E303: Communication Systems

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An Overview of Fundamentals: PN-codes/signals & Spread Spectrum

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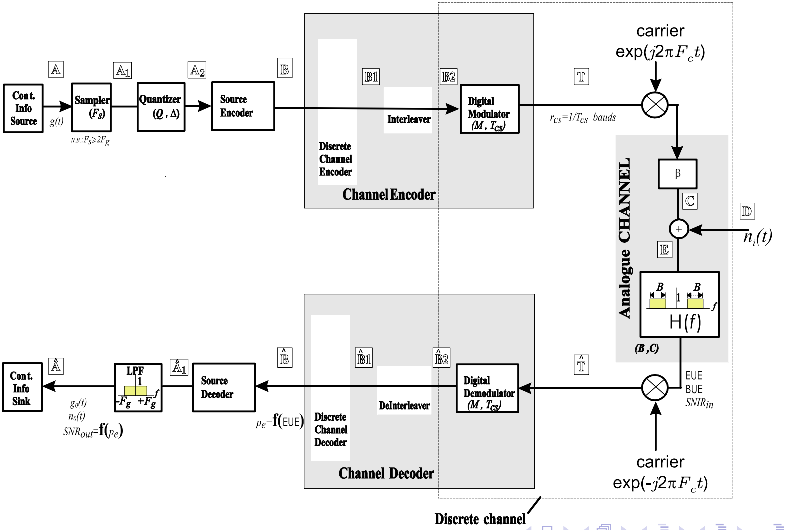
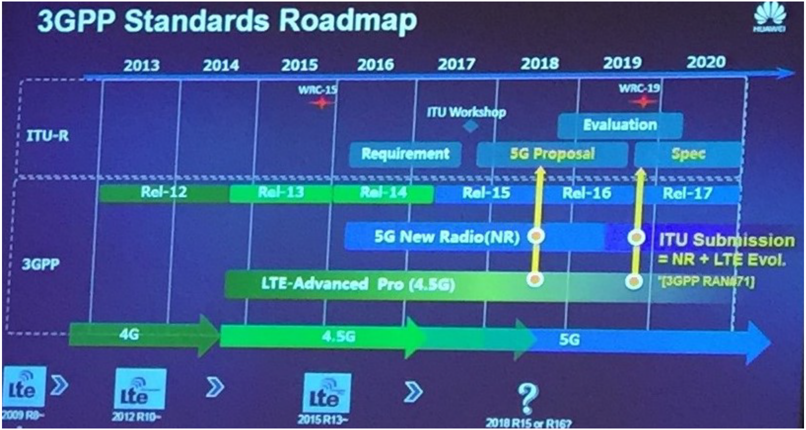
Introduction

- General Block Diagram of a Digital Comm. System (DCS)

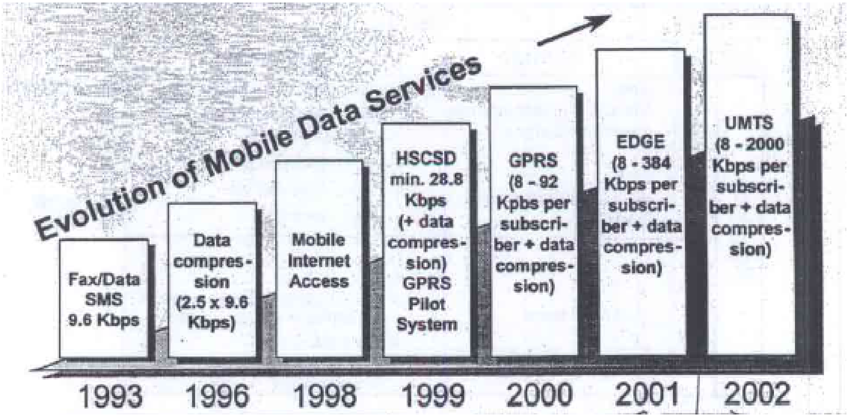


3GPP

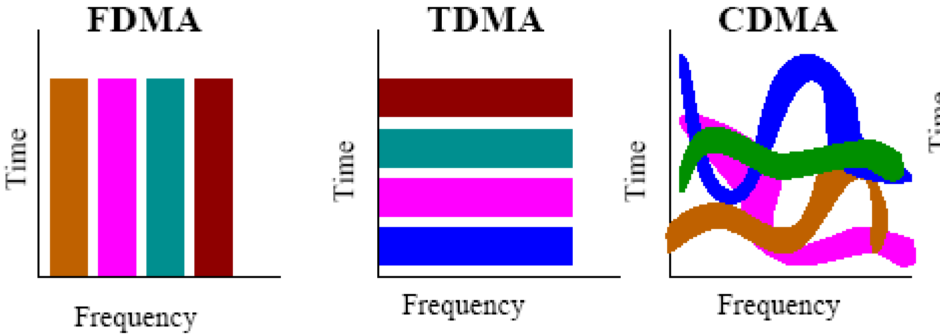
- 3GPP is a cellular communication standard development body (3GPP $\triangleq$ **3'd Generation Partnership Project**)
  - Found in 1998
  - Participated by over 100 companies and 1000s of communications experts
  - Globally dominant cellular standard
- 3GPP also
  - developed the 4G standards
  - is developing standards towards next generation (5G)


3GPP Standards Roadmap
2013
2014
2015
2016
2017
2018
2019
2020
ITU-R
WRC-15
ITU Workshop
WRC-19
Evaluation
Requirement
5G Proposal
Spec
3GPP
Rel-12
Rel-13
Rel-14
Rel-15
Rel-16
Rel-17
5G New Radio(NR)
ITU Submission
= NR + LTE Evol.
LTE-Advanced Pro (4.5G)
4G
4.5G
5G
2009 R8-
2012 R10-
2015 R13-
?
2018 R15 or R16?

Pre-4G Evolution

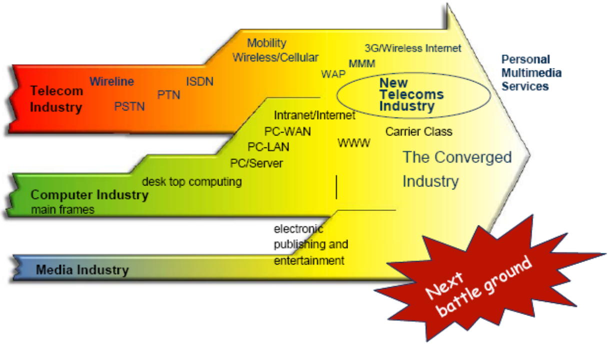


HSCDS: High Speed Circuit Switched Data
GPRS: General Packet Radio Systems (2+)
EDGE: Enhanced Data Rate GSM Evolution (2+)
UMTS:Universal Mobile Telecommunication Systems (3G)



Note: CDMA $\in$ Spread Spectrum Comms

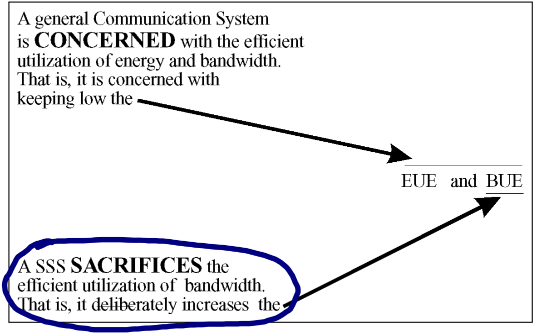
- Industry Transformation and Convergence [from Ericsson 2006, LZT 123 6208 R5B]



WCDMA (Wideband CDMA) is a 3G mobile comm system. It is a wireless system where the telecommunications, computing and **media** industry converge and is based on a Layered Architecture design. (Note: CDMA Systems $\in$ the class of SSS).

Definition of a SSS

- When a DCS becomes a Spread Spectrum System (SSS)



- LEMMA−1: CS $\triangleq$ SSS

iff
- ○ $B_{ss} \gg$ message bandwidth (i.e. BUE=large)
- ○ $B_{ss} \neq$ f{message}
- ○ spread is achieved by means of a code which is $\neq$f{message}

where B_{ss}=transmitted SS signal bandwidth

- our AIM: ways of accomplishing LEMMA-1.

SSS: $B_{Tx} \gg B_m$ { distributes energy over wide bandwidth / low SNIR

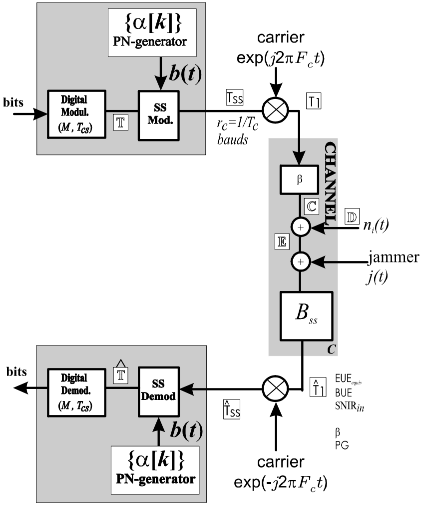
- NB:
  - PCM, FM, etc spread the signal bandwidth but do not satisfy the conditions to be called SSS
  - $B_{\text{transmitted-signal}} \gg B_{\text{message}}$

    $\Rightarrow$SSS distributes the transmitted energy over a wide bandwidth

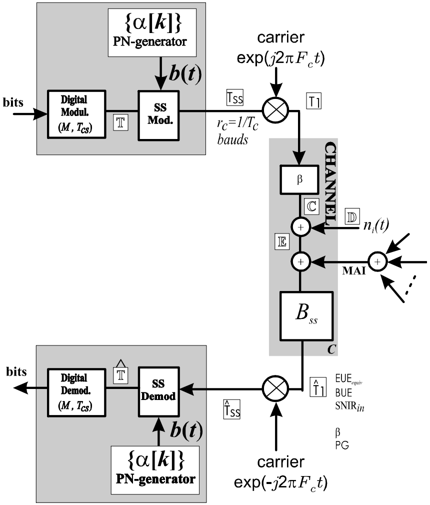
    $\Rightarrow$ SNIR at the receiver input is LOW.

    Nevertheless, the receiver is capable of operating successfully because the transmitted signal has distinct characteristics relative to the noise

(a) SSS:



(b) CDMA (K users):



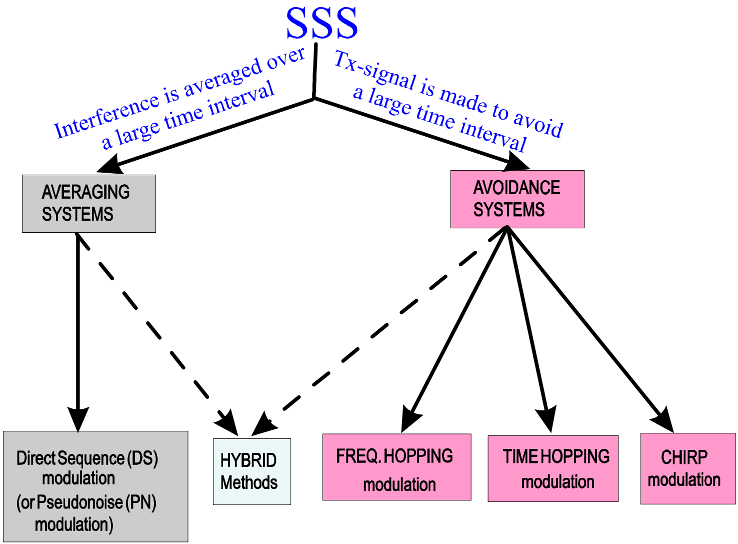
- The PN signal $b(t)$ is a function of a PN sequence of ± 1's $\{\alpha[n]\}$
  - The sequences $\{\alpha[n]\}$ must agreed upon in advance by Tx and Rx and they have status of password.
  - This implies that :
    - ★ knowledge of $\{\alpha[n]\} \Rightarrow$demodulation=possible
    - ★ without knowledge of $\{\alpha[n]\}$ $\Rightarrow$demod.=very difficult
  - If $\{\alpha[n]\}$ (i.e. "password") is purely random, with no mathematical structure, then
    - ★ without knowledge of $\{\alpha[n]\}$ $\Rightarrow$demodulation=impossible
  - However all practical random sequences have some periodic structure. This means:

$$\alpha[n] = \alpha[n + N_c] \tag{1}$$

where N_c =period of sequence

i.e. pseudo-random sequence (PN-sequence)

Classification of SSS

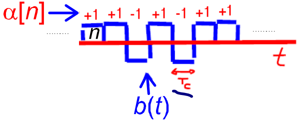


Modelling of b(t) in SSS

- DS-SSS (Examples: DS-BPSK, DS-QPSK):

$$b(t) = \sum_n \alpha[n].c(t - nT_c) \tag{2}$$

where $\{\alpha[n]\}$ is a sequence of ± 1's;
$c(t)$ is an energy signal of duration T_c $=\text{rect}\left\{\frac{t}{T_c}\right\}$



- FH-SSS (Examples: FH-FSK)

$$b(t) = \sum_n \exp\left\{j(2\pi \mathrm{k}[n]F_1 t + \phi[n])\right\}.c(t - nT_c) \tag{3}$$

where $\{\mathrm{k}[n]\}$ is a sequence of integers such that $\{\alpha[n]\} \mapsto \{\mathrm{k}[n]\}$
and $\{\alpha[n]\}$ is a sequence of ± 1's;
$c(t)$ is an energy signal of duration T_c
and with $\phi[n]$ = random: $\text{pdf}_{\phi[n]} = \frac{1}{2\pi}\text{rect}\left\{\frac{\varphi}{2\pi}\right\}$

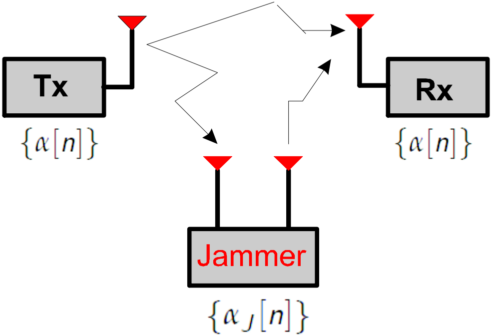
Applications of Spread Spectrum Techniques

1. Interference Rejection: to achieve interference rejection due to:
   - Jamming (hostile interference). N.B.: protection against cochannel interference is usually called anti-jamming (AJ)
   - Other users (Multiple Access Interefence - MAI): Spectrum shared by "coordinated " users.
   - Multipath: Self-Jamming by delayed signal
2. Energy Density Reduction (or Low Probability of Intercept LPI). LPI' main objectives:
   - to meet international allocations regulations
   - to reduce (minimize) the detectability of a transmitted signal by someone who uses **spectral analysis**
   - privacy in the presence of other listeners
3. Range or Time Delay Estimation

NB: interference rejection = most important application

- Jamming source, or, simply Jammer is defined as follows:

$$\boxed{\text{Jammer} \triangleq \text{intentional (hostile) interference}}$$

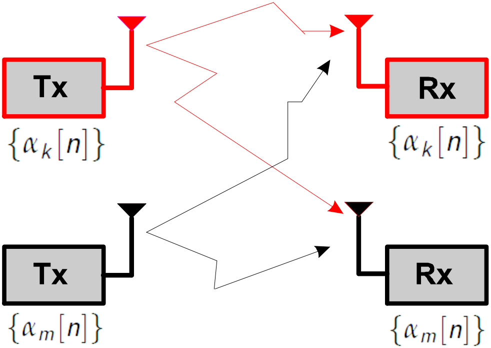


★ the jammer has full knowledge of SSS design except the jammer does not have the key to the PN-sequence generator,

★ i.e. the jammer may have full knowledge of the SSystem but it does know the PN sequence used.

- Multiple Access Interference (MAI) is defined as follows:

$$\boxed{\text{MAI} \triangleq \text{unintentional interference}}$$



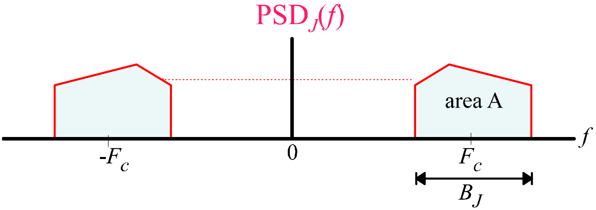
- PG: is a measure of the interference rejection capabilities
- definition:

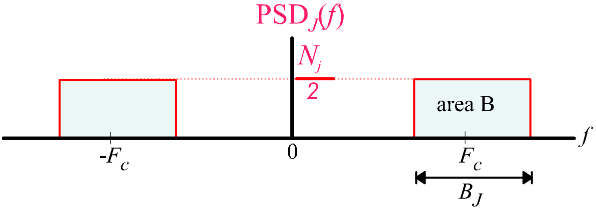
$$\text{PG} \triangleq \frac{B_{ss}}{B} = \frac{1/T_c}{1/T_{cs}} = \frac{T_{cs}}{T_c} \tag{4}$$

  where B=bandwidth of the conventional system

- PG is also known as "spreading factor" (SF)
- PG = very important in DS-SSS
- PG $\neq$ very important in FH-SSS

- Remember:
  - ★ Jamming source, or, simply Jammer = intentional interference
  - ★ Interfering source = unintentional interference





- ★ With $\boxed{\text{area-B} = \text{area-A}}$ we can find N_j
- ★ $P_j = 2\times\ \underline{\text{area}\mathbf{A}} = 2\times\ \underline{\text{area}\mathbf{B}} = N_j B_j \Rightarrow N_j = \frac{P_j}{B_j}$

- if

$$B_J = qB_{ss};\ 0 < q \leq 1 \tag{5}$$

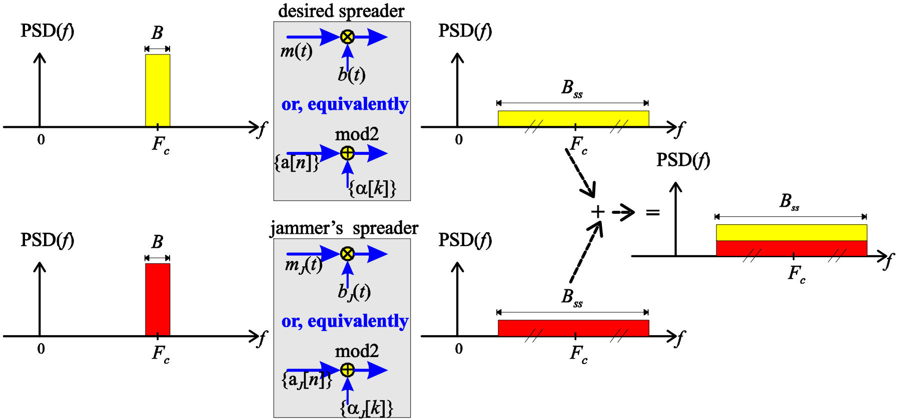
then

$$\begin{aligned} \text{EUE}_J &= \frac{E_b}{N_J} = \frac{P_s.B_J}{P_J.r_b} = \frac{P_s.q.B_{ss}}{P_J.B} = \text{PG} \times \text{SJR}_{in} \times q && (6) \\ \text{EUE}_{equ} &= \frac{E_b}{N_0 + N_J} && (7) \\ &= \text{PG} \times \text{SJR}_{in} \times q \times \left(\frac{N_0}{N_j} + 1\right)^{-1} && (8) \end{aligned}$$

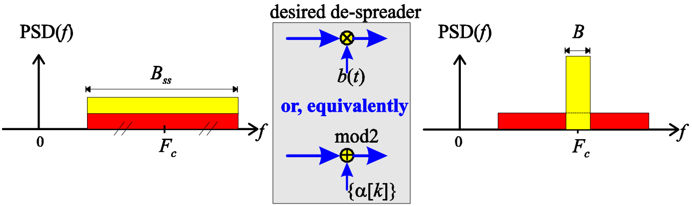
where

$$\text{SJR}_{in} \triangleq \frac{P_s}{P_J} \tag{9}$$

- SS Transmission in the presence of a Jammer (or MAI)



- SS Reception in the presence of a Jammer (or MAI)



- PN-codes (or PN-sequences, or spreading codes) are sequences of +1s and -1s (or 1s and 0s) having special correlation properties which are used to distinguish a number of signals occupying the same bandwidth.

$\{\alpha[n]\}$

- Five Properties of Good PN-sequences:

Property-1	easy to generate
Property-2	randomness
Property-3	long periods
Property-4	impulse-like auto-correlation functions
Property-5	low cross-correlation

Comments on PN-sequences Main Properties

- Comments on Properties 1, 2 & 3 PR sequence . . .|. . .| - -

  shift registers

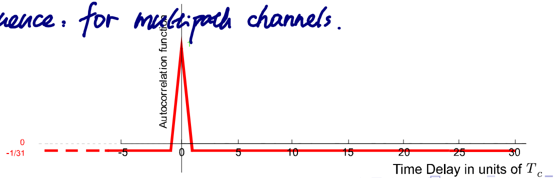
  - Property-1 is easily achieved with the generation of PN sequences by means of shift registers,
    while

    = . . . , . . . , . . .
    T : long period

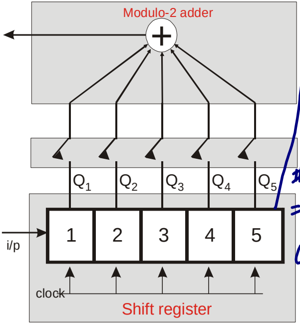
  - Property-2 & Property-3 are achieved by appropriately selecting the feedback connections of the shift registers.

- Comments on Property-4
  - to combat multipath, consecutive bits of the code sequences should be uncorrelated.
    i.e. code sequences should have impulse-like autocorrelation functions. Therefore it is desired that the auto-correlation of a PN-sequence is made as small as possible.
  - The success of any spread spectrum system relies on certain requirements for PN-codes. Two of these requirements are:
    1. the autocorrelation peak must be sharp and large (maximal) upon synchronisation (i.e. for time shift equal to zero)
    2. the autocorrelation must be minimal (very close to zero) for any time shift different than zero.
  - A code that meets the requirements (1) and (2) above is the m-sequence which is ideal for handling multipath channels.

impulse-like m-sequence: for multipath channels.



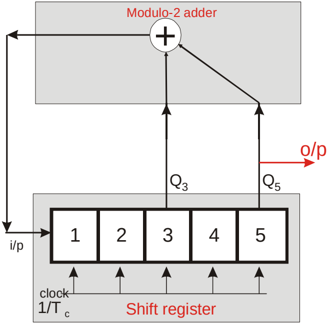
- The figure below shows a shift register of 5 stages together with a modulo-2 adder. By connecting the stages according to the coefficients of the polynomial $D^5 + D^2 + 1$ an m-sequence of length 31 is generated (output from Q5).
  The autocorrelation function of this m-sequence signal is shown in the previous page



(a)

\# max power = # stages

length $= 2^n - 1$



(b)

- Comments on Property-5
  - If there are a number of PN-sequences

$$\{\alpha_1[k]\},\ \{\alpha_2[k]\}, ..., \{\alpha_K[k]\} \tag{10}$$

    then if these code sequences are not totally uncorrelated, there is always an interference component at the output of the receiver which is proportional to the cross-correlation between different code sequences.

  - Therefore it is desired that this cross-correlation is made as small as possible.

An Important "Trade-off"

- There is a trade-off between Properties-4 and 5.
- In a CDMA communication environment there are a number of PN-sequences

$$\{\alpha_1[k]\}, \{\alpha_2[k]\}, ..., \{\alpha_K[k]\}$$

of period N_c which are used to distinguish a number of signals occupying the same bandwidth.

- Therefore, based on these sequences, we should be able to
  - ★ combat multipath
    (which implies that the auto-correlation of a PN-sequence $\{\alpha_i[k]\}$ should be made as small as possible) → impulse-like
  - ★ remove interference from other users/signals,
    (which implies that the cross-correlation should be made as small as possible). → steady small

- However

$$\boxed{R_{auto}^2 + R_{cross}^2 > \text{a constant which is a function of period } N_c} \quad (11)$$

i.e. there is a trade-off between the peak autocorrelation and cross-correlation parameters.
Thus, the autocorrelation and cross-correlation functions cannot be both made small simultaneously.

- The design of the code sequences should be therefore very careful.

m-seq: excellent autocorrelation

- N.B.: gold-seq: trade-off between auto and crosscorrelation.
  - A code with excellent autocorrelation is the m-sequence.
  - A code that provides a trade-off between auto and cross correlation is the gold-sequence.

m-sequences - definition

- m-seq.: widely used in SSS because of their very good autocorrelation properties.
- PN code generator: is periodic
  - i.e. the sequence that is produced repeats itself after some period of time
- **Definition** : A sequence generated by a linear *m*-stages Feedback shift register is called a maximal length, a maximal sequence, or simply m-sequence, if its period is

$$N_c = 2^m - 1 \tag{12}$$

(which is the maximum period for the above shift register generator)

- The initial contents of the shift register are called initial conditions.

Shift Registers and Primitive Polynomials

- The period N_c depends on the feedback connections (i.e. coefficients c_i) and $N_c = max$, i.e. $N_c = 2^m - 1$, when the characteristic polynomial

$$c(D) = c_m D^m + c_{m-1} D^{m-1} + ... + c_1 D + c_0 \quad \text{with} \quad c_0 = 1 \qquad (13)$$

is a primitive polynomial of degree m.

$\begin{cases} C_0 = 1 \\ C_m = 1 \end{cases}$

$$\boxed{\text{rule: if } c_i = \begin{cases} 0 \Longrightarrow & \text{no connection} \\ 1 \Longrightarrow & \text{there is connection} \end{cases}} \qquad (14)$$

- Definition of PRIMITIVE polynomial = very important (see Appendix C)

- Some Examples of Primitive Polynomials

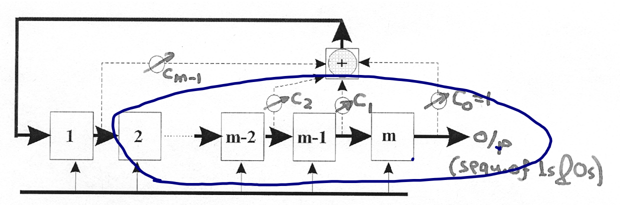
degree-m	polynomial
3	$D^3 + D + 1$
4	$D^4 + D + 1$
5	$D^5 + D^2 + 1$
6	$D^6 + D + 1$
7	$D^7 + D + 1$

- Please see Comm Systems LNs (Spread Spectrum Topic) for some tables of irreducible & primitive polynomial over GF(2).

Implementation of an m-sequence

- use a maximal length shift register
  i.e. in order to construct a shift register generator for sequences of any permissible length, it is only necessary to know the coefficients of the primitive polynomial for the corresponding value of m

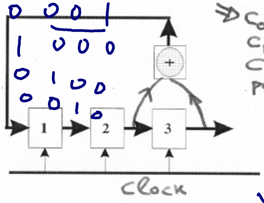
$$f_c = \frac{1}{T_c} = \text{chip-rate} = \text{clock-rate} \tag{15}$$



$$c(D) = c_m D^m + c_{m-1} D^{m-1} + ... + c_1 D + c_0 \tag{16}$$

$$\text{with} \quad c_0 = 1 \tag{17}$$

- Example: $c(D) = D^3 + D + 1$ = primitive $\Longrightarrow$ power $= m = 3$
  - coefficients $=(1,0,1,1) \Rightarrow N_c = 7 = 2^m - 1$ i.e. period $= 7T_c$



$\Rightarrow C_0 = 1$
$C_1 = 1$
$C_2 = 0$
power $= 3 = m$

1101

	1st	2nd	o/p 3rd
initial condition	1	1	1
clock pulse No.**1**	0	1	1
clock pulse No.**2**	0	0	1
clock pulse No.**3**	1	0	0
clock pulse No.**4**	0	1	0
clock pulse No.**5**	1	0	1
clock pulse No.**6**	1	1	0
clock pulse No.**7**	1	1	1

- Note that the sequence of 0's and 1's is transformed to a sequence of ± 1s by using the following function

$$o/p = 1 - 2 \times i/p \qquad (18)$$

Auto-Correlation Properties

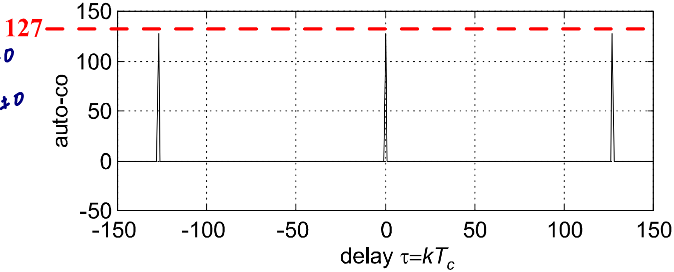
- An m-sequ. $\{\alpha[n]\}$ has a two valued auto-correlation function:

$$R_{\alpha\alpha}[k] = \sum_{n=1}^{N_c} \alpha[n]\alpha[n+k] = \begin{cases} \underline{N_c \quad k = 0 \bmod N_c} \\ \underline{-1 \quad k \neq 0 \bmod N_c} \end{cases} \qquad (19)$$

- This implies that $R_{bb}(\tau)$ is also a "two-valued"

$R_{bb}(\tau)$:

$R(\tau) = \begin{cases} 1, & \tau = 0 \\ -\frac{1}{2^m-1}, & \tau \neq 0 \end{cases}$



- Remember that a sequence $\{\alpha[n]\}$ of period $N_c = 2^m - 1$, generated by a linear FB shift register, is called a maximal length sequence.

Some Properties of m-sequences

- There is an appropriate balance of -1s and +1s
  - In any period there are $\left\{ \begin{array}{lll} N_{c-} & = 2^{m-1} & \text{No. of -1s} \\ N_{c+} & = 2^{m-1} - 1 & \text{No. of +1s} \end{array} \right\}$

    i.e.

$$\Pr(+1) \simeq \Pr(-1) \tag{20}$$

$\# 0 = 2^{m-1} - 1$

$\# 1 = 2^{m-1}$

- shift-property of m-sequences:
  - if $\{\alpha[n]\}$ is an m-sequence then

$$\{\alpha[n]\} + \underbrace{\{\alpha[n+m]\}}_{\text{shift by } m} = \underbrace{\{\alpha[n+k]\}}_{\text{shift by } k \neq m} \tag{21}$$

- In a complete SSS we use more than one different m-sequences
  - Thus the number of m-sequs of a given length is an IMPORTANT property
    - ★ because in a CDMA system several users communicate over a common channel so that different -sequences are necessary to distinguish their signals
  - Number of m-sequs of length N_c :

$$\text{No. of m-sequs of length } N_c \triangleq \frac{1}{m}\Phi\{N_c\} \tag{22}$$

where

$$\begin{aligned} \Phi\{N_c\} &\triangleq \text{Euler totient function} \qquad (23) \\ &= \text{No of } (+)\text{ve integers} < N_c \text{ and relative prime to } N_c \end{aligned}$$

  - Note: if $N_c = p.q$ where p, q are prime numbers then

$$\Phi\{N_c\} = (p-1).(q-1) \tag{24}$$

Cross-Correlation Properties and Preferred m-sequences

- sequences of period N_c are used to distinguish two signals occupying the same bandwidth.
- A measure of interaction between these signals is their cross-correlation:

$$R_{\alpha_i\alpha_j}[k] = \sum_{n=1}^{N_c} \alpha_i[n]\alpha_j[n+k]$$

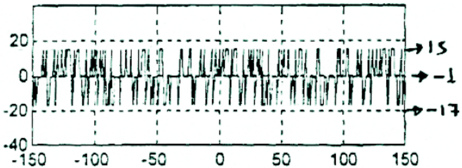
- However,
  - there exist **certain pairs of sequences** that have large peaks and noise-like behaviour in their cross-correlation
  - while others exhibit a rather smooth three valued cross-correlation.
- The latter are called **preferred sequences**.

- It can be shown that the cross-correlation of **preferred sequences** takes on values from the set

$$\{-1, -R_{cross}, R_{cross} - 2\} \tag{25}$$

where
$$R_{cross} = \begin{cases} 2^{\frac{m+1}{2}} + 1 & m = odd \\ 2^{\frac{m+2}{2}} + 1 & m = even \end{cases} \tag{26}$$

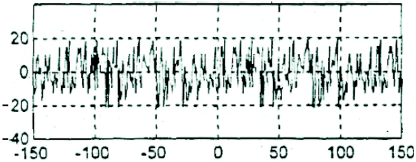
$R_{b_i b_j}(\tau) =$preferred:



delay $\tau = kT_c$

m

$R_{b_i b_j}(\tau) =$ *non*-preferred:



m

A Note on m-sequences for CDMA

- Because of the high cross-correlation between m-sequences, the interference between different users in a CDMA environment will be large.
  - Therefore, m-sequences are not suitable for CDMA applications.
- However, in a complete synchronised CDMA system, different offsets of the same m-sequence can be used by different users.
  - In this case the excellent autocorrelation properties (rather than the poor cross-correlation) are employed.
  - Unfortunately this approach cannot operate in an asynchronous environment.

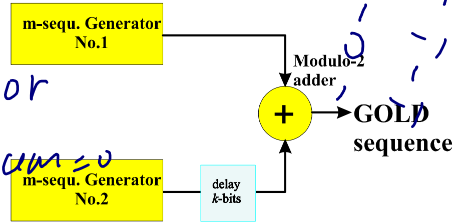
Gold Sequences

- Although m-sequences possess excellent randomness (and especially autocorrelation) properties, they are not generally used for CDMA purposes as it is difficult to find a set of m-sequences with low cross-correlation for all possible pairs of sequences within the set.
- However, by slightly relaxing the conditions on the autocorrelation function, we can obtain a family of code sequences with lower cross-correlation.
- Such an encoding family can be achieved by Gold sequences or Gold codes which are generated by the modulo-2 sum of two m-sequences of equal period.

- The Gold sequence is actually obtained by the modulo-2 sum of two m-sequences with different phase shifts for the first m-sequence relative to the second.
- Since there are $N_c = 2^m - 1$ different relative phase shifts, and since we can also have the two m-sequences alone, the actual number of different Gold-sequences that can be generated by this procedure is $2^m + 1$

$0 \rightarrow 1$ xor

$1 \rightarrow -1$ sum=0



Auto-Correlation Properties

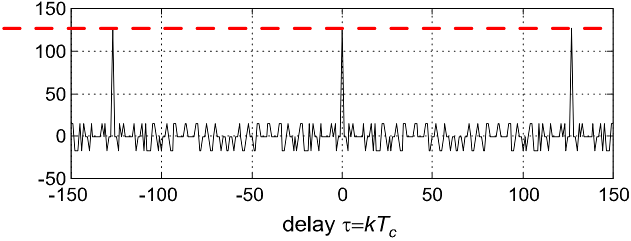
- Gold sequences, however, are not maximal length sequences.
- Therefore, their auto-correlation function **is not** the two valued one given by Equ. (19), i.e.

$$\{N_c, -1\} \tag{27}$$

- The auto-correlation still has the periodic peaks, but between the peaks the auto-correlation is no longer flat.

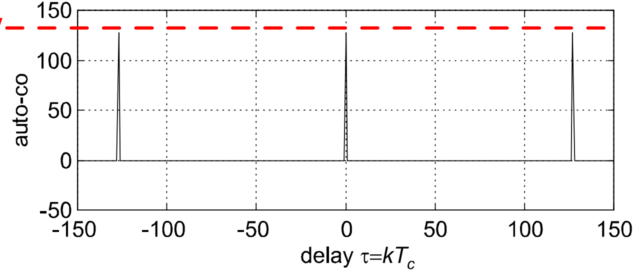
- Example of a Gold Sequence of $N_c = 127 = 2^7 - 1$

$R_{bb}(\tau)$: 127



- Example of an m-sequence of $N_c = 127 = 2^7 - 1$

$R_{bb}(\tau)$: 127



Cross-Correlation Properties

- Gold-sequences have the same cross-correlation characteristics as preferred m-sequences,
  i.e. their cross-correlation is three valued.
- Gold sequences have higher R_{auto} and lower R_{cross} than m-sequences, and the trade-off (see Equ. 11) between these parameters is thus verified.

Balanced Gold codes.

- Balanced Gold Sequence: The number of "-1s" in a code period exceed the number of "1s" by one as is the case for m-sequences.
- We should note that not all Gold codes (generated by modulo-2 addition of 2 m-sequences) are balanced, i.e. the number of "-1s" in a code period does not always exceed the number of "1s" by one.
- For example, for $m = odd$ only $2^{m-1}+1$ code sequences of the total 2^m+1 are balanced, while the rest code $2^{m-1}-1$ sequences have an excess or a deficiency of -1s.
- For $m = 7$, for instance, only 65 **balanced** Gold codes can be produced, out of a total possible of 129. Of these, 63 are non-maximal and two are maximal length sequences.
- Balanced Gold codes have more desirable spectral characteristics than non-balanced.
- Balanced Gold codes are generated by appropriately selecting the relative phases of the two original m-sequences.
- SUMMARY: By selecting any preferred pair of primitive polynomials it is easy to construct a very large set of PN-sequences (Gold-sequences). Thus, by assigning to each user one sequence from this set, the interference from other users is minimised.

Appendices

1. Appendix A:
   Properties of a purely random sequence

2. Appendix B:
   Auto and Cross Correlation functions of two PN-sequences

3. Appendix C:
   The concept of a 'Primitive Polynomial' in GF(2)

4. Appendix D:
   Finite Field - Basic Theory

5. Appendix E:
   Table of Irreducible Polynomials over GF(2)

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