# CHAPTER 1

# ASSIGNMENT 1

## 1.1 Question 1

(a) Write down the definition of **norm** for a vector space.

(b) Given $\mathbf{x} \in \mathbb{R}^n$, show that followings are norm on $\mathbb{R}^n$.

i. $\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$;

ii. $\|\mathbf{x}\|_1 = \sum_{1\le i\le n} |x_i|$;

iii. $\|\mathbf{x}\|_2 = \left(\sum_{1\le i\le n} |x_i|^2\right)^{1/2}$.

Plot the regions for $\|\mathbf{x}\|_p \le 1$ in $\mathbb{R}^2$ for $p = \infty, 1, 2$ respectively.

(c) Given the vector norm $\|\cdot\|_p$ for $\mathbb{R}^n$, the induced norm for matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ is defined as

$$\|\mathbf{A}\|_p = \max_{\mathbf{x}\neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p}.$$

Show that

i. $\|\mathbf{A}\|_\infty = \max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|$;

ii. $\|\mathbf{A}\|_1 = \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|$;

iii. $\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}(\mathbf{A}^T\mathbf{A})}$.

## 1.2 Question 2

(a) Use the method of undetermined coefficients to design third order accurate approximation to $u'(\bar{x})$ by using the discrete points $\bar{x}+h$, $\bar{x}$, $\bar{x}-h$, $\bar{x}-2h$.

(b) Assuming $u(x)$ is smooth enough, compute the truncation error (leading term) for the finite difference formula above.

(c) What is the truncation error if $u(x) = 4x^4 + 12x^3 + 6x^2 + x^2/2 + \pi$.

## 1.3 Question 3

(a) For the following $N \times N$ matrix, show that all eigenvalues are given by $\lambda_p = a + 2b \cos \frac{\pi p}{N+1}$ for $p = 1, 2, \dots, N$.

$$\begin{bmatrix} a & b & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & b \\ & & b & a \end{bmatrix}_{N \times N}$$

## 1.4 Question 4

(a) For the following $N \times N$ matrix, show that all eigenvalues are given by $\lambda_p = a + 2\sqrt{bc} \cos \frac{\pi p}{N+1}$ for $p = 1, 2, \dots, N$ and $bc > 1$.

$$\begin{bmatrix} a & c & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & c \\ & & b & a \end{bmatrix}_{N \times N}$$

## 1.5 Question 5

For the 2-point BVP with Neumann boundary conditions:

$$\begin{cases} u''(x) = f(x), & x \in (0, 1) \\ u'(0) = u'(1) = 0 \end{cases}$$

(a) Set the grid points as $x_j = jh$ for $j = 0, 1, \dots, N$ and $h = 1/N$. Write down the central FD scheme for the main equation with using $u_j$ to approximate $u(x_j)$.

(b) Add two "ghost" points as $x_{-1} = -h$ and $x_{N+1} = 1 + h$, and two more variables $u_{-1}$ and $u_{N+1}$. Treat the boundary condition as $(u_1 - u_{-1})/2h = 0$, $(u_{N+1} - u_{N-1})/2h = 0$. Assemble all the equations as $\mathbf{AU} = \mathbf{F}$ where $\mathbf{U} = [u_0, u_1, \dots, u_N]^T$.

(c) Show that $\mathbf{A}$ is singular, and find out when the system $\mathbf{AU} = \mathbf{F}$ has solutions.

(d) Find the kernel space of $\mathbf{A}$.

(e) Find out all eigenvalues and eigenvectors for the matrix $\mathbf{A}$ given in (b).