EE4-65/EE9-SO27 Wireless Communications Problem Sheets - Solutions

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Note: For references to equations and figures, please refer to the MIMO Wireless Networks reference book.

On a sheet of paper, ...

1. Verify the expression (1.22) and the high SNR approximation (1.23).

   Answer: We write
$$\bar{P} = \int_0^\infty \mathcal{Q}(\sqrt{2\rho}s) p_s(s)\, ds$$
   where
$$p_s(s) = \frac{s}{\sigma^2} \exp\left(-\frac{s^2}{2\sigma^2}\right),$$
   and
$$\mathcal{Q}(x) \triangleq P(y \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy.$$
   such that (with the normalization $2\sigma^2 = 1$)
$$\begin{aligned}
\bar{P} &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_{\sqrt{2\rho}s}^\infty \exp\left(-\frac{y^2}{2}\right) 2s \exp\left(-s^2\right)\, dy\, ds \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^{y/\sqrt{2\rho}} \exp\left(-\frac{y^2}{2}\right) 2s \exp\left(-s^2\right)\, ds\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{y^2}{2}\right) \left[-\exp\left(-s^2\right)\right]_0^{y/\sqrt{2\rho}}\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{y^2}{2}\right) \left[1 - \exp\left(-\frac{y^2}{2\rho}\right)\right] dy \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{y^2}{2}\right) dy - \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-y^2\left[\frac{1}{2} + \frac{1}{2\rho}\right]\right) dy.
\end{aligned}$$
   Since
$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{y^2}{2\eta^2}\right) dy = \frac{\eta}{2},$$

for any variance η^2, we further get

$$\begin{aligned}
\bar{P} &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{y^2}{2}\right) dy - \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-y^2 \left[\frac{1}{2} + \frac{1}{2\rho}\right]\right) dy \\
&= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\rho}{1+\rho}}.
\end{aligned}$$

Taking the series expansion, we can write

$$\sqrt{\frac{\rho}{1+\rho}} = \left(\frac{1}{1+x}\right)^{1/2} = (1+x)^{-1/2} = 1 - \frac{1}{2}x + \ldots = 1 - \frac{1}{2\rho} + \ldots$$

where $x = 1/\rho$. At high SNR,

$$\sqrt{\frac{\rho}{1+\rho}} \approx 1 - \frac{1}{2\rho}$$

and

$$\bar{P} = \frac{1}{2}\left(1 - \sqrt{\frac{\rho}{1+\rho}}\right) \overset{\rho\nearrow}{\approx} \frac{1}{4\rho}.$$

□

2. Verify the expression (1.47) and the high SNR approximation (1.48).

   Answer: See derivations in the book/lecture notes and appendix B for the MGF of a Chi-square. Make the assumption that the channel has been normalized such that $2\sigma^2 = 1$. Generalization of the approach will be used later on to derive error probability over MIMO channels. □

3. Assuming the channel is normalized such that $\mathcal{E}\{\|\mathbf{h}\|^2\} = 1$, derive an estimate of the error probability of MRC over i.i.d. Rayleigh fading channels. Take the limit for a very large number of receive antennas. What do you get? Interpret the result.

   Hint: Recall that (1.47) was found assuming $2\sigma^2 = 1$, i.e. $\mathcal{E}\{\|\mathbf{h}\|^2\} = n_r$.

   Answer: We can use the same derivations as for (1.47) but note that the MGF will be different because $2\sigma^2 = 1/n_r$ due to the normalization $\mathcal{E}\{\|\mathbf{h}\|^2\} = 1$. We get

$$\bar{P} \leq \bar{N}_e \left(\frac{1}{1 + \frac{\rho d_{min}^2}{4n_r}}\right)^{n_r}.$$

Since

$$\exp(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n,$$

we have

$$\exp(-x) = \lim_{n\to\infty}\left(\frac{1}{1+\frac{x}{n}}\right)^n.$$

Hence,

$$\lim_{n_r\to\infty} \bar{P} \approx \lim_{n_r\to\infty} \bar{N}_e \left(\frac{1}{1 + \frac{\rho d_{min}^2}{4n_r}}\right)^{n_r} \approx \bar{N}_e \exp\left(-\frac{\rho d_{min}^2}{4}\right).$$

The error probability decreases exponentially with the SNR, as on an AWGN channel. Indeed, with a large number of antennas, the diversity gain is very large and the faded channel behaves like an AWGN channel due to channel hardening. This can also be viewed by simply making use of the law of large number and write

$$\mathcal{Q}\left(d_{min}\sqrt{\frac{\rho u}{2}}\right) = \mathcal{Q}\left(d_{min}\sqrt{\frac{\rho n_r u}{2n_r}}\right),$$

$$\lim_{n_r\to\infty} \mathcal{Q}\left(d_{min}\sqrt{\frac{\rho n_r}{2}\frac{u}{n_r}}\right) = \mathcal{Q}\left(d_{min}\sqrt{\frac{\rho}{2}}\right) \approx \exp\left(-\frac{\rho d_{min}^2}{4}\right)$$

since $\lim_{n_r\to\infty}\frac{u}{n_r} = \mathcal{E}\left\{|h|^2\right\} = 1/n_r$ (recall that we assume iid Rayleigh fading with $\mathcal{E}\left\{\|\mathbf{h}\|^2\right\} = 1$). □

4. Assume a transmission of a signal c from a single antenna transmitter to a multi-antenna receiver through a SIMO channel $\mathbf{h}$. The transmission is subject to the interference from another transmitter sending signal x through the interfering SIMO channel $\mathbf{h}_i$. The received signal model writes as

$$\mathbf{y} = \mathbf{h}c + \mathbf{h}_i x + \mathbf{n}$$

where $\mathbf{n}$ is the zero mean complex additive white Gaussian noise (AWGN) vector with $\mathcal{E}\{\mathbf{n}\mathbf{n}^H\} = \sigma_n^2\mathbf{I}_{n_r}$.

We apply a combiner $\mathbf{g}$ at the receiver to obtain the observation $z = \mathbf{g}\mathbf{y}$. Derive the expression of the MMSE combiner and the SINR at the output of the combiner.

Answer: The MMSE combiner $\mathbf{g}$ is given by

$$\mathbf{g} = \mathbf{h}^H\mathbf{R}_{\mathbf{n}_i}^{-1}$$

where $\mathbf{R}_{\mathbf{n}_i} = \mathcal{E}\left\{\mathbf{n}_i\mathbf{n}_i^H\right\}$ with $\mathbf{n}_i = \mathbf{h}_i x + \mathbf{n}$. Hence $\mathbf{R}_{\mathbf{n}_i} = \mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2\mathbf{I}_{n_r}$ with $P_x = \mathcal{E}\left\{\mathbf{x}\mathbf{x}^*\right\} = \mathcal{E}\left\{|\mathbf{x}|^2\right\}$, the power of the interfering signal. Hence,

$$\mathbf{g} = \mathbf{h}^H\left(\mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2\mathbf{I}_{n_r}\right)^{-1}.$$

At the receiver, we obtain

$$z = \mathbf{g}\mathbf{y} = \mathbf{h}^H\mathbf{R}_{\mathbf{n}_i}^{-1}\mathbf{h}c + \mathbf{h}^H\mathbf{R}_{\mathbf{n}_i}^{-1}\mathbf{n}_i.$$

The output SINR writes

$$
\begin{aligned}
\rho_{out} &= \frac{\left|\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{h}\right|^2 P_c}{\mathcal{E}\left\{\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{n}_i \left(\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{n}_i\right)^H\right\}} \\
&= \frac{\left|\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{h}\right|^2 P_c}{\mathcal{E}\left\{\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{n}_i \mathbf{n}_i^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{h}\right\}} \\
&= \frac{\left|\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{h}\right|^2 P_c}{\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{h}} \\
&= \mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{h} P_c \\
&= P_c \mathbf{h}^H \left(\mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2 \mathbf{I}_{n_r}\right)^{-1} \mathbf{h} \\
&= \mathrm{SNR}\, \mathbf{h}^H \left(\mathrm{INR}\, \mathbf{h}_i \mathbf{h}_i^H + \mathbf{I}_{n_r}\right)^{-1} \mathbf{h}
\end{aligned}
$$

with $P_c = \mathcal{E}\{\, |c|^2 \,\}$, $\mathrm{SNR} = P_c/\sigma_n^2$ (the average SNR), $\mathrm{INR} = P_x/\sigma_n^2$ (the average INR - Interference to Noise Ratio). □

5. Verify the high SNR assumption (1.56).

   Answer: At the receiver, the signal reads as

   $$y = \sqrt{E_s}\mathbf{h}\mathbf{c}' + n = \sqrt{E_s}\mathbf{h}\mathbf{w}c + n,$$

   where $\mathbf{w} = \frac{\mathbf{h}^H}{\|\mathbf{h}\|}$. The SNR varies as $E_s \|\mathbf{h}\|^2 / \sigma_n^2 = \rho \|\mathbf{h}\|^2$ and the rest of the derivation is the same as that of (1.47), except that n_r is now replaced by n_t. □

6. Verify the high SNR assumption (1.62). Explain the difference with (1.56).

   Answer: The SNR varies as $E_s \|\mathbf{h}\|^2 / 2\sigma_n^2 = \frac{\rho}{2} \|\mathbf{h}\|^2$ due to half of the transmit power allocated to each symbol c_1 and c_2. Comparing with (1.56), a loss of 3dB is experienced because of the lack of CSI at the transmitter. Intuitively, the symbol cannot be beamformed in a specific direction because the transmitter does not have knowledge of the channel. Power is therefore spread isotropically. □

7. A system is made of one single-antenna transmitter and K single-antenna receivers. The transmitter decides to send data in a TDMA manner, i.e. one user a a time. The user selected to receive data is the one with the largest instantaneous channel magnitude. Assuming all K users experience independent and identically distributed Rayleigh fading channels, derive the expression of the average SNR after user selection. How does this average SNR change as a function of K?

   Hint: View a user as an antenna ...

   Answer: View a user as an antennas and the user selection as a receive diversity with selection combining. The average SNR at the output of the combiner (user selection) $\bar{\rho}_{out}$ is eventually given by

   $$\bar{\rho}_{out} = \int_0^\infty \rho s^2 p_{s_{max}}(s)\, ds = \rho \sum_{n=1}^{K} \frac{1}{n} \overset{K\nearrow}{\approx} \rho \left[\gamma + \log(K) + \frac{1}{2K}\right].$$

The average SNR increases with $\log(K)$. Hence by dynamically selecting the best user based on the channel magnitude, the average SNR is increased. This forms the basis of the multiuser diversity to be discussed in multiuser communications. □

8. Explain what happens if Alamouti is applied to a MISO channel that is varying over two consecutive symbol durations?

   Answer: The decoupling property of Alamouti does not hold, i.e. by applying a matched filter on the effective channel, the two transmitted symbols are not decoupled (i.e. still subject to interference from other symbol) and each symbol cannot be simply detected by using conventional SISO detector. □

9. Show that the optimum (in the sense of SNR maximization) transmit precoder and combiner in dominant eigenmode transmission is given by the dominant right and left singular vector of the channel matrix, respectively.

   Answer: Dominant eigenmode transmission is the extension of the matched beamforming (or maximum ratio transmission) seen for MISO channels to the MIMO channels. Let us write

$$\begin{aligned} \mathbf{y} &= \sqrt{E_s}\mathbf{H}\mathbf{c}' + \mathbf{n} = \sqrt{E_s}\mathbf{H}\mathbf{w}c + \mathbf{n}, \\ z &= \mathbf{g}\mathbf{y} = \sqrt{E_s}\mathbf{g}\mathbf{H}\mathbf{w}c + \mathbf{g}\mathbf{n}. \end{aligned}$$

   where $\|\mathbf{w}\|^2 = 1$ (power constraint). We decompose

$$\begin{aligned} \mathbf{H} &= \mathbf{U_H}\mathbf{\Sigma_H}\mathbf{V_H}^H, \\ \mathbf{\Sigma_H} &= \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_{r(\mathbf{H})}\}. \end{aligned}$$

   In order to maximize the SNR, we choose $\mathbf{g}$ as a matched filter, i.e. $\mathbf{g} = (\mathbf{H}\mathbf{w})^H$ such that

$$\begin{aligned} \mathbf{g}\mathbf{H}\mathbf{w} &= \mathbf{w}^H\mathbf{H}^H\mathbf{H}\mathbf{w} \\ &= \mathbf{w}^H\mathbf{V_H}\mathbf{\Sigma}_\mathbf{H}^2\mathbf{V_H}^H\mathbf{w} \\ &= \sum_{i=1}^{r(\mathbf{H})} \sigma_i^2 \left|\mathbf{v}_i^H\mathbf{w}\right|^2 \\ &\leq \sigma_{max}^2 \end{aligned}$$

   where $\mathbf{v}_i$ is the i column of $\mathbf{V_H}$ and $\sigma_{max} = \max_{i=1,\ldots,r(\mathbf{H})} \sigma_i$. The inequality is replaced by an equality if $\mathbf{w} = \mathbf{v}_{max}$. By choosing $\mathbf{w} = \mathbf{v}_{max}$,

$$\begin{aligned} \mathbf{g} = \mathbf{w}^H\mathbf{H}^H &= \mathbf{v}_{max}^H\mathbf{V_H}\mathbf{\Sigma_H}\mathbf{U_H}^H \\ &= \sigma_{max}\mathbf{u}_{max}^H \end{aligned}$$

   where $\mathbf{u}_{max}$ is the column of $\mathbf{U_H}$ corresponding to the dominant singular value σ_{max} of $\mathbf{H}$. If we normalize $\mathbf{g}$ such that $\|\mathbf{g}\|^2 = 1$, we can write $\mathbf{g} = \mathbf{u}_{max}$. □

10. Verify (1.80) and the high SNR assumption (1.81). What is the difference with the output SNR and error rate achieved in a SIMO channel with 4 receive antennas?

Answer: With Alamouti, the received signal vector writes as

$$\mathbf{y}_1 = \sqrt{E_s}\mathbf{H}\begin{bmatrix} c_1/\sqrt{2} \\ c_2/\sqrt{2} \end{bmatrix} + \mathbf{n}_1, \qquad \text{(first symbol period)}$$
$$\mathbf{y}_2 = \sqrt{E_s}\mathbf{H}\begin{bmatrix} -c_2^*/\sqrt{2} \\ c_1^*/\sqrt{2} \end{bmatrix} + \mathbf{n}_2. \qquad \text{(second symbol period)}.$$

Applying the matched filter $\mathbf{H}_{eff}^H$ to $\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2^* \end{bmatrix}^T$ and noting that $\mathbf{H}_{eff}^H\mathbf{H}_{eff} = \|\mathbf{H}\|_F^2\,\mathbf{I}_2$, we get

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \sqrt{E_s}\mathbf{H}_{eff}^H\mathbf{y} = \|\mathbf{H}\|_F^2\,\mathbf{I}_2\,\mathbf{c} + \mathbf{n}'$$

where $\mathbf{n}'$ is such that $\mathcal{E}\{\mathbf{n}'\} = \mathbf{0}_{2\times 1}$ and $\mathcal{E}\{\mathbf{n}'\mathbf{n}'^H\} = \|\mathbf{H}\|_F^2\;\sigma_n^2\mathbf{I}_2$. The above equation illustrates that the transmission of c_1 and c_2 is fully decoupled, i.e.,

$$z_k = \sqrt{E_s/2}\;\|\mathbf{H}\|_F^2\,c_k + \tilde{n}_k \quad k = 1, 2\ .$$

Hence the output SNR for each transmission is given by

$$\rho_{out} = \frac{E_s/2\left[\,\|\mathbf{H}\|_F^2\,\right]^2}{\mathcal{E}\left\{|\tilde{n}_k|^2\right\}} = \frac{E_s/2\left[\,\|\mathbf{H}\|_F^2\,\right]^2}{\|\mathbf{H}\|_F^2\,\sigma_n^2} = \frac{E_s\,\|\mathbf{H}\|_F^2}{2\sigma_n^2}.$$

Assuming the channel normalization $\mathcal{E}\{\,\|\mathbf{H}\|_F^2\,\} = 4$, $\bar{\rho}_{out} = \mathcal{E}\{\rho_{out}\} = 2\frac{E_s}{\sigma_n^2} = 2\rho$. A factor 2 is obtained compared to Alamouti over MISO channels thanks to the presence of 2 receive antennas, leading to an array gain of 3dB.

(1.81) follows from the same derivations as for (1.47), namely upper bounding Q by the Chernoff bound and then use the MGF of a Chi-square random variables with 8 degrees of freedom.

With a SIMO with 4 receive antennas ($\mathbf{h}$ is 4×1 vector) applying maximum ratio combining, we would get

$$\rho_{out} = \frac{E_s\,\|\mathbf{h}\|^2}{\sigma_n^2}$$
$$\bar{\rho}_{out} = \mathcal{E}\{\rho_{out}\} = 4\frac{E_s}{\sigma_n^2} = 4\rho$$

and

$$\bar{P} \leq \bar{N}_e\left(\frac{\rho d_{min}^2}{4}\right)^{-4}.$$

Hence the diversity gain is the same in both cases (= 4), but the array gain is twice as large in SIMO with 4 receive antennas compared to the one achieved by Alamouti with 2×2 MIMO. $\square$

11. Assume a MISO system with two transmit antennas. The channel gains are identically distributed circularly symmetric complex Gaussian (with

real and imarginary parts having a variance 1/2) but can be correlated and are denoted as h_1 and h_2. Write the expression of the transmit correlation matrix $\mathbf{R}_t$ and derive the eigenvalues and eigenvectors of $\mathbf{R}_t$ as a fonction of the transmit correlation coefficient t.

Answer: We write

$$\begin{aligned}
\mathbf{R}_t &= \mathcal{E}\left\{\left[\begin{array}{c} h_1^* \\ h_2^* \end{array}\right]\left[\begin{array}{cc} h_1 & h_2 \end{array}\right]\right\} \\
&= \left[\begin{array}{cc} \mathcal{E}\left\{|h_1|^2\right\} & \mathcal{E}\left\{h_1^* h_2\right\} \\ \mathcal{E}\left\{h_1 h_2^*\right\} & \mathcal{E}\left\{|h_2|^2\right\} \end{array}\right] \\
&= \left[\begin{array}{cc} 1 & t^* \\ t & 1 \end{array}\right]
\end{aligned}$$

where $t = \mathcal{E}\left\{h_1 h_2^*\right\}$ is the transmit correlation coefficient. The SVD leads to

$$\mathbf{R}_t = \left[\begin{array}{cc} 1 & 1 \\ t/|t| & -t/|t| \end{array}\right]\left[\begin{array}{cc} 1+|t| & 0 \\ 0 & 1-|t| \end{array}\right]\left[\begin{array}{cc} 1 & 1 \\ t/|t| & -t/|t| \end{array}\right]^H.$$

The eigenvalues are only function of the magnitude of t while the eigenvectors are only function of the phase of t.

□

12. Assume a dominant eigenmode transmission in a MIMO channel $\mathbf{H} = \mathbf{H}_w \mathbf{R}_t^{1/2}$ with $\mathbf{H}_w$ a random fading matrix made of i.i.d. circularly symmetric complex Gaussian entries (with real and imaginary parts having a variance 1/2) and $\mathbf{R}_t$ the transmit correlation matrix. Considering two transmit antennas, discuss how the array gain behaves as a function of the number of receive antennas n_r and the transmit correlation coefficient. For simplicity, look at the case where $n_r = 1$ and $n_r \to \infty$.

   Answer: Let us look at $n_r = 1$ and $n_r \to \infty$.

   For $n_r = 1$, there is a single receive antenna. Hence we simply write $\mathbf{h} = \mathbf{h}_w \mathbf{R}_t^{1/2}$. The dominant eigenmode transmission boils down to matched beamforming (or maximum ratio transmission/transmit MRC) and $\mathbf{w} = \mathbf{h}^H / \|\mathbf{h}\|$. We get

$$\begin{aligned}
\rho_{out} &= \frac{E_s\,|\mathbf{h}\mathbf{w}|^2}{\sigma_n^2} = \rho\,\|\mathbf{h}\|^2 \\
\bar{\rho}_{out} &= \rho n_t
\end{aligned}$$

   This is valid whatever the spatial correlation.

For $n_r \to \infty$, assuming a matched filter $\mathbf{g} = (\mathbf{H}\mathbf{w})^H$, we write

$$\begin{aligned}
\rho_{out} &= \rho \left\|\mathbf{H}\mathbf{w}\right\|^2 \\
&= \rho \mathbf{w}^H \mathbf{H}^H \mathbf{H}\mathbf{w} \\
&= \rho \mathbf{w}^H \mathbf{R}_t^{H/2} \mathbf{H}_w^H \mathbf{H}_w \mathbf{R}_t^{1/2} \mathbf{w} \\
&= \rho n_r \mathbf{w}^H \mathbf{R}_t^{H/2} \frac{\mathbf{H}_w^H \mathbf{H}_w}{n_r} \mathbf{R}_t^{1/2} \mathbf{w} \\
&\approx \rho n_r \mathbf{w}^H \mathbf{R}_t \mathbf{w} \\
&\leq \rho n_r (1 + |t|)
\end{aligned}$$

where $(1 + |t|)$ is the dominant eigenvalue of $\mathbf{R}_t = \begin{bmatrix} 1 & t^* \\ t & 1 \end{bmatrix}$. Equality is achieved if $\mathbf{w}$ is chosen as the dominant eigenvector of $\mathbf{R}_t$, i.e. $\mathbf{w} = \begin{bmatrix} 1 & t/|t| \end{bmatrix}^T$. As we can see the array gain increases with the spatial correlation. The larger $|t|$, the larger the array gain. This is intuitive as a larger t suggests a more directive propagation environment.

□

13. Is the rate achievable in a MIMO channel with multiple eigenmode transmission and uniform power allocation across modes always larger than that achievable with dominant eigenmode transmission?

Answer: No! The achievable rate with multiple eigenmode transmission in the MIMO channel is the sum of the SISO channel achievable rates

$$R = \sum_{k=1}^{r(\mathbf{H})} \log_2(1 + \rho s_k \sigma_k^2),$$

where $\{s_1, \ldots, s_{r(\mathbf{H})}\}$ is the power allocation on each of the channel eigenmodes. For a total power constraint $\sum_{k=1}^{r(\mathbf{H})} s_k = 1$, uniform power allocation consists in choosing $s_k = 1/r(\mathbf{H})$. For a uniform power allocation, the total achievable rate writes as

$$R_u = \sum_{k=1}^{r(\mathbf{H})} \log_2(1 + \rho 1/r(\mathbf{H})\sigma_k^2),$$

while with the dominant eigenmode transmission, we get

$$R_d = \log_2(1 + \rho \sigma_{max}^2),$$

where the whole transmit power is allocated to the dominant mode, i.e. along the singular vector of $\mathbf{H}$ corresponding to the largest singular value σ_{max}. It is clear that depending on the channel matrix, R_u could be either greater or smaller than R_d. For instance, if $\sigma_1 >> 0$ and $\sigma_k \approx \epsilon$ for $k > 1$, i.e. one large singular value and the other ones are small, the power allocated to σ_k, $k > 1$, will not contribute much to the rate at low SNR. Indeed we would have $R_u \approx \log_2(1 + \rho\sigma_1^2/r(\mathbf{H})) \leq R_d$ for small values of ρ. At very high SNR, on the other hand, despite the little contributions of $\sigma_k \approx \epsilon$, R_u will become higher than R_d.

Generally speaking, uniform power allocation across all modes is suitable at high SNR while dominant eigenmode transmission is suitable at low SNR.

□

14. Using Jensen inequality, upper bound the ergodic capacity of Rayleigh fast fading MIMO channels with transmit correlation only ($\mathbf{R}_r = \mathbf{I}_{n_r}$) and derive a power allocation strategy that maximizes this upper bound. Interpret the results.

    Answer: See discussion on Page 151 of the book □

15. Discuss the validity of the statement "Spatial correlations degrade the achievable rate of MIMO channels." Give examples and/or counter-examples.

    Answer: We need to distinguish two cases: the case where channel covariance matrix is unknown at the transmitter and the case where it is known at the transmitter.

    In the former case, uniform power allocation is commonly conducted as the transmitter does not have more information to take a more clever decision. In such case, at high SNR, spatial correlation degrades the performance. At low SNR, on the other hand, spatial correlation does not affect performance as the rate is a function of the Frobenius norm of the channel.

    In the latter case, the transmitter has partial channel knwoledge at the transmitter in the form of channel distribution knowledge. Receive correlation is detrimental while transmit correlation can be helpful at high SNR or when $n_t > n_r$. □

16. Derive the asymptotic (infinite SNR) multiplexing-diversity trade-off of a SIMO i.i.d. Rayleigh fading channel.

    Hint: For u that is χ^2_{2n} distributed, $P(u \leq \epsilon) \approx \epsilon^n$.

    Answer: For a total rate R that scales with ρ such as $R = g_s \log_2(\rho)$, the outage probability of this SIMO channel is written as

$$\begin{aligned} P_{\text{out}}(R) &= P\left(\log_2\left[1 + \frac{\rho}{n_t}\|\mathbf{h}\|^2\right] < g_s \log_2(\rho)\right) \\ &= P\left(1 + \rho\|\mathbf{h}\|^2 < \rho^{g_s}\right) \end{aligned}$$

    such that at high SNR, P_{out} is of the order of

$$\begin{aligned} P_{\text{out}}(R) &\approx P\left(\|\mathbf{h}\|^2 \leq \rho^{-(1-g_s)}\right) \\ &\approx \rho^{-n_r(1-g_s)} \end{aligned}$$

    The last expression comes from the fact that $\|\mathbf{h}\|^2$ is $\chi^2_{2n_r}$ distributed, i.e., $P\big(\|\mathbf{h}\|^2 \leq \epsilon\big) \approx \epsilon^{n_r}$ for small ϵ. Hence, the diversity-multiplexing trade-off of this SIMO channel is given by $g_d^\star(g_s, \infty) = n_r(1 - g_s)$, for $g_s \in [0, 1]$. □

17. Discuss the validity of the statement "The achievable rate of a $n_r \times n_t$ MIMO channel is increased by transmitting a larger number of streams." Give examples and/or counter-examples.

    Answer: Not necessarily. This was already discussed in a previous problem about "Is the rate achievable in a MIMO channel with multiple eigenmode transmission and uniform power allocation across modes always larger than that achievable with dominant eigenmode transmission?" The water-filling solution allocates power to an increasing number of streams as the SNR increases. At low SNR, it is preferable to transmit a small number of streams and only allocate power to the dominant modes. □

18. Discuss the validity of the statement "In a point to point SISO system where only the receiver has knowledge of the channel and the codewords span many coherence times, fading is detrimental to channel capacity."

    Answer: Looking at fast fading, the ergodic capacity writes as

$$\bar{C} = \mathcal{E}\left\{\log_2\left(1 + \rho\left|h\right|^2\right)\right\}$$

which becomes

$$\bar{C} \approx \rho\mathcal{E}\left\{\left|h\right|^2\right\}\log_2(e) \approx C_{AWGN}$$

at low SNR, and

$$\begin{aligned}\bar{C} &\approx \mathcal{E}\left\{\log_2\left(\rho\left|h\right|^2\right)\right\}\\ &= \log_2(\rho) + \mathcal{E}\left\{\log_2\left(\left|h\right|^2\right)\right\}\\ &= \log_2(\rho) - 0.83\\ &< \log_2(\rho) \approx C_{AWGN}\end{aligned}$$

at high SNR. Hence fading is detrimental at high SNR. □

19. Compare the capacity with perfect CSIT of a deterministic $n \times m$ MIMO channel $\mathbf{H}$ with that of the $m \times n$ MIMO channel $\mathbf{H}^T$ under a fixed total transmit power constraint. What do you observe? Would this observation also be true for the ergodic capacities (with CDIT) of i.i.d. $n \times m$ and $m \times n$ MIMO Rayleigh fading channels?

    Answer: For $n \times m$ MIMO channel $\mathbf{H}$

$$C(\mathbf{H}) = \max_{\mathbf{Q}_1 \geq 0 : \mathrm{Tr}\{\mathbf{Q}_1\}=1} \log_2 \det\left[\mathbf{I}_n + \rho\mathbf{H}\mathbf{Q}_1\mathbf{H}^H\right].$$

For $m \times n$ MIMO channel $\mathbf{H}^T$

$$C\left(\mathbf{H}^T\right) = \max_{\mathbf{Q}_2 \geq 0 : \mathrm{Tr}\{\mathbf{Q}_2\}=1} \log_2 \det\left[\mathbf{I}_m + \rho\mathbf{H}^T\mathbf{Q}_2\mathbf{H}^*\right].$$

we take

$$\begin{aligned}\mathbf{H} &= \mathbf{U}_\mathbf{H}\mathbf{\Sigma}_\mathbf{H}\mathbf{V}_\mathbf{H}^H,\\ \mathbf{H}^T &= \mathbf{V}_\mathbf{H}^*\mathbf{\Sigma}_\mathbf{H}^T\mathbf{U}_\mathbf{H}^T.\end{aligned}$$

Optimum input covariance matrix $\mathbf{Q}_1^\star$ writes as

$$\mathbf{Q}_1^\star = \mathbf{V_H}\text{diag}\left\{s_1^\star, \ldots, s_n^\star\right\}\mathbf{V_H}^H.$$

Optimum input covariance matrix $\mathbf{Q}_2^\star$ writes as

$$\mathbf{Q}_2^\star = \mathbf{U_H}^*\text{diag}\left\{s_1^\star, \ldots, s_n^\star\right\}\mathbf{U_H}^T.$$

The capacity writes in both cases as

$$C(\mathbf{H}) = \max_{\{s_k\}_{k=1}^n} \sum_{k=1}^{n} \log_2\left[1 + \rho s_k \lambda_k\right] = \sum_{k=1}^{n} \log_2\left[1 + \rho s_k^\star \lambda_k\right].$$

Hence both channels achieve the same capacity. The same is not true if the transmitter does not have perfect CSIT. The ergodic capacities (with CDIT) are different because the power is uniformly split among m and n antennas, respectively. If $m < n$, the ergodic capacities (with CDIT) of i.i.d. $n \times m$ MIMO Rayleigh fading channels outperforms that of $m \times n$ MIMO channels.

□

20. Assume $n_t = n_r$ and full rank transmit/receive correlation matrices, verify the expression (5.87).

Answer: It results from a direct application of the determinant of a product of full rank matrices. □

21. Derive the asymptotic (at infinite SNR) diversity-multiplexing tradeoff of a $1 \times n_t$ MISO i.i.d. Rayleigh slow fading channel. Assume that the input covariance matrix that minimizes the outage probability at high SNR is the identity matrix $\mathbf{Q} = \frac{1}{n_t}\mathbf{I}_{n_t}$. Explain your reasoning and the meaning of the result.
Hint: Note that for a χ^2_{2n} random variable u, $P\left[u \leq \delta\right] \approx \delta^n$ for small δ.

Answer: The outage probability of a MIMO channel writes as

$$P_{out} = \min_{\mathbf{Q}, \text{Tr}\{\mathbf{Q}\} \leq 1} P\left(\log_2 \det\left(\mathbf{I}_{n_r} + \rho \mathbf{H}\mathbf{Q}\mathbf{H}^H\right) \leq R\right).$$

We are told that the $\mathbf{Q}$ that minimizes the outage probability at high SNR is $\mathbf{Q} = \frac{1}{n_t}\mathbf{I}_{n_t}$ such that, for a MISO channel at high SNR,

$$P_{out} \approx P\left(\log_2\left(\frac{\rho}{n_t}\left\|\mathbf{h}\right\|^2\right) \leq R\right).$$

Let us write the total transmission rate as $R = g_s \log_2(\rho)$ with $0 \leq g_s \leq 1$, i.e. the rate scales with the SNR with a multiplexing gain of g_s. At high SNR,

$$P_{out} \approx P\left(\left\|\mathbf{h}\right\|^2 \leq \rho^{g_s - 1}\right).$$

Given that $\left\|\mathbf{h}\right\|^2$ is a $\chi^2_{2n_t}$ random variable, at high SNR (when ρ^{g_s-1} is small) $P\left(\left\|\mathbf{h}\right\|^2 \leq \rho^{g_s-1}\right) \approx \rho^{-n_t(1-g_s)}$. The diversity gain is obtained as the exponent of the SNR, therefore leading to the diversity-multiplexing tradeoff of $n_t(1-g_s)$ for $0 \leq g_s \leq 1$. A diversity gain n_t is achievable when the transmission rate is kept fixed and is not scaled with the SNR. When the rate is scaled with the SNR as $R = \log_2(\rho)$, the diversity gain is 0. Hence in a MISO slow fading channel, we cannot achieve simultaneously a diversity gain of n_t and a multiplexing gain of 1. $\square$

22. Consider the transmission $\mathbf{y} = \mathbf{H}\mathbf{c}' + \mathbf{n}$ with perfect CSIT over a deterministic point to point MIMO channel whose matrix is given by

$$\mathbf{H} = \begin{bmatrix} a & 0 & a & 0 \\ 0 & b & 0 & b \end{bmatrix}$$

where a and b are complex scalars with $|a| \geq |b|$. The input covariance matrix is given by $\mathbf{Q} = \mathcal{E}\left\{\mathbf{c}'\mathbf{c}'^H\right\}$ and is subject to the transmit power constraint $\mathrm{Tr}\left\{\mathbf{Q}\right\} \leq P$.

- Compute the capacity with perfect CSIT of that deterministic channel. Particularize to the case $a = b$. Explain your reasoning.
- Explain how to achieve that capacity.
- In which deployment scenario, could such channel matrix structure be encountered?

Answer:

- Let us write $\mathbf{Q} = \mathbf{V}\mathbf{P}\mathbf{V}^H$ with the diagonal element of $\mathbf{P}$ denoted as P_k (satisfying $\sum_{k=1}^{n_t} P_k = P$) refers to the power allocated to stream k. The capacity with perfect CSIT over the deterministic channel $\mathbf{H}$ is given by

$$C(\mathbf{H}) = \max_{P_1,\ldots,P_k} \sum_{k=1}^{\min\{2,4\}} \log_2\left(1 + \frac{P_k}{\sigma_n^2}\lambda_k\right)$$

where λ_k refers the non-zero eigenvalue of $\mathbf{H}^H\mathbf{H}$, respectively equal to $2|a|^2$ and $2|b|^2$. Hence,

$$C(\mathbf{H}) = \max_{P_1,P_2}\left(\log_2\left(1 + \frac{P_1}{\sigma_n^2}2|a|^2\right) + \log_2\left(1 + \frac{P_2}{\sigma_n^2}2|b|^2\right)\right).$$

The optimal power allocation is given by the water-filling solution

$$P_1^\star = \left(\mu - \frac{\sigma_n^2}{2|a|^2}\right)^+$$

and

$$P_2^\star = \left(\mu - \frac{\sigma_n^2}{2|b|^2}\right)^+$$

with μ computed such that $P_1^\star + P_2^\star = P$. Assuming $P_1^\star$ and $P_2^\star$ are positive, $\mu = \frac{P}{2} + \frac{\sigma_n^2}{4}\left(\frac{1}{|a|^2} + \frac{1}{|b|^2}\right)$. If $\mu - \frac{\sigma_n^2}{2|b|^2} \leq 0$, i.e. $\frac{P}{2} + \frac{\sigma_n^2}{4|a|^2} - \frac{\sigma_n^2}{4|b|^2} \leq 0$, $P_2^\star = 0$ and $P_1^\star = P$. The capacity writes as

$$C(\mathbf{H}) = \log_2\left(1 + \frac{P}{\sigma_n^2}2|a|^2\right).$$

If $\frac{P}{2}+\frac{\sigma_n^2}{4|a|^2}-\frac{\sigma_n^2}{4|b|^2}>0$, $P_1^\star=\frac{P}{2}-\frac{\sigma_n^2}{4|a|^2}+\frac{\sigma_n^2}{4|b|^2}$ and $P_2^\star=\frac{P}{2}+\frac{\sigma_n^2}{4|a|^2}-\frac{\sigma_n^2}{4|b|^2}$. The capacity writes as

$$C\left(\mathbf{H}\right)=\log_2\left(1+\frac{P_1^\star}{\sigma_n^2}2\left|a\right|^2\right)+\log_2\left(1+\frac{P_2^\star}{\sigma_n^2}2\left|b\right|^2\right).$$

In the particular case where $a=b$, uniform power allocation $P_1^\star=P_2^\star=\frac{P}{2}$ is optimal and

$$C\left(\mathbf{H}\right)=2\log_2\left(1+\frac{P}{\sigma_n^2}\left|a\right|^2\right).$$

- Transmit along $\mathbf{V}$, given by the two dominant eigenvector of $\mathbf{H}^H\mathbf{H}$. They are easily computed given the orthogonality of the channel matrix $\mathbf{H}$ as

$$\mathbf{V}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 0\\ 0 & 1\end{bmatrix}.$$

  The power allocated to the two streams is given by $P_1^\star$ and $P_2^\star$. At the receiver, the precoded channel is already decoupled and no further combiner is necessary. Each stream can be decoded using the corresponding SISO decoder.

- Dual-polarized antenna deployment (e.g. VHVH-VH) with LoS and good antenna XPD.

□

23. A common expression for the ZF applied to Spatial Multiplexing is given by $\mathbf{G}_{ZF}=\mathbf{H}^\dagger$ (see (6.91) where we simply omit additional scaling factor) where $\mathbf{H}$ is the MIMO channel matrix. Write an alternative expression for the ZF filter/combiner illustrating that ZF receiver actually maximizes the SNR under the constraint that the interferences from all other layers are nulled out. Specifically, focus on a given layer q and write the expression of the combiner such that this layer is detected through a projection of the output vector $\mathbf{y}$ onto the direction closest to $\mathbf{H}\left(:,q\right)$ within the subspace orthogonal to the one spanned by the set of vectors $\mathbf{H}\left(:,p\right)$, $p\neq q$.

    Answer: Let us assume the following system model with $n_r\geq n_t$

$$\begin{aligned}\mathbf{y}&=\mathbf{Hc}+\mathbf{n},\\ &=\mathbf{h}_q c_q+\sum_{p\neq q}\mathbf{h}_p c_p+\mathbf{n}\end{aligned}$$

    where $\mathbf{h}_q$ is the q^{th} column of $\mathbf{H}$. We aim at finding the combiner $\mathbf{g}_q$ to maximize the SNR of layer/stream q under the constraint that the interferences from all other layers/streams are nulled out. Let us build the following $n_r \times (n_t - 1)$ matrix by collecting all h_p with $p \neq q$:

$$\begin{aligned} \mathbf{H}' &= \left[\begin{array}{ccc} \dots & \mathbf{h}_p & \dots \end{array} \right]_{p \neq q}, \\ &= \left[\begin{array}{cc} \mathbf{U}' & \tilde{\mathbf{U}} \end{array} \right] \mathbf{\Lambda} \mathbf{V}^H \end{aligned}$$

where $\tilde{\mathbf{U}}$ is the matrix containing the left singular vectors corresponding to the null singular values. Similarly we define

$$\mathbf{c}' = \left[\begin{array}{c} \vdots \\ c_p \\ \vdots \end{array} \right]_{p \neq q}.$$

By multiplying by $\tilde{\mathbf{U}}^H$, we project the output vector onto the subspace orthogonal to the one spanned by the columns of $\mathbf{H}'$

$$\begin{aligned} \tilde{\mathbf{U}}^H \mathbf{y} &= \tilde{\mathbf{U}}^H \mathbf{h}_q c_q + \tilde{\mathbf{U}}^H \mathbf{H}' \mathbf{c}' + \tilde{\mathbf{U}}^H \mathbf{n} \\ &= \tilde{\mathbf{U}}^H \mathbf{h}_q c_q + \tilde{\mathbf{U}}^H \mathbf{n}. \end{aligned}$$

Now to maximize the SNR, noting the noise is still white, we match to the effective channel $\tilde{\mathbf{U}}^H \mathbf{h}_q$ such that

$$z = \left(\tilde{\mathbf{U}}^H \mathbf{h}_q \right)^H \tilde{\mathbf{U}}^H \mathbf{h}_q c_q + \left(\tilde{\mathbf{U}}^H \mathbf{h}_q \right)^H \tilde{\mathbf{U}}^H \mathbf{n}$$

and the ZF combiner is $g_q = \left(\tilde{\mathbf{U}}^H \mathbf{h}_q \right)^H \tilde{\mathbf{U}}^H = \mathbf{h}_q^H \tilde{\mathbf{U}} \tilde{\mathbf{U}}^H$. □

24. Show that at low SNR, the pairwise error probability $P(\mathbf{C} \to \mathbf{E})$ in slow fading i.i.d. channel is mainly dominated by the trace of error matrix $\mathrm{Tr}\{\tilde{\mathbf{E}}\}$. Derive a space-time code design criterion for low SNR. Explain the meaning and relate to SISO AWGN channel.

    Answer: See P179. □

25. Relying on the rank-determinant criterion, show that delay diversity achieves full diversity. Assume for simplicity two transmit antennas. Recall that delay diversity is characterized by the fact that the sequence of symbols transmitted on the first antenna also appears on the second antenna with a delay of one symbol duration. Generalize to n_t transmit antennas. Discuss the pros and cons of such scheme versus O-STBC.

    Answer: The codeword for delay diversity can be written as

$$\mathbf{C} = \frac{1}{\sqrt{2}} \left[\begin{array}{ccccc} c_1 & c_2 & \dots & c_{T-1} & 0 \\ 0 & c_1 & c_2 & \dots & c_{T-1} \end{array} \right].$$

Taking another codeword $\mathbf{E}$, different from $\mathbf{C}$,

$$\mathbf{E} = \frac{1}{\sqrt{2}} \left[\begin{array}{ccccc} e_1 & e_2 & \dots & e_{T-1} & 0 \\ 0 & e_1 & e_2 & \dots & e_{T-1} \end{array} \right].$$

The diversity gain is given by the minimum rank of the error matrix over all possible pairs of (different) codewords, i.e.

$$r_{min} = \min_{\substack{\mathbf{C},\mathbf{E} \\ \mathbf{C}\neq\mathbf{E}}} r\big(\tilde{\mathbf{E}}\big) = \min_{\substack{\mathbf{C},\mathbf{E} \\ \mathbf{C}\neq\mathbf{E}}} r\big(\mathbf{C}-\mathbf{E}\big).$$

With delay diversity, we have

$$\mathbf{C}-\mathbf{E} = \frac{1}{\sqrt{2}} \begin{bmatrix} c_1 - e_1 & c_2 - e_2 & \dots & c_{T-1} - e_{T-1} & 0 \\ 0 & c_1 - e_1 & c_2 - e_2 & \dots & c_{T-1} - e_{T-1} \end{bmatrix}.$$

Obviously, $r\big(\mathbf{C}-\mathbf{E}\big) \leq 2$. Actually, $r\big(\mathbf{C}-\mathbf{E}\big) = 2$ as long as $\mathbf{C} \neq \mathbf{E}$. Indeed even in the case where all $c_k - e_k = 0$ except for one index k (in order to keep $\mathbf{C} \neq \mathbf{E}$), e.g. $k = 1$,

$$\mathbf{C}-\mathbf{E} = \frac{1}{\sqrt{2}} \begin{bmatrix} c_1 - e_1 & 0 & \dots & 0 & 0 \\ 0 & c_1 - e_1 & 0 & \dots & 0 \end{bmatrix},$$

the rank is equal to 2. Extension to the general n_t case follows the same derivations. The main benefits of delay diversity vs. O-STBC is that delay diversity is a scalable scheme that achieves a full diversity gain of $n_t n_r$ at a spatial multiplexing rate equal to one in the limit of large codeword length ($T >> 0$). O-STBC achieves full diversity with a spatial multpexing rate of 1 only for $n_t = 2$. On the other hand, the codeword of O-STBC is much shorter. The receiver of an O-STBC is also much simpler than the one of delay diversity that requires equalization based on Maximum Likelihood Sequence Estimation (MLSE). □

26. Let us assume that c_1, c_2, c_3 and c_4 are constellation symbols taken from a unit average energy QAM constellation. A narrowband transmission using a Linear Space-Time Block Code, characterized by codewords

$$\mathbf{C} = \frac{1}{2} \begin{bmatrix} c_1 + c_3 & c_2 + c_4 \\ c_2 - c_4 & c_1 - c_3 \end{bmatrix},$$

is considered for low velocity deployments where the transmit/receive antennas are widely spaced and the transmitter/receiver are surrounded by a large number of scatterers. You are asked to provide a recommendation on the suitability of that code in slow fading scenarios. Is that a good space-time code? Justify your answer.

Answer: Low velocity deployments with large antenna spacing and a large number of scatterers can be fairly well modeled by an i.i.d. slow Rayleigh flat fading channel.

To minimize the error probability and provide full diversity in i.i.d. Rayleigh slow fading channels with PSK/QAM, the code needs to be full rank. To do so, we can check the rank of its error matrix

$$\mathbf{C}-\mathbf{E} = \frac{1}{2} \begin{bmatrix} d_1 + d_3 & d_2 + d_4 \\ d_2 - d_4 & d_1 - d_3 \end{bmatrix}$$

where $d_k = c_k - e_k$ for $k = 1, ..., 4$. This code is rank deficient. It is easily seen that by taking two codewords $\mathbf{C}$ and $\mathbf{E}$ such that $d_3 = d_4 = 0$ and $d_1 = d_2 = d$ (which is encountered for any constellations), $r\left(\mathbf{C}-\mathbf{E}\right) = 1$.

Its error rate performance is quite poor as it only provides a transmit diversity gain of 1. It requires joint ML decoding to extract a total diversity gain of n_r. Its error rate and throughput performance are expected to be similar as that of SM with ML decoding but at the cost of a higher receiver complexity. □

27. What is the capacity region of a Three-User SISO Multiple Access Channel?

Answer: Three-user SISO: $\mathcal{C}_{MAC}$ is the set of all rates pair (R_1, R_2, R_3) satisfying to

$$\begin{aligned}
R_q &\leq \log_2\left(1 + \eta_q \left|h_{ul,q}\right|^2\right), q = 1, 2, 3 \\
R_1 + R_2 &\leq \log_2\left(1 + \eta_1 \left|h_{ul,1}\right|^2 + \eta_2 \left|h_{ul,2}\right|^2\right), \\
R_2 + R_3 &\leq \log_2\left(1 + \eta_2 \left|h_{ul,2}\right|^2 + \eta_3 \left|h_{ul,3}\right|^2\right), \\
R_1 + R_3 &\leq \log_2\left(1 + \eta_1 \left|h_{ul,1}\right|^2 + \eta_3 \left|h_{ul,3}\right|^2\right), \\
R_1 + R_2 + R_3 &\leq \log_2\left(1 + \eta_1 \left|h_{ul,1}\right|^2 + \eta_2 \left|h_{ul,2}\right|^2 + \eta_3 \left|h_{ul,3}\right|^2\right).
\end{aligned}$$

□

28. Explain why in a two-user SISO BC, the channels have to be ordered in order to achieve the capacity region with superposition coding with SIC.

Answer: For user 1 to be able to correctly cancel user 2os signal, user 1os channel has to be good enough to support R2. See page 443 for more details. □

29. Show that the capacity region of a two-user SISO BC is a triangle if the normalized channel gains are equal, i.e. $|\mathsf{h}_1|^2 = |\mathsf{h}_2|^2 = |\mathsf{h}|^2$.

Answer: See pages 443-444. □

30. Show that the sum-rate capacity of a two-user SISO BC is achieved by allocating the transmit power to the strongest user.

Answer: Define $\mathsf{h}_q = \Lambda_q^{-1/2} h_q / \sigma_{n,q}$, $q = 1, 2$. Assume $|\mathsf{h}_1|^2 \geq |\mathsf{h}_2|^2$. The achievable rates are

$$\begin{aligned}
R_1 &\leq \log_2\left(1 + |\mathsf{h}_1|^2 s_1\right) \\
R_2 &\leq \log_2\left(1 + \frac{|\mathsf{h}_2|^2 s_2}{1 + |\mathsf{h}_2|^2 s_1}\right)
\end{aligned}$$

subject to $s_1 + s_2 = E_s$. The sum-rate writes as

$$\begin{aligned}
R_1 + R_2 &\leq \log_2\left(1 + |\mathsf{h}_1|^2 s_1\right) + \log_2\left(1 + \frac{|\mathsf{h}_2|^2 s_2}{1 + |\mathsf{h}_2|^2 s_1}\right) \\
&= \log_2\left(1 + |\mathsf{h}_1|^2 s_1\right) - \log_2\left(1 + |\mathsf{h}_2|^2 s_1\right) + \log_2\left(1 + |\mathsf{h}_2|^2 [s_2 + s_1]\right) \\
&= \log_2\left(1 + |\mathsf{h}_1|^2 s_1\right) - \log_2\left(1 + |\mathsf{h}_2|^2 s_1\right) + \log_2\left(1 + |\mathsf{h}_2|^2 E_s\right).
\end{aligned}$$

It should be clear from the last expression that the power s_1 that maximizes the difference $\log_2\left(1+|\mathsf{h}_1|^2 s_1\right) - \log_2\left(1+|\mathsf{h}_2|^2 s_1\right)$ is given by $s_1 = E_s$ since $|\mathsf{h}_1|^2 > |\mathsf{h}_2|^2$. Hence, the sum-rate capacity of the SISO BC is achieved by allocating the transmit power to the strongest user and

$$C_{BC} = \log_2\left(1+E_s\,|\mathsf{h}_1|^2\right).$$

□

31. Consider a MU-MIMO transmission with K streams transmitted to K users. Each receiver is equipped with multiple antennas and a single stream is intended to each user. The transmitter precodes user data information using a general precoder $\mathbf{P}$ (size $n_t \times K$, $K \leq n_t$). Assume that each receiver is equipped with a MMSE receiver. Write the expression of the MMSE receiver of user 1.

Answer: The received signal of terminal 1 writes as

$$\mathbf{y}_1 = \mathbf{H}_1\mathbf{P}\mathbf{c} = \mathbf{H}_1\mathbf{p}_1 c_1 + \sum_{k\neq 1}\mathbf{H}_1\mathbf{p}_k c_k + \mathbf{n}_1$$

where $\mathbf{y}_1$ is the $[n_r \times 1]$ received signal at user 1, $\mathbf{H}_1$ is the channel between transmitter and user 1, $\mathbf{p}_k$ is the $[n_t \times 1]$ precoder at the transmitter attached to user k (precoding data for terminal k), and $\mathbf{c} = [c_1, c_2, ..., c_k]^T$ is the $[K \times 1]$ transmit symbol vector whose entries are unit-average energy independent symbols. The transmit power at the transmitter writes as $\mathrm{Tr}\left\{\mathcal{E}\left\{\mathbf{P}\mathbf{c}\,(\mathbf{P}\mathbf{c})^H\right\}\right\} = \mathrm{Tr}\left\{\mathbf{P}\mathbf{P}^H\right\} = P$. Hence the power allocation to each stream is naturally accounted for in the precoder. The first term is the intended signal and the second term refers to the intra-cell interference.

We can simply write the received signal as follows

$$\mathbf{y}_1 = \mathbf{H}_1\mathbf{p}_1 c_1 + \mathbf{n}_1'$$

with $\mathbf{n}_1' = \sum_{k\neq 1}\mathbf{H}_1\mathbf{p}_k c_k + \mathbf{n}_1$. The term $\mathbf{n}_1'$ now expresses the combined noise plus interference (intra-cell interference) seen by stream 1. The covariance matrix of $\mathbf{n}_1'$ is given by $\mathbf{R}_{\mathbf{n}_1'} = \mathcal{E}\left\{\mathbf{n}_1'\mathbf{n}_1'^H\right\}$

$$\mathbf{R}_{\mathbf{n}_1'} = \sigma_n^2\mathbf{I}_{n_r} + \sum_{k=2}^{K}\mathbf{H}_1\mathbf{p}_k\left(\mathbf{H}_1\mathbf{p}_k\right)^H.$$

The receive combiner for terminal 1 writes as

$$\begin{aligned}\mathbf{g}_{MMSE,1} &= \left(\mathbf{H}_1\mathbf{p}_1\right)^H \mathbf{R}_{\mathbf{n}_1'}^{-1} \\ &= \mathbf{p}_1^H\mathbf{H}_1^H\left(\sigma_n^2\mathbf{I}_{n_r} + \sum_{k=2}^{K}\mathbf{H}_1\mathbf{p}_k\mathbf{p}_k^H\mathbf{H}_1^H\right)^{-1}.\end{aligned}$$

□

32. Consider a multi-cell network where each transmitter is equipped with 4 antennas and a user in the centre cell is equipped with 2 receive antennas. Assume this is the only scheduled user in the centre cell and two streams are transmitted to that user using percoded spatial multiplexing. Write a system model, the expression of the MMSE receiver for that user and the expression of the rate achievable by that user. Do the same for a ZF, ZF-SIC and MMSE-SIC receiver.

Answer: The received signal of terminal 1 in cell 1 writes as

$$\mathbf{y}_1 = \mathbf{H}_{1,1}\mathbf{P}_1\mathbf{c}_1 + \sum_{i=2}^{n_c} \mathbf{H}_{1,i}\mathbf{P}_i\mathbf{c}_i + \mathbf{n}_1$$

where $\mathbf{y}_1$ is the $[2 \times 1]$ received signal at user 1 in cell 1, $\mathbf{H}_{1,i}$ is the channel between transmitter i and user 1, $\mathbf{P}_i$ is the precoder at transmitter i. $\mathbf{P}_1$ of size $[4 \times 2]$ is the precoder for terminal 1 in cell 1, and $\mathbf{c}_1$ is the $[2 \times 1]$ transmit symbol vector whose entries are unit-average energy independent symbols. The transmit power at each transmitter writes as $\mathrm{Tr}\left\{\mathcal{E}\left\{\mathbf{P}_i\mathbf{c}_i\left(\mathbf{P}_i\mathbf{c}_i\right)^H\right\}\right\} = \mathrm{Tr}\left\{\mathbf{P}_i\mathbf{P}_i^H\right\} = P$. Hence the power allocation to each stream is naturally accounted for in the precoder. The first term is the intended signal and the second term refers to the inter-cell interference.

We can further expand the received signal as follows

$$\mathbf{y}_1 = \mathbf{H}_{1,1}\mathbf{p}_{1,1}c_{1,1} + \mathbf{n}_1'$$

with $\mathbf{n}_1' = \mathbf{H}_{1,1}\mathbf{p}_{1,2}c_{1,2} + \mathbf{n}_1''$, with $\mathbf{n}_1'' = \sum_{i=2}^{n_c} \mathbf{H}_{1,i}\mathbf{P}_i\mathbf{c}_i + \mathbf{n}_1$, where $\mathbf{P}_1 = [\mathbf{p}_{1,1}, \mathbf{p}_{1,2}]$ and $\mathbf{c}_1 = [c_{1,1}, c_{1,2}]^T$. The term $\mathbf{n}_1'$ now expresses the combined noise plus interference (inter-stream and inter-cell interference) seen by stream 1. The covariance matrix of $\mathbf{n}_1'$ is given by $\mathbf{R}_{\mathbf{n}_1'} = \mathcal{E}\left\{\mathbf{n}_1'\mathbf{n}_1'^H\right\}$

$$\mathbf{R}_{\mathbf{n}_1'} = \sigma_n^2\mathbf{I}_2 + \mathbf{H}_{1,1}\mathbf{p}_{1,2}\left(\mathbf{H}_{1,1}\mathbf{p}_{1,2}\right)^H + \sum_{i=2}^{n_c} \mathbf{H}_{1,i}\mathbf{P}_i\left(\mathbf{H}_{1,i}\mathbf{P}_i\right)^H.$$

The MMSE receive combiner for stream 1 of terminal 1 in cell 1 writes as

$$\mathbf{g}_{MMSE,1} = \left(\mathbf{H}_{1,1}\mathbf{p}_{1,1}\right)^H \mathbf{R}_{\mathbf{n}_1'}{}^{-1}$$

and the output SINR of stream 1 is given by

$$\rho_1 = \left(\mathbf{H}_{1,1}\mathbf{p}_{1,1}\right)^H \mathbf{R}_{\mathbf{n}_1'}{}^{-1}\mathbf{H}_{1,1}\mathbf{p}_{1,1}.$$

Rate of stream 1 is given by $R_1 = \log_2(1 + \rho_1)$. Similar calculation can be performed for the rate of stream 2.

The ZF receive combiner for terminal 1 in cell 1 writes as

$$\mathbf{G}_{ZF} = \left(\mathbf{H}_{1,1}\mathbf{P}_1\right)^\dagger = \left(\mathbf{P}_1^H\mathbf{H}_{1,1}^H\mathbf{H}_{1,1}\mathbf{P}_1\right)^{-1}\mathbf{P}_1^H\mathbf{H}_{1,1}^H.$$

The covariance matrix of the noise at the output of the ZF filter reads as

$$\mathbf{R} = \mathcal{E}\left\{\mathbf{G}_{ZF}\mathbf{n}_1''\left(\mathbf{G}_{ZF}\mathbf{n}_1''\right)^H\right\} = \mathbf{G}_{ZF}\left(\sigma_n^2\mathbf{I}_2 + \sum_{i=2}^{n_c} \mathbf{H}_{1,i}\mathbf{P}_i\left(\mathbf{H}_{1,i}\mathbf{P}_i\right)^H\right)\mathbf{G}_{ZF}^H.$$

The SNR of stream q writes as $\rho_q = 1/\mathbf{R}(q, q)$.

□