

Signals and Systems

Lecture 3

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DFT Properties

DFT: $X[k] = \sum_0^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$

DTFT: $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$

Case 1: $x[n] = 0$ for $n \notin [0, N-1]$

DFT is the same as DTFT at $\omega_k = \frac{2\pi}{N}k$.

The $\{\omega_k\}$ are uniformly spaced from $\omega = 0$ to $\omega = 2\pi \frac{N-1}{N}$.

DFT is the z -Transform evaluated at N equally spaced points around the unit circle beginning at $z = 1$.

Case 2: $x[n]$ is periodic with period N

DFT equals the normalized DTFT

$$X[k] = \lim_{K \rightarrow \infty} \frac{N}{2K+1} \times X_K(e^{j\omega_k})$$

where $X_K(e^{j\omega}) = \sum_{-K}^K x[n] e^{-j\omega n}$

Number of samples kept symmetrically around the origin.

Proof of Case 2

We want to show that if $x[n] = x[n + N]$ (i.e. $x[n]$ is periodic with period N) then

$$\lim_{K \rightarrow \infty} \frac{N}{2K+1} \times X_K(e^{j\omega_k}) \triangleq \lim_{K \rightarrow \infty} \frac{N}{2K+1} \times \sum_{-K}^K x[n]e^{-j\omega_k n} = X[k]$$

where $\omega_k = \frac{2\pi}{N}k$. We assume that $x[n]$ is bounded with $|x[n]| < B$.

We first note that the summand is periodic:

$$x[n + N]e^{-j\omega_k(n+N)} = x[n]e^{-j\omega_k n}e^{-jk\frac{2\pi}{N}N} = x[n]e^{-j\omega_k n}e^{-j2\pi k} = x[n]e^{-j\omega_k n}.$$

We now define M and R so that $2K + 1 = MN + R$ where $0 \leq R < N$ (i.e. MN is the largest multiple of N that is $\leq 2K + 1$). We can now write

$$\frac{N}{2K+1} \times \sum_{-K}^K x[n]e^{-j\omega_k n} = \frac{N}{MN+R} \times \sum_{-K}^{K-R} x[n]e^{-j\omega_k n} + \frac{N}{MN+R} \times \sum_{K-R+1}^K x[n]e^{-j\omega_k n}$$

$$(K - R) - (-K) + 1 = 2K + 1 - R = MN \text{ terms}$$

The first sum contains MN consecutive terms of a periodic summand and so equals M times the sum over one period. The second sum contains R bounded terms and so its magnitude is $< RB < NB$.

$$\text{So } \frac{N}{2K+1} \times \sum_{-K}^K x[n]e^{-j\omega_k n} = \frac{MN}{MN+R} \times \sum_0^{N-1} x[n]e^{-j\omega_k n} + P = \frac{1}{1+\frac{R}{MN}} \times X[k] + P$$


$$\text{where } |P| < \frac{N}{MN+R} \times NB \leq \frac{N}{MN+0} \times NB = \frac{NB}{M}.$$

As $M \rightarrow \infty$, $|P| \rightarrow 0$ and $\frac{1}{1+\frac{R}{MN}} \rightarrow 1$ so the whole expression tends to $X[k]$.

$$K - (K - R + 1) + 1 = R \text{ terms}$$

Symmetries

If $x[n]$ has a special property then $X(e^{j\omega})$ and $X[k]$ will have corresponding properties as shown in the table (and vice versa):



One domain	Other domain
Discrete	Periodic
Symmetric	Symmetric
Antisymmetric	Antisymmetric
Real	Conjugate Symmetric
Imaginary	Conjugate Antisymmetric
Real + Symmetric	Real + Symmetric
Real + Antisymmetric	Imaginary + Antisymmetric

Symmetric: $x[n] = x[-n]$
 $X(e^{j\omega}) = X(e^{-j\omega})$
 $X[k] = X[(-k)_{\text{mod } N}] = X[N - k]$ for $k > 0$

Conjugate Symmetric: $x[n] = x^*[-n]$
 Conjugate Antisymmetric: $x[n] = -x^*[-n]$

Parseval's Theorem

Fourier transforms preserve “energy”

CTFT $\int |x(t)|^2 dt = \frac{1}{2\pi} \int |X(j\Omega)|^2 d\Omega$

DTFT $\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

DFT $\sum_0^{N-1} |x[n]|^2 = \frac{1}{N} \sum_0^{N-1} |X[k]|^2$

Hermitian: A complex matrix that is equal to its own conjugate transpose.

More generally, they actually preserve complex inner products:

$$\sum_0^{N-1} x[n]y^*[n] = \frac{1}{N} \sum_0^{N-1} X[k]Y^*[k]$$

Unitary matrix viewpoint for DFT:

$$\begin{aligned} \mathbf{G}^H \mathbf{G} &= \frac{1}{\sqrt{N}} \mathbf{F}^H \frac{1}{\sqrt{N}} \mathbf{F} = \frac{1}{N} \mathbf{F}^H \mathbf{F} \\ &= \frac{1}{N} \mathbf{N} \mathbf{F}^{-1} \mathbf{F} = \mathbf{I} \end{aligned}$$

If we regard \mathbf{x} and \mathbf{X} as vectors, then $\mathbf{X} = \mathbf{F}\mathbf{x}$ where \mathbf{F} is a symmetric matrix defined by $f_{k+1,n+1} = e^{-j2\pi \frac{kn}{N}}$.

The inverse DFT matrix is $\mathbf{F}^{-1} = \frac{1}{N} \mathbf{F}^H$
equivalently, $\mathbf{G} = \frac{1}{\sqrt{N}} \mathbf{F}$ is a **unitary matrix** with $\mathbf{G}^H \mathbf{G} = \mathbf{I}$.

result (length = $H+X-1$)

(linear convolution): sum delayed version of x with weight h .

$$(h * x)[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

k^{th} entry of h delayed k version of x

(circular convolution): result (length = $\max(H, X)$)

sum circularly shifted version of x with weight h .

Convolution

$$(h \circledast x_N)[n] = \sum_{k=-\infty}^{\infty} h[k] x_N[n-k]$$

k^{th} entry of h circularly shifted version of x

DTFT: Convolution \rightarrow Product

$$x[n] = g[n] * h[n] = \sum_{k=-\infty}^{\infty} g[k] h[n-k]$$

$$h[n] * x[n] = (-1) \begin{Bmatrix} 2.0. -1 \\ 0.2.0. -1 \\ 0.0.2.0. -1 \end{Bmatrix} \Rightarrow X(e^{j\omega}) = G(e^{j\omega}) H(e^{j\omega})$$

DFT: Circular convolution \rightarrow Product

$$x[n] = g[n] \circledast_N h[n] = \sum_{k=0}^{N-1} g[k] h[(n-k) \bmod N]$$

$$x[n] * h[n] = (2) \begin{Bmatrix} -1. -1 \\ -1. -1 \\ 1.1. -1 \end{Bmatrix} \Rightarrow X[k] = G[k] H[k]$$

DTFT: Product \rightarrow Circular Convolution $\div 2\pi$

$$u[n] = g[n] h[n]$$

$$\Rightarrow Y(e^{j\omega}) = \frac{1}{2\pi} G(e^{j\omega}) \circledast_{\pi} H(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) H(e^{j(\omega-\theta)}) d\theta$$

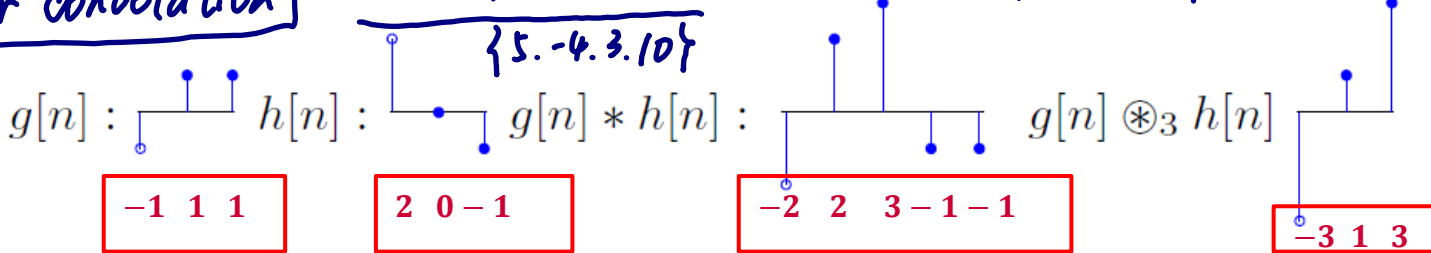
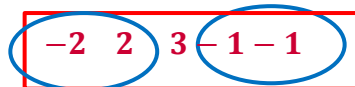
DFT: Product \rightarrow Circular Convolution $\div N$

$$y[n] = g[n] h[n] \Rightarrow Y[k] = \frac{1}{N} G[k] \circledast_N H[k]$$

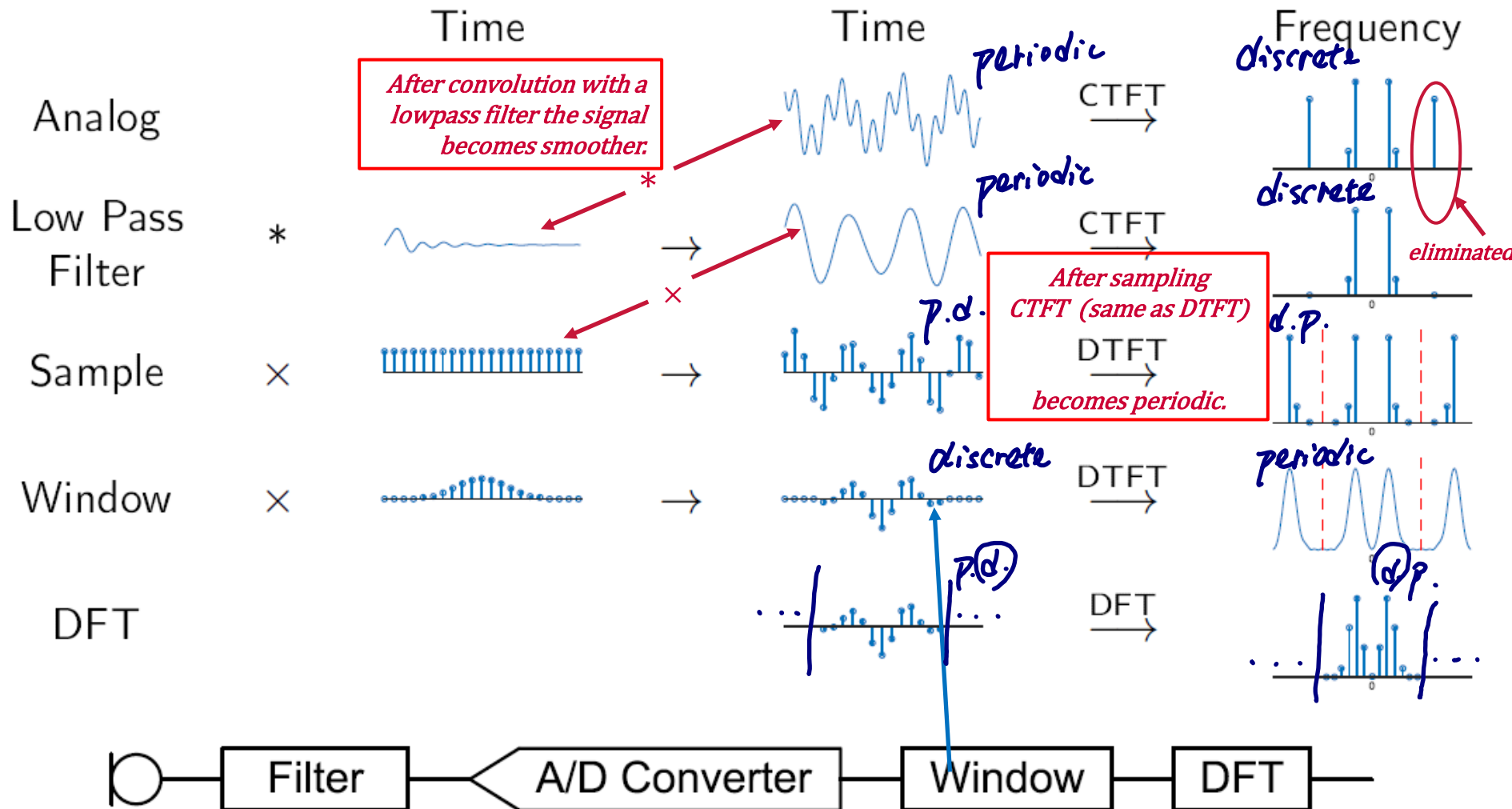
periodic
 \downarrow
circular convolution

$$h[n] = \{-1.0.1.2\} \quad x[n] = \{3.4\}$$

$$h[n] \circledast x_4[n] = (-1) \begin{Bmatrix} 3.4.0.0 \\ 0.3.4.0 \\ 0.0.3.4 \\ 4.0.0.3 \end{Bmatrix}$$



Sampling Process



Lowpass filter the signal in order to make it bandlimited for sampling.

Window the signal to make it of finite duration.

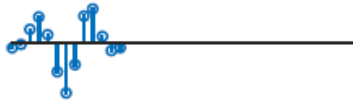
Zero Padding

Zero padding means added extra zeros onto the end of $x[n]$ before performing the DFT.

zero-padding: $T_0 \uparrow \Rightarrow f_0 \downarrow = \frac{1}{T_0}$

same resolution

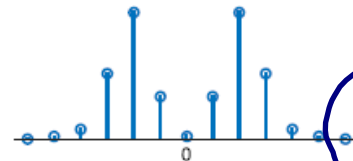
Windowed
Signal



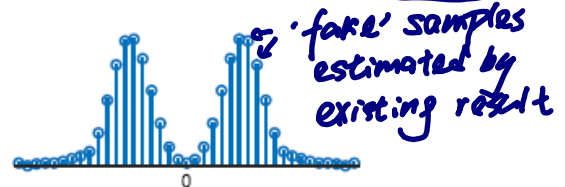
With zero-
padding



Frequency $|X[k]|$



same resolution



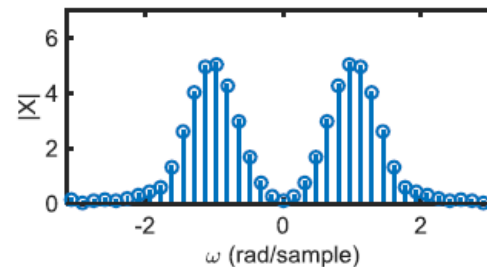
- Zero-padding causes the DFT to evaluate the DTFT at more values of ω_k . Denser frequency samples.
- Width of the peaks remains constant: determined by the length and shape of the window.
- Smoother graph but increased frequency resolution is an illusion.

Phase Unwrapping

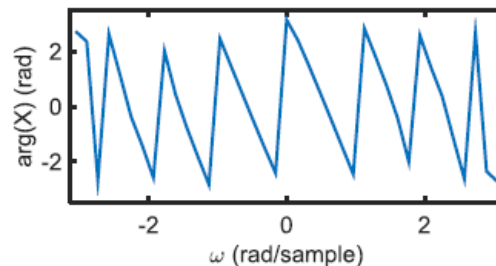
Phase of a DTFT is only defined to within an integer multiple of 2π .



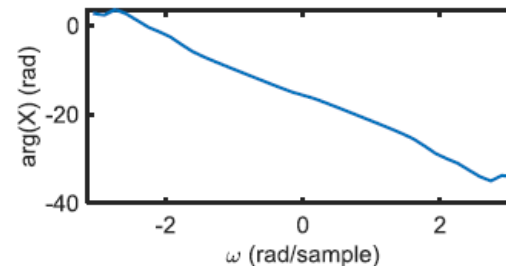
$x[n]$



$|X[k]|$



$\angle X[k]$



$\angle X[k]$ unwrapped

Phase unwrapping adds multiples of 2π onto each $\angle X[k]$ to make the phase as continuous as possible.

Uncertainty Principle

CTFT uncertainty principle:
$$\left(\frac{\int t^2 |x(t)|^2 dt}{\int |x(t)|^2 dt} \right)^{\frac{1}{2}} \left(\frac{\int \omega^2 |X(j\omega)|^2 d\omega}{\int |X(j\omega)|^2 d\omega} \right)^{\frac{1}{2}} \geq \frac{1}{2}$$

The first term measures the “width” of $x(t)$ around $t = 0$.

It is like σ if $|x(t)|^2$ was a zero-mean probability distribution.

The second term is similarly the “width” of $X(j\omega)$ in frequency.

A signal cannot be concentrated in both time and frequency.

Proof Outline: $u=t \quad dv=x dx$
 $du=dt \quad v=\frac{1}{2}x^2$
 Assume $\int |x(t)|^2 dt = 1 \Rightarrow \int |X(j\omega)|^2 d\omega = 2\pi$ [Parseval]
 Set $v(t) = \frac{dx}{dt} \Rightarrow V(j\omega) = j\omega X(j\omega)$ [by parts]
 Now $\int t x \frac{dx}{dt} dt = \frac{1}{2} t x^2(t) \Big|_{t=-\infty}^{\infty} - \int \frac{1}{2} x^2 dt = 0 - \frac{1}{2}$ [by parts]
 So $\frac{1}{4} = \left| \int t x \frac{dx}{dt} dt \right|^2 \leq \left(\int t^2 x^2 dt \right) \left(\int \left| \frac{dx}{dt} \right|^2 dt \right)$ [Schwartz]

$$\begin{aligned} &= \left(\int t^2 x^2 dt \right) \left(\int |v(t)|^2 dt \right) = \left(\int t^2 x^2 dt \right) \left(\frac{1}{2\pi} \int |V(j\omega)|^2 d\omega \right) \\ &= \left(\int t^2 x^2 dt \right) \left(\frac{1}{2\pi} \int \omega^2 |X(j\omega)|^2 d\omega \right) \end{aligned}$$

No exact equivalent for DTFT/DFT but a similar effect is true

Uncertainty Principle Proof Steps

- (1) Suppose $v(t) = \frac{dx}{dt}$. Then integrating the CTFT definition by parts w.r.t. t gives

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt = \left[\frac{-1}{j\Omega} x(t)e^{-j\Omega t} \right]_{-\infty}^{\infty} + \frac{1}{j\Omega} \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\Omega t} dt = 0 + \frac{1}{j\Omega} V(j\Omega)$$

- (2) Since $\frac{d}{dt} \left(\frac{1}{2} x^2 \right) = x \frac{dx}{dt}$, we can apply integration by parts to get

$$\int_{-\infty}^{\infty} tx \frac{dx}{dt} dt = \left[t \times \frac{1}{2} x^2 \right]_{t=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{dt}{dt} \times \frac{1}{2} x^2 dt = -\frac{1}{2} \int_{-\infty}^{\infty} x^2 dt = -\frac{1}{2} \times 1 = -\frac{1}{2}$$

It follows that $\left| \int_{-\infty}^{\infty} tx \frac{dx}{dt} dt \right|^2 = \left(-\frac{1}{2} \right)^2 = \frac{1}{4}$ which we will use below.

- (3) The Cauchy-Schwarz inequality is that in a complex inner product space $|\mathbf{u} \cdot \mathbf{v}|^2 \leq (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v})$. For the inner-product space of real-valued square-integrable functions, this becomes $\left| \int_{-\infty}^{\infty} u(t)v(t)dt \right|^2 \leq \int_{-\infty}^{\infty} u^2(t)dt \times \int_{-\infty}^{\infty} v^2(t)dt$. We apply this with $u(t) = tx(t)$ and $v(t) = \frac{dx(t)}{dt}$ to get

$$\frac{1}{4} = \left| \int_{-\infty}^{\infty} tx \frac{dx}{dt} dt \right|^2 \leq \left(\int_{-\infty}^{\infty} t^2 x^2 dt \right) \left(\int_{-\infty}^{\infty} \left(\frac{dx}{dt} \right)^2 dt \right) = \left(\int_{-\infty}^{\infty} t^2 x^2 dt \right) \left(\int_{-\infty}^{\infty} v^2(t) dt \right)$$

- (4) From Parseval's theorem for the CTFT, $\int v^2(t)dt = \frac{1}{2\pi} \int |V(j\Omega)|^2 d\Omega$. From step (1), we can substitute $V(j\Omega) = j\Omega X(j\Omega)$ to obtain $\int v^2(t)dt = \frac{1}{2\pi} \int \Omega^2 |X(j\Omega)|^2 d\Omega$. Making this substitution in (3) gives

$$\frac{1}{4} \leq \left(\int_{-\infty}^{\infty} t^2 x^2 dt \right) \left(\int_{-\infty}^{\infty} v^2(t) dt \right) = \left(\int_{-\infty}^{\infty} t^2 x^2 dt \right) \left(\frac{1}{2\pi} \int \omega^2 |X(j\Omega)|^2 d\Omega \right)$$

Summary

- Three types: CTFT, DTFT, DFT
 - DTFT = CTFT of continuous signal \times impulse train
 - DFT = DTFT of periodic or finite support signal
 - ▷ DFT is a scaled unitary transform
- DTFT: Convolution \rightarrow Product; Product \rightarrow Circular Convolution
- DFT: Product \leftrightarrow Circular Convolution
- DFT: Zero Padding \rightarrow Denser freq sampling but same resolution
- Phase is only defined to within a multiple of 2π .
- Whenever you integrate over frequency you need a scale factor
 - $\frac{1}{2\pi}$ for CTFT and DTFT or $\frac{1}{N}$ for DFT
 - e.g. Inverse transform, Parseval, frequency domain convolution

For further details see Mitra: 3 & 5.