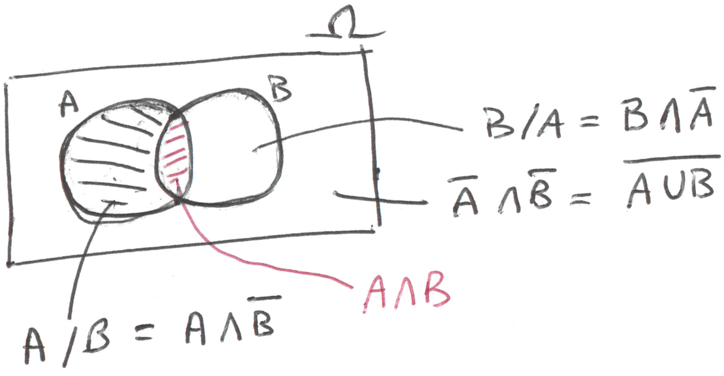
Lecture 3 and 4

Set



disjoint union

$$A = (A \wedge \bar{B}) \cup (A \wedge B)$$

$$\underline{A \cup B} = B \cup (A \wedge \bar{B})$$

Probability $\quad P(\underline{E})$

3 Axioms

1) $0 \leq P(E) \leq 1$
2) $P(S) = 1$
3) If $E \wedge F = \phi$, then $P(E \cup F) = P(E) + P(F)$

$\hookrightarrow$ $P(E \cup F) = P(E) + P(F) - P(E \wedge F)$

Conditional Probability

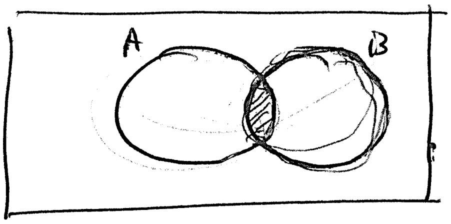
A, B

$$P(A \mid B) \geq P(A)$$

A pass
B study

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

when $P(B) > 0$



- $B \subseteq A$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1 \underset{Ax\,1}{\geq} P(A)$$

- $A \cap B = \emptyset$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = 0 \leq P(A)$$

Multiplication Law

$$\boxed{P(A \mid B)\,P(B) = P(A \cap B) = P(B \cap A) = P(B \mid A)\,P(A)}$$

A pass
B study

if $P(A) = P(B)$ $\rightarrow$ $P(A \mid B) = P(B \mid A)$

- A, B independent

$$\boxed{P(A|B) = P(A)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A|B)\,P(B) \underset{\text{ind.}}{=} P(A)\,P(B)$$

$$= P(B|A)\,P(A) \Rightarrow \boxed{P(B|A) = P(B)}$$

- n events mutually independent

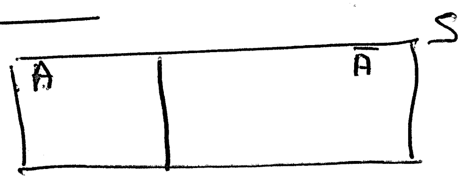
$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)\,P(A_2) \dots P(A_n)$$

ex $A = \{1, 2, 3\}$ $\quad B = \{3, 4\}$ $\quad S = \{1, 2, 3, 4, 5, 6\}$

$\boxed{P(A) = \frac{3}{6} = \frac{1}{2}}$ $\quad P(B) = \frac{2}{6} = \frac{1}{3}$ $\quad P(A \cap B) = \frac{1}{6}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{2/6} = \frac{1}{2} = P(A)$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{1/6}{3/6} = \frac{1}{3} = P(B)$$



$$A \cup \bar{A} = S$$

$$P(A \cup \bar{A}) \underset{Ax\,3}{=} P(A) + P(\bar{A}) = P(S) \underset{Ax\,2}{=} 1$$

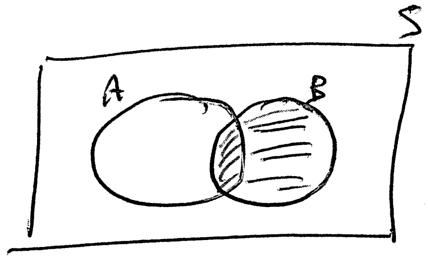
$A \cap \bar{A} = \emptyset$

A and $\bar{A}$ form a partition of S

$$P(\bar{A}) = 1 - P(A)$$

$$P(\bar{A}|B) = 1 - P(A|B)$$

Show that $P(\bar{A}|B) = 1 - P(A|B)$



$B = \underline{(B\wedge A)} \cup \underline{(B\wedge\bar{A})}$

$\frac{P(B)}{P(B)} = \frac{P(B\wedge A)}{P(B)} + \frac{P(B\wedge\bar{A})}{P(B)}$ Ax 3

$1 = P(A|B) + P(\bar{A}|B)$

$A\cup\bar{A} = S \longrightarrow P(A) + P(\bar{A}) = 1$

$B\cup\bar{B} = S \longrightarrow P(B) + P(\bar{B}) = 1$

$A = (A\wedge B)\cup(A\wedge\bar{B}) \longrightarrow P(A) = P(A\wedge B) + P(A\wedge\bar{B})$

$B = (B\wedge A)\cup(B\wedge\bar{A}) \longrightarrow \underline{P(B)} = \underline{P(B\wedge A)} + \underline{P(B\wedge\bar{A})}$

$\bar{A} = (\bar{A}\wedge B)\cup(\bar{A}\wedge\bar{B}) \longrightarrow P(\bar{A}) = P(\bar{A}\wedge B) + P(\bar{A}\wedge\bar{B})$

$\bar{B} = (\bar{B}\wedge A)\cup(\bar{B}\wedge\bar{A}) \longrightarrow P(\bar{B}) = P(\bar{B}\wedge A) + P(\bar{B}\wedge\bar{A})$

Probability Tables

	A	$\bar{A}$	
B	$P(A\wedge B)$	$P(\bar{A}\wedge B)$	$P(B)$
$\bar{B}$	$P(A\wedge\bar{B})$	$P(\bar{A}\wedge\bar{B})$	$P(\bar{B})$
	$P(A)$	$P(\bar{A})$	1

$P(A|B) = \frac{P(A\wedge B)}{P(B)}$

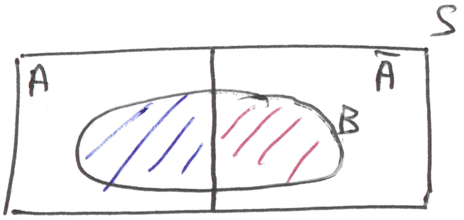
ex 100 components two defects A, B

2	components	have defect	A and B	$A \wedge B$ $\quad P(A \wedge B) = \frac{2}{100}$
6	"	" "	A only	$A \wedge \bar{B}$ $\quad P(A \wedge \bar{B}) = \frac{6}{100}$
4	"	" "	B only	$B \wedge \bar{A}$ $\quad P(B \wedge \bar{A}) = \frac{4}{100}$

	A	$\bar{A}$	
B	$\frac{2}{100}$	$\frac{4}{100}$	$\frac{6}{100}$
$\bar{B}$	$\frac{6}{100}$	$\frac{88}{100}$	$\frac{94}{100}$
	$\frac{8}{100}$	$\frac{92}{100}$	1

$$P(A|B) = \frac{P(A \wedge B)}{P(B)} = \frac{2/100}{6/100} = \frac{1}{3}$$

Total Probability



A and $\bar{A}$ form a partition of S

A, B

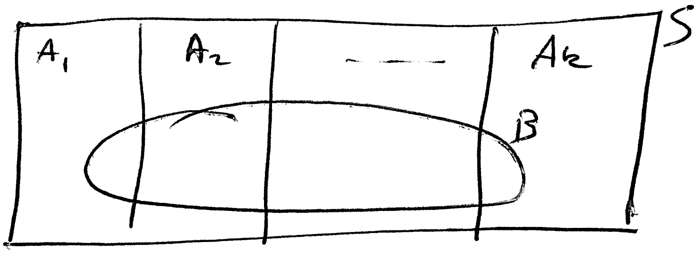
$$B = \underline{(B \wedge A)} \cup \underline{(B \wedge \bar{A})}$$

$$P(B) = P(B \wedge A) + P(B \wedge \bar{A})$$

$$= P(B|A)\,P(A) + P(B|\bar{A})\,P(\bar{A})$$

$$\begin{tikzcd}
P(A) & A \arrow[dr, "P(B/A)"] & & \\
& & B & P(B) \\
P(\bar{A}) & \bar{A} \arrow[ur, "P(B|\bar{A})"'] & &
\end{tikzcd}$$

causes effect



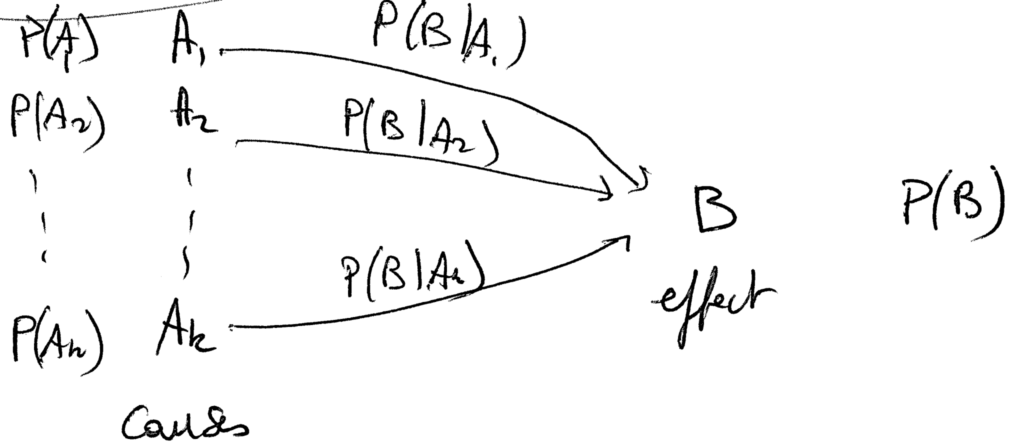
$A_1, A_2, \dots, A_k$ form a partition of S

$$B = (B \cap A_1) \cup (B \cap A_2) \cup \dots \cup (B \cap A_k)$$

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \dots + P(B \cap A_k)$$

$$= P(B|A_1)\,P(A_1) + P(B|A_2)\,P(A_2) + \dots + P(B|A_k)\,P(A_k)$$

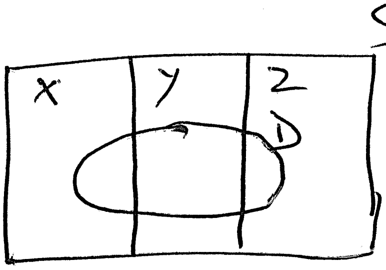
$$= \sum_{i=1}^{k} P(B|A_i)\,P(A_i)$$



ex 3 machines X, Y, Z

X produces 50% of the components, 3% are defective
Y " 30% " " , 4% "
Z " 20% " " , 5% "

D defective component $P(D) = ?$



$$D = (D \cap X) \cup (D \cap Y) \cup (D \cap Z)$$

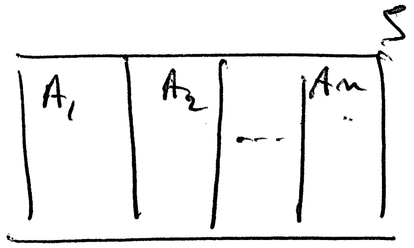
$$P(D) = P(D|X)\,P(X) + P(D|Y)\,P(Y) + P(D|Z)\,P(Z)$$

$$= 0.03 \times 0.5 + 0.04 \times 0.3 + 0.05 \times 0.2$$

$$
\begin{tikzcd}
P(X) & X \arrow[drr, "P(D|X)"] & & \\
P(Y) & Y \arrow[rr, "P(D|Y)"] & & D \\
P(Z) & Z \arrow[urr, "P(D|Z)"] & & \text{effect}
\end{tikzcd}
$$

causes

$A_1, A_2, \ldots, A_n$ partition of S



$P(A_k)$
$P(B|A_k)$ $\Longrightarrow P(B)$

$P(A_k | B)$ Bayes Theorem

$$P(A_k \cap B) = P(B|A_k)\,P(A_k) = P(A_k|B)\,P(B)$$

$$P(A_k|B) = \frac{P(B|A_k)\,P(A_k)}{P(B)}$$

$$\boxed{P(A_k|B) = \frac{P(B|A_k)\,P(A_k)}{\sum_{i=1}^{n} P(B|A_i)\,P(A_i)}}$$

$$
\begin{tikzcd}
P(A_1) & A_1 \arrow[drr, "P(B|A_1)"] & & \\
 & \vdots & & B \\
P(A_n) & A_n \arrow[urr, "P(B|A_n)"] & &
\end{tikzcd}
$$

$P(A_k|B) =$

causes effect

ex Communication

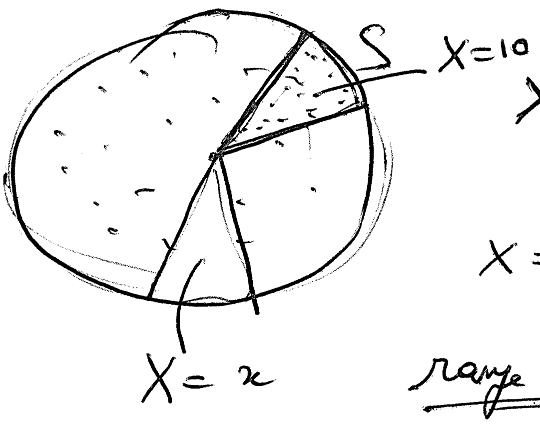
$$P(1 \mid X) = \frac{P(X \mid 1)\,P(1)}{P(X \mid 0)\,P(0) + P(X \mid 1)\,P(1)}$$

Random Variable (RV)

$X : S \longrightarrow \mathbb{R}$

$X = x$

ex darts



X score

$X = 1, 2, 3, \ldots 60$

range of $X = \{1, 2, 3, \ldots 60\}$

Discrete

- Probability mass function (PMF)

x	x_1	x_2	x_3	--	x_n
$f_X(x)$ $P(X=x)$	$P(X=x_1)$ $f_X(x_1)$ p_1	$P(X=x_2)$ $f_X(x_2)$ p_2	---		$P(X=x_n)$ $f_X(x_n)$ p_n

$f_X(x)$

$P(X = x)$

Properties

1) $\underline{f_X(x_i)} = P(X = x_i) = p_i \geq 0$

2) $\sum_{i=1}^{\infty} f_X(x_i) = \sum_{i=1}^{\infty} P(X = x_i) = 1$

ex

x	1	2	3	4
$f_X(x)$	0.2	0.3	0.3	0.2

- Cumulative distribution function (CDF)

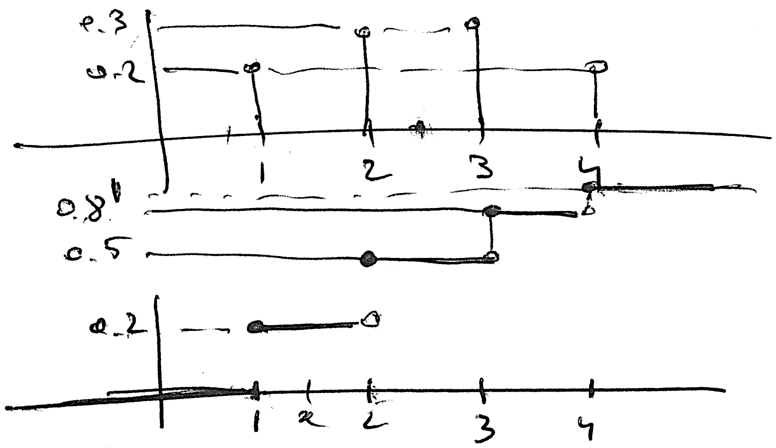
$P(X \leq x)$

$x_1, x_2, x_3 \dots x_n$
in increasing order

$$P(X \leq x_j) = \sum_{i=1}^{j} P(X = x_i) = \sum_{i=1}^{j} P_i$$

PMF

CDF



$P(X \leq 2$

- Expectation / Theoretical mean

$$\mu = \underline{E(X)} = \sum_x x\, f_X(x)$$

x	1	2	3	4
$f_X(x)$	0.2	0.3	0.3	0.2

$$\mu = E(X) = 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.3 + 4 \times 0.2 = 2.5$$

Property

- $E(aX+b) = \sum_x (ax+b)\, f_X(x)$

$$= \sum_x a\, x\, f_X(x) + \sum_x b\, f_X(x)$$

$$= a\underbrace{\left[\sum_x x\, f_X(x)\right]}_{E(X)} + b\underbrace{\left[\sum_x f_X(x)\right]}_{1}$$

$$= a\, E(X) + b$$

- $E(g(X)) = \sum_x g(x)\, f_X(x)$ $\qquad g(X) = aX+b$

$g(X) = X^2$

$$E(X^2) = \sum_x x^2 f_X(x) \neq [E(X)]^2 = \left[\sum_x x\, f_X(x)\right]^2$$

Variance

$$\sigma^2 = Var(X) = E\left([X-\mu]^2\right) = \sum_x (x-\mu)^2 f_X(x)$$

x	1	2	3	4
$f_X(x)$	0.2	0.3	0.3	0.2
$f_X(x)$	0.1	0.4	0.4	0.1

$\mu = 2.5$

$$Var(X) = (1-\mu)^2\, 0.2 + (2-\mu)^2\, 0.3 + (3-\mu)^2\, 0.3 + (4-\mu)^2\, 0.2$$

σ standard deviation

$$\mathrm{Var}(X) = \sum_x (x-\mu)^2 \, f_X(x) \geq 0$$

$$= \sum_x (x^2 - 2\mu x + \mu^2) f_X(x)$$

$$= \underbrace{\sum_x x^2 f_X(x)}_{E(X^2)} - \underbrace{\sum_x 2\mu x \, f_X(x)}_{-2\mu \underbrace{\sum x f_X(x)}_{E(X)=\mu} = -2\mu^2} + \underbrace{\sum \mu^2 f_X(x)}_{\mu^2 \underbrace{\sum f_X(x)}_{1} = \mu^2}$$

$$= E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2$$

$$= E(X^2) - E(X)^2 \geq 0$$

$$E(X^2) \geq E(X)^2$$

$$\mathrm{Var}(aX+b) = E\left(\left[(aX+b) - E(aX+b)\right]^2\right)$$

$$= E\left(\left[aX+\cancel{b} - aE(X) - \cancel{b}\right]^2\right)$$

$$= E\left(a^2 (X - E(X))^2\right)$$

$$= a^2 \underbrace{E\left((X-\mu)^2\right)}_{\mathrm{Var}(X)} = a^2 \, \mathrm{Var}(X)$$

function of a! not a function of b!

$$\sum_x a^2 (x-\mu)^2 f_X(x)$$

$$= a^2 \underbrace{\sum_x (x-\mu)^2 f_X(x)}_{\mathrm{Var}(X)}$$