# EE401: Advanced Communication Theory

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**Multi-Antenna Wireless Communications**
Part-B: SIMO, MISO and MIMO

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# Notation

matrix exponential

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^A \otimes e^B = e^{A \oplus B}$$

⇓ matrix A

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{AA}{2!} + \frac{AAA}{3!} + \cdots \text{ (converge)}$$

e.g. $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$, $e^A = \begin{bmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{bmatrix}$

$B = \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}$, $e^B = \begin{bmatrix} 1 & x & z+\frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$

| | |
|---|---|
| $\underline{a}, \underline{A}$ | denotes a column vector |
| $\mathbb{A}$ (or $\underline{\underline{A}}$) | denotes a matrix |
| $\mathbb{I}_N$ | $N \times N$ identity matrix |
| $\underline{1}_N$ | vector of $N$ ones |
| $\underline{0}_N$ | vector of $N$ zeros |
| $\mathbb{O}_{N,M}$ | $N \times M$ matrix of zeros |
| $(.)^T$ $[A^T]_{ij} = [A]_{ji}$ | transpose $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ |
| $(.)^H$ $[A^H]_{ij} = [\bar{A}]_{ji}$ | Hermitian transpose $\begin{bmatrix} 1+j \\ 2j \end{bmatrix} \Rightarrow \begin{bmatrix} 1-j & -2j \end{bmatrix}$ |
| $\mathbb{A}^{\#}$ | pseudo-inverse of $\mathbb{A}$ $\quad A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $A^{+} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ |
| $\odot$ $[A \odot B]_{ij} = [A]_{ij}[B]_{ij}$ | Hadamard product |
| $\oslash$ $[A \oslash B]_{ij} = [A]_{ij}/[B]_{ij}$ | Hadamard division |
| $\otimes$ $A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$ | Kronecker product $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 2 \cdot \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \\ 3 \cdot \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 4 \cdot \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \end{bmatrix}$ |
| $\exp(\underline{a}), \exp(\mathbb{A})$ | element by element exponential |
| $\mathcal{L}[\mathbb{A}]$ | linear space/subspace spanned by the columns of $\mathbb{A}$ |
| $\mathcal{L}[\mathbb{A}]^{\perp}$ | complement subspace to $\mathcal{L}[\mathbb{A}]$ |
| $\mathcal{P}[\mathbb{A}]$ (or $\mathbb{P}_{\mathbb{A}}$) | projection operator on to $\mathcal{L}[\mathbb{A}]$ |
| $\mathcal{P}[\mathbb{A}]^{\perp}$ (or $\mathbb{P}_{\mathbb{A}}^{\perp}$) | projection operator on to $\mathcal{L}[\mathbb{A}]^{\perp}$ |

pseudo-inverse $A^{\#}$

- equals matrix inverse $A^{-1}$ if invertible
- is defined even $A$ is not invertible
- for $A$ $(m \times n)$, $A^{\#}$ $(n \times m)$ satisfies:
  - $AA^{\#}A = A$
  - $A^{\#}AA^{\#} = A^{\#}$
  - $(AA^{\#})' = AA^{\#}$
  - $(A^{\#}A)' = A^{\#}A$
- if $A$ full rank $(=n)$, $A^{\#} = (A'A)^{-1}A'$

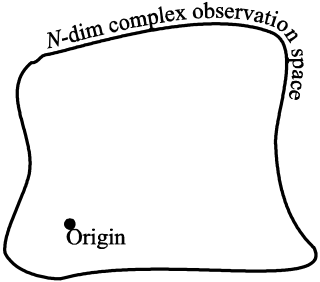
$Ax = b \Rightarrow x = A^{\#}b$ (not exact, closest in least squares)

Kronecker sum

$$A \oplus B = A \otimes I_b + B \otimes I_a$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \oplus \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The expression "**$N$-dimensional complex (or real) observation space** " is denoted by the symbol $\mathcal{H}$ and pictorially represented as follows




Note that any vector in this space has $N$ elements.

- The expression "one-dimensional subspace spanned by the $(N \times 1)$ vector $\underline{a}$" is mathematically denoted by $\mathcal{L}[\underline{a}]$ and pictorially represented by

$\mathcal{L}[\underline{a}]$

- The expression "$M$-dimensional subspace (with $M \geq 2$) spanned by the columns of the $(N \times M)$ matrix $\mathbb{A}$" is mathematically denoted by $\mathcal{L}[\mathbb{A}]$ and pictorially represented by

$\mathcal{L}[\mathbb{A}]$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad$ $Aa = x$

$a = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$x = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ $\quad$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$= \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$= -\frac{1}{2}\begin{bmatrix} 20-12 \\ -15+6 \end{bmatrix}$

$= \begin{bmatrix} -4 \\ \frac{9}{2} \end{bmatrix}$

- Note that any vector $\underline{x} \in \mathcal{L}[\mathbb{A}]$ can be written as a linear combination of the columns of the matrix $\mathbb{A}$

i.e.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -4 \\ \frac{9}{2} \end{bmatrix} = \begin{bmatrix} -4+9 \\ -12+18 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$$\underbrace{\underline{x}}_{N\times 1} = \underbrace{\mathbb{A}}_{N\times M}.\underbrace{\underline{a}}_{M\times 1} \qquad (2)$$

where $\underline{a}$ is an $(M \times 1)$ vector with elements that are the coefficients of this linear combination.

# Common Symbols

$$\underline{u}^T \cdot \underline{u} = [\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi]\begin{bmatrix}\cos\phi\cos\theta\\ \cos\phi\sin\theta\\ \sin\phi\end{bmatrix}$$
$$= \cos^2\phi\cos^2\theta + \cos^2\phi\sin^2\theta + \sin^2\phi$$
$$= \cos^2\phi(\cos^2\theta + \sin^2\theta) + \sin^2\phi = 1$$

| | |
|---|---|
| $N$ | number of array elements |
| $\phi$ | elevation angle |
| $\theta$ | azimuth angle |
| $\underline{u}$ | $[\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi]^T$ |
| | $(3 \times 1)$ real unit-vector pointing towards the direction $(\theta, \phi)$ |
| | $\underline{u}^T\underline{u} = 1$ |
| $c$ | velocity of light |
| $F_c$ | carrier frequency |
| $\lambda$ | wavelength |
| $k$ | wavenumber |
| $\underline{k}$ | wavevector |

wavenumber: number of waves per unit distance (2π).

$k = \frac{2\pi}{\lambda}$

wavevector
- magnitude: wavenumber
- direction: wave propagation

# The Concept of the Projection Operator

- Consider an $(N \times M)$ matrix $\mathbb{A}$ with $M \leq N$ tall
  (i.e. the matrix has $M$ columns)
- Let the columns of $\mathbb{A}$ be linearly independent
  (i.e. a column of $\mathbb{A}$ cannot be written as a linear combination of the remaining $M-1$ columns)
- Then the columns of $\mathbb{A}$ span a subspace $\mathcal{L}[\mathbb{A}]$ of dimensionality $M$ (i.e. $\dim\{\mathcal{L}[\mathbb{A}]\}=M$) lying in an $N$-dimensional space $\mathcal{H}$ (observation space), and this is shown below:

$\underset{\dim(\mathcal{H})}{A\,(N} \times \underset{\text{sub}\ \dim(\mathcal{L}[A])}{M)}$

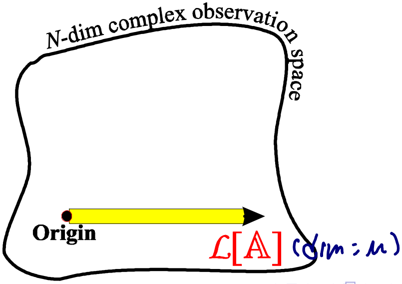
A determined

$\downarrow$

dim(N). sub dim(M) fixed

$\downarrow$

L[A] fixed. $P_A$ fixed.



- Any vector $\underline{x} \in \mathcal{H}$ can be projected on to $\mathcal{L}[\mathbb{A}]$ by using the concept of the projection operator $\mathcal{P}[\mathbb{A}]$ (or $\mathbb{P}_{\mathbb{A}}$). That is:

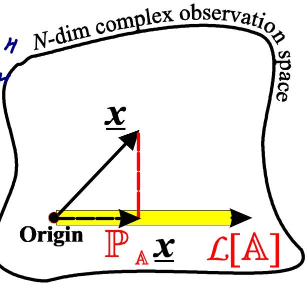
$$\mathcal{P}[\mathbb{A}] = \mathbb{P}_{\mathbb{A}} = \text{projection operator on to } \mathcal{L}[\mathbb{A}] \quad (3)$$

$$= \mathbb{A}(\mathbb{A}^H\mathbb{A})^{-1}\mathbb{A}^H \quad (4)$$

$P_A = \underset{N\times M}{A}(\overset{M\times N}{A^H}\overset{N\times M}{A})^{-1}\underset{M\times N}{A^H}$ $\quad N\times N$ matrix

$P_A \cdot P_A = A(A^HA)^{-1}A^H \cdot A(A^HA)^{-1}A^H$
$= A(A^HA^{-1})(A^HA)(A^HA)^{-1}A^H$
$= A(A^HA^{-1})A^H = P_A$

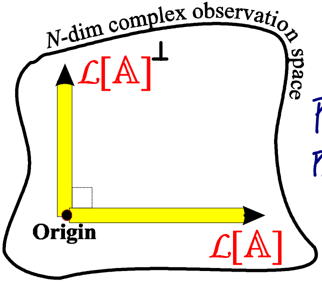
$P_A^H = (A^H)^H[(A^HA)^{-1}]^H A^H$
$= A[(A^HA)^H]^{-1}A^H$
$= A[A^H(A^H)^H]^{-1}A^H$
$= A(A^HA)^{-1}A^H = P_A$




- Properties of Projection Operator

$$\mathbb{P}_{\mathbb{A}} : \begin{cases} (N \times N) \text{ matrix} \\ \mathbb{P}_{\mathbb{A}}\mathbb{P}_{\mathbb{A}} = \mathbb{P}_{\mathbb{A}} \\ \mathbb{P}_{\mathbb{A}} = \mathbb{P}_{\mathbb{A}}^H \end{cases} \quad (5)$$

- $\mathcal{L}[\mathbb{A}]^{\perp}$ denotes the complement subspace to $\mathcal{L}[\mathbb{A}]$




$P_A = A(A^H A)^{-1} A^H$

$P_A^{\perp} = I_N - P_A$

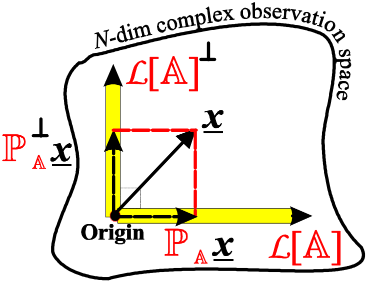
$$\dim\left(\mathcal{L}[\mathbb{A}]\right) = M \tag{6}$$

$$\dim\left(\mathcal{L}[\mathbb{A}]^{\perp}\right) = N - M \tag{7}$$

- $\mathbb{P}_{\mathbb{A}}^{\perp}$ represents the projection operator of $\mathcal{L}[\mathbb{A}]^{\perp}$ and is defined as

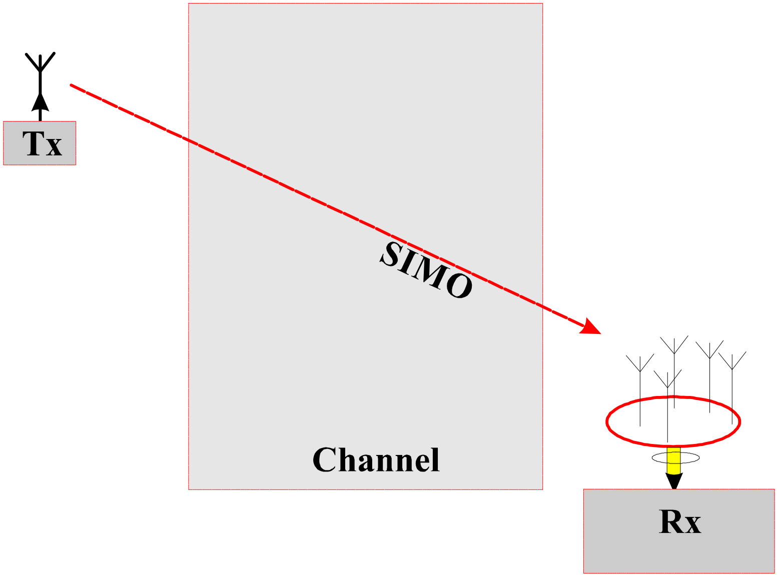
$$\mathbb{P}_{\mathbb{A}}^{\perp} = \mathbb{I}_N - \mathbb{P}_{\mathbb{A}} \tag{8}$$

- Any vector $\underline{x} \in \mathcal{H}$ can be projected on to $\mathcal{L}[\mathbb{A}]^{\perp}$ and $\mathcal{L}[\mathbb{A}]$ as follows




- Comment:

$$\text{if } \underline{x} \in \mathcal{L}[\mathbb{A}] \text{ then } \begin{cases} \mathbb{P}_{\mathbb{A}}\underline{x} = \underline{x} \\ \mathbb{P}_{\mathbb{A}}^{\perp}\underline{x} = \underline{0}_{N-M} \end{cases} \tag{9}$$


Tx
SIMO
Channel
Rx

# Space-Selective Fading

f-selective fading

- A wireless receiver is located (and moves) in our 3D real space.

  $T_{spread}$ interference to the next slot

  $T_{spread} < T_c$ flat fading

- In addition to delay-spread (causing frequency-selective fading) and Doppler-spread (causing time-selective fading) there is also angle-spread

  $T_{spread} > T_c$ frequency selective fading

  $T_{coh} < T_c$ fast fading

  t-selective fading (fast fading)

  $T_{coh} > T_c$ slow fading

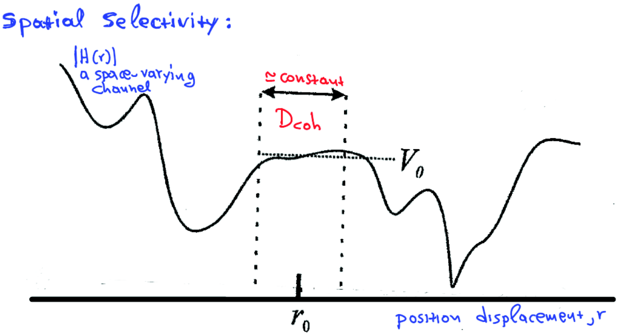
  Over one symbol period, signal not correlated.

- Angle Spread causes Space-Selective fading
- Note that a channel has
  - space-selectivity if it **varies** (i.e. its transfer function varies) as a function of space and
  - spatial-coherence if its transfer function **does not vary** as a function of space over a specified distance ($D_{coh}$) of interest

    where

$$D_{coh} = \text{is known as coherence distance.} \qquad (10)$$

Spatial Selectivity:

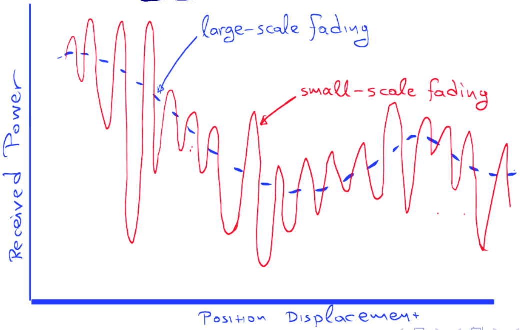



$$|H(r)| \approx V_0 \text{ for } |r - r_0| \leq \frac{D_{\text{coh}}}{2} \tag{11}$$

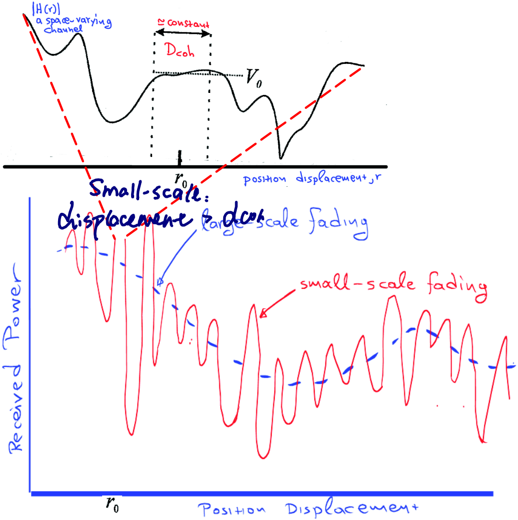
- In other words a wireless channel has spatial coherence if the envelope of the carrier remains constant over a spatial displacement of the receiver.
- $D_{\text{coh}}$ represents the largest distance that a wireless receiver can move with the channel appearing to be static/constant.

# Small-Scale and Large-Scale fading

- If the displacement of the receiver is greater than $D_{coh}$ then the channel experiences **small-scale fading** (this is due to multipaths added constructively or destructively).
- If this displacement is very large (i.e. corresponds to a very large number of wavelengths) then the channel experiences **large-scale fading** (this is due to path loss over large distances and shadowing by large objects - it is typically independent of frequency)



Spatial Selectivity:



# Scattering Function - Wireless Channel Analysis

avg. o/p power.

Auto correlation $\overset{FT}{\underset{IFT}{\Longleftrightarrow}}$ scattering func. power density function

$T_{coh}^{\Delta t} = \frac{1}{B_{Dop}}$

$B_{coh}^{\Delta f} = \frac{1}{T_{spread}}$

$h(t)$ impulse response

$f \rightarrow$ FT $\rightarrow \tau$

$H(f, t, \underline{r})$ transfer function (varies)

$\Phi \begin{vmatrix} \Delta f \\ \Delta t \\ \Delta \underline{r} \end{vmatrix} \rightleftarrows S \begin{vmatrix} \tau \text{ (time delay)} \\ f \text{ (Doppler frequency)} \\ \underline{k} \text{ (wavelength vector)} \end{vmatrix}$

Autocorr

$|\Phi_H(\Delta f; \Delta t; \Delta \underline{r})|$

spaced-frequency
spaced-time
spaced-position
Autocorr. function of the channel

multidim. FT

$S_H(\tau, f, \underline{k})$

Scattering function of the channel

It provides a measure of the average power at the o/p of the channel as a function of $\tau$, $f$ and $\underline{k}$

time delay, Doppler frequ., wavelength vecto

$\Delta f = 0;\ \Delta t = 0$

$\Delta \underline{r} = 0$

$|\Phi_H(\Delta f; \Delta t)|$

spaced-frequency
spaced-time
Autocorr. function of the channel

double FT

$S_H(\tau, f)$

Scattering function of the channel

It provides a measure of the average power at the o/p of the channel as a function of $\tau$ and $f$

$\Delta t = 0$

$\Delta f = 0$

$|\Phi_H(\Delta t)|$

$\Delta t$

$T_{coh}$

$T_{coh} = \frac{1}{B_{Dop}}$

FT

$S_H(f)$

Doppler power spread of the channel

$f$

$B_{Dop}$

Doppler spread

vector Spatial Autocorr. function

$|\Phi_H(\Delta \underline{r})|$

$\Delta \underline{r}$

$D_{coh}$

FT

wavevector Spectrum

$S_H(\underline{k})$

$\underline{k}$

$\underline{k}_{spread}$

wavevector sprea

$|\Phi_H(\Delta f)|$

tail approaches inf. use criteria as -3dB.

$\Delta f$

$B_{coh}$

FT

$B_{coh} = \frac{1}{T_{spread}}$

$S_H(\tau)$

multipath intensity profile or, delay power spectrum

$\tau$

0

$T_{spread}$

multipath spread of the channel (or delay spread)

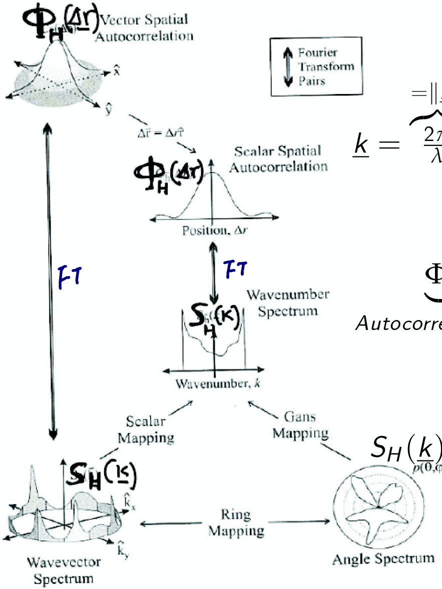
# Wavenumber Spectrum and Angle Spectrum

- If the transfer function of the channel includes "space" then we have: Channel

$$\underbrace{H(f,t,\underline{r})}_{Transfer-function} \Longrightarrow \underbrace{\Phi_H(\Delta f,\Delta t,\Delta \underline{r})}_{\substack{Autocorrelation-function\\(frequency,time,space)}} \xrightarrow{FT} \underbrace{S_H(\tau,\mathfrak{f},\underline{k})}_{Scattering-function} \tag{12}$$

$$\text{where} \left\{ \begin{array}{ll} f & frequency \\ t & time \\ \underline{r} & location\ \text{(x,y,z)} \\ \tau & delay \\ \mathfrak{f} & Doppler\ \ frequency \\ \underline{k} & wavenumber\ \ vector = \|\underline{k}\|\ .\underline{u}(\theta,\phi) \end{array} \right\} \tag{13}$$

- Note that if $\Delta f = 0$ and $\Delta t = 0$ then



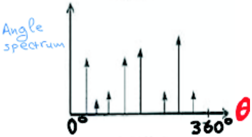

$$\underline{k} = \overbrace{\frac{2\pi}{\lambda}}^{=\|\underline{k}\|} [\cos\theta\cos\phi, \quad \sin\theta\cos\phi, \quad \sin\phi]^T \quad (14)$$

$$\underbrace{\Phi_H(\Delta \underline{r})}_{\substack{Autocorrelation-function \\ (space)}} \quad \overset{FT}{\Longrightarrow} \quad \underbrace{S_H(\underline{k})}_{\substack{Scattering-function \\ (wavenumber\ spectrum)}} \quad (15)$$

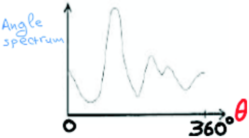
$$S_H(\underline{k}) = \frac{(2\pi)^3 \delta(\|\underline{k}\| - \frac{2\pi}{\lambda})}{(\frac{2\pi}{\lambda})^2} \times \underbrace{p(\theta,\phi)}_{angle\ spectrum} \quad (16)$$

# Types (and examples) of Angle Spectrum

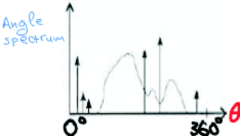
1. **Specular** Angle-Spectrum:




2. **Diffuse** Angle-Spectrum:




3. **Combination of Specular & Diffused:**



# Correspondence between Frequ, Time & Space Parameters

- Consider a wireless channel transfer function $H(f, t, \underline{r})$

| | Frequency | Time | Space |
|---|---|---|---|
| Dependency | $f$ | $t$ | $\underline{r}$ |
| Coherence | $B_{coh}$ coherence bandwidth | $T_{coh}$ coherence time | $D_{coh}$ coherence distance |
| Spectral domain | $\tau$ delay | $\mathfrak{f}$ Doppler freq | $\underline{k}$ wavevector |
| Spectral width | $T_{spread}$ delay spread | $B_{Dop}$ Doppler spread | $\underline{k}_{spread}$ wavevector spread |

- Remember the various types of coherence:

$$\begin{cases} \textit{temporal}\text{ coherence} & \text{-coherence time } T_{\mathrm{coh}} \\ \textit{frequency}\text{ coherence} & \text{-coherence bandwidth } B_{\mathrm{coh}} \\ \textit{spatial}\text{ coherence} & \text{-coherence distance } D_{\mathrm{coh}} \end{cases}$$

# The Relationship between Coherence-Time and Bandwidth

- We have seen that one of the wireless channels classification is "slow" and "fast fading - as this is shown below:

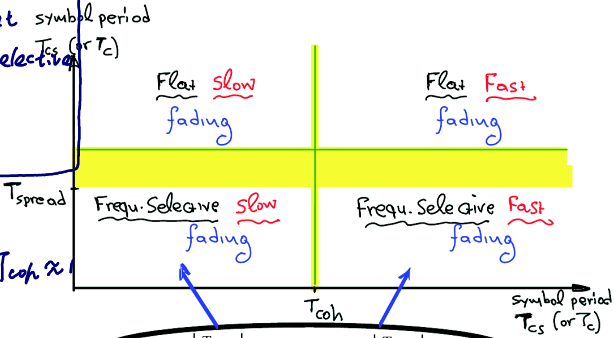
$T_{spread} < T_c$ flat
$T_{spread} > T_c$ f-selective

$T_{coh} < T_c$ fast
$T_{coh} > T_c$ slow

when $T_c \approx \frac{1}{B} \approx T_{coh}$.

$\frac{1}{B} \approx T_{coh} \Rightarrow B \cdot T_{coh} \approx 1$

path number
$L = \left\lfloor \frac{T_{spread}}{T_c} \right\rfloor + 1$




$L = \left\lfloor \frac{T_{spread}}{T_{cs}} \right\rfloor + 1$ or $L = \left\lfloor \frac{T_{spread}}{T_c} \right\rfloor + 1$

- A trade-off between "slow" and "fast" fading is when $T_{coh} \simeq T_{cs}$ (or, for CDMA, $T_{coh} \simeq T_c$). This implies

$$\boxed{T_{coh} \times B \simeq 1 \text{ (or, for CDMA: } T_{coh} \times B_{ss} \simeq 1)} \quad (17)$$

- An electromagnetic wave of frequency $F_c$ (carrier frequency) travelling for the time $T_{\text{coh}}$ with velocity $c$ (speed of light, i.e. $3 \times 10^8 m/s$) will cover a distance $d$, which is given as follows:

$d_0 = c \cdot T_{coh}$

$= \lambda_c F_c \cdot T_{coh}$

$\therefore T_{coh} = \frac{d_0}{\lambda_c F_c}$

$\because T_{coh} \cdot B \approx 1$ at threshold

$\therefore \frac{d_0}{\lambda_c F_c} \cdot B \approx 1$

$\therefore \boxed{d_0 \approx \frac{\lambda_c F_c}{B} = \frac{c}{B}}$

$T_c$ ($T_{cs}$) - channel symbol duration
$B$ - channel system bandwidth

$$\begin{aligned} d &= c.T_{\text{coh}} \\ &= \underbrace{F_c\lambda_c}_{=c} T_{\text{coh}} \Longrightarrow T_{\text{coh}} = \frac{d}{F_c\lambda_c} \end{aligned} \qquad (18)$$

slow fading: $d < d_0$ ($t < T_{coh}$)
fast fading: $d > d_0$ ($t > T_{coh}$)

Using Equations 17 and 18 we have

$$\frac{d}{F_c\lambda_c}B \simeq 1 \Longrightarrow d \simeq \frac{F_c}{B}\lambda_c(=\frac{c}{B}) \qquad (19)$$

- We have seen that in slow fading region $T_{\text{coh}} > T_{cs}$ (or, for CDMA, $T_{\text{coh}} > T_c$) and this implies that $T_{\text{coh}} \times B > 1$ and, finally,

$$d < \frac{F_c}{B}\lambda_c(=\frac{c}{B}) \qquad (20)$$

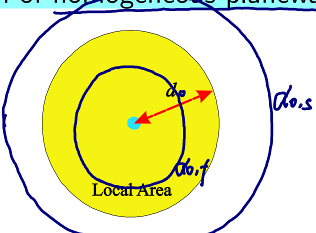
- Thus overall, for slow fading:

$$\boxed{T_{\text{coh}} \times B \gtrsim 1 \Longrightarrow d \lesssim \frac{F_c}{B}\lambda_c} \qquad (21)$$

# The Concept of the "Local Area"

for Rx

- "Local Area" is the largest volume of free-space (i.e. $\frac{4}{3}\pi d^3$) about a specific point in space $\underline{r}_o = [r_{x_0}, r_{y_0}, r_{z_0}]^T$ (e.g. the reference point of the Rx-array) in which the wireless channel can be modelled as the summation of homogeneous planewaves, where




$\begin{cases} d > d_o & \text{fast fading} \\ d \le d_o & \text{slow fading} \end{cases}$

Local area (slow fading)

$$d \lesssim \frac{F_c}{B}\lambda_c \qquad (22)$$

- Example: Local area for WiFi electromagnetic waves ($F_c = 2.4\text{GHz}, B = 5\text{MHz}$)

$$d \lesssim \frac{F_c}{B}\lambda_c \Longrightarrow d \lesssim \frac{2.4 \times 10^9}{5 \times 10^6} \times 0.125 = 60m$$

# Antenna Array

- Consider a single path from Tx single antenna system to an array of $N$ antennas
- An array system is a collection of $N > 1$ sensors (transducing elements, receivers, antennas, etc) distributed in the 3-dimensional Cartesian space, with a common reference point.
- Consider an antenna-array Rx with locations given by the matrix

$$\left[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N\right] = \left[\underline{r}_x, \underline{r}_y, \underline{r}_z\right]^T \quad (3 \times N) \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n \end{bmatrix} \tag{23}$$

  where $\underline{r}_k$ is a $3{\times}1$ real vector denoting the location of the $k^{th}$ sensor $\forall k = 1, 2, \ldots, N$

  and $\underline{r}_x$,$\underline{r}_y$ and $\underline{r}_z$ are $N \times 1$ vectors with elements the x, y and z coordinates of the $N$ antennas
- The region over which the sensors are distributed is called the aperture of the array. In particular the array aperture is defined as follows

$$\boxed{\text{array aperture} \doteq \max_{ij} \left\| \underline{r}_i - \underline{r}_j \right\|} \tag{24}$$

# Array Manifold Vector

$\underline{S}(\theta,\phi) = e^{-j\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} \cdot k(\theta,\phi)}$

$k(\theta,\phi) = \begin{cases} \frac{2\pi}{\lambda_c} u(\theta,\phi) & \text{- meters} \\ \pi u(\theta,\phi) & \text{- } \frac{\lambda}{2}\text{s} \end{cases}$

- It is also know as Array Response Vector.
- Modelling of Array Manifold Vectors (see Chapter 1 of my book):

$(3 \times N)^T = N \times 3$ $\quad 3 \times 1$

$$\underline{S}(\theta,\phi) = \exp(-j[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N]^T \underline{k}(\theta,\phi)) \quad (25)$$

$$\overset{or}{=} \exp(-j[\underline{r}_x, \underline{r}_y, \underline{r}_z]\underline{k}(\theta,\phi)) \quad (26)$$

$$= (N \times 1) \text{ complex vector}$$

where

$$\boxed{\underline{k}(\theta,\phi) = \begin{cases} \frac{2\pi F_c}{c}.\underline{u}(\theta,\phi) = \frac{2\pi}{\lambda_c}\underline{u}(\theta,\phi) & \text{in meters} \\ \pi \underline{u}(\theta,\phi) & \text{in units of halfwavelength} \end{cases}}$$

$$= \text{wavenumber vector}$$

$$\underline{u}(\theta,\phi) = [\cos\theta\cos\phi, \ \sin\theta\cos\phi, \ \sin\phi]^T \quad (27)$$

$$= (3 \times 1) \text{ real unit-vector pointing towards the direction } (\theta,\phi)$$

$$\|\underline{u}(\theta,\phi)\| = 1 \quad (28)$$

- In many cases the signals are assumed to be on the (x,y) plane (i.e. $\phi = 0^\circ$). In this case the manifold vector is simplified to

$K(\theta,0) = \pi u(\theta,0) = (\cos\theta, \sin\theta, 0)$

$$\begin{aligned} \underline{S}(\theta) &= \exp(-j[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N]^T \underline{k}(\theta, 0^\circ)) \\ &= \exp(-j\pi(\underline{r}_x \cos\theta + \underline{r}_y \sin\theta)) \end{aligned} \quad (29)$$

$N\times 3$ $\quad 3\times 1$

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_N & y_N & z_N \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta + y_1\sin\theta \\ x_2\cos\theta + y_2\sin\theta \\ \vdots \\ x_N\cos\theta + y_N\sin\theta \end{bmatrix} = r_x\cos\theta + r_y\sin\theta$$

- A popular class of arrays is that of linear arrays: $\underline{r}_y = \underline{r}_z = \underline{0}_N$ in this case, Equation 25 is simplified to

$$\underline{S}(\theta) = \exp(-j\pi\underline{r}_x \cos\theta) \quad (30)$$

- Summary: An array maps one or more real directional parameters $p$ or $(p,q)$ to an $(N \times 1)$ complex vector $\underline{S}(p)$ or $\underline{S}(p,q)$, known as array manifold vector, or array response vector, or source position vector. That is

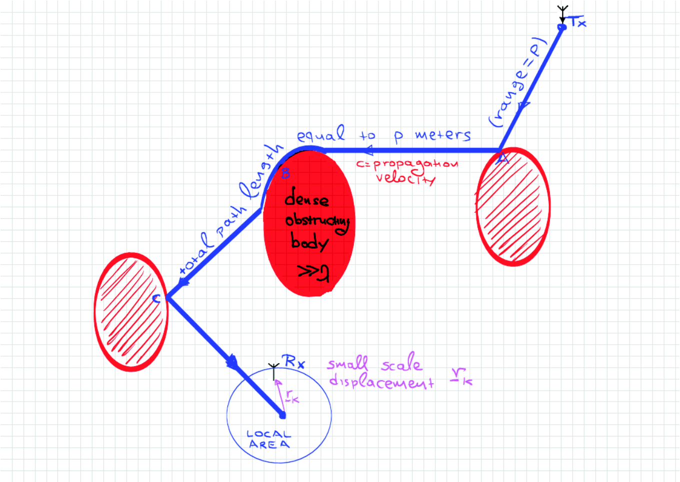
Rx coordinates

array manifold vector.

$$\begin{aligned} p &\in \mathcal{R}^1 \overset{\underline{r}}{\longmapsto} \underline{S}(p) \in \mathcal{C}^N && (31) \\ \text{or } (p,q) &\in \mathcal{R}^1 \overset{\underline{r}}{\longmapsto} \underline{S}(p,q) \in \mathcal{C}^N && (32) \end{aligned}$$

- Note: Ideally Equations 31 and 32 should be an '*one-to-one*' mapping

# Proof Equation 25 (Array Manifold Vector)



- With reference to the previous figure, consider at the Tx a single uniform planewave $\delta(t)\exp(j2\pi F_c t)$ that travels with the velocity of light $c$ for time $\tau$ and covers a distance $d$ arriving at the Rx., i.e. $d = c\tau$

$$\begin{aligned}
\text{at the Tx} &= \exp\left(j2\pi F_c t\right)\delta(t) \\
\text{at the Rx's reference point} &= \left(\frac{1}{d}\right)^{\alpha}\exp\left(j\phi\right)\exp\left(j2\pi F_c(t-\tau)\right)\delta(t-\tau) \\
&= \underbrace{\left(\frac{1}{d}\right)^{\alpha}\exp\left(j\phi\right)\exp\left(-j2\pi F_c\frac{d}{c}\right)}_{\triangleq\beta}\exp\left(j2\pi F_c t\right)\delta(t-\tau) \\
&= \beta.\exp\left(j2\pi F_c t\right).\delta(t-\tau) \qquad (33)
\end{aligned}$$

- If the $k$-th antenna of the array is in the "Local Area" about the reference point of the array then

$$\begin{aligned}
\text{at the } k\text{-th antenna of the Rx} &= \beta.\exp\left(j2\pi F_c(t-\Delta\tau_k)\right).\delta(t-\Delta\tau_k-\tau) \\
&= \beta.\exp\left(-j2\pi F_c\Delta\tau_k\right).\exp\left(j2\pi F_c t\right).\delta(t-\tau) \qquad (34)
\end{aligned}$$

$\tau >> \Delta\tau_k$

$\delta(t-\Delta\tau_k-\tau)$

$\approx \delta(t-\tau)$

- That is, the baseband signal is:

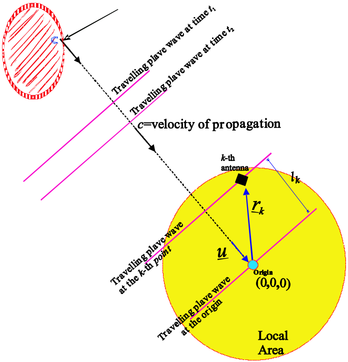
$$\text{at the } k^{\text{th}} \text{ antenna of the Rx} = \beta.\exp\left(-j2\pi F_c\Delta\tau_k\right).\delta(t-\tau) \qquad (35)$$

- However, $\Delta\tau_k$ is given as follows:

$$\Delta\tau_k = \frac{\underline{r}_k^T \underline{u}(\theta,\phi)}{c} \tag{36}$$

where $\underline{r}_k \in R^{3\times 1}$ denotes the Cartesian coordinates of the $k$-th Rx-antenna in metre.

- Proof: of Equation 36: With reference to the following figure we have:




$$\begin{aligned}\Delta\tau_k &= \frac{\sqrt{\underline{r}_k^T\underline{u}\left(\underline{u}^T\underline{u}\right)^{-1}\underline{u}^T\underline{r}_k}}{c} \\ &= \frac{\sqrt{\underline{r}_k^T\,\underline{u}\,\underline{u}^T\,\underline{r}_k}}{c} \\ &= \frac{\sqrt{\left(\underline{r}_k^T\underline{u}\right)^2}}{c} \\ &= \frac{\underline{r}_k^T\underline{u}}{c}\end{aligned} \tag{37}$$

- Thus, Equation 35 can be expressed as follows:

baseband k-th Rx $= \beta e^{-j2\pi F_c \Delta\tau_k} \delta(t-\tau)$

$$\text{at the } k\text{-th antenna of the receiver (baseband)} = \beta.\exp\left(-j\frac{2\pi F_c}{c}\underline{r}_k^T\underline{u}(\theta,\phi)\right).\delta(t-\tau)$$

$\Delta\tau_k = \frac{\underline{r}_k^T \cdot \underline{u}(\theta,\phi)}{c}$

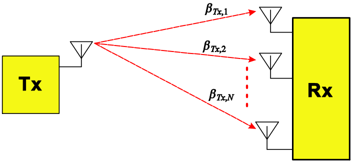
- That is, the Tx planewave $\exp(j2\pi F_c t)\,\delta(t)$ arrives at each antenna of the array and produces, at time $t=\tau$, a constant-amplitude voltage-vector as follows:

Rx

$$\begin{bmatrix} 1^{\text{st}} \text{ ant.} \\ 2^{\text{nd}} \text{ ant.} \\ \vdots \\ N^{\text{th}} \text{ ant.} \end{bmatrix} = \begin{bmatrix} \beta\exp\left(-j\frac{2\pi F_c}{c}\underline{r}_1^T\underline{u}(\theta,\phi)\right).\delta(t-\tau) \\ \beta\exp\left(-j\frac{2\pi F_c}{c}\underline{r}_2^T\underline{u}(\theta,\phi)\right).\delta(t-\tau) \\ \vdots \\ \beta\exp\left(-j\frac{2\pi F_c}{c}\underline{r}_N^T\underline{u}(\theta,\phi)\right).\delta(t-\tau) \end{bmatrix} = \beta \underbrace{\begin{bmatrix} \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_1^T\underline{u}(\theta,\phi)\right) \\ \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_2^T\underline{u}(\theta,\phi)\right) \\ \vdots \\ \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_N^T\underline{u}(\theta,\phi)\right) \end{bmatrix}}_{\text{array manifold vector } \underline{S}(\theta,\phi)} \delta(t-\tau)$$

$$= \beta.\underline{S}(\theta,\phi).\delta(t-\tau) \tag{38}$$

- i.e.




$$\Leftrightarrow \begin{bmatrix} \beta_{Tx,1} \\ \beta_{Tx,2} \\ \vdots \\ \beta_{Tx,N} \end{bmatrix} = \beta.\underline{S}(\theta,\phi) \tag{39}$$

- where

$$\underline{r}_n = \begin{pmatrix} r_{nx} \\ r_{ny} \\ r_{nz} \end{pmatrix} \quad \underline{u} = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi \end{pmatrix}$$

$$\underline{r}_1^T = (r_{nx} \; r_{ny} \; r_{nz})$$
$$\vdots$$
$$\underline{r}_N^T = (r_{Nx} \; r_{NY} \; r_{Nz})$$
$$\underline{r}_x$$

$$\underline{S}(\theta,\phi) = \begin{bmatrix} \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_1^T\underline{u}(\theta,\phi)\right) \\ \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_2^T\underline{u}(\theta,\phi)\right) \\ \cdots \\ \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_k^T\underline{u}(\theta,\phi)\right) \\ \cdots \\ \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_N^T\underline{u}(\theta,\phi)\right) \end{bmatrix} \quad (40)$$

$$= \exp\left(-j\frac{2\pi F_c}{c}\left[\underline{r}_1, \quad \underline{r}_2, \quad \ldots \quad \underline{r}_N\right]^T \underline{u}(\theta,\phi)\right) \quad (41)$$

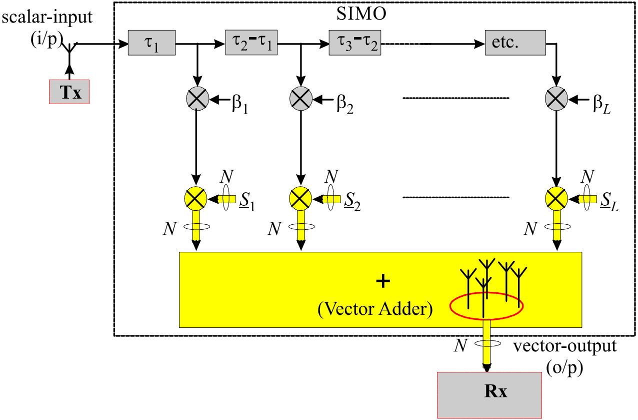
$$= \exp\left(-j\frac{2\pi F_c}{c}\underbrace{\left[\underline{r}_x, \quad \underline{r}_y, \quad \underline{r}_z\right]}_{N\times 3}\underbrace{\underline{u}(\theta,\phi)}_{3\times 1}\right) \quad (42)$$

# Multipath SIMO Channel

- Let us assume that the transmitted signal arrives at the reference point of an array receiver via $L$ paths (multipaths).
- Consider that the $\ell^{th}$ path arrives at the array from direction $(\theta_\ell, \phi_\ell)$ with channel propagation parameters $\beta_\ell$ and $\tau_\ell$ representing the complex path gain and path-delay, respectively.
- Note that $\theta_\ell$ and $\phi_\ell$ represent the azimuth and elevation angles respectively associated with $\ell$-th path.
- Let us assume that the $L$ paths are arranged such that

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_\ell \leq \ldots \leq \tau_L \tag{43}$$

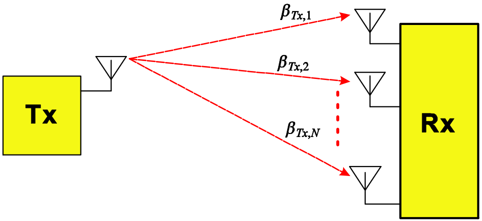
- Furthermore, the path coefficients $\beta_\ell$ model the effects of path losses and shadowing, in addition to random phase shifts due to reflection; they also encompass the effects of the phase offset between the modulating carrier at the transmitter and the demodulating carrier at the receiver, as well as differences in the transmitter powers.




- The vector $\underline{S}_\ell = \underline{S}(\theta_\ell, \phi_\ell) \in C^N$, is the array manifold vector of the $\ell$-th path .
- The impulse response (vector) of the SIMO multipath channel is

$$\boxed{\text{SIMO: } \underline{h}(t) = \sum_{\ell=1}^{L} \underline{S}(\theta_\ell, \phi_\ell).\beta_\ell.\delta(t-\tau_\ell)} \quad (44)$$

- i.e. multipath SIMO channel



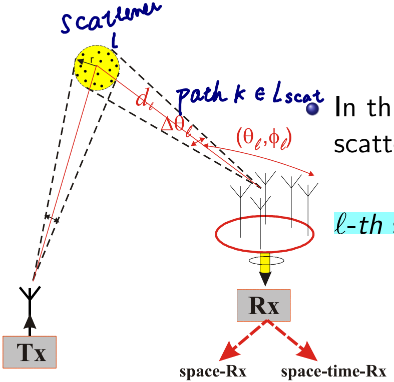

$$\Updownarrow$$

$$\begin{bmatrix} \beta_{Tx,1} \\ \beta_{Tx,2} \\ \vdots \\ \beta_{Tx,N} \end{bmatrix} = \sum_{\ell=1}^{L} \beta_{\ell} . \underline{S}(\theta_{\ell}, \phi_{\ell}) \qquad (45)$$

# Scatterers in SIVO channels

- Consider a scatterer ($\ell$-*th* scatterer, say) which can be seen as a large number of paths ($L_{scat}$, say) around the direction $(\theta_\ell, \phi_\ell)$. That is, the direction-of-arrival of the $k$-th path satisfies the condition

$$(\theta_\ell, \phi_\ell) - \frac{\Delta\theta_\ell}{2} \leq (\theta_{\ell k}, \phi_{\ell k}) \leq (\theta_\ell, \phi_\ell) + \frac{\Delta\theta_\ell}{2}, \forall k \tag{46}$$




- In this case the impulse response for this scatterer can be written as follows:

$$\ell\text{-}th \text{ scatterer} = \sum_{k=1}^{L_{scat}} \underline{S}(\theta_{\ell k}, \phi_{\ell k}).\beta_{\ell k}.\delta(t - \tau_{\ell k}) \tag{47}$$

- If $\boxed{\Delta\theta_\ell = \text{relatively small}}$ then

$$(\theta_{\ell 1}, \phi_{\ell 1}) \simeq (\theta_{\ell 2}, \phi_{\ell 2}) \simeq ... \simeq (\theta_{\ell L_{scat}}, \phi_{\ell L_{scat}}) \triangleq (\theta_\ell, \phi_\ell) \quad (48)$$

$$\tau_{\ell 1} \simeq \tau_{\ell k} \simeq ... \simeq \tau_{\ell L_{scat}} \triangleq \tau_\ell \quad (49)$$

and, thus, $\Delta\theta$ small $\rightarrow \begin{cases} (\theta_{\ell i}, \phi_{\ell i}) \approx (\theta_\ell, \phi_\ell) \\ \tau_{\ell i} \approx \tau_\ell \end{cases}$

$$\begin{aligned}
\ell\text{-}th \text{ scatterer} &= \sum_{k=1}^{L_{scat}} \underline{S}(\theta_{\ell k}, \phi_{\ell k}).\beta_{\ell k}.\delta(t - \tau_{\ell k}) \\
&= \sum_{k=1}^{L_{scat}} \underline{S}(\theta_\ell, \phi_\ell).\beta_{\ell k}.\delta(t - \tau_\ell) \\
&= \underline{S}(\theta_\ell, \phi_\ell).\delta(t - \tau_\ell) \underbrace{\sum_{k=1}^{L_{scat}} \beta_{\ell k}}_{\triangleq \beta_\ell} \\
&= \underline{S}(\theta_\ell, \phi_\ell).\beta_\ell.\delta(t - \tau_\ell) \\
&= \underline{S}_\ell.\beta_\ell.\delta(t - \tau_\ell) \quad (50)
\end{aligned}$$

- In other words a scatterer can be seen as a single path with direction $(\theta_\ell, \phi_\ell)$ and fading coefficient the term $\beta_\ell$ that represents the addition/combination of the fading coefficients of all paths of this scatterer i.e.

$$\beta_\ell = \sum_{k=1}^{L_{scat}} \beta_{\ell k} \tag{51}$$

.

- N.B.: Another way to represent a scatterer is using Taylor Series Expansion. This will involve $\dot{\underline{S}}_\ell$ (i.e the first derivative of the manifold vector $\underline{S}_\ell$) and $\Delta\theta_\ell$.

# Modelling of the Received Vector-Signal

- Consider a single Tx transmitting a baseband signal $m(t)$ via an $L$-path SIVO channel.
- The received $(N \times 1)$ vector-signal $\underline{x}(t)$ can be modelled as follows:

$S(\theta,\phi) = e^{-j2\pi F_c \tau} = e^{-j2\pi \frac{F_c}{c}[r_x\ r_y\ r_z]\, u(\theta,\phi)}$

$\underline{S}_i = e^{j \underline{r}_i^T \underline{k}}$

$$
\begin{aligned}
\underline{x}(t) &= \underline{h}(t) * m(t) + \underline{n}(t) \\
&= \left( \sum_{\ell=1}^{L} \underline{S}(\theta_\ell, \phi_\ell).\beta_\ell.\delta(t-\tau_\ell) \right) * m(t) + \underline{n}(t) \\
\Rightarrow \underline{x}(t) &= \underline{\sum_{\ell=1}^{L} \underline{S}(\theta_\ell, \phi_\ell).\beta_\ell}.\underline{m(t-\tau_\ell)} + \underline{n}(t) \\
&= \underline{\mathbb{S}}\,\underline{\underline{m}(t)} + \underline{n}(t) \qquad (52)
\end{aligned}
$$

$\mathbb{S} = [\underline{S}_1\ \underline{S}_2 \cdots \underline{S}_L]$ $\quad \underline{m}(t) = \begin{bmatrix} \beta_1 m(t-\tau_1) \\ \beta_2 m(t-\tau_2) \\ \vdots \\ \beta_L m(t-\tau_L) \end{bmatrix}$

where

$$
\begin{aligned}
\mathbb{S} &= \left[\underline{S}_1, \quad \underline{S}_2, \quad \cdots, \quad \underline{S}_L\right], \text{ with } \underline{S}_\ell \triangleq \underline{S}(\theta_\ell, \phi_\ell), \ell = 1, .., L \qquad (53) \\
\underline{m}(t) &= \left[\beta_1 m(t-\tau_1), \quad \beta_2 m(t-\tau_2), \quad \cdots, \quad \beta_L m(t-\tau_L)\right]^T \qquad (54) \\
\underline{n}(t) &= \left[\mathrm{n}_1(t), \quad \mathrm{n}_2(t), \quad \cdots, \quad \mathrm{n}_N(t)\right]^T \qquad (55)
\end{aligned}
$$

# Multi-user SIMO

- Next consider an Rx array of $N$ antennas operating the the presence of $M$ co-channel transmitters/users.
- In this case we have added the subscript $i$ to refer to the $i$-th Tx.
- The received $(N \times 1)$ vector-signal $\underline{x}(t)$ from all $M$ transmitters/users (transmitting at the same time on the same frequency band) can be modelled as follows:.

$$\begin{aligned}\underline{x}(t) &= \sum_{i=1}^{M}\sum_{\ell=1}^{L}\underline{S}_{i\ell}.\beta_{i\ell}.m_i(t-\tau_\ell)+\underline{\mathrm{n}}(t) \\ &= \mathbb{S}\underline{m}(t)+\underline{\mathrm{n}}(t) \qquad (56)\end{aligned}$$

with

$$\underline{S}_{il} \triangleq \underline{S}(\theta_{il},\phi_{il})$$

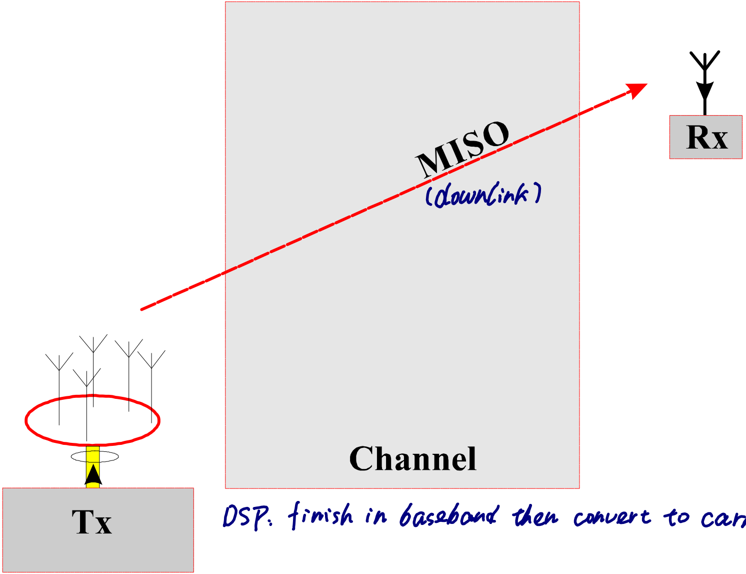
where $\mathbb{S}$, $\underline{m}(t)$ and $\underline{\mathrm{n}}(t)$ defined as follows:↘↘

$$\mathbb{S} \triangleq \Big[ \overbrace{\underline{S}_{11}, \underline{S}_{12}, ..., \underline{S}_{1L}}^{\substack{\text{user 1} \\ \triangleq \mathbb{S}_1}}, \overbrace{\underline{S}_{21}, \underline{S}_{22}, ..., \underline{S}_{2L}}^{\substack{\text{user 2} \\ \triangleq \mathbb{S}_2}}, ..., \overbrace{\underline{S}_{M1}, \underline{S}_{M2}, ..., \underline{S}_{ML}}^{\substack{\text{user } M \\ \triangleq \mathbb{S}_M}} \Big] \quad (57)$$

$$\underline{m}(t) \triangleq \begin{bmatrix} \left.\begin{matrix} \beta_{11} m_1(t-\tau_{11}) \\ \beta_{12} m_1(t-\tau_{12}) \\ \vdots \\ \beta_{1L} m_1(t-\tau_{1L}) \end{matrix}\right\} \triangleq \underline{m}_1(t) \quad \text{user 1} \\ \begin{matrix} \beta_{21} m_2(t-\tau_{21}) \\ \vdots \end{matrix} \qquad \vdots \\ \left.\begin{matrix} \beta_{M1} m_M(t-\tau_{M1}) \\ \beta_{M2} m_M(t-\tau_{M2}) \\ \vdots \\ \beta_{ML} m_M(t-\tau_{ML}) \end{matrix}\right\} \triangleq \underline{m}_M(t) \quad \text{user } M \end{bmatrix} \quad (58)$$

$$\underline{\mathrm{n}}(t) \triangleq \left[\mathrm{n}_1(t), \quad \mathrm{n}_2(t), \quad ..., \quad \mathrm{n}_N(t)\right]^T \quad (59)$$

# Wireless MISO Channels




DSP: finish in baseband then convert to carriers.

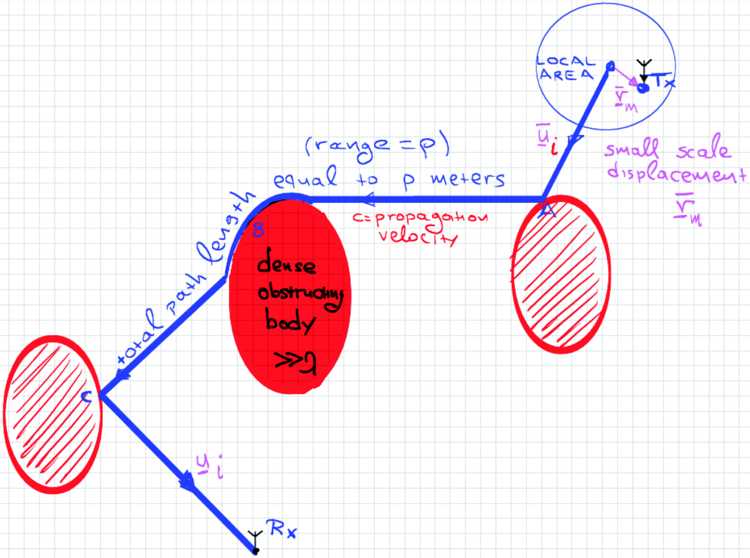
# Reciprocity Theorem

- Antenna characteristics are independent of the direction of energy flow.
  - The impedance & radiation pattern are the same when the antenna radiates a signal and when it receives it.
- The Tx and Rx array-patterns are the same.
- The Tx-array is an array of $\overline{N}$ bar → Tx. elements/sensors/antennas with locations $\underline{\underline{\overline{\mathbf{r}}}}$

$$\underline{\underline{\overline{\mathbf{r}}}} = [\underline{\overline{r}}_1, \underline{\overline{r}}_2, ..., \underline{\overline{r}}_{\overline{N}}] = [\underline{\overline{r}}_x, \underline{\overline{r}}_y, \underline{\overline{r}}_z]^T \; (3 \times \overline{N})$$

with $\underline{\overline{r}}_m$ denoting the location of the $m^{th}$ Tx-sensor $\forall m = 1, 2, ..., \overline{N}$

- Notation:
  - the bar at the top of a symbol, i.e. $\overline{(.)}$, denotes a Tx-parameter.




$$\beta \exp\left(+j\frac{2\pi}{\lambda_c}\underline{u}_i^T\underline{r}_m\right)\delta\left(t-\frac{\rho}{c}\right)$$

- That is, if the Tx is displaced at a specific point $\overline{\underline{r}}_m$ (within its local area $\overline{L}_A$) and the direction of the planewave propagation is described by the vector $\overline{\underline{u}}_i = \underline{u}(\overline{\theta}, \overline{\phi})$ where

$$\underline{u}_i(\overline{\theta},\overline{\phi}) = \left[\cos\overline{\theta}\cos\overline{\phi},\quad \sin\overline{\theta}\cos\overline{\phi},\quad \sin\overline{\phi}\right]^T$$

- **If** the Tx employs an array of $\overline{N}$ elements/sensors/antennas with locations $\overline{\underline{\mathbf{r}}}$

  **then** the channel impulse response (single path) is

SIMO (Rx)

$$\underline{S}(\theta,\phi) = e^{-j\frac{2\pi F_c}{c}[r_x\ r_y\ r_z]\,\underline{u}(\theta,\phi)}$$

$$\underline{u}(\theta,\phi) = \begin{bmatrix}\cos\theta\cos\phi\\ \sin\theta\cos\phi\\ \sin\phi\end{bmatrix}$$

$$\underline{h}(t) = \sum_{i=1}^{L} \underline{S}(\theta_i,\phi_i)\beta_i\,\delta(t-\tau_i)$$

MISO (Tx)

$$\overline{\underline{S}}(\overline{\theta},\overline{\phi}) = e^{-j\frac{2\pi F_c}{c}[\overline{r}_x\ \overline{r}_y\ \overline{r}_z]\,\underline{u}(\overline{\theta},\overline{\phi})}$$

$$\underline{k}(\overline{\theta},\overline{\phi}) = \begin{cases}\frac{2\pi F_c}{c}\underline{u}(\overline{\theta},\overline{\phi}) = \frac{2\pi}{\lambda_c}\underline{u}(\overline{\theta},\overline{\phi})\\ \pi\,\underline{u}(\overline{\theta},\overline{\phi})\end{cases}$$

$$h(t) = \beta\overline{\underline{S}}^H\underline{\delta}\left(t-\frac{\rho}{c}\right) \tag{60}$$

- where

$$\overline{\underline{S}}(\overline{\theta},\overline{\phi}) = \exp\left(+j[\overline{\underline{r}}_1,\overline{\underline{r}}_2,\ldots,\overline{\underline{r}}_N]^T\,\underline{k}(\overline{\theta},\overline{\phi})\right) \tag{61}$$

$$\overset{or}{=} \exp\left(-j[\overline{\underline{r}}_x,\overline{\underline{r}}_y,\overline{\underline{r}}_z]\,\underline{k}(\overline{\theta},\overline{\phi})\right) = e^{-j[r_x\ r_y\ r_z]\underline{k}(\overline{\theta},\overline{\phi})} \tag{62}$$

$$= (N\times 1) \text{ complex vector}$$

meters

$$\underline{k}(\overline{\theta},\overline{\phi}) = \begin{cases}\frac{2\pi F_c}{c}.\underline{u}(\overline{\theta},\overline{\phi}) = \frac{2\pi}{\lambda_c}.\underline{u}(\overline{\theta},\overline{\phi}) & \text{in meters}\\ \pi.\underline{u}(\overline{\theta},\overline{\phi}) & \text{in units of halfwavelength}\end{cases}$$

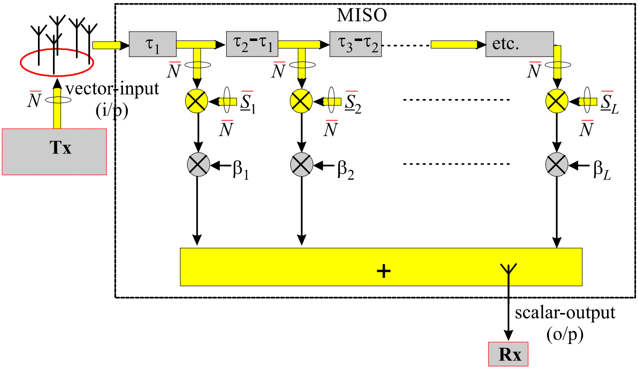
wavenumber vector

# Multipath MISO Channel

- Let us assume that the transmitted signal(s) from the Tx-array arrives at the receiver via $L$ resolvable paths (multipaths).
- Consider that the $\ell^{th}$ path's direction-of-departure is $(\overline{\theta}_\ell, \overline{\phi}_\ell)$ and propagates to the Rx with channel propagation parameters $\beta_\ell$ and $\tau_\ell$ representing the complex path gain and path-delay, respectively.
- Let us assume that the $L$ paths are arranged such that

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_\ell \leq \ldots \leq \tau_L \tag{63}$$

- remember that the path coefficients $\beta_\ell$ model the effects of path losses and shadowing, in addition to random phase shifts due to reflection; they also encompass the effects of the phase offset between the modulating carrier at the transmitter and the demodulating carrier at the receiver, as well as differences in the transmitter power



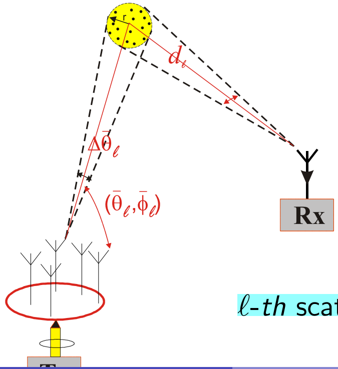

- The vector $\underline{\overline{S}}_\ell = \underline{\overline{S}}(\overline{\theta}_\ell, \overline{\phi}_\ell) \in C^{\overline{N}}$ is the array manifold vector of the $\ell$-th path .
- The impulse response of the MISO (VISO) channel is

$$\boxed{\text{MISO (VISO): } h(t) = \sum_{\ell=1}^{L} \beta_\ell . \underline{\overline{S}}_\ell^H \underline{\delta}(t-\tau_\ell)} \tag{64}$$

# Scatterers

- Consider a scatterer ($\ell$-*th* scatterer, say) which can be seen as a large number of paths ($L_\ell$, say) around the direction $(\overline{\theta}_\ell, \overline{\phi}_\ell)$. That is, the direction-of-arrival of the $k$-th path satisfies the condition

$$(\overline{\theta}_\ell, \overline{\phi}_\ell) - \frac{\overline{\Delta\theta_\ell}}{2} \leq (\overline{\theta}_{\ell k}, \overline{\phi}_{\ell k}) \leq (\overline{\theta}_\ell, \overline{\phi}_\ell) + \frac{\overline{\Delta\theta_\ell}}{2}, \forall k \qquad (65)$$




- $L_\ell$= number of paths ($\ell^{th}$ scatterer)
- In this case the impulse response for this scatterer can be written as follows:

$$\ell\text{-}th \text{ scatterer} = \sum_{k=1}^{L_\ell} \beta_{\ell k} . \underline{S}^H (\overline{\theta}_{\ell k}, \overline{\phi}_{\ell k}) . \underline{\delta}(t - \tau_{\ell k}) \qquad (66)$$

- **If** $\overline{\Delta\theta}_\ell$ = relatively small **then**

$$\begin{aligned}(\overline{\theta}_{\ell 1},\overline{\phi}_{\ell 1}) &\simeq (\overline{\theta}_{\ell 2},\overline{\phi}_{\ell 2}) \simeq \ldots \simeq (\overline{\theta}_{\ell L_\ell},\overline{\phi}_{\ell L_\ell}) \triangleq (\overline{\theta}_{\ell},\overline{\phi}_{\ell}) \quad (67a)\\ \tau_{\ell 1} &\simeq \tau_{\ell k} \simeq \ldots \simeq \tau_{\ell L_\ell} \triangleq \tau_\ell \quad (67b)\end{aligned}$$

and, thus,

$$\begin{aligned}\ell\text{-}th \text{ scatterer} &= \sum_{k=1}^{L_\ell} \beta_{\ell k}\underline{\overline{S}}^H(\overline{\theta}_{\ell k},\overline{\phi}_{\ell k}).\underline{\delta}(t-\tau_{\ell k})\\ &= \sum_{k=1}^{L_\ell} \beta_{\ell k}.\underline{\overline{S}}^H(\overline{\theta}_{\ell},\overline{\phi}_{\ell}).\underline{\delta}(t-\tau_{\ell})\\ &= \underline{\overline{S}}^H(\overline{\theta}_{\ell},\overline{\phi}_{\ell}).\underline{\delta}(t-\tau_{\ell})\underbrace{\sum_{k=1}^{L_\ell}\beta_{\ell k}}_{\triangleq\beta_\ell}\\ &= \underline{\overline{S}}^H(\overline{\theta}_{\ell},\overline{\phi}_{\ell})\underline{\delta}(t-\tau_{\ell}).\beta_\ell\\ &= \beta_\ell.\underline{\overline{S}}_\ell^H.\underline{\delta}(t-\tau_\ell) \quad (68)\end{aligned}$$

- In other words a scatterer can be seen as a single path with direction of departure $(\overline{\theta}_\ell, \overline{\phi}_\ell)$ and fading coefficient the term $\beta_\ell$ that represents the addition/combination of the fading coefficients of all paths of this scatterer i.e. $\beta_\ell = \sum_{k=1}^{L_\ell} \beta_{\ell k}$.

- N.B.: Another way to represent a scatterer is using Taylor Series Expansion. This will involve $\dot{\underline{\overline{S}}}$ (which is the first derivative of the manifold vector $\underline{\overline{S}}_\ell$) and $\Delta\overline{\theta}_\ell$

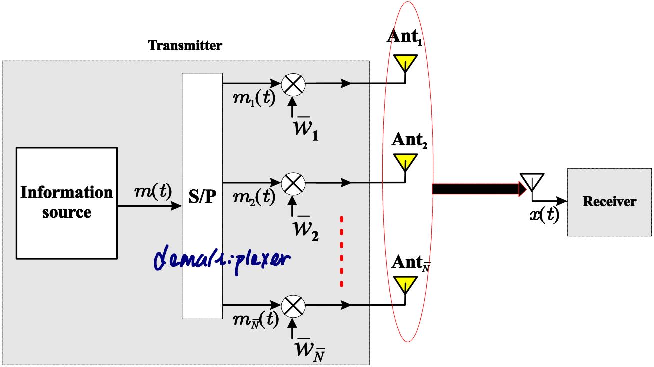
# Modelling of the Rx Scalar-Signal x(t)

- Consider a Tx-array of $\overline{N}$ antennas transmitting a baseband signal $m(t)$ via VISO multipath channel of $L$ resolvable paths (frequency selective VISO). We will consider the following two cases:
  - Case-1: The $m(t)$ is demultiplexed to $\overline{N}$ different signals (one signal per antenna element) forming the vector $\underline{m}(t)$.
  - Case-2: All Tx-array elements transmit the same signal $m(t)$.
- In both cases the transmitted signals may, or may not, be weighted.

- Case-1: The received scalar-signal $x(t)$ can be modelled as follows:

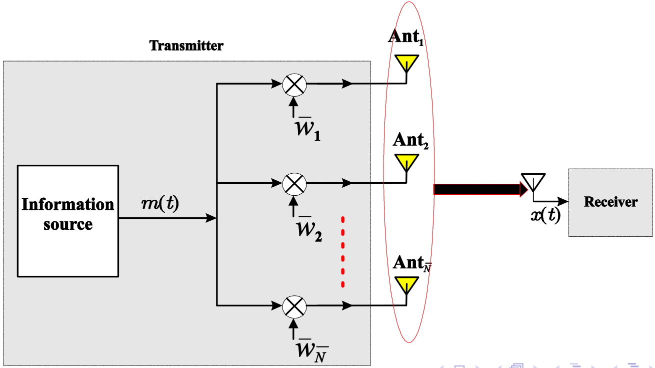
$$\begin{aligned} x(t) &= \sum_{\ell=1}^{L} \beta_\ell \underline{S}^H(\theta_\ell, \phi_\ell).\left(\underline{\delta}(t-\tau_\ell) \circledast (\underline{w} \odot \underline{m}(t))\right) + \underline{n}(t) \\ &= \sum_{\ell=1}^{L} \beta_\ell \underline{S}_\ell^H.\left(\underline{w} \odot \underline{m}(t-\tau_\ell)\right) + \underline{n}(t) \end{aligned} \tag{69}$$

.* element by element



- Case-2: The signal is "copied" to each antenna and the received scalar-signal $x(t)$ can be modelled as follows:

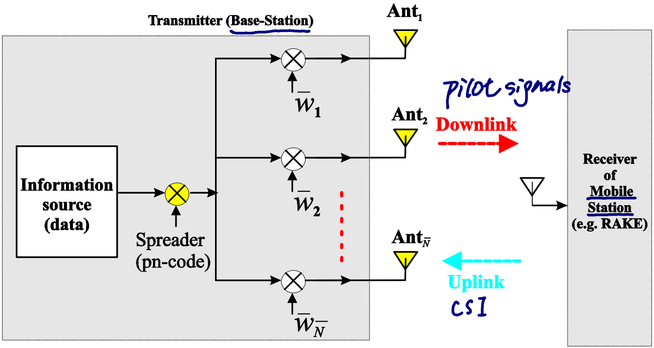
$$\begin{aligned} x(t) &= \sum_{\ell=1}^{L} \beta_\ell \underline{\overline{S}}^H(\overline{\theta}_\ell, \overline{\phi}_\ell).\left(\underline{\delta}(t-\tau_\ell) \circledast (\underline{\overline{w}} m(t))\right) + \underline{n}(t) \\ &= \sum_{\ell=1}^{L} \beta_\ell \underbrace{\underline{\overline{S}}_\ell^H \underline{\overline{w}}}_{\text{inner product}}.m(t-\tau_\ell) + \underline{n}(t) \end{aligned} \quad (70)$$



# Transmit Diversity

- Provides diversity benefits to a mobile using base station antenna array for frequency division duplexing (FDD) schemes. Cost is shared among different users.
- Order of diversity can be increased when used with other conventional forms of diversity (e.g. multipath diversity).
- Two main types of diversity combining techniques in 3G:
  - Transmit diversity with feedback from receiver (close loop)
  - Transmit diversity without feedback from receiver (open loop)

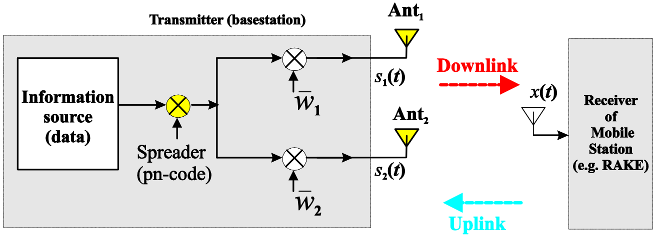
# Transmit Diversity: "Close Loop"



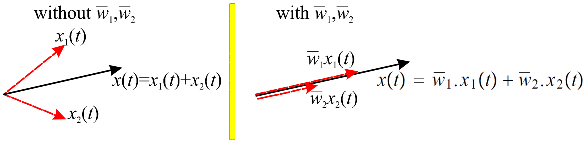

- The transmitter transmits some pilot signals
- The mobile (based on this pilot signals) estimates the Channel State Information (CSI), i.e. channel parameters.
- The mobile transmits the CSI to the BS (uplink)
- The base station generates the weights and transmits data to the mobile.

# UMTS 3GPP Standard (Close Loop)

- $\overline{N} = 2$ **Tx-antennas**



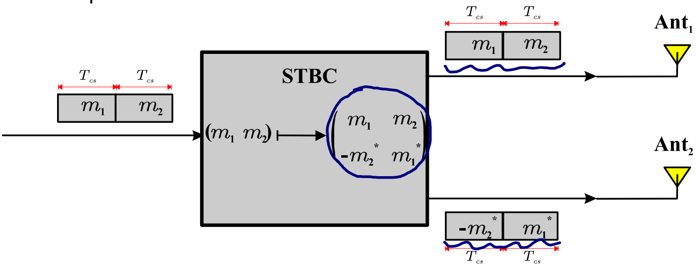

- where
  - $\overline{w}_1, \overline{w}_2$ are adjusted such as $|x(t)|^2$ is maximised, e.g.



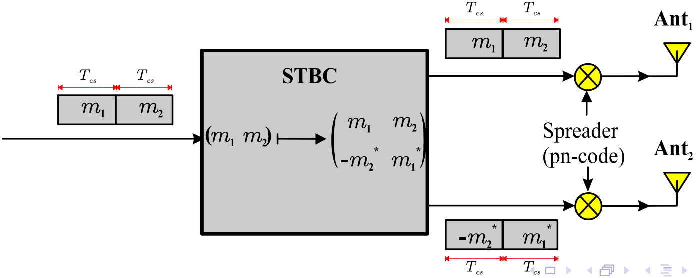

  - $\overline{w}_1, \overline{w}_2$ are adjusted based on the feedback information from the receiver (CSI)

# Transmit Diversity: "Open Loop"

- without spreader:




- with spreader:

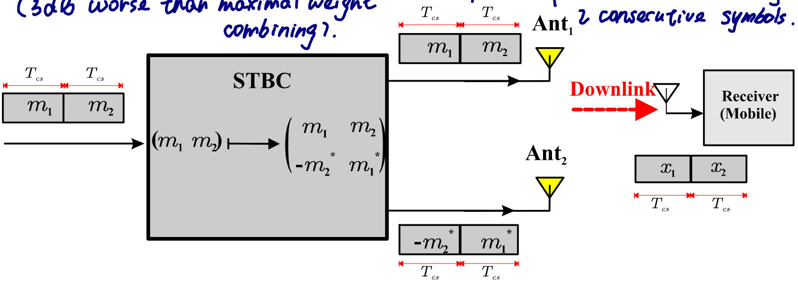


- Example of STBC Downlink Equations (without spreader):

  Relies on: the channel being constant for the period of transmitting 2 consecutive symbols.

  - STBC = Space-Time Block Code
  - No geometric/space information is used.

    (3dB worse than maximal weight combining).



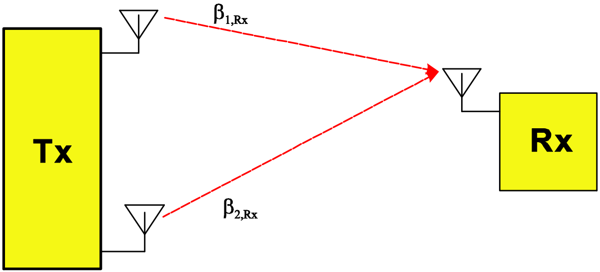

  - Receiver input ($L$ unresolvable multipaths - flat fading):

$$
\begin{cases}
\text{1st interval: } x_1 = \sum_{j=1}^{L} \left( \beta_{1j,Rx} m_1 - \beta_{2j,Rx} m_2^* \right) + n_1 \\
\text{2nd interval: } x_2 = \sum_{j=1}^{L} \left( \beta_{1j,Rx} m_2 + \beta_{2j,Rx} m_1^* \right) + n_2
\end{cases}
\tag{71}
$$

$$\Rightarrow \begin{cases} x_1 = \beta_{1,Rx} m_1 - \beta_{2,Rx} m_2^* + n_1 \\ x_2 = \beta_{1,Rx} m_2 + \beta_{2,Rx} m_1^* + n_2 \end{cases} \quad (72)$$

where

$$\beta_{1,Rx} \equiv \sum_{j=1}^{L} \beta_{1j,Rx} \quad \text{and} \quad \beta_{2,Rx} \equiv \sum_{j=1}^{L} \beta_{2j,Rx} \quad (73)$$



- Equivalently,

$$\begin{cases} x_1 = \beta_{1,Rx} m_1 - \beta_{2,Rx} m_2^* \\ x_2 = \beta_{1,Rx} m_2 + \beta_{2,Rx} m_1^* \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} m_1, & -m_2^* \\ m_2, & m_1^* \end{bmatrix} \underbrace{\begin{bmatrix} \beta_{1,Rx} \\ \beta_{2,Rx} \end{bmatrix}}_{\triangleq \underline{\beta}_{Rx}} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \tag{74}$$

$$x_2^* = \beta_{2,Rx}^* m_1 + \beta_{1,Rx}^* m_2^*$$

or, in an alternative format:

$$\underbrace{\begin{bmatrix} x_1 \\ x_2^* \end{bmatrix}}_{\triangleq \underline{x}} = \underbrace{\begin{bmatrix} \beta_{1,Rx} & -\beta_{2,Rx} \\ \beta_{2,Rx}^* & \beta_{1,Rx} \end{bmatrix}}_{\triangleq H} \underbrace{\begin{bmatrix} m_1 \\ m_2^* \end{bmatrix}}_{\triangleq \underline{m}}$$

$$\underline{x} \triangleq \begin{bmatrix} x_1 \\ x_2^* \end{bmatrix} = \underbrace{\begin{bmatrix} \beta_{1,Rx}, & -\beta_{2,Rx} \\ \beta_{2,Rx}^*, & \beta_{1,Rx}^* \end{bmatrix}}_{\triangleq \mathbb{H}} \underbrace{\begin{bmatrix} m_1 \\ m_2^* \end{bmatrix}}_{\triangleq \underline{m}} + \underbrace{\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}}_{\triangleq \underline{n}} \tag{75}$$

$$H^H = \begin{bmatrix} \beta_{1,Rx}^* & \beta_{2,Rx} \\ -\beta_{2,Rx}^* & \beta_{1,Rx}^* \end{bmatrix} \qquad H = \begin{bmatrix} \beta_{1,Rx} & -\beta_{2,Rx} \\ \beta_{2,Rx}^* & \beta_{1,Rx}^* \end{bmatrix}$$

i.e.

$$\underline{x} = \mathbb{H}\underline{m} + \underline{n} \tag{76}$$

where

$$H^T H = \left(|\beta_{1,Rx}|^2 + |\beta_{2,Rx}|^2\right) I$$

$$\parallel \triangle \quad \|\underline{\beta}_{Rx}\|^2$$

$$\mathbb{H}^H\mathbb{H} = \underbrace{\left(\left|\beta_{1,Rx}\right|^2 + \left|\beta_{2,Rx}\right|^2\right)}_{=\left\|\underline{\beta}_{Rx}\right\|^2} \mathbb{I}_2$$

- Decoder (Rx): This is denoted by the matrix $\mathbb{H}$

$\underline{G} = \underline{H}^H \underline{x}$

$= \underline{H}^H(\underline{H}\,\underline{m} + \underline{n}) = \underline{H}^H\underline{H}\,\underline{m} + \underline{H}^H\underline{n} \triangleq \tilde{\underline{n}}$

$= \|\underline{\beta}_{Rx}\|^2 \underline{m} + \tilde{\underline{n}}$

$$\text{decoder's o/p} : \underline{G} = \mathbb{H}^H \underline{x} \tag{77}$$

$$= \mathbb{H}^H \mathbb{H} \underline{m} + \underbrace{\mathbb{H}\underline{\mathrm{n}}}_{\triangleq \tilde{\underline{\mathrm{n}}}} \tag{78}$$

- That is, the decision variables are

$$\boxed{\underline{G} = \left\| \underline{\beta}_{Rx} \right\|^2 \underline{m} + \tilde{\underline{\mathrm{n}}}} \tag{79}$$

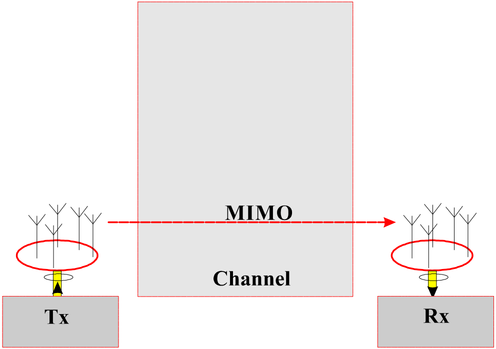
i.e.

$$\begin{cases} G_1 = \left\| \underline{\beta}_{Rx} \right\|^2 m_1 + \tilde{\mathrm{n}}_1 \\ G_2 = \left\| \underline{\beta}_{Rx} \right\|^2 m_2^* + \tilde{\mathrm{n}}_2 \end{cases} \tag{80}$$

- Note:
  1. the receiver (mobile station) needs to know (estimate) the channel weights $\beta_{1,Rx}$ and $\beta_{2,Rx}$ but there is no need to send them back to the transmitter (i.e. open loop)
  2. $\beta_{1,Rx}$ and $\beta_{2,Rx}$ can be estimated by transmitting some pilot symbols as $m_1$ and $m_2$ and, then, using Equation 74

pilot needed but don't send CSI back!

# Wireless MIMO (or VIVO) Channels




- Consider a single path from a Tx-array of $\overline{N}$ antennas to an Rx-array of $N$ antennas with locations given by the matrices

$$\text{Tx-array:} \qquad \overline{\underline{\underline{\mathbf{r}}}} = \left[\underline{\overline{r}}_1, \underline{\overline{r}}_2, \ldots, \underline{\overline{r}}_{\overline{N}}\right] = \left[\underline{\overline{r}}_{\mathbf{x}}, \underline{\overline{r}}_{\mathbf{y}}, \underline{\overline{r}}_{\mathbf{z}}\right]^T \quad (3 \times \overline{N})$$

$$\text{Rx-array:} \qquad \underline{\underline{\mathbf{r}}} = \left[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N\right] = \left[\underline{r}_{\mathbf{x}}, \underline{r}_{\mathbf{y}}, \underline{r}_{\mathbf{z}}\right]^T \quad (3 \times N)$$

- If the direction-of-departure of this single path planewave propagation is $(\overline{\theta}, \overline{\phi})$ and the direction-of-arrival is $(\theta, \phi)$ then the impulse response is

$$\underline{h}(t) = \beta \underline{S}(\theta, \phi).\overline{\underline{S}}^H(\theta, \phi).\underline{\delta}(t - \frac{\rho}{c})$$

where

direction of departure

$$\text{Tx:} \qquad \overline{\underline{S}} = \overline{\underline{S}}\left(\overline{\theta}, \overline{\phi}\right) = \exp\left(+j\underline{\underline{\mathbf{r}}}^T \underline{k}(\overline{\theta}, \overline{\phi})\right) \quad (81)$$

direction of arrival

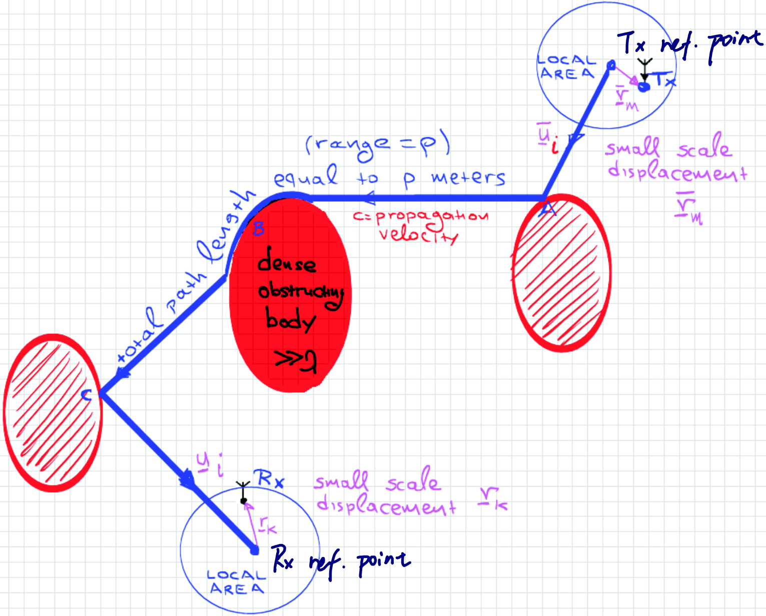
$$\text{Rx:} \qquad \underline{S} = \underline{S}\left(\theta, \phi\right) = \exp\left(-j\underline{\underline{\mathbf{r}}}^T \underline{k}(\theta, \phi)\right) \quad (82)$$

with $\underline{k}(\overline{\theta}, \overline{\phi})$ and $\underline{k}(\theta, \phi)$ denote the wavenumber vectors of the Tx-array and Rx-array respectively

- For instance $\underline{k}(\overline{\theta}, \overline{\phi})$ is defined as

$$\underline{k}(\overline{\theta}, \overline{\phi}) = \left\{ \begin{array}{ll} \frac{2\pi F_c}{c}.\underline{u}(\overline{\theta}, \overline{\phi}) = \frac{2\pi}{\lambda_c}.\underline{u}(\overline{\theta}, \overline{\phi}) & \text{in meters} \\ \pi.\underline{u}(\overline{\theta}, \overline{\phi}) & \text{in units of halfwavelength} \end{array} \right\}$$

$$\underline{u}\left(\overline{\theta}, \overline{\phi}\right) = \left[\cos\overline{\theta}\cos\overline{\phi}, \quad \sin\overline{\theta}\cos\overline{\phi} \quad \sin\overline{\phi}\right]$$



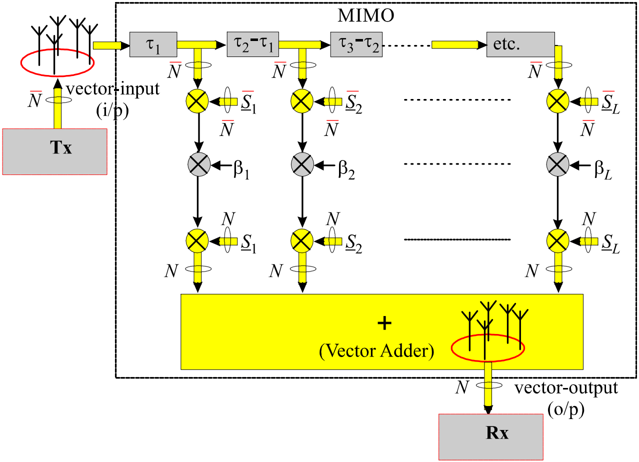
Tx ref. point
LOCAL AREA
Tx
$\underline{r}_M$
$\underline{u}_i$
small scale displacement $\underline{r}_M$
(range = ρ)
equal to ρ meters
c = propagation velocity
A
B
dense obstructing body
≫λ
total path length
C
$\underline{u}_i$
Rx
small scale displacement $\underline{r}_k$
$\underline{r}_k$
Rx ref. point
LOCAL AREA

# Multipath MIMO Channel

- Let us assume that the transmitted signal(s) from the Tx-array arrives at the Rx-array via $L$ resolvable paths (multipaths).
- Consider that the $\ell^{th}$ path's direction-of-departure is $(\overline{\theta}_\ell, \overline{\phi}_\ell)$ and propagates to the Rx with channel propagation parameters $\beta_\ell$ and $\tau_\ell$ representing the complex path gain and path-delay, respectively.
- Let us assume that the $L$ paths are arranged such that

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_\ell \leq \ldots \leq \tau_L \tag{83}$$

- once again, remember that the path coefficients $\beta_\ell$ model the effects of path losses and shadowing, in addition to random phase shifts due to reflection; they also encompass the effects of the phase offset between the modulating carrier at the transmitter and the demodulating carrier at the receiver, as well as differences in the transmitter power.
- In the following figure the vectors $\overline{\underline{S}}_\ell = \overline{\underline{S}}(\overline{\theta}_\ell, \overline{\phi}_\ell) \in C^{\overline{N}}$ and $\underline{S}_\ell = \underline{S}(\theta_\ell, \phi_\ell) \in C^N$ are the Tx- and Rx- array manifold vectors of the $\ell$-th path.

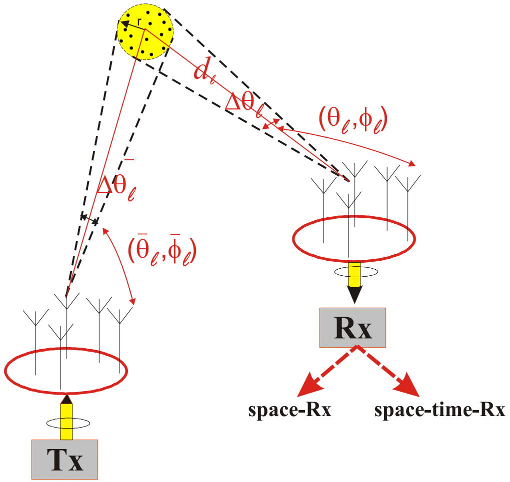



- The impulse response of the MIMO (VIVO) channel is

(reference point to reference point)

$$\boxed{\text{MIMO (VIVO): } \underline{h}(t) = \sum_{\ell=1}^{L} \beta_\ell . \underline{S}_\ell . \underline{\overline{S}}_\ell^H \delta(t - \tau_\ell)} \tag{84}$$

# Scatterers




- Consider a scatterer ($\ell$-*th* scatterer, say) which can be seen as a large number of paths ($L_\ell$, say) with directions-of-departure around the direction $(\bar{\theta}_\ell, \bar{\phi}_\ell)$ and directions-of-arrival around the direction $(\theta_\ell, \phi_\ell)$.

- That is, the direction-of-departure and direction-of-arrival of the $k$-th path satisfies the condition

$$\begin{aligned}(\overline{\theta}_\ell,\overline{\phi}_\ell)-\frac{\overline{\Delta\theta}_\ell}{2} &\leq (\overline{\theta}_{\ell k},\overline{\phi}_{\ell k})\leq(\overline{\theta}_\ell,\overline{\phi}_\ell)+\frac{\overline{\Delta\theta}_\ell}{2},\forall k \qquad (85)\\ (\theta_\ell,\phi_\ell)-\frac{\Delta\theta_\ell}{2} &\leq (\theta_{\ell k},\phi_{\ell k})\leq(\theta_\ell,\phi_\ell)+\frac{\Delta\theta_\ell}{2},\forall k \qquad (86)\end{aligned}$$

- In this case the impulse response for this scatterer can be written as follows:

$$\ell\text{-th scatterer}=\sum_{k=1}^{L_\ell}\beta_{\ell k}.\underline{S}(\theta_{\ell k},\phi_{\ell k})\underline{\overline{S}}^H(\overline{\theta}_{\ell k},\overline{\phi}_{\ell k}).\underline{\delta}(t-\tau_{\ell k}) \qquad (87)$$

- If $\boxed{\Delta\theta_\ell \text{ and } \overline{\Delta\theta}_\ell \text{ are relatively small}}$ then

$$\begin{aligned}(\overline{\theta}_{\ell 1},\overline{\phi}_{\ell 1}) &\simeq (\overline{\theta}_{\ell 2},\overline{\phi}_{\ell 2})\simeq\ldots\simeq(\overline{\theta}_{\ell L_\ell},\overline{\phi}_{\ell L_\ell})\triangleq(\overline{\theta}_\ell,\overline{\phi}_\ell) \qquad (88)\\ (\theta_{\ell 1},\phi_{\ell 1}) &\simeq (\theta_{\ell 2},\phi_{\ell 2})\simeq\ldots\simeq(\theta_{\ell L_\ell},\phi_{\ell L_\ell})\triangleq(\theta_\ell,\phi_\ell) \qquad (89)\\ \tau_{\ell 1} &\simeq \tau_{\ell k}\simeq\ldots\simeq\tau_{\ell L_\ell}\triangleq\tau_\ell \qquad (90)\end{aligned}$$

- Thus, if $L_\ell$ denotes the No. of paths of the $\ell$-*th* scatterer,

$$
\begin{aligned}
\ell\text{-}th\ \text{scatterer} &= \sum_{k=1}^{L_\ell} \beta_{\ell k}.\underline{S}(\theta_{\ell k},\phi_{\ell k})\overline{\underline{S}}^H(\overline{\theta}_{\ell k},\overline{\phi}_{\ell k}).\underline{\delta}(t-\tau_{\ell k}) \\
&= \sum_{k=1}^{L_\ell} \beta_{\ell k}.\underline{S}(\theta_{\ell},\phi_{\ell})\overline{\underline{S}}^H(\overline{\theta}_{\ell},\overline{\phi}_{\ell}).\underline{\delta}(t-\tau_{\ell}) \\
&= \underline{S}(\theta_{\ell},\phi_{\ell})\overline{\underline{S}}^H(\overline{\theta}_{\ell},\overline{\phi}_{\ell}).\underline{\delta}(t-\tau_{\ell})\underbrace{\sum_{k=1}^{L_\ell} \beta_{\ell k}}_{\triangleq \beta_\ell} && (91) \\
&= \beta_\ell.\underline{S}(\theta_{\ell},\phi_{\ell})\overline{\underline{S}}^H(\overline{\theta}_{\ell},\overline{\phi}_{\ell}).\underline{\delta}(t-\tau_{\ell}) \\
&= \beta_\ell.\underline{S}_\ell.\overline{\underline{S}}_\ell^H.\underline{\delta}(t-\tau_\ell) && (92)
\end{aligned}
$$

- In other words a scatterer can be seen as a single path with direction $(\theta_\ell, \phi_\ell)$ and fading coefficient the term $\beta_\ell$ that represents the addition/combination of the fading coefficients of all paths of this scatterer i.e. $\beta_\ell = \sum_{k=1}^{L_\ell} \beta_{\ell k}$.
- N.B.: Again, another way to represent a scatterer is using Taylor Series Expansion. This will involve $\underline{\dot{S}}_\ell$, $\overline{\underline{\dot{S}}}_\ell$(i.e the first derivative of the manifold vector $\underline{S}_\ell$), $\Delta\theta_\ell$ and $\overline{\Delta\theta}_\ell$.

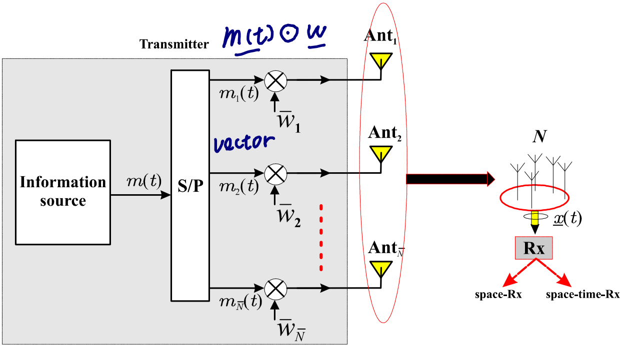
# Modelling of the Rx Vector-Signal x(t)

- Consider a Tx-array of $\overline{N}$ antennas transmitting a baseband signal $m(t)$ via VIVO multipath channel of $L$ resolvable paths (frequency selective VIVO).

create diversity.
complexity is trivial.

- We will consider the following two cases:
  - Case-1: The $m(t)$ is demultiplexed to $\overline{N}$ different signals (one signal per antenna element) forming the vector $\underline{m}(t)$.
  - Case-2: All Tx-array elements transmit the same signal $m(t)$.
    scalar
- In both cases the transmitted signals may, or may not, be weighted. The Rx is also equipped with an array of $N$ antennas.

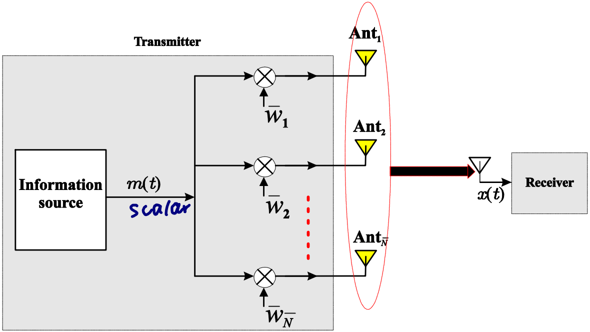
# Case-1:




- The received vector-signal $\underline{x}(t)$ can be modelled as follows:

$$\begin{aligned} \underline{x}(t) &= \sum_{\ell=1}^{L} \beta_\ell . \underline{S}_\ell . \underline{\overline{S}}_\ell^H . \left(\delta(t-\tau_\ell) \circledast (\underline{\overline{w}} \odot \underline{m}(t)\right) + \underline{n}(t) \\ &= \sum_{\ell=1}^{L} \beta_\ell . \underline{S}_\ell . \underline{\overline{S}}_\ell^H . \left(\underline{\overline{w}} \odot \underline{m}(t-\tau_\ell)\right) + \underline{n}(t) \end{aligned} \tag{93}$$

# Case-2:

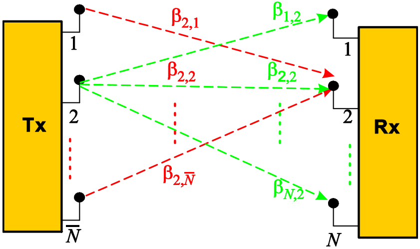



- In this case the signal is "copied" to each antenna and may be weighted by a complex weight.
- The received vector-signal $\underline{x}(t)$ can be modelled as follows:

$$\begin{aligned}\underline{x}(t) &= \sum_{\ell=1}^{L} \beta_\ell \underline{S}_\ell \overline{\underline{S}}_\ell^H . \left(\delta(t-\tau_\ell) \circledast (\overline{\underline{w}} m(t))\right) + \underline{n}(t) \\ &= \sum_{\ell=1}^{L} \beta_\ell \underline{S}_\ell \overline{\underline{S}}_\ell^H \overline{\underline{w}}.m(t-\tau_\ell) + \underline{n}(t) \qquad (94)\end{aligned}$$

# MIMO Systems (without geometric information)

- Let us consider a comm. system with multiple antennas at both the Tx and the Rx.




$$\mathbb{H} = \begin{array}{c} \left[\begin{array}{cccc} \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,\overline{N}} \\ \beta_{2,1} & \beta_{2,2} & \cdots & \beta_{2,\overline{N}} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N,1} & \beta_{N,2} & \cdots & \beta_{N,\overline{N}} \end{array}\right] \\ \begin{array}{cccc} \underline{\beta}_{1,\mathrm{Tx}} & \underline{\beta}_{2,\mathrm{Tx}} & & \underline{\beta}_{\overline{N},\mathrm{Tx}} \end{array} \end{array}$$

- $\beta_{i,j}^{\text{Rx}\,\text{Tx}}$ = gain from the $j^{\text{th}}$ Tx-antenna to the $i^{\text{th}}$ Rx-antenna

  $\underline{\beta}_{j,Tx}$ = gain-vector with its $i^{\text{th}}$ element the gain $\beta_{ij}$

  $j$-th transmitter

  - (i.e. from the $j$-th Tx antenna to all the Rx antennas)

- If Rx is synchronised to the Tx then for the $n^{th}$ data symbol interval then we have the following received vector-signal: matrix

  received    message

  $$\underline{x}[n] = \mathbb{H}\underline{m}[n] + \underline{\text{n}}[n] \quad (N \times 1)$$

  inner product

  $\langle \underline{A}, \underline{B} \rangle = \sum_{1<i,j<n} a_{ij} b_{ij}$

  outer product

  $\underline{u} \otimes \underline{v} = \underline{u}\,\underline{v}^*$

  where

  $$\mathbb{H} = \left[\underline{\beta}_{1,Tx}, \underline{\beta}_{2,Tx}, \ldots, \underline{\beta}_{\overline{N},Tx}\right]$$

- Second order statistics (covariance matrix) of $\underline{x}[n]$ is as follows:

$\underline{\underline{R}}_{xx}$ : outer product of $x$ with itself

$$\mathbb{R}_{xx} = \mathbb{H}\mathbb{R}_{mm}\mathbb{H}^H + \underbrace{\mathbb{R}_{\text{nn}}}_{\sigma_n^2 \mathbb{I}_N} \quad (N \times N)$$

- In a MIMO system it is assumed that the Matrix $\mathbb{H}$ is known
- Note:
  - The matrix $\mathbb{H}$ and the manifold vectors are related according to the following expression

$$\mathbb{H} = \sum_{\ell=1}^{L} \overbrace{\beta_\ell}^{\text{gain}} \underline{S}_\ell \overbrace{\underline{\bar{S}}_\ell^H}^{\text{geometry}} \tag{95}$$

$$= \mathbb{S} \begin{bmatrix} \beta_1, & 0, & ..., & 0 \\ 0, & \beta_2, & ..., & 0 \\ ..., & ..., & ..., & ... \\ 0, & 0, & ..., & \beta_L \end{bmatrix} \bar{\mathbb{S}}^H \tag{96}$$

where

$$\mathbb{S} = [\underline{S}_1, \quad \underline{S}_2, \quad ..., \quad \underline{S}_L]$$
$$\bar{\mathbb{S}} = [\underline{\bar{S}}_1, \quad \underline{\bar{S}}_2, \quad ..., \quad \underline{\bar{S}}_L]$$

# Capacity - of MIMO Channels (without space info)

- SISO Capacity:

$$C = B \log_2 \left(1 + \mathrm{SNR}_{in}\right) \text{ bits/s} \tag{97}$$

- General MIMO Capacity expression:

received signal outer product with itself

$$C = B \log_2 \left( \frac{\det\left(\mathbb{R}_{xx}\right)}{\det\left(\mathbb{R}_{\mathrm{nn}}\right)} \right) \text{ bits/s} \tag{98}$$

- Based on Equation 98 it can be easily proven that

$$C = B\log_2\left(\det\left(\mathbb{I}_N + \frac{1}{\underset{P_n}{\sigma_n^2}}\overset{P_s}{\mathbb{H}\mathbb{R}_{mm}\mathbb{H}^H}\right)\right) \text{ bits/s} \tag{99}$$

- Furthermore, for independent parallel channels (i.e. using a multiplexer at Tx, Case-1) :

msg pieces are orthogonal

$$\mathbb{R}_{mm} = \text{diagonal} = \begin{bmatrix} P_1, & 0, & \ldots, & 0 \\ 0, & P_2, & \ldots, & 0 \\ \ldots, & \ldots, & \ldots, & \ldots \\ 0, & 0, & \ldots, & P_{\overline{N}} \end{bmatrix}$$

and thus Equation 99 is simplified to

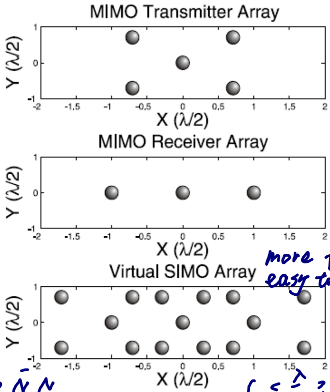
$$C = B\log_2\left(\prod_{j=1}^{\overline{N}}\left(1 + \frac{\left\|\underline{\beta}_{j,Tx}\right\|^2 P_j}{\sigma_n^2}\right)\right) \tag{100}$$

$$= B\sum_{j=1}^{\overline{N}}\log_2\left(1 + \frac{\left\|\underline{\beta}_{j,Tx}\right\|^2 P_j}{\sigma_n^2}\right) \text{ bits/sec} \tag{101}$$

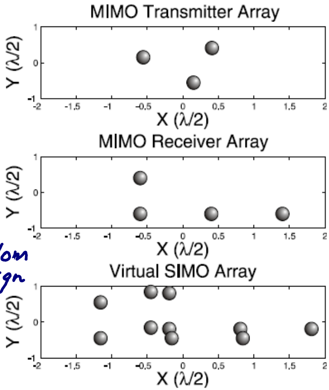
# Equivalence between MIMO and SIMO

- spatial convolution (ensure no overlapping)

| MIMO<br>Tx antennas: $\overline{N}$ , Rx antennas: $N$ | $\Longleftrightarrow$ | virtual-SIMO or MISO<br>$\overline{N} \times N$ Rx /Tx antennas |
|---|---|---|




more freedom
easy to design




$Rx \rightarrow \bar{N} N$

$( < \frac{\lambda}{2} ?)$

Tx: $\underline{\bar{r}} \triangleq [\bar{r}_1 \ \bar{r}_2 \ \cdots \ \bar{r}_{\bar{N}}] = \begin{bmatrix} \bar{r}_x \\ \bar{r}_y \\ \bar{r}_z \end{bmatrix}$ $3 \times \bar{N}$

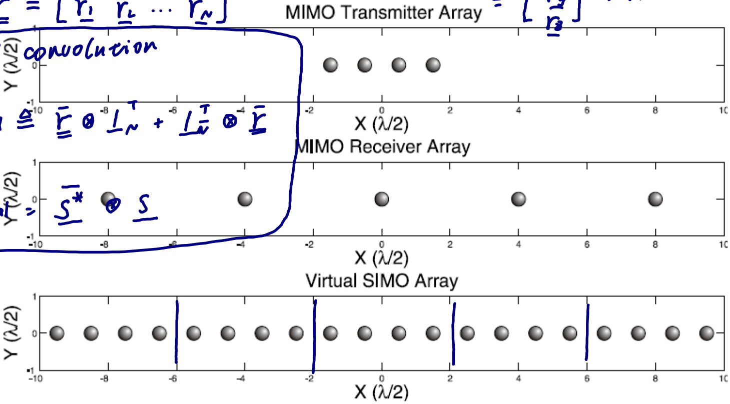
Rx: $\underline{r} \triangleq [r_1 \ r_2 \ \cdots \ r_N] = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$ $3 \times N$

spatial convolution

$\underline{r}_{virtual} \triangleq \underline{\bar{r}} \otimes \underline{1}_N^T + \underline{1}_{\bar{N}}^T \otimes \underline{\bar{r}}$

$\Updownarrow$

$\underline{S}_{virtual} = \underline{\bar{S}}^* \circledast \underline{S}$



# Equivalence between MIMO and SIMO (cont.)

Tx-array: $\quad \underline{\underline{\overline{\mathbf{r}}}} \triangleq [\underline{\overline{r}}_1, \underline{\overline{r}}_2, ..., \underline{\overline{r}}_N] = [\underline{\overline{r}}_{\mathbf{x}}, \underline{\overline{r}}_{\mathbf{y}}, \underline{\overline{r}}_{\mathbf{z}}]^T \quad (3 \times \overline{N})$

Rx-array: $\quad \underline{\underline{\mathbf{r}}} \triangleq [\underline{r}_1, \underline{r}_2, ..., \underline{r}_N] = [\underline{r}_{\mathbf{x}}, \underline{r}_{\mathbf{y}}, \underline{r}_{\mathbf{z}}]^T \quad (3 \times N)$

$$\text{virtual-array} \quad : \quad \underline{\underline{\mathbf{r}}}_{\text{virtual}} \triangleq \underline{\underline{\overline{\mathbf{r}}}} \otimes \underline{1}_N^T + \underline{1}_N^T \otimes \underline{\underline{\mathbf{r}}} \qquad (102)$$

$$\Updownarrow$$

$$\underline{S}_{\text{virtual}} \triangleq \underline{\overline{S}}^* \otimes \underline{S} \qquad (103)$$

**Note** : Equation 102 is known as "spatial convolution"