THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. This is a question to check your understanding of basic principles. Most students got it right.

a) $P(X+Y \leq 0.5) = 3\int_{x=0}^{0.5}\int_{y=0}^{0.5-x}(x+y)dydx$ [2 - E]
$P(X+Y \leq 0.5) = \frac{3}{2}\int_{x=0}^{0.5}\frac{1}{4} - x^2 dx = \frac{1}{8}$ [2 - E]

b) $f_X(x) = \begin{cases} \int_0^{1-x} 3(x+y)dy, & 0 < x < 1, \\ 0, & otherwise. \end{cases}$ [2 - E]

$f_X(x) = \begin{cases} \frac{3}{2}(1-x^2), & 0 < x < 1, \\ 0, & otherwise. \end{cases}$ [2 - E]

c) $E(X) = \int_0^1 x\frac{3}{2}(1-x^2)dx = \frac{3}{8}$ [2 - E]
$E(X^2) = \int_0^1 x^2\frac{3}{2}(1-x^2)dx = \frac{1}{5}$ [1 - E]
$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{19}{320}$ [1 - E]

d) In view of the joint pdf and the domain, we can easily reuse b) to write
$f_Y(y) = \begin{cases} \frac{3}{2}(1-y^2), & 0 < y < 1, \\ 0, & otherwise. \end{cases}$ [2 - E]

e) We find the same values as in c).
$E(Y) = \frac{3}{8}$ [1 - E]
$\text{Var}(Y) = \frac{19}{320}$ [1 - E]

f) $E(XY) = \int_{x=0}^{1}\int_{y=0}^{1-x} xy3(x+y)dydx = \int_{x=0}^{1}\frac{x}{2}(2-3x+x^3)dx = \frac{1}{10}$
$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{10} - \frac{3}{8}\frac{3}{8} = \frac{-26}{640} = -0.0406$ [1 - E]
$\text{Corr}(X,Y) = \frac{-26/640}{19/320} = -\frac{13}{19} = -0.684$ [1 - E]

g) X and Y are correlated since $\text{Corr}(X,Y) \neq 0$. [1 - E]
Since they are correlated, they are also dependent. [1 - E]

h) $f_{Y|X}(y|x) = \begin{cases} \frac{3(x+y)}{\frac{3}{2}(1-x^2)}, & 0 < x < 1,\ 0 < y < 1,\ 0 < x+y < 1, \\ 0, & otherwise. \end{cases}$ [2 - E]

i) $E[Y|X=x] = \begin{cases} \int_{y=0}^{1-x} y\frac{2(x+y)}{(1-x^2)}dy & 0 < x < 1, \\ 0, & otherwise. \end{cases}$ [2 - E]

$$E[Y|X=x] = \begin{cases} \frac{(1-x)(2+x)}{3(1+x)} & 0 < x < 1, \\ 0, & otherwise. \end{cases}$$

[1 - E]

2. This is a more complicated question and a large portion of students had some troubles answering all subquestions correctly. The main difficulties are in a)i), a)ii) and b)i). Questions similar to a)ii) and b)i) were nevertheless covered in the lectures.

a) i) The first challenge is to write the left hand side of the inequality into a sum of RVs so as to make use of the CLT. The second challenge is then to apply CLT properly. The common mistake is to apply CLT directly on the product of RVs.

Let us take $Y_i = \log X_i$ and write the probability

$$P\left(\prod_{i=1}^{n} X_i \le e^a\right) = P\left(\sum_{i=1}^{n} Y_i \le a\right)$$

with $a = -\frac{n}{2} + 0.5\sqrt{n}$. [1 - A]

We can then use the CLT to approximate the probability as

$$P\left(\sum_{i=1}^{n} Y_i \le a\right) \approx P\left(Z \le \frac{a - nE(Y)}{\sqrt{n\mathrm{Var}(Y)}}\right)$$

with $Z \sim N(0,1)$. [2 - A]

We can compute $E(Y)$ and $\mathrm{Var}(Y)$ as follows.
$E(Y) = \int_0^1 2x \log x dx = \left[x^2 \log x\right]_0^1 - \int_0^1 x dx = -\frac{1}{2}$.
(Use integration by part with $u = \log x$ and $dv = 2xdx$. Use L'Hospital rule to compute $\lim_{x\to 0} x^2 \log x$).

[1 - A]

$E(Y^2) = \int_0^1 2x(\log x)^2 dx = \left[x^2(\log x)^2\right]_0^1 - \int_0^1 2x \log x dx = \frac{1}{2}$.
(Use integration by part with $u = (\log x)^2$ and $dv = 2xdx$. Use L'Hospital rule to compute $\lim_{x\to 0} x^2(\log x)^2$).
$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. [1 - A]

We finally get $P\left(Z \le \frac{a-nE(Y)}{\sqrt{n\mathrm{Var}(Y)}}\right) = P\left(Z \le \frac{-n/2+0.5\sqrt{n}+n/2}{\sqrt{n}/2}\right) = P(Z \le 1) =$ 0.841 from the tables. [2 - A]

ii) This is similar to what we have done in the lecture. We use change of variables by introducing another RV V, compute the joint pdf of U and V and then the marginal of U. The common mistake is to write the pdf of U as being equal to the product of the marginals of X_1 and X_2. The joint pdf of X_1 and X_2 is equal to the product of their marginals (because of independence) but the pdf of $U = X_1X_2$ is certainly not the same as the joint pdf of X_1 and X_2.

We can make the change of variables $U = X_1X_2$ and $V = X_1$. [1 - A]

We can compute the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial X_1}{\partial U} & \frac{\partial X_1}{\partial V} \\ \frac{\partial X_2}{\partial U} & \frac{\partial X_2}{\partial V} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{V} & \frac{-U}{V^2} \end{bmatrix}$$

and make use of the change of variables relationship

$$f_{U,V}(u,v) = |\det J|\, f_{X_1,X_2}(x_1,x_2).$$

[2 - A]

We first get the joint pdf of X_1,X_2. Since X_1 and X_2 are independent, we have

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} (2x_1)(2x_2), & 0 < x_1 < 1,\ 0 < x_2 < 1, \\ 0, & otherwise. \end{cases}$$

Since $X_1 = V$ and $X_2 = \frac{U}{V}$, we get from the change of variables relationship

$$f_{U,V}(u,v) = \begin{cases} \left(\frac{1}{v}\right)(2v)\left(2\frac{u}{v}\right) = 4\frac{u}{v}, & 0 < v < 1,\ 0 < u < v, \\ 0, & otherwise. \end{cases}$$

Note the change of variables implies that the new domain is characterized by $0 < v < 1$ and $0 < u < v$ (Some mistakes were done on identifying the new domain). [2 - A]

The probability density function of U is finally obtained as

$$f_U(u) = \begin{cases} \int_u^1 4\frac{u}{v}dv = -4u\log u, & 0 < u < 1, \\ 0, & otherwise. \end{cases}$$

[1 - A]

iii) $E(U) = \int_0^1 u(-4u\log u)du = \left[-4(u^3/3\log u - u^3/9)\right] = \frac{4}{9}$. We could have obtained the same result by noting that since X_1 and X_2 are independent, $E(U) = E(X_1)E(X_2)$. [2 - A]

b) i) This was not well answered even though this is very similar to what we have done in the lectures and what is in the notes. In the notes, there is an example with $Y = X^2$. The difference here is that $Y = X^4$.

X and Y are clearly not independent since $Y = X^4$, i.e. if we know X, we also know Y. [2 - E]

To see whether X and Y are uncorrelated, we compute $\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = E(X^5) - E(X)E(X^4) = E(X^5)$ since $E(X) = 0$. Since the distribution of X is symmetric around 0, $E(X^5) = 0$ and $\text{Cov}(X,Y) = \text{Corr}(X,Y) = 0$. X and Y are not independent but are uncorrelated. [3 - E]

ii) No, it is incorrect. The correct statement is: If X is a continuous random variable with first moment m_1 and second moment m_2, then we have $m_1^2 \leq m_2$. [2 - B]

Recall that $\text{Var}(X) = E\left[(X - m_1)^2\right] \geq 0$. Moreover $\text{Var}(X) = m_2 - m_1^2$. Hence $m_2 - m_1^2 \geq 0$ and $m_1^2 \leq m_2$. [3 - B]