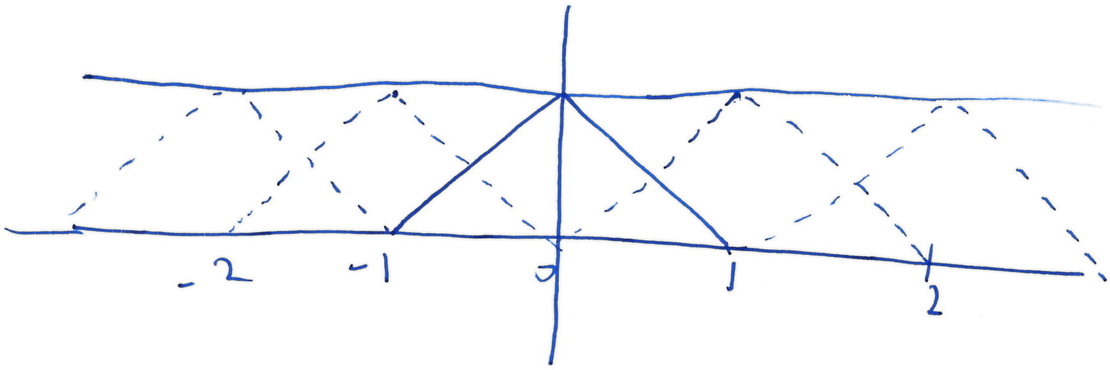
EXERCISE 3.1

(a)

i TWO-SCALE RELATION IS SATISFIED WITH

$$\begin{cases} g_0[1] = g_0[-1] = \frac{1}{2\sqrt{2}} \\ g_0[0] = \frac{1}{\sqrt{2}} \\ g_0[n] = 0 \quad \text{OTHERWISE} \end{cases}$$

i.i. GRAPHICALLY:



i.i.i.

$$x[n] = \langle \varphi(x), \varphi(x-n) \rangle = \begin{cases} \frac{2}{3} & \text{FOR } n=0 \\ \frac{1}{6} & \text{FOR } n=\pm 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$\sum_k \left| \Phi(\omega + 2\pi k) \right|^2 = \sum_n x[n] e^{j\omega n} = \frac{2}{3} + \frac{1}{3}\cos\omega$$

THUS

$$A = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} > 0$$

$$B = \frac{2}{3} + \frac{1}{3} = 1 < +\infty$$

$\Rightarrow$ RIESZ CRITERION SATISFIED

(b) THE DERIVATIVE OF $\varphi(x)$ DOES NOT SATISFY PARTITION OF UNITY THUS IT IS NOT A VALID SCALING FUNCTION

QUESTION 3.2

(a)

$$a[n] = \begin{cases} 2/3 & \text{FOR } n=0 \\ 1/6 & \text{FOR } n=\pm 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$A\left(e^{jw}\right) = \sum_{k=-\infty}^{\infty} \left|\hat{\varphi}\left(w+2k\pi\right)\right|^2 \quad (1)$$

THUS, IF $\hat{\phi}(w) = \dfrac{\hat{\varphi}(w)}{\sqrt{A(e^{jw})}}$ (2)

WE HAVE THAT

$$\sum_{k=-\infty}^{\infty} \left|\hat{\phi}\left(w+2k\pi\right)\right|^2 \overset{(a)}{=} \sum_{k=-\infty}^{\infty} \left|\frac{\hat{\varphi}\left(w+2k\pi\right)}{\sqrt{A(e^{jw})}}\right|^2 =$$

$$= \frac{1}{A(e^{jw})} \sum_{k=-\infty}^{\infty} \left|\hat{\varphi}\left(w+2k\pi\right)\right|^2 \overset{(b)}{=} 1$$

WHERE (a) FOLLOWS FROM EQ. (2) AND FROM THE FACT THAT $A(e^{jw})$ IS PERIODIC OF PERIOD 2π, AND (b) FOLLOWS FROM (1).

(b) FIRST NOTICE THAT $A\left(e^{j2\pi k}\right) = A(1) = 1$ AND THAT SINCE $\varphi(t)$ IS A VALID SCALING FUNCTION WE HAVE THAT:

$$\sum_n \varphi(t-n) = \sum_k \hat{\varphi}(2\pi k)\, e^{j2\pi k t} = 1 .$$

FOR THESE REASONS IT FOLLOWS THAT

$$\sum_n \phi(t-n) = \sum_k \frac{\hat{\varphi}(2\pi k)}{A(e^{j2\pi k})}\, e^{j2\pi k t} =$$

$$= \sum_k \hat{\varphi}(2\pi k)\, e^{j2\pi k t} = 1$$

(c) $$\hat{\varphi}(\omega) = \frac{G_0(e^{j\omega/2})}{\sqrt{2}}\,\hat{\varphi}(\omega/2) \Rightarrow$$

$$\hat{\phi}(\omega) = \frac{\hat{\varphi}(\omega)}{\sqrt{A(e^{j\omega})}} = \frac{G_0(e^{j\omega/2})}{\sqrt{2A(e^{j\omega})}}\,\hat{\varphi}\left(\frac{\omega}{2}\right) \Rightarrow$$

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2}} G_0(e^{j\omega/2}) \sqrt{\frac{A(e^{j\frac{\omega}{2}})}{A(e^{j\omega})}} \cdot \hat{\phi}\left(\frac{\omega}{2}\right).$$

THUS

$$H_0(e^{j\omega/2}) = G_0(e^{j\frac{\omega}{2}}) \sqrt{\frac{A(e^{j\frac{\omega}{2}})}{A(e^{j\omega})}} \quad \text{AND}$$

$$H_0(z) = G_0(z) \cdot \sqrt{\frac{A(z)}{A(z^2)}}$$

NOW

$$G_0(z) = \frac{1}{\sqrt{2}}\left(\frac{1}{2}z^{-1} + 1 + \frac{1}{2}z\right) = \frac{1}{2\sqrt{2}}(1+z)(1+z^{-1})$$

AND

$$A(z) = \frac{1}{3}\left(\frac{1}{2}z^{-1} + 2 + \frac{1}{2}z\right)$$

THUS

$$H_0(z) = \frac{1}{2\sqrt{2}}(1+z)(1+z^{-1})\sqrt{\frac{(z^{-1}+4+z)}{(z^{-2}+4+z^{2})}}$$

(d)

$$H_1(z) = -z^{-1} H_0(-z^{-1})$$

EXERCISE 3.3

SOLUTIONS NOT PROVIDED

EXERCISE 3.4

(a) $$\psi_{m,n}(t) = \frac{1}{\sqrt{2^m}}\,\psi\left(2^{-m}t - n\right)$$

$$d_{m,n} = \begin{cases} \dfrac{1}{\sqrt{2^m}} & \text{FOR } m>0 \ \& \ n=0 \\[2ex] 0 & \text{OTHERWISE} \end{cases}$$

(b) $\|f\|^2 = 1$

$$\sum_n \sum_m \left|\langle \psi_{m,n}, f\rangle\right|^2 = \sum_{m>0} |d_{m,n}|^2$$

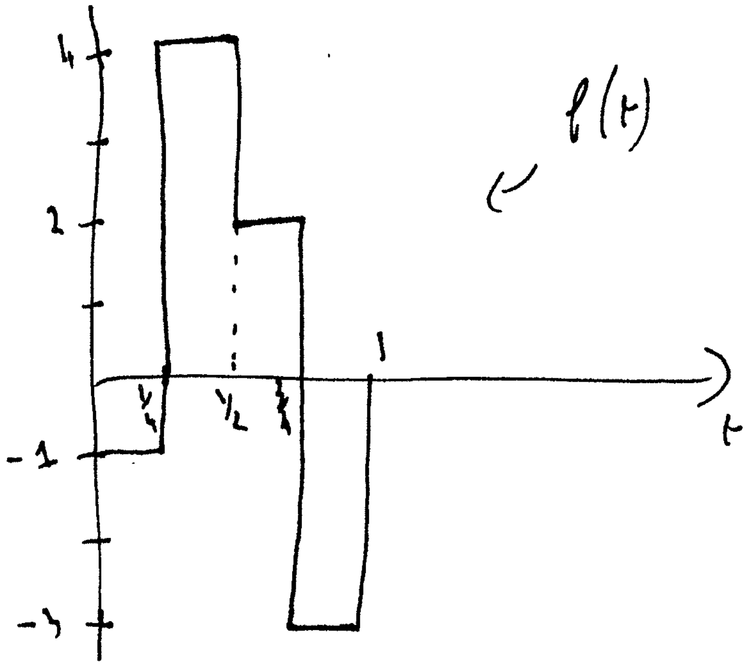
$$= \sum_{m=1}^{\infty} \frac{1}{2^m}$$

$$= \frac{1}{1-1/2} = 1$$

(c) FROM $m = -1$ TO $m = +\infty$

(d) $f(t) = 1$ FOR $t \in [0,2]$ IS NOT A VALID SCALING FUNCTION, SINCE IT DOES NOT SATISFY PARTITION OF UNITY

QUESTION 3.5



a) $\varphi_{-2,m}(t) = 2\varphi(4t-m)$, $c_{j,m} = \langle f(t), \varphi_{j,m}(t) \rangle$.

$c_{-2,0} = -1/2$

$c_{-2,1} = 2$

$c_{-2,2} = 1$

$c_{-2,3} = -3/2$

$c_{-2,m} = 0 \quad m \neq 0, 1, 2, 3$

b) SINCE THE BASIS IS ORTHO-NORMAL, WE HAVE THAT

$c_{j,m} = \langle f(t), \varphi_{j,m}(t) \rangle$ AND

$d_{j,m} = \langle f(t), \psi_{j,m}(t) \rangle$

b)

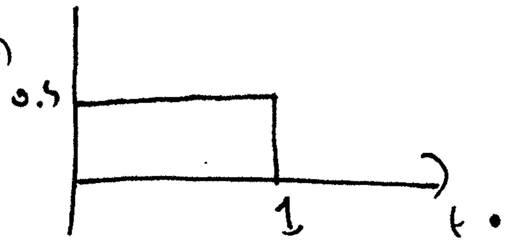
$$\begin{cases} c_{0,0} = \dfrac{-1+4+2-3}{4} = \dfrac{1}{2} \\ c_{0,m} = 0 \quad m \neq 0 \end{cases}$$

$$\begin{cases} d_{0,0} = 1 \\ d_{0,m} = 0 \quad m \neq 0 \end{cases}$$

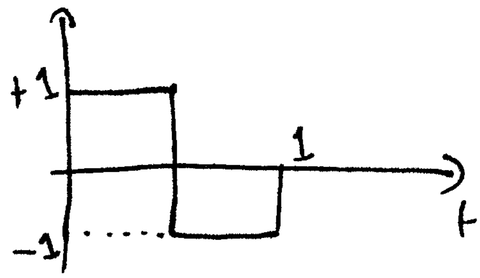
$$\begin{cases} d_{-1,0} = -\dfrac{5}{4} \cdot \sqrt{2} \\ d_{-1,1} = \dfrac{5}{4} \cdot \sqrt{2} \\ d_{-1,m} = 0 \quad m \neq 0, 1 \end{cases}$$

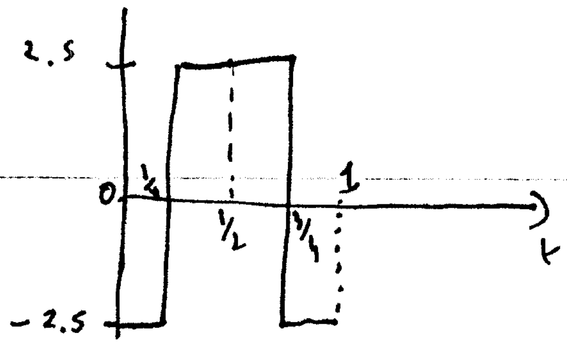
c)

$V_0 \rightarrow$



W_0





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d)

PARSEVAL $\Rightarrow$ $\|f\|^2 = \sum_n |c_{n,0}|^2 + \sum_{j=0}^{-1} \sum_n |d_{j,n}|^2$

$\|f\|^2 = \frac{1}{4} + \frac{16}{4} + \frac{4}{4} + \frac{9}{4} = \frac{15}{2}$

$|c_{0,0}|^2 + |d_{0,0}|^2 + |d_{-1,0}|^2 + |d_{-1,1}|^2 =$

$= \frac{1}{4} + 1 + \frac{25}{16} \cdot 2 + \frac{25}{16} \cdot 2 = \frac{15}{2}$ ✓

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QUESTION 3.6

(a) (TEXT BOOK)

$$\hat{X}(z) = \frac{1}{2} G_0(z)\left(H_0(z)X(z) + H_0(-z)X(-z)\right) +$$
$$\frac{1}{2} G_1(z)\left(H_1(z)X(z) + H_1(-z)X(-z)\right).$$

PR CONDITIONS:

$$\begin{cases} G_0(z)H_0(z) + G_1(z)H_1(z) = 2 \\ G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0 \end{cases}$$

(b) (NOVEL EXAMPLE)

(i) THE CHOSEN $P(z)$ IS SYMMETRIC.

$P(z) + P(-z) = 2$ IS SATISFIED

WHEN $p[0] = 1$, AND $p[2n] = 0$.

SINCE $p[0] = 1$ BY CONSTRUCTION

AND $p[2n] = \frac{\sin(\pi n)}{2\pi n} = 0$

CONDITIONS ARE SATISFIED.

iii) (ii) (NEW EXAMPLE)

$M=1$

$$P(z) = \frac{1}{\pi} z + 1 + \frac{1}{\pi} z^{-1} = \frac{1}{\pi} z^{-1}\left(z^{+2} + \pi z^{+1} + 1 \right)$$

ROOTS OF $P(z)$

$$\alpha_{1,2} = \frac{-\pi \pm \sqrt{\pi^2 - 4}}{2} \cong \begin{cases} -2.78 \\ -0.36 \end{cases}$$

$\alpha_1 = -0.36$, $\alpha_2 = -2.78$. NOTICE $\alpha_1 = 1/\alpha_2$

$$P(z) = \frac{z^{-1}}{\pi}(z - \alpha_1)(z - \alpha_2)$$

$$G_0(z) = \frac{1}{\sqrt{\pi}}\left(1 - \alpha_1 z^{-1}\right)$$

$$H_0(z) = \frac{1}{\sqrt{\pi}}(z - \alpha_2) = \frac{1}{\sqrt{\pi}}\left(z - \frac{1}{\alpha_1}\right)$$

$$G_1(z) = z^{-1} H_0(-z) = -\frac{1}{\sqrt{\pi}}\left(1 + \frac{z^{-1}}{\alpha_1}\right)$$

$$H_1(z) = z\, G_0(-z) = \frac{1}{\sqrt{\pi}}(z + \alpha_1)$$

1(iii) (NEW EXAMPLE)

THE LIMIT DOES NOT CONVENGE

SINCE $G_0(e^{j\omega}) \neq 0$ FOR $\omega = \pi$

AND $G_0(e^{j\omega}) \neq \sqrt{2}$ FOR $\omega = 0$

3.7

[redacted]

(a)

$$P_i = \prod_{k=0}^{i} a^{b^k} = a^{\sum_{k=0}^{i} b^k}$$

USING $\sum_{k=0}^{\infty} \rho^k = \frac{1}{1-\rho}$ $\quad |\rho|<1$

WE OBTAIN

$$P = \lim_{i\to\infty} P_i = \lim_{i\to\infty} a^{\sum_{k=0}^{i} b^k} = a^{\sum_{k=0}^{\infty} b^k} = a^{\frac{1}{1-b}} \qquad (1)$$

#

(b)

$$\prod_{k=1}^{i} m_0\left(\frac{\omega}{2^k}\right) = \prod_{k=1}^{i} e^{-\frac{j\omega}{2^{k+1}}}\left(\frac{e^{\frac{j\omega}{2^{k+1}}} + e^{-\frac{j\omega}{2^{k+1}}}}{2}\right) =$$

$$= \prod_{k=1}^{i} e^{-\frac{j\omega}{2^{k+1}}} \prod_{k=1}^{i} \cos\left(\frac{\omega}{2^{k+1}}\right)$$

USING THE RESULT OF PART (i),

(a) $\lim_{i\to\infty} \prod_{k=1}^{i} e^{-j\frac{\omega}{2^{k+1}}} = e^{-\frac{j\omega}{2}}$

$$\prod_{k=1}^{i} \cos\frac{\omega}{2^{k+1}} = \prod_{k=1}^{i} \frac{\sin\left(\frac{\omega}{2^k}\right)}{2\sin\left(\frac{\omega}{2^{k+1}}\right)} = \frac{1}{2^i}\frac{\sin\frac{\omega}{2}}{\sin\left(\frac{\omega}{2^{i+1}}\right)}$$

(b) $\lim_{i\to\infty} \frac{1}{2^i}\frac{\sin\frac{\omega}{2}}{\sin\left(\frac{\omega}{2^{i+1}}\right)} = \frac{\sin \omega/2}{\omega/2}$

BY COMBINING (a) WITH (b) WE OBTAIN THE RESULT #

QUESTION 3.8 #34

(a)

$$H_0(z) = \frac{1}{4\sqrt{2}}(1+z^{-1})^2(1+z), \quad G_0(z) = \frac{1}{2\sqrt{2}}(1+z)(-z^{-1}+4-z)$$

$$H_1(z) = z\,G_0(-z), \quad G_1(z) = z^{-1}H_0(-z)$$

(b) BOTH FUNCTIONS SATISFY THE NECESSARY CONVERGE CONDITIONS.

USING DAUBECHIES CRITERION WE HAVE THAT

$$\tilde{H}_0(w) = \left(\frac{1+e^{-jw}}{2}\right)^3 = D$$

$$N=3 \qquad B=1 \qquad B < 2^{N-1-p} = 2^{3-1-p}$$

THUS $P=1$ AND $\varphi(t) \in C^{(1)}$ THAT IS $\varphi(t)$ IS CONTINUOUS ~~AND~~ WITH ITS FIRST ORDER DERIVATIVE. IN FACT $\varphi(t)$ IS A QUADRATIC SPLINE.

$$H_0(w) = \left(\frac{1+e^{jw}}{2}\right)R(w) \quad \text{WHERE } R(w) = \frac{(4-2\cos w)}{2}$$

$$B = \max_w R(w) = 3$$ THUS SUFFICIENT

REGULARITY CONDITIONS ARE NOT SATISFIED. WE CANNOT GUARANTEE CONVERGENCE, ~~AND~~

AND REGULARITY

(c)

(i)

$$\hat{\tilde{\psi}}(\omega) = \frac{1}{\sqrt{2}} H_1\left(e^{j\frac{\omega}{2}}\right) \hat{\tilde{\varphi}}\left(\frac{\omega}{2}\right)$$

$\hat{\tilde{\varphi}}(\omega) \neq 0$ FOR $\omega = 0$. THAT IS BECAUSE $\varphi(t)$ IS A VALID SCALING FUNCTION.

$H_1(e^{j\frac{\omega}{2}})$ HAS THREE ZEROS AT $\omega = 0$.

THEREFORE $\tilde{\psi}(t)$ HAS THREE VANISHING MOMENTS.

THE OTHER WAVELET, $\psi(t)$, HAS ONE VANISHING MOMENT.

(ii)

$\langle f(t), \tilde{\psi}_{m,n}(t) \rangle$ DECAYS AS

$$2^{+m(2+1/2)} = 2^{m(2.5)}$$

$\langle f(t), \psi_{m,n}(t) \rangle$ DECAYS AS

$$2^{m(1+1/2)} = 2^{1.5m}.$$

THUS THE DECOMPOSITION

$f(t) = \sum_m \sum_n \langle f, \tilde{\psi}_{m,n} \rangle \psi_{m,n}$ IS BETTER

AND THE ROLES OF ANALYSIS AND SYNTHESIS FILTERS SHOULD BE SWAPPED.