6: Window Filter Design

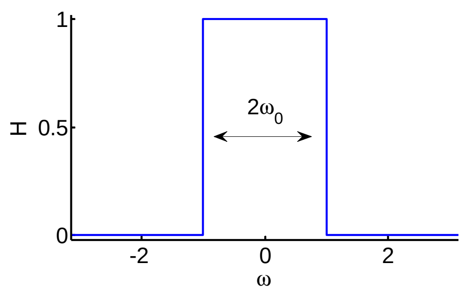
For any BIBO stable filter, $H(e^{j\omega})$ is the DTFT of $h[n]$

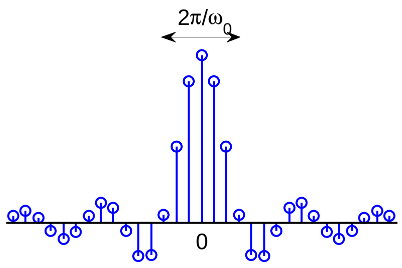
$$H(e^{j\omega}) = \sum_{-\infty}^{\infty} h[n]e^{-j\omega n} \quad \Leftrightarrow \quad h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

If we know $H(e^{j\omega})$ exactly, the IDTFT gives the ideal $h[n]$

Example: Ideal Lowpass filter

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases} \quad \Leftrightarrow \quad h[n] = \frac{\sin \omega_0 n}{\pi n}$$





Note: Width in ω is $2\omega_0$, width in n is $\frac{2\pi}{\omega_0}$: product is 4π always
Sadly $h[n]$ is infinite and non-causal. Solution: multiply $h[n]$ by a window

Rectangular window

Truncate to $\pm\frac{M}{2}$ to make finite; $h_1[n]$ is now of length $M+1$

MSE Optimality:

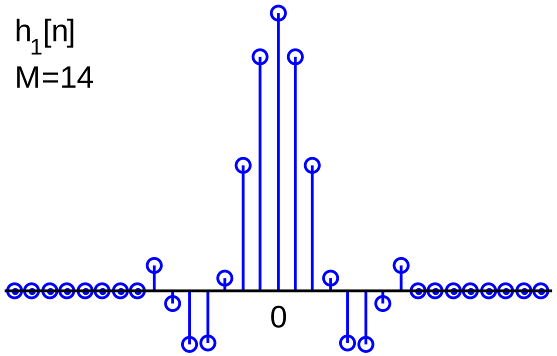
Define mean square error (MSE) in frequency domain

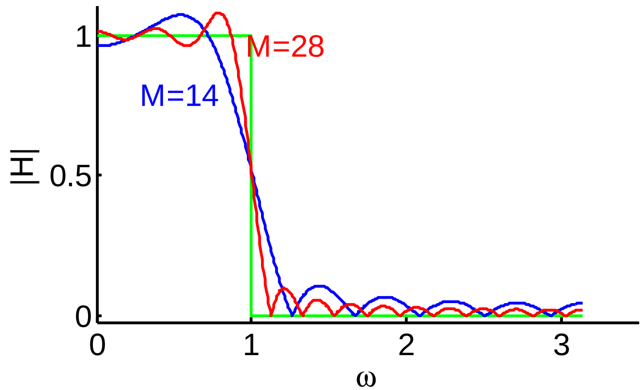
$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H(e^{j\omega}) - H_1(e^{j\omega})\right|^2 d\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H(e^{j\omega}) - \sum_{-\frac{M}{2}}^{\frac{M}{2}} h_1[n]e^{-j\omega n}\right|^2 d\omega$$

Minimum E is when $h_1[n] = h[n]$.

Proof: From Parseval: $E = \sum_{-\frac{M}{2}}^{\frac{M}{2}} \left|h[n] - h_1[n]\right|^2 + \sum_{|n|>\frac{M}{2}} \left|h[n]\right|^2$

However: 9% overshoot at a discontinuity even for large n.





Normal to delay by $\frac{M}{2}$ to make causal. Multiplies $H(e^{j\omega})$ by $e^{-j\frac{M}{2}\omega}$.

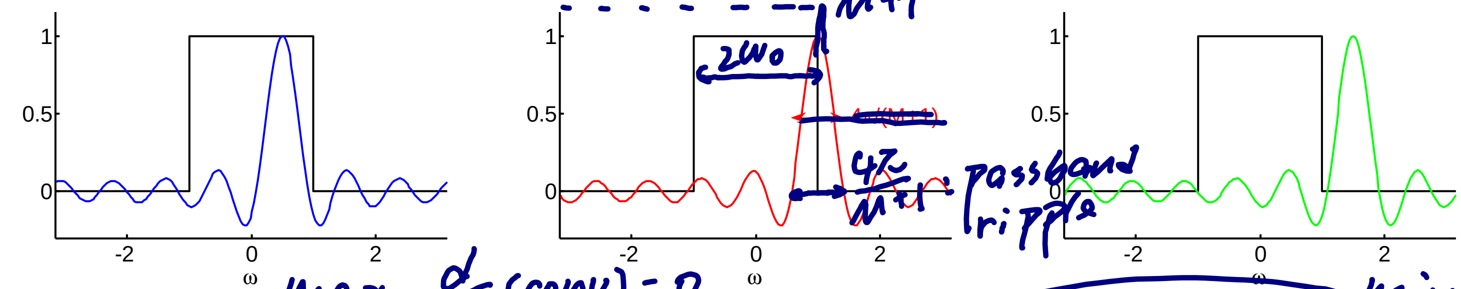
Truncation $\Leftrightarrow$ Multiply $h[n]$ by a rectangular window, $w[n] = \delta_{-\frac{M}{2} \le n \le \frac{M}{2}}$

$\Leftrightarrow$ Circular Convolution $H_{M+1}(e^{j\omega}) = \frac{1}{2\pi} H(e^{j\omega}) \circledast W(e^{j\omega})$

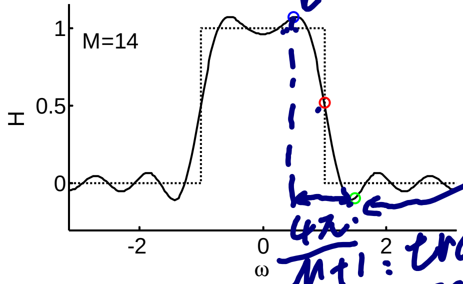
$$W(e^{j\omega}) = \sum_{-\frac{M}{2}}^{\frac{M}{2}} e^{-j\omega n} \overset{\text{(i)}}{=} 1 + 2\sum_{1}^{0.5M} \cos(n\omega) \overset{\text{(ii)}}{=} \frac{\sin 0.5(M+1)\omega}{\sin 0.5\omega}$$

Proof: (i) $e^{-j\omega(-n)} + e^{-j\omega(+n)} = 2\cos(n\omega)$ (ii) Sum geom. progression

Effect: convolve ideal freq response with Dirichlet kernel (aliassed sinc)



max: $\frac{d}{d\omega}$(conv) = 0



$\frac{4\pi}{M+1}$: transition peak-to-peak

min

Provided that $\frac{4\pi}{M+1} \ll 2\omega_0 \Leftrightarrow M+1 \gg \frac{2\pi}{\omega_0}$:

Passband ripple: $\Delta\omega \approx \frac{4\pi}{M+1}$, stopband $\frac{2\pi}{M+1}$ $M\uparrow \Rightarrow \Delta\omega\downarrow$

(width of mainlobe)

Transition pk-to-pk: $\Delta\omega \approx \frac{4\pi}{M+1}$

(max-to-min width)

Transition Gradient: $\left.\frac{d|H|}{d\omega}\right|_{\omega=\omega_0} \approx \frac{M+1}{2\pi}$

[Dirichlet Kernel]

$\frac{dH_{M+1}(e^{j\omega})}{d\omega} = \frac{1}{2\pi}\frac{d(H(e^{j\omega}))}{d\omega} \circledast W(e^{j\omega})$

$= \frac{1}{2\pi}(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)) \circledast W(e^{j\omega})$

$= \frac{1}{2\pi}[W(e^{j\omega_0}) + W(e^{-j\omega_0})]$

Other properties of $W(e^{j\omega})$:

The DTFT of a symmetric rectangular window of length $M+1$ is $W(e^{j\omega}) = \sum_{-\frac{M}{2}}^{\frac{M}{2}} e^{-j\omega n} = e^{j\omega\frac{M}{2}} \sum_0^M e^{-j\omega n} = e^{j\omega\frac{M}{2}} \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}} = \frac{e^{j0.5\omega(M+1)}-e^{-j0.5\omega(M+1)}}{e^{j0.5\omega}-e^{-j0.5\omega}} = \frac{\sin 0.5(M+1)\omega}{\sin 0.5\omega}$.

For small x we can approximate $\sin x \approx x$; the error is $< 1\%$ for $x < 0.25$. So, for $\omega < 0.5$, we have $W(e^{j\omega}) \approx 2\omega^{-1}\sin 0.5(M+1)\omega$. $= \frac{2\sin\frac{M+1}{2}\omega}{\omega} \overset{\omega\to 0}{\approx} M+1$

The peak value is at $\omega = 0$ and equals $M+1$; this means that the peak gradient of $H_{M+1}(e^{j\omega})$ will be $\frac{M+1}{2\pi}$.

$\xrightarrow{\min} -\frac{M+1}{1.5\pi}$ $(\omega = \frac{3\pi}{M+1})$ (M is ~50)

The minimum value of $W(e^{j\omega})$ is approximately equal to the minimuum of $2\omega^{-1}\sin 0.5(M+1)\omega$ which is when $\sin 0.5(M+1)\omega = -1$ i.e. when $\omega = \frac{1.5\pi}{0.5(M+1)} = \frac{3\pi}{M+1}$.

Hence $\min W(e^{j\omega}) \approx \min 2\omega^{-1}\sin 0.5(M+1)\omega = -\frac{M+1}{1.5\pi}$.

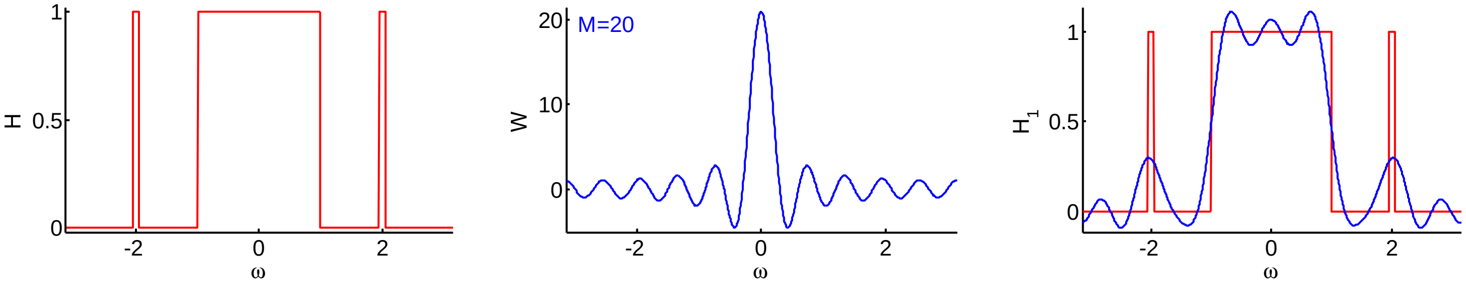
Passband and Stopband ripple:

The ripple in $W(e^{j\omega}) = \frac{\sin 0.5(+1)\omega}{\sin 0.5\omega}$ has a period of $\Delta\omega = \frac{2\pi}{0.5(+1)} = \frac{4\pi}{M+1}$ and this gives rise to ripple with this period in both the passband and stopband of $H_{M+1}(e^{j\omega})$.

However the stopband ripple takes the value of $H_{M+1}(e^{j\omega})$ alternately positive and negative. If you plot the magnitude response, $\left|H_{M+1}(e^{j\omega})\right|$ then this ripple will be full-wave rectified and will double in frequency so its period will now be $\frac{2\pi}{M+1}$.

When you multiply an impulse response by a window $M+1$ long

$$H_{M+1}(e^{j\omega}) = \frac{1}{2\pi} H(e^{j\omega}) \circledast W(e^{j\omega})$$



(a) passband gain $\approx w[0]$; peak$\approx \frac{w[0]}{2} + \frac{0.5}{2\pi}\int_{\text{mainlobe}} W(e^{j\omega})d\omega$
rectangular window: passband gain $= 1$; peak gain $= 1.09$

(b) transition bandwidth, $\Delta\omega =$ width of the main lobe
transition amplitude, $\Delta H =$ integral of main lobe$\div 2\pi$
rectangular window: $\Delta\omega = \frac{4\pi}{M+1}$, $\Delta H \approx 1.18$

(c) stopband gain is an integral over oscillating sidelobes of $W(e^{j\omega})$
rect window: $\left|\min H(e^{j\omega})\right| = 0.09 \ll \left|\min W(e^{j\omega})\right| = \frac{M+1}{1.5\pi}$

(d) features narrower than the main lobe will be broadened and attenuated

Common Windows

Rectangular: $w[n] \equiv 1$
don't use

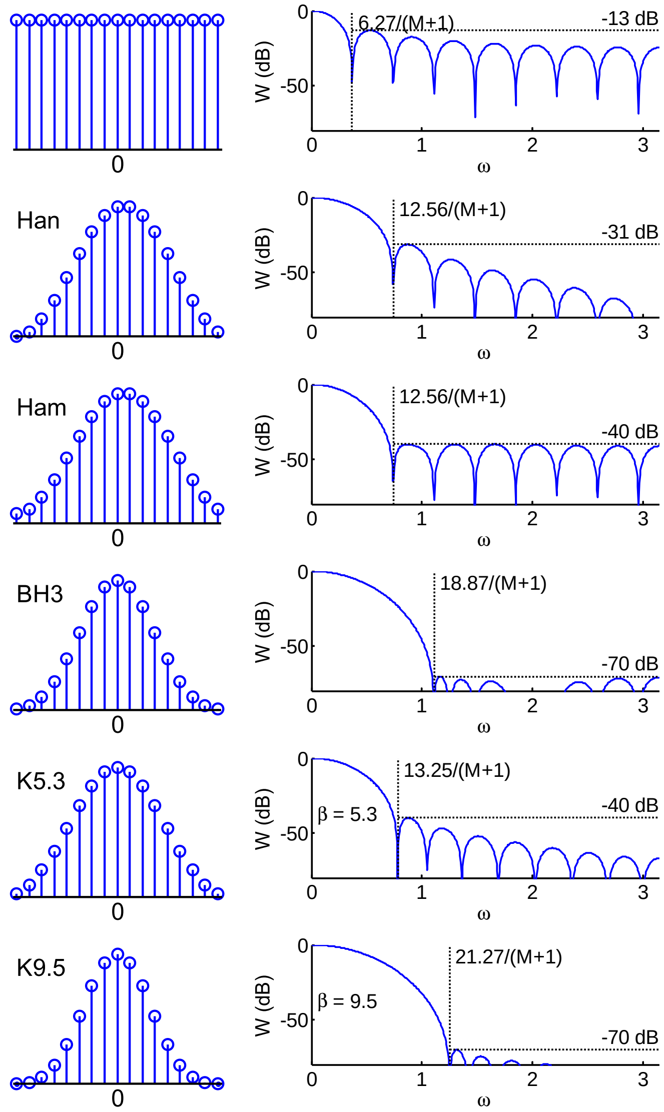
Hanning: $0.5 + 0.5c_1$
$c_k = \cos \frac{2\pi kn}{M+1}$
rapid sidelobe decay

Hamming: $0.54 + 0.46c_1$
best peak sidelobe

Blackman-Harris 3-term:
$$0.42 + 0.5c_1 + 0.08c_2$$
best peak sidelobe

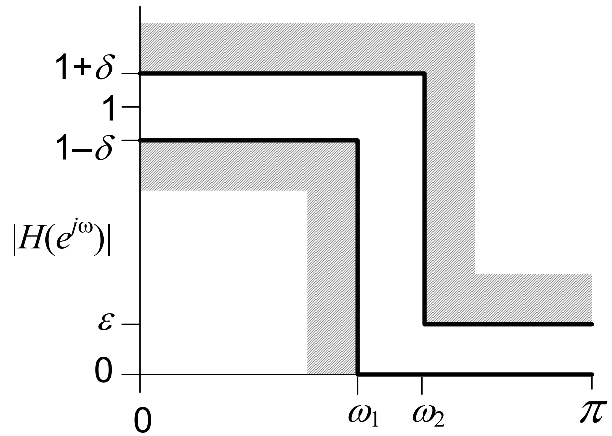
Kaiser: $\frac{I_0\left(\beta\sqrt{1-\left(\frac{2n}{M}\right)^2}\right)}{I_0(\beta)}$

β controls width v sidelobes
Good compromise:
Width v sidelobe v decay



Several formulae estimate the required order of a filter, M.

E.g. for lowpass filter



Estimated order is

$$M \approx \frac{-5.6-4.3\log_{10}(\delta\epsilon)}{\omega_2-\omega_1} \approx \frac{-8-20\log_{10}\epsilon}{2.2\Delta\omega}$$

Required M increases as either the transition width, $\omega_2-\omega_1$, or the gain tolerances δ and ϵ get smaller.

Only approximate.

Example:

Transition band: $f_1 = 1.8$ kHz, $f_2 = 2.0$ kHz, $f_s = 12$ kHz,.

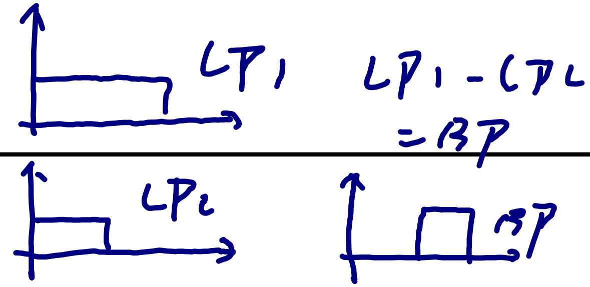
$$\omega_1 = \frac{2\pi f_1}{f_s} = 0.943,\ \omega_2 = \frac{2\pi f_2}{f_s} = 1.047$$

Ripple: $20\log_{10}(1+\delta) = 0.1$ dB, $20\log_{10}\epsilon = -35$ dB

$$\delta = 10^{\frac{0.1}{20}} - 1 = 0.0116,\ \epsilon = 10^{\frac{-35}{20}} = 0.0178$$

$$M \approx \frac{-5.6-4.3\log_{10}\left(2\times10^{-4}\right)}{1.047-0.943} = \frac{10.25}{0.105} = 98 \qquad \text{or } \frac{35-8}{2.2\Delta\omega} = 117$$

Example Design



Specifications:

Bandpass: $\omega_1 = 0.5$, $\omega_2 = 1$

Transition bandwidth: $\Delta\omega = 0.1$

Ripple: $\delta = \epsilon = 0.02$

$20 \log_{10} \epsilon = -34$ dB

$20 \log_{10} (1 + \delta) = 0.17$ dB

Order:

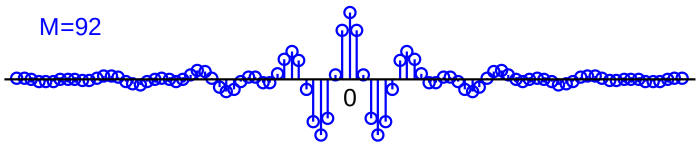
$$M \approx \frac{-5.6 - 4.3 \log_{10}(\delta\epsilon)}{\omega_2 - \omega_1} = 92$$

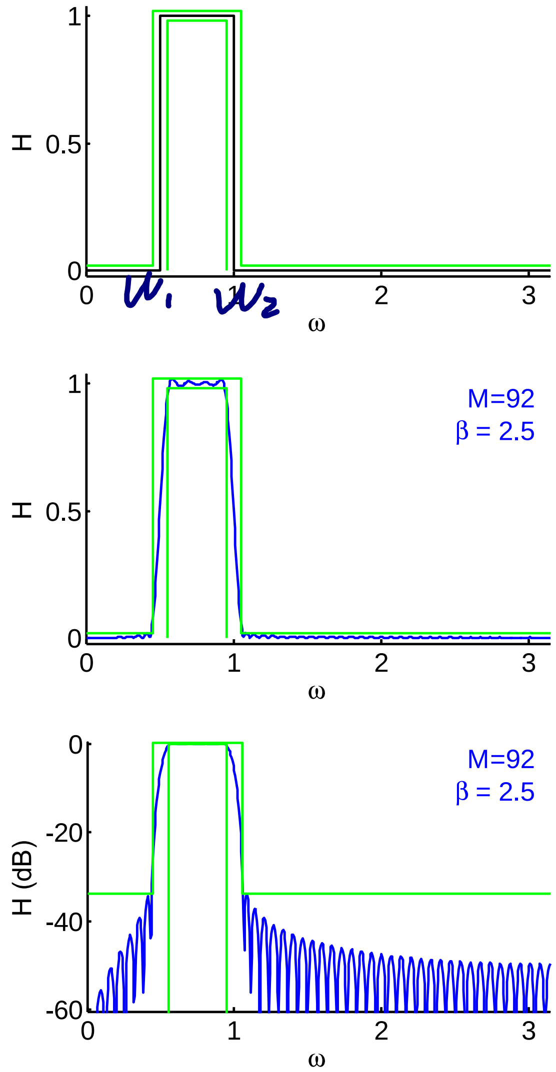
Ideal Impulse Response:

Difference of two lowpass filters

$$h[n] = \frac{\sin \omega_2 n}{\pi n} - \frac{\sin \omega_1 n}{\pi n}$$

Kaiser Window: $\beta = 2.5$





Take $M+1$ uniform samples of $H(e^{j\omega})$; take IDFT to obtain $h[n]$

Advantage:

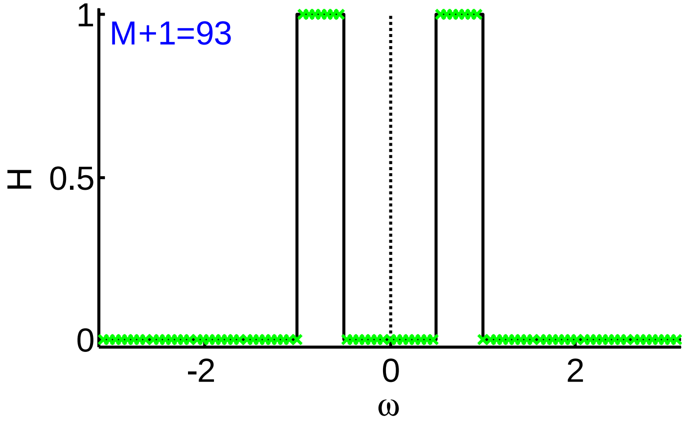
exact match at sample points

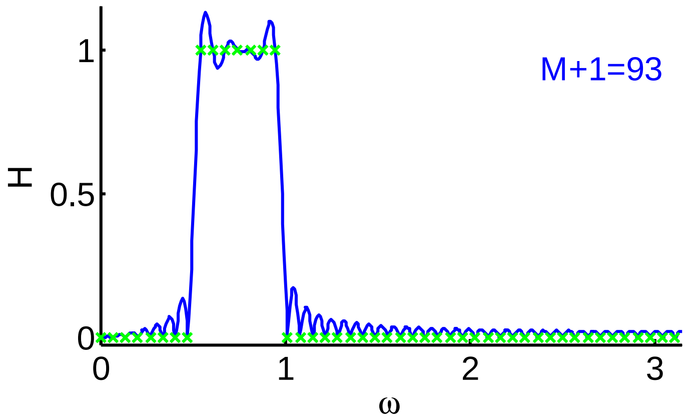
Disadvantage:

poor intermediate approximation if spectrum is varying rapidly

Solutions:

(1) make the filter transitions smooth over $\Delta\omega$ width
(2) oversample and do least squares fit (can't use IDFT)
(3) use non-uniform points with more near transition (can't use IDFT)





- Make an FIR filter by windowing the IDTFT of the ideal response
  - Ideal lowpass has $h[n] = \frac{\sin \omega_0 n}{\pi n}$
  - Add/subtract lowpass filters to make any piecewise constant response
- Ideal filter response is $\circledast$ with the DTFT of the window
  - Rectangular window ($W(z) =$ Dirichlet kernel) has -13 dB sidelobes and is always a bad idea
  - Hamming, Blackman-Harris are good
  - Kaiser good with β trading off main lobe width v. sidelobes
- Uncertainty principle: cannot be concentrated in both time and frequency
- Frequency sampling: IDFT of uniform frequency samples: not so great

For further details see Mitra: 7, 10.

MATLAB routines

diric(x,n)	Dirichlet kernel: $\frac{\sin 0.5nx}{\sin 0.5x}$
hanning hamming kaiser	Window functions (Note 'periodic' option)
kaiserord	Estimate required filter order and β