Wavelets, Sparsity and their Applications

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Session Five: Multiresolution Analysis and Splines

Multi-Resolution Analysis

Definition By a multi-resolution analysis we mean a sequence of embedded closed subspaces

$$\ldots V_2 \subset V_1 \subset V_0 \subset V_{-1} \ldots$$

such that

1. Upward Completeness: $\lim_{m\to-\infty} V_m = \bar{\bigcup}_{m\in\mathbb{Z}} V_m = L_2(\mathbb{R})$. (finite energy)

2. Downward Completeness: $\lim_{m\to\infty} V_m = \bigcap_{m\in\mathbb{Z}} V_m = \{0\}$.

3. Scale Invariance: $f(t) \in V_m \leftrightarrow f(2^m t) \in V_0$.

4. Shift Invariance: $f(t) \in V_0 \rightarrow f(t-n) \in V_0$ for all $n \in \mathbb{Z}$.

5. Existence of a Basis. There exists $\varphi(t) \in V_0$, such that $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for V_0.

Consequences of the Multi-Resolution Analysis

First notice that:[1]

$$\boxed{\langle \varphi(t-n), \varphi(t-m) \rangle = \delta_{m,n} \Longleftrightarrow \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2k\pi)|^2 = 1.}$$

Since V_1 is included in V_0, if $\varphi(t/2)$ belongs to V_1, it belongs to V_0 as well. Thus:

$$\varphi(x/2) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(x-n)$$

or

$$\boxed{\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t-n).}$$

This is the two-scale relation.

[1] see Appendix

$$\varphi(t) = \sqrt{2}\sum_{n=-\infty}^{+\infty} g_0[n]\varphi(2t-n)$$

$\Downarrow$ FT

Consequences of the Multi-Resolution Analysis

$$\hat{\varphi}(\omega) = \sqrt{2}\sum_{n=-\infty}^{+\infty} g_0[n]\int_{-\infty}^{+\infty}\varphi(2t-n)e^{-j\omega t}dt$$

$$\xrightarrow[t=\frac{x+n}{2}]{x=2t-n} \frac{\sqrt{2}}{2}\sum_{n=-\infty}^{+\infty} g_0[n]\int_{-\infty}^{+\infty}\varphi(x)e^{-j\omega\frac{x}{2}}e^{-j\omega\frac{n}{2}}dx$$

By taking the Fourier transform of both sides of the two-scale relation, we obtain

$$= \frac{\sqrt{2}}{2}\sum_{n=-\infty}^{+\infty} g_0[n]e^{-j\omega\frac{n}{2}}\int_{-\infty}^{+\infty}\varphi(x)e^{-j\omega\frac{x}{2}}dx$$

$$\hat{\varphi}(\omega) = \frac{1}{\sqrt{2}}G_0(e^{j\omega/2})\hat{\varphi}(\omega/2)$$

$$= \frac{\sqrt{2}}{2}G_0(e^{j\frac{\omega}{2}})\hat{\varphi}(\tfrac{\omega}{2})$$

where

$$\because \sum_{k=-\infty}^{+\infty}|\hat{\varphi}(2\omega+2k\pi)|^2 = 1$$

$$G_0(e^{j\omega}) = \sum_n g_0[n]e^{-j\omega n}.$$

$$G_0(z)G_0(z^{-1}) + G_0(-z)G_0(-z^{-1}) = 2$$

and because of the orthogonality of $\varphi(t)$, we obtain

$$\therefore \frac{1}{2}\sum_k |G_0(e^{j(\omega+k\pi)})|^2|\hat{\varphi}(\omega+k\pi)|^2 = 1$$

$$|G_0(e^{j\omega})|^2 + |G_0(e^{j(\omega+\pi)})|^2 = 2.$$

$$\therefore \frac{1}{2}\left(\sum_k |G_0(e^{j(\omega+2k\pi)})|^2|\hat{\varphi}(\omega+2k\pi)|^2 + \sum_k |G_0(e^{j(\omega+(2k+1)\pi)})|^2|\hat{\varphi}(\omega+(2k+1)\pi)|^2\right) = 1$$

$$\therefore \frac{1}{2}|G_0(e^{j\omega})|^2 \underbrace{\sum_k|\hat{\varphi}(\omega+2k\pi)|^2}_{=1} + \frac{1}{2}|G_0(e^{j(\omega+\pi)})|^2\underbrace{\sum_k|\hat{\varphi}(\omega+(2k+1)\pi)|^2}_{=1} = 1$$

$$\therefore |G_0(e^{j\omega})|^2 + |G_0(e^{j(\omega+\pi)})|^2 = 2$$

Consequences of the Multi-Resolution Analysis

Theorem 1. *Let* $\{V_n\}$, $n \in \mathbb{Z}$ *be a multiresolution analysis with the scaling function* $\varphi(t)$. *There exists an orthonormal basis for* $L_2(\mathbb{R})$:

$$\psi_{m,n}(t) = 2^{-m/2}\psi(2^{-m}t - n) \qquad m, n \in \mathbb{Z}$$

φ: scaling function (basis of V_0)

and

$$\boxed{\psi(t) = \sum_{n=-\infty}^{\infty} (-1)^n g_0[1-n]\varphi(2t-n)}$$

such that $\{\psi_{m,n}\}$, $n \in \mathbb{Z}$ *is an orthonormal basis for* W_m, *where* W_m *is the orthogonal complement of* V_m *in* V_{m-1}.

Construct $\psi(t)$ by
- φ (should satisfy 3 conditions)
- $g_1[n]$ (choose $g_1[n] = (-1)^n g_0[1-n]$ for orthogonality of $\psi(t-n)$ that span $W_0 \perp V_0$)

$$\begin{array}{ccc} V_1 & \subset & V_0 \\ \oplus & & \oplus \\ W_1 & & W_0 \leftarrow \psi(t) = \sum_n g_1[n]\varphi(t-n) \end{array}$$

$W_1 \leftarrow \psi(\tfrac{t}{2}) = \sum_n g_1[n]\varphi(t-n)$

Scaling Function and Splines

Central to multiresolution analysis is the design of a proper scaling function. It is possible to show that $\varphi(t)$ is an admissible scaling function of $L_2(\mathbb{R})$ if and only if it satisfies the three following conditions: existense of a (biorthogonal) basis:

$f(t) \in V_0$. $\{f(t-n)\}_{n\in\mathbb{Z}}$ is a (orthogonal) basis for V_0.

1. Riesz basis criterion: There exists two constants $A > 0$ and $B < +\infty$ such that

$$A \leq \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi k)|^2 \leq B \tag{1}$$

orthogonal basis $\langle f(t), f(t-n)\rangle = \delta_n$

$\Updownarrow$ FT

$\sum_{k\in\mathbb{Z}} |\hat{f}(\omega+2\pi k)|^2 = 1$

2. Two scale relation $\Leftrightarrow$ scale invariance: $f(t) \in V_m \Leftrightarrow f(2^m t) \in V_0$.
   shift invariance: $f(t) \in V_0 \Leftrightarrow f(t-n) \in V_0$ for all $n \in \mathbb{Z}$.

$$\varphi(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} g_0[k]\varphi(2t-k) \tag{2}$$

3. Partition of unity $\Leftrightarrow$ upward completeness $\lim_{m\to+\infty} V_m = \bigcap_{m\in\mathbb{Z}} V_m = \{0\}$.
   downward completeness $\lim_{m\to-\infty} V_m = \overline{\bigcup_{m\in\mathbb{Z}} V_m} = L_2(\mathbb{R})$

$$\sum_{k\in\mathbb{Z}} \varphi(t-k) = 1. \tag{3}$$

spline: have a certain number of continuous derivatives.

$\beta_N(t)$. $\begin{cases} (N-1)\text{-order derivatives is continuous.} \\ N\text{-order derivatives is not continuous.} \end{cases}$

Scaling Function and Splines

$\hat{\beta}_0^+(w) = \frac{1-e^{-jw}}{jw}$ $\quad \hat{\beta}_N^+(w) = \left(\frac{1-e^{-jw}}{jw}\right)^{N+1}$

$\hat{\beta}_0(w) = \text{sinc}\frac{w}{2}$ $\quad \hat{\beta}_N(w) = \left(\text{sinc}\frac{w}{2}\right)^{N+1}$

$\beta_0(t)$: box function

$\beta_1(t)$: triangular function

A remarkable example of scaling functions is given by the family of B-splines. A B-spline $\beta_N(t)$ of order N is obtained from the (N+1)-fold convolution of the box function $\beta_0(t)$ or

All B-splines $\beta_i(t)$ are valid scaling function.

$$\beta_N(t) = \underbrace{\beta_0(t) * \beta_0(t) \ldots * \beta_0(t)}_{N+1 \text{ times}} \quad \text{with } \hat{\beta}_0(\omega) = \frac{1-e^{-j\omega}}{j\omega}$$

· Riesz basis criterion

$A \le \sum_{k\in\mathbb{Z}} |\hat{\varphi}(w+2\pi k)|^2 \le B$

· Two scale relation

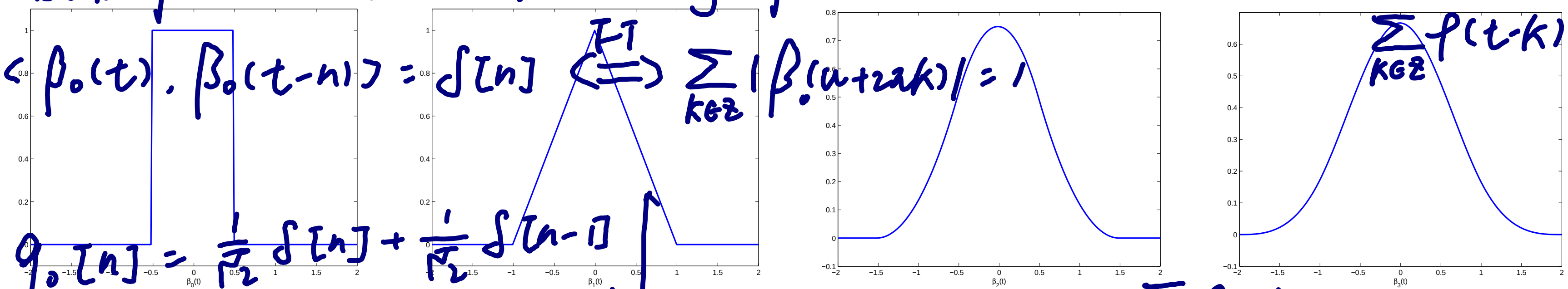
$\varphi(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} g_0(k)\varphi(2t-k)$

· Partition of unity

$\sum_{k\in\mathbb{Z}} \varphi(t-k) = 1$

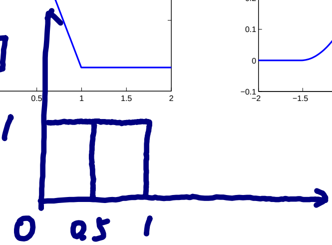
where $\hat{\beta}(\omega)$ is the Fourier transform of $\beta(t)$.

$\beta_0(t)$ box function is a valid scaling function:



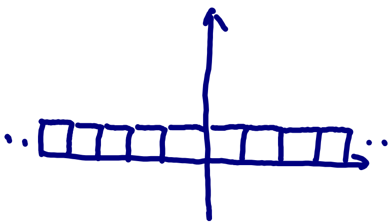
1. $\langle \beta_0(t), \beta_0(t-n)\rangle = \delta[n] \overset{FT}{\Longleftrightarrow} \sum_{k\in\mathbb{Z}} |\hat{\beta}_0(w+2\pi k)| = 1$

2. $g_0[n] = \frac{1}{\sqrt{2}}\delta[n] + \frac{1}{\sqrt{2}}\delta[n-1]$



$\varphi[n] = \sqrt{2}\sum_{k\in\mathbb{Z}} g_0[k]\varphi[2n-k]$

3.

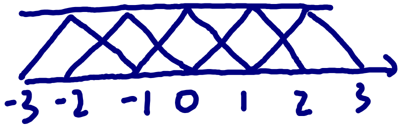


$\Rightarrow \sum_{k\in\mathbb{Z}} \beta(t-k) = 1$

$\beta_1(t)$ linear spline $\wedge$ is a valid scaling function.

3.

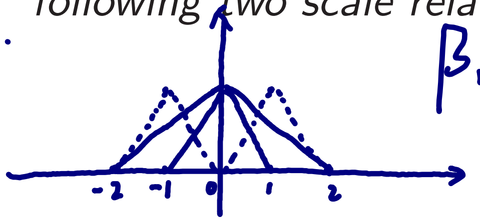
Scaling Function and Splines



$$\sum_{k\in\mathbb{Z}} \beta_1(t-k) = 1$$

Theorem 2. *Given two valid biorthogonal scaling functions $\varphi(t)$ and $\tilde{\varphi}(t)$ satisfying the following two scale relations*

2.



$$\beta_1\left(\frac{t}{2}\right) = \frac{1}{2}\beta_1(t+1) + \beta(t) + \frac{1}{2}\beta(t-1)$$

$$P(z) = \frac{1}{16\alpha}(1+z)^2(1+z^{-1})^2(1-\alpha z^{-1})(1-\alpha z)$$

$$\varphi(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} g_0[k]\varphi(2t-k)$$

$g_0[-1] = \frac{1}{2\sqrt{2}}$

$g_0[0] = \frac{1}{\sqrt{2}}$

$g_0[1] = \frac{1}{2\sqrt{2}}$

$$G_0(z) = \frac{1}{2\sqrt{2}}(1+z)(1+z^{-1})$$

$$\varphi\left(\frac{t}{2}\right) = \sqrt{2}\sum_{k=-\infty}^{+\infty} g_0[k]\varphi(t-k)$$

$$\tilde{\varphi}(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_0[k]\tilde{\varphi}(2t-k).$$

$$\therefore H_0(z) = \frac{\sqrt{2}}{8\alpha}(1+z)(1+z^{-1})(1-\alpha z^{-1})(1-\alpha z)$$

There exist two biorthogonal wavelets ψ and $\tilde{\psi}$ such that

1. finite compact support: check in ~~time then~~ FT.

$$\psi(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} \boxed{(-1)^{k-1}h_0}[1-k]\varphi(2t-k)$$

$$A(e^{j\omega}) = \sum_n |\Phi(\omega+2\pi n)|^2$$

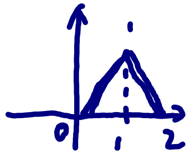
$a[n] = \langle \varphi(t), \varphi(t-n)\rangle \overset{FT}{\Longleftrightarrow} \sum_n |\Phi(\omega+2\pi n)|^2$

calculate $a[n]$ then FT.

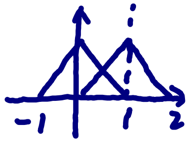
$$\therefore \frac{1}{3} \le \sum_n |\Phi(\omega+2\pi n)|^2 \le 1$$

$$\tilde{\psi}(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} (-1)^{k-1} g_0[1-k]\tilde{\varphi}(2t-k)$$

linear spline: biorthogonal



$$a[0] = 2\int_0^1 t^2\,dt = \frac{2}{3}$$



$$a[-1] = a[1] = \int_0^1 t(1-t)\,dt = \frac{1}{6}$$

$$\therefore A(e^{j\omega}) = \frac{2}{3} + \frac{1}{6}(e^{-j\omega} + e^{j\omega}) = \frac{2}{3} + \frac{1}{3}\cos\omega$$

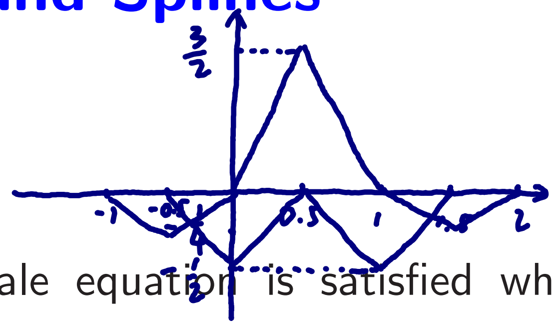
$$\boxed{\psi(t) = \sqrt{2}\sum_{n\in\mathbb{Z}}(-1)^{n-1}h_0[1-n]\beta_1(2t-n)}$$

Scaling Function and Splines

$h_0[-n] \Leftrightarrow H_0(z^{-1}) = \frac{\sqrt{2}}{8}(-z^{-2}+2z^{-1}+6+2z-z^{2})$

$h_0[1-n] \Leftrightarrow \frac{\sqrt{2}}{8}(-z^{-3}+2z^{-2}+6z^{-1}+2-z)$

$(-1)^{n-1}h_0[1-n] \Leftrightarrow \frac{\sqrt{2}}{8}(-z^{-3}-2z^{-2}+6z^{-1}-2-z)$



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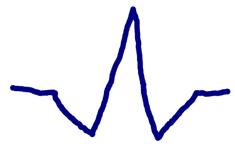


Example

Assume that $\varphi(t)$ is a linear spline. The two scale equation is satisfied when $G_0(z) = (\frac{1}{2}z^{-1} + 1 + \frac{1}{2}z)/\sqrt{2}$.

The biorthogonality relation says that

$$\langle\tilde{\varphi}(t), \varphi(t-n)\rangle = \delta_n.$$

Since both $\varphi(t)$ and $\tilde{\varphi}(t)$ satisfy a two scale relation, it follows that

$$\langle\tilde{\varphi}(t), \varphi(t-n)\rangle = \langle h_0[k], g_0[k-2n]\rangle = \delta_n.$$

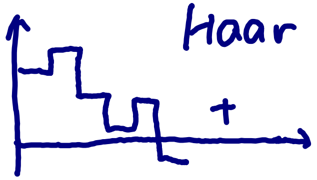
The above relation is equivalent to the condition $P(z) + P(-z) = 2$ with $P(z) = H_0(z^{-1})G_0(z)$. Here $G_0(z)$ is known and has two zeros at $\omega = \pi$, the shortest $H_0(z)$ with the same number of zeros at π is then

$$H_0(z) = \frac{\sqrt{2}}{8}(1+z)(1+z^{-1})(-z+4-z^{-1}) = \frac{\sqrt{2}}{8}(z^{-1}+2+z)(-z+4-z^{-1}).$$

$$f(t) = \sum_n c_{J,n} \varphi_{J,n}(t) + \sum_{m=-\infty}^{J} \sum_n d_{n,m} \psi_{m,n}(t)$$

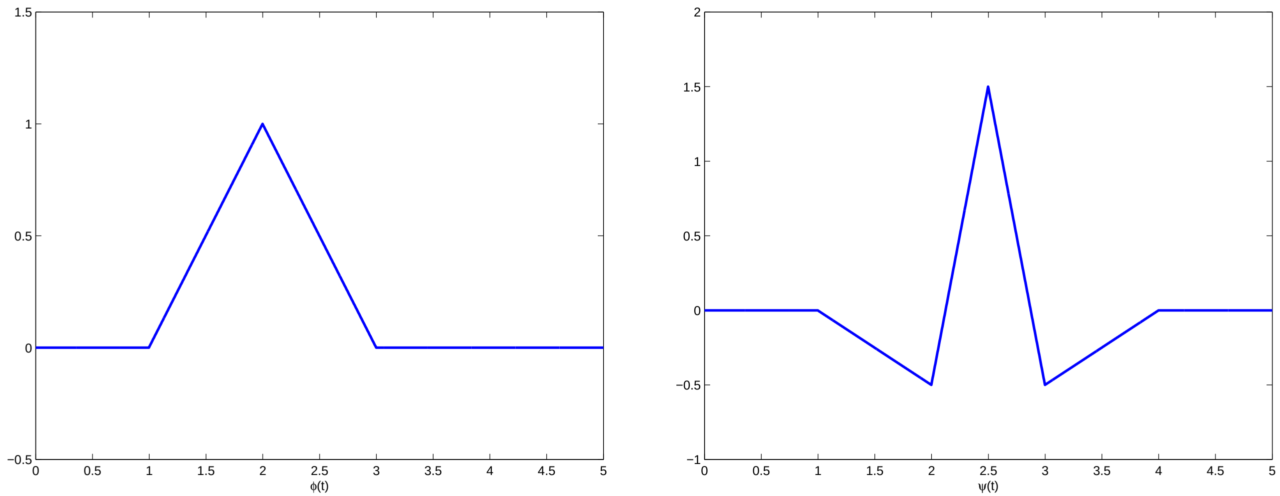
Scaling Function and Splines

Haar (orthogonal)



Linear spline (biorthogonal)

Given $H_0(z)$ the construction of the wavelet $\psi(t)$ is then straightforward. The scaling function $\varphi(t)$ and wavelet $\psi(t)$ for this example are shown below.



$A[n] = < f(t). f(t-n) > = \delta_n$

$A(\tau) = < f(t). f(t-\tau) > = \int_{-\infty}^{+\infty} f^*(t) f(t-\tau) dt$

$\Downarrow FT$

$A(\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f^*(t) f(t-\tau) dt\, e^{-j\omega\tau} d\tau$

Appendix

$\underset{\tau = t-x}{\underline{x = t-\tau}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f^*(t) e^{-j\omega t} dt\, f(x)\, e^{j\omega x} dx = \left[\int_{-\infty}^{+\infty} f(t) e^{j\omega t} dt\right]^* \left[\int_{-\infty}^{+\infty} f(x) e^{j\omega x} dx\right]$

Claim:

$$\langle \varphi(t-n), \varphi(t-m) \rangle = \delta_{m,n} \Longleftrightarrow \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2k\pi)|^2 = 1.$$

$= \hat{f}^*(-\omega) \hat{f}(-\omega) = |\hat{f}(-\omega)|^2$

Proof:

$\because f(t) * f(t) = f(-t) * f(-t)$

Define $p(\tau) = \langle \varphi(t), \varphi(t-\tau) \rangle$. Then $\langle \varphi(t), \varphi(t-m) \rangle$ is obtained by sampling $p(\tau)$ with sampling period $T = 1$. The Fourier transform of $p(\tau)$ is given by:

$\therefore ans = \hat{f}^*(\omega) \hat{f}(\omega) = |\hat{f}(\omega)|^2.$

$$\int_{-\infty}^{\infty} p(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} \langle \varphi(t), \varphi(t-\tau) \rangle e^{-j\omega\tau} d\tau = |\hat{\varphi}(\omega)|^2.$$

Applying the rule that sampling in time corresponds to replica in frequency leads to the desired equality.