EE4-45
EE9-SO22

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected copy

WAVELETS AND APPLICATIONS

Monday, 8 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	P.L. Dragotti
	Second Marker(s) :	A. Manikas

Special Information for the Invigilators: NONE

Information for Candidates:

Sub-sampling by an integer N

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N}) = \frac{1}{N}\sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

Dual Basis:
Given a basis $\{\varphi_i(t)\}_{i\in\mathbb{Z}}$, the dual basis is given by the set of elements $\{\tilde{\varphi}_i(t)\}_{i\in\mathbb{Z}}$ satisfying:

$$\langle \varphi_i(t), \tilde{\varphi}_j(t)\rangle = \delta_{i,j}.$$

Haar scaling and wavelet functions
The Haar scaling function is defined as:

$$\varphi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

The Haar wavelet is defined as:

$$\psi(t) = \begin{cases} 1 & 0 \le t < 1/2 \\ -1 & 1/2 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

The Questions

1. Consider the system shown in Fig. 1a. This is a two-channel filter bank where the analysis part contains also two modulators. The modulator in the upper branch modulates the input $x[n]$ with the sequence $p[n] = 1, \forall n$ and then filters $p[n]x[n]$ with $H_0(z)$. In the lower branch, the input $x[n]$ is modulated with the sequence $q[n] = (-1)^n$ and the resulting sequence is then filtered with $H_1(z)$.

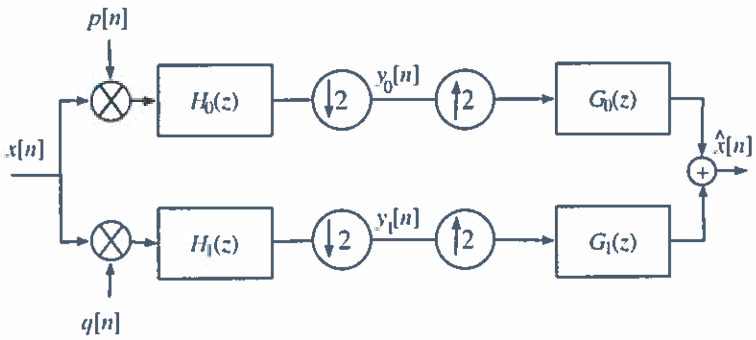


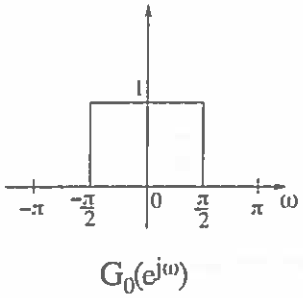
Figure 1a: Two-Channel Filter Bank with Modulation.

(a) Express $\hat{X}(z)$ as a function of $X(z)$ and the filters. Then derive the two perfect reconstruction (PR) conditions the filters have to satisfy. [7]

(b) Assume that $G_0(z) = (1+z^{-1})/\sqrt{2}$ and that $H_0(z) = (1+z)/\sqrt{2}$, find the shortest filters $H_1(z)$ and $G_1(z)$ that would lead to a perfect reconstruction filter bank. [6]

Question continues on next page.

(c) Assume now that $G_0(z)$ and $G_1(z)$ are half-band ideal low-pass and high-pass filters respectively as shown in Fig. 1b. Also assume that $H_0(z) = 2G_0(z^{-1})$. Sketch and dimension the Fourier transform of the filter $H_1(z)$ that would lead to a perfect reconstruction filter bank.



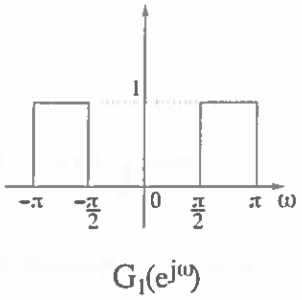


Figure 1b: Half-band synthesis filters.

[6]

(d) Given the filters of part (c), sketch and dimension the Fourier transform of $y_0[n]$ and $y_1[n]$, assuming that $x[n]$ has the spectrum shown in Fig. 1c

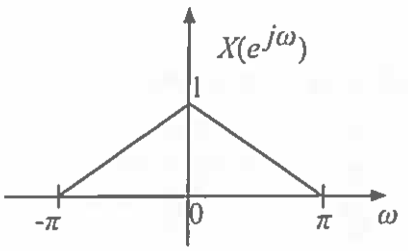


Figure 1c: Discrete-time Fourier transform of $x[n]$.

[6]

2. Consider the two systems shown in Fig. 2a and Fig. 2b.

Figure 2a: First multirate system.

Figure 2b: Second multirate system.

(a) Find the filters $c[n]$ and $d[n]$ and integers M and N so that the system in Fig. 2b has input-output relationship equivalent to the system in Fig 2a. (Either a time domain or a z domain answer is acceptable.) [5]

(b) Given that the filter $b[n]$ satisfies $\langle b[n], b[n-3k]\rangle = \delta_k$, find (and prove) a sufficient condition on $a[n]$ such that the mapping from x to y is idempotent. Idempotent means that if the mapping was applied again to the sequence $y[n]$, the output would stay $y[n]$. [10]

(c) Consider a filter bank specified by the following signal equations:

$$\begin{aligned} y_0 &= D_2 G D_2 G x \\ y_1 &= D_2 G D_2 H D_2 G x \\ y_2 &= D_2 H D_2 H D_2 G x \\ y_3 &= D_2 H x, \end{aligned}$$

where G and H are the infinite matrix representations for filtering with a lowpass filter $g[n]$ and a highpass filter $h[n]$, respectively, and D_2 is the matrix representation of downsampling by 2.

i. Draw a block diagram of the system using two-channel filter banks. [5]

ii. Draw the equivalent single-level four-channel filter bank clearly specifying the downsampling factors and transfer functions of the filters in each branch. [5]

3. Consider the interval $t \in [0,1]$ and the function

$$x(t) = \begin{cases} e^t, & \text{for } t \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

The aim is to find a polynomial of degree one that best approximates $x(t)$ over the interval $[0,1]$. Specifically, let

$$\varphi_1(t) = \begin{cases} 1, & \text{for } t \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

and

$$\varphi_2(t) = \begin{cases} t, & \text{for } t \in [0,1] \\ 0, & \text{otherwise,} \end{cases}$$

the degree-one polynomial is given by $p(t) = \sum_{k=1}^{2} a_k \varphi_k(t)$ and a_1, a_2 can be found by minimizing

$$\|\epsilon(t)\|^2 = \int_0^1 \epsilon^2(t)dt$$

with $\epsilon(t) = x(t) - p(t)$.

(a) Using the projection theorem, show that the first-order Taylor series of $x(t)$ at $t = 0$ given by:

$$p_s(t) = x(0) + x'(0)t$$

does not minimise $\|\epsilon(t)\|^2$. [5]

(b) Denote with $V = \text{span}\{\varphi_1(t), \varphi_2(t)\}$ the sub-space generated by $\varphi_i(t)$ with $i = 1,2$ over the interval $t \in [0,1]$, the minimization of the error is achieved by computing the orthogonal projection of $x(t)$ onto V. Recall that this is given by:

$$p_v(t) = \sum_{i=1}^{2} \langle x(t), \tilde{\varphi}_i(t) \rangle \varphi_i(t)$$

where $\{\tilde{\varphi}_i(t)\}_{i=1}^2$ are the two dual-basis functions.

i. Since $\tilde{\varphi}_i(t) \in V$ we can write $\tilde{\varphi}_i(t) = \sum_{k=1}^{2} \alpha_{i,k} \varphi_k(t)$. Using this fact, determine the two dual-basis functions $\tilde{\varphi}_i(t)$, $i = 1,2$. That is, find the coefficients $\alpha_{i,k}$, $i = 1,2$; $k = 1,2$. [5]

Question continues on next page.

ii. Given the dual basis

A. Compute the inner products $\langle x(t), \tilde{\varphi}_i(t) \rangle$, $i = 1, 2$. [5]

B. Write the exact expression for $p_v(t) = \sum_{i=1}^{2} \langle x(t), \tilde{\varphi}_i(t) \rangle \varphi_i(t)$. [5]

C. Verify that the error $\epsilon_v(t) = x(t) - p_v(t)$ is orthogonal to V. [5]

4. Let $\varphi(t)$ be the Haar scaling function and $\psi(t)$ be the Haar wavelet function. Let V_j be the subspace generated by $\varphi_{j,n}(t) = \sqrt{2^{-j}}\varphi(2^{-j}t - n)$, $n \in \mathbb{Z}$ and let W_j be the subspace generated by $\psi_{j,n}(t) = \sqrt{2^{-j}}\psi(2^{-j}t - n)$, $n \in \mathbb{Z}$. Consider the function defined on $0 \leq t < 1$ given by

$$f(t) = \begin{cases} 0 & 0 \leq t < 1/4 \\ 1 & 1/4 \leq t < 3/8 \\ 0 & 3/8 \leq t < 1. \end{cases}$$

(a) Express $f(t)$ in terms of the basis elements of V_k. In other words, find the coefficients $c_{k,n}$, $n \in \mathbb{Z}$ that lead to the decomposition

$$f(t) = \sum_{n \in \mathbb{Z}} c_{k,n}\varphi_{k,n}(t). \tag{1}$$

Pick a negative k so that only one coefficient in Eq. (1) is non-zero. [6]

(b) Given the k chosen in part (a), decompose $f(t)$ into its component parts that belong to the subspace $W_{k+1}, W_{k+2}, \cdots, W_{-1}, W_0$, and V_0. In other words, find the coefficients $d_{k+1,n}$, $d_{k+2,n}$, $\cdots$ $d_{0,n}$ and $c_{0,n}$, $n \in \mathbb{Z}$ that lead to the following decomposition

$$f(t) = \sum_{n \in \mathbb{Z}} c_{0,n}\varphi_{0,n}(t) + \sum_{j=k+1}^{0} \sum_{n \in \mathbb{Z}} d_{j,n}\psi_{j,n}(t).$$

[6]

(c) Sketch and dimension each of the decompositions of part (b). That is, sketch

$$f_{V_0}(t) = \sum_{n \in \mathbb{Z}} c_{0,n}\varphi_{0,n}(t)$$

and the components

$$f_{W_j}(t) = \sum_{n \in \mathbb{Z}} d_{j,n}\psi_{j,n}(t) \qquad \text{for} \qquad j = k, k+1, ..., 0.$$

[7]

(d) Verify the Parseval equality. That is, verify that:

$$\|f(t)\|^2 = \sum_n |c_{0,n}|^2 + \sum_{j=k+1}^{0} \sum_n |d_{j,n}|^2.$$

[6]