DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

*********** Solutions ***********

Information for Candidates:

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.

Abbreviations

BIBO	Bounded Input, Bounded Output
CTFT	Continuous-Time Fourier Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
FIR	Finite Impulse Response

IIR	Infinite Impulse Response
LTI	Linear Time-Invariant
MDCT	Modified Discrete Cosine Transform
PSD	Power Spectral Density
SNR	Signal-to-Noise Ratio

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

z:	$X(z)=\sum_{-\infty}^{\infty}x[n]z^{-n}$	$x[n]=\frac{1}{2\pi j}\oint X(z)z^{n-1}dz$
CTFT:	$X(j\Omega)=\int_{-\infty}^{\infty}x(t)e^{-j\Omega t}dt$	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\Omega)e^{j\Omega t}d\Omega$
DTFT:	$X(e^{j\omega})=\sum_{-\infty}^{\infty}x[n]e^{-j\omega n}$	$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$
DFT:	$X[k]=\sum_{0}^{N-1}x[n]e^{-j2\pi\frac{kn}{N}}$	$x[n]=\frac{1}{N}\sum_{0}^{N-1}X[k]e^{j2\pi\frac{kn}{N}}$
DCT:	$X[k]=\sum_{n=0}^{N-1}x[n]\cos\frac{2\pi(2n+1)k}{4N}$	$x[n]=\frac{X[0]}{N}+\frac{2}{N}\sum_{n=1}^{N-1}X[k]\cos\frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k]=\sum_{n=0}^{2N-1}x[n]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n]=\frac{1}{N}\sum_{0}^{N-1}X[k]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n]=x[n]*y[n]\triangleq\sum_{r=-\infty}^{\infty}x[r]y[n-r] \quad\Leftrightarrow\quad V\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$

$v[n]=x[n]y[n] \quad\Leftrightarrow\quad V\left(e^{j\omega}\right)=\frac{1}{2\pi}X\left(e^{j\omega}\right)\circledast Y\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\theta}\right)Y\left(e^{j(\omega-\theta)}\right)d\theta$

DFT: $v[n]=x[n]\circledast_N y[n]\triangleq\sum_{r=0}^{N-1}x[r]y[(n-r)_{\bmod N}] \quad\Leftrightarrow\quad V[k]=X[k]Y[k]$

$v[n]=x[n]y[n] \quad\Leftrightarrow\quad V[k]=\frac{1}{N}X[k]\circledast_N Y[k]\triangleq\frac{1}{N}\sum_{r=0}^{N-1}X[r]Y[(k-r)_{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega})=-\frac{d\angle H(e^{j\omega})}{d\omega}=\Re\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}}=\Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M\approx\frac{a}{3.5\Delta\omega}$
2. $M\approx\frac{a-8}{2.2\Delta\omega}$
3. $M\approx\frac{a-1.2-20\log_{10}b}{4.6\Delta\omega}$

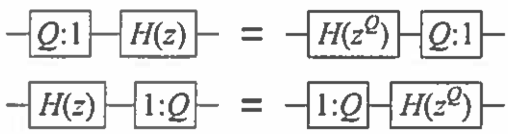
where a =stop band attenuation in dB, b = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in normalized rad/s.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$, $\rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$, $\rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities



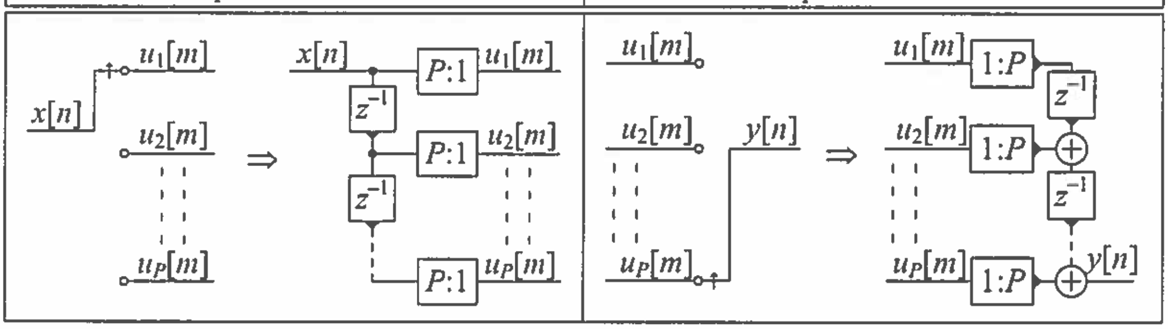
Multirate Spectra

Upsample $v[n]$ by Q: $x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases} \Rightarrow X(z) = V(z^Q)$

Downsample $v[n]$ by Q: $y[m] = v[Qm] \Rightarrow Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$

Multirate Commutators

Input Commutator	Output Commutator



********* Questions and Solutions *********

1. a) The finite length signals $u[0], \cdots, u[M-1]$ and $v[0], \cdots, v[N-1]$ are of length M and N respectively where $M < N$.

The signals $x[n] = u[n] * v[n]$ and $y[n] = u[n] \circledast_N v[n]$ are respectively the convolution and circular convolution of $u[n]$ and $v[n]$ as defined in the data sheet.

i) Prove that $y[n] = x[n]$ for $M-1 \leq n \leq N-1$. [3]

From the data sheet $y[n] = \sum_{r=0}^{N-1} u[r]v[(n-r)_{\bmod N}]$. Since $u[r] = 0$ for $r > M-1$, we can change the upper summation limit $M-1$.

For n in the range $M-1 \leq n \leq N-1$, the unwrapped index of $v[]$, $n-r$, therefore ranges from a minimum of n ranges from a minimum of $\min(n) - \max(r) = 0$ to a maximum of $\max(n) - \min(r) = N-1$. For this entire range $(n-r)_{\bmod N} = n-r$, so we may write $y[n] = \sum_{r=0}^{M-1} u[r]v[n-r]$. Since $u[r] = 0$ outside $0 \leq r \leq M-1$, we may expand the summation range as $y[n] = \sum_{r=-\infty}^{\infty} u[r]v[n-r] \triangleq x[n]$.

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ii) Determine an expression for $y[n]$ in terms of the $\{x[n]\}$ that is valid for $0 \leq n \leq M-2$. [2]

From the answer to part i), $y[n] = \sum_{r=0}^{M-1} u[r]v[(n-r)_{\bmod N}]$. We can split the summation up into two parts $y[n] = \sum_{r=0}^{n} u[r]v[(n-r)_{\bmod N}] + \sum_{r=n+1}^{M-1} u[r]v[(n-r)_{\bmod N}]$ and for n in the range $0 \leq n \leq M-2$ both summations include at least one term. For the first summation, $n-r$ is always ≥ 0 since $r \leq n$ and so it follows that $(n-r)_{\bmod N} = n-r$. For the second summation, $n-r$ ranges from a minimum of $\min(n) - \max(r) = -(M-1)$ to a maximum of -1 since $r > n$ always. For this range, $(n-r)_{\bmod N} = n-r+N$ since $M < N$ implies that $\min(n-r) = -(M-1) > -N$. Thus we can write $y[n] = \sum_{r=0}^{n} u[r]v[n-r] + \sum_{r=n+1}^{M-1} u[r]v[n-r+N]$. The first term equals $x[n]$ since for r outside the summing range, either $u[r]$ or $v[n-r]$ is zero. The second term equals $x[n+N] = \sum_{r=n+1}^{M-1} u[r]v[n+N-r]$ for the same reason. Thus $y[n] = x[n] + x[n+N]$. This result may also be determined graphically by considering the overlap between $v[n]$ and a time-reversed, time-shifted version of $x[n]$.

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iii) If $M = 3$ and $N = 4$ with $u[n] = [1, 2, -1]$ and $v[n] = [1, 1, -1, -1]$ determine both $x[n]$ and $y[n]$ for $0 \leq n \leq 7$. [3]

$x[n] = [1, 3, 1, -2, -2, 1, 0, 0]$ *and* $y[n] = [-1, 4, 1, -2, -1, 4, 1, -2]$.

Note that the convolution is a finite signal but that circular convolution is periodic.

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b) i) Show that, if $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$ and a is a complex-valued constant, then $x[n] = a^n u[n]$ and $y[n] = -a^n u[-n-1]$ have the same z-transform but with different regions of convergence. You may use without proof the geometric progression formulae given in the datasheet. [3]

Using the formula in the datasheet, $X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n} = \sum_0^\infty a^n z^{-n} = \frac{1}{1-az^{-1}}$ *provided that* $|az^{-1}| < 1 \Leftrightarrow |z| > |a|$.

Similarly, $Y(z) = \sum_{-\infty}^{\infty} y[n]z^{-n} = \sum_{-\infty}^{-1} -a^n z^{-n} = \sum_1^\infty -a^{-n}z^n = -a^{-1}z\sum_0^\infty a^{-n}z^n = \frac{-a^{-1}z}{1-a^{-1}z}$ *provided that* $|a^{-1}z| < 1 \Leftrightarrow |z| < |a|$.

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ii) The z-transform $H(z)$ is given by

$$H(z) = \frac{2+17z^{-1}}{(2-z^{-1})(1+4z^{-1})}.$$

By expressing $H(z)$ in partial fraction form, determine the sequence, $h[n]$, whose z-transform is $H(z)$ and whose region of convergence includes $|z| = 1$. [4]

We wish to write $H(z) = \frac{b}{2-z^{-1}} + \frac{c}{1+4z^{-1}} = \frac{(b+2c)+(4b-c)z^{-1}}{(2-z^{-1})(1+4z^{-1})}$. *By matching coefficients, we obtain* $b+2c = 2$ *and* $4b-c = 17$ *from which* $b = 4$ *and* $c = -1$. *These coefficients can also be derived using the residue theorem. Hence* $H(z) = \frac{2}{1-0.5z^{-1}} - \frac{1}{1+4z^{-1}}$.

The corresponding poles are at $z = 0.5$ *and* $z = -4$, *so the sequence we need is* $2 \times 0.5^n u[n] - (-(-4)^n u[-n-1]) = 2^{1-n}u[n] + (-4)^n u[-n-1]$.

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c) i) The frequency response of an ideal lowpass filter is given by

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}.$$

By taking the inverse DTFT of $H(e^{j\omega})$, show that the corresponding impluse response is $h[n] = \frac{\sin \omega_0 n}{\pi n}$ [3]

From the datasheet, $h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n}d\omega = \frac{1}{j2n\pi}\left[e^{j\omega}\right]_{-\omega_0}^{\omega_0} = \frac{2j\sin\omega_0}{j2n\pi} = \frac{\sin\omega_0 n}{\pi n}$.

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ii) By multiplying an ideal filter response by a Hamming window, determine an expression for the coefficients of an FIR causal bandpass filter of even order M whose passband is $1 \leq \omega \leq 2$.

For even M, a symmetric Hamming window is given by $w[n] = 0.54 + 0.46\cos\frac{2\pi n}{M+1}$ for $-0.5M \leq n \leq 0.5M$. [3]

The windowed response is $h[n]w[n]$ for $-0.5M \le n \le 0.5M$ where the ideal impulse response is given by the difference of two lowpass filters: $h[n] = \frac{\sin 2n - \sin n}{\pi n}$. In order to make the filter causal, we need to delay the impulse response by $0.5M$ samples, and so we need $w[n - 0.5M]h[n-0.5M]$ for $0 \le n \le M$. Thus the coefficients are

$$g[n] = \left(0.54 + 0.46\cos\frac{2\pi n - \pi M}{M+1}\right)\frac{\sin(2n-M) - \sin(n-0.5M)}{\pi n - 0.5\pi M}.$$

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d) i) Show that if

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{r=0}^{M} a[M-r]z^{-r}}{\sum_{r=0}^{M} a[r]z^{-r}}$$

then $|H(e^{j\omega})| \equiv 1$ and $\angle H(e^{j\omega}) = -M\omega - 2\angle A(e^{j\omega})$. [3]

We can express $B(z) = z^{-M}A(z^{-1})$. Hence $H(e^{j\omega}) = e^{-jM\omega}\frac{A(e^{-j\omega})}{A(e^{j\omega})} = e^{-jM\omega}\frac{A^*(e^{j\omega})}{A(e^{j\omega})}$.

Hence

$$|H(e^{j\omega})| = |e^{-jM\omega}|\frac{|A^*(e^{j\omega})|}{|A(e^{j\omega})|} = 1 \times \frac{|A(e^{j\omega})|}{|A(e^{j\omega})|} = 1$$

and

$$\angle H(e^{j\omega}) = \angle e^{-jM\omega} + \angle A^*(e^{j\omega}) - \angle A(e^{j\omega}) = -M\omega - 2\angle A(e^{j\omega}).$$

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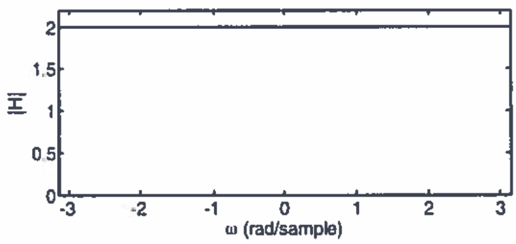
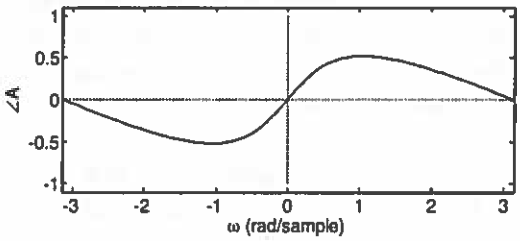
ii) If $H(z) = \frac{2 - 4z^{-1}}{2 - z^{-1}}$, sketch graphs of the magnitude and phase of $H(e^{j\omega})$ for $-\pi \le \omega \le \pi$. [3]

We can write $H(z) = -2z^{-1}\frac{2-z}{2-z^{-1}}$. Hence $|H(z)| = 2\forall\omega$.

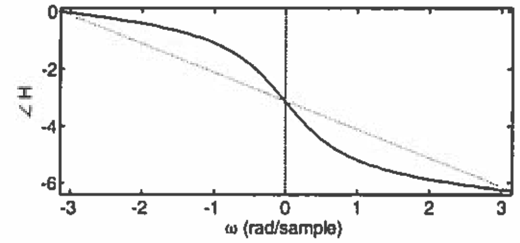


We can write $\angle A(e^{j\omega}) = \angle\left(2 - e^{-j\omega}\right) = \angle(2 - \cos\omega + j\sin\omega) = \tan^{-1}\frac{\sin\omega}{2-\cos\omega}$. The denominator of the fraction varies between 1 (at $\omega = 0$) and 3 (at $\omega = \pm\pi$) and, for $x < 1$, $\tan^{-1}x \approx x$, so the graph looks like a distorted sine wave:



Note that since $A(z)$ has one pole and one zero and both are within the unit circle, the total phase change over $-\pi \le \omega \le \pi$ is equal to zero.

Since $M = 1$, we have $\angle H(z) = -\pi - \omega - 2\angle\left(2 - e^{-j\omega}\right)$. The first two terms are plotted as the dashed line in the lower graph below, and onto this we add $-2\angle A(e^{j\omega})$ to get the final answer.



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e) Figure 1.1 shows the power spectral density (PSD) of a real-valued signal $x[n]$. The horizontal portions of the PSD have values 3, 2 and 1 respectively. The signal $y[n]$ is then obtained by downsampling $x[n]$ by a factor of 3.

Draw a dimensioned sketch showing the PSD of $y[n]$ for $0 \le \omega \le \pi$. You should assume that components of $x[n]$ at different frequencies are uncorrelated and may assume without proof that $Y(z) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{\frac{-j2\pi k}{3}} z^{\frac{1}{3}}\right)$.

Determine the value of each horizontal portion of the PSD and each of the angular frequencies at which its value changes. [5]

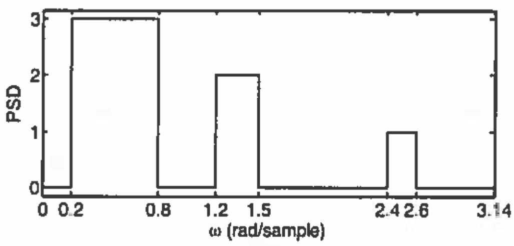

Figure 1.1

Each portion of the original PSD will be expanded horizontally by a factor of 3 and its amplitude reduced by a factor of 3 (i.e. the energy per second is re-

duced by 3^2 but since there are now fewer samples, the energy per sample is reduced only by a factor of 3). Thus the portion between $(0.2, 0.8)$ *will be mapped to* $(\mathrm{a}, \mathrm{e}) = (0.6, 2.4)$ *with amplitude* $\frac{3}{3}$. *The portion between* $(1.2, 1.5)$ *will be mapped to* $(3.6, 4.5) = (3.6 - 2\pi, 4.5 - 2\pi) = (-2.683, -1.783)$. *The symmetric part of this image will therefore be at* $(\mathrm{d}, \mathrm{f}) = (1.783, 2.683)$ *with an amplitude of* $\frac{2}{3}$. *Finally, the portion between* $(2.4, 2.6)$ *will be mapped to* $(7.2, 7.8) = (7.2 - 2\pi, 7.8 - 2\pi) = (0.917, 1.517) = (\mathrm{b}, \mathrm{c})$ *with an amplitude of 30,* $\frac{3}{3}, \frac{4}{3}, \frac{3}{3}, \frac{5}{3}, \frac{2}{3}, 0$. *The frequencies at which the value changes are* $0.6, 0.917, 1.517, 1.783, 2.4, 2.683$.

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f) Figure 1.2 shows the block diagram of a two-band analysis and synthesis processor. You may assume without proof that, for $m = 0$ or 1, $W_m(z) = U_m(z^2)$ and $U_m(z) = \frac{1}{2}\left\{V_m(z^{\frac{1}{2}}) + V_m(-z^{\frac{1}{2}})\right\}$.

i) Derive an expression for $Y(z)$ in terms of $X(z)$. [4]

We can write

$$\begin{aligned} W_0(z) &= U_0(z^2) = \frac{1}{2}\{V_0(z) + V_0(-z)\} \\ &= \frac{1}{2}\{H(z)X(z) + H(-z)X(-z)\} \end{aligned}$$

Similarly

$$\begin{aligned} W_1(z) &= U_1(z^2) = \frac{1}{2}\{V_1(z) + V_1(-z)\} \\ &= \frac{1}{2}\{H(-z)X(z) + H(z)X(-z)\} \end{aligned}$$

Therefore

$$\begin{aligned} Y(z) &= H(z)W_0(z) - H(-z)W_1(z) \\ &= \frac{1}{2}\{H^2(z)X(z) + H(z)H(-z)X(-z) - H^2(-z)X(z) - H(-z)H(z)X(-z)\} \\ &= \frac{1}{2}\{H^2(z) - H^2(-z)\}X(z) \end{aligned}$$

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ii) Explain the relationship between the magnitude responses of the filters $H(z)$ and $H(-z)$. [2]

The magnitude response of $H(-e^{j\omega})$ *is the same as that of* $H(e^{j\omega})$ *but reflected around the frequency* $\omega = \frac{\pi}{2}$.

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iii) Explain what is meant by saying that the analysis-synthesis processor shown in Figure 1.2 is "alias-free". [2]

The analuysis-synthesis process is alias free because the term $X(-z)$ does not appear in the expression for $Y(z)$. The power spectrum of $X(-z)$ is the same as that of $X(z)$but reflected around $\omega = \frac{\pi}{2}$.

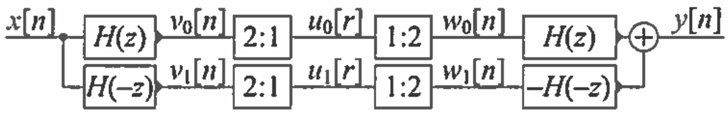

Figure 1.2

2. In this question, filters should be expressed in the standard form $g \times \frac{1+b_1z^{-1}+\cdots}{1+a_1z^{-1}+\cdots}$ with numerical values given for all coefficients.

a) A bilinear transformation of the z-plane is given by $z = \frac{\hat{z}-\lambda}{1-\lambda\hat{z}}$ where the real-valued constant λ satisfies $|\lambda| < 1$.

i) Show that $|z|^2 = 1 + \frac{\left(|\hat{z}|^2-1\right)\left(1-\lambda^2\right)}{|1-\lambda\hat{z}|^2}$.

Hence show that $|z| < 1$ if and only if $|\hat{z}| < 1$. [4]

We can write

$$\begin{aligned} zz^* &= \frac{(\hat{z}-\lambda)(\hat{z}^*-\lambda)}{(1-\lambda\hat{z})(1-\lambda\hat{z}^*)} \\ &= \frac{|\hat{z}|^2-2\lambda(\hat{z}+\hat{z}^*)+\lambda^2}{1-2\lambda(\hat{z}+\hat{z}^*)+\lambda^2|\hat{z}|^2} \\ &= 1+\frac{\left(|\hat{z}|^2-1\right)\left(1-\lambda^2\right)}{|1-\lambda\hat{z}|^2} \end{aligned}$$

Since $|\lambda| < 1$, the numerator term $\left(1-\lambda^2\right)$ must be strictly positive. In addition, the denominator term satisfies $|1-\lambda\hat{z}|^2 \geq 0$. Hence, assuming for the moment that $1-\lambda\hat{z} \neq 0$, the sign of the fraction is equal to the sign of $\left(|\hat{z}|^2-1\right)$ and is positive or negative according to whether $|\hat{z}| > 1$ or $|\hat{z}| < 1$. Clearly $|\hat{z}| = 1$ makes the fraction zero and hence $|z| = 1$. Putting all this together, we have shown that $|\hat{z}| < 1 \Rightarrow |z| < 1$ and $|\hat{z}| \geq 1 \Rightarrow |z| \geq 1$ which is equivalent to $|z| < 1 \Rightarrow |\hat{z}| < 1$.

The special case, $1-\lambda\hat{z} = 0$, arises when $\hat{z} = \lambda^{-1} > 1$. In this case, the numerator of the fraction is strictly positive and $|z| = +\infty \not< 1$ so the proposition is satisfied.

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ii) Explain why the property shown in part i) is important when using the transformation for filter design. [2]

A filter is stable iff all its poles lie strictly inside the unit circle. If this transformation is applied to a stable filter, the property proved in part i) ensure that the transformed filter is also stable. In the same way, it also ensures that a minimum phase filter will transform into another minimum phase filter.

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b) A first-order lowpass filter has the transfer function $G(z) = 1+z^{-1}$.

i) Determine the gain of the filter at $\omega = 0$ and show that the magnitude of the gain has decreased by a factor of $\sqrt{2}$ at the cutoff frequency, $\omega_G = \frac{\pi}{2}$. [2]

For $\omega = 0$, the filter gain is $G(e^{j\omega}) = G(1) = 2$.

At $\omega_1 = \frac{\pi}{2}$, the filter gain is $G(e^{j\omega_1}) = G(j) = 1 - j$. Hence $|G(e^{j\omega_1})| = |1-j| = \sqrt{2} = \frac{G(1)}{\sqrt{2}}$.

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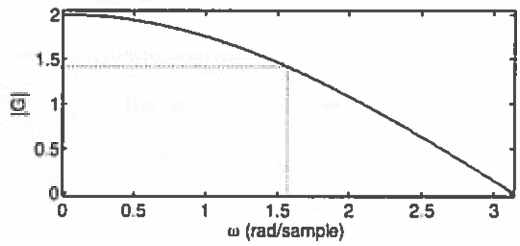
ii) By considering the value of $z^{\frac{1}{2}}G(z)$, determine a trigonometrical expression for $|G(e^{j\omega})|$ and draw a dimensioned sketch of its value over the range $0 \le \omega \le \pi$. [4]

For $z = e^{j\omega}$, we can write

$$\begin{aligned} z^{\frac{1}{2}}G(z) &= z^{\frac{1}{2}} + z^{-\frac{1}{2}} \\ e^{j\frac{\omega}{2}}G(e^{j\omega}) &= e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}} \\ &= 2\cos\frac{\omega}{2} \end{aligned}$$

Taking the magnitude of each side gives $|G(e^{j\omega})| = 2\cos\frac{\omega}{2}$ for $|\omega| \le \pi$.



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iii) Using the appropriate z-plane transformation from the datasheet, transform $G(z)$ to a lowpass filter, $H(z)$, with a cutoff frequency of $\omega_H = 0.2$. Calculate the numerical values of the filter coefficients when expressed in the standard form given in the first line of the question. [5]

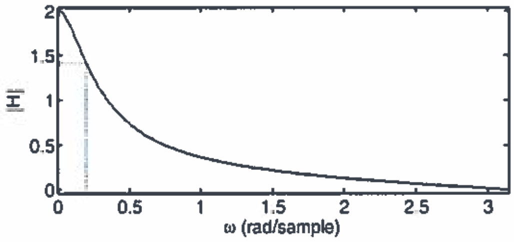
iv) Draw a dimensioned sketch of $|H(e^{j\omega})|$ over the range $0 \le \omega \le \pi$. [2]

We want a lowpass-to-lowpass transformation with $\omega_0 = \frac{\pi}{2}$ and $\hat{\omega}_1 = 0.2$. So

$$\begin{aligned} \lambda &= \frac{\sin\left(\frac{\omega_1 - \hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_1 + \hat{\omega}_1}{2}\right)} \\ &= \frac{\sin 0.6854}{\sin 0.8854} \\ &= 0.8176 \end{aligned}$$

Substituting for z^{-1} in $G(z)$ gives

$$
\begin{aligned}
H(\tilde{z}) &= 1 + \frac{\tilde{z}^{-1} - \lambda}{1 - \lambda \tilde{z}^{-1}} \\
&= \frac{1 - \lambda \tilde{z}^{-1} + \tilde{z}^{-1} - \lambda}{1 - \lambda \tilde{z}^{-1}} \\
&= (1 - \lambda) \frac{1 + \tilde{z}^{-1}}{1 - \lambda \tilde{z}^{-1}} \\
&= 0.1824 \frac{1 + \tilde{z}^{-1}}{1 - 0.8176 \tilde{z}^{-1}}
\end{aligned}
$$



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c) A quadratic transformation of the z-plane is given by $z = -\tilde{z}^2$.

i) Show that $|z| < 1$ if and only if $|\tilde{z}| < 1$. [2]

We see that $|\tilde{z}| < 1 \Rightarrow |\tilde{z}|^2 < 1 \Rightarrow |\tilde{z}^2| < 1 \Rightarrow |-\tilde{z}^2| < 1 \Rightarrow |z| < 1$.

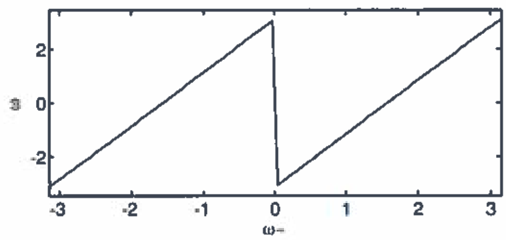
Also $|\tilde{z}| \geq 1 \Rightarrow |\tilde{z}|^2 \geq 1 \Rightarrow |\tilde{z}^2| \geq 1 \Rightarrow |-\tilde{z}^2| \geq 1 \Rightarrow |z| \geq 1$ *from which* $|z| < 1 \Rightarrow |\tilde{z}| < 1$.

Hence the transformation preserves stability.

---

ii) If $z = e^{j\omega}$ and $\tilde{z} = e^{j\tilde{\omega}}$ sketch a graph of ω versus $\tilde{\omega}$ over the range $-\pi \leq \tilde{\omega} \leq \pi$. The value of ω should be restricted to $-\pi < \omega \leq \pi$. [2]

If $z = -\tilde{z}^2$ *then* $e^{j\omega} = -e^{j2\tilde{\omega}} = e^{j\pi} \times e^{j2\tilde{\omega}} = e^{j(2\tilde{\omega}+\pi)}$ *from which* $\omega = (2\tilde{\omega} + \pi)_{\text{ mod } 2\pi}$



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iii) A new filter is defined by $P(\tilde{z}) = H(z)$. Determine the numerical values of the coefficients of $P(\tilde{z})$ when expressed in the standard form given in the first line of the question. [3]

We have $H(z) = (1-\lambda)\frac{1+z^{-1}}{1-\lambda z^{-1}}$.

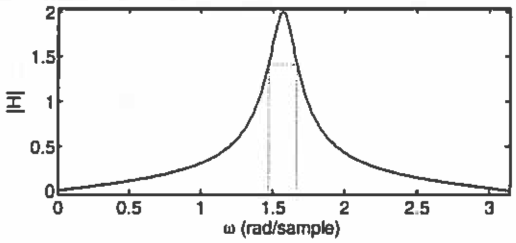
Making the substitution $z^{-1} = -\tilde{z}^{-2}$ *gives*

$$\begin{aligned} P(\tilde{z}) &= (1-\lambda)\frac{1-\tilde{z}^{-2}}{1+\lambda\tilde{z}^{-2}} \\ &= 0.1824\frac{1-\tilde{z}^{-2}}{1+0.8176\hat{z}^{-1}} \end{aligned}$$

---

iv) Draw a dimensioned sketch of $\left|P(e^{j\omega})\right|$ over the range $0 \leq \omega \leq \pi$ and determine the values of ω within this range for which $\left|P(e^{j\omega})\right| = \sqrt{2}$.

Explain the relationship between the bandwidth of the filter $P(e^{j\omega})$ and the cutoff frequency of the filter $H(e^{j\omega})$. [4]



$\left|H(e^{j\omega})\right| = \sqrt{2}$ *for* $\omega = \pm 0.2$. *Hence* $\left|P(e^{j\tilde{\omega}})\right| = \sqrt{2}$ *when* $(2\tilde{\omega}+\pi)_{\bmod 2\pi} = \omega = \pm 0.2$. *Solving this equation gives*

$$\begin{aligned} 2\tilde{\omega}+\pi &= \pm 0.2 + 2n\pi \\ \tilde{\omega} &= \pm 0.1 + \left(n - \frac{1}{2}\right)\pi \\ &= \cdots, -\frac{3\pi}{2} \pm 0.1, -\frac{\pi}{2} \pm 0.1, \frac{\pi}{2} \pm 0.1, \frac{3\pi}{2} \pm 0.1,, \cdots \end{aligned}$$

The two values of ω *in the range* $0 \leq \omega \leq \pi$ *for which* $\left|P(e^{j\omega})\right| = \sqrt{2}$ *are therefore* $\omega = \frac{\pi}{2} \pm 0.1 = \{1.4708, 1.6708\}$. *The bandwidth of the filter is* 0.2 *and is equal to the cutoff frequency of* $H(z)$.

---

3. a) The filter $H(z) = \frac{1}{1+a_1z^{-1}+a_2z^{-2}}$ where $a_1 = -1.56$ and $a_2 = 0.64$.

i) By multiplying $H(z)$ by its complex conjugate and using the identity $\cos 2\omega = 2\cos^2\omega - 1$, express $\left|H(e^{j\omega})\right|^{-2}$ as a polynomial in $\cos\omega$ giving the coefficients to 5 significant figures. [4]

If $z = e^{j\omega}$, then

$$\begin{aligned}\left|H(e^{j\omega})\right|^{-2} &= \left(1+a_1z^{-1}+a_2z^{-2}\right)\left(1+a_1z^{1}+a_2z^{2}\right)\\ &= 1+a_1^2+a_2^2+2a_1(1+a_2)\cos\omega+2a_2\cos 2\omega\\ &= 1+a_1^2+a_2(a_2-2)+2a_1(1+a_2)\cos\omega+4a_2\cos^2\omega\\ &= 2.560\cos^2\omega-5.1168\cos\omega+2.5632\end{aligned}$$

---

ii) The filter $H_1(z)$ is the same as $H(z)$ but with coefficient a_1 increased in magnitude by 1% (i.e. multiplied by 1.01). Similarly, the filter $H_2(z)$ is the same as $H(z)$ but with coefficient a_2 increased in magnitude by 1% .

For $\omega_0 = 0.2$, determine the ratios $\left|\frac{H_1(e^{j\omega_0})}{H(e^{j\omega_0})}\right|$ and $\left|\frac{H_2(e^{j\omega_0})}{H(e^{j\omega_0})}\right|$ in dB. [6]

At $\omega_0 = 0.2$, $\cos\omega_0 = 0.9801$ *and* $\left|H(e^{j\omega_0})\right|^{-2} = 0.007353$ *and* $\left|H(e^{j\omega_0})\right| = 11.66 = 21.34\,\text{dB}$.

We have $H_1(z)^{-1} = 1-1.5756z^{-1}+0.64z^{-2}$ *and so*

$$\begin{aligned}\left|H_1(e^{j\omega_0})\right|^{-2} &= 1+a_1^2+a_2(a_2-2)+2a_1(1+a_2)\cos\omega+4a_2\cos^2\omega\\ &= 2.56\cos^2\omega-5.1680\cos\omega+2.6121\end{aligned}$$

Evaluating this at $\cos\omega = 0.9801$ *gives* $\left|H_1(e^{j\omega_0})\right|^{-2} = 0.006121$ *and* $\left|H_1(e^{j\omega_0})\right| = 12.78 = 22.13\,\text{dB}$. *This is an error of* $0.797\,\text{dB}$.

Similarly $H_2(z)^{-1} = 1-1.56z^{-1}+0.6464z^{-2}$ *and so*

$$\begin{aligned}\left|H_2(e^{j\omega_0})\right|^{-2} &= 1+a_1^2+a_2(a_2-2)+2a_1(1+a_2)\cos\omega+4a_2\cos^2\omega\\ &= 2.5856\cos^2\omega-5.1368\cos\omega+2.5586\end{aligned}$$

Evaluating this at $\cos\omega_0 = 0.9801$ *gives* $\left|H_2(e^{j\omega_0})\right|^{-2} = 0.007806$ *and* $\left|H_2(e^{j\omega_0})\right| = 11.32 = 21.08\,\text{dB}$. *This is an error of* $-0.259\,\text{dB}$.

---

b) In the block diagram of Figure 3.1 the outputs of all adders are on the right and solid arrows indicate the direction of information flow. Multiplier gains are written adjacent to each multiplier symbol. The parameter p is strictly positive.

i) Show that $G(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+(p^2-pq-2)z^{-1}+(pq+1)z^{-2}}$. [6]

From the diagram, we can write down that $Y(z) = z^{-1}Y(z) + pU(z)$ *from which* $U(z) = p^{-1}\left(1-z^{-1}\right)Y(z)$.

We can also write $U(z) = z^{-1}U(z) + p\left(p^{-2}X(z) + qz^{-1}U(z) - z^{-1}Y(z)\right)$ *from which* $\left(1-z^{-1}-pqz^{-1}\right)U(z) = p^{-1}X(z) - pz^{-1}Y(z)$.

Substituting for $U(z)$ *gives*

$$\begin{aligned}\left(1-z^{-1}-pqz^{-1}\right)p^{-1}\left(1-z^{-1}\right)Y(z) &= p^{-1}X(z) - pz^{-1}Y(z)\\ \left(1-z^{-1}-pqz^{-1}-z^{-1}+z^{-2}+pqz^{-2}+p^2z^{-1}\right)Y(z) &= X(z)\end{aligned}$$

Hence $G(z) = \frac{1}{1+(p^2-pq-2)z^{-1}+(1+pq)z^{-2}}$ *as required.*

---

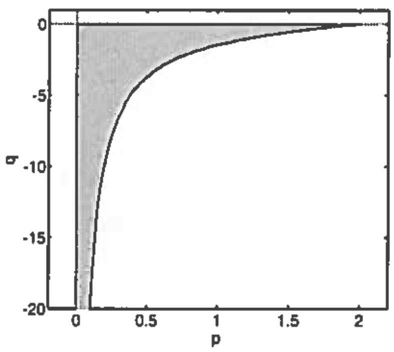
ii) Determine the conditions on p and q for the filter $G(z)$ to be BIBO stable.

You may assume without proof that the filter $\frac{1}{1+b_1z^{-1}+b_2z^{-2}}$ is BIBO stable if and only if $|b_1| - 1 < b_2 < 1$. [6]

We can express the condition $|p^2-pq-2|-1 < pq+1 < 1$ *as two separate inequalities* $\left(p^2-pq-2\right)-1 < pq+1 < 1$ *and* $-\left(p^2-pq-2\right)-1 < pq+1 < 1$ *which simplify to* $p^2-pq-4 < pq < 0$ *and* $-p^2+pq < pq < 0$. *From the rightmost inequality we have* $pq < 0 \Rightarrow q < 0$ *since* $p > 0$ *is stated in the question. The condition* $-p^2+pq < pq \Leftrightarrow p^2 > 0$ *is always satisfied, so the only remaining condition is*

$$\begin{aligned}p^2-pq-4 &< pq\\ 2pq &> p^2-4\\ 0 > q &> \frac{p}{2}-\frac{2}{p}\end{aligned}$$

where I have added in the condition $q < 0$ *in the final line. The condition* $\frac{p}{2}-\frac{2}{p} < 0$ *implies* $|p| < 2$ *so we know that* $0 < p < 2$ *and* $0 > q > \frac{p}{2}-\frac{2}{p}$. *This is the shaded region below (plot not requested).*



Note that the constraint $p > 0$ *is not actually necessary since changing the sign of both* p *and* q *leaves the transfer function unaltered.*

iii) If

$$G(z) = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2}},$$

determine expressions for p and q as functions of b_1 and b_2. Calculate the numerical values of p and q if $b_1 = -1.56$ and $b_2 = 0.64$.

[3]

We have $b_1 = p^2 - pq - 2$ and $b_2 = pq + 1$. Adding these equations together gives $b_1 + b_2 = p^2 - 1$ from which $p = \sqrt{b_1 + b_2 + 1}$ (always the positive root since $p > 0$ is given in the question). From the second equation, it is then possible to determine $q = \frac{b_2 - 1}{p} = \frac{b_2 - 1}{\sqrt{b_1 + b_2 + 1}}$.

For the specific values $b_1 = -1.56$ and $b_2 = 0.64$, we get $p = 0.2828$ and $q = -1.2728$.

iv) The filter $G_p(z)$ is the same as $G(z)$ but with coefficient p increased by 1% (i.e. multiplied by 1.01) from the value determined in part iii). Similarly, the filter $G_q(z)$ is the same as $G(z)$ but with coefficient q increased by 1% from the value determined in part iii).

For $\omega_0 = 0.2$, determine the ratios $\left|\frac{G_p(e^{j\omega_0})}{G(e^{j\omega_0})}\right|$ and $\left|\frac{G_q(e^{j\omega_0})}{G(e^{j\omega_0})}\right|$ in dB.

[5]

From part ii), at $\omega_0 = 0.2$, $\cos\omega_0 = 0.9801$ and $|G(e^{j\omega_0})|^{-2} = 0.007353$ and $|G(e^{j\omega_0})| = 11.66 = 21.34$ dB.

For G_p, $p = 0.2828 \times 1.01 = 0.2856$ and $q = -1.2728$. This gives $G_p(z)^{-1} = 1 - 1.5549z^{-1} + 0.6365z^{-2}$ and so

$$\begin{aligned}|G_p(e^{j\omega_0})|^{-2} &= 1 + b_1^2 + b_2(b_2 - 2) + 2b_1(1 + b_2)\cos\omega + 4b_2\cos^2\omega \\ &= 2.5458\cos^2\omega - 5.0889\cos\omega + 2.5498\end{aligned}$$

Evaluating this at $\cos\omega = 0.9801$ gives $|G_p(e^{j\omega_0})|^{-2} = 0.007618$ and $|G_p(e^{j\omega_0})| = 11.457 = 21.18$ dB. This is an error of -0.15 dB.

Similarly, for G_q, $p = 0.2828$ and $q = -1.2728 * 1.01 = -1.2855$. This gives $G_q(z)^{-1} = 1 - 1.5565z^{-1} + 0.6365z^{-2}$ and so

$$\begin{aligned}|G_q(e^{j\omega_0})|^{-2} &= 1 + b_1^2 + b_2(b_2 - 2) + 2b_1(1 + b_2)\cos\omega + 4b_2\cos^2\omega \\ &= 2.5458\cos^2\omega - 5.0942\cos\omega + 2.5548\end{aligned}$$

Evaluating this at $\cos\omega_0 = 0.9801$ gives $|G_q(e^{j\omega_0})|^{-2} = 0.007463$ and $|G_q(e^{j\omega_0})| = 11.58 = 21.27$ dB. This is an error of -0.064 dB.

Thus $G(z)$ is significantly less sensitive to coefficient errors (at least at ω_0).

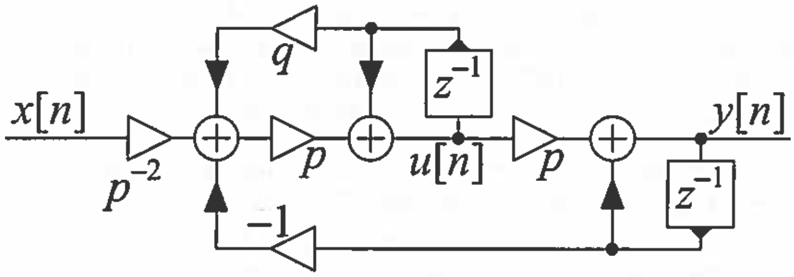
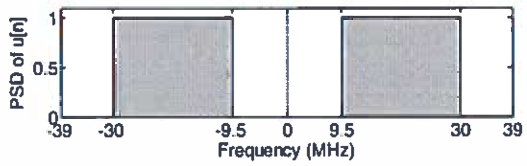


Figure 3.1

4. The FM radio band extends from 87.5 to 108 MHz. Within this band, an FM channel occupies ± 100 kHz around a centre frequency of $c \times 100$ kHz where the channel index, c, is an integer in the range $876 \leq c \leq 1079$. Figure 4.1 shows the block diagram of an FM radio front-end in which bold lines denote complex-valued signals. The diagram includes a bandpass filter (BPF) whose passband is 87.5 to 108 MHz and an analogue-to-digital converter (ADC) with a sample rate of 78 MHz.

a) Assume the bandpass filter is ideal and the power spectral density of the received signal is constant within the FM band. Sketch the power spectrum of $u[n]$ over the unnormalized frequency range -39 to $+39$ MHz. Determine the maximum width of both the lower transition region and the upper transition region of the BPF block in order to ensure that the FM band image is uncorrupted by aliasing. [3]

The FM band of 87.5 *to* 108 Mhz *will be aliased down by the sample frequency to an image covering* 9.5 *to* 30 Mhz. *Frequencies of* $78 - 9.5 = 68.5$ Mhz *and* $2 \times 78 - 30 = 129$ Mhz *will be aliased onto the edges of this image and so the widest possible transition bands for the bandpass filter (BPF) are* $68.5 - 87.5 = 19$ Mhz *and* $108 - 129 = 21$ Mhz.



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b) In Figure 4.1, $u[n]$ is multiplied by the complex-valued $v[n] = \exp(-j\omega_c n)$ where ω_c is the normalized centre frequency of the wanted channel.

i) Give a formula for ω_c in terms of c and state how many multiplications are required per second to multiply $u[n]$ and $v[n]$ (where one multiplication calculates the product of two real numbers). [2]

The unnormalized centre frequency is $\Omega_c = 2\pi c \times 10^5$ *so the normalized centre frequency is* $\omega_c = \frac{\Omega_c}{f_s} = \frac{2\pi c \times 10^5}{78 \times 10^6} = \frac{2\pi}{780} c$ *meaning that* $v[n] = e^{-j\omega_c n} = e^{-j2\pi \frac{cn}{780}}$.

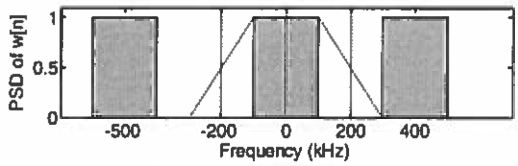
Multiplying a real number, $u[n]$, *by a complex number,* $v[n]$, *requires two multiplications and so the multiplication rate is* $2f_s = 156 \times 10^6 = 1.56 \times 10^8$.

---

ii) Assume now that only the FM channels with centre frequencies 99.5, 100 and 100.4 MHz are present. Using an unnormalized frequency axis in kHz, draw a dimensioned sketch of the power spectrum of $w[n]$ when $c = 1000$ covering the range -700 to $+700$ kHz. On your sketch, label the centre frequency of each of the occupied spectral regions. [3]

When $c = 1000$, *the spectrum of* $u[n]$ *is shifted down by* 100 MHz *to become that of* $w[n]$. *The shifted centre frequencies of the active*

FM channels are -0.5, 0 *and* $+0.4$ MHz. *Also marked on the sketch below, but not requested in the question, is the gain of* $H(z)$ *and* $\pm$ *the Nyquist frequency of the sample rate at* $y[n]$.



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c) i) Explain the purpose of the lowpass FIR filter, $H(z)$ in Figure 4.1.[2]

The lowpass filter must remove frequencies above 200 kHz *in order to prevent aliasing by the downsampler.*

---

ii) Assuming that the centre frequencies of active channels are always at least 400 kHz apart, determine the cutoff frequency and maximum transition width of the filter $H(z)$ in radians/sample. Hence use the formula $M = \dfrac{a}{3.5\Delta\omega}$ from the datasheet to determine the order of the filter to give a stopband attenuation of 50 dB. [3]

The response of $H(z)$ *is shown in the answer to part ii) above. The unnormalized cutoff frequency and transition width are* 100 kHz *and* 200 kHz *respectively. Multiplying by* $\frac{2\pi}{78\,\text{MHz}}$, *the normalized values are therefore* 8.06×10^{-3} *and* 1.61×10^{-2} *rad/sample. Thus the formula gives* $M = \frac{50}{3.5\times1.61\times10^{-2}} = 887$.

---

iii) Suppose that $H(z)$ is implemented as a polyphase filter as shown in Figure 4.3. Determine the order of the sub-filters assuming they all have the same order. Give an expression for $h_p[r]$, the impulse response of the sub-filter $H_p(z)$, in terms of $h[n]$, the impulse response of $H(z)$. [2]

The order of an FIR filter is one less than the number of coefficients, so since $H(z)$ *has M+1 coefficients, the order of* $H_p(z)$ *will be* $\left\lceil \frac{M+1}{195} - 1 \right\rceil = 4$ *where the brackets denote the ceiling function.*

$h_p[r] = h[p+195r]$ *for* $0 \le p < 195$ *and* $0 \le r < 4$.

---

iv) Calculate the number of multiplications per second needed to implement Figure 4.3 assuming that all sub-filters have the same order. [3]

The filter coefficients are real but the filer input signal, $w[n]$, *is complex. Therefore each of the sub-filters requires* $2\times(4+1) = 10$ *multiplications for each of its input samples. Therefore, for each* *input sample, $u[n]$, we need two multiplications for $u[n] \times v[n]$ and 10 for the selected sub-filter giving a total of 12. The total rate of multiplications per second is therefore $12 \times 78 \times 10^6 = 936 \times 10^6 = 9.36 \times 10^8$.*

---

d) i) Determine the impulse response of $G_c(z)$ such that Figures 4.1 and 4.2 are functionally identical. [3]

From Figure 4.1,

$$\begin{aligned} x[n] &= \sum_{m=0}^{M} h[m]w[n-m] \\ &= \sum_{m=0}^{M} h[m]u[n-m]e^{-j\omega_c(n-m)} \\ &= e^{-j\omega_c n} \sum_{m=0}^{M} \left(e^{j\omega_c m}h[m]\right) u[n-m] \\ &= e^{-j\omega_c n} \sum_{m=0}^{M} g_c[m]u[n-m] \end{aligned}$$

where $g_c[m] = e^{j\omega_c m}h[m]$. The final expression directly implements Figure 4.2 with $g_c[m]$ the impulse response of $G_c(z)$.

---

ii) If $G_c(z)$ is implemented as a conventional polyphase filter, give an expression for the impulse response, $g_{c,p}[r]$, of the sub-filter $G_{c,p}(z)$.

Show that if $\alpha_c = \exp\left(\frac{j2\pi c}{780}\right)$, then each coefficient, $\alpha_c^{-p} g_{c,p}[r]$, of $\alpha_c^{-p} G_{c,p}(z)$ is either purely real or purely imaginary. [3]

From the previous part, we have $g_c[n] = e^{j\omega_c n}h[n] = e^{j2\pi\frac{cn}{780}}h[n]$. The polyphase implementation therefore has

$$\begin{aligned} g_{c,p}[r] &= g_c[p+195r] \\ &= e^{j2\pi\frac{c(p+195r)}{780}}h[p+195r] \\ &= e^{j2\pi\frac{cp}{780}}e^{j2\pi\frac{cr}{4}}h[p+195r] \\ &= \alpha_c^p j^{cr} h[p+195r] \end{aligned}$$

which, as required, is α_c^p times a quantity that is either purely real or purely imaginary.

---

iii) In Figure 4.4, the subfilter $G_{c,p}(z)$ is implemented as $\alpha_c^{-p}G_{c,p}(z)$ followed by a multiplication by α_c^p. Determine a simplified expression for $s[r]$ in so that Figure 4.4 is functionally equivalent to Figure 4.3. [3]

In Figure 4.3 we multiply by $v[n] = e^{-j2\pi\frac{cn}{780}}$ *immediately before downsampling. From the noble identities, the is equivalent to multiplying by* $s[r] = v[195r] = e^{-j2\pi\frac{195cr}{780}} = e^{-j2\pi\frac{cr}{4}} = -j^{cr}$ *after the downsampler as in Figure 4.4.*

---

iv) Giving your reasons fully, determine the number of multiplications per second required to implement Figure 4.4. You may exclude negation operations from the multiplication count. [3]

Although the subfilter coefficients in Figure 4.4, only one multiplication per coefficient is required because the input signal is real and the coefficient is either real of purely imaginary. Therefore, for each input sample at $u[n]$, *we require 5 multiplies for the filter and 4 for the multiplication by* α^p. *Since* $s[r]$ *is a power of* j, *it does not involve any actual multiplications. Hence the total number of multiplications is* $9 \times 78 \times 10^6 = 702 \times 10^6 = 7.02 \times 10^8$. *The reduction relative to part iv) would be larger for larger values of M.*

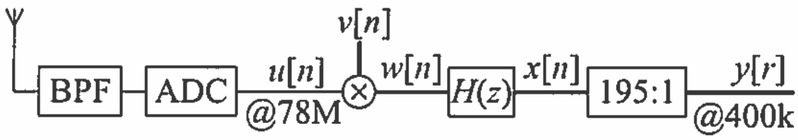

Figure 4.1

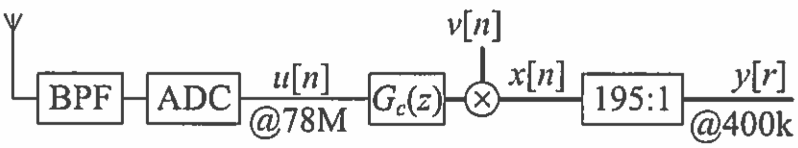

Figure 4.2

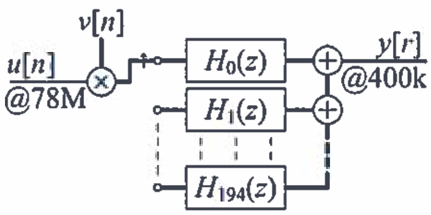

Figure 4.3

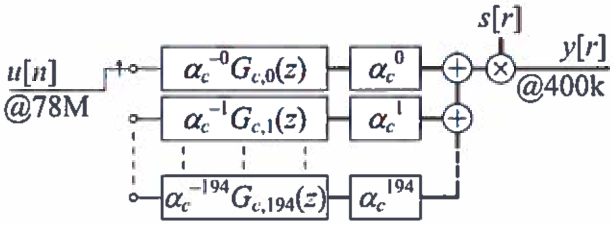

Figure 4.4