THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. a) i) $P(X \leq Y) = 0.05 + 0.05 + 0.15 + 0.05 + 0.25 + 0.05 = 1 - 0.05 - 0.15 - 0.20 = 0.60$

[1 - E]

$P(X < Y) = 0.05 + 0.15 + 0.25 = 0.45 = P(X \leq Y) = P(X = Y)$

[1 - E]

ii)

x	0	1	2
$P(X=x)$	0.25	0.35	0.40

[1 - E]

y	0	1	2
$P(Y=y)$	0.25	0.30	0.45

[1 - E]

iii) $E(X) = 0 \times 0.25 + 1 \times 0.35 + 2 \times 0.40 = 1.15$ [1 - E]

$E(Y) = 1.20$ [1 - E]

iv) $\text{Var}(X) = E(X^2) - E(X)^2 = 1 \times 0.35 + 4 \times 0.40 - (1.15)^2 = 0.6275,$

[1 - E]

$\text{Var}(Y) = 0.66,$ [1 - E]

$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = 1 \times 0.05 + 2 \times 0.20 + 2 \times 0.25 + 4 \times 0.05 - 1.15 \times 1.20 = -0.23$ [1 - E]

$\text{Corr}(X,Y) = \frac{-0.23}{\sqrt{0.6275 \times 0.66}} = -0.3574.$ [1 - E]

v) X and Y are correlated since $\text{Corr}(X,Y) \neq 0$. [1 - E]

Since they are correlated, they are also dependent. Dependency can also be seen from $P(X=1, Y=1) - 0.05 \neq P(X=1)P(Y=1) = 0.35 \times 0.30$ [1 - E]

vi) Compute the conditional probability mass function of X given that $Y = 0, 1, 2$.

x	0	1	2
$P(X=x \mid Y=0)$	$\frac{0.05}{0.25} = 0.20$	$\frac{0.05}{0.25} = 0.20$	$\frac{0.15}{0.25} = 0.60$

[1 - E]

x	0	1	2
$P(X=x\|Y=1)$	$\frac{0.05}{0.30}=1/6$	$\frac{0.05}{0.30}=1/6$	$\frac{0.20}{0.30}=2/3$

[1 - E]

x	0	1	2
$P(X=x\|Y=2)$	$\frac{0.15}{0.45}=1/3$	$\frac{0.25}{0.45}=0.5555$	$\frac{0.05}{0.45}=0.1111$

[1 - E]

vii) Compute the conditional expectation of X given that $Y=0,1,2$.

$E(X|Y=0)=0\times 0.20+1\times 0.20+2\times 0.60=1.4$ [1 - E]

$E(X|Y=1)=0\times \frac{0.05}{0.30}+1\times \frac{0.05}{0.30}+2\times \frac{0.20}{0.30}=1.5$ [1 - E]

$E(X|Y=2)=0\times \frac{0.15}{0.45}+1\times \frac{0.25}{0.45}+2\times \frac{0.05}{0.45}=7/9$ [1 - E]

viii) $E(X)=E(E(X|Y))=1.4\times 0.25+1.5\times 0.30+7/9\times 0.45=1.15$

[2 - E]

b) We can re-express the argument as the pdf of a Normal distribution

$$\int_{-\infty}^{2.35}\sqrt{\frac{2}{\pi}}e^{-2(u-2)^2}du=\int_{-\infty}^{2.35}\frac{1}{\sqrt{2\pi\frac{1}{4}}}e^{-\frac{1}{2}(\frac{u-2}{\frac{1}{2}})^2}du.$$

Hence this is the CDF of a normal distribution with mean $\mu=2$ and $\sigma^2=\frac{1}{4}$.

[2 - A]

By standardizing the normal distribution, we can write

$$\int_{-\infty}^{2.35}\frac{1}{\sqrt{2\pi\frac{1}{4}}}e^{-\frac{1}{2}(\frac{u-2}{\frac{1}{2}})^2}du=\int_{-\infty}^{\frac{2.35-2}{1/2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz.$$

[2 - A]

Last integral is obtained from the table

$$\int_{-\infty}^{0.7}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz=0.758.$$

[1 - E]

2. a) i) $F_P(S) = P(P \leq S) = P(P_1 \leq S \cap P_2 \leq S)$. [1 - A]

From independence, we write $P(P_1 \leq S \cap P_2 \leq S) = P(P_1 \leq S)P(P_2 \leq S)$ [1 - A]

From the exponential distribution, we get $F_P(S) = \begin{cases} (1-e^{-\lambda S})^2 & S > 0 \\ 0 & \text{otherwise} \end{cases}$

[2 - A]

ii) $f_P(p) = \frac{dF_P(p)}{dp}$ [2 - A]

$f_P(p) = \begin{cases} 2\lambda(1-e^{-\lambda p})e^{-\lambda p} & p > 0 \\ 0 & \text{otherwise} \end{cases}$

[2 - A]

iii) The error probability approximates as $m_P(-d) = E(e^{-dP})$. [1 - A]

Hence $m_P(-d) = \int_0^\infty e^{-dP} 2\lambda(1-e^{-\lambda p})e^{-\lambda p} dp = \frac{2\lambda^2}{(d+\lambda)(d+2\lambda)}$. [3 - A]

iv) $E(P) = m'_P(0)$. [2 - A]

$E(P) = m'_P(0) = \frac{3}{2\lambda}$. [2 - A]

b) i) The MGF of a Normal random variable $X \sim N(\mu, \sigma^2)$ is given as

$$\begin{aligned} m_X(t) = E(e^{tX}) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx \\ &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \end{aligned}$$

[1 - B]

Hence

$$\begin{aligned} m_X(t) &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x^2-2(\mu+t\sigma^2)x+\mu^2}{\sigma^2}} dx \\ &= e^{t\mu+t^2\sigma^2/2} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x^2-2(\mu+t\sigma^2)x+\mu^2+2t\mu\sigma^2+t^2\sigma^4}{\sigma^2}} dx. \end{aligned}$$

[2 - B]

Hence

$$m_X(t) = e^{t\mu+t^2\sigma^2/2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-(\mu+t\sigma^2)}{\sigma}\right)^2}}_{N(\mu+t\sigma^2,\sigma^2)} dx = e^{t\mu+t^2\sigma^2/2}.$$

[1 - B]

ii) No, it is not correct. If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ and X_1, X_2 are independent random variables, we have $2X_1 - X_2 \sim N(2\mu_1 - \mu_2, 4\sigma_1^2 + \sigma_2^2)$.

[1 - A]

Now, we can show (using independence)

$$m_{2X_1-X_2} = E(e^{t(2X_1-X_2)}) = E(e^{t2X_1})E(e^{-tX_2})$$

[2 - A]

Hence,

$$m_{2X_1-X_2} = e^{2t\mu_1+4t^2\sigma_1^2/2}e^{-t\mu_2+t^2\sigma_2^2/2} = e^{t(2\mu_1-\mu_2)+t^2(4\sigma_1^2/2+\sigma_2^2/2)}.$$

This is the MGF of a Normal distribution with mean $2\mu_1 - \mu_2$ and variance $4\sigma_1^2 + \sigma_2^2$. Since the pdf uniquely identifies the pdf, $2X_1 - X_2 \sim N(2\mu_1 - \mu_2, 4\sigma_1^2 + \sigma_2^2)$. [2 - A]