$ [2T]

$$\leq \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} M \sum_{i+j+k=n} \frac{(n)!}{3^n(i!\,j!\,k!)}$$ (*)

Part c is hard. Very few students got the correct answer.

where

$$M = \max\{\sum_{i+j+k=n} \frac{(n)!}{3^n(i!\,j!\,k!)}\} \approx \frac{(n)!}{3^n\left(\left(\frac{n}{3}\right)!\right)^3}$$

Further, the sum in (*) equals 1, since the summands form a probability distribution. It follows that

$$P\{X_{2n} = (0,0,0)\} \leq \left(\frac{1}{12}\right)^n \frac{(2n)!}{n!\left(\left(\frac{n}{3}\right)!\right)^3}$$ [2T]

Using Stirling's formula, we obtain

$$P\{X_{2n} = (0,0,0)\} \leq \left(\frac{1}{12}\right)^n \frac{\sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}}{\sqrt{2\pi(n)}\left(\frac{n}{e}\right)^n\left(\sqrt{2\pi\left(\frac{n}{3}\right)}\left(\frac{\frac{n}{3}}{e}\right)^{\frac{n}{3}}\right)^3} = Cn^{-\frac{3}{2}}$$

for some constant C. Finally, we find that [2T]

$$\sum_n P\{X_{2n} = (0,0,0)\} = C\sum_{n>1} n^{-\frac{3}{2}} < C\zeta\left(\frac{3}{2}\right) < \infty$$

because the sum converges. Here $\zeta(s)$ is the Riemann zeta function, which is well-known to converge for $s > 1$. We therefore conclude that the origin is a transient state.

Of course, this is not required; what matters is to recognize that the series converges.

PS. The famous Riemann hypothesis, considered one of the greatest unsolved problems in mathematics, asserts that any non-trivial zero s of $\zeta(s)$ on the complex plane has Re(s) = 1/2.