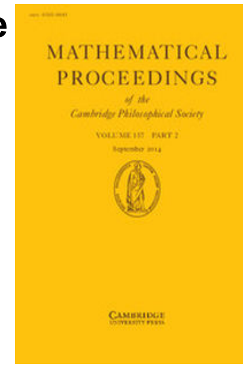

The numerical solution of Poisson's and the bi-harmonic equations by matrices

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THE NUMERICAL SOLUTION OF POISSON'S AND THE BI-HARMONIC EQUATIONS BY MATRICES

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This note describes a simple method of solving the matrix forms of the customary families of finite difference equations which approximate to Poisson's equation (in three dimensions as well as two) and to the bi-harmonic equation. It is intended for use where the boundary of the region over which a solution is wanted is comparatively simple, that is to say, can be subdivided into a comparatively few rectangular areas or rectangular parallelepipeds.

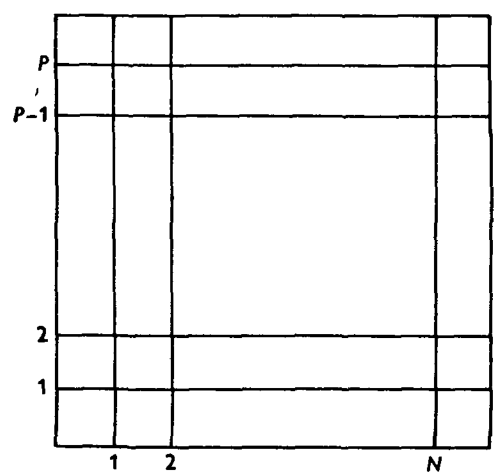

Fig. 1

1. *Poisson's equation in two dimensions.* Consider the rectangular region of sides $(N+1)h$, $(P+1)h$, shown in Fig. 1, covered (in the customary way) by a square grid of side h, a solution of Poisson's equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=g(x,y) \tag{1·1}$$

being desired which assumes prescribed values on the boundary. Then, indicating a function value at the grid point (ph, qh) by suffix p, q, the simplest customary finite difference approximation replaces (1·1) by the family of linear algebraic equations

$$v_{p-1,q}+v_{p+1,q}+v_{p,q-1}+v_{p,q+1}-4v_{p,q}=h^2g_{p,q} \quad (1\leqslant p\leqslant N, 1\leqslant q\leqslant P), \tag{1·2}$$

in which v's lying on the boundary are given their prescribed values.

Taking, for simplicity, $N = P = 3$, and writing equations (1·2) in matrix form after removing prescribed boundary values to the right-hand side of each equation, the equations assume the form

$$\left[\begin{array}{ccc:ccc:ccc} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hdashline -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ \hdashline 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{array}\right] \left[\begin{array}{c} v_{11} \\ v_{12} \\ v_{13} \\ \hdashline v_{21} \\ v_{22} \\ v_{23} \\ \hdashline v_{31} \\ v_{32} \\ v_{33} \end{array}\right] = \left[\begin{array}{c} \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \hdashline \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \\ \hdashline \gamma_{31} \\ \gamma_{32} \\ \gamma_{33} \end{array}\right] \qquad (1{\cdot}3)$$

if each line of the grid, from bottom to top, is successively traversed in order from left to right. That is to say, as pointed out by Karlqvist (1), the matrix equation, when partitioned into submatrices as indicated by the dotted lines, has the form

$$\begin{bmatrix} \mathbf{A} & -\mathbf{I} & \mathbf{O} \\ -\mathbf{I} & \mathbf{A} & -\mathbf{I} \\ \mathbf{O} & -\mathbf{I} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{\Gamma}_1 \\ \mathbf{\Gamma}_2 \\ \mathbf{\Gamma}_3 \end{bmatrix}, \qquad (1{\cdot}4)$$

and similarly, for the $N \times P$ grid of Fig. 1, the matrix form of the equation, when partitioned into submatrices, is

$$\mathbf{MV} = \begin{bmatrix} \mathbf{A} & -\mathbf{I} & \mathbf{O} & \dots & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{I} & \mathbf{A} & -\mathbf{I} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & -\mathbf{I} & \mathbf{A} & -\mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & -\mathbf{I} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \\ \mathbf{V}_{P-1} \\ \mathbf{V}_P \end{bmatrix} = \begin{bmatrix} \mathbf{\Gamma}_1 \\ \mathbf{\Gamma}_2 \\ \\ \mathbf{\Gamma}_{P-1} \\ \mathbf{\Gamma}_P \end{bmatrix} = \mathbf{\Gamma}, \qquad (1{\cdot}5)$$

in which $\mathbf{A}$ is the $N \times N$ submatrix

$$\begin{bmatrix} 4 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 4 \end{bmatrix}, \qquad (1{\cdot}6)$$

$\mathbf{I}$ is the $N \times N$ unit submatrix, and (using $\{\}$ brackets to indicate that a matrix, though written in line form, is a column matrix) $\mathbf{V}_r$ and $\mathbf{\Gamma}_r$ are respectively the column submatrix $\{v_{r,1}, v_{r,2}, \dots, v_{r,N}\}$ and the column submatrix $\{\gamma_{r,1}, \gamma_{r,2}, \dots, \gamma_{r,N}\}$.

The more general equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + f(x,y)\, v = g(x,y) \qquad (1{\cdot}7)$$

similarly reduces, on an $N \times 5$ grid (for simplicity), to the matrix equation

$$\mathbf{M}_1 \mathbf{V} = \mathbf{\Gamma}, \qquad (1{\cdot}8)$$

i.e.
$$\begin{bmatrix} \mathbf{A}_1 & -\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{I} & \mathbf{A}_2 & -\mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} & \mathbf{A}_3 & -\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -\mathbf{I} & \mathbf{A}_4 & -\mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & -\mathbf{I} & \mathbf{A}_5 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \\ \mathbf{V}_4 \\ \mathbf{V}_5 \end{bmatrix} = \begin{bmatrix} \mathbf{\Gamma}_1 \\ \mathbf{\Gamma}_2 \\ \mathbf{\Gamma}_3 \\ \mathbf{\Gamma}_4 \\ \mathbf{\Gamma}_5 \end{bmatrix}. \tag{1·8·1}$$

Karlqvist (1) purposes, in §8 of his paper, to factorize the matrix $\mathbf{M}$ (Choleski-wise) into the product of a lower-triangular and an upper-triangular matrix (both being partitioned in the same way as $\mathbf{M}$). The calculation of the submatrices lying on the principal diagonals of the matrix factors involves the inversion of one submatrix for each row; but none the less the work of inversion is much less than would be entailed in the inversion of the main matrix. In the process about to be described, both sides of equation (1·5) or (1·8·1), as the case may be, are premultiplied by a suitable auxiliary matrix so constructed that (for a rectangular region) complete inversion of any submatrix is unnecessary, it being necessary only to solve one set of equations of the order of one of the submatrices. The rest of the work consists of matrix multiplications and back substitution.

Accordingly, one forms from (1·8·1) the equation

$$\mathbf{aM}_1\mathbf{V} = \mathbf{a\Gamma} = \{\mathbf{\Delta}_1, \mathbf{\Delta}_2, \mathbf{\Delta}_3, \mathbf{\Delta}_4, \mathbf{\Delta}_5\} \tag{1·9}$$

by premultiplication by

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} & \mathbf{a}_{14} & \mathbf{a}_{15} \\ \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23} & \mathbf{a}_{24} & \mathbf{a}_{25} \\ \mathbf{a}_{31} & \mathbf{a}_{32} & \mathbf{a}_{33} & \mathbf{a}_{34} & \mathbf{a}_{35} \\ \mathbf{a}_{41} & \mathbf{a}_{42} & \mathbf{a}_{43} & \mathbf{a}_{44} & \mathbf{a}_{45} \\ \mathbf{a}_{51} & \mathbf{a}_{52} & \mathbf{a}_{53} & \mathbf{a}_{54} & \mathbf{a}_{55} \end{bmatrix}, \tag{1·10}$$

where the $N \times N$ submatrices $\mathbf{a}_{pq}$ are to be so chosen that $\mathbf{aM}_1$ is of lower triangular form, viz.

$$\mathbf{aM}_1 = \begin{bmatrix} \mathbf{B} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{B}_{21} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{B}_{31} & \mathbf{B}_{32} & \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{B}_{41} & \mathbf{B}_{42} & \mathbf{B}_{43} & \mathbf{I} & \mathbf{O} \\ \mathbf{B}_{51} & \mathbf{B}_{52} & \mathbf{B}_{53} & \mathbf{B}_{54} & \mathbf{I} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{a}_{11}\mathbf{A}_1 - \mathbf{a}_{12} & -\mathbf{a}_{11} + \mathbf{a}_{12}\mathbf{A}_2 - \mathbf{a}_{13} & -\mathbf{a}_{12} + \mathbf{a}_{13}\mathbf{A}_3 - \mathbf{a}_{14} & -\mathbf{a}_{13} + \mathbf{a}_{14}\mathbf{A}_4 - \mathbf{a}_{15} & -\mathbf{a}_{14} + \mathbf{a}_{15}\mathbf{A}_5 \\ \mathbf{a}_{21}\mathbf{A}_1 - \mathbf{a}_{22} & -\mathbf{a}_{21} + \mathbf{a}_{22}\mathbf{A}_2 - \mathbf{a}_{23} & -\mathbf{a}_{22} + \mathbf{a}_{23}\mathbf{A}_3 - \mathbf{a}_{24} & -\mathbf{a}_{23} + \mathbf{a}_{24}\mathbf{A}_4 - \mathbf{a}_{25} & -\mathbf{a}_{24} + \mathbf{a}_{25}\mathbf{A}_5 \\ \mathbf{a}_{31}\mathbf{A}_1 - \mathbf{a}_{32} & -\mathbf{a}_{31} + \mathbf{a}_{32}\mathbf{A}_2 - \mathbf{a}_{33} & -\mathbf{a}_{32} + \mathbf{a}_{33}\mathbf{A}_3 - \mathbf{a}_{34} & -\mathbf{a}_{33} + \mathbf{a}_{34}\mathbf{A}_4 - \mathbf{a}_{35} & -\mathbf{a}_{34} + \mathbf{a}_{35}\mathbf{A}_5 \\ \mathbf{a}_{41}\mathbf{A}_1 - \mathbf{a}_{42} & -\mathbf{a}_{41} + \mathbf{a}_{42}\mathbf{A}_2 - \mathbf{a}_{43} & -\mathbf{a}_{42} + \mathbf{a}_{43}\mathbf{A}_3 - \mathbf{a}_{44} & -\mathbf{a}_{43} + \mathbf{a}_{44}\mathbf{A}_4 - \mathbf{a}_{45} & -\mathbf{a}_{44} + \mathbf{a}_{45}\mathbf{A}_5 \\ \mathbf{a}_{51}\mathbf{A}_1 - \mathbf{a}_{52} & -\mathbf{a}_{51} + \mathbf{a}_{52}\mathbf{A}_2 - \mathbf{a}_{53} & -\mathbf{a}_{52} + \mathbf{a}_{53}\mathbf{A}_3 - \mathbf{a}_{54} & -\mathbf{a}_{53} + \mathbf{a}_{54}\mathbf{A}_4 - \mathbf{a}_{55} & -\mathbf{a}_{54} + \mathbf{a}_{55}\mathbf{A}_5 \end{bmatrix}. \tag{1·11}$$

Since there are more submatrices at one's disposal in $\mathbf{a}$ than are needed to satisfy the condition that $\mathbf{aM}_1$ shall be of the triangular form shown in (1·11), all the $\mathbf{a}_{p,5}$'s can be assigned arbitrarily, subject to the condition that $\mathbf{a}$ be not singular. The $\mathbf{a}_{p,5}$'s can conveniently be taken as unit submatrices, and the remaining $\mathbf{a}_{1,q}$'s are obtained by equating successive submatrices in the top line of $\mathbf{aM}_1$ to zero, beginning at the right-hand end.

$$\mathbf{B} = \mathbf{a}_{11}\mathbf{A}_1 - \mathbf{a}_{12}$$

is then found. $\mathbf{a}_{24}$, $\mathbf{a}_{23}$ and $\mathbf{a}_{22}$ are similarly found by equating terms in the second line of $\mathbf{aM}_1$ to zero, and $\mathbf{a}_{21}$ is found from

$$-\mathbf{a}_{21} + \mathbf{a}_{22}\mathbf{A}_2 - \mathbf{a}_{23} = \mathbf{I},$$

and so on with the remaining terms lying on, or to the right of, the principal diagonal of $\mathbf{aM}_1$. (It will be noted that $\mathbf{a}_{24} = \mathbf{a}_{14}$, $\mathbf{a}_{23} = \mathbf{a}_{13}$, etc., only $\mathbf{a}_{21}$ needing to be calculated.) It will also be obvious that the $\mathbf{B}_{pq}$ lying to the left of the principal diagonal could be made zero, except those in the first column. Unless, however, the labour of matrix multiplication is justified by subsequent savings in the amount of back substitution, the simplest procedure is to take as zero the $\mathbf{a}_{pq}$'s remaining undetermined after reduction of $\mathbf{aM}_1$ to lower triangular form.

The method would, of course, fail in a case in which the $\mathbf{A}$'s were such that $\mathbf{B}$ was singular. This has not been investigated. It does not arise with Poisson's equation.

It will be noted that the reduction of $\mathbf{aM}_1$ to triangular form involves only matrix multiplications and no inversions. The equation

$$\mathbf{BV}_1 = \mathbf{\Delta}_1 \tag{1·12}$$

must be solved, but even the submatrix $\mathbf{B}$ does not need to be completely inverted. Since the evaluation of the v's generally involves taking differences between larger quantities it is desirable that (1·12) should be solved (or inverted) by one of the exact methods as far as possible, approximations and loss of significant figures during the process being deferred to as late a stage as possible. The method of systematic addition and subtraction described by Rosser (2) is recommended, since it is extremely simple and makes surprisingly small demands on the register capacity of the calculating machine used.

2. *Poisson's equation in three dimensions.* The application of this premultiplication procedure to Poisson's equation in three dimensions,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = g(x, y, z), \tag{2·1}$$

extended over a parallelepipedal region, is extremely simple.

Comparison of the terms in (1·2) and (1·5) shows that, in the two-dimensional case, the elements of any $\mathbf{A}$ submatrix are the numerical coefficients attached to $v_{p,q}$, $v_{p-1,q}$, $v_{p+1,q}$, i.e. terms lying on the same horizontal grid line, whilst the terms of the submatrix $-\mathbf{I}$ to the left of any $\mathbf{A}$ arise from terms with suffix $q-1$, i.e. terms lying on the immediately preceding grid line, and the terms of the submatrix $-\mathbf{I}$ to the right of any $\mathbf{A}$ arise from terms with suffix $q+1$ (which lie on the immediately succeeding grid line). For convenience one can refer to the submatrices $\mathbf{A}$ as the line submatrices, and the submatrices $\mathbf{I}$ as coupling matrices, since they express the effect, on a mesh point in any grid line, of the immediately neighbouring points in the two adjacent grid lines to which it is 'coupled' by the equation (rewriting (1·2))

$$4v_{p,q} - v_{p-1,q} - v_{p+1,q} - v_{p,q-1} - v_{p,q+1} = \gamma_{p,q}. \tag{2·2}$$

In effect, therefore, the procedure for constructing the matrix $\mathbf{M}$ (or $\mathbf{M}_1$) for Poisson's equation for a rectangular region consists of the following steps:

(i) writing down line submatrices (for each line of the grid) as the elements of the principal diagonal of the main matrix,

(ii) inserting unit coupling submatrices on the adjacent diagonals, and

(iii) completing the main matrix by filling in with null matrices.

Now, the finite difference approximation to (2·1) is

$$6v_{j,k,l} - v_{j-1,k,l} - v_{j+1,k,l} - v_{j,k-1,l} - v_{j,k+1,l} - v_{j,k,l-1} - v_{j,k,l+1} = \gamma_{j,k,l}, \tag{2·3}$$

and this latter family of equations, extended over a rectangular parallelepipedal lattice region bounded by $j = 0$, $j = N+1$, $k = 0$, $k = P+1$, $l = 0$, $l = Q+1$, gives, in place of (1·5), the matrix equation

$$\mathbf{SV} = \mathbf{\Gamma}, \tag{2·4}$$

the 'super-matrix' $\mathbf{S}$ being of the form

$$\mathbf{S} = \begin{bmatrix} \mathbf{M}_2 & -\mathbf{I} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{I} & \mathbf{M}_2 & -\mathbf{I} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} & \mathbf{M}_2 & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & -\mathbf{I} & \mathbf{M}_2 \end{bmatrix} \tag{2·5}$$

when partitioned into Q rows and columns of matrices. In this super-matrix, $\mathbf{M}_2$ is a matrix (partitioned in P rows and columns of submatrices) of the same form as $\mathbf{M}$ in (1·5), i.e.

$$\mathbf{M}_2 = \begin{bmatrix} \mathbf{A}_1 & -\mathbf{I} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{I} & \mathbf{A}_1 & -\mathbf{I} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & -\mathbf{I} & \mathbf{A}_1 \end{bmatrix}, \tag{2·6}$$

where $\mathbf{A}_1$ is the modified $N \times N$ 'line' matrix

$$\mathbf{A}_1 = \begin{bmatrix} 6 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 6 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 6 & -1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 6 \end{bmatrix} \tag{2·7}$$

(owing to the appearance of the principal coefficient 6 in (2·3) in place of the principal coefficient 4 in (2·2)). The $\mathbf{I}$'s in (2·5) are unit matrices of the same order as $\mathbf{M}_2$, while the $\mathbf{I}$'s in (2·6) are unit matrices of the same order as $\mathbf{A}_1$; and $\mathbf{V}$ and $\mathbf{\Gamma}$ are of course partitioned suitably to make them compatible with the partitioning of $\mathbf{S}$. By analogy with the line submatrices $\mathbf{A}$ and $\mathbf{A}_1$, the matrices $\mathbf{M}_2$ can be called 'plane' matrices.

This form of partitioned matrix for the three-dimensional Poisson equation is very briefly mentioned by Karlqvist in § 13 of his paper.

Thus the procedure for constructing the super-matrix $\mathbf{S}$ for a parallelepiped is precisely analogous to constructing the matrix $\mathbf{M}$ for a rectangle. The steps are

(i) construct the typical line submatrix $\mathbf{A}_1$ for each line;

(ii) position the line submatrices on the principal diagonal of the partitioned plane matrix $\mathbf{M}_2$ of each plane of the lattice;

(iii) couple each line matrix to its two adjacent lines by the appropriate unit coupling submatrices, by arranging these unit coupling submatrices on the two subdiagonals adjacent to the main diagonal;

(iv) fill in the remainder of the plane matrix $\mathbf{M}_2$ by null submatrices;

(v) then construct the final partitioned super-matrix $\mathbf{S}$ by writing down $\mathbf{M}_2$'s (for each plane of the lattice) to fill the principal diagonal of the super-matrix, and

(vi) couple each plane to its neighbours by appropriate unit matrices lying on the subdiagonals immediately above and below the principal diagonal of the super-matrix;

(vii) complete $\mathbf{S}$ by filling in with null matrices.

Also, since the structure of $\mathbf{S}$ in (2·5) is of precisely the same type as that of $\mathbf{M}$ in (1·5), the family of equations represented by (2·4) can be dealt with by premultiplication in precisely the same way as was (1·8·1). The result being of the form

$$\mathbf{aS} = \begin{bmatrix} \mathbf{B}_2 & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{B}_{21} & \mathbf{I} & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{B}_{31} & \mathbf{B}_{32} & \mathbf{I} & \dots & \mathbf{O} \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \\ \dots \end{bmatrix} = \begin{bmatrix} \mathbf{\Delta}_1 \\ \mathbf{\Delta}_2 \\ \mathbf{\Delta}_3 \\ \dots \end{bmatrix},$$

there remains merely the solution of

$$\mathbf{B}_2\mathbf{V}_1 = \mathbf{\Delta}_1$$

and the subsequent back-substitutions.

It will be noted that the premultiplication procedure in effect reduces the solution of the two-dimensional case to the solution of a set of equations of the order of a line matrix, and reduces the solution of the three-dimensional case to the order of the inversion of a plane matrix.

3. *Extension of non-rectangular two-dimensional regions.* The procedure specified in the second paragraph of § 2 for the rectangular two-dimensional grid has an obvious extension to non-rectangular two-dimensional regions. The modification that results from the non-rectangularity is that some of the coupling submatrices cease to be square, becoming rectangular submatrices in which rows or columns of zeros have been added to the square unit submatrix. The coupling submatrices, however, still lie on the subdiagonals immediately above and below the principal diagonal of the main matrix. Extensions to multiply-connected regions and to 'graded' nets do not appear worth discussing since, in addition to the occurrence of the non-square submatrices adjacent to the principal diagonal, some of the matrices lying on the subdiagonals immediately above and below the principal diagonal become zero and the coupling terms are displaced to positions in the main matrix which are more remote from the line submatrices which they couple. The method of extension is, however, simple, and will become clear from the following example of the construction of a matrix for a non-rectangular simply connected area.

Taking the non-rectangular region shown in Fig. 2, it will be seen that the top five horizontal lines each contain 4 inner points, and the next three lines contain 7 inner points each. The submatrices on the principal diagonal of the complete matrix $\mathbf{M}$ will, therefore, consist of four 4×4 $\mathbf{A}$ submatrices and three 7×7 $\mathbf{A}$ submatrices. The coupling submatrices between two grid lines of the same length will, as before, be unit submatrices (of the same order as the neighbouring submatrices) which lie on the diagonals neighbouring the principal diagonal. The coupling submatrices (again lying on the diagonals neighbouring the principal diagonal) between two grid lines of different lengths are non-square submatrices formed by extending a unit matrix (of the order of the smaller adjacent **A** matrix) by adding to it a number of rows (or columns) of zeros, the number of the added rows (or columns) being equal to the difference between the number of inner points on the longer grid line and the number of inner points on the shorter grid line.

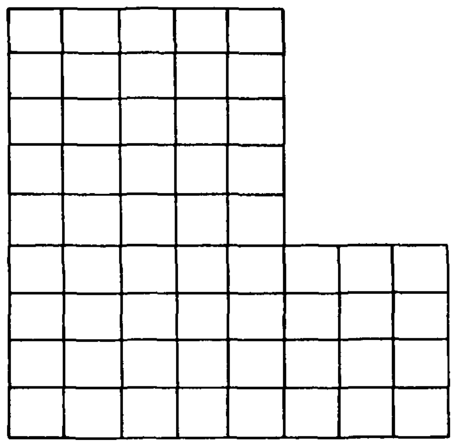

Fig. 2

The sizes of the submatrices will in what follows be shown by the addition of a parenthetical subscript (rc) to indicate the possession of r rows and c columns—this parenthetical subscript following the subscript pq (when used) which indicates the position of the submatrix in the main matrix—save that, when the submatrix is square, having r rows and columns, a single parenthetical subscript (r) will be used. In addition, the non-square coupling submatrices will be symbolized by $\mathbf{I}_{(r)}\mathbf{O}_{(rn)}$ to indicate a unit matrix of order r to which n columns of zeros have been added on the right and by $\mathbf{O}_{(rn)}\mathbf{I}_{(r)}$ to indicate that n columns of zeros have been added on the left; and $\mathbf{I}_{(r)}/\mathbf{O}_{(nr)}$ will indicate that n rows of zeros have been added below $\mathbf{I}_{(r)}$, while $\mathbf{O}_{(nr)}/\mathbf{I}_{(r)}$ will indicate that n rows of zeros have been added above $\mathbf{I}_{(r)}$.

With this symbolism, the matrix for the grid shown in Fig. 2 is

$$\mathbf{M} = \begin{bmatrix} \mathbf{A}_{(4)} & -\mathbf{I}_{(4)} & \mathbf{O}_{(4)} & \mathbf{O}_{(4)} & \mathbf{O}_{(4)} & \mathbf{O}_{(47)} & \mathbf{O}_{(47)} & \mathbf{O}_{(47)} \\ -\mathbf{I}_{(4)} & \mathbf{A}_{(4)} & -\mathbf{I}_{(4)} & \mathbf{O}_{(4)} & \mathbf{O}_{(4)} & \mathbf{O}_{(47)} & \mathbf{O}_{(47)} & \mathbf{O}_{(47)} \\ \mathbf{O}_{(4)} & -\mathbf{I}_{(4)} & \mathbf{A}_{(4)} & -\mathbf{I}_{(4)} & \mathbf{O}_{(4)} & \mathbf{O}_{(47)} & \mathbf{O}_{(47)} & \mathbf{O}_{(47)} \\ \mathbf{O}_{(4)} & \mathbf{O}_{(4)} & -\mathbf{I}_{(4)} & \mathbf{A}_{(4)} & -\mathbf{I}_{(4)} & \mathbf{O}_{(47)} & \mathbf{O}_{(47)} & \mathbf{O}_{(47)} \\ \mathbf{O}_{(4)} & \mathbf{O}_{(4)} & \mathbf{O}_{(4)} & -\mathbf{I}_{(4)} & \mathbf{A}_{(4)} & -\mathbf{I}_{(4)}\mathbf{O}_{(43)} & \mathbf{O}_{(47)} & \mathbf{O}_{(47)} \\ \mathbf{O}_{(74)} & \mathbf{O}_{(74)} & \mathbf{O}_{(74)} & \mathbf{O}_{(74)} & -\mathbf{I}_{(4)}/\mathbf{O}_{(34)} & \mathbf{A}_{(7)} & -\mathbf{I}_{(7)} & \mathbf{O}_{(7)} \\ \mathbf{O}_{(74)} & \mathbf{O}_{(74)} & \mathbf{O}_{(74)} & \mathbf{O}_{(74)} & \mathbf{O}_{(74)} & -\mathbf{I}_{(7)} & \mathbf{O}_{(7)} & -\mathbf{I}_{(7)} \\ \mathbf{O}_{(74)} & \mathbf{O}_{(74)} & \mathbf{O}_{(74)} & \mathbf{O}_{(74)} & \mathbf{O}_{(7)} & \mathbf{O}_{(7)} & \mathbf{O}_{(7)} & \mathbf{A}_{(7)} \end{bmatrix}, \tag{3·1}$$

where the 'irregular' coupling submatrices are, in full,

$$\mathbf{I}_{(4)}\mathbf{O}_{(43)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and } \mathbf{I}_{(4)}/\mathbf{O}_{(34)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix $\mathbf{M}$ for the non-rectangular region shown in Fig. 2 calls for only a slightly modified treatment when solving the equations. It is premultiplied by a matrix $\mathbf{a}$ which is partitioned into submatrices of precisely the same orders as the corresponding submatrices of $\mathbf{M}$, i.e. a matrix of the form

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_{11(4)} & \mathbf{a}_{12(4)} & \mathbf{a}_{13(4)} & \mathbf{a}_{14(4)} & \mathbf{a}_{15(4)} & \mathbf{a}_{16(47)} & \mathbf{a}_{17(47)} & \mathbf{a}_{18(47)} \\ \mathbf{a}_{21(4)} & \mathbf{a}_{22(4)} & \mathbf{a}_{23(4)} & \mathbf{a}_{24(4)} & \mathbf{a}_{25(4)} & \mathbf{a}_{26(47)} & \mathbf{a}_{27(47)} & \mathbf{a}_{28(47)} \\ \mathbf{a}_{31(4)} & \mathbf{a}_{32(4)} & \mathbf{a}_{33(4)} & \mathbf{a}_{34(4)} & \mathbf{a}_{35(4)} & \mathbf{a}_{36(47)} & \mathbf{a}_{37(47)} & \mathbf{a}_{38(47)} \\ \mathbf{a}_{41(4)} & \mathbf{a}_{42(4)} & \mathbf{a}_{43(4)} & \mathbf{a}_{44(4)} & \mathbf{a}_{45(4)} & \mathbf{a}_{46(47)} & \mathbf{a}_{47(47)} & \mathbf{a}_{48(47)} \\ \mathbf{a}_{51(4)} & \mathbf{a}_{52(4)} & \mathbf{a}_{53(4)} & \mathbf{a}_{54(4)} & \mathbf{a}_{55(4)} & \mathbf{a}_{56(47)} & \mathbf{a}_{57(47)} & \mathbf{a}_{58(47)} \\ \mathbf{a}_{61(74)} & \mathbf{a}_{62(74)} & \mathbf{a}_{63(74)} & \mathbf{a}_{64(74)} & \mathbf{a}_{65(74)} & \mathbf{a}_{66(7)} & \mathbf{a}_{67(7)} & \mathbf{a}_{68(7)} \\ \mathbf{a}_{71(74)} & \mathbf{a}_{72(74)} & \mathbf{a}_{73(74)} & \mathbf{a}_{74(74)} & \mathbf{a}_{75(74)} & \mathbf{a}_{76(7)} & \mathbf{a}_{77(7)} & \mathbf{a}_{78(7)} \\ \mathbf{a}_{81(74)} & \mathbf{a}_{82(74)} & \mathbf{a}_{83(74)} & \mathbf{a}_{84(74)} & \mathbf{a}_{85(74)} & \mathbf{a}_{86(7)} & \mathbf{a}_{87(7)} & \mathbf{a}_{88(7)} \end{bmatrix}. \tag{3·2}$$

The element $\mathbf{a}_{18(47)}$ can no longer be a unit matrix, but is taken as a unit matrix to which three columns have been added

$$\mathbf{a}_{18(47)} = \begin{bmatrix} 1 & 0 & 0 & 0 & c_{15} & c_{16} & c_{17} \\ 0 & 1 & 0 & 0 & c_{25} & c_{26} & c_{27} \\ 0 & 0 & 1 & 0 & c_{35} & c_{36} & c_{37} \\ 0 & 0 & 0 & 1 & c_{45} & c_{46} & c_{47} \end{bmatrix}, \tag{3·3}$$

whose elements are to be so chosen as to make the premultiplication process self-consistent. Thus, considering the submatrices in the first row of the product $\mathbf{aM}$ and equating all save the first submatrix to zero, one has (starting from the right-hand end of the row)

$$\left.\begin{aligned} \mathbf{a}_{17(47)}\mathbf{I}_{(7)} &= \mathbf{a}_{18(47)}\mathbf{A}_{(7)}, \\ \mathbf{a}_{16(47)}\mathbf{I}_{(7)} &= \mathbf{a}_{17(47)}\mathbf{A}_{(7)} - \mathbf{a}_{18(47)}, \\ \mathbf{a}_{15(4)}[\mathbf{I}_{(4)}\mathbf{O}_{(43)}] &= \mathbf{a}_{16(47)}\mathbf{A}_{(7)} - \mathbf{a}_{17(47)}, \\ \mathbf{a}_{14(4)} &= \mathbf{a}_{15(4)}\mathbf{A}_{(4)} - \mathbf{a}_{16(47)}[\mathbf{I}_{(4)}/\mathbf{O}_{(34)}], \\ \mathbf{a}_{13(4)} &= \mathbf{a}_{14(4)}\mathbf{A}_{(4)} - \mathbf{a}_{15(4)}, \\ \mathbf{a}_{12(4)} &= \mathbf{a}_{13(4)}\mathbf{A}_{(4)} - \mathbf{a}_{14(4)}, \\ \mathbf{a}_{11(4)} &= \mathbf{a}_{12(4)}\mathbf{A}_{(4)} - \mathbf{a}_{13(4)}. \end{aligned}\right\} \tag{3·4}$$

Accordingly, $$\mathbf{a}_{16(47)} = \mathbf{a}_{18(47)}[\mathbf{A}^2_{(7)} - \mathbf{I}_{(7)}]$$

and $$\mathbf{a}_{15(47)}[\mathbf{I}_{(4)}\mathbf{O}_{(43)}] = \mathbf{a}_{18(47)}[\mathbf{A}^3_{(7)} - 2\mathbf{A}_{(7)}]$$

$$= \mathbf{a}_{18(47)} \begin{bmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{17} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{27} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{71} & \delta_{72} & \cdots & \delta_{77} \end{bmatrix} \text{(say)}. \tag{3·5}$$

Now if

$$\mathbf{a}_{15(4)} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix},$$

$$\mathbf{a}_{15(4)}[\mathbf{I}_{(4)}\,\mathbf{O}_{(43)}] = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & 0 & 0 & 0 \\ d_{21} & d_{22} & d_{23} & d_{24} & 0 & 0 & 0 \\ d_{31} & d_{32} & d_{33} & d_{34} & 0 & 0 & 0 \\ d_{41} & d_{42} & d_{43} & d_{44} & 0 & 0 & 0 \end{bmatrix},$$

so that if (3·5) is to be self-consistent, the last three columns in $\mathbf{a}_{18(47)}[\mathbf{A}^3_{(7)} - 2\mathbf{A}_{(7)}]$ must vanish, i.e.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & c_{15} & c_{16} & c_{17} \\ 0 & 1 & 0 & 0 & c_{25} & c_{26} & c_{27} \\ 0 & 0 & 1 & 0 & c_{35} & c_{36} & c_{37} \\ 0 & 0 & 0 & 1 & c_{45} & c_{46} & c_{47} \end{bmatrix} \begin{bmatrix} \delta_{15} & \delta_{16} & \delta_{17} \\ \delta_{25} & \delta_{26} & \delta_{27} \\ \delta_{35} & \delta_{36} & \delta_{37} \\ \delta_{45} & \delta_{46} & \delta_{47} \\ \delta_{55} & \delta_{56} & \delta_{57} \\ \delta_{65} & \delta_{66} & \delta_{67} \\ \delta_{75} & \delta_{76} & \delta_{77} \end{bmatrix} = 0. \tag{3·6}$$

That is to say, writing $\mathbf{D}$ for the matrix

$$\begin{bmatrix} \delta_{55} & \delta_{65} & \delta_{75} \\ \delta_{56} & \delta_{66} & \delta_{76} \\ \delta_{57} & \delta_{67} & \delta_{77} \end{bmatrix}$$

(i.e. the transpose of the last three lines in the right-hand matrix in (3·6)), the c's satisfy the four equations represented by

$$\mathbf{D} \begin{bmatrix} c_{15} & c_{25} & c_{35} & c_{45} \\ c_{16} & c_{26} & c_{36} & c_{46} \\ c_{17} & c_{27} & c_{37} & c_{47} \end{bmatrix} = - \begin{bmatrix} \delta_{15} & \delta_{25} & \delta_{35} & \delta_{45} \\ \delta_{16} & \delta_{26} & \delta_{36} & \delta_{46} \\ \delta_{17} & \delta_{27} & \delta_{37} & \delta_{47} \end{bmatrix}, \tag{3·7}$$

so that one extra 'de-coupling' matrix $\mathbf{D}$ (of order $7-4 = 3$, in this case) has to be inverted in order to reduce $\mathbf{aM}$ to lower triangular form.

The rest of the calculation proceeds substantially as with equation (1·11), for the first five rows. $\mathbf{a}_{18}$ having been found as described in the preceding paragraph, all the $\mathbf{a}_{1,r}$'s in the first row of $\mathbf{aM}$ are found by equating the successive submatrices in the first row to zero, so that $\mathbf{a}_{17}, \mathbf{a}_{16}, \ldots, \mathbf{a}_{11}$ are successively determined. $\mathbf{a}_{11}\mathbf{A}_{(4)} - \mathbf{a}_{(12)} = \mathbf{B}$ is then computed. $\mathbf{a}_{27}, \mathbf{a}_{26}, \ldots, \mathbf{a}_{22}$ are similarly found by equating to zero the submatrices in the second row of $\mathbf{aM}$; and $\mathbf{a}_{21}$ is found from the condition that

$$-\mathbf{a}_{21} + \mathbf{a}_{22}\mathbf{A}_{(4)} - \mathbf{a}_{23} = \mathbf{I}_{(4)},$$

so as to make the second submatrix on the principal diagonal of $\mathbf{aM}$ equal to the unit submatrix. This process is repeated, row by row of $\mathbf{aM}$, until finally one has made

$$-\mathbf{a}_{54} + \mathbf{a}_{55}\mathbf{A}_{(4)} - \mathbf{a}_{56}[\mathbf{I}_{(4)}/\mathbf{O}_{(34)}] = \mathbf{I}_{(4)}.$$

The remaining three lines of the matrix are treated precisely as for equation (1·11), taking

$$\mathbf{a}_{68} = \mathbf{a}_{78} = \mathbf{a}_{88} = \mathbf{I}_{(7)}.$$

The matrix product $\mathbf{aM}$ is thus reduced to exactly the same lower triangular type as the product $\mathbf{aM}_1$ in (1·11) and there remains only one set of equations (of the order of a line submatrix) to be solved.

Once again it will be seen that the premultiplication technique enables the inversion of the main matrix to be replaced by the solution of a comparatively few matrix equations of substantially lower order with a considerable saving of work, unless the region has too many changes in its width when the number of de-coupling matrices to be inverted becomes considerable.

4. *The bi-harmonic equation*. The customary finite difference approximation to the bi-harmonic equation

$$\nabla^4\psi = g(x,y) \tag{4·1}$$

is

$$\begin{aligned}\nabla^4\psi \doteqdot 20\psi_{j,k} &- 8(\psi_{j,k+1}+\psi_{j,k-1}+\psi_{j+1,k}+\psi_{j-1,k})\\ &+2(\psi_{j+1,k+1}+\psi_{j+1,k-1}+\psi_{j-1,k+1}+\psi_{j-1,k-1})\\ &+(\psi_{j+2,k}+\psi_{j-2,k}+\psi_{j,k+2}+\psi_{j,k-2}) = g_{j,k},\end{aligned} \tag{4·2}$$

and the corresponding matrix equation for the rectangular region shown in Fig. 1 (with $P = 7$) is

$$\mathbf{MR} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}\\ \mathbf{B} & \mathbf{A} & \mathbf{B} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O}\\ \mathbf{I} & \mathbf{B} & \mathbf{A} & \mathbf{B} & \mathbf{I} & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & \mathbf{I} & \mathbf{B} & \mathbf{A} & \mathbf{B} & \mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{B} & \mathbf{A} & \mathbf{B} & \mathbf{I}\\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{B} & \mathbf{A} & \mathbf{B}\\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{B} & \mathbf{A} \end{bmatrix} \begin{bmatrix}\mathbf{R}_1\\ \mathbf{R}_2\\ \mathbf{R}_3\\ \mathbf{R}_4\\ \mathbf{R}_5\\ \mathbf{R}_6\\ \mathbf{R}_7\end{bmatrix} = \begin{bmatrix}\mathbf{\Gamma}_1\\ \mathbf{\Gamma}_2\\ \mathbf{\Gamma}_3\\ \mathbf{\Gamma}_4\\ \mathbf{\Gamma}_5\\ \mathbf{\Gamma}_6\\ \mathbf{\Gamma}_7\end{bmatrix}, \tag{4·3}$$

where $\mathbf{I}$ is the $N \times N$ unit submatrix, the successive $\mathbf{R}$'s are column submatrices whose elements are the ψ's at points lying on successive horizontal grid lines, and the $\mathbf{\Gamma}$'s are corresponding column submatrices arising in an obvious way from the values of $g_{j,k}$ and the boundary values and gradients prescribed for ψ. In addition, $\mathbf{A}$ is the $N \times N$ submatrix of the form

$$\mathbf{A} = \begin{bmatrix} 20 & -8 & 1 & 0 & \dots & 0 & 0 & 0\\ -8 & 20 & -8 & 1 & \dots & 0 & 0 & 0\\ 1 & -8 & 20 & -8 & \dots & 0 & 0 & 0\\ 0 & 1 & -8 & 20 & \dots & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & \dots & 20 & -8 & 1\\ 0 & 0 & 0 & 0 & \dots & -8 & 20 & -8\\ 0 & 0 & 0 & 0 & \dots & 1 & -8 & 20 \end{bmatrix},$$

and $\mathbf{B}$ is the $N \times N$ submatrix of the form

$$\mathbf{B} = \begin{bmatrix} -8 & 2 & 0 & 0 & \dots & 0 & 0 & 0\\ 2 & -8 & 2 & 0 & \dots & 0 & 0 & 0\\ 0 & 2 & -8 & 2 & \dots & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & \dots & 2 & -8 & 2\\ 0 & 0 & 0 & 0 & \dots & 0 & 2 & -8 \end{bmatrix}.$$

The method of constructing the matrix for non-rectangular regions is obvious.

It may be noted that where (4·1) can conveniently be replaced by the two equations

$$\nabla^2\psi = \zeta, \quad \nabla^2\zeta = g(x,y),$$

it is probably advantageous to do so, and the methods of the preceding sections can be applied immediately. If, however, it is desired to treat (4·3) directly, a modification of the foregoing may be used when the boundary is rectangular. The process is again best described with reference to an example—generalizations being obvious.

Consider, then, the solution for the region shown in Fig. 1, with $P = 7$. Premultiplying $\mathbf{M}$ in (4·3) by the matrix

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} & \cdots & \mathbf{a}_{17} \\ \mathbf{a}_{21} & \mathbf{a}_{22} & \cdots & \cdots & \mathbf{a}_{27} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{a}_{71} & \mathbf{a}_{72} & \cdots & \cdots & \mathbf{a}_{77} \end{bmatrix} \tag{4·4}$$

(in which the $\mathbf{a}_{pq}$'s are again $N \times N$ submatrices),

$$\mathbf{aM} = \begin{bmatrix} \mathbf{a}_{11}\mathbf{A}+\mathbf{a}_{12}\mathbf{B}+\mathbf{a}_{13} & \mathbf{a}_{11}\mathbf{B}+\mathbf{a}_{12}\mathbf{A}+\mathbf{a}_{13}\mathbf{B} & \mathbf{a}_{11}+\mathbf{a}_{12}\mathbf{B}+\mathbf{a}_{13}\mathbf{A}+\mathbf{a}_{14}\mathbf{B}+\mathbf{a}_{15} & \cdots & \mathbf{a}_{15}+\mathbf{a}_{16}\mathbf{B}+\mathbf{a}_{17}\mathbf{A} \\ \mathbf{a}_{21}\mathbf{A}+\mathbf{a}_{22}\mathbf{B}+\mathbf{a}_{23} & \mathbf{a}_{21}\mathbf{B}+\mathbf{a}_{22}\mathbf{A}+\mathbf{a}_{23}\mathbf{B} & \mathbf{a}_{21}+\mathbf{a}_{22}\mathbf{B}+\mathbf{a}_{23}\mathbf{A}+\mathbf{a}_{24}\mathbf{B}+\mathbf{a}_{25} & \cdots & \mathbf{a}_{25}+\mathbf{a}_{26}\mathbf{B}+\mathbf{a}_{27}\mathbf{A} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{a}_{71}\mathbf{A}+\mathbf{a}_{72}\mathbf{B}+\mathbf{a}_{73} & \mathbf{a}_{71}\mathbf{B}+\mathbf{a}_{72}\mathbf{A}+\mathbf{a}_{73}\mathbf{B} & \mathbf{a}_{71}+\mathbf{a}_{72}\mathbf{B}+\mathbf{a}_{73}\mathbf{A}+\mathbf{a}_{74}\mathbf{B}+\mathbf{a}_{75} & \cdots & \mathbf{a}_{75}+\mathbf{a}_{76}\mathbf{B}+\mathbf{a}_{77}\mathbf{A} \end{bmatrix},$$

and it will be seen that the terms in all columns except the first two begin with a 'free' submatrix, and it is therefore possible (without matrix inversion) to choose the $\mathbf{a}_{pq}$'s so that $\mathbf{aMR} = \mathbf{a\Gamma}$ reduces to the form

$$\begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{C}_{41} & \mathbf{C}_{42} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{C}_{51} & \mathbf{C}_{52} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{C}_{61} & \mathbf{C}_{62} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} \\ \mathbf{C}_{71} & \mathbf{C}_{72} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \\ \mathbf{R}_4 \\ \mathbf{R}_5 \\ \mathbf{R}_6 \\ \mathbf{R}_7 \end{bmatrix} = \begin{bmatrix} \mathbf{\Delta}_1 \\ \mathbf{\Delta}_2 \\ \mathbf{\Delta}_3 \\ \mathbf{\Delta}_4 \\ \mathbf{\Delta}_5 \\ \mathbf{\Delta}_6 \\ \mathbf{\Delta}_7 \end{bmatrix} \tag{4·5}$$

In so doing, only five of the seven submatrices $\mathbf{a}_{pq}$ in each row of $\mathbf{a}$ have been used, and two submatrices in each row can be assigned arbitrarily, subject to the condition that $\mathbf{a}$ must not be singular. The simplest way of doing this appears to be to make

$$\left.\begin{aligned} \mathbf{a}_{17} = \mathbf{a}_{77} = \mathbf{I}, \quad \mathbf{a}_{16} = \mathbf{a}_{76} = \mathbf{O}, \\ \mathbf{a}_{m,6} = \mathbf{I} \quad (m \neq 1 \text{ or } 7), \quad \mathbf{a}_{n,7} = \mathbf{O} \quad (n \neq 1 \text{ or } 7). \end{aligned}\right\} \tag{4·6}$$

It then remains to solve the system

$$\begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{\Delta}_1 \\ \mathbf{\Delta}_2 \end{bmatrix} \tag{4·7}$$

which calls only for the inversion of two $N \times N$ matrices to give $\mathbf{R}_1$ and $\mathbf{R}_2$, and the remaining $\mathbf{R}$'s are obtained by back-substitution.

It would, of course, be possible to assign arbitrarily only the submatrices in the last column of $\mathbf{a}$ and to solve a pair of equations for $\mathbf{a}_{15}$ and $\mathbf{a}_{16}$, for example, so as to make $\mathbf{C}_{12} = \mathbf{O}$, and similarly to solve for $\mathbf{a}_{25}$ and $\mathbf{a}_{26}$ so as to make $\mathbf{C}_{22} = \mathbf{I}$; but this seems more troublesome than the course proposed above.

The extension to more irregular boundaries, such as that of Fig. 2, is obvious, but it is equally obvious that the complexity of the process increases rapidly as the boundary becomes more irregular.

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