EE4-45

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

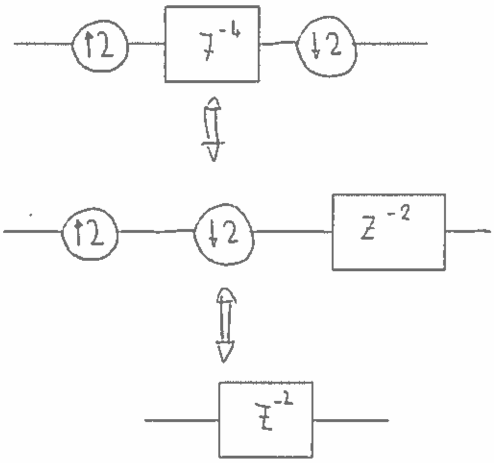
DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2018

WAVELETS AND APPLICATIONS SOLUTIONS

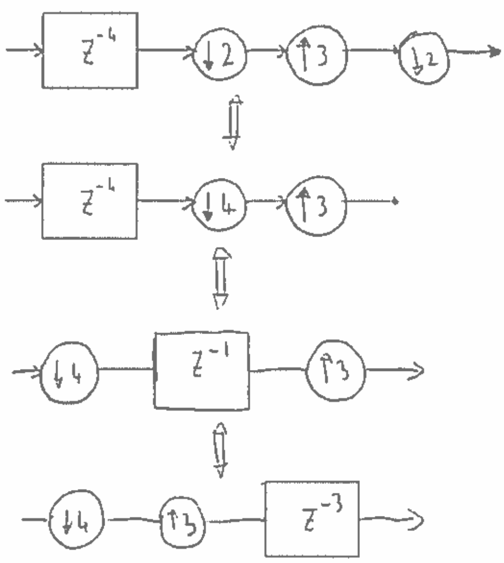
SOLUTIONS

1. (a) By applying Nobel identities as shown in the figure below, we realise that the system is implementing a delay by 2, so $y[n] = x[n-2]$.



(b) i. We apply repeatedly Nobel identities and fractional sampling rules as shown in the figure below in order to show that:

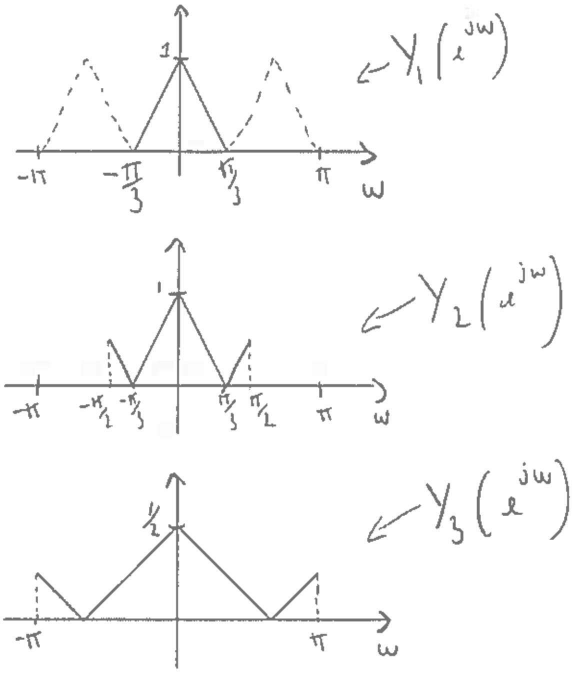
$$Y(z) = \frac{z^{-3}}{4}[X(z^{3/4}) + X(jz^{3/4}) + X(-z^{-3/4}) + X(-jz^{3/4})]$$



ii. $y[n] = \delta[n-3]$

iii. $y[n] = \{...1, 0, 0, 1, ..., 1, 0, 0, 1...\}$

(c) The three spectra are shown in the figure below.



2. (a)

$$Y_0(z) = \frac{1}{2}[H_0(z^{1/2})X(z^{1/2}) + H_0(-z^{1/2})X(-z^{1/2}]$$

and

$$Y_1(z) = \frac{1}{2}[H_1(z^{1/2})X(z^{1/2}) + H_1(-z^{1/2})X(-z^{1/2}].$$

Therefore

$$\hat{X}(z) = \frac{G_0(z)}{2}[H_0(z)X(z) + H_0(-z)X(-z)] + \frac{G_1(z)}{2}[H_1(z)X(z) + H_1(-z)X(-z)].$$

This implies that PR is achieved when

$$G_0(z)H_0(z) + G_1(z)H_1(z) = 2$$

and

$$G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0.$$

(b) We need to satisfy the two (bio)-orthogonal relations: $\langle h_0^T[n], g_1[n-2k]\rangle = 0$ and $\langle h_1^T[n], g_0[n-2k]\rangle = 0$ Consequently, by using 'shift and modulation' we obtain

$$H_0(z) = -zG_1(-z) = \sqrt{2}(z^2 - 2z + 3 - 2z^{-1} + z^{-2})$$

and

$$H_1(z) = zG_0(-z) = (-z^2 + 2z - 1)/(2\sqrt{2})$$

.

(c) Since $Q(z) = \frac{1}{256}(3z^2 - 18z + 38 - 18z^{-1} + 3z^{-2})$, we first get the following quadratic equation

$$3x^2 - 18x + 32 = 0,$$

where $x = z + \frac{1}{z}$. This gives $z + \frac{1}{z} = 3 \pm j\frac{\sqrt{15}}{3}$ which can then be solved to give us the four roots of $Q(z)$: z_0, z_0^*, $1/z_0$ and $1/z_0^*$ where

$$z_0 = \frac{1}{2}\left(3 - j\frac{\sqrt{15}}{3} - \sqrt{\frac{10}{3} - 2j\sqrt{15}}\right).$$

The ten roots of $P(z)$ are then given by these four roots together with $z_1 = -1$ with multiplicity 6.

(d) Since $P(z)$ satisfies the half-band condition, through spectral factorization, we find $G_0(z)$ and $H_0(z)$ such that $H_0(z^{-1}) = G_0(z)$ and this yields

$$G_0(z) = \frac{1}{128}(1 + z^{-1})^3(1 + z_0 z)(1 + z_0^* z).$$

Finally, $G_1(z) = -z^{-1}G_0(-z^{-1})$ and $H_1(z) = G_1(z^{-1})$

(e) We need $H_1(z)$ to have six roots at $\omega = 0$. Since $H_1(z) = zG_0(-z)$, it is enough that $G_0(z)$ has six roots at $\omega = \pi$. Therefore we pick $G_0(z) = (z^{-1} + 2 + z)^3/(32\sqrt{2})$. Consequently $H_0(z) = \frac{\sqrt{2}}{8}(3z^2 - 18z + 38 - 18z^{-1} + 3z^{-2})$.

3. (a) We have:

$$\langle \varphi_1, \varphi_2 \rangle = \langle \varphi_2, \varphi_1 \rangle = \int_0^1 \sin \pi t dt = \frac{2}{\pi}.$$

Moreover, $\|\varphi_1\|^2 = 1$ and $\|\varphi_2\|^2 = \frac{1}{2}$. Consequently,

$$1 = \langle \varphi_1(t), \tilde{\varphi}_1(t) \rangle = a_{1,1}\langle \varphi_1, \varphi_1 \rangle + a_{1,2}\langle \varphi_1, \varphi_2 \rangle = a_{1,1} + \frac{2a_{1,2}}{\pi}$$

$$0 = \langle \varphi_2(t), \tilde{\varphi}_1(t) \rangle = a_{1,1}\langle \varphi_2, \varphi_1 \rangle + a_{1,2}\langle \varphi_2, \varphi_2 \rangle = \frac{2a_{1,1}}{\pi} + \frac{a_{1,2}}{2}$$

which yields

$$a_{1,1} = \frac{\pi^2}{\pi^2 - 8}$$

and

$$a_{1,2} = -\frac{4\pi}{\pi^2 - 8}.$$

Similarly, we have:

$$a_{2,1} = -4\frac{\pi}{\pi^2 - 8},$$

and

$$a_{2,2} = 2\frac{\pi^2}{\pi^2 - 8}.$$

The two dual-basis functions can then be written as:

$$\tilde{\varphi}_1(t) = \frac{\pi^2}{\pi^2 - 8} - \frac{4\pi}{\pi^2 - 8}\sin \pi t$$

and

$$\tilde{\varphi}_2(t) = 2\frac{\pi^2}{\pi^2 - 8}\sin \pi t - 4\frac{\pi}{\pi^2 - 8}$$

(b) Clearly $\langle x(t), \varphi_1(t) \rangle = 0$ therefore,

$$\langle x(t), \tilde{\varphi}_1(t) \rangle = -\frac{4\pi}{\pi^2 - 8}\int_0^1 \cos 2\pi t \sin \pi t dt = \frac{8}{3(\pi^2 - 8)}$$

and

$$\langle x(t), \tilde{\varphi}_2(t) \rangle = 2\frac{\pi^2}{\pi^2 - 8}\int_0^1 \cos 2\pi t \sin \pi t dt = -\frac{4\pi}{3(\pi^2 - 8)}$$

(c)

$$x_v(t) = \sum_{i=1}^{2} \langle x(t), \tilde{\varphi}_i(t) \rangle \varphi_i(t) = \frac{8}{3(\pi^2 - 8)} - \frac{4\pi}{3(\pi^2 - 8)}\sin \pi t$$

(d) We need to verify that $\langle \epsilon_v(t), \varphi_1(t) \rangle = 0$ and that $\langle \epsilon_v(t), \varphi_2(t) \rangle = 0$.
We have

$$\langle \epsilon_v(t), \varphi_1(t) \rangle = \langle x(t), \varphi_1(t) \rangle - \frac{8}{3(\pi^2 - 8)} + \frac{4\pi}{3(\pi^2 - 8)}\int_0^1 \sin \pi t dt = 0$$

where we have used the fact that $\langle x(t), \varphi_1(t) \rangle = 0$. Moreover,

$$\langle \epsilon_v(t), \varphi_2(t) \rangle = \int_0^1 \cos 2\pi t \sin \pi t dt - \frac{8}{3(\pi^2 - 8)}\int_0^1 \sin \pi t dt + \frac{4\pi}{3(\pi^2 - 8)}\int_0^1 \sin^2 \pi t dt = 0.$$

So orthogonality condition of the error is satisfied.

4. (a) $G_0(z) = \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^2$, therefore $G_0(e^{j\pi}) = 0$ and $G_0(1) = \sqrt{2}$, so the necessary conditions for convergence are satisfied.

(b) If the limit exists, we can write the Fourier transform of $\varphi(t)$ as

$$\hat{\varphi}(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right),$$

where $M_0(\omega) = G_0(e^{j\omega})/\sqrt{2}$. Using Poisson summation formula the partition of unity condition

$$\sum_n \varphi(t-n) = 1$$

becomes

$$\sum_k \hat{\varphi}(2\pi k)e^{j2\pi kt} = 1$$

which is satisfied given that $G_0(z)$ satisfies the necessary conditions for convergence.

(c) We write the two-scale equation in the Fourier domain:

$$\hat{\varphi}(\omega) = \frac{1}{\sqrt{2}}G_0(e^{j\omega/2})\hat{\varphi}(\frac{\omega}{2}).$$

If the limit exist then

$$\hat{\varphi}(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right)$$

and we can write

$$\hat{\varphi}(\omega) = M_0(\omega/2)\prod_{k=2}^{\infty} M_0\left(\frac{\omega}{2^k}\right) = M_0(\omega/2)\prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2\cdot 2^k}\right) = \frac{1}{\sqrt{2}}G_0(e^{j\omega/2})\hat{\varphi}(\frac{\omega}{2}),$$

where in the last equality we have used the fact that $M_0(\omega) = G_0(e^{j\omega})/\sqrt{2}$.

(d) We can write $M_0(\omega)$ as follows

$$M_0(\omega)) = \left(\frac{1+e^{-j\omega}}{2}\right)^N R(\omega),$$

where $R(\omega) = 1$ and $N = 2$. This means that $B = \max_\omega R(\omega) = 1$ and since $B < 2^{N-1}$ we know that the sufficient condition for $\varphi(t)$ to be continuous is satisfied.

(e) $G_0(z)$ has two zeros at $\omega = \pi$, therefore the wavelet has two vanishing moments.