Answers (2015)

B — Book work
E — New example
T — New theory

1. a) The pdf of X is

$$f_X(x) = \frac{1}{2\pi} \qquad x \in [-\pi, \pi]$$

i) $y = g(x) = x^3 \qquad x = y^{1/3} \qquad y \in [-\pi^3, \pi^3]$ [1 E]

$g'(x) = 3x^2 = 3y^{2/3}$ [1 E]

$$f_Y(y) = \frac{1}{|g'(x)|} f_X(x) = \frac{1}{3y^{2/3}} f_X(x)$$

$$= \frac{1}{6\pi y^{2/3}} \qquad y \in [-\pi^3, \pi^3]$$ [1 E]

ii) $y = g(x) = x^4 \qquad x_1 = y^{1/4} \qquad y \in [0, \pi^4]$ [1 E]

$g'(x) = 4x^3 \qquad x_2 = -y^{1/4}$

[1 E]

$$f_Y(y) = \frac{1}{|g'(x_1)|} f_X(x_1) + \frac{1}{|g'(x_2)|} f_X(x_2)$$

$$= \frac{1}{4y^{3/4}} [f_X(x_1) + f_X(x_2)]$$

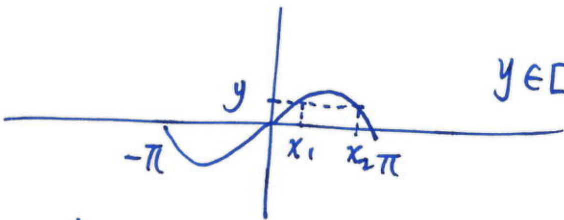
[1 E]

$$= \frac{1}{4\pi y^{3/4}} \qquad y \in [0, \pi^4]$$

iii) $y = g(x) = \sin(x) \qquad x_1 = \sin^{-1}(y) \qquad x_2 = \pi - \sin^{-1}(y)$

$g'(x) = \cos x$ [2E]

$= \sqrt{1-y^2}$



$y \in [-1, 1]$

$$f_Y(y) = \frac{1}{|g'(x_1)|} f_X(x_1) + \frac{1}{|g'(x_2)|} f_X(x_2)$$ [2E]

$$= \frac{1}{\sqrt{1-y^2}} \left(\frac{1}{2\pi} + \frac{1}{2\pi} \right)$$

$$= \frac{1}{\pi\sqrt{1-y^2}} \qquad y \in [-1, 1]$$

b) i) Let $X' = 2X$

Then $$f_{X'}(x') = \frac{1}{2} f_X(x) = \frac{1}{2} e^{-x'/2} \qquad x' > 0 \qquad [2E]$$

$$f_Z(z) = \int_0^z f_{X'}(z-y) f_Y(y)\, dy$$
$$= \int_0^z \frac{1}{2} e^{-(z-y)/2} e^{-y} dy$$
$$= \frac{1}{2} e^{-z/2} \int_0^z e^{-y/2} dy \qquad [3E]$$
$$= \frac{1}{2} e^{-z/2} \cdot 2(1-e^{-z/2})$$
$$= e^{-z/2} - e^{-z} \qquad z > 0$$

ii) $$F_Z(z) = P\{\min(X, Y) \le z\}$$
$$= 1 - P\{X > z, Y > z\}$$
$$= 1 - (1-F_X(z))(1-F_Y(z)) \qquad [2E]$$
$$= F_X(z) + F_Y(z) - F_X(z) F_Y(z)$$

$$f_Z(z) = f_X(z) + f_Y(z) - f_X(z) F_Y(z) - F_X(z) f_Y(z)$$

We have $$f_X(z) = f_Y(z) = e^{-z} \qquad z > 0 \qquad [1E]$$
$$F_Y(z) = F_X(z) = 1 - e^{-z} \qquad z > 0$$

Thus,
$$f_Z(z) = 2e^{-z} - 2e^{-z}(1-e^{-z}) \qquad [2E]$$
$$= 2e^{-2z} \qquad z > 0$$

iii) $Z = \max(X, Y)$

$$F_Z(z) = P\{\max(X, Y) \le z\}$$ [2E]

$$= P\{X \le z,\ Y \le z\}$$

$$= F_X(z)\, F_Y(z)$$

$$f_Z(z) = f_X(z)\, F_Y(z) + F_X(z)\, f_Y(z)$$ [3E]

$$= 2\, e^{-z}(1 - e^{-z}) \qquad z > 0$$

2. a) Let $\bar{x}$ denote the average.

The joint density

$$f(X,c) = c^n e^{-cn(\bar{x}-x_0)}$$ [3E]

has maximum if

$$\frac{\partial f(X,c)}{\partial c} = 0 \quad \Rightarrow \hat{c} = \frac{1}{\bar{x}-x_0}$$ [2E]

obviously, $\bar{x} = 9$ in this problem. So

$$\hat{c} = \frac{1}{9-5} = \frac{1}{4}$$ [3E]

b) From the Wiener-Hopf equation,

$$c = R^{-1} r$$ [2B]

$$\sigma^2 = r_0 - r^T R^{-1} r$$

i) When $n=1$, we have

$$R = 1$$ [2E]

$$r = r_1 = 0.643$$

Thus,

$$c_1 = r_1 = 0.643$$

$$\sigma^2 = 1 - r_1^2 = 1 - 0.643^2 = 0.587$$ [3E]

ii) When $n=2$, we have

$$R = \begin{bmatrix} 1 & 0.643 \\ 0.643 & 1 \end{bmatrix}$$ [2E]

$$r = \begin{bmatrix} -0.055 \\ 0.643 \end{bmatrix}$$

Thus,

$$c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = R^{-1} r$$

$$= \begin{bmatrix} 1 & 0.643 \\ 0.643 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -0.055 \\ 0.643 \end{bmatrix} \qquad [4E]$$

$$= \begin{bmatrix} -0.797 \\ 1.154 \end{bmatrix}$$

$$\sigma^2 = 1 - \begin{bmatrix} -0.055 & 0.643 \end{bmatrix} \begin{bmatrix} -0.797 \\ 1.154 \end{bmatrix} \qquad [4E]$$

$$= 0.214$$

3. a) i) $X(n) = \cos(nU)$

$$E[X(n)] = E[\cos(nU)] = 0$$ [2 E]

$$E[X^2(n)] = E[\cos^2(nU)]$$

$$= E[\tfrac{1}{2}(1+\cos(2nU))] = \tfrac{1}{2}$$

$$E[X(m)X(n)] = E[\cos(mU)\cos(nU)]$$ [3 E]

$$= E[\tfrac{1}{2}(\cos((m+n)U) + \cos((m-n)U))]$$

$$= 0 \qquad \text{if } m \neq n$$

Therefore, $X(n)$ is wide-sense stationary.

ii) Here, the answer is not unique.

For example, one may check

$$E[X(m)X(n)X(r)] = E[\cos(mU)\cos(nU)\cos(rU)]$$

$$= \tfrac{1}{2}E[(\cos((m+n)U) + \cos((m-n)U))\cos(rU)]$$

$$= \tfrac{1}{4}E[\cos((m+n+r)U) + \cos((m+n-r)U) + \cos((m-n+r)U) + \cos((m-n-r)U)]$$

$$= \tfrac{1}{4}[\delta(m+n+r) + \delta(m+n-r) + \delta(m-n+r) + \delta(m-n-r)]$$ [3 T]

where $\delta(n) = 1$ if $n = 0$

$\delta(n) = 0$ if $n \neq 0$

Consider two cases $(m, n, r) = (1, 2, 3), (2, 3, 4)$. [2 T]

They take different values $\frac{1}{4}$ and 0.

So it doesn't satisfy the definition of strict-sense stationarity (which would require the same values).

b)

i) This is the same as ~~the~~ average time when the third patient arrives.

$$E[T_3] = \frac{3}{\lambda} = \frac{3}{0.1} = 30 \quad \text{minutes}$$ [3E]

ii) This means that the number of patients arrived in the first hour is less than three

$$P(N(t) < 3) \qquad t = 60 \text{ minutes}$$

$$= P(N(t)=0) + P(N(t)=1) + P(N(t)=2)$$ [3E]

$$= e^{-60/10} + \left(\frac{60}{10}\right) e^{-60/10} + \frac{1}{2}\left(\frac{60}{10}\right)^2 e^{-60/10}$$

$$= 25 \cdot e^{-6}$$ [3E]

$$= 0.062$$

Recall Poisson: $P(N(t)=k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!} \quad k=0,1,2,\ldots$

iii) Poisson process is memoryless, so this probability is given by

$$P(N(t_1) \geq 2) \cdot P(N(t_2-t_1) \leq 2) \qquad \begin{matrix} t_1 = 60 \\ t_2 = 120 \end{matrix} \text{ minutes}$$

$$= [1 - P(N(t_1) < 2)]\, P(N(t_2-t_1) \leq 2)$$ [3E]

$$= \{1 - [P(N(t_1)=0) + P(N(t_1)=1)]\}\, P(N(t_2-t_1) \leq 2)$$

$$= [1-(e^{-6} + 6e^{-6})] \cdot 25 e^{-6}$$ (from ii)) [3E]

$$= 0.061$$

4. a) i) For any sequence of states $i_0, i_1, \cdots$

$$P(Y_{r+1} = i_{r+1} \mid Y_r = i_r, Y_{r-1} = i_{r-1}, \cdots, Y_0 = i_0)$$

$$= \frac{P(Y_{r+1} = i_{r+1}, Y_r = i_r, \cdots, Y_0 = i_0)}{P(Y_r = i_r, \cdots, Y_0 = i_0)}$$ [2 T]

$$= \frac{\prod_{s=0}^{r} P_{i_s, i_{s+1}}(n_{s+1} - n_s)}{\prod_{s=0}^{r-1} P_{i_s, i_{s+1}}(n_{s+1} - n_s)}$$

$$= P_{i_r, i_{r+1}}(n_{r+1} - n_r)$$ [2 T]

$$= P(Y_{r+1} = i_{r+1} \mid Y_r = i_r)$$

ii) The transition matrix is given by

$$P^2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$ [3 E]

b) Let $S_{n+1} = S_n + Z_{n+1}$ where

$P(Z_{n+1} = 1) = p$, $\quad P(Z_{n+1} = -1) = q$.

Thus, [2 B]

$$E[Y_{n+1} \mid Y_n, \cdots, Y_0] = E\left[\left(\frac{q}{p}\right)^{S_{n+1}} \middle| S_n, S_{n-1}, \cdots S_0\right]$$

$$= E\left[\left(\frac{q}{p}\right)^{S_n + Z_{n+1}} \middle| S_n\right]$$ Markov

$$= \left(\frac{q}{p}\right)^{S_n}\left[\frac{q}{p} \cdot p + \left(\frac{q}{p}\right)^{-1} \cdot q\right]$$

$$= \left(\frac{q}{p}\right)^{S_n}$$ [2 B]

$$= Y_n$$

i.e., $\{Y_n\}$ is a martingale.

ii) Let T denote the stopping time.

$$E[Y_T] = E[Y_0] = (q/p)^i$$

since $Y_0 = i$.

We also have

[2B]

$$E[Y_T] = P_i \left(\frac{q}{p}\right)^0 + (1-P_i)\left(\frac{q}{p}\right)^N$$
$$= P_i + (1-P_i)\left(\frac{q}{p}\right)^N$$

Thus

$$P_i = \frac{1-\left(\frac{p}{q}\right)^{N-i}}{1-\left(\frac{p}{q}\right)^N} \qquad \text{if } \frac{p}{q} \neq 1$$ [2B]

$$P_i = 1 - \frac{i}{N} \qquad \text{if } p = q = \frac{1}{2}$$

c) From the iteration $T_k = 1 + p\,T_{k+1} + q\,T_{k-1}$, we have

$$p\,(T_{i+1} - T_i) = q\,(T_i - T_{i-1}) - 1$$ [2T]

Let $M_{i+1} = T_{i+1} - T_i$, we obtain iteration

$$M_{i+1} = \frac{q}{p} M_i - \frac{1}{p}$$
$$= \begin{cases} \left(\frac{q}{p}\right)^i M_1 - \dfrac{1-(q/p)^i}{p-q} & p \neq q \\ M_1 - \dfrac{i}{p} & p = q \end{cases}$$ [2T]

This gives

$$T_i = \sum_{k=0}^{i-1} M_{k+1}$$
$$= \begin{cases} \left(M_1 + \dfrac{1}{p-q}\right)\displaystyle\sum_{k=0}^{i-1}\left(\frac{q}{p}\right)^k - \dfrac{i}{p-q} & p \neq q \\ i M_1 - \dfrac{i(i-1)}{2p} & p = q \end{cases}$$ [2T]

We yet need to determine M_1 from initial conditions

$$T_0 = 0$$

$$T_N = 0$$

Which gives

$$M_1 + \frac{1}{p-q} = \frac{N}{p-q} \cdot \frac{1-q/p}{1-(q/p)^N} \qquad p \neq q$$

Thus [2T]

$$T_i = \frac{N}{p-q} \cdot \frac{1-q/p}{1-(q/p)^N} \cdot \frac{1-(q/p)^i}{1-q/p} - \frac{i}{p-q}$$

$$= \begin{cases} \frac{N}{p-q} \cdot \frac{(1-q/p)^i}{(1-q/p)^N} - \frac{i}{p-q} & p \neq q \\ \end{cases}$$

Similarly, if $p = q$

$$T_N = 0 = N M_1 - \frac{N(N-1)}{2p} = 0$$

$$M_1 = \frac{N-1}{2p}$$

[2T]

$$T_i = \frac{i(N-1)}{2p} - \frac{i(i-1)}{2p}$$

$$= \frac{i(N-i)}{2p} \qquad p = \frac{1}{2}$$

$$= i(N-i)$$