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DSP Design of IIR Filters in Continuous Time

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Effect on poles and zeros on frequency response

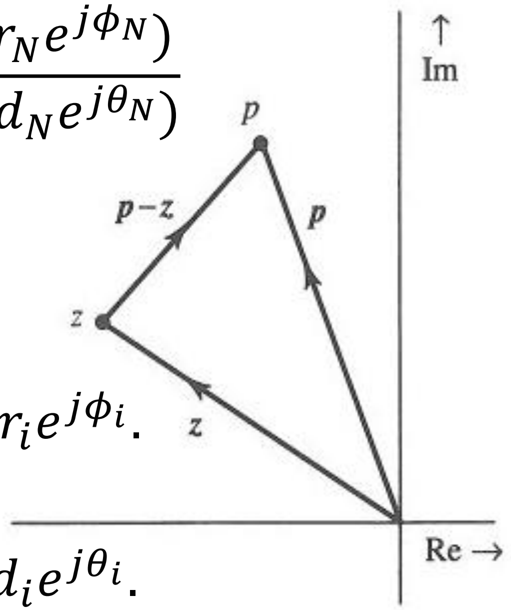
- Consider a generic system transfer function

$$H(s) = \frac{P(s)}{Q(s)} = b_0 \frac{(s - z_1)(s - z_2) \dots (s - z_N)}{(s - \lambda_1)(s - \lambda_2) \dots (s - \lambda_N)}$$

- The value of the transfer function at some complex frequency $s = p$ is:

$$H(p) = \frac{P(p)}{Q(p)} = b_0 \frac{(p - z_1)(p - z_2) \dots (p - z_N)}{(p - \lambda_1)(p - \lambda_2) \dots (p - \lambda_N)}$$

$$H(p) = \frac{P(p)}{Q(p)} = b_0 \frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2}) \dots (r_N e^{j\phi_N})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2}) \dots (d_N e^{j\theta_N})}$$



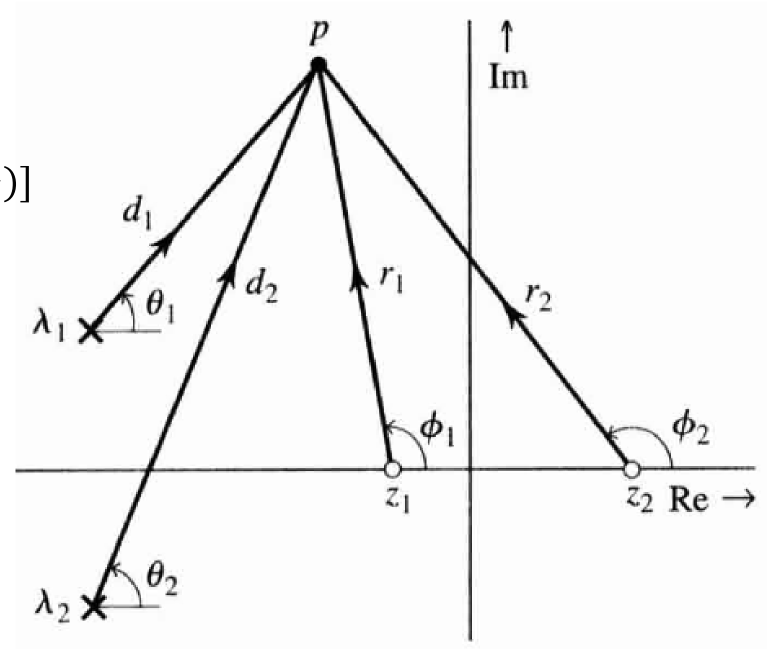
- The factor $p - z$ is a complex number.
  - It is represented by a vector drawn from point z to point p in the complex plane.
  - Using polar coordinates we can write $p - z_i = r_i e^{j\phi_i}$. with $r_i = |p - z_i|$ and $\phi_i = \angle(p - z_i)$
- Same comments are valid for the factor $p - \lambda_i = d_i e^{j\theta_i}$.
- Note that z_i and λ_i is a pole.

Effect on poles and zeros on frequency response cont.

- The previous form can be further modified as:

$$H(p) = b_0 \frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2}) \ldots (r_N e^{j\phi_N})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2}) \ldots (d_N e^{j\theta_N})}$$

$$= b_0 \frac{r_1 r_2 \ldots r_N}{d_1 d_2 \ldots d_N} e^{j[(\phi_1+\phi_2+\cdots+\phi_N)-(\theta_1+\theta_2+\cdots+\theta_N)]}$$



- Therefore, the magnitude and phase at $s = p$ are given by:

$$|H(s)|_{s=p} = b_0 \frac{r_1 r_2 \ldots r_N}{d_1 d_2 \ldots d_N}$$

$$= b_0 \frac{\text{product of the distances of zeros to } p}{\text{product of the distances of poles to } p}$$

$$\angle H(s)_{s=p} = (\phi_1 + \phi_2 + \cdots + \phi_N) - (\theta_1 + \theta_2 + \cdots + \theta_N)$$

$$= \text{sum of zeros}' \text{ angles to } p - \text{sum of poles}' \text{ angles to } p$$

- If b_0 is negative, there is an additional phase π since in that case $b_0 = -|b_0| = |b_0| e^{j\pi}$

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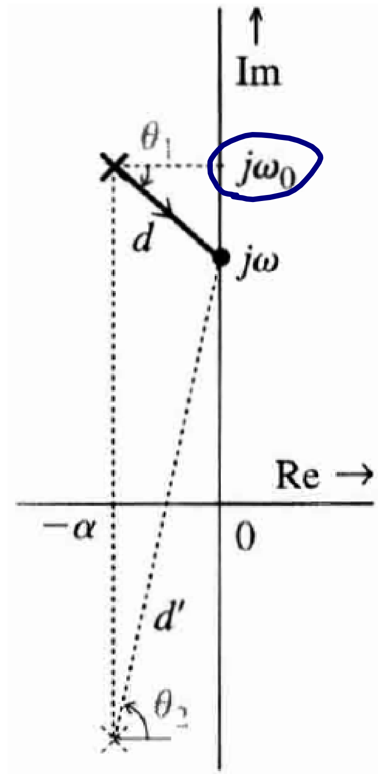
Gain enhancement by a single pole

- Consider the hypothetical case of a single pole at $-a + j\omega_0$. d
- The amplitude response at a specific value of ω, $|H(j\omega)|$, is found by measuring the length of the line that connects the pole to the point $j\omega$
- If the length of the above mentioned line is d, then $|H(j\omega)|$ is proportional to $\frac{1}{d}$.

$$|H(j\omega)| = \frac{K}{d}$$

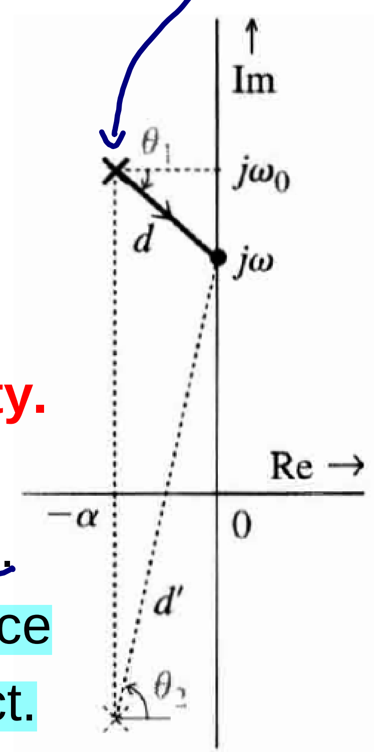


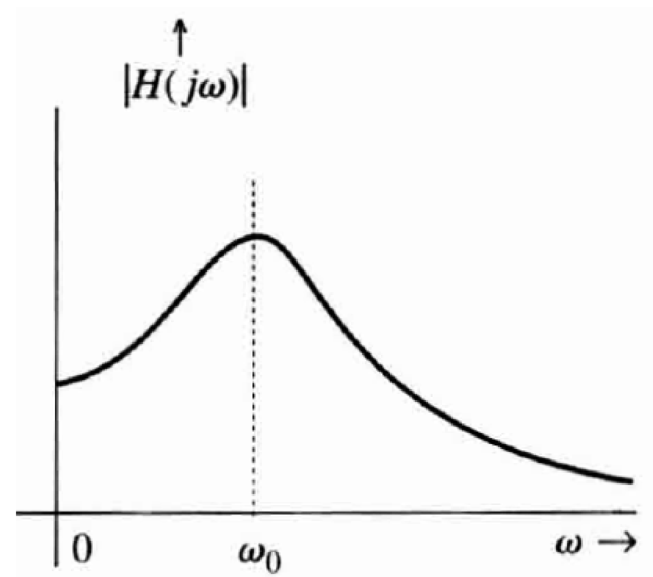
- As ω increases from zero, d decreases progressively until ω reaches the value ω_0.
- As ω increases beyond ω_0, d increases progressively.
- Therefore, the peak of $|H(j\omega)|$ occurs at ω_0.

  As a becomes smaller, i.e., as the pole moves closer to the imaginary axis the gain enhancement at ω_0 becomes more prominent (d becomes very small.)

Gain enhancement by a single pole cont.

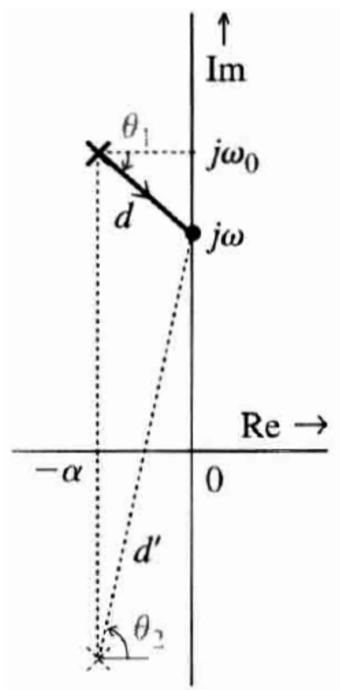
- In conclusion, we can enhance a gain at a frequency ω_0 by placing a pole opposite the point $j\omega_0$.
- The closer the pole is to $j\omega_0$, the higher is the gain at ω_0 and furthermore, the enhancement is more prominent around ω_0.
- **In the extreme case of $a = 0$ (pole on the imaginary axis) the gain at ω_0 goes to infinity.**
- Recall that poles must lie on the left half of the s –plane.
- Repeated poles further enhance the **frequency selective** effect.

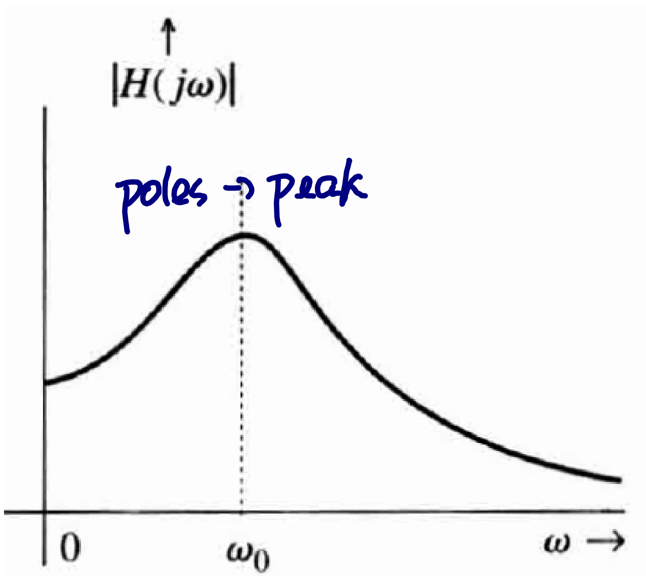




Gain enhancement by a pair of complex conjugate poles

- In a real system, a complex pole at $-a + j\omega_0$ must be accompanied by its conjugate pole $-a - j\omega_0$.
- The amplitude response at a specific value of ω, $|H(j\omega)|$, is found by measuring the length of the two lines that connect the poles to the point $j\omega$.
- If the lengths of the above mentioned lines are d, d' then $|H(j\omega)| = \frac{K}{dd'}$.
- We can see graphically that the presence of the conjugate pole does not affect substantially the behaviour of the system around ω_0. This is because as we move around ω_0, d' does not change dramatically.

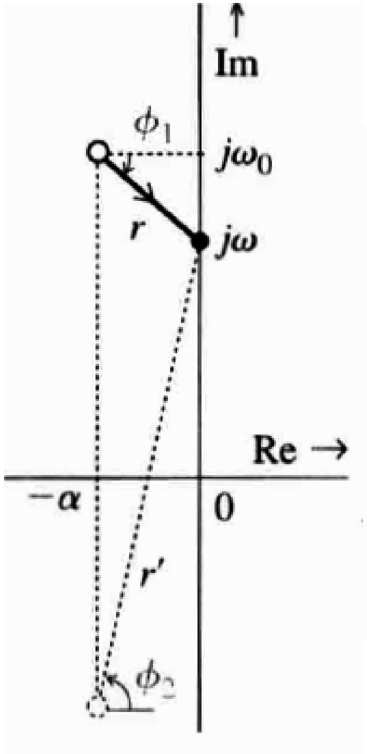


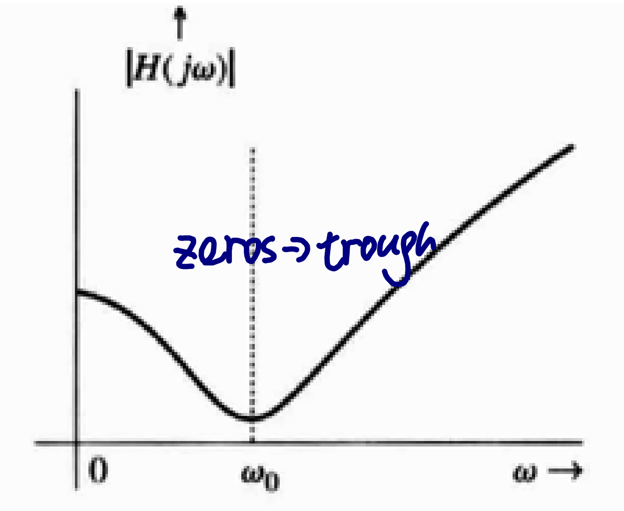


Gain suppression by a pair of complex conjugate zeros

- Consider a real system with a pair of complex conjugate zeros at $-a + j\omega_0$ and $-a - j\omega_0$.
- The amplitude response at a specific value of ω, $|H(j\omega)|$ is again found by measuring the length of the two lines that connect the zeros to the point $j\omega$.
- If the lengths of the above mentioned lines are r, r' then $|H(j\omega)| = Krr'$.
- In that case,
  the minimum of $|H(j\omega)|$
  occurs at ω_0.
- As a becomes smaller,
  i.e., as the zero moves closer
  to the imaginary axis,
  the gain suppression at ω_0
  becomes more prominent.

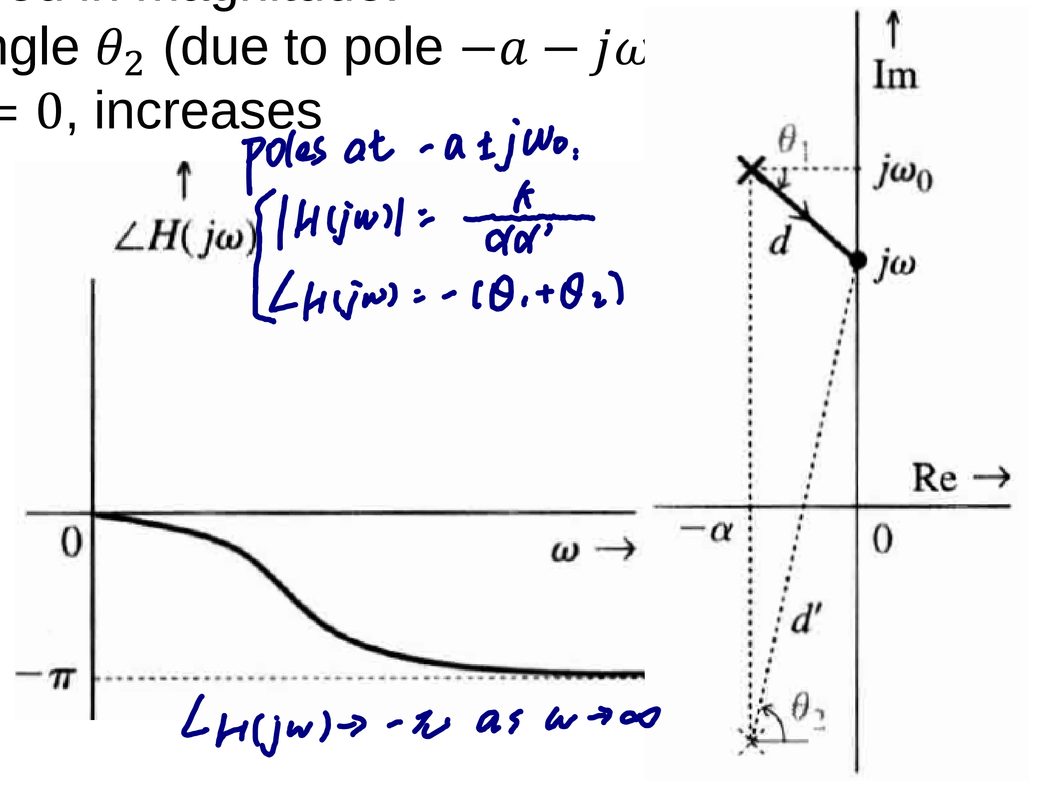
▪ **In the extreme case of $a = 0$ (zero on the imaginary axis) the gain at ω_0 goes to zero.**





Phase response due to a pair of complex conjugate poles

- Angles formed by the poles $-a + j\omega_0$ and $-a - j\omega_0$ at $\omega = 0$ are equal and opposite.
- Their contribution to the phase response is $\angle H(j\omega) = -(\theta_1 + \theta_2)$.
- As ω increases from 0 up, the angle θ_1 (due to pole $-a + j\omega_0$), which has a negative value at $\omega = 0$, is reduced in magnitude.
- As ω increases from 0 up, the angle θ_2 (due to pole $-a - j\omega$ which has a positive value at $\omega = 0$, increases in magnitude.
- As a result, both θ_1, θ_2, increase continuously and approaches a value of $\pi/2$ as $\omega \to \infty$.
- Therefore, $\theta_1 + \theta_2$, the sum of the two angles, increases continuously and approaches the value of π as $\omega \to \infty$.

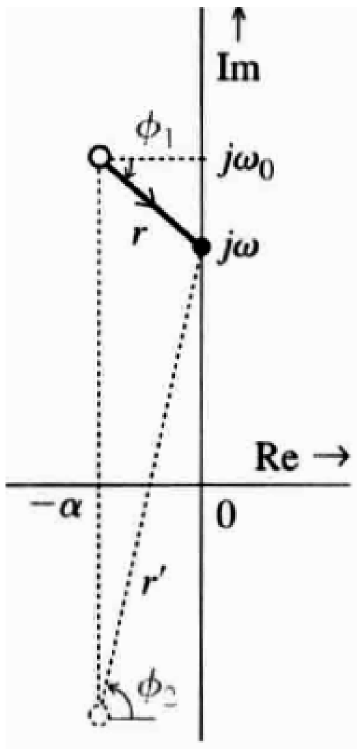


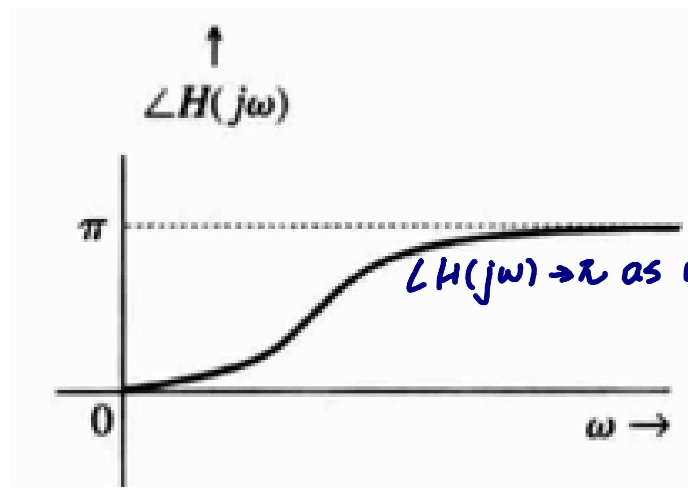
Phase response due to a pair of complex conjugate zeros

- Similar arguments regarding the phase are applied for a pair of complex conjugate zeros $-a + j\omega_0$ and $-a - j\omega_0$.
- $\angle H(j\omega) = +(\phi_1 + \phi_2)$

zeros at $-a \pm j\omega_0$:

$$\begin{cases} H(j\omega) = K r r' \\ \angle H(j\omega) = \phi_1 + \phi_2 \end{cases}$$





$\angle H(j\omega) \to \pi$ as $\omega \to \infty$

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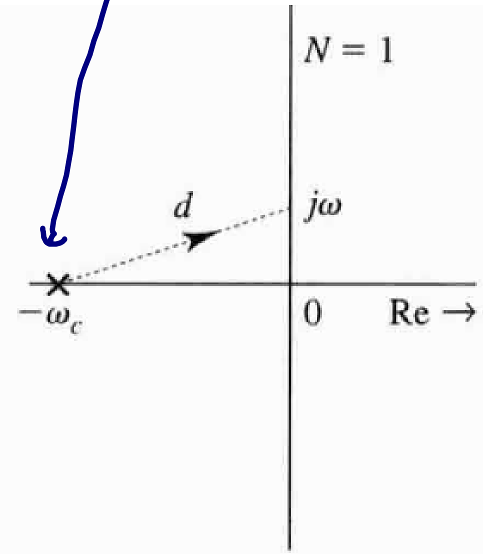
Lowpass filters. The simplest case.

- A lowpass filter is a system with a frequency response that has its maximum gain at $\omega = 0$.
- We showed in detail previously that a pole enhances the gain of the frequency response at frequencies which are within its close neighbourhood.
- Therefore, for a maximum gain at $\omega = 0$, we must place pole(s) on the real axis, within the left half plane, opposite the point $\omega = 0$.
- The simplest lowpass filter can be described by the transfer function:

$$H(s) = \frac{\omega_c}{s + \omega_c}$$

  - Observe that by putting ω_c to the numerator we achieve $H(0) = 1$.
  - If the distance from the pole to a point $j\omega$ is d then $|H(j\omega)| = \frac{\omega_c}{d}$.



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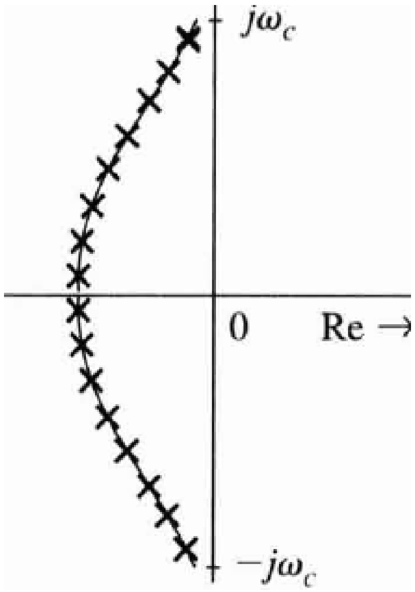
Lowpass filters. Wall of poles – Butterworth filters

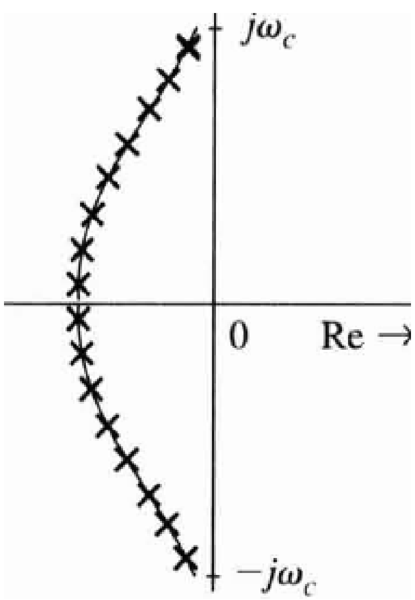
- An ideal lowpass filter has a constant gain of 1 up to a desired frequency ω_c and then the gain drops to 0.
- Therefore, for an ideal lowpass filter an enhanced gain is required within the frequency range 0 to ω_c. This implies that a pole must be placed opposite every single frequency within the range 0 to ω_c.
- We require ideally a continuous "wall of poles" facing the imaginary axis opposite the range 0 to ω_c , and consequently, their complex conjugates facing the imaginary axis opposite the range 0 to $-\omega_c$.
  - At this stage we are not interested in investigating the optimal shape of this wall of poles.
  - We can prove that for a maximally flat response within the range 0 to ω_c, the wall is a semicircle.
  - A maximally flat amplitude response implies:

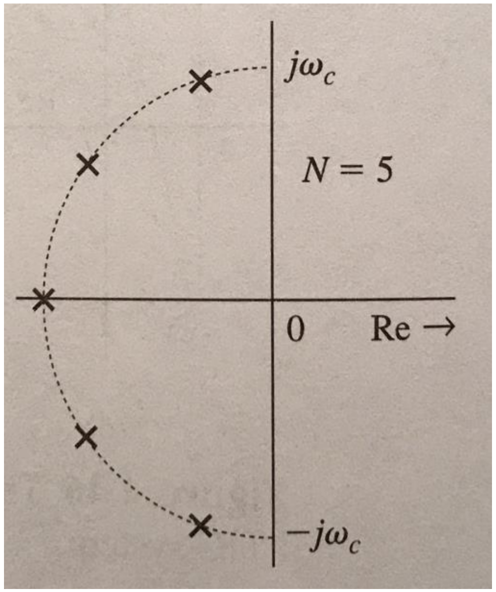
$$\left.\frac{d^i|H(\omega)|}{d\omega^i}\right|_{\omega=0} = 0, i = 0,1,2, \ldots$$

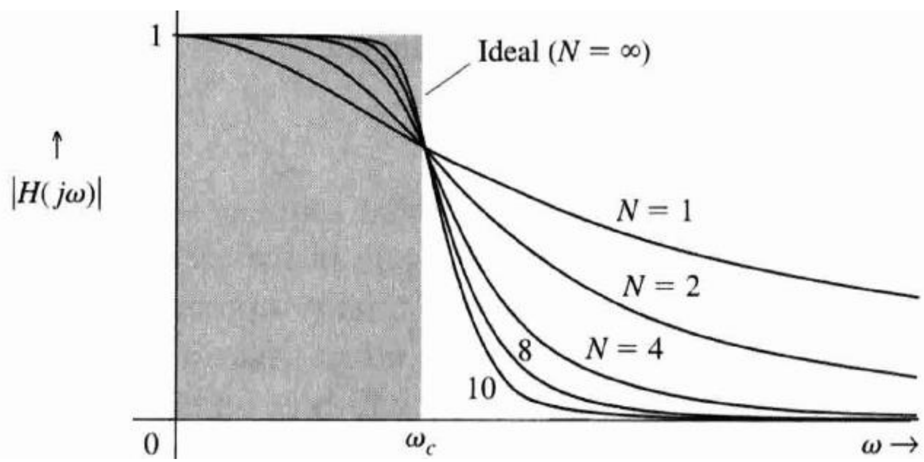


Lowpass filters. Wall of poles – Butterworth filters.

- We can prove that for a maximally flat response within the range 0 to ω_c, the wall is a semicircle with infinite number of poles.
- In practice we use N poles and we end up with a filter with non-ideal characteristics.
- Observe the response as a function of N.
- This family of filters are called Butterworth filters.
- There are families of filters with different characteristics (Chebyshev etc.)

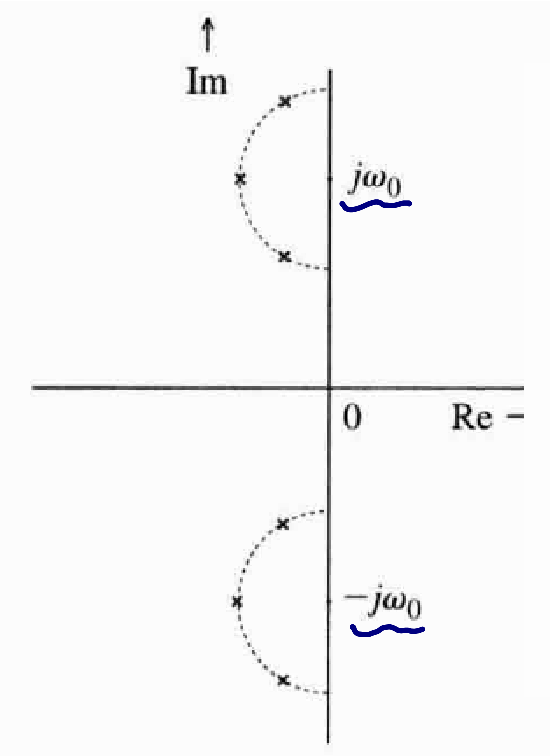


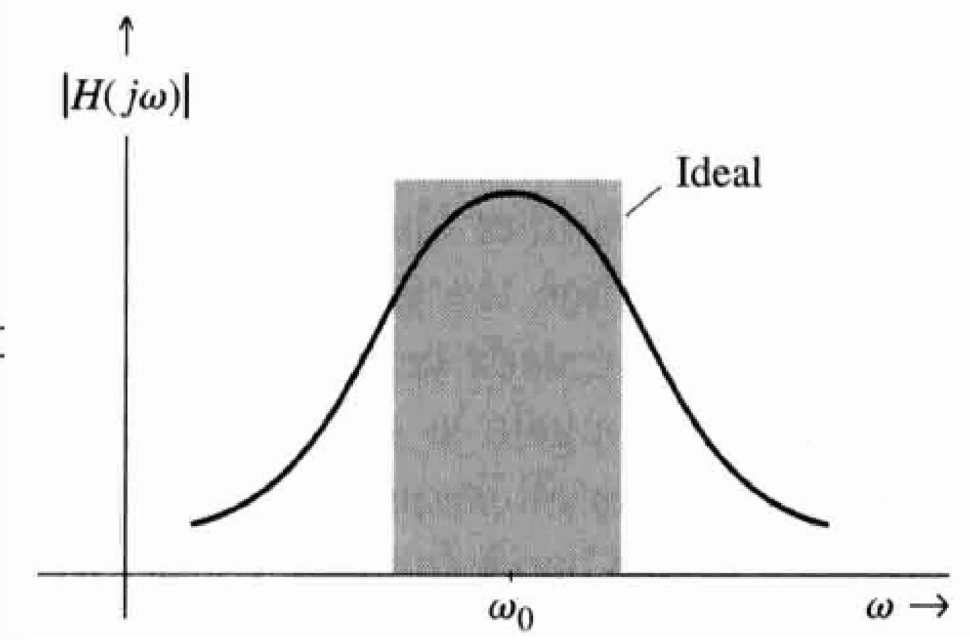




Bandpass filters

- An ideal bandpass filter has a constant gain of 1 placed symmetrically around a desired frequency ω_0; otherwise the gain drops to 0.
- Therefore, we require ideally a continuous wall of poles facing the imaginary axis opposite ω_0, and consequently, their complex conjugates facing the imaginary axis opposite $-\omega_0$.





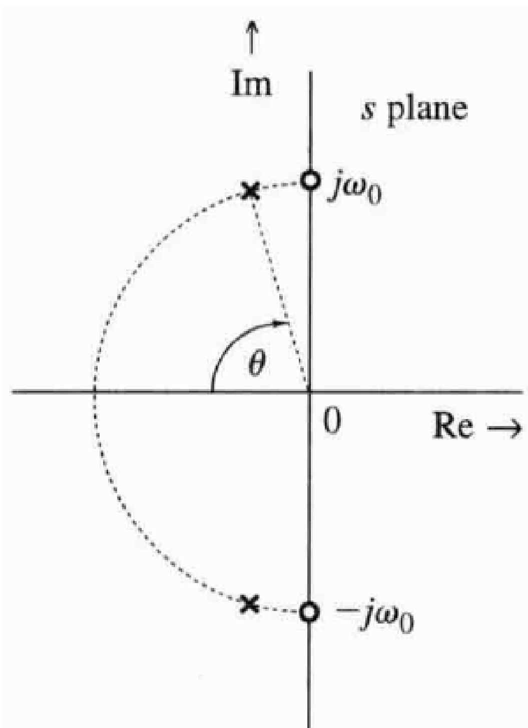
Bandstop (Notch) filters

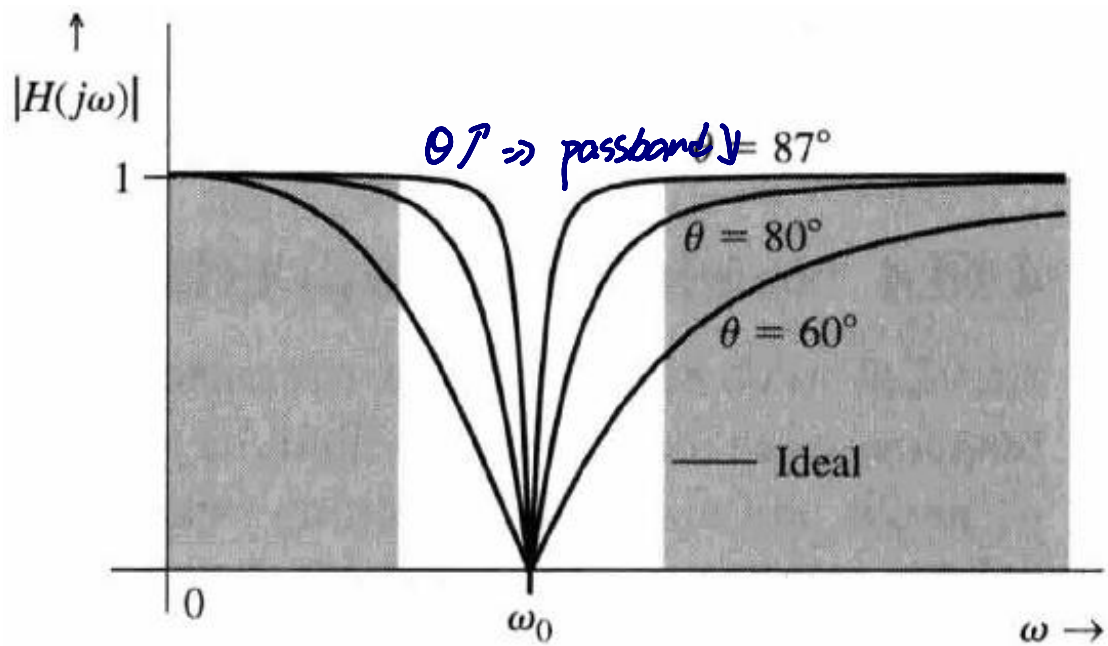
- An ideal bandstop (notch) filter has 0 amplitude response placed symmetrically around a desired frequency ω_0; otherwise the gain is 1.
- Realization in theory requires infinite number of zeros and poles.
- Let us consider a second order notch filter with zero gain at ω_0.
  - We must have zeros at $\pm j\omega_0$.

    1. zeros at $\pm j\omega_0$
    2. # poles = # zeros
  - For $\lim_{\omega\to\infty} |H(j\omega)| = 1$ the number of poles must be equal to the number of zeros. (For $\omega \to \infty$ the distance of all poles and zeros from ω is basically the same.) $\frac{\prod d_z}{\prod d_p} = 1$
  - Based on the above two points, we must have two poles.
  - In order to have $|H(0)| = 1$ each pole much pair up with a zero and their distances from the origin must be the same.
    - This requirement can be satisfied if we place the two conjugate poles along a semicircle of radius ω_0 that lies within the left half plane.

Bandstop (Notch) filters cont.

- Based on the previous statements, the pole-zero configuration and the amplitude response of a bandstop filter are shown in the two figures below.
- Observe the behaviour of the amplitude response as a function of θ, the angle that the pole vector forms with the negative real axis.





Notch filter example

- Design a second-order notch filter to suppress $60Hz$ hum in a radio receiver.
- Make $\omega_0 = 120\pi$. Place zeros are at $s = \pm j\omega_0$, and poles at $-\omega_0\cos\theta \pm j\omega_0\sin\theta$. We obtain:

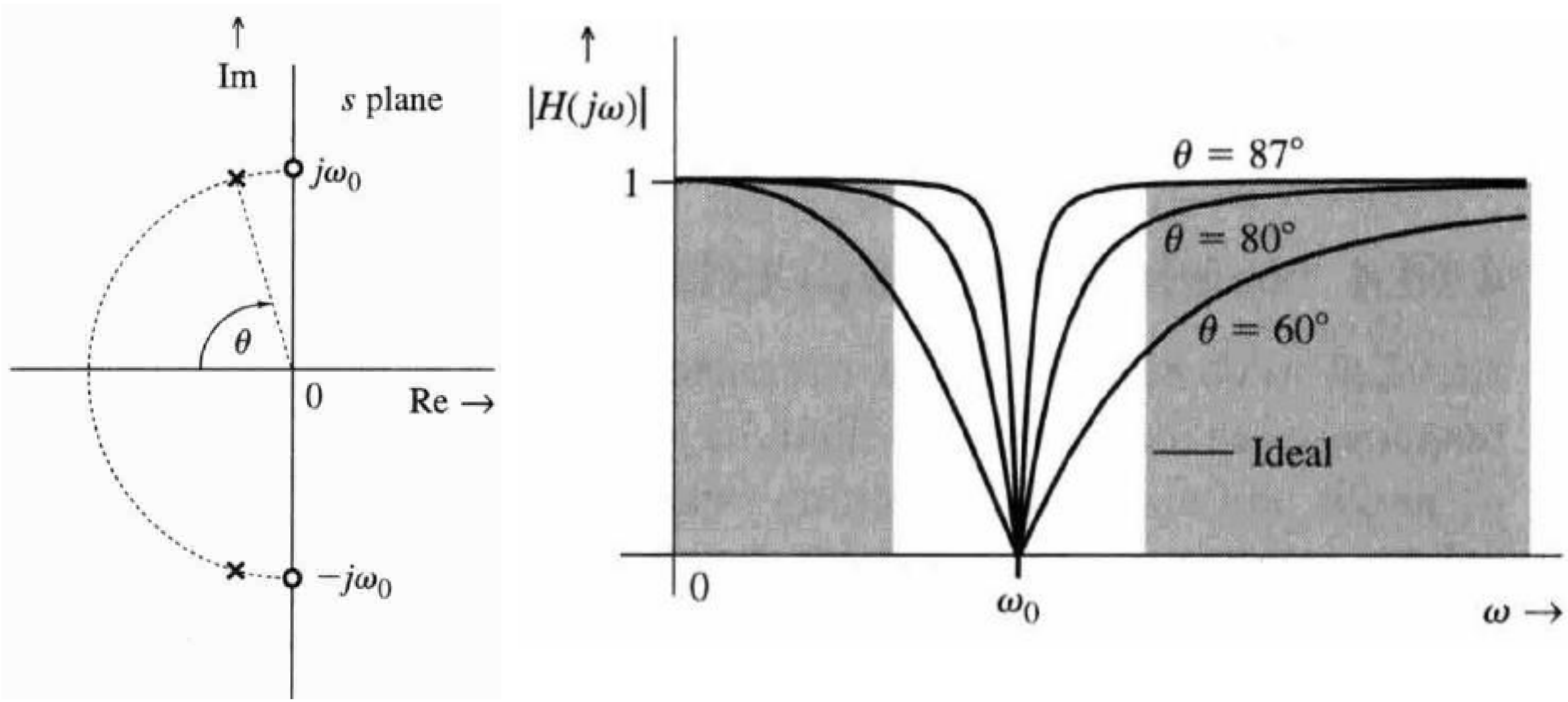
$$H(s) = \frac{(s - j\omega_0)(s + j\omega_0)}{(s + \omega_0\cos\theta + j\omega_0\sin\theta)(s + \omega_0\cos\theta - j\omega_0\sin\theta)}$$

$$= \frac{s^2+\omega_0{}^2}{s^2+(2\omega_0\cos\theta)s+\omega_0{}^2} = \frac{s^2+142122.3}{s^2+(753.98\cos\theta)s+142122.3}$$

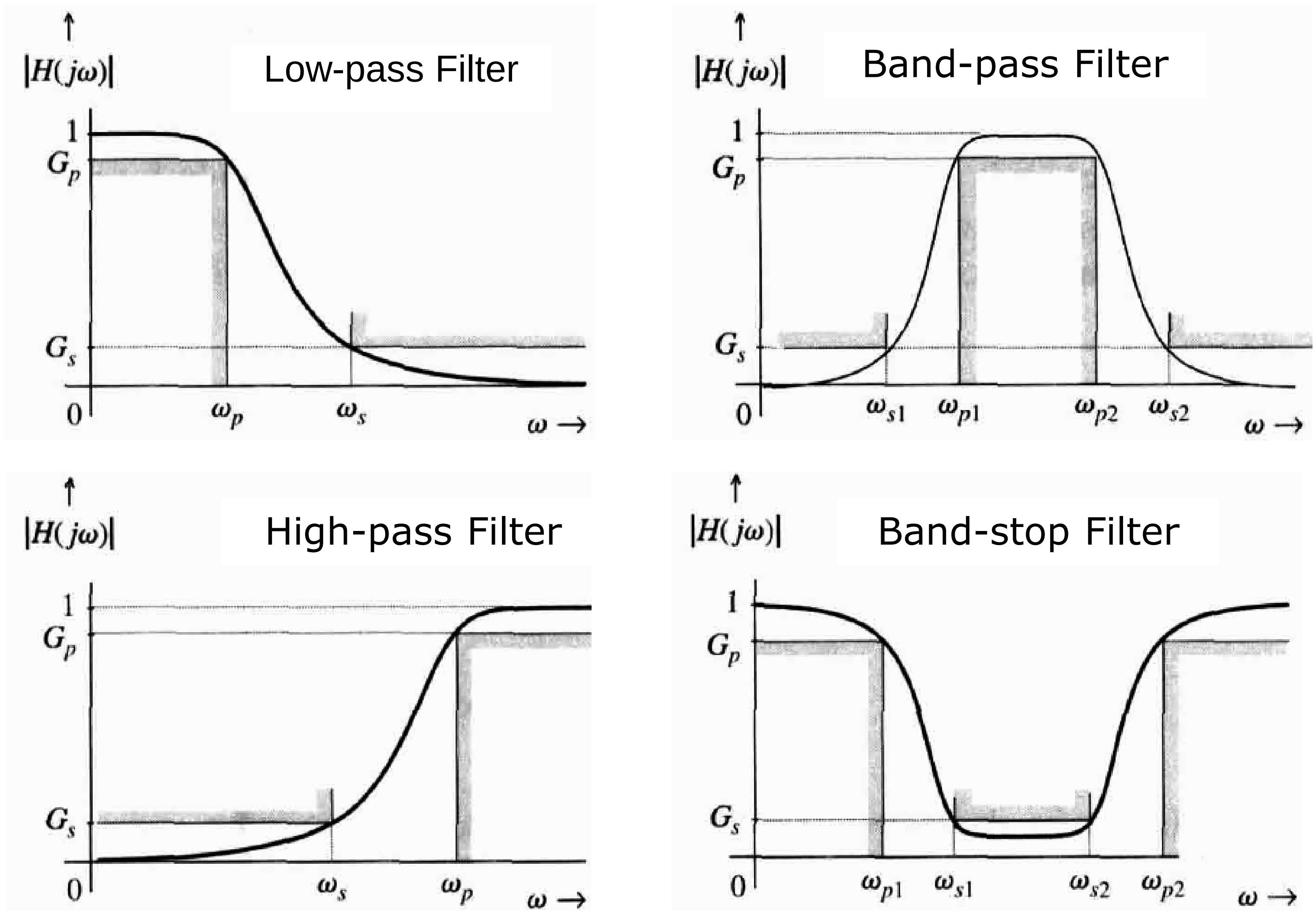
$$|H(j\omega)| = \frac{-\omega^2 + 142122.3}{\sqrt{(-\omega^2 + 142122.3)^2+(753.98\omega\cos\theta)^2}}$$

Notch filter example cont.

- The figures below depict the location of poles and zeros within the plane and the amplitude response.



Practical filter specification

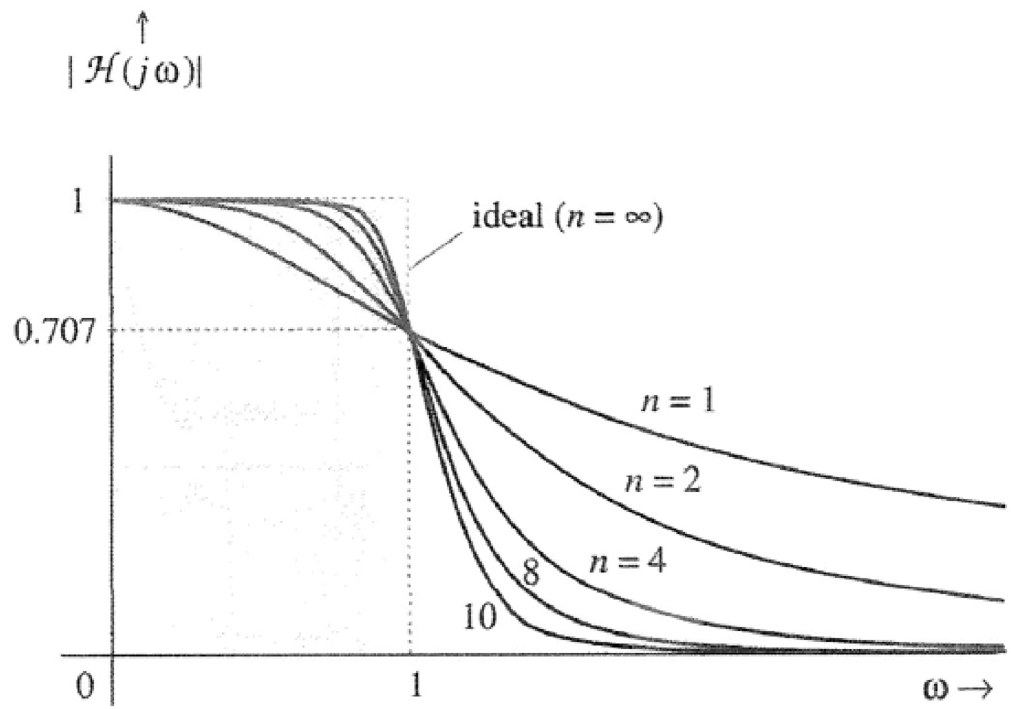


Butterworth filters again

- Let us consider a normalised low-pass filter (i.e., one that has a cut-off frequency at 1) with an amplitude characteristic given by the equation:

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}}$$

- As $n \to \infty$, this gives a ideal LPF response:
  - $|H(j\omega)| = 1$ if $\omega \leq 1$
  - $|H(j\omega)| = 0$ if $\omega > 1$



Butterworth filters cont.

- In the previous amplitude response we replace ω with $\frac{s}{j}$ and we obtain:

$$|H(j\omega)|^2 = H(j\omega)H^*(j\omega) = H(j\omega)H(-j\omega) = \frac{1}{1+\omega^{2n}} \Rightarrow H(s)H(-s) = \frac{1}{1+\left(\frac{s}{j}\right)^{2n}}$$

$s^{2n} = e^{j\frac{\pi}{2}\cdot 2n} \cdot e^{j\pi(2k-1)}$

- The poles of $H(s)H(-s)$ are given by $1+\left(\frac{s}{j}\right)^{2n} = 0 \Rightarrow \left(\frac{s}{j}\right)^{2n} = -1$.
- We know that $-1 = e^{j\pi(2k-1)}$ and $j = e^{j\frac{\pi}{2}}$.
- $\left(\frac{s}{j}\right)^{2n} = -1 \Rightarrow s^{2n} = j^{2n}\cdot(-1) = e^{\left(j\frac{\pi}{2}\right)2n}\cdot e^{j\pi(2k-1)} = e^{j\pi n}\cdot e^{j\pi(2k-1)}$

  $\Rightarrow s^{2n} = e^{j\pi(2k-1+n)} \Rightarrow s = e^{\frac{j\pi(2k-1+n)}{2n}}$, k integer.
- Therefore, the poles of $H(s)H(-s)$ lie along the unit circle (a circle around the origin with radius equal to 1). There are $2n$ distinct poles given by:

$$s_k = e^{\frac{j\pi(2k-1+n)}{2n}},\ k = 1,2,\ldots,2n$$

Butterworth filters cont.

- We are only interested in $H(s)$, not $H(-s)$. Therefore, we choose the poles of the low-pass filter to be those lying on the left half plane only. These poles are:

$$s_k = e^{\frac{j\pi(2k-1+n)}{2n}} = \cos\frac{\pi}{2n}(2k-1+n) + j\sin\frac{\pi}{2n}(2k-1+n),\ k = 1,2,\dots,n$$

- The transfer function of the filter is:

$$H(s) = \frac{1}{(s-s_1)(s-s_2)\dots(s-s_N)}$$

- This is a class of filters known as Butterworth filters.

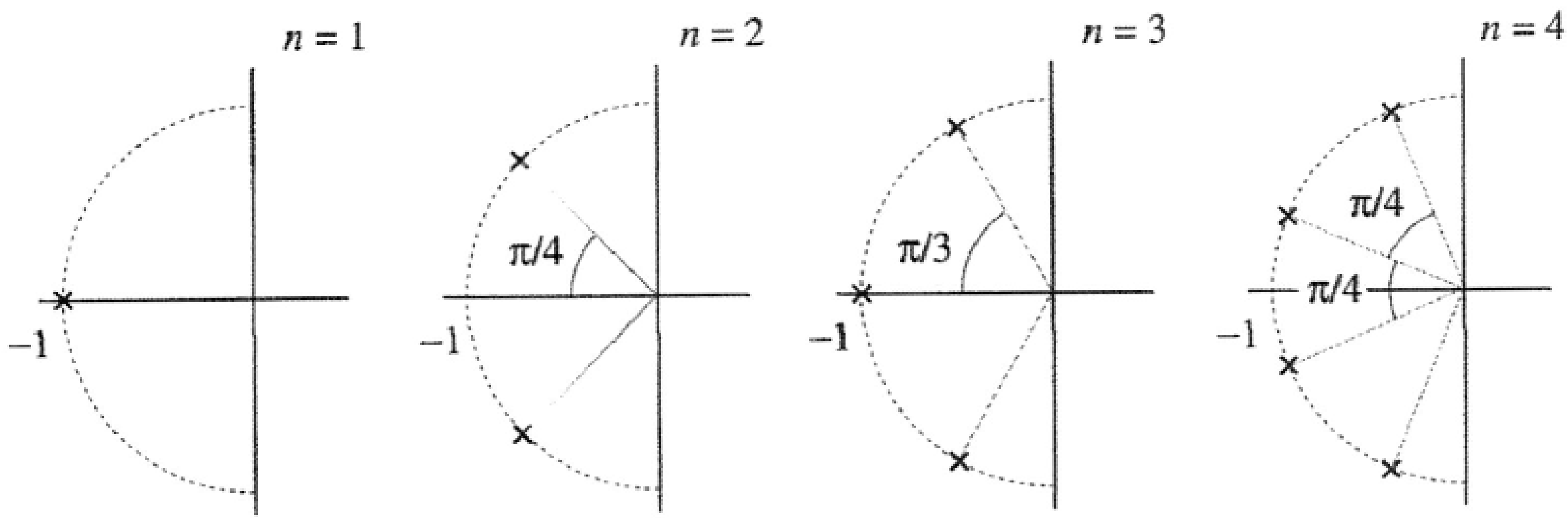
Butterworth filters cont.

- To resume, Butterworth filters are a family of filters with poles distributed evenly around the left half of the unit circle. The poles are given by:

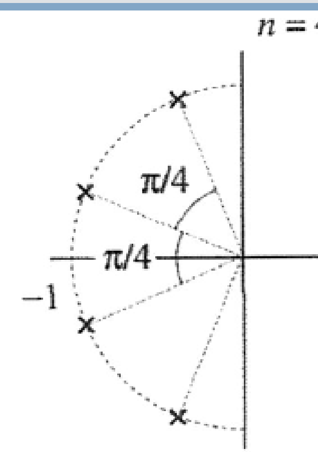
$$s_k = e^{\frac{j\pi(2k+n-1)}{2n}}, \; k = 1,2,\ldots,n$$

- We assume $\omega_c = 1$.
- Here are the pole locations for Butterworth filters for orders $n = 1$ to 4.

$$H(s) = \frac{1}{(s-s_1)(s-s_2)\ldots(s-s_N)}$$



Butterworth filters cont.



- Consider a fourth-order Butterworth filter (i.e., $n = 4$).
- The poles are at angles $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$.
- Therefore, the pole locations are:
  $-0.3827 \pm j0.9239$, $-0.9239 \pm j0.3827$.

Therefore, $H(s) = \dfrac{1}{(s^2+0.7654s+1)(s^2+1.8478s+1)} = \dfrac{1}{s^4+2.6131s^3+3.4142s^2+2.6131s+1}$

Coefficients of Butterworth polynomial: $B_n(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1$

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	1.41421356								
3	2.00000000	2.00000000							
4	2.61312593	3.41421356	2.61312593						
5	3.23606798	5.23606798	5.23606798	3.23606798					
6	3.86370331	7.46410162	9.14162017	7.46410162	3.86370331				
7	4.49395921	10.09783468	14.59179389	14.59179389	10.09783468	4.49395921			
8	5.12583090	13.13707118	21.84615097	25.68835593	21.84615097	13.13707118	5.12583090		
9	5.75877048	16.58171874	31.16343748	41.98638573	41.98638573	31.16343748	16.58171874	5.75877048	
10	6.39245322	20.43172909	42.80206107	64.88239627	74.23342926	64.88239627	42.80206107	20.43172909	6.39245322