Wavelets, Sparsity and their Applications

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Session Six: Wavelets from Iterated Filter-banks

Equivalent Filters and Functions

Let $g_0[n]$ and $g_1[n]$ denote low-pass and high-pass filters, respectively, and, for simplicity, assume that this is an orthogonal filter bank. The equivalent filters $g_0^{(i)}[n], g_1^{(i)}[n]$ after i steps of iteration are given by:

$$G_0^{(i)}(z) = \prod_{k=0}^{i-1} G_0(z^{2^k})$$

$$G_1^{(i)}(z) = G_1(z^{2^{i-1}}) \prod_{k=0}^{i-2} G_0(z^{2^k}).$$

Let us define a continuous-time function associated with $g_0^{(i)}[n]$ and $g_1^{(i)}[n]$ in the following way:

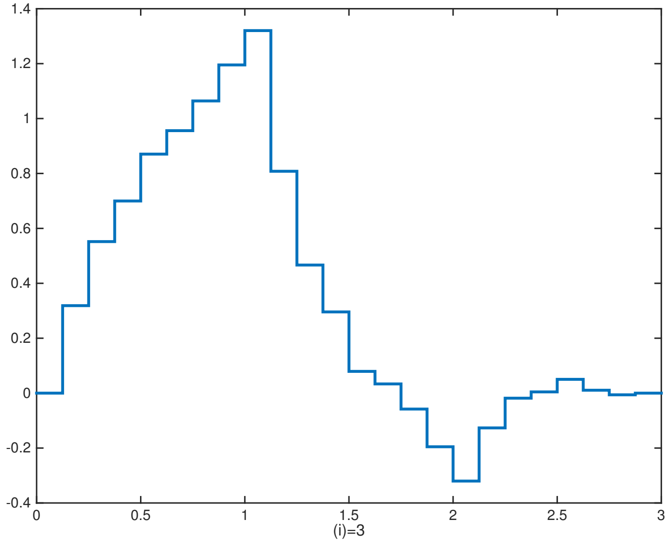
$$\varphi^{(i)}(t) = 2^{i/2} g_0^{(i)}[n], \quad n/2^i \leq t < (n+1)/2^i$$

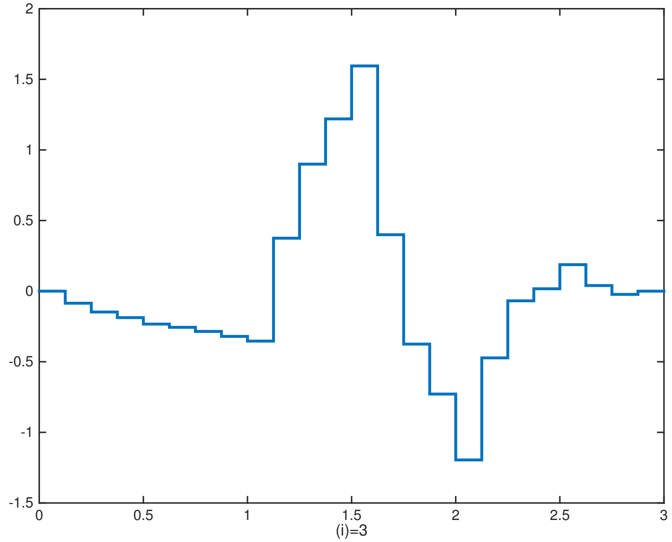
$$\psi^{(i)}(t) = 2^{i/2} g_1^{(i)}[n], \quad n/2^i \leq t < (n+1)/2^i$$

Equivalent Filters and Functions (example)

$$\varphi^{(i)}(t) = 2^{i/2} g_0^{(i)}[n], \qquad n/2^i \le t < (n+1)/2^i$$

$$\psi^{(i)}(t) = 2^{i/2} g_1^{(i)}[n], \qquad n/2^i \le t < (n+1)/2^i$$





Equivalent Filters and Functions

Since

$$\varphi^{(i)}(t) = \sum_n g_0^{(i)}[n] 2^{i/2} \delta(2^i t - n) * \mathbf{1}[0, 2^{-i}],$$

in the Fourier domain, using $M_0(\omega) = G_0(e^{j\omega})/\sqrt{2}$ and $M_1(\omega) = G_1(e^{j\omega})/\sqrt{2}$, we have that

$$\hat{\varphi}^{(i)}(\omega) = \Theta^{(i)}(\omega) \prod_{k=1}^{i} M_0\left(\frac{\omega}{2^k}\right)$$

where

$$\Theta^{(i)}(\omega) = e^{-j\omega/2^{i+1}} \frac{\sin(\omega/2^{i+1})}{\omega/2^{i+1}}$$

and

$$\hat{\psi}^{(i)}(\omega) = M_1\left(\frac{\omega}{2}\right) \Theta^{(i)}(\omega) \prod_{k=2}^{i} M_0\left(\frac{\omega}{2^k}\right).$$

Equivalent Filters and Functions

Now assume that the limit for $i \to \infty$ of the two functions $\varphi^{(i)}(t), \psi^{(i)}(t)$ exists and is well defined. Let $\varphi(t), \psi(t)$ denote the two limit functions, that is

$$\varphi(t) = \lim_{i\to\infty} \varphi^{(i)}(t),$$

$$\psi(t) = \lim_{i\to\infty} \psi^{(i)}(t).$$

In the Fourier domain

$$\hat{\varphi}(\omega) = \lim_{i\to\infty} \hat{\varphi}^{(i)}(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right)$$

$$\hat{\psi}(\omega) = \lim_{i\to\infty} \hat{\psi}^{(i)}(\omega) = M_1\left(\frac{\omega}{2}\right) \prod_{k=2}^{\infty} M_0\left(\frac{\omega}{2^k}\right)$$

since $\Theta^{(i)}(\omega)$ tends to 1 as $i \to \infty$.

Equivalent Filters and Functions

We first need to establish some necessary conditions for the limit to exist.

Theorem 1. *For the limit $\varphi(t) = \lim_{i\to\infty} \varphi^{(i)}(t)$ to exist, it is necessary that $G_0(e^{j\omega}) = \sqrt{2}$ for $\omega = 0$ and $G_0(e^{j\omega}) = 0$ for $\omega = \pi$.*

Claim: Given that the limits exist, the so obtained functions are indeed a scaling function and a wavelet.

Proof:
We show that $\varphi(t)$ satisfies the criteria of a valid scaling function. First of all it satisfies the two scale equation. In frequency domain the equation can be written as:

$$\varphi(t) = \sqrt{2}\sum_n g_0[n]\varphi(2t-n) \Leftrightarrow \frac{1}{\sqrt{2}}G_0\left(e^{j\omega/2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right).$$

By construction

$$\hat{\varphi}(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right) = M_0\left(\frac{\omega}{2}\right)\prod_{k=2}^{\infty} M_0\left(\frac{\omega}{2^k}\right) = M_0\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right) = \frac{1}{\sqrt{2}}G_0\left(e^{j\omega/2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right).$$

So the two scale relationship is satisfied.

Equivalent Filters and Functions

Proof (cont'd):
Note that we started with an orthogonal filter bank. Therefore:

$$|G_0(e^{j\omega})|^2 + |G_0(e^{j\omega+\pi})|^2 = 2.$$

Moreover, for a function satisfying a two scale relationship, the Riesz criterion for orthogonality:

$$\sum_{l} |\hat{\varphi}(\omega + 2k\pi)|^2 = 1$$

is satisfied if and only if

$$|G_0(e^{j\omega})|^2 + |G_0(e^{j\omega+\pi})|^2 = 2.$$

Thus our function satisfies the criterion.
Finally, by using Poisson summation formula, we write partition of unity as follows:

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k) e^{j2\pi k t} = 1,$$

which is satisfied since we assumed $G_0(1) = \sqrt{2}$ and $G_0(-1) = 0$.

Regularity

Given a filter $G_0(e^{j\omega})$, the limit function $\varphi(t)$ depends on the behaviour of the product

$$\prod_{k=1}^{i} M_0(\frac{\omega}{2^k})$$

as $i \to \infty$.

Daubechies studied the regularity of iterated filter banks in detail and provided sufficient conditions for regularity.

Factor $M_0(\omega)$ as follows

$$M_0(\omega) = \left(\frac{1 + e^{j\omega}}{2}\right)^N R(\omega).$$

Because of the necessary condition, we know that N must be at least equal to 1. Define B as

$$B = \sup_{\omega \in [0, 2\pi]} |R(\omega)|$$

Regularity

Theorem 2. *If*

$$B < 2^{N-1}$$

then the limit $\lim_{i\to\infty} \varphi_i(t)$ *converges pointwise to a continuous function* $\varphi(t)$ *with Fourier transform*

$$\hat{\varphi}(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right).$$

Moreover, if

$$B < 2^{N-1-n} \quad n = 1, 2, \ldots$$

then $\varphi(t)$ *is* n*-times continuously differentiable.*

Regularity

Proof:

- First remember that if a constant K and $\epsilon > 0$ exists such that $|\hat{f}(\omega)| \leq \frac{K}{1+|\omega|^{n+1+\epsilon}}$, then $f(t)$ is n-times continuously differentiable.
- Just need to prove $|\hat{\varphi}(\omega)| \leq \frac{K}{1+|\omega|^{n+1+\epsilon}}$
- $$\hat{\varphi}(\omega) = \prod_{k=1}^{\infty} \left(\frac{1 + e^{j\omega/2^k}}{2} \right)^N \prod_{k=1}^{\infty} R\left(\frac{\omega}{2^k}\right)$$
- The first product converges to sinc $\left(\frac{\omega}{2}\right)^N$ which decays like $(1 + |\omega|^{-N})$
- The second product is upper bounded by a constant when $|\omega| < 1$. Therefore for $2^{J-1} < \omega < 2^J$, we have:

$$\prod_{k=1}^{\infty} \left| R\left(\frac{\omega}{2^k}\right) \right| = \prod_{k=1}^{J} \left| R\left(\frac{\omega}{2^k}\right) \right| \prod_{k=1}^{\infty} \left| R\left(\frac{\omega}{2^k 2^J}\right) \right|$$

Regularity

Proof (cont'd):

- The first term is smaller or equal to B^J while the second term is upper bounded by a constant, so we can write

$$\prod_{k=1}^{\infty}\left|R\left(\frac{\omega}{2^k}\right)\right| \leq c_0 B^J \leq c_1 2^{J(N-1-n-\epsilon)} < c_2(1+|\omega|)^{N-1-n-\epsilon},$$

  where we have used the fact that $B < 2^{N-1-n}$.
- Putting all together gives us the desired decay.

Properties of the Wavelet series

A key property of the wavelet transform is that of *vanishing moments*. We know that $g_0[n]$ has at least one zero at $\omega = \pi$ and thus $g_1[n]$ has at least one zeros at $\omega = 0$. Since $\hat{\varphi}(0) = 1$ (from the normalization of $M_0(\omega)$), it follows that

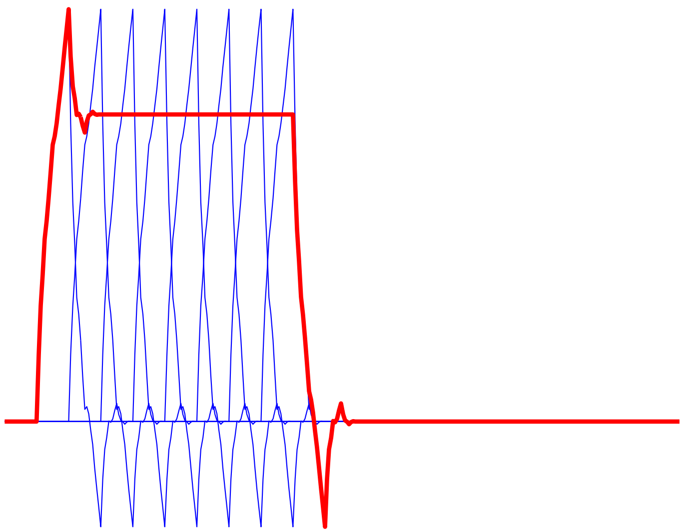
$$\int_{-\infty}^{\infty} \psi(t)dt = \hat{\psi}(0) = \underbrace{\frac{G_1(1)}{\sqrt{2}}}_{=0} \hat{\varphi}(0) = 0.$$

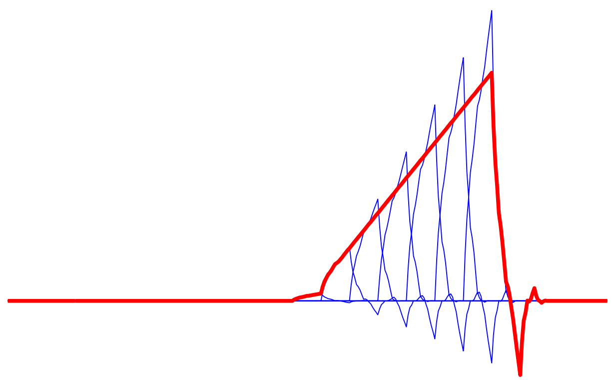
In general, if $g_0[n]$ has a zero of order N at π then $\hat{\psi}(\omega)$ has N zeros at $\omega = 0$ and using the moment property of the Fourier transform we have that

$$\int_{-\infty}^{\infty} t^k \psi(t)dt = 0 \quad k = 0, ..., 1, N-1. \tag{1}$$

Properties of the Wavelet series

As a consequence of the vanishing moment property, scaling functions reproduce polynomials.





Properties of the Wavelet series

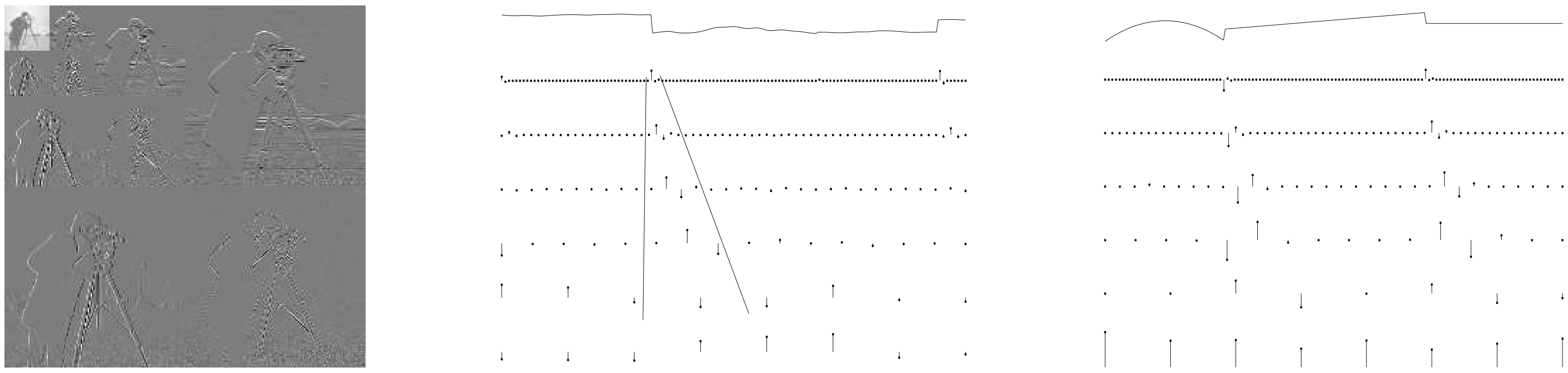
The restriction of $f(t)$ to $[a,b]$ is uniformly Lipschitz $\alpha \geq 0$ over $[a,b]$ if there exists $K > 0$ such that for all $\nu \in [a,b]$ there exists a polynomial $p_\nu(t)$ of degree $m = \lfloor \alpha \rfloor$ such that

$$\forall t \in (a,b), \;\; |f(t) - p_\nu(t)| \leq K|t-\nu|^\alpha.$$

Assume that $f(t)$ is uniformly α-Lipshitz around t_0 and that $\psi(t)$ has at least $\lfloor \alpha \rfloor + 1$ vanishing moments. Assume that $\psi(t)$ is of compact support C, then:

$$\begin{aligned}
\langle f, \psi_{m,n} \rangle &= \underbrace{\langle p_{t_0}(t), \psi_{m,n}(t) \rangle}_{=0} + \langle \epsilon(t), \psi_{m,n}(t) \rangle \\
&\leq K 2^{-m/2} \textstyle\int_{-\infty}^{\infty} |t - t_0|^\alpha \psi(2^{-m}t - n) dt \\
&= K 2^{m/2} \textstyle\int_{-\infty}^{\infty} |x 2^m + n 2^m - t_0|^\alpha \psi(x) dx \\
&\leq K C 2^{m(\alpha+1/2)} \underbrace{\int_{-\infty}^{\infty} (|x| + |C|)^\alpha \psi(x) dx}_{=A} \\
&= C_1 2^{m(\alpha+1/2)}.
\end{aligned}$$

Signals of Interest and Wavelet Representations



- Wavelet coefficients around smooth parts of the signal are small and have fast decay ($\sim 2^{-j(\alpha+1/2)}$).
- Wavelet coefficients around polynomial parts of the signal are exactly zero.
- Discontinuities generate a finite number of large wavelet coefficients.

Wavelets vs Fourier

Preview 1: Approximation

Consider signals in the vector space $\mathbb{R}^N$

There are many orthonormal expansions

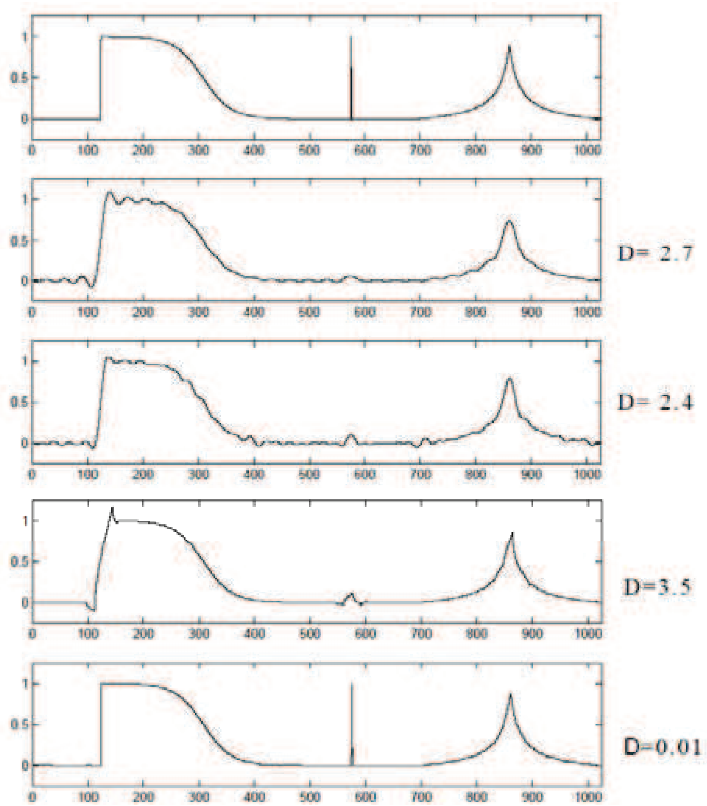
$$x = \sum_{i=1}^{N} \alpha_i \varphi_i$$

How do they differ for *approximation*?

$$x \approx \hat{x}_K = \sum_{i=1}^{K} \beta_i \varphi_{j_i}, \qquad K < N$$

- Different bases $\{\varphi_i\}_{i=1}^N$
- Different orders $j_1, j_2, j_3, \ldots$

Wavelets vs Fourier



Original signal, N=1024

M=64, linear
Fourier

M=64, nonlinear
Fourier

M=64, linear
Daubechies wavelet

M=64, nonlinear
Daubechies wavelet

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Wavelets vs Fourier

Preview 2: Estimation

Consider signal x observed through noisy observation

$$y = x + d \qquad d \sim \mathcal{N}(0, \sigma^2 I)$$

MMSE estimate $\hat{x}_{\mathrm{MMSE}} = E[x \mid y]$ can be complicated (if even known)

For simplicity, we often insist on a diagonal estimator in a transform domain:

$$\hat{x} = T^{-1} D(Ty), \qquad D(\,\cdot\,) \text{ diagonal}$$

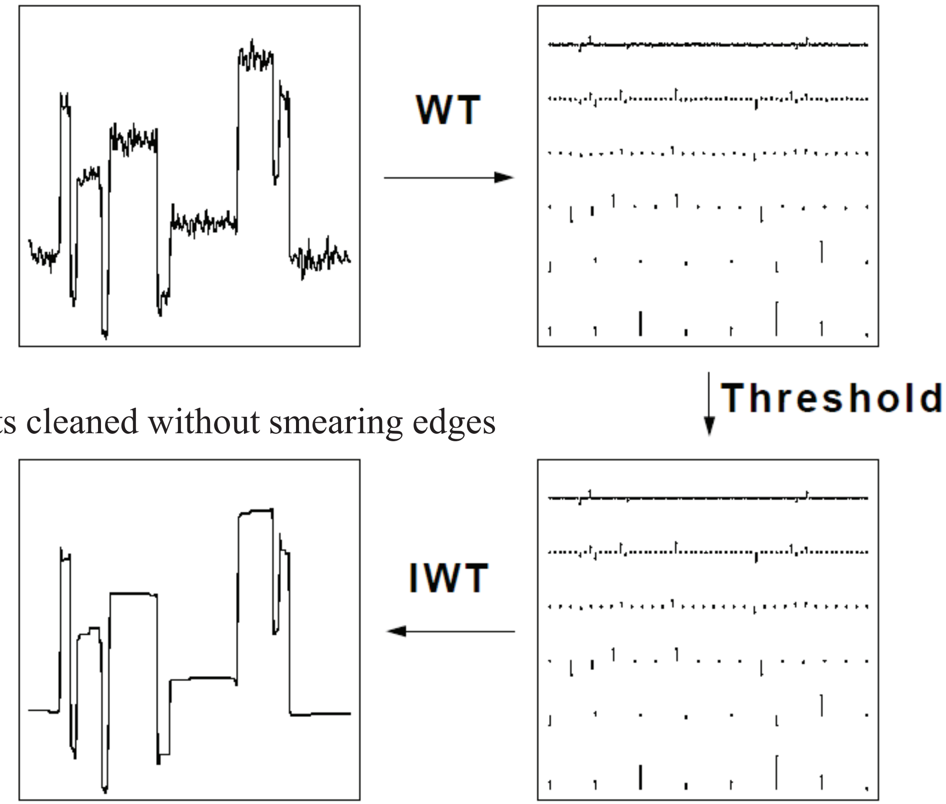
What are good choices for T and D?

Wavelets vs Fourier

Original image	Original image + i.i.d. Gaussian noise	Denoised in Fourier domain	Denoised in Wavelet domain

Wavelets vs Fourier

1-D Example



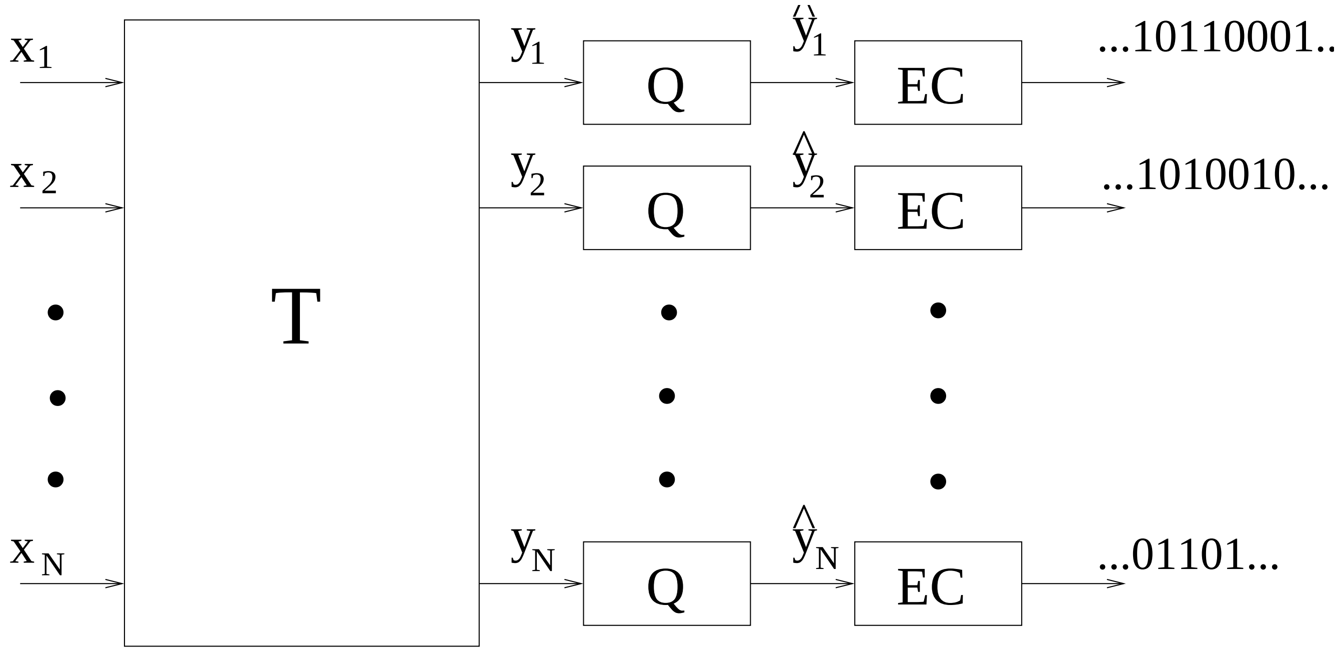
Smooth parts cleaned without smearing edges

Wavelets vs Fourier

Fourier	Wavelet
linear, time invariant	not linear, time invariant
Gaussian	non-Gaussian
stationary	non-stationary
continuous	discontinuous
smooth	containing edges
known structure	(partly) unknown structure
global	locally “adaptive”
single resolution	multiple resolutions
modeled by sinusoids	modeled by piecewise polynomials

[10]

Application in Compression: Transform Coding



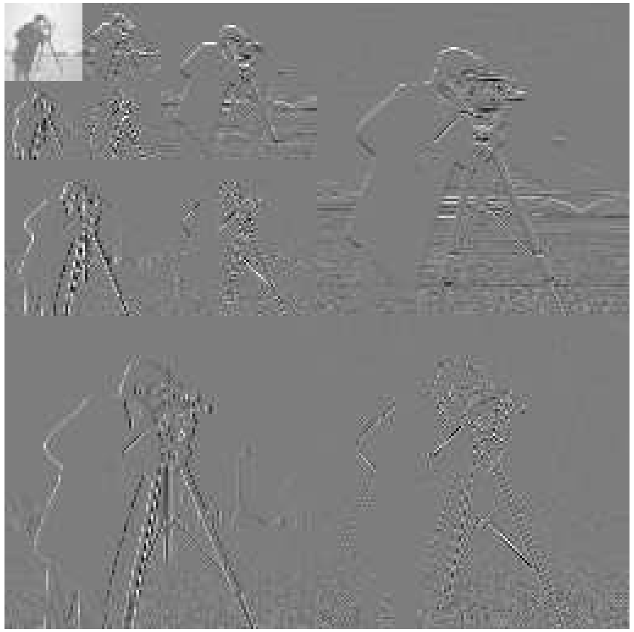
A transform coder is made of a linear transform, scalar quantisers and entropy encoders.

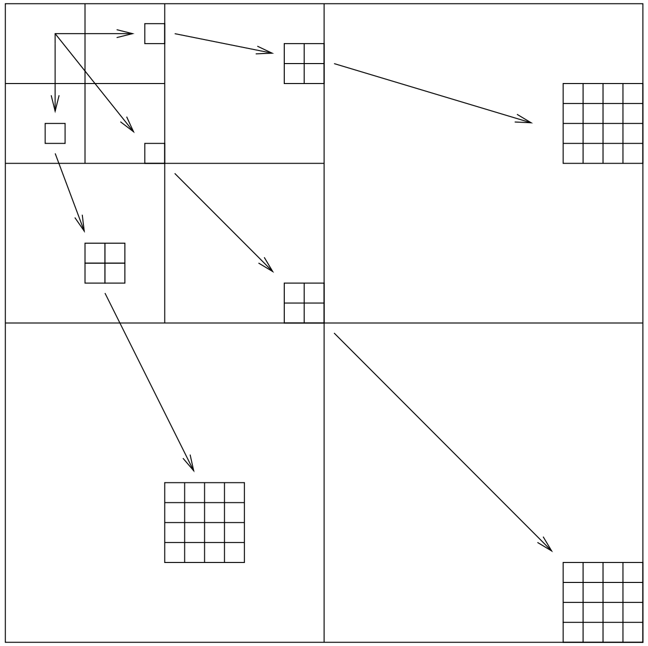
Approximation and Compression

More formally:

- Non-linear approximation of piecewise smooth signals
  - With Fourier $MSE \sim M^{-1}$
  - With Wavelets $MSE \sim M^{-2\alpha}$
- Compression of piecewise smooth signals
  - With Fourier $D(R) \leq c_0 R^{-1}$
  - With Wavelets $D(R) \leq c_1 R^{-2\alpha} + c_2\sqrt{R}2^{-c_2\sqrt{R}}$ ([CohenDGO:02])
  - With Footprints $D(R) \leq c_3 R^{-2\alpha} + c_4 2^{-c_5 R}$ ([DragottiV:03])
- Compression of piecewise polynomial signals
  - With Wavelets $D(R) \leq \ c_2\sqrt{R}2^{-c_2\sqrt{R}}$
  - With Footprints $D(R) \leq c_4 2^{-c_5 R}$

Application in Compression





Zerotree algorithm.

Application in Compression

63	−34	49	10	7	13	−12	7
−31	23	14	−13	3	4	6	−1
15	14	3	−12	5	−7	3	9
−9	−7	−14	8	4	−2	3	2
−5	9	−1	47	4	6	−2	2
3	0	−3	2	3	−2	0	4
2	−3	6	−4	3	6	3	6
5	11	5	6	0	3	−4	4

Coefficient	Symbol	Reconstruction
63	POS	48
-34	NEG	-48
-31	IZ	0
23	ZTR	0
49	POS	48
10	ZTR	0
14	ZTR	0
-13	ZTR	0
15	ZTR	0
14	IZ	0
-9	ZTR	0
-7	ZTR	0
7	Z	0
13	Z	0
3	Z	0
4	Z	0
-1	Z	0
47	POS	48

A Success Story

Wavelets are in the new image compression standard (JPEG2000)

Original Lena Image (256×256 pixels)

JPEG (Compression Ratio 43:1)

JPEG2000 (Compression Ratio 43:1)

Note: images courtesy of dspworx.com