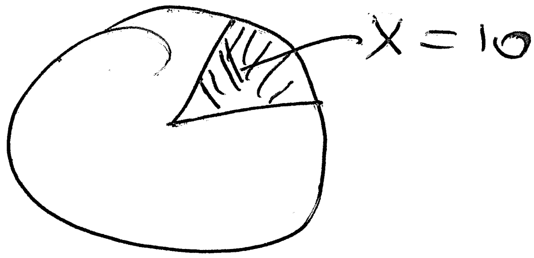
Lecture 5 and 6

RV $X : S \to R$

discrete and continuous



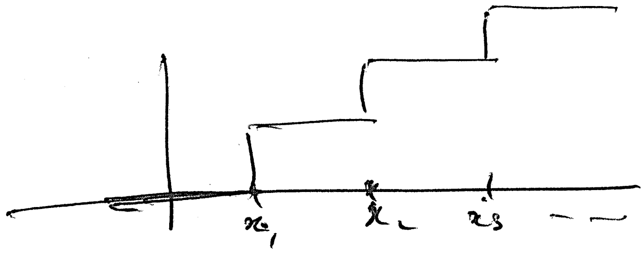
- PMF

$$f_X(x) = P(X = x)$$

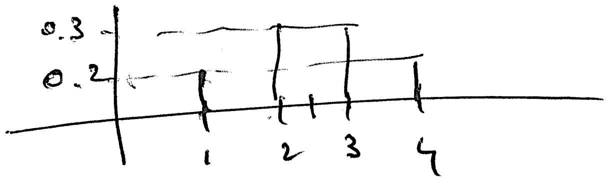
x	x_1	x_2	x_3	...	x_m
$f_X(x)$	$f_X(x_1)$	$f_X(x_2)$	$f_X(x_3)$	...	$f_X(x_m)$

1) $f_X(x) \geq 0$
2) $\sum_i f_X(x_i) = 1$

- CDF $\quad P(X \leq x)$



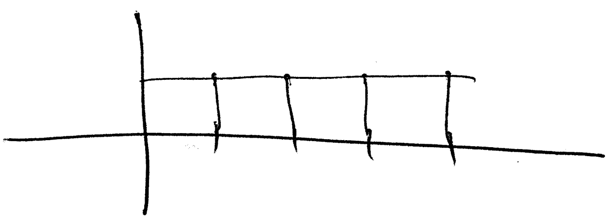
- $\mu = E(X) = \sum_x x \, f_X(x)$



- $\sigma^2 = Var(X) = E((X-\mu)^2)$

$$= E(X^2) - E(X)^2$$

$$\geq 0$$



$$E(X^2) = \sum_x x^2 f_X(x)$$

$$Y = X^2$$

x	x_1	x_2	x_3	$\dots$	x_n
$f_X(x)$	$f_X(x_1)$	$f_X(x_2)$		$\dots$	$f_X(x_n)$

$$E(Y) = \sum_y y\, f_Y(y)$$

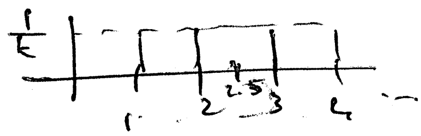
$$= \sum_i x_i^2 f_Y(y) = \sum_i x_i^2 f_X(x_i)$$

y	$y_1 = x_1^2$	$y_2 = x_2^2$	$y_3 = x_3^2$	$\dots$
$f_Y(y)$	$f_Y(y_1) = f_X(x_1)$			

Uniform

X

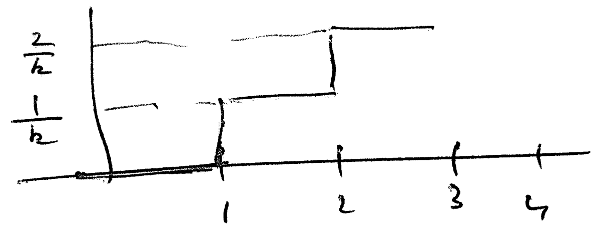
PMF



- PMF $f_X(x) = \frac{1}{k}$ $\quad x = 1, \dots, k$

- CDF $F_X(x) = \sum_{i=1}^{x} \frac{1}{k} = \frac{x}{k}$ $\quad$ for $x = 1, 2, \dots k$

CDF



- $E(X) = \sum_x x \, f_X(x) = \sum_{x=1}^{k} x \frac{1}{k} = \frac{1}{k}\left[\sum_{x=1}^{k} x\right]$

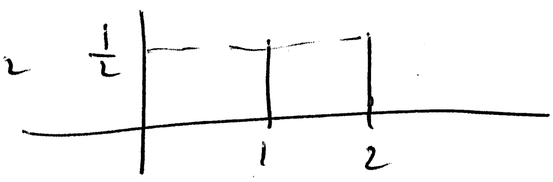
$$= \frac{1}{k} \frac{k(k+1)}{2}$$

$$= \frac{k+1}{2}$$

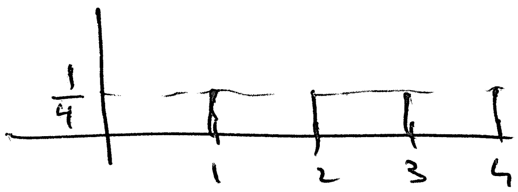
- $Var(X) = E(X^2) - \underset{\frac{k+1}{2}}{E(X)}^2 = \frac{k^2-1}{12}$

$$\sum_{x=1}^{k} x^2 \frac{1}{k}$$

$k=2$



$k=4$

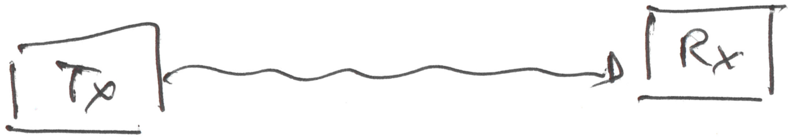


Binomial

$$X \sim \mathrm{Bin}(n, p)$$

experiment :
- n independent and identical trials
- binary outcome : success or failure
- probability of success p

X is the number of success in n trials

ex 001010 

PMF ?

$$P(X = x) = P\Big(\overbrace{(S_1 \cap S_2 \cap \dots \cap S_x \cap F_{x+1} \cap \dots \cap F_n)}^{A} \cup$$
$$\underbrace{(S_1 \cap S_2 \cap \dots S_{x-1} \cap F_x \cap S_{x+1} \cap F_{x+2} \dots \cap F_n)}_{B}$$
$$\cup \dots \Big)$$

$$P(A) = P(S_1 \cap S_2 \cap \dots \cap S_x \cap F_{x+1} \cap \dots \cap F_n)$$
$$= \underbrace{\underbrace{P(S_1)}_{p} \underbrace{P(S_2)}_{p} \dots P(S_x)}_{p^x} \underbrace{\underbrace{P(F_{x+1})}_{1-p} \dots \underbrace{P(F_n)}_{1-p}}_{(1-p)^{n-x}}$$

$$P(A|B) = P(A)$$
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

independence
$$P(A \cap B) = P(A)P(B)$$

$$P(B) = p^x (1-p)^{n-x}$$

$$P(X=x) = P(A) + P(B) + \cdots$$

$$= \binom{n}{x} p^x (1-p)^{n-x}$$

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

$$P(X=x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, \ldots, n \\ 0 & \text{oth} \end{cases}$$

<u>ex</u> toss a coin 6 times $(n=6)$ $\quad p = \frac{1}{2}$

Success = head

$$P(X=2) = \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4$$

$$P(X \geq 4) = P(X=4) + P(X=5) + P(X=6)$$

Valid PMF? <u>Yes</u>

1) $f_X(x) \geq 0$

2) $\sum_x f_X(x) = 1$

$$\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = (p + 1 - p)^n = 1$$

$a = p$
$b = 1-p$

<u>Binomial theorem</u>

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$n=2$

$$(a+b)^2 = \binom{2}{0} a^0 b^2 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^2 b^0$$

b^2
$+2ab$
$+a^2$

$$E(X) = \sum_x x\, f_X(x)$$

$$= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$x \binom{n}{x} = x \frac{n!}{x!\,(n-x)!} = n \frac{(n-1)!}{(x-1)!\,(n-1-(x-1))!} = n \binom{n-1}{x-1}$$

$$= \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x}$$

$y = x-1$

$x = y+1$

$$= np \underbrace{\sum_{y=0}^{n-1} \underbrace{\binom{n-1}{y} p^y (1-p)^{n-y-1}}_{\text{PMF of } Y \sim Bin(n-1,\,p)}}_{1}$$

$$= np$$

- $\sigma^2 = Var(X) = np(1-p)$

Geometric

$X \sim Geo(p)$

$$f_X(x) = P(X=x) = P(F_1 \cap F_2 \cdots F_{x-1} \cap S_x)$$
$$= \underbrace{P(F_1)}_{1-p} P(F_2) \cdots P(F_{x-1}) \underbrace{P(S_x)}_{p}$$
$$= \begin{cases} p(1-p)^{x-1} & x = 1, 2, 3, \ldots \\ 0 & \text{oth} \end{cases}$$

Valid PMF?

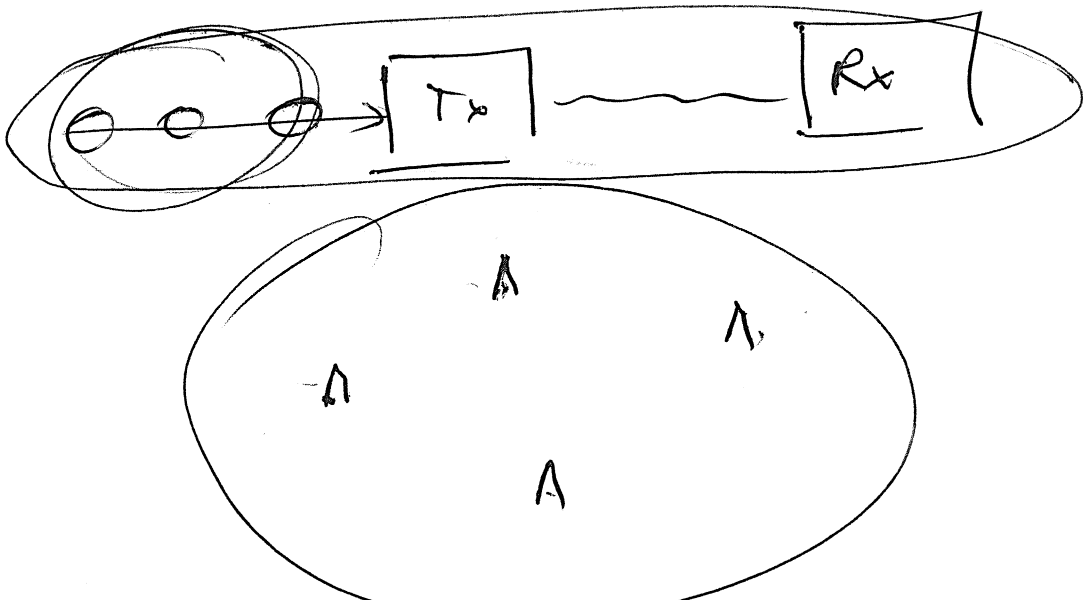
1) $f_X(x) \geq 0$

2) $\sum_x f_X(x) = \sum_{x=1}^{\infty} p(1-p)^{x-1} = p\left[\sum_{x=1}^{\infty} (1-p)^{x-1}\right]$ Geometric series

$$= \frac{p}{1-(1-p)} = 1$$

Poisson

$X \sim Poisson(\lambda)$ $\lambda > 0$



$$f_X(x) = \frac{e^{-\lambda} \lambda^x}{x!} \qquad x = 0, 1, 2 \dots$$

- Valid PMF ?

1) $f_X(x) \geq 0$

2) $\sum_x f_X(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \underbrace{\left[\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \right]}_{e^{\lambda}}$

$$= e^{-\lambda} e^{\lambda} = \underline{1}$$

- $E(X) = \sum x \, f_X(x)$.

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \left[\sum_{x=0}^{\infty} \frac{x \lambda^x}{x!} \right]$$

$$= e^{-\lambda} \left[0 + \lambda + 2\frac{\lambda^2}{2!} + 3\frac{\lambda^3}{3!} + \dots \right]$$

$$= e^{-\lambda} \lambda \underbrace{\left[1 + \lambda + \frac{\lambda^2}{2!} + \dots \right]}_{e^{\lambda}}$$

$$= \lambda$$

- $Var(X) = E(X^2) - E(X)^2 = E(X^2) - E(X) - E(X)^2 + E(X)$

$$= \underline{E(X(X-1))} - \underbrace{E(X)}_{\lambda}(\underbrace{E(X)}_{\lambda} - 1)$$

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \left[\sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x}{x!} \right]$$

$$= e^{-\lambda} \left[0 + 0 + 2\frac{\lambda^2}{2!} + 3 \times 2 \frac{\lambda^3}{3!} + 4 \times 3 \frac{\lambda^4}{4!} + \dots \right]$$

$$= e^{-\lambda} \lambda^2 \underbrace{\left[1 + \lambda + \frac{\lambda^2}{2!} + \dots \right]}_{\lambda} = \lambda^2$$

$$E(X^2) - E(X) = E(X(X-1))$$

$$E(X(X-1)) = E(X^2 - X)$$
$$= \sum_x (x^2 - x) f_X(x)$$
$$= \underbrace{\sum_x x^2 f_X(x)}_{E(X^2)} - \underbrace{\sum_x x f_X(x)}_{E(X)}$$

$$Var(X) = \underbrace{E(X(X-1))}_{\lambda^2} - \underbrace{E(X)}_{\lambda}(\underbrace{E(X)}_{\lambda} - 1)$$
$$= \lambda$$

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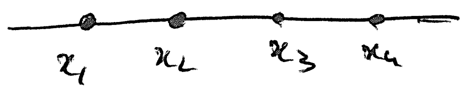
ex $\quad f_X(x) = \dfrac{e^{-\lambda} \lambda^x}{x!}$

$X \sim \text{Poisson}(\lambda = 2)$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$
$$= 1 - \frac{e^{-2}\lambda^0}{0!} - \frac{e^{-2}\lambda^1}{1!}$$

Continuous RV

X



$\longrightarrow$ PMF

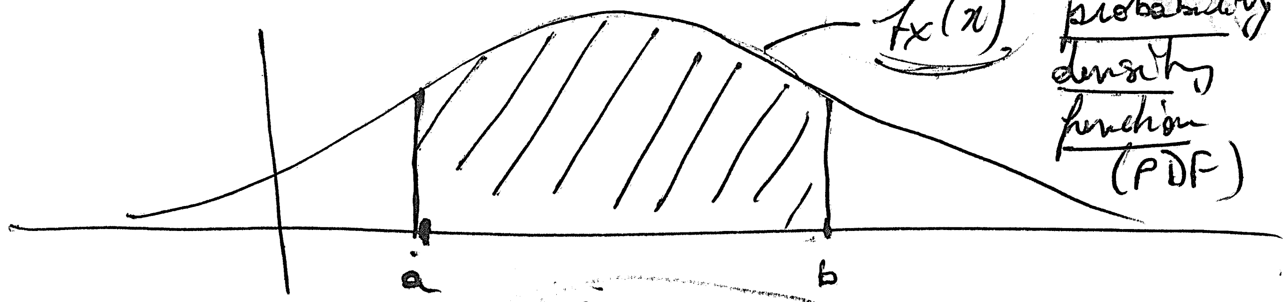
Probability mass function

x	x_1	x_2	...
$f_X(x)$	$f_X(x_1)$	$f_X(x_2)$	

$f_X(x_1) \downarrow P(X = x_1)$

X

1)



$f_X(x)$ probability density function (PDF)

$P(a \le X \le b)$

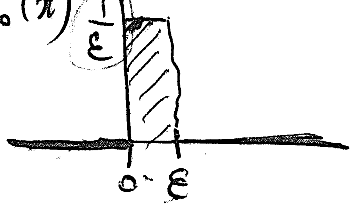
properties

1) $f_X(x) \geq 0$

2) $\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$

$f_X(x) \geqslant 1$

$f_X(x)$



PMF

$\sum_i f_X(x_i) = 1$

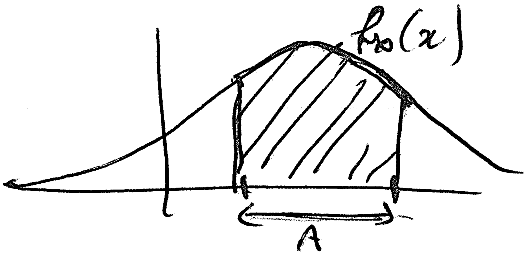
$f_X(x_i) \geq 0$

$\downarrow$

$f_X(x_i) \leq 1$

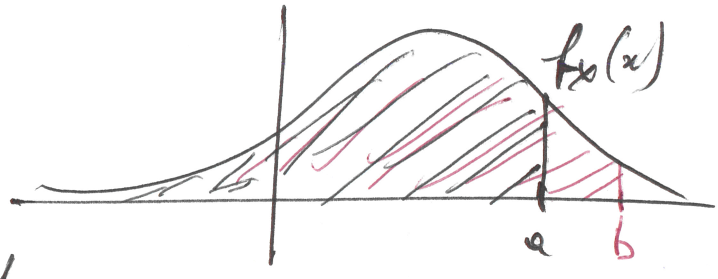
$P(X = x_i) \leq 1$

$$P(X \in A) = \int_A f_X(x)\, dx$$



2) CDF

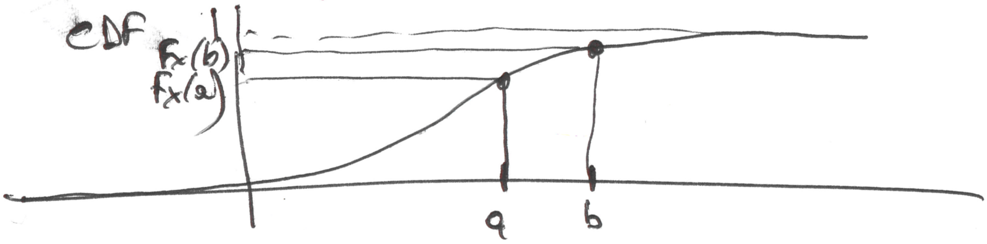
$$F_X(a) = P(X \leq a)$$



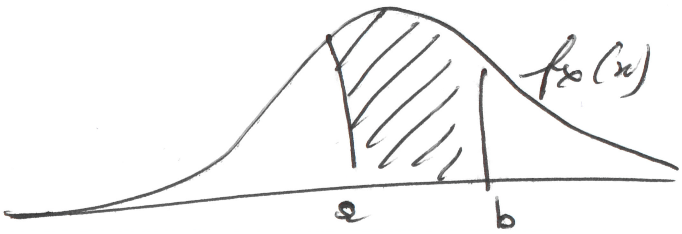
$$= \int_{-\infty}^{a} f_X(x)\,dx$$

$$F_X(b) = P(X \leq b) = \int_{-\infty}^{b} f_X(x)\,dx$$

$$a \leq b \Rightarrow F_X(a) \leq F_X(b)$$



$$f_X(x) = \frac{d\,F_X(x)}{dx}$$ when derivative exists



$$P(a < X \leq b) = \int_{a}^{b} f_X(x)\,dx$$

$$= F_X(b) - F_X(a)$$

ex

$$f_X(x) = \begin{cases} cx^2, & 0 \leq x \leq 2 \\ 0, & \text{oth} \end{cases}$$

- find c such that $f_X(x)$ is a valid PDF.

1) $f_X(x) \geq 0 \rightarrow c \geq 0$
2) $\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$

$$\int_{-\infty}^{+\infty} f_X(x)\, dx = \int_0^2 cx^2\, dx = c\left[\frac{x^3}{3}\right]_0^2$$
$$= c\,\frac{8}{3}$$

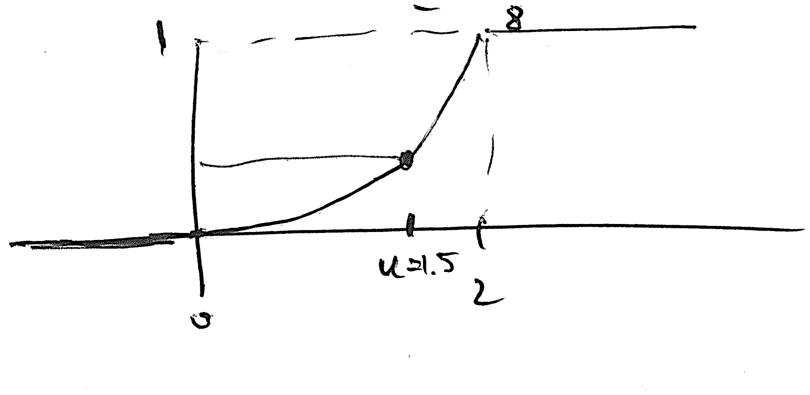
choose c such that $c\,\frac{8}{3} = 1$

$$c = \frac{3}{8}$$

- CDF of X

$$F_X(u) = P(X \leq u) = \begin{cases} 0 & u < 0 \\ \int_{-\infty}^{u} f_X(x)\,dx = \frac{u^3}{8} & 0 \leq u \leq 2 \\ 1 & u > 2 \end{cases}$$

$$\int_{-\infty}^{u} f_X(x)\, dx = \int_0^u \frac{3}{8} x^2\, dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_0^u$$
$$= \frac{u^3}{8}$$



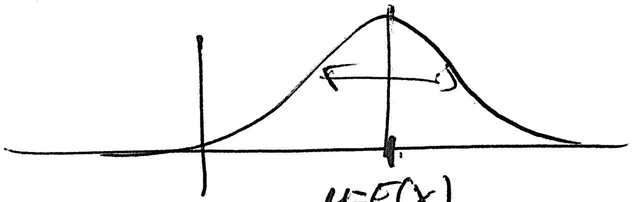
Expectation

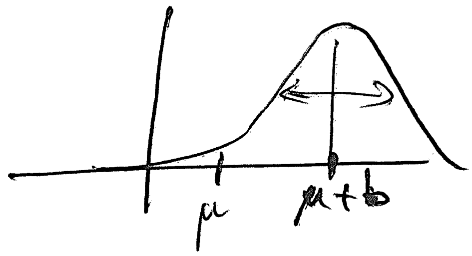
$$\mu = E(X) = \int_{-\infty}^{+\infty} x \, f_X(x) \, dx$$

$$E(aX+b) = \int_{-\infty}^{+\infty} (ax+b) \, f_X(x) \, dx$$

$$= a \underbrace{\int_{-\infty}^{+\infty} x \, f_X(x) \, dx}_{E(X)} + b \underbrace{\int_{-\infty}^{+\infty} f_X(x) \, dx}_{1}$$

$$= a \, E(X) + b$$





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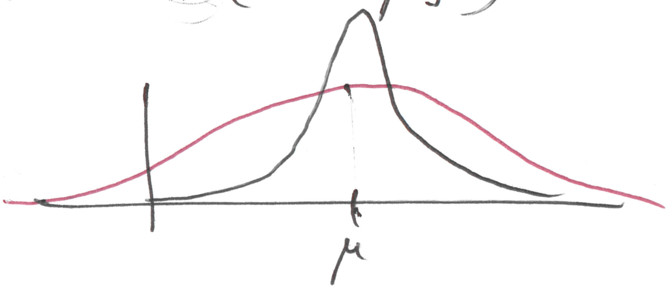
ex
$$f_X(x) = \begin{cases} \frac{3}{8} x^2 & 0 \le x \le 2 \\ 0 & \text{oth} \end{cases}$$

$$E(X) = \int_{-\infty}^{+\infty} x \, f_X(x) \, dx = \int_0^2 \frac{3}{8} x^3 \, dx$$

$$= \frac{3}{8} \left[\frac{x^4}{4} \right]_0^2 = \frac{3}{8} \cdot \frac{16}{4} = \frac{3}{2}$$

Variance

$$\sigma^2 = Var(X) = E((X-\mu)^2)$$



σ = standard deviation

$$E((X-\mu)^2) = \int_{-\infty}^{+\infty} (x-\mu)^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{+\infty} (x^2 - 2\mu x + \mu^2) f_X(x)\,dx$$

$$= \underbrace{\int_{-\infty}^{+\infty} x^2 f_X(x)}_{E(X^2)} - 2\mu \underbrace{\int_{-\infty}^{+\infty} x f_X(x)\,dx}_{\mu} + \mu^2 \underbrace{\int_{-\infty}^{+\infty} f_X(x)\,dx}_{1}$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2 = E(X^2) - E(X)^2 \geq 0$$

$$E(X^2) \geq E(X)^2$$

$$Var(aX+b) = E\left(((aX+b) - (a\mu+b))^2\right)$$

$$= E((aX + \cancel{b} - a\mu - \cancel{b})^2)$$

$$= a^2 \underbrace{E((X-\mu)^2)}_{\int_{-\infty}^{+\infty} (x-\mu)^2 f_X(x)\,dx} = \underline{\underline{a^2}}\, Var(X)$$

ex $f_X(x) = \begin{cases} \frac{3}{8} x^2 & 0 \le x \le 2 \\ 0 & \text{oth} \end{cases}$

$$\mathrm{Var}(X) = \underbrace{E(X^2)}_{12/5} - \underbrace{E(X)}_{3/2}{}^2 = 3/20$$

$$E(X^2) = \int_0^2 x^2 \, \frac{3}{8} x^2 \, dx = \frac{12}{5}$$