THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. a) This is the joint pdf of two independent Gaussian RVs with zero mean and variance 1. Hence $P(X \leq 0.5 \cap Y \leq 0.7) = P(X \leq 0.5)P(Y \leq 0.7) = 0.691 * 0.758 = 0.5238$. [2 - E]

b) $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. [2 - E]

c) $E(X) = 0$, [2 - E]

$\mathrm{Var}(X) = 1$, [2 - E]

We can find these results by directly computing the integrals but it would be simpler to note form the marginal PDF that $X \sim N(0,1)$.

d) $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$. [2 - E]

e) $E(Y) = 0$,

[2 - E]

$\mathrm{Var}(Y) = 1$ [2 - E]

f) $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0$. [1 - E]
$\mathrm{Corr}(X,Y) = 0$ [1 - E]

g) X and Y are uncorrelated since $\mathrm{Corr}(X,Y) = 0$. [1 - E]
They are also independent since the joint pdf is written as the product of marginals.
[1 - E]

h) We can first compute the Jacobian and write

$$\begin{vmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{vmatrix} = \begin{vmatrix} \cos V & -U \sin V \\ \sin V & U \cos V \end{vmatrix} = U$$

[2 - B]

We then write

$$f_{U,V}(u,v) = \frac{u}{2\pi} e^{-\frac{u^2}{2}}, \quad u > 0, -\pi \leq v \leq \pi.$$

[2 - B]

i) The marginal are obtained by integration of the joint pdf as follows

$$f_U(u) = \int_{-\pi}^{\pi} f_{U,V}(u,v)dv = ue^{-\frac{u^2}{2}}, \;\; u > 0$$
$$f_V(v) = \int_{0}^{\infty} f_{U,V}(u,v)du = \frac{1}{2\pi}, -\pi \leq v \leq \pi$$

U is Rayleigh distributed and V is uniformly distributed over $[-\pi, \pi]$.

[2 - B]

j) Since $f_{U,V}(u,v) = f_U(u) f_V(v)$, U and V are two independent random variables.

[2 - A]

k) The conditional pdf $f_{U|V}(u|v)$ is given as

$$f_{U|V}(u|v) = f_U(u) = ue^{-\frac{u^2}{2}}, \;\; u > 0$$

[2 - A]

l) $E(U|V) = E(U) = \sqrt{\pi/2}$.

[2 - A]

2. a) The pdf is valid since $f_X(x) \geq 0$ and $\int_{-\infty}^{\infty} f_X(x)dx = 1$. This can be easily verified by noting that $X \sim \mathrm{EXPO}(2)$. [4 - A]

b) The CDF is given by $F_X(x) = \int_{-\infty}^{x} f_X(x)dx$ which leads to

$$F_X(x) = 1 - e^{-2x}, \quad x \geq 0$$

[4 - A]

c) $E(X) = \int_{-\infty}^{\infty} x f_X(x)dx$. We get $E(X) = 1/2$.

[2 - A]

$\mathrm{Var}(X) = E(X^2) - E(X)^2 = 1/4$.

[2 - A]

d) We write $m_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$. [1 - A]

By integration,

$$m_X(t) = \int_0^{\infty} e^{tx} 2e^{-2x} dx = 2\int_0^{\infty} e^{(t-2)x} dx = \frac{2}{2-t}$$

[1 - A]

We can compute $E(X) = m_X'(0)$ and $E(X^2) = m_X''(0)$.
We get $m_X'(0) = \frac{2}{(2-t)^2}\,|_{t=0} = 1/2$.

[1 - A]

Similarly $m_X''(0) = 1/2$ such that $\mathrm{Var}(X) = 1/2 - (1/2)^2 = 1/4$.

[1 - A]

e) By Chebyshev's inequality $P\left(\left|X - \frac{1}{3}\right| \geq \frac{1}{4}\right) \leq \frac{1}{1/16} E\left[(X - 1/3)^2\right] = 16 * 5/18 = 4.44$.

[2 - A]

The exact value can be computed as follows

$$\begin{aligned}
P\left(\left|X - \frac{1}{3}\right| \geq \frac{1}{4}\right) &= 1 - P\left(\left|X - \frac{1}{3}\right| \leq \frac{1}{4}\right) \\
&= 1 - P\left(-\frac{1}{4} \leq X - \frac{1}{3} \leq \frac{1}{4}\right) \\
&= 1 - P\left(\frac{1}{12} \leq X \leq \frac{7}{12}\right) \\
&= 1 - F_X(\frac{7}{12}) + F_X(\frac{1}{12}) \\
&= 1 - (1 - e^{-7/6} + (1 - e^{-1/6})) \\
&= 0.4649
\end{aligned}$$

[2 - A]