Information for students

Each of the four questions has 25 marks.

The Questions

1. Random variables.

a) The random variable X is uniformly distributed in the interval $[-\pi, \pi]$. Find the probability density function of the following random variables

i) $Y = X^3$ [3]

ii) $Y = X^4$ [3]

iii) $Y = \sin(X)$ [4]

b) X and Y are independent, identically distributed (i.i.d.) random variables with common probability density function

$$f_X(x) = e^{-x}, \qquad x > 0$$

$$f_Y(y) = e^{-y}, \qquad y > 0$$

Find the probability density function of the following random variables:

i) $Z = 2X + Y$. [5]

ii) $Z = \min(X, Y)$. [5]

iii) $Z = \max(X, Y)$. [5]

2. Estimation.

a) The random variable X has the truncated exponential density $f(x) = ce^{-c(x-x_0)}, \; x > x_0$, and $f(x) = 0$ otherwise. Let $x_0 = 5$. We observe the i.i.d. samples $x_i = 7, 8, 9, 10, 11$. Find the maximum-likelihood estimate of parameter c.

[8]

b) Consider the Rayleigh fading channel in wireless communications, where the channel gains $Y(n)$ has autocorrelation function

$$R_Y(n) = J_0(2\pi f_d n)$$

where J_0 denotes the zeroth-order Bessel function of the first kind, and f_d represents the normalized Doppler frequency shift. Suppose we wish to predict $Y(n+1)$ from $Y(n)$, $Y(n-1)$, …, $Y(1)$ using the linear MMSE estimator

$$Y(n+1) = \sum_{i=1}^{n} c_i Y(i)$$

Let $f_d = 0.2$ and given the following values of J_0 for $f_d = 0.2$:

$$J_0(2\pi f_d n) = \begin{cases} 1 & n = 0 \\ 0.643 & n = 1 \\ -0.055 & n = 2 \end{cases}$$

i) Compute the coefficient and mean-square error of the first-order linear MMSE estimator, i.e., $n = 1$. [7]

ii) Compute the coefficients and mean-square error of the second-order linear MMSE estimator, i.e., $n = 2$. [10]

3. Random processes.

a) Consider the random process $X(n) = \cos(nU)$ for $n \geq 1$, where U is uniformly distributed on interval $[-\pi, \pi]$.

i) Show that $\{X(n)\}$ is wide-sense stationary. [5]

ii) Show that $\{X(n)\}$ is not strict-sense stationary. [5]

b) The number of patients $N(t)$ arriving at the doctor's office over the time interval $[0, t)$ can be modelled by a Poisson process $\{N(t), t \geq 0\}$. On the average, there is a new patient arriving after every 10 minutes, i.e., the intensity of the process is equal to $\lambda = 0.1$. The doctor will not see a patient until at least three patients are in the waiting room.

i) Find the expected waiting time until the first patient is admitted to see the doctor.

[3]

ii) What is the probability that nobody is admitted to see the doctor in the first hour?

[6]

iii) What is the probability that at least two patient arrive in the first hour while at most two patients arrive in the second hour?

[6]

4. Markov chains and martingales.

a) Let $\{X_n\}$ be a Markov chain and let $\{n_r\colon r \geq 0\}$ be an unbounded increasing sequence of positive integers.

i) Show that $\{Y_r = X_{n_r}\}$ constitutes a (possibly inhomogeneous) Markov chain.

[4]

ii) Find the transition matrix of $\{Y_r\}$ when $n_r = 2r$ and $\{X_n\}$ has transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

[3]

b) Consider the gambler's ruin with state space $E = \{0,1,2,\ldots,N\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & & & 0 & 0 \\ q & 0 & p & & & 0 \\ & q & 0 & p & & \\ & & . & . & . & \\ 0 & & & q & 0 & p \\ 0 & 0 & & & 0 & 1 \end{pmatrix}$$

where $0 < p < 1$, $q = 1 - p$. This Markov chain models a gamble where the gambler wins with probability p and loses with probability q at each step. Reaching state 0 corresponds to the gambler's ruin.

i) Denote by S_n the gambler's capital at step n. Show that $Y_n = \left(\frac{q}{p}\right)^{S_n}$ is a martingale (known as DeMoivre's martingale). [4]

ii) Using the theory of stopping time, derive the ruin probability for initial capital i $(0 < i < N)$. [4]

c) Derive the average duration T_i of the game for the gambler starting from state i.

[Hint: Show that T_k satisfies the iteration $T_k = 1 + pT_{k+1} + qT_{k-1}$ under the initial conditions $T_0 = T_N = 0$.]

[10]