C477: Computing for Optimal Decisions
Tutorial 1: Convexity

Exercise 1. Intersections & Unions of Convex Sets

(a) Show that if C_1 and C_2 are convex then their intersection S_1 is also convex:

$$S_1 = C_1 \bigcap C_2.$$

(b) Show that if C_i, $i = 1, \ldots, n$ are convex sets then their intersection S_2 is also convex:

$$S_2 = \bigcap_{i=1}^{n} C_i.$$

(c) Use part (b) to argue that if the m sets defined by each of the m constraints in an optimisation problem are convex, the feasible space of the optimisation problem must itself be a convex set.

(d) The union of two convex sets C_1 and C_2 is not necessarily convex. Give an example where $C_1 \bigcup C_2$ is convex and another example where $C_1 \bigcup C_2$ is **not** convex.

(e) The intersection of a convex set C_1 and a nonconvex set N_1 may be convex or not. Give an example where $C_1 \bigcap N_1$ is convex and another where $C_1 \bigcap N_1$ is **not** convex.

(f) Use part (e) to argue that the feasible space of an optimisation problem may be convex even if some (or all!) of the sets defined by the constraints are nonconvex.

Note: In general it is difficult to determine if the feasible space of a generic optimisation problem is convex, so we often require the condition in (c) for the definition of a *convex optimisation problem*. Since each of the constraints define convex sets, the feasible space of the optimisation problem is also convex.

Exercise 2. Fermat-Weber problem (Human-Cyborg Relations)

C-3PO has a list of people and robots he serves who live in different locations throughout the universe. He would like to choose a location to live such that he is as close to all of them as possible. In other words, C-3PO wants to minimise the sum of Euclidean distances to each person.[1]

(a) Formulate C-3PO's choice as an unconstrained minimisation problem where the location, $\boldsymbol{x_i} \in \mathbb{R}^3$, is given data for each person $i \in \{\text{General Organa, R2-D2, } \ldots\}$ and the decision variable $\boldsymbol{y} \in \mathbb{R}^3$ is the location of C-3PO.

(b) Consider the following data points and find (using any optimisation function in Matlab or other software) where C-3PO should locate himself.

i	$x_{i,1}$	$x_{i,2}$	$x_{i,3}$
General Organa	0.9172	0.2858	0.7572
R2-D2	0.7537	0.3804	0.5678
Finn	0.0759	0.0540	0.5308
Chewbacca	0.7792	0.9340	0.1299
Rey	0.5688	0.4694	0.0119

(c) Is it possible that a different algorithm finding a local minimum may find a different solution? Is there any guarantee that this is the *only* optimum point for C-3PO? Is it possible that there is a better point for the droid?

Exercise 3. Lasso & Sparse regressor selection

For each of the two following formulations, prove that it is a convex optimisation problem or provide a counter example. Feel free to use the most restrictive definition of *convex optimisation problem*, that is that the objective and sets defined by each of the constraints must be convex. Assume that $N \in \mathbb{N}$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$, $t \in \mathbb{R}$ are all given as parameters, $\lambda \geq 0$, and that $\boldsymbol{x} \in \mathbb{R}^n$ is the vector of decision variables.

(a) Sparse regularized regression (Lasso),

$$\min_{\boldsymbol{x}} \ \frac{1}{N} \left\| \boldsymbol{A}\,\boldsymbol{x} - \boldsymbol{b} \right\|_2^2 + \lambda \left\| \boldsymbol{x} \right\|_1$$

[1] Please assume that locations in space can be measured as points in $\mathbb{R}^3$; this problem is about optimisation, not physics.

(b) Sparse regressor selection,

$$\begin{aligned}&\min_{\boldsymbol{x}} \|\boldsymbol{A}\,\boldsymbol{x} - \boldsymbol{b}\|_2\\ &\text{s.t. } \|\boldsymbol{x}\|_1 \leq t\end{aligned}$$

Exercise 4. Sums & Differences of Convex Functions

(a) Show that if $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are convex functions, then their linear combination: $\lambda\, f(\boldsymbol{x}) + \mu\, g(\boldsymbol{x})$, is convex if $\lambda \geq 0$, $\mu \geq 0$.

(b) A function $f : C \to \mathbb{R}$ defined on a convex set C is called a *Difference of Convex* (DC) functions on C if there exists a pair of convex functions $F_1,\, F_2 : C \to \mathbb{R}$ such that $f(\boldsymbol{x})$ is the difference:

$$f(\boldsymbol{x}) = F_1(\boldsymbol{x}) - F_2(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in C.$$

Using DC functions is important in global optimisation because DC optimisation separates each function into an easily-manageable (F_1) and a difficult ($-F_2$) bit. For example, the function $f_a(x_1,\, x_2) = x_1\, x_2$ can be written as the difference:

$$f_a(x_1,\, x_2) = x_1\, x_2 = \frac{1}{4}(x_1 + x_2)^2 - \frac{1}{4}(x_1 - x_2)^2.$$

Show that the functions $F_{1,\,a}(x_1,\, x_2) = \frac{1}{4}(x_1 + x_2)^2$ and $F_{2,\,a}(x_1,\, x_2) = \frac{1}{4}(x_1 - x_2)^2$ are both convex functions.

Exercise 5. Euclidean norm for vectors $\boldsymbol{x},\, \boldsymbol{y} \in \mathbb{R}^n$

(a) Show that the triangle inequality holds for the 2-norm, i.e.,

$$\|\boldsymbol{x} + \boldsymbol{y}\|_2 \leq \|\boldsymbol{x}\|_2 + \|\boldsymbol{y}\|_2.$$

(b) Show that for any two vectors $\boldsymbol{x},\, \boldsymbol{y} \in \mathbb{R}^n$, the following holds,

$$\big|\|\boldsymbol{x}\|_2 - \|\boldsymbol{y}\|_2\big| \leq \|\boldsymbol{x} - \boldsymbol{y}\|_2.$$

Hint: Write $\boldsymbol{x} = (\boldsymbol{x} - \boldsymbol{y}) + \boldsymbol{y}$ and use the triangle inequality derived in part (a). Do the same for $\boldsymbol{y}$.

Exercise 6. Convexity of constraint sets

(a) Show that $\{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\|_2 \leq r\}$ is a convex set, where $r \in \mathbb{R}$ and $r > 0$.

(b) Show that $\{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\|_2 \geq r\}$ is not a convex set, where $r \in \mathbb{R}$ and $r > 0$.

Exercise 7. Use the gradient inequality on the function $f(x) = -x^2 + x$ at the points $X = \{-1, 0, 1, 2\}$ to derive valid bounds on the function f.

Hint: f is not convex, but is concave. The gradient inequality will have to be flipped.