C477: Computational Optimisation
Tutorial 2: Optimality Conditions

Exercise 1. Consider the problem,

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^2} \; & f(\boldsymbol{x}) \\ \text{s.t. } & \boldsymbol{x} \in \Omega, \end{aligned}$$

where $f : \mathbb{R}^2 \mapsto \mathbb{R}$ and $f \in \mathcal{C}^2$. For each of the four following specifications for Ω, $\boldsymbol{x}^*$ and f, determine if the given point $\boldsymbol{x}^*$ is: (i) definitely a local minimiser; (ii) definitely not a local minimiser, or (iii) possibly a local minimiser. Justify your answers.

(a)	Set:	$\Omega = \{\boldsymbol{x} = [x_1,\, x_2]^\top \mid x_1 \geq 1\}$,
	Given point:	$\boldsymbol{x}^* = [1,\, 2]^\top$,
	Gradient:	$\nabla f(\boldsymbol{x}^*) = [1,\, 1]^\top$.
(b)	Set:	$\Omega = \{\boldsymbol{x} = [x_1,\, x_2]^\top \mid x_1 \geq 1, x_2 \geq 2\}$,
	Given point:	$\boldsymbol{x}^* = [1,\, 2]^\top$,
	Gradient:	$\nabla f(\boldsymbol{x}^*) = [1,\, 0]^\top$.
(c)	Set:	$\Omega = \{\boldsymbol{x} = [x_1,\, x_2]^\top \mid x_1 \geq 0, x_2 \geq 0\}$,
	Given point:	$\boldsymbol{x}^* = [1,\, 2]^\top$,
	Gradient:	$\nabla f(\boldsymbol{x}^*) = [0,\, 0]^\top$,
	Hessian:	$\nabla^2 f(\boldsymbol{x}^*) = \boldsymbol{I}$ (identity matrix).
(d)	Set:	$\Omega = \{\boldsymbol{x} = [x_1,\, x_2]^\top \mid x_1 \geq 1, x_2 \geq 2\}$,
	Given point:	$\boldsymbol{x}^* = [1,\, 2]^\top$,
	Gradient:	$\nabla f(\boldsymbol{x}^*) = [1,\, 0]^\top$,
	Hessian:	$\nabla^2 f(\boldsymbol{x}^*) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Exercise 2. Consider the following function $f :\to \mathbb{R}^2 \to \mathbb{R}$:

$$f(\boldsymbol{x}) = \boldsymbol{x}^\top \begin{bmatrix} 2 & 5 \\ -1 & 1 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^\top \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 7$$

(a) Find the directional derivative of f at $[0,\, 1]^\top$ in the direction $[1,\, 0]^\top$.

(b) Find all the points that satisfy the first order necessary condition for f.

(c) Does f have a minimiser? If it does, then find all minimiser(s); otherwise explain why it does not.

Exercise 3. Consider the problem,

$$\begin{aligned} \min \quad & -x_2^2 \\ \text{s.t.} \quad & |x_2| \leq x_1^2 \\ & x_1 \geq 0 \end{aligned}$$

(a) Does the point $[x_1,\, x_2]^\top = \mathbf{0}$ satisfy the first order necessary condition for a minimiser? That is, if $f(\boldsymbol{x}) = -x_2^2$ is the objective function, is it true that $\boldsymbol{d}^\top \nabla f(\boldsymbol{x}) \geq 0$ for all feasible directions $\boldsymbol{d}$ at $\mathbf{0}$?

(b) Is the point $[x_1,\, x_2]^\top = \mathbf{0}$ a local minimiser, a strict local minimiser, a local maximiser, a strict local maximiser, or none of the above?

Exercise 4. Suppose that we are given n real numbers, $x_1, \ldots, x_n$. Find the number $\bar{x} \in \mathbb{R}$ such that the sum of the squared difference between $\bar{x}$ and the numbers $x_1, \ldots, x_n$ above is minimised (assuming that the solution $\bar{x} \in \mathbb{R}$ exists).

Exercise 5. Suppose that we are given p vectors, $\{\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(p)}\}$, where $\boldsymbol{x}^{(i)} \in \mathbb{R}^n$, $i = 1, \dots, p$. Find the vector $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ such that the average squared distance (2-norm) between $\bar{\boldsymbol{x}}$ and $\boldsymbol{x}^{(1)}$, $\dots$, $\boldsymbol{x}^{(p)}$,

$$\frac{1}{p} \sum_{i=1}^{p} \|\bar{\boldsymbol{x}} - \boldsymbol{x}^{(i)}\|_2^2,$$

is minimised. Use the SOSC to prove that the vector $\bar{\boldsymbol{x}}$ found above is a strict local minimiser.
