1.

Kraft's ineq: $\sum_{i=1}^{|X|} r^{-l_i} \leq 1$

$\sum r^{-l_i} = 4^{-1} \times 2 + 4^{-2} \times 7 = 0.9375 \leq 1$ ✓

To construct such a code we should make sure
that any code is not a prefix of any other
codeword in the code. We have

$\{0 . 1 . 20 . 21 . 22 . 30 . 31 . 32 . 33\}$ instantaneous code

$\{0.15\ 0.5\ 0.1\ 0.1\ 0.1\ 0.1\ 0.1\ 0.1\ 0.1\}$ prob

Expected length:

$E(l) = \sum P(l_i) l_i = 1 \times 0.15 \times 2 + 2 \times 0.1 \times 7$

2.

To check a codebook we first verify kraft's ineq:

if it satisfies $\Rightarrow$ there exists an instantaneous code.

if a code is <u>instantaneous</u>
(no codeword is a prefix of others)

$\downarrow$

uniquely decodable
(its extension must be non-singular)

$\downarrow$

non-singular $(x_1 \neq x_2 \Rightarrow C(x_1) \neq C(x_2))$

2.

a) $\{1, 01, 000, 001\}$

Kraft's ineq:

$$2^{-1} + 2^{-2} + 2^{-3} \times 2 \leq 1$$

inst. → uniquely decodable → non-singular

b) $\{0.10, 000, 100\}$

nonsingular (no codewords equal)?   ✓

uniquely decodable (no ambiguity/different
                    interpretation)?   ✗

instantaneous (decode without ref. to future?) ✗

c) $\{01, 01, 110, 100\}$

$$\sum r^{-l_i} = 2^{-2} \times 2 + 2^{-3} \times 2 < 1$$

two codewords equal ⇒ not nonsingular
                              ↓
                       not u.d.
                              ↓
                       not inst.

d) $\{0. 01, 011, 0111\}$

$2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} < 1$

non-singular ✓

u.d. ✓ (only one possible string. c^+ is non-singular)

inst. ✗ (0 is prefix of the others)

e) $\{10, 01, 0010, 0111\}$
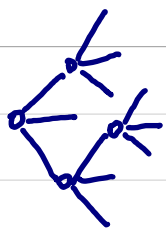
non-singular ✓

u.d. ✓

inst. ✗ (01 is prefix of 0111)

3.

$M(D-1) + 1$

m: integer

D: number of leaves at each node

$\xleftarrow{} \Rightarrow$ $\underline{-1 + D}$
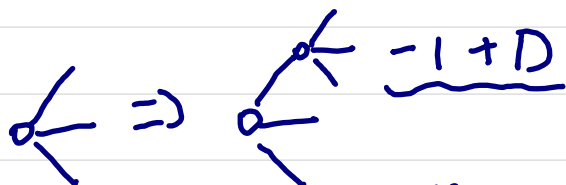
every time remove one leaf.

add D leaves

$+$

initial leaf $= 1$

$1 + m(D-1)$

4.

a) $P = \{0.25, 0.2, 0.15, 0.1, 0.1, 0.1, 0.1\}$

$D = 3 \Rightarrow m(D-1) + 1 = 2m + 1$

0.25   0.3   0.45 — 0 — 1
0.2   0.25   0.3 — 1, 2
0.15   0.2 — 0   0.25
0.1   0.15 — 1
0.1 — 0   0 — 1, 2
0.1 — 1
0.1 — 2

c) $D = 5 \Rightarrow m(D-1) + 1 = 4m + 1$

0.25   0.3 — 0 — 1
0.2   0.25 — 1
0.15   0.2 — 2
0.1   0.15 — 3
0.1 — 0   0.1 — 4
0.1 — 1
0.1 — 2
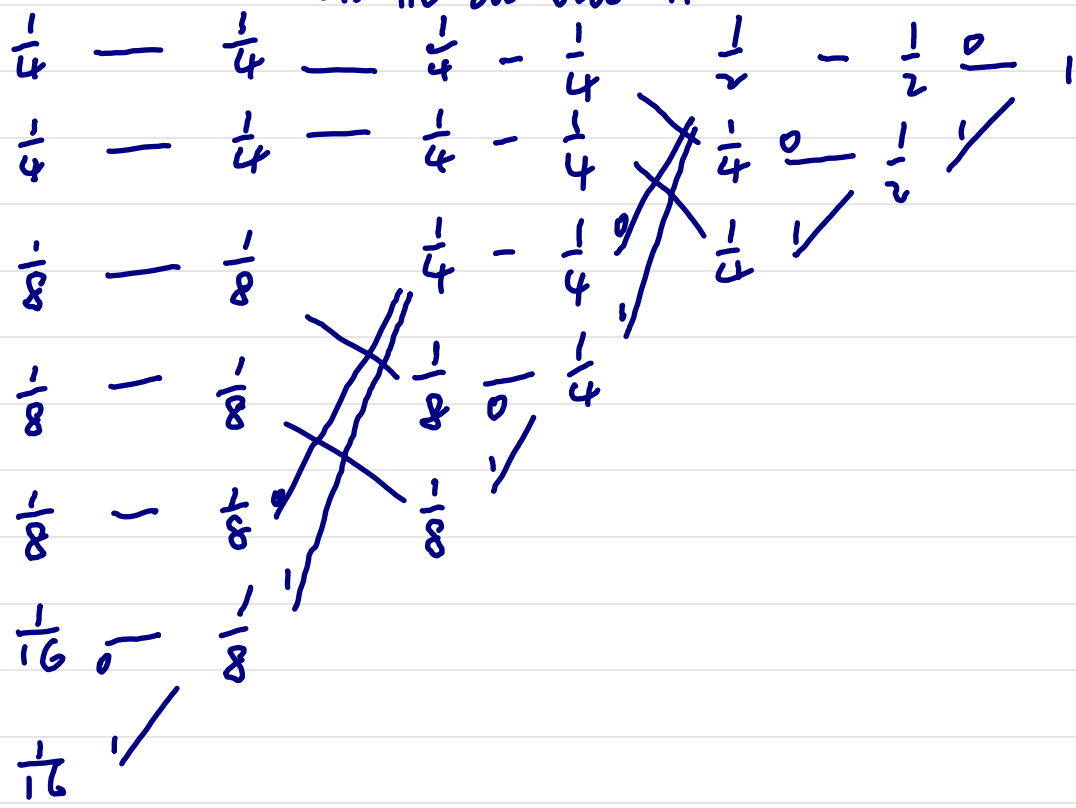0.1 — 3
[0, 0] — 4
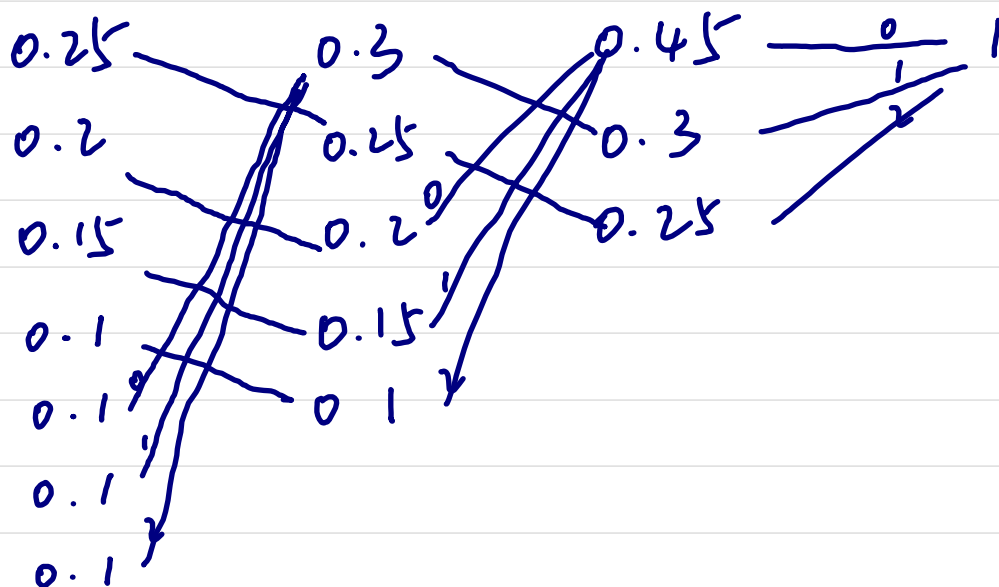
dummy symbols to satisfy $4m + 1$
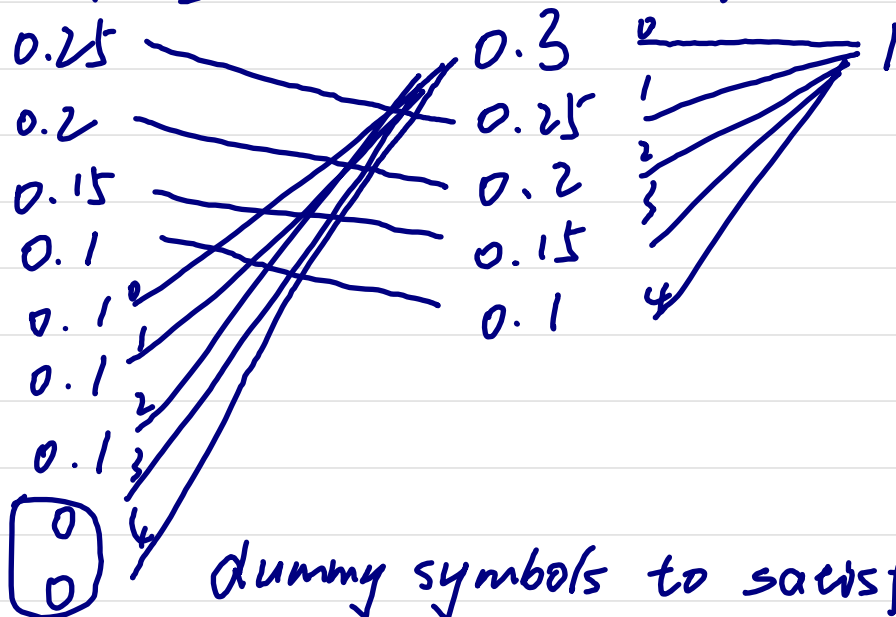
5.

The entropy of a source (prob. vector) is equal
to the expected value of codelength if
it satisfies:

$$\sum l \cdot 2^l = H(p)$$

For this we need the probabilities equal 2^{-l}
for each l

$$\{2.2, 3. 3. 3. 4. 4\} \quad (binary \to D = 2)$$

$$\{\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{16}\} \to \text{sums up to 1}$$

01   11   010   110   000   0000   1100   to be a prob. vec.

$\tfrac{1}{4}$ — $\tfrac{1}{4}$ — $\tfrac{1}{4}$ - $\tfrac{1}{4}$ $\tfrac{1}{2}$ - $\tfrac{1}{2}$ $\tfrac{0}{}$ 1

$\tfrac{1}{4}$ — $\tfrac{1}{4}$ — $\tfrac{1}{4}$ - $\tfrac{1}{4}$ $\tfrac{1}{4}$ 0 $\tfrac{1}{2}$

$\tfrac{1}{8}$ — $\tfrac{1}{8}$ $\tfrac{1}{4}$ - $\tfrac{1}{4}$ $\tfrac{1}{2}$

$\tfrac{1}{8}$ — $\tfrac{1}{8}$ $\tfrac{1}{8}$ $\tfrac{1}{4}$

$\tfrac{1}{8}$ — $\tfrac{1}{8}$ $\tfrac{1}{8}$

$\tfrac{1}{16}$ $\tfrac{1}{8}$

$\tfrac{1}{16}$

6.

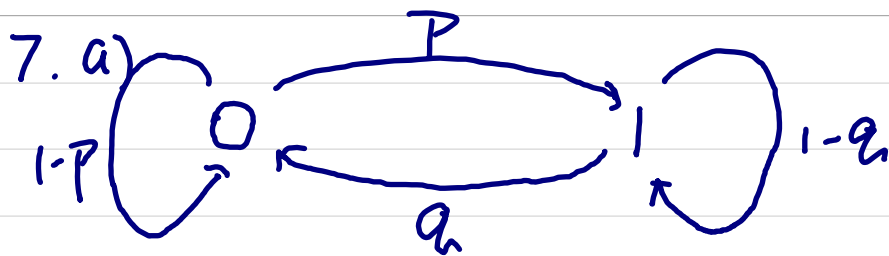$$\underline{H(X_i \mid X_{i-1}, X_{i-2} \cdots X_{i-n})} = \underline{H(X_i \mid X_{i+1}, X_{i+2} \cdots X_{i+n})}$$

$$H(X_i, X_{i-1}, \cdots X_{i-n}) - H(X_{i-1}, X_{i-2} \cdots X_{i-n})$$

$$\overset{(a)}{=} H(X_{i+n}, X_{i+n-1}, \cdots X_i) - H(X_{i+n-1}, X_{i+n-2} \cdots X_i)$$

$$= H(X_i \mid X_{i+1} \cdots X_{i+n})$$

(a) because of stationary we shift the seq.
n times forward.

7. a)

$$T = \begin{pmatrix} 1-P & P \\ q & 1-q \end{pmatrix} \begin{matrix} 0 \\ 1 \end{matrix}$$
$$\qquad\quad \begin{matrix} 0 & \;\; 1 \end{matrix}$$

To find the stationary distribution we should have.

$$\begin{pmatrix} 1-P & P \\ q & 1-q \end{pmatrix} \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} \Rightarrow \alpha = \frac{q}{P+q}$$

b)

$$H(X) = H(X_n | X_{n-1}) \quad \longrightarrow \text{two-state Markov chain}$$

    in order to maximise $H(X)$ we have

$$H(X_n | X_{n-1}) \leq H(X_n) \leq 1$$

    We can attain $H(X) = 1$ if $P = q = \frac{1}{2}$

* Another approach would have to differentiate
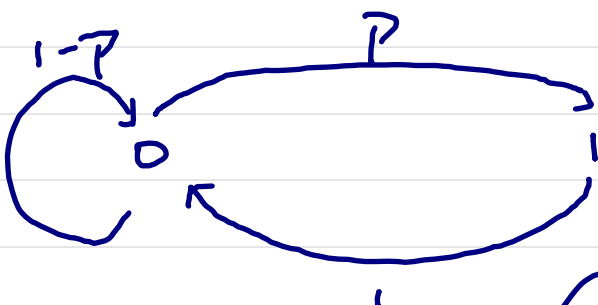   w.r.t. q and P and equate with zero
   to solve it.

c)

(i) $H(X) = H(X_2 | X_1) = H(X_2 | X_1 = 0) \underset{\text{prob. state 0 is chosen}}{P(X_1 = 0)} + H(X_2 | X_1 = 1) \underset{\text{}}{P(X_1 = 1)}$

$= \alpha H(p) + (1-\alpha) H(q)$

$\underset{\text{entropy of going to state } X_2 = 0 \text{ or } 1 \text{ when we observe state } 1}{}$

$\underset{q=1}{=} \alpha H(p)$

$1-p$
p
0
1
1

$S_1(n) = S_0(n-1)$
$S_0(n) = S(n-1)$

(once you see 1 the next r.v. is 0

(ii)

the next r.v. can be either 0 or 1.

$S_1(n) = S_0(n-1) = S(n-2)$

using this we have

$S(n) = S_0(n) + S_1(n) = S(n-1) + S(n-2)$

the total number
of 0/1 sequences

$f(1) = S(1) = 2$
$f(2) = S(2) = 3$
$f(3) = S(3) = 2+3 = 5$
$f(4) = S(4) = 8$
$f(5) = S(5) = 13$

Fibonacci

$\varphi = \dfrac{1 + \sqrt{5}}{2}$

golden number

$f(n) = \dfrac{\varphi^n + (1-\varphi)^n}{\sqrt{5}}$

(iii) $H(X) = \frac{1}{n} H(X^n)$

entropy rate / $(X_1, \dots X_n)$

The maximum is attained when X_n follows a uniform distribution over its space

The space of X^n is the sequence of 0/1 with length n which is S_n.

$$\Rightarrow \max H(X) \leq \max \frac{H(X^n)}{n} = \frac{1}{n} \log |S_n| = \frac{1}{n} \log \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$$

$$n \to \infty : \max_{n \to \infty} H(X) = \frac{1}{n} (n \log \varphi - \frac{1}{2} \log 5) = \log \varphi = \log \frac{1 + \sqrt{5}}{2}$$

8. $X_i \in \{A, B\} \sim \{0.9 \ 0.1\}$

$H(X) = 0.9 \log 0.9 + 0.1 \log 0.1$

$\{AA \quad AB \quad BA \quad BB\} \rightarrow$ use Huffman coding to design
$\ \ 0.81 \quad 0.09 \quad 0.09 \quad 0.01 \qquad$ the codebook

obtain the average length of the codebook by $\sum_{i=1}^{4} l_i P(i)$

redun. $H(P) - E(L)$