

THE QUESTIONS

1. [25]

- a) Consider two discrete random variables X and Y that have joint probability mass function represented by the joint probability table

		Y		
		0	1	2
X	0	0.05	0.05	0.15
	1	0.05	0.05	0.25
	2	0.15	0.20	0.05

- i) Compute the probability that X is smaller or equal to Y , i.e. $P(X \leq Y)$ and the probability that X is strictly smaller than Y , i.e. $P(X < Y)$. [2]
 - ii) Compute the marginal probability mass function of X and Y . [2]
 - iii) Compute the expectation of X , i.e. $E(X)$, and the expectation of Y , i.e. $E(Y)$. [2]
 - iv) Compute the variance of X and the variance of Y , i.e. $\text{Var}(X)$ and $\text{Var}(Y)$, the covariance between X and Y , i.e. $\text{Cov}(X, Y)$, and the correlation coefficient between X and Y , i.e. $\text{Corr}(X, Y)$. [4]
 - v) Are X and Y uncorrelated? independent? Provide your reasoning. [2]
 - vi) Compute the conditional probability mass function of X given that $Y = 0, 1, 2$. [3]
 - vii) Compute the conditional expectation of X given that $Y = 0, 1, 2$. [3]
 - viii) Relying on your result in vii), compute the expectation of X , i.e. $E(X)$. [2]
- b) By making use of the standard normal table, compute the following integral

$$\int_{-\infty}^{2.35} \sqrt{\frac{2}{\pi}} e^{-2(u-2)^2} du.$$

Provide your reasoning. [5]

- a) We consider a communication system where the transmitter is equipped with one antenna and the receiver with two antennas. The power of the signal received at antenna i is denoted as P_i , $i = 1, 2$, and is modeled as an exponentially distributed random variable with parameter $\lambda > 0$. The receiver only uses one antenna at a time and selects the antenna with the largest power. Hence, the power of the signal after selection is given by $P = \max_{i=1,2} P_i$. We assume the receive antennas are deployed such that P_1 and P_2 are independent.
- i) Find the probability that the power of the signal after selection, P , falls below a certain level S . Provide your reasoning. [4]
 - ii) Find the probability density function of P . Provide your reasoning. [4]
 - iii) We are interested in computing the error probability of this communication system. The error probability is the probability of wrongly decoding the transmitted signal and can be approximated as the moment generating function of P evaluated at the point $t = -d$ for $d > 0$. Making use of such approximation, find the error probability. Provide your reasoning. [4]
 - iv) From the results in iii), find the expected value of the received power after selection. Provide your reasoning. [4]
- b)
- i) Assume $X \sim N(\mu, \sigma^2)$. Find the moment generating function $m_X(t)$ of X . Provide your reasoning. [4]
 - ii) Consider the following statement: If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ and X_1, X_2 are independent random variables, we have $2X_1 - X_2 \sim N(2\mu_1 - \mu_2, 2\sigma_1^2 - \sigma_2^2)$. Is the statement correct? If yes, provide a proof. If not, correct the statement and provide a proof. [5]

Mathematical Formulae

1. Probabilities for events

For events A , B , and C

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

More generally $P(\cup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \dots$

The odds in favour of A

$$P(A) / P(\bar{A})$$

Conditional probability

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \quad \text{provided that } P(B) > 0$$

Chain rule

$$P(A \cap B \cap C) = P(A) P(B | A) P(C | A \cap B)$$

Bayes' rule

$$P(A | B) = \frac{P(A) P(B | A)}{P(A) P(B | A) + P(\bar{A}) P(B | \bar{A})}$$

A and B are independent if

$$P(B | A) = P(B)$$

A , B , and C are independent if

$$P(A \cap B \cap C) = P(A)P(B)P(C), \quad \text{and}$$

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable X is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of x , $E\{g(X)\} = \sum_x g(x) p_x$, so $E(X^2) = \sum_x x^2 p_x$

Sample mean $\bar{x} = \frac{1}{n} \sum_k x_k$ estimates μ from random sample x_1, x_2, \dots, x_n

Variance $\text{var}(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$

Sample variance $s^2 = \frac{1}{n-1} \left\{ \sum_k x_k^2 - \frac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ estimates σ^2

Standard deviation $\text{sd}(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\beta_1 = E\left(\frac{X - \mu}{\sigma}\right)^3$ is estimated by $\frac{1}{n-1} \sum \left(\frac{x_i - \bar{x}}{s}\right)^3$

Kurtosis $\beta_2 = E\left(\frac{X - \mu}{\sigma}\right)^4 - 3$ is estimated by $\frac{1}{n-1} \sum \left(\frac{x_i - \bar{x}}{s}\right)^4 - 3$

Sample median \tilde{x} or x_{med} . Half the sample values are smaller and half larger

If the sample values x_1, \dots, x_n are ordered as $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$,

then $\tilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\tilde{x} = \frac{1}{2}(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})})$ if n is even

α -quantile $Q(\alpha)$ is such that $P(X \leq Q(\alpha)) = \alpha$

Sample α -quantile $\hat{Q}(\alpha)$ Proportion α of the data values are smaller

Lower quartile $Q1 = \hat{Q}(0.25)$ one quarter are smaller

Upper quartile $Q3 = \hat{Q}(0.75)$ three quarters are smaller

Sample median $\tilde{x} = \hat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $F(x) = P(X \leq x) = \int_{x_0=-\infty}^x f(x_0)dx_0$

The probability density function (pdf) $f(x) = \frac{dF(x)}{dx}$

$E(X) = \mu = \int_{-\infty}^{\infty} x f(x)dx$, $\text{var}(X) = \sigma^2 = E(X^2) - \mu^2$, where $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$

4. Discrete probability distributions

Discrete Uniform $Uniform(n)$

$$p_x = \frac{1}{n} \quad (x = 1, 2, \dots, n) \quad \mu = (n+1)/2, \quad \sigma^2 = (n^2 - 1)/12$$

Binomial distribution $Binomial(n, \theta)$

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \dots, n) \quad \mu = n\theta, \quad \sigma^2 = n\theta(1-\theta)$$

Poisson distribution $Poisson(\lambda)$

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \dots) \quad (\text{with } \lambda > 0) \quad \mu = \lambda, \quad \sigma^2 = \lambda$$

Geometric distribution $Geometric(\theta)$

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \dots) \quad \mu = \frac{1}{\theta}, \quad \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution $Uniform(\alpha, \beta)$

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \quad \mu = (\alpha + \beta)/2, \quad \sigma^2 = (\beta - \alpha)^2/12$$

Exponential distribution $Exponential(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0). \end{cases} \quad \mu = 1/\lambda, \quad \sigma^2 = 1/\lambda^2$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \text{var}(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

$$\text{If } X \text{ is } N(\mu, \sigma^2), \text{ then } Y = \frac{X-\mu}{\sigma} \text{ is } N(0,1)$$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ ($t > 0$)

$$\text{The reliability function at time } t \quad R(t) = P(T > t)$$

$$\text{The failure rate or hazard function} \quad h(t) = f(t)/R(t)$$

$$\text{The cumulative hazard function} \quad H(t) = \int_0^t h(t_0) dt_0 = -\ln\{R(t)\}$$

$$\text{The Weibull}(\alpha, \beta) \text{ distribution has} \quad H(t) = \beta t^\alpha$$

7. System reliability

For a system of k devices, which operate independently, let

$$R_i = P(D_i) = P(\text{"device } i \text{ operates"})$$

The system reliability, R , is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \dots \cap D_k) = R_1 R_2 \dots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \dots \cup D_k) = 1 - (1 - R_1)(1 - R_2) \dots (1 - R_k)$$

8. Covariance and correlation

$$\text{The covariance of } X \text{ and } Y \quad \text{cov}(X, Y) = E(XY) - \{E(X)\}\{E(Y)\}$$

$$\text{From pairs of observations } (x_1, y_1), \dots, (x_n, y_n) \quad S_{xy} = \sum_k x_k y_k - \frac{1}{n} \left(\sum_i x_i \right) \left(\sum_j y_j \right)$$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n} \left(\sum_i x_i \right)^2, \quad S_{yy} = \sum_k y_k^2 - \frac{1}{n} \left(\sum_j y_j \right)^2$$

$$\text{Sample covariance} \quad s_{xy} = \frac{1}{n-1} S_{xy} \quad \text{estimates } \text{cov}(X, Y)$$

$$\text{Correlation coefficient} \quad \rho = \text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\text{sd}(X) \cdot \text{sd}(Y)}$$

$$\text{Sample correlation coefficient} \quad r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}} \quad \text{estimates } \rho$$

9. Sums of random variables

$$E(X + Y) = E(X) + E(Y)$$

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2 \text{cov}(X, Y)$$

$$\text{cov}(aX + bY, cX + dY) = (ac) \text{var}(X) + (bd) \text{var}(Y) + (ad + bc) \text{cov}(X, Y)$$

If X is $N(\mu_1, \sigma_1^2)$, Y is $N(\mu_2, \sigma_2^2)$, and $\text{cov}(X, Y) = c$, then $X + Y$ is $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

$$\text{Bias of } t \quad \text{bias}(t) = E(T) - \theta$$

$$\text{Standard error of } t \quad \text{se}(t) = \text{sd}(T)$$

$$\text{Mean square error of } t \quad \text{MSE}(t) = E\{(T - \theta)^2\} = \{\text{se}(t)\}^2 + \{\text{bias}(t)\}^2$$

If \bar{x} estimates μ , then $\text{bias}(\bar{x}) = 0$, $\text{se}(\bar{x}) = \sigma/\sqrt{n}$, $\text{MSE}(\bar{x}) = \sigma^2/n$, $\widehat{\text{se}}(\bar{x}) = s/\sqrt{n}$

Central limit property If n is fairly large, \bar{x} is from $N(\mu, \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ .

For a random sample x_1, x_2, \dots, x_n

$$\ell(\theta; x_1, x_2, \dots, x_n) = P(X_1 = x_1 | \theta) \cdots P(X_n = x_n | \theta) \quad (\text{discrete distribution})$$

$$\ell(\theta; x_1, x_2, \dots, x_n) = f(x_1 | \theta) f(x_2 | \theta) \cdots f(x_n | \theta) \quad (\text{continuous distribution})$$

The maximum likelihood estimator (MLE) is $\hat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If x_1, x_2, \dots, x_n are a random sample from $N(\mu, \sigma^2)$ and σ^2 is known, then

the 95% confidence interval for μ is $(\bar{x} - 1.96 \frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0 = t_{n-1, 0.05}$

The 95% confidence interval for μ is $(\bar{x} - t_0 \frac{s}{\sqrt{n}}, \bar{x} + t_0 \frac{s}{\sqrt{n}})$

13. Standard normal table Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	$p=$	0.10	0.05	0.02	0.01	m	$p=$	0.10	0.05	0.02	0.01
1		6.31	12.71	31.82	63.66	9		1.83	2.26	2.82	3.25
2		2.92	4.30	6.96	9.92	10		1.81	2.23	2.76	3.17
3		2.35	3.18	4.54	5.84	12		1.78	2.18	2.68	3.05
4		2.13	2.78	3.75	4.60	15		1.75	2.13	2.60	2.95
5		2.02	2.57	3.36	4.03	20		1.72	2.09	2.53	2.85
6		1.94	2.45	3.14	3.71	25		1.71	2.06	2.48	2.78
7		1.89	2.36	3.00	3.50	40		1.68	2.02	2.42	2.70
8		1.86	2.31	2.90	3.36	∞		1.645	1.96	2.326	2.576

15. Chi-squared table Values $\chi_{k,p}^2$ of x for which $P(X > x) = p$, when X is χ_k^2 and $p = .995, .975, etc$

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	18.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency \hat{n}_y for every group exceeds about 5.

$$X^2 = \sum_y \frac{(n_y - \hat{n}_y)^2}{\hat{n}_y} \quad \text{is referred to the table of } \chi_k^2 \text{ with significance point } p,$$

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching \bar{x} with μ

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x | Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

$$\text{Joint cdf} \quad F(x, y) = P(\{X \leq x\} \cap \{Y \leq y\}) = \int_{x_0=-\infty}^x \int_{y_0=-\infty}^y f(x_0, y_0) dx_0 dy_0$$

$$\text{Joint pdf} \quad f(x, y) = \frac{d^2 F(x, y)}{dx dy}$$

$$\text{Marginal pdf of } X \quad f_X(x) = \int_{-\infty}^{\infty} f(x, y_0) dy_0$$

$$\text{Conditional pdf of } X \text{ given } Y = y \quad f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} \quad (\text{provided } f_Y(y) > 0)$$

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\hat{y}_x = \hat{\alpha} + \hat{\beta}x$ from observations

$$(x_1, y_1), \dots, (x_n, y_n), \quad \text{the least squares fit is} \quad \hat{\alpha} = \bar{y} - \bar{x}\hat{\beta}, \quad \hat{\beta} = \frac{S_{xy}}{S_{xx}}$$

$$\text{The residual sum of squares} \quad \text{RSS} = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$$

$$\hat{\sigma}^2 = \frac{\text{RSS}}{n-2} \quad \frac{n-2}{\sigma^2} \hat{\sigma}^2 \text{ is from } \chi_{n-2}^2$$

$$E(\hat{\alpha}) = \alpha, \quad E(\hat{\beta}) = \beta,$$

$$\text{var}(\hat{\alpha}) = \frac{\sum x_i^2}{n S_{xx}} \sigma^2, \quad \text{var}(\hat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \text{cov}(\hat{\alpha}, \hat{\beta}) = -\frac{\bar{x}}{S_{xx}} \sigma^2$$

$$\hat{y}_x = \hat{\alpha} + \hat{\beta}x, \quad E(\hat{y}_x) = \alpha + \beta x, \quad \text{var}(\hat{y}_x) = \left\{ \frac{1}{n} + \frac{(x - \bar{x})^2}{S_{xx}} \right\} \sigma^2$$

$$\frac{\hat{\alpha} - \alpha}{\text{se}(\hat{\alpha})}, \quad \frac{\hat{\beta} - \beta}{\text{se}(\hat{\beta})}, \quad \frac{\hat{y}_x - \alpha - \beta x}{\text{se}(\hat{y}_x)} \quad \text{are each from } t_{n-2}$$