Lecture 13 & 14

X, Y independent RVs

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \qquad \forall x, y$$

$$E(g(X)\, h(Y)) = E(g(X))\, E(h(Y))$$

$$\mathrm{Cov}(X,Y) = 0$$

$$\rho = 0$$

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

$$\mathrm{Var}(aX+bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y)$$

$$(m_{X+Y}(t) = m_X(t)\, m_Y(t))$$

$$f_{X|Y}(x|y) = f_X(x) \qquad \forall x, y$$

X, Y independent RVs

$$X \sim N(\mu_X, \sigma_X^2)$$
$$Y \sim N(\mu_Y, \sigma_Y^2)$$

$$Z = aX + bY$$

$$Z \sim N(a\mu_X + b\mu_Y,\; a^2\sigma_X^2 + b^2\sigma_Y^2)$$

$X_1, \ldots, X_n$ independent RVs

$$X_i \sim N(\mu_i, \sigma_i^2)$$

$$Z = \sum a_i X_i$$

$$\sim N\Big(\sum_i a_i \mu_i,\; \sum_i a_i^2 \sigma_i^2\Big)$$

$X \sim Bin(n, p)$ $\quad\quad E(X) = np$

$\quad\quad\quad\quad\quad\quad\quad Var(X) = np(1-p)$

$X = X_1 + \cdots + X_n$ independent

$$X_i = \begin{cases} 1 & \text{success} \quad p \\ 0 & \text{failure} \quad 1-p \end{cases} \quad \text{Bernoulli}$$

$$E(X) = \sum_i E(X_i) = \underline{\underline{np}}$$

$$E(X_i) = 1 \times p + 0 \times (1-p) = p$$

$$Var(X) = \sum_i Var(X_i) = \underline{\underline{np(1-p)}}$$

$$Var(X_i) = (1-p)^2 p + (0-p)^2 (1-p)$$
$$= (1-p)p$$

$X_1 \sim N(\mu_1, \sigma_1^2)$

$X_2 \sim N(\mu_2, \sigma_2^2)$

$$2X_1 - X_2 \sim N(2\mu_1 - \mu_2, 4\sigma_1^2 + \sigma_2^2)$$

Change of Variable

$$X \longrightarrow Y = g(X)$$

$f_X(x)$ $\qquad$ $f_Y(y) = ?$ $\qquad\qquad$ $Y = aX + b$

$$X, Y \longrightarrow Z = g(X, Y)$$

$f_{X,Y}(x,y)$ $\qquad$ $f_Z(z) = ?$

- $Z = g(X, Y) = \max(X, Y)$

$X \sim EXPO(\lambda)$
$Y \sim EXPO(\lambda)$
X, Y independent

$$F_Z(z) = P(Z \le z) = P(\max(X,Y) \le z)$$

$$= P(X \le z \underset{\text{AND}}{\cap} Y \le z)$$

$$\overset{\text{ind.}}{=} P(X \le z)\, P(Y \le z)$$

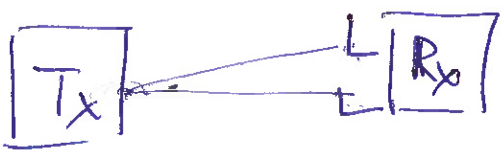
$$= \begin{cases} (1 - e^{-\lambda z})^2 & z > 0 \\ 0 & \text{oth} \end{cases}$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{oth} \end{cases}$$

$$F_X(u) = \begin{cases} 1 - e^{-\lambda u} & u > 0 \\ 0 & \text{oth} \end{cases}$$

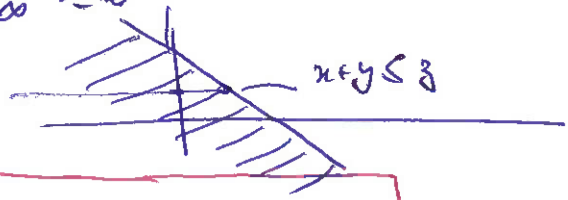
$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} \boxed{2(1 - e^{-\lambda z})\, \lambda e^{-\lambda z}} & z > 0 \\ 0 & \text{oth} \end{cases}$$

ex



- $Z = X + Y \qquad f_Z(z) = ?$

$$F_Z(z) = P(X+Y \leq z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{z-y} f_{X,Y}(x,y)\, dx\, dy$$



$$f_Z(z) = \frac{d}{dz} \int_{-\infty}^{+\infty} \boxed{\int_{-\infty}^{z-y} f_{X,Y}(x,y)\, dx}\, dy$$

Leibnitz's differentiation rule

$$H(z) = \int_{a(z)}^{b(z)} h(x,z)\, dx$$

$$\frac{d}{dz} H(z) = \frac{d\, b(z)}{dz} h\big(b(z), z\big) - \frac{d\, a(z)}{dz} h\big(a(z), z\big) + \int_{a(z)}^{b(z)} \frac{\partial}{\partial z} h(x,z)\, dx$$

$$f_Z(z) = 0 - 0 + \int_{-\infty}^{+\infty} \frac{\partial}{\partial z} \left(\int_{-\infty}^{z-y} f_{X,Y}(x,y)\, dx \right) dy$$

$$= \int_{-\infty}^{+\infty} \left(f_{X,Y}(z-y, y) - 0 + \underbrace{\int_{-\infty}^{z-y} \frac{\partial}{\partial z} f_{X,Y}(x,y)\, dx}_{0} \right) dy$$

$$= \int_{-\infty}^{+\infty} f_{X,Y}(z-y, y)\, dy$$

$$\overset{ind.}{=} \int_{-\infty}^{+\infty} f_X(z-y)\, f_Y(y)\, dy \qquad f_{X,Y}(x,y) \overset{ind}{=} f_X(x)\, f_Y(y)$$

$$= f_X(x) \otimes f_Y(y)$$

ex X, Y independent

$X, Y \sim EXPO(\lambda)$

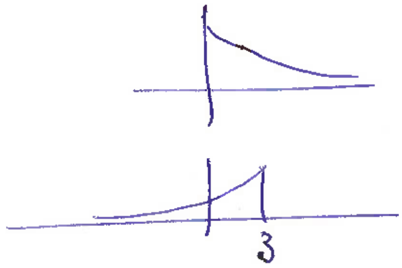
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x>0 \\ 0 & \text{oth} \end{cases}$$

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y>0 \\ 0 & \text{oth} \end{cases}$$

$Z = X+Y$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y)\, f_Y(y)\, dy$$

$$= \begin{cases} \int_0^z \lambda^2 e^{-\lambda(z-y)} e^{-\lambda y}\, dy = z\lambda^2 e^{-\lambda z} & z>0 \\ 0 & \text{oth} \end{cases}$$



$X, Y \longrightarrow U, V$

$U = R(X,Y)\ ,\ V = S(X,Y)$

if 1) $X = L(U,V)$ (one-to-one correspondence)
$Y = T(U,V)$

2) $\det\begin{pmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{pmatrix} \neq 0$

<u>then</u> $f_{U,V}(u,v) = |\det(J)|\, f_{X,Y}(x,y)$ $X,Y \longrightarrow U,V$

Reminiscent $f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right|$ $X \longrightarrow Y$

ex 1

$X \sim EXPO(1)$ $\quad X, Y$ independent

$Y \sim EXPO(\lambda)$

$$U = \frac{X}{X+Y} \qquad f_U(u) = ?$$

$$f_{X,Y} \longrightarrow f_{U,V} \longrightarrow f_U$$

$$\begin{cases} U = \frac{X}{X+Y} \\ V = X+Y \end{cases} \Rightarrow \begin{array}{l} X = UV \\ Y = V - X = V - UV = V(1-U) \end{array} \qquad \begin{array}{l} 0 < u < 1 \\ v > 0 \end{array}$$

$$\det(J) = \det\begin{pmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} \end{pmatrix} = \det\begin{pmatrix} V & U \\ -V & 1-U \end{pmatrix}$$

$$= V(1-U) + UV = V$$

$$f_{U,V}(u,v) = |\det(J)| \underbrace{f_{X,Y}(x,y)}_{\cancel{\lambda} e^{-\cancel{\lambda}(x+y)}} \qquad \begin{array}{l} f_X(x) = \lambda e^{-\lambda x} \quad x > 0 \\ f_Y(y) = \lambda e^{-\lambda y} \quad y > 0 \end{array}$$

$$= \begin{cases} v \cancel{\lambda} e^{-\cancel{\lambda} v} & v > 0,\ 0 < u < 1 \\ 0 & \text{oth} \end{cases}$$

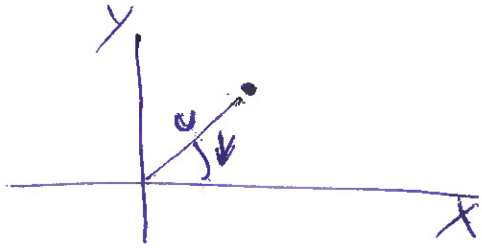
$$f_U(u) = \int_0^{+\infty} f_{U,V}(u,v)\, dv = \int_0^{+\infty} v e^{-v}\, dv = \begin{cases} 1 & 0 < u < 1 \\ 0 & \text{oth} \end{cases}$$

$$f_V(v) = \int_0^1 v e^{-v}\, du = \begin{cases} v e^{-v} & v > 0 \\ 0 & \text{oth} \end{cases}$$

ex 2 $\quad X, Y \sim N(0, \sigma^2)$ independent

$U = \sqrt{X^2 + Y^2}$

$V = \tan^{-1}\left(\frac{Y}{X}\right)$



$X = U \cos V$

$Y = U \sin V$

$f_U(u)$?

$f_{X,Y} \to f_{U,V} \to f_U$

$$f_{U,V}(u,v) = |\det(J)| \, f_{X,Y}(x,y)$$

$$\det(J) = \det\begin{pmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{pmatrix} = \det\begin{pmatrix} \cos V & -U \sin V \\ \sin V & U \cos V \end{pmatrix}$$

$$= U \qquad u > 0$$

$$f_{U,V}(u,v) = u \; \underbrace{f_{X,Y}(x,y)}_{\frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}} \qquad u^2$$

$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$

$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}$

$\boxed{Tx} \sim\sim \boxed{Rx}$

$$= \begin{cases} \frac{u}{2\pi\sigma^2} e^{-\frac{u^2}{2\sigma^2}} & u > 0 \\ & -\pi \le v < \pi \\ 0 & \text{oth} \end{cases}$$

Rayleigh

$$f_U(u) = \int_{-\pi}^{+\pi} \frac{u}{2\pi\sigma^2} e^{-\frac{u^2}{2\sigma^2}} \, dv = \begin{cases} \frac{u}{\sigma^2} e^{-\frac{u^2}{2\sigma^2}} & u > 0 \\ 0 & \text{oth} \end{cases}$$

$$f_V(v) = \int_0^{\infty} \frac{u}{2\pi\sigma^2} e^{-\frac{u^2}{2\sigma^2}} \, du = \frac{1}{2\pi} \left[-e^{-\frac{u^2}{2\sigma^2}} \right]_0^{+\infty}$$

$$= \frac{1}{2\pi}$$

LLN and CLT

RVs $X_1, \ldots X_n$ independent

$E(X_i) = \mu \quad , \forall i$

$Var(X_i) = \sigma^2 \quad , \forall i$

$S = X_1 + \cdots + X_n \qquad E(S) = n\mu \qquad Var(S) = n\sigma^2$

$\bar{X} = \dfrac{X_1 + \cdots + X_n}{n} \qquad E(\bar{X}) = \dfrac{n\mu}{n} = \mu \qquad Var(\bar{X})$

Sample Mean

$= \dfrac{1}{n^2} \, Var(X_1) + \cdots + Var(X_n)$

$= \dfrac{n\sigma^2}{n^2} = \dfrac{\sigma^2}{n}$

LLN (Law of Large Number)

$\bar{X}$ converges towards μ $\quad (\lim n \to \infty)$

$$\forall \varepsilon > 0, \ \lim_{n \to \infty} P(|\bar{X} - \mu| > \varepsilon) = 0$$

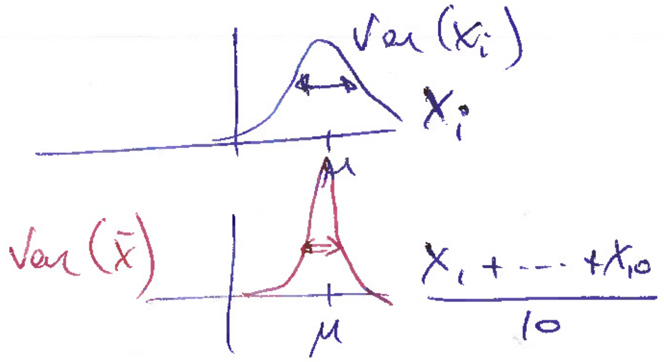
Proof Remind Chebyshev

$$\forall a > 0, \ P(|X - \mu| > a) \leq \frac{\sigma^2}{a^2}$$

Apply Chebyshev

$$\forall \varepsilon > 0, \ P(|\bar{X} - \mu| > \varepsilon) \leq \frac{Var(\bar{X})}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \xrightarrow{n \to \infty} 0$$

Ex



CLT (Central Limit Theorem)

$X_1, \dots, X_n$ independent and identically distributed $\sim f_X(x)$

$E(X_i) = \mu$
$Var(X_i) = \sigma^2$

$$S = X_1 + \dots + X_n \xrightarrow{n \to \infty} N(n\mu, n\sigma^2)$$

CLT

$X_1 \dots X_n$ iid $E(X_i) = \mu$, $Var(X_i) = \sigma^2$

$$P(S \leq a) = P\left(\frac{S - n\mu}{\sqrt{n\sigma^2}} \leq \frac{a - n\mu}{\sqrt{n\sigma^2}} \right)$$

$$\underset{n \to \infty}{\approx} P\left(\underset{\sim N(0,1)}{Z} < \frac{a - n\mu}{\sqrt{n\sigma^2}} \right)$$

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statistics

$50\ \Omega$ $\quad 50 + \underset{\sim N(0,\sigma^2)}{\text{noise}}$

$50.2\ \Omega$	51	49.5	48	51.5
x_1	x_2	x_3	x_4	x_5
$X_1 \sim N(\underset{=50}{\mu}, \sigma^2)$	$X_2 \sim N(\mu, \sigma^2)$	X_3	X_4	X_5

$$\bar{X} = \frac{X_1 \dots X_5}{5} \qquad \bar{x} = \frac{x_1 + \dots + x_5}{5}$$

statistic: quantity calculated from sample data
- RV

Random sample $X_1 \dots X_n$ independent and identically distributed $f_X(x)$

$\bar{X}$ $\quad E(\bar{x}) = \mu$
$\quad Var(\bar{x}) = \frac{\sigma^2}{n}$

$X_1 \dots X_n \sim N(\mu, \sigma^2)$ random sample

$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$f_X \sim N(\mu, \sigma^2)$ $\qquad \hat{\mu} \qquad \hat{\sigma}^2$

use $\bar{X}$ as an estimator of μ

option 1 $\quad \hat{\mu} = \bar{X} = \dfrac{X_1 + \dots + X_n}{n}$ estimator

$\bar{x} = \dfrac{x_1 + \dots + x_n}{n}$ estimate

option 2 $\quad \dfrac{X_1 + X_{10}}{2}$ estimator

$\dfrac{x_1 + x_{10}}{2}$ estimate

Properties

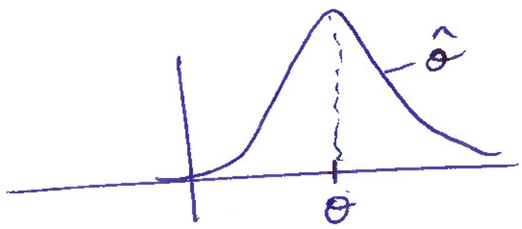
$\hat{\theta}$ estimator of θ

$\hat{\mu}$ " μ

1) unbiased vs biased estimator

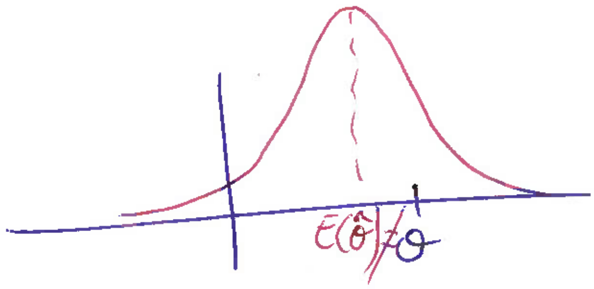
unbiased $\hat{\theta}$ $\qquad E(\hat{\theta}) = \theta$

bias $E(\hat{\theta}) - \theta = 0$



biased $\hat{\theta}$ $\qquad E(\hat{\theta}) \neq \theta$

bias $E(\hat{\theta}) - \theta \neq 0$



- use $\bar{X}$ as an estimator of μ

$X_1 \dots X_n$ random sample f_X

$E(X_i) = \mu$

$\bar{X} = \dfrac{X_1 + \dots + X_n}{n}$

$E(\bar{X}) = \dfrac{\sum E(X_i)}{n} = \dfrac{n\mu}{n} = \mu$

$\bar{X}$ is an unbiased estimator of μ

• S^2 estimator of σ^2

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$$ is an unbiased estimator of σ^2

$$= \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\mu+\mu-\bar{X})^2$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}(X_i-\mu)^2 + \underbrace{\sum_{i=1}^{n}(\mu-\bar{X})^2}_{n(\mu-\bar{X})^2} + \underbrace{2\sum_{i=1}^{n}(X_i-\mu)(\mu-\bar{X})}_{2n(\mu-\bar{X})(\bar{X}-\mu) = -2n(\mu-\bar{X})^2}\right]$$

$$(\mu-\bar{X})\underbrace{\left(\sum_{i=1}^{n}X_i-\mu\right)}_{=n\bar{X}-n\mu = n(\bar{X}-\mu)}$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}(X_i-\mu)^2 - n(\mu-\bar{X})^2\right]$$

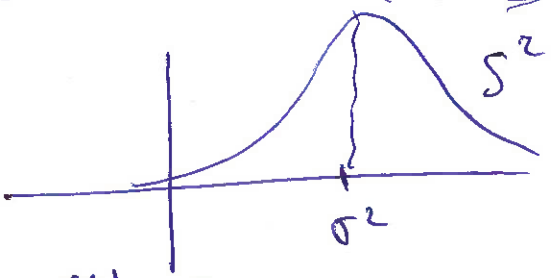
$$E(S^2) = \frac{1}{n-1}\left[\sum_{i=1}^{n}\underbrace{E((X_i-\mu)^2)}_{\substack{Var(X_i)\\ =\sigma^2}} - n\,E((\underset{E(\bar{X})}{\mu}-\bar{X})^2)\right]$$

~~Var(X)=~~ $E(\bar{X}) = \mu$

$$= \frac{1}{n-1}\left[n\sigma^2 - n\,Var(\bar{X})\right]$$

$$Var(\bar{X}) = \frac{\sigma^2}{n}$$

$$= \frac{1}{n-1}\left[n\sigma^2 - \sigma^2\right] = \sigma^2$$



Minimum variance unbiased estimator (MVUE)

$E(\hat{\theta}_1) = \theta$

$E(\hat{\theta}_2) = \theta$