440 Information theory.

# The Solutions to Exam 2016

B—bookwork, E—new example, T—new theory

1.

a) The possible outcomes of $X$ and $Y$ are given in the table below (O-Odd, E-Even):

| X | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Y | O | E | O | E | O | E |

i) Obviously,
$H(X) = \log 6$, [2E]
$H(Y) = 1$, [2E]
because they are uniform.

ii) $H(X|Y) = ½ H(X|Y{=}E) + ½ H(X|Y{=}O) = ½ \log 3 + ½ \log 3 = \log 3$ [2E]
$H(Y|X) = 1/6\, H(Y|X{=}1) + 1/6\, H(Y|X{=}2) + \ldots + 1/6\, H(Y|X{=}6)$
$= 0 + 0 + \ldots + 0 = 0$ [2E]
$H(X, Y) = H(X) + H(Y|X) = \log 6$

iii) $I(X; Y) = H(X) - H(X|Y) = \log 6 - \log 3 = 1$ [2E]

b)

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$
$$= E \log \frac{p(X,Y)}{p(X)p(Y)} = D(P_{X,Y} \| P_X \otimes P_Y) \geq 0$$
[3B]

Equality holds iff P(X,Y) = P(X)P(Y), i.e., X and Y are independent. [2B]

c) Any $n - 1$ or fewer of these random variables are independent of each other. Thus, for $k \leq n - 1$,

$$I(X_{k-1}; X_k | X_1, X_2, \ldots, X_{k-2}) = 0.$$ [3E]

However, given $X_1, X_2, \ldots, X_{n-2}$, once we know either $X_{n-1}$ or $X_n$, we know the other. Therefore, [2E]
$I(X_{n-1}; X_n | X_1, X_2, \ldots, X_{n-2})$
$= H(X_n | X_1, X_2, \ldots, X_{n-2}) - H(X_n | X_1, X_2, \ldots, X_{n-1})$ [3E]
$= 1 - 0 = 1$ bit. [2E]


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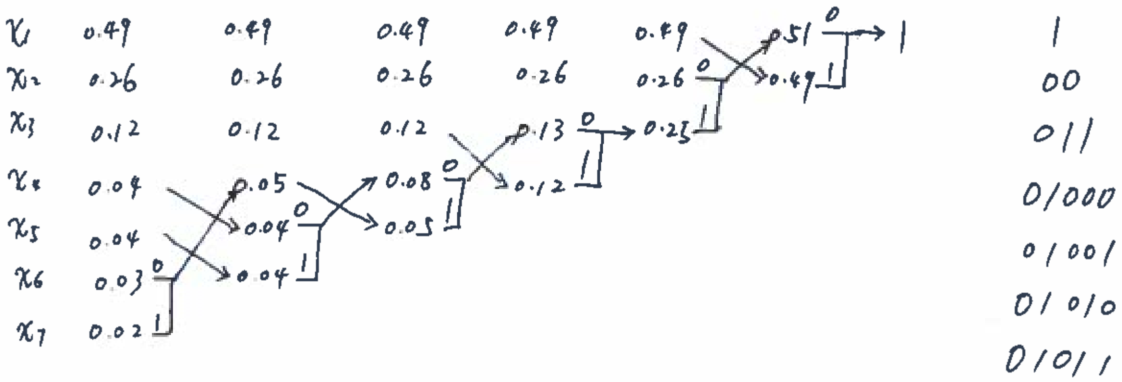
2.

a) i) [5E]




ii)

$$L = \sum p(x_i)\ell(x_i) = 1\times 0.49 + 2\times 0.26 + 3\times 0.12 + 5\times 0.13$$
$$= 2.02$$

[3E]

b) Phrases: 1, 0, 10, 01, 100, 101, 00, 1000, 1010, 11 [2E]

Locations: 1,2, 3, 4, 5, 6, 7, 8, 9, 10 [2E]

There are 10 phrases, so 4 bits are needed to represent locations.

Encodings: (0000,1); (0000,0); (0001,0); (0010,1); (0011,0)
(0011,1); (0010,0); (0101,0); (0110,0); (0001,1)

[3E]

c) i)

The transition matrix is

$$P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 0 & 3/4 \end{pmatrix}$$

The stationary distribution is found from

$$\mu P = \mu$$

$$\Rightarrow \begin{cases} -\frac{1}{4}\mu_1 \qquad + \frac{1}{4}\mu_3 = 0 \\ \frac{1}{4}\mu_1 - \frac{1}{2}\mu_2 \qquad = 0 \\ \qquad \frac{1}{2}\mu_2 - \frac{1}{4}\mu_3 = 0 \end{cases}$$

Together with $\sum_i \mu_i = 1$ we get $\mu_1 = \frac{2}{5}$, $\mu_2 = \frac{1}{5}$, $\mu_3 = \frac{2}{5}$.

[3E]

The entropy rate is

$$H_\infty(U) = \sum_i \mu_i H(S_i) = \frac{2}{5}h\left(\frac{1}{4}\right) + \frac{1}{5}h\left(\frac{1}{2}\right) + \frac{2}{5}h\left(\frac{1}{4}\right)$$
$$= \frac{4}{5}\left(2 - \frac{3}{4}\log 3\right) + \frac{1}{5} = \frac{1}{9} - \frac{3}{4}\log 3 \approx 0.8490$$

[4E]

ii)

$$H\left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right) = -\frac{2}{5}\log\frac{2}{5} - \frac{1}{5}\log\frac{1}{5} - \frac{2}{5}\log\frac{2}{5} = \log 5 - \frac{4}{5} \approx 1.5219$$

That is, we gain in uncertainty if we take into consideration the memory of the source.

[3E]

3.

a)

(1) definition of mutual info [1B]
(2) Markov chain w – x – y, and data processing theorem [1B]
(3) definition of mutual info [1B]
(4) chain rule, translation-invariant [1B]
(5) indep. bound, Fano's inequality [1B]
(6) mutual info. <= capacity [1B]
(7) algebra [1B]
(8) taking limit and $P_e^{(n)} \to 0$ [1B]

b)

i)

$$h(X) = -\int_0^\infty f(x)\log f(x)dx = -\log e\int_0^\infty f(x)\ln f(x)dx$$

$$= -\log e\int_0^\infty \lambda e^{-\lambda x}(\ln\lambda - \lambda x)dx = -\log e(\ln\lambda - \int_0^\infty \lambda x e^{-\lambda x} d\lambda x)$$ [2E]

$$= -\log e(\ln\lambda - 1) = -\log e\cdot\ln(\lambda / e) = \log(e/\lambda) \text{ bits}$$

[2E]

ii) The sum of two normal random variables is also normal, so applying the result derived in the class for the normal distribution,

$$h(f) = \frac{1}{2}\log 2\pi e(\sigma_1^2 + \sigma_2^2) \text{ bits}$$

[3E]

c) Recall the definition of R(D):

$$R(D) = \min_{p(\hat{X}|X)} I(X;\hat{X}) \text{ such that } E\left[d(X,\hat{X})\right] \le D$$ [2B]

We have

$$R(D) = \min h(X) - h(X|\hat{X}) \ge h(X) - \tfrac{1}{2}\log(2\pi e D)$$ [3T]

because

$$h(X|\hat{X}) = h(X-\hat{X}|\hat{X}) \le h(X-\hat{X}) \le \tfrac{1}{2}\log(2\pi e D)$$ [3T]

Therefore,

$$D \ge \frac{2^{2h(X)}}{2\pi e}2^{-2R}$$ [2T]

4.

a)

Under no interference, each sender has power $P$ and rate $C(P/N)$.

Each user independently sends a codeword from a Gaussian codebook. [2B]

Consider receiver 1 (receiver 2 is the same). Under very strong interference,

- Treats sender 1 as interference, and decode sender 2 at rate $C(a^2P/(P+N))$; [2B]
- Subtracting it from received signal, he sees a clean channel for sender 1 with capacity $C(P/N)$. [2B]

This is possible if $C(a^2P/(P+N)) > C(P/N)$, i.e., crosslink is better:

$$\frac{1}{2}\log\left(1+\frac{P}{N}\right) < \frac{1}{2}\log\left(1+\frac{a^2P}{P+N}\right) \quad [2B]$$

$$\frac{P}{N} < \frac{a^2P}{P+N}$$

$$a^2 > \frac{P+N}{N} = 1+\frac{P}{N} \quad [2B]$$

b)

i) [3E]

The joint distribution of $X$ and $Y$ is shown in following table

| $Z_1$ | $Z_2$ | $X$ | $Y$ | probability |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $(1-p_1)(1-p_2)$ |
| 0 | 1 | 1 | -1 | $(1-p_1)p_2$ |
| 1 | 0 | 1 | 1 | $p_1(1-p_2)$ |
| 1 | 1 | 2 | 0 | $p_1p_2$ |

and hence we can calculate

$$H(X) = H(p_1p_2, p_1+p_2-2p_1p_2, (1-p_1)(1-p_2))$$

$$H(Y) = H(p_1p_2+(1-p_1)(1-p_2), p_1-p_1p_2, p_2-p_1p_2)$$

and

$$H(X,Y) = H(Z_1,Z_2) = H(p_1)+H(p_2)$$

[4E]

Therefore,

$$H(X|Y) = H(p_1)+H(p_2) - H(p_1p_2+(1-p_1)(1-p_2), p_1-p_1p_2, p_2-p_1p_2)$$
$$H(Y|X) = H(p_1)+H(p_2) - H(p_1p_2, p_1+p_2-2p_1p_2, (1-p_1)(1-p_2))$$

Slepian-Wolf region is

$$R_X \geq H(X|Y)$$

$$R_Y \geq H(Y|X)$$

$$R_X + R_Y \geq H(X,Y) = H(p_1)+H(p_2)$$

[3E]

ii)

The Slepian Wolf region for $(Z_1, Z_2)$ is

$$\begin{aligned} R_1 &\geq H(Z_1|Z_2) = H(p_1) \\ R_2 &\geq H(Z_2|Z_1) = H(p_2) \\ R_1 + R_2 &\geq H(Z_1, Z_2) = H(p_1) + H(p_2) \end{aligned}$$

which is a rectangular region. [3E]

The minimum sum rate is the same in both cases, since if we knew both X and Y, we could find $Z_1$ and $Z_2$ and vice versa. However, the region in part (i) is usually pentagonal in shape, and is larger than the region in (ii).

[2E]