THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. a)

$$\begin{aligned} P(X \le 3 \cap Y \le 2) &= P(X \le 2 \cap Y \le 2) \\ &= \int_{x=0}^{2}\int_{y=x}^{2} 2e^{-(x+y)}dxdy \\ &= \int_{x=0}^{2} 2e^{-x}[-e^{-y}]_x^2 dx \\ &= 2\int_0^2 e^{-2x}dx - 2e^{-2}\int_0^2 e^{-x}dx \\ &= [-e^{-2x}]_0^2 - 2e^{-2}[-e^{-x}]_0^2 = 1-2e^{-2}+e^{-4} = 0.747 \end{aligned}$$

[2 - E]

b)

$$\begin{aligned} P(Y \le 2) &= \int_{y=0}^{2}\int_{x=0}^{y} 2e^{-(x+y)}dxdy \\ &= \int_{y=0}^{2} 2e^{-y}[-e^{-x}]_0^y dy \\ &= \int_{y=0}^{2} 2e^{-y}(1-e^{-y})dy \\ &= 2(1-e^{-2}) + [e^{-2y}]_0^2 \\ &= 1-2e^{-2}+e^{-4} = 0.747 \end{aligned}$$

[2 - E]

c) We find the same value in a) and b) since we actually evaluated the probability over the same domain. [1 - E]

d)

$$f_X(x) = \begin{cases} 2e^{-x}\int_x^\infty e^{-y}dy = 2e^{-2x}, & 0 < x, \\ 0, & otherwise. \end{cases}$$

[2 - E]

e) $E(X) = \int_0^\infty x2e^{-2x}dx = 1/2,$ [2 - E]

$\mathrm{Var}(X) = \int_0^\infty (x-E(X))^2 2e^{-2x}dx = 1/4,$ [2 - E]

We can find these results by directly computing the integrals but it would be simpler to note form the marginal PDF that $X \sim \text{EXPO}(2)$.

f)

$$f_y(y) = \begin{cases} 2e^{-y}\int_0^y e^{-x}dx = 2e^{-y}(1-e^{-y}), & 0 < y, \\ 0, & otherwise. \end{cases}$$

[2 - E]

g) $E(Y) = \int_0^\infty y2e^{-y}(1-e^{-y})dy = 3/2,$

[2 - E]

$E(Y^2) = \int_0^\infty y^2 2e^{-y}(1-e^{-y})dy = 4 - 1/2 = 7/2,$

$\text{Var}(Y) = 7/2 - 9/4 = 5/4$ [2 - E]

h)

$$\begin{aligned} E(XY) &= \int_{x=0}^{\infty}\int_{y=x}^{\infty} xy2e^{-(x+y)}dydx \\ &= \int_{x=0}^{\infty} 2xe^{-x}\int_{y=x}^{\infty} ye^{-y}dydx \\ &= \int_{x=0}^{\infty} 2xe^{-x}[-ye^{-y}-e^{-y}]_x^\infty dx \\ &= \int_{x=0}^{\infty} 2x^2e^{-2x}dx + \int_{x=0}^{\infty} 2xe^{-2x}dx \quad = 1/2 + 1/2 = 1 \end{aligned}$$

$\text{Cov}(X,Y) = 1 - 3/4 = 1/4$ [1 - E]

$\text{Corr}(X,Y) = \frac{1/4}{\sqrt{1/45/4}} = 0.447$ [1 - E]

i) X and Y are correlated since $\text{Corr}(X,Y) \neq 0$. [1 - E]

Since they are correlated, they are also dependent. [1 - E]

j)

$$f_{X|Y}(x|y) = \begin{cases} \frac{2e^{-(x+y)}}{2e^{-2x}} = e^{-(y-x)}, & 0 < x < y, \\ 0, & otherwise. \end{cases}$$

[2 - E]

k) $E[Y|X=x] = \int_{y=x}^\infty ye^{-(y-x)}dy = x+1$ [2 - E]

2. a) i) $F_P(S) = P(P \leq S) = P(\text{ant. 1 selected and} P_1 \leq S) + P(\text{ant. 2 selected and} P_2 \leq S)$. [1 - A]

We then write $P(\text{ant. 1 selected and} P_1 \leq S) = P(P_1 \leq S)\alpha_1$ and $P(\text{ant. 2 selected and} P_2 \leq S) = P(P_2 \leq S)\alpha_2$. [2 - A]

From the exponential distribution, we get

$$F_P(S) = \begin{cases} \alpha_1(1-e^{-\lambda_1 S}) + \alpha_2(1-e^{-\lambda_2 S}) & S > 0 \\ 0 & \text{otherwise} \end{cases}$$

[2 - A]

ii) $f_P(p) = \frac{dF_P(p)}{dp}$ [2 - A]

$$f_P(p) = \begin{cases} \alpha_1\lambda_1 e^{-\lambda_1 p} + \alpha_2\lambda_2 e^{-\lambda_2 p} & p > 0 \\ 0 & \text{otherwise} \end{cases}$$

[2 - A]

iii) The MGF writes as $m_P(t) = E(e^{tP})$. [1 - A]

Hence $m_P(t) = \int_0^\infty e^{tp}(\alpha_1\lambda_1 e^{-\lambda_1 p} + \alpha_2\lambda_2 e^{-\lambda_2 p})dp = \frac{\alpha_1\lambda_1}{t-\lambda_1} + \frac{\alpha_2\lambda_2}{t-\lambda_2}$. [1 - A]

The expected value of the received power after selection writes as $E(P) = m_P'(0)$. [1 - A]

$E(P) = m_P'(0) = \frac{\alpha_1}{\lambda_1} + \frac{\alpha_2}{\lambda_2}$. [1 - A]

iv) Since $\alpha_1 + \alpha_2 = 1$, $E(P) = \frac{\alpha_1(\lambda_2 - \lambda_1) + \lambda_1}{\lambda_1\lambda_2}$, $E(P)$ is a linearly increasing function of α_1 if $\lambda_2 > \lambda_1$ and decreasing otherwise. To maximize $E(P)$, we would prefer taking $\alpha_1 = 1$ if $\lambda_2 > \lambda_1$, such that antenna 1 is always selected, and $\alpha_1 = 0$ (antenna 2 always selected) otherwise. [2 - A]

b) i) Yes, it is correct

$$E_B[E[A|B]] = \int_{-\infty}^{+\infty} E[A|B=b]\, f_B(b)db$$

[2 - A]

Hence,

$$E_B[E[A|B]] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} a\, f_{A|B}(a|b)da\, f_B(b)db$$

[2 - A]

Finally,

$$E_B[E[A|B]] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} a\, f_{A,B}(a,b)dadb = E(A)$$

[1 - A]

ii) No, it is uncorrect. The correct statement is: If A, B are two continuous independent random variables, then $E[A|B] = E(A)$. [2 - B]

$$\begin{aligned} \forall b,\ \ E[A|B=b] &= \int_{-\infty}^{+\infty} a\, f_{A|B}(a|b)da \\ &= \int_{-\infty}^{+\infty} a\, f_A(a)da \ \ (\text{by independence}) \\ &= E(A) \end{aligned}$$

[3 - B]