Study Group

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Comms-1

Fourier Series & Fourier Transform

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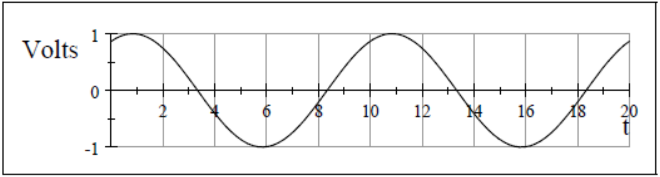
Introduction

- Consider the sinusoid $g(t)$ of amplitude A, frequency F_0 and phase ϕ. This signal can be represented as follows:

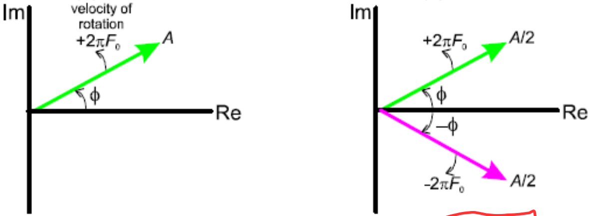
$$\begin{aligned} g(t) &= A.\cos(2\pi F_0 t+\phi) && (1)\\ &= A.\mathrm{Re}\left(\exp\{j.(2\pi F_0 t+\phi)\}\right) && (2)\\ &= \frac{A}{2}.\left(\exp\{j.(2\pi F_0 t+\phi)\} + \exp\{-j.(2\pi F_0 t+\phi)\}\right) && (3)\\ &= \frac{A}{2}.\cos\phi.\left(\exp[j.2\pi F_0 t] + \exp\{-j.2\pi F_0 t\}\right)\\ &\quad +j.\frac{A}{2}.\sin\phi.\left(\exp\{j.2\pi F_0 t\} - \exp\{-j.2\pi F_0 t\}\right) && (4) \end{aligned}$$

i) Temporal Representation (Equ.1): $g(t) = A.\cos(2\pi F_0 t+\phi)$



ii) Phasor Representation (Equs 2 and 3): we represent exponentials as rotating phasors in the complex plane. There are two different phasor representations:

ii(a): $g(t) =$Equ.2 ii(b): $g(t) =$Equ.3

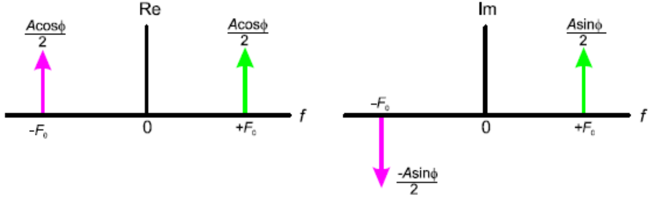


MOST IMPORTANT *

- ★ above, both diagrams, represent a "snapshot" at $t = 0$
- ★ angular velocity of phasors $= \pm 2\pi F_0 \ \frac{\text{rad}}{\text{sec}}$ i.e. always real
- ★ The second phasor representation (ii.b) leads to the SPECTRAL REPRESENTATION.

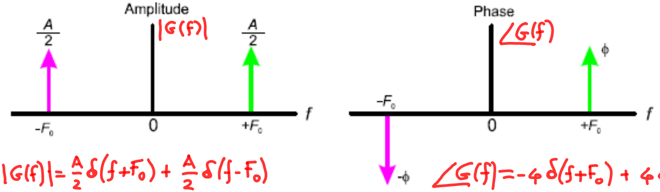
iii) Spectral Representation (Equ.4)

★ Line spectrum:



The spectrum indicates the sizes and starting angles of the phasors (resolved intro Re and Im parts), and also that they should rotate at $2\pi F_0 \ \frac{\text{rad}}{\text{sec}}$

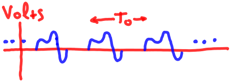
Most Important → ★ More usual representation: (based on Equ.3)



$|G(f)| = \frac{A}{2}\delta(f+F_0) + \frac{A}{2}\delta(f-F_0)$

$\angle G(f) = -\phi\,\delta(f+F_0) + \phi\,\delta(f-F_0)$

★ N.B.: to specify spectrum fully, we need two quantities at each frequency, that is *amplitude* and *phase*.

$g(t) =$ 

$=$ periodic waveform of period T_0

1st Form

- Joseph Fourier (1768-1830) Theorem: Any periodic waveform can be represented by a sum of sine and cosine terms having frequencies $F_0, 2F_0, 3F_0$, etc. (in general to ∞).
- Let us define

$$T_0 = \frac{1}{F_0}$$

- **1^{st}- Form:**

nF0 nF0

$$g(t) = a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(2\pi \frac{n}{T_0} t\right) + b_n \sin\left(2\pi \frac{n}{T_0} t\right) \right\}$$

$$a_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g(t).dt$$

DC level

$$a_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g(t) \cos\left(2\pi \frac{n}{T_0} t\right) dt$$

$$b_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g(t) \sin\left(2\pi \frac{n}{T_0} t\right) dt$$

- N.B.: The reason why the DC level is an a and not a b is that $\cos 0° = 1$.

2nd Form

- by using

$$A.\cos\theta + B.\sin\theta = \sqrt{A^2+B^2}\cos\left(\theta - \tan^{-1}\frac{B}{A}\right)$$

we get the **2^nd Form**:

$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(2\pi \frac{n}{T_0} t - \phi_n\right)$	$\underset{\substack{\uparrow \\ DC \\ level}}{C_0} = a_0$ $C_n = \sqrt{a_n^2 + b_n^2}$ $\phi_n = \tan^{-1}\frac{b_n}{a_n}$

- N.B: the coefficients C_n are called Single-Sided Spectral-Amplitudes

3rd Form (Most Important)

- The 3^{rd} **Form** can be derived by using $\cos\theta = \frac{\exp(j\theta)+\exp(-j\theta)}{2}$.
  This form, which finds extensive application in communication theory, is given as follows:

$$g(t) = \sum_{n=-\infty}^{\infty} G_n \exp\left(j2\pi\frac{n}{T_0}t\right) \qquad G_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\exp\left(-j2\pi\frac{n}{T_0}t\right)dt$$

- Remember: FS provides a **Discrete** *Spectral Representation for* **Periodic** *Signals* (i.e we sum spectral components to return to the time domain).
- N.B.:
  - G_n = are in general complex - even with a real input signal, i.e. amplitude of a component at $n.F_0$ is $|G_n|$ and phase angle is $\angle G_n$
  - $G_{-n} = G_n^*$
  - $G_0 = C_0$
  - $G_n = \frac{C_n}{2}.\exp\{-j.\phi_n\}$

 BASIC THEORY - FT

Fourier Transform Tables

$g(t) \xrightarrow{FT} G(f) = FT\{g(t)\}$

IMPORTANT

	Description	Function	Transformation
1	Definition	$g(t)$	$G(f)=\int_{-\infty}^{\infty} g(t).e^{-j2\pi ft}dt$
2	Scaling	$g\left(\frac{t}{T}\right)$	$\lvert T\rvert .G(fT)$
3	Time shift	$g(t-T)$	$G(f).e^{-j2\pi fT}$
4	Frequency shift	$g(t).e^{j2\pi Ft}$	$G(f-F)$
5	Complex conjugate	$g^*(t)$	$G^*(-f)$
6	Temporal derivative	$\frac{d^n}{dt^n}g(t)$	$(j2\pi f)^n.G(f)$
7	Spectral derivative	$(-j2\pi t)^n.g(t)$	$\frac{d^n}{df^n}G(f)$
8	Reciprocity	$G(t)$	$g(-f)$
9	Linearity	$A.g(t)+B.h(t)$	$A.G(f)+B.H(f)$
10	Multiplication	$g(t).h(t)$	$G(f)*H(f)$
11	Convolution	$g(t)*h(t)$	$G(f).H(f)$
12	Delta function	$\delta(t)$	1
13	Constant	1	$\delta(f)$
14	Rectangular function	$\mathbf{rect}\{t\} \triangleq \begin{cases} 1 & \text{if } \lvert t\rvert<\frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$	$\mathbf{sinc}\{f\} \triangleq \frac{\sin(\pi f)}{\pi f}$
15	Sinc function	$\mathbf{sinc}(t)$	$\mathbf{rect}\{f\}$
16	Unit step function	$u(t) \triangleq \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$	$\frac{1}{2}\delta(f)-\frac{j}{2\pi f}$
17	Signum function	$\mathrm{sgn}(t) \triangleq \begin{cases} 1 & t>0 \\ -1 & t<0 \end{cases}$	$-\frac{j}{\pi f}$
18	decaying exp (two-sided)	$e^{-\lvert t\rvert}$	$\frac{2}{1+(2\pi f)^2}$
19	decaying exp (one-sided)	$e^{-\lvert t\rvert}.u(t)$	$\frac{1-j2\pi f}{1+(2\pi f)^2}$
20	Gaussian function	$e^{-\pi t^2}$	$e^{-\pi f^2}$
21	Lambda function	$\Lambda\{t\} \triangleq \begin{cases} 1-t & \text{if } 0\le t\le 1 \\ 1+t & \text{if } -1\le t\le 0 \end{cases}$	$\mathbf{sinc}^2\{f\}$
22	Repeated function	$\mathbf{rep}_T\{g(t)\}=g(t)*\mathbf{rep}_T\{\delta(t)\}$	$\lvert\frac{1}{T}\rvert\mathbf{comb}_{\frac{1}{T}}\{G(f)\}$
23	Sampled function	$\mathbf{comb}_T\{g(t)\}=g(t).\mathbf{rep}_T\{\delta(t)\}$	$\lvert\frac{1}{T}\rvert\mathbf{rep}_{\frac{1}{T}}\{G(f)\}$

$\boxed{\text{These replace FS}}$

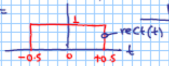
FOURIER TRANSFORM (NON PERIODIC SIGNALS)

Definition: $G(f) = \int_{-\infty}^{+\infty} g(t)\exp(-j2\pi f t)\,dt$

inverse: $g(t) = \int_{-\infty}^{\infty} G(f)\exp(+j2\pi f t)\,df$

important examples:

$g(t) =$



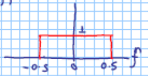
$\xrightarrow{FT}$ $G(f) =$

$g(t) =$

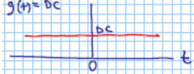
$\xrightarrow{FT}$ $G(f) =$

$g(t) =$

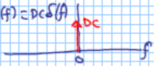
$\xrightarrow{FT}$ $G(f)$



$g(t) = DC$



$\xrightarrow{FT}$ $G(f) = DC\,\delta(f)$



$g\left(\frac{t}{T}\right) \xrightarrow{FT} |T|\,G(fT)$

$g_1(t) \cdot g_2(t) \xrightarrow{FT} G_1(f) * G_2(f)$ (conv)

$Ag_1(t) + Bg_2(t) \xrightarrow{FT} AG_1(f) + BG_2(f)$

PROPERTIES OF SYMMETRY for Aperiodic Signals

1) if $g(t) = \text{real}$ then $G(-f) = G^*(f)$

$\Updownarrow$

$$\boxed{\begin{aligned} |G(-f)| &= |G(f)| \\ \angle G(-f) &= -\angle G(f) \end{aligned}}$$

i.e.
amplitude spectrum = has EVEN symmetry

phase spectrum = has ODD symmetry

2) if $g(t) = \text{real}$ and has EVEN time symmetry (like a cosine)

then $G(f) = \text{entirely real}$ (ie $\angle G(f) = 0$)

$= 2\int_0^{\infty} g(t)\cos(2\pi f t)\,dt$

3) if $g(t) = \text{real}$ and has ODD time symmetry (like a sin)

then $G(f) = \text{entirely imaginary}$ (ie $\angle G(f) = \mp\frac{\pi}{2}$)

$= -2j\int_0^{\infty} g(t)\sin(2\pi f t)\,dt$

Note: Problem Sheet 2, Q3 and Q4 are related to FS

$\boxed{\text{PS2-Q3}}$

* $\int \underbrace{[\text{even function of time}] \times [\text{odd function of time}]}\, dt = 0$

this is a function with (+)ve area under the curve equal to the (−)ve area under the curve ∴ its integral is equal to 0

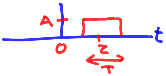
* $\cos(2\pi F_0 t)$ = even function of time
* $\sin(2\pi F_0 t)$ = odd —"— —"—
* $\int [\text{even function of time}] . \cos(2\pi F_0 t)\, dt \neq 0$
* $\int [\text{even function of time}] . \sin(2\pi F_0 t)\, dt = 0 \Rightarrow b_n = 0$
* $\int [\text{odd function of time}] \cdot \cos(2\pi F_0 t)\, dt = 0 \Rightarrow a_n = 0$
* $\int [\text{odd function of time}] \cdot \sin(2\pi F_0 t)\, dt \neq 0$

$\boxed{\text{PS2-Q4}} \Rightarrow x(t) = \text{even} \Rightarrow b_n = 0 \Rightarrow$ only cosines

$\boxed{\text{PS3-Q1}}$: see page 10, properties of symmetry: (2) and (3)

$\boxed{\text{PS3-Q2}}$: see page 10, properties of symmetry: (1)

$\boxed{\text{PS3-Q3}}$: $g(t) =$

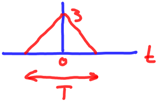


$= A.\text{rect}\left(\frac{t-\tau}{T}\right)$

$$G(f) = FT\{g(t)\} = AT\,\text{sinc}\{fT\}\exp(-j2\pi f\tau)$$

$A=1,\ T=1,\ \tau=5$

$g(t) =$



$= 3\,\Lambda\left\{\frac{t}{T/2}\right\}$

$$G(f) = FT\{g(t)\} = 3\,\frac{T}{2}\,\text{sinc}^2\left\{f\,\frac{T}{2}\right\}$$

PS3-Q4 :

Example: $g(t) = \mathrm{rect}\left(\frac{t}{T}\right) =$

$$G(f) \triangleq \int_{-\infty}^{\infty} g(t)\exp(-j2\pi f t)\,dt$$

$$= \int_{-T/2}^{T/2} 1\exp(-j2\pi f t)\,dt$$

$$= -\frac{1}{j2\pi f}\Big[\exp(-j2\pi f t)\Big]_{-T/2}^{T/2}$$

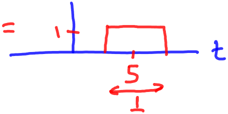
$$= -\frac{1}{j2\pi f}\left\{\exp\left(-j2\pi f\frac{T}{2}\right) - \exp\left(+j2\pi f\frac{T}{2}\right)\right\}$$

$$= \frac{1}{\pi f}\underbrace{\frac{\exp(j\pi f T) - \exp(-j\pi f T)}{2j}}_{\sin(\pi f T)}$$

$$= \frac{\sin(\pi f T)}{\pi f} = T\underbrace{\frac{\sin(\pi f T)}{\pi f T}}_{\mathrm{sinc}(fT)}$$

$$= T\,\mathrm{sinc}(fT)$$

Q4

$g(t) =$ 

$= \mathrm{rect}\{t-5\}$