# Wavelets, Sparsity and their Applications

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Session Five: Multiresolution Analysis and Splines

# Multi-Resolution Analysis

**Definition** By a multi-resolution analysis we mean a sequence of embedded closed subspaces

$$\ldots V_2 \subset V_1 \subset V_0 \subset V_{-1} \ldots$$

such that

1. Upward Completeness: $\lim_{m\to-\infty} V_m = \bar{\bigcup}_{m\in\mathbb{Z}} V_m = L_2(\mathbb{R})$.
2. Downward Completeness: $\lim_{m\to\infty} V_m = \bigcap_{m\in\mathbb{Z}} V_m = \{0\}$.
3. Scale Invariance: $f(t) \in V_m \leftrightarrow f(2^m t) \in V_0$.
4. Shift Invariance: $f(t) \in V_0 \rightarrow f(t-n) \in V_0$ for all $n \in \mathbb{Z}$.
5. Existence of a Basis. There exists $\varphi(t) \in V_0$, such that $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $V_0$.

# Consequences of the Multi-Resolution Analysis

First notice that:[1]

$$\boxed{\langle \varphi(t-n), \varphi(t-m) \rangle = \delta_{m,n} \Longleftrightarrow \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2k\pi)|^2 = 1.}$$

Since $V_1$ is included in $V_0$, if $\varphi(t/2)$ belongs to $V_1$, it belongs to $V_0$ as well. Thus:

$$\varphi(x/2) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(x-n)$$

or

$$\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t-n).$$

This is the two-scale relation.

[1]see Appendix

$$f(t) = \sqrt{2}\sum_{n=-\infty}^{+\infty} g_0[n] f(2t-n)$$

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# Consequences of the Multi-Resolution Analysis

$$\hat{f}(\omega) = \sqrt{2}\sum_{n=-\infty}^{+\infty} g_0[n]\int_{-\infty}^{+\infty} f(2t-n)e^{-j\omega t}dt$$

$$\xrightarrow[t=\frac{x+n}{2}]{x=2t-n} \frac{\sqrt{2}}{2}\sum_{n=-\infty}^{+\infty} g_0[n]\int_{-\infty}^{+\infty} f(x)e^{-j\omega\frac{x}{2}}e^{-j\omega\frac{n}{2}}dx$$

By taking the Fourier transform of both sides of the two-scale relation, we obtain

$$= \frac{\sqrt{2}}{2}\sum_{n=-\infty}^{+\infty} g_0[n]e^{-j\omega\frac{n}{2}}\int_{-\infty}^{+\infty} f(x)e^{-j\omega\frac{x}{2}}dx$$

$$\hat{\varphi}(\omega) = \frac{1}{\sqrt{2}}G_0(e^{j\omega/2})\hat{\varphi}(\omega/2)$$

$$= \frac{\sqrt{2}}{2} G_0(e^{j\frac{\omega}{2}})\hat{f}(\tfrac{\omega}{2})$$

where

$$\because \sum_{k=-\infty}^{+\infty}|\hat{f}(2\omega+2k\pi)|^2 = 1$$

$$G_0(e^{j\omega}) = \sum_n g_0[n]e^{-j\omega n}.$$

$$G_0(z)G_0(z^{-1}) + G_0(-z)G_0(-z^{-1}) = 2$$

and because of the orthogonality of $\varphi(t)$, we obtain

$$\therefore \frac{1}{2}\sum_k |G_0(e^{j(\omega+k\pi)})|^2|\hat{f}(\omega+k\pi)|^2 = 1$$

$$|G_0(e^{j\omega})|^2 + |G_0(e^{j(\omega+\pi)})|^2 = 2.$$

$$\therefore \frac{1}{2}\left(\sum_k |G_0(e^{j(\omega+2k\pi)})|^2|\hat{f}(\omega+2k\pi)|^2 + \sum_k |G_0(e^{j(\omega+(2k+1)\pi)})|^2|\hat{f}(\omega+(2k+1)\pi)|^2\right) = 1$$

$$\therefore \frac{1}{2}|G_0(e^{j\omega})|^2\underbrace{\sum_k|\hat{f}(\omega+2k\pi)|^2}_{=1} + \frac{1}{2}|G_0(e^{j(\omega+\pi)})|^2\underbrace{\sum_k|\hat{f}(\omega+(2k+1)\pi)|^2}_{=1} = 1$$

$$\therefore |G_0(e^{j\omega})|^2 + |G_0(e^{j(\omega+\pi)})|^2 = 2$$

# Consequences of the Multi-Resolution Analysis

**Theorem 1.** *Let $\{V_n\}$, $n \in \mathbb{Z}$ be a multiresolution analysis with the scaling function $\varphi(t)$. There exists an orthonormal basis for $L_2(\mathbb{R})$:*

$$\psi_{m,n}(t) = 2^{-m/2}\psi(2^{-m}t - n) \qquad m, n \in \mathbb{Z}$$

*and*

$$\psi(t) = \sum_{n=-\infty}^{\infty} (-1)^n g_0[1-n]\varphi(2t - n)$$

*such that $\{\psi_{m,n}\}$, $n \in \mathbb{Z}$ is an orthonormal basis for $W_m$, where $W_m$ is the orthogonal complement of $V_m$ in $V_{m-1}$.*

# Scaling Function and Splines

Central to multiresolution analysis is the design of a proper scaling function. It is possible to show that $\varphi(t)$ is an admissible scaling function of $L_2(\mathbb{R})$ if and only if it satisfies the three following conditions:

1. Riesz basis criterion: There exists two constants $A > 0$ and $B < +\infty$ such that

$$A \leq \sum_{k \in \mathbb{Z}} |\hat{\varphi}(\omega + 2\pi k)|^2 \leq B \tag{1}$$

2. Two scale relation

$$\varphi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} g_0[k] \varphi(2t - k) \tag{2}$$

3. Partition of unity

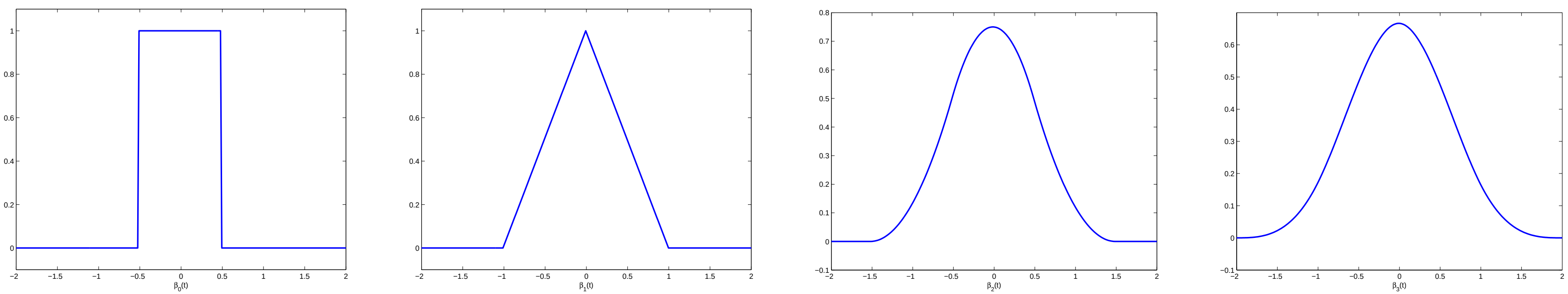
$$\sum_{k \in \mathbb{Z}} \varphi(t - k) = 1. \tag{3}$$

# Scaling Function and Splines

A remarkable example of scaling functions is given by the family of B-splines. A B-spline $\beta_N(t)$ of order $N$ is obtained from the (N+1)-fold convolution of the box function $\beta_0(t)$ or

$$\beta_N(t) = \underbrace{\beta_0(t) * \beta_0(t) \ldots * \beta_0(t)}_{N+1 \text{ times}} \quad \text{with } \hat{\beta}_0(\omega) = \frac{1 - e^{-j\omega}}{j\omega}$$

where $\hat{\beta}(\omega)$ is the Fourier transform of $\beta(t)$.



# Scaling Function and Splines

**Theorem 2.** *Given two valid biorthogonal scaling functions* $\varphi(t)$ *and* $\tilde{\varphi}(t)$ *satisfying the following two scale relations*

$$\varphi(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} g_0[k]\varphi(2t-k)$$

$$\tilde{\varphi}(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_0[k]\tilde{\varphi}(2t-k).$$

*There exist two biorthogonal wavelets* $\psi$ *and* $\tilde{\psi}$ *such that*

$$\psi(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} (-1)^{k-1} h_0[1-k]\varphi(2t-k)$$

$$\tilde{\psi}(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} (-1)^{k-1} g_0[1-k]\tilde{\varphi}(2t-k)$$

# Scaling Function and Splines

**Example**

Assume that $\varphi(t)$ is a linear spline. The two scale equation is satisfied when $G_0(z) = (\frac{1}{2}z^{-1} + 1 + \frac{1}{2}z)/\sqrt{2}$.

The biorthogonality relation says that

$$\langle \tilde{\varphi}(t), \varphi(t-n) \rangle = \delta_n.$$

Since both $\varphi(t)$ and $\tilde{\varphi}(t)$ satisfy a two scale relation, it follows that

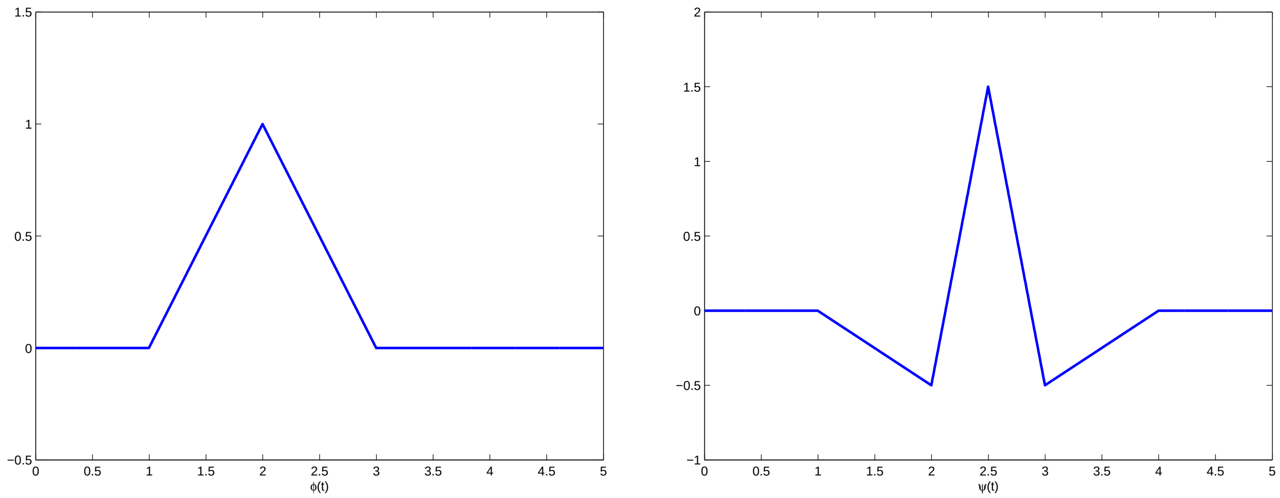
$$\langle \tilde{\varphi}(t), \varphi(t-n) \rangle = \langle h_0[k], g_0[k-2n] \rangle = \delta_n.$$

The above relation is equivalent to the condition $P(z) + P(-z) = 2$ with $P(z) = H_0(z^{-1})G_0(z)$. Here $G_0(z)$ is known and has two zeros at $\omega = \pi$, the shortest $H_0(z)$ with the same number of zeros at $\pi$ is then

$$H_0(z) = \frac{\sqrt{2}}{8}(1+z)(1+z^{-1})(-z+4-z^{-1}) = \frac{\sqrt{2}}{8}(z^{-1}+2+z)(-z+4-z^{-1}).$$

# Scaling Function and Splines

Given $H_0(z)$ the construction of the wavelet $\psi(t)$ is then straightforward. The scaling function $\varphi(t)$ and wavelet $\psi(t)$ for this example are shown below.



$A[n] = \langle f(t), f(t-n) \rangle = \delta_n$

$A(\tau) = \langle f(t), f(t-\tau) \rangle = \int_{-\infty}^{+\infty} f^*(t) f(t-\tau) dt$

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$A(\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f^*(t) f(t-\tau) dt\, e^{-j\omega\tau} d\tau$

$\overset{x=t-\tau}{\underset{\tau=t-x}{=}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f^*(t) e^{-j\omega t} dt\, f(x) e^{j\omega x} dx = \left[\int_{-\infty}^{+\infty} f(t) e^{j\omega t} dt\right]^* \left[\int_{-\infty}^{+\infty} f(x) e^{j\omega x} dx\right]$

# Appendix

Claim:

$$\langle \varphi(t-n), \varphi(t-m) \rangle = \delta_{m,n} \Longleftrightarrow \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2k\pi)|^2 = 1.$$

$= \hat{f}^*(-\omega)\hat{f}(-\omega) = |\hat{f}(-\omega)|^2$

Proof:

$\because f(t) * f(t) = f(-t) * f(-t)$

Define $p(\tau) = \langle \varphi(t), \varphi(t-\tau) \rangle$. Then $\langle \varphi(t), \varphi(t-m) \rangle$ is obtained by sampling $p(\tau)$ with sampling period $T = 1$. The Fourier transform of $p(\tau)$ is given by:

$\therefore$ ans $= \hat{f}^*(\omega)\hat{f}(\omega) = |\hat{f}(\omega)|^2.$

$$\int_{-\infty}^{\infty} p(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} \langle \varphi(t), \varphi(t-\tau) \rangle e^{-j\omega\tau} d\tau = |\hat{\varphi}(\omega)|^2.$$

Applying the rule that sampling in time corresponds to replica in frequency leads to the desired equality.