EE4-45
EE9-SO22

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2016

MSc and EEE/EIE PART IV: MEng and ACGI

WAVELETS AND APPLICATIONS

Corrected copy

Friday, 20 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.L. Dragotti

Second Marker(s) : A. Manikas

Special Information for the Invigilators: NONE

Information for Candidates:

Sub-sampling by an integer N

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N}) = \frac{1}{N}\sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

Dual Basis:
Given a basis $\{\varphi_i(t)\}_{i\in\mathbf{Z}}$, the dual basis is given by the set of elements $\{\tilde{\varphi}_i(t)\}_{i\in\mathbf{Z}}$ satisfying:

$$\langle \varphi_i(t), \tilde{\varphi}_j(t)\rangle = \delta_{i,j}.$$

Poisson Summation Formula

$$\sum_{n=-\infty}^{\infty} \varphi(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \hat{\varphi}\left(\frac{2\pi k}{T}\right) e^{-j2\pi kt/T}$$

The Questions

1. *Multirate Signal Processing*

    (a) With reference to Fig. 1a, given an input $x[n]$, consider upsampling by 2 followed by interpolation with a filter $H(z)$. Then to recover the original signal, apply filtering with a filter $G(z)$ followed by downsampling by 2 in order to obtain a reconstruction $\hat{x}[n]$.

Figure 1a: Interpolation followed by decimation.

i. What does the product $P(z) = H(z)G(z)$ have to satisfy in order for $\hat{x}[n]$ to be equal to $x[n]$? [5]

ii. Assume that $H(z) = (z^{-1} + 1)$. Find a 2-tap filter $G(z)$ such that perfect reconstruction is achieved. [5]

(b) Consider the system in Fig. 1b. Prove that, if $G(z) = E_0(z^2) + z^{-1}E_1(z^2)$, then $Y(z) = E_0(z)X(z)$.

Figure 1b: A multirate system.

[5]

(c) Consider the system in Fig. 1c where the output $y^{(1)}[n]$ is an interpolated version of the input sequence $x[n]$. We would like $y^{(1)}[2n] = x[n]$, while $y^{(1)}[2n+1]$ is

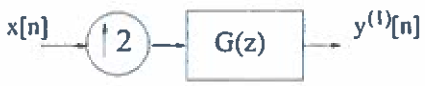

Figure 1c: Interpolation of $x[n]$.

interpolated.

i. What conditions does that impose on $G(z)$? [5]

ii. Design a 7-tap symmetric filter $g[n]$ that satisfies the conditions you have just derived. [5]

2. Consider the two-channel filter bank of Figure 2.

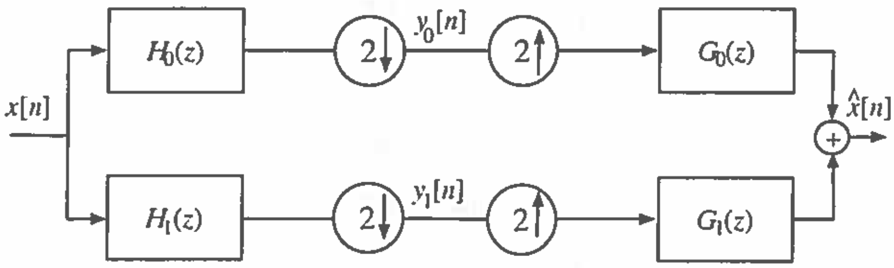

Figure 2: Two-channel filter bank.

(a) Express $\hat{X}(z)$ as a function of $X(z)$ and the filters. Then derive the two perfect reconstruction (PR) conditions the filters have to satisfy. [5]

(b) Consider $P(z) = H_0(z)G_0(z) = (z+2+z^{-1})(z^2-2z+3-2z^{-1}+z^{-2})/2$ and assume that $P(z)$ satisfies the half-band condition: $P(z)+P(-z)=2$. Find the roots of $P(z)$, knowing that one root is $z_1 = 1/2 + j\sqrt{3}/2$.
[Hint: Note that $P(z)$ is symmetric and has real-valued coefficients]. [7]

(c) Given $P(z) = H_0(z)G_0(z)$ of part (b), design the filters $H_0(z), H_1(z), G_0(z), G_1(z)$ in order to have a perfect reconstruction orthogonal filter-bank. [7]

(d) Using again $P(z)$ of part (b), design a biorthogonal perfect-reconstruction filter bank with symmetric $h_0[n]$ and $g_0[n]$. [6]

3. Consider the interval $t \in [-0.5, 0.5]$ and let

$$\varphi_1(t) = \begin{cases} 1, & \text{for } t \in [-0.5, 0.5] \\ 0, & \text{otherwise.} \end{cases}$$

$$\varphi_2(t) = \begin{cases} t, & \text{for } t \in [-0.5, 0.5] \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\varphi_3(t) = \begin{cases} t^2 - \frac{1}{12}, & \text{for } t \in [-0.5, 0.5] \\ 0, & \text{otherwise.} \end{cases}$$

Denote with $V = \text{span}(\{\varphi_1(t), \varphi_2(t), \varphi_3(t)\})$ the sub-space generated by $\varphi_i(t)$ with $i = 1, 2, 3$ over the interval $t \in [-0.5, 0.5]$.

Given a signal $x(t)$ defined for $t \in [-0.5, 0.5]$, the aim is to compute the orthogonal projection of $x(t)$ onto V. Recall that this is given by:

$$x_v(t) = \sum_{i=1}^{3} \langle x(t), \tilde{\varphi}_i(t) \rangle \varphi_i(t)$$

where $\{\tilde{\varphi}_i(t)\}_{i=1}^3$ are the three dual-basis functions.

(a) Since $\tilde{\varphi}_i(t) \in V$ we can write $\tilde{\varphi}_i(t) = \sum_{k=1}^3 \alpha_{i,k} \varphi_k(t)$. Using this fact

i. Determine the three dual-basis functions $\tilde{\varphi}_i(t)$, $i = 1, 2, 3$. That is, find the coefficients $\alpha_{i,k}$, $i = 1, 2, 3$; $k = 1, 2, 3$. [5]

ii. Sketch and dimension $\tilde{\varphi}_i(t)$, $i = 1, 2, 3$. [5]

(b) Given the dual basis and the signal

$$x(t) = \begin{cases} \sin \pi t, & \text{for } t \in [-0.5, 0.5] \\ 0 & \text{otherwise .} \end{cases}$$

i. Compute the inner products $\langle x(t), \tilde{\varphi}_i(t) \rangle$, $i = 1, 2, 3$. [5]

ii. Sketch and dimension $x_v(t) = \sum_{i=1}^{3} \langle x(t), \tilde{\varphi}_i(t) \rangle \varphi_i(t)$. [5]

iii. Verify that the error $e(t) = x(t) - x_v(t)$ is orthogonal to V. [5]

4. Consider an arbitrary real-valued function $\varphi(t)$ with Fourier transform $\hat{\varphi}(\omega)$.

(a) Show that the Fourier transform of the sequence $a[n] = \langle \varphi(t), \varphi(t-n) \rangle$ is:

$$A(e^{j\omega}) = \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2\pi k)|^2.$$

[5]

(b) Using the Poisson summation formula, show that a function $\varphi(t)$ satisfies partition of unity:

$$\sum_{k=-\infty}^{\infty} \varphi(t-k) = 1$$

when $\hat{\varphi}(2\pi n) = 1$ for $n = 0$ and $\hat{\varphi}(2\pi n) = 0$ for $n \in \mathbb{Z}$ and $n \neq 0$.

[5]

(c) Assume now that $\hat{\varphi}(2\pi n) = 1$ for $n = 0$ and that $\hat{\varphi}^{(i)}(2\pi n) = 0$ for $n \in \mathbb{Z}$ and $n \neq 0$, where $\hat{\varphi}^{(i)}(\omega)$ indicates the i-th derivative of $\hat{\varphi}(\omega)$ and $i = 0, 1$. Show that, under the above assumptions, there exists coefficients c_k such that:

$$\sum_{k=-\infty}^{\infty} c_k \varphi(t-k) = t.$$

Hint: use the Poisson summation formula and the fact that

$$\hat{\varphi}^{(1)}(\omega) = -j \int_{-\infty}^{\infty} t\varphi(t) e^{-j\omega t} dt.$$

[10]

(d) Show that the Fourier transform of the two-scale equation

$$\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t-n)$$

is:

$$\hat{\varphi}(\omega) = \frac{1}{\sqrt{2}} G_0(e^{j\omega/2}) \hat{\varphi}(\omega/2).$$

[5]