Numerical Analysis

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目录

1 Cubic Spline Interpolation

定义 1.1 *Given a function f defined on $[a,b]$, $a = x_0 < x_1 < \cdots, < x_n = b$, a cubic spline interpolation S for f is a function that satisfies the following conditions.*

1. $S_j(x)$ *is a cubic polynomial, on the subinterval* $[x_j, x_{j+1}]$ *for* $j = 0 : n-1$.
2. $S_j(x_j) = f(x_j)$, $S_j(x_{j+1}) = f(x_{j+1})$ *for* $j = 0 : n-2$.
3. $S_j'(x_{j+1}) = S_{j+1}'(x_{j+1})$ *for* $j = 0 : n-2$.
4. $S_j''(x_{j+1}) = S_{j+1}''(x_{j+1})$ *for* $j = 0 : n-2$.
5. $\begin{cases} \textit{natural boundary:} & S''(x_0) = S''(x_n) = 0. \\ \textit{clamped boundary:} & S'(x_0) = f'(x_0),\ S'(x_n) = f'(x_n). \end{cases}$

1.1 Construction of a Cubic Spline

Let $h_j = x_{j+1} - x_j$ (forward):

(1)

$$
\begin{aligned}
S_j(x) &= a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 \quad \text{for } j = 0 : n-1 \\
\Rightarrow a_{j+1} &= S_{j+1}(x_{j+1}) = S_j(x_{j+1}) \\
&= a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 = f(x_{j+1}) \quad \text{for } j = 0 : n-1
\end{aligned}
$$

(2)

$$
\begin{aligned}
S_j'(x) &= b_j + 2c_j(x - x_j) + 3d_j(x - x_j)^2 \\
\Rightarrow b_{j+1} &= S_{j+1}'(x_{j+1}) = S_j'(x_{j+1}) \\
&= b_j + 2c_j h_j + 3d_j h_j^2 \quad \text{for } j = 0 : n-1
\end{aligned}
$$

(3)

$$\begin{aligned} S_j''(x) &= 2c_j + 6d_j(x - x_j) \\ \Rightarrow 2c_{j+1} &= S_{j+1}''(x_{j+1}) = S_j''(x_{j+1}) \\ &= 2c_j + 6d_jh_j \quad \text{for } j = 0 : n-1 \end{aligned}$$

Above all, the linear system to be solved is:

$Ax = b$

$$\begin{cases} A = diag([1, 2(h_0 + h_1), \cdots, 2(h_{n-2} + h_{n-1}), 1]) \\ \qquad + diag([0, h_1, \cdots, h_{n-1}], 1) + diag([h_0, \cdots, h_{n-2}, 0], -1) \\ x = [c_0; c_1; \cdots; c_n] \\ b = \left[0; \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0); \cdots; \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}); 0\right] \end{cases}$$

Then we will get b_j, d_j by

$$\begin{cases} b_j &= \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}) \\ d_j &= \frac{1}{3h_j}(c_{j+1} - c_j) \end{cases}$$

1.2 Clamped Splines

$$Ax = b$$

$$\begin{cases} A = \begin{pmatrix} 2h_0 & h_0 & 0 & \cdots & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & \ddots & \vdots \\ 0 & h_1 & 2(h_1 + h_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & h_{n-1} \\ 0 & \cdots & \cdots & h_{n-1} & 2h_{n-1} \end{pmatrix} \\ x = \begin{pmatrix} c_0 & c_1 & \cdots & c_n \end{pmatrix}^T \\ b = \begin{pmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f'(a) \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1}) \end{pmatrix} \end{cases}$$