1. Solution:

(a)

$$E[X(t)] = E[A_t \cos(\omega t + \theta)]$$
$$= E[A_t] \cdot \cos(\omega t + \theta) \qquad [2\,A]$$
$$= 0$$

$$E[X^2(t)] = E[A_t^2] \cos^2(\omega t + \theta) \qquad [2\,A]$$
$$= \sigma^2 \cos^2(\omega t + \theta) = Var[X(t)]$$

Since the variance is a function of t, it is not stationary. [1 A]

(b)

$$E[X(t)] = E[A_t] \cdot E[\cos(\omega t + \theta)]$$
$$= 0 \qquad [1\,A]$$

$$E[X(t)X(t+\tau)] = E[A_t A_{t+\tau}]\, E[\cos(\omega t + \theta)\cos(\omega(t+\tau) + \theta)]$$
$$= \begin{cases} 0 & \tau \neq 0 \\ \sigma^2 E[\cos^2(\omega t + \theta)] & \tau = 0 \end{cases}$$
$$= \begin{cases} 0 & \tau \neq 0 \\ \frac{\sigma^2}{2} & \tau = 0 \end{cases} \qquad [3\,A]$$
$$= \frac{\sigma^2}{2}\delta(\tau) \quad \text{only a function of } \tau$$

So it is wide-sense stationary. [1 A]

2. Solution:

(a)

$$E[X(n)] = E[\cos(nU)] = 0$$

$$E[X^2(n)] = E[\cos^2(nU)]$$

$$= E[\tfrac{1}{2}(1+\cos(2nU))] = \tfrac{1}{2}$$

$$E[X(m)X(n)] = E[\cos(mU)\cos(nU)]$$

$$= E[\tfrac{1}{2}(\cos((m+n)U) + \cos((m-n)U))]$$

$$= 0 \qquad \text{if } m \neq n$$

Therefore, $X(n)$ is wide-sense stationary.

(b)

ii) Here, the answer is not unique.

For example, one may check

$$E[X(m)X(n)X(r)] = E[\cos(mU)\cos(nU)\cos(rU)]$$

$$= \tfrac{1}{2}E[(\cos((m+n)U) + \cos((m-n)U))\cos(rU)]$$

$$= \tfrac{1}{4}E[\cos((m+n+r)U) + \cos((m+n-r)U) + \cos((m-n+r)U) + \cos((m-n-r)U)]$$

$$= \tfrac{1}{4}[\delta(m+n+r) + \delta(m+n-r) + \delta(m-n+r) + \delta(m-n-r)] \qquad [3T]$$

where $\delta(n) = 1$ if $n = 0$

$\delta(n) = 0$ if $n \neq 0$

Consider two cases $(m, n, r) = (1, 2, 3), (2, 3, 4)$. [2T]

They take different values $\frac{1}{4}$ and 0.

So it doesn't satisfy the definition of strict-sense stationarity (which would require the same values).

3. Solution:

(a)

By Chebyshev's inequality [1

$$P\{|X(t+\tau)-X(t)|>a\} \le \frac{E[|X(t+\tau)-X(t)|^2]}{a^2}$$

$$= \frac{2[R(0)-R(\tau)]}{a^2}$$

(b)

$$\sum_{i,k} a_i a_k^* R(\tau_i-\tau_k)$$

$$= \sum_{i,k} a_i a_k^* \frac{1}{2\pi}\int S(w) e^{jw(\tau_i-\tau_k)} dw$$

$$= \frac{1}{2\pi}\int S(w) \left|\sum_i a_i e^{jw\tau_i}\right|^2 dw$$

$$\geq 0$$

(c)

If we suppress w_2 in $\phi(w_1, w_2)$, we recover the Characteristic function of a Gaussian r.v.

$$\phi(w) = \exp\left(-\frac{\sigma^2 w^2}{2}\right) = E[e^{jwX}]$$

Then

$$E[Y(t)] = E[I e^{aX(t)}]$$

$$= I\ E[e^{aX(t)}] \qquad [2E]$$

$$= I\ \exp\left(\frac{\sigma^2 a^2}{2}\right) \qquad \text{definition of C.F.}$$

$$= I\ \exp\left(\frac{\sigma^2 R(0)}{2}\right) \qquad \sigma^2 = R(0)$$

Meanwhile,

$$R_y(\tau) = E[Y(t)\,Y(t+\tau)]$$

$$[2E]$$

$$= I^2\ E[e^{aX(t)} e^{aX(t+\tau)}]$$

$$= I^2\ \exp\left(\frac{\sigma^2 a^2 + 2R(\tau)a^2 + \sigma^2 a^2}{2}\right) \qquad \text{definition of C.}$$

$$= I^2\ \exp\{\sigma^2[R(0) + R(\tau)]\} \qquad [1E]$$

$$\phi(w_1, w_2) = E[e^{j(X_1 w_1 + X_2 w_2)}]$$