Problem Sheets: Advanced Communication Theory

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Decision Rules

1. The receiver of a binary communication system was designed in an optimum way based on the following information:

   - $C_{00} = C_{11} = 0; C_{10} = 3; C_{01} = 1$
     where C_{ij} is the cost associated with choosing hypothesis H_i when in fact H_j is true
   - the likelihood functions are

   $$\mathrm{pdf}_{\mathrm{r}/H_0}(r) = \frac{1}{3}\mathrm{rect}\left\{\frac{r}{3}\right\}$$

   and

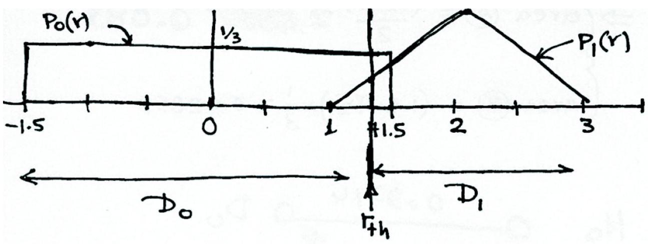
   $$\mathrm{pdf}_{\mathrm{r}/H_1}(r) = \Lambda\left\{r-2\right\}$$

   where r is the observed signal at the output of the channel.

   (a) Design an optimum receiver. 15%

   (b) Find the forward transition matrix $\mathbb{F}$ of this binary channel. 10%

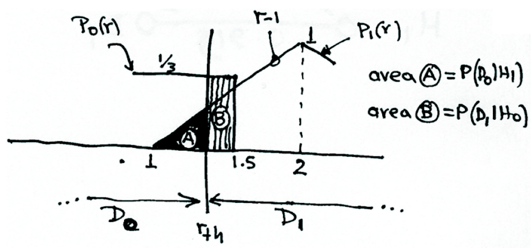
Solution



(a)

$C_{00}.\Pr(D_0|H_0) + C_{10}.\Pr(D_1|H_0) = C_{11}.\Pr(D_1|H_1) + C_{01}.\Pr(D_0|H_1)$

$\Rightarrow 3\Pr(D_1|H_0) = \Pr(D_0|H_1)$ Equ(1)



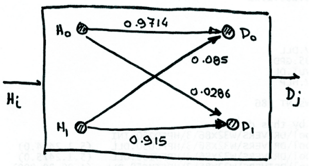
$$\text{Equ(1)} \Rightarrow 3\underbrace{(\mathbf{1.5}-r_{\text{th}})\frac{\mathbf{1}}{\mathbf{3}}}_{\text{area B}} = \underbrace{\frac{(r_{\text{th}}-\mathbf{1})^{\mathbf{2}}}{\mathbf{2}}}_{\text{area A}}$$

$\Rightarrow(1.5-r_{\text{th}})=\frac{r_{\text{th}}^2-2r_{\text{th}}+1}{2}$
$\Rightarrow 3-2r_{\text{th}}=r_{\text{th}}^2-2r_{\text{th}}+1$
$\Rightarrow r_{\text{th}}^2=2$
$\Rightarrow r_{\text{th}}=\sqrt{2}=1.41$

(b) $\mathbb{F}$=?

$$r_{\text{th}}=\sqrt{2}=1.41 \Rightarrow \begin{cases}\text{area A}=\frac{(\sqrt{2}-1)^2}{2}\simeq \mathbf{0.085}\\ \text{area B}=(\mathbf{1.5}-\sqrt{\mathbf{2}})\frac{1}{3}\simeq\mathbf{0.02859}\end{cases}$$

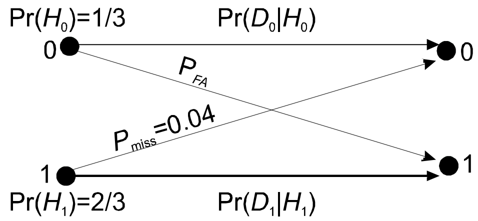
i.e. $\mathbb{F}=\begin{bmatrix}0.9714, & 0.085\\ 0.0286, & 0.915\end{bmatrix}$



2. Consider a binary communication system in which the channel noise is additive Gaussian of zero mean and variance 1, that is N(0,1). The system employs two correlated signals (with time-cross correlation $\rho=0.5$), and a correlation receiver which operates on the Bayes-decision criterion with the following costs:

$$C_{00}=C_{11}=0; C_{10}=1.858; C_{01}=0.5$$

If the communication system is modelled as follows:



(a) estimate its energy utilization efficiency EUE. 30%

(b) What is the False Alarm Probability, p_{FA}, and the bit error probabilty, p_{e}, for the above system? 10%

Solution

(a) One way to solve this problem is to use the Bayes decision variables (see Lecture Notes):
$G_1=\int_0^{T_{cs}} r(t)s_1(t)dt+\frac{N_0}{2}\ln\left(\Pr(H_1)(C_{01}-C_{11})\right)-\frac{1}{2}E_1$
$G_0=\int_0^{T_{cs}} r(t)s_0(t)dt+\frac{N_0}{2}\ln\left(\Pr(H_0)(C_{10}-C_{00})\right)-\frac{1}{2}E_0$
at threshold $(r_{th})\Rightarrow$

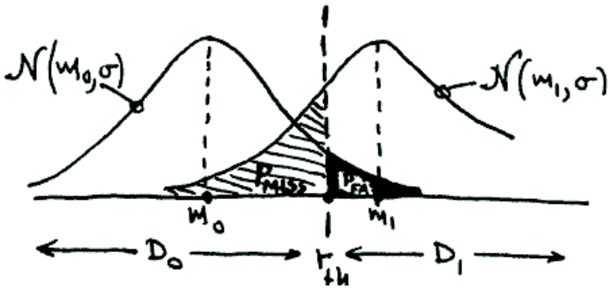
$$\begin{aligned}
& & G_1 &= G_0\\
& \Rightarrow & \int_0^{T_{cs}} r(t)s_1(t)dt+\frac{N_0}{2}\ln\left(\Pr(H_1)(C_{01}-C_{11})\right)-\frac{1}{2}E_1 &\\
& = & \int_0^{T_{cs}} r(t)s_0(t)dt+\frac{N_0}{2}\ln\left(\Pr(H_0)(C_{10}-C_{00})\right)-\frac{1}{2}E_0 &\\
\Rightarrow \underbrace{\int_0^{T_{cs}} r(t)s_1(t)dt-\int_0^{T_{cs}} r(t)s_0(t)dt}_{\triangleq G} & = & \underbrace{\frac{N_0}{2}\ln\left(\frac{\Pr(H_0)(C_{10}-C_{00})}{\Pr(H_1)(C_{01}-C_{11})}\right)+\frac{1}{2}(E_1-E_0)}_{\triangleq r_{th}} &
\end{aligned}$$

Let us define $\frac{\Pr(H_0)(C_{10}-C_{00})}{\Pr(H_1)(C_{01}-C_{11})} \triangleq \lambda$.Then

$$\lambda = \frac{\frac{1}{3}\times 1.858}{\frac{2}{3}\times 0.5} = 1.858$$

- Thus the optimum decisison rule is equivalent:

$$\begin{aligned} D_1 \text{ (i.e. choose } H_1) \text{ iff } G &\geqslant r_{thr}, \\ \text{otherwise } D_0 & \quad \text{(i.e. choose } H_0) \end{aligned}$$



$$p_e = \overbrace{\Pr(D_1|H_0)\Pr(H_0)}^{=\Pr(D_1,H_0)} + \overbrace{\Pr(D_0|H_1)\Pr(H_1)}^{=\Pr(D_0,H_1)}$$

where

- False Alarm probability: $p_{\mathrm{FA}} \triangleq \Pr(D_1|H_0) = \mathbf{T}\{\frac{r_{th}-m_0}{\sigma}\}$
- Probability of a "miss": $p_{\mathrm{miss}} \triangleq \Pr(D_0|H_1) = \mathbf{T}\{\frac{r_{th}-m_1}{\sigma}\} = 0.04$

Next let us define

$$\begin{aligned} E_b &\triangleq \frac{1}{2}(E_1+E_0) \\ \rho &\triangleq \frac{1}{E_b}\int_0^{T_{cs}} s_0(t).s_1(t)dt \end{aligned}$$

and estimate the parameters m_0, m_1 and σ:

- mean $m_0 = \mathcal{E}\{G|H_0\} = ... = \int_0^{T_{cs}} \left(s_0(t).s_1(t) - s_0^2(t)\right) dt$
- mean $m_1 = \mathcal{E}\{G|H_1\} = ... = \int_0^{T_{cs}} \left(s_1^2(t) - s_0(t).s_1(t)\right) dt$
- $\sigma^2 = var\{G\} = ... = \frac{N_0}{2}\int \left(s_1^2(t) + s_0^2(t) - 2s_1(t).s_0(t)\right) dt = N_0.E_b(1-\rho)$
  where

$$\begin{aligned} T\left\{\frac{r_{th}-m_1}{\sigma}\right\} &= 0.04 \\ \Rightarrow \text{(fromTail function graph} & \quad \text{- inverse)} \\ \Rightarrow \frac{r_{th}-m_1}{\sigma} &= 1.75 \end{aligned}$$

$$\begin{aligned} r_{th} - m_1 &= \underbrace{\frac{N_0}{2}\ln(\lambda) + \frac{1}{2}(E_1 - E_0)}_{r_{th}} - \underbrace{\int_0^{T_{cs}} \left(s_1^2(t) - s_0(t).s_1(t)\right) dt}_{m_1} \\ &= \frac{N_0}{2}\ln(\lambda) - \underbrace{\frac{1}{2}(E_1+E_0)}_{=E_b} + \underbrace{\int_0^{T_{cs}} s_0(t).s_1(t)dt}_{\rho E_b} \\ &= \frac{N_0}{2}\ln(\lambda) - E_b(1-\rho) \end{aligned}$$

Thus

$$\begin{aligned}\frac{r_{th}-m_1}{\sigma} &= \frac{\frac{N_0}{2}\ln(\lambda) - E_b(1-\rho)}{\sqrt{N_0.E_b(1-\rho)}} \\ &= \frac{\frac{1}{2}\ln(\lambda)}{\sqrt{(1-\rho)\mathrm{EUE}}} - \sqrt{(1-\rho)\mathrm{EUE}}\end{aligned}$$

In a similar fashion

$$\frac{r_{th}-m_0}{\sigma} = \frac{\frac{1}{2}\ln(\lambda)}{\sqrt{(1-\rho)\mathrm{EUE}}} + \sqrt{(1-\rho)\mathrm{EUE}}$$

- However,

$$\begin{aligned}\frac{r_{th}-m_1}{\sigma} &= 1.75 \\ \frac{\frac{1}{2}\ln(\lambda)}{\sqrt{(1-\rho)\mathrm{EUE}}} - \sqrt{(1-\rho)\mathrm{EUE}} &= 1.75 \\ (\text{Let us define } \sqrt{(1-\rho)\mathrm{EUE}} &\triangleq A \text{ with } A>0) \\ \frac{\frac{1}{2}\ln(\lambda)}{A} - A = 1.75 &\Rightarrow A^2 + 1.75A - \frac{1}{2}\ln(\lambda) = 0 \\ \text{with } \lambda &= 1.858 \Rightarrow A = 0.162 \\ \Rightarrow \sqrt{(1-\rho)\mathrm{EUE}} &= 0.162 \text{ (with } \rho = 0.5) \\ &\Rightarrow \mathrm{EUE} = 5.25\times 10^{-2}\end{aligned}$$

- $p_{\mathrm{FA}} = \Pr(D_1|H_0) = \mathrm{T}\{\frac{r_{th}-m_0}{\sigma}\}$

$$\begin{aligned}p_{\mathrm{FA}} &= \mathrm{T}\left\{\frac{\frac{1}{2}\ln(\lambda)}{\sqrt{(1-\rho)\mathrm{EUE}}} + \sqrt{(1-\rho)\mathrm{EUE}}\right\} \\ &= \mathrm{T}\left\{\frac{\frac{1}{2}\ln(1.858)}{0.162} + 0.162\right\} \\ &= \mathrm{T}\{2.074\} \simeq 1.7\times 10^{-2}\end{aligned}$$

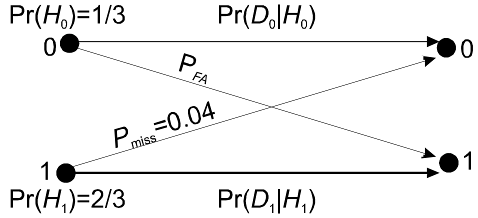
- $p_e = \overbrace{\Pr(D_1|H_0)\Pr(H_0)}^{=\Pr(D_1,H_0)} + \overbrace{\Pr(D_0|H_1)\Pr(H_1)}^{=\Pr(D_0,H_1)}$

$$p_e = 1.7\times 10^{-2}\times\frac{1}{3} + 4\times 10^{-2}\times\frac{2}{3} = 3.2333\times 10^{-2}$$

3. Consider a binary communication system in which the channel noise is additive Gaussian of zero mean and variance 1, that is N(0,1). The system employs two correlated signals with cross-correlation coefficient ρ, and a correlation receiver which operates on the Bayes-decision criterion with the following costs:

$$C_{00} = C_{11} = 0; C_{10} = 1.858; C_{01} = 0.5$$

If the communication system has an energy utilisation efficiency EUE$= 5.25\times 10^{-2}$ and is modelled as follows:



(a) estimate the cross correlation coefficient ρ. 30%

(b) What is the False Alarm Probability, p_{FA}, and the bit error probabilty, p_{e}, for the above system? 10%

Solution

(a) see solution to Problem 2(a)

(b) as in Problem 2(b) but

- $$\begin{aligned}
\frac{r_{th}-m_1}{\sigma} &= 1.75 \\
\frac{\frac{1}{2}\ln(\lambda)}{\sqrt{(1-\rho)\mathrm{EUE}}} - \sqrt{(1-\rho)\mathrm{EUE}} &= 1.75 \\
(\text{Let us define } \sqrt{(1-\rho)\mathrm{EUE}} &\triangleq A \text{ with } A>0) \\
\frac{\frac{1}{2}\ln(\lambda)}{A} - A = 1.75 &\Rightarrow A^2 + 1.75A - \frac{1}{2}\ln(\lambda) = 0 \\
\text{with } \lambda &= 1.858 \Rightarrow A = 0.162 \\
\Rightarrow \sqrt{(1-\rho)\mathrm{EUE}} &= 0.162 \text{ (with EUE=}5.25\times 10^{-2}) \\
\Rightarrow \rho &= 0.5
\end{aligned}$$

(c) see solution Problem 2(c)

4. Consider a binary pulse-code-modulation (binary-PCM) system where the digital modulation scheme being used is described as follows:
"The input to the digital modulator is a binary sequence of 1's and 0's with the number of 1s being twice the number of zeros. The binary sequence is transmitted as a pulse signal $s(t)$ with a *one* being sent as $2.\mathrm{rect}\left\{\frac{t}{T_b}\right\} + 4\Lambda\left\{\frac{t}{T_b/2}\right\}$ and *zero* being sent as $0.\mathrm{rect}(\frac{t}{T_b})$."
and the channel noise is assumed to be additive and uniformly distributed between −2 Volts and +2 Volts

(a) plot the probability density function of $s(t)$ 15%

(b) plot the probability density function of $r(t) = s(t) + n(t)$ 10%

(c) identify the likelihood functions $p_0(r)$ and $p_1(r)$ 15%

(d) design a Bayes Detector (i.e. decision rule) when the following costs apply:

$$C_{00} = C_{11} = 0; C_{10} = 0.8; C_{01} = 1$$

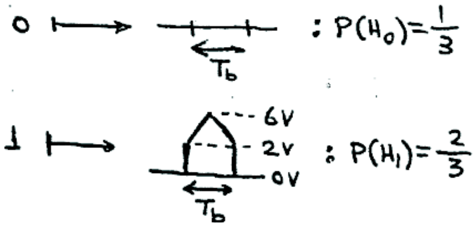
30%

(e) Find

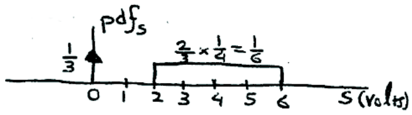
- the forward trans. probability matrix $\mathbb{F}$ 10%
- the joint-probability matrix $\mathbb{J}$ (i.e. the matrix with elements the probabilities $\Pr(H_i, D_j)\ \forall i,j$ 5%
- the amount of information (bits per channel symbol) delivered at the output of the system/channel. 10%
- the bit error probability, p_{e}. 10%

Solution

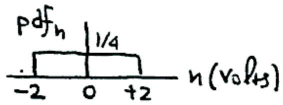
(a) pdf of $s(t)$:



$\Rightarrow \text{pdf}_s(s) = \frac{1}{3}\delta(s) + \frac{2}{3} \times \frac{1}{4} \times \text{rect}\{\frac{s-4}{4}\}$
i.e.

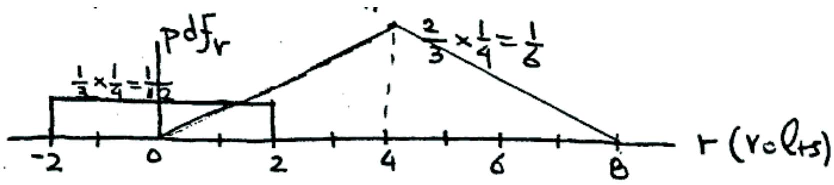


(b) pdf_n :



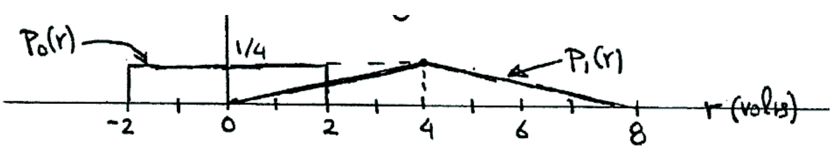
$$\begin{aligned} r(t) &= s(t) + n(t) \\ \Rightarrow \text{pdf}_r &= \text{pdf}_s * \text{pdf}_n \\ &= \underset{\Pr(H_0)}{\frac{1}{3}} \times \underbrace{\frac{1}{4} \times \text{rect}\left\{\frac{r}{4}\right\}}_{\text{pdf}_{r|H_0}} + \underset{\Pr(H_1)}{\frac{2}{3}} \times \underbrace{\frac{1}{4} \times 4 \times \Lambda\left\{\frac{r-4}{4}\right\}}_{\text{pdf}_{r|H_1}} \end{aligned}$$

i.e.



i.e. $\text{pdf}_r(r) = \frac{1}{12}\text{rect}\frac{r}{4} + \frac{2}{12}\Lambda\left\{\frac{r-4}{4}\right\}$

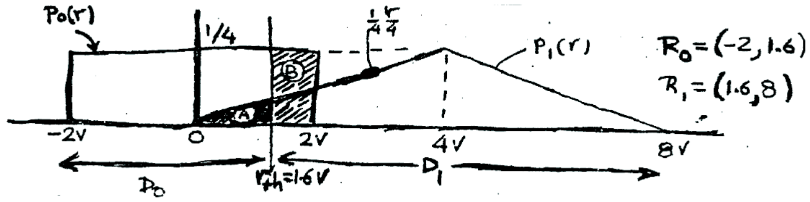
(c) likelihood functions placed together on the same graph:



i.e.

$p_0(r) \triangleq \text{pdf}_{r/H_0}(r) = \frac{1}{4}\ \text{rect}\{\frac{r}{4}\}, \qquad p_1(r) \triangleq \text{pdf}_{r/H_1}(r) = \frac{1}{4}\ \Lambda\left\{\frac{r-4}{4}\right\}$

(d) Bayes detector:
Choose H_1 iff
$(C_{10} - C_{00}) \Pr(H_0)\text{pdf}_{r/H_0}(r) < (C_{01} - C_{11}) \Pr(H_1)\text{pdf}_{r/H_1}(r)$
$\Rightarrow 0.8 \times \frac{1}{3} \times \frac{1}{4}\ \text{rect}\{\frac{r}{4}\} < 1 \times \frac{2}{3} \times \frac{1}{4}\ \Lambda\left\{\frac{r-4}{4}\right\}$
$\Rightarrow 0.4\ \text{rect}\{\frac{r}{4}\} < \Lambda\left\{\frac{r-4}{4}\right\} \Rightarrow 0.4 < \frac{r}{4} \Rightarrow r > 1.6$Volts
i.e. $\boxed{\text{choose } H_1 \text{ iff } r > 1.6\text{V, otherwise choose } H_0}$



(e) $P_{FA} = \Pr(D_1|H_0) = \text{area } \boxed{\text{B}} = \frac{1}{4} \times 0.4 = 0.1$

$P_{miss} = \Pr(D_0|H_1) = \text{area } \boxed{\text{A}} = \int_0^{1.6} \frac{1}{4}\frac{r}{4} dr = \frac{1}{16}\frac{r^2}{2}]_0^{1.6} = \frac{1}{32} \times 2.56 = 0.08$

- $\mathbb{F} = \begin{bmatrix} 0.9, & 0.08 \\ 0.1, & 0.92 \end{bmatrix}, \underline{p} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$
- $p_e = ?$

$$\begin{aligned} p_e &= \Pr(D_1|H_0).\Pr(H_0) + \Pr(D_0|H_1).\Pr(H_1) \\ &= 0.1 \times \frac{1}{3} + 0.08 \times \frac{2}{3} \\ &= 0.0867 \end{aligned}$$

- $\mathbb{J} = ?$

$$\begin{aligned} \mathbb{J} = \mathbb{F} \quad .diag\left(\underline{p}\right) = & \begin{bmatrix} 0.9, & 0.08 \\ 0.1, & 0.92 \end{bmatrix} \begin{bmatrix} 1/3, & 0 \\ 0 & 2/3 \end{bmatrix} \\ = & \begin{bmatrix} 0.3, & 0.0533 \\ 0.0333, & 0.6133 \end{bmatrix} \end{aligned}$$

- $H_{mut} = -\underline{1}_2^T \left(\mathbb{J} \odot \log_2 \left(\frac{\overbrace{\underline{q}}^{=\mathbb{F}.\underline{p}} \ \underline{p}^T}{\mathbb{J}} \right) \right) \underline{1}_2 = 0.5126 \frac{\text{bits}}{symbol}$

Costellation Diagram

5. The two signals $s_0(t)$ and $s_1(t)$ of a binary communication system each have energy equal to 93.3 and cross correlation coefficient 0.866.

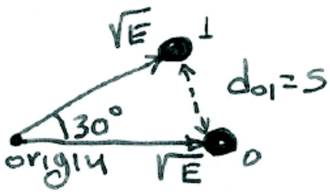
   (a) Draw the constellation diagram of the system properly labeled. 10%

   (b) What is the distance of these two signals? 10%

   Solution

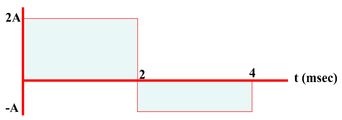
   (a) $\rho_{01} = \cos\phi \Rightarrow 0.866 = \cos\phi \Rightarrow \phi = 30^\circ$

   (b) $d_{01} = 2E - 2 \times 0.866 \times \sqrt{E \times E} \Rightarrow d_{01}^2 = 25 \Rightarrow d_{01} = 5$

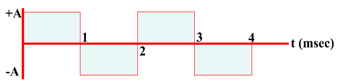


6. Consider a binary communication system which uses the following two equiprobable signals

   $s_0(t)$ in Volts:



   $s_1(t)$ in Volts:



   These signals are transmitted over a communication channel which adds white Gaussian noise having a double-sided power spectral density of 10^{-3} W/Hz.

   (a) Calculate the values of the associated signal-vectors $\underline{w}_{s_i}, \forall i$ for the above two signals as a function of the amplitude A. 15%

   (b) Draw a labelled block diagram of the MAP correlation receiver based on the signals vectors $\underline{w}_{s_i}$, $\forall i$. 15%

   (c) Plot the constellation diagram and properly label the decision regions as a function of the amplitude A. 15%

   (d) Calculate the amplitude A needed to achieve a minimum-bit-error probability of 6×10^{-3}. 20%

   (e) Find the forward transition matrix $\mathbb{F}$ of the equivalent discrete channel. 10%

   Solution

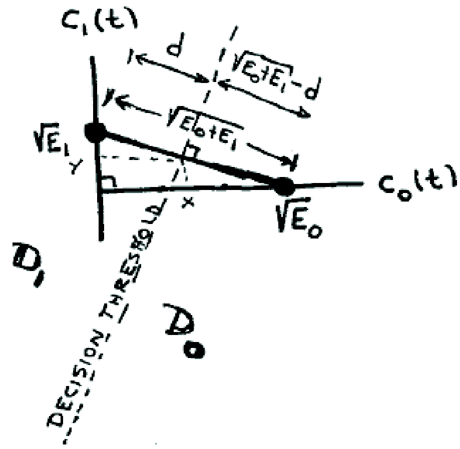
$$s_0(t) \perp s_1(t) \Rightarrow \begin{array}{l} c_0(t) = \frac{1}{\sqrt{E_0}} s_0(t) \\ c_1(t) = \frac{1}{\sqrt{E_1}} s_1(t) \end{array}, c_0(t) \perp c_1(t)$$

   $T_{cs} = 4 \times 10^{-3}$

   $E_0 = 10A^2 \times 10^{-3}$

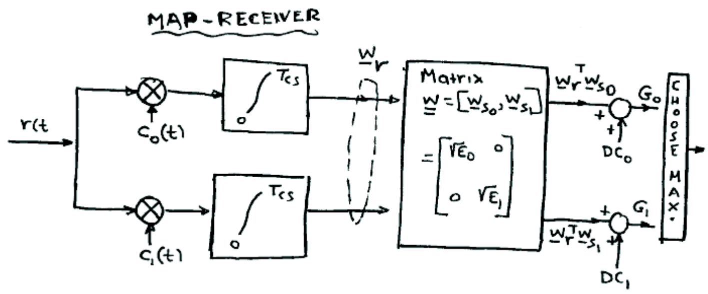
   $E_1 = 4A^2 \times 10^{-3}$

   - $\underline{w}_{s_0} = [\sqrt{E_0}, 0]^T$
   - $\underline{w}_{s_1} = [0, \sqrt{E_1}]^T$
   - $\mathrm{DC}_0 = \frac{N_0}{2} \ln(\Pr(H_0)) - \frac{1}{2} E_0$
   - $\mathrm{DC}_1 = \frac{N_0}{2} \ln(\Pr(H_1)) - \frac{1}{2} E_1$



Note: $\sqrt{E_0} = 0.1A$ and $\sqrt{E_1} = 0.06A$

- MAP Rx



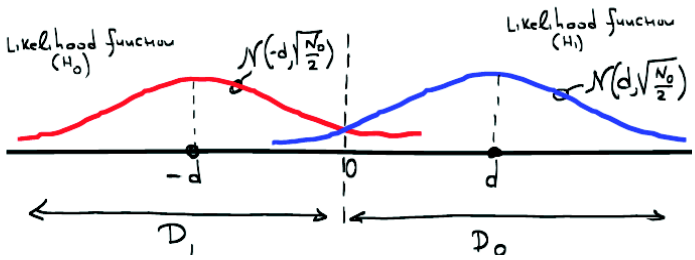
- let $\underline{w}_r^T = [x, y]$.Then, at decision threshold⇒

$$
\begin{aligned}
G_0 &= G_1 \\
\Rightarrow \underline{w}_r^T \underline{w}_{s_0} + \mathrm{DC}_0 &= \underline{w}_r^T \underline{w}_{s_1} + \mathrm{DC}_1 \\
\Rightarrow x\sqrt{E_0} + \frac{N_0}{2}\ln(\Pr(H_0)) - \frac{1}{2}E_0 &= y\sqrt{E_1} + \frac{N_0}{2}\ln(\Pr(H_1)) - \frac{1}{2}E_1 \\
x\sqrt{E_0} - \frac{1}{2}E_0 &= y\sqrt{E_1} - \frac{1}{2}E_1
\end{aligned}
$$

However, from the constellation diagram, we

$$
\begin{aligned}
\frac{x}{\sqrt{E_0}} &= \frac{d}{\sqrt{E_0+E_1}} \\
\frac{x}{\sqrt{E_1}} &= 1 - \frac{d}{\sqrt{E_0+E_1}}
\end{aligned}
$$

$$d = 5.91 \times 10^{-2} A$$



$$\begin{aligned}
\Pr(D_0|H_1) &= \Pr(D_1|H_0) = \mathbf{T}\left\{\frac{d}{\sqrt{N_0/2}}\right\} \\
\Pr(D_1|H_1) &= \Pr(D_0|H_0) = 1 - \mathbf{T}\left\{\frac{d}{\sqrt{N_0/2}}\right\} \\
p_e &= \frac{1}{2}\Pr(D_0|H_1) + \frac{1}{2}\Pr(D_1|H_0) \\
&= \mathbf{T}\left\{\frac{d}{\sqrt{N_0/2}}\right\} = \mathbf{T}\left\{\sqrt{\frac{E_0+E_1}{2N_0}}\right\} = \\
&= \mathbf{T}\left\{\sqrt{\frac{14A^2\times 10^{-3}}{2\times 2\times 10^{-3}}}\right\} = \mathbf{T}\left\{\sqrt{\frac{7}{2}A^2}\right\}
\end{aligned}$$

$$\begin{aligned}
p_e &= \mathbf{T}\left\{\sqrt{\frac{7}{2}A^2}\right\} \Rightarrow 6\times 10^{-3} = \mathbf{T}\left\{\sqrt{\frac{7}{2}A^2}\right\} \\
& \text{(using tail function, inverse)} \\
2.5 &= \sqrt{\frac{7}{2}A^2} \Rightarrow 6.25 = \frac{7}{2}A^2 \\
\Rightarrow & \; A^2 = 1.7857 \Rightarrow A = \sqrt{1.7857} = 1.3363V
\end{aligned}$$

$$\mathbb{F} = \begin{bmatrix} \Pr(D_0|H_0) = 0.994 & \Pr(D_0|H_1) = 0.006 \\ \Pr(D_1|H_0) = 0.006 & \Pr(D_1|H_1) = 0.994 \end{bmatrix}$$

7. Consider an M-ary Communication System with its signal set described as follows:

$$s_i(t) = A_i\left(\Lambda\left\{\frac{2t}{Tcs}\right\} + \text{rect}\left\{\frac{t}{T_{cs}}\right\}\right); i = 1,2,..,M \quad (1)$$

$$\text{with}\begin{cases} M = 4 \\ A_i = (2i-1-M)\times 10^{-3}\text{Volts} \\ T_{cs} = 6 \text{ sec} \\ \Pr(H_1) = \Pr(H_4) = 0.2 \\ \Pr(H_2) = \Pr(H_3) = 0.3 \end{cases}$$

The signals are transmitted over a communication channel which adds white Gaussian noise having a double-sided power spectral density of 10^{-6} W/Hz.

(a) Calculate the values of the signal-vectors $\underline{w}_{s_i}$, $i = 1,2,3,4$ for the above signal-set. 20%

(b) plot the constellation diagram and label the decision regions. 20%

(c) Draw an optimum receiver, based on the signals vectors $w_{s_i}, i =$ 1,2,3,4. 10%

(d) Model the whole system as a discrete communication channel. 15%

(e) Find

- the symbol error probability $p_{e,cs}$ at the output of the receiver 10%
- the joint-probability matrix $\mathbb{J}$ (i.e. the matrix with elements the probabilities $\Pr(H_i, D_j)$ $\forall i$,j 5%
- the amount of information (bits per channel symbol) delivered at the output of the system/channel. 10%

Solution

(a)

$$\begin{aligned}
A_1 &= -3mV; \Pr(A_1) = 0.2 \\
A_2 &= -1mV; \Pr(A_2) = 0.3 \\
A_3 &= +1mV; \Pr(A_3) = 0.3 \\
A_4 &= +3mV; \Pr(A_4) = 0.2
\end{aligned}$$

$$
\begin{aligned}
E_i &= \int_{-T_{cs}/2}^{T_{cs}/2} A_i^2 \left(\Lambda \left\{ \frac{2t}{T_{cs}} \right\} + \operatorname{rect}\left\{ \frac{t}{T_{cs}} \right\} \right)^2 dt \\
&= 2A_i^2 \int_{0}^{T_{cs}/2} \left(\Lambda \left\{ \frac{2t}{T_{cs}} \right\} + \operatorname{rect}\left\{ \frac{t}{T_{cs}} \right\} \right)^2 dt \\
& \quad \text{note that } \Lambda \left\{ \frac{2t}{T_{cs}} \right\} |_0^{T_{cs}/2} \text{ is equal to } \frac{-t + T_{cs}/2}{T_{cs}/2} \\
& \quad \text{and } \Lambda \left\{ \frac{2t}{T_{cs}} \right\} + \operatorname{rect}\left\{ \frac{t}{T_{cs}} \right\} = 1 + \frac{-t + T_{cs}/2}{T_{cs}/2} = 2 - \frac{2}{T_{cs}} t \\
& \quad \text{and } \left(\Lambda \left\{ \frac{2t}{T_{cs}} \right\} + \operatorname{rect}\left\{ \frac{t}{T_{cs}} \right\} \right)^2 = 4 + \frac{4}{T_{cs}^2} t^2 - 8 \frac{1}{T_{cs}} t \\
&= 2A_i^2 \int_{0}^{T_{cs}/2} \left(4 + \frac{4}{T_{cs}^2} t^2 - 8 \frac{1}{T_{cs}} t \right) dt = 2A_i^2 \left(\int_{0}^{T_{cs}/2} 4dt + \int_{0}^{T_{cs}/2} \frac{4}{T_{cs}^2} t^2 dt - \int_{0}^{T_{cs}/2} 8 \frac{1}{T_{cs}} t dt \right) \\
&= 2A_i^2 \left(2T_{cs} + \frac{T_{cs}}{6} - 2T_{cs} \right) = 2A_i^2 T_{cs} (2 + \frac{1}{6} - 1) = 2A_i^2 T_{cs} \frac{7}{6} \\
&= 14A_i^2
\end{aligned}
$$

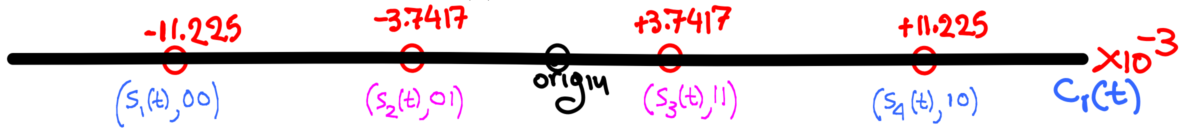
i.e.

$$
E_i = 14A_i^2 \Longrightarrow \left\{ \begin{array}{c} E_1 = E_4 = 126 \times 10^{-6} \\ E_1 = E_2 = 14 \times 10^{-6} \end{array} \right.
$$

$$
D = 1; c_1(t) = \frac{1}{\sqrt{E_1}} s_1(t) = \underbrace{\frac{1}{\sqrt{126 \times 10^{-6}}} 3 \times 10^{-3}}_{=0.26726} \left(\Lambda \left\{ \frac{2t}{T_{cs}} \right\} + \operatorname{rect}\left\{ \frac{t}{T_{cs}} \right\} \right)
$$

$$
\begin{aligned}
w_{s_1} &= -\sqrt{E_1} = -3\sqrt{14A_1^2} = -\sqrt{126} \times 10^{-3} = -11.225 \times 10^{-3} \\
w_{s_2} &= -\sqrt{E_2} = -\sqrt{14A_2^2} = -\sqrt{14} \times 10^{-3} = -3.7417 \times 10^{-3} \\
w_{s_3} &= +\sqrt{E_3} = +3.7417 \times 10^{-3} \\
w_{s_3} &= +3\sqrt{E_3} = +11.225 \times 10^{-3}
\end{aligned}
$$

(b)

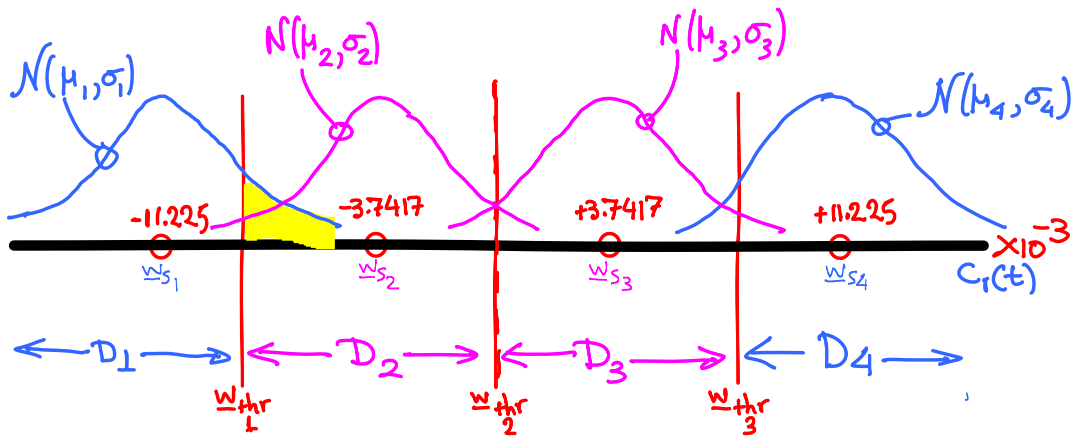


Likelihood functions

$$
\begin{aligned}
\text{LF}_1 &= \text{pdf}_{w_r|w_1} = N(\mu_1, \sigma_1) \\
\text{LF}_2 &= \text{pdf}_{w_r|w_2} = N(\mu_2, \sigma_2) \\
\text{LF}_3 &= \text{pdf}_{w_r|w_3} = N(\mu_3, \sigma_3) \\
\text{LF}_4 &= \text{pdf}_{w_r|w_4} = N(\mu_4, \sigma_4)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_1 &= w_{s_1} = -\sqrt{E_1}; \mu_2 = w_{s_2} = -\sqrt{E_2}; \mu_3 = w_{s_3} = \sqrt{E_3}; \mu_4 = w_{s_4} = \sqrt{E_4} \\
\sigma_1 &= \sigma_2 = \sigma_3 = \sigma_4 = \sqrt{\text{noise energy}} = \sqrt{\underbrace{\frac{N_0}{2} \times \overbrace{2B}^{=\frac{1}{T_{cs}}}}_{noise \text{ power}} \times T_{cs}} = \sqrt{\frac{N_0}{2}} = 10^{-3}
\end{aligned}
$$



$$
\begin{aligned}
\mathrm{DC}_1 &= \frac{N_0}{2}\ln(\Pr(H_1)) - \frac{1}{2}E_1 = 10^{-6}\ln(0.2) - \frac{1}{2}126 \times 10^{-6} = -64.609 \times 10^{-6} \\
\mathrm{DC}_2 &= \frac{N_0}{2}\ln(\Pr(H_2)) - \frac{1}{2}E_2 = 10^{-6}\ln(0.3) - \frac{1}{2}14 \times 10^{-6} = -8.2040 \times 10^{-6} \\
\mathrm{DC}_3 &= \mathrm{DC}_2 = -8.2040 \times 10^{-6}; \\
\mathrm{DC}_4 &= \mathrm{DC}_1 = -64.609 \times 10^{-6}
\end{aligned}
$$

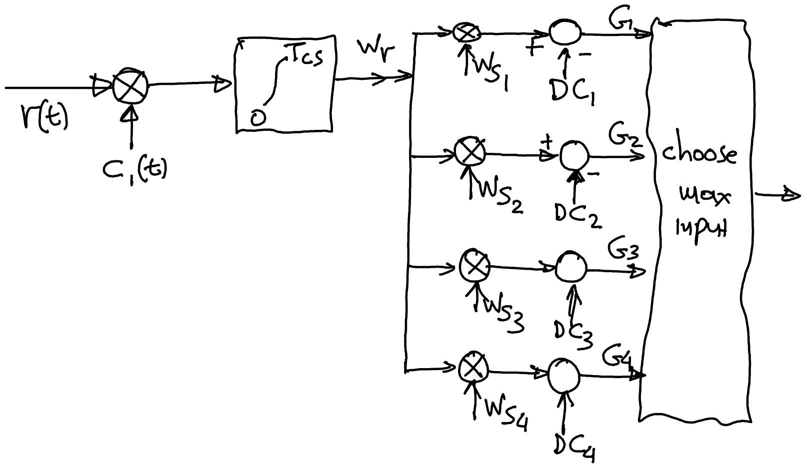
$$
\begin{aligned}
G_1 &= w_r w_{s_1} + \mathrm{DC}_1 = -3\sqrt{14} \times 10^{-3} w_r - 64.609 \times 10^{-6} \\
G_2 &= w_r w_{s_2} + \mathrm{DC}_2 = -\sqrt{14} \times 10^{-3} w_r - 8.2040 \times 10^{-6} \\
G_3 &= w_r . w_{s_3} + \mathrm{DC}_3 = \sqrt{14} \times 10^{-3} w_r - 8.2040 \times 10^{-6} \\
G_4 &= w_r . w_{s_4} + \mathrm{DC}_4 = 3\sqrt{14} \times 10^{-3} w_r - 64.609 \times 10^{-6}
\end{aligned}
$$

$$
\begin{aligned}
G_1 &= G_2 \Rightarrow \\
-3\sqrt{14} \times 10^{-3} w_r - 64.609 \times 10^{-6} &= -\sqrt{14} \times 10^{-3} w_r - 8.2040 \times 10^{-6} \\
w_{r,th1} &= \frac{-64.609 + 8.2040}{2\sqrt{14}} \times 10^{-3} \\
&= -7.5374 \times 10^{-3}
\end{aligned}
$$

$$
\begin{aligned}
G_2 &= G_3 \Rightarrow \\
-\sqrt{14} \times 10^{-3} w_r - 8.2040 \times 10^{-6} &= \sqrt{14} \times 10^{-3} w_r - 8.2040 \times 10^{-6} \\
w_{r,th2} &= 0
\end{aligned}
$$

$$
\begin{aligned}
G_3 &= G_4 \Rightarrow \\
\sqrt{14} \times 10^{-3} w_r - 8.2040 \times 10^{-6} &= 3\sqrt{14} \times 10^{-3} w_r - 64.609 \times 10^{-6} \\
w_{r,th3} &= \frac{64.609 - 8.2040}{2\sqrt{14}} \times 10^{-3} \\
&= 7.5374 \times 10^{-3}
\end{aligned}
$$

(c) Optimun Receiver:



(d) Forward Transition Matrix $\mathbb{F}$:

$$\mathbb{F} = \begin{bmatrix} \Pr(\boldsymbol{D_1}|\boldsymbol{H_1}), & \Pr(\boldsymbol{D_1}|\boldsymbol{H_2}), & \Pr(\boldsymbol{D_1}|\boldsymbol{H_3}), & \Pr(\boldsymbol{D_1}|\boldsymbol{H_4}) \\ \Pr(\boldsymbol{D_2}|\boldsymbol{H_1}), & \Pr(\boldsymbol{D_2}|\boldsymbol{H_2}), & \Pr(\boldsymbol{D_2}|\boldsymbol{H_3}), & \Pr(\boldsymbol{D_2}|\boldsymbol{H_4}) \\ \Pr(\boldsymbol{D_3}|\boldsymbol{H_1}), & \Pr(\boldsymbol{D_3}|\boldsymbol{H_2}), & \Pr(\boldsymbol{D_3}|\boldsymbol{H_3}), & \Pr(\boldsymbol{D_3}|\boldsymbol{H_4}), \\ \Pr(\boldsymbol{D_4}|\boldsymbol{H_1}), & \Pr(\boldsymbol{D_4}|\boldsymbol{H_2}), & \Pr(\boldsymbol{D_4}|\boldsymbol{H_3}), & \Pr(\boldsymbol{D_4}|\boldsymbol{H_4}) \end{bmatrix}$$

$$\begin{aligned} \Pr(\boldsymbol{D_1}|\boldsymbol{H_1}) &= \Pr(\boldsymbol{D_4}|\boldsymbol{H_4}) \\ &= 1 - \mathrm{T}\{\frac{|-7.5374 + 11.2249| \times 10^{-3}}{10^{-3}}\} = 1 - \mathrm{T}\{|-7.5374 + 11.2249|\} \\ &= 1 - \mathrm{T}\{3.6875\} = 1 - 1.1323 \times 10^{-4} \\ &= 0.99989 \end{aligned}$$

$$\begin{aligned} \Pr(\boldsymbol{D_1}|\boldsymbol{H_2}) &= \Pr(\boldsymbol{D_4}|\boldsymbol{H_3}) \\ &= \mathrm{T}\{\frac{|-7.5374 + 3.7417| \times 10^{-3}}{10^{-3}}\} = \mathrm{T}\{|-7.5374 + 3.7417|\} \\ &= \mathrm{T}\{3.7957\} \\ &= 7.3614 \times 10^{-5} \end{aligned}$$

$$\begin{aligned} \Pr(\boldsymbol{D_1}|\boldsymbol{H_3}) &= \Pr(\boldsymbol{D_4}|\boldsymbol{H_2}) \\ &= \mathrm{T}\{\frac{|-7.5374 - 3.7417| \times 10^{-3}}{10^{-3}}\} = \mathrm{T}\{|-7.5374 - 3.7417|\} \\ &= \mathrm{T}\{11.2791\} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \Pr(\boldsymbol{D_1}|\boldsymbol{H_4}) &= \Pr(\boldsymbol{D_4}|\boldsymbol{H_1}) \\ &= \mathrm{T}\{\frac{|-7.5374 - 11.2249| \times 10^{-3}}{10^{-3}}\} = \mathrm{T}\{|-7.5374 - 11.2249|\} \\ &= \mathrm{T}\{18.7623\} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \Pr(\boldsymbol{D_2}|\boldsymbol{H_1}) &= \Pr(\boldsymbol{D_3}|\boldsymbol{H_4}) \\ &= \mathrm{T}\{|11.2249 - 7.5374|\} - \mathrm{T}\{|11.2249|\} \\ &= \mathrm{T}\{3.6875\} - \mathrm{T}\{11.2249\} = 1.1323 \times 10^{-4} - 0 \\ &= 1.1323 \times 10^{-4} \end{aligned}$$

$$\begin{aligned}
\Pr\left(\boldsymbol{D_2}|\boldsymbol{H_3}\right) &= \Pr\left(\boldsymbol{D_3}|\boldsymbol{H_2}\right) \\
&= \mathrm{T}\{|3.7417|\} - \mathrm{T}\{|7.5374 + 3.7417|\} \\
&= \mathrm{T}\{3.7417\} - \mathrm{T}\{11.2791\} = 9.1390 \times 10^{-5} \\
&= 9.1390 \times 10^{-5}
\end{aligned}$$

$$\begin{aligned}
\Pr\left(\boldsymbol{D_2}|\boldsymbol{H_4}\right) &= \Pr\left(\boldsymbol{D_3}|\boldsymbol{H_1}\right) \\
&= \mathrm{T}\{|11.2249|\} - \mathrm{T}\{|11.2249 + 7.5374|\} \\
&= 1 - \mathrm{T}\{11.2249\} - \mathrm{T}\{18.7623\} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Pr\left(\boldsymbol{D_2}|\boldsymbol{H_2}\right) &= \Pr\left(\boldsymbol{D_3}|\boldsymbol{H_3}\right) \\
&= 1 - \mathrm{T}\{|-7.5374 + 3.7417|\} - \mathrm{T}\{|0 + 3.7417|\} \\
&= 1 - \mathrm{T}\{3.7957\} - \mathrm{T}\{3.7417\} = 1 - 7.3614 \times 10{-}5 - 9.139 \times 10^{-5} \\
&= 0.99983
\end{aligned}$$

$$\mathbb{F} = \begin{bmatrix} 0.99989, & 7.3614 \times 10^{-5}, & 0, & 0 \\ 1.1323 \times 10^{-4}, & 0.99983, & 9.1390 \times 10^{-5}, & 0 \\ 0, & 9.1390 \times 10^{-5}, & 0.99983, & 1.1323 \times 10^{-4}, \\ 0, & 0, & 7.3614 \times 10^{-5}, & 0.99989 \end{bmatrix}$$

(e)
- Channel symbol error rate:

$$\begin{aligned}
p_{e,cs} &= 1 - 2\Pr(D_1|H_1).\Pr(H_1) - 2\Pr(D_2|H_2)\Pr(H_2) \\
&= 1 - 2 \times 0.99989 \times 0.2 - 2 \times 0.99983 \times 0.3 \\
&= 1 - 0.399956 - 0.59898 \\
&= 1.46 \times 10^{-4}
\end{aligned}$$

or

$$p_{e,cs} = 1 - \underline{1}_4^T diag(\mathbb{J})\underline{1}_4$$

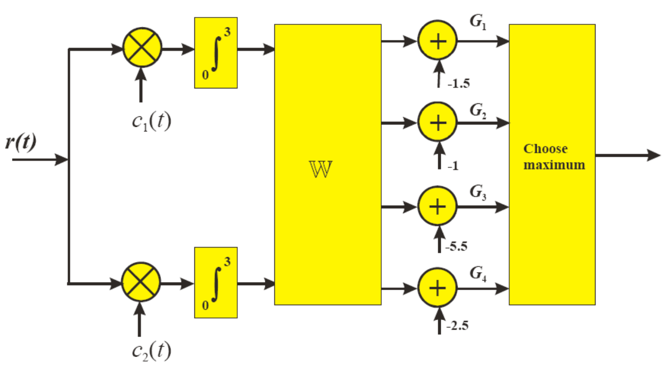
- Joint probability matrix $\mathbb{J}$
$\underline{p} = [0.2, 0.3, 0.3, 0.2]^T$

$$\mathbb{J} = \mathbb{F}\mathrm{diag}\{\underline{p}\} = \begin{bmatrix} 0.19998, & 2.2084 \times 10^{-5}, & 0, & 0 \\ 2.2646 \times 10^{-5}, & 0.29995, & 2.7417 \times 10^{-5}, & 0 \\ 0, & 2.7417 \times 10^{-5}, & 0.29995, & 2.2646 \times 10^{-4}, \\ 0, & 0, & 2.2084 \times 10^{-5}, & 0.19998 \end{bmatrix}$$

-

$$\begin{aligned}
H_{mut} &= -\underline{1}_4^T(\mathbb{J} \odot log_2(\frac{\mathbb{F}\underline{P}\underline{P}^T}{\mathbb{J}}))\underline{1}_4 \\
&= 1.9709 \text{ bits/channel-symbol}
\end{aligned}$$

8. Consider an M-ary communication system involving $M = 4$ signals $s_1(t), s_2(t), s_3(t)$ and $s_4(t)$ with energies E_1, E_2, E_3 and E_4 respectively. If the system uses the following MAP receiver



where $r(t)$ denoted the received signal, $c_1(t)$ and $c_2(t)$ are two orthonormal signals and the matrix $\mathbb{W}$ is defines as follows:

$$\mathbb{W} = \begin{bmatrix} \sqrt{3}, & 0, & -\sqrt{3}, & -\sqrt{3} \\ 0, & \sqrt{2}, & -2\sqrt{2}, & \sqrt{2} \end{bmatrix}$$

(a) Plot the constellation diagram. 5%

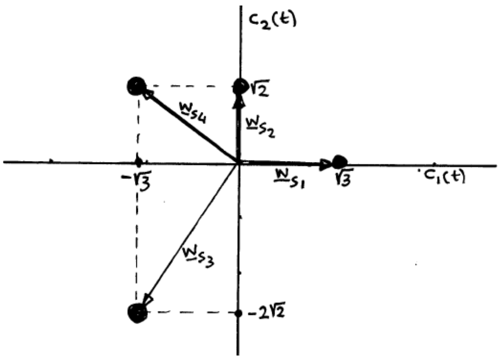
(b) Find the cross-correlation coefficients $\rho_{2,4}$ and $\rho_{3,4}$ 5%

(c) Find the probabilities $\Pr(s_1)$, $\Pr(s_2)$,$\Pr(s_3)$ and $\Pr(s_4)$; 5%

(d) Find the energies E_1, E_2, E_3 and E_4. 5%

Solution

(a) Constellation diagram:



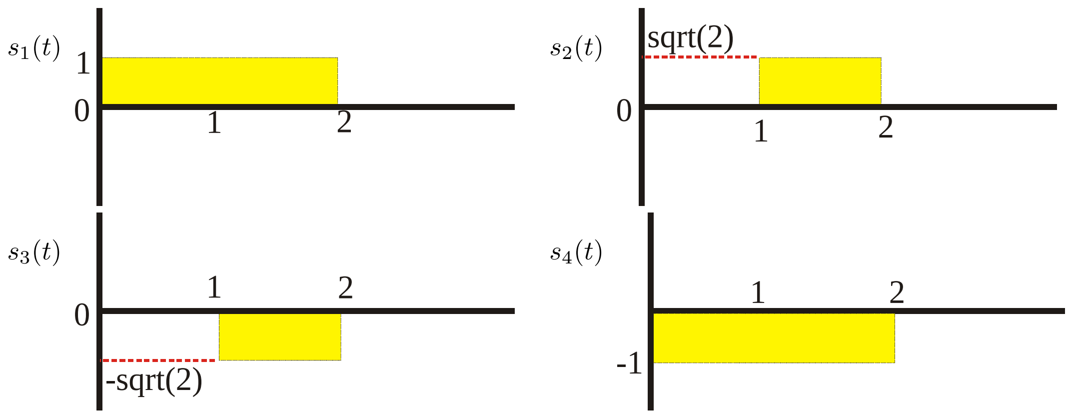
(b) The cross-correlation coefficients: $\rho_{2,4} = \frac{w_{s_2}^T w_{s_4}}{\|w_{s_2}\|\|w_{s_4}\|} = \cos(50.8°) = 0.632$;

$\rho_{3,4} = \frac{w_{s_3}^T w_{s_4}}{\|w_{s_3}\|\|w_{s_4}\|} = \cos(97.8°) = -0.135$

(c) $\Pr(s_1) = \Pr(s_2) = \Pr(s_3) = \Pr(s_4) = \frac{1}{4} = 0.25$

(d) $E_1 = 3\,; E_2 = 2; E_3 = 11; E_4 = 5$

9. Given the signalling waveforms $s_1(t), s_2(t), s_3(t)$ and $s_4(t)$ as shown below



find:

(a) The number D of a set of orthonormal signals $\{c_i(t)\}$ that can be used to represent the above waveforms. 10%

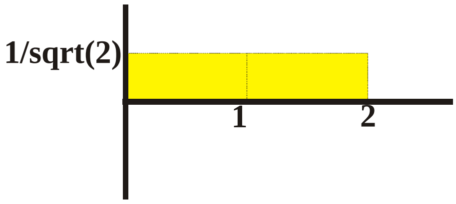
(b) The cross-correlation coefficients $\rho_{2,3}$ and $\rho_{3,4}$. 10%

(c) The weight vector $\underline{w}_{s_2}$. 10%

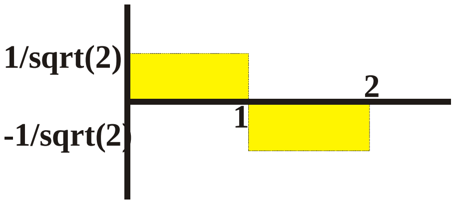
(d) The minimum distance of this set of signals. 5%

Solution

(a) Using Gram-Schmidt Orthogonalisation:
$c_1(t)$:



$c_2(t)$:



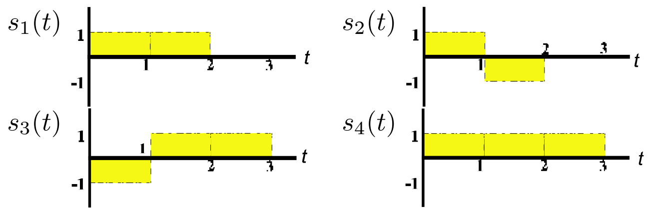
i.e. $M = 4,\ D = 2$

(b) $\rho_{23} = -1$ and $\rho_{34} = \cos(45°) = 0.707$

(c) $\underline{\boldsymbol{w}}_{\boldsymbol{s_2}} = [\mathbf{1},\ \mathbf{1}]^{\boldsymbol{T}}$

(d) $\underline{\boldsymbol{w}}_{\boldsymbol{s_1}} = [\sqrt{\mathbf{2}}, \mathbf{0}]^{\boldsymbol{T}} \Rightarrow$min distance$= \|\underline{\boldsymbol{w}}_{\boldsymbol{s_2}} - \underline{\boldsymbol{w}}_{\boldsymbol{s_1}}\| = \left\| \begin{bmatrix} \sqrt{\mathbf{2}} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \right\| = \mathbf{1.08}$

10. Given the signalling waveforms $s_1(t), s_2(t), s_3(t)$ and $s_4(t)$ as shown below



find:

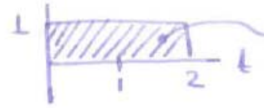
(a) the minimum number of dimensions required to represent these waveforms in N-dimensional vector space 10%

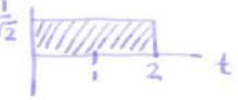
(b) an orthonormal set of of N signals $\{c_i(t)\}$ that can be used to represent the above waveforms 30%

(c) the values of the associated signal-vectors $\underline{w}_{s_i}$, $\forall i$ 10%

(d) the minimum distance between the signal-vectors $\underline{w}_{s_i}$, $\forall i$ 10%

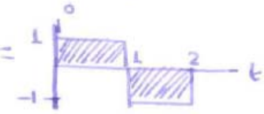
Solution

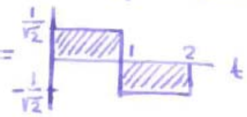
$\hat{c}_1(t) = s_1(t) =$  $\boxed{E_1 = 2}$ energy of $\hat{c}_1(t)$

$\therefore \; c_1(t) = \dfrac{\hat{c}_1(t)}{\sqrt{E_1}} =$ 

$\hat{c}_2(t) = s_2(t) - W_{21} c_1(t)$

$W_{21} = \int_0^3 s_2(t) c_1(t)\, dt = \left(1 \times \frac{1}{\sqrt{2}} + (-1) \times \frac{1}{\sqrt{2}}\right) = 0$

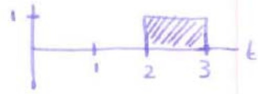
$\hat{c}_2(t) = s_2(t) =$  $\boxed{E_2 = 2}$

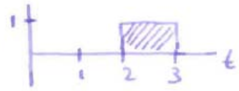
$\therefore \; c_2(t) = \dfrac{\hat{c}_2(t)}{\sqrt{E_2}} =$ 

$\hat{c}_3(t) = s_3(t) - W_{31} c_1(t) - W_{32} c_2(t)$

$W_{32} = \int_0^3 s_3(t) c_2(t)\, dt = \left(-1 \times \frac{1}{\sqrt{2}} + 1 \times \left(-\frac{1}{\sqrt{2}}\right) + 0\right) = -\sqrt{2}$

$W_{31} = \int_0^3 s_3(t) c_1(t)\, dt = \left(-1 \times \frac{1}{\sqrt{2}} + 1 \times \frac{1}{\sqrt{2}} + 0\right) = 0$

$\hat{c}_3(t) = s_3(t) + \sqrt{2}\, c_2(t) =$  $\boxed{E_3 = 1}$

$\therefore \; c_3(t) = \dfrac{\hat{c}_3(t)}{\sqrt{E_3}} =$ 

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$\hat{c}_4(t) = s_4(t) - W_{41} c_1(t) - W_{42} c_2(t) - W_{43} c_3(t)$

$W_{43} = \int_0^3 s_4(t) c_3(t)\, dt = 0 + 0 + 1 = 1$

$W_{42} = \int_0^3 s_4(t) c_2(t)\, dt = \frac{1}{\sqrt{2}} \times 1 + \left(-\frac{1}{\sqrt{2}}\right) \times 1 + 0 = 0$

$W_{41} = \int_0^3 s_4(t) c_1(t)\, dt = 1 \times \frac{1}{\sqrt{2}} + 1 \times \frac{1}{\sqrt{2}} + 0 = \sqrt{2}$

8

$\hat{c}_4(t) = s_4(t) - \sqrt{2}\, c_1(t) + c_3(t) = 0$

$\therefore \; \boxed{c_4(t) = 0}$

Collection of ACT Exam Papers & TESTS 28

$\therefore \boxed{N=3}$ ✓ $\underline{c}(t) = [c_1(t), c_2(t), c_3(t)]^T$

$$\underline{\underline{W}} = [\underline{w}_{s_1}, \underline{w}_{s_2}, \underline{w}_{s_3}, \underline{w}_{s_4}] = \begin{bmatrix} \sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$d_{12} = |\underline{w}_{s_1} - \underline{w}_{s_2}| = |[\sqrt{2}, -\sqrt{2}, 0]^T| = \sqrt{4} = 2$

$d_{13} = |\underline{w}_{s_1} - w_{s_3}| = |[\sqrt{2}, \sqrt{2}, 0]^T| = \sqrt{4} = 2$

$d_{14} = |\underline{w}_{s_1} - w_{s_4}| = |[0, 0, -1]^T| = 1$

$d_{23} = |\underline{w}_{s_2} - \underline{w}_{s_3}| = |[0, 2\sqrt{2}, -1]^T| = \sqrt{9} = 3$

$d_{24} = |\underline{w}_{s_2} - \underline{w}_{s_4}| = |[-\sqrt{2}, \sqrt{2}, -1]^T| = \sqrt{5}$

$d_{34} = |\underline{w}_{s_3} - \underline{w}_{s_4}| = |[-\sqrt{2}, -\sqrt{2}, 0]^T| = \sqrt{4} = 2$

minimum distance $= \min(d_{ij}) \ \forall i,j.$

$= d_{14} = 1.$

Matched Filters

11. A matched filter is used to detect the signal $s(t)$

$$s(t) = 3\text{rect}\left\{\frac{t}{10^{-6}}\right\}$$

which is corrupted by additive white Gaussian noise of double-sided power spectral density 0.5×10^{-6}.

What is the maximum Signal-to-Noise ratio at the filter output? 10%

Solution:

$T = 10^{-6}$; $A = 3$; $\frac{N_0}{2} = 0.5 \times 10^{-6} \Rightarrow N_0 = 10^{-6}$

$$\begin{aligned} SNR_{out,max} &= \int_0^T h_{opt}(\tau)s(T-\tau)d\tau \\ &= \int_0^T \frac{2}{N_0}A.A.dt = \frac{2}{N_0}A^2.T \\ &= \frac{2}{10^{-6}}3^2 \times 10^{-6} = 18 \end{aligned}$$

12. Find the impulse response of an approximate-matched filter matched to the signal $\Lambda\left(\frac{\tau}{T}\right)$ in the presence of non-white noise of autocorrelation function 10%

$$R_{nn}(\tau) = \Lambda\left(\frac{\tau}{T}\right)$$

Solution

$$\hat{h}_o(t) = FT^{-1}\{\hat{H}_o(f)\} \quad \boxed{1}$$

$$\hat{H}_o(f) = \frac{S(f)\cdot\exp(-j2\pi f T_o)}{PSD_n(f)} \quad \boxed{2}$$

However, $R_{nn}(\tau) = \Lambda\left\{\frac{\tau}{T}\right\} \Rightarrow PSD_n(f) = T\cdot\text{sinc}^2(fT)$

$$s(t) = \Lambda\left\{\frac{\tau}{T}\right\} \Rightarrow S(f) = T\cdot\text{sinc}^2(fT)$$

$$\therefore \boxed{2} \Rightarrow \hat{H}_o(f) = \frac{\cancel{T\cdot\text{sinc}^2(fT)}\cdot\exp(-j2\pi f T_o)}{\cancel{T\cdot\text{sinc}^2(fT)}}$$

$$= \exp(-j2\pi f T_o)$$

$$\boxed{1} \Rightarrow \hat{h}_o(t) = FT^{-1}\{\exp(-j2\pi f T_o)\} = \delta(t-T_o)$$

13. A matched filter is used to detect the signal $s(t)$

$$s(t) = \begin{cases} A\,, & \text{if } 0 \le t \le T \\ 0\,, & \text{otherwise} \end{cases}$$

which is corrupted by additive white Gaussian noise. What is the peak Signal-to-Noise ratio at the filter output? 10%

Solutions

$$\begin{aligned} SNR_{out,max} &= \int_0^T h_{opt}(\tau)s(T-\tau)d\tau \\ &= \int_0^T \frac{2}{N_0}A.A.dt = \frac{2}{N_0}A^2.T \end{aligned}$$

14. Consider an M-ary communication system involving M equiprobable orthogonal signals $s_i(t)$, $i = 1, 2, .., M$, $0 < t < T_{cs}$ of equal energy E. The system operates in the presence of additive white Gaussian noise of double sided power spectral density $N_0/2$ which is bandlimited to B Hz.

    If $r(t)$ represents the received signal at the input of a correlation receiver and G_j is the output of its j-th correlator (decision variable), defined as

$$G_j = \int_0^{T_{cs}} r(t).s_j(t).dt$$

    find the quantity

$$\mathcal{E}\{G_j|H_k\}$$

    where H_k denotes the hypothesis that the signal $s_k(t)$ was sent and $\mathcal{E}\{.\}$ is the expectation operator. 10%

    Solution

    $H_k : r(t) = s_k(t) + n(t)$

$$\mathcal{E}\{G_j|H_k\} = \mathcal{E}\left\{\int_0^{T_{cs}} r(t).s_j(t).dt|H_k\right\} = \mathcal{E}\left\{\int_0^{T_{cs}} (s_k(t) + n(t)).s_j(t).dt\right\} =$$

$$= \mathcal{E}\left\{\int_0^{T_{cs}} s_k(t).s_j(t).dt\right\} + \mathcal{E}\left\{\int_0^{T_{cs}} n(t).s_j(t).dt\right\}$$

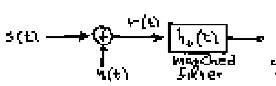
$$= \begin{cases} if\ k \neq j & \text{then } 0+0=0 \\ if\ k = j & \text{then } \mathcal{E}\left\{\int_0^{T_{cs}} s_k(t)^2.dt\right\} + 0 = E \end{cases}$$

15. Prove that the maximum signal-to-noise ratio $\mathrm{SNR}_{\underset{out}{\max}}$ at the output of a matched filter is given by:

$$\mathrm{SNR}_{\underset{out}{\max}} = \int_0^T h_o(z).s(T-z).dz$$

    where $h_o(t)$ is the impulse response of the filter matched to the signal $s(t)$ 10%

    Solution



$$\to \underbrace{\int_0^T h_o(\tau).s(T-\tau)\,d\tau}_{\tilde{s}(T)} + \underbrace{\int_0^T h_o(\tau).n(T-\tau)\,d\tau}_{\tilde{n}(T)}$$

$$\therefore \mathrm{SNR}_{\underset{\max}{out}} = \frac{\mathcal{E}\{\tilde{s}^2(T)\}}{\mathcal{E}\{\tilde{n}(T)^2\}} = \frac{\tilde{s}^2(T)}{\mathcal{E}\{\tilde{n}^2(T)\}} = \frac{\int_0^T\int_0^T h_o(\tau)h_o(u)\,s(T-\tau)\,s(T-u)\,d\tau.du}{\mathcal{E}\left\{\int_0^T\int_0^T h_o(\tau)h_o(u)\,n(T-\tau)\,n(T-u)\,d\tau\,du\right\}}$$

$$= \frac{\int_0^T\int_0^T \ldots}{\int\int h_o(\tau)h_o(u)\,\underbrace{\mathcal{E}\{n(T-\tau)n(T-u)\}}_{R_{nn}(\tau-u)}\,d\tau\,du}$$

$$= \frac{\int_0^T h_o(\tau)\,s(T-\tau)\,d\tau \,.\, \int_0^T h_o(u)\,s(T-u)\,du}{\int_0^T h_o(\tau)\underbrace{\left\{\int_0^T h_o(u)\,R_{nn}(\tau-u)\,du\right\}}_{s(T-\tau)}d\tau}$$

$$= \frac{\cancel{\int_0^T h_o(\tau).s(T-\tau)\,d\tau} \,.\, \int_0^T h_o(u)\,s(T-u)\,du}{\cancel{\int_0^T h_o(\tau).s(T-\tau)\,d\tau}}$$

$$= \int_0^T h_o(u)\,s(T-u)\,du.$$

More on MAP Receivers & Constellation Diagram

16. Consider an M-ary communication system with its signal set described as follows:
$s_i(t) = A_i \mathbf{cos}(2\pi F_c t)$, $i = 1, 2, .., M, 0 < t < 2\,\text{sec}$

$$\text{with} \begin{cases} M = 4 \\ A_i = (2i - 1 - M) \times 10^{-3}\text{Volts} \\ \Pr(\text{H}_1) = \Pr(\text{H}_4) = 0.2 \text{ and } \Pr(\text{H}_2) = \Pr(\text{H}_3) = 0.3 \end{cases}$$

The signals are transmitted over a communication channel which adds white Gaussian noise having a double-sided power spectral density of 10^{-6} W/Hz.

(a) Draw a labelled block diagram of the MAP receiver. [5 marks]

(b) Plot the constellation diagram and label the decision regions. [5 marks]

Solution

(a) MAP Rx:

$N_o = 2 \times 10^{-6}$

$T_{cs} = 2$

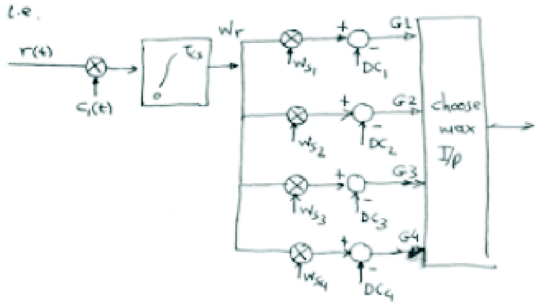
$D = 1$ (dim) $\longrightarrow c_1(t) = \frac{2}{T_{cs}} \cos(2\pi F_c t) = \cos(2\pi F_c t)$

$$A_i = (2i - 1 - 4) \times 10^{-3} \Rightarrow \begin{cases} A_1 = -3\text{mV} \\ A_2 = -1\text{mV} \\ A_3 = 1\text{mV} \\ A_4 = 3\text{mV} \end{cases}$$

$$w_{s_i} = -\sqrt{E_i} = -\sqrt{\frac{A_i^2}{2} T_{cs}} \Rightarrow \begin{cases} w_{s_1} = -3 \times 10^{-3} \\ w_{s_2} = -10^{-3} \\ w_{s_3} = +10^{-3} \\ w_{s_4} = 3 \times 10^{-3} \end{cases}$$

$$DC_i = \frac{N_o}{2} \ln(P_i) - \frac{1}{2} E_i \Rightarrow \begin{cases} DC_1 = -6.109 \times 10^{-6} \\ DC_2 = -1.704 \times 10^{-6} \\ DC_3 = -1.704 \times 10^{-6} \\ DC_4 = -6.109 \times 10^{-6} \end{cases}$$

i.e.



Note: $G_i = w_r . w_{s_i} + DC_i$

(b) 1st threshold, w_{r,thr_1} :

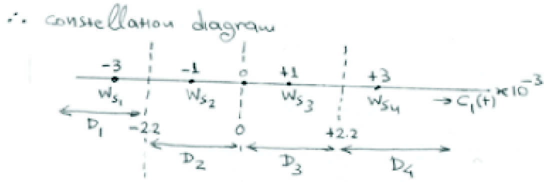
$$\begin{aligned} G_1 &= G_2 \Rightarrow \\ w_{r,thr_1} . w_{s_1} + DC_1 &= w_{r,thr_1} . w_{s_2} + DC_2 \\ w_{r,thr_1} &= \frac{DC_2 - DC_1}{w_{s_1} - w_{s_2}} = -2.2 \times 10^{-3} \end{aligned}$$

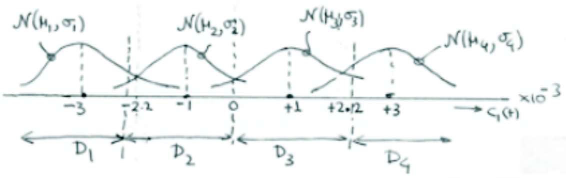
similarly, 2nd threshold w_{r,thr_2} :

$$\begin{aligned} G_2 &= G_3 \Rightarrow \\ w_{r,thr_2} &= \frac{DC_3 - DC_2}{w_{s_2} - w_{s_3}} = 0 \end{aligned}$$

and 3rd threshold w_{r,thr_3} :

$$\begin{aligned} G_3 &= G_4 \Rightarrow \\ w_{r,thr_3} &= \frac{DC_4 - DC_3}{w_{s_3} - w_{s_4}} = 2.2 \times 10^{-3} \end{aligned}$$





$\mu_1 = -3 \times 10^{-3}$
$\mu_2 = -1 \times 10^{-3}$
$\mu_3 = +1 \times 10^{-3}$
$\mu_4 = +3 \times 10^{-3}$
$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma$
where $\sigma^2 = \frac{N_0}{2} 2 \underset{\substack{\uparrow \\ 2T_{cs}}}{B}\, T_{cs} = \frac{N_0}{2} = 10^{-6}$ (this is the noise energy over T_{cs}) $\Rightarrow \sigma = 10^{-3}$