## THE QUESTIONS

[30]

1. Consider two continuous random variables $X$ and $Y$ characterized by the following joint probability density function

$$f_{X,Y}(x,y) = \frac{1}{4\pi} e^{-\frac{(x^2+y^2)}{4}}, \quad -\infty < x,y < +\infty,$$

a) Compute the probability that $X$ is smaller than or equal to 0.5 and $Y$ is smaller than or equal to 0.7, i.e. $P(X \leq 0.5 \cap Y \leq 0.7)$.

[2]

b) Compute the marginal probability density function of $X$.

[2]

c) Compute the expectation of $X$, i.e. $E(X)$, and the variance of $X$, i.e. $\mathrm{Var}(X)$.

[4]

d) Compute the marginal probability density function of $Y$.

[2]

e) Compute the expectation of $Y$, i.e. $E(Y)$, and the variance of $Y$, i.e. $\mathrm{Var}(Y)$.

[4]

f) Compute the covariance between $X$ and $Y$, i.e. $\mathrm{Cov}(X,Y)$, and the correlation coefficient between $X$ and $Y$, i.e. $\mathrm{Corr}(X,Y)$.

[2]

g) Are $X$ and $Y$ uncorrelated? Independent? Provide your reasoning.

[2]

h) Make the change of variables $U = \sqrt{X^2+Y^2}$, $V = \tan^{-1}\left(\frac{Y}{X}\right)$ and compute the joint probability density function $f_{U,V}(u,v)$.

[4]

i) Compute the marginal probability density function of $U$ and $V$, i.e. $f_U(u)$ and $f_V(v)$.

[2]

j) Are $U$ and $V$ independent? Provide your reasoning.

[2]

k) Compute the conditional probability density function of $U$ given $V$, i.e. $f_{U|V}(u|v)$.

[2]

l) Compute the conditional expectation of $U$ given $V$, i.e. $E(U|V)$.

[2]

[20]

2. Consider the continuous random variable $X$ characterized by the following probability density function

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}}, & x > 0, \\ 0, & otherwise. \end{cases}$$

a) Show that $f_X(x)$ is a valid probability density function.

[4]

b) Compute the cumulative distribution function of $X$, i.e. $F_X(x)$.

[4]

c) Compute the expectation of $X$, i.e. $E(X)$, and the variance of $X$, i.e. $\mathrm{Var}(X)$.

[4]

d) Compute the moment generating function of $X$, i.e. $m_x(t)$, assuming $t < \frac{1}{2}$. Explain how to make use this function to find the expectation and variance of a random variable. Apply this principle to $X$.

[4]

e) Compute

$$P\left(\left|X - \frac{1}{3}\right| \geq \frac{1}{4}\right).$$

[4]