

11: Multirate Systems

- Multirate Systems
- Building blocks
- Resampling Cascades
- Noble Identities
- Noble Identities Proof
- Upsampled z-transform
- Downsampled z-transform
- Downsampled Spectrum
- Power Spectral Density +
- Perfect Reconstruction
- Commutators
- Summary
- MATLAB routines

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Multirate systems include more than one sample rate

Why bother?:

- May need to change the sample rate
e.g. Audio sample rates include 32, 44.1, 48, 96 kHz
MP3 CD studio
- Can relax analog or digital filter requirements
e.g. Audio DAC increases sample rate so that the reconstruction filter can have a more gradual cutoff
- Reduce computational complexity *→ battery power*
FIR filter length $\propto \frac{f_s}{\Delta f}$ where Δf is width of transition band
Lower $f_s \Rightarrow$ shorter filter + fewer samples \Rightarrow computation $\propto f_s^2$

Building blocks

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Downsample

$$x[n] \xrightarrow{K:1} y[m]$$

$$y[m] = x[Km]$$

skip (K-1) samples then take.

Upsample

$$u[m] \xrightarrow{1:K} v[n]$$

$$v[n] = \begin{cases} u\left[\frac{n}{K}\right] & K \mid n \\ 0 & \text{else} \end{cases}$$

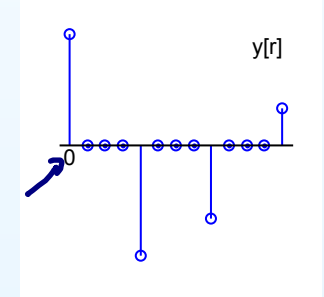
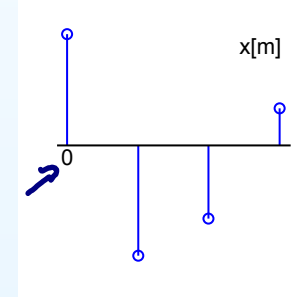
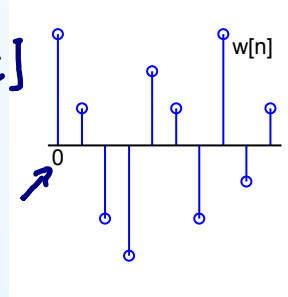
padding (K-1) zeros

Example:

Downsample by 3 then upsample by 4

$$x[m] = w[3m] \quad y[r] = x\left[\frac{m}{4}\right]$$

$$w[n] \xrightarrow{3:1} x[m] \xrightarrow{1:4} y[r]$$



- We use different index variables (n, m, r) for different sample rates
- Use different colours for signals at different rates (sometimes)
- **Synchronization:** all signals have a sample at $n = 0$.

Resampling Cascades

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Successive downsamplers or upsamplers can be combined

$$\begin{array}{c} \boxed{P:1} \quad \boxed{Q:1} \\ \hline \boxed{1:P} \quad \boxed{1:Q} \end{array} = \begin{array}{c} \boxed{PQ:1} \\ \hline \boxed{1:PQ} \end{array}$$

Upsampling can be exactly inverted

$$\begin{array}{c} \boxed{1:P} \quad \boxed{P:1} \\ \hline \boxed{1:P} \quad \boxed{P:1} \end{array} = \text{identity}$$

padding removing

$$\boxed{P:1} \quad \boxed{1:P} \neq \text{identity}$$

Downsampling destroys information permanently \Rightarrow uninvertible

Resampling can be interchanged iff P and Q are coprime (surprising!)

$$\begin{array}{c} w[n] = x[Pn] \quad y[n] = w[\frac{n}{Q}] \\ x \quad \boxed{P:1} \quad w \quad \boxed{1:Q} \quad y \\ u[n] = x[\frac{n}{Q}] \quad v[n] = u[Pn] \end{array} = \begin{array}{c} x \quad \boxed{1:Q} \quad u \quad \boxed{P:1} \quad v \end{array}$$

Proof: Left side: $y[n] = w[\frac{n}{Q}] = x[\frac{P}{Q}n]$ if $Q \mid n$ else $y[n] = 0$.

Right side: $v[n] = u[Pn] = x[\frac{P}{Q}n]$ if $Q \mid Pn$.

But $\{Q \mid Pn \Rightarrow Q \mid n\}$ iff P and Q are coprime.

if P, Q are not coprime, information lost do not overlap.
 [Note: $a \mid b$ means " a divides into b exactly"]

Noble Identities

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Resamplers commute with addition and multiplication

Delays must be multiplied by the resampling ratio

Noble identities: !
Exchange resamplers and filters

Corollary

Example: $H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \dots$
 $H(z^3) = h[0] + h[1]z^{-3} + h[2]z^{-6} + \dots$

$$\left[\begin{array}{c} \oplus \\ \oplus \end{array} \right] P:Q = \left[\begin{array}{c} P:Q \\ P:Q \end{array} \right] \oplus$$

$$\triangleleft P:Q = P:Q \triangleleft$$

$$Q:1 \cdot z^{-1} = z^{-Q} \cdot Q:1$$

$$z^{-1} \cdot 1:Q = 1:Q \cdot z^{-Q}$$

$$\begin{aligned} Q:1 \cdot H(z) &= H(z^Q) \cdot Q:1 \\ H(z) \cdot 1:Q &= 1:Q \cdot H(z^Q) \end{aligned}$$

$$H(z) = 1:Q \cdot H(z^Q) \cdot Q:1$$

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Noble Identities Proof

$$w[r] = v[Qr] = \sum_{s=0}^{QM} h_Q[s] x[Qr-s]$$

$$= \sum_{m=0}^M h_Q[Qm] x[Q(r-m)]$$

$$= \sum_{m=0}^M h[m] u[r-m] = y[r]$$

$$\boxed{x[n]} \xrightarrow{Q:1} \boxed{H(z)} \xrightarrow{y[r]} = \boxed{x[n]} \xrightarrow{H(z^Q)} \boxed{v[n]} \xrightarrow{Q:1} \boxed{w[r]}$$

Define $h_Q[n]$ to be the impulse response of $H(z^Q)$.

Assume that $h[r]$ is of length $M + 1$ so that $h_Q[n]$ is of length $QM + 1$.

We know that $h_Q[n] = 0$ except when $Q \mid n$ and that $h[r] = h_Q[Qr]$.

down

$$\begin{aligned} w[r] &= v[Qr] = \sum_{s=0}^{QM} h_Q[s] x[Qr-s] \\ &= \sum_{m=0}^M \underbrace{h_Q[Qm]}_{\substack{\downarrow s=Qm \\ h_Q[n]=0 \text{ except } Q \mid n}} x[Qr-Qm] = \sum_{m=0}^M h[m] x[Q(r-m)] \\ &= \sum_{m=0}^M h[m] u[r-m] = y[r] \end{aligned}$$



Upsampled Noble Identity: !

$$\boxed{x[r]} \xrightarrow{H(z)} \boxed{u[r]} \xrightarrow{1:Q} \boxed{y[n]} = \boxed{x[r]} \xrightarrow{1:Q} \boxed{v[n]} \xrightarrow{H(z^Q)} \boxed{w[n]}$$

We know that $v[n] = 0$ except when $Q \mid n$ and that $v[Qr] = x[r]$.

$$\begin{aligned} w[n] &= \sum_{s=0}^{QM} h_Q[s] v[n-s] = \sum_{m=0}^M h_Q[Qm] v[n-Qm] \\ &= \sum_{m=0}^M h[m] v[n-Qm] \end{aligned}$$

$$\begin{aligned} Q \nmid n &\Rightarrow v[n-Qm] = 0 \\ &\Rightarrow w[n] = 0 = y[n] \end{aligned}$$

$$n = Qr \Rightarrow w[Qr] = \sum_{m=0}^M h[m] v[Qr-Qm]$$

If $Q \nmid n$, then $v[n-Qm] = 0 \forall m$ so $w[n] = 0 = y[n]$

$$\begin{aligned} &= \sum_{m=0}^M h[m] x[r-m] = u[r] \\ &= \sum_{m=0}^M h[m] x[Qr-m] = y[Qr] \end{aligned}$$



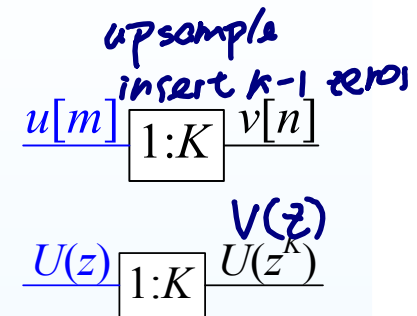
Upsampled z-transform

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$$V(z) = \sum_n v[n] z^{-n} = \sum_{n \text{ s.t. } K|n} u\left[\frac{n}{K}\right] z^{-n}$$

$$= \sum_m u[m] z^{-Km} = U(z^K)$$



Spectrum: $V(e^{j\omega}) = U(e^{jK\omega})$

Spectrum is horizontally shrunk and replicated K times.

△ Total energy (integral) unchanged; power (= energy/sample) multiplied by $\frac{1}{K}$

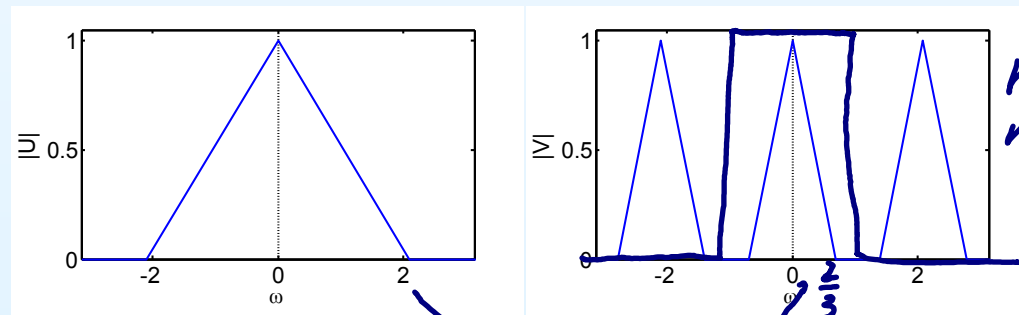
Upsampling normally followed by a LP filter to remove images.

Example:

$K = 3$: three images of the original spectrum in all.

Energy unchanged: $\frac{1}{2\pi} \int |U(e^{j\omega})|^2 d\omega = \frac{1}{2\pi} \int |V(e^{j\omega})|^2 d\omega$

upsampling \rightarrow LP



no overlap
no aliasing

Downsampled z-transform

impulse every K
 (. . . 1 . . . 1 . . .)

Define $c_K[n] = \delta_{K|n}[n] = \frac{1}{K} \sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}}$

$$x[n] \xrightarrow{K:1} y[m] \xrightarrow{1:K} x_K[n]$$

Now define $x_K[n] = \begin{cases} x[n] & K | n \\ 0 & K \nmid n \end{cases} = c_K[n]x[n]$

$$\begin{aligned} X_K(z) &= \sum_n x_K[n] z^{-n} = \frac{1}{K} \sum_n \sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}} x[n] z^{-n} \\ &= \frac{1}{K} \sum_{k=0}^{K-1} \sum_n x[n] \left(e^{\frac{-j2\pi k}{K}} z \right)^{-n} = \frac{1}{K} \sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}} z) \end{aligned}$$

From previous slide:

$$\begin{aligned} X_K(z) &= Y(z^K) \quad \begin{matrix} X_K(z) \\ \downarrow K:1 \\ Y(z) \end{matrix} \\ \Rightarrow Y(z) &= X_K(z^{\frac{1}{K}}) = \frac{1}{K} \sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}} z^{\frac{1}{K}}) \end{aligned}$$

Frequency Spectrum:

LP \rightarrow downsampling

$$\begin{aligned} Y(e^{j\omega}) &= \left(\frac{1}{K} \right) \sum_{k=0}^{K-1} X(e^{j(\frac{\omega - 2\pi k}{K})}) \\ &= \frac{1}{K} \left(X(e^{j\frac{\omega}{K}}) + X(e^{j\frac{\omega}{K} - \frac{2\pi}{K}}) + X(e^{j\frac{\omega}{K} - \frac{4\pi}{K}}) + \dots \right) \end{aligned}$$

shifted $\frac{2\pi}{K}$
shifted $\frac{4\pi}{K}$

Average of K aliased versions, each expanded in ω by a factor of K .

Downsampling is normally preceded by a LP filter to prevent aliasing.

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Downsampled Spectrum

downsampling:
 energy $\propto \frac{1}{K}$ (samples gone)
 power \approx unchanged $x[n] \xrightarrow{K:1} y[m]$

$$Y(e^{j\omega}) = \frac{1}{K} \sum_{k=0}^{K-1} X(e^{j(\omega - 2\pi k)})$$

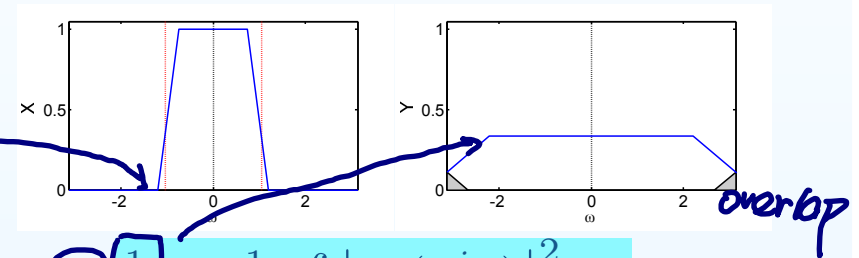
Example 1:

$$K = 3$$

Not quite limited to $\pm \frac{\pi}{K}$

Shaded region shows aliasing

$$\text{Energy decreases: } \frac{1}{2\pi} \int |Y(e^{j\omega})|^2 d\omega \approx \frac{1}{K} \times \frac{1}{2\pi} \int |X(e^{j\omega})|^2 d\omega$$

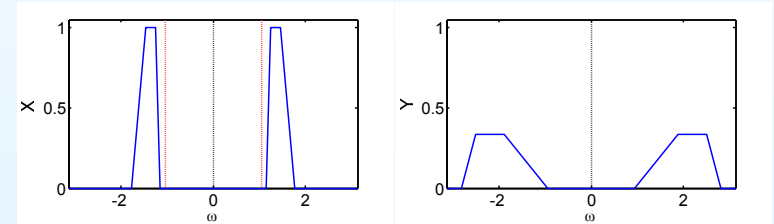


Example 2:

$$K = 3$$

Energy all in $\frac{\pi}{K} \leq |\omega| < 2\frac{\pi}{K}$

No aliasing: 😊



No aliasing: If all energy is in $r\frac{\pi}{K} \leq |\omega| < (r+1)\frac{\pi}{K}$ for some integer r

Normal case ($r = 0$): If all energy in $0 \leq |\omega| \leq \frac{\pi}{K}$

Downsampling: Total **energy** multiplied by $\approx \frac{1}{K}$ ($= \frac{1}{K}$ if no aliasing)

Average **power** \approx unchanged ($=$ energy/sample)

Power Spectral Density

+

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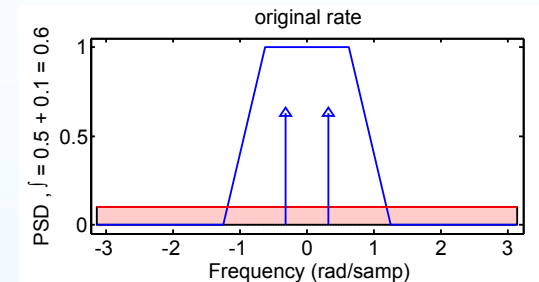
Example: Signal in $\omega \in \pm 0.4\pi$ + Tone @ $\omega = \pm 0.1\pi$ + White noise

Power = Energy/sample = Average PSD

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{PSD}(\omega) d\omega = 0.6$$

Component powers:

Signal = 0.3, Tone = 0.2, Noise = 0.1

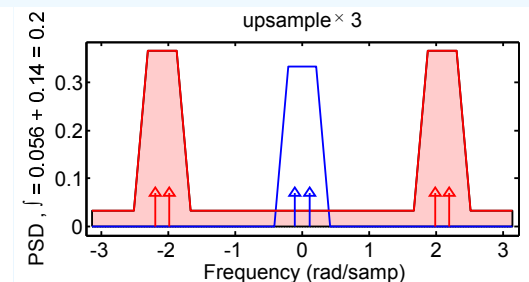
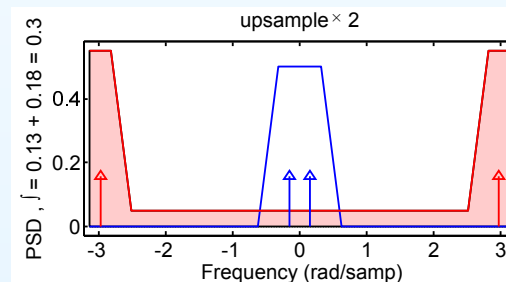


Upsampling:

Same energy

per second

⇒ Power is $\div K$



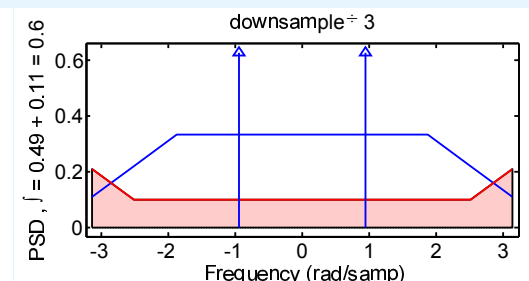
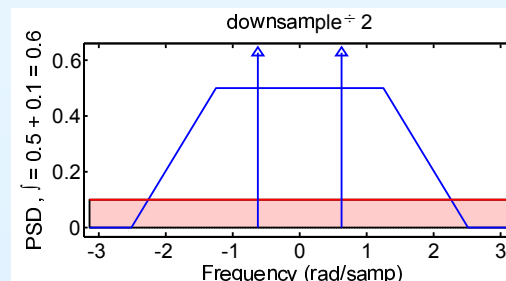
Downsampling:

Average power

is unchanged.

⊂ aliasing in

the $\div 3$ case.



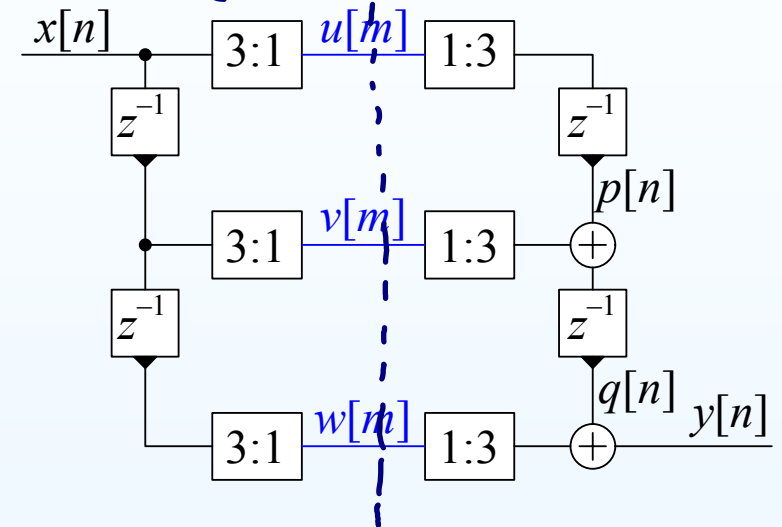
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Perfect Reconstruction

$x[n]$ c d e f g h i j k l m n
 $u[m]$ c f i l
 $p[n]$ - c - - f - - i - - l
 $v[m]$ b e h k
 $q[n]$ - b c - e f - h i - k l
 $w[m]$ a d g j
 $y[n]$ a b c d e f g h i j k l
 accept delay!

only $\frac{1}{3}$ sampling rate
but repeat 3 times



Input sequence $x[n]$ is split into three streams at $\frac{1}{3}$ the sample rate:

$$u[m] = x[3m], v[m] = x[3m - 1], w[m] = x[3m - 2]$$

Following upsampling, the streams are aligned by the delays and then added to give:

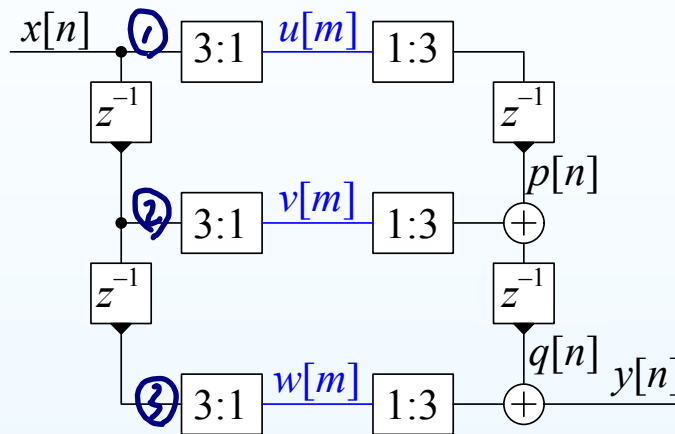
$$y[n] = x[n - 2]$$

Perfect Reconstruction: output is a delayed scaled replica of the input

Commutators

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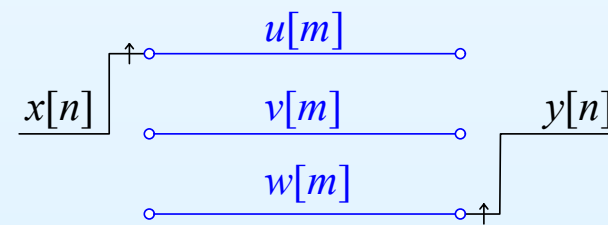
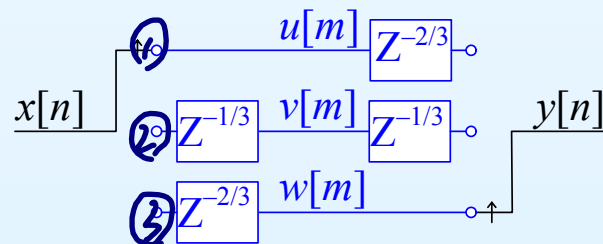
$x[n]$	c	d	e	f	g	h	i	j	k	l	m	n
$u[m]$	c			f			i			l		
$v[m]$	b			e			h			k		
$w[m]$	a			d			g			j		
$v[m] + \frac{1}{3}$				e			h			k		l
$w[m] + \frac{2}{3}$				d			g			j		m
$y[n]$	a	b	c	d	e	f	g	h	i	j	k	l

The combination of delays and downsamplers can be regarded as a **commutator** that **distributes values in sequence** to u , w and v .

Fractional delays, $z^{-\frac{1}{3}}$ and $z^{-\frac{2}{3}}$ are needed to synchronize the streams.

The **output commutator** takes values from the streams in sequence.

For clarity, we omit the fractional delays and regard each terminal, \circ , as holding its value until needed. **Initial commutator position has zero delay.**



The commutator direction is **against the direction** of the z^{-1} delays.

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• Multirate Building Blocks

- **Upsample:** $X(z) \xrightarrow{1:K} X(z^K)$

Invertible, Inserts $K - 1$ zeros between samples

Shrinks and replicates spectrum

Follow by LP filter to remove images

- **Downsample:** $X(z) \xrightarrow{K:1} \frac{1}{K} \sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}} z^{\frac{1}{K}})$

Destroys information and energy, keeps every K^{th} sample

Expands and aliases the spectrum

Spectrum is the average of K aliased expanded versions

Precede by LP filter to prevent aliases

$$\boxed{Q:1} \boxed{H(z)} = \boxed{H(z^Q)} \boxed{Q:1}$$

• Equivalences

- **Noble Identities:** $H(z) \longleftrightarrow H(z^K)$
- **Interchange** $P : 1$ and $1 : Q$ iff P and Q coprime

• Commutators

- Combine delays and down/up sampling

For further details see Mitra: 13.

MATLAB routines

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resample

change sampling rate