

## 8: IIR Filter

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# 8: IIR Filter Transformations

# Continuous Time Filters

Classical continuous-time filters optimize tradeoff: passband ripple v stopband ripple v transition width  
**There are explicit formulae for pole/zero positions.**

Butterworth:  $\tilde{G}^2(\Omega) = \left| \tilde{H}(j\Omega) \right|^2 = \frac{1}{1 + \Omega^{2N}}$

- Monotonic  $\forall \Omega$
- $\tilde{G}(\Omega) = 1 - \frac{1}{2}\Omega^{2N} + \frac{3}{8}\Omega^{4N} + \dots$   
“Maximally flat”:  $2N - 1$  derivatives are zero

Chebyshev:  $\tilde{G}^2(\Omega) = \frac{1}{1 + \epsilon^2 T_N^2(\Omega)}$

where polynomial  $T_N(\cos x) = \cos Nx$

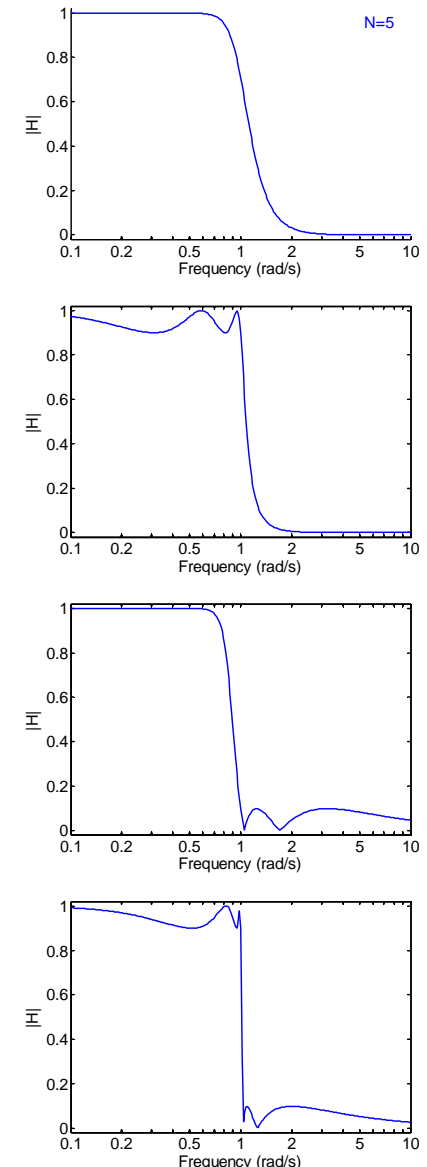
- passband equiripple + very flat at  $\infty$

Inverse Chebyshev:  $\tilde{G}^2(\Omega) = \frac{1}{1 + (\epsilon^2 T_N^2(\Omega^{-1}))^{-1}}$

- stopband equiripple + very flat at 0

Elliptic: [no nice formula]

- Very steep + equiripple in pass and stop bands



# Bilinear Mapping

Change variable:  $z = \frac{\alpha+s}{\alpha-s} \Leftrightarrow s = \alpha \frac{z-1}{z+1}$ : a one-to-one invertible mapping

- $\Re$  axis ( $s$ )  $\leftrightarrow \Re$  axis ( $z$ )
- $\Im$  axis ( $s$ )  $\leftrightarrow$  Unit circle ( $z$ )

Proof:  $z = e^{j\omega} \Leftrightarrow s = \alpha \frac{e^{j\omega} - 1}{e^{j\omega} + 1} = \alpha \frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}} = j\alpha \tan \frac{\omega}{2} = j\Omega$

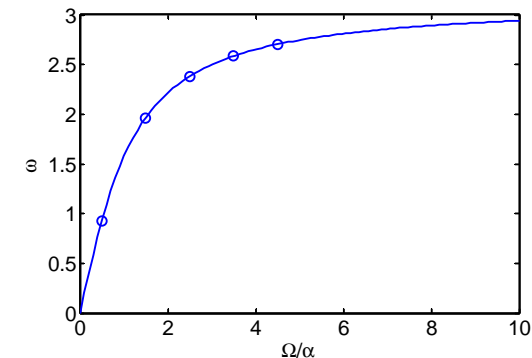
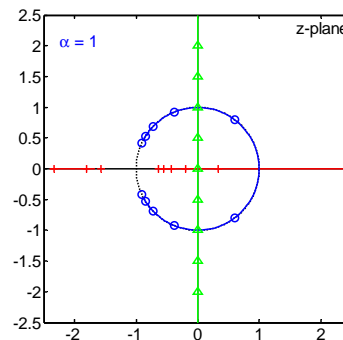
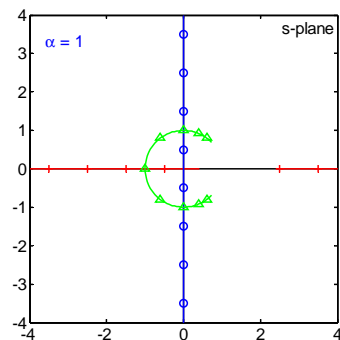
- Left half plane( $s$ )  $\leftrightarrow$  inside of unit circle ( $z$ )

Proof:  $s = x + jy \Leftrightarrow |z|^2 = \frac{|(\alpha+x)+jy|^2}{|(\alpha-x)-jy|^2}$   

$$= \frac{\alpha^2 + 2\alpha x + x^2 + y^2}{\alpha^2 - 2\alpha x + x^2 + y^2} = 1 + \frac{4\alpha x}{(\alpha-x)^2 + y^2}$$

$$x < 0 \Leftrightarrow |z| < 1$$

- Unit circle ( $s$ )  $\leftrightarrow \Im$  axis ( $z$ )



# Continuous Time Filters

Take  $\tilde{H}(s) = \frac{1}{s^2 + 0.2s + 4}$  and choose  $\alpha = 1$

Substitute:  $s = \alpha \frac{z-1}{z+1}$  [extra zeros at  $z = -1$ ]

$$\begin{aligned} H(z) &= \frac{1}{\left(\frac{z-1}{z+1}\right)^2 + 0.2 \frac{z-1}{z+1} + 4} \\ &= \frac{(z+1)^2}{(z-1)^2 + 0.2(z-1)(z+1) + 4(z+1)^2} \\ &= \frac{z^2 + 2z + 1}{5.2z^2 + 6z + 4.8} = 0.19 \frac{1 + 2z^{-1} + z^{-2}}{1 + 1.15z^{-1} + 0.92z^{-2}} \end{aligned}$$

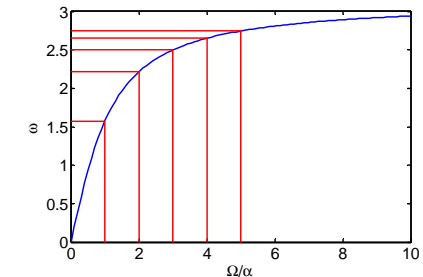
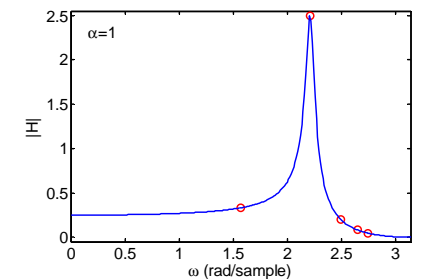
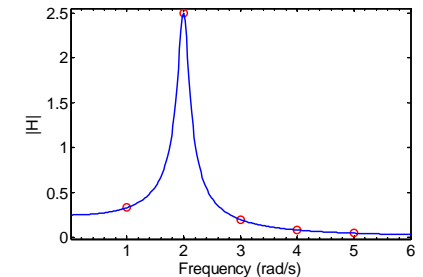
Frequency response is identical (both magnitude and phase) but with a distorted frequency axis:

Frequency mapping:  $\omega = 2 \tan^{-1} \frac{\Omega}{\alpha}$

$$\begin{aligned} \Omega &= [\alpha \quad 2\alpha \quad 3\alpha \quad 4\alpha \quad 5\alpha] \\ \rightarrow \omega &= [1.6 \quad 2.2 \quad 2.5 \quad 2.65 \quad 2.75] \end{aligned}$$

Choosing  $\alpha$ : Set  $\alpha = \frac{\Omega_0}{\tan \frac{1}{2}\omega_0}$  to map  $\Omega_0 \rightarrow \omega_0$

Set  $\alpha = 2f_s = \frac{2}{T}$  to map low frequencies to themselves



# Mapping Poles and Zeros

Alternative method:  $\tilde{H}(s) = \frac{1}{s^2 + 0.2s + 4}$

Find the poles and zeros:  $p_s = -0.1 \pm 2j$

Map using  $z = \alpha \frac{1+s}{1-s} \Rightarrow p_z = -0.58 \pm 0.77j$

After the transformation we will always end up with the **same number of poles as zeros**:

Add extra poles or zeros at  $z = -1$

$$H(z) = g \times \frac{(1+z^{-1})^2}{(1+(0.58-0.77j)z^{-1})(1+(0.58+0.77j)z^{-1})}$$

$$= g \times \frac{1+2z^{-1}+z^{-2}}{1+1.15z^{-1}+0.92z^{-2}}$$

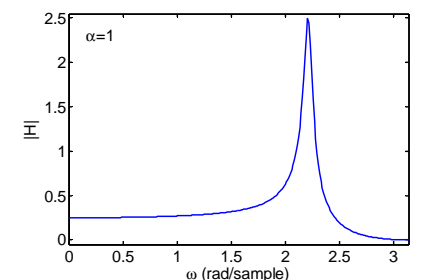
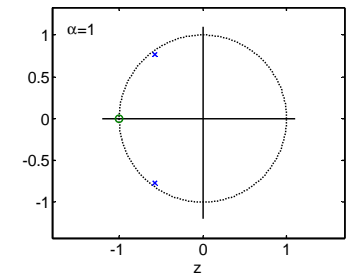
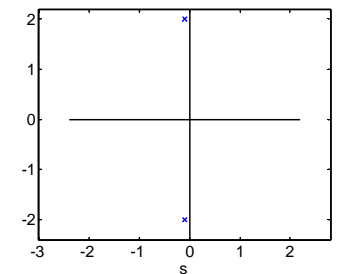
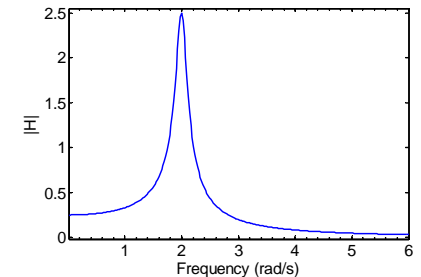
Choose overall **scale factor,  $g$** , to give the same gain at any convenient pair of mapped frequencies:

$$\text{At } \Omega_0 = 0 \Rightarrow s_0 = 0 \Rightarrow |\tilde{H}(s_0)| = 0.25$$

$$\Rightarrow \omega_0 = 2 \tan^{-1} \frac{\Omega_0}{\alpha} = 0 \Rightarrow z_0 = e^{j\omega_0} = 1$$

$$\Rightarrow |H(z_0)| = g \times \frac{4}{3.08} = 0.25 \Rightarrow g = 0.19$$

$$H(z) = 0.19 \frac{1+2z^{-1}+z^{-2}}{1+1.15z^{-1}+0.92z^{-2}}$$



# Spectral Transformations

We can transform the z-plane to change the cutoff frequency by substituting

$$z = \frac{\hat{z} - \lambda}{1 - \lambda \hat{z}} \Leftrightarrow \hat{z} = \frac{z + \lambda}{1 + \lambda z}$$

Frequency Mapping:

If  $z = e^{j\omega}$ , then  $\hat{z} = z \frac{1 + \lambda z^{-1}}{1 + \lambda z}$  has modulus 1 since the numerator and denominator are complex conjugates.

Hence the **unit circle is preserved**.

$$\Rightarrow e^{j\hat{\omega}} = \frac{e^{j\omega} + \lambda}{1 + \lambda e^{j\omega}}$$

Some algebra gives:  $\tan \frac{\omega}{2} = \left( \frac{1 + \lambda}{1 - \lambda} \right) \tan \frac{\hat{\omega}}{2}$

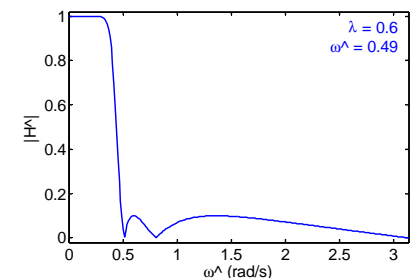
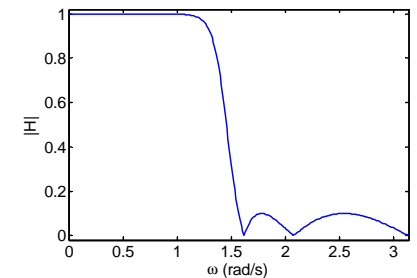
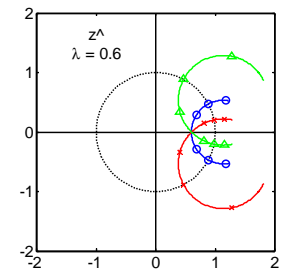
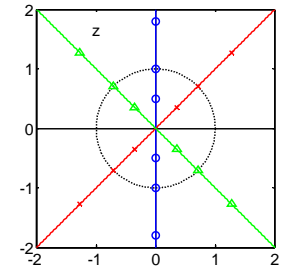
Equivalent to:

$$z \longrightarrow s = \frac{z-1}{z+1} \longrightarrow \hat{s} = \frac{1-\lambda}{1+\lambda} s \longrightarrow \hat{z} = \frac{1+\hat{s}}{1-\hat{s}}$$

Lowpass Filter example:

Inverse Chebyshev

$$\omega_0 = \frac{\pi}{2} = 1.57 \xrightarrow{\lambda=0.6} \hat{\omega}_0 = 0.49$$

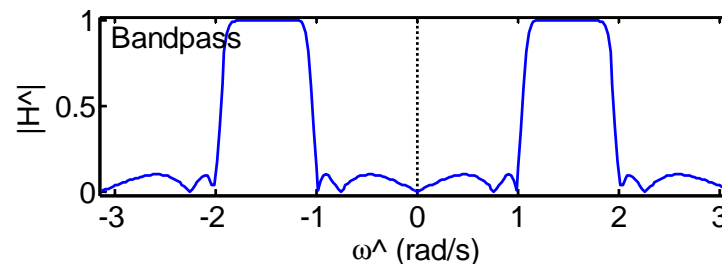
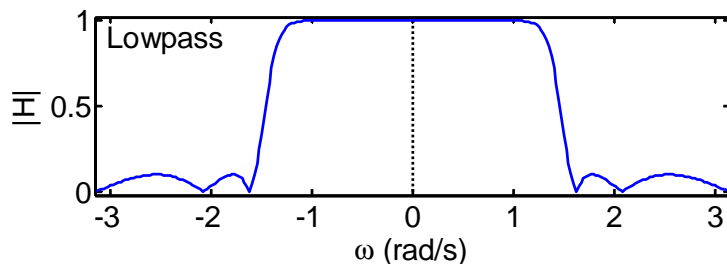


# Constantinides Transformations

Transform any lowpass filter with cutoff frequency  $\omega_0$  to:

Target	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1} - \lambda}{1 - \lambda \hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0 - \hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0 + \hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1} + \lambda}{1 + \lambda \hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0 + \hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0 - \hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1) - 2\lambda\rho\hat{z}^{-1} + (\rho+1)\hat{z}^{-2}}{(\rho+1) - 2\lambda\rho\hat{z}^{-1} + (\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2 + \hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2 - \hat{\omega}_1}{2}\right)}$ $\rho = \cot\left(\frac{\hat{\omega}_2 - \hat{\omega}_1}{2}\right) \tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \nless \hat{\omega} \nless \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho) - 2\lambda\hat{z}^{-1} + (\rho+1)\hat{z}^{-2}}{(\rho+1) - 2\lambda\hat{z}^{-1} + (1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2 + \hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2 - \hat{\omega}_1}{2}\right)}$ $\rho = \tan\left(\frac{\hat{\omega}_2 - \hat{\omega}_1}{2}\right) \tan\left(\frac{\omega_0}{2}\right)$

Bandpass and bandstop transformations are quadratic and so will double the order:



# Impulse Invariance

Bilinear transform works well for a lowpass filter but the non-linear compression of the frequency distorts any other response.

**Alternative method:**  $\tilde{H}(s) \xrightarrow{\mathcal{L}^{-1}} h(t) \xrightarrow{\text{sample}} h[n] = T \times h(nT) \xrightarrow{\mathcal{Z}} H(z)$   
Express  $\tilde{H}(s)$  as a sum of partial fractions  $\tilde{H}(s) = \sum_{i=1}^N \frac{g_i}{s - \tilde{p}_i}$

Impulse response is  $\tilde{h}(t) = u(t) \times \sum_{i=1}^N g_i e^{\tilde{p}_i t}$

Digital filter  $\frac{H(z)}{T} = \sum_{i=1}^N \frac{g_i}{1 - e^{\tilde{p}_i T} z^{-1}}$  has identical impulse response

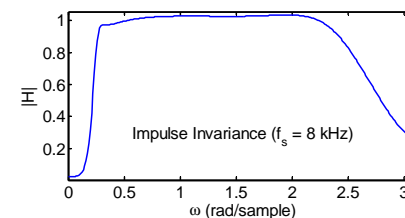
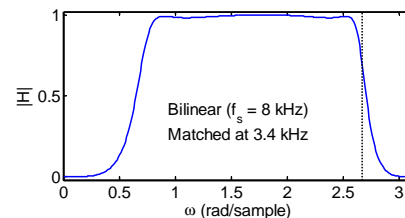
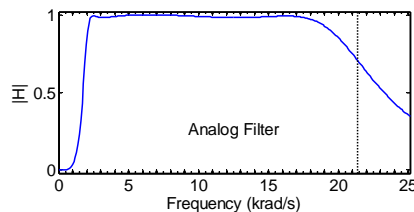
Poles of  $H(z)$  are  $p_i = e^{\tilde{p}_i T}$  (where  $T = \frac{1}{f_s}$  is sampling period)

Zeros do not map in a simple way

**Properties:**

- 😊 Impulse response correct.
- 😊 No distortion of frequency axis.
- 😞 Frequency response is aliased.

**Example:** Standard telephone filter - 300 to 3400 Hz bandpass





# Summary

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Continuous Time Filters  
Bilinear Mapping  
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Mapping Poles and Zeros  
Spectral Transformations  
Constantinides Transformations  
Impulse Invariance  
▷ Summary  
MATLAB routines

- **Classical filters** have optimal tradeoffs in continuous time domain
  - Order  $\leftrightarrow$  transition width  $\leftrightarrow$  pass ripple  $\leftrightarrow$  stop ripple
  - Monotonic passband and/or stopband
- **Bilinear mapping**
  - Exact preservation of frequency response (mag + phase)
  - non-linear frequency axis distortion
  - can choose  $\alpha$  to map  $\Omega_0 \rightarrow \omega_0$  for one specific frequency
- **Spectral transformations**
  - lowpass  $\rightarrow$  lowpass, highpass, bandpass or bandstop
  - bandpass and bandstop double the filter order
- **Impulse Invariance**
  - Aliasing distortion of frequency response
  - preserves frequency axis and impulse response

For further details see Mitra: 9.

# MATLAB routines

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▷ MATLAB routines

bilinear	Bilinear mapping
impinvar	Impulse invariance
butter butterord	Analog or digital Butterworth filter
cheby1 cheby1ord	Analog or digital Chebyshev filter
cheby2 cheby2ord	Analog or digital Inverse Chebyshev filter
ellip ellipord	Analog or digital Elliptic filter