EE4-45
Wavelets + Applications

SOLUTIONS EE4-45

QUESTION 1

(a) WE KNOW THAT FOR $\omega_0 = 0$, $x[n] = 1$ AND THAT THE CONSTANT SEQUENCE IS ANNIHILATED BY A FILTER WITH A ZERO AT $\omega = 0$.

WE THEN EXPECT $x[n] = e^{j\omega_0 n}$ TO BE ANNIHILATED BY $H(z) = (z - e^{j\omega_0})$

PROOF:

$$x[n] * h[n] = \sum_k h[k]\, x[n-k] = e^{j\omega_0 n} \sum_k h[k] e^{-j\omega_0 k}$$

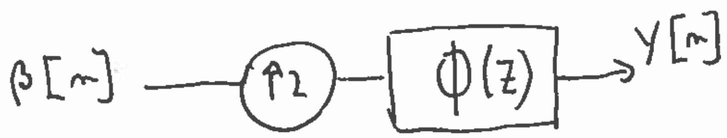
$$= e^{j\omega_0 n} \underbrace{H(e^{j\omega_0})}_{=0} = 0$$

$\square$

CONSEQUENTLY SINCE $x[n] = e^{j\omega_0 n} \cos\omega_0 n = \dfrac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}$,

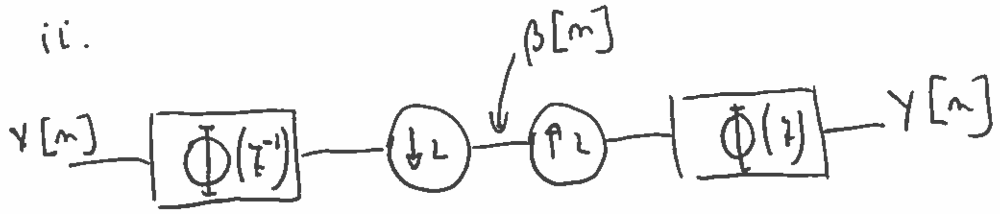
WE HAVE THAT $x[n]$ IS ANNIHILATED BY $H(z) = (z - e^{j\omega_0})(z - e^{-j\omega_0})$

(b) i.



SOLUTIONS

Q1.(b) ii.



SINCE $\langle \psi[n], \phi[n-2k] \rangle = \delta_k$ THE SYSTEM PERFORMS AN ORTHOGONAL PROJECTION OF $x[n]$ ON $V = \text{SPAN}\{\phi[n-2k]\}_{k \in \mathbb{Z}}$.

iii.

WE NEED $\psi[n]$ TO BE SUCH THAT

$$\langle \psi[n-2k], \phi[n-2\ell] \rangle = \delta_{k,\ell}$$

THIS IS ACHIEVED BY SETTING

$$\psi(z) = -z^{-1}\Phi(-z^{-1}) = (-z^{-1}+3-3z-z^2)/2\sqrt{5}$$

MOREOVE

$$\alpha[k] = \langle x[n], \psi^T[n-2k] \rangle$$

WHERE $\psi^T[n] = \psi[-n]$.

SOLUTIONS

QUESTION 2

(a)
$$\hat{X}(z) = \frac{1}{2} G_0(z) H_0(z) X(z) + \frac{1}{2} G_0(z) H_0(-z) X(-z)$$
$$+ \frac{1}{2} G_1(z) H_1(z) X(z) - \frac{1}{2} G_1(z) H_1(-z) X(-z)$$

THIS LEADS TO THE FOLLOWING PR CONDITIONS:

$$\begin{cases} G_0(z) H_0(z) + G_1(z) H_1(z) = 2 & \text{(DISTORTION-FREE CONDITION)} \\ G_0(z) H_0(-z) - G_1(z) H_1(-z) = 0 & \text{(ALIAS-FREE CONDITION)} \end{cases}$$

NOTE THE MINUS SIGN IN THE 'ALIAS-FREE' CONDITION.

(b) FOR THE UPPER BRANCH WE IMPOSE THE USUAL ORTHOGONALITY CONDITIONS:

$$\begin{cases} G_0(z) G_0(z^{-1}) + G_0(-z) G_0(-z^{-1}) = 2 \\ H_0(z) = G_0(z^{-1}) \end{cases}$$

FOR THE LOWER BRANCH, BECAUSE OF THE DELAY WE NOW IMPOSE

$$\langle g_0[n], g_1[n-2k-1] \rangle = 0$$

WHICH LEADS TO

$$G_1(z) = G_0(-z^{-1})$$

AND THEN TO $H_1(z) = G_1(z^{-1})$

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SOLUTIONS

Q2.

(c) $$G_0(z) = \frac{1}{\sqrt{2}}\left(1+z^{-1}\right)$$

GIVEN $G_0(z)$ WE HAVE THAT

$$H_0(z) = G_0(z^{-1}) = \frac{1}{\sqrt{2}}(1+z)$$

$$G_1(z) = G_0(-z^{-1}) = \frac{1}{\sqrt{2}}(1-z)$$

$$H_1(z) = G_1(z^{-1}) = \frac{1}{\sqrt{2}}\left(1-z^{-1}\right)$$

(d) WE FIRST NOTE THAT THE SOLUTION TO THIS PROBLEM IS NOT UNIQUE, MOREOVER BY CONSTRUCTION

$$G_0(z)\,H_0(z) + G_0(-z)\,H_0(-z) = 2$$

THROUGH SPECTRAL FACTORIZATION WE PICK

$$H_0(z) = \frac{\sqrt{2}}{8}(1+z)\left(1+z^{-1}\right)\left(-z+4-z^{-1}\right)$$

AND $$G_0(z) = \frac{1}{2\sqrt{2}}(1+z)\left(1+z^{-1}\right).$$

THE ABOVE FILTERS ARE SYMMETRIC AS REQUESTED.

SOLUTIONS

Q2 (d) CONTINUED

BECAUSE OF THE DELAY. THE NEW

BIORTHOGONAL CONDITIONS ARE:

$$G_1(z) = H_0(-z) = \frac{\sqrt{2}}{8}(1-z)(1-z^{-1})(z+4+z^{-1})$$

AND

$$H_1(z) = G_0(-z) = \frac{1}{2\sqrt{2}}(1-z)(1-z^{-1}).$$

SOLUTIONS

QUESTION 3

(a) i. SINCE $\tilde{\varphi}_i(t) = \sum_{k=1}^{3} \alpha_{i,k} \varphi_k(t)$

AND GIVEN THAT

$$\langle \tilde{\varphi}_i(t), \varphi_j(t) \rangle = \delta_{i,j}$$

WE CAN WRITE

$$\langle \tilde{\varphi}_i(t), \varphi_j(t) \rangle = \sum_{k=1}^{3} \alpha_{i,k} \langle \varphi_k(t), \varphi_j(t) \rangle = \delta_{i,j} \quad (1)$$

MOREOVER

$$\langle \varphi_1, \varphi_2 \rangle = 0.5, \quad \langle \varphi_1, \varphi_3 \rangle = 0.5, \quad \langle \varphi_2, \varphi_3 \rangle = 0.5$$

AND $\langle \varphi_i, \varphi_i \rangle = 1.5 \qquad i = 1, 2, 3.$

COMBINING (1) WITH THE ABOVE EQUATIONS LEADS TO:

$$\begin{cases} \frac{3}{2}\alpha_{1,1} + \frac{\alpha_{1,2}}{2} + \frac{\alpha_{1,3}}{2} = 1 \\ \frac{\alpha_{1,1}}{2} + \frac{3}{2}\alpha_{1,2} + \frac{\alpha_{1,3}}{2} = 0 \\ \frac{\alpha_{1,1}}{2} + \frac{\alpha_{1,2}}{2} + \frac{3}{2}\alpha_{1,3} = 0 \end{cases}$$

SOLUTIONS

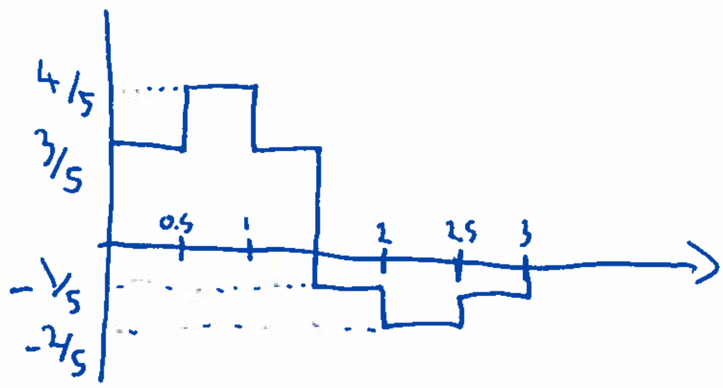
Q3 (a) i.

THEREFORE

$$\alpha_{1,1} = \frac{4}{5}, \quad \alpha_{1,2} = -\frac{1}{5}, \quad \alpha_{1,3} = -\frac{1}{5},$$

BY SYMMETRY WE CAN FIND THE OTHER $\alpha_{i,k}$'S.

ii.



$\tilde{\varphi}_1(t)$

$\tilde{\varphi}_2(t)$ AND $\tilde{\varphi}_3(t)$ ARE GENERATED BY COMPUTING CIRCULAR SHIFTS BY ONE OF $\tilde{\varphi}_1(t)$.

SOLUTIONS

Q.3 (b)

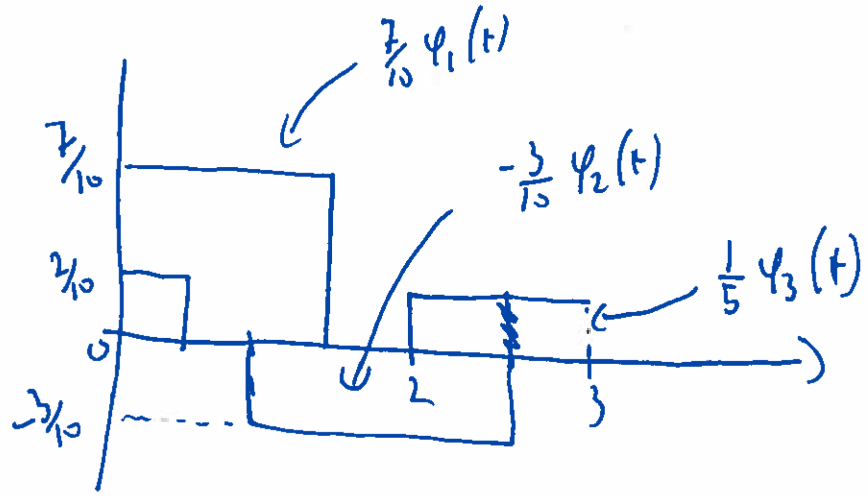
i.

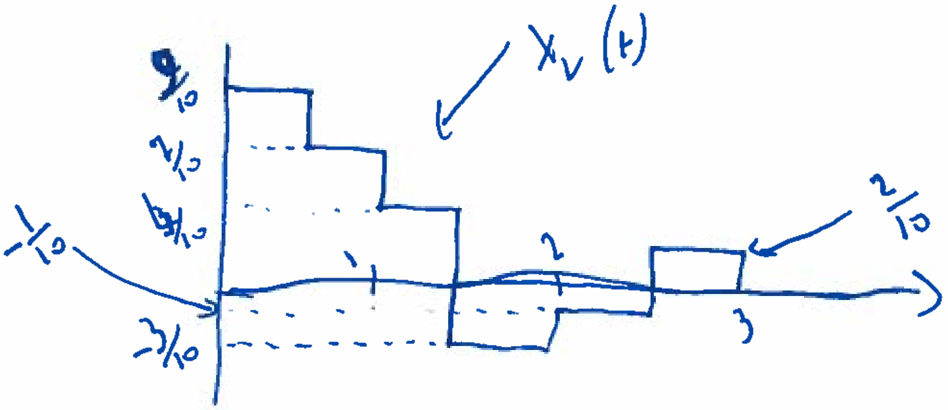
$$\langle x(t), \tilde{\varphi}_1(t)\rangle = \frac{1}{2}\left(\frac{3}{5}+\frac{4}{5}\right) = \frac{7}{10}$$

$$\langle x(t), \tilde{\varphi}_2(t)\rangle = -\frac{1}{2}\left(\frac{2}{5}+\frac{1}{5}\right) = -\frac{3}{10}$$

$$\langle x(t), \tilde{\varphi}_3(t)\rangle = \frac{1}{2}\left(\frac{3}{5}-\frac{1}{5}\right) = \frac{1}{5}$$

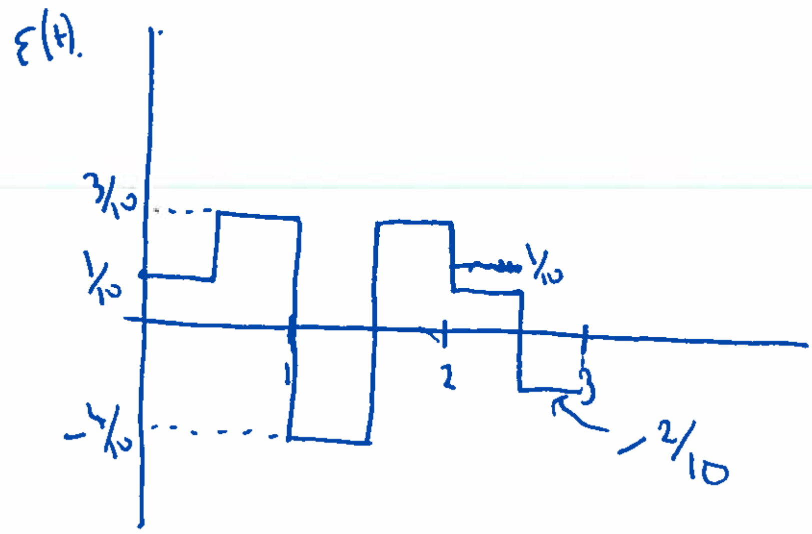
ii.





SOLUTIONS

Q3 (b) iii



CONSEQUENTLY

$$\langle \varepsilon(t), \varphi_1(t) \rangle = \langle \varepsilon(t), \varphi_2(t) \rangle = \langle \varepsilon(t), \varphi_3(t) \rangle = 0$$

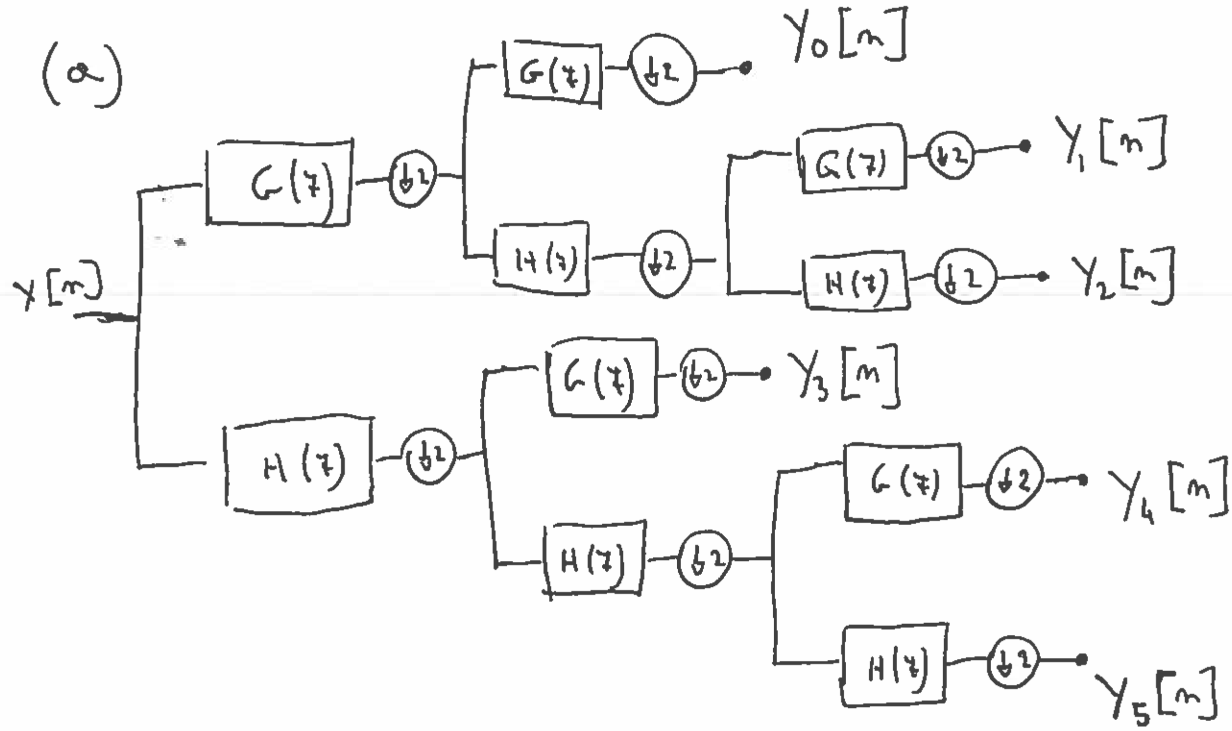
THIS MEANS THAT

$$\varepsilon(t) \perp V \qquad \#$$

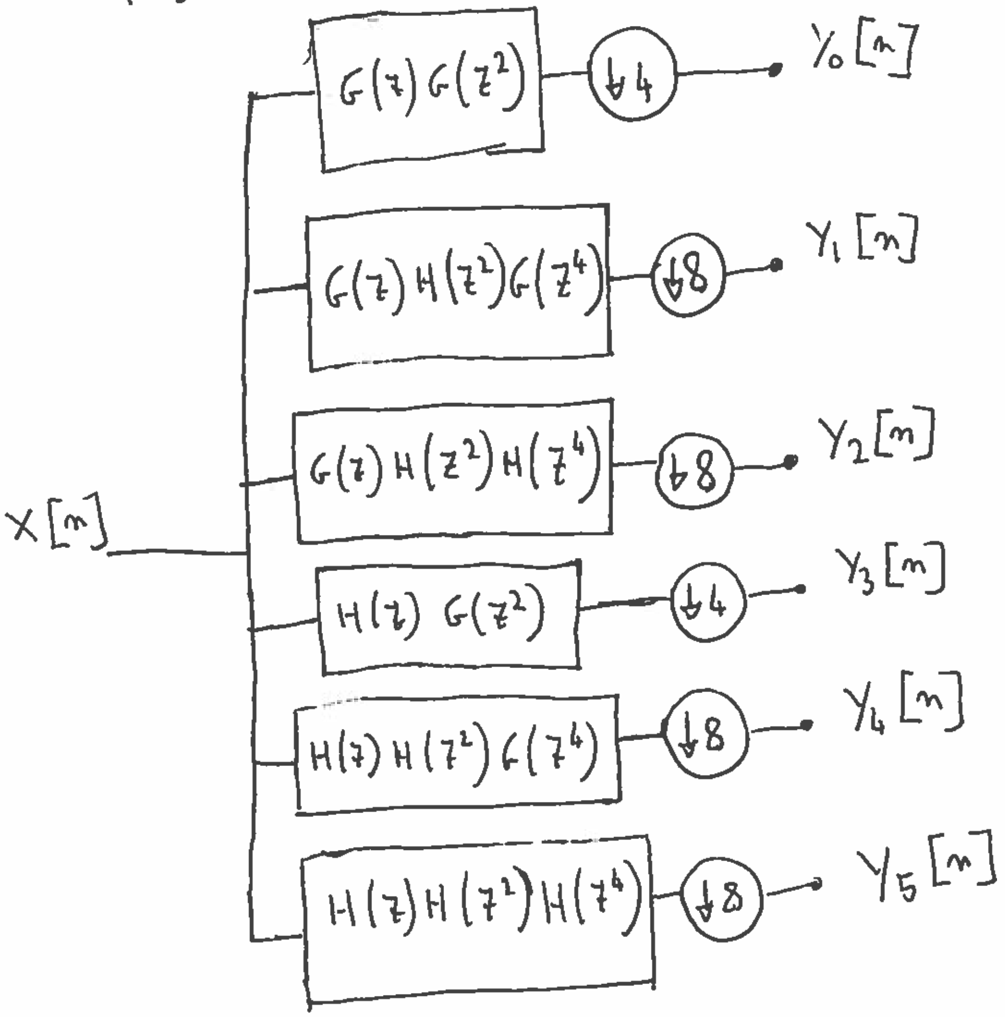
SOLUTIONS

QUESTION 4

(a)



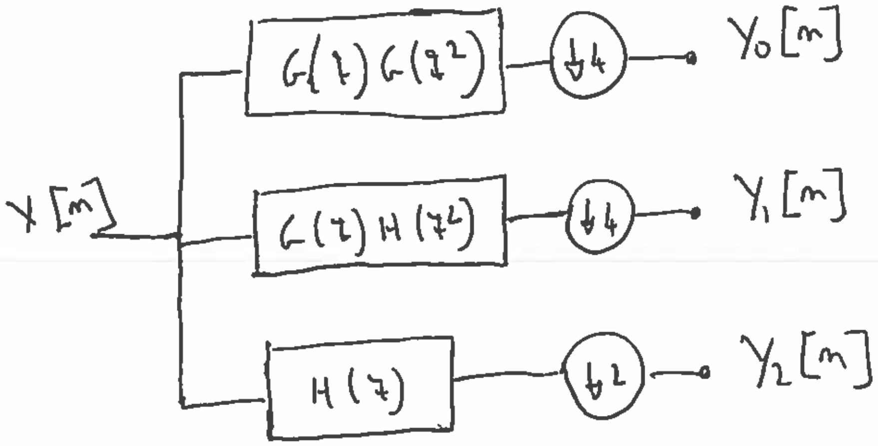
(b)



SOLUTIONS

Q.4 (c)

i.



ii.

WE FIRST NOTE THAT

$$H(1) = 0.$$

MOREOVER

$$\frac{dH(z)}{dz} = \sqrt{2}\left(2z + 2 - 2z^{-2} - 2z^{-3}\right)$$

THEREFORE

$$\left.\frac{dH(z)}{dz}\right|_{z=1} = 0$$

THIS MEANS THAT $H(z)$ HAS TWO ROOT AT $\omega = 0$ AND SO ANNIHILATES LINEAR POLYNOMIALS. THIS IMPLIES $Y_1[n] = Y_2[n] = 0$. SINCE $G(z)$ HAS NO ZEROS AT $\omega = 0$, $Y_0[n] \neq 0$.