1. Solution:

The distribution of Y is

$$F_Y(y) = P(Y \le y) = P\left(X^2 \le y\right)$$

$$= P\left(-\sqrt{y} \le x \le \sqrt{y}\right) = F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)$$

Since density is the differentiation of distribution, we have

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}\left(f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right), & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-y/2}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

2. Solution:

According to the definition:

$$\left|J(y_1, y_2)\right| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{y_2}{1+y_2} & \frac{1}{1+y_2} \\ \frac{y_1}{y_2+1} - \frac{y_1 y_2}{(y_2+1)^2} & \frac{-y_1}{(y_2+1)^2} \end{vmatrix} = -\frac{y_1}{(y_2+1)^2}$$

then the joint distribution of (y_1, y_2) can be calculated as

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\left(x_1(y_1, y_2), x_2(y_1, y_2)\right)\left|J(y_1, y_2)\right|$$

$$= \lambda^2 e^{-\lambda y_1} \frac{y_1}{\left(1+y^2\right)^2} \quad \text{if} \quad y_1, y_2 \ge 0$$

Note that y_1, y_2 are separated variables, so they are independent and we have

$$f_{Y_1}(y_1) = \lambda^2 e^{-\lambda y_1} y_1, \quad y_1 \ge 0$$

$$f_{Y_2}(y_2) = \frac{1}{\left(1+y_2\right)^2}, \quad y_2 \ge 0$$

3. Solution:

(a)

It is not difficult to obtain the joint law of (U, Y)

U \ Y	1	2	3	4	5
1	*p*=1/25	*p*=1/25	*p*=1/25	*p*=1/25	*p*=1/25
2	0	*p*=2/25	*p*=1/25	*p*=1/25	*p*=1/25
3	0	0	*p*=3/25	*p*=1/25	*p*=1/25
4	0	0	0	*p*=4/25	*p*=1/25
5	0	0	0	0	*p*=5/25

From the table, we can see that: $p = \begin{cases} \frac{n}{25}, & U = Y, \ n = 1, \cdots, 5 \\ \frac{1}{25}, & U < Y \end{cases}$

(b)

$$E(U|Y = n) = \sum_k kP(U = k|Y = n)$$

So we need to calculate $P(U = k|Y = n) = \frac{P(U = k, Y = n)}{P(Y = n)}$

where $P(Y = n) = \sum_{k=1}^{n} P(U = k, Y = n)$

$= (n-1)\frac{1}{25} + \frac{n}{25}$ (from the table)

$= \frac{2n-1}{25}$

Then, $P(U = k|Y = n) = \begin{cases} \frac{1}{2n-1}, & 1 \le k \le n-1 \\ \frac{n}{2n-1}, & k = n \end{cases}$

Finally, $E(U|Y = n) = \sum_{k=1}^{n-1} k\frac{1}{2n-1} + n\frac{n}{2n-1} = \frac{n(3n-1)}{2(2n-1)}$

(c)

$E(U|Y) = \frac{Y(3Y-1)}{2(2Y-1)}$ (just replace n by Y)

(d)

Similarly, $E\left(Y \middle| U=n\right)=\sum_{k} kP\left(Y=k \middle| U=n\right)$ and we need to calculate the conditional probability by calculating the marginal and joint distribution.
As for the marginal distribution:

$P\left(U=n\right)=\frac{1}{5}$ (sum each row of the table)

then, $P\left(Y=k \middle| U=n\right)=\frac{P\left(Y=k,U=n\right)}{P\left(U=n\right)}=\begin{cases}1/5, & (n+1)\le k\le 5\\ n/5, & k=n\end{cases}$

Finally,

$$E\left(Y \middle| U=n\right)=\sum_{k} kP\left(Y=k \middle| U=n\right)$$

$$=n\frac{n}{5}+\left(\left(n+1\right)+\cdots+5\right)\frac{1}{5}$$

$$=\frac{n^{2}-n+30}{10}$$

(where $\left(n+1\right)+\cdots+5=\frac{\left(n+6\right)\left(5-\left(n+1\right)+1\right)}{2}=\frac{\left(n+6\right)\left(5-n\right)}{2}$)

and $E\left(Y \middle| U\right)=\frac{U^{2}-U+30}{10}$.

(e)

Joint low of $\left(U,X\right)$

U \ X	1	2	3	4	5
1	p=5/25	0	0	0	0
2	p=1/25	p=4/25	0	0	0
3	p=1/25	p=1/25	p=3/25	0	0
4	p=1/25	p=1/25	p=1/25	p=2/25	0
5	p=1/25	p=1/25	p=1/25	p=1/25	p=1/25

$$E\left(U \middle| X=n\right)=\sum_{n}^{5} kP\left(Y=k \middle| U=n\right)$$

$P\left(X=n\right)=\frac{11-2n}{5}$ (sum each column of the table)

$$P\left(Y=k \middle| U=n\right)=\frac{P\left(Y=k,U=n\right)}{P\left(U=n\right)}=\begin{cases}\dfrac{1}{11-2n}, & (n+1)\le k\le 5\\ \dfrac{6-n}{11-2n}, & k=n\end{cases}$$

Finally, $E\left(U \middle| X=n\right)=\sum_{n}^{5} kP\left(Y=k \middle| U=n\right)$

$$=n\frac{6-n}{11-2n}+\left((n+1)+\cdots+5\right)\frac{1}{11-2n}$$

$$=\frac{30+11n-3n^2}{22-4n}$$

$$E\left(U \middle| X\right)=\frac{30+11X-3X^2}{22-4X}$$

$E\left(X+Y \middle| U\right)=E\left(U+V \middle| U\right)$ (it is easy to verify $X+Y=U+V$)

$$=E\left(U \middle| U\right)+E\left(V \middle| U\right)$$

where $E\left(U \middle| U\right)=U$ and $E\left(V \middle| U\right)=E\left(V\right)$.

So, $E\left(X+Y \middle| U\right)=E\left(X \middle| U\right)+E\left(Y \middle| U\right)=U+E\left(V\right)$

$$\Rightarrow \quad E\left(X \middle| U\right)=U+E\left(V\right)-E\left(Y \middle| U\right)$$

where $E\left(V\right)=3$ and $E\left(Y \middle| U\right)$ is known as above

Finally, $E\left(X \middle| U\right)=\dfrac{11U-U^2}{10}$.