11: Multirate Systems

Multirate Systems

Multirate systems include more than one sample rate

Why bother?:

- May need to change the sample rate
  e.g. Audio sample rates include 32, 44.1, 48, 96 kHz
  MP3 CD studio
- Can relax analog or digital filter requirements
  e.g. Audio DAC increases sample rate so that the reconstruction filter can have a more gradual cutoff
- Reduce computational complexity → battery power
  FIR filter length $\propto \frac{f_s}{\Delta f}$ where Δf is width of transition band
  Lower $f_s \Rightarrow$ shorter filter + fewer samples $\Rightarrow$ computation $\propto f_s^2$

Building blocks

Downsample $x[n] \rightarrow \boxed{K{:}1} \rightarrow y[m]$ $\quad y[m] = x[Km]$ skip (K-1) samples then take.

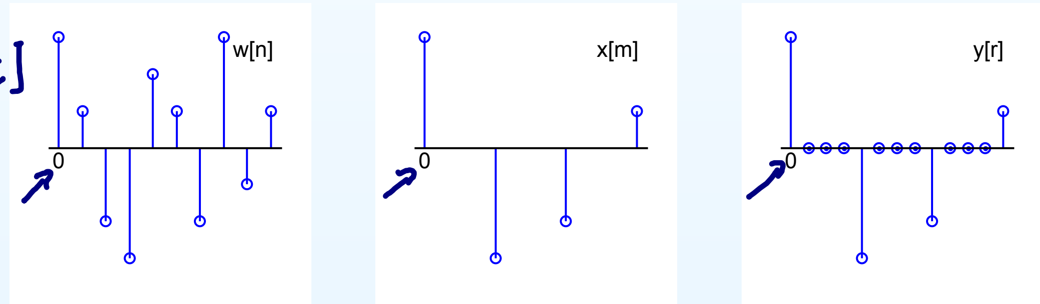
Upsample $u[m] \rightarrow \boxed{1{:}K} \rightarrow v[n]$ $\quad v[n] = \begin{cases} u\left[\frac{n}{K}\right] & K \mid n \\ 0 & \text{else} \end{cases}$ padding (k-1) zeros

Example:

Downsample by 3 then upsample by 4

$x[m] = w[3n]$ $\quad y[r] = x\left[\frac{m}{4}\right]$

$w[n] \rightarrow \boxed{3{:}1} \rightarrow x[m] \rightarrow \boxed{1{:}4} \rightarrow y[r]$



- We use different index variables (n, m, r) for different sample rates
- Use different colours for signals at different rates (sometimes)
- Synchronization: all signals have a sample at $n = 0$.

Resampling Cascades

Successive downsamplers or upsamplers can be combined

$\rightarrow \boxed{P{:}1} \rightarrow \boxed{Q{:}1} \rightarrow \;=\; \rightarrow \boxed{PQ{:}1} \rightarrow$

$\rightarrow \boxed{1{:}P} \rightarrow \boxed{1{:}Q} \rightarrow \;=\; \rightarrow \boxed{1{:}PQ} \rightarrow$

Upsampling can be exactly inverted

padding removing

$\rightarrow \boxed{1{:}P} \rightarrow \boxed{P{:}1} \rightarrow \;=\; \longrightarrow$

Downsampling destroys information permanently $\Rightarrow$ uninvertible

$\rightarrow \boxed{P{:}1} \rightarrow \boxed{1{:}P} \rightarrow \;\neq\; \longrightarrow$

Resampling can be interchanged iff P and Q are coprime (surprising!)

$w[n]=x[Pn]$ $y[n]=w[\frac{n}{Q}]$

$x \rightarrow \boxed{P{:}1} \xrightarrow{w} \boxed{1{:}Q} \xrightarrow{y} \;=\; x \rightarrow \boxed{1{:}Q} \xrightarrow{u} \boxed{P{:}1} \xrightarrow{v}$

$u[n]=x[\frac{n}{Q}]$ $v[n]=u[Pn]$

1 0 1 0
⇑ 3:1
1 0 1 0 1 0 1 0 1 0 1 0
⇑ 1:2
1 1 1 1 1 1
⇓ 3:1
1 1
⇓ 1:2
1 0 1 0 ✓

1 1 1
⇑ 4:1
1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0
⇑ 1:2
1 1 1 1 1 1 1 1
⇓ 4:1
1 1
⇓ 1:2
1 0 1 0 ✗

Proof: Left side: $y[n] = w\left[\frac{1}{Q}n\right] = x\left[\frac{P}{Q}n\right]$ if $Q \mid n$ else $y[n] = 0$.

Right side: $v[n] = u\left[Pn\right] = x\left[\frac{P}{Q}n\right]$ if $Q \mid Pn$.

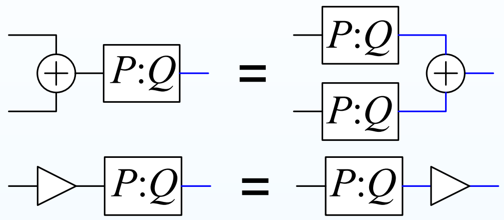
But $\{Q \mid Pn \;\Rightarrow\; Q \mid n\}$ iff P and Q are coprime.

if P·Q are not coprime, information lost do not overlap.

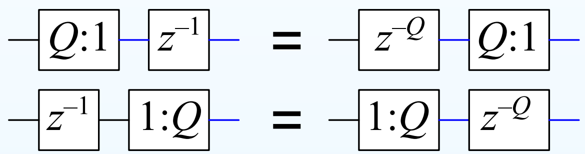
[Note: $a \mid b$ means "a divides into b exactly"]

Noble Identities

Resamplers commute with addition and multiplication

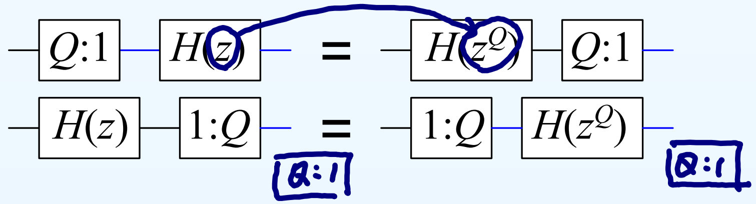


Delays must be multiplied by the resampling ratio

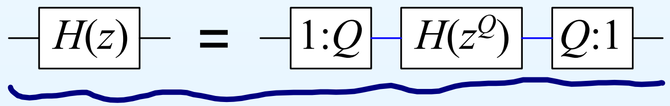


Noble identities: !

Exchange resamplers and filters



Corrollary



Example: $H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \cdots$

$$H(z^3) = h[0] + h[1]z^{-3} + h[2]z^{-6} + \cdots$$

Noble Identities Proof

$w[r] = v[Qr] = \sum_{s=0}^{QM} h_Q[s]\, x[Qr-s]$
$= \sum_{m=0}^{M} h_Q[Qm]\, x[Q(r-m)]$
$= \sum_{m=0}^{M} h[m]\, u[r-m] = y[r]$

Define $h_Q[n]$ to be the impulse response of $H(z^Q)$.

$x[n] \rightarrow \boxed{Q{:}1} \xrightarrow{u[r]} \boxed{H(z)} \xrightarrow{y[r]} \quad = \quad x[n] \rightarrow \boxed{H(z^Q)} \xrightarrow{v[n]} \boxed{Q{:}1} \xrightarrow{w[r]}$

Assume that $h[r]$ is of length $M+1$ so that $h_Q[n]$ is of length $QM+1$.

We know that $h_Q[n] = 0$ except when $Q \mid n$ and that $\underline{h[r] = h_Q[Qr]}$.

down

$$w[r] = v[Qr] = \sum_{s=0}^{QM} h_Q[s]x[Qr-s]$$

$s = Qm$ ($h_Q[n] = 0$ except $Q|n$)

$$= \sum_{m=0}^{M} \underline{h_Q[Qm]}x[Qr-Qm] = \sum_{m=0}^{M} h[m]x[Q(r-m)]$$

$$= \sum_{m=0}^{M} h[m]u[r-m] = y[r]$$ ☺

$w[n] = \sum_{s=0}^{QM} h_Q[s]\, v[n-s]$
$= \sum_{m=0}^{M} h_Q[QM]\, v[n-QM]$
$= \sum_{m=0}^{M} h[m]\, v[n-QM]$
$Q \nmid n \Rightarrow v[n-QM] = 0$
$\Rightarrow w[n] = 0 = y[n]$
$n = Qr \Rightarrow w[Qr] = \sum_{m=0}^{M} h[m]\, v[Q(r-m)]$
$= \sum_{m=0}^{M} h[m]\, x[r-m] = u[r]$
$= y[Qr]$

Upsampled Noble Identity: !

$x[r] \rightarrow \boxed{H(z)} \xrightarrow{u[r]} \boxed{1{:}Q} \xrightarrow{y[n]} \quad = \quad x[r] \rightarrow \boxed{1{:}Q} \xrightarrow{v[n]} \boxed{H(z^Q)} \xrightarrow{w[n]}$

We know that $v[n] = 0$ except when $Q \mid n$ and that $v[Qr] = x[r]$.

$$w[n] = \sum_{s=0}^{QM} h_Q[s]v[n-s] = \sum_{m=0}^{M} h_Q[Qm]v[n-Qm]$$

$$= \sum_{m=0}^{M} h[m]v[n-Qm]$$

If $Q \nmid n$, then $v[n-Qm] = 0\ \forall m$ so $w[n] = 0 = y[n]$

If $Q \mid n = Qr$, then $w[Qr] = \sum_{m=0}^{M} h[m]v[Qr-Qm]$

$$= \sum_{m=0}^{M} h[m]x[r-m] = u[r] = y[Qr]$$ ☺

Upsampled z-transform

upsample
insert K-1 zeros

$u[m] \rightarrow \boxed{1:K} \rightarrow v[n]$

$U(z) \rightarrow \boxed{1:K} \rightarrow U(z^K)$ V(z)

$$V(z) = \sum_n v[n]z^{-n} = \sum_{n \text{ s.t. } K|n} u[\tfrac{n}{K}]z^{-n}$$
$$= \sum_m u[m]z^{-Km} = U(z^K)$$

Spectrum: $V(e^{j\omega}) = U(e^{jK\omega})$

Spectrum is horizontally shrunk and replicated K times.

Total energy (integral) unchanged; power (= energy/sample) multiplied by $\frac{1}{K}$

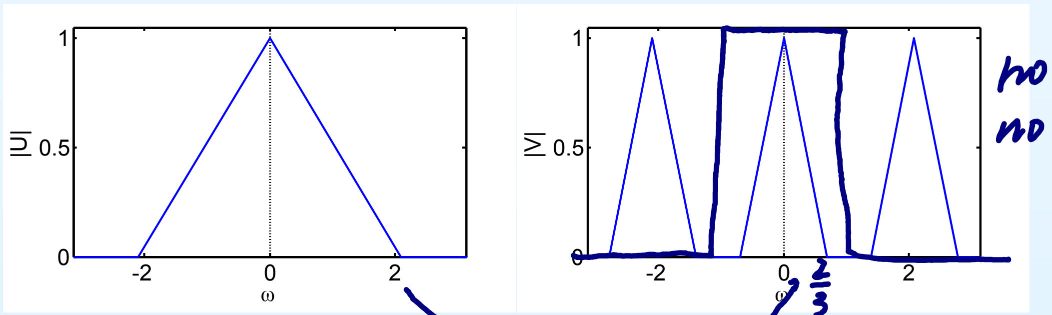
Upsampling normally followed by a LP filter to remove images.

Example:

$K = 3$: three images of the original spectrum in all.

Energy unchanged: $\frac{1}{2\pi}\int \left|U(e^{j\omega})\right|^2 d\omega = \frac{1}{2\pi}\int \left|V(e^{j\omega})\right|^2 d\omega$

upsampling → LP



no overlap
no aliasing

Downsampled z-transform

impulse every K

$$x[n] \rightarrow \boxed{K{:}1} \xrightarrow{y[m]} \boxed{1{:}K} \rightarrow x_K[n]$$

Define $c_K[n] = \delta_{K|n}[n] = \frac{1}{K}\sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}}$

Now define $x_K[n] = \begin{cases} x[n] & K \mid n \\ 0 & K \nmid n \end{cases} = c_K[n]x[n]$

$$X_K(z) = \sum_n x_K[n]z^{-n} = \frac{1}{K}\sum_n \sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}} x[n]z^{-n}$$

$$= \frac{1}{K}\sum_{k=0}^{K-1}\sum_n x[n]\left(e^{\frac{-j2\pi k}{K}}z\right)^{-n} = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}}z)$$

$$X(z) \rightarrow \boxed{K{:}1} \rightarrow \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}}z^{\frac{1}{K}})$$

From previous slide:

$$X_K(z) \rightarrow \boxed{K{:}1} \rightarrow Y(z) = X_K(z^{\frac{1}{K}})$$

$$X_K(z) = Y(z^K)$$

$$\Rightarrow Y(z) = X_K(z^{\frac{1}{K}}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}}z^{\frac{1}{K}})$$

Frequency Spectrum:

LP → downsampling

$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$

to be filtered

$$= \frac{1}{K}\left(X(e^{\frac{j\omega}{K}}) + X(e^{\frac{j\omega}{K}-\frac{2\pi}{K}}) + X(e^{\frac{j\omega}{K}-\frac{4\pi}{K}}) + \cdots\right)$$

shifted $\frac{2\pi}{K}$ shifted $\frac{4\pi}{K}$

Average of K aliased versions, each expanded in ω by a factor of K.

Downsampling is normally preceded by a LP filter to prevent aliasing.

Downsampled Spectrum

downsampling:
energy $\times = \frac{1}{K}$ (samples gone)
power $\approx$ unchanged

$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$

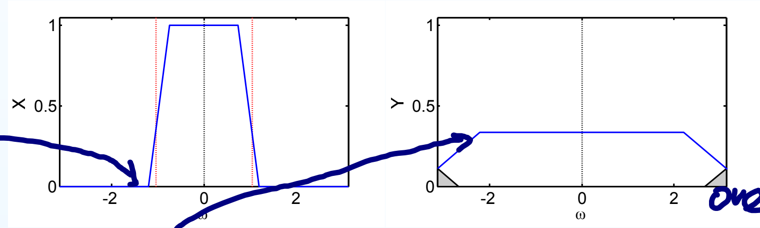
$x[n] \rightarrow$ K:1 $\rightarrow y[m]$

Example 1:

$K = 3$

Not quite limited to $\pm\frac{\pi}{K}$

Shaded region shows aliasing



overlap

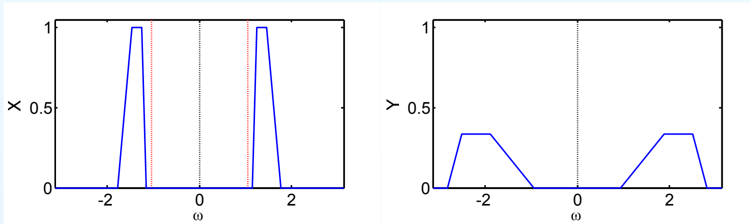
Energy decreases: $\frac{1}{2\pi}\int \left|Y(e^{j\omega})\right|^2 d\omega \approx \boxed{\frac{1}{K}} \times \frac{1}{2\pi}\int \left|X(e^{j\omega})\right|^2 d\omega$

Example 2:

$K = 3$

Energy all in $\frac{\pi}{K} \le |\omega| < 2\frac{\pi}{K}$

No aliasing: ☺



No aliasing: If all energy is in $r\frac{\pi}{K} \le |\omega| < (r+1)\frac{\pi}{K}$ for some integer r

Normal case ($r = 0$): If all energy in $0 \le |\omega| \le \frac{\pi}{K}$

Downsampling: Total energy multiplied by $\approx \frac{1}{K}$ ($= \frac{1}{K}$ if no aliasing)

Average power $\approx$ unchanged ($=$ energy/sample)

Power Spectral Density

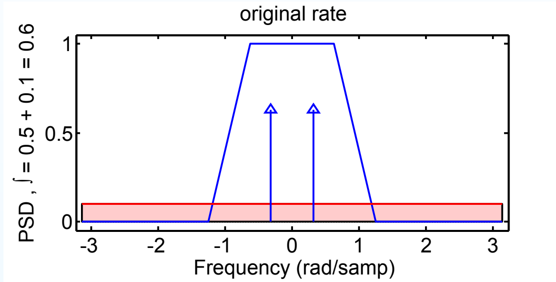
Example: Signal in $\omega \in \pm 0.4\pi$ + Tone @ $\omega = \pm 0.1\pi$ + White noise

Power $=$ Energy/sample $=$ Average PSD

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \text{PSD}(\omega)d\omega \;= 0.6$$

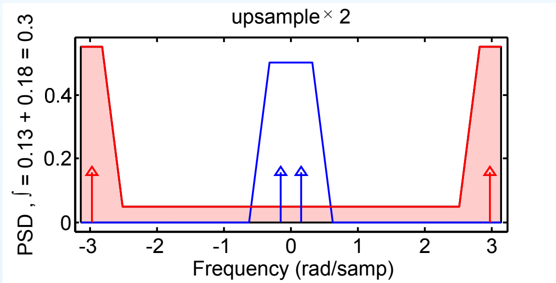
Component powers:

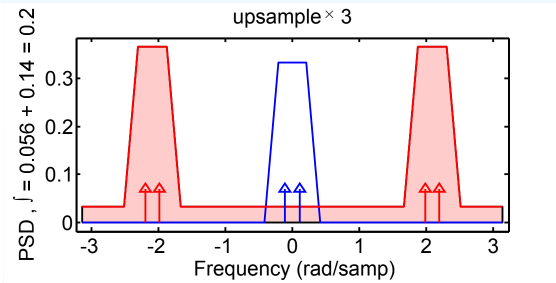
Signal = 0.3, Tone = 0.2, Noise = 0.1



Upsampling:

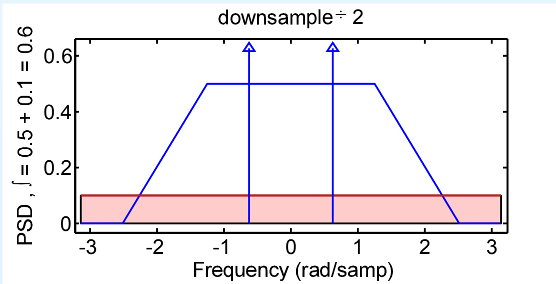
Same energy per second $\Rightarrow$ Power is $\div K$

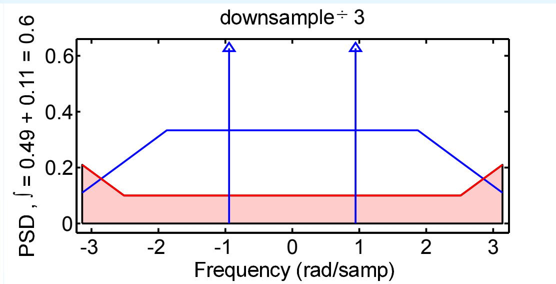




Downsampling:

Average power is unchanged. $\exists$ aliasing in the $\div 3$ case.



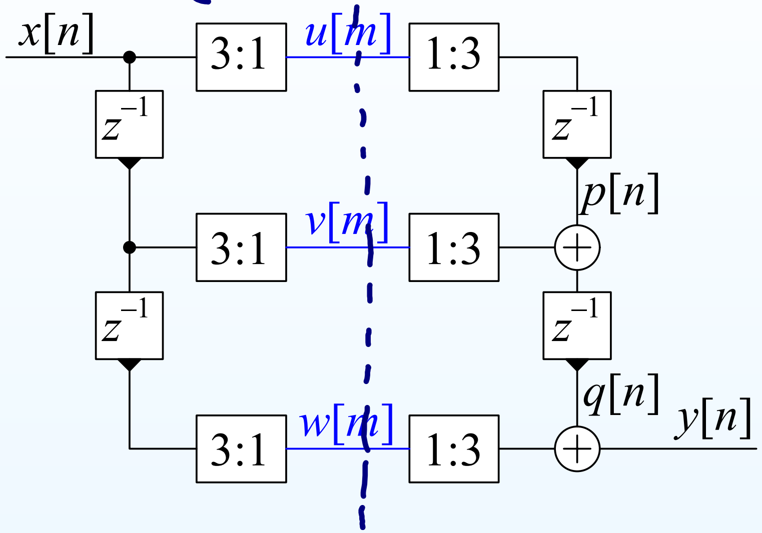


Perfect Reconstruction

only $\frac{1}{3}$ sampling rate but repeat 3 times

$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$p[n]$	`-c--f--i--l`
$v[m]$	`b  e  h  k`
$q[n]$	`-bc-ef-hi-kl`
$w[m]$	`a  d  g  j`
$y[n]$	`abcdefghijkl`

accept delay!



Input sequence $x[n]$ is split into three streams at $\frac{1}{3}$ the sample rate:

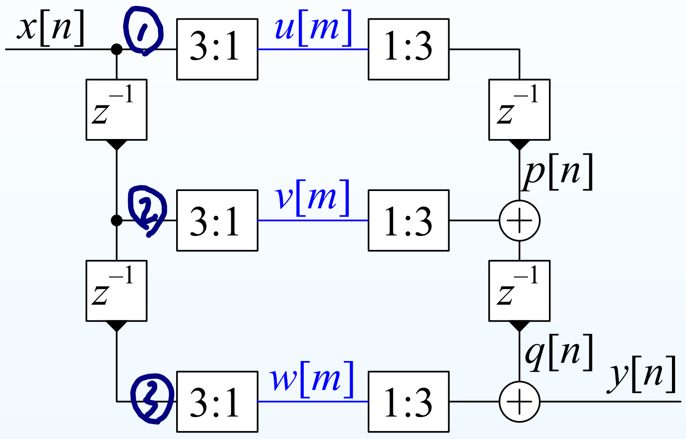
$$u[m] = x[3m],\ v[m] = x[3m-1],\ w[m] = x[3m-2]$$

Following upsampling, the streams are aligned by the delays and then added to give:

$$y[n] = x[n-2]$$

Perfect Reconstruction: output is a delayed scaled replica of the input

Commutators



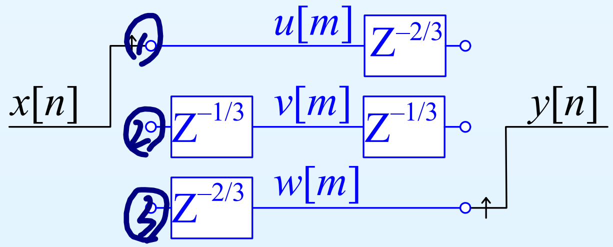
$x[n]$	cdefghijklmn
$u[m]$	c f i l
$v[m]$	b e h k
$w[m]$	a d g j
$v[m+\frac{1}{3}]$	e h k l
$w[m+\frac{2}{3}]$	d g j m
$y[n]$	abcdefghijkl

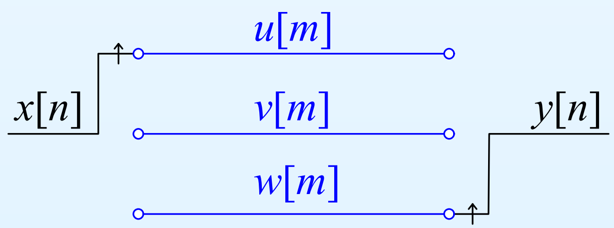
The combination of delays and downsamplers can be regarded as a commutator that distributes values in sequence to u, w and v.

Fractional delays, $z^{-\frac{1}{3}}$ and $z^{-\frac{2}{3}}$ are needed to synchronize the streams.

The output commutator takes values from the streams in sequence.

For clarity, we omit the fractional delays and regard each terminal, ○, as holding its value until needed. Initial commutator position has zero delay.





The commutator direction is against the direction of the z^{-1} delays.

Summary

- Multirate Building Blocks
  - Upsample: $X(z) \overset{1:K}{\rightarrow} X(z^K)$
    Invertible, Inserts $K-1$ zeros between samples
    Shrinks and replicates spectrum
    Follow by LP filter to remove images
  - Downsample: $X(z) \overset{K:1}{\rightarrow} \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}} z^{\frac{1}{K}})$
    Destroys information and energy, keeps every K^{th} sample
    Expands and aliasses the spectrum
    Spectrum is the average of K aliased expanded versions
    Precede by LP filter to prevent aliases

$$-\boxed{Q:1}-\boxed{H(z)}- \;=\; -\boxed{H(z^Q)}-\boxed{Q:1}-$$

- Equivalences
  - Noble Identities: $H(z) \longleftrightarrow H(z^K)$
  - Interchange $P:1$ and $1:Q$ iff $P$and Q coprime
- Commutators
  - Combine delays and down/up sampling

For further details see Mitra: 13.

MATLAB routines

resample	change sampling rate