THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. a) i) The CDF is given by $F_X(x) = \int_{-\infty}^{x} f_X(x)dx$ which leads to

$$F_X(x) = \begin{cases} 0, & x \leq 0, \\ 2x - x^2, & 0 < x < 1, \\ 1, & x \geq 1. \end{cases}$$

[2 - A]

ii) $E(X) = \int_{-\infty}^{\infty} x f_X(x)dx$. We get $E(X) = \int_0^1 x2(1-x)dx = 1/3$.

[2 - A]

$\mathrm{Var}(X) = E(X^2) - E(X)^2$. $E(X^2) = 1/6$. So $\mathrm{Var}(X) = 1/6 - 1/9 = 1/18$.

[2 - A]

iii) We write $m_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$. By integration by part,

$$m_X(t) = \begin{cases} \frac{2e^t}{t^2} - \frac{2}{t^2} - \frac{2}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

[1 - A]

We can compute $E(X) = m_X'(0)$ and $E(X^2) = m_X''(0)$.

[1 - A]

We get $m_X'(t) = \frac{2e^t t - 4e^t + 4 + 2t}{t^3}$. Applying L'hospital rule 3 times, we get $E(X) = 1/3$.

[1 - A]

Similarly $m_X'(t) = \frac{2e^t t^2 - 8te^t + 12e^t - 12 - 4t}{t^4}$. Applying L'hospital rule 4 times, we get $E(X) = 1/6$ such that $\mathrm{Var}(X) = 1/6 - 1/9 = 1/18$.

[1 - A]

iv) By Chebyshev's inequality $P\left(\left|X - \frac{1}{3}\right| \geq \frac{1}{4}\right) \leq \frac{1}{1/16} E\left[(X - 1/3)^2\right] = 16\mathrm{Var}(X) = 8/9$.

[2 - A]

The exact value can be computed as follows

$$
\begin{aligned}
P\left(\left|X-\frac{1}{3}\right| \geq \frac{1}{4}\right) &= 1-P\left(\left|X-\frac{1}{3}\right| \leq \frac{1}{4}\right) \\
&= 1-P\left(-\frac{1}{4} \leq X-\frac{1}{3} \leq \frac{1}{4}\right) \\
&= 1-P\left(\frac{1}{12} \leq X \leq \frac{7}{12}\right) \\
&= 1-F_X(\frac{7}{12})+F_X(\frac{1}{12}) \\
&= \frac{1}{3}
\end{aligned}
$$

[2 - A]

b) i) By the method of moments, we aim at finding the estimator by equating the sample mean with corresponding population mean, i.e. $E(X)=\overline{X}$.

[2 - A]

We can compute

$$E(X)=\int_0^\theta \frac{2x^2}{\theta^2}dx=\frac{2}{3}\theta.$$

We choose $\tilde{\theta}$ as $\frac{2}{3}\tilde{\theta}=\overline{X}$, i.e. $\tilde{\theta}=\frac{3}{2}\overline{X}$.

[2 - A]

ii) Expectation

$$E(\tilde{\theta})=E(\frac{3}{2}\overline{X})=\frac{3}{2}E(\overline{X})=\frac{3}{2}E(X)=\theta.$$

[2 - A]

Variance

$$\mathrm{Var}(\tilde{\theta})=\mathrm{Var}(\frac{3}{2}\overline{X})=\frac{9}{4}\mathrm{Var}(\overline{X})=\frac{9}{4}\frac{\mathrm{Var}(X)}{n}.$$

We can compute $E(X^2)=\int_0^\theta \frac{2x^3}{\theta^2}dx=\frac{1}{2}\theta^2$ such that $\mathrm{Var}(X)=\frac{1}{2}\theta^2-\frac{4}{9}\theta^2=\frac{1}{18}\theta^2$. This leads to $\mathrm{Var}(\tilde{\theta})=\frac{\theta^2}{8n}$.

[2 - A]

iii) Since $E(\tilde{\theta})=\theta$, the estimator is unbiased.

[4 - A]

iv) For large n, we can make use of the Central Limit Theorem and write

$$
\begin{aligned}
P\left(\tilde{\theta} \geq \theta\right) &= P(\frac{3}{2}\overline{X} \geq \theta)=P(\overline{X} \geq \frac{2}{3}\theta) \\
&= P(Z \geq \frac{\frac{2}{3}\theta-E(\overline{X})}{\sqrt{\mathrm{Var}(\overline{X})}})=P(Z \geq 0)=1/2
\end{aligned}
$$

where Z is a standard normal random variable.

[4 - A]

2. a) We need to compute $E(X)$. We can first compute the marginal of X as

$$f_X(x) = \begin{cases} 2e^{-x}\int_x^\infty e^{-y}dy = 2e^{-2x}, & x > 0, \\ 0, & otherwise. \end{cases}$$

[2 - A]

This is an EXPO(2). Hence $E(X) = \frac{1}{2}$.

[2 - A]

b) We can first compute the Jacobian and write

$$\begin{vmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

[2 - A]

We then write

$$f_{U,V}(u,v) = \begin{cases} 2e^{-(u+2v)}, & u > 0, v > 0, \\ 0, & otherwise. \end{cases}$$

[2 - A]

c) The marginal are obtained by integration of the joint pdf as follows

$$f_U(u) = \begin{cases} e^{-u}, & u > 0, \\ 0, & otherwise. \end{cases}$$

$$f_V(v) = \begin{cases} 2e^{-2v}, & v > 0, \\ 0, & otherwise. \end{cases}$$

U is EXPO(1) and V is EXPO(2).

[2 - A]

d) Since $f_{U,V}(u,v) = f_U(u)f_V(v)$, U and V are two independent random variables.

[2 - A]

e) The conditional pdf $f_{U|V}(u|v)$ is given as

$$f_{U|V}(u|v) = f_U(u) = \begin{cases} e^{-u}, & u > 0, \\ 0, & otherwise. \end{cases}$$

[2 - A]

f) $E(U|V) = E(U) = 1$.

[2 - A]

g) $1 = E(U|V) = E(Y-X|X) = E(Y|X) - E(X|X) = E(Y|X) - X$. Hence $E(Y|X) = 1 + X$.

[2 - A]

h) $E(Y) = E_X E(Y|X) = E(1+X) = 1 + E(X) = 1 + \frac{1}{2} = \frac{3}{2}$.

[2 - A]