EXERCISE 2.1

(a) $P(z) = H_0(z)\,G_0(z)$

IF $H_0(z) = \left(z - 1 + z^{-1}\right)$

THEN $G_0(z) = \left(\frac{1}{2}z^{-2} + \frac{1}{2}z^{-1} + \frac{1}{2}z + \frac{1}{2}z^{2}\right)$

THE OTHER TWO FILTERS ARE:

$G_1(z) = z^{-1} H_0(-z)$ $\qquad$ $H_1(z) = z\,G_0(-z)$

(b) $P(z) = \frac{1}{2}\left(z - 1 + z^{-1}\right)\left(z - 1 + z^{-1}\right)\left(1 + z\right)\left(1 + z^{-1}\right)$

THEREFORE

$$G_0(z) = \frac{1}{\sqrt{2}}\left(1 + z^{-1}\right)\left(z - 1 + z^{-1}\right)$$

$H_0(z)$ MUST BE EQUAL TO $G_0(z^{-1})$

INDEED

$$H_0(z) = \frac{1}{\sqrt{2}}\left(1 + z\right)\left(z - 1 + z^{-1}\right)$$

FINALLY

$G_1(z) = -z^{-1} G_0(z^{-1})$ AND

$H_1(z) = G_1(z^{-1})$

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QUESTION 2.2

(a)

$$\hat{X}(z) = \frac{1}{3}\Big[G_0(z)\Big(X(z)H_0(z) + X(W_3^1 z)H_0(W_3^1 z) + X(W_3^2 z)H_0(W_3^2 z)\Big) +$$
$$G_1(z)\Big(X(z)H_1(z) + X(W_3^1 z)H_1(W_3^1 z) + X(W_3^2 z)H_1(W_3^2 z)\Big)$$
$$+ G_2(z)\Big(X(z)H_2(z) + X(W_3^1 z)H_2(W_3^1 z) + X(W_3^2 z)H_2(W_3^2 z)\Big)\Big].$$

THUS FOR PERFECT RECONSTRUCTION WE REQUIRE:

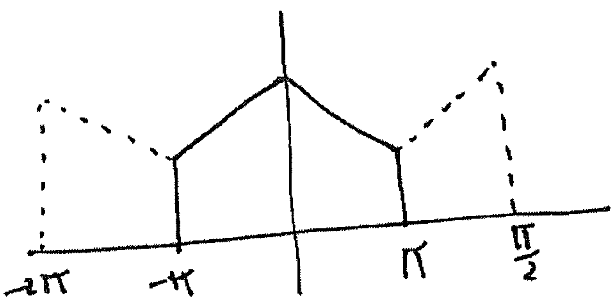
$$G_0(z)H_0(z) + G_1(z)H_1(z) + G_2(z)H_2(z) = 3$$

AND THE TWO FOLLOWING NO-ALIASING CONDITIONS:

$$\begin{cases} G_0(z)H_0(W_3^1 z) + G_1(z)H_1(W_3^1 z) + G_2(z)H_2(W_3^1 z) = 0 \\ G_0(z)H_0(W_3^2 z) + G_1(z)H_1(W_3^2 z) + G_2(z)H_2(W_3^2 z) = 0 \end{cases}$$

WHERE $W_N^k = e^{-j\frac{2\pi k}{N}}$

(b) $Y_0(w)$



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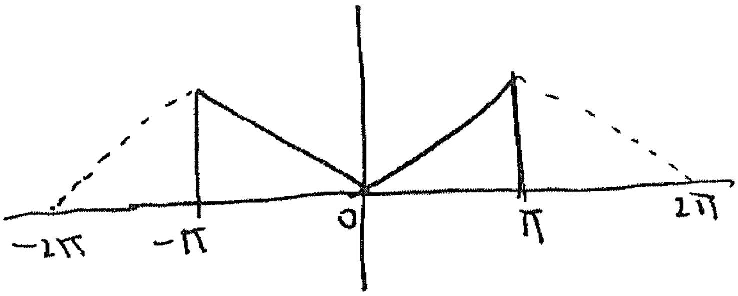
$Y_1(e^{jw})$

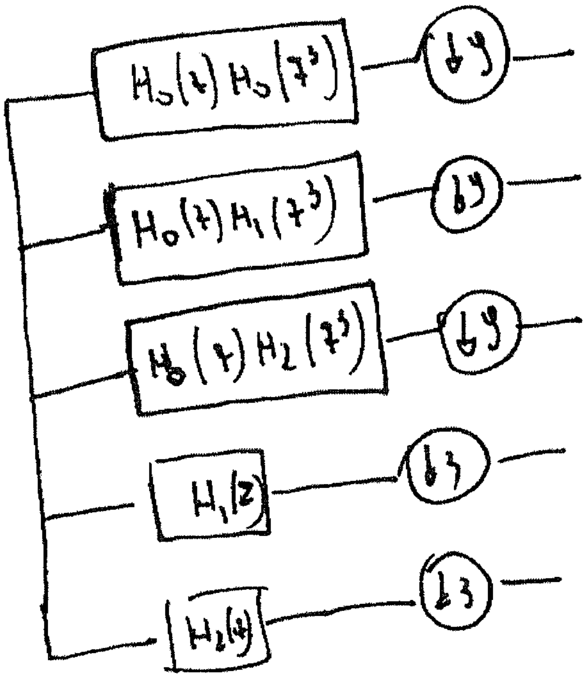


$Y_2(e^{jw}) = 0$

$\hat{X}(e^{jw}) = X(e^{jw})$

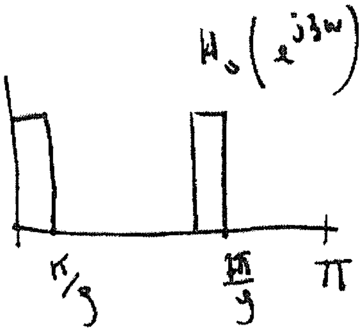
(c)

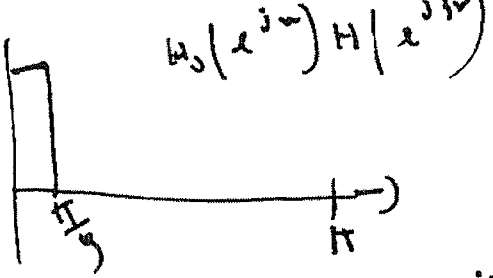
(i)

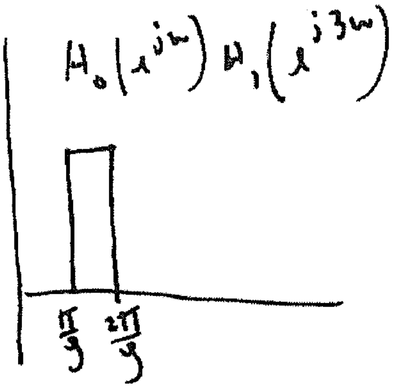


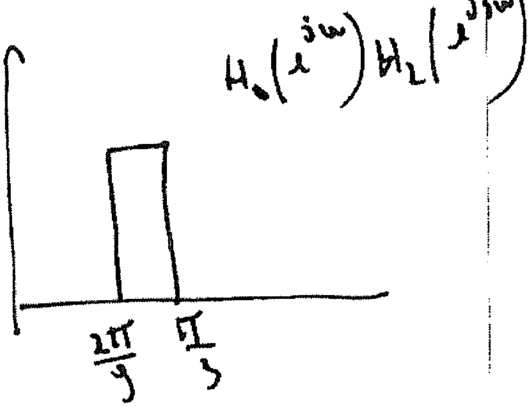
(ii)

(ii)


$H_0(e^{j3w})$
$\pi/9$
$\frac{5\pi}{9}$
π


$H_0(e^{jw})H(e^{j3w})$
$\pi/9$
π


$H_0(e^{jw})H_1(e^{j3w})$
$\frac{\pi}{9}$
$\frac{2\pi}{9}$


$H_0(e^{jw})H_2(e^{j3w})$
$\frac{2\pi}{9}$
$\frac{\pi}{3}$

QUESTION 2.3

(a) $G_0(z) = a + (a+b)z^{-1} + (a+b)z^{-2} + az^{-3}$;

THE CONDITION

$\langle g_0[n], g_0[n] \rangle = 1 \iff 2a^2 + 2(a+b)^2 = 1$ (1)

AND THE CONDITION

$\langle g_0[n], g_0[n-2] \rangle = 0 \iff 2a(a+b) = 0$ SINCE $a \neq 0$ AND $b \neq 0$ WE HAVE $a = -b$ AND FROM (1) WE GET

$$a = \frac{1}{\sqrt{2}}, \quad b = -\frac{1}{\sqrt{2}}$$

AND

$$G_0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} z^{-3}$$

(b)

$$G_1(z) = -z^{-1} G_0(-z^{-1}) = \frac{1}{\sqrt{2}} z^{2} - \frac{1}{\sqrt{2}} z^{-1}$$

$$H_0(z) = G_0(z^{-1}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} z^{3}$$

$$H_1(z) = G_1(z^{-1}) = \frac{1}{\sqrt{2}} z^{-2} - \frac{1}{\sqrt{2}} z$$

(c)

$$\hat{X}(z) = \{H_0(z) G_0(z) + H_1(z) G_1(z)\} X(z)$$

SINCE $\{H_0, G_0, H_1, G_1\}$ ARE ORTHOGONAL, IT FOLLOWS THAT $\hat{X}(z) = 2X(z)$.

(d) $\hat{X}(z) = X(z) \Rightarrow H_0(z) G_0(z) + H_1(z) G_1(z) = 1$.

IF $H_0(z)$ AND $G_0(z)$ ARE GIVEN, WE HAVE THAT $H_1(z) G_1(z) = 1 - H_0(z) G_0(z)$.

IN THE EXAMPLE

$$H_1(z) G_1(z) = 1 - z^{-1} - 4 - z = -z^{-1} - 3 - z =$$

$$= z^{-1}(z - a_0)(z - a_1) \quad \text{WITH} \quad \begin{cases} a_1 = \frac{-3+\sqrt{5}}{2} \\ a_0 = \frac{-3-\sqrt{5}}{2} \end{cases}$$

THUS POSSIBLE SOLUTIONS ARE

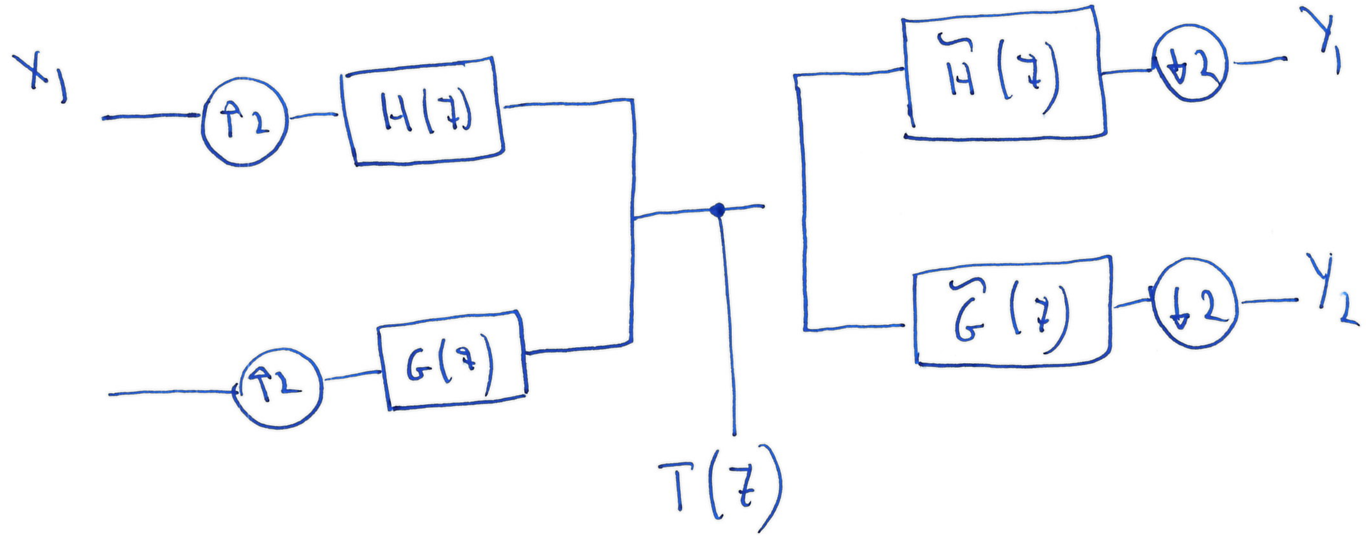
$H_1(z) = 1$, $G_1(z) = z^{-1}(z - a_0)(z - a_1)$

$H_1(z) = (z - a_0)$, $G_1(z) = (z - a_1) z^{-1}$

$H_1(z) = (z - a_1) z^{-1}$, $G_1(z) = (z - a_0)$

$H_1(z) = z^{-1}(z - a_0)(z - a_1)$, $G_1(z) = 1$

EXERCISE 2.4



(a)

$$T(z) = X_1(z^2)H(z) + X_2(z^2)G(z)$$

$$Y_1(z) = \frac{1}{2}\left[T(z^{1/2})\tilde{H}(z^{1/2}) + T(-z^{1/2})\tilde{H}(-z^{1/2})\right]$$

$$Y_2(z) = \frac{1}{2}\left[T(z^{1/2})\tilde{G}(z^{1/2}) + T(-z^{1/2})\tilde{G}(-z^{1/2})\right]$$

$$Y_1(z^2) = \frac{1}{2}\Big[X_1(z^2)H(z)\tilde{H}(z) + X_2(z^2)G(z)\tilde{H}(z) + X_1(z^2)H(-z)\tilde{H}(-z) + X_2(z^2)G(-z)\tilde{H}(-z)\Big]$$

$$Y_2(z^2) = \frac{1}{2}\Big[X_1(z^2)H(z)\tilde{G}(z) + X_2(z^2)G(z)\tilde{G}(z) + X_1(z^2)H(-z)\tilde{G}(-z) + X_2(z^2)G(-z)\tilde{G}(-z)\Big]$$

PR : $Y_1(z) = X_1(z)$ & $Y_2(z) = X_2(z)$

$$\Longrightarrow \begin{cases} H(z)\tilde{H}(z) + H(-z)\tilde{H}(-z) = 2 \\ G(z)\tilde{G}(z) + G(-z)\tilde{G}(-z) = 2 \end{cases}$$

NO CROSS-TALK
$$\begin{cases} G(z)\tilde{H}(z) + G(-z)\tilde{H}(-z) = 0 \\ \tilde{G}(z)H(z) + \tilde{G}(-z)H(-z) = 0 \end{cases}$$

(b) AS TRANSMULTIPLEXER IS STRUCTURALLY EQUIVALENT TO A 2-CHANNEL FILTER BANK, WE HAVE THAT PR CONDITIONS ARE SATISFIED WHEN

$$G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 2$$

$$\tilde{G}(z) = G(z^{-1})$$

$$H(z) = -z^{-1}G(-z^{-1})$$

$$\tilde{H}(z) = H(z^{-1})$$

QUESTION 2.5

a) $$C(z) = \frac{1}{2}\left[H\left(z^{1/2}\right)X\left(z^{1/2}\right) + H\left(-z^{1/2}\right)X\left(-z^{1/2}\right)\right]$$

$$Y(z) = \frac{1}{2}\,G(z)\left[H(z)X(z) + H(-z)\,X(-z)\right]$$

$$D(z) = X(z) - Y(z)$$

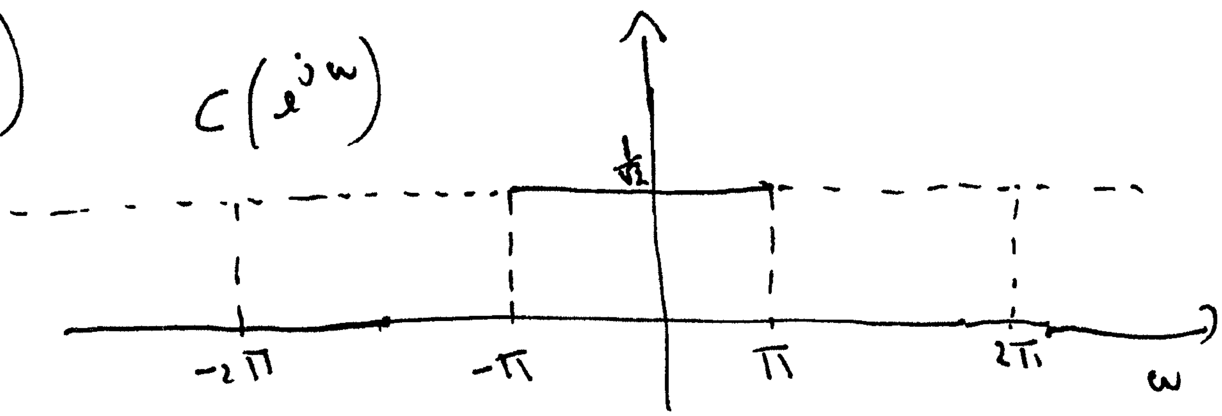
$$\hat{X}(z) = X(z) - \frac{1}{2}G(z)\left[H(z)X(z) + H(-z)\,X(-z)\right] + \frac{1}{2}\,F(z)\left[H(z)X(z) + H(-z)\,X(-z)\right]$$

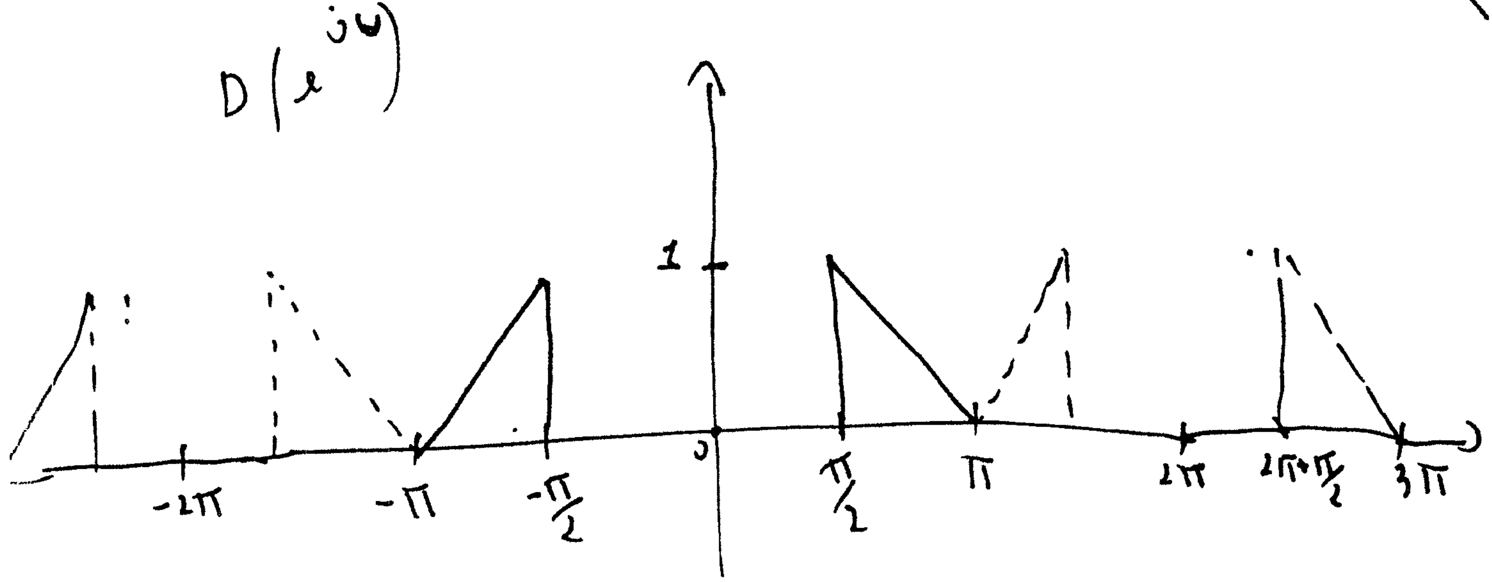
$$\Downarrow$$

PR: $\quad F(z) = G(z)$

NOTICE THAT PR CONDITION DOES NOT DEPEND ON $H(z)$.

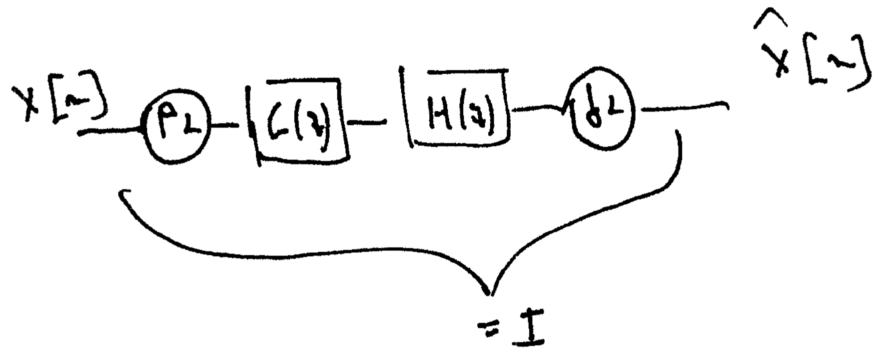
b) $C\left(e^{j\omega}\right)$





c) THE SYSTEM IS NOT IDEMPOTENT : $P^2 \neq P$

IN FACT THE SYSTEM IS IDEMPOTENT IF



THAT IS $\hat{x}[n] = x[n]$

CHECK:

$$\hat{X}(z) = \downarrow H(z)G(z)X(z^2)\Big|_{\downarrow 2} =$$

$$= \frac{X(z)}{2}\left(H(z^{1/2})G(z^{1/2}) + H(-z^{1/2})G(-z^{1/2})\right)$$

$$= \frac{X(z)}{2}\underbrace{\left(z^{-1} + 6 + z\right)}_{\neq I}$$

d) THE SYSTEM IS IDEMPOTENT IF AND ONLY IF

$$H(z)\cdot G(z) + H(-z)\,G(-z) = 2 \qquad (1)$$

$$H(z) = (z+1)\left(z^{-1}+1\right)\left(a + bz + bz^{-1}\right)$$

THE COEFFICIENTS a AND b MUST BE SUCH THAT CONDITION (1) IS SATISFIED.

THUS $a = \frac{1}{4}$ AND $b = -\frac{1}{16}$.

NOTICE THAT THE UPPER-BRANCH IS NOW PERFORMING AN OBLIQUE PROJECTION WHICH IS GOOD NEWS. HOWEVER, THE PROJECTION IS NOT OPTIMAL IN THAT IT IS NOT ORTHOGONAL.

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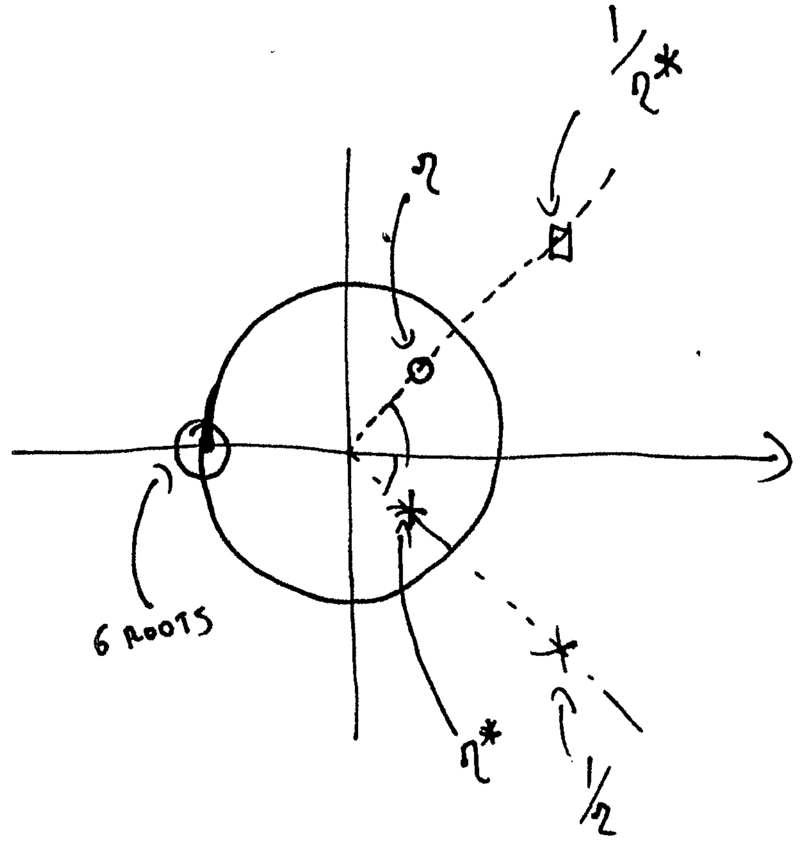
QUESTION 2.6

a)



b) $$P(z) = (1+z)^3 (1+z^{-1})^3 (z-\rho)(z-\rho^{-1})(z-\rho^*)\left(z-\frac{1}{\rho^*}\right)$$

$$G_0(z) = (1+z^{-1})^3 (z-\rho)(z-\rho^*)$$

$$H_0(z) = (1+z)^3 \left(z-\frac{1}{\rho}\right)\left(z-\frac{1}{\rho^*}\right)$$

$$G_1(z) = -z^{-1} G_0(-z^{-1})$$

$$H_1(z) = G_1(z^{-1})$$

c) $$H_1(z) = -z(1-z^{-1})^3 (z+\rho)(z+\rho^*)$$

THEREFORE $H_1(z)$ HAS A ZERO OF ORDER 3 AT $w=0$:

$$H_1(e^{jw})\Big|_{w=0}=0 \qquad H_1^{(1)}(e^{jw})\Big|_{w=0}=0 \qquad H_1^{(11)}(e^{jw})\Big|_{w=0}=0$$

NOW

$$\sum_{k}(n-k)^2 h_1[k] = n^2\sum_{k} h_1[k] - 2n\sum_{k} k\, h_1[k] + \sum_{k} k^2 h_1[k]$$

$$= 0$$ ~~SINCE~~

SINCE $$\sum_{k} h_1[k] = H_1(e^{jw})\Big|_{w=0} = 0$$

$$\sum_{k} k\, h_1[k] = j\frac{dH_1}{dw}\Big|_{w=0} = 0$$

$$\sum_{k} k^2 h_1[k] = -\frac{d^2 H_1(e^{jw})}{dw}\Big|_{w=0} = 0$$

d) POSSIBLE FACTORIZATION:

$$G_0(z) = (1+z)^2(1+z^{-1})^2$$

$$H_0(z) = (1+z)(1+z^{-1})\Phi(z)$$