Will Bitcoin hit 100k in 2024?

Izaac Mammadov
~diblud-ricbet

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Abstract

The title is the name of one of the markets on Polymarket. The market at the time of writing has 3 million dollars worth of traded volume. Now I'm not in any better position to answer that question directly than anyone else. What I can say is that not enough people realise that the question is already (roughly) being answered on any other venue which trades vanilla options (calls and puts). There exists a (not well-enough known among retail-traders) semi-arbitrage between vanilla options and one-touch options (which is what is actually being traded in the Polymarket market).

Contents

1 Prerequisites

I make the assumption that you are already familar and comfortable with vanilla options (calls and puts) and their payoffs. I also assume you are comfortable with the idea that you can trade combinations of different calls/puts to make

Figure 9.1 Profit from buying a European call option on one share of a stock. Option price = \$5; strike price = \$100.

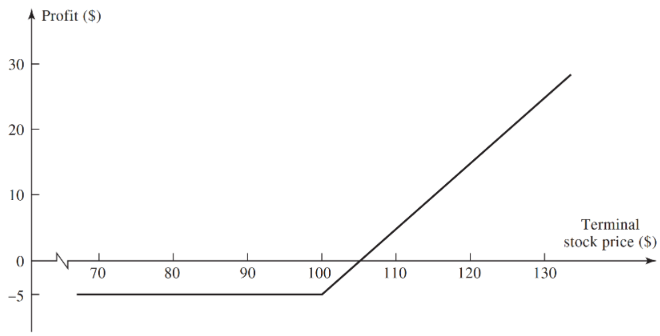

Figure 1: European Call Profit (taken from Hull: Options Futures and other Derivatives).

many different kinds of payoffs. I've attached diagrams of the payoffs of a call, put and a call-spread in Figures 1–3. A call-spread is created by buying a call and selling a call at the same expiry but with a higher strike price.

2 Digital Options

2.1 Definition

Digital Options (called Digis for short, or Binary Options) are a type of exotic option. As you might guess from their name, they have a 0/1 payoff. Being an exotic option, digital options for crypto tend not to be traded on exchanges, but can be bought directly from crypto market-makers. For example, a digital call on Bitcoin with a strike of \$80,000 and an expiry of 1st Dec 2024 will pay \$1 if Bitcoin is trading above \$80,000 on the 1st Dec 2024, and nothing otherwise. The formula for the payoff of a digital option can be given as:

$$\text{Payoff of a Digital Call} = \begin{cases} 0 & \text{if } S_T < K \\ 1 & \text{if } S_T \geq K \end{cases} \tag{1}$$

$$\text{Payoff of a Digital Put} = \begin{cases} 1 & \text{if } S_T \leq K \\ 0 & \text{if } S_T > K \end{cases} \tag{2}$$

where S_T is the price of the underlying at expiry time T, and K is the strike price of the digital options. The payoffs for digital call options are just Heaviside step functions as can be seen in Figure 4.

Figure 9.2 Profit from buying a European put option on one share of a stock. Option price = \$7; strike price = \$70.

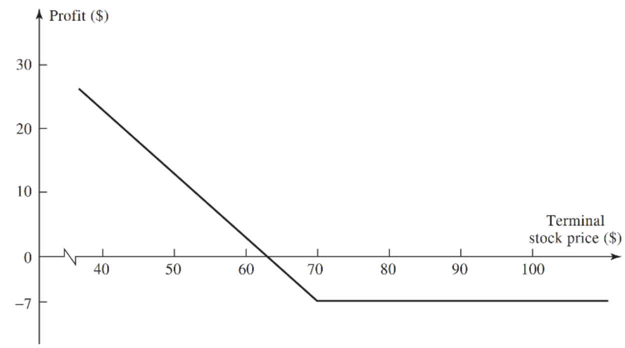

Figure 2: European Put Profit (taken from Hull: Options Futures and other Derivatives).

Figure 11.2 Profit from bull spread created using call options.

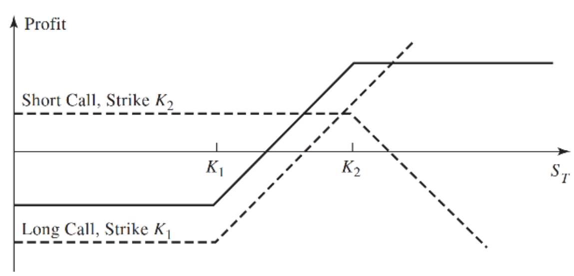

Figure 3: Call-Spread Profit (taken from Hull: Options Futures and other Derivatives). The overall payoff (solid line) is the sum of the two individual option payouts (dashed lines).

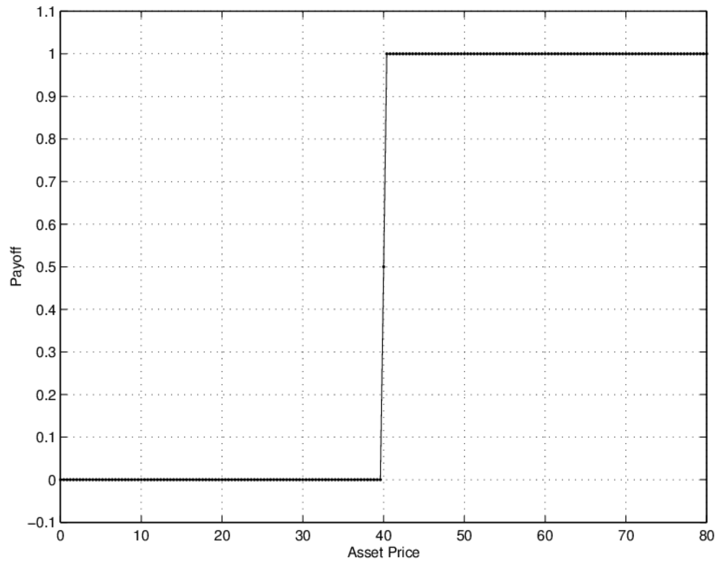

Figure 4: Digital Call Payoff with a strike price of 40. Note this graph doesn't include the price of buying the digital call and is strictly the payoff at expiry, not the overall profit.

The price of digital options are always between 0 and 1. This price repesents in some sense[1] the probability that the price of the underlying at expiry is in the money ($\geq K$ for a digital call, or $\leq K$ for a digital put), discounted to today. This is actually just an application of a simple result in probability theory saying that for a random event A, the expectation of the indicator function[2] on A is just the probability of A:

$$\mathbb{E}(\mathbf{1}_A) = \mathbb{P}(A). \tag{3}$$

To specifically relate this back to a digital call, we set $A = \{S_T \geq K\}$ since the payoff of the digital call is precisely $\mathbf{1}_{\{S_T \geq K\}}$, so assuming no discounting:[3]

$$\begin{aligned}
\text{Price of Digital Call} &= \text{Expected Payoff at Expiry} && (4)\\
&= \mathbb{E}(\mathbf{1}_{\{S_T \geq K\}}) && (5)\\
&= \mathbb{P}(S_T \geq K). && (6)
\end{aligned}$$

[1] In particular, it is a market-implied risk-neutral probability. In layman's terms, this is a probability assuming you remove all risk-premia out of the market (e.g. the expected returns from holding risky S&P is set to be same as holding near risk-free treasuries).

[2] The indicator function on a random event is just a random variable which is 1 when the event occurs, and 0 otherwise.

[3] The result in Equation 4 is only true in the sense that the risk-neutral price (which is equivalent to the no-arbitrage price) must be equal to the risk-neutral expected payoff. Thus the $\mathbb{E}$ and $\mathbb{P}$ in Equations 5–6 should really be $\mathbb{E}^{\mathbb{Q}}$ and $\mathbb{Q}$ respectively, where $\mathbb{Q}$ is the risk-neutral probability measure.

This result can also be generalised to all digital contracts (where the payoff is 0/1 based on some condition).

2.2 Digital Contracts in General on Polymarket

All contracts on Polymarket are Digital contracts (though not necessarily digital calls or puts). It is the piece of obscure, but perhaps common-sense maths above that means the price of all Polymarket digital contracts actually correspond to bona-fide probabilities of the underlying event occuring. If a Polymarket contract trades at 60 cents, the market-implied probability of the underlying event happening is 60%, well except, for two important simplifications we made.

Firstly, we assumed no discounting. This means we assume a dollar today is worth just as much to as as a dollar in ten years. That's obviously not true. For example, assume there existed a Polymarket market for a fair 50/50 coin flip happening in 5 years. What price would that contract trade at? At first glance, and using the logic above, you'd answer 50 cents. The actual answer is actually (as of writing) around 40.4 cents. This is because the contract only pays out after 5 years, and I can have a guaranteed 50 cents in 5 years time by investing only 40.4 cents in five-year US Treasury Notes. It would actually make a lot of sense if Polymarket introduced these "dummy" markets that always resolved Yes, or were 50/50 coinflips for calibration purposes, so that we can properly convert prices to probabilities.[4]

Secondly, we glossed over points related to risk-neutral measures and shoved them into the footnotes. This is a much more mathematically technical point, but still has significant real-world implications. To avoid 50 pages of measure theory, let me try to give an illustrative example. Let's say we have some Polymarket digital contract, which is very highly correlated with the S&P Index (perhaps it is a digital call on the S&P, or something similar). Say further that the probability of the contract being in-the-money is 50%. Even assuming a dollar today is worth a dollar when this contract expires, would I pay 50 cents for this contract? I wouldn't, because I could earn on average 8% per year by investing in an S&P Index Fund. Given this digital contract is very correlated to the S&P, I would be taking all the risk on as-if I was invested in an index fund, but making none of the returns. Nobody would ever do that.

Neither of the two points above are mere technicalities. They're serious issues that force you to be careful in pricing digital contracts. I would even argue that the statement that Polymarket places in its documentation "prices are probabilities" is untrue if you don't mention the two points above (which they don't). A more accurate statement would be "assuming no arbitrage, prices are time-discounted risk-neutral probabilities". It's not as catchy.

[4]This would be analogous to how Deribit is one of the few platforms to list fixed-expiry crypto futures. They only do so to find fair forward prices to be used in pricing their raison d'etre: crypto options.

2.3 Digital Call/Put Replication

An important thing to realise is that the payoff of a digital call (seen in Figure 4) is similar to the payoff a call-spread (seen in Figure 3). You just have to make the non-flat part of a call-spread steeper (infinitely steep?). In case you haven't guessed it by now, assuming you can trade options of any strike price, you can create an "artificial" digital call by trading a very steep and narrow call spread.

Specifically, to replicate a digital call, with a strike price of K, you can buy $1/\varepsilon$ of a vanilla call at a strike price of $K - \varepsilon/2$ and sell $1/\varepsilon$ of a vanilla call at a strike price of $K + \varepsilon/2$. The payoff of this replication is precisely:

$$\text{Payoff} = \begin{cases} 0 & \text{if } S_T \leq K - \frac{\varepsilon}{2} \\ \frac{1}{\varepsilon}(S_T - K) + \frac{1}{2} & \text{if } K - \frac{\varepsilon}{2} < S_T < K + \frac{\varepsilon}{2} \\ 1 & \text{if } S_T \geq K + \frac{\varepsilon}{2} \end{cases} \tag{7}$$

As $\varepsilon \to 0$, this call spread has a payoff which tends pointwise to the payoff of the digital call (seen in Equation 1). All of these results can also be applied analogously to digital puts, which can be replicated with put spreads. In some sense, a digital call isn't a brand new type of option, but merely taking the limit of this replicating call-spread (which only uses vanilla options).

This means if we have prices for vanilla calls for all strikes on a given expiry, then from them we automatically find prices for digital calls of all strikes on that expiry. There are no new degrees of freedom in a digital call, its price is precisely defined based on the prices of vanilla calls. Based on the replication described above, we can say exactly:

$$DC(K_0) = \lim_{\varepsilon \to 0} \frac{C\left(K_0 - \frac{\varepsilon}{2}\right) - C\left(K_0 + \frac{\varepsilon}{2}\right)}{\varepsilon} \tag{8}$$

$$= -\left.\frac{\partial C}{\partial K}\right|_{K=K_0} \tag{9}$$

where $DC(K)$ is the price of a digital call with strike K and $C(K)$ is the price of a vanilla call with strike K. However, in our eponymous Polymarket contract, we don't actually have a "proper" digital call. If you read the conditions of the market, you'll see that actually the contract with resolve to Yes if the price of Bitcoin touches the price of \$100,000 at any time in 2024, not just at expiry (T). So we're sadly not in the home-straight yet.

3 One-Touch Options

3.1 Definition

A one-touch option is an even more exotic option that pays out the full \$1, as soon as the price of the underlying touches the strike price, regardless of what its final price is at expiry. Mathematically, the payoff is given by:

$$\text{One-Touch Call Payoff} = \begin{cases} 1 & \text{if } \exists t \in [0,T] : S_t \geq K \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Our Polymarket contract is actually just a one-touch vanilla call on Bitcoin with a strike price of \$100,000 and expiry on 31st Dec 2024 23:59. Note that you can receive the payout regardless of what the final price of Bitcoin is at the end of the year, it just needs to touch \$100,000 at some point.

At first glance, it may seem utterly implausible to come up with a price for one-touch options based on vanilla options. They're just so different in so many ways. But making a few assumptions we have a saving grace.

3.2 Pricing

We have, what I believe to be one of the most beautiful equations in Quant Finance. It states, assuming no discounting (r=0):

$$OT(K_0) = 2 \times DC(K_0) \tag{11}$$

where $OT(K)$ is the price of a one-touch call option with strike K and $DC(K)$ is the price of a digital call option with strike K.

The proof of this equation comes from the fact that we can replicate the payout of the one-touch, by instead buying two digital calls. Now that's a statement that deserves some justifying. Let's say Alice buys a one-touch call with strike K and Bob buys $2\times$ digital calls with strike K. Lets consider if Bob can make his profits exactly match up with Alice's profits.

If the underlying never touches K, it certainly can't be $\geq K$ at expiry. Thus in this case, both the one-touch and digital expire worthless. Alice and Bob both have the same payoff of 0.

If the underlying does touch K, then Alice will receive a payout of 1. At the precise moment that the underlying touches K, Bob can sell both of his digital calls. What price will he get for his digital calls? Given the price of the underlying is exactly at K, there is a 50% probability that the underlying will have price above K at expiry (by symmetry).[5] In the same way that if the price of Bitcoin is \$80,000 right now, the price tomorrow has a 50% chance of being greater than \$80,000, and 50% chance of being less than \$80,000. Thus he receives 50 cents for each of his digital calls, and so in total receives a payout of $2 \times 0.50 = 1$.

In both of these scenarios, the payoff for Alice and Bob are the same, and thus whatever Alice bought must cost the same as whatever Bob bought.

Combining Equations 9 and 11, we find under the assumption that $r = 0$:

$$OT(K_0) = -2 \times \left. \frac{\partial C}{\partial K} \right|_{K=K_0} \tag{12}$$

[5] I used the Physicist's cop-out of arguing "by symmetry". In reality, you have to consider the reflection principle of a Brownian Motion under the risk-neutral measure, yadda yadda yadda...

Bid	Mid	Ask
0.53	0.535	0.54

Table 1: Polymarket Market Data for a Bitcoin One-Touch Call Option with Strike \$100,000 and expiry 31st Dec 2024 23:59 ET.

Strike	Bid	Mid	Ask
\$95,000	\$4022.94	\$4066.20	\$4109.46
\$100,000	\$2811.74	\$2854.995	\$2898.25
\$105,000	\$1946.56	\$1989.815	\$2033.07

Table 2: Deribit Market Data for a Bitcoin Vanilla Call Options with expiry 27th Dec 2024 08:00 UTC.

where $C(K)$ is the price of a vanilla call option with strike price K. Thus the no-arbitrage price of this incredibly exotic derivative can actually be determined precisely by the price of call options. "No-arbitrage price" means that if the left-hand side doesn't equal the right-hand side, there is risk-free money to be made.

4 The Trade

4.1 Replication

The Polymarket one-touch call contract and relevant Deribit option markets are given in Tables 1–3.

Unfortunately, the expiry for the one-touch on Polymarket doesn't line up with any of the listed expiries on Deribit Bitcoin options. To mananage this, we instead use a weighted sum of options split between the two closest expiries either side of the one-touch expiry. To roughly replicate a 31st Dec 2024 expiry, we take 13.9% of the 31st Jan 2025 expiry and 86.1% of the 27th Dec 2024 expiry.

Further, we cannot trade all possible strikes on Deribit options and only certain round number striked options are tradeable. Thus we are forced to replicate the call-spread with a relatively large ε of \$5,000 for the 27th Dec expiry and \$2,000 for the 31st Jan expiry.

Overall we trade the following:

Strike	Bid	Mid	Ask
\$98,000	\$5537.00	\$5580.26	\$5623.52
\$100,000	\$5018.04	\$5061.30	\$5104.56
\$102,000	\$4498.78	\$4563.665	\$4628.55

Table 3: Deribit Market Data for a Bitcoin Vanilla Call Options with expiry 31st Jan 2025 08:00 UTC.

- Buy $0.861 \times 2 \div 10,000$ Bitcoin 2024-12-27 C95000
- Sell $0.861 \times 2 \div 10,000$ Bitcoin 2024-12-27 C105000
- Buy $0.139 \times 2 \div 4,000$ Bitcoin 2025-01-31 C98000
- Sell $0.139 \times 2 \div 4,000$ Bitcoin 2025-01-31 C102000

This very closely replicates the one-touch option on Polymarket, and yet has an overall price of 0.428 at mid. Crossing the bid-ask spread with each trade only raises this price to 0.451.

We can thus buy call-spreads which replicate the one-touch on Deribit for a price of 0.451, and sell the one-touch on Polymarket at 0.53. This represents a profit of close to 8 cents per Polymarket contract.

4.2 Managing the Trade

Note that replicating the one-touch call with digital calls is not completely passive. Specifically, if Bitcoin ever hits 100,000, you have to immediately close all of the option positions on Deribit, hoping to ideally net to a payoff of close to 1.00 for all your trades. In fact, because of discounting effects, you should actually net more than 1.00 for all of these closing trades. This will offset your losses from being short the Polymarket one-touch.

As we move closer to the 31st Dec 2024, more expiries and strikes will be added and listed on Deribit. You should always try to move your expiries to be the expiries closest to the one-touch's expiry and the strikes to be as close as possible to 100,000. This is to mitigate any strike/expiry risk and make the replication as accurate as possible.