**Information for students**

*Each of the four questions has 25 marks.*

# The Questions

1. Random variables.

a) Let the joint probability density function of $X$ and $Y$ be given by

$$f_{XY}(x,y) = \begin{cases} e^{-x} & 0 < y \leq x \leq \infty \\ 0 & \text{otherwise} \end{cases}$$

Define $Z = X + Y, W = X - Y$.

i) Find the joint probability density function of $Z$ and $W$. [4]

ii) Derive the probability density function of $Z$. [3]

b) There are 17 fenceposts around the perimeter of a field, exactly 5 of which are rotten. Show that irrespective of which these 5 are, there necessarily exists a run of 7 consecutive posts at least 3 of which are rotten.

[6]

c) Let random variable $X$ take non-negative integer values.

i) Show that

$$E[X] = \sum_{n=0}^{\infty} P(X > n) \qquad (1.1)$$

[6]

ii) An urn contains 10 red and 10 blue balls. Draw the balls uniformly at random until the first red ball is seen. Using the formula (1.1) given above, find the expected number of draws (until the first red ball is seen). You may use another approach if you wish. [6]

  

2. Estimation.

a) A random variable $X$ has a Poisson distribution with parameter $\lambda = 1$

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \;\; k = 0,1,2,\ldots$$

We wish to estimate $\lambda$ from $n$ = 10 i.i.d. samples of $X$. Compute the Cramer-Rao lower bound $\mathrm{Var}[\hat{\lambda}] = \frac{1}{I(\lambda)}$ where the Fisher information

$$I(\lambda) = -E\left[\frac{\partial^2}{\partial\lambda^2}\ln f_X(x_1,\cdots,x_n;\lambda)\right].$$

[10]

b) Consider the Rayleigh fading channel in wireless communications, where the channel gains $Y(n)$ have autocorrelation function

$$R_Y(n) = J_0(2\pi f_d n)$$

where $J_0$ denotes the zeroth-order Bessel function of the first kind, and $f_d$ represents the normalized Doppler frequency shift. Suppose we wish to predict $Y(n+1)$ from $Y(n)$, $Y(n-1)$, …, $Y(1)$ using the linear MMSE estimator

$$Y(n+1) = \sum_{i=1}^{n} c_i Y(i)$$

Let $f_d$ = 0.3 and given the following values of $J_0$ for $f_d$ = 0.3:

$$J_0(2\pi f_d n) = \begin{cases} 1 & n = 0 \\ 0.291 & n = 1 \\ -0.402 & n = 2 \end{cases}$$

i) Compute the coefficient and mean-square error of the first-order linear MMSE estimator, i.e., $n$ = 1. [6]

ii) Compute the coefficients and mean-square error of the second-order linear MMSE estimator, i.e., $n = 2$. [9]

3. Random processes.

a) Consider the random process $X(n) = A\cos(n\lambda) + B\sin(n\lambda)$ for parameter $\lambda > 0$, where $A$ and $B$ are uncorrelated random variables with zero means and unit variances.

i) Find the mean and autocorrelation function of $X(n)$. [4]

ii) Is $\{X(n)\}$ wide-sense stationary? Justify your answer. [2]

iii) Is $\{X(n)\}$ strict-sense stationary? Justify your answer. [4]

b) Let $Z_1, Z_2, \ldots$ be uncorrelated random variables with zero mean and unit variance. You may assume the period $T = 1$ without loss of generality.

i) Define the moving average process $X_n = Z_n + \alpha Z_{n-1}$ where $\alpha$ is a constant. Find the power spectrum density of $\{X_n\}$. [6]

ii) More generally, let $Y_n = \sum_{i=0}^{r} \alpha_i Z_{n-i}$ where $\alpha_0 = 1$ and $\alpha_1, \ldots, \alpha_r$ are constants. Find the power spectrum density of $\{Y_n\}$. [9]

4. Markov chains and martingales.

a) Calculate the stationary distribution for a Markov chain with state space $E = \{1, 2, 3, \ldots, M\}$, whose transitional probability matrix is

$$\begin{pmatrix} 3/4 & 1/4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 1/2 & 1/4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1/4 & 1/2 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1/4 & 3/4 \end{pmatrix}$$

[5]

b) Let $X_1, X_2, \ldots$ be independent random variables with zero means and finite variances. Define $S_n = \sum_{i=1}^{n} X_i$ and $T_n = S_n^2 = (\sum_{i=1}^{n} X_i)^2$.

i) Is $\{S_n\}$ a martingale? Justify your answer. [3]

ii) Is $\{T_n\}$ a martingale? Justify your answer. [4]

iii) Is $\{T_n\}$ a super or submartingale? Justify your answer. [3]

c) Consider a symmetric random walk on the three-dimensional grid $\{(x, y, z): x, y, z \in 0, \pm 1, \pm 2, \ldots\}$. This Markov chain is a sequence $\{X_n\}$ where $P\{X_{n+1} = X_n + \epsilon\} = \frac{1}{6}$ where the vector $\epsilon \in \{(\pm 1,0,0), (0, \pm 1, 0), (0,0, \pm 1)\}$. Suppose the chain starts at the origin $X_0 = (0,0,0)$.

i) Derive the probability $P\{X_{2n} = (0,0,0)\}$. [4]

ii) Using the Stirling formula $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$, show that the origin is a transient state. [6]

Hint: Being transient requires $\sum_n P\{X_{2n} = (0,0,0)\} < \infty$.