THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. a) This is a basic question to check your understanding of basic principles. Most students got it right.

i) $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$

[1 - B]

$E(XY) = \int_0^1 \int_0^1 2x^2 y dx dy = \frac{1}{3}$

[1 - A]

$E(X) = \int_0^1 2x^2 dx = \frac{2}{3}$

[1 - A]

$E(Y) = \int_0^1 y dy = \frac{1}{2}$

[1 - A]

$\text{Cov}(X,Y) = \frac{1}{3} - \frac{2}{3}\frac{1}{2} = 0$. Hence $\text{Corr}(X,Y) = 0$. Note that some students first found out that the random variables are independent and then argued that because of independence, they are uncorrelated. This is a valid answer as well.

ii) $P(X \leq 0.25 \,|\, Y \geq 1/3) = \frac{P(X \leq 0.25 \cap Y \geq 1/3)}{P(Y \geq 1/3)}$

[1 - A]

$P(Y \geq 1/3) = \int_{1/3}^1 dy = \frac{2}{3}$

[1 - A]

$P(X \leq 0.25 \cap Y \geq 1/3) = \int_0^{0.25} \int_{1/3}^1 2x dy dx = \frac{1}{24}$

[1 - A]

$P(X \leq 0.25 \,|\, Y \geq 1/3) = \frac{1}{16}$

[1 - A]

iii) $\text{Var}(3X - 2Y + 5) = 9\text{Var}(X) + 4\text{Var}(Y)$.

[1 - A]

$\text{Var}(X) = E\left(X^2\right) - E(X)^2 = \frac{1}{18}$.

[1 - A]

$\text{Var}(Y) = E\left(Y^2\right) - E(Y)^2 = \frac{1}{12}$.

[1 - A]

$\text{Var}(3X - 2Y + 5) = \frac{5}{6}$.

[1 - A]

iv) independent if $f_{X,Y}(x,y) = f_X(x) f_Y(y)$, $\forall x, y$.

[1 - A]

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & otherwise. \end{cases}$$

[1 - A]

$$f_Y(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & otherwise. \end{cases}$$

[1 - A]

Yes, they are independent.

[1 - A]

b) The overall methodology was explained by most students. However the methodology was often not correctly applied. It is a common mistake to replace $x_1, \ldots, x_n$ by a sample mean. A ML estimator may not always be written as a function of a sample mean.

i) Consider the random sample $X_1, \ldots, X_n$. Likelihood function $L(\theta) = f_{X_1,\ldots,X_n}(x_1,\ldots,x_n|\theta) = \prod_{i=1}^n f_{X_i}(x_i|\theta)$.

[1 - B]

$L(\theta) = \prod_{i=1}^n \theta\,(1-x_i)^{\theta-1} = \theta^n \left(\prod_{i=1}^n (1-x_i)\right)^{\theta-1}$.

[1 - A]

Log-likelihood function becomes $\log L(\theta) = n\log\theta + (\theta-1)\sum_{i=1}^n \log(1-x_i)$.

[1 - A]

Derivative to zero $\frac{d}{d\theta}\log L(\theta) = \frac{n}{\theta} + \sum_{i=1}^n \log(1-x_i) = 0$.

[1 - A]

Estimator $\hat{\theta} = \frac{-n}{\sum_{i=1}^n \log(1-X_i)}$.

[1 - A]

Concavity $\frac{d^2}{d\theta^2}\log L(\theta) = -\frac{n}{\theta^2} < 0$.

[1 - A]

ii) Evaluate the estimator using data such that the estimate of θ is given by $\hat{\theta} = \frac{-20}{\sum_{i=1}^{20}\log(1-x_i)} = 4.59$. [3 - A]

2. a) $P(X_1+X_2 \le 1) = \int_0^1 \int_0^{1-x_1} f_{X_1,X_2}(x_1,x_2)dx_2dx_1.$

[1 - A]

Using independence,

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 4(1-x_1)(1-x_2), & 0<x_1<1, 0<x_2<1, \\ 0, & otherwise. \end{cases}$$

[1 - A]

$P(X_1+X_2 \le 1) = \int_0^1 \int_0^{1-x_1} 4(1-x_1)(1-x_2)dx_2dx_1 = \frac{5}{6}.$

[1 - A]

$P(X_1+X_2 \ge 1) = 1 - P(X_1+X_2 \le 1) = \frac{1}{6}.$

[1 - A]

This questions was surprisingly not well answered by many students. Another approach is to use the convolution of the pdf of the two independent random variables. It is another approach that gives the same results.

b) i) $f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x)$

[1 - B]

$$f_{X,Y}(x,y) = \begin{cases} \frac{2(1-x)}{x}\exp\left(-\frac{y}{x}\right), & 0<x<1, 0<y<\infty, \\ 0, & otherwise. \end{cases}$$

[1 - A]

ii) $E(Y|X=x) = \int_{-\infty}^{+\infty} y f_{Y|X}(y|x)dy$ for $0<x<1$.

[1 - B]

$E(Y|X=x) = \int_0^{+\infty} \frac{y}{x}\exp\left(-\frac{y}{x}\right)dy = x$. Hence $E(Y|X) = X$.

[2 - A]

iii) The question here clearly asks to make use of ii) for solving iii) and iv).

$E_X E(Y|X) = E(Y)$.

[2 - A]

$E(Y) = \int_0^1 x2(1-x)dx = \frac{1}{3}.$ [2 - A]

iv) $\mathrm{Var}(Y) = E(\mathrm{Var}(Y|X)) + \mathrm{Var}(E(Y|X)).$

[2 - A]

$\mathrm{Var}(Y|X=x) = \int_0^{+\infty} (y - E(Y|X=x))^2 f_{Y|X}(y|x)dy = \int_0^{+\infty} (y-x)^2 \frac{1}{x}e^{-\frac{y}{x}}dy = x^2$

[2 - A]

$E(\mathrm{Var}(Y|X)) = \int_0^1 x^2 2(1-x)dx = \frac{1}{6}.$

[1 - A]

$\mathrm{Var}(E(Y|X)) = \int_0^1 (x-\frac{1}{3})^2 2(1-x)dx = \frac{1}{18}.$

[1 - A]

$\mathrm{Var}(Y) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}.$

c) This question checks your understanding of CDF and its relationship with PDF.
Write $W = 1-\sqrt{1-U}$.
$F_W(w) = P(W \le w) = P\left(1-\sqrt{1-U} \le w\right) = P\left(1-w \le \sqrt{1-U}\right).$

[2 - A]

$$F_W(w) = \begin{cases} P\left((1-w)^2 \leq 1-U\right), & 0 \leq w \leq 1, \\ 1, & w > 1. \end{cases}$$

[2 - A]

$$F_W(w) = \begin{cases} -w^2 + 2w, & 0 \leq w \leq 1, \\ 0, & w < 0, \\ 1, & w > 1. \end{cases}$$

[1 - A]

$$f_W(w) = \frac{d}{dw} F_W(w) = \begin{cases} 2(1-w), & 0 \leq w \leq 1, \\ 0, & otherwise. \end{cases}$$

[1 - A]