1. Solution:

(a)

Let A, B and C indicating the player that is not playing in the given game. The transition matrix is

$$\begin{bmatrix} P_{A\to A} & P_{A\to B} & P_{A\to C} \\ P_{B\to A} & P_{B\to B} & P_{B\to C} \\ P_{C\to A} & P_{C\to B} & P_{C\to C} \end{bmatrix} = \begin{bmatrix} 0 & \frac{S_C}{S_B+S_C} & \frac{S_B}{S_B+S_C} \\ \frac{S_C}{S_A+S_C} & 0 & \frac{S_A}{S_A+S_C} \\ \frac{S_B}{S_A+S_B} & \frac{S_A}{S_A+S_B} & 0 \end{bmatrix}$$

where $P_{A\to B} = \frac{S_C}{S_B+S_C}$ means A is not playing in the first game and B is beaten in this game and will not play in the following game.

The process is Markov chain because the results of the subsequent game are independent.

(b)

We are looking for the probability that after three steps the chain returns to a given initial state.

For example, if the initial state is A, then there are two way to return to A after three steps: $A \to B \to C \to A$ and $A \to C \to B \to A$. The probability is

$$P_{A\to B}P_{B\to C}P_{C\to A} + P_{A\to C}P_{C\to B}P_{B\to A}$$
$$= \frac{2S_A S_B S_C}{(S_A+S_B)(S_B+S_C)(S_C+S_A)}$$

The probabilities of states B and C are similar. Obviously, this probability doesn't depend on who play in the first game.

2. Solution:

(a)

(i) Starting from 2, we jump to 1 with probability 1/2, and to 3 with probability 1/2.

So, $h_2 = \frac{1}{2}h_1 + \frac{1}{2}h_3$. (where $\frac{1}{2}h_1$ means we jump to 1 with probability 1/2 and then is absorbed in state 4 given the initial state is 1)

Similarly, $h_3 = \frac{1}{2}h_2 + \frac{1}{2}h_4$

(ii) If we state from 1, we stay in 1, therefore $h_1 = 0$. If we start from 4, we are already in 4, therefore $h_4 = 1$. So,

$$h_2 = \frac{1}{2}h_3 = \frac{1}{2}\left(\frac{1}{2}h_2 + \frac{1}{2}h_4\right) = \frac{1}{2}\left(\frac{1}{2}h_2 + \frac{1}{2}\right)$$

$$\Rightarrow h_2 = \frac{1}{3},\ h_3 = \frac{2}{3}$$

(b)

(i) Starting from 2 after 1 step, we jump to 1 with probability 1/2 or to 3 with probability 1/2, then:

$k_2 = 1 + \frac{1}{2}k_1 + \frac{1}{2}k_3$ (where 1 means there is 1 step if we jump from 2 to 1 or 3).

Similarly, $k_3 = 1 + \frac{1}{2}k_2 + \frac{1}{2}k_4$

(ii) It is easy to see that $k_1 = k_4 = 0$.

Finally, $k_2 = \left(1 + \frac{1}{2}k_3\right) = 1 + \frac{1}{2}\left(1 + \frac{1}{2}k_2\right) \Rightarrow k_2 = 2, k_3 = 2$

3. Solution:

Firstly, we know that:

$P(\text{seat } j \text{ is free}) = \frac{1}{N}$

$P\left(X_1 = i_1, \cdots, X_{N-1} = i_{N-1} \middle| \text{seat } j \text{ is free}\right) = \frac{1}{(N-1)!}$ for any given (fixed) sequence

$i_1, \cdots, i_{N-1} \in \{1, \cdots, N\} / \{j\}$ (i.e. given j is free, the probability that the first spectator takes seat i_1, the second spectator takes seat i_2, …, the (N-1)-th takes seat i_{N-1})

Now, consider a discrete time Markov chain with states $0, 1, 2, \ldots, N-1, N$. Here, state 0 means that the spectator attempting the free seat "succeeds", and state $1 \le n \le N-1$ means that there are *n* seats left and one of them is what the spectator wants (i.e. the $(N-n)$-th move is "unsuccessful"), and *N* is the initial state.

The probability of transition $n \to 0$ is $\frac{1}{n}$

The probability of transition $n \to (n-1)$ is $\frac{n-1}{n}$

The probability of transition $0 \to 0$ is 1

Let $T(n)$ denote the expected number of transitions for the Markov chain to go from state n to state 0. The key fact is the following recursion

$$T(n) = \frac{n-1}{n}\left(1 + T(n-1)\right) + \frac{1}{n} \cdot 0 \quad \text{(with } T(1) = 0\text{)}$$

$$\Rightarrow T(N) = \frac{1}{N}(N-1+N-2+\cdots+1) = \frac{N(N-1)}{2N} = \frac{N-1}{2}$$

If N=121, the expected delay will be $45 \cdot \frac{120}{2} = 45$ min.