EE4-10
EE9-CS5-1
EE9-SC3

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2015

MSc and EEE PART IV: MEng and ACGI

Corrected Copy

PROBABILITY AND STOCHASTIC PROCESSES

Friday, 15 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions. All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : C. Ling
Second Marker(s) : D. Angeli

Information for students

Each of the four questions has 25 marks.

The Questions

1. Random variables.

a) The random variable X is uniformly distributed in the interval [$-\pi$, π]. Find the probability density function of the following random variables

i) $Y = X^3$ [3]

ii) $Y = X^4$ [3]

iii) $Y = \sin(X)$ [4]

b) X and Y are independent, identically distributed (i.i.d.) random variables with common probability density function

$$f_X(x) = e^{-x}, \qquad x > 0$$
$$f_Y(y) = e^{-y}, \qquad y > 0$$

Find the probability density function of the following random variables:

i) $Z = 2X + Y$. [5]

ii) $Z = \min(X, Y)$. [5]

iii) $Z = \max(X, Y)$. [5]

2. Estimation.

a) The random variable X has the truncated exponential density $f(x) = ce^{-c(x-x_0)},\ x > x_0$, and $f(x) = 0$ otherwise. Let $x_0 = 5$. We observe the i.i.d. samples $x_i = 7, 8, 9, 10, 11$. Find the maximum-likelihood estimate of parameter c.

[8]

b) Consider the Rayleigh fading channel in wireless communications, where the channel gains $Y(n)$ has autocorrelation function

$$R_Y(n) = J_0(2\pi f_d n)$$

where J_0 denotes the zeroth-order Bessel function of the first kind, and f_d represents the normalized Doppler frequency shift. Suppose we wish to predict $Y(n+1)$ from $Y(n)$, $Y(n-1)$, …, $Y(1)$ using the linear MMSE estimator

$$Y(n+1) = \sum_{i=1}^{n} c_i Y(i)$$

Let $f_d = 0.2$ and given the following values of J_0 for $f_d = 0.2$:

$$J_0(2\pi f_d n) = \begin{cases} 1 & n = 0 \\ 0.643 & n = 1 \\ -0.055 & n = 2 \end{cases}$$

i) Compute the coefficient and mean-square error of the first-order linear MMSE estimator, i.e., $n = 1$. [7]

ii) Compute the coefficients and mean-square error of the second-order linear MMSE estimator, i.e., $n = 2$. [10]

3. Random processes.

a) Consider the random process $X(n) = \cos(nU)$ for $n \geq 1$, where U is uniformly distributed on interval $[-\pi, \pi]$.

i) Show that $\{X(n)\}$ is wide-sense stationary. [5]

ii) Show that $\{X(n)\}$ is not strict-sense stationary. [5]

b) The number of patients $N(t)$ arriving at the doctor's office over the time interval $[0, t)$ can be modelled by a Poisson process $\{N(t), t \geq 0\}$. On the average, there is a new patient arriving after every 10 minutes, i.e., the intensity of the process is equal to $\lambda = 0.1$. The doctor will not see a patient until at least three patients are in the waiting room.

i) Find the expected waiting time until the first patient is admitted to see the doctor.

[3]

ii) What is the probability that nobody is admitted to see the doctor in the first hour?

[6]

iii) What is the probability that at least two patient arrive in the first hour while at most two patients arrive in the second hour?

[6]

4. Markov chains and martingales.

a) Let $\{X_n\}$ be a Markov chain and let $\{n_r: r \geq 0\}$ be an unbounded increasing sequence of positive integers.

i) Show that $\{Y_r = X_{n_r}\}$ constitutes a (possibly inhomogeneous) Markov chain.

[4]

ii) Find the transition matrix of $\{Y_r\}$ when $n_r = 2r$ and $\{X_n\}$ has transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

[3]

b) Consider the gambler's ruin with state space $E = \{0,1,2,\ldots,N\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & & & 0 & 0 \\ q & 0 & p & & & 0 \\ & q & 0 & p & & \\ & & \cdot & \cdot & \cdot & \\ 0 & & & q & 0 & p \\ 0 & 0 & & & 0 & 1 \end{pmatrix}$$

where $0 < p < 1$, $q = 1 - p$. This Markov chain models a gamble where the gambler wins with probability p and loses with probability q at each step. Reaching state 0 corresponds to the gambler's ruin.

i) Denote by S_n the gambler's capital at step n. Show that $Y_n = \left(\frac{q}{p}\right)^{S_n}$ is a martingale (known as DeMoivre's martingale). [4]

ii) Using the theory of stopping time, derive the ruin probability for initial capital i $(0 < i < N)$. [4]

c) Derive the average duration T_i of the game for the gambler starting from state i.

[Hint: Show that T_k satisfies the iteration $T_k = 1 + pT_{k+1} + qT_{k-1}$ under the initial conditions $T_0 = T_N = 0$.]

[10]