Revision 1

29-04-2014

Question 1

subpart 1 (taken from Question 3 Exam 2009)

Consider the discrete random variables X and Y with joint probability mass function $p(x,y) = \mathrm{P}(X = x, Y = y)$ given by the table below.

$p(x,y)$		$y=0$	$y=1$
x	0	0.4k	0.2k
	1	0.2k	0.4k
	2	0.6k	0.2k

(i) Show that $k = 1/2$.

(ii) Find the marginal distribution of X and the marginal distribution of Y.

(iii) Find $\mathrm{E}(X)$ and $\mathrm{E}(Y)$.

(iv) Find $\mathrm{Var}(X)$.

(v) Find $\mathrm{cov}(X, Y)$.

(vi) Are X and Y uncorrelated? Give your reasoning.

(vii) Are X and Y independent? Give your reasoning.

subpart 2 (taken from Question 4 Exam 2009)

(i) Let X be *Exponential* (1).

Find the cumulative distribution function (cdf) of the random variable X^2.

(ii) Let X_1, X_2, X_3, X_4 be independent *Exponential* (λ) distributed random variables.

Find the cdf of $\min(X_1, X_2, X_3, X_4)$.

Question 2

subpart 1 (taken from Question 4 Exam 2010)

The daily electrical consumption in a town has a normal distribution with mean 500 megawatts and standard deviation 150 megawatts. The power station, which provides power for this town alone, has an output of $Y = 650$ megawatts with probability 0.7 and an output of $Y = 800$ megawatts with probability 0.3, on any particular day.

(i) If the output is 650 megawatts on a particular day, calculate the probability that the demand cannot be met from the power station.

(ii) Show that the probability that the demand cannot be met on a particular day is 0.118.

(iii) If the demand cannot be met on a particular day, there is a power cut (but a maximum of only one power cut per day). Assuming that outputs and demands on different days are independent, what is the distribution of the number of power cuts in any seven day period?

(iv) What is the probability that there are no more than 2 power cuts caused by unsatisfied demand in a seven day period?

subpart 2 (taken from Question 3 Exam 2010)

For events A and B, where $P(A) > 0$ and $P(B) > 0$, show that

(a) $P(A|B) = P(B|A)$ if $P(A) = P(B)$.

(b) $P(A) = P(A \cap B) + P(A \cap B')$.

(c) Using the result in (b), and assuming that A and B are independent, show that A and B' are also independent.

Solution 1

Question 1

subpart 1

(i) Since $1 = \sum_{x,y} p(x,y) = 2k$, we have $k = 1/2$.

(ii) $\mathrm{P}(X=0) = 0.4k + 0.2k = 0.3$
$\mathrm{P}(X=1) = 0.2k + 0.4k = 0.3$
$\mathrm{P}(X=2) = 0.6k + 0.2k = 0.4$

$\mathrm{P}(Y=0) = 0.4k + 0.2k + 0.6k = 0.6$
$\mathrm{P}(Y=1) = 0.2k + 0.4k + 0.2k = 0.4$

(iii) $\mathrm{E}(X) = 0 \cdot 0.3 + 1 \cdot 0.3 + 2 \cdot 0.4 = 1.1$
$\mathrm{E}(Y) = 0 \cdot 0.6 + 1 \cdot 0.4 = 0.4$

(iv) $\mathrm{E}(X^2) = 0^2 \cdot 0.3 + 1^2 \cdot 0.3 + 2^2 \cdot 0.4 = 1.9$
Hence, $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = 1.9 - 1.1^2 = 1.9 - 1.21 = 0.69$

(v) $\mathrm{E}(XY) = 0 + 1 \cdot 0.4k + 2 \cdot 0.2k = 0.8k = 0.4$
$\mathrm{cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y) = 0.4 - 1.1 \cdot 0.4 = -0.04$

(vi) No, they are not uncorrelated since $\mathrm{cov}(X,Y) \neq 0$.

(vii) No, they are not independent since they are not uncorrelated.

subpart 2

(i) For $x \geq 0$:

$$F_{X^2}(x) = \mathrm{P}(X^2 \leq x) = \mathrm{P}(X \leq \sqrt{x}) = 1 - e^{-\sqrt{x}}$$

$F_{X^2}(x) = 0$ for $x < 0$.

(ii) For $x \geq 0$:

$$F_{\min_i X_i}(x) = \mathrm{P}(\min_i X_i \leq x) = 1 - \mathrm{P}(X_i > x, i = 1, \ldots, 4)$$
$$= 1 - \prod_{i=1}^{4} e^{-\lambda x} = 1 - e^{-4\lambda x}$$

For $x < 0$: $F_{\min_i X_i}(x) = 0$.

Question 2

subpart 1

i. Let $D = \{\text{demand cannot be met}\}$, and let X = daily consumption, and Y=output.

$$P(D|Y = 650) = P(X > 650)$$

where $X \sim N(500, (150)^2)$. So,

$$\begin{aligned} P(D|Y = 650) &= P\left(\frac{X - 500}{150} > \frac{650 - 500}{150}\right) \\ &= P\left(Z > \frac{150}{150}\right) \qquad \text{where } Z \sim N(0, 1) \\ &= 1 - P\left(Z \leq \frac{150}{150}\right) \\ &= 1 - \Phi(1.0) \approx 1 - 0.841 = \underline{0.159} \end{aligned}$$

ii. Using total probability

$$P(D) = P(D|Y = 650)P(Y = 650) + P(D|Y = 800)P(Y = 800)$$

So we also require $P(D|Y = 800) = P(X > 800)$

$$\begin{aligned} P(D|Y = 800) &= P\left(\frac{X - 500}{150} > \frac{800 - 500}{150}\right) \\ &= P\left(Z > \frac{300}{150}\right) \\ &= 1 - P(Z \leq 2) \\ &= 1 - \Phi(2.0) \approx 1 - 0.977 = \underline{0.023} \end{aligned}$$

Hence the probability that the demand cannot be met on a particular day is

$$P(D) = (0.159 \times 0.7) + (0.023 \times 0.3) = \underline{0.1182}.$$

iii. In any week, we have a sequence of Bernoulli trials that are independent and have a common probability of success. Since we are interested in the number of successes in a week, a Binomial distribution is the appropriate model.

Thus, let C =number of power cuts in a week. Then $C \sim \text{Bin}(n, \theta)$, where $n = 7$ and $\theta = P(D) = 0.1182$.

iv. Require $P(C \leq 2)$:

$$\begin{aligned}
P(C \leq 2) &= P(C=0) + P(C=1) + P(C=2) \\
&= \sum_{j=0}^{2} \binom{7}{j} \theta^j (1-\theta)^{7-j} \\
&= \binom{7}{0}(0.1182)^0(0.8818)^7 + \binom{7}{1}(0.1182)^1(0.8818)^6 + \binom{7}{2}(0.1182)^2(0.8818)^5 \\
&= (0.8818)^7 + 7(0.1182)(0.8818)^6 + 21(0.1182)^2(0.8818)^5 \approx \underline{0.960}.
\end{aligned}$$

subpart 2

a. From the definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B)} \frac{P(A)}{P(A)} = P(B|A)\frac{P(A)}{P(B)}.$$

so $P(A|B) = P(B|A)$ if and only if $P(A) = P(B)$.

b.

$$\begin{aligned}
A &= A \cap \Omega \\
&= A \cap (B \cup B') \\
&= (A \cap B) \cup (A \cap B') \quad \text{by the distributive rule}
\end{aligned}$$

This is a disjoint union, so Axiom III says

$$P(A) = P(A \cap B) + P(A \cap B')$$

c. From the previous result

$$\begin{aligned}
P(A \cap B') &= P(A) - P(A \cap B) \\
&= P(A) - P(A)P(B) \\
&= P(A)(1 - P(B)) \\
&= P(A)P(B')
\end{aligned}$$