Information for students

Each of the four questions has 25 marks.

The Questions

1. Random variables.

a) A rare disease affects one person in 10^4. A test for this disease shows positive with probability 9/10 when applied to an ill person, and with probability 1/10 when applied to a healthy person. What is the probability that you have the disease given that the test shows positive?

[5]

b) Suppose the random variable X has a Cauchy density

$$f_X(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}$$

and $Y = \tan^{-1} X$, derive the probability density function of Y, and determine the value of α such that Y is uniformly distributed.

[5]

c) X and Y are independent, identically distributed (i.i.d.) random variables with common probability density function

$$f_X(x) = e^{-x}, \qquad x > 0$$
$$f_Y(y) = e^{-y}, \qquad y > 0$$

Find the probability density function of the following random variables:

i) $Z = X + Y$. [5]

ii) $Z = X - Y$. [5]

iii) $Z = XY$. [5]

2. Estimation.

a) The random variable X has the density $f(x) \sim c^4 x^3 e^{-cx},\ \ x > 0$. We observe the i.i.d. samples x_i = 6.1, 5.7, 6.3, 5.7, 6.2. Find the maximum-likelihood estimate of parameter c.

[8]

b) Consider the auto-regressive process

$$Y(n) = \alpha Y(n-1) + Z(n)$$

where is α a real number satisfying $|\alpha| < 1$, and $Z(n)$ is an i.i.d. sequence with zero mean and unit variance.

i) Show that the autocorrelation function of $Y(n)$ is given by

$$R_Y(n) = \frac{\alpha^{|n|}}{1-\alpha^2}$$

[7]

ii) Suppose we wish to predict $Y(n+1)$ from $Y(n)$, $Y(n-1)$, …, $Y(1)$. The coefficients of the linear MMSE estimator

$$Y(n+1) = \sum_{i=1}^{n} c_i Y(i)$$

are given by the Wiener-Hopf equation

$$\boldsymbol{Rc} = \boldsymbol{r}$$

where $\boldsymbol{c} = [c_1, c_2, \dots, c_n]^T$, $\boldsymbol{r} = [R_Y(n), R_Y(n-1), \dots, R_Y(1)]^T$, and $\boldsymbol{R}$ is a n-by-n matrix whose (i, j)th entry is $R_Y(i-j)$. Find the best coefficients and the associated mean-square error.

[10]

3. Random processes.

a) Consider the random process

$$X(t) = A_t \cos(\omega t + \theta)$$

where t is continuous time and A_t are i.i.d. random variables with $E[A_t] = 0, Var[A_t] = \sigma^2$.

i) Let θ be a constant. Calculate the mean, variance of $X(t)$ and determine whether it is stationary or not.

[5]

ii) Now let θ be uniformly distributed on $[-\pi, \pi]$, and also independent of A_t. Calculate the mean, autocorrelation function of $X(t)$ and determine whether it is wide-sense stationary or not.

[5]

b) The random process $X(t)$ has autocorrelation $R(\tau)$.

i) If $X(t)$ is real-valued, show that

$$P\{|X(t+\tau) - X(t)| \geq a\} \leq 2[R(0) - R(\tau)]/a^2.$$

[5]

ii) From the fact that $R(\tau)$ is the inverse Fourier transform of the power spectral density $S(\omega)$, show that $R(\tau)$

$$\sum_{i,k} a_i a_k^* R(\tau_i - \tau_k) \geq 0$$

for all a_i.

[5]

iii) If $X(t)$ is a normal (i.e., Gaussian) process with zero mean and $Y(t) = Ie^{aX(t)}$, show that

$$E[Y(t)] = I\exp\left\{\frac{a^2}{2}R(0)\right\}$$

$$R_Y(\tau) = I^2 \exp\{a^2[R(0) + R(\tau)]\}$$

Hint: Use the characteristic function of two jointly Gaussian random variables $N(0,0,\sigma_1^2,\sigma_2^2,\rho)$, which is given by

$$\Phi(\omega_1, \omega_2) = \exp\left\{-\frac{\sigma_1^2\omega_1^2 + 2\rho\sigma_1\ \sigma_2\ \omega_1\ \omega_2\ + \sigma_2^2\omega_2^2}{2}\right\}$$

[5]

4. Martingale and Markov chains.

a) Show that the sums $S_n = X_1 + X_2 + \cdots + X_n$ of independent zero mean random variables form a martingale.

[5]

b) Consider a Markov chain with states $e_1, e_2, \dots, e_m$ and the following transition matrix

$$P = \begin{pmatrix} q & p & 0 & \cdots & 0 \\ 0 & q & p & \cdots & 0 \\ \vdots & \vdots & q & \cdots & 0 \\ 0 & 0 & \cdots & q & p \\ p & 0 & \cdots & 0 & q \end{pmatrix}$$

Find the limiting distribution.

[5]

c) Consider a stationary Markov chain $\dots, X_{n-1}, X_n, X_{n+1}, \dots$ with transition probabilities $\{p_{ij}\}$.

i) Assuming the chain has reached the steady state with limiting distribution $\{q_i\}$, show that the reversed sequence is also a stationary Markov chain with transition probabilities

$$P(X_n = j | X_{n+1} = i) \triangleq p_{ij}^* = \frac{q_j p_{ji}}{q_i}$$

[5]

ii) A Markov chain is said to be reversible if $p_{ij}^* = p_{ij}$ for all i, j. Show that a necessary condition for reversibility is

$$p_{ij} p_{jk} p_{ki} = p_{ik} p_{kj} p_{ji}, \qquad \text{for all } i, j, k.$$

[5]

iii) In general, a Markov chain may or may not have a steady state distribution. Yet, show that if it is reversible for some distribution $\{q_i\}$, then $\{q_i\}$ is just the steady state distribution.

[5]