

DSP & Digital Filters

Lecture 1 z-Transform

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Continuous-time signals

Laplace transform:
generalised frequency
transformation.

(all signals have LT but not all have FT).

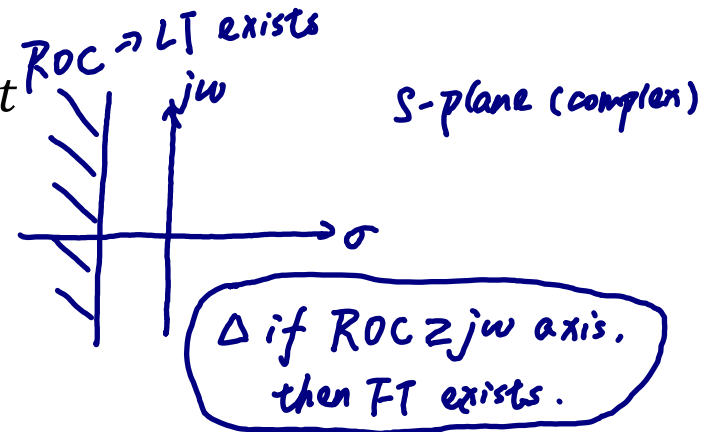
- Recall that in order to describe a continuous-time signal $x(t)$ in frequency domain we use:

□ The Continuous-Time Fourier Transform (or Fourier Transform):

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

□ The Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$



- The above transforms and their basic properties are considered known in this course.
- If you have doubts please consult any book on Signals and Systems.

Discrete-time signals

The z-transform derived from the Laplace transform

$$x[n] = x(nT)$$

$$x(t) = \sum_n x[n] \delta(t - nT)$$

- Consider a discrete-time signal $x(t)$ sampled every T seconds.

$$x(t) = x_0 \delta(t) + x_1 \delta(t - T) + x_2 \delta(t - 2T) + x_3 \delta(t - 3T) + \dots$$

- Recall that in the Laplace domain we have:

$$\mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

$$\mathcal{L}\{\delta(t)\} = 1$$

$$\mathcal{L}\{\delta(t - T)\} = e^{-sT}$$

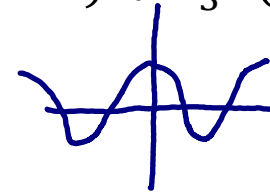
- Therefore, the Laplace transform of $x(t)$ is:

$$X(s) = x_0 + x_1 e^{-sT} + x_2 e^{-s2T} + x_3 e^{-s3T} + \dots$$

- Now define $z = e^{sT} = e^{(\sigma + j\omega)T} = e^{\sigma T} \cos \omega T + j e^{\sigma T} \sin \omega T$.

- Finally, define

$$X[z] = x_0 + x_1 z^{-1} + x_2 z^{-2} + x_3 z^{-3} + \dots$$



$$x(t) = \frac{1}{K} \sum_{k=-K}^K \cos(\omega_k t)$$

$$= \begin{cases} 1, & t=0 \\ \rightarrow 0, & t \neq 0 \end{cases}$$

if $K \rightarrow \infty$, $x(t) \rightarrow \delta(t)$.

z^{-1} : the sampling period delay operator

$\delta(t-T) \leftrightarrow e^{-sT} = z^{-1}$: delay by a sampling period.

- From the Laplace time-shift property, we know that an additional term $z = e^{sT}$ in the Laplace domain, corresponds to time-advance by T seconds (T is the sampling period) of the original function in time.
- Accordingly, $z^{-1} = e^{-sT}$ corresponds to a time-delay of one sampling period.
- As a result, all sampled data (and discrete-time systems) can be expressed in terms of the variable z .
- More formally, the unilateral z – transform of a causal sampled sequence:

$$x[n] = \{x[0], x[1], x[2], x[3], \dots\}$$

is given by:

$$X[z] = x_0 + x_1 z^{-1} + x_2 z^{-2} + x_3 z^{-3} + \dots = \sum_{n=0}^{\infty} x[n] z^{-n}, \quad x_n = x[n]$$

- The bilateral z – transform for any sampled sequence is:

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X[z] = \sum_{n=0}^{\infty} r^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{r}{z}\right)^n = \frac{1}{1 - \frac{r}{z}} = \frac{z}{z-r}, \quad \left|\frac{r}{z}\right| < 1$$

Example: Find the z – transform of $x[n] = \gamma^n u[n]$ ROC: $|z| > |\gamma|$

- Find the z – transform of the **causal** signal $\gamma^n u[n]$, where γ is a constant.
- By definition:

$$\begin{aligned} X[z] &= \sum_{n=-\infty}^{\infty} \gamma^n u[n] z^{-n} = \sum_{n=0}^{\infty} \gamma^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^n \\ &= 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \dots \end{aligned}$$

- We apply the geometric progression formula:

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}, \quad |x| < 1$$

- Therefore,

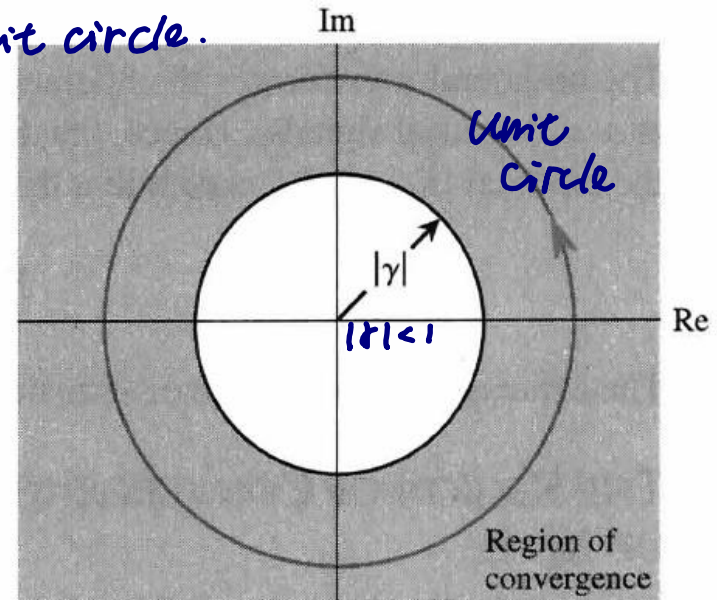
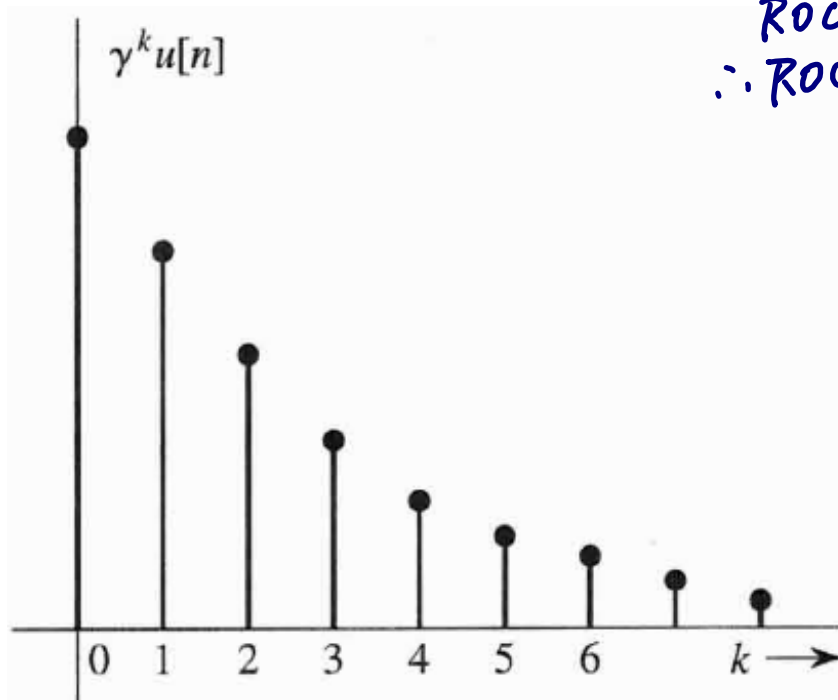
$$\begin{aligned} X[z] &= \frac{1}{1 - \frac{\gamma}{z}}, \quad \left|\frac{\gamma}{z}\right| < 1 \\ &= \frac{z}{z-\gamma}, \quad |z| > |\gamma| \end{aligned}$$

- We notice that the z – transform exists for certain values of z . These values form the so called Region-of-Convergence (ROC) of the transform.

Example: Find the z — transform of $x[n] = \gamma^n u[n]$ cont.

- Observe that a simple rational equation in z -domain corresponds to an infinite sequence of samples in time-domain.
- The figures below depict the signal in time (left) for $|\gamma| < 1$ and the ROC, shown with the shaded area, within the z — plane.

Handwritten notes:
 $x[n]$ absolutely summable: $|\gamma| < 1$
 ROC: $|z| > |\gamma|$
 \therefore ROC \supseteq unit circle.



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} \sum_{i=1}^K r_i^n z^{-n} = \sum_{i=1}^K \frac{z}{z - r_i} \quad |z| > |r_i|_{\max}$$

Generic form of a causal signal poles at $z = r_i$

- Consider the causal signal $x[n] = \sum_{i=1}^K \gamma_i^n u[n]$ with $X(z) = \sum_{i=1}^K \frac{z}{z - \gamma_i}$.
- In that case the ROC is the intersection of the ROCs of the individual terms, i.e., the intersection of the sets $|z| > |\gamma_i|$ i.e., ROC: $|z| > |\gamma_{\max}|$
- In case that $x[n]$ is the impulse response of a system, the transfer function of the system is the rational function $X(z) = \sum_{i=1}^K \frac{z}{z - \gamma_i}$ with poles γ_i .
- The above analysis yields the following properties regarding the ROC:

PROPERTY:

If $x[n]$ is a causal signal, the ROC of its z -transform is $|z| > |\gamma_{\max}|$ with γ_{\max} the maximum magnitude pole of the z -transform.

- In the general case of $x[n]$ being a right-sided signal (RSS) the ROC is as above but might not include ∞ (think why?).

PROPERTY:

No pole can exist in ROC.



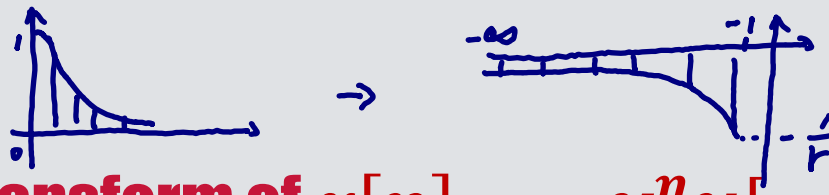
Generic form of a causal signal cont.

- absolutely summable \Leftrightarrow converges for $|z|=1$* *bounded: $|r_i| < 1$* *ROC: $|z| > |r_i|_{\max}$* *\Rightarrow ROC must include the unit circle.*
- The signal $x[n] = \sum_{i=1}^K \gamma_i^n u[n]$ is bounded only if $|\gamma_i| < 1 \forall i$ or $|\gamma_{\max}| < 1$.
 - In that case the ROC includes a circle with radius equal to 1. This is known as the **unit circle**.
 - The above observation yields the following property:

PROPERTY:

If the ROC of $X(z)$ includes the unit circle in z -plane, then the signal in time is bounded and its Discrete Time Fourier Transform exists.

- In case that $\gamma^n u[n]$ is part of a causal system's impulse response, we see that the condition $|\gamma| < 1$ must hold. This is because, since $\lim_{n \rightarrow \infty} (\gamma)^n = \infty$, for $|\gamma| > 1$, the system will be unstable in that case.
- Therefore, in causal systems, stability requires that the ROC of the system's transfer function includes the unit circle.



Example: Find the z – transform of $x[n] = -\gamma^n u[-n - 1]$

- Find the z – transform of the anti-causal signal $-\gamma^n u[-n - 1]$, where γ is a constant.

- By definition:

$$\begin{aligned}
 X[z] &= \sum_{n=-\infty}^{\infty} -\gamma^n u[-n - 1] z^{-n} = \sum_{n=-\infty}^{-1} -\gamma^n z^{-n} = -\sum_{n=1}^{\infty} \gamma^n z^n = -\sum_{n=1}^{\infty} \left(\frac{z}{\gamma}\right)^n \\
 &= -\frac{z}{\gamma} \frac{1}{1 - \frac{z}{\gamma}} = -\frac{z}{\gamma} \frac{1}{1 - \frac{z}{\gamma}} = -\frac{z}{z - \gamma} \quad |z| < |\gamma|
 \end{aligned}$$

$$\begin{aligned}
 X[z] &= \sum_{n=-\infty}^{\infty} -\gamma^n u[-n - 1] z^{-n} = \sum_{n=-\infty}^{-1} -\gamma^n z^{-n} = -\sum_{n=1}^{\infty} \gamma^{-n} z^n = -\sum_{n=1}^{\infty} \left(\frac{z}{\gamma}\right)^n \\
 &= -\frac{z}{\gamma} \sum_{n=0}^{\infty} \left(\frac{z}{\gamma}\right)^n = -\left(\frac{z}{\gamma}\right) \left[1 + \left(\frac{z}{\gamma}\right) + \left(\frac{z}{\gamma}\right)^2 + \left(\frac{z}{\gamma}\right)^3 + \dots \right]
 \end{aligned}$$

- Therefore,

anticausal: working offline

$$\begin{aligned}
 X[z] &= -\left(\frac{z}{\gamma}\right) \frac{1}{1 - \frac{z}{\gamma}}, \quad \left|\frac{z}{\gamma}\right| < 1 \\
 &= \frac{z}{z - \gamma}, \quad |z| < |\gamma|
 \end{aligned}$$

- We notice that the z – transform exists for certain values of z , which consist the complement of the ROC of the function $\gamma^n u[n]$ with respect to the z – plane.

$$\begin{aligned}
 x[n] &= \sum_{i=1}^K -\gamma_i^n u[-n-1] \\
 X(z) &= \sum_{n=-\infty}^{+\infty} \sum_{i=1}^K -\gamma_i^n u[-n-1] z^{-n} = - \sum_{n=-\infty}^{-1} \sum_{i=1}^K \left(\frac{\gamma_i}{z}\right)^n = - \sum_{n=1}^{+\infty} \sum_{i=1}^K \left(\frac{z}{\gamma_i}\right)^n \\
 &= - \sum_{i=1}^K \frac{z}{\gamma_i} \sum_{n=0}^{+\infty} \left(\frac{z}{\gamma_i}\right)^n = - \sum_{i=1}^K \frac{z}{\gamma_i} \frac{1}{1 - \frac{z}{\gamma_i}} = - \sum_{i=1}^K \frac{z}{\gamma_i} \frac{\gamma_i}{\gamma_i - z} = \sum_{i=1}^K \frac{z}{z - \gamma_i}
 \end{aligned}$$

- Consider the anti-causal signal $x[n] = \sum_{i=1}^K -\gamma_i^n u[-n-1]$ with $|z| < |\gamma_{i_{\min}}|$
z-transform $X(z) = \sum_{i=1}^K \frac{z}{z - \gamma_i}$.
- In that case the ROC is the intersection of the sets $|z| < |\gamma_i|$, i.e., ROC: $|z| < |\gamma_{\min}|$
- In case that $x[n]$ is the impulse response of a system, the transfer function of the system is the rational function $X(z) = \sum_{i=1}^K \frac{z}{z - \gamma_i}$ with poles γ_i .
- The above analysis yield the following property regarding ROCs:

PROPERTY:

If $x[n]$ is an anti-causal signal, the ROC of its z-transform is $|z| < |\gamma_{\min}|$ with γ_{\min} the minimum magnitude pole of the z-transform.

□ In the general case of $x[n]$ being a left-sided signal (LSS) the ROC is as above but might not include 0 (think why).

Summary of previous examples

- We proved that the following two functions:
 - The causal function $\gamma^n u[n]$ and
 - the anti-causal function $\ominus \gamma^n u[-n - 1]$ have:
 - ❖ The same analytical expression for their z –transforms.
 - ❖ Complementary ROCs. More specifically, the union of their ROCS forms the entire z –plane.
- The above observations verify that the analytical expression alone is not sufficient to define the z –transform of a signal. The ROC is also required.

causal vs. anticausal:
• same z -transform
• complementary ROCs

$$\begin{array}{ccc} r^n u[n] & \xrightarrow{\frac{z}{z-r}} & -r^n u[-n-1] \\ |z| > |r| & & |z| < |r| \end{array}$$

Two-sided signals

- Example:** Find the z –transform of the two-sided signal:

$$x[n] = 2^n u[n] - 4^n u[-n - 1]$$

Based on the previous analysis we have:

$$X[z] = \frac{z}{z-2} + \frac{z}{z-4}, \text{ ROC: } |z| > 2 \cap |z| < 4 \text{ or ROC: } 2 < |z| < 4$$

- Example:** Find the z –transform of the two-sided signal:

$$x[n] = 4^n u[n] - 2^n u[-n - 1]$$

Based on the previous analysis we have:

$$X[z] = \frac{z}{z-2} + \frac{z}{z-4}, \text{ ROC: } |z| > 4 \cap |z| < 2 \text{ or ROC: } \emptyset$$

PROPERTY:

If $x[n]$ is two-sided signal then the ROC of its z –transform is of the form:

□ $\gamma_1 < |z| < \gamma_2$ with γ_1, γ_2 poles of the system or

□ \emptyset

$$\delta[n] \xrightarrow{\frac{zT}{1zT}} 1$$

$$u[n] \xrightarrow{\frac{zT}{1zT}} \frac{z}{z-1}, |z| > 1$$

Example: Find the z -transform of $\delta[n]$ and $u[n]$

- By definition $\delta[0] = 1$ and $\delta[n] = 0$ for $n \neq 0$.

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = z^0 = 1$$

$$X[z] = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = \delta[0] z^{-0} = 1$$

- By definition $u[n] = 1$ for $n \geq 0$.

$$X(z) = \sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}, |z| > 1$$

$$X[z] = \sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-\frac{1}{z}}, \left| \frac{1}{z} \right| < 1$$

$$= \frac{z}{z-1}, |z| > 1$$

$$\cos \beta n u[n] \xleftrightarrow{ZT} \frac{z(z - \cos \beta)}{z^2 - 2z \cos \beta + 1}, |z| > 1$$

Example: Find the z — transform of $\cos \beta n u[n]$

$$x[n] = \cos \beta n u[n] = \frac{1}{2} (e^{j\beta n} + e^{-j\beta n}) u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \frac{1}{2} (e^{j\beta n} + e^{-j\beta n}) u[n] = \frac{1}{2} \left[\sum_{n=0}^{\infty} (z e^{j\beta})^{-n} + \sum_{n=0}^{\infty} \left(\frac{e^{j\beta}}{z} \right)^n \right]$$

- We write $\cos \beta n = \frac{1}{2} (e^{j\beta n} + e^{-j\beta n})$. $= \frac{1}{2} \left(\frac{1}{1 - (z e^{j\beta})^{-1}} + \frac{1}{1 - \frac{e^{j\beta}}{z}} \right)$

- From previous analysis we showed that:

$$\gamma^n u[n] \Leftrightarrow \frac{z}{z - \gamma}, |z| > |\gamma|$$

$$= \frac{1}{2} \left(\frac{z}{z - e^{j\beta}} + \frac{z}{z - e^{-j\beta}} \right)$$

$$= \frac{1}{2} \frac{z^2 - z e^{j\beta} + z^2 - z e^{-j\beta}}{z^2 - z(e^{j\beta} + e^{-j\beta}) + 1}$$

- Hence,

$$e^{\pm j\beta n} u[n] \Leftrightarrow \frac{z}{z - e^{\pm j\beta}}, |z| > |e^{\pm j\beta}| = 1$$

$$= \frac{1}{2} \frac{z^2 - z \cdot 2 \cos \beta}{z^2 - z \cdot 2 \cos \beta + 1} = \frac{z^2 - z \cos \beta}{z^2 - 2z \cos \beta + 1}$$

- Therefore,

$$X[z] = \frac{1}{2} \left[\frac{z}{z - e^{j\beta}} + \frac{z}{z - e^{-j\beta}} \right] = \frac{z(z - \cos \beta)}{z^2 - 2z \cos \beta + 1}, |z| > 1 \quad |z| > |e^{\pm j\beta}| = 1$$

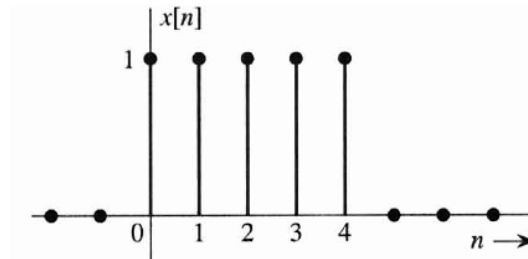
$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$$

z — transform of 5 impulses

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = z^0 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$$

$$= \frac{1 - (\frac{1}{z})^5}{1 - \frac{1}{z}}$$

- Find the z — transform of the signal depicted in the figure.




- By definition:

$$X[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} = \sum_{k=0}^4 (z^{-1})^k = \frac{1 - (z^{-1})^5}{1 - z^{-1}} = \frac{z}{z-1} (1 - z^{-5})$$

Inverse z – transform

- As with other transforms, inverse z – transform is used to derive $x[n]$ from $X[z]$, and is formally defined as:

$$x[n] = \frac{1}{2\pi j} \oint X[z] z^{n-1} dz$$


- Here the symbol \oint indicates an integration in counter-clockwise direction around a circle within the ROC and $z = Re^{j\theta}$.
- Such contour integral is difficult to evaluate (but could be done using Cauchy's residue theorem), therefore we often use other techniques to obtain the inverse z – transform.
- One such technique is to use a z – transform pairs Table shown in the last two slides with partial fraction expansion.

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz = \frac{1}{2\pi j} \oint \sum_{m=-\infty}^{\infty} x[m] z^{-m} z^{n-1} dz = \frac{1}{2\pi j} \sum_{m=-\infty}^{\infty} x[m] \oint z^{n-m-1} dz$$

$$\oint z^{k-1} dz \stackrel{z=Re^{j\theta}}{dz=jRe^{j\theta}d\theta} \int_0^{2\pi} R^{k-1} e^{j\theta(k-1)} j R e^{j\theta} d\theta = j \int_0^{2\pi} R^k e^{jk\theta} d\theta = R^k j \left. \frac{1}{jk} e^{jk\theta} \right|_0^{2\pi} = 0, k \neq 0$$

Inverse Z-transform: Proof

Proof: $\therefore x[n] = \frac{1}{2\pi j} \sum_{m=-\infty}^{\infty} x[m] 2\pi j \delta[n-m] = x[n].$

$j \int_0^{2\pi} 1 d\theta = 2\pi j, k=0$

$$\frac{1}{2\pi j} \oint X(z) z^{n-1} dz = \frac{1}{2\pi j} \oint \left(\sum_{m=-\infty}^{\infty} x[m] z^{-m} \right) z^{n-1} dz = 2\pi j \delta[k]$$

$$= \sum_{m=-\infty}^{\infty} x[m] \frac{1}{2\pi j} \oint z^{n-m-1} dz = \sum_{m=-\infty}^{\infty} x[m] \delta[n-m] = x[n]$$

□ For the above we used the Cauchy's theorem:

$$\frac{1}{2\pi j} \oint z^{k-1} dz = \delta[k] \text{ for } z = Re^{j\theta} \text{ anti-clockwise.}$$

$$\frac{dz}{d\theta} = jRe^{j\theta} \Rightarrow \frac{1}{2\pi j} \oint z^{k-1} dz = \frac{1}{2\pi j} \int_{\theta=0}^{2\pi} R^{k-1} e^{j(k-1)\theta} j R e^{j\theta} d\theta =$$

$$\frac{R^k}{2\pi} \int_{\theta=0}^{2\pi} e^{jk\theta} d\theta = R^k \delta[k]$$

$\int_0^{2\pi} e^{jk\theta} d\theta = \left. \frac{1}{jk} e^{jk\theta} \right|_0^{2\pi} = 0, k \neq 0$

$$\left[\frac{R^k}{2\pi} \int_{\theta=0}^{2\pi} e^{jk\theta} d\theta = \begin{cases} 0 & k \neq 0 \\ \frac{R^k}{2\pi} 2\pi = R^k & k = 0 \end{cases} \right] \int_0^{2\pi} 1 d\theta = 2\pi, k=0$$

$$X(z) = \frac{8z-19}{z(z-2)(z-3)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$xz, z=0 \Rightarrow \frac{0-19}{(0-2)(0-3)} = A = -\frac{19}{6}$$

Find the inverse z — transform in the case of real unique poles

$$x(z-2), z=2 \Rightarrow \frac{16-19}{2(-1)} = B = \frac{3}{2}$$

$$x(z-3), z=3 \Rightarrow \frac{24-19}{3 \cdot 1} = C = \frac{5}{3}$$

assume causal!

- Find the inverse z — transform of $X[z] = \frac{8z-19}{(z-2)(z-3)}$ $\rightarrow \frac{5}{3} 3^n u[n], |z| > 3$ ③

Solution

$$\therefore X(z) = -\frac{19}{6} + \frac{3}{2} \frac{z}{z-2} + \frac{5}{3} \frac{z}{z-3}$$

Target $\frac{X(z)}{z}$
to recover
 z in the nominators.

$$\frac{X[z]}{z} = \frac{8z-19}{z(z-2)(z-3)} = \frac{(-\frac{19}{6})}{z} + \frac{3/2}{z-2} + \frac{5/3}{z-3}$$

① ③ causal. unstable $|z| > 3$
② ④ anticasual stable $|z| < 2$
① ④ two-sided unstable $3 > |z| > 2$

$$X[z] = -\frac{19}{6} + \frac{3}{2} \left(\frac{z}{z-2} \right) + \frac{5}{3} \left(\frac{z}{z-3} \right)$$

By using the simple transforms that we derived previously we get:

$$x[n] = -\frac{19}{6} \delta[n] + \left[\frac{3}{2} 2^n + \frac{5}{3} 3^n \right] u[n]$$

Solution 1:
(derivative)

$$\frac{X(z)}{z} = \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{z-2}$$

$$X(z-1) \cdot z=1 \Rightarrow k = \frac{2(-1)+12}{(1-2)^3} = -3$$

Find the inverse z-transform in the case of real repeated poles

$$X(z-2) \cdot \frac{d}{dz} \cdot z=2 \Rightarrow a_0 = \frac{2z^2 - 11z + 12}{z-1} \Big|_{z=2} = -2$$

$$X(z-2) \cdot \frac{d^2}{dz^2} \cdot z=2 \Rightarrow \frac{d}{dz} \left(\frac{2z^2 - 11z + 12}{z-1} \right) = \frac{d}{dz} \left(k \frac{(z-2)^3}{z-1} + a_2(z-2)^2 + a_1(z-2) + a_0 \right)$$

- Find the inverse z-transform of $X[z] = \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3}$

Solution

$$\frac{(4z-11)(z-1) - (2z^2 - 11z + 12)}{(z-1)^2} = \frac{d}{dz} \left(k \frac{(z-2)^3}{z-1} + a_2(z-2)^2 + a_1(z-2) + a_0 \right)$$

$$\Rightarrow \frac{4z^2 - 11z - 4z + 11 - 2z^2 + 11z - 12}{(z-1)^2} = \frac{2z^2 - 4z - 1}{(z-1)^2} = a_1 \Rightarrow a_1 = -1$$

$$\frac{X[z]}{z} = \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

- We use the so called **covering method** to find k and a_0

$$X(z-2) \cdot \frac{d}{dz} \cdot z=2$$

$$\frac{(4z-4)(z^2-2z+1) - (2z^2-4z-1) \cdot 2(z-1)}{(z-1)^4} \cdot \frac{(2z^2 - 11z + 12)}{(z-1)(z-2)^3} \Big|_{z=1} = -3$$

$$= 6 = 2a_2 \Rightarrow a_2 = 3$$

$$a_0 = \frac{(2z^2 - 11z + 12)}{(z-1)(z-2)^3} \Big|_{z=2} = -2$$

The shaded areas above indicate that they are excluded from the entire function when the specific value of z is applied.

Solution 2. (limit)

$$\frac{X(z)}{z} = \frac{(2z^2 - 11z + 12)}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

$$X(z-1), z=1 \Rightarrow k = \frac{2z^2 - 11z + 12}{(z-2)^3} \Big|_{z=1} = -3$$

$$X(z-2)^3, z=2 \Rightarrow a_0 = \frac{2z^2 - 11z + 12}{z-1} \Big|_{z=2} = -2$$

Find the inverse z-transform in the case of real repeated poles cont.

$$\therefore \frac{X(z)}{z} = \frac{-3}{z-1} + \frac{-2}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_1}{z-2} = \frac{(2z^2 - 11z + 12)}{(z-1)(z-2)^3}$$

• Find the inverse z-transform of $X[z] = \frac{z(2z^2 - 11z + 12)}{(z-1)(z-2)^3}$

$$X[z] \Rightarrow -3 \frac{z}{z-1} - 2 \frac{z}{(z-2)^3} + a_1 \frac{z}{(z-2)^2} + a_2 \frac{z}{z-2} = \frac{(2z^2 - 11z + 12)z}{(z-1)(z-2)^3}$$

Solution

$$z \rightarrow \infty \Rightarrow -3 + 0 + 0 + a_2 = 0 \Rightarrow a_2 = 3$$

$$\frac{X[z]}{z} = \frac{(2z^2 - 11z + 12)}{(z-1)(z-2)^3} = \frac{-3}{z-1} + \frac{-2}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

$$z \rightarrow 0 \Rightarrow 3 + \frac{1}{4} + \frac{a_1}{4} - \frac{2}{8} = \frac{3}{8} \Rightarrow a_1 = -1$$

■ To find a_2 we multiply both sides of the above equation with z and let $z \rightarrow \infty$.

$$0 = -3 - 0 + 0 + a_2 \Rightarrow a_2 = 3$$

■ To find a_1 let $z \rightarrow 0$.

$$\frac{12}{8} = 3 + \frac{1}{4} + \frac{a_1}{4} - \frac{3}{2} \Rightarrow a_1 = -1$$

$$\frac{X[z]}{z} = \frac{(2z^2 - 11z + 12)}{(z-1)(z-2)^3} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{(z-2)} \Rightarrow$$

$$X[z] = \frac{-3z}{z-1} - \frac{2z}{(z-2)^3} - \frac{z}{(z-2)^2} + \frac{3z}{(z-2)}$$

$\therefore X(z) = -3 \frac{z}{z-1} - 2 \frac{z}{(z-2)^3} - \frac{z}{(z-2)^2} + 3 \frac{z}{z-2}$. Assume $x[n]$ casual.

Find the inverse z -transform in the case of real repeated poles cont.

$$X[z] = -3 \frac{z}{z-1} - 2 \frac{z}{(z-2)^3} - \frac{z}{(z-2)^2} + 3 \frac{z}{z-2}$$

$$X[z] = \frac{-3z}{z-1} - \frac{2z}{(z-2)^3} - \frac{z}{(z-2)^2} + \frac{3z}{(z-2)}$$

- We use the following properties:

$$\gamma^n u[n] \Leftrightarrow \frac{z}{z-\gamma}$$

$$\frac{n(n-1)(n-2)\dots(n-m+1)}{\gamma^m m!} \gamma^n u[n] \Leftrightarrow \frac{z}{(z-\gamma)^{m+1}}$$

$$\left[-\frac{2z}{(z-2)^3} = (-2) \frac{z}{(z-2)^{2+1}} \Leftrightarrow (-2) \frac{n(n-1)}{2^2 2!} \gamma^n u[n] = -2 \frac{n(n-1)}{8} \cdot 2^n u[n] \right]$$

- Therefore,

$$x[n] = \left[-3 \cdot 1^n - 2 \frac{n(n-1)}{8} \cdot 2^n - \frac{n}{2} \cdot 2^n + 3 \cdot 2^n \right] u[n]$$

$$= - \left[3 + \frac{1}{4} (n^2 + n - 12) 2^n \right] u[n]$$

$$\frac{X(z)}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{k}{z-1} + \frac{Az+B}{z^2-6z+25}$$

$$k = \frac{2(3z+17)}{z^2-6z+25} \Big|_{z=1} = 2$$

Find the inverse z — transform in the case of complex poles

$$X(z) \cdot z \rightarrow \infty \quad \frac{2z(3z+17)}{(z-1)(z^2-6z+25)} = 2 \frac{z}{z-1} + \frac{Az^2+Bz}{z^2-6z+25}$$

$$0 = 2 + A \rightarrow A = -2$$

- Find the inverse z — transform of $X[z] = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$

Solution

$$\frac{X(z)}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

$$z \rightarrow 0 : -\frac{34}{25} = -2 + \frac{B}{25} \Rightarrow B = 16$$

$$X[z] = \frac{2z(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

$$\frac{X[z]}{z} = \frac{(2z^2-11z+12)}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

Whenever we encounter a complex pole we need to use a special partial fraction method called **quadratic factors method**.

$$\frac{X[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{Az+B}{z^2-6z+25}$$

We multiply both sides with z and let $z \rightarrow \infty$:

$$0 = 2 + A \Rightarrow A = -2$$

Therefore,

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

$$X(z) = 2 \frac{z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$$

$$r|\gamma|^n \cos(\beta n + \theta) u[n] \leftrightarrow \frac{z(Az+B)}{z^2+2az+|\gamma|^2}$$

Find the inverse z — transform in the case of complex poles cont.

$$\therefore \begin{cases} A = -2 \\ B = 16 \\ a = -3 \\ |\gamma| = 5 \end{cases} \quad r = \frac{\sqrt{A^2|\gamma|^2 + B^2 - 2AaB}}{|\gamma|^2 - a^2} = 3.2$$

$$\frac{z(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

To find B we let $z = 0$:

$$\theta = \tan^{-1} \frac{Aa-B}{A\sqrt{|\gamma|^2-a^2}}$$

$$\frac{-34}{25} = -2 + \frac{B}{25} \Rightarrow B = 16$$

$$\frac{X[z]}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2-6z+25} \Rightarrow X[z] = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$$

We use the following property.

$$r|\gamma|^n \cos(\beta n + \theta) u[n] \Leftrightarrow \frac{z(Az+B)}{z^2+2az+|\gamma|^2} \text{ with } A = -2, B = 16, a = -3, |\gamma| = 5.$$

$$r = \frac{\sqrt{A^2|\gamma|^2 + B^2 - 2AaB}}{|\gamma|^2 - a^2} = \frac{\sqrt{4 \cdot 25 + 256 - 2 \cdot (-2) \cdot (-3) \cdot 16}}{25 - 9} = 3.2, \beta = \cos^{-1} \frac{-a}{|\gamma|} = 0.927 \text{ rad},$$

$$\theta = \tan^{-1} \frac{Aa-B}{A\sqrt{|\gamma|^2-a^2}} = -2.246 \text{ rad}.$$

$$\text{Therefore, } x[n] = [2 + 3.2 \cos(0.927n - 2.246)]u[n]$$

z — transform Table

No.	$x[n]$	$X[z]$
1	$\delta[n - n]$	z^{-k}
2	$u[n]$	$\frac{z}{z - 1}$
3	$nu[n]$	$\frac{z}{(z - 1)^2}$
4	$n^2u[n]$	$\frac{z(z + 1)}{(z - 1)^3}$
5	$n^3u[n]$	$\frac{z(z^2 + 4z + 1)}{(z - 1)^4}$
6	$\gamma^n u[n]$	$\frac{z}{z - \gamma}$
7	$\gamma^{n-1} u[n - 1]$	$\frac{1}{z - \gamma}$
8	$n\gamma^n u[n]$	$\frac{\gamma z}{(z - \gamma)^2}$

z — transform Table

No.	$x[n]$	$X[z]$
10	$\frac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!} \gamma^n u[n]$	$\frac{z}{(z-\gamma)^{m+1}}$
11a	$ \gamma ^n \cos \beta n u[n]$	$\frac{z(z- \gamma \cos \beta)}{z^2 - (2 \gamma \cos \beta)z + \gamma ^2}$
11b	$ \gamma ^n \sin \beta n u[n]$	$\frac{z \gamma \sin \beta}{z^2 - (2 \gamma \cos \beta)z + \gamma ^2}$
12a	$r \gamma ^n \cos (\beta n + \theta) u[n]$	$\frac{rz[z \cos \theta - \gamma \cos (\beta - \theta)]}{z^2 - (2 \gamma \cos \beta)z + \gamma ^2}$
12b	$r \gamma ^n \cos (\beta n + \theta) u[n] \quad \gamma = \gamma e^{j\beta}$	$\frac{(0.5re^{j\theta})z}{z-\gamma} + \frac{(0.5re^{-j\theta})z}{z-\gamma^*}$
12c	$r \gamma ^n \cos (\beta n + \theta) u[n]$	$\frac{z(Az+B)}{z^2 + 2az + \gamma ^2}$

$$r = \sqrt{\frac{A^2|\gamma|^2 + B^2 - 2AaB}{|\gamma|^2 - a^2}}$$

$$\beta = \cos^{-1} \frac{-a}{|\gamma|}$$

$$\theta = \tan^{-1} \frac{Aa - B}{A\sqrt{|\gamma|^2 - a^2}}$$