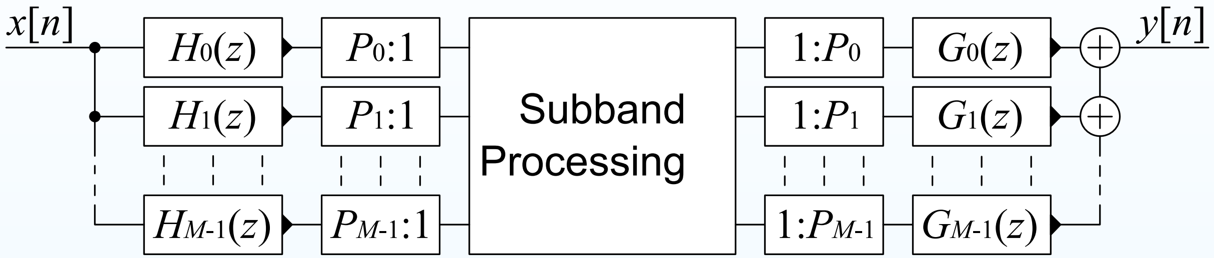
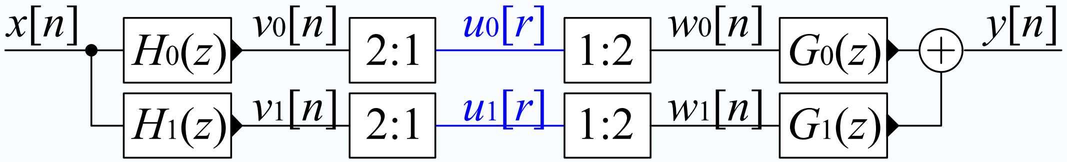
15: Subband Processing

Subband processing



- The $H_m(z)$ are bandpass *analysis filters* and divide $x[n]$ into frequency bands
- Subband processing often processes frequency bands independently
- The $G_m(z)$ are *synthesis filters* and together reconstruct the output
- The $H_m(z)$ outputs are bandlimited and so can be subsampled without loss of information
    - Sample rate multiplied overall by $\sum \frac{1}{P_i}$
      $\sum \frac{1}{P_i} = 1 \Rightarrow$ *critically sampled*: good for coding
      $\sum \frac{1}{P_i} > 1 \Rightarrow$ *oversampled*: more flexible
- Goals:
    (a) good frequency selectivity in $H_m(z)$
    (b) *perfect reconstruction*: $y[n] = x[n-d]$ if no processing
- Benefits: Lower computation, faster convergence if adaptive

2-band Filterbank



$$V_m(z) = H_m(z)X(z) \qquad [m \in \{0,\ 1\}]$$

$$U_m(z) = \frac{1}{K}\sum_{k=0}^{K-1} V_m(e^{\frac{-j2\pi k}{K}} z^{\frac{1}{K}}) = \frac{1}{2}\left\{V_m\left(z^{\frac{1}{2}}\right) + V_m\left(-z^{\frac{1}{2}}\right)\right\}$$

$$W_m(z) = U_m(z^2) = \frac{1}{2}\{V_m(z) + V_m(-z)\} \qquad [K=2]$$

$$= \frac{1}{2}\{H_m(z)X(z) + H_m(-z)X(-z)\}$$

$$X(-z) = X(e^{-j\omega}) = \underbrace{X(e^{j(\omega-\pi)})}_{\text{aliasing : shifted by } \pi}$$

$$Y(z) = \begin{bmatrix} W_0(z) & W_1(z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} X(z) & \underbrace{X(-z)}_{\text{aliasing}} \end{bmatrix} \begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix}$$

$$= \begin{bmatrix} X(z) & X(-z) \end{bmatrix} \begin{bmatrix} T(z) \\ A(z) \end{bmatrix}$$

[$X(-z)A(z)$ is "aliased" term]

We want (a) $T(z) = \frac{1}{2}\{H_0(z)G_0(z) + H_1(z)G_1(z)\} = z^{-d}$

and (b) $A(z) = \frac{1}{2}\{H_0(-z)G_0(z) + H_1(-z)G_1(z)\} = 0$

Perfect Reconstruction

For perfect reconstruction without aliasing, we require

$$\frac{1}{2}\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} z^{-d} \\ 0 \end{bmatrix}$$

Hence:

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}^{-1}\begin{bmatrix} 2z^{-d} \\ 0 \end{bmatrix}$$

$$= \frac{2z^{-d}}{H_0(z)H_1(-z)-H_0(-z)H_1(z)}\begin{bmatrix} H_1(-z) & -H_1(z) \\ -H_0(-z) & H_0(z) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{2z^{-d}}{H_0(z)H_1(-z)-H_0(-z)H_1(z)}\begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix}$$

$\frac{2}{c}$: compensate downsample/upsample scaling

For all filters to be FIR, we need the denominator to be

c: scale factor

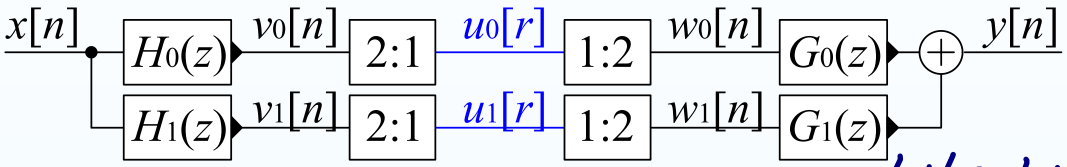
$H_0(z)H_1(-z) - H_0(-z)H_1(z) = cz^{-k}$, which implies

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{2}{c}z^{k-d}\begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix} \overset{d=k}{=} \frac{2}{c}\begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix}$$

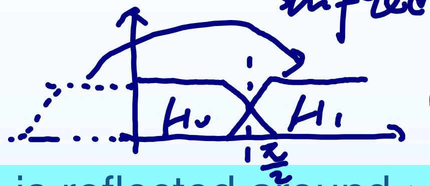
Note: c just scales $H_i(z)$ by $c^{\frac{1}{2}}$ and $G_i(z)$ by $c^{-\frac{1}{2}}$.

Quadrature Mirror Filterbank (QMF)

LP → HP: coef * $(-1)^n$



shifted by π



$H(-z) = H(e^{j(\omega-\pi)})$

QMF satisfies:

(a) $H_0(z)$ is causal and real

(b) $H_1(z) = H_0(-z)$: i.e. $\left|H_0(e^{j\omega})\right|$ is reflected around $\omega = \frac{\pi}{2}$

(c) $G_0(z) = 2H_1(-z) = 2H_0(z)$

(d) $G_1(z) = -2H_0(-z) = -2H_1(z)$

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = 2 \begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix}$$

QMF is alias-free:

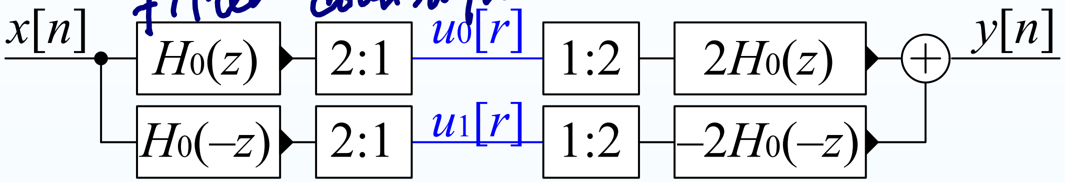
$$A(z) = \tfrac{1}{2}\left\{H_0(-z)G_0(z) + H_1(-z)G_1(z)\right\}$$
$$= \tfrac{1}{2}\left\{2H_1(z)H_0(z) - 2H_0(z)H_1(z)\right\} = 0$$

QMF Transfer Function:

$$T(z) = \tfrac{1}{2}\left\{H_0(z)G_0(z) + H_1(z)G_1(z)\right\}$$
$$= H_0^2(z) - H_1^2(z) = H_0^2(z) - H_0^2(-z)$$

Polyphase QMF

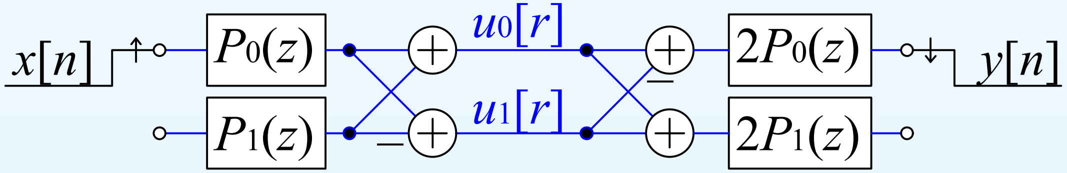
filter downsampler



Polyphase decomposition:

only need calculate coef of H. G once
(H. G: different input.
need separate calculation)

$$H_0(z) = P_0(z^2) + z^{-1}P_1(z^2)$$
$$H_1(z) = H_0(-z) = P_0(z^2) - z^{-1}P_1(z^2)$$
$$G_0(z) = 2H_0(z) = +2P_0(z^2) + 2z^{-1}P_1(z^2)$$
$$G_1(z) = -2H_0(-z) = -2P_0(z^2) + 2z^{-1}P_1(z^2)$$



Transfer Function: $T(z) = H_0^2(z) - H_1^2(z) = 4z^{-1}P_0(z^2)P_1(z^2) = z^{-d}$

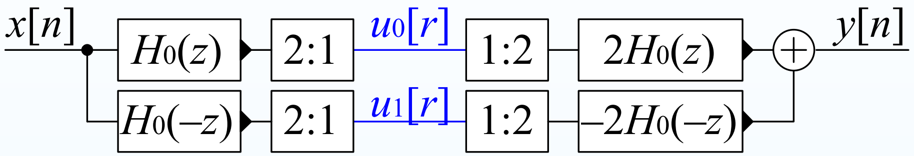
$T(z) = H_0^2(z) - H_1^2(z) = 4z^{-1}P_0(z^2)P_1(z^2)$ $\Downarrow$

we want $T(z) = z^{-d} \Rightarrow P_0(z^2) = a_0 z^{-k}, P_1(z^2) = a_1 z^{k+1-d}$ perfect construction is

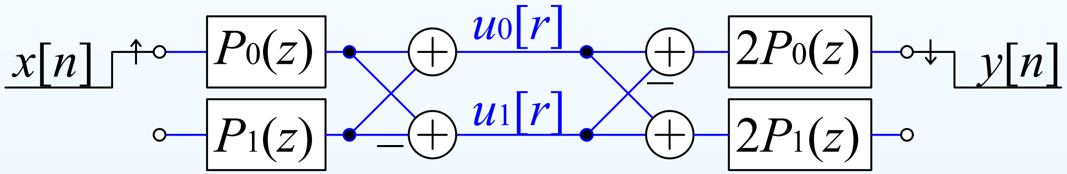
$\Rightarrow H_0(z)$ has only two non-zero taps $\Rightarrow$ poor freq selectivity freq. selectivity
(a_0, a_1)

$\therefore$ Perfect reconstruction QMF filterbanks cannot have good freq selectivity

QMF Options



Polyphase decomposition:



$A(z) = 0 \Rightarrow$ no alias term

$T(z) = H_0^2(z) - H_1^2(z) = H_0^2(z) - H_0^2(-z) = 4z^{-1}P_0(z^2)P_1(z^2)$

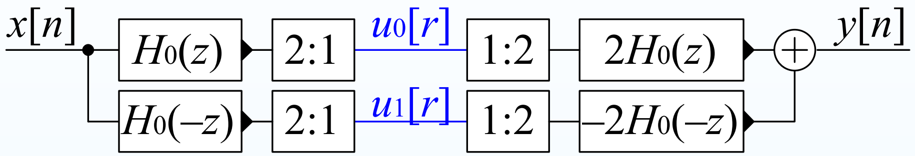
Options:

(A) Perfect Reconstruction: $T(z) = z^{-d} \Rightarrow H_0(z)$ is a bad filter.

(B) $T(z)$ is Linear Phase FIR:
$\Rightarrow$ Tradeoff: $\left|T(e^{j\omega})\right| \approx 1$ versus $H_0(z)$ stopband attenuation

(C) $T(z)$ is Allpass IIR: $H_0(z)$ can be Butterworth or Elliptic filter
$\Rightarrow$ Tradeoff: $\angle T(e^{j\omega}) \approx \tau\omega$ versus $H_0(z)$ stopband attenuation

Option (B): Linear Phase QMF



$T(z) \approx 1$

$H_0(z)$ order M, linear phase $\Rightarrow H_0(e^{j\omega}) = \pm e^{-j\omega\frac{M}{2}} \left|H_0(e^{j\omega})\right|$

$$
\begin{aligned}
T(e^{j\omega}) &= H_0^2(e^{j\omega}) - H_1^2(e^{j\omega}) = H_0^2(e^{j\omega}) - H_0^2(-e^{j\omega}) \\
&= e^{-j\omega M}\left|H_0(e^{j\omega})\right|^2 - e^{-j(\omega-\pi)M}\left|H_0(e^{j(\omega-\pi)})\right|^2 \\
&= e^{-j\omega M}\left(\left|H_0(e^{j\omega})\right|^2 - (-1)^M\left|H_0(e^{j(\pi-\omega)})\right|^2\right)
\end{aligned}
$$

stopband attenuation

M even $\Rightarrow T(e^{j\frac{\pi}{2}}) = 0$ ☹ so choose M odd $\Rightarrow -(-1)^M = +1$

passband ripple

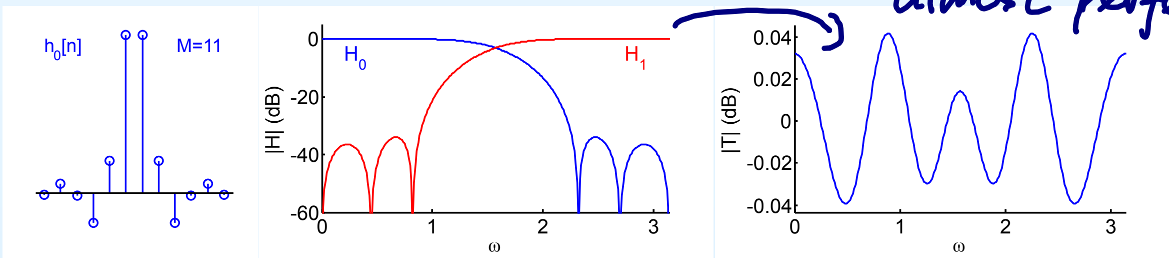
Select $h_0[n]$ by numerical iteration to minimize

$$\alpha \int_{\frac{\pi}{2}+\Delta}^{\pi} \left|H_0(e^{j\omega})\right|^2 d\omega + (1-\alpha)\int_0^{\pi} \left(\left|T(e^{j\omega})\right| - 1\right)^2 d\omega$$

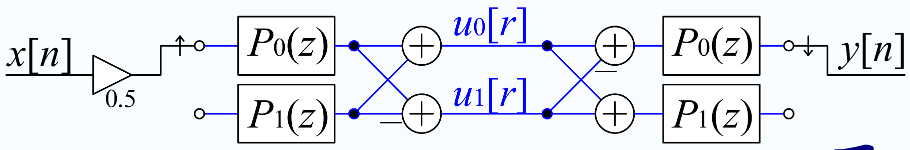
$\alpha \rightarrow$ balance between $H_0(z)$ being lowpass and $T(e^{j\omega}) \approx 1$

almost perfect

Johnston filter ($M = 11$):



Option (C): IIR Allpass QMF



$|T(z)| = 1$

$P_0(z), P_1(z)$: all pass
phase differ.

Choose $P_0(z)$ and $P_1(z)$ to be allpass IIR filters:

$$H_{0,1}(z) = \frac{1}{2}\left(P_0(z^2) \pm z^{-1}P_1(z^2)\right), \qquad G_{0,1}(z) = \pm 2H_{0,1}(z)$$

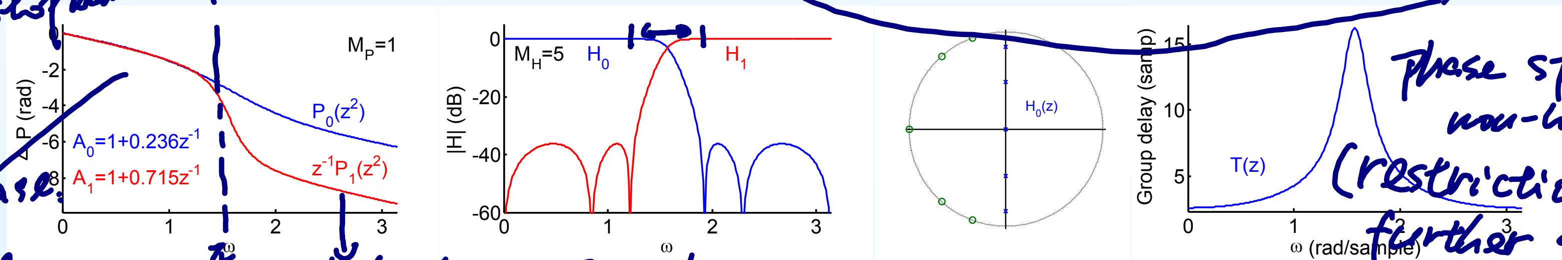
$A(z) = 0 \Rightarrow$ No aliasing

allpass · allpass = allpass

$T(z) = H_0^2 - H_1^2 = \ldots = z^{-1}P_0(z^2)P_1(z^2)$ is an allpass filter.

phase:
inphase → passband
out of phase → stopband
by π

$H_0(z)$ can be made a Butterworth or Elliptic filter with $M_H = 4M_P + 1$:



inphase: double

π/2

shift by π: cancel

phase spectrum: non-linear
(restriction in further splitting)

Phase cancellation: $\angle z^{-1}P_1 = \angle P_0 + \pi$; Ripples in H_0 and H_1 cancel.

phase transition: narrower transition band

perfect but nonlinear.

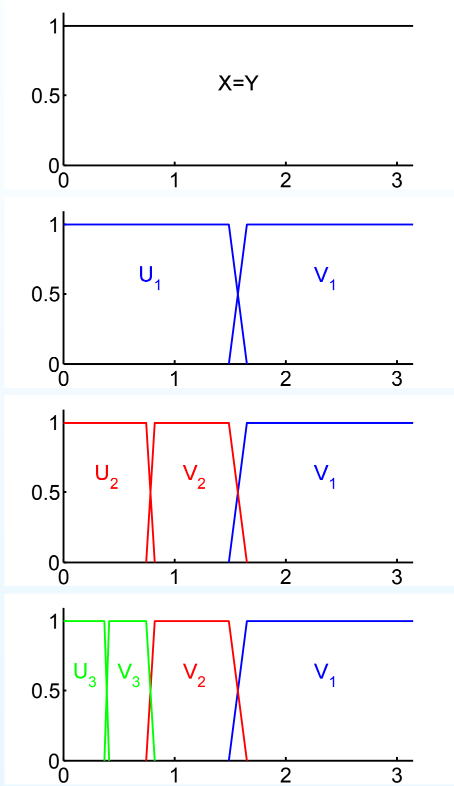
Tree-structured filterbanks

A *half-band filterbank* divides the full band into two equal halves.

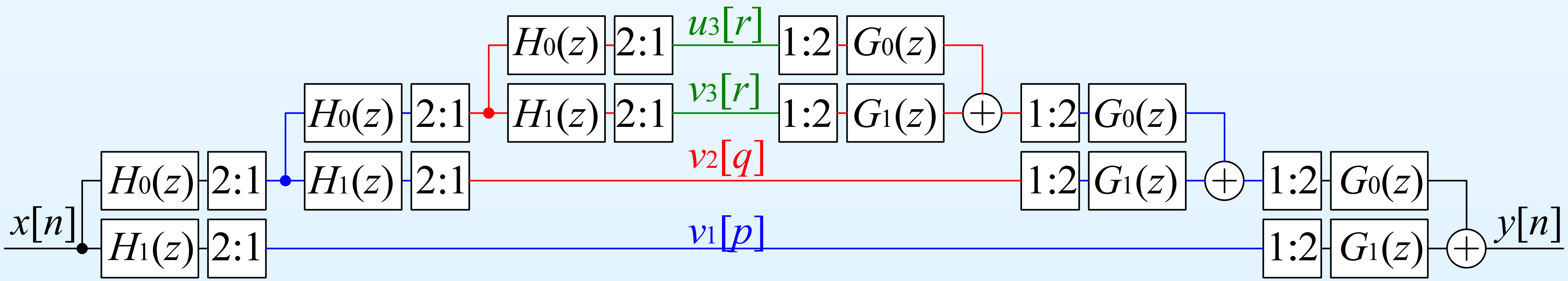
You can repeat the process on either or both of the signals $u_1[p]$ and $v_1[p]$.

Dividing the lower band in half repeatedly results in an *octave band filterbank*. Each subband occupies one octave (= a factor of 2 in frequency) except the first subband.

The properties "perfect reconstruction" and "allpass" are preserved by the iteration.



iterate in tree structure



Summary

- Half-band filterbank:
  - Reconstructed output is $T(z)X(z) + A(z)X(-z)$
  - Unwanted alias term is $A(z)X(-z)$
- Perfect reconstruction: imposes strong constraints on analysis filters $H_i(z)$ and synthesis filters $G_i(z)$.
- Quadrature Mirror Filterbank (QMF) adds an additional symmetry constraint $H_1(z) = H_0(-z)$.
  - Perfect reconstruction now impossible except for trivial case.
  - Neat polyphase implementation with $A(z) = 0$
  - Johnston filters: Linear phase with $T(z) \approx 1$
  - Allpass filters: Elliptic or Butterworth with $|T(z)| = 1$
- Can iterate to form a tree structure with equal or unequal bandwidths.

See Mitra chapter 14 (which also includes some perfect reconstruction designs).