EE4-12

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

*********** Solutions ***********

Information for Candidates:

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.
- Where necessary, the sample rate of a signal in a block diagram is indicated in the form "@f".

Abbreviations

BIBO	Bounded Input, Bounded Output
CTFT	Continuous-Time Fourier Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
FIR	Finite Impulse Response

IIR	Infinite Impulse Response
LTI	Linear Time-Invariant
MDCT	Modified Discrete Cosine Transform
PSD	Power Spectral Density
SNR	Signal-to-Noise Ratio

A datasheet is included at the end of the examination paper.

********* Questions and Solutions *********

1. a) The signals $x[n]$ and $y[n]$ are defined as

$$x[n] = \begin{cases} 2^{-n} & n > 0 \\ 0 & n \leq 0 \end{cases}, \quad \text{and} \quad y[n] = \begin{cases} 0 & n > 0 \\ 5^{-n} & n \leq 0 \end{cases}.$$

i) Determine the z-transform of $x[n]$ and its region of convergence. [3]

$$X(z) = \sum_{n=1}^{\infty} 2^{-n}z^{-n} = 2z^{-1}\sum_{n=0}^{\infty} 2^{-n}z^{-n} = \frac{2z^{-1}}{1-2^{-1}z^{-1}} = \frac{4}{2z-1}$$

provided that $\|2^{-1}z^{-1}\| < 1 \Leftrightarrow |z| > 0.5$.

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ii) Determine the z-transform of $y[n]$ and its region of convergence.[3]

$$Y(z) = \sum_{n=-\infty}^{0} 5^{-n}z^{-n} = \sum_{m=0}^{\infty} 5^{m}z^{m} = \frac{1}{1-5z}$$

provided that $\|5z\| < 1 \Leftrightarrow |z| < 0.2$.

You may assume without proof that $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$ provided that $|\alpha| < 1$.

b) Consider the convolution $y[n] = h[n] * x[n] = \sum_{m=0}^{M} h[m]x[n-m]$ where $h[n]$ is the impulse response of an FIR filter of order M (i.e. $n \in [0, M]$) and $x[n]$ is a signal defined for $n \in [-\infty, \infty]$.

In the overlap-save method of convolution, $y[n]$ is divided into blocks of length K and a circular convolution of length $K+M$ is used to calculate each block. To calculate block b, the circular convolution evaluates

$$y[bK+n] = \sum_{m=0}^{M} h[m]x[bK-M+(n-m+M)_{\text{mod}(K+M)}]$$

for $n \in [0, K-1]$. The notation $P_{\text{mod}\,Q}$ denotes the remainder when P is divided by Q and satisfies $0 \leq P_{\text{mod}\,Q} < Q$.

i) Show that the expression given above is equivalent to the direct convolution given by $y[bK+n] = \sum_{m=0}^{M} h[m]x[bK+n-m]$. [3]

For $m \in [0, M]$ and $n \in [0, K-1]$, the value of $n-m+M$ lies in the range $0-M+M = 0$ to $K-1-0+M = K+M-1$ (where the first expression takes the lowest value of n and the highest of m and the second expression takes the opposite). It follows that, since $(n-m+M) \in [0, K+M-1]$, the modulus operation has no effect *and* $(n-m+M)_{\text{mod},K+M} = n-m+M$. *Thus we can write*

$$\begin{aligned} y[bK+n] &= \sum_{m=0}^{M} h[m]x[(n-m+M)_{\text{mod}\,(K+M)} - M + bK] \\ &= \sum_{m=0}^{M} h[m]x[(n-m+M) - M + bK] \\ &= \sum_{m=0}^{M} h[m]x[bK+n-m] \end{aligned}$$

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ii) Suppose that a circular convolution of length R requires approximately $5R\log_2 R$ multiplications. If $M = 200$, estimate the number of multiplications per output sample required by the overlap-save method when $K = 20$, 1500 and 10^4 and compare these results with the number of multiplications required for implementing the direct convolution. [4]

The overlap-save method uses a circular convolution of length $M+K$ *to calculate* K *values of* $y[n]$. *The number of multiplications per output sample is therefore* $5\frac{M+K}{K}\log_2(M+K)$. *For* $M = 200$ *and* $K = \{20, 1500, 10000\}$ *this gives* $\{428.0, 60.8, 67.9\}$. *A direct convolution requires* $M+1 = 201$ *multiplications per output sample, so the overlap save method is more efficient for moderate K. For large* K *the number of multiplications is approximately* $5\log_2 K$ *which increases slowly with K. Although not requested from candidates, the optimum value of K is the root of* $M\ln(M+K) = K$ *which, for* $M = 200$, *is* $K = 1486$.

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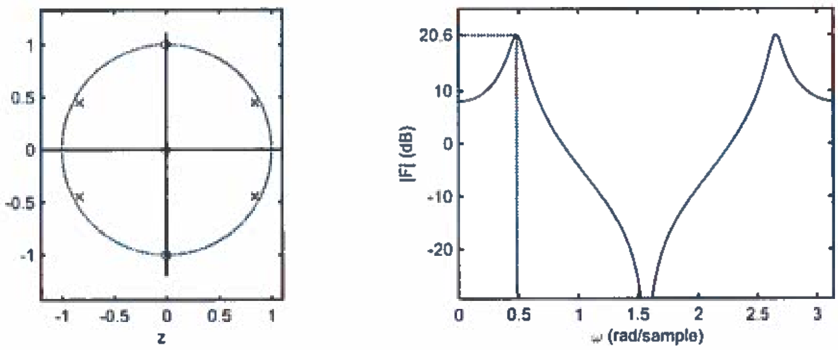
c) The filter $H(z)$ is given by

$$H(z) = \frac{1+z^{-1}}{1-z^{-1}+0.8z^{-2}}.$$

The pole-zero diagram and magnitude response (in dB) of $H(z)$ are shown in Figures 1.1 and 1.2 respectively.

i) Determine the transfer function $F(z) = H(z^2)$ and sketch its pole-zero diagram and magnitude response in dB. It is not necessary to determine exact values of the magnitude response. [4]

The transfer function is $F(z) = \frac{1+z^{-2}}{1-z^{-2}+0.8z^{-4}}$. The poles and zeros of $F(z)$ are the square root of the poles and zeros of $H(z)$. Therefore the zero at $z = -1$ becomes a zero pair at $\pm j$ and the pole pair at $0.5 \pm 0.74j$ becomes two pole pairs with half the argument at $\pm 0.835 \pm 0.444j$ (candidates are not required to calculate the exact values). These poles move closer to the unit circle and the peak bandwidths are correspondingly halved. The magnitude response is shrunk horizontally by a factor of 2 and replicated but is otherwise unchanged.



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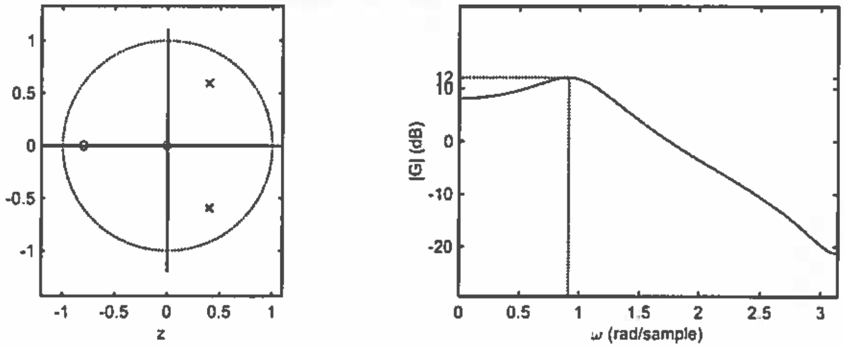
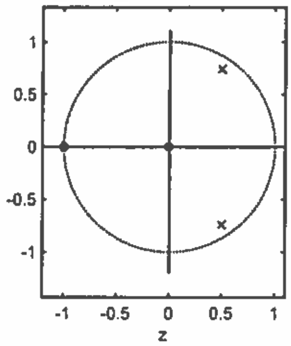
ii) Determine the transfer function $G(z) = H(1.25z)$ and sketch its pole-zero diagram and magnitude response in dB. It is not necessary to determine exact values of the magnitude response. [4]

The coefficients of $H(z)$ are multiplied by $1.25^{-n} = \{1, 0.8, 0.64\}$ so the transfer function is $G(z) = \frac{1+0.8z^{-1}}{1-0.8z^{-1}+0.512z^{-2}}$. The poles and zeros of $F(z)$ are the poles and zeros of $H(z)$ but multiplied by $\frac{1}{1.25} = 0.8$. Therefore they have the same arguments as befor but are moved inwards towards the origin. Since the pole pair is further from the unit circle, the magnitude response peak is lower and has a larger bandwidth.





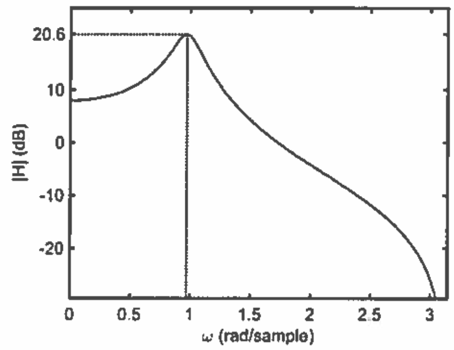


Figure 1.1

Figure 1.2

d) A bilinear transformation, $s = \alpha \frac{z-1}{z+1}$, is used to convert a continuous-time filter into a discrete-time filter.

i) Show that if $\alpha = \frac{\Omega_0}{\tan(0.5\omega_0)}$ then $z = e^{j\omega_0} \Leftrightarrow s = j\Omega_0$. [3]

We will calculate $s \times \frac{z+1}{z-1}$ and show that it equals α.

$$
\begin{aligned}
s \times \frac{z+1}{z-1} &= j\Omega_0 \times \frac{e^{j\omega_0}+1}{e^{j\omega_0}-1} = j\Omega_0 \times \frac{e^{j0.5\omega_0}\left(e^{0.5j\omega_0}+e^{-0.5j\omega_0}\right)}{e^{j0.5\omega_0}\left(e^{0.5j\omega_0}-e^{-0.5j\omega_0}\right)} \\
&= j\Omega_0 \times \frac{e^{0.5j\omega_0}+e^{-0.5j\omega_0}}{e^{0.5j\omega_0}-e^{-0.5j\omega_0}} \\
&= j\Omega_0 \times \frac{2\cos(0.5\omega_0)}{2j\sin(0.5\omega_0)} = \frac{\Omega_0}{\tan(0.5\omega_0)}
\end{aligned}
$$

ii) A continuous-time highpass filter with a cutoff frequency of 1 kHz is given by $H(s) = \frac{s}{s+\Omega_0}$ where $\Omega_0 = 2000\pi$ rad/s. Using the bilinear transformation given above, determine the coefficients (to 3 decimal places) of a discrete-time filter having an unnormalized cutoff frequency of 1 kHz. The sample frequency is 8 kHz. [4]

The normalized cutoff frequency of the discrete-time filter is $\omega_0 = \frac{\Omega_0}{f_s} = \frac{2000\pi}{8000} = \frac{\pi}{4} = 0.785$. *The filter is*

$$
\begin{aligned}
H(z) &= \alpha\frac{z-1}{z+1} \div \left(\alpha\frac{z-1}{z+1} + \Omega_0\right) (\alpha(z-1) + \Omega_0(z+1)) \\
&= \alpha(z-1) \div (\alpha(z-1) + \Omega_0(z+1)) \\
&= \frac{z-1}{\left(\frac{\Omega_0}{\alpha}+1\right)z + \left(\frac{\Omega_0}{\alpha}-1\right)} \\
&= \frac{1-z^{-1}}{\left(\frac{\Omega_0}{\alpha}+1\right) + \left(\frac{\Omega_0}{\alpha}-1\right)z^{-1}}
\end{aligned}
$$

Since $\alpha = \frac{\Omega_0}{\tan(0.5\omega_0)}$, $\frac{\Omega_0}{\alpha} = \tan(0.5\omega_0) = 0.414$. *Thus the filter is*

$$
H(z) = \frac{1-z^{-1}}{\left(\frac{\Omega_0}{\alpha}+1\right) + \left(\frac{\Omega_0}{\alpha}-1\right)z^{-1}} = \frac{1-z^{-1}}{1.414 - 0.586z^{-1}}
$$

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e) i) Explain why the average power of a discrete time signal (i.e. the average energy per sample) is always decreased by upsampling but is normally unchanged by downsampling. Give an example of a signal for which the latter statement is untrue. [3]

Upsampling by Q inserts $Q-1$ zero-valued samples between each of the original samples. Thus in any given time interval, the energy stays the same but the number of samples is multiplied by Q. It follows that the power of the signal has been reduced by a factor of Q.

Downsampling by Q removes $Q-1$ out of every Q samples but the average energy of the remaining smples will be unchanged. Therefore, if the average energy of the samples that are removed is the same as the average energy of the retained samples, the signal power will be unchanged.

If we define a signal $x[n] = \begin{cases} 1 & n\,\text{even} \\ 0 & n\,\text{odd} \end{cases}$ *then the average power of* $x[n]$ *is 0.5. However, if we downsample by 2 then the resultant signal is always 1 and its average power is 1.*

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ii) Figure 1.3 shows the power spectral density (PSD) of a real-valued signal, $x[n]$; the horizontal portions of the PSD have values 1 or 4. The signal $y[m]$ is obtained by upampling $x[n]$ by a factor of 3 as shown. Draw a dimensioned sketch of the PSD of $y[m]$ giving the values of all horizontal portions of the graph and the values of all frequencies at which there is a discontinuity in the PSD. [3]

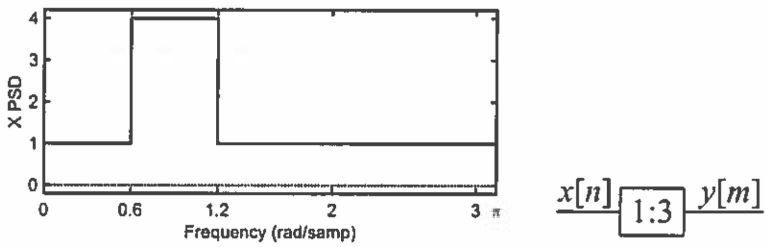
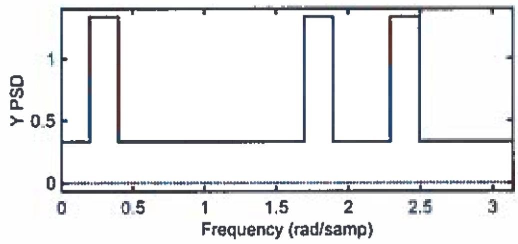


Figure 1.3

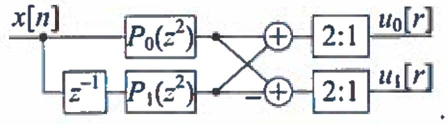
The upsampled PSD is shrunk horizontally by a factor of 3 and replicated at intervals of $\frac{2\pi}{3}$. The total power, which equals $\frac{1}{2\pi}\int S_{xx}(e^{j\omega})d\omega$, has been reduced by a factor of 3 for the reason given n part i). The discontinuities are at $\omega = \{0.2, 0.4, \frac{2\pi}{3}-0.4, \frac{2\pi}{3}-0.2, \frac{2\pi}{3}+0.2, \frac{2\pi}{3}+0.4\}$ which equal $\omega = \{0.2, 0.4, 1.694, 1.894, 2.294, 2.494\}$. The horizontal levels are 0.333 and 1.333.



f) Figure 1.4 shows the block diagram of a two-band analysis processor. The inputs to the adder/subtractor blocks, $\oplus$, are additive unless labelled with a minus sign in which case they are subtractive.

i) By using the Noble identities or otherwise, determine $H_0(z)$ and $H_1(z)$ so that Figure 1.5 is equivalent to Figure 1.4. [3]

Using the Noble identities we can move the downsampler modules to the right to obtain an equivalent diagram



This is equivalent to Figure 1.5 with $H_0(z) = P_0(z^2) + z^{-1}P_1(z^2)$ *and* $H_1(z) = P_0(z^2) - z^{-1}P_1(z^2)$.

ii) Assuming that $P_0(z)$ and $P_1(z)$ are FIR filters with real-valued coefficients, show that $\left|H_1\left(e^{j\omega}\right)\right| = \left|H_0\left(e^{j(\pi-\omega)}\right)\right|$ and explain the significance of this relationship. [3]

We can see that $H_1(z) = P_0(z^2) - z^{-1}P_1(z^2) = P_0((-z)^2) + -(-z)P_1((-z)^2) = H_0(-z)$.

It follows that $H_1\left(e^{j\omega}\right) = H_0\left(-e^{j\omega}\right) = H_0\left(e^{j(\omega-\pi)}\right) = H_0^*\left(e^{j(\pi-\omega)}\right)$ *where the final equivalence uses the fact that all the coefficients of* $H_0(z)$ *are real-valued. It follows that* $\left|H_1\left(e^{j\omega}\right)\right| = \left|H_0\left(e^{j(\pi-\omega)}\right)\right|$. *Thus the magnitude response of* $H_1(z)$ *is the same as that of* $H_0(z)$ *but reflected around* $\omega = \frac{\pi}{2}$.

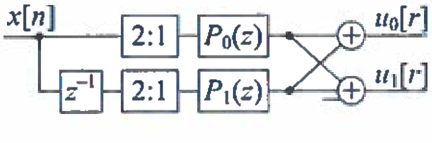


Figure 1.4

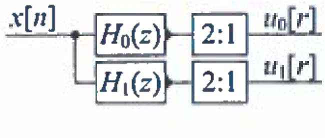


Figure 1.5

2. In the block diagram of Figure 2.1 the outputs of all adders and delay elements are on the right and solid arrows indicate the direction of information flow. The real-valued gain of each multiplier is written adjacent to its triangular symbol.

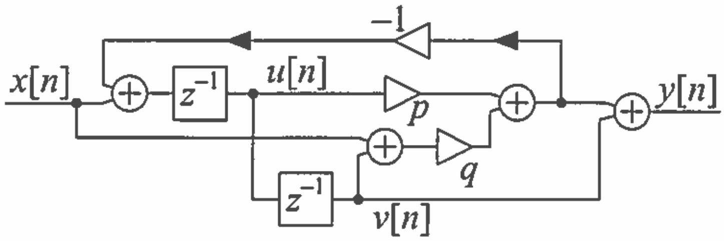


Figure 2.1

a) Show that [8]

$$G(z) = \frac{Y(z)}{X(z)} = \frac{q + pz^{-1} + z^{-2}}{1 + pz^{-1} + qz^{-2}}.$$

From the diagram, we can write

$$\begin{aligned}
U &= z^{-1}(X - (pU + q(X+V))) = z^{-1}((1-q)X - pU - qV) \\
&\Rightarrow (1-q)X - (z+p)U - qV = 0 \\
V &= z^{-1}U \\
Y &= V + pU + q(X+V) = qX + pU + (1+q)V.
\end{aligned}$$

We now need to eliminate U and V from these equations. Substituting for V gives

$$\begin{aligned}
0 &= (1-q)X - (z+p)U - qz^{-1}U \\
&\Rightarrow U = \frac{1-q}{z+p+qz^{-1}}X \\
Y &= qX + pU + (1+q)z^{-1}U \\
&= qX + \frac{(p+z^{-1}+qz^{-1})(1-q)}{z+p+qz^{-1}}X \\
&= \frac{qz+pq+q^2z^{-1}+p+z^{-1}+qz^{-1}-pq-qz^{-1}-q^2z^{-1}}{z+p+qz^{-1}}X \\
&= \frac{qz+p+z^{-1}}{z+p+qz^{-1}}X = \frac{q+pz^{-1}+z^{-2}}{1+pz^{-1}+qz^{-2}}X
\end{aligned}$$

---

b) Prove that $|G(e^{j\omega})| = 1$ for all ω. [6]

If $z = e^{j\omega}$, then $z^{-1} = z^*$ and $|z| = 1$. We can write

$$
\begin{aligned}
|G(z)| &= \frac{|q + pz^{-1} + z^{-2}|}{|z^{-2}(z^2 + pz + q)|} \\
&= \frac{\left|q + pz^* + (z^*)^2\right|}{|z^{-2}||z^2 + pz + q|} \\
&= \frac{\left|(z^2 + pz + q)^*\right|}{1 \times |z^2 + pz + q|} \\
&= \frac{|z^2 + pz + q|}{|z^2 + pz + q|} = 1.
\end{aligned}
$$

---

c) Figure 2.2 shows a graph of $\angle G(e^{j\omega})$ when $p = -1.2$ and $q = 0.8$. The dotted lines indicate the frequencies, $\omega = \{a, b, c\}$, at which $G(e^{j\omega}) = \{-j, -1, +j\}$ respectively. Given that $G(e^{jb}) = -1$, show that $\cos b = \frac{-p}{q+1}$ and find the numerical value of b for the given values of p and q. [6]

If $z = e^{jb} = \cos b + j \sin b$, then $G(z) = -1$. We can therefore write

$$
\begin{aligned}
G(z) = -1 &= \frac{qz^2 + pz + 1}{z^2 + pz + q} \\
-(z^2 + pz + q) &= qz^2 + pz + 1 \\
(q+1)z^2 + 2pz + (1+q) &= 0 \\
z &= \frac{-2p \pm \sqrt{4p^2 - 4(q+1)^2}}{2(q+1)} \\
\Rightarrow \Re(z) = \cos b &= \frac{-2p}{2(q+1)} = \frac{-p}{q+1}.
\end{aligned}
$$

For the given values of p and q,

$$
\begin{aligned}
b &= \cos^{-1} \frac{-p}{q+1} \\
&= \cos^{-1} 0.667 \\
&= 0.8411
\end{aligned}
$$

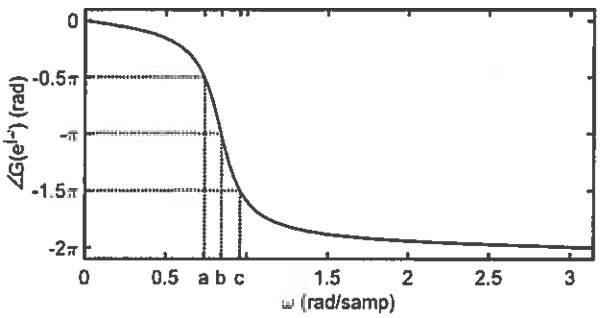
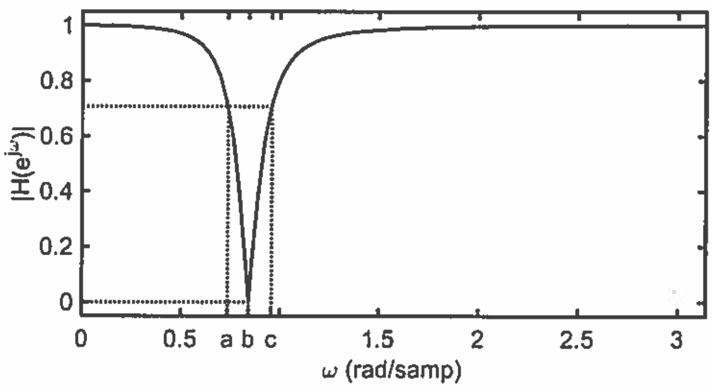


Figure 2.2

d) The filter $H(z)$ is defined as $H(z) = \frac{1}{2}(1+G(z))$.

i) Determine the value of $H(e^{j\omega})$ for each of $\omega = \{a, b, c\}$ defined above.
Hence, for $p = -1.2$ and $q = 0.8$, sketch a graph of $|H(e^{j\omega})|$ for $\omega \in [0, \pi]$ using linear scales for both axes. [6]

For $\omega = \{a, b, c\}$ we know that $G(e^{j\omega}) = \{-j, -1, +j\}$ and so $H(e^{j\omega}) = \frac{1}{2}\{1-j, 1-1, 1+j\} = \{0.5-0.5j, 0, 0.5+0.5j\}$. From this, $|H(e^{j\omega})| = \{0.707, 0, 0.707\}$. In addition, for $\omega = \{0, \pi\}$, $z = \pm 1$ and $G(z) = \frac{q \pm p + 1}{1 \pm p + q} = 1$ and so $H(z) = \frac{1}{2}(1+G(z)) = 1$ as well. Thus we have a notch filter



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ii) For $p = -1.2$ and $q = 0.8$, determine the poles and zeros of $H(z)$ in polar form and sketch a diagram of the complex plane that includes the unit circle and the poles and zeros of $H(z)$ (indicated by × and ○ respectively). [4]

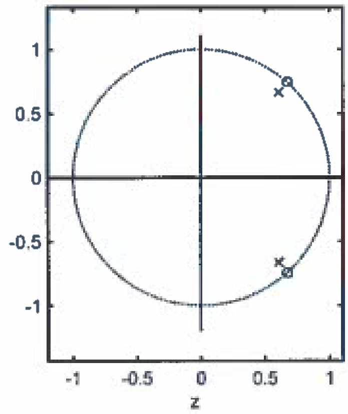
We have

$$
\begin{aligned}
H(z) = \frac{1}{2}(1+G(z)) &= \frac{1}{2}\left(1+\frac{qz^2+pz+1}{z^2+pz+q}\right) \\
&= \frac{(q+1)z^2+2pz+(q+1)}{2(z^2+pz+q)}
\end{aligned}
$$

Thus the poles are the roots of $z^2+pz+q=0$*, i.e.* $z = \frac{-p\pm\sqrt{p^2-4q}}{2} = 0.6\pm j\sqrt{0.44} = 0.6\pm 0.6633j = 0.8944\angle\pm 0.8355 = 0.8944\angle\pm 47.9°$

The zeros are the roots of $(q+1)z^2+2pz+(q+1)=0$*, i.e.* $z = \frac{-p\pm\sqrt{p^2-(q+1)^2}}{q+1} = \frac{1.2\pm j\sqrt{1.8}}{1.8} = \frac{1.2\pm 1.342}{1.8} = 0.667\pm 0.745 = 1\angle\pm 0.841 = 1\angle\pm 48.2°$.

The pole-zero diagram is therefore



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3. a) A symmetric Hanning window of odd length $M+1$ is defined as

$$w[n] = 0.5 + 0.5\cos\omega_M n,$$

where $\omega_M = \frac{2\pi}{(M+1)}$ and $-0.5M \leq n \leq 0.5M$.

i) Show that the DTFT of $w[n]$ is given by

$$\begin{aligned} W(e^{j\omega}) &= 0.5\frac{\sin 0.5\,(M+1)\,\omega}{\sin 0.5\omega} \\ &\quad +0.25\frac{\sin 0.5\,(M+1)\,(\omega-\omega_M)}{\sin 0.5\,(\omega-\omega_M)} \\ &\quad +0.25\frac{\sin 0.5\,(M+1)\,(\omega+\omega_M)}{\sin 0.5\,(\omega+\omega_M)}. \end{aligned}$$

You may assume without proof that $\sum_{n=-0.5M}^{0.5M} e^{j\alpha n} = \frac{\sin 0.5(M+1)\alpha}{\sin 0.5\alpha}$ provided that $\alpha \neq 0$. [5]

From the DTFT formula in the datasheet. we have

$$\begin{aligned} W(e^{j\omega}) &= \sum_{n=-0.5M}^{0.5M} w[n]e^{-jn\omega} \\ &= \sum_{n=-0.5M}^{0.5M} \left(0.5 + 0.25e^{j\omega_M n} + 0.25e^{-j\omega_M n}\right) e^{-jn\omega} \\ &= \sum_{n=-0.5M}^{0.5M} 0.5e^{-jn\omega} + 0.25e^{-j(\omega-\omega_M)n} + 0.25e^{-j(\omega+\omega_M)n} \end{aligned}$$

The summation formula given in the question can now be applied to each of these terms to obtain the desired result.

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ii) We define $S(\omega)$ to be the integrated spectrum

$$S(\omega) = \frac{1}{2\pi}\int_{\theta=0}^{\omega} W(e^{j\theta})d\theta.$$

Using the inverse DTFT formula or otherwise, show that $S(\pi) = 0.5$. [3]

Since $W(e^{j\omega})$ is even, we can write

$$\begin{aligned} S(\pi) &= \frac{1}{2\pi}\int_{\theta=0}^{\pi} W(e^{j\theta})d\theta \\ &= \frac{1}{2}\times\frac{1}{2\pi}\int_{\omega=-\pi}^{\pi} W(e^{j\omega})d\omega \\ &= \frac{1}{2}\times w[0] = 0.5 \end{aligned}$$

where we use the inverse DTFT formula for $n = 0$.

---

b) i) Show that, if $G(z)$ is an ideal lowpss filters with

$$G(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & \omega_0 < |\omega| \leq \pi \end{cases}, \quad \text{then} \quad g[n] = \frac{\sin \omega_0 n}{\pi n}.$$

[4]

From the inverse DTFT formula

$$\begin{aligned} g[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n}d\omega \\ &= \frac{1}{2\pi jn}\left[e^{j\omega n}\right]_{-\omega_0}^{\omega_0} \\ &= \frac{1}{2\pi jn}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right) \\ &= \frac{1}{2\pi jn} \times 2j\sin \omega_0 n = \frac{\sin \omega_0 n}{\pi n}. \end{aligned}$$

---

ii) By combining the ideal response from part b) i) with the Hanning window from part a), use the window method to design a causal lowpass filter, $H(z)$, of order M with a cutoff frequency of ω_0.

Give a formula for the impulse response, $h[n]$, of the filter where $0 \leq n \leq M$. [3]

The impulse response is $h[n] = g\left[n - \frac{M}{2}\right] w\left[n - \frac{M}{2}\right]$ *where the shift of* $\frac{M}{2}$ *samples is needed to make a causal filter. As an explicit formula, this is*

$$h[n] = \frac{\sin \omega_0\left(n - \frac{M}{2}\right)}{\pi\left(n - \frac{M}{2}\right)} \times \left(0.5 + 0.5\cos \omega_M\left(n - \frac{M}{2}\right)\right).$$

---

iii) Show that $H(e^{j\omega}) = e^{-0.5j\omega M}\left(S(\omega+\omega_0) - S(\omega-\omega_0)\right)$. [5]

From the datasheet, the DTFT of $h[n + \frac{M}{2}] = g[n]w[n]$ *is*

$$\begin{aligned} e^{0.5j\omega M}H(e^{j\omega}) &= \frac{1}{2\pi}\int_{\theta=-\pi}^{\pi} G(e^{j\theta})W(e^{j(\omega-\theta)})d\theta \\ &= \frac{1}{2\pi}\int_{\theta=-\omega_0}^{\omega_0} W(e^{j(\omega-\theta)})d\theta \\ &= \frac{-1}{2\pi}\int_{\phi=\omega+\omega_0}^{\omega-\omega_0} W(e^{j\phi})d\phi \\ &= -\left[S(\phi)\right]_{\omega+\omega_0}^{\omega-\omega_0} \\ &= S(\omega+\omega_0) - S(\omega-\omega_0) \end{aligned}$$

where in the second line we restrict the integral to the support of $G(e^{j\omega})$ and in the third line we substitute $\theta = \omega - \phi$.

[This question is continued on the next page]

c) Figures 3.1 and 3.2 show $W(e^{j\omega})$ and $S(\omega)$ for a Hanning window of length $M+1=41$. The first few values of ω for which $S(\omega)$ either equals 0.5 or has a turning point are listed in the following table:

ω	0.2565	0.3065	0.4003	0.4598	0.5490
$S(\omega)$	0.5	0.5064	0.5	0.4981	0.5

For the case $\omega_0 = 1$, the magnitude response, $|H(e^{j\omega})|$, of the resultant lowpass FIR filter from part b) is shown in Figure 3.3 plotted in dB. The ideal response, $G(e^{j\omega})$, is shown on the graph as a dashed line. Using appropriate values from the table given above,

i) estimate the smallest positive ω (marked "a" in Figure 3.3) for which $H(e^{j\omega}) = 0$; [3]

From Figure 3.3, the answer is in the region of $\omega \approx 1.2$. Thus $\omega + \omega_0 \approx 2.2$ and, from 3.2, $S(\omega+\omega_0) \approx 0.5$. We therefore need to find the smallest ω for which $S(\omega-\omega_0) = S(\omega+\omega_0) - H(e^{j\omega}) = 0.5 - 0 = 0.5$. From the table, this happes when $\omega - \omega_0 = 0.2565$ i.e. when $\omega = 1.2565$.

---

ii) estimate the magnitude in dB (marked "b" in Figure 3.3) of the first peak in the stopband; [3]

As in the previous part, we can assume that $S(\omega+\omega_0) \approx 0.5$ which means that $H(e^{j\omega}) = 0.5 - S(\omega-\omega_0)$. First peak in $\left\|H(e^{j\omega})\right\|$ in the stopband will therefore be at the peak of $S(\omega-\omega_0)$ when $\omega - \omega_0 =$ 0.3065, i.e. $\omega = 1.3065$. At this frequency, we have $H(e^{j\omega}) = 0.5 - 0.5064 = -0.0064$. It follows that $\left|H(e^{j\omega})\right| = 20\log_{10} 0.0064 = -43.9\,\text{dB}$.

---

iii) estimate the peak passband gain in dB. [4]

Since $S(\omega)$ is an odd function, we can write $H(e^{j\omega}) = S(\omega+\omega_0) + S(\omega_0-\omega)$ when $\omega < \omega_0$. If we still assume that $S(\omega+\omega_0) \approx 0.5$, we have the peak gain when $\omega_0 - \omega = 0.3065$ i.e. when $\omega = 0.6035$. At this frequency, $H(e^{j\omega}) = 0.5 + 0.5064 = 1.0064 = +0.055\,\text{dB}$.

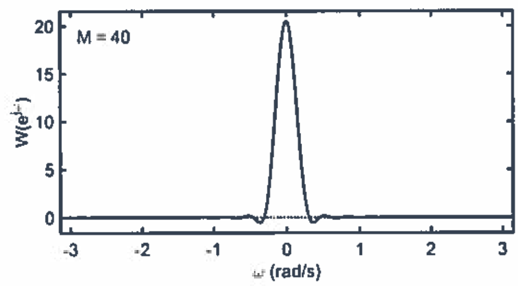


Figure 3.1

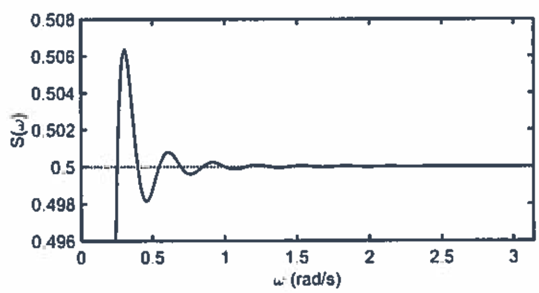


Figure 3.2

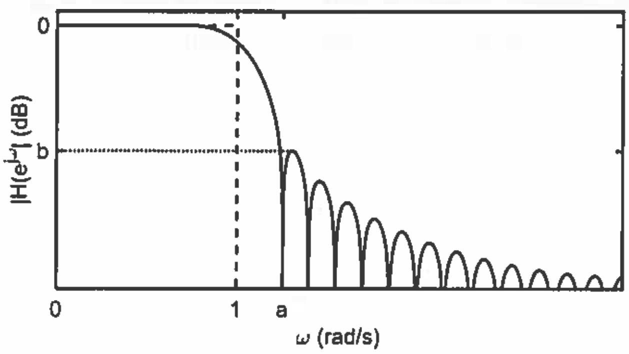


Figure 3.3

4. Figure 4.1 shows a system intended to apply a fractional-sample delay to its input signal, $x[n]$, where the delay is an integer multiple of $\frac{1}{P}$ samples.

Figure 4.1

Figure 4.2

a) i) Explain the purpose of the lowpass filter, $H(z)$, in Figure 4.1. [2]

The filter removes frequency components above $\frac{\pi}{P}$ from the signal. It is needed because the upsampler, $1:P$, introduces $P-1$ images of the baseband which must be removed before downsampling, $P:1$, to prevent aliasing.

---

ii) The input signal, $x[n]$, contains frequency components in the range $0 \le \omega \le 0.8\pi$. Using the datasheet formula, $M \approx \frac{a}{3.5\Delta\omega}$, estimate the order required for $H(z)$ to give a stopband attenuation of 60dB. [3]

The passband edge for $x[n]$ is given as 0.8π. After upsampling, this becomes $\frac{0.8\pi}{P}$ and the edge of the first image is at $\frac{2\pi-0.8\pi}{P} = \frac{1.2\pi}{P}$. Thus the transition bandwidth for the filter is $\Delta\omega = \frac{0.4\pi}{P}$. Substituting this into the formula, $M \approx \frac{60P}{3.5\times0.4\pi} = 13.6P$.

---

iii) For a direct implementation of Figure 4.1, estimate as a function of P the number of multiplications required per input sample, $x[n]$. [3]

The number of multiplications per sample of $v[m]$ is $M+1 = 13.6P+1$. The number per input sample, $x[n]$, is therefore $(13.6P+1)P = 13.6P^2+P$.

---

iv) If $H(z)$ is a causal symmetric FIR filter of order M, determine the delay of $y[n]$ relative to $x[n]$ as a function of M and d. [3]

If $H(z)$ is a symmetric causal filter of order M, it will introduce a delay of $\frac{M}{2}$ samples and the z^{-d} delay will introduce a further d samples. Thus, at the input sampling rate, we have a delay of $\frac{M+2d}{2P}$ samples.

---

b) i) The signal $w[m]$ in Figure 4.1 is given by $w[m] = \sum_{s=0}^{M} h[s]u[m-s]$ where $h[s]$ is the impulse response of $H(z)$.
If $m = Pn+p$ where $0 \le p < P$, show that $w[m]$ may be written in the form $w[m] = \sum_{r=0}^{R} h_p[r]x[n-r]$. Determine the value of R and give an expression for $h_p[r]$ in terms of $h[s]$. [6]

We have

$$\begin{aligned} w[m] &= \sum_{s=0}^{M} h[s]u[m-s] \\ &= \sum_{s=0}^{M} h[s]u[Pn+p-s]. \end{aligned}$$

However, we know that

$$u[Pn+p-s] = \begin{cases} x[n+\frac{p-s}{P}] & P \mid (p-s) \\ 0 & \textit{otherwise} \end{cases}.$$

Hence, $u[Pn+p-s]$ is zero unless s is of the form $s = Pr+p$ (to ensure that $p-s$ is a multiple of P. We know that $0 \le s \le M$ so $0 \le Pr+p \le M \Rightarrow 0 \le r \le \frac{M-p}{P} \le \frac{M}{P}$ where the last inequality is an equality when p has its minimum value of 0. We can therefore write

$$\begin{aligned} w[m] &= \sum_{s=0}^{M} h[s]u[Pn+p-s] \\ &= \sum_{r=0}^{R} h[Pr+p]u[Pn+p-Pr-p] \\ &= \sum_{r=0}^{R} h[Pr+p]u[Pn+p-Pr-p] \\ &= \sum_{r=0}^{R} h[Pr+p]x[n-r] \\ &= \sum_{r=0}^{R} h_p[r]x[n-r] \end{aligned}$$

where $h_p[r] = h[Pr+p]$. The summation limit is $R = \lceil \frac{M}{P} \rceil$ where $\lceil \ldots \rceil$ denotes the ceiling function.

---

ii) Derive expressions for q and k in Figure 4.2 as functions of d in Figure 4.1 so that that the two figures are equivalent. You may assume that $y[n] = w[Pn-d]$ in Figure 4.1. [4]

We can write d uniquely as $d = Pk - q$ where $k = \lceil \frac{d}{P} \rceil$ and $0 \le q = Pk - d < P$ (or, equivalently, $q = (-d) \mod P$). Using the identity given in the question, we can then write $y[n] = w[Pn-d] = w[P(n-k)+q]$. From the previous part, with $m = P(n-k)+q$, we therefore have

$$\begin{aligned} y[n] &= w[P(n-k)+q] = \sum_{r=0}^{R} h_q[r]x[(n-k)-r] \\ y[n+k] &= \sum_{r=0}^{R} h_q[r]x[n-r] \end{aligned}$$

(where in the last line, we substitute $n \to n+k$) which is in the required form for Figure 4.2.

---

c) Suppose now that, for each r, the coefficients $h_q[r]$ may be closely approximated using a polynomial of order T as $h_q[r] \approx \sum_{t=0}^{T} f_t[r] \left(\frac{q}{P}\right)^t$ where the polynomial argument, $\frac{q}{P}$, lies in the range $0 \le \frac{q}{P} < 1$.

i) The Farrow filter shown in Figure 4.3 calculates its output, $y[n]$, from $y[n+k] = \sum_{t=0}^{T} \left(\frac{q}{P}\right)^t v_t[n]$ where each of the signals $v_t[n]$ is obtained from $x[n]$ by applying a filter, $G_t(z)$, whose coefficients do not depend on q. Derive an expression for the coefficients of $G_t(z)$ so that Figure 4.3 is approximately equivalent to Figure 4.2. [4]

Figure 4.2 calculates $y[n+k] = \sum_{r=0}^{R} h_q[r]x[n-r]$ so we can write

$$
\begin{aligned}
y[n+k] &= \sum_{r=0}^{R} h_q[r]x[n-r] \\
&= \sum_{r=0}^{R}\sum_{t=0}^{T} f_t[r] \left(\frac{q}{P}\right)^t x[n-r] \\
&= \sum_{t=0}^{T} \left(\frac{q}{P}\right)^t \sum_{r=0}^{R} f_t[r]x[n-r] \\
&= \sum_{t=0}^{T} \left(\frac{q}{P}\right)^t v_t[n]
\end{aligned}
$$

where $v_t[n] = \sum_{r=0}^{R} f_t[r]x[n-r]$.

It follows that the coefficients of $G_t(z)$ are $g_t[n] = f_t[n]$.

---

ii) Suppose that $P = 20$, $M = 199$ and $T = 4$. For each of Figure 4.2 and Figure 4.3, estimate the number of multiplications required per input sample, $x[n]$. [3]

For Figure 4.2, we require $R+1 = \frac{M+1}{P} = 10$ multiplications per input sample. For Figure 4.3, we have $T+1 = 5$ filters of length $R+1 = 10$ and in addition, we have $T = 4$ mutiplications per input sample to evaluate the polynomial. This gives a total of 54 multiplications per input sample which is much more than Figure 4.2.

---

iii) Explain why the implementation of Figure 4.3 may be preferable to that of Figure 4.2 under some circumstances. [2]

For a fixed fractional delay, Figure 4.2 is clearly preferable since it entails far fewer multiplications. However Figure 4.3 has the advantage that the fractional part of the delay, q/P can be varied on a sample-by-sample basis without needing to change any filter coefficients. Also, q is not restricted to being an integer and so the delay can be varied continuously.

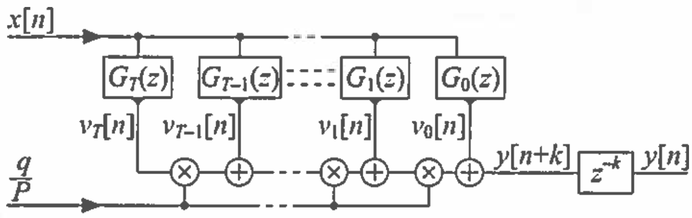


Figure 4.3

Datasheet:

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$	$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
DFT:	$X[k] = \sum_{0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$	$x[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n]\cos\frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N}\sum_{n=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r]y[n-r] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) Y\left(e^{j\omega}\right)$

$v[n] = x[n]y[n] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\omega-\theta)}\right) d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r]y[(n-r)_{\bmod N}] \quad \Leftrightarrow \quad V[k] = X[k]Y[k]$

$v[n] = x[n]y[n] \quad \Leftrightarrow \quad V[k] = \frac{1}{N} X[k] \circledast_N Y[k] \triangleq \frac{1}{N} \sum_{r=0}^{N-1} X[r]Y[(k-r)_{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)} \frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10} b}{4.6\Delta\omega}$

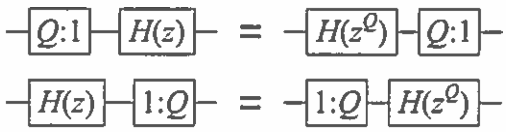
where a =stop band attenuation in dB, b = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in radians per sample.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities



Multirate Spectra

Upsample: 

$$x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases} \quad \Rightarrow \quad X(z) = V(z^Q)$$

Downsample: 

$$y[m] = v[Qm] \quad \Rightarrow \quad Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$$

Multirate Commutators