8: IIR Filter Transformations

Continuous Time Filters

Classical continuous-time filters optimize tradeoff:
passband ripple v stopband ripple v transition width
There are explicit formulae for pole/zero positions.

Butterworth: $\widetilde{G}^2(\Omega) = \left|\widetilde{H}(j\Omega)\right|^2 = \frac{1}{1+\Omega^{2N}}$

- Monotonic $\forall\Omega$
- $\widetilde{G}(\Omega) = 1 - \frac{1}{2}\Omega^{2N} + \frac{3}{8}\Omega^{4N} + \cdots$
  "Maximally flat": $2N-1$ derivatives are zero

Chebyshev: $\widetilde{G}^2(\Omega) = \frac{1}{1+\epsilon^2 T_N^2(\Omega)}$

where polynomial $T_N(\cos x) = \cos Nx$

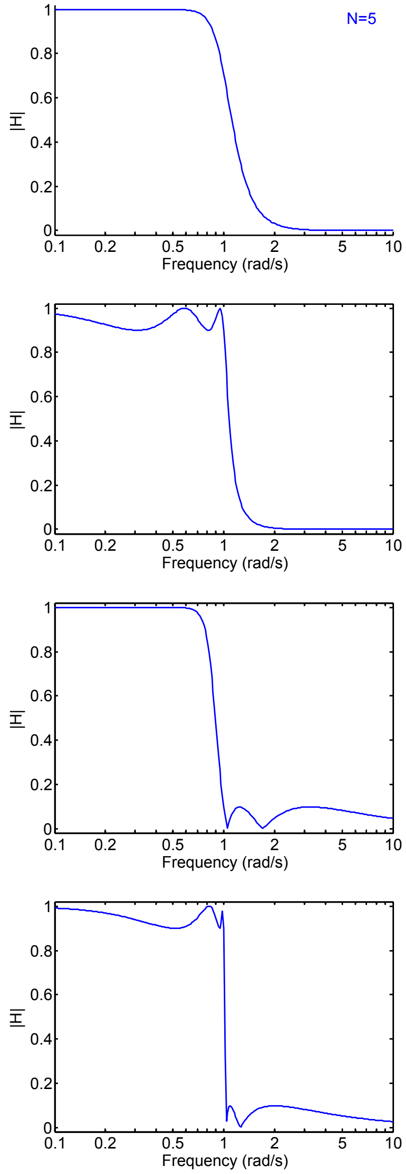
- passband equiripple + very flat at ∞

Inverse Chebyshev: $\widetilde{G}^2(\Omega) = \frac{1}{1+\left(\epsilon^2 T_N^2(\Omega^{-1})\right)^{-1}}$

- stopband equiripple + very flat at 0

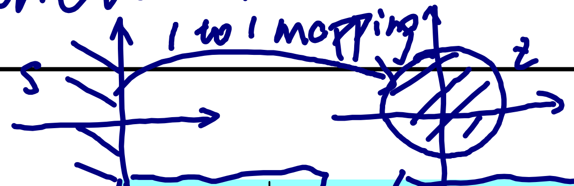
Elliptic: [no nice formula]

- Very steep + equiripple in pass and stop bands



Bilinear Mapping

continuous → discrete



$z = \frac{\alpha + s}{\alpha - s}$

$s = \alpha \frac{z-1}{z+1}$

$\frac{\omega}{2} = \tan^{-1}\frac{\Omega}{\alpha}$

$\omega = 2\tan^{-1}\frac{\Omega}{\alpha}$

Change variable: $\boxed{z = \frac{\alpha+s}{\alpha-s}} \Leftrightarrow \boxed{s = \alpha\frac{z-1}{z+1}}$ a one-to-one invertible mapping

- $\Re$ axis $(s) \leftrightarrow \Re$ axis (z)
- $\Im$ axis $(s) \leftrightarrow$ Unit circle (z)

  $\alpha \cdot \tan\frac{\omega}{2} = \Omega$

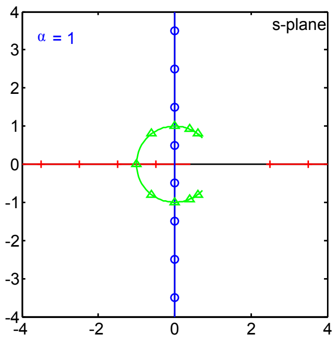
  Proof: $z = e^{j\omega} \Leftrightarrow s = \alpha\frac{e^{j\omega}-1}{e^{j\omega}+1} = \alpha\frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}} = j\alpha\tan\frac{\omega}{2} = j\Omega$
- Left half plane$(s) \leftrightarrow$inside of unit circle (z)

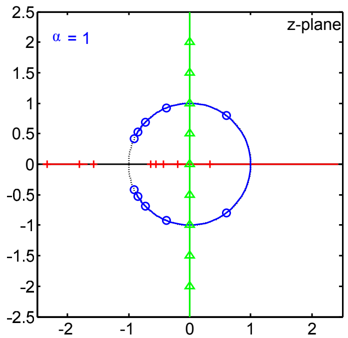
  Proof: $s = x + jy \Leftrightarrow |z|^2 = \frac{|(\alpha+x)+jy|^2}{|(\alpha-x)-jy|^2}$

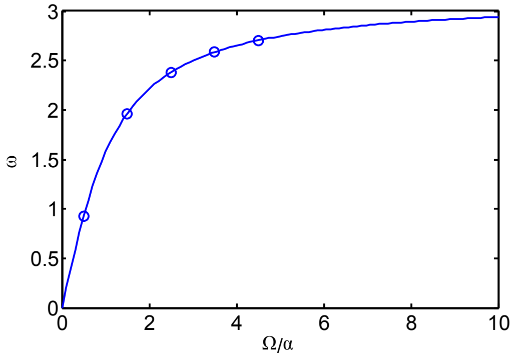
  $$= \frac{\alpha^2+2\alpha x+x^2+y^2}{\alpha^2-2\alpha x+x^2+y^2} = 1 + \frac{4\alpha x}{(\alpha-x)^2+y^2}$$

  $$x < 0 \Leftrightarrow |z| < 1$$

  $S = e^{j\Omega}$ $\quad z = \frac{\alpha + e^{j\Omega}}{\alpha - e^{j\Omega}} = \frac{\alpha^2 + 2e^{j\Omega} + e^{j2\Omega}}{\alpha^2 - 1}$
- Unit circle $(s) \leftrightarrow \Im$ axis (z)







S → z : direct substitution (extra zeros at z=-1)

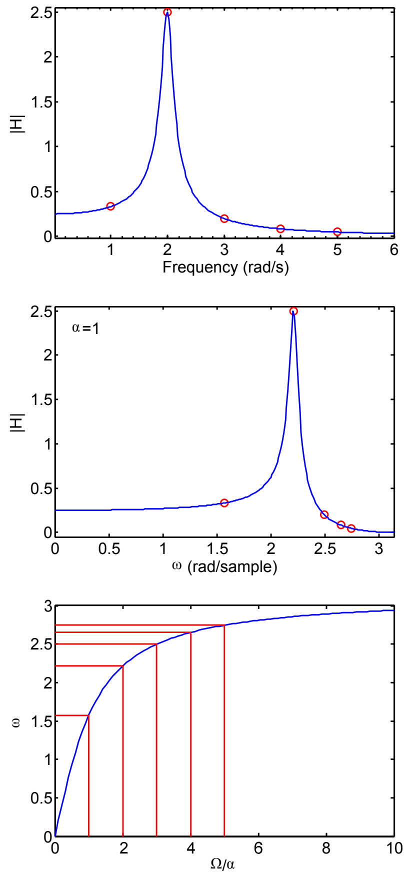
Take $\widetilde{H}(s) = \frac{1}{s^2+0.2s+4}$ and choose $\alpha = 1$

Substitute: $s = \alpha\frac{z-1}{z+1}$ [extra zeros at $z = -1$]

$$H(z) = \frac{1}{\left(\frac{z-1}{z+1}\right)^2+0.2\frac{z-1}{z+1}+4}$$

$$= \frac{(z+1)^2}{(z-1)^2+0.2(z-1)(z+1)+4(z+1)^2}$$

$$= \frac{z^2+2z+1}{5.2z^2+6z+4.8} = 0.19\frac{1+2z^{-1}+z^{-2}}{1+1.15z^{-1}+0.92z^{-2}}$$



Frequency response is identical (both magnitude and phase) but with a distorted frequency axis:

Frequency mapping: $\boxed{\omega = 2\tan^{-1}\frac{\Omega}{\alpha}}$

$$\Omega = \begin{bmatrix} \alpha & 2\alpha & 3\alpha & 4\alpha & 5\alpha \end{bmatrix}$$ evenly spaced (x)

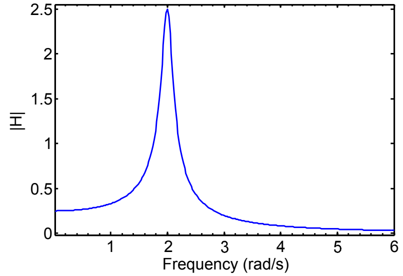
$$\to \omega = \begin{bmatrix} 1.6 & 2.2 & 2.5 & 2.65 & 2.75 \end{bmatrix}$$ unevenly spaced (y)

Choosing α: Set $\alpha = \frac{\Omega_0}{\tan\frac{1}{2}\omega_0}$ to map $\Omega_0 \to \omega_0$

Set $\alpha = 2f_s = \frac{2}{T}$ to map low frequencies to themselves

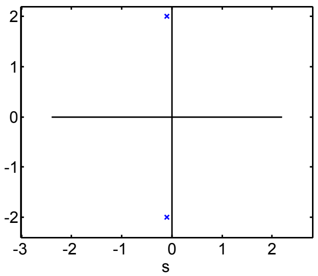
$S \to z$: roots mapping (need to add extra poles or zeros at $z=-1$)

Alternative method: $\widetilde{H}(s) = \frac{1}{s^2+0.2s+4}$



Find the poles and zeros: $p_s = -0.1 \pm 2j$

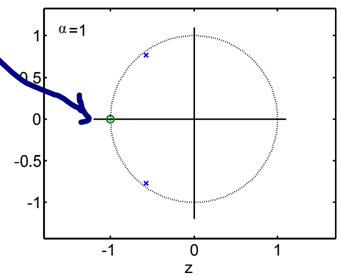
Map using $z = \frac{\alpha+s}{\alpha-s} \Rightarrow p_z = -0.58 \pm 0.77j$



After the transformation we will always end up with the same number of poles as zeros:

▷ Add extra poles or zeros at $z=-1$

$$H(z) = g \times \frac{(1+z^{-1})^2}{(1+(0.58-0.77j)z^{-1})(1+(0.58+0.77j)z^{-1})}$$
$$= g \times \frac{1+2z^{-1}+z^{-2}}{1+1.15z^{-1}+0.92z^{-2}}$$

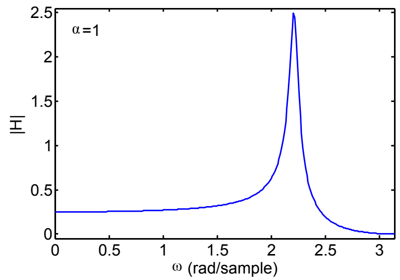


Choose overall scale factor, g, to give the same gain at any convenient pair of mapped frequencies:

At $\Omega_0 = 0 \Rightarrow s_0 = 0 \Rightarrow \left|\widetilde{H}(s_0)\right| = 0.25$

$$\Rightarrow \omega_0 = 2\tan^{-1}\frac{\Omega_0}{\alpha} = 0 \Rightarrow z_0 = e^{j\omega_0} = 1$$
$$\Rightarrow |H(z_0)| = g \times \frac{4}{3.08} = 0.25 \Rightarrow g = 0.19$$

$$H(z) = 0.19\frac{1+2z^{-1}+z^{-2}}{1+1.15z^{-1}+0.92z^{-2}}$$



Spectral Transformations

$\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$ (2)

$e^{j\hat\omega} = \dfrac{e^{j\omega}+\lambda}{1+\lambda e^{j\omega}} = \dfrac{(e^{j\omega}+\lambda)(1+\lambda e^{-j\omega})}{(1+\lambda e^{j\omega})(1+\lambda e^{-j\omega})} = \dfrac{e^{j\omega}+\lambda^2 e^{-j\omega}+2\lambda}{1+\lambda^2+2\lambda\cos\omega}$

$s \to z \to \hat z$

$\underbrace{\quad}$

$\hat s \to \hat z$

cutoff freq. $\div 3$

We can transform the z-plane to change the cutoff frequency by substituting

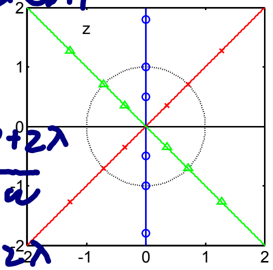
$$z = \frac{\hat z-\lambda}{1-\lambda\hat z} \Leftrightarrow \hat z = \frac{z+\lambda}{1+\lambda z}$$

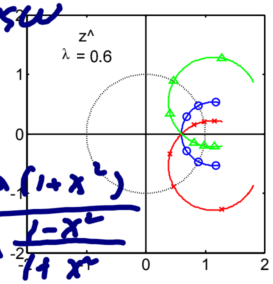
mapping within discrete

z inside unit circle $\Updownarrow$ $\hat z$ inside unit circle

|num| = |den|

$\Rightarrow \cos\hat\omega = \dfrac{\cos\omega+\lambda^2\cos\omega+2\lambda}{1+\lambda^2+2\lambda\cos\omega} = \dfrac{(1+\lambda^2)\cos\omega+2\lambda}{(1+\lambda^2)+2\lambda\cos\omega}$





Frequency Mapping:

If $z = e^{j\omega}$, then $\hat z = z\frac{1+\lambda z^{-1}}{1+\lambda z}$ has modulus 1 since the numerator and denominator are complex conjugates.

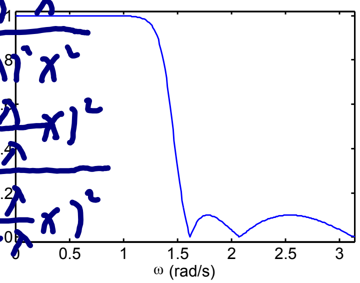
Hence the unit circle is preserved.

$$\Rightarrow e^{j\hat\omega} = \frac{e^{j\omega}+\lambda}{1+\lambda e^{j\omega}}$$

$\tan\frac{\omega}{2} = x$, $\tan\frac{\hat\omega}{2} = \hat x$ (2) $\dfrac{1-\hat x^2}{1+\hat x^2} = \dfrac{(1+\lambda^2)\frac{1-x^2}{1+x^2}+2\lambda}{(1+\lambda^2)+2\lambda\frac{1-x^2}{1+x^2}} = \dfrac{(1+\lambda^2)(1-x^2)+2\lambda(1+x^2)}{(1+x^2)(1+\lambda^2)+2\lambda(1-x^2)} = \dfrac{(1+\lambda^2)-(1-\lambda)^2x^2}{(1+\lambda^2)+(1-\lambda)^2x^2}$

Some algebra gives: $\tan\frac{\omega}{2} = \left(\frac{1+\lambda}{1-\lambda}\right)\tan\frac{\hat\omega}{2}$

$\dfrac{1-\hat x^2}{1+\hat x^2} = \dfrac{1-\left(\frac{1-\lambda}{1+\lambda}x\right)^2}{1+\left(\frac{1-\lambda}{1+\lambda}x\right)^2}$



Equivalent to:

$$z \longrightarrow s = \frac{z-1}{z+1} \longrightarrow \hat s = \frac{1-\lambda}{1+\lambda}s \longrightarrow \hat z = \frac{1+\hat s}{1-\hat s}$$

Lowpass Filter example:

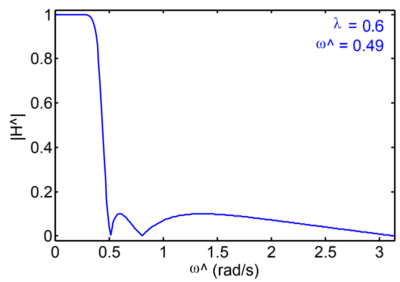
$\omega_0 = \frac{\pi}{2}$ $z = e^{j\omega_0} = e^{j\frac{\pi}{2}} = j$

Inverse Chebyshev

$s = \dfrac{z-1}{z+1} = \dfrac{j-1}{j+1}$

$$\omega_0 = \tfrac{\pi}{2} = 1.57 \overset{\lambda=0.6}{\longrightarrow} \hat\omega_0 = 0.49$$

$\hat s = \dfrac{1-\lambda}{1+\lambda}s = \dfrac{0.4}{1.6}\,\dfrac{j-1}{j+1}$



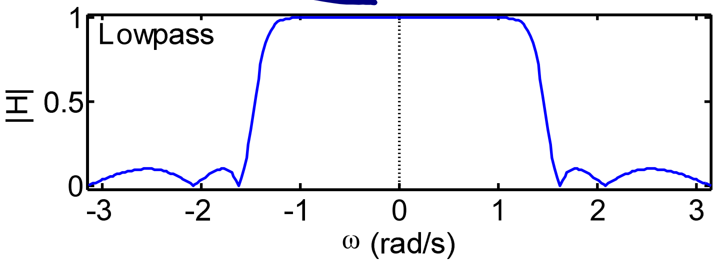
$\hat z = \dfrac{1+\hat s}{1-\hat s} = \dfrac{1+0.25\frac{j-1}{j+1}}{1-0.25\frac{j-1}{j+1}} = \dfrac{(j+1)+0.25(j-1)}{(j+1)-0.25(j-1)}$

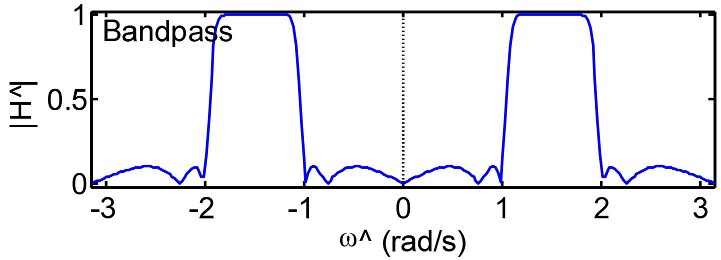
Constantinides Transformations

Transform any lowpass filter with cutoff frequency ω_0 to:

Target	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$ $\rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$ $\rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Bandpass and bandstop transformations are quadratic and so will double the order:





Impulse Invariance

Bilinear transform works well for a lowpass filter but the non-linear compression of the frequency distorts any other response.

Alternative method: $\widetilde{H}(s) \xrightarrow{\mathscr{L}^{-1}} h(t) \xrightarrow{\text{sample}} h[n] = T \times h(nT) \xrightarrow{\mathscr{Z}} H(z)$

Express $\widetilde{H}(s)$ as a sum of partial fractions $\widetilde{H}(s) = \sum_{i=1}^{N} \frac{g_i}{s - \widetilde{p}_i}$

Impulse response is $\tilde{h}(t) = u(t) \times \sum_{i=1}^{N} g_i e^{\tilde{p}_i t}$

Digital filter $\frac{H(z)}{T} = \sum_{i=1}^{N} \frac{g_i}{1 - e^{\tilde{p}_i T} z^{-1}}$ has identical impulse response

Poles of $H(z)$ are $p_i = e^{\tilde{p}_i T}$ (where $T = \frac{1}{f_s}$ is sampling period)

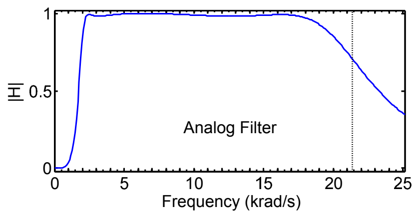
Zeros do not map in a simple way

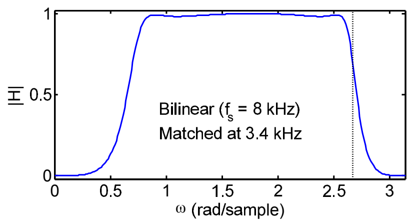
Properties:

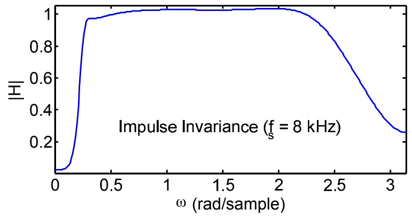
☺ Impulse response correct. ☺ No distortion of frequency axis.

☹ Frequency response is aliased.

Example: Standard telephone filter - 300 to 3400 Hz bandpass







Summary

- Classical filters have optimal tradeoffs in continuous time domain
  - Order $\leftrightarrow$ transition width$\leftrightarrow$ pass ripple$\leftrightarrow$ stop ripple
  - Monotonic passband and/or stopband
- Bilinear mapping
  - Exact preservation of frequency response (mag + phase)
  - non-linear frequency axis distortion
  - can choose α to map $\Omega_0 \rightarrow \omega_0$ for one specific frequency
- Spectral transformations
  - lowpass $\rightarrow$ lowpass, highpass, bandpass or bandstop
  - bandpass and bandstop double the filter order
- Impulse Invariance
  - Aliassing distortion of frequency response
  - preserves frequency axis and impulse response

For further details see Mitra: 9.

MATLAB routines

bilinear	Bilinear mapping
impinvar	Impulse invariance
butter butterord	Analog or digital Butterworth filter
cheby1 cheby1ord	Analog or digital Chebyshev filter
cheby2 cheby2ord	Analog or digital Inverse Chebyshev filter
ellip ellipord	Analog or digital Elliptic filter