Probability and Stochastic Processes
Problem Sheet 1

1. Consider an experiment which includes flipping a fair coin. Suppose that this experiment succeeds if the outcome is "heads" and fails otherwise.

   (a) Let Y be a random variable such that $Y = 1$ if the experiment succeeds and $Y = 0$ otherwise. What is the expected value of Y?

   (b) What is the probability of having j successes (j heads) knowing that you have flipped the coin for n times?

   (c) Suppose that we flip a coin until the first heads comes up. What is the probability distribution of the number of flips?

2. A traditional fair die is tossed twice, what is the probability that

   (a) A six turns up exactly once?

   (b) Both numbers are odd?

   (c) The sum of the scores is 4?

   (d) The sum of the scores is divisible by 3?

3. A fair coin is thrown repeatedly. What is the probability that on the nth throw:

   (a) A head happens for the first time?

   (b) The number of heads and tails are equal?

   (c) Exactly two heads have appeared?

   (d) At least two heads have appeared?

4. An urn contains 10 identical balls numbered 0, 1, ..., 9. A random experiment involves selecting a ball from the urn and noting the number of the ball. Find the probability of the following events:

   $A =$ "number of ball selected is odd,"
   $B =$ "number of ball selected is a multiple of 3,"
   $C =$ "number of ball selected is less than 5,"
   $A \cup B \cup C$.

   Hint: Use the following formula.

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{j=1}^{n} P[A_j] - \sum_{j<k} P[A_j \cap A_k] + \cdots + (-1)^{n+1} P[A_1 \cap \cdots \cap A_n]$$

5. A communication system transmits binary information over a channel that introduces random bit errors with probability $P_e = 10^{-3}$. The transmitter transmits each information bit three times, and a decoder takes a majority vote of the received bits to decide on what the transmitted bit was. Find the probability that the receiver will make an incorrect decision.

6. Consider an experiment where we pick two numbers x and y uniformly at random between zero and one. Suppose that all pairs of numbers are equally likely to be selected. Find the probability of the following:

   $A = \{x > 0.5\}$
   $B = \{y > 0.5\}$
   $C = \{x > y\}$
   $D = \{x = 0.5\}$

7. (*) At the station there are three payphones which accept 20p pieces. One never works, another always works, while the third works with probability 1/2. On my way to the metropolis for the day, I wish to identify the reliable phone, so that I can use it on my return. The station is empty and I have just three 20p pieces. I try one phone and it does not work. I try another twice in succession and it works both times. What is the probability that this second phone is the reliable one?

Note: Questions with the (*) might be a little tricky. Think about them but try not to waste too much time on them.

Probability and Stochastic Processes
Problem Sheet 2

1. A rare disease affects one person in 10^5. A test for a disease shows positive with probability $\frac{99}{100}$ when applied to an ill person, and with probability $\frac{1}{100}$ when applied to a healthy person. What is the probability that you have the disease given that the test shows positive?

2. There are two roads from A to B and two roads from B to C. Each of the four roads has probability p of being blocked by snow, independently of all others. What is the probability that there is an open road from A to C? Now suppose that there is also a direct road from A to C, which is independently blocked with probability p. What is the probability that there is an open road from A to C in this case?

3. Calculate the probability that a hand of 13 cards dealt from a normal shuffled pack of 52 contains exactly two kings and one ace. What is the probability that it contains exactly one ace given that it contains exactly two kings?

4. Two fair coins are rolled. Show that the event that their sum is 7 is independent of the score shown by the first die.

5. A pack contains m cards, labelled 1, 2, ..., m. The cards are dealt out in a random order, one by one. Given that the label of the kth card dealt is the largest of the first k cards dealt, what is the probability that it is also the largest in the pack?

6. (*) Suppose that n absent-minded professors attend the opera one night. They each leave their hat at the cloakroom at the beginning of the night, but by the end of the night they have all lost their cloakroom ticket.
   They are the last n people to return to the cloakroom at the end of the opera, and none of them can recognize his own hat; so they decide to each pick a hat at random. What is the probability that none of the professors leaves the opera wearing the same hat he came in with?

   Hints: First try to use DeMorgan's laws and the basic laws of probability in order to be able to use inclusion-exclusion principle:
   $$P(A_1^c \cap \cdots \cap A_n^c) = 1 - P((A_1^c \cap \cdots \cap A_n^c)^c) = 1 - P(A_1 \cup \cdots \cup A_n).$$
   The following is also useful:
   $$e^{-1} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}$$

7. (*) ('Gambler's ruin') A man is saving up to buy a Jaguar at a cost of N units of money. He starts with k units where $0 < k < N$, and tries to win the remainder by the following gamble with his bank manager. He tosses a fair coin repeatedly; if it comes up heads then the manager pays him one unit, but if it comes up tails then he pays the manager one unit. He plays this game repeatedly until one of two events occurs: either he runs out of money and is bankrupted or he wins enough to buy the Jaguar. What is the probability that he is ultimately bankrupted?

Probability and Stochastic Processes
Problem Sheet 3

Functions of Random Variables

1. Let X be a random variable with normal distribution $N(0, 1)$. Find the p d f (probability density function) of $Y = X^2$.

   Hint: Try to find the (cumulative) distribution function of Y first.

2. Consider the joint density function $f_{X_1,X_2}(x_1, x_2)$. Let Y_1, Y_2 be other two random variables which are the functions of X_1, X_2:

$$(Y_1, Y_2) = T(X_1, X_2)$$

   Correspondingly,

$$(X_1, X_2) = T^{-1}(Y_1, Y_2)$$

   where T^{-1} is the inverse function for T. Then the following is true,

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} f\big(x_1(y_1, y_2), x_2(y_1, y_2)\big)|J(y_1, y_2)|, & \text{if } y_1, y_2 \in \text{Range}(T) \\ 0, & \text{otherwise,} \end{cases}$$

   where Range(T) is the set of all output values produced by function T and $|J(y_1, y_2)|$ is

$$\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}.$$

   Now, consider X_1, X_2 as two independent random variables with distribution Exp(λ) and Y_1 and Y_2 as random variables which are the function of X_1, X_2 as follows,

$$Y_1 = X_1 + X_2$$
$$Y_2 = \frac{X_1}{X_2}$$

Correspondingly,

$$(x_1, x_2) \to (\frac{y_1 y_2}{1 + y_2}, \frac{y_1}{1 + y_2})$$

Compute the joint distribution of Y_1, Y_2 and then marginal distributions.

3. Assume that random variables U and V are chosen independently and uniformly out of the set $\{1,2,3,4,5\}$. From this, we derive random variables $X = \min(U,V)$ and $Y = \max(U,V)$.

(a) Determine the joint law of (U, Y)

(b) Determine $E[U|Y = n]$, for $n \in \{1, 2, 3, 4, 5\}$.

(c) Derive $E[U|Y]$.

(d) Derive $E[Y \mid U]$.

(e) Derive also $E[U|X]$ and $E[X|U]$.

Probability and Stochastic Processes
Problem Sheet 4

Sequences of Random Variables

1. Try to prove *Markov's Inequality* which is as following:
   Let X be a random variable that assumes only non-negative values. Then, for all $a > 0$,
$$P(X \geq a) \leq \frac{E[X]}{a}$$
   Hint: Use the indicator function I:
$$I = \begin{cases} 1, & \text{if } X \geq a, \\ 0, & \text{otherwise,} \end{cases}$$

2. The moment generating function (Laplace transform) of a random variable X is defined as follows
$$\phi_X(s) = \mathbb{E}(e^{sX}) = \sum_{k \geq 0} \frac{s^k}{k!} \mathbb{E}(X^k).$$
   (a) Prove
$$\phi'_X(0) = \mathbb{E}[X]$$
   (b) Prove
$$\frac{d^k \phi_X}{dx^k}(0) = \mathbb{E}[\mathrm{X}^k]$$
   (c) Compute the moment generating function of *Binomial and Exponential* distributions.

3. Using Markov's Inequality prove
   (a) *Chebyshev's Inequality* which is as follows:
   For any $a > 0$,
$$P(|X - E[X]| \geq a) \leq \frac{\mathrm{Var}[X]}{a^2}$$
   Hint: Use the following
$$P(|X - E[X]| \geq a) = P((X - E[X])^2 \geq a^2)$$
   (b) *Chernoff Bound* which is the following:
$$P(X \geq a) \leq \min_{t>0} \frac{E[e^{tX}]}{e^{ta}}$$

4. Consider tossing a coin for n times. The variable X_i describes the outcome of the i-th toss: $X_i = 1$ if heads shows and $X_i = 0$ if tail shows. Let $X = \sum_{i=1}^{n} X_i$.

   (a) State the distribution of X, and then compute its expectation and its variance.

(b) i. Show that

$$P\left(X \geq \frac{3n}{4}\right) \leq P\left(\left|X - \frac{n}{2}\right| \geq \frac{n}{4}\right).$$

ii. Using Chebyshev's inequality, prove that

$$P(X \geq \frac{3n}{4}) \leq \frac{4}{n}$$

iii. Find $\lim_{n\to\infty} P(X \geq 3n/4)$

(c) We now derive a tighter bound for convergence of $P(X \geq 3n/_4)$ as n goes to ∞.

i. Let $x, \theta \geq 0$. Combining Markov's inequality and the fact that

$$\{X \geq x\} = \{e^{\theta X} \geq e^{\theta x}\}$$

Prove that

$$P(X \geq x) \leq \exp(\frac{n}{2}\left(e^{\theta} - 1\right) - \theta x)$$

Hint: Use the inequality $1 + \alpha \leq e^{\alpha}$, for $\alpha \geq 0$.

ii. Choose θ so that $\frac{1}{2}\left(e^{\theta} - 1\right) - \frac{3}{4}\theta \leq -0.01$.

iii. Prove that for the choice of θ in the previous question

$$P\left(X \geq \frac{3n}{4}\right) \leq e^{-0.01n}.$$

5. There are 17 fenceposts around the perimeter of a field, exactly 5 of which are rotten. Show that irrespective of which these 5 are, there necessarily exists a run of 7 consecutive posts at least 3 of which are rotten. Hint: Try to find the expected value of rotten ones in a run of 7 consecutive posts.

6. Alice and Bob agree to meet in the Copper Kettle after their Friday lectures. They arrive at times that are independent and uniformly distributed between 12:00 and 13:00. Each is prepared to wait s minutes before leaving. Find a minimal s such that the probability that they meet is at least 1/2.

7. (*) *Bertrand's Paradox.* A chord has been chosen at random in a circle of radius r. What is the probability that it is longer than the side of the equilateral triangle inscribed in the circle?
Hint: The answer is different in different approaches! Consider following three cases:

(a) the middle point of the cord is distributed uniformly inside the circle;

(b) one endpoint is fixed and the second is uniformly distributed over the circumference;

(c) The distance between the middle point of the chord and the centre of the circle is uniformly distributed over the interval $[0, r]$.

Probability and Stochastic Processes
Problem Sheet 5

Estimation

1. The random variable X has the density $f(x) \sim c^4 x^3 e^{-cx}, \ \ x > 0$. We observe the i.i.d. samples x_i = 6.1, 5.7, 6.3, 5.7, 6.2. Find the maximum-likelihood estimate of parameter c.

2. The random variable X has the truncated exponential density
$$f(x) = ce^{-c(x-x_0)}, \ \ x > x_0,$$
$$f(x) = 0 \text{ otherwise.}$$
Let x_0 = 5. We observe the i.i.d. samples x_i = 7, 8, 9, 10, 11. Find the maximum-likelihood estimate of parameter c.

3. Consider the auto-regressive process
$$Y(n) = \alpha Y(n-1) + Z(n)$$
where is α a real number satisfying $|\alpha| < 1$, and $Z(n)$ is an i.i.d. sequence with zero mean and unit variance.

   (a) Show that the autocorrelation function of $Y(n)$ is given by
$$R_Y(n) = \frac{\alpha^{|n|}}{1-\alpha^2}$$

   (b) Suppose we wish to predict $Y(n+1)$ from $Y(n)$, $Y(n-1)$, …, $Y(1)$. The coefficients of the linear MMSE estimator
$$Y(n+1) = \sum_{i=1}^{n} c_i Y(i)$$
are given by the Wiener-Hopf equation
$$\boldsymbol{Rc} = \boldsymbol{r}$$
where $\boldsymbol{c} = [c_1, c_2, \dots, c_n]^T$, $\boldsymbol{r} = [R_Y(n), R_Y(n-1), \dots, R_Y(1)]^T$, and $\boldsymbol{R}$ is a n-by-n matrix whose (i, j)th entry is $R_Y(i-j)$. Find the best coefficients and the associated mean-square error.

Probability and Stochastic Processes
Problem Sheet 6

Random Processes

1. Consider the random process

$$X(t) = A_t \cos(\omega t + \theta)$$

where t is continuous time and A_t are i.i.d. random variables with

$$E[A_t] = 0, Var[A_t] = \sigma^2.$$

   (a) Let θ be a constant. Calculate the mean, variance of $X(t)$ and determine whether it is stationary or not.

   (b) Now let θ be uniformly distributed on [-π, π], and also independent of A_t. Calculate the mean, autocorrelation function of $X(t)$ and determine whether it is wide-sense stationary or not.

2. Consider the random process $X(n) = \cos(nU)$ for $n \geq 1$, where U is uniformly distributed on interval [-π, π].

   (a) Show that $\{X(n)\}$ is wide-sense stationary.

   (b) Show that $\{X(n)\}$ is not strict-sense stationary.

3. The random process $X(t)$ has autocorrelation $R(\tau)$.

   (a) If $X(t)$ is real-valued, show that

$$P\{|X(t+\tau) - X(t)| \geq a\} \leq 2[R(0) - R(\tau)]/a^2.$$

   (b) From the fact that $R(\tau)$ is the inverse Fourier transform of the power spectral density $S(\omega)$, show that $R(\tau)$

$$\sum_{i,k} a_i a_k^* R(\tau_i - \tau_k) \geq 0$$

   for all a_i.

   (c) If $X(t)$ is a normal (i.e., Gaussian) process with zero mean and $Y(t) = Ie^{aX(t)}$, show that

$$E[Y(t)] = I\exp\left\{\frac{a^2}{2}R(0)\right\}$$

$$R_Y(\tau) = I^2\exp\{a^2[R(0) + R(\tau)]\}$$

   Hint: Use the characteristic function of two jointly Gaussian random variables $N(0,0,\sigma_1^2,\sigma_2^2,\rho)$, which is given by

$$\Phi(\omega_1,\omega_2) = \exp\left\{-\frac{\sigma_1^2\omega_1^2 + 2\rho\sigma_1\ \sigma_2\ \omega_1\ \omega_2\ + \sigma_2^2\omega_2^2}{2}\right\}$$

Probability and Stochastic Processes
Problem Sheet 7

Markov Chains

1. Let X be a Markov chain. Which of the following are Markov chains?

   (a) $(X_{m+r}, r \geq 0)$.

   (b) $(X_{2m},\ m \geq 0)$.

   (c) The sequences of pairs $((X_n, X_{n+1}), n \geq 0)$.

2. A die is rolled repeatedly. Which of the following are Markov chains? For those that are, supply the transition matrix.

   (a) The largest number X_n shown up to the nth roll.

   (b) The number N_n of sixes in n rolls.

   (c) At time r, the time C_r since the most recent six.

   (d) At time r, the time B_r until the next six.

3. Let X be a Markov chain and let $n_r : r \geq 0$ be an unbounded increasing sequence of positive integers. Show that $Y_r = X_{n_r}$ constitutes a (possibly inhomogeneous) Markov chain. Find the transition matrix of Y when $n_r = 2r$ and X is simple random walk.

4. *Virus mutation.* Suppose that a virus can exist in N different strains and each generation either stays the same, or with probability α mutates to another strain, chosen at random. What is the probability that the strain in the nth generation is the same as the initial strain?

5. *Markov's other chain.* Let $Y_1, Y_3, Y_5, \ldots$ a sequence of iid r.v. such that, for $k = 0, 1, \ldots$

$$P(Y_{2k+1} = -1) = P(Y_{2k+1} = 1) = 1/2\,.$$

   Let $Y_{2k} = Y_{2k-1}Y_{2k+1}$, for $k = 1, 2 \ldots$. It is not difficult to see that $Y_2, Y_4, \ldots$ is a sequence of iid r.v. with the same distribution as $Y_1, Y_3, Y_5, \ldots$.

   Is $Y_1, Y_2, Y_3, \ldots$ a Markov chain?

Probability and Stochastic Processes
Problem Sheet 8

Markov Chains

1. Three girls A, B and C are playing table tennis. In each game, two of the girls play against each other and the third girl does not play. The winner of any given game n plays again in game $n+1$. The probability that girl x will beat girl y in any game that they play against each other is $s_x/(s_x+s_y)$ for $x, y \in \mathrm{A, B, C}, x \neq y$, where s_A, s_B, s_C represent the playing strengths of the three girls.

   (a) Represent this process as a discrete-time Markov chain by defining the possible states and constructing the transition matrix.

   (b) Determine the probability that two girls who play each other in the first game will play each other again in the fourth game. Show that this probability does not depend in which two girls play in the first game.

2. Consider the Markov chain with the following transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   We define the probability of absorption (in state 4) by

$$h_i = P(X_n \text{ is absorded in state } 4 | X_0 = i)$$

   and the expected time to absorption by

$$k_i = E(\text{time for } X_n \text{ to be absorded in state 1 or } 4 | X_0 = i)$$

   where $i = 1, 2, 3, 4$.

   (a) First compute the probabilities of absorption h_i:

      i. Show that

$$h_2 = \frac{1}{2}h_1 + \frac{1}{2}h_3 \text{ and } h_3 = \frac{1}{2}h_2 + \frac{1}{2}h_4$$

      ii. After deriving the values of h_1 and h_4, show that

$$h_2 = \frac{1}{3} \text{ and } h_3 = \frac{2}{3}$$

   (b) Now compute the expected times to absorption k_i:

      i. Show that

$$k_2 = 1 + \frac{1}{2}k + \frac{1}{2}k_3 \text{ and } k_3 = 1 + \frac{1}{2}k_2 + \frac{1}{2}k$$

      ii. After deriving the values of k_1 and k_4, show that

$$k_2 = 2 \text{ and } k_3 = 2$$

3. A rock concert held in a hall with N numbered seats attracted a huge crowd of spectators. The lights have been dimmed and $N-1$ seats have already been taken, and now the last spectator enters the hall. The first $N-1$ spectators were advised by ushers, rather imprudently, to take their seats completely at random, but the last spectator is determined to sit in the place indicated on her ticket. If her place is free, she takes it, and the concert is ready to begin. However, if her seat is taken, she loudly insists that the occupier vacates it. In this case the occupier decides to follow the same rule: if the free seat is his, he takes it, otherwise he insists on his place being vacated. The same policy is then adopted by the next unfortunate spectator, and so on. Each move takes 45 seconds. What is the expected duration of the delay caused by these displacements?