1. a) X can take $1 \to \infty$

$$\left.\begin{aligned} P(X=1) &= \frac{1}{2} \\ P(X=2) &= \frac{1}{2}\cdot\frac{1}{2} \\ P(X=k) &= \left(\frac{1}{2}\right)^k \end{aligned}\right\} H(X) = -\sum_{n=1}^{\infty} P(n)\log_2 P(n)$$

$$H(X) = -\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \log_2\left(\frac{1}{2}\right)^n$$

$$= -\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\cdot(-n)$$

$$= \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2 \text{ bits}$$

b) question sequence : most likely $\Rightarrow$ unlikely

Δ ask questions that | have probability $\to \frac{1}{2}$

| reduce uncertainty most efficiently

| greedy algorithm

eg: 1. 1st ?

2. 2nd ?

⋮

$$E(k) = \sum_{k=1}^{\infty} P(n)\cdot n$$

$$= \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^n\cdot n = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$$

2. r.v. X, $Y = g(X)$ $\quad = 0$: $X \to g(X)$

$$H(X, g(X)) = H(X) + \underline{H(g(X) \mid X)}$$
$$= \underbrace{H(g(X))}_{\geq 0} + \underbrace{H(X \mid g(X))}_{\geq 0}$$

△ for <u>discrete</u> r.v., entropy is positive:

$$\Rightarrow \boxed{H(X) \geq H(g(X))}$$

△ equality when $g(X)$ is a 1-1 function of X.

a) $Y = X^2$ $\quad H(Y) < H(X)$ depends on distribution.

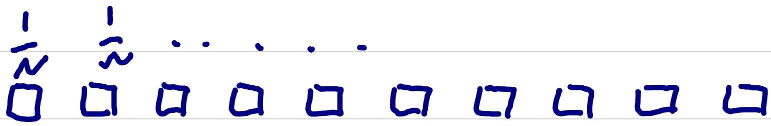
b) $Y = X^3$ $\quad H(Y) = H(X)$

3. vector:

$$H(\vec{p}) = -\sum_{i=1}^{n} P(n) \log_2 P(n)$$

min: 0

max: $-\sum_{i=1}^{N} \frac{1}{N} \log_2 \frac{1}{N} = \log_2 N$



$$H'(P) = -\log P - P\cdot\frac{1}{P}\cdot\log e + \log(1-P) + (1-P)\frac{1}{1-P}\log e$$
$$= -\log P + \log(1-P)$$

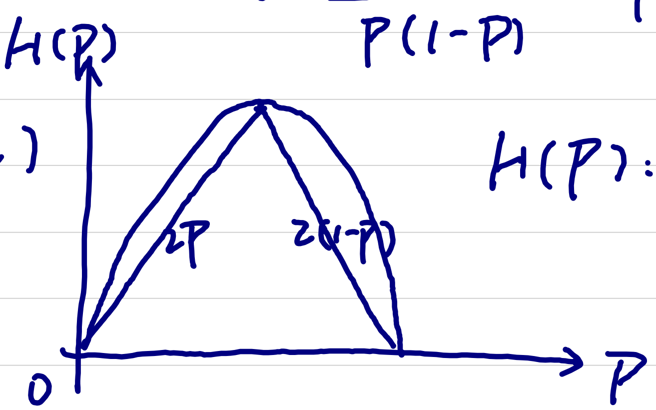
4. a) $H(P) = -P\log P - (1-P)\log(1-P)$

$$H'(P) = -\log P + \log(1-P) = \log\frac{1-P}{P}$$

b) $H''(P) = -\frac{1}{P}\log e - \frac{1}{1-P}\log e$

$$= \frac{-(1-P)-P}{P(1-P)}\log e = \frac{-\log e}{P(1-P)}$$

c)



H(P): concave

d) $D(P) = H(P) - 1 + 4(P-\frac{1}{2})^2$

prove $D(P)$ is positive in $P\in[0.1]$

$D'(P) = \log\frac{1-P}{P} + 8(P-\frac{1}{2})$ $D'(P)=0 \Rightarrow P=\frac{1}{2}$ (min)

$$D''(P) = \frac{-\log e}{P(1-P)} + 8 = 0$$

$$-8P^2 + 8P - \log e = 0 \Rightarrow P = 0.24.\ 0.76.$$

e) write as difference.
show bounds by concavity or convexity.

$D(P) = 1 - 2\log e \cdot (P - \frac{1}{2})^2 - H(P)$

$D(P)$ is symmetrical over $P = \frac{1}{2}$. consider $P \in [0, \frac{1}{2}]$

$D(0) = 1 - \frac{\log e}{2}$, $D(\frac{1}{2}) = 0$.

$D'(P) = -4\log e \cdot (P - \frac{1}{2}) - \log \frac{1-P}{P}$

$D'(0) = -\infty$. $D'(\frac{1}{2}) = 0$.

$D''(P) = -4\log e + \frac{\log e}{P(1-P)}$

$D''(P) = -4P + 4P^2 + 1 = 0$

$\Rightarrow P = \frac{1}{2}$