Discriminant Analysis

Fisherfaces

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https://labicvl.github.io/

Some references

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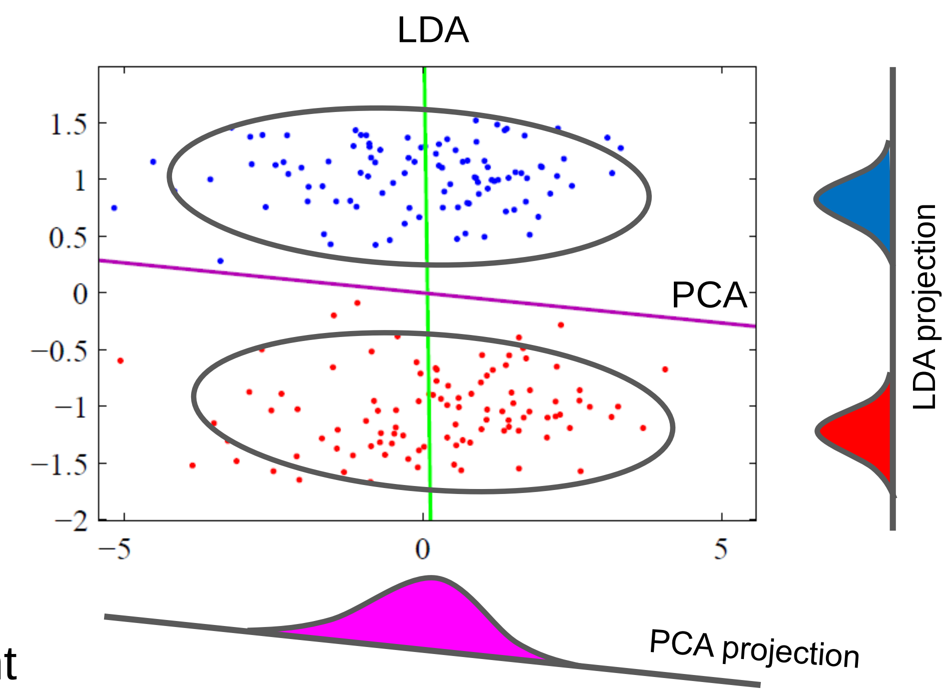
Motivation

Projection that best separates the data in a least-squares sense:

- PCA finds components that are useful for representing data.
- Pooling (or projecting) data may discard essential information for discriminating between data in different classes.
- PCA finds the direction for maximum data variance (unsupervised/generative).
- LDA (Linear Discriminant Analysis) or MDA (Multiple Discriminant Analysis) finds the direction that optimally separates data of different classes (supervised/discriminative).



Fisher Linear Discriminant (FLD)

- We first consider ***2-class problem*** i.e. binary-classification.
- Data are projected from *D* dimensions onto a line, i.e. one-dimensional subspace.

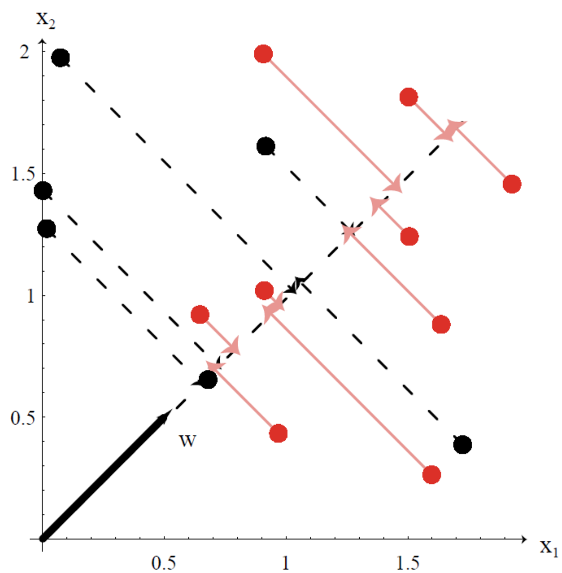
- Given a set of N *D*-dimensional samples $\mathbf{x}_1, \dots, \mathbf{x}_N$,
  where N_1 samples belong to the class c_1 and N_2 to the class c_2.

- We wish to form a linear combination of the components of $\mathbf{x}$ as
  $y = \mathbf{w}^T \mathbf{x}$
  and a corresponding set of N samples $y_1, \dots, y_N$.

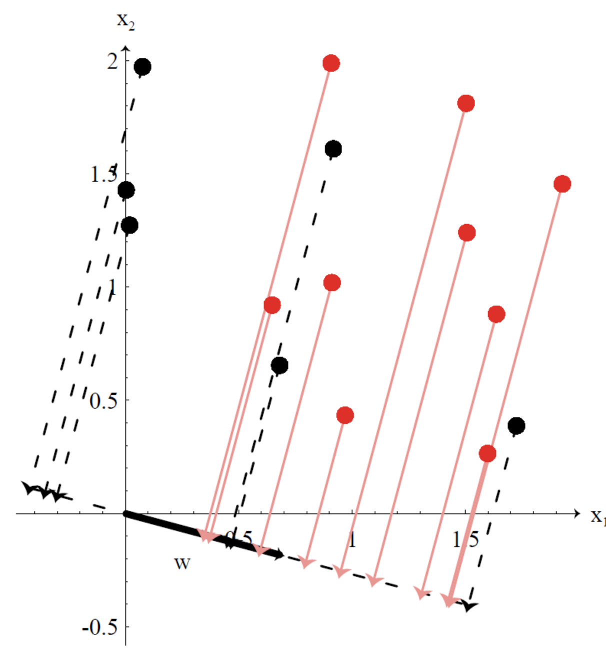
FLD: two-dimensional example

– Projection of same set of two-class samples onto two different lines in the direction marked **w**.

Classes mixed



Better separation



https://goo.gl/m6Mnov for questions

Finding best direction **w**

- ***Class mean*** in D-dimensional space:

$$\mathbf{m}_i = \frac{1}{N_i}\sum_{\mathbf{x}\in c_i}\mathbf{x}$$

- Class mean of projected points:

$$\widetilde{\mathbf{m}}_i = \frac{1}{N_i}\sum\nolimits_{y\in c_i} y = \frac{1}{N_i}\sum\nolimits_{\mathbf{x}\in c_i}\mathbf{w}^T\mathbf{x} = \mathbf{w}^T\mathbf{m}_i$$

- Distance between projected class means is

$$|\widetilde{\mathbf{m}}_1 - \widetilde{\mathbf{m}}_2| = |\mathbf{w}^T(\mathbf{m}_1 - \mathbf{m}_2)|$$

Criterion for Fisher Linear Discriminant

– Define the scatter of the projected samples as

$$\tilde{s}_i^2 = \sum_{y \in c_i} (\mathbf{y} - \tilde{\mathbf{m}}_i)^2$$

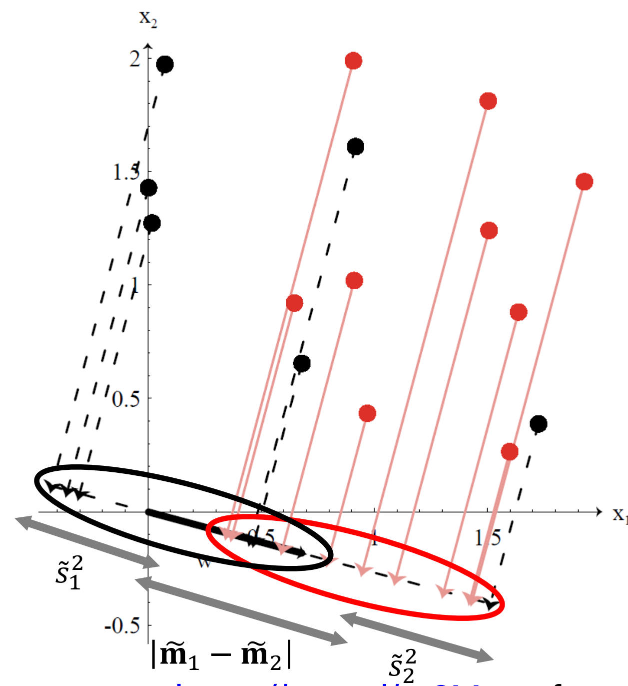
– Thus $(1/N)(\tilde{s}_1^2 + \tilde{s}_2^2)$ is the variance of the pooled (or projected) data.

– **Total within-class scatter** is $\tilde{s}_1^2 + \tilde{s}_2^2$.

– Find that linear function $\mathbf{w}^T\mathbf{x}$ for which

$$J(\mathbf{w}) = \frac{|\tilde{\mathbf{m}}_1 - \tilde{\mathbf{m}}_2|^2}{\tilde{s}_1^2 + \tilde{s}_2^2}$$

is maximum and independent of ||**w**||.



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Scatter matrices

– To obtain *J(·)* as an explicit function of **w**, we define scatter matrices $\mathbf{S}_i$ and $\mathbf{S}_W$

$$\mathbf{S}_i = \sum_{\mathbf{x}\in c_i} (\mathbf{x} - \mathbf{m}_i)(\mathbf{x} - \mathbf{m}_i)^T$$

And Within-class scatter matrix $\mathbf{S}_W = \mathbf{S}_1 + \mathbf{S}_2$.

– We can then write

$$\tilde{s}_i^2 = \sum_{\mathbf{x}\in c_i} (\mathbf{w}^T\mathbf{x} - \mathbf{w}^T\mathbf{m}_i)^2$$

$$= \sum_{\mathbf{x}\in c_i} \mathbf{w}^T(\mathbf{x} - \mathbf{m}_i)(\mathbf{x} - \mathbf{m}_i)^T\mathbf{w}$$

$$= \mathbf{w}^T\mathbf{S}_i\mathbf{w}$$

Therefore, $\tilde{s}_1^2 + \tilde{s}_2^2 = \mathbf{w}^T(\mathbf{S}_1 + \mathbf{S}_2)\mathbf{w} = \mathbf{w}^T\mathbf{S}_W\mathbf{w}$

– The within-class scatter matrix $\mathbf{S}_W \in \mathrm{R}^{DxD}$ is symmetric and positive semidefinite, and is nonsingular if N>D.

Scatter matrices

– Similarly, the separation of the projected class means is

$$|\tilde{\mathbf{m}}_1 - \tilde{\mathbf{m}}_2|^2 = (\mathbf{w}^T\mathbf{m}_\mathbf{1} - \mathbf{w}^T\mathbf{m}_\mathbf{2})^2$$
$$= \mathbf{w}^T(\mathbf{m}_\mathbf{1} - \mathbf{m}_\mathbf{2})(\mathbf{m}_\mathbf{1} - \mathbf{m}_\mathbf{2})^T\mathbf{w}$$
$$=\mathbf{w}^T\mathbf{S}_B\mathbf{w}$$

Where Between-class scatter matrix $\mathbf{S}_B = (\mathbf{m}_\mathbf{1} - \mathbf{m}_\mathbf{2})(\mathbf{m}_\mathbf{1} - \mathbf{m}_\mathbf{2})^T$.

– The between-class scatter matrix $\mathbf{S}_B$ is also symmetric and positive semidefinite.
– Its rank is at most one, since it is the outer product of two vectors.

Criterion function in terms of scatter matrices and optimisation

- The criterion function is written as

$$J(\mathbf{w}) = \frac{\mathbf{w}^T \mathbf{S}_B \mathbf{w}}{\mathbf{w}^T \mathbf{S}_W \mathbf{w}}$$

- This is well known the generalised Rayleigh quotient.

- Maximizing the ratio is equivalent to maximizing the numerator while keeping the denominator constant, i.e.

$$\max_{\mathbf{w}} \mathbf{w}^T \mathbf{S}_B \mathbf{w} \quad \text{subject to} \quad \mathbf{w}^T \mathbf{S}_W \mathbf{w} = \mathrm{k}$$

- This can be accomplished using Lagrange multipliers as

$$L = \mathbf{w}^T \mathbf{S}_B \mathbf{w} + \lambda(\mathrm{k} - \mathbf{w}^T \mathbf{S}_W \mathbf{w})$$

maximize *L* with respect to both **w** and λ.

Optimisation for Fisher Discriminant

– Setting the gradient of

$$L = \mathbf{w}^T(\mathbf{S}_B - \lambda\mathbf{S}_W)\mathbf{w} + \lambda \mathrm{k}$$

with respect to **w** to zero, we get

$$2(\mathbf{S}_B - \lambda\mathbf{S}_W)\mathbf{w} = 0$$

then

$$\mathbf{S}_B\mathbf{w} = \lambda\mathbf{S}_W\mathbf{w}$$

– This is a generalized eigenvalue problem.
– The solution is easy, when $\mathbf{S}_W$ is *nonsingular*:

$$\mathbf{S}_W^{-1}\mathbf{S}_B\mathbf{w}=\lambda\mathbf{w}$$

where $\mathbf{w}$ and λ are the eigenvector and eigenvalue of $\mathbf{S}_W^{-1}\mathbf{S}_B$.

Multiple Discriminant Analysis

– Generalization of Fisher's Linear Discriminant, for multiple *c* classes, involves M discriminant functions $\mathbf{w}_{\mathrm{i}}, \quad \mathrm{i} = 1, \ldots, \mathrm{M}$.

– Projection is from a D-dimensional space to a M-dimensional subspace.

– The Within-class and Between-class scatter matrices are defined as

$$\mathbf{S}_W = \sum_{i=1}^{c} \mathbf{S}_i$$

where $\mathbf{S}_i = \sum_{\mathbf{x} \in c_i} (\mathbf{x} - \mathbf{m}_i)(\mathbf{x} - \mathbf{m}_i)^T$,

$$\mathbf{S}_B = \sum_{i=1}^{c} (\mathbf{m}_i - \mathbf{m})(\mathbf{m}_i - \mathbf{m})^T.$$

– The desired projections are found as generalised eigenvectors:

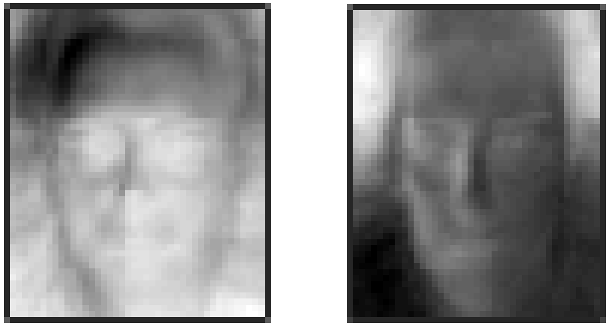
$$\mathbf{S}_B \mathbf{w}_i = \lambda_i \mathbf{S}_W \mathbf{w}_i, \quad i = 1, \ldots, \mathrm{M}$$

for eigenvalues λ_i.

– If $\mathbf{S}_W$ has full rank, the solutions are generalized eigenvectors of $\mathbf{S}_W^{-1}\mathbf{S}_B$ with largest M eigenvalues.

Fisherfaces

Eigenfaces



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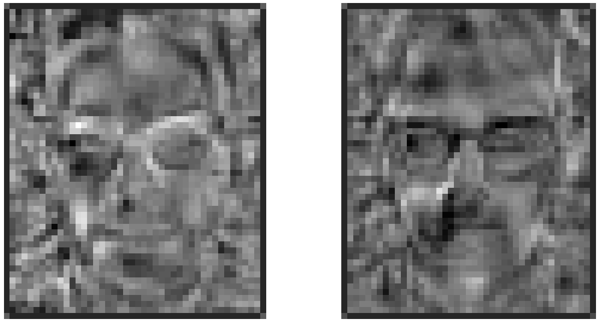
Fisherfaces



P. Belhumeur, J. Hespanha, D. Kriegman, Eigenfaces vs. Fisherfaces: recognition using class specific linear projection, TPAMI, 1997.

Recognition using class specific linear projection

- Let us consider N sample images $\{\mathbf{x}_n\}$, n = 1,...,N and $\mathbf{x}_n \in R^D$ in an D-dimensional image space, and assume that each image belongs to one of c classes $\{c_i\}$, i = 1,...,c.
- We consider a linear transformation mapping the D-dimensional image space into an M-dimensional feature space, where M < D.
- The feature vectors $\mathbf{y}_n \in R^M$ are defined by the following linear transformation:

$$\mathbf{y}_n = \mathbf{W}^T \mathbf{x}_n$$

where $\mathbf{W} \in R^{DxM}$ is a matrix with orthonormal columns.

Eigenfaces

- The total scatter matrix $\mathbf{S}_T$ (or the covariance matrix) is defined as

$$\mathbf{S}_T = \sum_n (\mathbf{x}_n - \mathbf{m})(\mathbf{x}_n - \mathbf{m})^T$$

where $\mathbf{m} \in R^D$ is the mean of all samples.

Recognition using class specific linear projection

– After applying the linear transformation $\mathbf{W}^{T}$, the scatter matrix of the feature vectors $\mathbf{y}_n \in R^M$, n = 1,...,N, is $\mathbf{W}^{T}\mathbf{S}_{T}\mathbf{W}$

– In PCA, the projection $\mathbf{W}_{opt}$ is chosen to maximize the determinant of the total scatter matrix of the projected samples, i.e.,

$$\mathbf{W}_{\mathrm{opt}} = \arg\max_{\mathbf{W}} |\mathbf{W}^{T}\mathbf{S}_{T}\mathbf{W}| = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{\mathrm{M}}]$$

where $\mathbf{w}_{\mathrm{i}}$, i = 1,...,M is the set of D-dimensional eigenvectors of $\mathbf{S}_{T}$ corresponding to the M largest eigenvalues.

– A drawback of this approach is that both the between-class and within-class scatter are maximized, since $\mathbf{S}_{T} = \mathbf{S}_{B} + \mathbf{S}_{W}$.

Recognition using class specific linear projection

Fisherfaces

- Since the learning set is class-labelled, we use this information to build a more discriminative method for reducing the feature space dimensionality.
- Using class specific linear methods for dimensionality reduction and NN classifiers in the reduced feature space, we may get better recognition rates than with the Eigenface method.
- FLD is a class specific method that selects **W** in such a way that the ratio of the between-class scatter and the within-class scatter is maximized.
- Let the between-class scatter matrix be defined as

$$\mathbf{S}_B = \sum_{i=1}^{c} (\mathbf{m}_i - \mathbf{m})(\mathbf{m}_i - \mathbf{m})^T ,$$

the within-class scatter matrix be defined as

$$\mathbf{S}_W = \sum_{i=1}^{c} \sum_{\mathbf{x} \in c_i} (\mathbf{x} - \mathbf{m}_i)(\mathbf{x} - \mathbf{m}_i)^T$$

where $\mathbf{m}_i$ is the mean image of class c_i , and N_i is the number of samples in class c_i.

Fisherfaces

– If $\boldsymbol{S}_W$ is nonsingular, the optimal projection $\mathbf{W}_{\text{opt}}$ is chosen as the matrix with orthonormal columns which maximizes the ratio of the determinant of the between-class scatter matrix to the determinant of the within-class scatter matrix of the projected samples, i.e.,

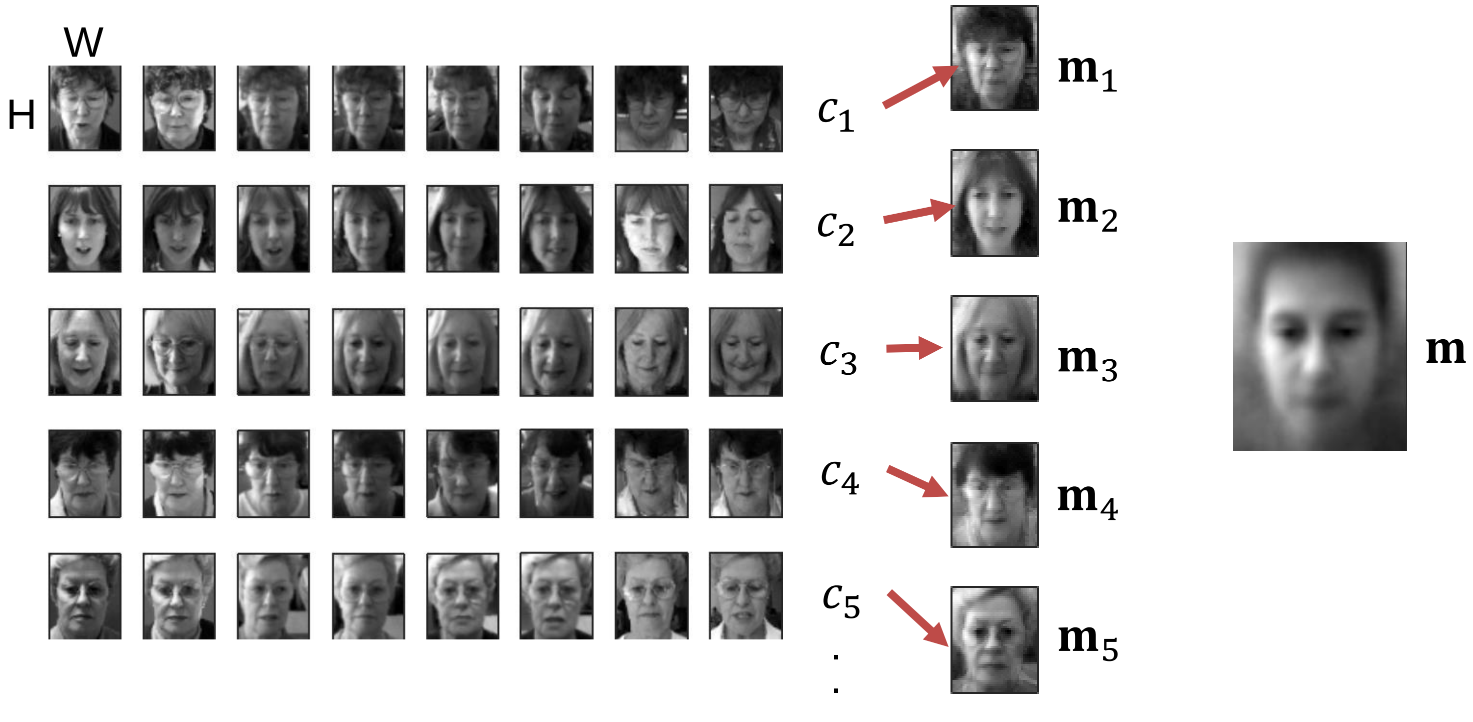
$$\mathbf{W}_{\text{opt}} = \arg\max_{\mathbf{W}} \frac{|\mathbf{W}^T \boldsymbol{S}_B \mathbf{W}|}{|\mathbf{W}^T \mathbf{S}_W \mathbf{W}|} = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{\mathrm{M}}]$$

where $\mathbf{w}_{\mathrm{i}}$ is the set of generalized eigenvectors of $\boldsymbol{S}_B$ and $\boldsymbol{S}_W$ corresponding to the M largest eigenvalues:

$$\mathbf{S}_B \mathbf{w}_i = \lambda_i \mathbf{S}_W \mathbf{w}_i, \quad \mathrm{i} = 1, \dots, \mathrm{M}$$

Procedures: Fisherfaces

- Collect training images $\mathbf{x}_n$ of c classes (c=26, N=208, D=2576)
- Compute the class means $\mathbf{m}_i$, i = 1,…,c, and the global mean $\mathbf{m}$

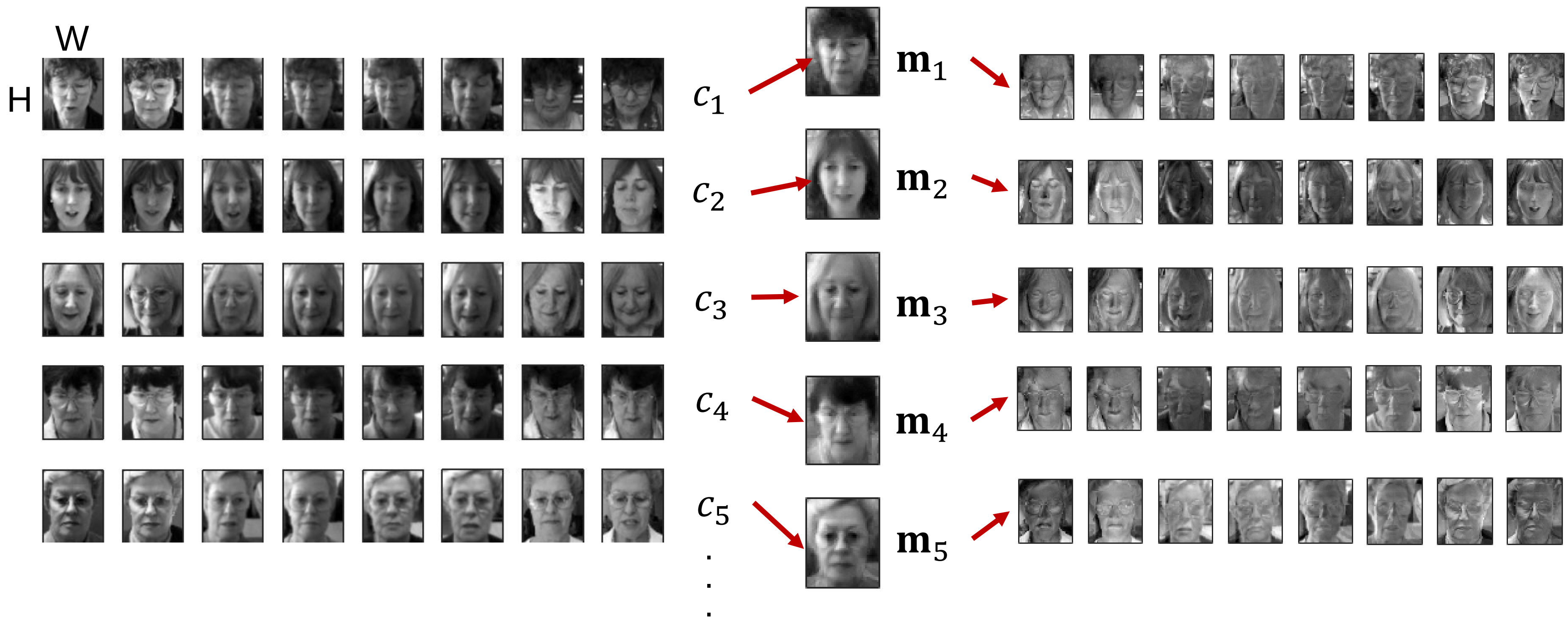


Procedures: Fisherfaces

– Compute $\mathbf{m}_i - \mathbf{m}$, and $\mathbf{S}_B$, where rank($\mathbf{S}_B$) = c -1.

Procedures: Fisherfaces

– Compute $\mathbf{x} - \mathbf{m}_i$, and $\mathbf{S}_W$, where rank($\mathbf{S}_W$) is N – c.



Fisherfaces

– Given the generalized eigenvalue/vector problem of $\boldsymbol{S}_B$ and $\boldsymbol{S}_W$:

$$\mathbf{S}_B\mathbf{w}_i=\lambda_i\mathbf{S}_W\mathbf{w}_i, \quad \mathrm{i} = 1, \dots, \mathrm{M}$$

– Note that there are at most c - 1 nonzero generalized eigenvalues i.e. the rank of $\boldsymbol{S}_B$, and so an upper bound on M is c - 1.

– The within-class scatter matrix $\boldsymbol{S}_W \in \mathrm{R}^{DxD}$ is often singular, since the rank of $\boldsymbol{S}_W$ is at most N - c, and, in general, N is smaller than D.

Fisherfaces

– In order to overcome the singular $\boldsymbol{S}_W$, we propose an alternative to the criterion.

– This method, which we call Fisherfaces, avoids the problem by projecting the image set to a lower dimensional space.

– We use PCA to reduce the dimension of the feature space M_{pca} (<=N–c), and then apply the standard FLD to reduce the dimension to M_{lda} (<=c–1).

– Formally, $\mathbf{W}_{opt}$ is given by

$$\mathbf{W}_{\mathrm{opt}}^T = \mathbf{W}_{\mathrm{lda}}^T \mathbf{W}_{\mathrm{pca}}^T$$

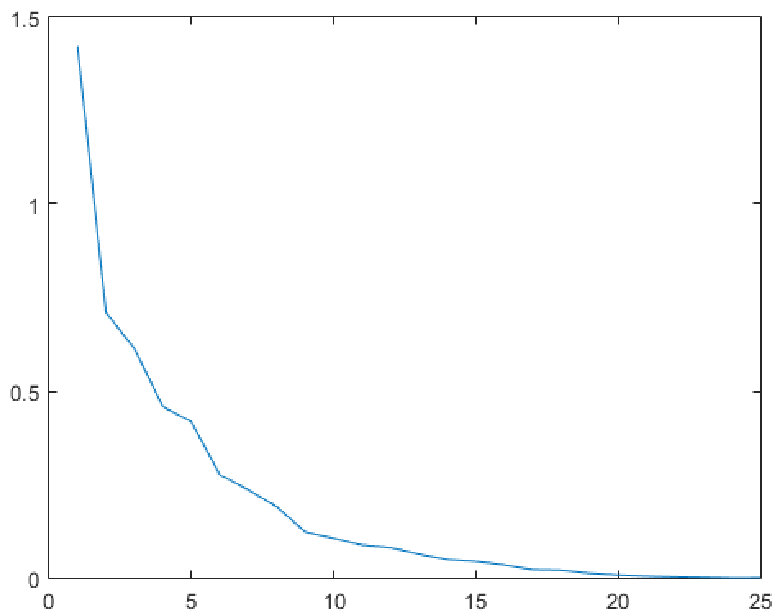
$$\mathbf{W}_{\mathrm{pca}} = \arg\max_{\mathbf{W}} |\mathbf{W}^T \mathbf{S}_T \mathbf{W}|$$

$$\mathbf{W}_{\mathrm{lda}} = \arg\max_{\mathbf{W}} \frac{\left|\mathbf{W}^T \mathbf{W}_{\mathrm{pca}}^T \mathbf{S}_B \mathbf{W}_{\mathrm{pca}} \mathbf{W}\right|}{\left|\mathbf{W}^T \mathbf{W}_{\mathrm{pca}}^T \mathbf{S}_W \mathbf{W}_{\mathrm{pca}} \mathbf{W}\right|}$$

– There are other ways of reducing the withinclass scatter while preserving between-class scatter e.g. Direct LDA, Null LDA, etc.

Procedures: Fisherfaces

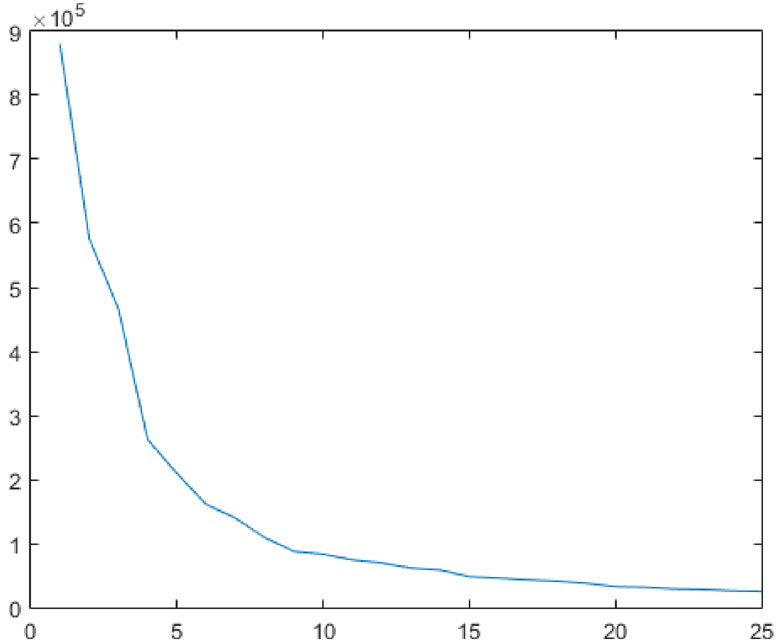
- rank(Sw) = 182 (=N – c), rank(Sb) = 25 (=c – 1)
- Perform PCA to get $\mathbf{W}_{\text{pca}}$ (Mpca=25), and compute $\mathbf{W}_{\text{pca}}^T\mathbf{S}_B\mathbf{W}_{\text{pca}}$ and $\mathbf{W}_{\text{pca}}^T\mathbf{S}_W\mathbf{W}_{\text{pca}}$.
- Get the generalized eigenvectors of $(\mathbf{W}_{\text{pca}}^T\mathbf{S}_W\mathbf{W}_{\text{pca}})^{-1}(\mathbf{W}_{\text{pca}}^T\mathbf{S}_B\mathbf{W}_{\text{pca}})$ with largest Mlda eigenvalues.



Generalized eigenvalues

Top 25 generalized eigenvectors

Comparison to Eigenfaces

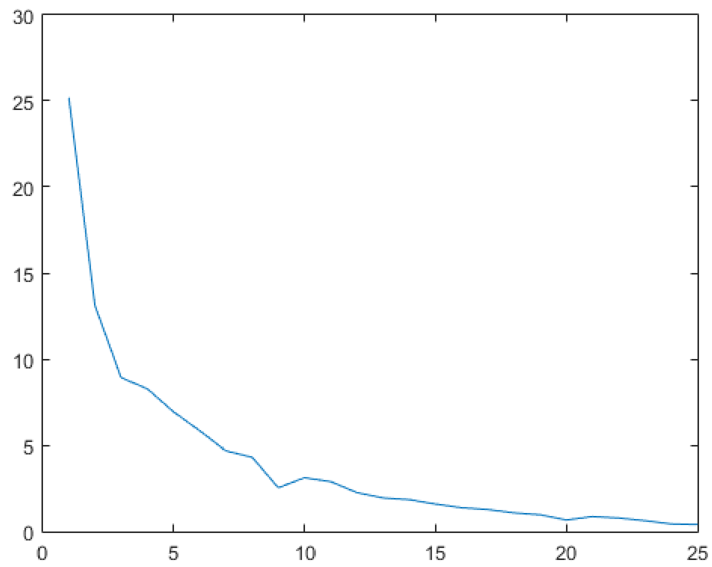


Eigenvalues

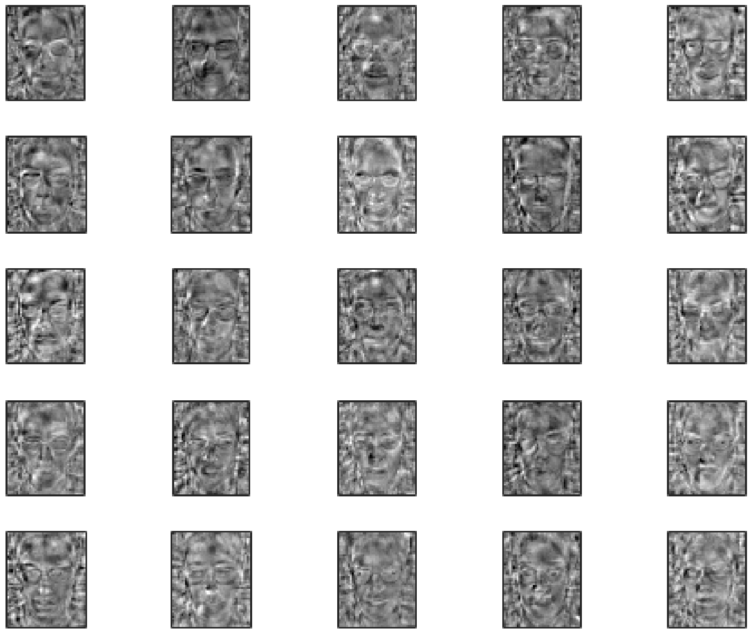
Top 25 eigenvectors

Procedures: Fisherfaces

- rank(Sw) = 182 (=N – c), rank(Sb) = 25 (=c – 1)
- Perform PCA to get $\mathbf{W}_{\text{pca}}$ (Mpca=150), and compute $\mathbf{W}_{\text{pca}}^T \mathbf{S}_B \mathbf{W}_{\text{pca}}$ and $\mathbf{W}_{\text{pca}}^T \mathbf{S}_W \mathbf{W}_{\text{pca}}$.
- Get the generalized eigenvectors of $(\mathbf{W}_{\text{pca}}^T \mathbf{S}_W \mathbf{W}_{\text{pca}})^{-1}(\mathbf{W}_{\text{pca}}^T \mathbf{S}_B \mathbf{W}_{\text{pca}})$ with largest Mlda eigenvalues.

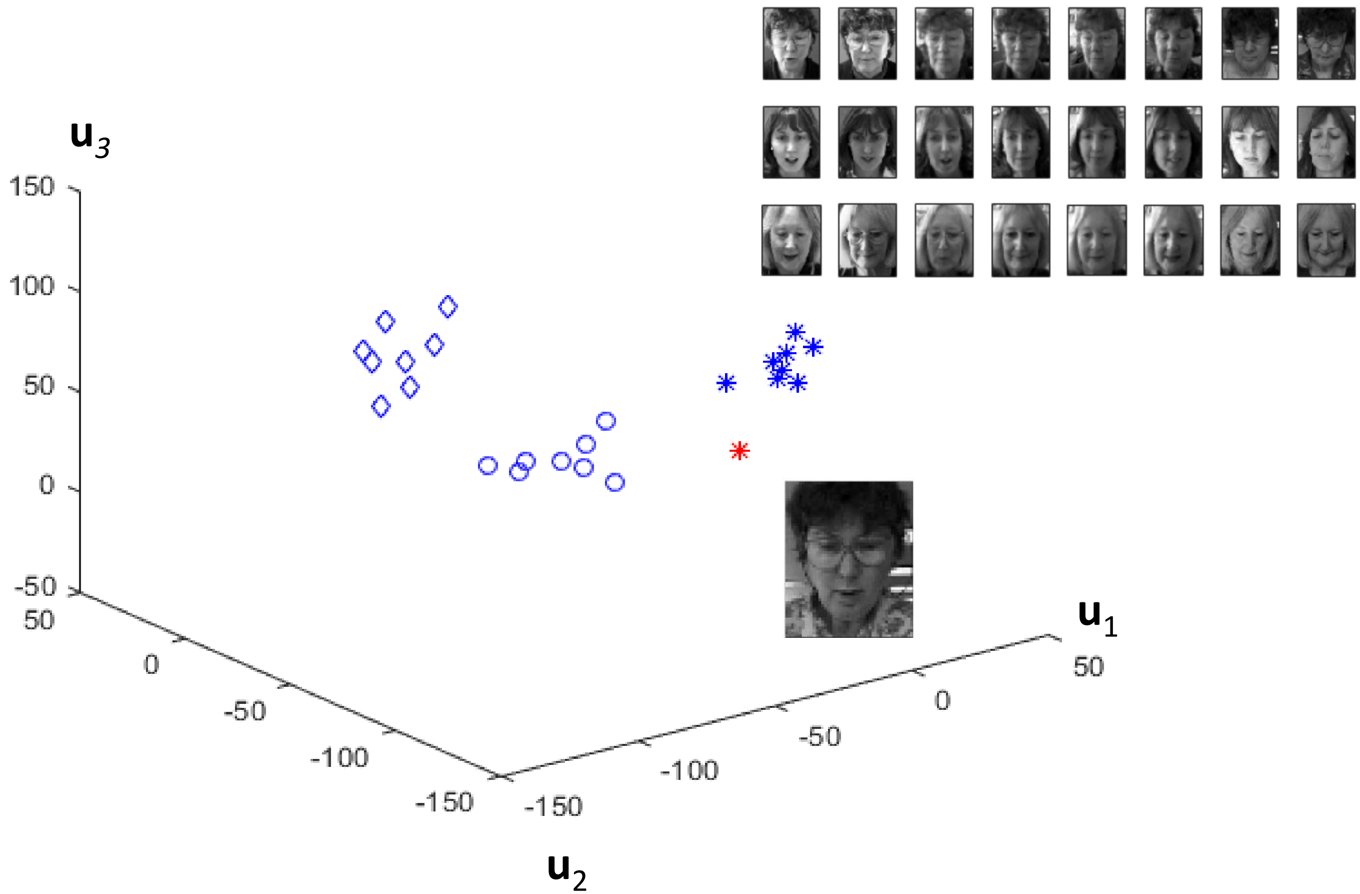


Generalized eigenvalues



Top 25 generalized eigenvectors

Procedures: Fisherfaces

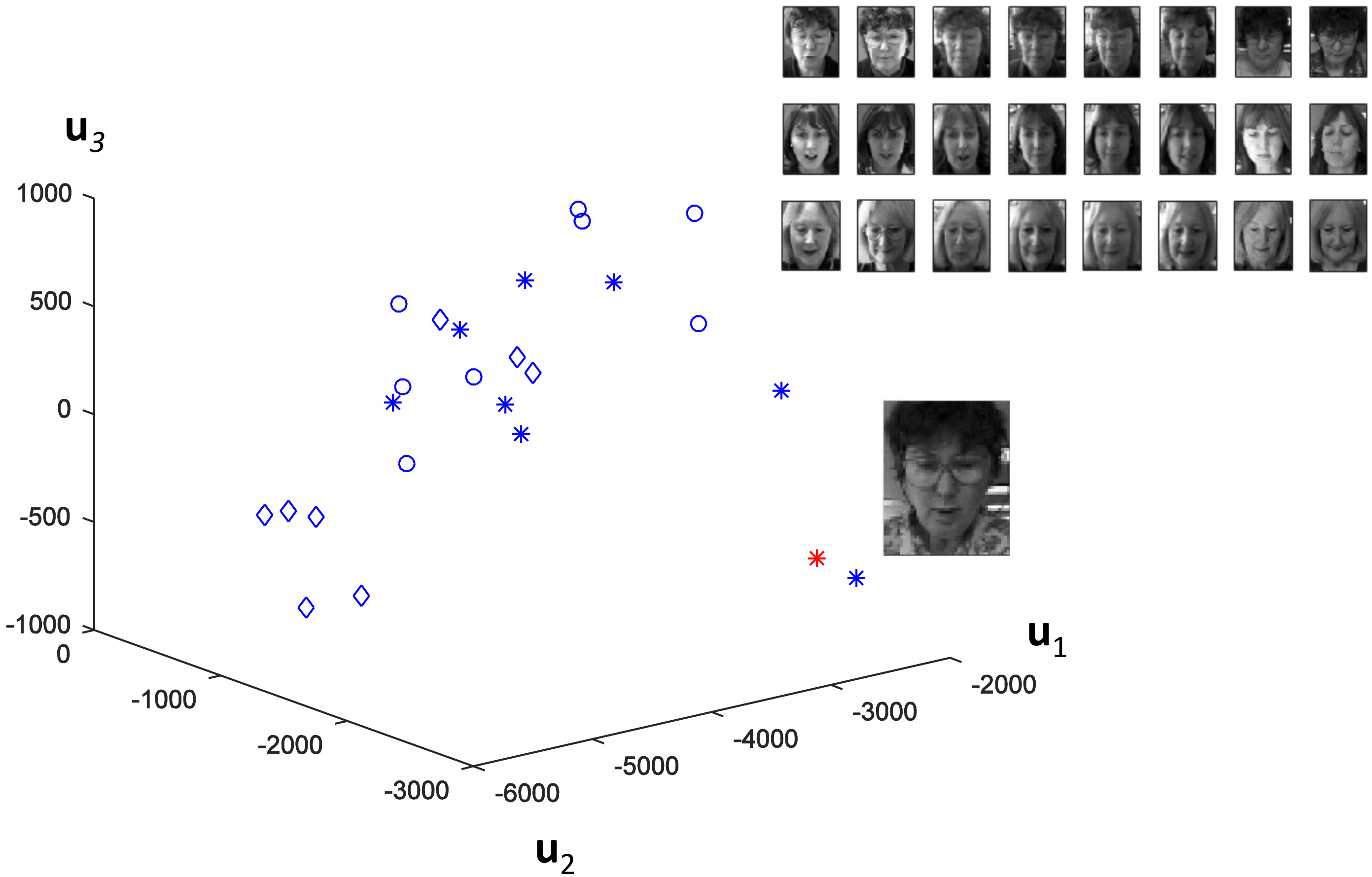


Face images in 3-dimensional fisher-subspace

24 training images of 3 different face classes (star, diamond, circle, "in blue") are projected. A query image projection is "in red".

https://goo.gl/m6Mnov for questions

Comparison to Eigenfaces



Face images in 3-dimensional eigen-subspace

24 training images of 3 different face classes (star, diamond, circle, “in blue”) are projected. A query image projection is “in red”.

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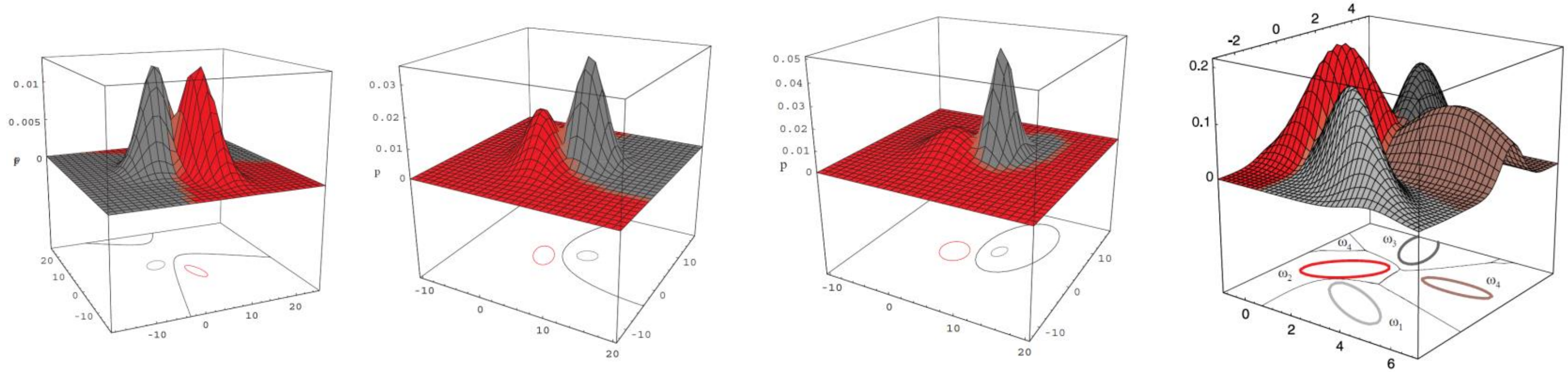
Relation to optimal bayesian decision theory

Bayes Decision Theory

- Fundamental statistical approach to pattern classification
- Quantifies trade-offs between classification using probabilities and costs of decisions
- Assumes all relevant probabilities are known

- Σ_i (data covariance matrix of class i) = arbitrary
  - Arbitrary Gaussian distributions lead to Bayes decision boundaries that are general hyperquadrics

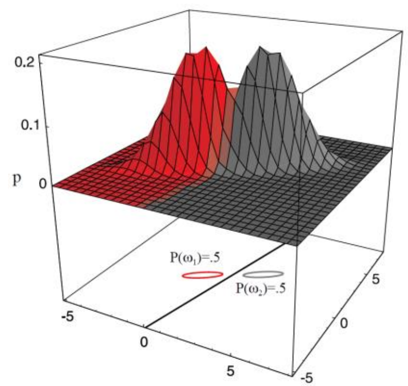


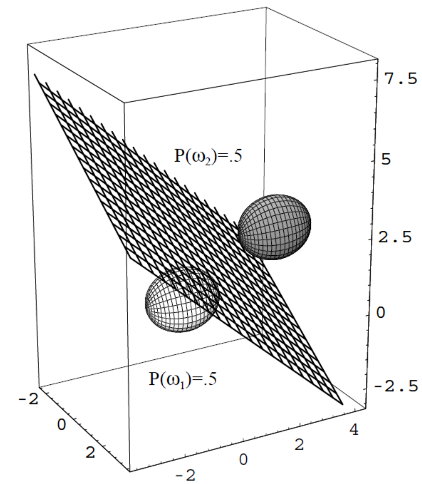
Relation to optimal bayesian decision theory

– $\Sigma_i = \Sigma$
  - For a classification problem with Gaussian classes of equal covariance $\Sigma_i = \Sigma$, the Bayes decision boundaries (or the discriminant function) is the plane of normal

$$\mathbf{w} = \Sigma^{-1}(\mathbf{m}_1 - \mathbf{m}_2)$$

  - The hyperplane is generally not orthogonal to the line between the means.





Relation to optimal bayesian decision theory

- If $\Sigma_1 = \Sigma_2$, this is also the FLD solution.
- In FLD, $\mathbf{S}_W = \mathbf{S}_1 + \mathbf{S}_2, \;\; \mathbf{w} = \mathbf{S}_W^{-1}(\mathbf{m}_1 - \mathbf{m}_2)$

- This gives some interpretations of FLD/LDA
  - It is optimal if and only if the classes are Gaussian and have equal covariance.
  - The extension from two-classes to multiple classes in LDA is ad-hoc.

T-K Kim, PhD dissertation: Discriminant Analysis of Patterns in Images, Image Ensembles, and Videos, Univ. of Cambridge, 2008.