Numerical Analysis

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目录

1   Cubic Spline Interpolation

> **定义 1.1** *Given a function f defined on $[a,b]$, $a = x_0 < x_1 < \cdots , < x_n = b$, a cubic spline interpolation S for f is a function that satisfies the following conditions.*
>
> 1. *$S_j(x)$ is a cubic polynomial, on the subinterval $[x_j, x_{j+1}]$ for $j = 0 : n - 1$.*
>
> 2. *$S_j(x_j) = f(x_j)$, $S_j(x_{j+1}) = f(x_{j+1})$ for $j = 0 : n - 2$.*
>
> 3. *$S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$ for $j = 0 : n - 2$.*
>
> 4. *$S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$ for $j = 0 : n - 2$.*
>
> 5. $\begin{cases} \textit{natural boundary:} & S''(x_0) = S''(x_n) = 0. \\ \textit{clamped boundary:} & S'(x_0) = f'(x_0),\ S'(x_n) = f'(x_n). \end{cases}$

1.1   Construction of a Cubic Spline

Let $h_j = x_{j+1} - x_j$ (forward):

(1)

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 \quad \text{for } j = 0 : n - 1$$
$$\Rightarrow a_{j+1} = S_{j+1}(x_{j+1}) = S_j(x_{j+1})$$
$$= a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 = f(x_{j+1}) \quad \text{for } j = 0 : n - 1$$

(2)

$$S'_j(x) = b_j + 2c_j(x - x_j) + 3d_j(x - x_j)^2$$
$$\Rightarrow b_{j+1} = S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$$
$$= b_j + 2c_j h_j + 3d_j h_j^2 \quad \text{for } j = 0 : n - 1$$

(3)

$$S_j''(x) = 2c_j + 6d_j(x - x_j)$$
$$\Rightarrow 2c_{j+1} = S_{j+1}''(x_{j+1}) = S_j''(x_{j+1})$$
$$= 2c_j + 6d_j h_j \quad \text{for } j = 0 : n - 1$$

Above all, the linear system to be solved is:

$Ax = b$

$$\begin{cases} A = diag([1, 2(h_0 + h_1), \cdots, 2(h_{n-2} + h_{n-1}), 1]) \\ \quad + diag([0, h_1, \cdots, h_{n-1}], 1) + diag([h_0, \cdots, h_{n-2}, 0], -1) \\ x = [c_0; c_1; \cdots; c_n] \\ b = \left[0; \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0); \cdots; \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}); 0 \right] \end{cases}$$

Then we will get b_j, d_j by

$$\begin{cases} b_j &= \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}) \\ d_j &= \frac{1}{3h_j}(c_{j+1} - c_j) \end{cases}$$

1.2   Clamped Splines

$Ax = b$

$$
\begin{cases}
A = \begin{pmatrix}
2h_0 & h_0 & 0 & \cdots & 0 \\
h_0 & 2(h_0 + h_1) & h_1 & \ddots & \vdots \\
0 & h_1 & 2(h_1 + h_2) & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & h_{n-1} \\
0 & \cdots & \cdots & h_{n-1} & 2h_{n-1}
\end{pmatrix} \\[2em]
x = \begin{pmatrix} c_0 & c_1 & \cdots & c_n \end{pmatrix}^T \\[2em]
b = \begin{pmatrix}
\frac{3}{h_0}(a_1 - a_0) - 3f'(a) \\
\frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\
\vdots \\
\frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\
3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1})
\end{pmatrix}
\end{cases}
$$