# Math 2 Part B: Probability and Statistics

BRUNO CLERCKX LEVEL 8 (CSP)

15 hours lecture (pre-recorded)

7 hours classes (live)

3 hours of revision lectures (2 pre-recorded and 1 live)

4 Q&A sessions with GTAs

Exam 2 questions
closed book
formula sheet + calculator

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Probability uncertainty

Statistics data + uncertainty


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## Set Theory

element

set $\quad A = \{\underline{1}, \underline{2}, \underline{3}, \alpha, \beta\}$

collection of objects distincts and unordered

$1 \in A$ "1 belongs to A"

$4 \notin A$ "4 does not belong to A"

$A = \{1\}$ singleton

$A = \{\} = \emptyset$ empty set

$\Omega = \{1, 2, 3, 4, 5, 6\}$ universal set

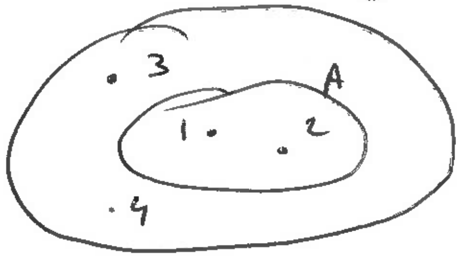
### relations

Sets $A$, $B$

- $A \subseteq B$ $\quad$ A is a subset of B

$A = \{1, 2\}$

$B = \{1, 2, 3, 4\}$

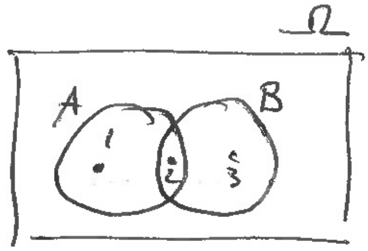



- if $A \subseteq B$ and $B \subseteq A$, then $A = B$
- $\emptyset \subseteq A$

## Union

Sets $A, B$

$$A \cup B = \{ \omega \in \Omega : \ \omega \in A \ \underline{\text{or}} \ \omega \in B \}$$




$$A \cup B = \{1, 2, 3\}$$

## Properties

$$A \cup \phi = A$$

$$A \cup A = A$$

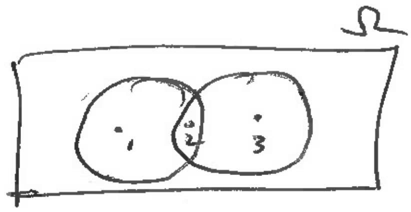
$$A \cup \Omega = \Omega$$

$$A \cup B = B \cup A$$

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^{n} A_i = \{ \omega \in \Omega : \text{for some } i, \ \omega \in A_i \}$$

## Intersection

Sets $A, B$




$$A \cap B = \{ \omega \in \Omega : \ \omega \in A \ \underline{\text{and}} \ \omega \in B \} \qquad A \cap B = \{2\}$$

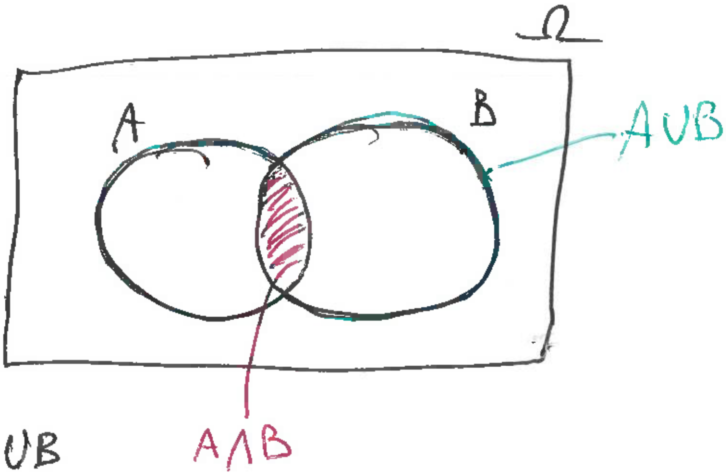
## Properties

$$A \cap \phi = \phi$$

$$A \cap A = A$$

$$A \cap \Omega = A$$

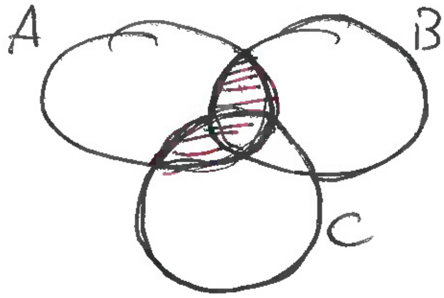
$$A \cap B = \cancel{A \cap B} \ B \cap A$$




$$A \cap B \subseteq A \subseteq A \cup B$$

Sets $A, B, C$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$




$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

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- $A_1, A_2, \dots, A_n$

$$A_i \cap A_j = \phi \qquad \text{for all } i, j,\ i \neq j$$

Those sets $A_1, \dots A_n$ are <u>disjoints</u>

<u>ex</u>

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$A_1 = \{1, 2\}$

$A_2 = \{3, 4\}$

$A_3 = \{4\} \longrightarrow A_3 = \{5\}$

Not disjoint $\qquad$ disjoint

- $A_1, A_2 \dots A_n$ disjoint and are such that

$$\bigcup_{i=1}^{n} A_i = \Omega$$

$A_1, A_2 \dots A_n$ form a partition of $\Omega$

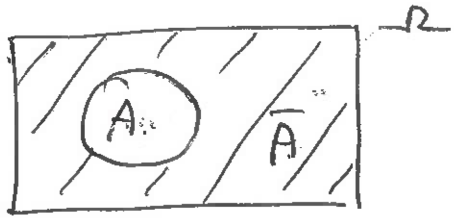
$\Omega = \{1, 2, 3, 4, 5, 6\}$

$A_1 = \{1, 2, 3\}$ $\qquad$ $A_1 = \{1, 2, 3\}$

$A_2 = \{4, 5, 6\}$ $\qquad$ $A_2 = \{3, 4, 5, 6\}$

## Complement of A

$\bar{A} = \{ \omega \in \Omega : \omega \notin A \}$




## Properties

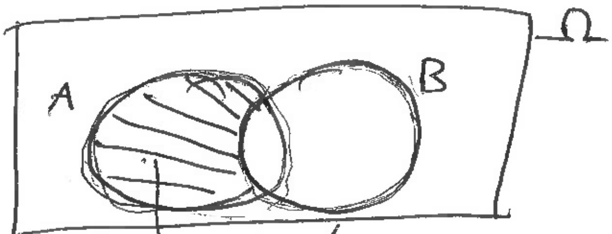
$\bar{\bar{A}} = A$

$\bar{\phi} = \Omega$

$$\left.\begin{array}{l} A \cup \bar{A} = \Omega \\ A \cap \bar{A} = \phi \end{array}\right\} \quad A \text{ and } \bar{A} \text{ form a partition of } \Omega$$

## Difference

Sets $A, B$



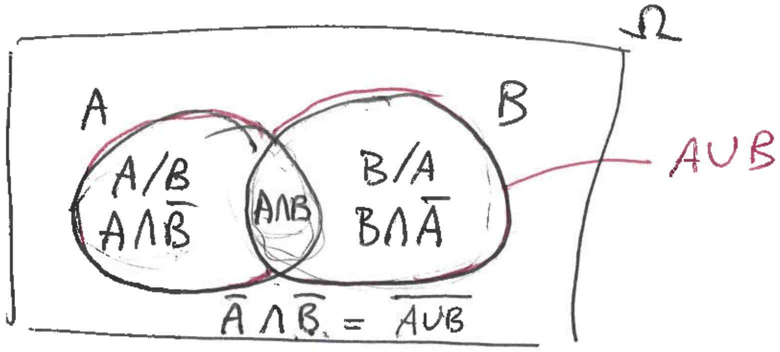

$A / B = \{ \omega \in \Omega : \omega \in A \ \underline{\text{and}} \ \omega \notin B \}$

$A/B$

A and B are disjoint $A \cap B = \phi$

$A / B = A$

$A / B = A \cap \bar{B}$




## identities

$$\begin{aligned} A &= A \cap \Omega \\ &= A \cap (B \cup \bar{B}) \\ &= \underline{(A \cap B)} \cup \underline{(A \cap \bar{B})} \end{aligned}$$

disjoint union

$$A \cup B = (A \cup B) \cap \Omega$$

$$= (A \cup B) \cap (B \cup \bar{B})$$

$$= (A \cap (B \cup \bar{B})) \cup (B \cap (B \cup \bar{B}))$$

$$= (A \cap B) \cup (A \cap \bar{B}) \cup \underbrace{(\underbrace{(B \cap B)}_{B} \cup \underbrace{(B \cap \bar{B})}_{\phi})}_{B}$$

$$= \underbrace{(A \cap B) \cup B}_{B} \cup (A \cap \bar{B}) = \underbrace{B \cup (A \cap \bar{B})}_{\text{disjoint union}}$$

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De Morgan's laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

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example

3 components $A, B, C$

$A$ means "A works"

$\bar{A}$ means "A fails"

- 3 components work $A \cap B \cap C$
- All 3 components fail $\bar{A} \cap \bar{B} \cap \bar{C} = \overline{A \cup B \cup C}$
- exactly one component works

$$(A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap \bar{C})$$

- At least two components work

$$(A \cap B) \cup (A \cap C) \cup (B \cap C)$$

$(A \cap B \cap C) \cup (A \cap B \cap \bar{C})$

exercise

a) $A \cap \bar{B} \cap \bar{C}$

b) $A \cap B \cap \bar{C}$

c) $A \cap B \cap C$

d) $A \cup B \cup C$

e) $(A \cap B) \cup (A \cap C) \cup (B \cap C)$

f) $(A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)$

g) $(A \cap B \cap \bar{C}) \cup (\bar{A} \cap B \cap C) \cup (A \cap \bar{B} \cap C)$

h) $\bar{A} \cap \bar{B} \cap \bar{C} = \overline{A \cup B \cup C}$

i) $\overline{A \cap B \cap C}$

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Sample space and events

random experiment

- sample space $S$ $\longleftrightarrow$ universal set $\Omega$

$S = \{1, 2, 3, 4, 5, 6\}$ discrete

$S = \{y : a < y < b\}$ continuous

- event is a subset of $S$

$E = \{1, 2\}$

- null event $\emptyset$ $\longleftrightarrow$ empty set $\emptyset$
is never occurs
- elementary event $\longleftrightarrow$ singleton
$E = \{1\}$
- universal event always occurs
$E = \{1, 2, 3, 4, 5, 6\}$

random experiment

outcome is $\omega^* \in S$

event $E \subseteq S$ occurs iff $\omega^* \in E$

throw a die

$S = \{1, 2, 3, 4, 5, 6\}$ $\quad \omega^* = \{2\}$

$E = \{1, 3\}$ does not occur

$E = \{1, 2\}$ occurs

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## Probability Axioms

$P(E) \qquad E \subseteq S$

1) $0 \leq P(E) \leq 1$
2) $P(S) = 1$
3) if $E \cap F = \phi$, then $P(E \cup F) = P(E) + P(F)$

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$E \cup \overline{E} = S$

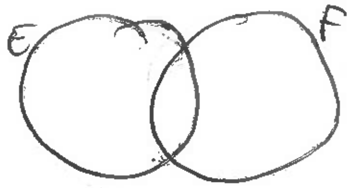
$$P(E \cup \overline{E}) \underset{\text{Ax3}}{=} P(E) + P(\overline{E}) \underset{\text{Ax2}}{=} 1$$

$P(\overline{E}) = 1 - P(E)$

$E = \cancel{\phi}\, S$

$P(\overline{S}) = P(\phi) = 1 - P(S) = 1 - 1 = 0$

$E \cup F$ $\qquad\qquad$ $P(E \cup F) = ?$




$E \cup F = E \cup (\bar{E} \wedge F)$

Ax 3 $\quad P(E \cup F) = P(E) + \underline{P(\bar{E} \wedge F)}$

$F = (F \wedge E) \cup (F \wedge \bar{E})$

Ax 3 $\quad P(F) = P(F \wedge E) + \underline{P(F \wedge \bar{E})}$

$$\boxed{P(E \cup F) = P(E) + P(F) - P(F \wedge E)}$$

if $F \wedge E = \phi \Rightarrow P(F \wedge E) = 0 \Rightarrow P(E \cup F) = P(E) + P($

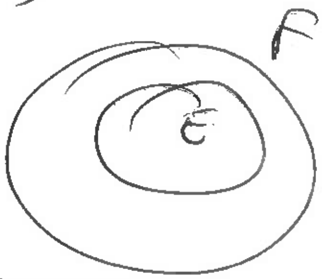
- Ax 1 $\quad P(E \wedge F) \geq 0$

$$P(E \cup F) = P(E) + P(F) - P(E \wedge F) \leq \underline{P(E) + P(F)}$$

- $\underline{E \subset F}$

$$F = (F \wedge E) \cup (F \wedge \bar{E}) = E \cup (\bar{E} \wedge F)$$

$$P(F) = P(E) + \underbrace{P(\bar{E} \wedge F)}_{\geq 0} \geq P(E)$$



$$\max(P(E),P(F)) \le P(E\cup F) \le P(E)+P(F)$$

$$\left.\begin{array}{l} E \subseteq E\cup F \Rightarrow P(E) \le P(E\cup F) \\ F \subseteq E\cup F \Rightarrow P(F) \le P(E\cup F) \end{array}\right\} \max(P(E),P(F)) \le P(E\cup F)$$

$$P(E)+P(F)-1 \le P(E\cap F) \le \min(P(E),P(F))$$

$$P(E\cap F) = P(E)+P(F) - \underbrace{P(E\cup F)}_{\substack{\le 1 \\ Ax\,1}} \ge P(E)+P(F)-1$$

$$\left.\begin{array}{l} E\cap F \subseteq E \Rightarrow P(E\cap F) \le P(E) \\ E\cap F \subseteq F \Rightarrow P(E\cap F) \le P(F) \end{array}\right\} P(E\cap F) \le \min(P(E),P(F))$$

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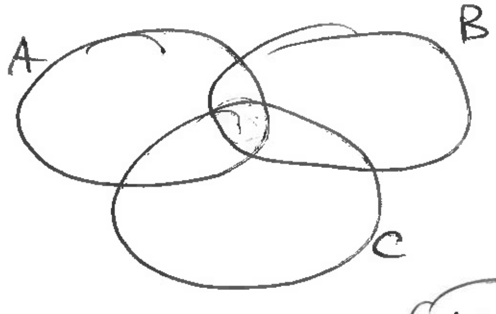
$$P(A\cup B\cup C) = P(A\cup B) + P(C) - P((A\cup B)\cap C)$$

$$= P(A)+P(B)-P(A\cap B)+P(C) - P((A\cap C)\cup(B\cap C))$$

$$= P(A)+P(B)+P(C) - P(A\cap B) - \left[P(A\cap C)+P(B\cap C) - P(A\cap B\cap C)\right]$$

$$= P(A)+P(B)+P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$$



$n$ equally likely elementary events

$E \subseteq S$

$$P(E) = \frac{\#\text{ elementary events in } E}{n}$$

$S = \{1, 2, 3, 4, 5, 6\}$

$E = \{1, 2\}$

$P(E) = \frac{2}{6} = \frac{1}{3}$

A drawing a heart B drawing a face

$P(A) = \frac{13}{52}$ $P(B) = \frac{12}{52}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52}$$

$P(A \cap B) = \frac{3}{52}$