Sparsity, Wavelets and their Applications

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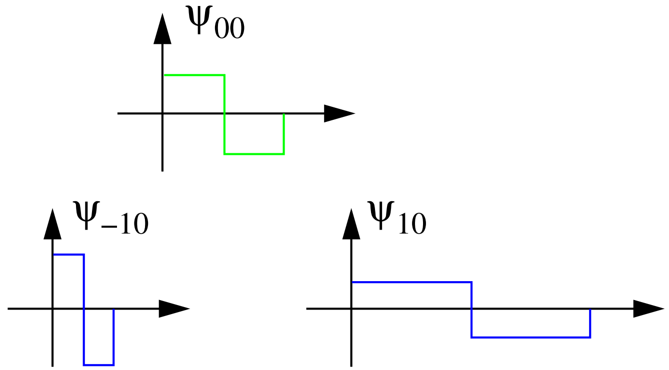
Session four: Haar Expansion

The Haar decomposition

$$\psi_{mn}(t) = 2^{-\frac{m}{2}}\psi(2^{-m}t - n)$$

Basis functions

$$\psi(t) = \begin{cases} 1 & 0 \le t < 0.5 \\ -1 & 0.5 \le t < 1 \\ 0 & \text{else} \end{cases}$$

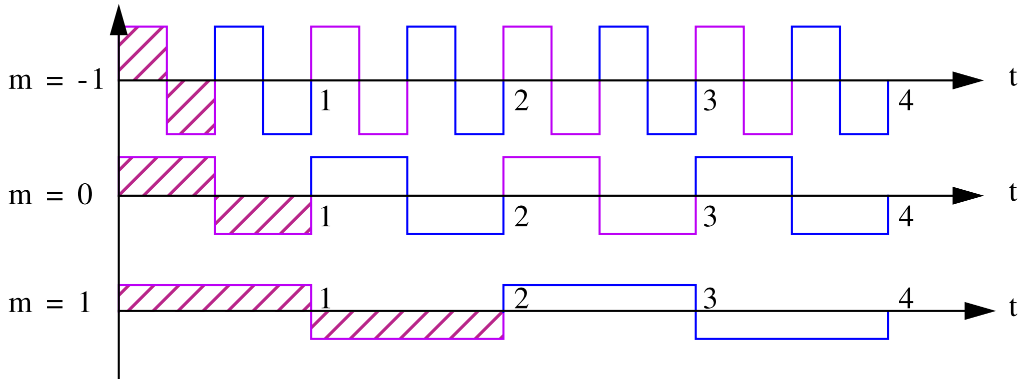


$$\psi_{mn}(t) = 2^{-m/2}\psi(2^{-m}t - n)$$

normalisation shape

$$\begin{cases} m > 0 & \text{dilact} \\ m < 0 & \text{suppress} \end{cases}$$

Basis functions across scales (note orthogonality)



Haar scaling function (LP filter)

$$\phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

The Haar decomposition

$G_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$ $\quad g_0[n] = \frac{1}{\sqrt{2}}(1,1)$

$G_1(z) = \frac{1}{\sqrt{2}}(1-z^{-1})$ $\quad g_1[n] = \frac{1}{\sqrt{2}}(1,-1)$

Haar wavelet function (HP filter)

$$\psi(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Haar system...

... scaling function and wavelet

$$\begin{cases} \phi(t) = \sqrt{2}\sum_n g_0[n]\,\phi(2t-n) \\ \psi(t) = \sqrt{2}\sum_n g_1[n]\,\phi(2t-n) \end{cases}$$

- The Haar scaling function (indicator of unit interval)

$$\varphi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{else} \end{cases}$$

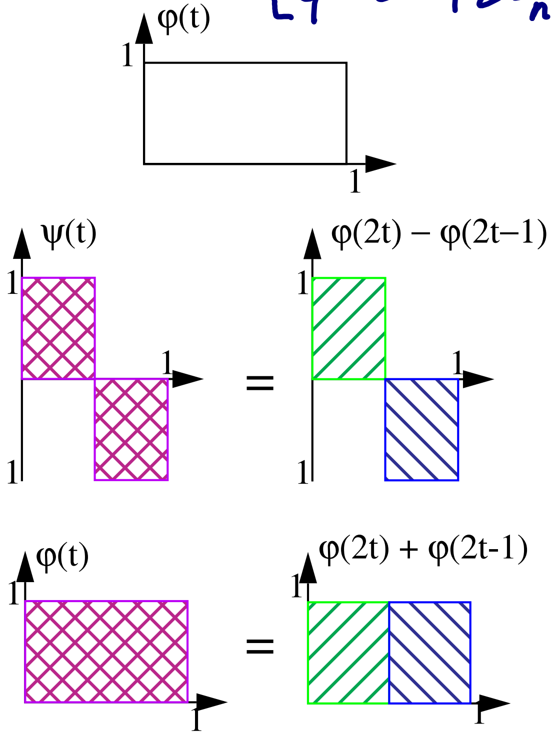
$\psi_{mn}(t) = 2^{-\frac{m}{2}}\,\psi(2^{-m}t - n)$ $\quad m, n \in \mathbb{Z}$.

helps in the construction of the wavelet, since

$$\psi(t) = \varphi(2t) - \varphi(2t-1)$$

and satisfies a two-scale equation

$$\varphi(t) = \varphi(2t) + \varphi(2t-1)$$



Note:

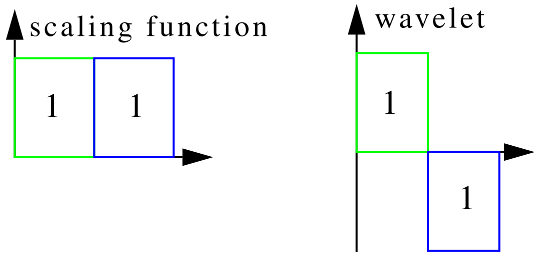
Haar wavelet a bit too trivial to be useful...

The Haar decomposition

Recall Haar in discrete time

$$\varphi_0[n] = \frac{1}{\sqrt{2}}(1, 1) \qquad \varphi_1[n] = \frac{1}{\sqrt{2}}(1, -1)$$

Now look at Haar in continuous time



$$\varphi(t) = 1 \cdot \varphi(2t) + 1 \cdot \varphi(2t-1) \qquad \psi(t) = 1 \cdot \varphi(2t) - 1 \cdot \varphi(2t-1)$$

Is there a connections?

(lowpass filter $\Leftrightarrow$ scaling function

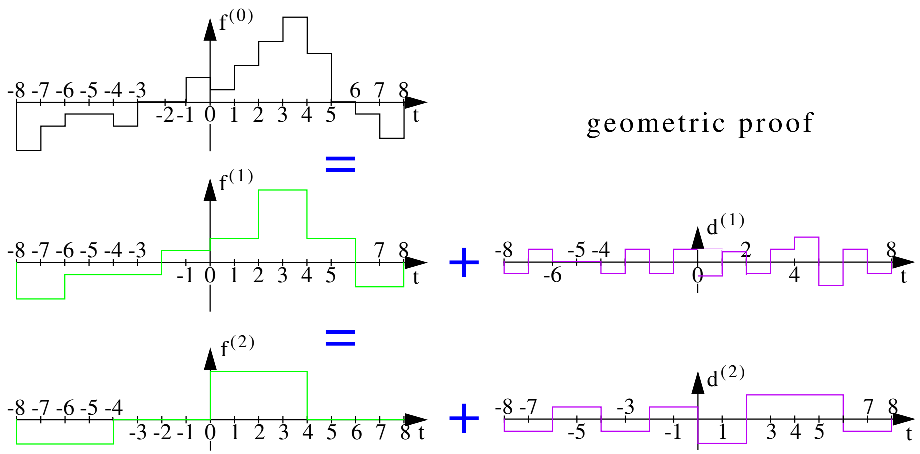
highpass filter $\Leftrightarrow$ wavelet)

The Haar decomposition

Haar system...

... proof it is an orthonormal basis for $L_2(\Re)$

- piecewise constant functions are dense in $L_2(\Re)$
- write $f^{(i)} = f^{(i+1)} + d^{(i+1)}$
- show that $\left|f^{(i)}\right| \to 0$ as $i \to \infty$

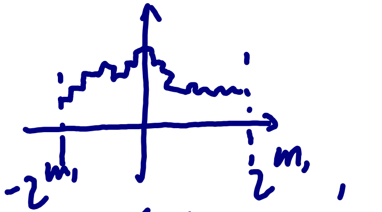


geometric proof

Note: two functions are involved

- scaling function $\varphi^{(i)}(t)$, to go from $f^{(i-1)}$ to $f^{(i)}$
- wavelet $\psi^{(i)}(t)$ to represent the difference $d^{(i)}$

The Haar decomposition



resolution $= \frac{1}{2^{m_0}}$

$$f(t) = \sum_{n=2^{-m_0+m_1}}^{n=2^{m_0+m_1}} \alpha_{n.-m_0}\, \varphi_{n.-m_0}(t)$$

$$= \sum_n C_{n.-m_0+1}\, \varphi_{n.-m_0+1}(t) \quad \text{avg.}$$

$$+ \sum_n d_{n.-m_0+1}\, \varphi_{n.-m_0+1}(t) \quad \text{diff.}$$

$$= \sum_n C_{n.-m_0+2}\, \varphi_{n.-m_0+2}(t) + \sum_{j=1}^{2}\sum_n d_{n.-m_0+j}\, \varphi_{n.-m_0+j}(t)$$

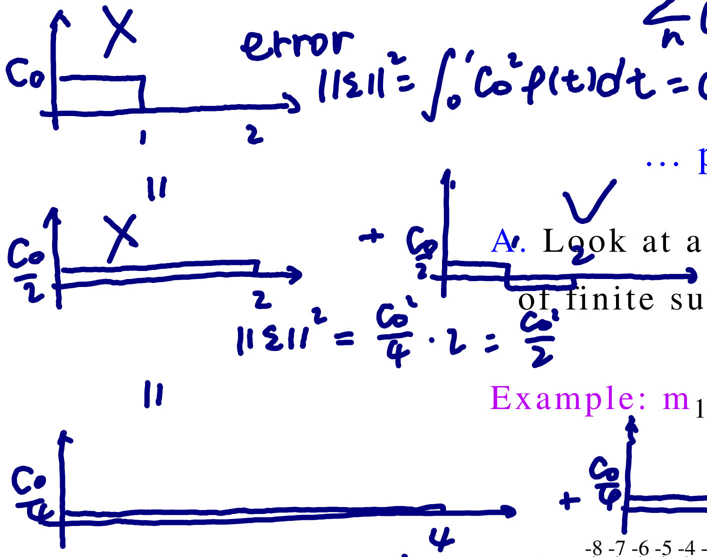
$$= \sum_n C_{n.-m_0+J}\, \varphi_{n.-m_0+J}(t) + \sum_{j=1}^{J}\sum_n d_{n.-m_0+j}\, \varphi_{n.-m_0+j}(t) \quad [\text{discrete}]$$

$\downarrow m_0 \to \infty$. $V_{-\infty} \to l_2$

$$= \sum_n C_{n.J}\, \varphi_{n.J}(t) + \sum_{j=-\infty}^{J}\sum_n d_{n.j}\, \psi_{n.j}(t) \quad (\text{continous})$$

$$= C_0 f(t) + \sum_{m=-\infty}^{J}\sum_n d_{m.n}\, \psi_{m.n}(t)$$

error



$\|\varepsilon\|^2 = \int_0^1 C_0^2 \varphi(t)\,dt = C_0^2$

$\|\varepsilon\|^2 = \frac{C_0^2}{4}\cdot 2 = \frac{C_0^2}{2}$

$\|\varepsilon\|^2 = \frac{C_0^2}{16}\, 4 = \frac{C_0^2}{4}$

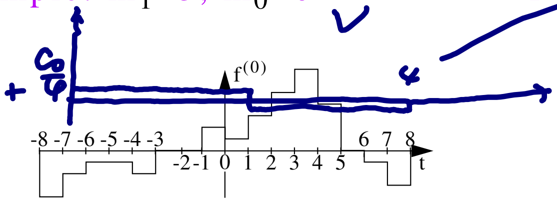
Haar system...

... proof it is an orthonormal basis for $L_2(\Re)$

A. Look at a function constant on intervals $[n2^{-m_0}, (n+1)2^{-m_0})$ and of finite support $[-2^{-m_1}, 2^{-m_1})$, call it $f^{(-m_0)}(t)$

Example: $m_1=3$, $m_0=0$

$$f^{(0)}(t) = \sum_{n=-8}^{8} f^{(0)}[n]\varphi_{0,n}(t)$$



If we choose m_0, m_1, large enough, we can approximate any function from $L_2(\Re)$

$$f^{(-m_0)}(t) = \sum_{n=-N}^{N} f_n^{(-m_0)}\varphi_{-m_0,n}(t) \quad \text{with } N = 2^{m_0+m_1}$$

$$f_n^{(-m_0)} = 2^{-m_0/2} f^{(-m_0)}\left(n2^{-m_0}\right), \quad \varphi_{-m_0,n}(t) = \begin{cases} 2^{m_0/2} & n2^{-m_0} \le t < (n+1)2^{-m_0} \\ 0 & \text{otherwise} \end{cases}$$

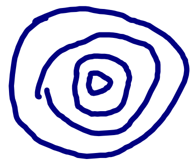
$\vdots$

$J \nearrow \Rightarrow \|\varepsilon\|^2 \searrow$. first term reduced.

$J \to \infty \Rightarrow \|\varepsilon\|^2 \to 0$. first term eliminated. $\Rightarrow$ subspace by $\text{span}\{w\} = l_2$.

$\therefore$ as $J \to \infty$. $f(t) = \sum_n \sum_m d_{m.n}\psi_{m.n}(t)$

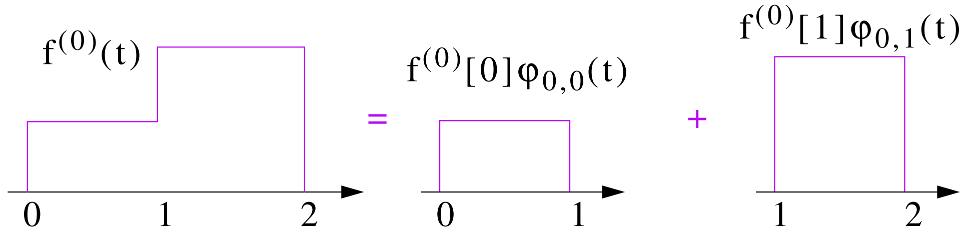
$\ldots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots$



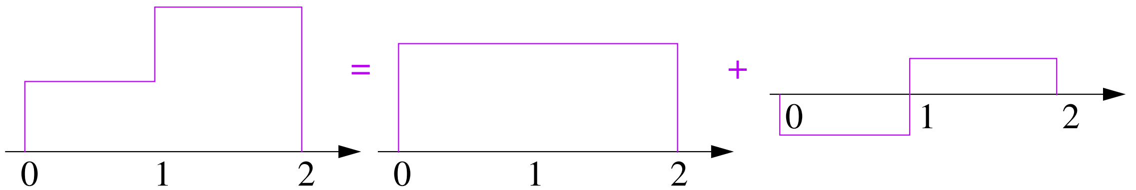
φ: reshape + shift to cover subspace.

The Haar decomposition

B. Zoom into two consecutive intervals



Express it as average and difference

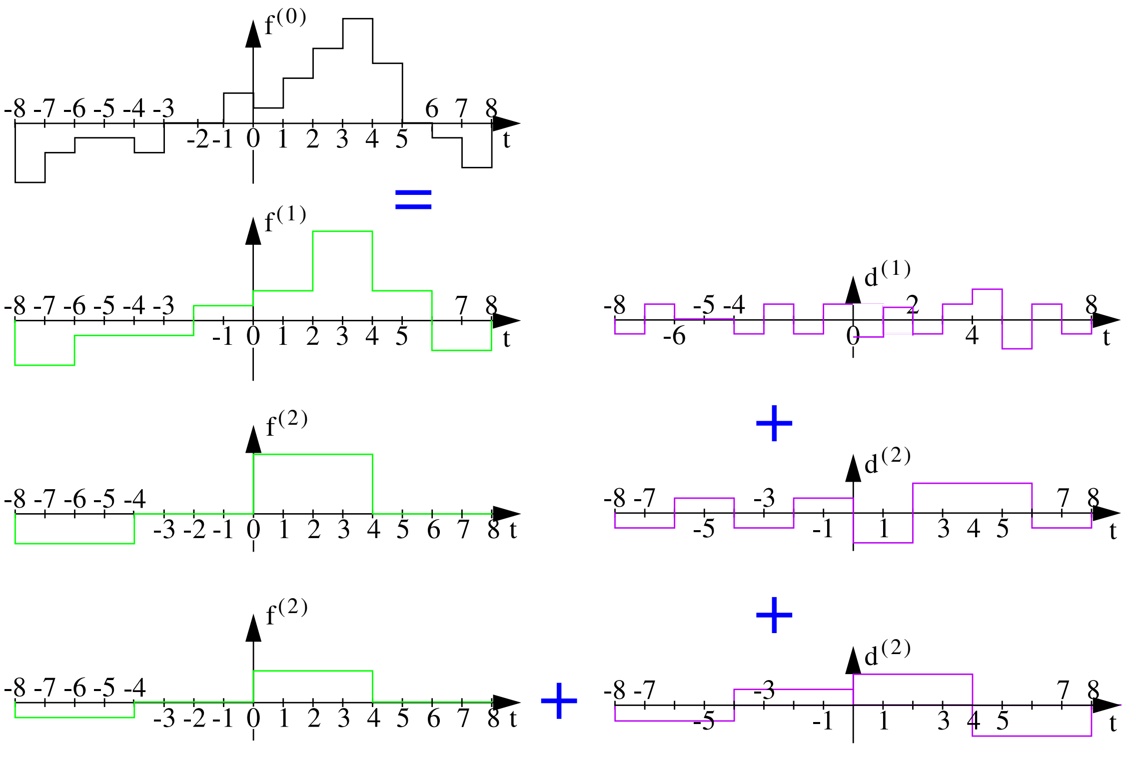


$f^{(0)}(t)$ $\qquad \dfrac{f_0^{(0)} + f_1^{(0)}}{2}\sqrt{2}\varphi_{1,0}(t)$ $\qquad \dfrac{f_0^{(0)} - f_1^{(0)}}{2}\sqrt{2}\psi_{1,0}(t)$

$f_0^{(1)}\varphi_{1,0}(t)$ $\qquad d_0^{(1)}\psi_{1,0}(t)$

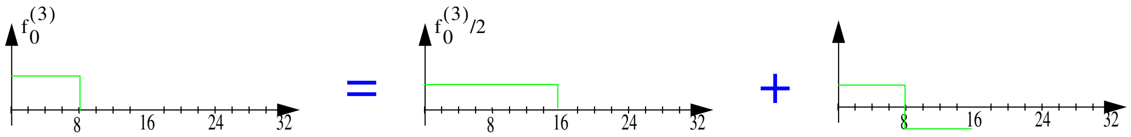
The Haar decomposition

D. Pictorially

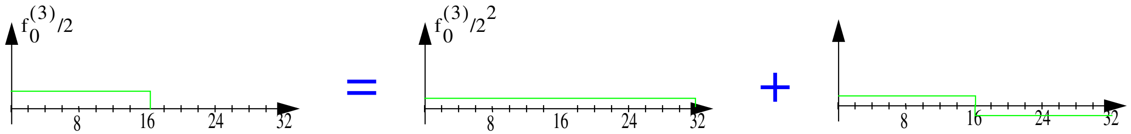


The Haar decomposition

E. Convergence



norms $\quad |f_0^{(3)}| 2^{3/2} \qquad |f_0^{(3)}| 2^{2/2}$



norms $\quad |f_0^{(3)}| 2^{2/2} \qquad |f_0^{(3)}| 2^{1/2}$

M times

$|f_0^{(3)}| 2^{(3-M)/2}$

$$f^{(0)}(t) = \sum_{m=1}^{M+3} \sum_{n=-2^{3-m}}^{2^{3-m}-1} d_n^{(m)} \cdot \psi_{m,n}(t) + \varepsilon_M$$

$$\|\varepsilon_M\| = (|f_{-1}^{(3)}(t)| + |f_0^{(3)}(t)|) \cdot 2^{(3-M)/2}, \ \|\varepsilon_M\| \to 0 \text{ as } M \to \infty$$