Information for students

This coursework is intended to be a sample exam paper. However, the level of difficulty may vary to some extent.

It accounts for 15% of the mark for this course.

Deadline: Friday, 5PM, December 14, 2017. Pease submit a hard (hand-written is fine) copy of your answers, as well as a PDF copy to Blackboard.

Do not submit the MATLAB codes.

The Questions

1. Random variables.

a) The random variable X has a Gaussian distribution with mean 5 and standard deviation 2, and $Y = 2X + 4$. Find the mean, standard deviation and probability density function of Y. [5]

b) Let X be a Gaussian random variable with zero mean and variance σ^2. Estimate the tail probability $P(|X| > a)$ where $a = 4\sigma$ using

i) Markov inequality [5]

ii) Chebyshev inequality [5]

iii) Chernoff bound [5]

iv) Markov inequality with the k-th absolute moment for $k = 3, 4, 5, \ldots$ Draw a figure showing your bound as k increases and discuss your findings. [5]

Hint: $E(|X|^n) = \begin{cases} 1 \cdot 3 \cdots (n-1)\sigma^n, & n \text{ even}, \\ 2^k k! \sigma^{2k+1}\sqrt{2/\pi}, & n = (2k+1), \text{ odd}. \end{cases}$

2. Random variables and estimation.

a) X and Y are independent, identically distributed (i.i.d.) random variables with common probability density function

$$f_X(x) = e^{-x}, \qquad x > 0$$

$$f_Y(y) = e^{-y}, \qquad y > 0$$

Find the probability density function of the following random variables:

i) $Z = X + Y$. [5]

ii) $Z = \min(X, Y)$. [5]

iii) $Z = \max(X, Y)$. [5]

b) If the autocorrelation function $R_S(\tau) = Ie^{-|\tau|/T}$ and the linear MMSE estimate of $S(t - T/2)$ is given by $aS(t) + bS(t - T)$. Find the coefficients a and b and the corresponding mean-square error. [10]

3. Random processes.

a) The number of failures $N(t)$, which occur in a computer network over the time interval $[0, t)$, can be modelled by a Poisson process $\{N(t), t \geqslant 0\}$. On the average, there is a failure after every 2 hours, i.e. the intensity of the process is equal to $\lambda = 0.5$.

i) What is the probability of at most 1 failure in time interval [0, 8), at least 2 failures in [8, 16), and at most 1 failure in [16, 24) ? (time unit: hour) [10]

ii) What is the probability that the third failure occurs after 8 hours? [5]

b) In the fair-coin experiment, we define the random process $X(t)$ as follows:

$X(t) = \sin \pi t$ if head shows;

$X(t) = 2t$ if tail shows.

i) Find the mean $E[X(t)]$. [4]

ii) Find the autocorrelation function of $X(t)$. [4]

iii) Is this a stationary process? [2]

4. Markov chains.

a) Classify the states of the Markov chain with the following transition matrix (i.e., is each state recurrent/transient, periodic/aperiodic? Are there absorbing states/closed sets? Is the chain ergodic?)

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

[3]

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

[3]

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \end{pmatrix}$$

[4]

b) Consider the random walk with state space $E = \{0,1,2,\ldots\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & & & & 0 \\ q & 0 & p & & & \\ & q & 0 & p & & \\ & & . & . & . & \\ 0 & & . & . & . & \end{pmatrix}$$

where $0 < p < 1$, $q = 1 - p$. Write a computer program to simulate the random walk and show the realizations of $X(t)$ as a function of t, for

i) $p = 1/3$; [4]

ii) $p = 1/2$; [4]

iii) $p = 2/3$. [4]

Discuss your findings. [3]

[Obviously, such a question cannot be tested in this way in the exam!]