EE2 Mathematics – Probability & Statistics

Exercise 7

1. The discrete random variables X and Y have joint PMF

$$f_{X,Y}(x,y) = \begin{cases} a\frac{x}{y} & \text{if } x, y \in \{1,2,3\} \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the value of a that makes this a valid joint PMF.

   (b) Construct a table of joint probabilities and include the marginal distributions.

   (c) Compute $E(XY)$, $\mathrm{Cov}(X,Y)$ and $\mathrm{Corr}(X,Y)$. Are X and Y uncorrelated? Are they independent?

   (d) Evaluate $P(X \leq 2 | Y \leq 2)$.

2. Consider the continuous random variables X and Y with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} k(x+y-2xy) & \text{if } 0 \leq x, y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Which value of k makes this a valid PDF?

   (b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$. What are these distributions? Are X and Y independent?

   (c) Evaluate $\mathrm{E}(X)$, $\mathrm{E}(Y)$, $\mathrm{E}(XY)$ and $\mathrm{Cov}(X,Y)$.

3. Consider the continuous random variables X and Y with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx(y-x^2+1) & \text{if } 0 \leq x, y \leq 1 \text{ and } y \geq x^2 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Which value of c makes this a valid PDF?

   (b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$. Compute $\mathrm{E}(X)$.

   (c) Write down the conditional PDF $f_{X|Y}(x|y)$ and use it to compute $\mathrm{E}(X|Y=y)$. What is $\mathrm{E}(X|Y)$?

   (d) Verify that $\mathrm{E}[\mathrm{E}(X|Y)] = \mathrm{E}(X)$. (This is known as the law of iterated expectation.)

4. Suppose that X, Y are random variables and a, b, c, d are constants, with $a, c \neq 0$. Find an expression for $\mathrm{Corr}(aX+b, cY+d)$ in terms of $\mathrm{Corr}(X,Y)$ and the constants.

5. Over the course of its operation a system may develop faults. Suppose that the distribution of the number of faults in one month is $\mathrm{Poisson}(\lambda)$. Each of these faults is either severe (with probability p) or not severe (with probability $1-p$), independently of all other faults.

(a) Let X denote the number of all faults and Y the number of severe faults in a one-month period. Write down $f_X(x)$ and $f_{Y|X}(y|x)$, and hence find an expression for the joint PDF $f_{X,Y}(x, y)$.

(b) Find the marginal PDF $f_Y(y)$. What is the distribution of Y?

6. Find the moment generating function of X and use it to compute the mean and variance of the binomial distribution.

   (a) $X \sim \text{Bin}(n, p)$

   (b) $X \sim \text{Gamma}(k, r)$ such that $f_X(x) = e^{-kx}x^{r-1}k^r/(r-1)!$, $x > 0$, $r \in \mathbb{N}^*$, $k > 0$