Wavelets, Sparsity and their Applications

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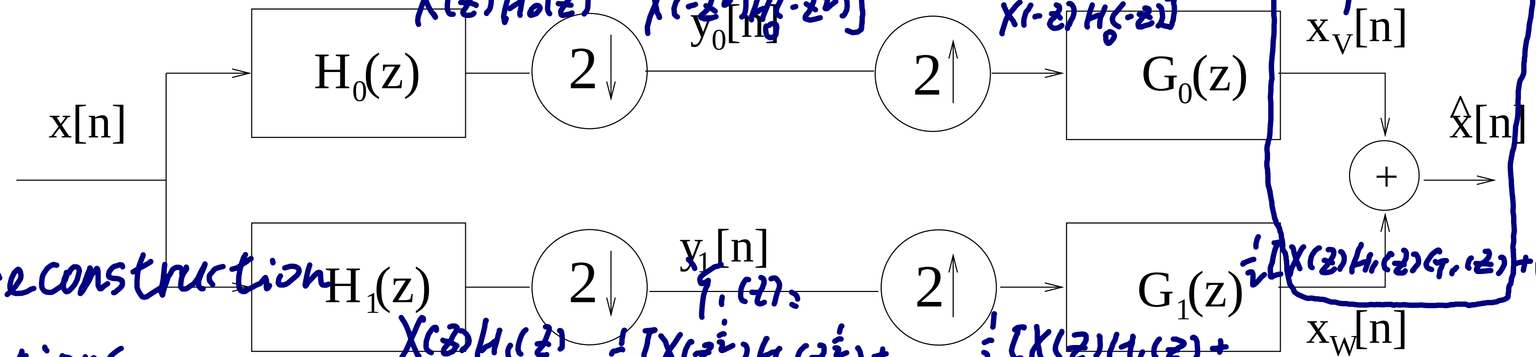
Session three: Filter Banks

$X(z) = (-1)^n \sum_n h[n] z^{-\frac{n}{2}} = H(-z^{\frac{1}{2}})$

$\downarrow\ Y(z) = X(z^2)$

$Y(z) = (-1)^n \sum_n h[n] z^{-n} = H(-z)$

Two-channel Filter Banks



$X(z)H_0(z)$ $\quad Y_0(z) = \frac{1}{2}[X(z^{\frac{1}{2}})H_0(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})H_0(-z^{\frac{1}{2}})]$ $\quad \frac{1}{2}[X(z)H_0(z) + X(-z)H_0(-z)]$

$X(z)$ $\quad \frac{1}{2}[X(z)H_0(z)G_0(z) + X(-z)H_0(-z)G_0(z)]$

V: subspace spanned by shifting $g_0[n]$

$X(z)H_1(z)$ $\quad Y_1(z) = \frac{1}{2}[X(z^{\frac{1}{2}})H_1(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})H_1(-z^{\frac{1}{2}})]$ $\quad \frac{1}{2}[X(z)H_1(z) + X(-z)H_1(-z)]$

$\frac{1}{2}[X(z)H_1(z)G_1(z) + X(-z)H_1(-z)G_1(z)]$

Perfect reconstruction conditions:

$\therefore \begin{cases} H_0(z)G_0(z) + H_1(z)G_1(z) = 2 \\ H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0 \end{cases}$

Perfect Reconstruction (PR) Conditions

$$\boxed{\begin{aligned} &H_0(z)G_0(z) + H_1(z)G_1(z) = 2 \qquad \text{(distortion-free)} \\ &H_0(-z)G_0(z) + H_1(-z)G_1(-z) = 0 \qquad \text{(aliasing-free)} \end{aligned}}$$

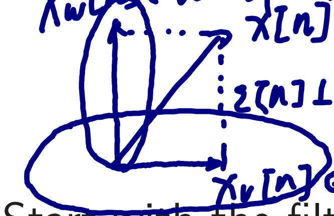
W: orthogonal subspace spanned by shifting $g_1[n]$

Goal: To find systematic ways to design h_0, h_1, g_0, g_1 such that the PR conditions are satisfied.

$\langle g_0[n], g_0[n-2k] \rangle = \delta_k$

$h_0[n] = g_0[-n]$ $\Rightarrow$ orthogonal projection

Orthogonal Filter Banks



inner product

$z[n] \perp V \Rightarrow \langle z[n], \text{any vector} \in V \rangle = 0$

$\Rightarrow \langle g_0[n-2l], g_1[n-2k] \rangle = 0$

$\therefore \langle g_0[n], g_1[n-2k] \rangle = 0$

as long as it holds, the 'error' of branch $g_0[n]$ is completely recovered by $g_1[n]$, since $g_0[n] \perp g_1[n]$.

Start with the filters we designed in the previous session:

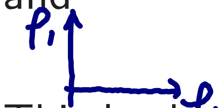
$$\langle g_0[n], g_0[n-2k]\rangle = \delta_k$$

$x[n] = x_V[n] + x_W[n]$

$= \sum_k \alpha_k g_0[k-2n] + \sum_l \beta_l g_1[k-2n]$

orthogonal

and



freedom only in $g_0[n]$.

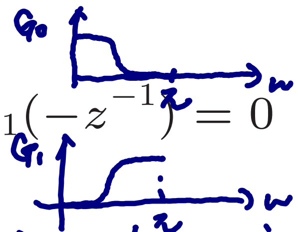
$$h_0[n] = g_0[-n].$$

This leads to an orthogonal projection. You want to recover the error of this projection with the lower branch of the filter bank.
Therefore design:

$$\langle g_0[n], g_1[n-2k]\rangle = 0 \longleftrightarrow G_0(z)G_1(z^{-1}) + G_0(-z)G_1(-z^{-1}) = 0$$

Curve $\Rightarrow$ bad LP filter

$\therefore$ desire derivative $= 0$



or choose $g_1[n] = (-1)^n g_0[1-n]$ (shift and modulation) and

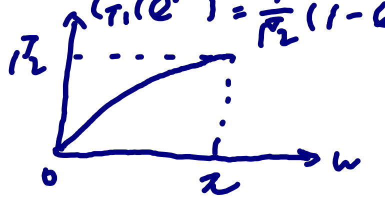
$$h_1[n] = g_1[-n].$$

$G_0(e^{jw}) = \frac{1}{\sqrt{2}}(1+e^{-jw})$

$g_0[n]$: low-pass filter

$\Rightarrow V$ 'smooth' subspace. main approximation of $x[n]$.

$G_1(e^{jw}) = \frac{1}{\sqrt{2}}(1-e^{jw})$



$g_1[n]$: high-pass filter

$\Rightarrow W$ complementary subspace. missing details of $x[n]$.

eg. $g_0[n]$ $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ ... 0 1 ... $\longleftrightarrow$ $G_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$

Orthogonal Filter Banks (cont'd)

$G_1(z) = -z^{-1}G_0(-z^{-1})$
$= -z^{-1}\frac{1}{\sqrt{2}}(1-z)$
$= \frac{1}{\sqrt{2}}(1-z^{-1})$

$H_0(z) = G_0(z^{-1}) = \frac{1}{\sqrt{2}}(1+z)$

$H_1(z) = G_1(z^{-1}) = \frac{1}{\sqrt{2}}(1-z)$

Orthogonal requirements:

To summarize, we have that:

$$\langle g_0[n], g_0[n-2k]\rangle = \delta_k \longleftrightarrow G_0(z)G_0(z^{-1}) + G_0(-z)G_0(-z^{-1}) = 2$$

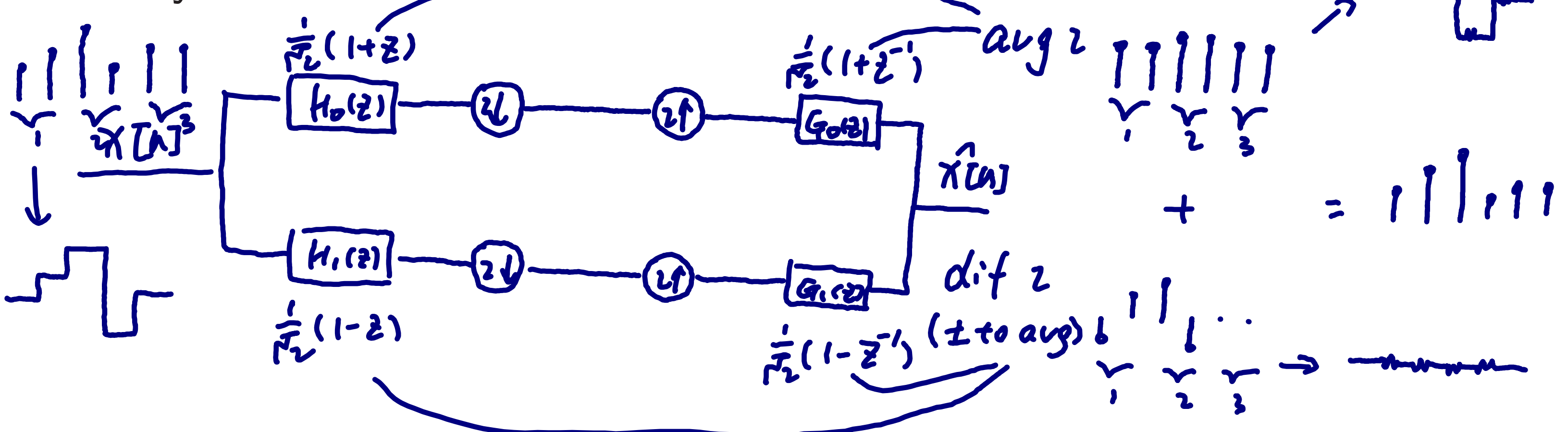
$$h_0[n] = g_0[-n] \longleftrightarrow H_0(z) = G_0(z^{-1})$$

$$g_1[n] = (-1)^n g_0[1-n] \longleftrightarrow G_1(z) = \underline{-z^{-1}}G_0(\underline{-z^{-1}})$$

$$h_1[n] = g_1[-n] \longleftrightarrow H_1(z) = G_1(z^{-1}).$$

$g_1[n] = (-1)^n g_0[1-n]$
$= (-1)^n g_0[-(n-1)]$
$\updownarrow$
$-z^{-1}G(-z^{-1})$
$n-1$ $(-1)^n$

By replacing the above equalities in the PR conditions, we can easily see that they are indeed satisfied!



$f(t) = \sum_n \alpha_n f_n(t)$

$\|f\|^2 = \sum_n |\alpha_n|^2$

$\Rightarrow \|x\|^2 = \|y_0[n]\|^2 + \|y_1[n]\|^2$

$P(z) = G_0(z) G_0(z^{-1})$

if $g_0[n]$ has $N+1$ taps.

$P(z) = \sum_{n=-N}^{N} P[n] z^{-n}$

Daubechies Filters

$P(z) = \cdots + z^2 P[-2] + z P[-1] + P[0] + z^{-1} P[1] + z^{-2} P[2] + \cdots$

$P(-z) = \cdots + (-1)^{-2} z^2 P[-2] + (-1)^{-1} z P[-1] + P[0] + (-1)^1 z^{-1} P[1] + (-1)^2 z^{-2} P[2] + \cdots$

We want to design $g_0[n]$ satisfying:

$P[n] z^n$

$\therefore P[0] = 1$

$P[2] = P[4] = \cdots = P[2n-2] = 0$

$$\langle g_0[n], g_0[n-2k] \rangle = \delta_k \longleftrightarrow G_0(z)G_0(z^{-1}) + G_0(-z)G_0(-z^{-1}) = 2.$$

Denote with $P(z) = G_0(z)G_0(z^{-1})$, it has to satisfy the *halfband* property:

$$P(z) + P(-z) = 2.$$

Moreover $P(z)$ is symmetric (i.e $p[n] = p[-n]$), since $P(z) = P(1/z)$. If $g_0[n]$ has $N+1$ taps, then the polynomial $z^N P(z)$ has degree $2N$ and can be factored as follows:

$G_0(z)$ $\quad$ $G_0(z^{-1})$ $\quad$ on unit circle

$$z^N P(z) = \alpha \prod_{i=1}^{M} (z - z_i)(z - \frac{1}{z_i}) \prod_{j=1}^{N-M} (z - z_j)^2.$$

$G_0(z)$

$G_0(z^{-1})$

i/o unit circle

eg. $G_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})^2 R(z)$

$G_1(z) = \frac{1}{\sqrt{2}}(1-z^{-1})^2 \tilde{R}(z)$

$H_0(z) = \frac{1}{\sqrt{2}}(1+z)^2 \tilde{Q}(z)$

$H_1(z) = \frac{1}{\sqrt{2}}(1-z)^2 Q(z)$

$H_1(z)|_{z=1} = \sum_n h_1[n] z^{-n}|_{z=1} = 0 \Rightarrow \sum_n h_1[n] = 0$

$z = e^{-jw}$ (HP)

Daubechies Filters (cont'd)

$\frac{dH_1(z)}{dz}|_{z=1} = -\sqrt{2}(1-z)Q(z) + \frac{1}{\sqrt{2}}(1-z)^2 Q'(z) = 0 \Rightarrow \sum -n h_1[n] z^{-(n-1)}|_{z=1} = -\sum_n n h_1[n] = 0$

i.e. split & extract $G_0(z)$ from $P(z)$.

Daubechies filters are obtained through spectral factorization of $P(z)$. They are the **shortest orthogonal FIR filters with maximum flat frequency responses at** $\omega = 0$ **and** $\omega = \pi$. The lowpass filters $g_0[n]$ have p zeros at π and have $2p$ coefficients.

$g_0[n]$: $2p$ coef.s $\begin{cases} p \text{ zeros at } \pi \\ P[0]=1,\ P[2]=P[4]=\cdots=P[2p-2]=0 \end{cases}$

The minimum number of requirements to satisfy the two conditions of orthogonality and flatness is $p + p = 2p$.

$x[n] = \alpha n + \beta \rightarrow \boxed{h_1[n]} \rightarrow y_n = x[n] * h_1[n] = \sum_k h_1[k](\alpha n + \beta - \alpha k) = (\alpha n + \beta)\sum_k h_1[k] - \alpha k \sum_k h_1[k] = 0$

- $G_0(e^{j\omega})$ has a zero of order p at π:

$G_0(z) = (\frac{1+z^{-1}}{2})^p R(z)$ has p zeros at π.

$H_1(z) = (\frac{1-z}{2})^p Q(z)$ has p zeros at 0.

$$G_0(e^{j\pi}) = G_0'(e^{j\pi}) = \ldots = G_0^{(p-1)}(e^{j\pi}) = 0.$$

In the z-domain this means that $G(z)$ must have a factor of the form $(\frac{1+z^{-1}}{2})^p$.

- $P(z)$ is halfband, namely

input, x, x_v, x_w

$$P(z) + P(-z) = 2.$$

[$g_0[n]$ should be as short to ensure locality]
[sparse]

Since $P(z)$ is symmetric this means that

$$p(0) = 1, \text{ and } p(2) = p(4) = \ldots = p(2p-2) = 0.$$

components of polinimial of order up to $p-1$ will be regarded as 'low-frequency'.

Daubechies Filters (cont'd)

Thus $G_0(e^{j\omega}) = \left(\frac{1+e^{-j\omega}}{2}\right)^p R(e^{j\omega})$, where $R(e^{j\omega})$ has degree $p-1$ and its p coefficients are chosen to satisfy the halfband condition.

$P(z)$ can be written as follows

$$P(z) = \left(\frac{1+z^{-1}}{2}\right)^p \left(\frac{1+z}{2}\right)^p R(z)R(z^{-1}).$$

Ingrid Daubechies found an explicit formula for $P(z)$ for any choice of p:

$$P(\omega) = 2\left(\frac{1+\cos\omega}{2}\right)^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} \left(\frac{1-\cos\omega}{2}\right)^k .$$

Example (p=2)

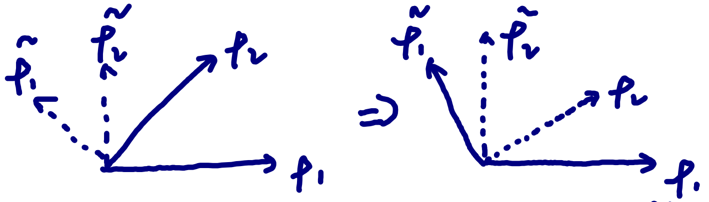
α: solution of $f(x)$

$$P(z) = \frac{1}{16}(1+z)^2(1+z^{-1})^2\underbrace{(-z+4-z^{-1})}_{f(x)} = \frac{1}{16\alpha}(1+z)^2(1+z^{-1})^2(1-\alpha z)(1-\alpha z^{-1})$$

$f(x)$: depends on Daubechies' formula.

with $\alpha = 2 \pm \sqrt{3}$ and

$$G_0(z) = \frac{1}{4\sqrt{\alpha}}(1+z^{-1})^2(1-\alpha z^{-1}) = \frac{1}{4\sqrt{2}}\left[(1+\sqrt{3})+(3+\sqrt{3})z^{-1}+(3-\sqrt{3})z^{-2}+(1-\sqrt{3})z^{-3}\right].$$



Biorthogonal Filters

design $f_1 \cdot f_2$ not aligned $\Rightarrow$ design $f_1 \cdot \tilde{f}_1$, $\langle f_1 \cdot \tilde{f}_1 \rangle = 1$ | $f_1 \to g_0[n]$, $\tilde{f}_1 \to g_1[n]$

Issue with orthogonal constructions:
With the exception of the Haar filter banks, it is not possible to design perfect-reconstruction real-valued linear-phase orthogonal filter banks.

Relax condition that $\boxed{h_0^T[n] = g_0[n] \quad H_0(z) = G_0(z^{-1})}$. $\Rightarrow \langle h_0^T[n-2l], g_0[n-2k] \rangle = \delta_{l,k}$ i.e. $\langle h_0^T[n], g_0[n-2k] \rangle = \delta_k$

notice! for bio. filters:
$\langle g_0[n], g_0[n-2k] \rangle \neq \delta_k$
$\langle g_0[n], g_1[n-2k] \rangle \neq 0$.
$\|x[n]\| \neq \|y_0[n]\| + \|y_1[n]\|$
not orthogonal, Parserval's theorem can't apply.

- Assume that $H_0(z)G_0(z) + H_0(-z)G_0(-z) = 2$.

- Design $g_1[n]$ orthogonal to $h_0^T[n]$ and $h_1^T[n]$ orthogonal to $g_0[n]$. That is, ~~$G_1(z) =$~~ $z^{-1}H_0(-z)$ and $H_1(z) = zG_0(-z)$.

$G_1(z) = -z^{-1}\tilde{G}_0(-z^{-1}) = \ominus z^{-1} H_0(-z)$
$H_1(z) = \tilde{G}_1(z^{-1}) = -z\tilde{G}_0(-z) = \ominus z G_0(-z)$
replace '−' with '+' at the same time.

- PR conditions again satisfied!

0 | bio

$P(z)=G_0(z)G_0(z^{-1})$ | $P(z)=H_0(z)G_0(z)$ [$P(z)$ can be the same].

$P(z)+P(-z)=2 \longleftrightarrow P(z)+P(-z)=2$

$P(z)=P(\frac{1}{z})$ | $P(z)$ don't have to be symmetric

SF is fixed | SF is arbitary.

(spectral factorisation)

Biorthogonal Filters (cont'd)

Biorthogonal filters can be constructed by spectral factorization of $P(z) = H_0(z)G_0(z)$. Here $P(z) + P(-z) = 2$

$g_0[n] \perp g_1[n]$ | $g_0[n] \perp h_1[n]$, $g_1[n] \perp h_0[n]$

In the biorthogonal case, we can assign the zeros arbitrarily and this leads to a variety of factorizations.

Example Consider the half-band filter of the previous example:

$$P(z) = \frac{1}{16\alpha}(1+z)^2(1+z^{-1})^2(1-\alpha z^{-1})(1-\alpha z).$$

In the biorthogonal case, we can assign the zeros arbitrarily. For instance we can have

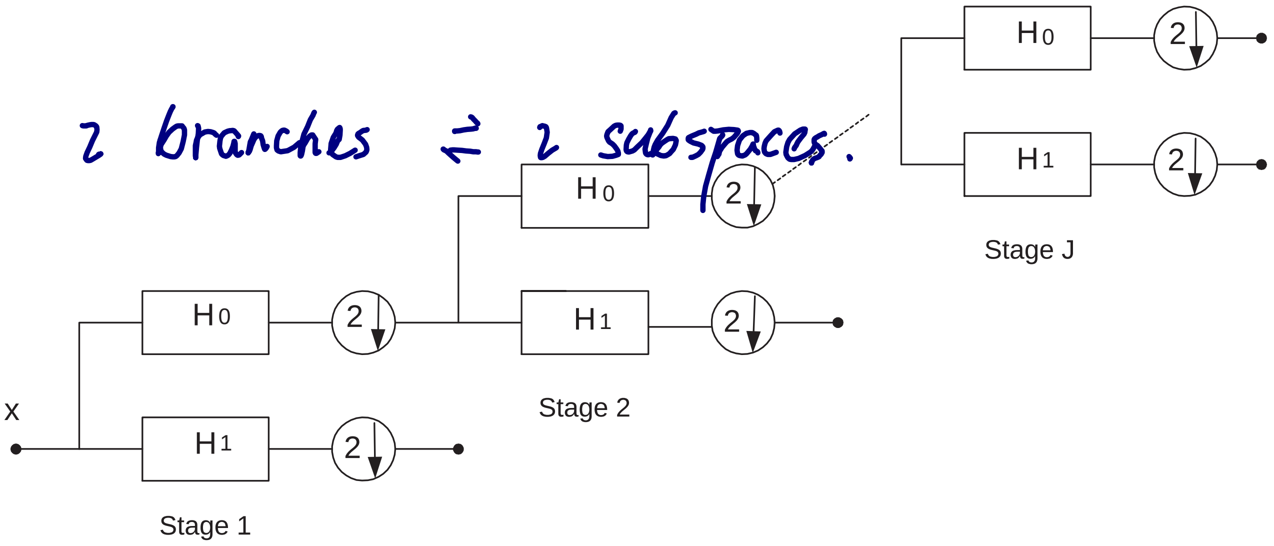
$$G_0(z) = \frac{1}{2\sqrt{2}}(1+z^{-1})^2 \text{ and } H_0(z) = \frac{\sqrt{2}}{8\alpha}(1+z)^2(1-\alpha z^{-1})(1-\alpha z).$$

These filters form the shortest symmetrical biorthogonal pair of order 2 (i.e., they have two zeros at π). They are known as the $5/3$ LeGall filters and are quite important since they are used in the new image compression standard (JPEG2000).

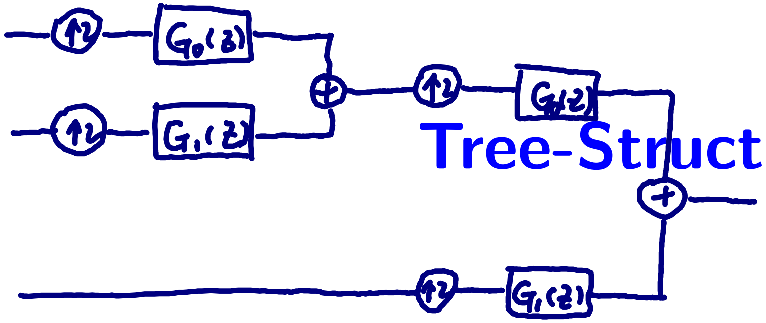
Tree-Structured Filter-Banks

A two-channel filter bank splits the input signal into two components. It is possible to iterate this process by splitting the two components again. Usually the process is iterated on the low-pass version.

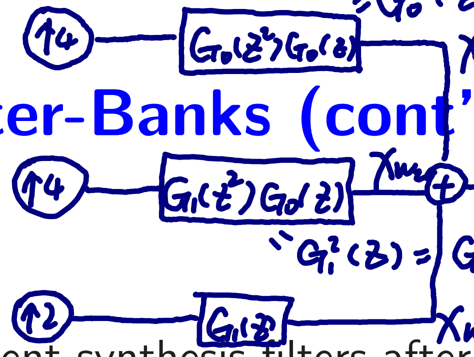
2 branches ⇄ 2 subspaces.



Tree-Structured Filter-Banks (cont'd)



Noble identities



$= G_0^2(z) = G_0(z)\cdot G_0(z^2)$

x_{V_2}: $V_2 = \mathrm{span}\{g_0^2[n-4k]\}$

x_{W_2}: $W_2 = \mathrm{span}\{g_1^2[n-4k]\}$

union(V_2, W_2) = $V_1 = \mathrm{span}\{g_0[n-2k]\}$

$= G_1^2(z) = G_0(z)G_1(z^2)$

x_{W_1}: $W_1 = \mathrm{span}\{g_1[n-2k]\}$

$= G_1^1(z) = G_1(z)$

Using Noble identities we can see that the equivalent synthesis filters after J iterations are

$V_3 \subset V_2 \subset V_1$

$W_3 \quad W_2 \quad W_1$

$$G_0^{(J)}(z) = G_0^{(J-1)}(z)G_0(z^{2^{J-1}}) = \prod_{k=0}^{J-1} G_0(z^{2^k})$$

last stage $\qquad$ prev. stages

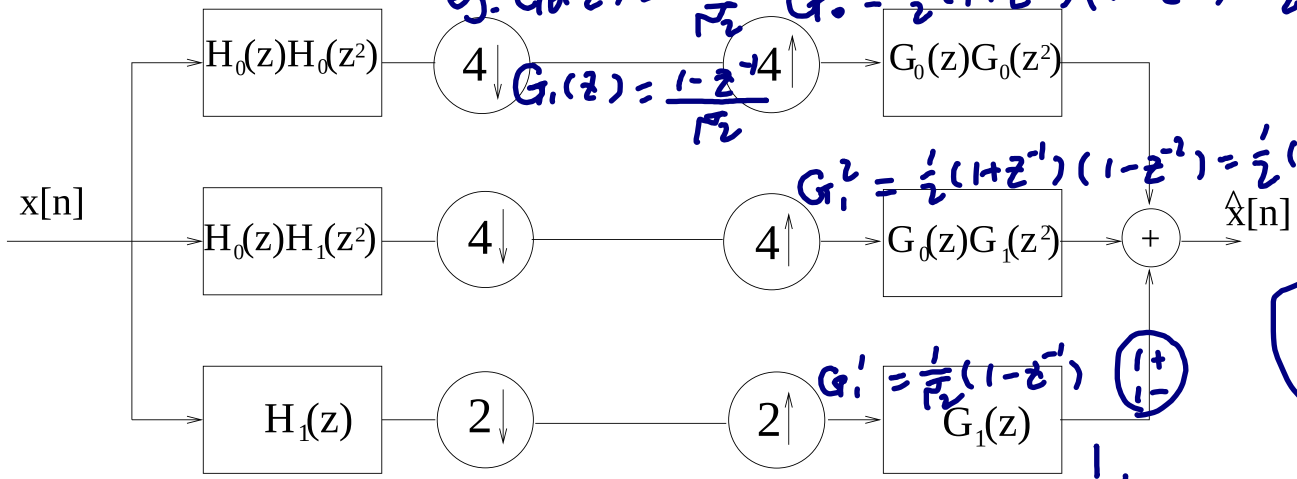
$$G_1^{(j)}(z) = G_0^{(j-1)}(z)G_1(z^{2^{j-1}}) = \underbrace{G_1(z^{2^{j-1}})}_{\text{last stage}} \underbrace{\prod_{k=0}^{j-2} G_0(z^{2^k})}_{\text{prev. stages}}, \quad j = 1, \ldots, J.$$

e.g. $G_0(z) = \frac{1+z^{-1}}{\sqrt{2}}$ $\quad G_0^2 = \frac{1}{2}(1+z^{-1})(1+z^{-2}) = \frac{1}{2}(1+z^{-1}+z^{-2}+z^{-3})$ (4+)

$G_1(z) = \frac{1-z^{-1}}{\sqrt{2}}$

$G_1^2 = \frac{1}{2}(1+z^{-1})(1-z^{-2}) = \frac{1}{2}(1+z^{-1}-z^{-2}-z^{-3})$ (2+ 2−)

$G_1^1 = \frac{1}{\sqrt{2}}(1-z^{-1})$ (1+ 1−)



G_1^j: same shape, formed by dilating.

Tree-Structured Filter-Banks (cont'd)

- Assume, for simplicity, orthogonal filter-banks.
- By construction, it follows that $V_j = \text{span}\{g_0^{(j)}[n - 2^j k]\}_{k\in\mathbb{Z}}$ and that $W_j = \text{span}\{g_1^{(j)}[n - 2^j k]\}_{k\in\mathbb{Z}}$ for $j = 1, \ldots, J$.
- Denote with $\varphi_{J,k}[n] = g_0^{(J)}[n - 2^J k]$ and with $\psi_{j,k} = g_1^{(j)}[n - 2^j k]$.
- We have that

$$x[n] = \sum_{k=-\infty}^{\infty} \langle x[n], \varphi_{J,k}[n]\rangle \varphi_{J,k}[n] + \sum_{j=1}^{J} \sum_{k=-\infty}^{\infty} \langle x[n], \psi_{j,k}[n]\rangle \psi_{j,k}[n].$$

$J \to \infty$: no LF component, the first part reduces to zero. ($V_\infty \Rightarrow 0$)

· The signal can be expressed by $\bigcup_i W_i$.

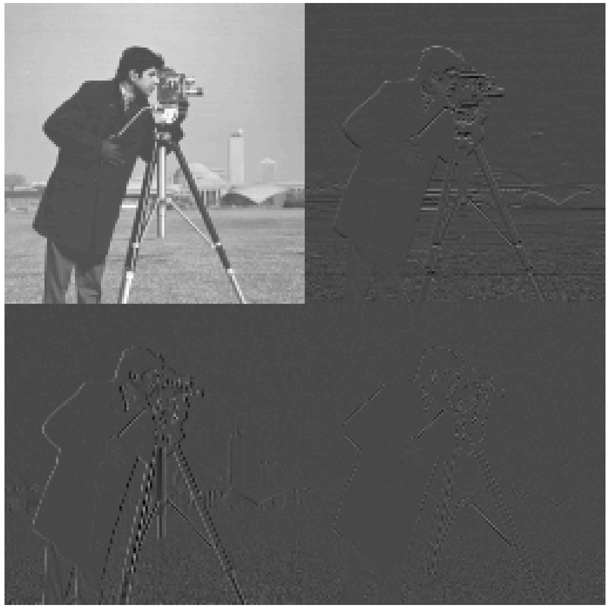
$x[n] = \sum_{j=1}^{+\infty} \sum_{k=-\infty}^{+\infty} \alpha_{n,m} \psi_{mn}(t)$ [discrete : $j = 1 \to +\infty$]

no need

$x(t) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \alpha_{n,m} \psi_{mn}(t)$ (concrete : $j = -\infty \to +\infty$)

The Wavelet Transform

The Wavelet Transform

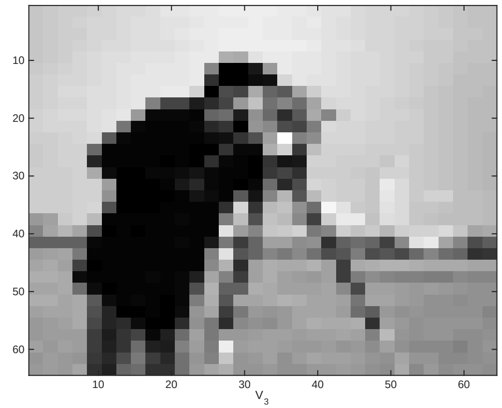


The Wavelet Transform

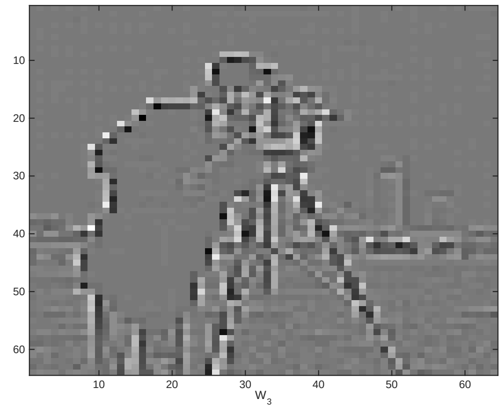
The Wavelet Transform

2-D 3-level Haar Decomposition of Cameraman

(a)Original



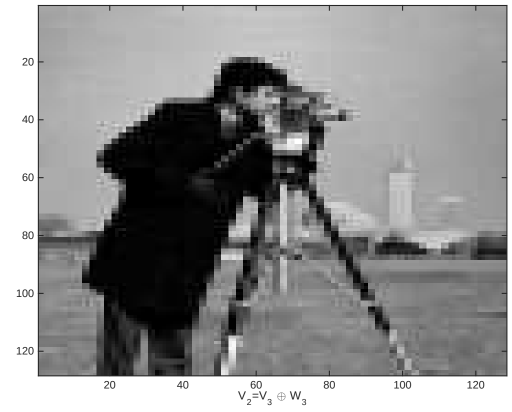
(b) V_3



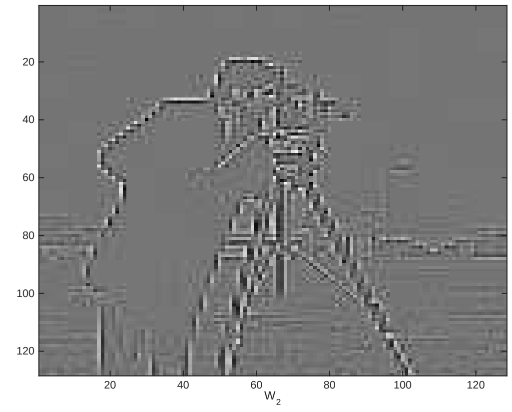
(c) W_3

2-D 3-level Haar Decomposition of Cameraman

(a)Original



(b) $V_2 = V_3 \oplus W_3$

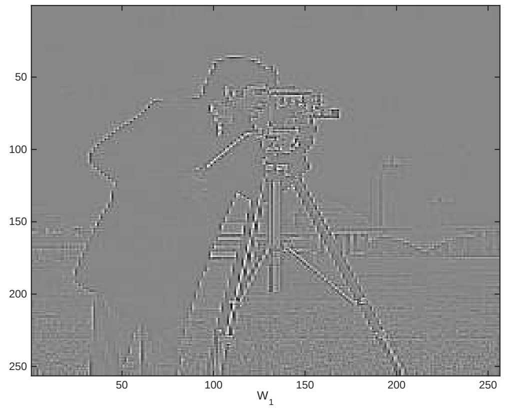


(c) W_2

2-D 3-level Haar Decomposition of Cameraman

(a)Original

(b) $V_1 = V_2 \oplus W_2$



(c) W_1