THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. This is a question to check your understanding of basic principles. A majority students got it right.

a) This is the joint pdf of two independent Gaussian RVs with zero mean and variance 1/4. Hence $P(X \leq 0.5 \cap Y \leq 0.7) = P(X \leq 0.5)P(Y \leq 0.7)$. After standardizing the two random variables, we find $P(X \leq 0.5) = P(Z_1 \leq 1) \approx 0.841$ and $P(Y \leq 0.7) = P(Z_2 \leq 1.4) \approx 0.919$ such that $P(X \leq 0.5 \cap Y \leq 0.7) \approx 0.773$. [2 - E]

b) $f_X(x) = \sqrt{\frac{2}{\pi}} e^{-2x^2}$. [2 - E]

c) $E(X) = 0,$ [2 - E]

$\mathrm{Var}(X) = 1/4,$ [2 - E]

We can find these results by directly computing the integrals but it would be simpler to note from the marginal PDF that $X \sim N(0, 1/4)$.

d) $f_Y(y) = \sqrt{\frac{2}{\pi}} e^{-2y^2}$. [2 - E]

e) $E(Y) = 0,$

[2 - E]

$\mathrm{Var}(Y) = 1/4$ [2 - E]

f) $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0.$ [1 - E]
$\mathrm{Corr}(X,Y) = 0$ [1 - E]

g) X and Y are uncorrelated since $\mathrm{Corr}(X,Y) = 0$. [1 - E]
They are also independent since the joint pdf is written as the product of marginals. [1 - E]

h) We can first compute the Jacobian and write

$$\begin{vmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{vmatrix} = \begin{vmatrix} \cos V & -U \sin V \\ \sin V & U \cos V \end{vmatrix} = U$$

[2 - B]

We then write

$$f_{U,V}(u,v) = \frac{2u}{\pi} e^{-2u^2}, \quad u > 0, -\pi \leq v \leq \pi.$$

[2 - B]

i) The marginal are obtained by integration of the joint pdf as follows

$$f_U(u) = \int_{-\pi}^{\pi} f_{U,V}(u,v)dv = 4ue^{-2u^2}, \;\; u > 0$$
$$f_V(v) = \int_0^{\infty} f_{U,V}(u,v)du = \frac{1}{2\pi}, -\pi \le v \le \pi$$

U is Rayleigh distributed and V is uniformly distributed over $[-\pi, \pi]$.

[2 - B]

j) Since $f_{U,V}(u,v) = f_U(u) f_V(v)$, U and V are two independent random variables.

[2 - A]

k) The conditional pdf $f_{U|V}(u|v)$ is given as

$$f_{U|V}(u|v) = f_U(u) = 4ue^{-2u^2}, \;\; u > 0$$

[2 - A]

l) $E(U|V) = E(U) = \frac{\sqrt{\pi}}{2\sqrt{2}}$.

[2 - A]

2. Compared to the first question, a larger portion of students had troubles answering this question. Part b of the question was not answered well despite the fact it directly comes from the lecture notes.

a) i) $P(P \geq S) = P(P_1 \geq S \cap P_2 \geq S)$. [1 - A]

From independence, we write $P(P_1 \geq S \cap P_2 \geq S) = P(P_1 \geq S)P(P_2 \geq S)$ [1 - A]

From the exponential distribution, we get $P(P \geq S) = \begin{cases} e^{-2\lambda S} & S > 0 \\ 0 & \text{otherwise} \end{cases}$

[2 - A]

ii) $f_P(p) = \frac{dF_P(p)}{dp}$ [1 - A]

$f_P(p) = \begin{cases} 2\lambda e^{-2\lambda p} & p > 0 \\ 0 & \text{otherwise} \end{cases}$

P is exponentially distributed with parameter 2λ. [2 - A]

iii) The MGF is given by $m_P(t) = E(e^{tP})$. [1 - A]

Hence $m_P(t) = \int_0^\infty e^{tp} 2\lambda e^{-2\lambda p} dp = \frac{2\lambda}{2\lambda - t}$ for $t < 2\lambda$. [2 - A]

iv) $E(P) = m'_P(0)$. [1 - A]

$E(P) = m'_P(0) = \frac{1}{2\lambda}$. [1 - A]

b) i) We define a set function P, called a probability function that takes a set as argument, and returns a value. For any event $E \subseteq S$ (with S the universal event), the three axioms of probability are given by

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. if $E \cap F = \emptyset$ then
   $P(E \cup F) = P(E) + P(F)$

[3 - B]

ii) Union of two arbitrary events A and B.

$$A \cup B = A \cup (\overline{A} \cap B)$$

$$P(A \cup B) = P(A) + P(\overline{A} \cap B) \qquad (1)$$

[2 - B]

Also

$$B = (A \cap B) \cup (\overline{A} \cap B)$$

$$P(B) = P(A \cap B) + P(\overline{A} \cap B) \qquad (2)$$

[2 - B]

Note that (1) and (2) are obtained from Axiom 3 of probability since $A \cup (\overline{A} \cap B)$ and $(A \cap B) \cup (\overline{A} \cap B)$ are written as disjoint unions. Rearrange (2) and substitute into (1), to obtain

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

[1 - B]