C477: Computational Optimisation
Tutorial 3: 1D Optimisation

Exercise 1. Let $f(x) = x^2 + 4\cos(x)$, $x \in \mathbb{R}$. We wish to find the minimiser x^* of f over the interval $[1, 2]$.

(a) Use the Golden Section method to locate x^* to within an uncertainty of 0.2. Display all intermediate steps using a table

Iteration k	a_k	b_k	$f(a_k)$	$f(b_k)$	New uncertainty interval
1	?	?	?	?	[?,?]
2	?	?	?	?	[?,?]
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(b) Repeat part (a) using the same number of iterations but using the Newton method with $x_0 = 1$

Exercise 2. Consider the problem of finding the zero of $g(x) = (e^x - 1)/(e^x + 1)$, $x \in \mathbb{R}$ (not that 0 is the unique zero of x).

(a) Write down the algorithm for Newton's method to find a zero of this function. Simplify your calculations using the identity $\sinh(x) = (e^x - e^{-x})/2$.

(b) Find an initial condition x_0 such that the algorithm cycles. You need not explicitly calculate the initial condition; it suffices to provide an equation that the initial condition must satisfy.

(c) For what values of the initial condition does the algorithm converge?

Exercise 3. Consider the function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) = x^{4/3}$$

The minimiser $x = 0$ gives a global minimum value of $f(x = 0) = 0$. Suppose we initialise 1-dimensional Newton's algorithm at a point other than the global solution. What happens? Please justify your argument.

Exercise 4 (Exam Question, Spring 2016)**.** Consider the function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) = x^3 - 3x^2 - 24x.$$

(a) Use the first-order necessary condition and the second-order sufficient condition to find the local minima and local maxima of $f(x)$.

(b) Use the first- and second-derivatives to show that $f(x)$ has neither a global maximum nor a global minimum.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, i.e., $f \in \mathcal{C}^1$. Suppose that $\boldsymbol{x}^*$ is a local minimum of f along every line that passes through $\boldsymbol{x}^*$, i.e., the function:

$$g(\alpha) = f(\boldsymbol{x}^* + \alpha \boldsymbol{d})$$

is minimized at $\alpha = 0$ for all $\boldsymbol{d} \in \mathbb{R}^n$.

(a) Show that $\nabla f(\boldsymbol{x}^*) = 0$.

(b) Is $\boldsymbol{x}^*$ a local minimum of f? Justify your answer.

(c) Consider the function:

$$f(x_1,\, x_2) = (x_2 - x_1^2)(x_2 - 2x_1^2).$$

Show that the point $(0,\, 0)$ is a local minimum of f along every line that passes through $(0,\, 0)$.

Hint: Consider a line that goes through $(0,\, 0)$, namely $x_2 = \gamma x_1$ where γ is a scalar. Calculate the function values of $f(x_1, \gamma x_1)$.

(d) Show that the point $(0, 0)$ is not a local minimum of f.

Hint: Consider the values of f for $x_1 = y$ and $x_2 = my^2$ and $m \in \mathbb{R}$.

Exercise 5 (Lipschitz Continuity)**.** Consider the function $f(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{b}^\top \boldsymbol{x} + c$.

(a) Is the gradient $\nabla f(\boldsymbol{x})$ Lipschitz continuous?

(b) If yes, what is a Lipschitz constant L of $\nabla f(\boldsymbol{x})$?