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EE9-SC1
EE9-FPN2-03

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected copy

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Wednesday, 3 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer Question 1 and any TWO other questions

Question 1 is worth 40% of the marks and other questions are worth 30%

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D.M. Brookes

Second Marker(s) : P.T. Stathaki

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Information for Candidates:

Where a question requires a numerical answer, it must be given as a fully evaluated decimal number and not as an unevaluated arithmetic expression.

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.
- The expected value of x is denoted $E\{x\}$.
- In block diagrams: solid arrows denote the direction of signal flow; an open triangle denotes a gain element with the gain indicated adjacently; a "+" in a circle denotes an adder/subtractor whose inputs may be labelled "+" or "−" according to their sign; the sample rate of a signal may be indicated in the form "@f".

Abbreviations

BIBO	Bounded Input, Bounded Output
CTFT	Continuous-Time Fourier Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
FIR	Finite Impulse Response

IIR	Infinite Impulse Response
LTI	Linear Time-Invariant
MDCT	Modified Discrete Cosine Transform
PSD	Power Spectral Density
SNR	Signal-to-Noise Ratio

A datasheet is included at the end of the examination paper.

1. a) A finite-length complex exponential signal is given by $x[n] = e^{j\omega n}$ for $n \in [0, N-1]$. The DFT of $x[n]$ satisfies

$$|X[k]| = \frac{\left|\sin \frac{2\pi k - N\omega}{2}\right|}{\left|\sin \frac{2\pi k - N\omega}{2N}\right|}.$$

i) By using the approximation $\sin\theta \approx \theta$ for $|\theta| < 0.2\,\text{rad}$, show that $|X[k]|$ is approximately bounded by $2\left(\frac{2\pi k}{N} - \omega\right)^{-1}$ for a suitable range of k. Give the range of k for which this bound applies and explain the significance of the term: $\left(\frac{2\pi k}{N} - \omega\right)$. [4]

ii) Explain why it is customary to multiply a signal, $x[n]$, by a window before performing a DFT and explain the tradeoffs that affect the choice of window function. [3]

b) i) Explain what is meant by saying that a linear time invariant system is "BIBO stable". [2]

ii) Prove that if a linear time invariant system is BIBO stable, then its impulse response, $h[n]$, satisfies $\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$. [3]

c) A first-order FIR filter is given by $H(z) = 1 - 0.5z^{-1}$.

i) Determine a simplified expression for the squared magnitude response, $\left|H(e^{j\omega})\right|^2$, and sketch its graph for $\omega \in [0, \pi]$. [4]

ii) Using the formula in the datasheet, or otherwise, determine the group delay of the filter, $\tau_H(e^{j\omega})$, and sketch its graph for $\omega \in [0, \pi]$. [4]

d) In the block diagram of Figure 1.1, all elements are drawn with their outputs on the right. The input and output signals are $x[n]$ and $y[n]$ respectively.

i) Determine the transfer function of the system, $H(z) = \frac{Y(z)}{X(z)}$. [3]

ii) Draw the transposed form of the block diagram. [4]

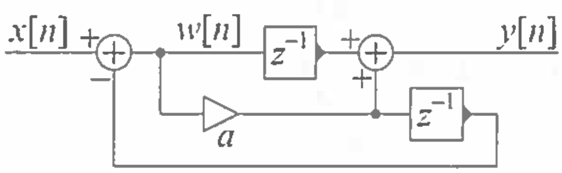

Figure 1.1

e) i) If a bounded discrete-time signal, $x[n]$, is stationary ergodic then $E\{x^2[n]\}$ for any n is equal to its average power (i.e. the average energy per sample). Explain why the average power of such a signal is unchanged by downsampling. [3]

ii) Figure 1.2 shows the power spectral density (PSD) of a real-valued stationary ergodic signal, $x[n]$; the horizontal portions of the PSD have values 1 or 4.

The signal $y[m] = x[3m]$ is obtained by downsampling $x[n]$ by a factor of 3. Draw a dimensioned sketch of the PSD of $y[m]$ giving the values of all horizontal portions of the graph and the values of all frequencies at which there is a discontinuity in the PSD. [4]

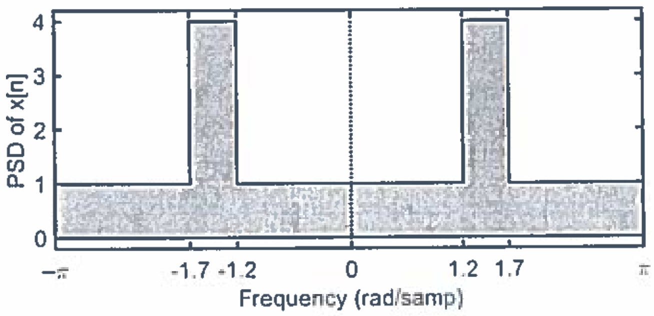

Figure 1.2

f) i) In the block diagram of Figure 1.3 the input is $x[m]$ and the output is $y[n]$. Determine a simplified expression for $Y(z)$ in terms of $X(z)$ and the filters $H_p(z)$ for $p \in [0, 2]$. [3]

ii) If $H_p(z) = \sum_{m=0}^{M} h_p[m]z^{-m}$, derive an expression for $g[n]$ in terms of the $h_p[m]$ so that the block diagram of Figure 1.4 is equivalent to that of Figure 1.3. [3]

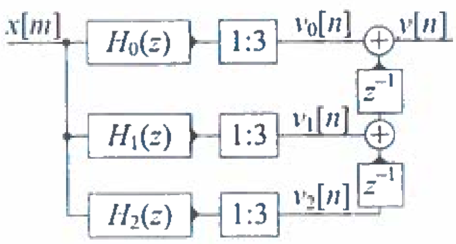

Figure 1.3

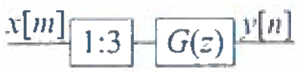

Figure 1.4

2. a) Outline the relative advantages of the bilinear and impulse-invariant transformations for converting a continuous-time filter into a discrete-time filter. [2]

b) If p is a complex-valued constant, show that the z-transform of the causal sequence $v[n] = e^{pn}$ is given by $V(z) = \left(1 - e^{p}z^{-1}\right)^{-1}$ and give its region of convergence. [3]

c) For $t \geq 0$, the impulse response of the causal continuous-time filter $H(s) = \frac{\Omega_0^2}{(s+\alpha)^2+\Omega_0^2}$ is given by

$$h(t) = \Omega_0 e^{-\alpha t}\sin(\Omega_0 t)$$
$$= -0.5j\Omega_0 e^{-\alpha t}\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right).$$

i) Use the result of part b) to find a simplified expression for the z-transform, $G(z)$, of the causal sequence given by $g[n] = T \times h(nT)$ where T is the sample period. Express $G(z)$ as a ratio of polynomials in z^{-1}. [6]

ii) If $T = 10^{-4}$ s, $\Omega_0 = 5000$ rad/s and $\alpha = 800\,\text{s}^{-1}$, give the numerical values of the coefficients of $G(z)$ to 3 decimal places after normalizing to make the leading denominator coefficient unity. [3]

d) i) Show that, under the mapping $s = \kappa\frac{z-1}{z+1}$, the value $s = j\Omega_0$ corresponds to $z = e^{j\omega_0}$ where $\Omega_0 = \kappa\tan(0.5\omega_0)$. Determine the numerical value of κ such that $\omega_0 = \Omega_0 T$ when T and Ω_0 have the values given in part c)ii). [4]

ii) Use the bilinear mapping from part d)i) to transform the filter $H(s)$ from part c) into a discrete time filter, $F(z)$, and give the numerical values of its coefficients to 3 decimal places after normalizing to make the leading denominator coefficient unity. [6]

e) Using the values given in part c)ii), determine the pole and zero positions of $H(s)$, $G(z)$ and $F(z)$ and comment on their relationship to the properties of the three filters. [6]

3. The FM radio baseband spectrum shown in Figure 3.1 comprises (i) a mono signal (L+R) with a bandwidth of 15 kHz, (ii) a 19 kHz pilot tone and (iii) stereo information (L–R) modulated on a suppressed 38 kHz subcarrier. To demodulate the stereo component it is necessary to regenerate the 38 kHz subcarrier by isolating the 19 kHz pilot tone and multiplying its frequency by 2. The baseband signal is sampled at $f_s = 200\,\text{kHz}$.

All filters in this question are lowpass FIR filters with a stopband attenuation of 60 dB whose order may be estimated using the datatsheet formula $M = \frac{60}{3.5\Delta\omega}$ where $\Delta\omega$ is the transition bandwidth in rad/sample.

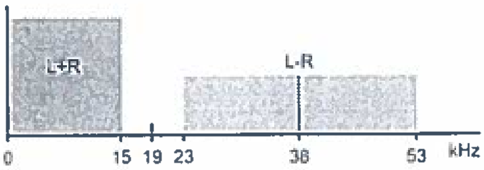

Figure 3.1

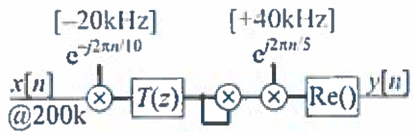

Figure 3.2

a) A block diagram for obtaining the 38 kHz subcarrier, $y[n]$, is shown in Figure 3.2 in which complex-valued signal paths are shown as bold lines. The baseband FM signal, $x[n]$, is translated down in frequency by 20 kHz and lowpass filtered by $T(z)$ to isolated the pilot tone component. The output of $T(z)$ is squared and translated up in frequency by 40 kHz and then the subcarrier, $y[n]$, is obtained by taking the real part of the signal.

i) The pilot tone component of $x[n]$ is given by $x_p[n] = \cos\omega_p n$ and has a frequency of $\omega_p = 2\pi \times \frac{19}{200} = 0.597\,\text{rad/sample}$.

Give the signed frequencies, in rad/sample, of the complex exponential components of the pilot tone signal at each stage of the processing, i.e. for each horizontal line segment in Figure 3.2. [3]

ii) Determine the passband edge frequency and the width of the transition band, $\Delta\omega$, for the lowpass filter, $T(z)$. Hence determine the required FIR filter order using the formula given at the start of the question. [3]

iii) Explain why squaring the output of $T(z)$ doubles the frequency of the pilot tone component. [3]

iv) Estimate the number of real multiplications per second needed to implement the block diagram of Figure 3.2. [3]

[This question is continued on the next page]

b) An alternative block diagram for generating the 38 kHz subcarrier is shown in Figure 3.3 in which $T(z)$ has been replaced by a lowpass filter, $G(z)$, operating at a sample frequency of 10 kHz.

i) Explain the reason that the lowpass filters $F(z)$ and $H(z)$ are needed. [2]

ii) Determine the passband edge, transition band width and filter order for each of the lowpass filters $F(z)$, $G(z)$ and $H(z)$. [6]

iii) Estimate the number of real multiplications per second needed to implement the block diagram assuming that $F(z)$ and $H(z)$ both use a polyphase implementation that incorporates the associated upsampler/downsampler. You may assume without proof that a polyphase filter of order M acting on a complex-valued signal requires $(2M+2)$ multiplications per sample at the lower of the two sample rates. [3]

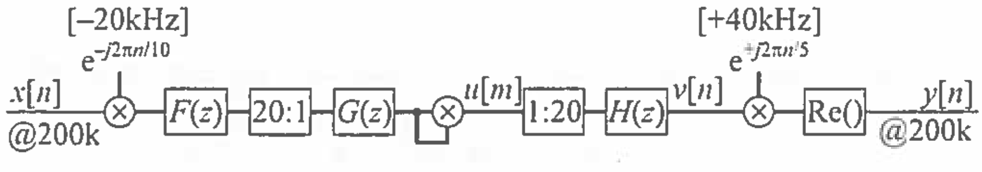

Figure 3.3

c) Suppose now that the upsampling is performed in two stages as illustrated in Figure 3.4 which replaces the blocks "1 : 20" and "$H(z)$" in Figure 3.3.

i) Determine the cutoff frequency, transition bandwidth and filter order for each of the lowpass filters $P(z)$ and $Q(z)$. [4]

ii) Estimate the number of real multiplications per second needed to implement the block diagram of Figure 3.4 assuming that a polyphase implementation is used for $P(z)$ and $Q(z)$. Compare this with the number of multiplications needed for the corresponding part of Figure 3.3. [3]

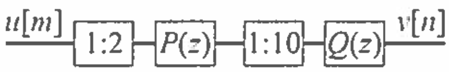

Figure 3.4

4. a) Explain briefly the advantages of processing signals in subbands. [2]

b) Figure 4.1 shows the analysis and synthesis stages of a 2-subband system. Show that $Y(z) = T(z)X(z)$ where $T(z) = \frac{1}{2}(H(z) - H(-z))(H(z) + H(-z))$. [4]

For $p \in [0, 1]$ you may assume without proof that $W_p(z) = U_p(z^2)$ and that $U_p(z) = \frac{1}{2}\left\{V_p\left(z^{\frac{1}{2}}\right) + V_p\left(-z^{\frac{1}{2}}\right)\right\}$.

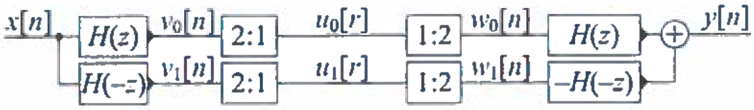

Figure 4.1

c) Given that the impulse response, $h[n]$, is causal and of odd order M, we define

$$t[n] = \frac{1}{2}\left(h[n] + (-1)^n h[n]\right) * \left(h[n] - (-1)^n h[n]\right)$$

where $*$ denotes convolution.

i) Show that the z-transform of $t[n]$ is [3]

$$T(z) = \frac{1}{2}(H(z) - H(-z))(H(z) + H(-z)).$$

ii) Show that, if $h[n]$ satisfies the symmetry condition $h[M-n] = h[n]$, then $t[n]$ satisfies the condition $t[2M-n] = t[n]$. [3]

iii) Deduce the group delay function, $\tau_T\left(e^{j\omega}\right)$, of the filter $T(z)$ from the symmetry condition of part ii). [2]

[This question is continued on the next page]

d) i) By using the inverse DTFT, show that the impulse response of an ideal lowpass filter whose frequency response is

$$G\left(e^{j\omega}\right) = \begin{cases} e^{-j0.5M\omega} & |\omega| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| \le \pi \end{cases}$$

is given by [3]

$$g[n] = \frac{\sin(0.5\pi(n-0.5M))}{\pi(n-0.5M)}.$$

ii) A causal Hamming window of length $M+1$ is given by

$$w[n] = 0.54 - 0.46\cos\left(\frac{2n\pi}{M}\right)$$

for $n \in [0, M]$. Using the window design method with $g[n]$ and $w[n]$, design a causal FIR filter, $H(z)$, of order $M = 7$ with a cutoff frequency of $\omega = \frac{\pi}{2}$. Determine the numerical values of the filter coefficients, $h[n]$, to 3 decimal places. [4]

iii) The filter, $H(z)$, from part ii) is used in the block diagram shown in Figure 4.1. If $T(z) = \frac{Y(z)}{X(z)}$, determine the magnitude gain, $\left|T(e^{j\omega})\right|$ for $\omega = 0$, $\frac{\pi}{2}$ and π. [4]

e) A "Johnston half-band filter" selects the coefficients, $h[n]$, to minimize the cost function

$$\alpha \int_{\frac{\pi}{2}+\Delta}^{\pi} \left|H(e^{j\omega})\right|^2 d\omega + (1-\alpha)\int_0^{\pi} \left(\left|H^2(e^{j\omega}) - H^2(-e^{j\omega})\right| - 1\right)^2 d\omega$$

for suitable choices of α and Δ.

i) Explain the significance of the two integrals in the cost function and hence explain the effect of reducing the value of α. [2]

ii) For $M = 7$, $\alpha = 0.5$ and $\Delta = 0.07$, the $h[n]$ are given by

$$h[0] = h[7] = 0.009, \qquad h[1] = h[6] = -0.071$$
$$h[2] = h[5] = 0.069, \qquad h[3] = h[4] = 0.490$$

Determine the magnitude gain, $\left|T(e^{j\omega})\right|$ for $\omega = 0$, $\frac{\pi}{2}$ and π. [3]

Datasheet:

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1} z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$	$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
DFT:	$X[k] = \sum_{0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$	$x[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N} \sum_{n=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r] y[n-r] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) Y\left(e^{j\omega}\right)$

$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\omega-\theta)}\right) d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r] y[(n-r)_{\bmod N}] \quad \Leftrightarrow \quad V[k] = X[k] Y[k]$

$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V[k] = \frac{1}{N} X[k] \circledast_N Y[k] \triangleq \frac{1}{N} \sum_{r=0}^{N-1} X[r] Y[(k-r)_{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)} \frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10}b}{4.6\Delta\omega}$

where a =stop band attenuation in dB, $b =$ peak-to-peak passband ripple in dB and $\Delta\omega =$ width of smallest transition band in radians per sample.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \nless \hat{\omega} \nless \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities

$$-\boxed{Q:1}-\boxed{H(z)}- \;=\; -\boxed{H(z^Q)}-\boxed{Q:1}-$$

$$-\boxed{H(z)}-\boxed{1:Q}- \;=\; -\boxed{1:Q}-\boxed{H(z^Q)}-$$

Multirate Spectra

Upsample: $v[n] \rightarrow \boxed{1:Q} \rightarrow x[r]$ $\qquad x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases} \quad \Rightarrow \quad X(z) = V(z^Q)$

Downsample: $v[n] \rightarrow \boxed{Q:1} \rightarrow y[m]$ $\qquad y[m] = v[Qm] \quad \Rightarrow \quad Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$

Multirate Commutators