Imperial College London

Signals and Systems

Lecture 4

DR TANIA STATHAKI

READER (ASSOCIATE PROFESSOR) IN SIGNAL PROCESSING
IMPERIAL COLLEGE LONDON

LTI Systems

Linear Time-invariant (LTI) systems have two properties:

Linear: $\mathscr{H}\left(\alpha u[n]+\beta v[n]\right)=\alpha\mathscr{H}\left(u[n]\right)+\beta\mathscr{H}\left(v[n]\right)$
Time Invariant: $y[n]=\mathscr{H}\left(x[n]\right)\Rightarrow y[n-r]=\mathscr{H}\left(x[n-r]\right)\,\forall r$

The behaviour of an LTI system is completely defined by its impulse response: $h[n]=\mathscr{H}\left(\delta[n]\right)$

Proof:
We can always write $x[n]=\sum_{r=-\infty}^{\infty}x[r]\delta[n-r]$

Hence

$$\begin{aligned}\mathscr{H}\left(x[n]\right)&=\mathscr{H}\left(\sum_{r=-\infty}^{\infty}x[r]\delta[n-r]\right)\\&=\sum_{r=-\infty}^{\infty}x[r]\mathscr{H}\left(\delta[n-r]\right)\\&=\sum_{r=-\infty}^{\infty}x[r]h[n-r]\\&=x[n]*h[n]\end{aligned}$$

Convolution Properties

Convolution: $x[n] * v[n] = \sum_{r=-\infty}^{\infty} x[r]v[n-r]$

Convolution obeys normal arithmetic rules for multiplication:

Commutative: $x[n] * v[n] = v[n] * x[n]$

Proof: $\sum_r x[r]v[n-r] \overset{(i)}{=} \sum_p x[n-p]v[p]$
(i) substitute $p = n - r$

Associative: $x[n] * (v[n] * w[n]) = (x[n] * v[n]) * w[n]$
$\Rightarrow x[n] * v[n] * w[n]$ is unambiguous

Proof: $\sum_{r,s} x[n-r]v[r-s]w[s] \overset{(i)}{=} \sum_{p,q} x[p]v[q-p]w[n-q]$
(i) substitute $p = n - r$, $q = n - s$

Distributive over $+$:
$x[n] * (\alpha v[n] + \beta w[n]) = (x[n] * \alpha v[n]) + (x[n] * \beta w[n])$

Proof: $\sum_r x[n-r] (\alpha v[r] + \beta w[r]) =$
$\alpha \sum_r x[n-r]v[r] + \beta \sum_r x[n-r]w[r]$

Identity: $x[n] * \delta[n] = x[n]$

Proof: $\sum_r \delta[r]x[n-r] \overset{(i)}{=} x[n]$ (i) all terms zero except $r = 0$.

BIBO Stability

BIBO Stability: Bounded Input, $x[n] \Rightarrow$ Bounded Output, $y[n]$

The following are equivalent:

(1) An LTI system is BIBO stable

(2) $h[n]$ is absolutely summable, i.e. $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

(3) $H(z)$ region of absolute convergence includes $|z| = 1$.

(1) and (2) consist and iff condition

Proof $(1) \Rightarrow (2)$: **Suppose that $\sum |h[n]| \not\to \infty$**

The approach to go from (1) to (2) is to show that if $h[n]$ is not absolutely summable then the system is not BIBO stable, i.e., we can find at least one input for which the output is not bounded.

Define $x[n] = \begin{cases} 1 & h[-n] \geq 0 \\ -1 & h[-n] < 0 \end{cases}$

$y[0] = \sum h[0-n]\, x[n] = \sum |h[n]|$ BIBO $\Rightarrow y[0] \not\to \infty$

then $y[0] = \sum x[0-n]h[n] = \sum |h[n]|$. $\Rightarrow \sum |h[n]| \not\to \infty$

But $|x[n]| \leq 1 \forall n$ so BIBO $\Rightarrow y[0] = \sum |h[n]| \not\to \infty$.

Proof $(2) \Rightarrow (1)$:

Suppose $\sum |h[n]| = S < \infty$ and $|x[n]| \leq B$ is bounded.

$$\begin{aligned} \text{Then } |y[n]| &= \left|\sum_{r=-\infty}^{\infty} x[n-r]h[r]\right| \\ &\leq \sum_{r=-\infty}^{\infty} |x[n-r]|\,|h[r]| \\ &\leq B\sum_{r=-\infty}^{\infty} |h[r]| \leq BS < \infty \end{aligned}$$

distortionless system $H(e^{j\omega}) = Ke^{-j\omega t_0} \leftarrow$ delay

$t_g = t_p$ $\quad \angle H(e^{j\omega}) = -\omega t_0$

Frequency Response

$$t_0 = \underbrace{-\frac{\angle H(e^{j\omega})}{\omega}\Big|_{\omega=\omega_0}}_{\text{phase delay (sec)}} = \underbrace{-\frac{d}{d\omega}\angle H(e^{j\omega})\Big|_{\omega=\omega_0}}_{\text{group delay (sec)}}$$

For a BIBO stable system $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
where $H(e^{j\omega})$ is the DTFT of $h[n]$ i.e. $H(z)$ evaluated at $z = e^{j\omega}$.

group delay $-\frac{d}{d\omega}\angle H(e^{j\omega})$
- delay of the frequency components (groups)

Example: $h[n] = [\ 1 \quad 1 \quad 1\ ]$

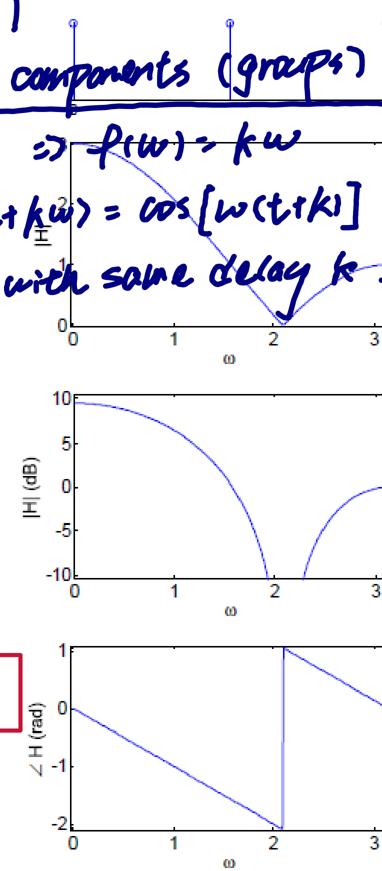
$$H(e^{j\omega}) = 1 + e^{-j\omega} + e^{-j2\omega}$$
$$= e^{-j\omega}\,(1 + 2\cos\omega)$$

if constant $\Rightarrow \frac{d\phi(\omega)}{d\omega} = k \Rightarrow \phi(\omega) = k\omega$

$\cos(\omega t + \phi(\omega)) = \cos(\omega t + k\omega) = \cos[\omega(t+k)]$

all frequency components with same delay k.

$$|H(e^{j\omega})| = |1 + 2\cos\omega|$$
$$\angle H(e^{j\omega}) = -\omega + \pi\frac{1 - \mathrm{sgn}(1+2\cos\omega)}{2}$$



Sign change in $(1 + 2\cos\omega)$ at $\omega = 2.1$ gives
(a) gradient discontinuity in $|H(e^{j\omega})|$
(b) an abrupt phase change of $\pm\pi$.

Group delay is $\boxed{-\frac{d}{d\omega}\angle H(e^{j\omega})}$: gives delay of the modulation envelope at each ω.

We will see that later with an example.

Normally varies with ω but for a symmetric filter it is constant: in this case $+1$ samples. !
Discontinuities of $\pm k\pi$ do not affect group delay.

$y(t) = m(t-t_g)\cdot\cos[w_c(t-t_p)]$

$t_g = -\frac{d}{dw}\angle H(e^{jw})\Big|_{w_0}$ $\quad t_p = -\frac{\angle H(e^{jw})}{w}\Big|_{w_0}$

Signal distortion during transmission
Distortionless transmission

$x(t) = \underbrace{m(t)}_{\text{envelope}}\underbrace{\cos(w_c t)}_{\text{HF carrier}} \longrightarrow \boxed{LTI}$

single freq of signal

group of freq.

- In certain types of systems we require the input to pass through the system without distortion. For example:
  - Signal transmission over a communication channel.
  - Amplifying systems.

phase delay: delay provided by the system to the single frequency (HF carrier)

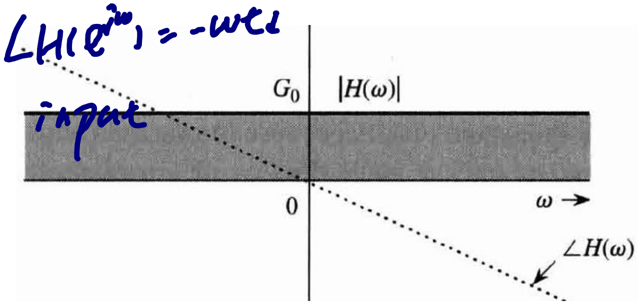
group delay: delay provided by the system to the envelope (group of freq).

- Distortionless transmission of an input $x(t)$ implies that $y(t) = G_0 x(t - t_d)$.
- Taking the Fourier transform of the above yields $Y(\omega) = G_0 X(\omega)e^{-j\omega t_d}$.
- Knowing that $Y(\omega) = H(\omega)X(\omega)$ we can write that the transfer function of a distortionless system is $H(\omega) = G_0 e^{-j\omega t_d}$.
  - $|H(\omega)| = G_0$ amplitude response must be a constant
  - $\angle H(\omega) = -\omega t_d$ phase response must be a linear function of ω with slope $-t_d$ which also passes through the origin

distortionless system:

$$\begin{cases} y(t) = G_0 x(t-t_d) \\ Y(\omega) = G_0 X(\omega) e^{-jwt_d} \end{cases} \Rightarrow \begin{cases} |H(e^{jw})| = G_0 \\ \angle H(e^{jw}) = -wt_d \end{cases}$$

only delay. no distortion to input



Group delay

- In order to assess phase linearity we can find the slope of $\angle H(\omega)$ as a function of frequency and see whether it is constant. We define:

$$t_g(\omega) = -\frac{d}{d\omega}\angle H(\omega)$$

- $t_g(\omega)$ is called group delay or envelope delay.

- Note that a phase response given by $\angle H(\omega) = \phi_0 - \omega t_d$ also has a constant group delay. From now on we can write $t_d = t_g$.

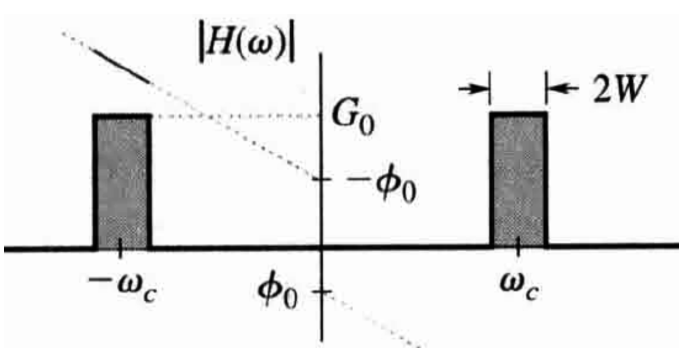
- Therefore, the condition for phase linearity by testing whether the group delay is constant is more relaxed.

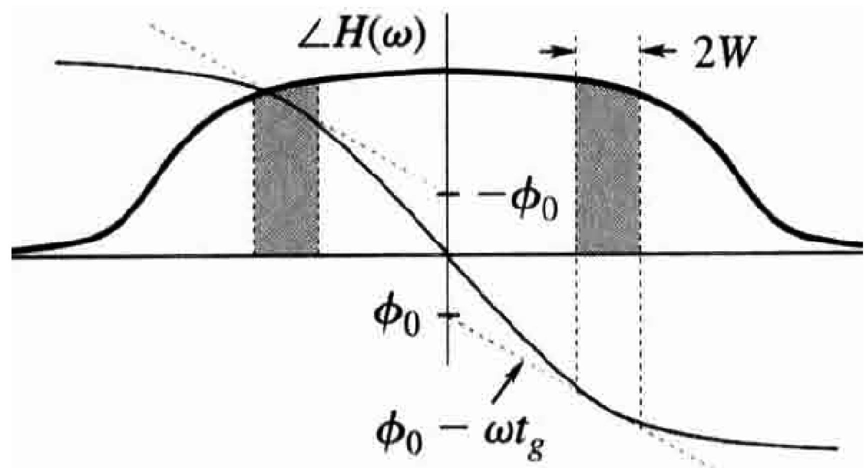
- Human ears are sensitive to amplitude distortion, but not phase distortion.

- Human eyes are sensitive to phase distortion, but not so much to amplitude distortion (recall the experiment where we have combined the amplitude of one image and the phase of another).

Bandpass systems and group delay

- For lowpass systems, the phase must be linear over the band of interest and also must pass through the origin.
  - Recall that phase is an odd function. Therefore, if it doesn't pass through the origin, it will have a jump at the origin; this means that the group delay will be a Dirac function.
  - Infinite group delay means that the input takes infinite time to arrive at the output, i.e., it doesn't practically get through.
- For bandpass systems, the phase must be linear over the band of interest but does not have to pass through the origin.
- Consider the following bandpass LTI system.

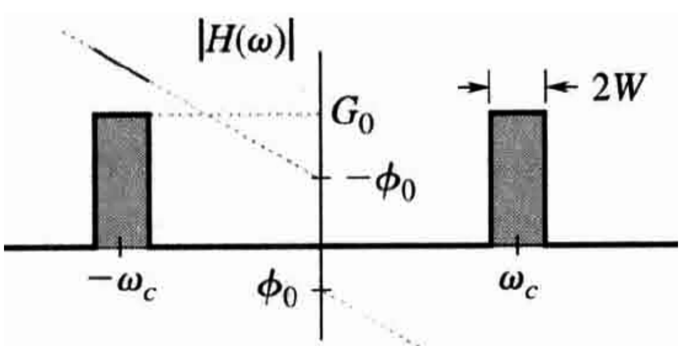


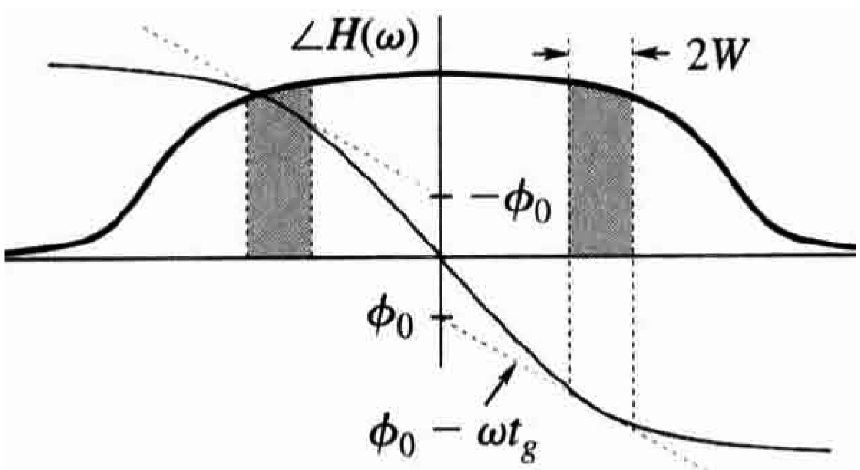


- The pass band is of width $2W$ centred at ω_c.

Bandpass systems and group delay cont.

- Within the pass band and for $\omega \geq 0$ the phase can be described as
$$\angle H(\omega) = \phi_0 - \omega t_g$$
- The phase is always an odd function, and therefore,
$$\angle H(-\omega) = -\angle H(\omega) = -(\phi_0 - \omega t_g) = -\phi_0 + \omega t_g$$
- We can write:
$$\boxed{\angle H(\omega) = \begin{cases} \phi_0 - \omega t_g & \omega \geq 0 \\ -\phi_0 - \omega t_g & \omega < 0 \end{cases} \text{(over band of interest)}}$$
- For a distortionless system we have $H(\omega) = G_o e^{j(\phi_0 - \omega t_g)}$, $\omega \geq 0$.





Bandpass systems and group delay cont.

envelope const.

linear phase / delay

- Consider the distortionless system $H(\omega) = G_o e^{j(\phi_0 - \omega t_g)}$, $\omega \geq 0$.
- Consider the bandpass modulated signal $z(t) = x(t)\cos\omega_c t$ centred at ω_c where $x(t)$ is a lowpass signal with bandwidth W.
  - $\cos\omega_c t$ is the carrier of $z(t)$
  - $x(t)$ is the envelope of $z(t)$
- Consider now the input $\hat{z}(t) = x(t)e^{j\omega_c t}$ with $\hat{Z}(\omega) = X(\omega - \omega_c)$.

$Z(\omega) = X(\omega - \omega_c)$

- The corresponding output is:

$Y(\omega) = H(\omega)Z(\omega) = H(\omega)X(\omega - \omega_c) = G_0 e^{j(\phi_0 - \omega t_g)} X(\omega - \omega_c)$

$= \underbrace{G_0 e^{j\phi_0}}_{\text{const}} X(\omega - \omega_c) e^{-j\omega t_g}$

$= G_0 e^{j\phi_0} x(t - t_g) e^{j\omega_c(t - t_g)}$

$= G_0 x(t - t_g) e^{j[\omega_c(t - t_g) + \phi_0]}$

$$\hat{Y}(\omega) = H(\omega)\,\hat{Z}(\omega) = H(\omega)X(\omega - \omega_c)$$

$$\hat{Y}(\omega) = G_0 X(\omega - \omega_c)e^{j(\phi_0 - \omega t_g)} = G_0 e^{j\phi_0} X(\omega - \omega_c)e^{-j\omega t_g}$$

- We use the properties:
  - If $x(t) \Leftrightarrow X(\omega)$ then:

    $x(t - t_0) \Leftrightarrow X(\omega)e^{-j\omega t_0}$ and $x(t)e^{j\omega_0 t} \Leftrightarrow X(\omega - \omega_0)$.
- We obtain: $\hat{y}(t) = G_0 e^{j\phi_0} x(t - t_g) e^{j\omega_c(t - t_g)} = G_0 x(t - t_g) e^{j[\omega_c(t - t_g) + \phi_0]}$

Imperial College London

Bandpass systems and group delay cont.

- Consider the distortionless system $H(\omega) = G_o e^{j(\phi_0 - \omega t_g)}$, $\omega \geq 0$.
- We showed that for the input $\hat{z}(t) = x(t)e^{j\omega_c t}$ the output is:

$$\hat{y}(t) = G_0 x(t - t_g) e^{j[\omega_c(t - t_g) + \phi_0]}$$

- For the input $z(t) = x(t)\cos\omega_c t = \mathrm{Re}\{\hat{z}(t)\}$ the output is

$$y(t) = \mathrm{Re}\{\hat{y}(t)\} = \mathrm{Re}\left\{G_0 x(t - t_g) e^{j[\omega_c(t - t_g) + \phi_0]}\right\}$$
$$= G_0 x(t - t_g)\cos[\omega_c(t - t_g) + \phi_0]$$

  - The output envelope $x(t - t_g)$ remains undistorted.
  - The output carrier acquires an extra phase ϕ_0.
  - In a modulation system the transmission is considered distortionless if the envelope $x(t)$ remains undistorted. This is because the signal information is contained solely in the envelope.
  - Therefore, the above type of transmission is considered distortionless.
  - We see that the group delay gives the delay of the modulation envelope at each ω.

Causality

Causal System: cannot see into the future

i.e. output at time n depends only on inputs up to time n.

Formal definition:

If $v[n] = x[n]$ for $n \leq n_0$ then $\mathscr{H}(v[n]) = \mathscr{H}(x[n])$ for $n \leq n_0$.

The following are equivalent:

(1) An LTI system is causal

(2) $h[n]$ is causal $\Leftrightarrow h[n] = 0$ for $n < 0$

(3) $H(z)$ converges for $z = \infty$

Any right-sided sequence can be made causal by adding a delay.
All the systems we will deal with are causal.

Imperial College London

Conditions on $h[n]$ and $H(z)$

Summary of conditions on $h[n]$ for LTI systems:

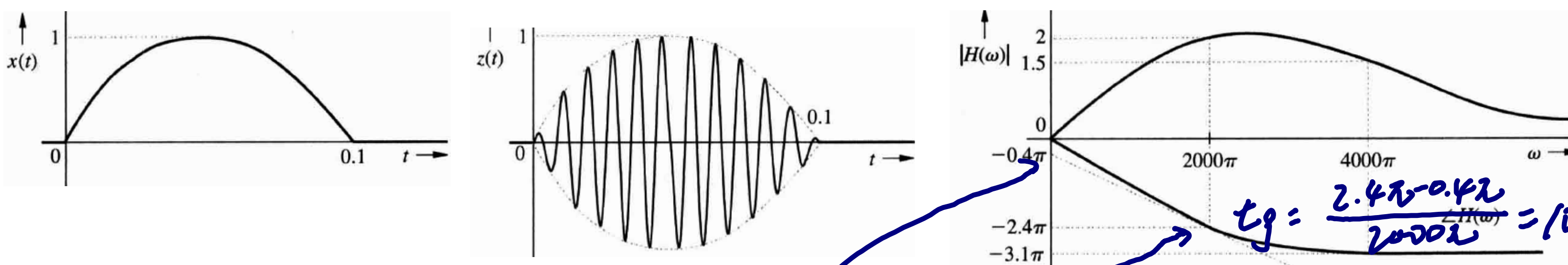
Causal	$\Leftrightarrow$	$h[n] = 0$ for $n < 0$
BIBO Stable	$\Leftrightarrow$	$\sum_{n=-\infty}^{\infty} \lvert h[n] \rvert < \infty$

Summary of conditions on $H(z)$ for LTI systems:

Causal	$\Leftrightarrow$	$H(\infty)$ converges
BIBO Stable	$\Leftrightarrow$	$H(z)$ converges for $\lvert z \rvert = 1$
Passive	$\Leftrightarrow$	$\lvert H(z) \rvert \le 1$ for $\lvert z \rvert = 1$
Lossless or Allpass	$\Leftrightarrow$	$\lvert H(z) \rvert = 1$ for $\lvert z \rvert = 1$

Problem on group delay

- A signal $z(t)$ shown below is given by $x(t)\cos\omega_c t$ where $\omega_c = 2000\pi$. The pulse $x(t)$ is a lowpass pulse of duration 0.1sec and has a bandwidth of about $10Hz$. This signal is passed through a filter whose frequency response is shown below. Find and sketch the filter output $y(t)$.



$t_g = \frac{2.4\pi - 0.4\pi}{2000\pi} = 10^{-3}$

- $z(t)$ is a narrow band signal with bandwidth of $20Hz$ centred around $f_c = \omega_c/2\pi = 1kHz$.

$y(t) = G_0 x(t-t_g)\cos[\omega_c(t-t_g)+\phi_0]$

$= 2x(t-10^{-3})\cos[2000\pi(t-10^{-3})-0.4\pi]$

- The gain at the centre frequency of $1kHz$ is 2.
- The group delay is: $t_g = \frac{2.4\pi - 0.4\pi}{2000\pi} = 10^{-3}$. It can be found by **drawing** (not calculating formally!) the tangent at ω_c.
- The intercept along the vertical axis by the tangent is $\phi_0 = -0.4\pi$.

Imperial College London

Problem on group delay cont.

- Based on the above analysis the output of the system is:

$$y(t) = G_0 x(t - t_g)\cos[\omega_c(t - t_g) + \phi_0]$$
$$= 2x(t - 10^{-3})\cos[2000\pi(t - 10^{-3}) - 0.4\pi]$$

distortionless system. linear phase delay.

envelope shape doesn't change.