Numerical Analysis

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目录

1 Euler's Method

The object of *Euler's method* is to obtain approximations to the well-posed initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

We will use Taylor's Theorem to derive Euler's method. Suppose that $y(t)$, the unique solution has two continuous derivations on $[a, b]$, so that for each $i = 0, 1, \cdots, N-1$

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2}y''(\xi_i)$$

for some number ξ_i in t_i, t_{i+1}. Because $h = t_{i+1} - t_i$, we have

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h)^2}{2}y''(\xi_i)$$

Euler's method constructs $\omega_i \approx y(t_i)$, for each $i = 1, 2, \cdots, N$, by deleting the remainder term, then Euler's method is

$$\begin{cases} \omega_0 = \alpha \\ \omega_{i+1} = \omega_i + hf(t_i, \omega_i) \quad for i = 0, 1, \cdots, N-1 \end{cases}$$

1.1 Errors Bounds for Euler's Method

定理 1.1 *Suppose f is continuous and satisfies a Lipschitz condition with constant L on*

$$D = \{(t, y) | a \le t \le b, y \in \mathbb{R}\}$$

and that a constant M exists with

$$|y''(t)| \le M, \quad \textit{for all } t \in [a, b]$$

where $y(t)$ denotes the unique solution to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

Let $\omega_0, \cdots, \omega_N$ be the approximations generated by Euler's method for *some positive integer N, then for each $i = 0, 1, \cdots, N$*

$$|y(t_i) - \omega_i| \le \frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right].$$

[Proof]

$$\begin{aligned}
|y_{i+1} - \omega_{i+1}| &\le |y_i - \omega_i| + h\,|f(t_i, y_i) - f(t_i, \omega_i)| + \frac{h^2}{2}\,|y''(\xi_i)| \\
&\le (1 + hL)\,|y_i - \omega_i| + \frac{h^2 M}{2} \\
&\le e^{(i+1)hL}(|y_0 - \omega_0| + \frac{h^2 M}{2hL}) - \frac{h^2 M}{2hL} \\
&= \frac{hM}{2L}\left(e^{(t_{i+1}-a)L} - 1\right).
\end{aligned}$$

□

定理 1.2 *If $u_0, u_1, \cdots, u_N$ be the approximations and $|\delta_i| < \delta$, then*

$$|y(t_i) - u_i| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left[e^{L(t_i-a)} - 1\right] + \delta e^{L(t_i-a)}$$

The minimal value of $E(f)$ occurs when $h = \sqrt{\frac{2\delta}{M}}$