1. Solution:

If we denote:

D={diseased}, H={healthy}, and +={positive}, then from the conditions, we know that:

$P(D) = 10^{-5}$, $P(H) = 1 - 10^{-5}$

$P(+|D) = 0.99$, $P(+|H) = 0.01$

Based on these, we can use Bayes' Formula to calculate

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|H)P(H)}$$

$$= \frac{0.99 \times 10^{-5}}{0.99 \times 10^{-5} + 0.01 \times (1 - 10^{-5})} = 9.89 \times 10^{-4}$$

The value is pretty small

Extension:

How about the probability of diseased, if two independent tests are both positive, i.e. $P(D|+,+)$?

$$P(D|+,+) = \frac{P(D)P(+,+|D)}{P(D)P(+,+|D) + P(H)P(+,+|H)}$$

$$= \frac{P(D)P(+|D)P(+|D)}{P(D)P(+|D)P(+|D) + P(H)P(+|H)P(+|H)}$$

2. Solution:

1) If we denote there is an open road between A and B as O_{AB} , and similarly O_{BC} and O_{AC} , then

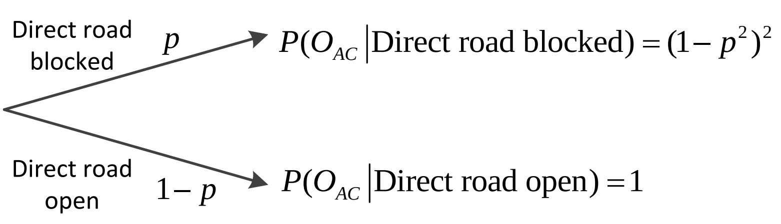
$$P(O_{AC}) = P(O_{AB} \cap O_{BC})$$

$$= P(O_{AB}) \cdot P(O_{BC}) \text{ (independence)}$$

Furthermore, $P(O_{AB}) = 1 - P(\text{two roads are blocked}) = 1 - P^2$

So, $P(O_{AC}) = (1 - P^2)^2$

2)



$$P(O_{AC}) = P(O_{AC}|\text{Direct road blocked}) \cdot p + P(O_{AC}|\text{Direct road open}) \cdot (1-p)$$

$$= (1-p^2)^2 \cdot p + (1-p)$$

3. Solution:

We denote:

A: exactly one Ace, and KK: exactly two Kings

We want to calculate: $P(A|KK) = \dfrac{P(A \cap KK)}{P(KK)}$,

where $P(A \cap KK) = \dfrac{\binom{4}{1}\binom{4}{2}\binom{44}{10}}{\binom{52}{13}}$, and $P(KK) = \dfrac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}}$.

So, $P(A|KK) \approx 0.44$

4. Solution:

We denote:

S: their sum is 7

X: the score of the first dice is X, where X=1, 2, 3, 4, 5, 6

We want to show that $P(S \cap X) = P(S) \cdot P(X)$ (so, S and X are independent)

$P(S \cap X) = \dfrac{1}{36}$ for any X

$$P(S) = \sum_{X=1}^{6} P(S|X)P(X) = \sum_{X=1}^{6} \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{6}$$

$P(X) = \dfrac{1}{6}$ for any X

So, $P(S \cap X) = P(S) \cdot P(X)$ is true.

5. Solution:

If we denote the label of the k-th card as L_k, then

“the label of the k-th card dealt is the largest of the first k cards dealt” can be described as

$\{L_k > L_r, \text{ for any } 1 \le r \le k-1\}$

“it is also the largest in the pack” means $L_k = m$

What we want to calculate is $P\left(L_k = m \middle| L_k > L_r, \text{ for any } 1 \le r \le k-1\right)$

$$= \frac{P\left(L_k = m, \text{ and } L_k > L_r, \text{ for any } 1 \le r \le k-1\right)}{P\left(L_k > L_r, \text{ for any } 1 \le r \le k-1\right)}$$

$$= \frac{1/m}{1/k} = \frac{k}{m}$$

6. We denote:

A_i : Professor i leaves with his own hat

A_i^c : Professor i leaves without his own hat

So, what we want to calculate is $P\left(A_1^c \cap \cdots \cap A_n^c\right)$, which equals to $1 - P(A_1 \cup \cdots \cup A_n)$, where

$$
\begin{aligned}
P(A_1 \cup \cdots \cup A_n) = {} & P(A_1) + P(A_2) + \cdots + P(A_n) \\
& - P(A_1A_2) - P(A_1A_3) - \cdots - P(A_{n-1}A_n) \\
& + P(A_1A_2A_3) + \cdots + P(A_{n-2}A_{n-1}A_n) \\
& - \cdots \\
& + (-1)^{n+1} P(A_1 \cdots A_n)
\end{aligned}
$$

$$
\begin{aligned}
= {} & \frac{(n-1)!}{n!} + \frac{(n-1)!}{n!} + \cdots + \frac{(n-1)!}{n!} \\
& - \frac{(n-2)!}{n!} - \frac{(n-2)!}{n!} - \cdots - \frac{(n-2)!}{n!} \\
& + \frac{(n-3)!}{n!} + \cdots + \frac{(n-3)!}{n!} \\
& - \cdots \\
& + (-1)^{n+1} \frac{(n-n)!}{n!}
\end{aligned}
$$

The term $\frac{(n-k)!}{n!}$ appears $\binom{n}{k}$ times, so

$$
P(A_1 \cup \cdots \cup A_n) = \binom{n}{1}\frac{(n-1)!}{n!} - \binom{n}{2}\frac{(n-2)!}{n!} + \cdots + (-1)^{n+1}\binom{n}{n}\frac{(n-n)!}{n!},
$$

$$
= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n+1}}{n!}
$$

then

$$
P\left(A_1^c \cap \cdots \cap A_n^c\right) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} - \cdots (-1)^n \frac{1}{n!} = e^{-1} \text{ when } n \to \infty .
$$

7. Solution:

Symmetric random walk: “Gambler’s ruin”

We denote:

A: the gambler is finally bankrupted

B: the first toss of coin shows heads (the manager pays the gambler one unit)

k: starting point (with k units money)

From the Total Probability Theorem, we know that

$$
P_k(A) = P_k(A|B)P(B) + P_k(A|B^c)P(B^c), \qquad (*)
$$

where $P_k(A)$ means “the gambler finally bankrupted when starting point is k”.

$P_k(A|B) = P_{k+1}(A)$ because event B happens means the money gambler has becomes k+1.

Similarly: $P_k(A|B^c) = P_{k-1}(A)$

So, (*)$\Rightarrow$ $P_k(A) = P_{k+1}(A)\cdot\frac{1}{2} + P_{k-1}(A)\cdot\frac{1}{2}$, where $P(B) = P(B^c) = \frac{1}{2}$

With boundary $P_0(A) = 1,\ P_N(A) = 0$, we will have

$\Rightarrow P_k(A) = 1 - \frac{k}{N}$.

Details: $P_k(A) = P_{k+1}(A)\cdot\frac{1}{2} + P_{k-1}(A)\cdot\frac{1}{2}$

$\Rightarrow 2P_k(A) = P_{k+1}(A) + P_{k-1}(A)$

$\Rightarrow P_{k+1}(A) - P_k(A) = P_k(A) - P_{k-1}(A)$

$\Rightarrow b_k = b_{k-1}$, where $b_k = P_{k+1}(A) - P_k(A)$

$\Rightarrow P_N(A) = Nb_1 + P_0(A)$

$\Rightarrow b_1 = -\frac{1}{N} \Rightarrow P_k(A) = 1 - \frac{k}{N}$.