Imperial College London

Signals and Systems

Lecture 3

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DFT Properties

DFT: $X[k] = \sum_0^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$

DTFT: $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$

Case 1: $x[n] = 0$ for $n \notin [0, N-1]$

DFT is the same as DTFT at $\omega_k = \frac{2\pi}{N}k$.

The $\{\omega_k\}$ are uniformly spaced from $\omega = 0$ to $\omega = 2\pi\frac{N-1}{N}$.
DFT is the z-Transform evaluated at N equally spaced points around the unit circle beginning at $z = 1$.

Case 2: $x[n]$ is periodic with period N

DFT equals the normalized DTFT

$$X[k] = \lim_{K\to\infty} \frac{N}{2K+1} \times X_K(e^{j\omega_k})$$

where $X_K(e^{j\omega}) = \sum_{-K}^{K} x[n]e^{-j\omega n}$

Number of samples kept symmetrically around the origin.

Proof of Case 2

We want to show that if $x[n] = x[n+N]$ (i.e. $x[n]$ is periodic with period N) then

$$\lim_{K\to\infty} \frac{N}{2K+1} \times X_K(e^{j\omega_k}) \triangleq \lim_{K\to\infty} \frac{N}{2K+1} \times \sum_{-K}^{K} x[n]e^{-j\omega_k n} = X[k]$$

where $\omega_k = \frac{2\pi}{N}k$. We assume that $x[n]$ is bounded with $|x[n]| < B$.

We first note that the summand is periodic:

$$x[n+N]e^{-j\omega_k(n+N)} = x[n]e^{-j\omega_k n}e^{-jk\frac{2\pi}{N}N} = x[n]e^{-j\omega_k n}e^{-j2\pi k} = x[n]e^{-j\omega_k n}.$$

We now define M and R so that $2K+1 = MN + R$ where $0 \le R < N$ (i.e. MN is the largest multiple of N that is $\le 2K+1$). We can now write

$(K-R)-(-K)+1 = 2K+1-R = MN$ ***terms***

$$\frac{N}{2K+1} \times \sum_{-K}^{K} x[n]e^{-j\omega_k n} = \frac{N}{MN+R} \times \sum_{-K}^{K-R} x[n]e^{-j\omega_k n} + \frac{N}{MN+R} \times \sum_{K-R+1}^{K} x[n]e^{-j\omega n}$$

The first sum contains MN consecutive terms of a periodic summand and so equals M times the sum over one period. The second sum contains R bounded terms and so its magnitude is $< RB < NB$.

So $\frac{N}{2K+1} \times \sum_{-K}^{K} x[n]e^{-j\omega_k n} = \frac{MN}{MN+R} \times \sum_{0}^{N-1} x[n]e^{-j\omega_k n} + P = \frac{1}{1+\frac{R}{MN}} \times X[k] + P$

where $|P| < \frac{N}{MN+R} \times NB \le \frac{N}{MN+0} \times NB = \frac{NB}{M}$.

As $M \to \infty$, $|P| \to 0$ and $\frac{1}{1+\frac{R}{MN}} \to 1$ so the whole expression tends to $X[k]$.

$K-(K-R+1)+1 =$ R ***terms***

Symmetries

If $x[n]$ has a special property then $X(e^{j\omega})$and $X[k]$ will have corresponding properties as shown in the table (and vice versa):

One domain	Other domain
Discrete	Periodic
Symmetric	Symmetric
Antisymmetric	Antisymmetric
Real	Conjugate Symmetric
Imaginary	Conjugate Antisymmetric
Real + Symmetric	Real + Symmetric
Real + Antisymmetric	Imaginary + Antisymmetric

Symmetric: $x[n] = x[-n]$

$$X(e^{j\omega}) = X(e^{-j\omega})$$

$$X[k] = X[(-k)_{\text{mod } N}] = X[N-k] \text{ for } k > 0$$

Conjugate Symmetric: $x[n] = x^*[-n]$

Conjugate Antisymmetric: $x[n] = -x^*[-n]$

Parseval's Theorem

Fourier transforms preserve "energy"

CTFT $\int |x(t)|^2\, dt = \frac{1}{2\pi} \int |X(j\Omega)|^2\, d\Omega$

DTFT $\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$

DFT $\sum_0^{N-1} |x[n]|^2 = \frac{1}{N} \sum_0^{N-1} |X[k]|^2$

Hermitian: A complex matrix that is equal to its own conjugate transpose.

More generally, they actually preserve complex inner products:

$$\sum_0^{N-1} x[n]y^*[n] = \frac{1}{N} \sum_0^{N-1} X[k]Y^*[k]$$

Unitary matrix viewpoint for DFT:

$$\boldsymbol{G^H G = \frac{1}{\sqrt{N}} F^H \frac{1}{\sqrt{N}} F = \frac{1}{N} F^H F}$$
$$\boldsymbol{= \frac{1}{N} N F^{-1} F = I}$$

If we regard $\mathbf{x}$ and $\mathbf{X}$ as vectors, then $\mathbf{X} = \mathbf{F}\mathbf{x}$ where $\mathbf{F}$ is a symmetric matrix defined by $f_{k+1,n+1} = e^{-j2\pi\frac{kn}{N}}$.

The inverse DFT matrix is $\mathbf{F}^{-1} = \frac{1}{N}\mathbf{F}^H$

equivalently, $\mathbf{G} = \frac{1}{\sqrt{N}}\mathbf{F}$ is a unitary matrix with $\mathbf{G}^H\mathbf{G} = \mathbf{I}$.

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Convolution

linear convolution: sum delayed version of x with weight h. — result length $= H+X-1$

$(h*x)[n] = \sum_{k=-\infty}^{+\infty} \underbrace{h[k]}_{k^{th}\text{ entry of } h} \underbrace{x[n-k]}_{\text{delayed } k \text{ version of } x}$

circular convolution: sum circularly shifted version of x with weight h. — result length $= \max(H, X)$

$(h \circledast x_N)[n] = \sum_{k=-\infty}^{+\infty} \underbrace{h[k]}_{k^{th}\text{ entry of } h} \underbrace{x_N[n-k]}_{\text{circularly shifted version of } x}$

DTFT: Convolution → Product

$h[n] = \{-1, 1, 1\}$ $x[n] = \{2, 0, -1\}$

$$x[n] = g[n] * h[n] = \sum_{k=-\infty}^{\infty} g[k]h[n-k]$$

$$\Rightarrow X(e^{j\omega}) = G(e^{j\omega})H(e^{j\omega})$$

$h[n] * x[n] = (-1)\{2, 0, -1\}$
$(1)\{0, 2, 0, -1\}$
$(1)\{0, 0, 2, 0, -1\}$
$\{-2, 2, 3, -1, -1\}$

$h[n] \circledast_3 x[n] = (-1)\{2, 0, -1\}$
$(1)\{-1, 2, 0\}$
$(1)\{0, -1, 2\}$
$\{-3, 1, 3\}$

DFT: Circular convolution → Product

$$x[n] = g[n] \circledast_N h[n] = \sum_{k=0}^{N-1} g[k]h[(n-k) \bmod N]$$

$$\Rightarrow X[k] = G[k]H[k]$$

$x[n] * h[n] = (2)\{-1, 1, 1\}$
$(0)\{0, -1, 1, 1\}$
$(-1)\{0, 0, -1, 1, 1\}$
$\{-2, 2, 3, -1, -1\}$

$x[n] \circledast_3 h[n] = (2)\{-1, 1, 1\}$
$(0)\{0, -1, 1\}$
$(-1)\{1, 1, -1\}$
$\{-3, 1, 3\}$

DTFT: Product → Circular Convolution $\div 2\pi$

$$y[n] = g[n]h[n]$$

$$\Rightarrow Y(e^{j\omega}) = \frac{1}{2\pi}G(e^{j\omega}) \circledast_\pi H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$$

$h[n] = \{-1, 0, 1, 2\}$ $x[n] = \{3, 4\}$

$h[n] \circledast_4 x[n] = (-1)\{3, 4, 0, 0\}$
$(0)\{0, 3, 4, 0\}$
$(1)\{0, 0, 3, 4\}$
$(2)\{4, 0, 0, 3\}$
$\{5, -4, 3, 10\}$

$x[n] \circledast_4 h[n] = (3)\{-1, 0, 1, 2\}$
$(4)\{2, -1, 0, 1\}$
$\{5, -4, 3, 10\}$

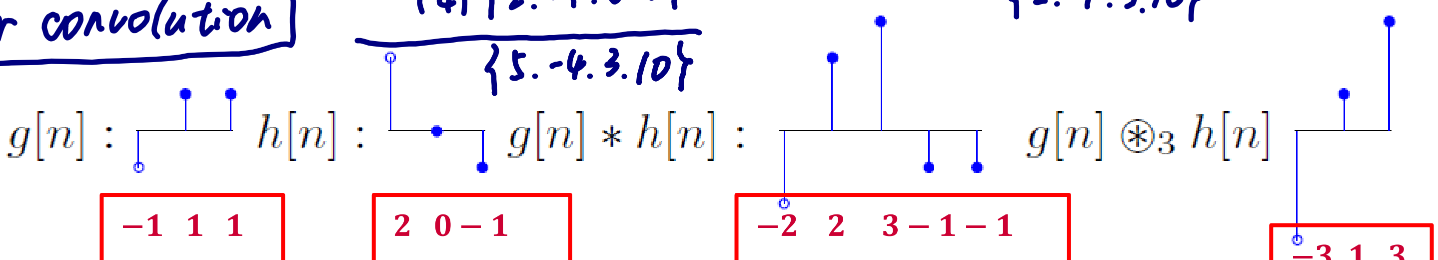
DFT: Product → Circular Convolution $\div N$

$$y[n] = g[n]h[n]$$

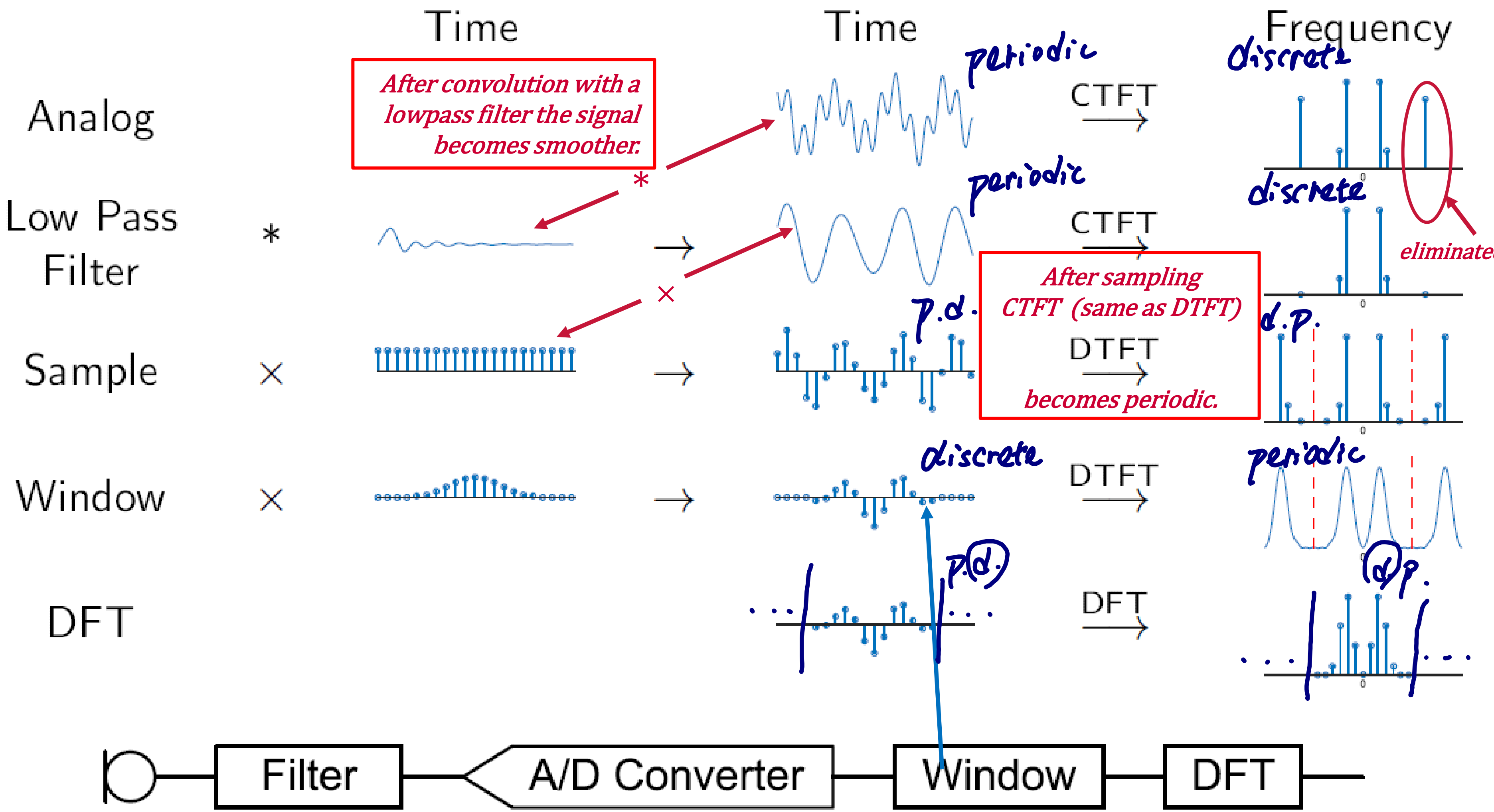
$$\Rightarrow Y[k] = \frac{1}{N}G[k] \circledast_N H[k]$$

periodic → circular convolution

−2 2 3 −1 −1



Sampling Process



Lowpass filter the signal in order to make it bandlimited for sampling. *Window the signal to make it of finite duration.*

Zero Padding

Zero padding means added extra zeros onto the end of $x[n]$ before performing the DFT.

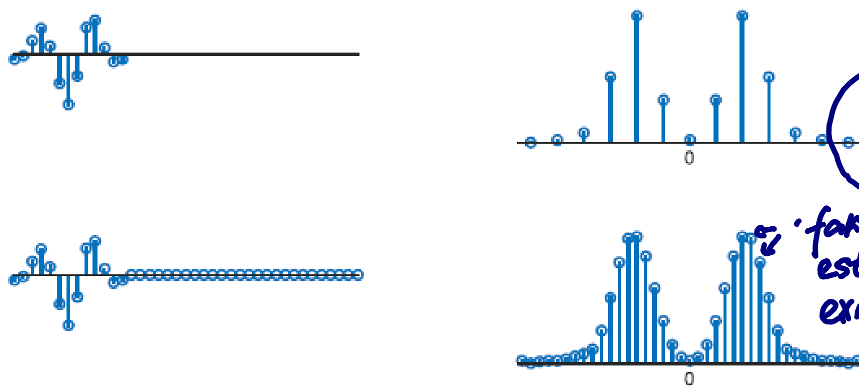
zero-padding : $T_0 \nearrow \Rightarrow f_0 \searrow = \frac{1}{T_0}$

same resolution

Time $x[n]$ Frequency $|X[k]|$

Windowed Signal

With zero-padding



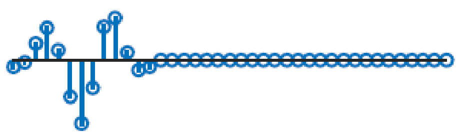
same resolution

'fake' samples estimated by existing result

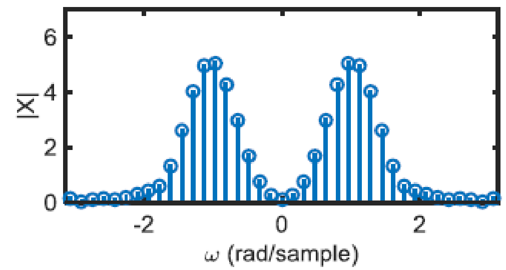
- Zero-padding causes the DFT to evaluate the DTFT at more values of ω_k. Denser frequency samples.
- Width of the peaks remains constant: determined by the length and shape of the window.
- Smoother graph but increased frequency resolution is an illusion.

Phase Unwrapping

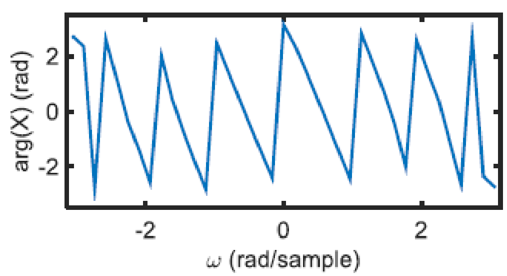
Phase of a DTFT is only defined to within an integer multiple of 2π.



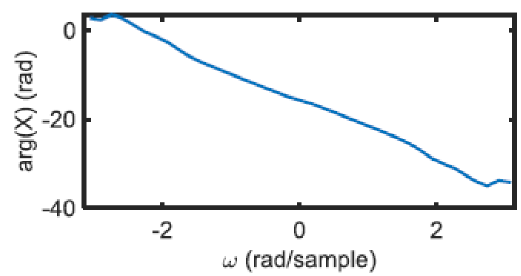
$x[n]$



$|X[k]|$



$\angle X[k]$



$\angle X[k]$ unwrapped

Phase unwrapping adds multiples of 2π onto each $\angle X[k]$ to make the phase as continuous as possible.

Uncertainty Principle

CTFT uncertainty principle: $\boxed{\left(\frac{\int t^2|x(t)|^2 dt}{\int |x(t)|^2 dt}\right)^{\frac{1}{2}} \left(\frac{\int \omega^2|X(j\omega)|^2 d\omega}{\int |X(j\omega)|^2 d\omega}\right)^{\frac{1}{2}} \geq \frac{1}{2}}$

The first term measures the "width" of $x(t)$ around $t = 0$.

It is like σ if $|x(t)|^2$ was a zero-mean probability distribution.

The second term is similarly the "width" of $X(j\omega)$ in frequency.

A signal cannot be concentrated in both time and frequency.

$u = t$ $\quad dv = x\,dx$

$du = dt$ $\quad v = \frac{1}{2}x^2$

Proof Outline:

Assume $\int |x(t)|^2 dt = 1 \Rightarrow \int |X(j\omega)|^2 d\omega = 2\pi$ [Parseval]

Set $v(t) = \frac{dx}{dt} \Rightarrow V(j\omega) = j\omega X(j\omega)$ [by parts]

Now $\int tx\frac{dx}{dt}dt = \frac{1}{2}t\underbrace{x^2(t)}_{\text{zero mean}}\Big|_{t=-\infty}^{\infty} - \int \frac{1}{2}x^2 dt = 0 - \frac{1}{2}$ [by parts]

So $\frac{1}{4} = \left|\int tx\frac{dx}{dt}dt\right|^2 \leq \left(\int t^2x^2dt\right)\left(\int \left|\frac{dx}{dt}\right|^2 dt\right)$ [Schwartz]

$= \left(\int t^2x^2dt\right)\left(\int |v(t)|^2 dt\right) = \left(\int t^2x^2dt\right)\left(\frac{1}{2\pi}\int |V(j\omega)|^2 d\omega\right)$

$= \left(\int t^2x^2dt\right)\left(\frac{1}{2\pi}\int \omega^2 |X(j\omega)|^2 d\omega\right)$

No exact equivalent for DTFT/DFT but a similar effect is true

Uncertainty Principle Proof Steps

(1) Suppose $v(t) = \frac{dx}{dt}$. Then integrating the CTFT definition by parts w.r.t. t gives
$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt = \left[\frac{-1}{j\Omega}x(t)e^{-j\Omega t}\right]_{-\infty}^{\infty} + \frac{1}{j\Omega}\int_{-\infty}^{\infty}\frac{dx(t)}{dt}e^{-j\Omega t}dt = 0 + \frac{1}{j\Omega}V(j\Omega)$

(2) Since $\frac{d}{dt}\left(\frac{1}{2}x^2\right) = x\frac{dx}{dt}$, we can apply integration by parts to get
$\int_{-\infty}^{\infty} tx\frac{dx}{dt}dt = \left[t \times \frac{1}{2}x^2\right]_{t=-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{dt}{dt} \times \frac{1}{2}x^2 dt = -\frac{1}{2}\int_{-\infty}^{\infty} x^2 dt = -\frac{1}{2} \times 1 = -\frac{1}{2}$
It follows that $\left|\int_{-\infty}^{\infty} tx\frac{dx}{dt}dt\right|^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$ which we will use below.

(3) The Cauchy-Schwarz inequality is that in a complex inner product space $|\mathbf{u}\cdot\mathbf{v}|^2 \le (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v})$. For the inner-product space of real-valued square-integrable functions, this becomes $\left|\int_{-\infty}^{\infty} u(t)v(t)dt\right|^2 \le \int_{-\infty}^{\infty} u^2(t)dt \times \int_{-\infty}^{\infty} v^2(t)dt$. We apply this with $u(t) = tx(t)$ and $v(t) = \frac{dx(t)}{dt}$ to get

$$\frac{1}{4} = \left|\int_{-\infty}^{\infty} tx\frac{dx}{dt}dt\right|^2 \le \left(\int t^2x^2dt\right)\left(\int\left(\frac{dx}{dt}\right)^2 dt\right) = \left(\int t^2x^2dt\right)\left(\int v^2(t)dt\right)$$

(4) From Parseval's theorem for the CTFT, $\int v^2(t)dt = \frac{1}{2\pi}\int |V(j\Omega|^2 d\Omega$. From step (1), we can substitute $V(j\Omega) = j\Omega X(j\Omega)$ to obtain $\int v^2(t)dt = \frac{1}{2\pi}\int \Omega^2 |X(j\Omega|^2 d\Omega$. Making this substitution in (3) gives

$$\frac{1}{4} \le \left(\int t^2x^2dt\right)\left(\int v^2(t)dt\right) = \left(\int t^2x^2dt\right)\left(\frac{1}{2\pi}\int \omega^2 |X(j\Omega|^2 d\Omega\right)$$

Summary

- Three types: CTFT, DTFT, DFT
  - DTFT = CTFT of continuous signal $\times$ impulse train
  - DFT = DTFT of periodic or finite support signal
    - DFT is a scaled unitary transform
- DTFT: Convolution $\rightarrow$ Product; Product $\rightarrow$ Circular Convolution
- DFT: Product $\leftrightarrow$ Circular Convolution
- DFT: Zero Padding $\rightarrow$ Denser freq sampling but same resolution
- Phase is only defined to within a multiple of 2π.
- Whenever you integrate over frequency you need a scale factor
  - $\frac{1}{2\pi}$ for CTFT and DTFT or $\frac{1}{N}$ for DFT
  - e.g. Inverse transform, Parseval, frequency domain convolution

For further details see Mitra: 3 & 5.