

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2015

MSc and EEE/EIE PART IV: MEng and ACGI

WAVELETS AND APPLICATIONS

Corrected Copy

Thursday, 30 April 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

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Second Marker(s) : A. Manikas

Special Information for the Invigilators: NONE

Information for Candidates:

Sub-sampling by an integer N

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j(\omega - 2\pi k)/N}) = \frac{1}{N} \sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

Dual Basis:

Given a basis $\{\varphi_i(t)\}_{i \in \mathbb{Z}}$, the dual basis is given by the set of elements $\{\tilde{\varphi}_i(t)\}_{i \in \mathbb{Z}}$ satisfying:

$$\langle \varphi_i(t), \tilde{\varphi}_j(t) \rangle = \delta_{i,j}.$$

The Questions

1. Multirate Signal Processing

- (a) Design a filter $H(z)$ that annihilates the sequence $x[n] = \cos(\omega_0 n)$, where ω_0 is an arbitrary constant. [7]
- (b) You are asked to approximate a sequence $x[n]$ with a sequence $y[n]$ having the following form:

$$y[n] = \sum_{k \in \mathbb{Z}} \beta[k] \phi[n - 2k]$$

where $\phi[n]$ is a length-4 sequence with z-transform given by

$$\Phi(z) = (1 + 3z^{-1} + 3z^{-2} - z^{-3}) / (2\sqrt{5}).$$

- i. Draw a block-diagram of a system having $\beta[n]$ as input and $y[n]$ as output. [6]
- ii. We compute $\beta[n] = (x[2n] + 3x[2n + 1] + 3x[2n + 2] - x[2n + 3]) / (2\sqrt{5})$. Draw a block diagram of a system having $x[n]$ as input and $y[n]$ as output. Explain what that system does. [6]
- iii. You wish to find a sequence $z[n]$ which, when added to $y[n]$ exactly recovers $x[n]$. That is, $y[n] + z[n] = x[n]$. The sequence $z[n]$ can be written as follows:

$$z[n] = \sum_{k \in \mathbb{Z}} \alpha[k] \psi[n - 2k].$$

Find the sequence $\psi[n]$ and give an expression for the coefficients $\alpha[k]$ in terms of $x[n]$. [6]

2. Consider the two-channel filter bank of Figure 2. The analysis high-pass branch has a delay since we want the down-sampler to retain the odd-indexed terms of the sequence.

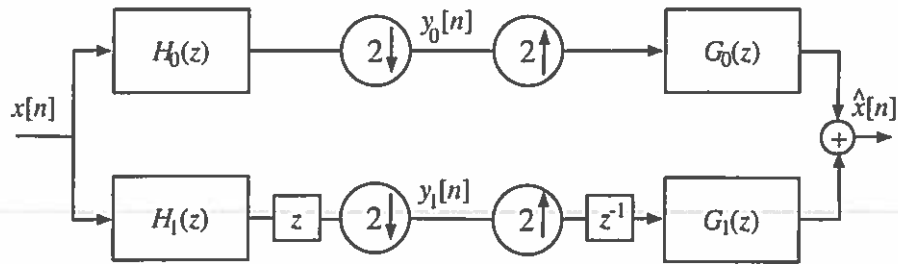


Figure 2: Two-channel filter bank with a delay.

- (a) Express $\hat{X}(z)$ as a function of $X(z)$ and the filters. Then derive the two perfect reconstruction (PR) conditions the filters have to satisfy. [7]
- (b) Derive the conditions the filters have to satisfy in order to have an orthogonal perfect reconstruction filter bank. [Hint: Use the condition $\langle g_0[n], g_1[n - 2k - 1] \rangle = 0$]. [6]
- (c) Using the conditions of part (b), design the shortest orthogonal filter bank with one vanishing moment. [6]
- (d) Assuming that the filters $H_0(z)$ and $G_0(z)$ of Figure 2 satisfy

$$H_0(z)G_0(z) = \frac{1}{16}(1+z)^2(1+z^{-1})^2(-z+4-z^{-1}),$$

design a biorthogonal perfect-reconstruction filter bank with symmetric $h_0[n]$ and $g_0[n]$. [6]

3. Consider the interval $t \in [0, 3]$ and let

$$\varphi_1(t) = \begin{cases} 1, & \text{for } t \in [0, 3/2) \\ 0, & \text{for } t \in [3/2, 3]. \end{cases}$$

Denote with $V = \text{span}(\{\varphi_1(t), \varphi_2(t), \varphi_3(t)\})$ the sub-space generated by $\varphi_1(t)$ and its circular shifts by 1 over the interval $t \in [0, 3]$. The three basis functions are shown in Fig. 3.

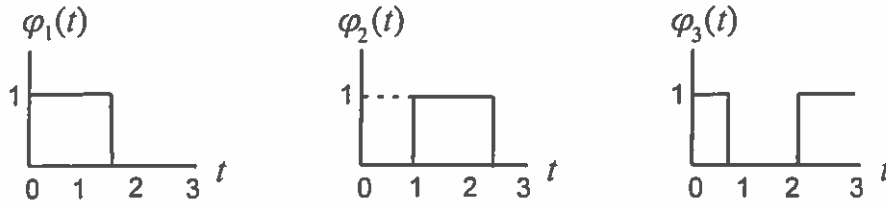


Figure 3: Three functions $\varphi_1, \varphi_2, \varphi_3$ defined for $t \in [0, 3]$ and related by circular shifts.

Given a signal $x(t)$ defined for $t \in [0, 3]$, the aim is to compute the orthogonal projection of $x(t)$ onto V . Recall that this is given by:

$$x_v(t) = \sum_{i=1}^3 \langle x(t), \tilde{\varphi}_i(t) \rangle \varphi_i(t)$$

where $\{\tilde{\varphi}_i(t)\}_{i=1}^3$ are the three dual-basis functions.

(a) Since $\tilde{\varphi}_i(t) \in V$ we can write $\tilde{\varphi}_i(t) = \sum_{k=1}^3 \alpha_{i,k} \varphi_k(t)$. Using this fact

i. Determine the three dual-basis functions $\tilde{\varphi}_i(t)$, $i = 1, 2, 3$. That is, find the coefficients $\alpha_{i,k}$, $i = 1, 2, 3$; $k = 1, 2, 3$. [5]

ii. Sketch and dimension $\tilde{\varphi}_i(t)$, $i = 1, 2, 3$. [5]

(b) Given the dual basis and the signal

$$x(t) = \begin{cases} 1, & \text{for } t \in [0, 1] \\ 0 & \text{for } t \in (1, 3]. \end{cases}$$

i. Compute the inner products $\langle x(t), \tilde{\varphi}_i(t) \rangle$, $i = 1, 2, 3$. [5]

ii. Sketch and dimension $x_v(t) = \sum_{i=1}^3 \langle x(t), \tilde{\varphi}_i(t) \rangle \varphi_i(t)$. [5]

iii. Verify that the error $e(t) = x(t) - x_v(t)$ is orthogonal to V . [5]

4. Consider a filter bank specified by the following signal equations:

$$\begin{aligned}y_0 &= D_2 G D_2 G x \\y_1 &= D_2 G D_2 H D_2 G x \\y_2 &= D_2 H D_2 H D_2 G x \\y_3 &= D_2 G D_2 H x \\y_4 &= D_2 G D_2 H D_2 H x \\y_5 &= D_2 H D_2 H D_2 H x,\end{aligned}$$

where G and H are the infinite matrix representations for filtering with a lowpass filter g_n and a highpass filter h_n , respectively, and D_2 is the matrix representation of down-sampling by 2.

(a) Draw a block diagram of the system using two-channel filter banks. [8]

(b) Draw the equivalent single-level six-channel filter bank clearly specifying the down-sampling factors and transfer functions of the filters in each branch. [7]

(c) Consider now a filter bank specified by the following signal equations:

$$\begin{aligned}y_0 &= D_2 G D_2 G x \\y_1 &= D_2 H D_2 G x \\y_2 &= D_2 H x.\end{aligned}$$

i. Draw the equivalent single-level three-channel filter bank clearly specifying the downsampling factors and transfer functions of the filters in each branch. [5]

ii. Assume now that

$$G(z) = (1 + z)(1 + z^{-1})/(2\sqrt{2})$$

and that

$$H(z) = \frac{\sqrt{2}}{8}(z^2 + 2z - 6 + 2z^{-1} + z^{-2}).$$

Moreover, assume that $x[n] = n$ and ignore any boundary effect. Which of the signals $y_0[n], y_1[n], y_2[n]$ is nonzero? (Justify your answer). [5]