1.1 Question 1

- (a) Write down the definition of **norm** for a vector space.
- (b) Given $\mathbf{x} \in \mathbb{R}^n$, show that followings are norm on \mathbb{R}^n .

i.
$$\|\mathbf{x}\|_{\infty} = \max_{1 \le i \le n} |x_i|$$
;

ii.
$$\left\|\mathbf{x}\right\|_1 = \sum_{1 \leq i \leq n} |x_i|;$$

iii.
$$\left\|\mathbf{x}\right\|_2 = \left(\sum_{1 \leq i \leq n} \left|x_i\right|^2\right)^{1/2}$$
.

Plot the regions for $\|\mathbf{x}\|_p \leq 1$ in \mathbb{R}^2 for $p = \infty, 1, 2$ respectively.

(c) Given the vector norm $\|\cdot\|_p$ for \mathbb{R}^n , the induced norm for matrices $\mathbf{A}\in\mathbb{R}^{n\times n}$ is defined as

$$\left\|\mathbf{A}\right\|_{p} = \max_{\mathbf{x} \neq 0} \frac{\left\|\mathbf{A}\mathbf{x}\right\|_{p}}{\left\|\mathbf{x}\right\|_{p}}.$$

Show that

i.
$$\|\mathbf{A}\|_{\infty} = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|;$$

ii.
$$\|\mathbf{A}\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|;$$

iii.
$$\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}(\mathbf{A}^T \mathbf{A})}$$
.

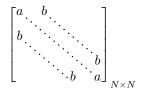
1.2 Question 2

- (a) Use the method of undetermined coefficients to design third order accurate approximation to $u'(\bar{x})$ by using the discrete points $\bar{x}+h$, \bar{x} , $\bar{x}-h$, $\bar{x}-2h$.
- (b) Assuming u(x) is smooth enough, compute the truncation error (leading term) for the finite difference formula above.
- (c) What is the truncation error if $u(x) = 4x^4 + 12x^3 + 6x^2 + x^2/2 + \pi$.

1.3 Question 3

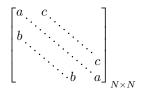
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(a) For the following $N \times N$ matrix, show that all eigenvalues are given by $\lambda_p = a + 2b\cos\frac{\pi p}{N+1}$ for $p=1,2,\ldots,N$.



1.4 Question 4

(a) For the following $N \times N$ matrix, show that all eigenvalues are given by $\lambda_p = a + 2\sqrt{bc}\cos\frac{\pi p}{N+1}$ for $p=1,2,\ldots,N$ and bc>1.



1.5 Question 5

For the 2-point BVP with Neumann boundary conditions:

$$\begin{cases} u''(x) = f(x), & x \in (0,1) \\ u'(0) = u'(1) = 0 \end{cases}$$

- (a) Set the grid points as $x_j=jh$ for $j=0,1,\ldots,N$ and h=1/N. Write down the central FD scheme for the main equation with using u_j to approximate $u(x_j)$.
- (b) Add two "ghost" points as $x_{-1}=-h$ and $x_{N+1}=1+h$, and two more variables u_{-1} and u_{N+1} . Treat the boundary condition as $(u_1-u_{-1})/2h=0$, $(u_{N+1}-u_{N-1})/2h=0$. Assemble all the equations as $\mathbf{A}\mathbf{U}=\mathbf{F}$ where $\mathbf{U}=[u_0,u_1,\ldots,u_N]^T$.
- (c) Show that \mathbf{A} is singular, and find out when the system $\mathbf{A}\mathbf{U} = \mathbf{F}$ has solutions.
- (d) Find the kernel space of **A**.
- (e) Find out all eigenvalues and eigenvectors for the matrix **A** given in (b).