VIP Refresher: Trigonometry

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Definitions

❒ **Trigonometric functions** – The following common trigonometric functions are 2π-periodic and are defined as follows:

Function	Domain and Image	Definition	Derivative
Cosine $\cos(\theta)$	$\theta \in \mathbb{R}$ $\cos(\theta) \in [-1,1]$	$\dfrac{\textrm{adjacent}}{\textrm{hypotenuse}}$	$-\sin(\theta)$
Sine $\sin(\theta)$	$\theta \in \mathbb{R}$ $\sin(\theta) \in [-1,1]$	$\dfrac{\textrm{opposite}}{\textrm{hypotenuse}}$	$\cos(\theta)$
Tangent $\tan(\theta)$	$\theta \in \mathbb{R}\backslash\{2k\pi\}$ $\tan(\theta) \in]-\infty,+\infty[$	$\dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{\textrm{opposite}}{\textrm{adjacent}}$	$1+\tan^2(\theta)$

❒ **Euler's formula** – The following formula establishes a fundamental relationship between the trigonometric functions and the complex exponential function as follows:

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$

Therefore, we have:

$$\boxed{\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}} \quad \textrm{and} \quad \boxed{\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}} \quad \textrm{and} \quad \boxed{\tan(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}}$$

❒ **Inverse trigonometric functions** – The common inverse trigonometric functions are defined as follows:

Function	Domain and Image	Definition	Derivative
Arccosine $\arccos(x)$	$x \in [-1,1]$ $\arccos(x) \in [0,\pi]$	$\cos(\arccos(x)) = x$	$-\dfrac{1}{\sqrt{1-x^2}}$
Arcsine $\arcsin(x)$	$x \in [-1,1]$ $\arcsin(x) \in [-\frac{\pi}{2},\frac{\pi}{2}]$	$\sin(\arcsin(x)) = x$	$\dfrac{1}{\sqrt{1-x^2}}$
Arctangent $\arctan(x)$	$x \in]-\infty,+\infty[$ $\arctan(x) \in]-\frac{\pi}{2},\frac{\pi}{2}[$	$\tan(\arctan(x)) = x$	$\dfrac{1}{1+x^2}$

Trigonometric identities

❒ **Pythagorean identity** – The following identity is commonly used:

$$\forall\theta, \quad \boxed{\cos^2(\theta)+\sin^2(\theta) = 1}$$

❒ **Inverse trigonometric identities** – The following identities are commonly used:

$$\forall x, \quad \boxed{\arccos(x)+\arcsin(x) = \frac{\pi}{2}}$$

$$\forall x, \quad \boxed{\arctan(x)+\arctan\left(\frac{1}{x}\right) = \begin{cases} \dfrac{\pi}{2} & (x>0) \\ -\dfrac{\pi}{2} & (x<0) \end{cases}}$$

❒ **Addition formulas** – The following identities are commonly used:

Name	Formula
Cosine addition	$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
Sine addition	$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$
Tangent addition	$\tan(a+b) = \dfrac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$

❒ **Symmetry identities** – The following identities are commonly used:

By $\alpha = 0$	By $\alpha = \frac{\pi}{4}$	By $\alpha = \frac{\pi}{2}$
$\cos(-\theta) = \cos(\theta)$	$\cos\left(\frac{\pi}{2}-\theta\right) = \sin(\theta)$	$\cos(\pi-\theta) = -\cos(\theta)$
$\sin(-\theta) = -\sin(\theta)$	$\sin\left(\frac{\pi}{2}-\theta\right) = \cos(\theta)$	$\sin(\pi-\theta) = \sin(\theta)$
$\tan(-\theta) = -\tan(\theta)$	$\tan\left(\frac{\pi}{2}-\theta\right) = \dfrac{1}{\tan(\theta)}$	$\tan(\pi-\theta) = -\tan(\theta)$

❒ **Shift identities** – The following identities are commonly used:

By $\frac{\pi}{2}$	By π
$\cos\left(\theta+\frac{\pi}{2}\right) = -\sin(\theta)$	$\cos(\theta+\pi) = -\cos(\theta)$
$\sin\left(\theta+\frac{\pi}{2}\right) = \cos(\theta)$	$\sin(\theta+\pi) = -\sin(\theta)$
$\tan\left(\theta+\frac{\pi}{2}\right) = -\dfrac{1}{\tan(\theta)}$	$\tan(\theta+\pi) = \tan(\theta)$

❒ **Product-to-sum and sum-to-product identities** – The following identities are commonly used:

Name	Formula
Product-to-sum	$\cos(a)\cos(b)=\frac{1}{2}(\cos(a-b)+\cos(a+b))$ $\sin(a)\sin(b)=\frac{1}{2}(\cos(a-b)-\cos(a+b))$ $\sin(a)\cos(b)=\frac{1}{2}(\sin(a+b)+\sin(a-b))$ $\cos(a)\sin(b)=\frac{1}{2}(\sin(a+b)-\sin(a-b))$ $\tan(a)\tan(b)=\frac{\cos(a-b)-\cos(a+b)}{\cos(a-b)+\cos(a+b)}$
Sum-to-product	$\cos(a)+\cos(b)=2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$ $\cos(a)-\cos(b)=-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$ $\sin(a)+\sin(b)=2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$ $\sin(a)-\sin(b)=2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$

Miscellaneous

❒ **Values for common angles** – The following table sums up the values for common angles to have in mind:

Angle θ (radians $\leftrightarrow$ degrees)	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta)$
$0 \leftrightarrow 0°$	1	0	0
$\frac{\pi}{6} \leftrightarrow 30°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4} \leftrightarrow 45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3} \leftrightarrow 60°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{2} \leftrightarrow 90°$	0	1	∞

❒ **Kashani theorem** – The Kashani theorem, also known as the law of cosines, states that in a triangle, the lengths a, b, c and the angle γ between sides of length a and b satisfy the following equation:

$$\boxed{c^2=a^2+b^2-2ab\cos(\gamma)}$$

Remark: for $\gamma=\frac{\pi}{2}$, the triangle is right and the identity is the Pythagorean theorem.

❒ **Law of sines** – In a given triangle of lengths a,b,c and opposite angles A,B,C, the law of sines states that we have:

$$\boxed{\frac{a}{\sin(A)}=\frac{b}{\sin(B)}=\frac{c}{\sin(C)}}$$