Imperial College London

# DSP & Digital Filters

## Lecture 1 z-Transform


**DR TANIA STATHAKI**

READER (ASSOCIATE PROFESSOR) IN SIGNAL PROCESSING

IMPERIAL COLLEGE LONDON

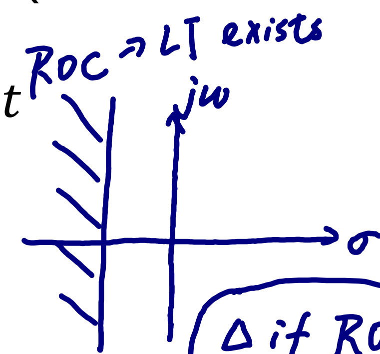
Laplace transform: generalised frequency transformation.

# Continuous-time signals

(all signals have LT but not all have FT).

- Recall that in order to describe a continuous-time signal $x(t)$ in frequency domain we use:
  - ❑ The Continuous-Time Fourier Transform (or Fourier Transform):

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$




s-plane (complex)

$\Delta$ if ROC $\ni$ jw axis, then FT exists.

  - ❑ The Laplace Transform

$s = \sigma + jw$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

- The above transforms and their basic properties are considered known in this course.
- If you have doubts please consult any book on Signals and Systems.

# Discrete-time signals

$x[n] = x(nT)$

$x(t) = \sum_n x[n]\delta(t-nT)$

## The z-transform derived from the Laplace transform

samples $x[n]$

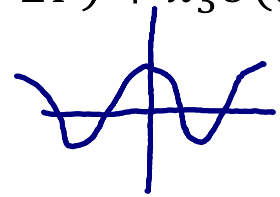
- Consider a discrete-time signal $x(t)$ sampled every $T$ seconds.

$$x(t) = x_0\delta(t) + x_1\delta(t-T) + x_2\delta(t-2T) + x_3\delta(t-3T) + \cdots$$

- Recall that in the Laplace domain we have:

LT: $X(s) = \int_{-\infty}^{+\infty} x[n] e^{-st} dt$

$$\mathcal{L}\{\delta(t)\} = 1$$
$$\mathcal{L}\{\delta(t-T)\} = e^{-sT}$$



$x(t) = \frac{1}{k}\sum_{i=0}^{k-1} \cos(\omega_i t)$

$= \begin{cases} 1, & t=0 \\ \to 0, & t\neq 0 \end{cases}$

if $k\to\infty$, $x(t) \to \delta(t)$.

- Therefore, the Laplace transform of $x(t)$ is:

$$X(s) = x_0 + x_1e^{-sT} + x_2e^{-s2T} + x_3e^{-s3T} + \cdots$$

- Now define $z = e^{sT} = e^{(\sigma+j\omega)T} = e^{\sigma T}\cos\omega T + je^{\sigma T}\sin\omega T$.
- Finally, define

$$X[z] = x_0 + x_1z^{-1} + x_2z^{-2} + x_3z^{-3} + \cdots$$

# $z^{-1}$: the sampling period delay operator

$\delta(t-T) \leftrightarrow e^{-sT} = z^{-1}$ : delay by a sampling period.

- From the Laplace time-shift property, we know that an additional term $z = e^{sT}$ in the Laplace domain, corresponds to time-advance by $T$ seconds ($T$ is the sampling period) of the original function in time.
- Accordingly, $z^{-1} = e^{-sT}$ corresponds to a time-delay of one sampling period.
- As a result, all sampled data (and discrete-time systems) can be expressed in terms of the variable $z$.
- More formally, the **unilateral $z-$transform** of a causal sampled sequence:

$$x[n] = \{x[0], x[1], x[2], x[3], \dots\}$$

is given by:

$$X[z] = x_0 + x_1 z^{-1} + x_2 z^{-2} + x_3 z^{-3} + \cdots = \sum_{n=0}^{\infty} x[n] z^{-n}, \; x_n = x[n]$$

- The **bilateral $z-$transform** for any sampled sequence is:

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$X[z] = \sum_{n=0}^{\infty} \gamma^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^n = \frac{1}{1-\frac{\gamma}{z}} = \frac{z}{z-\gamma}, \left|\frac{\gamma}{z}\right| < 1$

ROC: $|z| > |\gamma|$

# Example: Find the $z$ –transform of $x[n] = \gamma^n u[n]$

- Find the $z$ –transform of the **causal** signal $\gamma^n u[n]$, where $\gamma$ is a constant.
- By definition:

$$X[z] = \sum_{n=-\infty}^{\infty} \gamma^n u[n] z^{-n} = \sum_{n=0}^{\infty} \gamma^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^n$$
$$= 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots$$

- We apply the geometric progression formula:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, |x| < 1$$

- Therefore,

$$X[z] = \frac{1}{1-\frac{\gamma}{z}}, \left|\frac{\gamma}{z}\right| < 1$$
$$= \frac{z}{z-\gamma}, |z| > |\gamma|$$

- We notice that the $z$ –transform exists for certain values of $z$. These values form the so called Region-of-Convergence (ROC) of the transform.

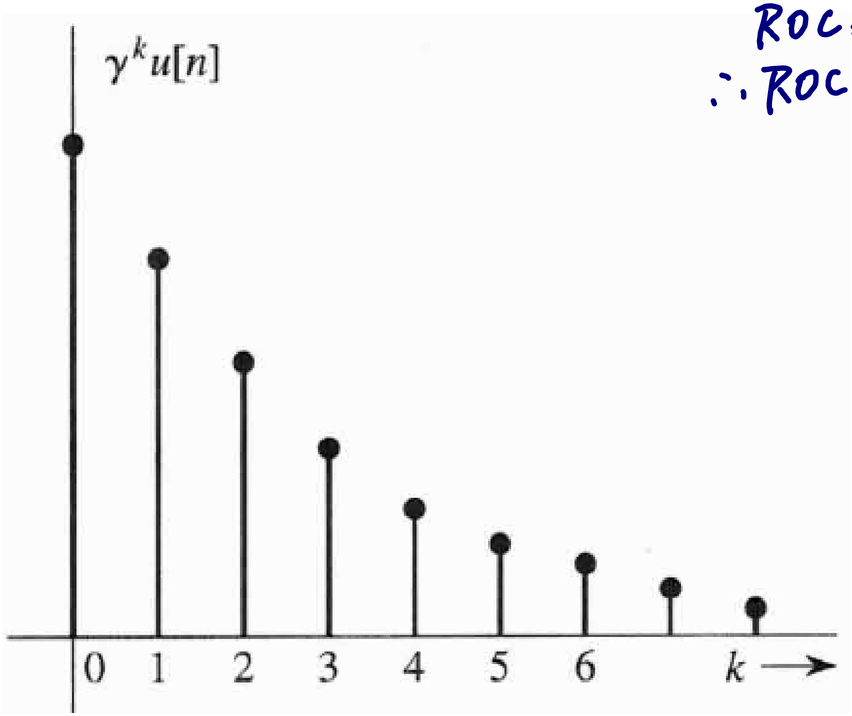
# Example: Find the $z$ –transform of $x[n] = \gamma^n u[n]$ cont.

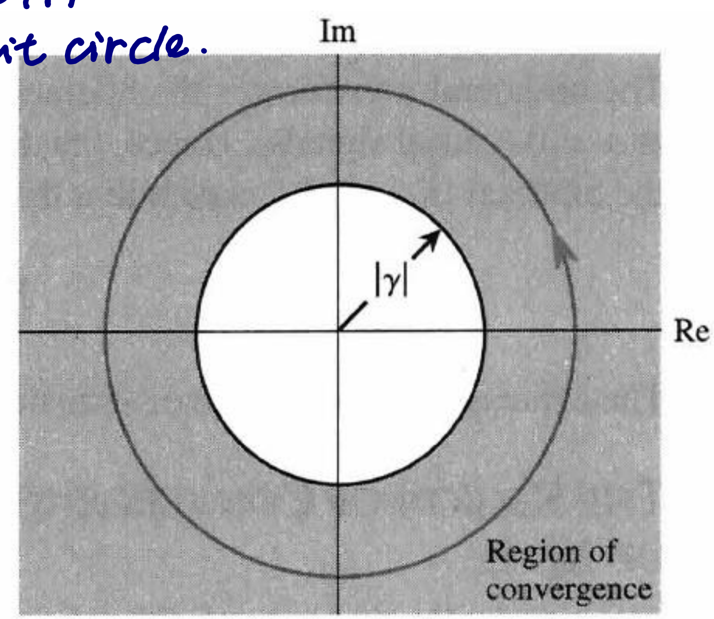
- Observe that a simple rational equation in $z$-domain corresponds to an infinite sequence of samples in time-domain.
- The figures below depict the signal in time (left) for $|\gamma| < 1$ and the ROC, shown with the shaded area, within the $z$ –plane.

$x[n]$ absolutely summable : $|\gamma| < 1$

ROC: $|z| > |\gamma|$

$\therefore$ ROC $\supseteq$ unit circle.



Imperial College London

$x[n] = \sum_{i=1}^{k} r_i^n u[n]$

$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = \sum_{n=0}^{+\infty} \sum_{i=1}^{k} r_i^n z^{-n} = \sum_{i=1}^{k} \frac{z}{z-r_i}$, $|z| > |r_i|_{max}$

# Generic form of a causal signal

poles at $z = r_i$

$r \neq i$

$r_i \sim i$

- Consider the causal signal $x[n] = \sum_{i=1}^{K} \gamma_i^n u[n]$ with $X(z) = \sum_{i=1}^{K} \frac{z}{z-\gamma_i}$.
- In that case the ROC is the intersection of the ROCs of the individual terms, i.e., the intersection of the sets $|z| > |\gamma_i|$ i.e., ROC: $|z| > |\gamma_{\max}|$
- In case that $x[n]$ is the impulse response of a system, the transfer function of the system is the rational function $X(z) = \sum_{i=1}^{K} \frac{z}{z-\gamma_i}$ with poles $\gamma_i$.
- The above analysis yields the following properties regarding the ROC:

**PROPERTY:**

**If $x[n]$ is a causal signal, the ROC of its $z$ −transform is $|z| > |\gamma_{\max}|$ with $\gamma_{\max}$ the maximum magnitude pole of the $z$ −transform.**

- ❑ **In the general case of $x[n]$ being a right-sided signal (RSS) the ROC is as above but might not include $\infty$ (think why).**

**PROPERTY:**

**No pole can exist in ROC.** ROC: $\sum_{n} x[n] z^{-n} \neq \pm\infty$

# Generic form of a causal signal cont.

bounded: $|\gamma_i| < 1$
ROC: $|z| > |\gamma_i|_{max}$ $\Leftrightarrow$ ROC must include the unit circle.

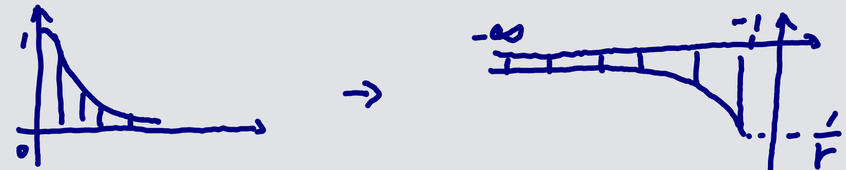
- The signal $x[n] = \sum_{i=1}^{K} \gamma_i^n u[n]$ is bounded only if $|\gamma_i| < 1\ \forall i$ or $|\gamma_{\max}| < 1$.
- In that case the ROC includes a circle with radius equal to 1. This is known as the **unit circle**.
- The above observation yields the following property:

**PROPERTY:**

**If the ROC of $X(z)$ includes the unit circle in $z-$plane, then the signal in time is bounded and its Discrete Time Fourier Transform exists.**

- In case that $\gamma^n u[n]$ is part of a causal system's impulse response, we see that the condition $|\gamma| < 1$ must hold. This is because, since $\lim_{n\to\infty} (\gamma)^n = \infty$, for $|\gamma| > 1$, the system will be unstable in that case.

- Therefore, in causal systems, stability requires that the ROC of the system's transfer function includes the unit circle.




# Example: Find the $z$ −transform of $x[n] = -\gamma^n u[-n-1]$

- Find the $z$ −transform of the anti-causal signal $-\gamma^n u[-n-1]$, where $\gamma$ is a constant.

$$X(z) = \sum_{n=-\infty}^{+\infty} -r^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} -r^n z^{-n} = -\sum_{n=1}^{\infty} r^{-n} z^n = -\sum_{n=1}^{\infty} \left(\frac{z}{r}\right)^n = -\frac{z}{r}\sum_{n=0}^{+\infty} \left(\frac{z}{r}\right)^n$$

$$= -\frac{z}{r}\frac{1}{1-\frac{z}{r}} = -\frac{z}{r}\frac{r}{r-z} = \frac{z}{z-r}, \quad |z| < |r|$$

- By definition:

$$X[z] = \sum_{n=-\infty}^{\infty} -\gamma^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} -\gamma^n z^{-n} = -\sum_{n=1}^{\infty} \gamma^{-n} z^n = -\sum_{n=1}^{\infty} \left(\frac{z}{\gamma}\right)^n$$

$$= -\frac{z}{\gamma}\sum_{n=0}^{\infty}\left(\frac{z}{\gamma}\right)^n = -\left(\frac{z}{\gamma}\right)\left[1 + \left(\frac{z}{\gamma}\right) + \left(\frac{z}{\gamma}\right)^2 + \left(\frac{z}{\gamma}\right)^3 + \cdots\right]$$

- Therefore,

anticausal: working offline

$$X[z] = -\left(\frac{z}{\gamma}\right)\frac{1}{1-\frac{z}{\gamma}}, \quad \left|\frac{z}{\gamma}\right| < 1$$

$$= \frac{z}{z-\gamma}, \quad |z| < |\gamma|$$

- We notice that the $z$ −transform exists for certain values of $z$, which consist the complement of the ROC of the function $\gamma^n u[n]$ with respect to the $z$ −plane.

$x[n] = \sum_{i=1}^{K} -r_i^n u[-n-1]$

$X(z) = \sum_{n=-\infty}^{+\infty}\sum_{i=1}^{K} -r_i^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1}\sum_{i=1}^{K}\left(\frac{r_i}{z}\right)^n = -\sum_{n=1}^{+\infty}\sum_{i=1}^{K}\left(\frac{z}{r_i}\right)^n = -\sum_{i=1}^{K}\sum_{n=1}^{+\infty}\left(\frac{z}{r_i}\right)^n$

# Generic form of an anti-causal signal

$= -\sum_{i=1}^{K}\frac{z}{r_i}\sum_{n=0}^{+\infty}\left(\frac{z}{r_i}\right)^n \underset{|z|<|r_i|}{=} -\sum_{i=1}^{K}\frac{z}{r_i}\frac{1}{1-\frac{z}{r_i}} = -\sum_{i=1}^{K}\frac{z}{r_i}\frac{r_i}{r_i-z} = \sum_{i=1}^{K}\frac{z}{z-r_i}.$

- Consider the anti-causal signal $x[n] = \sum_{i=1}^{K} -\gamma_i^n u[-n-1]$ with $|z|<|r_i|_{min}$ $z$ –transform $X(z) = \sum_{i=1}^{K}\frac{z}{z-\gamma_i}$.
- In that case the ROC is the intersection of the sets $|z| < |\gamma_i|$, i.e., ROC: $|z| < |\gamma_{\min}|$
- In case that $x[n]$ is the impulse response of a system, the transfer function of the system is the rational function $X(z) = \sum_{i=1}^{K}\frac{z}{z-\gamma_i}$ with poles $\gamma_i$.
- The above analysis yield the following property regarding ROCs:

**PROPERTY:**

**If $x[n]$ is an anti-causal signal, the ROC of its $z$ –transform is $|z| < |\gamma_{\min}|$ with $\gamma_{\min}$ the minimum magnitude pole of the $z$ –transform.**

- ❑ **In the general case of $x[n]$ being a left-sided signal (LSS) the ROC is as above but might not include 0 (think why).**

# Summary of previous examples

causal vs. anticausal:
· same ZT
· complementary ROCs

- We proved that the following two functions:
  - The causal function $\gamma^n u[n]$ and
  - the anti-causal function $-\gamma^n u[-n-1]$ have:
    - The same analytical expression for their $z$ –transforms.
    - Complementary ROCs. More specifically, the union of their ROCS forms the entire $z$ –plane.

- The above observations verify that the analytical expression alone is not sufficient to define the $z$ –transform of a signal. The ROC is also required.

$r^n u[n]$ — $\frac{z}{z-r}$ — $-r^n u[-n-1]$

$|z|>|r|$ $|z|<|r|$

# Two-sided signals

- **Example:** Find the $z$ –transform of the two-sided signal:

  $x[n] = 2^n u[n] - 4^n u[-n-1]$

  Based on the previous analysis we have:

  $$X[z] = \frac{z}{z-2} + \frac{z}{z-4}, \text{ ROC: } |z| > 2 \cap |z| < 4 \text{ or ROC: } 2 < |z| < 4$$

- **Example:** Find the $z$ –transform of the two-sided signal:

  $x[n] = 4^n u[n] - 2^n u[-n-1]$

  Based on the previous analysis we have:

  $$X[z] = \frac{z}{z-2} + \frac{z}{z-4}, \text{ ROC: } |z| > 4 \cap |z| < 2 \text{ or ROC: } \emptyset$$

**PROPERTY:**

**If $x[n]$ is two-sided signal then the ROC of its $z$ –transform is of the form:**

- ❑ $\gamma_1 < |z| < \gamma_2$ **with $\gamma_1, \gamma_2$ poles of the system or**
- ❑ $\emptyset$

$\delta[n] \xleftrightarrow[IZT]{ZT} 1$

$u[n] \xleftrightarrow[IZT]{ZT} \frac{z}{z-1}, \ |z|>1$

# Example: Find the $z$ −transform of $\delta[n]$ and $u[n]$

- By definition $\delta[0] = 1$ and $\delta[n] = 0$ for $n \neq 0$.

$X(z) = \sum_{n=-\infty}^{+\infty} \delta[n] z^{-n} = z^0 = 1$

$$X[z] = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = \delta[0] z^{-0} = 1$$

- By definition $u[n] = 1$ for $n \geq 0$.

$X(z) = \sum_{n=-\infty}^{+\infty} u[n] z^{-n} = \sum_{n=0}^{+\infty} z^{-n} = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}, \ |z|>1$

$$X[z] = \sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-\frac{1}{z}}, \ \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{z-1}, \ |z| > 1$$

Imperial College London

$\cos\beta n u[n] \underset{IZT}{\overset{ZT}{\longleftrightarrow}} \frac{z(z-\cos\beta)}{z^2-2z\cos\beta+1}, \ |z|>1$

# Example: Find the $z$ –transform of $\cos\beta n u[n]$

$x[n] = \cos\beta n u[n] = \frac{1}{2}(e^{-j\beta n} + e^{j\beta n})u[n]$

$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = \sum_{n=-\infty}^{+\infty} \frac{1}{2}(e^{-j\beta n} + e^{j\beta n})u[n] z^{-n} = \frac{1}{2}\left[\sum_{n=0}^{+\infty}(z e^{j\beta})^{-n} + \sum_{n=0}^{+\infty}\left(\frac{e^{j\beta}}{z}\right)^n\right]$

- We write $\cos\beta n = \frac{1}{2}\left(e^{j\beta n} + e^{-j\beta n}\right)$.
- From previous analysis we showed that:

$$\gamma^n u[n] \Leftrightarrow \frac{z}{z-\gamma}, \ |z| > |\gamma|$$

- Hence,

$$e^{\pm j\beta n}u[n] \Leftrightarrow \frac{z}{z-e^{\pm j\beta}}, \ |z| > \left|e^{\pm j\beta}\right| = 1$$

- Therefore,

$$X[z] = \frac{1}{2}\left[\frac{z}{z-e^{j\beta}} + \frac{z}{z-e^{-j\beta}}\right] = \frac{z(z-\cos\beta)}{z^2-2z\cos\beta+1}, \ |z| > 1$$

$|z| > |e^{\pm j\beta}| = 1$

$= \frac{1}{2}\left(\frac{1}{1-(ze^{j\beta})^{-1}} + \frac{1}{1-\frac{e^{j\beta}}{z}}\right)$

$= \frac{1}{2}\left(\frac{z}{z-e^{-j\beta}} + \frac{z}{z-e^{j\beta}}\right)$

$= \frac{1}{2}\frac{z^2-ze^{j\beta}+z^2-ze^{-j\beta}}{z^2-z(e^{j\beta}+e^{-j\beta})+1}$

$= \frac{1}{2}\frac{2z^2-z\cdot 2\cos\beta}{z^2-z\cdot 2\cos\beta+1} = \frac{z^2-z\cos\beta}{z^2-2z\cos\beta+1}$.

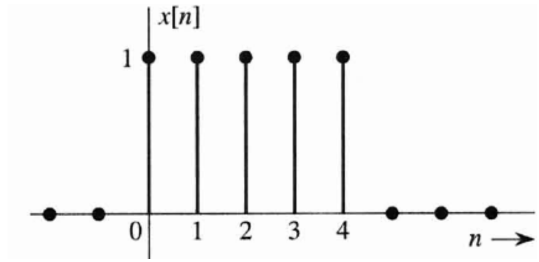
$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$

# $z$ –transform of 5 impulses

$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = z^{0} + z^{-1} + z^{-2} + z^{-3} + z^{-4}$

$= \frac{1-(\frac{1}{z})^5}{1-\frac{1}{z}}$

- Find the $z$ –transform of the signal depicted in the figure.




- By definition:

$$X[z] = 1 + \frac{1}{z} + \frac{1`}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} = \sum_{k=0}^{4} (z^{-1})^k = \frac{1-(z^{-1})^5}{1-z^{-1}} = \frac{z}{z-1}(1-z^{-5})$$

# Inverse $z$ –transform

- As with other transforms, inverse $z$ –transform is used to derive $x[n]$ from $X[z]$, and is formally defined as:

$$x[n] = \frac{1}{2\pi j}\oint X[z]z^{n-1}dz$$

- Here the symbol $\oint$ indicates an integration in counter-clockwise direction around a circle within the ROC and $z = Re^{j\theta}$.
- Such contour integral is difficult to evaluate (but could be done using Cauchy's residue theorem), therefore we often use other techniques to obtain the inverse $z$ –transform.
- One such technique is to use a $z$ –transform pairs Table shown in the last two slides with partial fraction expansion.

# Inverse $z$ −transform: Proof

$$x[n] = \frac{1}{2\pi j}\oint X(z) z^{n-1} dz = \frac{1}{2\pi j}\oint \sum_{m=-\infty}^{+\infty} x[m] z^{-m} z^{n-1} dz = \frac{1}{2\pi j}\sum_{m=-\infty}^{+\infty} x[m] \oint z^{n-m-1} dz$$

$$\oint z^{k-1} dz \; \underset{dz=jRe^{j\theta}d\theta}{\overset{z=Re^{j\theta}}{=}} \int_0^{2\pi} R^{k-1} e^{j\theta(k-1)} jRe^{j\theta} d\theta = j\int^{2\pi} R^k e^{j\theta k} d\theta = \begin{cases} R^k j \frac{1}{jk} e^{j\theta k}\Big|_0^{2\pi} = 0, & k \neq 0 \\ j\int_0^{2\pi} 1\, d\theta = 2\pi j, & k = 0 \end{cases} = 2\pi j\,\delta[k]$$

**Proof:** $\therefore x[n] = \frac{1}{2\pi j}\sum_{m=-\infty}^{+\infty} x[m]\, 2\pi j\, \delta[n-m] = x[n].$

$$\frac{1}{2\pi j}\oint X[z] z^{n-1} dz = \frac{1}{2\pi j}\oint \left(\sum_{m=-\infty}^{\infty} x[m] z^{-m}\right) z^{n-1} dz$$

$$= \sum_{m=-\infty}^{\infty} x[m] \frac{1}{2\pi j}\oint z^{n-m-1} dz = \sum_{m=-\infty}^{\infty} x[m]\, \delta[n-m] = x[n]$$

- For the above we used the Cauchy's theorem:

$\frac{1}{2\pi j}\oint z^{k-1} dz = \delta[k]$ for $z = Re^{j\theta}$ anti-clockwise.

$$\frac{dz}{d\theta} = jRe^{j\theta} \Rightarrow \frac{1}{2\pi j}\oint z^{k-1} dz = \frac{1}{2\pi j}\int_{\theta=0}^{2\pi} R^{k-1} e^{j(k-1)\theta} jRe^{j\theta} d\theta =$$

$$\frac{R^k}{2\pi}\int_{\theta=0}^{2\pi} e^{jk\theta}\, d\theta = R^k\, \delta[k]$$

$$\int_{\theta=0}^{2\pi} e^{jk\theta} d\theta = \begin{cases} \frac{1}{jk} e^{jk\theta}\Big|_0^{\theta=2\pi} = 0, & k \neq 0 \\ \int_{\theta=0}^{2\pi} 1\, d\theta = 2\pi, & k = 0 \end{cases}$$

$$\left[\frac{R^k}{2\pi}\int_{\theta=0}^{2\pi} e^{jk\theta}\, d\theta = \begin{cases} 0 & k \neq 0 \\ \frac{R^k}{2\pi} 2\pi = R^k & k = 0 \end{cases}\right]$$

$\frac{X(z)}{z} = \frac{8z-19}{z(z-2)(z-3)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z-3}$

$\times z,\ z=0 \Rightarrow \frac{0-19}{(0-2)(0-3)} = A = -\frac{19}{6}$

# Find the inverse $z$ −transform in the case of real unique poles

$\times (z-2),\ z=2 \Rightarrow \frac{16-19}{2(-1)} = B = \frac{3}{2}$

$\times (z-3),\ z=3 \Rightarrow \frac{24-19}{3\cdot 1} = C = \frac{5}{3}$

assume causal!

- Find the inverse $z$ −transform of $X[z] = \frac{8z-19}{(z-2)(Z-3)}$

$\frac{5}{3}3^n u[n],\ |z|>3$ ③

$-\frac{5}{3}3^n u[-n-1],\ |z|<3$ ④

$\therefore X(z) = -\frac{19}{6} + \frac{3}{2}\frac{z}{z-2} + \frac{5}{3}\frac{z}{z-3}$

$-\frac{19}{6}\delta[n]$ $\quad \frac{3}{2}2^n u[n]$ $|z|>2$ ① $\quad -\frac{3}{2}2^n u[-n-1]$ $|z|<2$ ②

**Solution**

Target $\frac{X(z)}{z}$ to recover $z$ in the nominators.

$$\frac{X[z]}{z} = \frac{8z-19}{z(z-2)(Z-3)} = \frac{(-\frac{19}{6})}{z} + \frac{3/2}{z-2} + \frac{5/3}{z-3}$$

①③ causal. unstable $|z|>3$

②④ anticausal stable $|z|<2$

①④ two-sided unstable $3>|z|>2$

$$X[z] = -\frac{19}{6} + \frac{3}{2}\left(\frac{z}{z-2}\right) + \frac{5}{3}\left(\frac{z}{z-3}\right)$$

By using the simple transforms that we derived previously we get:

$$x[n] = -\frac{19}{6}\delta[n] + \left[\frac{3}{2}2^n + \frac{5}{3}3^n\right]u[n]$$

Solution 1 : (derivative)

$\frac{X(z)}{z} = \frac{2z^2-11z+12}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{z-2}$

$\times (z-1).\ z=1 \Rightarrow k = \frac{2-11+12}{(1-2)^3} = -3$

# Find the inverse $z$ −transform in the case of real repeated poles

$\times (z-2)^3,\ \frac{d}{dz^0}.\ z=2 \Rightarrow a_0 = \frac{2\times 2^2-11\times 2+12}{2-1} = -2$

$\times (z-2)^3.\ \frac{d}{dz^1}$ : $\frac{d}{dz}\left(\frac{2z^2-11z+12}{z-1}\right) = \frac{d}{dz}\left(k\frac{(z-2)^3}{z-1} + a_2(z-2)^2 + a_1(z-2) + a_0\right)$

- Find the inverse $z$ −transform of $X[z] = \frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$

$\frac{(4z-11)(z-1)-(2z^2-11z+12)}{(z-1)^2} = \frac{d}{dz}\left(k\frac{(z-2)^3}{z-1} + a_2(z-2)^2 + a_1(z-2) + a_0\right)$

**Solution** $z=2$

$\Rightarrow \frac{4z^2-11z-4z+11-2z^2+11z-12}{(z-1)^2} = \frac{2z^2-4z-1}{(z-1)^2} = a_1 \Rightarrow a_1 = -1$

$$\frac{X[z]}{z} = \frac{(2z^2-11z+12)}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

- We use the so called **<u>covering method</u>** to find $k$ and $a_0$

$\times (z-2)^3.\ \frac{d}{dz^2}$ :

$\frac{(4z-4)(z^2-2z+1)-(2z^2-4z-1)\cdot 2(z-1)}{(z-1)^4}$

$$k = \frac{(2z^2-11z+12)}{(z-1)(z-2)^3}\bigg|_{z=1} = -3$$

$= 6 = 2a_2 \Rightarrow a_2 = 3.$

$$a_0 = \frac{(2z^2-11z+12)}{(z-1)(z-2)^3}\bigg|_{z=2} = -2$$

The shaded areas above indicate that they are excluded from the entire function when the specific value of $z$ is applied.

Imperial College London

# Find the inverse $z$ –transform in the case of real repeated poles cont.

- Find the inverse $z$ –transform of $X[z]=\frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$

**Solution**

$$\frac{X[z]}{z}=\frac{(2z^2-11z+12)}{(z-1)(z-2)^3}=\frac{-3}{z-1}+\frac{-2}{(z-2)^3}+\frac{a_1}{(z-2)^2}+\frac{a_2}{(z-2)}$$

- To find $a_2$ we multiply both sides of the above equation with $z$ and let $z\to\infty$.

$$0=-3-0+0+a_2\Rightarrow a_2=3$$

- To find $a_1$ let $z\to 0$.

$$\frac{12}{8}=3+\frac{1}{4}+\frac{a_1}{4}-\frac{3}{2}\Rightarrow a_1=-1$$

$$\frac{X[z]}{z}=\frac{(2z^2-11z+12)}{(z-1)(z-2)^3}=\frac{-3}{z-1}-\frac{2}{(z-2)^3}-\frac{1}{(z-2)^2}+\frac{3}{(z-2)}\Rightarrow$$

$$X[z]=\frac{-3z}{z-1}-\frac{2z}{(z-2)^3}-\frac{z}{(z-2)^2}+\frac{3z}{(z-2)}$$

Solution 2. (limit)

$$\frac{X(z)}{z}=\frac{(2z^2-11z+12)}{(z-1)(z-2)^3}=\frac{k}{z-1}+\frac{a_0}{(z-2)^3}+\frac{a_1}{(z-2)^2}+\frac{a_2}{(z-2)}$$

$\times(z-1)$, $z=1\Rightarrow k=\frac{2z^2-11z+12}{(z-2)^3}\Big|_{z=1}=-3$

$\times(z-2)^3$, $z=2\Rightarrow a_0=\frac{2z^2-11z+12}{z-1}\Big|_{z=2}=-2$

$$\therefore\frac{X(z)}{z}=\frac{-3}{z-1}+\frac{-2}{(z-2)^3}+\frac{a_1}{(z-2)^2}+\frac{a_2}{z-2}=\frac{(2z^2-11z+12)}{(z-1)(z-2)^3}$$

$\times z\Rightarrow -3\frac{z}{z-1}-2\frac{z}{(z-2)^3}+a_1\frac{z}{(z-2)^2}+a_2\frac{z}{z-2}=\frac{(2z^2-11z+12)z}{(z-1)(z-2)^3}$

$z\to\infty\Rightarrow -3+0+0+a_2=0\Rightarrow a_2=3$

$z\to 0\Rightarrow 3+\frac{1}{4}+\frac{a_1}{4}-\frac{3}{2}=\frac{12}{8}\Rightarrow a_1=-1$

$\therefore X(z) = -3\frac{z}{z-1} - 2\frac{z}{(z-2)^3} - \frac{z}{(z-2)^2} + 3\frac{z}{z-2}$. Assume $x[n]$ casual.

# Find the inverse $z$ –transform in the case of real repeated poles cont.

$x[n] = -3u[n] - 2\frac{n(n-1)}{2^2\cdot 2}2^n u[n] - \frac{n}{2^1\cdot 1}2^n u[n] + 3\cdot 2^n u[n]$

$\frac{z}{(z-\gamma)^{m+1}} \Leftrightarrow \frac{n(n-1)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$

$$X[z] = \frac{-3z}{z-1} - \frac{2z}{(z-2)^3} - \frac{z}{(z-2)^2} + \frac{3z}{(z-2)}$$

$= \left[-3 - \frac{1}{4}(n^2-n)2^n - \frac{n}{2}2^n + 3\,2^n\right]u[n]$

$= \left[-3 + \left(3 - \frac{n^2+n}{4}\right)2^n\right]u[n]$

- We use the following properties:
  - $\gamma^n u[n] \Leftrightarrow \frac{z}{z-\gamma}$
  - $\frac{n(n-1)(n-2)\ldots(n-m+1)}{\gamma^m m!}\gamma^n u[n] \Leftrightarrow \frac{z}{(z-\gamma)^{m+1}}$

$$[-\frac{2z}{(z-2)^3} = (-2)\frac{z}{(z-2)^{2+1}} \Leftrightarrow (-2)\frac{n(n-1)}{2^2 2!}\gamma^n u[n] = -2\frac{n(n-1)}{8}\cdot 2^n u[n]$$

- Therefore,

$$x[n] = [-3\cdot 1^n - 2\frac{n(n-1)}{8}\cdot 2^n - \frac{n}{2}\cdot 2^n + 3\cdot 2^n]u[n]$$

$$= -\left[3 + \frac{1}{4}(n^2 + n - 12)2^n\right]u[n]$$

Imperial College London

$$\frac{X(z)}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{K}{z-1} + \frac{Az+B}{z^2-6z+25}$$

$X(z-1)\,.\, z=1$

$$K = \left.\frac{2(3z+17)}{z^2-6z+25}\right|_{z=1} = 2$$

# Find the inverse $z$ −transform in the case of complex poles

$X\,z\,,\ z\to\infty$

$$\frac{2z(3z+17)}{(z-1)(z^2-6z+25)} = 2\frac{z}{z-1} + \frac{Az^2+Bz}{z^2-6z+25}$$

$0 = 2 + A \Rightarrow A = -2$

- Find the inverse $z$ −transform of $X[z] = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$

$$\frac{X(z)}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

**Solution**

$z\to 0: -\frac{34}{25} = -2 + \frac{B}{25} \Rightarrow B = 16$

$$X[z] = \frac{2z(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

$$\frac{X[z]}{z} = \frac{(2z^2-11z+12)}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

Whenever we encounter a complex pole we need to use a special partial fraction method called **quadratic factors method**.

$$\frac{X[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{Az+B}{z^2-6z+25}$$

We multiply both sides with $z$ and let $z \to \infty$:

$$0 = 2 + A \Rightarrow A = -2$$

Therefore,

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

$X(z) = 2\frac{z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$ $\quad r|\gamma|^n\cos(\beta n+\theta)\,u[n] \longleftrightarrow \frac{z(Az+B)}{z^2+2az+|\gamma|^2}$

# Find the inverse $z$ −transform in the case of complex poles cont.

$\therefore \begin{cases} A=-2 \\ B=16 \\ a=-3 \\ |\gamma|=5 \end{cases}$ $\quad r = \sqrt{\frac{A^2|\gamma|^2+B^2-2AaB}{|\gamma|^2-a^2}} = 3.2$

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

$\beta = \cos^{-1}\frac{-a}{|\gamma|} = 0.927\ rad$

To find $B$ we let $z = 0$:

$\theta = \tan^{-1}\frac{Aa-B}{A\sqrt{|\gamma|^2-a^2}}$

$$\frac{-34}{25} = -2 + \frac{B}{25} \Rightarrow B = 16$$

$= 0.896\ rad$

$$\frac{X[z]}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2-6z+25} \Rightarrow X[z] = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$$

$\therefore x[n] = [2 + 3.2\times 5^n\cos(0.927n+0.896)]\,u[n]$

- We use the following property:

$r|\gamma|^n\cos(\beta n+\theta)\,u[n] \Leftrightarrow \frac{z(Az+B)}{z^2+2az+|\gamma|^2}$ with $A=-2$, $B=16$, $a=-3$, $|\gamma|=5$.

$$r = \sqrt{\frac{A^2|\gamma|^2+B^2-2AaB}{|\gamma|^2-a^2}} = \sqrt{\frac{4\cdot 25+256-2\cdot(-2)\cdot(-3)\cdot 16}{25-9}} = 3.2,\ \beta = \cos^{-1}\frac{-a}{|\gamma|} = 0.927rad,$$

$$\theta = \tan^{-1}\frac{Aa-B}{A\sqrt{|\gamma|^2-a^2}} = -2.246rad.$$

Therefore, $x[n] = [2 + 3.2\cos(0.927n - 2.246)]u[n]$

Imperial College London

# $z$ —transform Table

| No. | $x[n]$ | $X[z]$ |
|---|---|---|
| 1 | $\delta[n-n]$ | $z^{-k}$ |
| 2 | $u[n]$ | $\frac{z}{z-1}$ |
| 3 | $nu[n]$ | $\frac{z}{(z-1)^2}$ |
| 4 | $n^2u[n]$ | $\frac{z(z+1)}{(z-1)^3}$ |
| 5 | $n^3u[n]$ | $\frac{z(z^2+4z+1)}{(z-1)^4}$ |
| 6 | $\gamma^n u[n]$ | $\frac{z}{z-\gamma}$ |
| 7 | $\gamma^{n-1}u[n-1]$ | $\frac{1}{z-\gamma}$ |
| 8 | $n\gamma^n u[n]$ | $\frac{\gamma z}{(z-\gamma)^2}$ |

# $z$ —transform Table

| No. | $x[n]$ | $X[z]$ |
|---|---|---|
| 10 | $\dfrac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$ | $\dfrac{z}{(z-\gamma)^{m+1}}$ |
| 11a | $\lvert\gamma\rvert^n \cos \beta n\, u[n]$ | $\dfrac{z(z-\lvert\gamma\rvert \cos \beta)}{z^2-(2\lvert\gamma\rvert \cos \beta)z+\lvert\gamma\rvert^2}$ |
| 11b | $\lvert\gamma\rvert^n \sin \beta n\, u[n]$ | $\dfrac{z\lvert\gamma\rvert \sin \beta}{z^2-(2\lvert\gamma\rvert \cos \beta)z+\lvert\gamma\rvert^2}$ |
| 12a | $r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$ | $\dfrac{rz[z\cos \theta-\lvert\gamma\rvert \cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert \cos \beta)z+\lvert\gamma\rvert^2}$ |
| 12b | $r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n] \qquad \gamma=\lvert\gamma\rvert e^{j\beta}$ | $\dfrac{(0.5re^{j\theta})z}{z-\gamma}+\dfrac{(0.5re^{-j\theta})z}{z-\gamma^*}$ |
| 12c | $r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$ | $\dfrac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$ |

$$r=\sqrt{\frac{A^2\lvert\gamma\rvert^2+B^2-2AaB}{\lvert\gamma\rvert^2-a^2}} \qquad \beta=\cos^{-1}\frac{-a}{\lvert\gamma\rvert} \qquad \theta=\tan^{-1}\frac{Aa-B}{A\sqrt{\lvert\gamma\rvert^2-a^2}}$$