## THE QUESTIONS

[25]

1. a) Consider two continuous random variable $X$ and $Y$ characterized by the following joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2x, & 0 < x < 1, 0 < y < 1 \\ 0, & otherwise. \end{cases}$$

i) Compute the correlation coefficient between $X$ and $Y$, i.e. $\mathrm{Corr}(X,Y)$.

[4]

ii) Compute the probability that $X$ is smaller than or equal to 0.25 given that $Y$ is larger than or equal to 1/3, i.e. $P(X \leq 0.25 \,|Y \geq 1/3)$.

[4]

iii) Compute the variance of $3X - 2Y + 5$, i.e. $\mathrm{Var}(3X - 2Y + 5)$.

[4]

iv) Are $X$ and $Y$ independent? Provide your reasoning.

[4]

b) Consider the continuous random variable $X$ characterized by the following probability density function

$$f_X(x) = \begin{cases} \theta (1-x)^{\theta - 1}, & 0 < x < 1, \\ 0, & otherwise. \end{cases}$$

with $\theta > 0$.

i) We observe a random sample of size $n$ drawn from the above distribution. Determine the maximum likelihood estimator of $\theta$.

[6]

ii) If the observed random sample is given by the following data, using results from i), compute the estimate of $\theta$.

0.214; 0.108; 0.015; 0.186; 0.054; 0.487; 0.062; 0.095; 0.052; 0.111;
0.519; 0.061; 0.046; 0.180; 0.351; 0.439; 0.088; 0.176; 0.041; 0.288

[3]

[25]

2. a) Consider two independent continuous random variables $X_1$ and $X_2$, each characterized by the probability density function

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & otherwise. \end{cases}$$

Compute the probability that $X_1 + X_2$ is larger or equal to 1, i.e. $P(X_1 + X_2 \geq 1)$.

[4]

b) Consider two continuous random variables $X$ and $Y$ characterized by the conditional probability density function

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x}\exp\left(-\frac{y}{x}\right), & 0 < y < \infty, 0 < x < 1 \\ 0, & otherwise. \end{cases}$$

and the marginal probability density function

$$f_X(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & otherwise. \end{cases}$$

i) Compute the joint probability density function $f_{X,Y}(x,y)$.

[2]

ii) Show that the conditional expectation of $Y$ given $X$, i.e. $E(Y|X)$, is given by $X$.

[3]

iii) By making use of ii), compute the expectation of $Y$, i.e. $E(Y)$.

[4]

iv) By making use of ii), compute the variance of $Y$, i.e. $\mathrm{Var}(Y)$.

[6]

c) Consider the random variable $U$, uniformly distributed between 0 and 1. Compute the probability density function of $1 - \sqrt{1-U}$.

[6]

## 1. Probabilities for events

For events $A$, $B$, and $C$ $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of $A$ $\quad P(A) \,/\, P(\overline{A})$

Conditional probability $\quad P(A \,|\, B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\, P(B \,|\, A)\, P(C \,|\, A \cap B)$

Bayes' rule $\quad P(A \,|\, B) = \dfrac{P(A)\, P(B \,|\, A)}{P(A)\, P(B \,|\, A) + P(\overline{A})\, P(B \,|\, \overline{A})}$

$A$ and $B$ are independent if $\quad P(B \,|\, A) = P(B)$

$A$, $B$, and $C$ are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

## 2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable $X$ is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of $x$, $\quad E\{g(X)\} = \sum_x g(x) p_x$, so $E(X^2) = \sum_x x^2 p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates $\mu$ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \mathrm{var}\,(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$

Sample variance $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left( \sum_j x_j \right)^2 \right\}$ estimates $\sigma^2$

Standard deviation $\quad \mathrm{sd}\,(X) = \sigma$

If value $y$ is observed with frequency $n_y$

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\quad \beta_1 = E\left(\dfrac{X - \mu}{\sigma}\right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X - \mu}{\sigma}\right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\quad \widetilde{x}$ or $x_{\mathrm{med}}$. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$,
then $\widetilde{x} = x_{(\frac{n+1}{2})}$ if $n$ is odd, and $\widetilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if $n$ is even

$\alpha$-quantile $Q(\alpha)$ is such that $P(X \le Q(\alpha)) = \alpha$

Sample $\alpha$-quantile $\widehat{Q}(\alpha)$ Proportion $\alpha$ of the data values are smaller

Lower quartile Q1 $= \widehat{Q}(0.25)$ one quarter are smaller

Upper quartile Q3 $= \widehat{Q}(0.75)$ three quarters are smaller

Sample median $\widetilde{x} = \widehat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $F(x) = P(X \le x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x\,,\quad \mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2,$ where $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$

4. Discrete probability distributions

Discrete Uniform *Uniform* $(n)$

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = (n+1)/2,\ \ \sigma^2 = (n^2-1)/12$$

Binomial distribution *Binomial* $(n, \theta)$

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta\,,\ \ \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* $(\lambda)$

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \quad (\text{with } \lambda > 0) \qquad \mu = \lambda\,,\ \ \sigma^2 = \lambda$$

Geometric distribution *Geometric* $(\theta)$

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta}\,,\ \ \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution *Uniform* $(\alpha, \beta)$

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \mu = (\alpha+\beta)/2\,,\ \ \sigma^2 = (\beta-\alpha)^2/12$$

Exponential distribution *Exponential* $(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \le 0). \end{cases} \qquad \mu = 1/\lambda\,,\ \ \sigma^2 = 1/\lambda^2$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \quad (-\infty < x < \infty), \quad E(X) = \mu\,, \quad \mathrm{var}\,(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If $X$ is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

## 6. Reliability

For a device in continuous operation with failure time random variable $T$ having pdf $f(t)$ $(t > 0)$

| | | |
|---|---|---|
| The reliability function at time $t$ | $R(t)$ | $= P(T > t)$ |
| The failure rate or hazard function | $h(t)$ | $= f(t)/R(t)$ |
| The cumulative hazard function | $H(t)$ | $= \int_0^t h(t_0)\,\mathrm{d}t_0 \;=\; -\ln\{R(t)\}$ |
| The Weibull($\alpha, \beta$) distribution has | $H(t)$ | $= \beta t^\alpha$ |

## 7. System reliability

For a system of $k$ devices, which operate independently, let

$R_i = P(D_i) = P($"device $i$ operates"$)$

The system reliability, $R$, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1-R_1)(1-R_2)\cdots(1-R_k)$$

## 8. Covariance and correlation

The covariance of $X$ and $Y$ $\quad \mathrm{cov}\,(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

| | | |
|---|---|---|
| Sample covariance | $s_{xy} = \dfrac{1}{n-1} S_{xy}$ | estimates $\mathrm{cov}\,(X,Y)$ |
| Correlation coefficient | $\rho = \mathrm{corr}\,(X,Y) = \dfrac{\mathrm{cov}\,(X,Y)}{\mathrm{sd}\,(X)\cdot\mathrm{sd}\,(Y)}$ | |
| Sample correlation coefficient | $r = \dfrac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$ | estimates $\rho$ |

9. Sums of random variables

$$E(X+Y) = E(X)+E(Y)$$

$$\mathrm{var}\,(X+Y) = \mathrm{var}\,(X)+\mathrm{var}\,(Y)+2\,\mathrm{cov}\,(X,Y)$$

$$\mathrm{cov}\,(aX+bY,\ cX+dY) = (ac)\,\mathrm{var}\,(X)+(bd)\,\mathrm{var}\,(Y)+(ad+bc)\,\mathrm{cov}\,(X,Y)$$

If $X$ is $N(\mu_1,\sigma_1^2)$, $Y$ is $N(\mu_2,\sigma_2^2)$, and $\mathrm{cov}\,(X,Y)=c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If $t$ estimates $\theta$ (with random variable $T$ giving $t$)

Bias of $t$ $\quad \mathrm{bias}\,(t) = E(T)-\theta$

Standard error of $t$ $\quad \mathrm{se}\,(t) = \mathrm{sd}\,(T)$

Mean square error of $t$ $\quad \mathrm{MSE}(t) = E\{(T-\theta)^2\} = \{\mathrm{se}\,(t)\}^2+\{\mathrm{bias}\,(t)\}^2$

If $\overline{x}$ estimates $\mu$, then $\mathrm{bias}\,(\overline{x})=0$, $\mathrm{se}\,(\overline{x})=\sigma/\sqrt{n}$, $\mathrm{MSE}(\overline{x})=\sigma^2/n$, $\widehat{\mathrm{se}}\,(\overline{x})=s/\sqrt{n}$

Central limit property $\quad$ If $n$ is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter $\theta$.

For a random sample $x_1,x_2,\ldots,x_n$

$$\ell(\theta;\,x_1,x_2,\ldots,x_n) = P(X_1=x_1\,|\,\theta)\ \cdots\ P(X_n=x_n\,|\,\theta) \qquad \text{(discrete distribution)}$$

$$\ell(\theta;\,x_1,x_2,\ldots,x_n) = f(x_1\,|\,\theta)\,f(x_2\,|\,\theta)\ \cdots\ f(x_n\,|\,\theta) \qquad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1,x_2,\ldots,x_n$ are a random sample from $N(\mu,\sigma^2)$ and $\sigma^2$ is known, then the 95% confidence interval for $\mu$ is $(\overline{x}-1.96\frac{\sigma}{\sqrt{n}},\ \overline{x}+1.96\frac{\sigma}{\sqrt{n}})$

If $\sigma^2$ is estimated, then from the Student t table for $t_{n-1}$ we find $t_0=t_{n-1,0.05}$

The 95% confidence interval for $\mu$ is $(\overline{x}-t_0\frac{s}{\sqrt{n}},\ \overline{x}+t_0\frac{s}{\sqrt{n}})$

13. Standard normal table — Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

| $y$ | $\phi(y)$ | $\Phi(y)$ | $y$ | $\phi(y)$ | $\Phi(y)$ | $y$ | $\phi(y)$ | $\Phi(y)$ | $y$ | $\Phi(y)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | .399 | .5 | .9 | .266 | .816 | 1.8 | .079 | .964 | 2.8 | .997 |
| .1 | .397 | .540 | 1.0 | .242 | .841 | 1.9 | .066 | .971 | 3.0 | .999 |
| .2 | .391 | .579 | 1.1 | .218 | .864 | 2.0 | .054 | .977 | 0.841 | .8 |
| .3 | .381 | .618 | 1.2 | .194 | .885 | 2.1 | .044 | .982 | 1.282 | .9 |
| .4 | .368 | .655 | 1.3 | .171 | .903 | 2.2 | .035 | .986 | 1.645 | .95 |
| .5 | .352 | .691 | 1.4 | .150 | .919 | 2.3 | .028 | .989 | 1.96 | .975 |
| .6 | .333 | .726 | 1.5 | .130 | .933 | 2.4 | .022 | .992 | 2.326 | .99 |
| .7 | .312 | .758 | 1.6 | .111 | .945 | 2.5 | .018 | .994 | 2.576 | .995 |
| .8 | .290 | .788 | 1.7 | .094 | .955 | 2.6 | .014 | .995 | 3.09 | .999 |

14. Student t table — Values $t_{m,p}$ of $x$ for which $P(|X| > x) = p$, when $X$ is $t_m$

| $m$ | $p=$ 0.10 | 0.05 | 0.02 | 0.01 | $m$ | $p=$ 0.10 | 0.05 | 0.02 | 0.01 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 6.31 | 12.71 | 31.82 | 63.66 | 9 | 1.83 | 2.26 | 2.82 | 3.25 |
| 2 | 2.92 | 4.30 | 6.96 | 9.92 | 10 | 1.81 | 2.23 | 2.76 | 3.17 |
| 3 | 2.35 | 3.18 | 4.54 | 5.84 | 12 | 1.78 | 2.18 | 2.68 | 3.05 |
| 4 | 2.13 | 2.78 | 3.75 | 4.60 | 15 | 1.75 | 2.13 | 2.60 | 2.95 |
| 5 | 2.02 | 2.57 | 3.36 | 4.03 | 20 | 1.72 | 2.09 | 2.53 | 2.85 |
| 6 | 1.94 | 2.45 | 3.14 | 3.71 | 25 | 1.71 | 2.06 | 2.48 | 2.78 |
| 7 | 1.89 | 2.36 | 3.00 | 3.50 | 40 | 1.68 | 2.02 | 2.42 | 2.70 |
| 8 | 1.86 | 2.31 | 2.90 | 3.36 | $\infty$ | 1.645 | 1.96 | 2.326 | 2.576 |

15. Chi-squared table — Values $\chi^2_{k,p}$ of $x$ for which $P(X > x) = p$, when $X$ is $\chi^2_k$ and $p =$ .995, .975, *etc*

| $k$ | .995 | .975 | .05 | .025 | .01 | .005 | $k$ | .995 | .975 | .05 | .025 | .01 | .005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | .000 | .001 | 3.84 | 5.02 | 6.63 | 7.88 | 18 | 6.26 | 8.23 | 28.87 | 31.53 | 34.81 | 37.16 |
| 2 | .010 | .051 | 5.99 | 7.38 | 9.21 | 10.60 | 20 | 7.43 | 9.59 | 31.42 | 34.17 | 37.57 | 40.00 |
| 3 | .072 | .216 | 7.81 | 9.35 | 11.34 | 12.84 | 22 | 8.64 | 10.98 | 33.92 | 36.78 | 40.29 | 42.80 |
| 4 | .207 | .484 | 9.49 | 11.14 | 13.28 | 14.86 | 24 | 9.89 | 12.40 | 36.42 | 39.36 | 42.98 | 45.56 |
| 5 | .412 | .831 | 11.07 | 12.83 | 15.09 | 16.75 | 26 | 11.16 | 13.84 | 38.89 | 41.92 | 45.64 | 48.29 |
| 6 | .676 | 1.24 | 12.59 | 14.45 | 16.81 | 18.55 | 28 | 12.46 | 15.31 | 41.34 | 44.46 | 48.28 | 50.99 |
| 7 | .990 | 1.69 | 14.07 | 16.01 | 18.48 | 20.28 | 30 | 13.79 | 16.79 | 43.77 | 46.98 | 50.89 | 53.67 |
| 8 | 1.34 | 2.18 | 15.51 | 17.53 | 20.09 | 21.95 | 40 | 20.71 | 24.43 | 55.76 | 59.34 | 63.69 | 66.77 |
| 9 | 1.73 | 2.70 | 16.92 | 19.02 | 21.67 | 23.59 | 50 | 27.99 | 32.36 | 67.50 | 71.41 | 76.15 | 79.49 |
| 10 | 2.16 | 3.25 | 13.31 | 20.48 | 23.21 | 25.19 | 60 | 35.53 | 40.48 | 79.08 | 83.30 | 88.38 | 91.95 |
| 12 | 3.07 | 4.40 | 21.03 | 23.34 | 26.22 | 28.30 | 70 | 43.28 | 48.76 | 90.53 | 95.02 | 100.4 | 104.2 |
| 14 | 4.07 | 5.63 | 23.68 | 26.12 | 29.14 | 31.32 | 80 | 51.17 | 57.15 | 101.9 | 106.6 | 112.3 | 116.3 |
| 16 | 5.14 | 6.91 | 26.30 | 28.85 | 32.00 | 34.27 | 100 | 67.33 | 74.22 | 124.3 | 129.6 | 135.8 | 140.2 |

16. The chi-squared goodness-of-fit test

The frequencies $n_y$ are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of $\chi^2_k$ with significance point $p$,

where $k$ is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with $\mu$

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

Joint cdf $F(x, y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0\,, y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0$

Joint pdf $f(x, y) = \frac{\mathrm{d}^2 F(x, y)}{\mathrm{d}x\,\mathrm{d}y}$

Marginal pdf of $X$ $f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\,\mathrm{d}y_0$

Conditional pdf of $X$ given $Y = y$ $f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}$, $\widehat{\beta} = \frac{S_{xy}}{S_{xx}}$

The residual sum of squares $\text{RSS} = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$

$\widehat{\sigma^2} = \frac{\text{RSS}}{n-2}$ $\qquad \frac{n-2}{\sigma^2}\widehat{\sigma^2}$ is from $\chi^2_{n-2}$

$$E(\widehat{\alpha}) = \alpha\,, \quad E(\widehat{\beta}) = \beta\,,$$

$$\operatorname{var}(\widehat{\alpha}) = \frac{\sum x_i^2}{n\,S_{xx}}\sigma^2\,, \quad \operatorname{var}(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}\,, \quad \operatorname{cov}(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta}x\,, \qquad E(\widehat{y}_x) = \alpha + \beta x\,, \qquad \operatorname{var}(\widehat{y}_x) = \left\{\frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}}\right\}\sigma^2$$

$\frac{\widehat{\alpha} - \alpha}{\widehat{\text{se}}\,(\widehat{\alpha})}\,, \quad \frac{\widehat{\beta} - \beta}{\widehat{\text{se}}\,(\widehat{\beta})}\,, \quad \frac{\widehat{y}_x - \alpha - \beta\,x}{\widehat{\text{se}}\,(\widehat{y}_x)}$ are each from $t_{n-2}$