Probability and Stochastic Processes
2014

[A] — new application
[B] — bookwork
[E] — new example
[T] — new theory

Q1

a)

$$p(ill \mid +) = \frac{p(ill, +)}{p(+)} \quad [1A]$$

$$= \frac{p(+\mid ill)\, p(ill)}{p(+\mid ill)\, p(ill) + p(+\mid healty)\, p(healthy)} \quad [1A]$$

$$= \frac{0.9 \times 10^{-4}}{0.9\times10^{-4} + 0.1\times(1-10^{-4})} \quad [1A]$$

$$= \frac{9}{9 + 10^4 - 1} \quad [1A]$$

$$= \frac{9}{10008} \approx 9\times10^{-4} \quad [1A]$$

b) If $y = \tan^{-1}x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$ [1A]

So $$f_Y(y) = \frac{1}{dy/dx} f_X(x) = (1+x^2)\frac{\alpha/\pi}{\alpha^2+x^2} \quad [1A]$$

$$= (1+x^2) f_X(\tan y) \quad [1A]$$

$$= [1+(\tan y)^2]\frac{\alpha/\pi}{\alpha^2+(\tan y)^2} \quad [1A]$$

We observe that if this is uniform, then [1A]

$\alpha = 1$

c) i) $f_Z(z)$ is the convolution of $f_X(x)$ and $f_Y(y)$.

$$f_Z(z) = \int_0^z f_X(z-y) f_Y(y)\, dy$$ [2E]

$$= \int_0^z e^{-(z-y)} e^{-y} dy$$ [2E]

$$= \int_0^z e^{-z} dy$$

$$= z e^{-z} \qquad z > 0$$ [1 E]

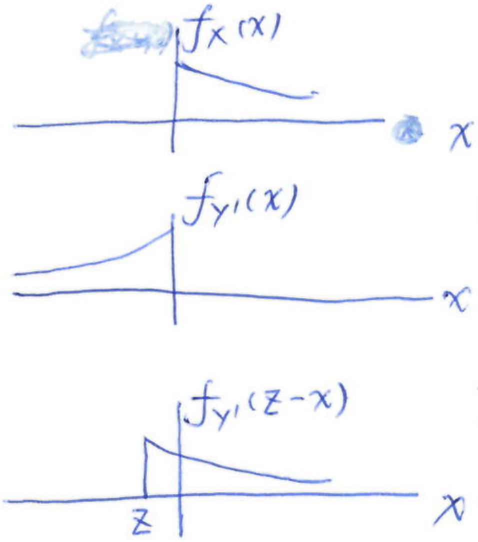
ii) Define $Y' = -Y$ so that $Z = X + Y'$.

We note the pdf of Y' is given by

$$f_{Y'}(y') = e^{y'} \qquad y' < 0$$

$f_Z(z)$ is the convolution of $f_X(x)$ and $f_{Y'}(y')$. [1 E]

$$f_Z(z) = f_X(z) \circledast f_{Y'}(z)$$



$$= \begin{cases} \int_0^\infty e^{-x} e^{z-x} dx, & z < 0 \\ \int_z^\infty e^{-x} e^{z-x} dx, & z > 0 \end{cases}$$ [2E]

$$= \begin{cases} \frac{1}{2} e^{z}, & z < 0 \\ \frac{1}{2} e^{-z}, & z > 0 \end{cases}$$ [1 E]

[1 E]

$$= \frac{1}{2} e^{-|z|} \qquad -\infty < z < \infty$$

iii) $F_Z(z) = P(Z \le z) = P(XY \le z)$ [1 E]

$$= \int_0^\infty \int_0^{z/y} f_X(x) f_Y(y)\, dx\, dy$$ [1 E]

$$f_Z(z) = \int_0^\infty \frac{1}{y} f_X\left(\frac{z}{y}\right) f_Y(y)\, dy$$ [2 E]

$$= \int_0^\infty \frac{1}{y} e^{-\left(\frac{z}{y} + y\right)} dy \qquad z > 0$$ [1 E]

Q2

a) For n samples we have

$$f(\underline{x}, c) = c^{4n}(x_1 \cdots x_n)^{3n} e^{-c(x_1 + \cdots + x_n)} \qquad [3E]$$

$$\frac{\partial f(\underline{x}, c)}{\partial c} = 4n \cdot c^{4n-1} \cdot (x_1 \cdots x_n)^{3n} e^{-c(x_1 + \cdots + x_n)} \qquad [2E]$$

$$- (x_1 + \cdots + x_n)\, c^{4n} (x_1 \cdots x_n)^{3n} e^{-c(x_1 + \cdots + x_n)}$$

$$= \left[\frac{4n}{c} - (x_1 + \cdots + x_n)\right] f(\underline{x}, c) \qquad [2E]$$

$$= 0$$

$$c = \frac{4n}{x_1 + \cdots + x_n} \qquad [2E]$$

In this problem, $n = 5$

$$c = \frac{4 \times 5}{30} = \frac{4}{6} = \frac{2}{3} \qquad [2E]$$

b) i) Note that the tranfer function is

$$H(z) = \frac{1}{1-\alpha z^{-1}} = \sum_{n=0}^{\infty} \alpha^n z^{-n} \qquad [2B]$$

so

$$h(n) = \alpha^n \qquad n \geq 0 \qquad [2B]$$

Therefore,

$$\begin{aligned} R_y(n) &= R_x(n) \circledast h(-n) \otimes h(n) \\ &= h(-n) \otimes h(n) \end{aligned} \qquad [2B]$$

Since $R_x(n) = \delta(n)$.

$$R_y(n) = \begin{cases} \sum_{k=0}^{\infty} \alpha^{-(n-k)} \alpha^k & n < 0 \\ \sum_{k=n}^{\infty} \alpha^{-(n-k)} \alpha^k & n > 0 \end{cases} \qquad [2B]$$

$$= \begin{cases} \alpha^{-n} \sum_{k=0}^{\infty} \alpha^{2k} & n < 0 \\ \alpha^{n} \sum_{k=0}^{\infty} \alpha^{2k} & n > 0 \end{cases} \qquad [1B]$$

$$= \begin{cases} \alpha^{-n} \frac{1}{1-\alpha^2} & n < 0 \\ \alpha^{n} \frac{1}{1-\alpha^2} & n > 0 \end{cases} \qquad [1B]$$

$$= \alpha^{|n|} \frac{1}{1-\alpha^2} \qquad [1B]$$

ii) The Wiener-Hopf equation reads

$$\begin{pmatrix} R_y(0) & R_y(1) & \cdots & R_y(n-1) \\ R_y(1) & R_y(0) & \cdots & R_y(n-2) \\ & \vdots & & \\ R_y(n-1) & \cdots & & R_y(0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \\ c_n \end{pmatrix} = \begin{pmatrix} R_y(n) \\ R_y(n-1) \\ \\ R_y(1) \end{pmatrix}$$

that is

$$\begin{pmatrix} 1 & \alpha & \cdots & \alpha^{n-1} \\ \alpha & 1 & \cdots & \alpha^{n-2} \\ & & \vdots & \\ \alpha^{n-1} & & \cdots & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \\ c_n \end{pmatrix} = \begin{pmatrix} \alpha^n \\ \alpha^{n-1} \\ \vdots \\ \alpha \end{pmatrix}$$ [1 E]

whose solution is

$$c_n = \alpha, \quad c_i = 0 \quad i < n$$

Consequently, the MMSE prediction is

$$y(n+1) = \alpha y(n)$$ [1 E]

The mean-square error is given by

$$\begin{aligned} MSE &= E[y(n+1) - \alpha y(n)]^2 \\ &= E[y^2(n+1) - 2\alpha y(n+1) y(n) + \alpha^2 y^2(n)] \\ &= R_y(0) - 2\alpha R_y(1) + \alpha^2 R_y(0) \\ &= \frac{1 - 2\alpha^2 + \alpha^2}{1 - \alpha^2} \\ &= 1 \end{aligned}$$ [1 E]

Q3

a) i) $E[X(t)] = E[A_t \cos(\omega t + \theta)]$

$= E[A_t] \cdot \cos(\omega t + \theta)$ [2 A]

$= 0$

$E[X^2(t)] = E[A_t^2] \cos^2(\omega t + \theta)$

$= \sigma^2 \cos^2(\omega t + \theta) = Var[X(t)]$ [2 A]

Since the variance is a function of t, it is not stationary. [1 A]

ii) $E[X(t)] = E[A_t] \cdot E[\cos(\omega t + \theta)]$

$= 0$ [1 A]

$E[X(t) X(t+\tau)] = E[A_t A_{t+\tau}] \, E[\cos(\omega t + \theta) \cos(\omega(t+\tau) + \theta)]$

$$= \begin{cases} 0 & \tau \neq 0 \\ \sigma^2 E[\cos^2(\omega t + \theta)] & \tau = 0 \end{cases}$$

$$= \begin{cases} 0 & \tau \neq 0 \\ \frac{\sigma^2}{2} & \tau = 0 \end{cases}$$ [3 A]

$= \frac{\sigma^2}{2} \delta(\tau)$ only a function of τ

So it is wide-sense stationary. [ ] A]

b) i) By Chebyshev's inequality [1 E]

$$P\{|X(t+\tau) - X(t)| > a\} \leq \frac{E[|X(t+\tau) - X(t)|^2]}{a^2}$$ [2 E]

$$= \frac{2[R(0) - R(\tau)]}{a^2}$$ [2 E]

ii) $$\sum_{i,k} a_i a_k^* R(\tau_i - \tau_k)$$

[2B]

$$= \sum_{i,k} a_i a_k^* \frac{1}{2\pi} \int S(w) e^{jw(\tau_i - \tau_k)} dw$$ inverse Fourier transform

[2B]

$$= \frac{1}{2\pi} \int S(w) \left| \sum_i a_i e^{jw\tau_i} \right|^2 dw$$ rearragement

[1B]

$$\geq 0$$

iii) If we suppress w_2 in $\phi(w_1, w_2)$, we recover the Characteristic function of a Gaussian r.v.

$$\phi(w) = \exp\left(-\frac{\sigma^2 w^2}{2}\right) = E[e^{jwX}]$$

Then

$$E[Y(t)] = E[I e^{aX(t)}]$$

[2E]

$$= I \, E[e^{aX(t)}]$$

$$= I \exp\left(\frac{\sigma^2 a^2}{2}\right)$$ definition of C.F.

$$= I \exp\left(\frac{\sigma^2 R(0)}{2}\right)$$ $\sigma^2 = R(0)$

Meanwhile,

$$R_Y(\tau) = E[Y(t) Y(t+\tau)]$$

[2E]

$$= I^2 E[e^{aX(t)} e^{aX(t+\tau)}]$$

$$= I^2 \exp\left(\frac{\sigma^2 a^2 + 2R(\tau) a^2 + \sigma^2 a^2}{2}\right)$$ definition of C.F.

$$= I^2 \exp\{\sigma^2 [R(0) + R(\tau)]\}$$

[1E]

$$\phi(w_1, w_2) = E[e^{j(X_1 w_1 + X_2 w_2)}]$$

Q4

a) $E[S_{n+1} \mid S_n, \cdots, S_1]$

$= E[X_1 + X_2 + \cdots + X_n + X_{n+1} \mid S_n, \cdots, S_1]$ [2 E]

$= E[S_n + X_{n+1} \mid S_n, \cdots, S_1]$

$= S_n + E[X_{n+1} \mid S_n, \cdots, S_1]$ [2 E]

$= S_n + 0$

$= S_n$ [2 E]

Therefore, $\{S_n\}$ is a martingale.

b) Denote by π the limiting distribution.

$$\pi = \pi P$$ [2 E]

$\pi_1 = \pi_1 q + \pi_m p \Rightarrow \pi_1 = \pi_m$

$\pi_2 = \pi_1 p + \pi_2 q \Rightarrow \pi_2 = \pi_1$ [3 E]

$\pi_3 = \pi_2 p + \pi_3 q \Rightarrow \pi_3 = \pi_2$

$\vdots$

$\pi_m = \pi_{m-1} p + \pi_m q \Rightarrow \pi_m = \pi_{m-1}$

Thus, $\pi_i = \frac{1}{m} \quad 1 \le i \le m$ [2 E]

c) i) $P(X_n = j \mid X_{n+1} = i)$

$= P(X_n = j \mid X_{n+1} = i, X_{n+2}, X_{n+3}, \cdots)$

This is needed to prove the reversed chain is Markov. (We need to prove this.)

$$P(X_n=j \mid X_{n+1}, X_{n+2}, X_{n+3}, \cdots)$$

$$= \frac{P(X_n=j, X_{n+1}=i, X_{n+2}=k, \cdots)}{P(X_{n+1}=i, X_{n+2}=k, \cdots)} \qquad [1T]$$

$$= \frac{q_j P_{ji} P_{ik} \cdots}{q_i P_{ik} \cdots}$$ forward chain is Markov and it is steady

$$= \frac{q_j P_{ji}}{q_i}$$

$$= \frac{P(X_n=j, X_{n+1}=i)}{P(X_{n+1}=i)} = P(X_n=j \mid X_{n+1}=i) \qquad [1T]$$

ii) If the chain is reversible,

$$P_{ij} P_{jk} P_{ki} = P^*_{ij} P^*_{jk} P^*_{ki}$$ reversible [1T]

$$= \frac{q_j P_{ji}}{q_i} \cdot \frac{q_k P_{kj}}{q_j} \cdot \frac{q_i P_{ik}}{q_k} \qquad [2T]$$

$$= P_{ik} P_{kj} P_{ji} \qquad [2T]$$

iii) We need show $\{q_i\}$ is invariant under transformation P, i.e.,

$$\underline{q} = \underline{q} P \qquad [1T]$$

The jth element of $\underline{q}P$ is

$$\sum_i q_i P_{ij} \overset{(1)}{=} \sum_i q_j P_{ji} \overset{(2)}{=} q_j \sum_i P_{ji} \overset{(3)}{=} q_j$$

where [3T]

(1) is due to reversibility $P_{ij} q_i = P_{ji} q_j$

(2) is by rearrangement

(3) is because $\sum_i P_{ji} = 1$

Therefore, $\{q_i\}$ is the steady state distribution [1T]