1. Solution:

According to the definition of expectation

$$E(X)=\int_0^{+\infty} xf(x)\mathrm{d}x$$

where X is non-negative, and $f(x)$ is the density of X .

$E(X)=\int_0^{+\infty} xf(x)\mathrm{d}x$

$\geq \int_a^{+\infty} af(x)\mathrm{d}x$ if $X \geq a$

$=a\int_a^{+\infty} f(x)\mathrm{d}x$

$=aP(X\geq a)$

$\Rightarrow P(X\geq a)\leq \dfrac{E(X)}{a}$.

2. Solution:

(a)

As we know:

$$\phi_X(s)=E\left(e^{sX}\right)=\sum_{k\geq 0}\frac{s^k}{k!}E\left(X^k\right)$$

Differentiating $\phi_X(s)$ with respect to s will result in

$$\phi_X'(s)=0+E(X)+sE\left(X^2\right)+\cdots$$

So: $\phi_X'(0)=E(X)$.

(b)

Similarly as question (a), we can get:

$$\frac{\mathrm{d}^k\phi_X(s)}{\mathrm{d}s^k}=\frac{k!}{k!}E\left(X^k\right)+s\frac{(k+1)!}{(k+1)!}E\left(X^{k+1}\right)+\cdots$$

$$\Rightarrow \left.\frac{\mathrm{d}^k\phi_X(s)}{\mathrm{d}s^k}\right|_{s=0}=E\left(X^k\right)$$

(c) First method

$$\phi_X(s) = E\left(e^{sX}\right)$$

$$= \sum_{X=0}^{n} \binom{n}{X} p^X (1-p)^{n-X} e^{sX} \quad \text{(The definition of expectation)}$$

$$= \sum_{X=0}^{n} \binom{n}{X} \left(pe^s\right)^X (1-p)^{n-X}$$

$$= \left(pe^s + (1-p)\right)^n$$

(c) Second method

If we note the fact that Binomial is the sum of Bernoulli trials, i.e.

$$X = \sum_{i=1}^{n} X_i\,,\ X_i = \begin{cases} 1, & \text{with prob. } p \\ 0, & \text{with prob. } 1-p \end{cases} \quad i = 1, \ldots, n$$

where X is Binomial and X_i is Bernoulli, then:

$$E\left(e^{sX}\right) = E\left(e^{s\sum_{i=1}^{n} X_i}\right)$$

$$= \left(E\left(e^{sX_1}\right)\right)^n \quad (X_i, i = 1, \ldots, n \text{ are independent and identically distributed})$$

$$= \left(pe^s + (1-p)\right)^n$$

For exponential distribution

$$E\left(e^{sX}\right) = \int_0^\infty \lambda e^{-\lambda u} e^{su} \mathrm{d}u = \lambda \frac{1}{s-\lambda} e^{(s-\lambda)u}\Big|_0^\infty = \frac{\lambda}{\lambda - s}$$

3. Solution:

(a)

$$P\left(\left|X-E(X)\right|\geq a\right)=P\left(\left|X-E(X)\right|^2\geq a^2\right)$$

Since $\left|X-E(X)\right|^2$ is non-negative, we can apply Markov inequality

$$P\left(\left|X-E(X)\right|^2\geq a^2\right)\leq\frac{E\left(\left|X-E(X)\right|^2\right)}{a^2}=\frac{\mathrm{Var}(X)}{a^2}$$

(b)

For any $t>0$

$$P(X\geq a)=P\left(e^{tX}\geq e^{ta}\right)$$

$$\leq\frac{E\left[e^{tX}\right]}{e^{ta}}\text{ (Markov inequality)}$$

Particular, $P(X\geq a)\leq\min\limits_{t>0}\frac{E\left[e^{tX}\right]}{e^{ta}}$

4. Solution:

(a)

Since $X_i, i=1,\ldots,n$ are independent Bernoulli variables, $X=\sum_{i=1}^{n}X_i$ is Binomial with

$$X\sim B\left(n,\frac{1}{2}\right)$$

$$E(X)=E\left(\sum_{i=1}^{n}X_i\right)=nE(X_1)=np\text{ (p=1/2)}$$

$$\mathrm{Var}(X)=\mathrm{Var}\left(\sum_{i=1}^{n}X_i\right)$$

$$=n\mathrm{Var}(X_1)\text{ (Independence)}$$

$$=np(1-p)\text{ (p=1/2)}$$

(b)

i. If we note that

$$\left\{\left|X-\frac{n}{2}\right|\geq\frac{n}{4}\right\}=\left\{X\geq\frac{3n}{4}\right\}\cup\left\{X\leq\frac{n}{4}\right\}$$

then

$$P\left(\left|X-\frac{n}{2}\right|\geq\frac{n}{4}\right)\geq P\left(X\geq\frac{3n}{4}\right)$$

ii.

$$P\left(X\geq\frac{3n}{4}\right)\leq P\left(\left|X-\frac{n}{2}\right|\geq\frac{n}{4}\right)\leq\frac{\operatorname{var}(X)}{(n/4)^2}=\frac{4}{n}$$

iii. $\lim_{n\to\infty}P\left(X\geq\frac{3n}{4}\right)=0$

(c)

i. $P(X\geq x)=P\left(e^{\theta X}\geq e^{\theta x}\right)\leq\frac{E\left(e^{\theta X}\right)}{e^{\theta x}}$ (Markov's inequality)

where $E\left(e^{\theta X}\right)=\left(\frac{1}{2}+\frac{1}{2}e^{\theta}\right)^n$ (see solutions of **2.(c)**)

$$=\left(1+\frac{1}{2}\left(e^{\theta}-1\right)\right)^n\leq\exp\left(\frac{1}{2}\left(e^{\theta}-1\right)\right)^n$$ (hint: $1+\alpha\leq e^{\alpha}$, where $\alpha=\frac{1}{2}\left(e^{\theta}-1\right)$)

$$=\exp\left(\frac{n}{2}\left(e^{\theta}-1\right)\right)$$

Therefore $P(X\geq x)=\exp\left(\frac{n}{2}\left(e^{\theta}-1\right)-\theta x\right)$ (*)

ii.set $\theta=\log 2$, then $\frac{1}{2}\left(e^{\theta}-1\right)-\frac{3}{4}\theta\approx-0.0199\leq-0.01$ (**)

iii. let $x=\frac{3}{4}n$ and using (*), we have

$$P\left(X \geq \frac{3}{4}n\right) = \exp\left(\frac{n}{2}\left(e^{\theta}-1\right)-\theta\frac{3}{4}n\right) \leq \exp(-0.01n)$$ (by using (**))

5. Solution:

We denote

$$I_k = \begin{cases} 1, & \text{the k-th post is rotten} \\ 0, & \text{otherwise} \end{cases}$$

$k+1, k+2, \ldots, k+7$ are 7 consecutive posts (if $k+i > 17$, then $k+i \Leftarrow k+i-17$)

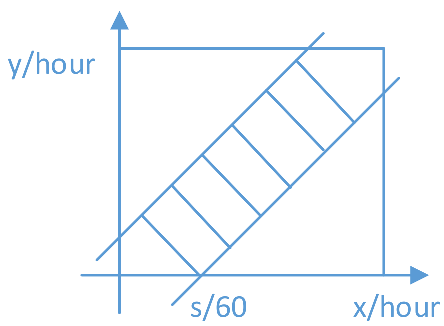
$R_k = \sum_{i=1}^{7} I_{k+i}$ is the number of rotten posts.

So, $E(R_k) = E\left(\sum_{i=1}^{7} I_{k+i}\right) = \sum_{i=1}^{7} E(I_{k+i}) = 7E(I_k) = \frac{35}{17} > 2$

where $E(I_k) = \frac{5}{17}$ (5 of 17 posts are rotten)

Since R_k should be integer, it must be the case that $P(R_k \geq 3) > 0$, for some k, i.e. there necessarily exists a run of 7 consecutive posts at least 3 of which are rotten.

6. Solution:



See the unit square in the figure, the origin is 12:00, x axis (unit: hour) represents the time Alice arrive and y axis represents the time Bob arrives.

So, “each is prepared to wait s/60 hours before leaving and they meet each other” can be mathematically described as

$\left\{|x-y|\le\frac{s}{60}\right\}$ (see the shadow area in the figure)

$$P\left(|x-y|\le\frac{s}{60}\right)=\iint_{|x-y|\le\frac{s}{60}} f(x,y)\mathrm{d}x\mathrm{d}y=1-\left(1-\frac{s}{60}\right)^2$$

where (x,y) is a two-dimensional variable stands for the time Alice and Bob arrive, and its density function is $f(x,y)=\begin{cases}1, & 0\le x\le 1, 0\le y\le 1\\ 0, & \text{otherwise}\end{cases}$

Finally, $1-\left(1-\frac{s}{60}\right)^2\ge\frac{1}{2}\Rightarrow s\ge 18\,\text{min}$

7. Solution:

See

https://en.wikipedia.org/wiki/Bertrand_paradox_(probability)

http://www.cut-the-knot.org/bertrand.shtml

http://web.mit.edu/tee/www/bertrand/problem.html

for more details.