

## THE QUESTIONS

[30]

1. a) Consider the continuous random variable  $X$  characterized by the following probability density function

$$f_X(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- i) Compute the cumulative distribution function of  $X$ , i.e.  $F_X(x)$ . [2]
- ii) Compute the expectation of  $X$ , i.e.  $E(X)$ , and the variance of  $X$ , i.e.  $\text{Var}(X)$ . [4]
- iii) Compute the moment generating function of  $X$ , i.e.  $m_X(t)$ . Explain how to make use this function to find the expectation and variance of a random variable. Apply this principle to  $X$ . [4]
- iv) By making use of Chebyshev's Inequality, determine a bound on

$$P\left(\left|X - \frac{1}{3}\right| \geq \frac{1}{4}\right).$$

Compute then the exact value of this probability.

[4]

- b) Consider the continuous random variable  $X$  characterized by the following probability density function

$$f_X(x) = \begin{cases} \frac{2x}{\theta^2}, & 0 \leq x \leq \theta, \\ 0, & \text{otherwise.} \end{cases}$$

We observe the random sample  $X_1, \dots, X_n$  (of size  $n$ ).

- i) Determine the method of moment estimator of  $\theta$  (denoted as  $\tilde{\theta}$ ). [4]
- ii) Compute the expectation and the variance of this estimator. [4]
- iii) Is this estimator biased or unbiased? Provide your reasoning. [4]
- iv) Assume  $n$  large. Compute an estimate of  $P(\tilde{\theta} \geq \theta)$ . Provide your reasoning. [4]

2. Consider two continuous random variables  $X$  and  $Y$  characterized by the following joint probability density function [20]

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-(x+y)}, & 0 < x < y < +\infty, \\ 0, & \text{otherwise.} \end{cases}$$

- a) Compute the expectation of  $X$ , i.e.  $E(X)$ . [4]
- b) Make the change of variables  $U = Y - X$ ,  $V = X$  and compute the joint probability density function  $f_{U,V}(u,v)$ . [4]
- c) Compute the marginal probability density function of  $U$  and  $V$ , i.e.  $f_U(u)$  and  $f_V(v)$ . [2]
- d) Are  $U$  and  $V$  independent? Provide your reasoning. [2]
- e) Compute the conditional probability density function of  $U$  given  $V$ , i.e.  $f_{U|V}(u|v)$ . [2]
- f) Compute the conditional expectation of  $U$  given  $V$ , i.e.  $E(U|V)$ . [2]
- g) By making use of f), compute the conditional expectation of  $Y$  given  $X$ , i.e.  $E(Y|X)$ . [2]
- h) By making use of g), compute the expectation of  $Y$ , i.e.  $E(Y)$ . [2]

## Mathematical Formulae

### 1. Probabilities for events

For events  $A$ ,  $B$ , and  $C$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

More generally  $P(\cup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \dots$

The odds in favour of  $A$

$$P(A) / P(\bar{A})$$

Conditional probability

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \quad \text{provided that } P(B) > 0$$

Chain rule

$$P(A \cap B \cap C) = P(A) P(B | A) P(C | A \cap B)$$

Bayes' rule

$$P(A | B) = \frac{P(A) P(B | A)}{P(A) P(B | A) + P(\bar{A}) P(B | \bar{A})}$$

$A$  and  $B$  are independent if

$$P(B | A) = P(B)$$

$A$ ,  $B$ , and  $C$  are independent if

$$P(A \cap B \cap C) = P(A)P(B)P(C), \quad \text{and}$$

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

### 2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable  $X$  is called the probability mass function (pmf) and is the complete set of probabilities  $\{p_x\} = \{P(X = x)\}$

Expectation  $E(X) = \mu = \sum_x x p_x$

For function  $g(x)$  of  $x$ ,  $E\{g(X)\} = \sum_x g(x) p_x$ , so  $E(X^2) = \sum_x x^2 p_x$

Sample mean  $\bar{x} = \frac{1}{n} \sum_k x_k$  estimates  $\mu$  from random sample  $x_1, x_2, \dots, x_n$

Variance  $\text{var}(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$

Sample variance  $s^2 = \frac{1}{n-1} \left\{ \sum_k x_k^2 - \frac{1}{n} \left( \sum_j x_j \right)^2 \right\}$  estimates  $\sigma^2$

Standard deviation  $\text{sd}(X) = \sigma$

If value  $y$  is observed with frequency  $n_y$

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness  $\beta_1 = E\left(\frac{X - \mu}{\sigma}\right)^3$  is estimated by  $\frac{1}{n-1} \sum \left(\frac{x_i - \bar{x}}{s}\right)^3$

Kurtosis  $\beta_2 = E\left(\frac{X - \mu}{\sigma}\right)^4 - 3$  is estimated by  $\frac{1}{n-1} \sum \left(\frac{x_i - \bar{x}}{s}\right)^4 - 3$

Sample median  $\tilde{x}$  or  $x_{\text{med}}$ . Half the sample values are smaller and half larger

If the sample values  $x_1, \dots, x_n$  are ordered as  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ ,

then  $\tilde{x} = x_{(\frac{n+1}{2})}$  if  $n$  is odd, and  $\tilde{x} = \frac{1}{2}(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})})$  if  $n$  is even

$\alpha$ -quantile  $Q(\alpha)$  is such that  $P(X \leq Q(\alpha)) = \alpha$

Sample  $\alpha$ -quantile  $\hat{Q}(\alpha)$  Proportion  $\alpha$  of the data values are smaller

Lower quartile  $Q1 = \hat{Q}(0.25)$  one quarter are smaller

Upper quartile  $Q3 = \hat{Q}(0.75)$  three quarters are smaller

Sample median  $\tilde{x} = \hat{Q}(0.5)$  estimates the population median  $Q(0.5)$

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf)  $F(x) = P(X \leq x) = \int_{x_0=-\infty}^x f(x_0)dx_0$

The probability density function (pdf)  $f(x) = \frac{dF(x)}{dx}$

$E(X) = \mu = \int_{-\infty}^{\infty} x f(x)dx$ ,  $\text{var}(X) = \sigma^2 = E(X^2) - \mu^2$ , where  $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$

4. Discrete probability distributions

Discrete Uniform  $Uniform(n)$

$$p_x = \frac{1}{n} \quad (x = 1, 2, \dots, n) \quad \mu = (n+1)/2, \quad \sigma^2 = (n^2-1)/12$$

Binomial distribution  $Binomial(n, \theta)$

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \dots, n) \quad \mu = n\theta, \quad \sigma^2 = n\theta(1-\theta)$$

Poisson distribution  $Poisson(\lambda)$

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \dots) \quad (\text{with } \lambda > 0) \quad \mu = \lambda, \quad \sigma^2 = \lambda$$

Geometric distribution  $Geometric(\theta)$

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \dots) \quad \mu = \frac{1}{\theta}, \quad \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution  $Uniform(\alpha, \beta)$

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \quad \mu = (\alpha + \beta)/2, \quad \sigma^2 = (\beta - \alpha)^2/12$$

Exponential distribution  $Exponential(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0). \end{cases} \quad \mu = 1/\lambda, \quad \sigma^2 = 1/\lambda^2$$

Normal distribution  $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2 \right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \text{var}(X) = \sigma^2$$

Standard normal distribution  $N(0,1)$

$$\text{If } X \text{ is } N(\mu, \sigma^2), \text{ then } Y = \frac{X-\mu}{\sigma} \text{ is } N(0,1)$$

## 6. Reliability

For a device in continuous operation with failure time random variable  $T$  having pdf  $f(t)$  ( $t > 0$ )

$$\text{The reliability function at time } t \quad R(t) = P(T > t)$$

$$\text{The failure rate or hazard function} \quad h(t) = f(t)/R(t)$$

$$\text{The cumulative hazard function} \quad H(t) = \int_0^t h(t_0) dt_0 = -\ln\{R(t)\}$$

$$\text{The Weibull}(\alpha, \beta) \text{ distribution has} \quad H(t) = \beta t^\alpha$$

## 7. System reliability

For a system of  $k$  devices, which operate independently, let

$$R_i = P(D_i) = P(\text{"device } i \text{ operates"})$$

The system reliability,  $R$ , is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \dots \cap D_k) = R_1 R_2 \dots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \dots \cup D_k) = 1 - (1 - R_1)(1 - R_2) \dots (1 - R_k)$$

## 8. Covariance and correlation

$$\text{The covariance of } X \text{ and } Y \quad \text{cov}(X, Y) = E(XY) - \{E(X)\}\{E(Y)\}$$

$$\text{From pairs of observations } (x_1, y_1), \dots, (x_n, y_n) \quad S_{xy} = \sum_k x_k y_k - \frac{1}{n} \left( \sum_i x_i \right) \left( \sum_j y_j \right)$$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n} \left( \sum_i x_i \right)^2, \quad S_{yy} = \sum_k y_k^2 - \frac{1}{n} \left( \sum_j y_j \right)^2$$

$$\text{Sample covariance} \quad s_{xy} = \frac{1}{n-1} S_{xy} \quad \text{estimates } \text{cov}(X, Y)$$

$$\text{Correlation coefficient} \quad \rho = \text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\text{sd}(X) \cdot \text{sd}(Y)}$$

$$\text{Sample correlation coefficient} \quad r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}} \quad \text{estimates } \rho$$

9. Sums of random variables

$$E(X + Y) = E(X) + E(Y)$$

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2 \text{cov}(X, Y)$$

$$\text{cov}(aX + bY, cX + dY) = (ac) \text{var}(X) + (bd) \text{var}(Y) + (ad + bc) \text{cov}(X, Y)$$

If  $X$  is  $N(\mu_1, \sigma_1^2)$ ,  $Y$  is  $N(\mu_2, \sigma_2^2)$ , and  $\text{cov}(X, Y) = c$ , then  $X + Y$  is  $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2c)$

10. Bias, standard error, mean square error

If  $t$  estimates  $\theta$  (with random variable  $T$  giving  $t$ )

$$\text{Bias of } t \quad \text{bias}(t) = E(T) - \theta$$

$$\text{Standard error of } t \quad \text{se}(t) = \text{sd}(T)$$

$$\text{Mean square error of } t \quad \text{MSE}(t) = E\{(T - \theta)^2\} = \{\text{se}(t)\}^2 + \{\text{bias}(t)\}^2$$

If  $\bar{x}$  estimates  $\mu$ , then  $\text{bias}(\bar{x}) = 0$ ,  $\text{se}(\bar{x}) = \sigma/\sqrt{n}$ ,  $\text{MSE}(\bar{x}) = \sigma^2/n$ ,  $\widehat{\text{se}}(\bar{x}) = s/\sqrt{n}$

Central limit property If  $n$  is fairly large,  $\bar{x}$  is from  $N(\mu, \sigma^2/n)$  approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter  $\theta$ .

For a random sample  $x_1, x_2, \dots, x_n$

$$\ell(\theta; x_1, x_2, \dots, x_n) = P(X_1 = x_1 | \theta) \cdots P(X_n = x_n | \theta) \quad (\text{discrete distribution})$$

$$\ell(\theta; x_1, x_2, \dots, x_n) = f(x_1 | \theta) f(x_2 | \theta) \cdots f(x_n | \theta) \quad (\text{continuous distribution})$$

The maximum likelihood estimator (MLE) is  $\hat{\theta}$  for which the likelihood is a maximum

12. Confidence intervals

If  $x_1, x_2, \dots, x_n$  are a random sample from  $N(\mu, \sigma^2)$  and  $\sigma^2$  is known, then

the 95% confidence interval for  $\mu$  is  $(\bar{x} - 1.96 \frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}})$

If  $\sigma^2$  is estimated, then from the Student t table for  $t_{n-1}$  we find  $t_0 = t_{n-1, 0.05}$

The 95% confidence interval for  $\mu$  is  $(\bar{x} - t_0 \frac{s}{\sqrt{n}}, \bar{x} + t_0 \frac{s}{\sqrt{n}})$

13. Standard normal table Values of pdf  $\phi(y) = f(y)$  and cdf  $\Phi(y) = F(y)$

$y$	$\phi(y)$	$\Phi(y)$	$y$	$\phi(y)$	$\Phi(y)$	$y$	$\phi(y)$	$\Phi(y)$	$y$	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table Values  $t_{m,p}$  of  $x$  for which  $P(|X| > x) = p$ , when  $X$  is  $t_m$

$m$	$p=$	0.10	0.05	0.02	0.01	$m$	$p=$	0.10	0.05	0.02	0.01
1		6.31	12.71	31.82	63.66	9		1.83	2.26	2.82	3.25
2		2.92	4.30	6.96	9.92	10		1.81	2.23	2.76	3.17
3		2.35	3.18	4.54	5.84	12		1.78	2.18	2.68	3.05
4		2.13	2.78	3.75	4.60	15		1.75	2.13	2.60	2.95
5		2.02	2.57	3.36	4.03	20		1.72	2.09	2.53	2.85
6		1.94	2.45	3.14	3.71	25		1.71	2.06	2.48	2.78
7		1.89	2.36	3.00	3.50	40		1.68	2.02	2.42	2.70
8		1.86	2.31	2.90	3.36	$\infty$		1.645	1.96	2.326	2.576

15. Chi-squared table Values  $\chi_{k,p}^2$  of  $x$  for which  $P(X > x) = p$ , when  $X$  is  $\chi_k^2$  and  $p = .995, .975, etc$

$k$	.995	.975	.05	.025	.01	.005	$k$	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	18.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies  $n_y$  are grouped so that the fitted frequency  $\hat{n}_y$  for every group exceeds about 5.

$$X^2 = \sum_y \frac{(n_y - \hat{n}_y)^2}{\hat{n}_y} \quad \text{is referred to the table of } \chi_k^2 \text{ with significance point } p,$$

where  $k$  is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching  $\bar{x}$  with  $\mu$

17. Joint probability distributions

Discrete distribution  $\{p_{xy}\}$ , where  $p_{xy} = P(\{X = x\} \cap \{Y = y\})$ .

Let  $p_{x\bullet} = P(X = x)$ , and  $p_{\bullet y} = P(Y = y)$ , then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x | Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

$$\text{Joint cdf} \quad F(x, y) = P(\{X \leq x\} \cap \{Y \leq y\}) = \int_{x_0=-\infty}^x \int_{y_0=-\infty}^y f(x_0, y_0) dx_0 dy_0$$

$$\text{Joint pdf} \quad f(x, y) = \frac{d^2 F(x, y)}{dx dy}$$

$$\text{Marginal pdf of } X \quad f_X(x) = \int_{-\infty}^{\infty} f(x, y_0) dy_0$$

$$\text{Conditional pdf of } X \text{ given } Y = y \quad f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} \quad (\text{provided } f_Y(y) > 0)$$

18. Linear regression

To fit the linear regression model  $y = \alpha + \beta x$  by  $\hat{y}_x = \hat{\alpha} + \hat{\beta}x$  from observations

$$(x_1, y_1), \dots, (x_n, y_n), \quad \text{the least squares fit is} \quad \hat{\alpha} = \bar{y} - \bar{x}\hat{\beta}, \quad \hat{\beta} = \frac{S_{xy}}{S_{xx}}$$

$$\text{The residual sum of squares} \quad \text{RSS} = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$$

$$\hat{\sigma}^2 = \frac{\text{RSS}}{n-2} \quad \frac{n-2}{\sigma^2} \hat{\sigma}^2 \text{ is from } \chi_{n-2}^2$$

$$E(\hat{\alpha}) = \alpha, \quad E(\hat{\beta}) = \beta,$$

$$\text{var}(\hat{\alpha}) = \frac{\sum x_i^2}{n S_{xx}} \sigma^2, \quad \text{var}(\hat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \text{cov}(\hat{\alpha}, \hat{\beta}) = -\frac{\bar{x}}{S_{xx}} \sigma^2$$

$$\hat{y}_x = \hat{\alpha} + \hat{\beta}x, \quad E(\hat{y}_x) = \alpha + \beta x, \quad \text{var}(\hat{y}_x) = \left\{ \frac{1}{n} + \frac{(x - \bar{x})^2}{S_{xx}} \right\} \sigma^2$$

$$\frac{\hat{\alpha} - \alpha}{\text{se}(\hat{\alpha})}, \quad \frac{\hat{\beta} - \beta}{\text{se}(\hat{\beta})}, \quad \frac{\hat{y}_x - \alpha - \beta x}{\text{se}(\hat{y}_x)} \quad \text{are each from } t_{n-2}$$