

Probability and Stochastic Processes

Exam 2017 solutions

B—bookwork, E—new example, T—new theory

1.

a) We denote by **students did well**

A: exactly one Ace, and KK: exactly two Kings

We want to calculate: $P(A|KK) = \frac{P(A \cap KK)}{P(KK)}$,

$$\text{where } P(A \cap KK) = \frac{\binom{4}{1} \binom{4}{2} \binom{44}{10}}{\binom{52}{13}} \approx 0.094, \quad [2B]$$

$$\text{and } P(KK) = \frac{\binom{4}{2} \binom{48}{11}}{\binom{52}{13}} \approx 0.214. \quad [3B]$$

So, $P(A|KK) \approx 0.44$

[2B]

In general, students did well

b) Recall the definition of the characteristic function

$$\begin{aligned} \Phi_X(\omega) &= E(e^{j\omega X}) = E\left[\sum_{k=0}^{\infty} \frac{(j\omega X)^k}{k!}\right] = \sum_{k=0}^{\infty} j^k \frac{E(X^k)}{k!} \omega^k \\ &= 1 + jE(X)\omega + j^2 \frac{E(X^2)}{2!} \omega^2 + \dots + j^k \frac{E(X^k)}{k!} \omega^k + \dots \end{aligned} \quad [2E]$$

It is readily verified that

$$E(X^k) = \frac{1}{j^k} \left. \frac{\partial^k \Phi_X(\omega)}{\partial \omega^k} \right|_{\omega=0}, \quad k \geq 1.$$

In particular,

[2E]

$$E(X^4) = \left. \frac{\partial^4 \Phi_X(\omega)}{\partial \omega^4} \right|_{\omega=0}$$

Now, let's go step by step.

$$\frac{\partial \Phi_X(\omega)}{\partial \omega} = -\sigma^2 \omega e^{-\sigma^2 \omega^2 / 2} \quad [1E]$$

$$\frac{\partial^2 \Phi_X(\omega)}{\partial \omega^2} = -\sigma^2 e^{-\sigma^2 \omega^2 / 2} + \sigma^4 \omega^2 e^{-\sigma^2 \omega^2 / 2} \quad [1E]$$

$$\begin{aligned} \frac{\partial^3 \Phi_X(\omega)}{\partial \omega^3} &= \sigma^4 \omega e^{-\sigma^2 \omega^2 / 2} + 2\sigma^4 \omega e^{-\sigma^2 \omega^2 / 2} - \sigma^6 \omega^3 e^{-\sigma^2 \omega^2 / 2} \\ &= 3\sigma^4 \omega e^{-\sigma^2 \omega^2 / 2} - \sigma^6 \omega^3 e^{-\sigma^2 \omega^2 / 2} \end{aligned} \quad [1E]$$

$$\frac{\partial^4 \Phi_X(\omega)}{\partial \omega^4} = 3\sigma^4 e^{-\sigma^2 \omega^2 / 2} - 3\sigma^6 \omega^2 e^{-\sigma^2 \omega^2 / 2} + \text{terms containing } \omega \quad [1E]$$

We only care the first term, since all others = 0 if $\omega = 0$. Thus

$$E(X^4) = \left. \frac{\partial^4 \Phi_X(\omega)}{\partial \omega^4} \right|_{\omega=0} = 3\sigma^4 \quad [2E]$$

c) The derivation is not unique. One approach is the following:

$$\Phi_Y(\omega) = E\{e^{j\omega X_1 X_2}\} = E\{E\{e^{j\omega X_1 X_2} | X_2\}\} \quad [1T]$$

Note that the inner expectation is

$$\Phi_{X_1}(\omega X_2) = e^{-\sigma^2 \omega^2 X_2^2 / 2} \quad [1T]$$

Now we average over X_2 to obtain

$$\Phi_Y(\omega) = E\left\{e^{-\frac{\sigma^2 \omega^2 X_2^2}{2}}\right\} = \int e^{-\frac{\sigma^2 \omega^2 x_2^2}{2}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x_2^2}{2\sigma^2}} dx_2 \quad [1T]$$

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}\sigma} \int e^{-\frac{(1+\sigma^4 \omega^2)x_2^2}{2\sigma^2}} dx_2 \\ &= \frac{\sigma}{\sqrt{1+\sigma^4 \omega^2}} \frac{1}{\sqrt{2\pi} \frac{\sigma}{\sqrt{1+\sigma^4 \omega^2}}} \int e^{-\frac{(1+\sigma^4 \omega^2)x_2^2}{2\sigma^2}} dx_2 \end{aligned} \quad [1T]$$

Notice that the integrand is a density, we have

$$= \frac{1}{\sqrt{1+\sigma^4 \omega^2}} \quad [1T]$$

This is the tricky part; very few students did it right. Derivation is not unique; students got marks as long as the approach is valid.

2. **Students did well.**

a) The joint distribution of the samples is given by

$$f_X(x_1, \dots, x_n; \lambda) = \prod_{i=1}^n \frac{1}{\lambda} e^{-x_i/\lambda} = \frac{1}{\lambda^n} e^{-\sum x_i/\lambda}$$

Then the log-likelihood function is

[3E]

$$\ln f_X(x_1, \dots, x_n; \lambda) = -n \ln \lambda - \sum x_i/\lambda$$

Now take the derivatives:

$$\frac{\partial}{\partial \lambda} \ln f_X(x_1, \dots, x_n; \lambda) = -\frac{n}{\lambda} + \frac{\sum x_i}{\lambda^2}$$

[3E]

$$\frac{\partial^2}{\partial \lambda^2} \ln f_X(x_1, \dots, x_n; \lambda) = \frac{n}{\lambda^2} - 2 \frac{\sum x_i}{\lambda^3}$$

Then the Fisher information is given by

$$I = -E \left[\frac{n}{\lambda^2} - 2 \frac{\sum x_i}{\lambda^3} \right] = -\frac{n}{\lambda^2} + \frac{2n\lambda}{\lambda^3} = \frac{n}{\lambda^2}$$

[2E]

So the Cramer-Rao bound is

$$\text{Var}[\hat{\lambda}] = \frac{1}{I(\lambda)} = \frac{\lambda^2}{n} = \frac{100}{100} = 1$$

[2E]

b) i) Since $X_i, i = 1, \dots, n$ are independent Bernoulli variables, $X = \sum_{i=1}^n X_i$ is Binomial with

$$X \sim B \left(n, \frac{1}{2} \right)$$

[1B]

$$E(X) = E \left(\sum_{i=1}^n X_i \right) = n E(X_1) = n/2$$

[2B]

$$\text{Var}(X) = \text{Var} \left(\sum_{i=1}^n X_i \right)$$

$$= n \text{Var}(X_1) \text{ (Independence)}$$

[2B]

$$= n/4$$

Part i) is easy

Students did less well in Part ii). Some didn't use Chebyshev inequality, which lost mark.

ii) If we note that

$$\left\{ \left| X - \frac{n}{2} \right| \geq \frac{n}{8} \right\} = \left\{ X \geq \frac{5n}{8} \right\} \cup \left\{ X \leq \frac{3n}{8} \right\} \quad [2E]$$

then

$$P \left(\left| X - \frac{n}{2} \right| \geq \frac{n}{8} \right) \geq P \left(X \geq \frac{5n}{8} \right) \quad [3E]$$

Thus

$$P \left(X \geq \frac{5n}{8} \right) \leq P \left(\left| X - \frac{n}{2} \right| \geq \frac{n}{8} \right) \quad [2E]$$

$$\leq \frac{\text{var}(X)}{\left(n/8\right)^2} = \frac{16}{n} \quad [3E]$$

The answer is not unique. Due to symmetry, the following bound also holds:

$$P \left(X \geq \frac{5n}{8} \right) = \frac{1}{2} P \left(\left| X - \frac{n}{2} \right| \geq \frac{n}{8} \right) \\ \leq \frac{1}{2} \frac{\text{var}(X)}{\left(n/8\right)^2} = \frac{8}{n}$$

3. **It's a proof problem, which requires some mathematical maturity.**

a) Firstly consider the mean

$$E[X(n)] = E[e^{j(U-Vn)}] = E[e^{j(U)}]E[e^{j(-Vn)}] = 0 \quad [3E]$$

because U is uniform and independent of V.

Secondly consider the autocorrelation function

$$\begin{aligned} R_X(m) &= E[X(n+m)X(n)^*] = E[e^{j(U-V(m+n))}e^{-j(U-Vn)}] \\ &= E[e^{j(-Vm)}] = \Phi_V(-m) \end{aligned} \quad [3E]$$

where Φ_V denotes the characteristic function of V:

$$\Phi_V(-m) = E[e^{-jVm}] = \int e^{-jVm} f_V(x) dx \quad [2E]$$

Hence it is wide-sense stationary.

Recall the Wiener-Khinchin relation

$$\text{Power spectral density} \Leftrightarrow \text{autocorrelation function} \quad [2E]$$

via the Fourier transform. Since $f_V(\omega)$ is the inverse Fourier transform of $R_X(m)$, it is exactly the power spectral density.

b)

The system transfer function is given by

$$H(z) = \frac{1}{1 - az^{-1}} = \sum_{n=0}^{\infty} a^n z^{-n} \quad [2B]$$

$$\text{provided } |a| < 1. \text{ Thus } h(n) = a^n, \quad |a| < 1 \quad [2B]$$

represents the impulse response of an AR(1) stable system.

We get the output autocorrelation sequence of an AR(1) process to be

$$R_y(n) = \delta(n) * \{a^{-n}\} * \{a^n\} = \sum_{k=0}^{\infty} a^{|n|+k} a^k = \frac{a^{|n|}}{1 - a^2} \quad [3B]$$

ii) Recall the Wiener-Hopf equation

$$\mathbf{c} = \mathbf{R}^{-1}\mathbf{r}$$

Be carefully it's a 2-step ahead predictor

$$\begin{pmatrix} 1 & \alpha & \cdots & \alpha^{n-1} \\ \alpha & 1 & & \alpha^{n-2} \\ & \vdots & \ddots & \vdots \\ \alpha^{n-1} & & \cdots & \alpha & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \alpha^{n+1} \\ \alpha^n \\ \vdots \\ \alpha^2 \end{pmatrix} \quad [2E]$$

Whose solution is given by

$$c_n = \alpha^2, \quad c_k = 0, \quad 1 \leq k < n \quad [2E]$$

The MMSE predictor turns out to be a one-tap predictor

$$\hat{Y}(n+2) = \alpha^2 Y(n) \quad [2E]$$

The mean-square error is

$$\begin{aligned} MSE &= E[(Y(n+2) - \alpha^2 Y(n))^2] \\ &= R_Y(0) - 2\alpha^2 R_Y(2) + \alpha^4 R_Y(0) = \frac{1 - \alpha^4}{1 - \alpha^2} = 1 + \alpha^2 \end{aligned} \quad [2E]$$

4.

a) **Just use the definition of martingale. However, some students forgot.**

$$E[S_{n+1}|X_1, \dots, X_n] = E[\beta X_{n+1} + X_n|X_1, \dots, X_n] \quad [2E]$$

$$= \beta(aX_n + bX_{n-1}) + X_n = (\beta a + 1)X_n + \beta bX_{n-1} \quad [3E]$$

If it's a martingale, then

$$(\beta a + 1)X_n + \beta bX_{n-1} = S_n = \beta X_n + X_{n-1} \quad [3E]$$

Hence

$$\beta = \frac{1}{b} = \frac{1}{1-a} \quad [2E]$$

Part b is quite hard. Few students worked it out.

b)

i) Of the $2n$ steps, suppose the chain goes upward for i steps, leftward for j steps. It returns to the origin if and only if it also goes downward for i steps, rightward for j steps. Here we must have $i + j = n$. [2E]

Therefore,

$$P\{X_{2n} = (0,0)\} = \left(\frac{1}{4}\right)^{2n} \sum_{i+j=n} \frac{(2n)!}{(i!j!)^2} \quad [3E]$$

ii) The above formula may be rewritten as

$$\begin{aligned} P\{X_{2n} = (0,0)\} &= \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \sum_{i+j=n} \frac{(n)!}{2^n(i!j!)} \frac{(n)!}{2^n(i!j!)} \\ &\leq \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} M \sum_{i+j=n} \frac{(n)!}{2^n(i!j!)} \end{aligned} \quad [3T]$$

(*)

where

$$M = \max\left\{ \sum_{i+j=n} \frac{(n)!}{2^n(i!j!)} \right\} \approx \frac{(n)!}{2^n \left(\left(\frac{n}{2}\right)!\right)^2}$$

Further, the sum in (*) equals 1, since the summands form a probability distribution. It follows that

$$P\{X_{2n} = (0,0)\} \leq \left(\frac{1}{8}\right)^n \frac{(2n)!}{n! \left(\left(\frac{n}{2}\right)!\right)^2} \quad [2T]$$

Using Stirling's formula, we obtain

$$P\{X_{2n} = (0,0)\} \leq \left(\frac{1}{8}\right)^n \frac{\sqrt{2\pi(2n)} \left(\frac{2n}{e}\right)^{2n}}{\sqrt{2\pi(n)} \left(\frac{n}{e}\right)^n \left(\sqrt{2\pi\left(\frac{n}{2}\right)} \left(\frac{n}{2}\right)^{\frac{n}{2}} \left(\frac{n}{e}\right)^{\frac{n}{2}}\right)^2} = Cn^{-1} \quad [2T]$$

for some constant C .

The key point is to use Stirling's formula

Finally, we find that

$$\sum_n P\{X_{2n} = (0,0)\} = C \sum_{n>1} n^{-1} = \infty \quad [3T]$$

because the sum diverges. We therefore conclude that the origin is a recurrent state.