

# **DSP Design of FIR Filters using the Remez Exchange Algorithm**

## **Lecture 7**

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## Computer-Aided Design of Linear-Phase FIR Filters

- In this section, we consider the application of computer-aided optimization techniques for the design of FIR filters.
- The basic idea behind the computer-based technique is to minimize iteratively an error measure that is function of the difference between the desired frequency response  $D(e^{j\omega})$  and the frequency response  $H(e^{j\omega})$  of the filter being designed.
- In the case of linear-phase FIR filter design,  $H(e^{j\omega})$  and  $D(e^{j\omega})$  are zero-phase frequency responses.
- For IIR filter design, these functions are replaced with their magnitude functions.

# Computer-Aided Design of Linear-Phase FIR Filters

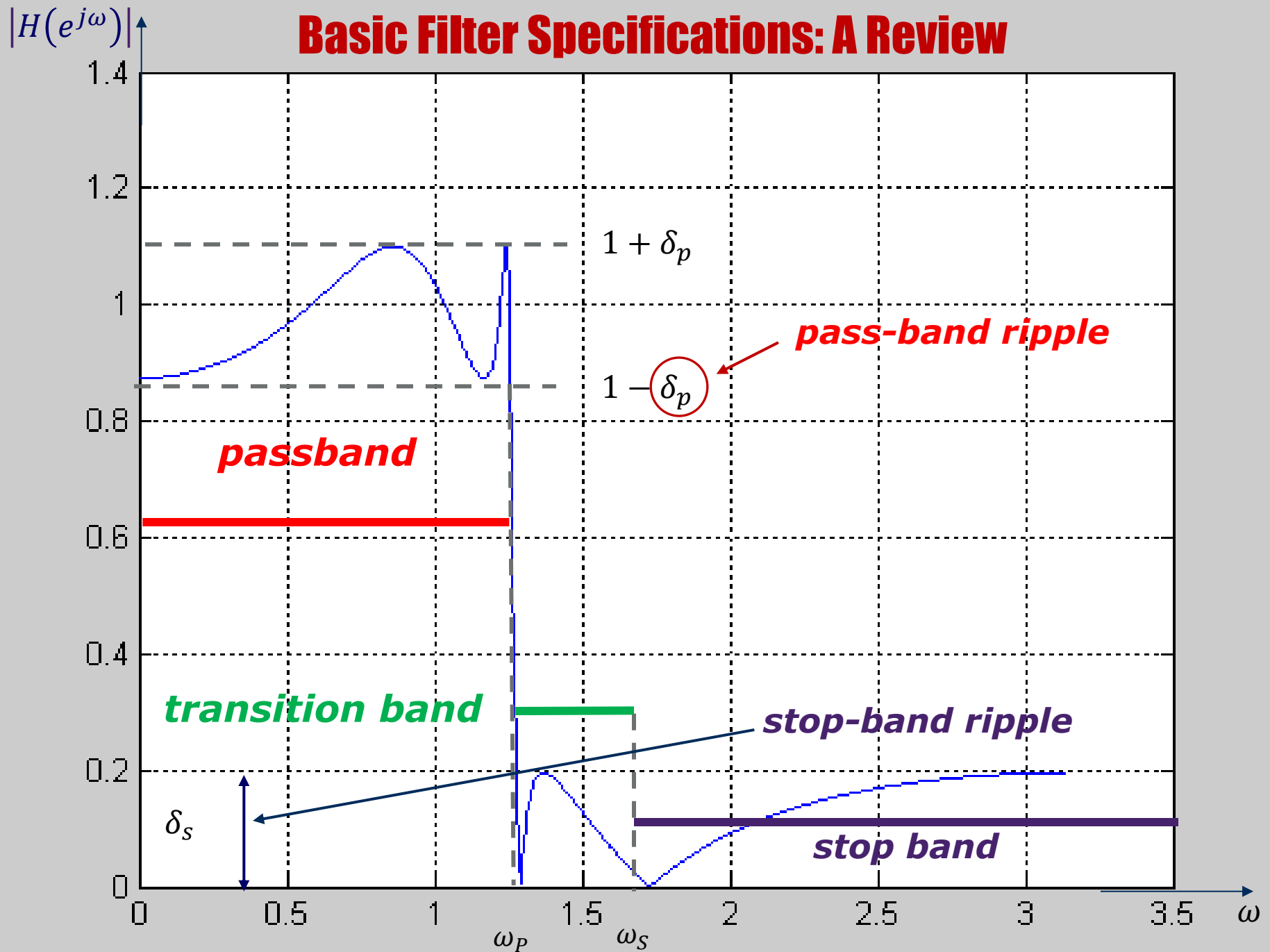
## Previous part

- The windowing method and the frequency-sampling method are relatively simple techniques for designing linear-phase FIR filters.
- Here, a major problem, is a lack of precise control of the critical frequencies such cut-off frequencies of pass band and stop band.

## This part

- The new filter design method described in this section is formulated as a so called **Chebyshev approximation problem**.
- It is viewed as an optimum design criterion in the sense that the maximum weighted approximation error between the desired frequency response and the actual frequency response is minimized.
- **The resulting filter designs have ripples in both the pass-band and the stop-band.**
- To describe the design procedure, let us recall the following basic filter specifications.

# Basic Filter Specifications: A Review



## Computer-Aided Design of Linear-Phase FIR Filters

- The design objective is to iteratively adjust the filter parameters so that the error function defined by the equation:

$$\varepsilon(\omega) = W(e^{j\omega})[H(e^{j\omega}) - D(e^{j\omega})]$$

is minimum according to some criterion.

$W(e^{j\omega})$  is some user-specified positive weighting function.

- The following criteria are popular:

### **Minimax criterion:**

minimize  $\max_{\omega \in R} |W(e^{j\omega})[H(e^{j\omega}) - D(e^{j\omega})]|$

### **Least squares criterion:**

Minimize  $\int_{\omega \in R} |W(e^{j\omega})[H(e^{j\omega}) - D(e^{j\omega})]|^p d\omega$

- $R$  is the set of disjoint frequency bands in the range  $0 \leq \omega \leq \pi$ . In filtering applications,  $R$  is composed of passbands and stopbands.

# Computer-Aided Design of Equiripple Linear-Phase FIR Filters

- The linear phase filter that is obtained by minimizing the peak absolute value of the weighted error  $\varepsilon$  given by

$$\varepsilon = \max_{\omega \in R} |\varepsilon(\omega)|$$

is usually called the ***equiripple FIR filter***, since, after  $\varepsilon$  has been minimized, the weighted error function  $\varepsilon(\omega)$  exhibits an equiripple behavior in the frequency range of interest.

- In this part we outline the ***weighted-Chebyshev approximation method*** advanced by Parks and McClellan for designing equiripple linear phase FIR filters.
- This method is more commonly known as the ***Parks-McClellan algorithm***.

## Computer-Aided Design of Equiripple Linear-Phase FIR Filters

- The general form of the frequency response  $H(e^{j\omega})$  of a causal linear-phase FIR filter of length  $N + 1$  is given by

$$H(e^{j\omega}) = e^{-jN\omega/2} e^{j\beta} \check{H}(\omega)$$

where  $\check{H}(\omega)$  is the amplitude response of  $H(e^{j\omega})$  and is a real function of  $\omega$ .

- The weighted error function in this case involves the amplitude response and is given by

$$\varepsilon(\omega) = W(\omega) [\check{H}(\omega) - D(\omega)]$$

*A positive weighting  
function*

*The desired  
amplitude response*

- The Parks-McClellan algorithm is based on iteratively adjusting the coefficients of the amplitude response until the peak absolute value of  $\varepsilon(\omega)$  is minimized.

# Computer-Aided Design of Equiripple Linear-Phase FIR Filters

- If the minimum value of the peak absolute value of  $\varepsilon(\omega)$  in a band  $\omega_a \leq \omega \leq \omega_b$  is  $\varepsilon_0$ , then the absolute error satisfies

$$|\check{H}(\omega) - D(\omega)| \leq \frac{\varepsilon_0}{|W(\omega)|}, \omega_a \leq \omega \leq \omega_b$$

- In typical filter design applications, the desired amplitude response is given by

$$D(\omega) = \begin{cases} 1, & \text{in the passband} \\ 0, & \text{in the stopband} \end{cases}$$

- The amplitude response  $\check{H}(\omega)$  is required to satisfy the above desired response with a ripple of  $\pm\delta_p$  in the passband and a ripple  $\delta_s$  in the stopband.
- As a result, it is evident from the weighted error function that the weighting function can be chosen either as

$$W(\omega) = \begin{cases} 1, & \text{in the passband} \\ \delta_p/\delta_s, & \text{in the stopband} \end{cases} \quad \text{or} \quad W(\omega) = \begin{cases} \delta_s/\delta_p, & \text{in the passband} \\ 1, & \text{in the stopband} \end{cases}$$



## Linear-Phase FIR Transfer Functions

- It is nearly impossible to design a linear-phase IIR transfer function.
- It is always possible to design an FIR transfer function with an exact linear-phase response.
- Consider a causal FIR transfer function  $H(z)$  of length  $N + 1$ , i.e., of order  $N$  as follows:

$$H(z) = \sum_{n=0}^N h[n]z^{-n}$$

- The above transfer function has a linear phase, if its impulse response  $h[n]$  is either **symmetric**, i.e.,

$$h[n] = h[N - n], 0 \leq n \leq N$$

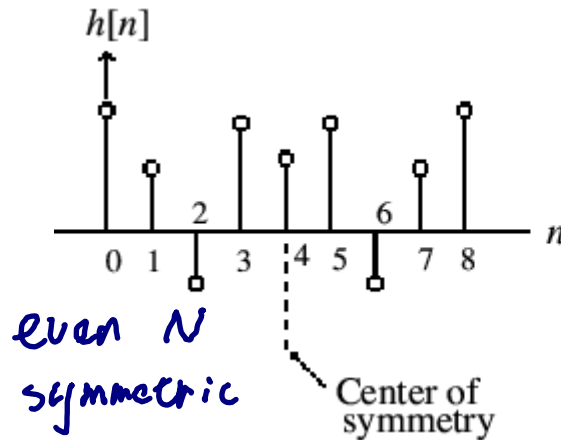
or is **antisymmetric**, i.e.,

$$h[n] = -h[N - n], 0 \leq n \leq N$$

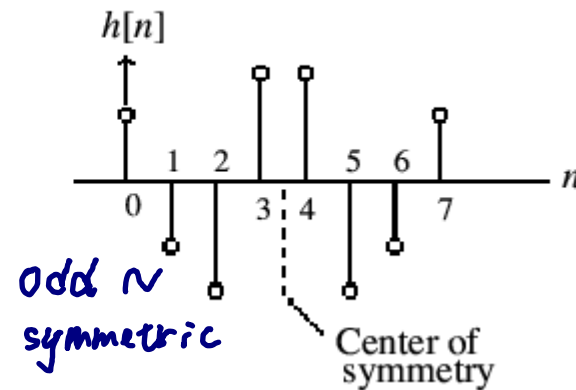
- Since the length of the impulse response can be either even or odd, we can define four types of linear-phase FIR transfer functions.
- For an antisymmetric FIR filter of odd length, i.e.,  $N$  even

$$h[N/2] = 0$$

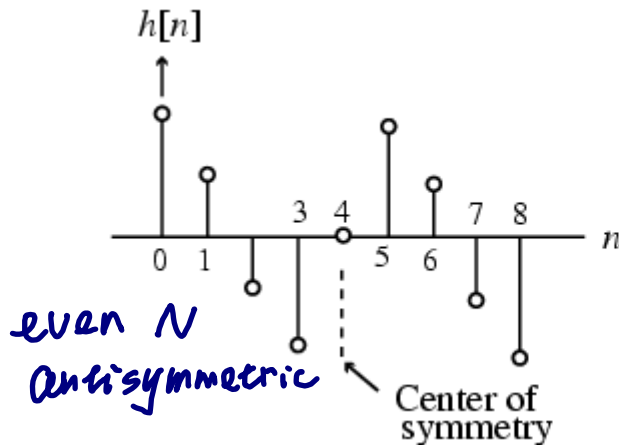
## 4 Types of Linear-Phase FIR Transfer Functions



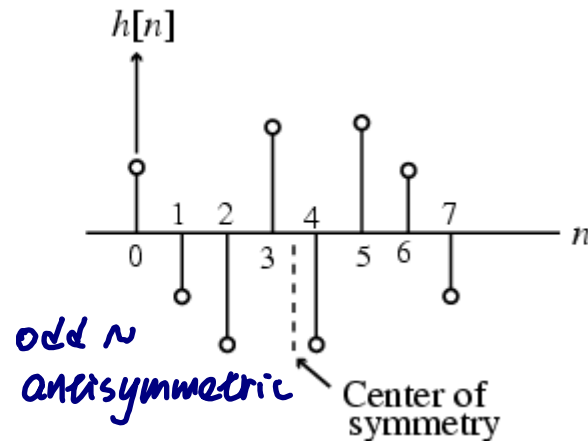
Type 1:  $N = 8$



Type 2:  $N = 7$



Type 3:  $N = 8$



Type 4:  $N = 7$

## 4 Types of Linear-Phase FIR Transfer Functions

### Amplitude Response of Type 1

- By a clever manipulation, the expression for the amplitude response for each of the four types of linear-phase FIR filters can be expressed in the same form.
- The same algorithm can be adapted to design any one of the four types of filters.
- To develop this general form for the amplitude response expression, we consider each of the four types of filters separately.
- For the **Type 1 linear-phase FIR filter**, the amplitude response can be rewritten using the notation  $N = 2M$  in the form

$$\tilde{H}(\omega) = \sum_{k=0}^M a[k] \cos(\omega k)$$

$$a[0] = h[M], \quad a[k] = 2h[M - k], \quad 1 \leq k \leq M$$

## 4 Types of Linear-Phase FIR Transfer Functions

### Amplitude Response of Type 2

- For the **Type 2 linear-phase FIR filter**, the amplitude response can be rewritten using the notation  $N = 2M$  in the form

$$\check{H}(\omega) = \sum_{k=1}^{(2M+1)/2} b[k] \cos\left(\omega\left(k - \frac{1}{2}\right)\right)$$

$$b[k] = 2h\left[\frac{2M+1}{2} - k\right], \quad 1 \leq k \leq \frac{2M+1}{2}$$

- The above can also be expressed in the form:

$$\check{H}(\omega) = \cos\left(\frac{\omega}{2}\right) \sum_{k=1}^{(2M-1)/2} \tilde{b}[k] \cos(\omega k)$$

where

$$b[1] = \frac{1}{2}(\tilde{b}[1] + 2\tilde{b}[0])$$

$$b[k] = \frac{1}{2}(\tilde{b}[k] + \tilde{b}[k-1]), \quad 2 \leq k \leq \frac{2M-1}{2}$$

$$b\left[\frac{2M+1}{2}\right] = \frac{1}{2}\tilde{b}\left[\frac{2M-1}{2}\right]$$

## 4 Types of Linear-Phase FIR Transfer Functions

### Amplitude Response of Type 3

- For the **Type 3 linear-phase FIR filter**, the amplitude response can be rewritten using the notation  $N = 2M$  in the form

$$\check{H}(\omega) = \sum_{k=1}^M c[k] \sin(\omega k)$$

$$c[k] = 2h[M - k], \quad 1 \leq k \leq M$$

- The above can also be expressed in the form:

$$\check{H}(\omega) = \sin(\omega) \sum_{k=0}^{M-1} \tilde{c}[k] \cos(\omega k)$$

where

$$c[1] = \tilde{c}[0] - \frac{1}{2} \tilde{c}[1]$$

$$c[k] = \frac{1}{2} (\tilde{c}[k-1] - \tilde{c}[k]), \quad 2 \leq k \leq M-1$$

$$c[M] = \frac{1}{2} \tilde{c}[M-1]$$

## 4 Types of Linear-Phase FIR Transfer Functions

### Amplitude Response of Type 4

- For the **Type 4 linear-phase FIR filter**, the amplitude response can be rewritten using the notation  $N = 2M$  in the form

$$\check{H}(\omega) = \sum_{k=1}^{(2M+1)/2} d[k] \sin\left(\omega\left(k - \frac{1}{2}\right)\right)$$

$$d[k] = 2h\left[\frac{2M+1}{2} - k\right], \quad 1 \leq k \leq \frac{2M+1}{2}$$

- The above can also be expressed in the form:

$$\check{H}(\omega) = \sin\left(\frac{\omega}{2}\right) \sum_{k=1}^{(2M-1)/2} \tilde{d}[k] \cos(\omega k)$$

where

$$d[1] = \tilde{d}[0] - \frac{1}{2}\tilde{d}[1]$$

$$d[k] = \frac{1}{2}(\tilde{d}[k-1] - \tilde{d}[k]), \quad 2 \leq k \leq \frac{2M-1}{2}$$

$$d\left[\frac{2M+1}{2}\right] = \tilde{d}\left[\frac{2M-1}{2}\right]$$

## Amplitude response of linear-phase FIR filters: Generic Form

- The amplitude response for all four types of linear-phase FIR filters can be expressed in the form

$$\check{H}(\omega) = Q(\omega)A(\omega)$$

$$Q(\omega) = \begin{cases} 1, & \text{for Type 1} \\ \cos(\omega/2), & \text{for Type 2} \\ \sin(\omega), & \text{for Type 3} \\ \sin(\omega/2), & \text{for Type 4} \end{cases}$$

$$A(\omega) = \sum_{k=0}^L \tilde{a}[k] \cos(\omega k)$$

$$\tilde{a}[k] = \begin{cases} a[k], & \text{for Type 1} \\ \tilde{b}[k], & \text{for Type 2} \\ \tilde{c}[k], & \text{for Type 3} \\ \tilde{d}[k], & \text{for Type 4} \end{cases} \quad L = \begin{cases} M, & \text{for Type 1} \\ \frac{2M-1}{2}, & \text{for Type 2} \\ M-1, & \text{for Type 3} \\ \frac{2M-1}{2}, & \text{for Type 4} \end{cases}$$

## Linear-Phase FIR Filter Design by Optimisation

- The amplitude response for all 4 types of linear-phase FIR filters can be expressed as

$$\check{H}(\omega) = Q(\omega)A(\omega)$$

- Before, we gave the weighted error function as

$$\varepsilon(\omega) = W(\omega)[\check{H}(\omega) - D(\omega)]$$

- The modified form of the weighted error function is now

$$\begin{aligned}\varepsilon(\omega) &= W(\omega)[Q(\omega)A(\omega) - D(\omega)] = W(\omega)Q(\omega)\left[A(\omega) - \frac{D(\omega)}{Q(\omega)}\right] \\ &= \tilde{W}(\omega)[A(\omega) - \tilde{D}(\omega)]\end{aligned}$$

where

$$\tilde{W}(\omega) = W(\omega)Q(\omega)$$

$$\tilde{D}(\omega) = D(\omega)/Q(\omega)$$



## Optimisation Problem

- **Problem formulation**

Determine  $\tilde{a}[k]$  which minimise the peak absolute value of

$$\varepsilon(\omega) = \tilde{W}(\omega) \left[ \sum_{k=0}^L \tilde{a}[k] \cos(\omega k) - \tilde{D}(\omega) \right]$$

over the specified frequency bands  $\omega \in R$ .

- After  $\tilde{a}[k]$  has been determined, construct the original  $A(e^{j\omega})$  and hence  $h[n]$ .
- Solution is obtained via the so called **Alternation Theorem**.
- The optimal solution has equiripple behavior, consistent with the total number of available parameters.
- Parks and McClellan used the **Remez** algorithm to develop a procedure for designing linear FIR digital filters.

## The Parks-McClellan Algorithm

- **Problem formulation**

Determine  $\tilde{a}[k]$  which minimise the peak absolute value of

$$\varepsilon(\omega) = \tilde{W}(\omega) \left[ \sum_{k=0}^L \tilde{a}[k] \cos(\omega k) - \tilde{D}(\omega) \right]$$

- Parks and McClellan solved the above problem applying the following theorem from the theory of Chebyshev approximation.

**Alternation Theorem:** The amplitude function  $A(\omega)$  is the best unique approximation of the desired amplitude response obtained by minimizing the peak absolute value  $\varepsilon$  of  $\varepsilon(\omega)$ , if and only if there exist at least  $L + 2$  extremal angular frequencies  $\omega_0, \omega_1, \dots, \omega_{L+1}$ , in a closed subset  $R$  of the frequency range  $0 \leq \omega \leq \pi$  such that  $\omega_0 < \omega_1 < \dots < \omega_L < \omega_{L+1}$  and  $\varepsilon(\omega_i) = -\varepsilon(\omega_{i+1})$ , with  $|\varepsilon(\omega_i)| = \varepsilon$  for all  $i$  in the range  $0 \leq i \leq L + 1$ .

## The Parks-McClellan Algorithm

- Let us examine the behaviour of the amplitude response for a Type I equiripple lowpass FIR filter whose approximation error  $\varepsilon(\omega)$  satisfies the condition of the alternation theorem.
- The peaks of  $\varepsilon(\omega)$  are at  $\omega = \omega_i$ ,  $0 \leq i \leq L + 1$ , where
 
$$\frac{d\varepsilon(\omega)}{d\omega} = 0$$
- Since in the passband and the stopband,  $\tilde{W}(\omega)$  and  $\tilde{D}(\omega)$  are piecewise constant, we see that

$$\left. \frac{d\varepsilon(\omega)}{d\omega} \right|_{\omega=\omega_i} = \left. \frac{dA(\omega)}{d\omega} \right|_{\omega=\omega_i} = 0$$

or, in other words, the amplitude response  $A(\omega)$  also has peaks at  $\omega = \omega_i$ .

- We use the relation  $\cos(\omega k) = T_k(\cos\omega)$  where  $T_k(x)$  is the  $k$ th order Chebyshev polynomial defined by

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad T_0(x) = 0, T_1(x) = 1$$

The amplitude response  $A(\omega)$  can be expressed as a power series in  $\cos\omega$

$$A(\omega) = \sum_{k=0}^L a[k](\cos\omega)^k$$

## Chebyshev Polynomial Revision

- Chebyshev polynomials of 1<sup>st</sup> kind:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

We know that

$$\cos 2\omega = 2\cos^2 \omega - 1 = T_2(\cos \omega)$$

$$\cos 3\omega = 4\cos^3 \omega - 3\cos \omega = T_3(\cos \omega)$$

It is proven that

$$\cos k\omega = T_k(\cos \omega)$$

The amplitude response  $A(\omega)$  can be expressed as a power series in  $\cos \omega$ .

$$A(\omega) = \sum_{k=0}^L a[k](\cos \omega)^k$$

## The Parks-McClellan Algorithm

- The amplitude response  $A(\omega)$  can be expressed as a power series in  $\cos\omega$

$$A(\omega) = \sum_{k=0}^L a[k](\cos\omega)^k$$

- It is an  $L$ th order polynomial in  $\cos\omega$ .
- As a result  $A(\omega)$  can have at most  $L - 1$  minima and maxima inside the specified passband and stopband.
- Moreover, at the band edges,  $\omega = \omega_p$  and  $\omega = \omega_s$ ,  $|\varepsilon(\omega)|$  is maximum and therefore,  $A(\omega)$  has extrema in these angular frequencies.
- In addition  $A(\omega)$  may also have extrema at  $\omega = 0$  and  $\omega = \pi$ .
- Therefore, there are, at most  $L + 3$  extremal frequencies of  $\varepsilon(\omega)$ .
- We can generalize and say that in the case of a linear phase FIR filter with  $K$  specified band edges and designed using the Remez exchange algorithm, there can be at most  $L + K + 1$  extremal frequencies.
- To arrive at the optimum solution we need to solve the set of  $L + 2$  equations:
 
$$\tilde{W}(\omega_i)[A(\omega_i) - \tilde{D}(\omega_i)] = (-1)^i \varepsilon, \quad 0 \leq i \leq L + 1$$
 for the unknowns  $\tilde{a}(i)$  and  $\varepsilon$ , provided the  $L + 2$  extremal angular frequencies are known.

## The Parks-McClellan Algorithm

- To arrive at the optimum solution we need to solve the set of  $L + 2$  equations:

$$\tilde{W}(\omega_i)[A(\omega_i) - \tilde{D}(\omega_i)] = (-1)^i \varepsilon, \quad 0 \leq i \leq L + 1$$

for the unknowns  $\tilde{a}(i)$  and  $\varepsilon$ , provided the  $L + 2$  extremal angular frequencies are known.

- The above is rewritten in matrix form as

$$\begin{bmatrix} 1 & \cos(\omega_0) & \dots & \cos(L\omega_0) & \frac{-1}{\tilde{W}(\omega_0)} \\ 1 & \cos(\omega_1) & \dots & \cos(L\omega_1) & \frac{-1}{\tilde{W}(\omega_1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\omega_L) & \dots & \vdots & \frac{(-1)^{L-1}}{\tilde{W}(\omega_L)} \\ 1 & \cos(\omega_{L+1}) & \dots & \cos(L\omega_{L+1}) & \frac{(-1)^{L-1}}{\tilde{W}(\omega_{L+1})} \end{bmatrix} \begin{bmatrix} \tilde{a}[0] \\ \tilde{a}[1] \\ \vdots \\ \tilde{a}[L] \\ \varepsilon \end{bmatrix} = \begin{bmatrix} \tilde{D}(\omega_0) \\ \tilde{D}(\omega_1) \\ \vdots \\ \tilde{D}(\omega_L) \\ \tilde{D}(\omega_{L+1}) \end{bmatrix}$$

- The **Remez Exchange Algorithm** is used to solve the above.

## The Parks-McClellan Algorithm

- The Remez exchange algorithm, a highly efficient iterative procedure, is used to determine the locations of the extremal frequencies and consists of the following steps at each iteration stage.
- **Step 1:** A set of initial values for the extremal frequencies are either chosen or are available from the completion of the previous iteration.
- **Step 2:** Solving the system of equations we obtain

$$\varepsilon = \frac{c_0 \tilde{D}(\omega_0) + c_1 \tilde{D}(\omega_1) + \dots + c_{L+1} \tilde{D}(\omega_{L+1})}{\frac{c_0}{\tilde{W}(\omega_0)} - \frac{c_1}{\tilde{W}(\omega_1)} + \dots + \frac{(-1)^{L-1} c_{L+1}}{\tilde{W}(\omega_{L+1})}}$$

$$c_n = \prod_{\substack{i=0 \\ i \neq n}}^{L+1} \frac{1}{\cos(\omega_n) - \cos(\omega_i)}$$

## The Parks-McClellan Algorithm

- **Step 3:** The values of the amplitude response  $A(\omega)$  at  $\omega = \omega_i$  are then computed using

$$A(\omega_i) = \frac{(-1)^i \varepsilon}{\tilde{W}(\omega_i)} + \tilde{D}(\omega_i), \quad 0 \leq i \leq L + 1$$

- **Step 4:** The polynomial  $A(\omega)$  is determined by interpolating the above values at the  $L + 2$  extremal frequencies using the Lagrange interpolation formula:

$$A(\omega) = \sum_{i=0}^{L+1} A(\omega_i) P_i(\cos \omega)$$

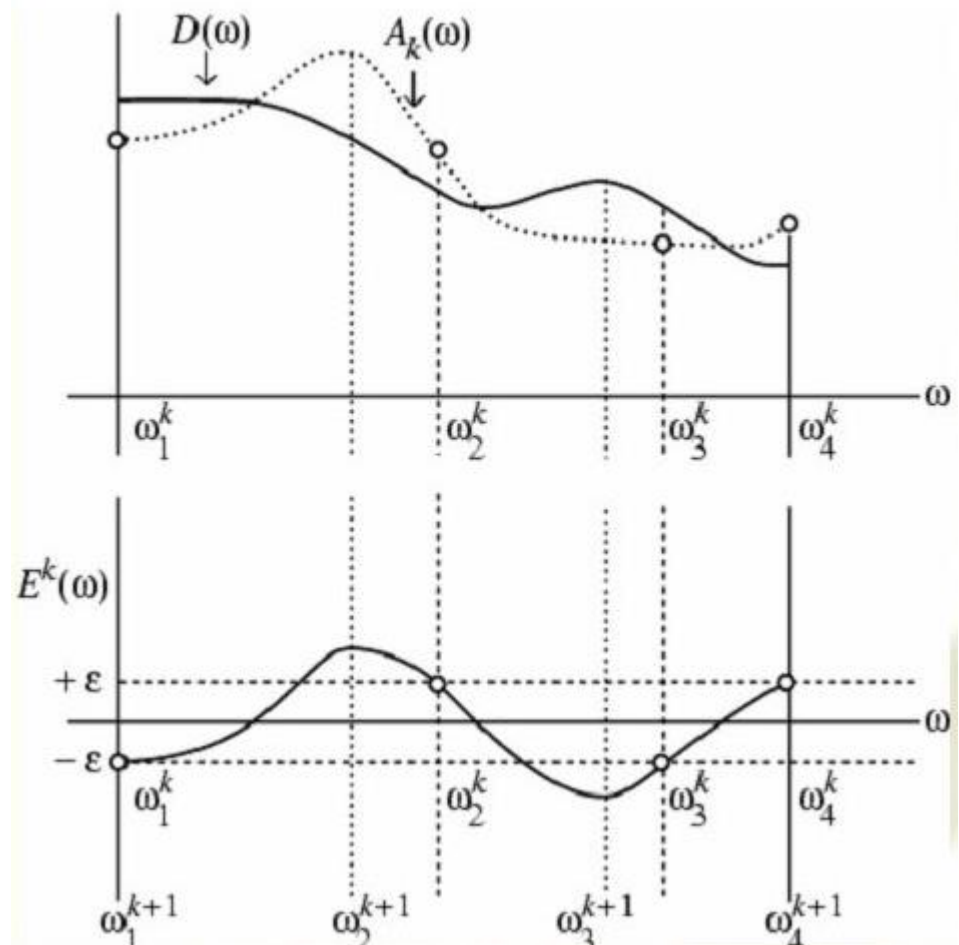
where  $P_i(\cos \omega) = \prod_{\substack{l=0 \\ l \neq i}}^{L+1} \left( \frac{\cos \omega - \cos \omega_l}{\cos \omega_i - \cos \omega_l} \right), \quad 0 \leq i \leq L + 1$

- **Step 5:** The new weighted error function  $\varepsilon(\omega)$  is computed at a dense set  $S (S \geq L)$  of frequencies. In practice,  $S = 16L$  is adequate. Determine the  $L + 2$  new extremal frequencies from the values of  $\varepsilon(\omega)$  evaluated at the dense set of frequencies.
- **Step 6:** If the peak values  $\varepsilon$  are equal in magnitude, the algorithm has converged. Otherwise, we go back to Step 2.



## The Parks-McClellan Algorithm

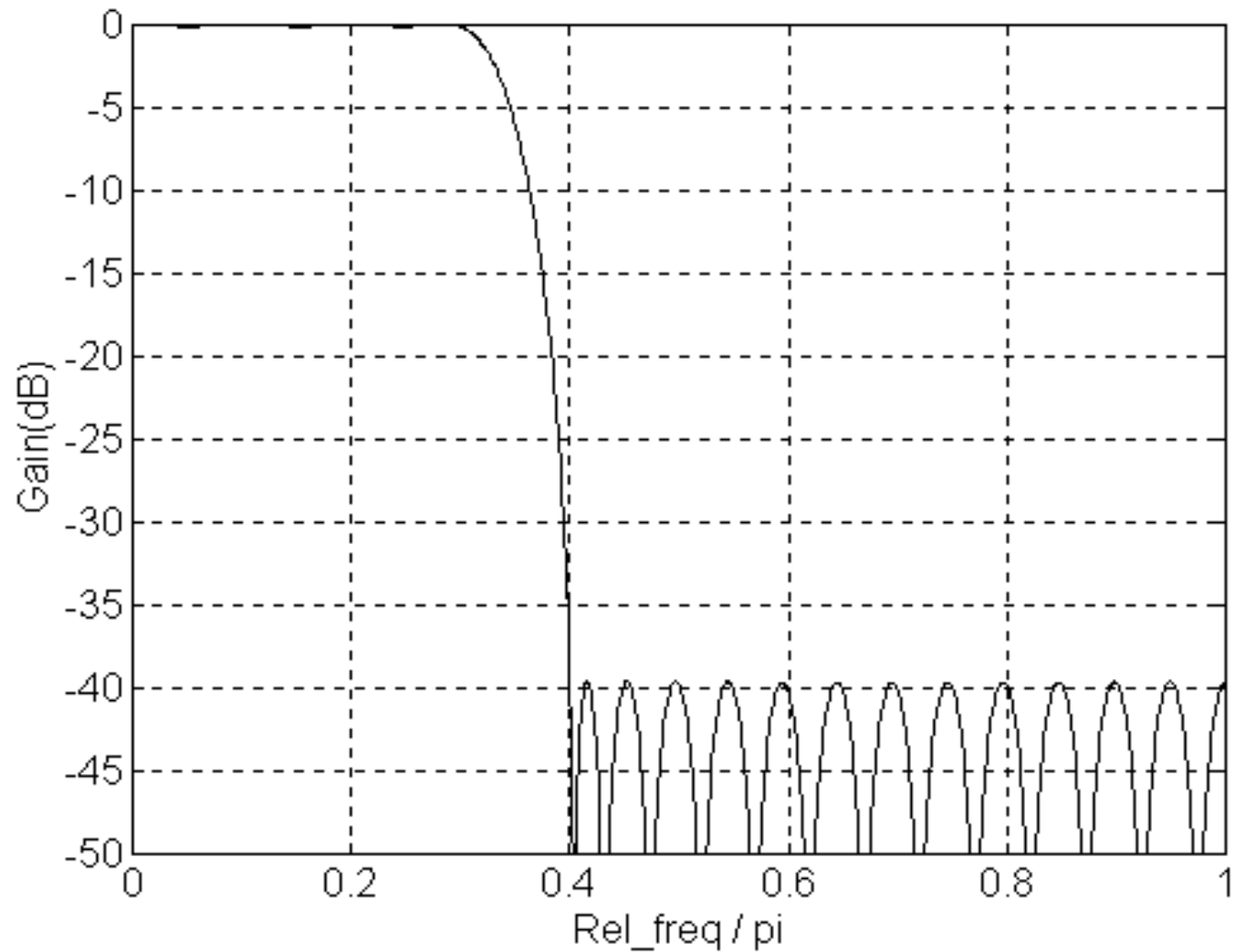
- Plots of the desired response  $D(\omega)$ , the amplitude response  $A_k(\omega)$  and the error  $\varepsilon^k(\omega)$  at the end of the  $k$ th iteration. The locations of the new extremal frequencies are given by  $\omega_i^{k+1}$ .
- The iteration process is stopped after the difference between the value of the peak error  $\varepsilon$  calculated at any stage and that at the previous stage is below a present threshold value, such as  $10^{-6}$ .
- In practice the process converges after very few iterations.



## Remez Exchange Algorithm

- Better than windowing technique, but more complicated.
- Available in MATLAB.
- Design 40<sup>th</sup> order FIR lowpass filter whose gain is unity (0 dB) in range 0 to  $0.3\pi$  radians/sample & zero in range  $0.4\pi$  to  $\pi$ .
- The 41 coefficients will be found in array 'a'.
- Produces equiripple gain-responses where peaks of stop-band ripples are equal rather than decreasing with increasing frequency.
- Highest peak in stop-band lower than for FIR filter of same order designed by windowing technique to have same cut-off rate.
- There are equiripple pass-band ripples.

```
a = remez (40, [0, 0.3, 0.4,1],[1, 1, 0, 0] );  
h = freqz (a,1,1000);  
plot([0:999]/1000,20*log10(abs(h)),'k');  
axis([0,1,-50,0]);  
grid on;  
xlabel('Rel_freq / pi');  
ylabel('Gain(dB)');
```



Gain of 40<sup>th</sup> order FIR lowpass filter designed by “Remez”