

UNIVERSITY OF BUEA

FACULTY OF SCIENCE

**DEPARTMENT OF COMPUTER
SCIENCE**

IMPLEMENTATION OF A HEART-SHAPED PRIMITIVE

By

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DEDICATION

To My Family

UNIVERSITY OF BUEA

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CERTIFICATION

The thesis of Isaac Kamga Mkounga (SC09B676) entitled: "**Implementation Of A Heart-Shaped Primitive**" submitted to the Department of Computer Science, Faculty of Science at the University of Buea in partial fulfillment of the requirements for the award of the Master of Science (M.Sc.) degree in Computer Science has been examined and approved by the examination panel composed of:

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ABSTRACT

This thesis titled Implementation Of A Heart-Shaped Primitive was aimed at demonstrating the engineering of a heart-shaped primitive within the BRL-CAD (Ballistic Research Laboratory Computer Aided Design) package. Using a case study approach, the heart-shaped primitive's data structure was designed, necessary callback functions were written and tested with BRL-CAD's testing infrastructure. It was shown that the Laguerre-based root solver is indeed a sure-fire iterative method for finding roots of polynomials and it's stability on sextic equations was ascertained. Also, this work provided a guideline for the development of primitives within open source Computer Aided Design (CAD) software and showed that the BRL-CAD package primarily developed for the military sector can be used in non-military locales for entertainment purposes.

Keywords: heart-shape, sextic, primitive, BRL-CAD

TABLE OF CONTENTS

| | |
|-----------------------------|------|
| Dedication | i |
| Certification | ii |
| Acknowledgements | iii |
| Abstract | iv |
| Table of Contents | v |
| List of Tables | vii |
| List of Figures | viii |
| List of Abbreviations | ix |

CHAPTER ONE INTRODUCTION

| | |
|-----------------------------------|---|
| 1.1 Historical Background | 1 |
| 1.2 Importance Of This Work | 3 |
| 1.3 Thesis Organisation | 4 |

CHAPTER TWO LITERATURE REVIEW OF GEOMETRIC MODELING

| | |
|---|----|
| 2.1 Representation Schemes | 6 |
| 2.2 Wireframe Modeling | 8 |
| 2.3 Surface Modeling | 9 |
| 2.3.1 Implicit Representation | 10 |
| 2.3.2 Parametric Representation | 10 |
| 2.3.3 Implicitization | 11 |
| 2.3.4 Parameterization | 12 |
| 2.4 Solid Modeling | 13 |
| 2.4.1 Boundary Representations (B-REPS) | 13 |
| 2.4.2 Constructive Solid Geometry (CSG) | 15 |
| 2.4.3 Spatial Subdivision | 16 |
| 2.5 Non-manifold Geometry | 21 |

CHAPTER THREE ANALYSIS & DESIGN

| | |
|--|----|
| 3.1 The Open Source Community | 23 |
| 3.2 Analysis Of The Work | 24 |
| 3.3 Design | 27 |
| 3.3.1 Tagging the heart-shaped primitive | 27 |
| 3.3.2 Data Structure of Heart-shaped primitive | 28 |

| | |
|--|----|
| 3.3.3 A bare bones heart-shape | 29 |
| 3.3.4 Formatted description of the heart-shaped primitive | 30 |
| 3.3.5 Database importation and exportation of the heart-shaped primitive ... | 30 |
| 3.3.6 Computing the bounding box of the heart-shaped primitive | 31 |
| 3.3.7 Plotting the wireframe of the heart-shaped primitive | 32 |
| 3.3.8 Surface representation and raytracing of the heart-shaped primitive | 33 |
| 3.3.9 Type in support for the heart-shaped primitive | 36 |

CHAPTER FOUR RESULTS & DISCUSSION

| | |
|--|----|
| 4.1 Type-in support for the heart-shaped primitive | 37 |
| 4.2 Formatted description of the heart-shaped primitive | 39 |
| 4.3 The bounding box of the heart-shaped primitive | 40 |
| 4.4 Plotting the wireframe of the heart-shaped primitive | 42 |
| 4.5 Ray tracing and surface representation of the heart-shaped primitive | 42 |

CHAPTER FIVE CONCLUSIONS

| | |
|-----------------------------|----|
| 5.1. Our Contribution | 45 |
| 5.2. Further Research | 45 |

| | |
|--------------------|----|
| Bibliography | 50 |
| APPENDIX | 50 |

LIST OF TABLES

Table 2.1: Implicit and parametric equations of some BRL-CAD primitives. . . . 12

LIST OF FIGURES

| | |
|--|----|
| Figure 1.1: Model of a Goliath tracked mine | 5 |
| Figure 2.1: A wireframe of a sphere | 8 |
| Figure 2.2: BRL-CAD Solid Primitive Shapes | 9 |
| Figure 2.3: The Utah Tea Pot model | 15 |
| Figure 2.4: CSG Boolean operations | 16 |
| Figure 2.5: An M1A1 tank on a pedestal in a mirrored showcase room | 17 |
| Figure 2.6: Model of a Sphere Flake | 20 |
| Figure 2.7: Examples of Non-Manifold Geometry | 21 |
| Figure 3.1: Diagram of the heart-shaped primitive | 28 |
| Figure 4.1: Testing type in support for the heart-shaped primitive in archer | 38 |
| Figure 4.2: Testing the formatted description of the heart-shaped primitive. | 39 |
| Figure 4.3: Testing the bounding box of the heart-shape. | 41 |
| Figure 4.4: Testing the wireframe of the heart-shaped primitive | 42 |
| Figure 4.5: Images rendered after raytracing the heart-shaped primitive | 44 |

LIST OF ABBREVIATIONS

- CAD :** Computer Aided Design
BRL-CAD : Ballistic Research Laboratory Computer Aided Design
CSG : Constructive Solid Geometry
NURBS : Non-Uniform Rational B-Splines
NMG : Non - Manifold Geometry
B-REPS : Boundary Representations

Chapter 1

INTRODUCTION

1.1 Historical Background

Throughout our long history, humans have always sought for means to express their creativity – ways to communicate ideas through writing, sculpting, painting, carving, architecture and drawing. As a matter of fact, paleolithic cave representations of animals at least 32,000 years ago in Southern France, ink drawings and paintings of human figures as well as writings in hieroglyphics on papyrus in the pyramids of ancient Egypt is indicative of the fact that humanity's need to express its individuality goes back to antiquity. Before the renaissance, drawing was treated as a preparatory stage for painting and sculpting. The wide availability of drawing instruments such as pens and pencils and most especially paper made master craftsmen like Leonardo Da Vinci, Raphael and Michelangelo around the world to lift drawing to an art in its own right. According to Wikipedia [?], drawing stood out as the most popular and fundamental means of public expression in human history and is one of the simplest and most efficient means of communicating visual ideas.

The drawing board era where paper, pens, pencils, rulers and ink prevailed has been relegated to the background in this information age powered by ubiquitous computer technology. Craving the unity of science and art, this essentially binary-sequenced revolutionary device called the computer married the artistic and engineering forms of drawing into an androgynous one called Computer Aided Geometric Design or Geometric Modeling for short. Geometric modeling currently involves the use of computers to aid in the creation, manipulation, maintenance and analysis of representations of the geometric shapes of two and three-dimensional objects [?]. It is the outgrowth of convergent motivations and developments from several works of life as outlined below.

- In the 1950s, the need to automate the engineering drawing process led to electronic drawings which could be archived and modified more easily, could be easily verified and errors could be eliminated from mechanical designs without introducing new ones. These computer drafting systems allowed designers to produce drawings of objects by projecting three-dimensional objects onto two-dimensional surfaces.
- In the 1960s, there was a pressing need for software in the automobile, shipbuilding and aircraft industries to produce computer-compatible descriptions of geometric shapes which can be machined from wood and steel into stamps and dies for the manufacturing and assembling of car parts, ship hulls as well as wings and fuselages using computer numerically controlled tools.
- Later in the 1970s, the growing need for computers to render realistic images of objects as well as animate solid objects pushed research institutes like Xerox Palo Alto Research Center (Xerox PARC) and Apple Computers to make significant contributions to graphical user interface design and computer graphics.

These needs and problems could only be solved by research in fields such as graphics, animation and applications from algebraic geometry. The work of various computer scientists and mathematicians led to the active development of several commercial packages sponsored by companies such as Renault, Citroen, Ford and Boeing who could afford the computers capable of performing such lengthy calculations.

Today, geometric modeling is also referred to as Computer Aided Design (CAD), is pronounced “kad” and is routinely used in the design and manufacturing of engineering and architectural structures such as buildings, car parts, ship hulls and aircraft artillery as well as to specify special effects in cartoon movies, music videos and television shows. Indeed, CAD packages provide facilities for designing shapes of solid physical objects and specifying their motion in a way that art and science can unite to create cool designs.

Even though significant progress had been made in basic research and the functionality of commercially available solid modelers like Apple Computer’s RenderMan, many solid modelers especially within the open source community are still limited in their geometric features. The open source community is a self organising collaborative social network of programmers driven by a passion to solve problems using computers. It has several thousands of its projects on sites that offer services like bug tracking, mailing lists and version control viz

Github and Source Forge. These projects are constantly being improved upon by thousands of programmers putting in time and effort to write and debug software without direct monetary pay.

In this thesis, we document the process of developing a heart-shaped primitive, a set of callback functions and procedures which compute geometrically useful properties of solids such as wireframe plotting, database importation and exportation, ray tracing, bounding box calculations, just to name a few, within the Ballistic Research Laboratory Computer Aided Design (BRL-CAD) software package.

BRL-CAD was initiated by the United States Army Research Laboratory in 1983, the same agency which created the Electronic Numerical Integrator And Computer (E.N.I.A.C.), the world's first general purpose computer in the 1940s, to model military systems for the United States Government. According to Morrison [?], BRL-CAD became born again in 2004 when it joined the open source community with portions of its source code licensed under the Lesser General Public License (LGPL) and Berkeley Software Distributions (BSD) licenses and has been credited as being the oldest open source repository in continuous development. It supports a wide variety of geometric representations including an extensive set of traditional implicit primitive shapes as well as explicit primitives made from collections of uniform B - spline surfaces, Non - Uniform Rational B - spline (NURBS) surfaces, Non - Manifold geometry (NMG) and purely faceted polygonal mesh geometry.

BRL-CAD also focuses on solid modeling aspects of Computer Aided Design. Figure 1.1 below shows a three-dimensional model of a Goliath tracked mine, a German engineered remote controlled vehicle used during World War II. This model was created by students new to BRL-CAD in the span of about 2 weeks, starting from actual measurements in a museum.

1.2 Importance Of This Work

This work is significant to several stakeholders for several reasons ;

- By raytracing the heart's surface, it shows that the Laguerre zerofinder is indeed a surefire iterative method for finding roots of polynomials and that Laguerre-based root solvers are stable on sextic equations.
- This work incorporates more geometric modeling functionality into the free and open source software community through BRL-CAD, which according

to Morrison [?] is the oldest open source repository in continuous development. It goes beyond traditional CSG primitives shapes such as tori, spheres, boxes and ellipsoids towards the more complex heart-shape based on a sextic equation. This work provides a guideline for the development of primitives within open source CAD software.

- Given that BRL-CAD is used within governments to model military artillery and for engineering and analysis purposes within academia, this heart-shaped primitive gives BRL-CAD a loving aura – an environment where artists can produce cartoon animations as well as design cards, royal seals and banners, gifts and presents for family and communal celebrations such as weddings, family reunions and Valentine’s day for entertainment purposes.

1.3 Thesis Organisation

This thesis is divided into five (5) chapters. Chapter 2 reviews the literature in the field of geometric modeling. Chapter 3 states the problems intended to solve and our project design. Chapter 4 discusses the results obtained. Finally, Chapter 5 states the contribution of the work and gives possible research directions which can proceed from it.



FIGURE 1.1: Model of a Goliath tracked mine

Chapter 2

LITERATURE REVIEW OF GEOMETRIC MODELING

With the advent of computers which perform millions of floating point operations in unit time and which are still growing faster, researchers who believed computers could aid the processes of mechanical design and manufacturing were faced with a critical issue – how to represent physical reality using computer software. They sought the best data structures to represent this reality and the most appropriate algorithms to manipulate these representations.

BRL-CAD supports a wide variety of geometric representations including an extensive set of traditional implicit primitive shapes as well as explicit primitives made from collections of uniform B - spline surfaces, non - uniform rational B - spline (NURBS) surfaces, non - manifold geometry (NMG) and purely faceted polygonal mesh geometry. Consequently, this chapter reviews the existing work done by scholars in the field of geometric modeling which have been applied to the development of BRL-CAD. First of all, it introduces the issue of representation and the notion of representation schemes. Then, it summarizes developments in wireframe modeling, surface modeling, solid modeling and non - manifold modeling (aka non - manifold geometry or nmg for short) with a keen eye on the algorithms underlying them.

2.1 Representation Schemes

Definition 2.1.1. A representation \mathbf{R} of a solid or representation for short is a subset of three-dimensional Euclidean space denoted \mathbb{E}^3 which models a physical solid.

According to Requicha and Tilove [?], point set topology provided a formal language for describing the geometric properties of solids and they also threw more light on the mathematical characteristics of solids such as a solid's interior, boundary, complement, closure, boundedness and regularity.

Requicha [?] insisted that to be computationally useful, a representation should formally have the following properties ;

- **Rigidity:** Representations should have an invariant configuration irrespective of their location and orientation.
- **Homogeneity:** A representation should have an interior.
- **Finiteness:** A representation must occupy a finite amount of space.
- **Boundary determinism:** A representation must unambiguously determine the interior of that solid.
- **Closure:** Representations of solids which are manipulated by rigid motions and regularized boolean operations should produce representations of solids too.

These formal characteristics left representations bounded, closed, regularized and semi - analytic, hence they were called **r - sets** according to Requicha [?].

Remark 2.1.1. An **r - set** is simply a regular and bounded set in \mathbb{E}^3 .

A representation scheme is simply a relation between physical solids and their representations which can be characterized by the following properties;

- **Domain:** A representation scheme must represent quite a number of useful geometric solids.
- **Unambiguity:** A representation scheme should produce representations which have the properties of the physical solid so that it can be easily distinguished from other representations.
- **Uniqueness:** A representation scheme should uniquely represent a solid object within a software's database.
- **Validity:** Representation schemes should yield representations of solids which do not exist or are valid.
- **Closure:** A Representation scheme which transforms (reflects, scales, rotates) a representation should yield other representations too.
- **Compactness:** Representation schemes should yield representations which save space and allow efficient algorithms to determine desirable physical characteristics.

2.2 Wireframe Modeling

For rectilinear objects whose edges are straight lines and whose faces are planar, the ordered pair of vertices (\mathbf{V}, \mathbf{E}) where $\mathbf{V} \in \mathbb{E}^3$ and $\mathbf{E} \in \mathbb{E}^3$ is the object's wireframe. In a practical sense, it is the skeleton of an object wherein joints are vertices and bones are edges. In Markowsky [?], a six step algorithm to generate an object's wireframe was developed wherein an object's wireframe was represented by a vertex table and an edge table. Although the work done by Markowsky [?] had drawbacks such as not checking the validity of edge data, wireframe modeling has always provided designers with a chance to experiment with the final result of their models through sketching and it is frequently used to preview complex models. However, the use of only edge information left wireframe models ambiguous on both rectilinear and topological polyhedra. Rectilinear polyhedra denote plane-faced solids while topological polyhedra denote solids obtained by an elastic deformation of rectilinear polyhedra. Figure 2.1 below shows the wireframe of a sphere in greyscale.

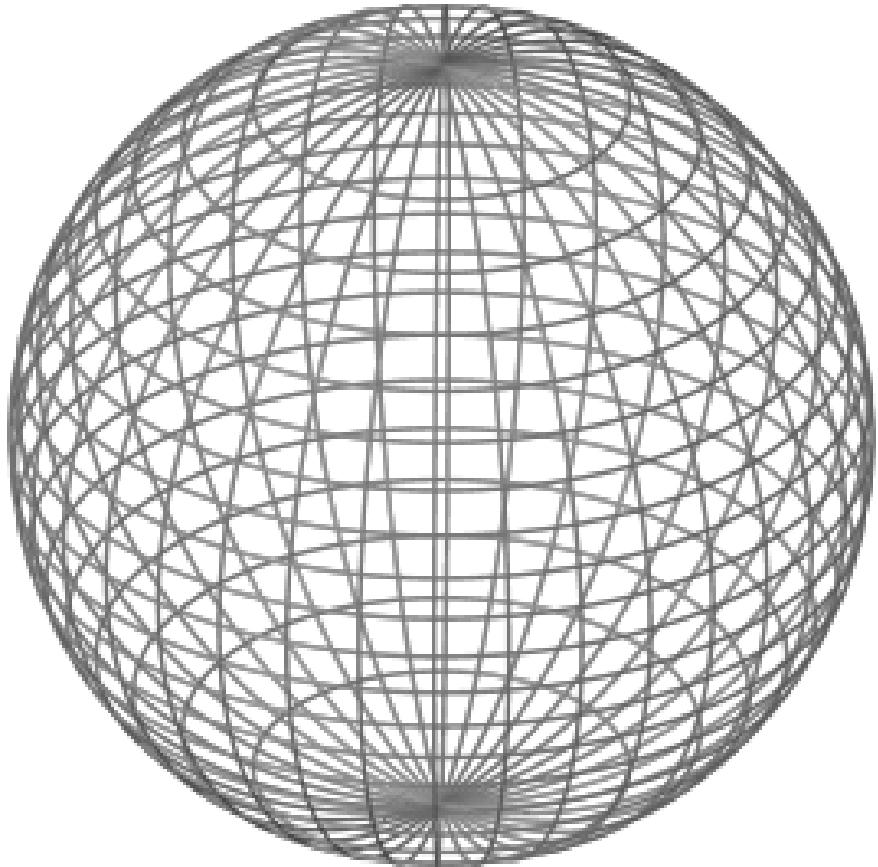


FIGURE 2.1: A wireframe of a sphere

2.3 Surface Modeling

After breakthroughs in wireframe modeling, research efforts in geometric modeling were directed towards extending the geometric coverage of CAD packages by incorporating complex freeform surfaces and curves. This section emphasizes algebraic surfaces and curves, the basis for Bezier surfaces and NURBS used within BRL-CAD.

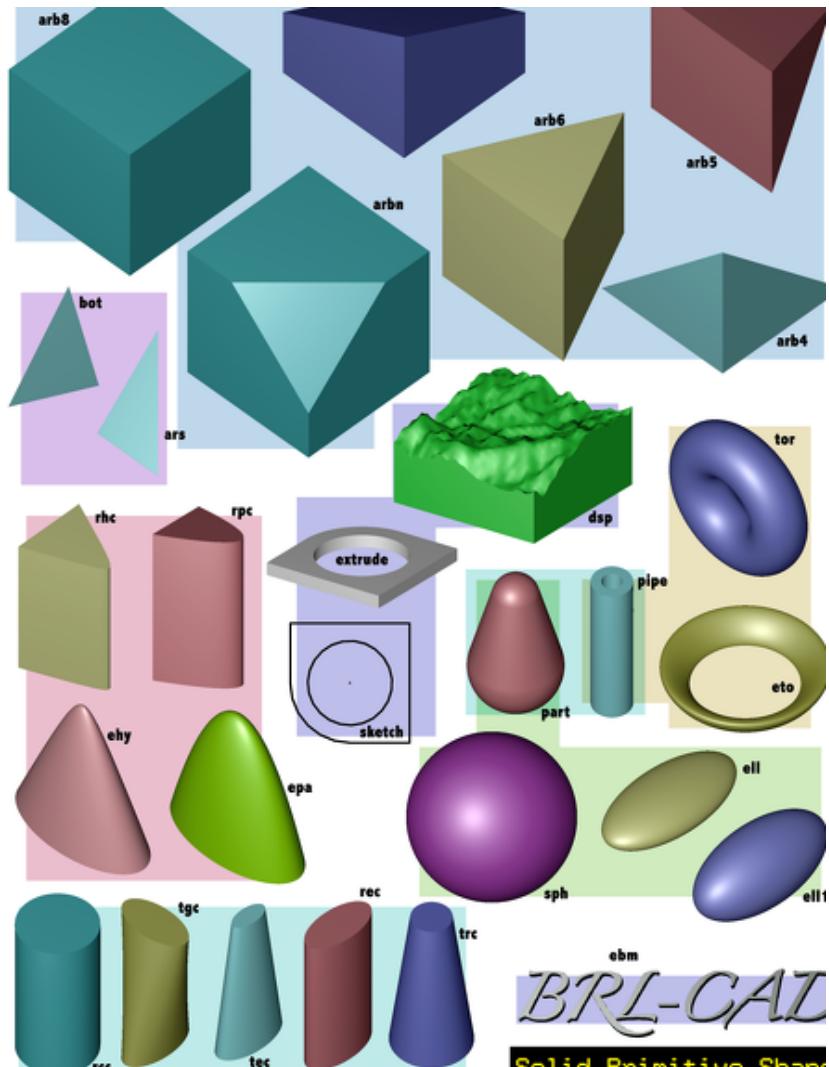


FIGURE 2.2: BRL-CAD Solid Primitive Shapes

Figure 2.2 above shows a collection of some primitives used within the BRL-CAD package before the heart-shaped primitive was developed – several of which are implicitly and/or parameterically represented by algebraic equations. On the BRL-CAD’s ideas page [?], there is a list of primitives which have not yet been implemented such as the Steiner surface, the ring cyclide surface, the quartoid, Wallis’ conical edge solid, etc.

2.3.1 Implicit Representation

An algebraic surface in \mathbb{E}^3 is expressed as the set of points satisfying an irreducible polynomial equation

$$g(x,y,z) = 0$$

in the unknowns x,y and z.

Definition 2.3.1. A polynomial $f(x,y,z)$ over a field \mathbf{F} is said to be irreducible over \mathbf{F} if the degree of $f(x,y,z)$ is positive and its only factors are c and $cf(x,y,z)$ where c is a nonzero constant in \mathbf{F} .

The requirement of irreducibility is so that a surface represented by an equation should not be decomposed into two separate surfaces, each of which can be described by an implicit equation.

2.3.2 Parametric Representation

Some algebraic surfaces possess a parametric representation which consists of a system of equations similar to the ones listed below;

$$\begin{aligned} x &= h_1(u, v) \\ y &= h_2(u, v) \\ z &= h_3(u, v) \end{aligned}$$

where h_i are rational functions and, u and v are restricted to particular closed intervals in \mathbb{R} .

As an example, the unit sphere given implicitly by $x^2 + y^2 + z^2 - 1 = 0$ can be parameterized by the following equation where $t \in [0, 1]$;

$$\begin{aligned} x &= (1 - s^2 - t^2)/(1 + s^2 + t^2) \\ y &= 2s / (1 + s^2 + t^2) \\ z &= 2t / (1 + s^2 + t^2) \end{aligned}$$

Also, some algebraic curves possess parametric forms. A parameterization of the unit circle is given by the system of equations below;

$$\begin{aligned} x &= (1 - t^2)/(1 + t^2) \\ y &= 2t / (1 + t^2) \end{aligned}$$

When the parametric representation is employed, it is easier to generate points on an algebraic surface or curve as compared to the implicit representation. Also, parametric equations are useful for interactive design since changes in their polynomial coefficients alter the surface's shape in an intuitive manner.

Lots of geometric operations could become faster if both aforementioned representations are made available within CAD packages. Thus, the problem of how to convert from one representation to the other is of great importance.

2.3.3 Implicitization

Implicitization is the process of converting a parametric representation into an implicit representation. Sederberg [?] demonstrated in his thesis that, in principle, it is always possible to convert a parametric surface or curve into implicit form using classical elimination theory developed in the early 20th century. In fact, Sylvester resultants require evaluating a determinant whose entries are coefficients of powers of the variable to be eliminated in several phases.

Although Sederberg's work stimulated lots of research interests in resultant-based methods, this sense of enjoyment within the research community was short-lived due to the following factors;

- Unfaithfulness: Polynomials derived using resultant-based methods give birth to phantom solutions.
- Ineffectiveness: Using floating point arithmetic, resultant-based methods become inaccurate.
- Inefficiency: Evaluating resultants entails huge amounts of computation and is expensive.

Another method of Implicitization is the Gröbner basis technique introduced by Buchberger [?] where it was learned that expressing a polynomial as a linear combination of a Gröbner basis solves the nonlinear system of equations just as an **Lower Upper** decomposition (**LU** - decomposition) does.

Definition 2.3.1. A polynomial basis is a set of polynomials which can be used to express any polynomial and can be viewed as a vector space over the field of coefficients \mathbb{F} .

Lazard [?] constructed a Gröbner basis with respect to a term ordering known as the elimination order. Hoffmann [?] improved upon a basis conversion algorithm developed in Faugere *et al* [?] by first constructing a Gröbner basis with respect to a term ordering different from the elimination order and finally built the final polynomial in which all variables have been eliminated term by term. This algorithm was known to be the fastest elimination technique for geometry applications that was implemented before the 1990s.

Although Gröbner basis methods are more efficient and effective than resultant-based ones, implicitization is fairly expensive and limited in practice.

2.3.4 Parameterization

Parameterization is the process of converting an implicit representation of an object into its parametric equivalent, if it exists. Parameterization is not always possible since not all implicit surfaces can be expressed as rational parameterizations. Noether's theorem states that a plane algebraic curve possesses a rational parameterization if and only if it has genus zero.

Definition 2.3.1. An algebraic surface is said to have genus zero if it cannot be cut along non-intersecting closed curve without disconnecting the resultant manifold.

Golub [?] used a numerically stable Jacobi rotation adapted from Abhyankar [?] to parameterize several conics and parametric surfaces. Table 2.1 below shows a list of parameterizations of some popular conics – circle, ellipse, hyperbola and parabola.

TABLE 2.1: Implicit and parametric equations of some BRL-CAD primitives

| | Implicit Form | Parametric Form |
|-----------|---|---|
| Circle | $x^2 + y^2 - r^2 = 0$ | $x = \frac{r(1-t^2)}{(1+t^2)}, y = \frac{2rt}{(1+t^2)}$ |
| Ellipse | $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ | $x = \frac{a(1-t^2)}{(1+t^2)}, y = \frac{2bt}{(1+t^2)}$ |
| Hyperbola | $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ | $x = \frac{a(1+t^2)}{(1-t^2)}, y = \frac{2bt}{(1-t^2)}$ |
| Parabola | $y^2 - 2px = 0$ | $x = \frac{t^2}{2p}, y = t$ |

2.4 Solid Modeling

A solid can be represented explicitly by its boundary, its volume, or implicitly by specifying operations on volumetric properties that construct it. BRL-CAD focuses on solid modeling by emphasizing physical accuracy and fully describing \mathbb{E}^3 .

There are three main well established paradigms for representing solids used within BRL-CAD based on their boundaries, volumes or constructing from primitives using regularized boolean operations, boolean set theoretic operations applied to r - sets.

This section reviews the literature around classical representation paradigms such as *Boundary Representations*, *Constructive Solid Geometry* and *Spatial Subdivision*.

2.4.1 Boundary Representations (B-REPS)

Just as Rome was not built in a day, the transition from surface modeling representation schemes to solid modeling representation schemes was a gradual process which started off with boundary representations. The boundary representations popularly known as B-REPS describe a solid as a set of surfaces which separate an object's interior from its exterior.

With B-REPS, the boundary of a solid consists of vertices, edges and faces. For each vertex, edge and face, the geometric section of the representation holds the shape and location of the object in space possibly using an equation while the topological section of the representation holds the adjacency relationships between the vertices, edges and faces as well as their orientation. There are nine (9) ordered adjacency relationships between the three (3) aforementioned topological entities. In order to build a complete representation, it is normal for one to allow the retrieval of any topological entity and any of the nine (9) ordered adjacency relationships. Kevin Weiler [?] showed that only three (3) ordered adjacency relationships are sufficient to obtain the others. From his work, a space/time tradeoff arose: Maintaining all 9 adjacency relationships required more space but little time for retrieval while maintaining only sufficient adjacency relationships required little space but more time for retrieval. A more indepth comparison of the space/time tradeoffs of different boundary representations was conducted by Woo [?].

For solids with a manifold surface, several boundary representations exist, the earliest of which was Baumgart's **winged edge** data structure [?]. Here, an edge node holds information about the edge orientation, face adjacencies and the clockwise/counterclockwise successor and predecessor edges about adjacent faces. This work was based on the assumption that faces are simply connected. The winged edge data structure has to be modified to accommodate solids with multiple connected faces which contain holes (genus). Braid *et al* [?] modified the winged edge data structure by introducing a fourth topological entity called a **loop** into a proposed data structure where each face consists of one or more edge loops surrounding each hole in that face. In the same vein, Yamaguchi and Tokieda [?] modified the winged edge data structure by introducing yet another topological entity called the **bridge edge** (or auxillary edge which is simply a double edge connecting two (2) edge cycles of a given face).

The reader is invited to explore other existing boundary representations such as the **half edge** data structure by Mantyla [?], Hanharan's **face edge** representation [?] and Ansaldi *et al*'s hierarchical **face adjacency** hypergraph [?]. In order to accomodate solids with internal cavities and permit single volumes to contain multiple faces, a new topological entity called the **shell** was introduced by Ansaldi [?]. As regards conversion, Shapiro and Vossler [?] developed an algorithm which converts a boundary representation to an equivalent CSG representation in two-dimensional space.

The *Euler - Poincare characterization* has been used as a necessary condition for the validity of a solid using its topological information viz edges (E), vertices (V), faces (F), loops (L), Holes (G) and shells (S). It is given by the equation (1) below.

$$V-E + F - (L-F) - 2(S-G) = 0 \quad (1)$$

Mantyla showed that any topologically valid polyhedron can be constructed from an initial polyhedron by a finite sequence of Euler operations viz **make** and **kill**.

Some representation schemes are hybrids such as the **B-rep index** which is a mixture of the boundary representation and spacial subdivision schemes. BRL-CAD is one of the few solid modeling systems which supports the B-REPS and NURBS representation format. The image shown in Figure 2.3 below is the classic computer graphics Utah teapot model prepared for 3D printing and rendered via BRL-CAD ray tracing.



FIGURE 2.3: The Utah Tea Pot model

2.4.2 Constructive Solid Geometry (CSG)

During the 18th century, George Boole's work on the algebra of logic gave birth to an algebra attributed to his name called Boolean algebra which possessed its own set theoretic concepts and operations like *union*, *intersection*, *difference* and *complement*. Although these operations seemed to be obvious candidates for combining solids, the conventional aforementioned boolean operations had been discovered to be inadequate to manipulate solids in \mathbb{E}^3 because the regularity and compactness of solids became compromised. In fact, the complement operator destroys the compactness of solids because the complement of an rset is unbounded. Thus, the need to develop new kinds of set theoretic operations which work well within the realm of solid modeling was of paramount importance. Requicha [?] developed a new set of operations peculiar to r-sets called *regularized union*, *regularized intersections* and *regularized difference*.

His work even went forward to show that the r-sets and the regularized boolean operations formed not just a boolean algebra but also a ring. Besides these regularized boolean operations on solids, there also existed other valid operations on r-sets which are linear transformations viz reflection, scaling, rotation, shearing, etc. The image in Figure 2.4 below illustrates how the regularized boolean operations result to new shapes.



FIGURE 2.4: CSG Boolean operations

One of the most widely used representation schemes of our time combines the volumes occupied by overlapping rsets using regularized boolean operations. This representation scheme is called Constructive Solid Geometry (CSG) and its signature is the CSG tree – a binary tree data structure whose leaves are primitives and whose internal nodes are either regularized boolean operations or linear transformations. By a primitive, we simply mean a regular set of points in \mathbb{E}^3 satisfying an irreducible polynomial equation $f(x,y,z) = 0$ in the unknowns x, y and z. Examples of primitives used in BRL-CAD include blocks, pyramids, circles, cones, cylinders, etc which can be seen in Figure 2.2 above and the recently developed heart-shaped primitive.

Although BRL-CAD has become a fully hybrid modeling system, it has always had its roots in CSG. The image below depicts a detailed M1A1 tank on a pedestal in a mirrored showcase room which is entirely constructed from implicit primitives and CSG boolean operations.

2.4.3 Spatial Subdivision

Despite the wide applicability and attention received by boundary representations, it gradually became exciting to look for new ways of representing objects explicitly by their volume – a collection of minute cells of a partition of \mathbb{E}^3 . Some widely used data structures under this representational paradigm are hierarchical in nature and recursively decompose space until they obtain atomic blocks for which no further decomposition is necessary. This representation scheme is used within the BRL-CAD package during the raytracing of Non - Uniform Rational B - Splines (NURBS) surfaces where the raytracer builds



FIGURE 2.5: An M1A1 tank on a pedestal in a mirrored showcase room

a Boundary Volume Hierarchy (BVH) tree which guides the ray to patches before the final iterative solution is obtained. Hence, the developments in this area of research were highlighted here. Due to the frequent use of hierarchical data structures in representing images in the domains of computer vision, image processing and computer graphics, the atomic blocks obtained from recursive decomposition are called *picture elements* or ***pixels*** for short.

The term ***quadtree*** has been used to describe hierarchical data structures which are based on the common property of recursive decomposition and can be differentiated by the type of data being represented, the principle guiding the decomposition process and the resolution of that decomposition (the number of times the decomposition is applied – whether fixed or variable). The most common quadtree is the region quadtree due to Klinger and Dyer [?] and was coined by Hunter [?]. It is based on the successive subdivision of the image array into four (4) equal-sized quadrants which may be recursively subdivided until pixels are obtained. In the tree representation, the root node corresponds to the entire array, each child of a node represents a quadrant of the region the parent node represents while a leaf node represents an atomic block (usually

a pixel). A leaf node may be black or white depending on whether its pixel is entirely inside or outside the represented region. It may be gray if it is an internal node of the tree.

As initiated by the aforementioned research carried out by Finkel and Bentley [?], hierarchical data structures can be used to represent images in three-dimensional space and even higher. The *octree* data structure is the three-dimensional analog of the quadtree. Here, an image is first represented in the form of a cubical volume and then recursively decomposed into eight (8) congruent disjoint cubes called *octants* until cubes are obtained or a uniform or predetermined decomposition is reached. The original structure of a quadtree encodes it as a tree structure using pointers. This additional memory overhead posed a problem and prompted two (2) approaches to curbing it. Although it is difficult to put a finger on who exactly developed the first approach to solving this problem, it is an efficient method which treats the image as a collection of leaf nodes each of which is encoded by a base 4 number termed a *locational code*.

The second approach to spacial subdivision due to Kawaguchi and Endo [?] is termed a ***DF-expression***. It represents the image in the form of a traversal of the nodes of its quadtree – a compact representation in which each node type is encoded with 2 bits. Although it is not easy to use when random access to nodes is required, it has been shown to be faster than the former for a static collection of nodes representing images in higher dimensional space using an efficient pointer-based implementation.

Rectangular data is often used to approximate objects in an image for which they serve as the minimum rectilinear enclosing object. Indeed, bounding rectangles are used in cartographic applications to approximate lakes, forests, hills, etc. The first representation of quadtrees holding rectangular data was due to Hinrichs and Nievergelt [?]. In this representation, each rectangle is seen as a cartesian product of two line intervals each analogous to an interval tree holding the interval's centroid and extent. The rectangles are reduced to points in three-dimensional space and then the problem is treated as if dealing with a collection of points. Another such representation is the region-based *MXCIF quadtree* in which each rectangle is associated with the quadtree node corresponding to the smallest block containing it entirely. The subdivision ceases whenever a node's block contains no rectangles or is smaller than a predetermined threshold size. Yet another such data structure arose called the *R-tree* developed by Guttman [?] in which each node in the R-tree is a d - dimensional rectangle enclosing its child nodes. Its leaf nodes are the rectangles in the database. Each rectangle may be contained in several nodes each of which had to be visited before ascertaining the presence or absence of a particular rectangle. This made the retrieval

of rectangles slow; a problem which was alleviated by the implementation of an R^+ -tree in which each rectangle is associated with several non-overlapping bounding rectangles. The retrieval time of the R-tree is sped up at the expense of an increase in the tree's height.

By looking at data structures which explicitly specify the boundaries of regions, the review of hierarchical data structures was concluded. These structures include Polygonal Maps (**PM**) – a collection of polygons which can either be vertex-based or edge-based. The vertex-based Polygon Maps quadtree or PM_1 quadtree in two-dimensions was due to Samet *et al* [?]. The PM_1 **quadtree** was based on a decomposition rule stipulating that positioning occurs as long as a block contained more than one line segment unless the line segments are all incident at the same vertex in the same block. The three-dimensional PM_1 octree data structure is quite useful as it is more forgiving on memory than the conventional octree when representing objects. The edge-based variant of the PM quadtree also dubbed the **PMR quadtree** uses a probabilistic splitting rule in which nodes contain a variable number of line segments.

Shown below in Figure 2.6 is a Sphere Flake drawn using BRL-CAD represented using an octree with five levels of recursion, specular reflections, multiple light sources, environment mapping, checkered texture synthesis, ambient occlusion, and soft shadows.



FIGURE 2.6: Model of a Sphere Flake

2.5 Non-manifold Geometry

So far, the modeling of manifold objects – objects with the property that each of their points have neighbourhoods which are homeomorphic to \mathbb{E}^3 has been explored. These manifolds can also be seen as r-sets which were defined by Requicha [?]. Lines and circles are one-dimensional manifolds while primitives such as tori, spheres, cones, cylinders, heart-shape, etc are examples of three-dimensional manifolds.

Now, some light is shed on geometric algorithms used to represent non-manifold objects (objects which have points with neighbourhoods which are not homeomorphic to \mathbb{E}^3). While lines and circles are one-dimensional manifolds, figure eights are one-dimensional non-manifolds. Examples of two-dimensional non-manifolds may include a cone touching another surface at a single point, faces meeting along a common edge, etc. Non-manifolds usually arise when topological structures such as vertices, edges, and faces hangoff the mapped boundary graph of an object. They also naturally result from regularized boolean operations in CSG even when input is restricted to manifolds only.

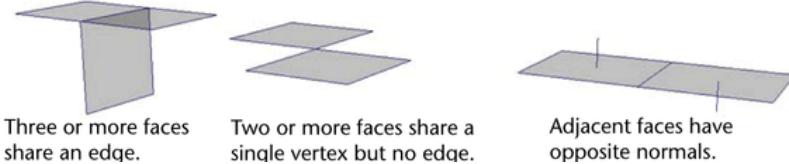


FIGURE 2.7: Examples of Non-Manifold Geometry

According to Weiler [?], non-manifold representations are representations which allow volume, manifold and non-manifold curves, surfaces and point elements in a single uniform environment. Thus, non-manifold modeling is seen as a hybrid representation paradigm which simultaneously encompasses wireframe modeling, surface modeling and solid modeling. It enables a smooth transition between several representation schemes and the automatic detection of solid enclosures without any need for restructuring or translation.

Also, Weiler [?] developed the ***radial-edge*** data structure which can be used to represent non-manifold geometry. NMG topological elements include vertices, edges, loops, faces, shells, regions and models. The topological information stored in a non-manifold model consists of adjacencies of topological elements. An adjacency relationship is the adjacency of a group of topological elements of one type around some other specific single element.

Thirty-six (36) adjacency relationships are possible in non manifold boundary representations. The radial-edge data structure employs *uppointers* and *downpointers* to depict the relative “*hierarchy*” which exists between topological elements. The reader can find more information about data structures used for non-manifold geometric modeling in Karasick [?] and Vanecek [?].

This chapter reviewed the work done by researchers in the field of geometric modeling which has been applied in the development of the BRL-CAD viz wireframe modeling, surface modeling, solid modeling and non-manifold modeling.

Chapter 3

ANALYSIS & DESIGN

This chapter states the aim of contributing to the BRL-CAD project and explains how a heart-shaped primitive was implemented in the project design section. First of all, it introduces the concept of free and open source software. Secondly, the BRL-CAD software package is overviewed. Later, it states the aim of contributing to the BRL-CAD project. And finally, it explains the design used to implement the heart-shaped primitive for BRL-CAD.

3.1 The Open Source Community

Depending on how it is termed, *Free/Libre/Open Source Software (FLOSS)*, *Free and Open Source Software (FOSS)* or simply *Open Source Software OSS*) is software for which users have access to both the source code and binary executables and is licensed under a license which permits its users to read, edit and distribute the software to anyone and for any reason. This distinguishes open source software from commercial software which is distributed by giving away its binary executable version only. Usually, OSS is distributed at no cost with limited restrictions on how it can be used. According to Eric S. Raymond [?], one of the most prominent evangelists of the open source movement, hackerdom can be likened to what anthropologists call a gift culture— a culture wherein members gain status and reputation by giving away their time, creativity and skills to reading, writing and debugging software, publishing useful information in blogs or documents like Frequently Asked Questions (FAQs) lists as well as handling unglamorous tasks like maintaining mailing lists, moderating news groups, etc without any monetary compensation. The word **hacker** was coined by a shared community of expert programmers and networking masters which traces its history back to the days of the earliest ARPAnet experiments and timesharing minicomputers who made the unix operating systems and the worldwide web work.

Today, the open source community has become a selforganizing collaborative social network of hackers driven by a passion to solve problems using free software with thousands of projects hosted on Sourceforge [?] and Github [?]. It has singularly developed some software packages and tools which are the best in the world such as the firefox web browser, Apache webserver, Linux operating systems like BSD, Ubuntu, Debian, etc, the MySQL database management system, the VLC media player, programming languages and tools like gcc, C, Perl, Python, Java, etc and much more. Some Examples of CAD packages within the open source community include BRL-CAD, Blender, FreeCAD, OpenSCAD and LibreCAD, etc.

3.2 Analysis Of The Work

BRL-CAD (pronounced Be-Are-El-CAD) was originally conceived and written by the late Mike Muss, a programmer and networking expert who also wrote the popular PING network program. In 1979, the United States Army's Ballistic Research Laboratory (BRL) (the agency responsible for creating ENIAC, the world's first general-purpose electronic computer in the 1940s) identified a need for tools that could assist with the computer simulations and analysis of combat vehicle systems and environments. When no existing CAD package was found to be adequate for this specialized purpose, Mike and fellow software developers began developing and assembling a unique suite of utilities capable of interactively displaying, editing, and interrogating geometric models. Those early efforts subsequently became the foundation on which BRL-CAD was built.

The initial architecture and design of BRL-CAD began in 1979 and its development as a unified software package kicked off in 1983 with its first public release the following year. As a software package with a mature code base which has been actively developed for decades, BRL-CAD pays close attention to design and maintainability. Like other FLOSS packages, BRL-CAD's source code and most of its project data are stored in a subversion version control system for change tracking, collaborative development and is redistributed as free and open source software under the Open Source Initiative license terms. The design of its system architecture is based on a UNIX methodology of command of the command line services, providing many tools that work in harmony to complete a specific task. These tools include geometry and image converters, signal and image processing tools, various ray tracing applications, geometry manipulators, and much more. They are used to test that the geometric properties of the heart-shaped primitive works in Chapter Four.

The basic layout of its code places public API headers in the toplevel *include/* directory and source code for both applications and libraries in the *src/* directory. The following is a partial listing of how BRL-CAD's source code is organised in a typical checkout or source distribution.

Applications and Resources

- *db/* for Example Geometry.
- *doc/* for project Documentation.
- *include/* for Public API headers.
- *regress/* for Regression test scripts
- *src/* for Application and library source.
- *src/conv* for Geometry converters.
- *src/fb* for Displaying data in windows.
- *src/mged* for the Multidevice geometry editor, the main GUI application.
- *src/rt* for Ray tracing applications.
- *src/util* for Image processing utilities.

Libraries

- *src/libbn* for Numerics library.
- *src/libbu* for Utility library.
- *src/libgcv* for Geometry conversion library.
- *src/libged* for Geometry Editing library.
- *src/icv* for Image conversion library.
- *src/libpkg* for Network Package library.
- *src/librt* for Raytracing library.
- *src/libwbd* for Geometry creation library.

The majority of BRL-CAD's source code is written in ANSI/POSIX C with the intent of strictly conforming with the C standard. The core libraries are all C API though several such as the Utility and Raytracing libraries use C++ for implementation details. Major components of the system are written in C, C++, Tcl/Tk, Bash and PHP with source code files using extensions such as *.c, *.h, *.cpp, *.tcl, *.tk, *.sh and *.php. BRL-CAD uses the CMake build system for compilation and an in - built infrastructure in *regress/* for testing.

BRL-CAD has a longlasting heritage of maintaining verifiable, validated and repeatable results in critical portions of the software package, particularly within the ray tracing library. It has an in built testing infrastructure which compares all program output against benchmark results during each build. The ray tracing library is a multi-representational library which lies at the heart of BRL-CAD and uses a suite of other libraries for other basic application functionality. Considerable attention is put into verification and validation throughout the package which includes regression tests that compare runtime behaviour against known results and reports any adverse variances from standard results as failures.

Despite this sophisticated infrastructure, performant design and longlasting heritage, many still coin BRL-CAD's aspiration of one day being the most widely used open source CAD package as rather lofty for the following reasons;

- With one of the fastest raytracers in existence (on several types of geometry) which is supported by an effective Laguerre-based root solver and used within academia for scientific instruction, computer graphics education and research, the stability of BRL-CAD's root solver on higher order polynomials such as quintics (of power 5) and sextics (of power 6) is still uncertain.
- As an open source CAD software which is deeply rooted in the Constructive Solid Geometry, BRL-CAD's set of primitives is still limited to traditional ones such as cones, cylinders, spheres, tori, etc.
- BRL-CAD is widely used within agencies within the United States Government for the modeling of military artillery and simulating combat vehicle systems and environments. This limits its user base to the military sector and repels users who would have used it for more entertainment purposes.

In a bid to contribute to BRL-CAD, a heart-shaped primitive was developed. The following section documents the development of various callback functions which compute useful geometric properties of the heart-shaped primitive such

as formatted description, database importation and exportation, computation of the bounding box and plotting the wireframe and ray tracing.

3.3 Design

After having stated the overall goal of contributing to the BRL-CAD project, the development of the heart-shape is explained. Currently, BRL-CAD aspires to become the most widely used open source CAD software package in existence. Presently, it is mostly used by the United States of America's Government agencies which fund its development as well as academic institutes which use it for computer graphics educations and scientific research. Aljazeera news Channel's recent revelation that less than 1% of the world's population works in the military sector is indicative of the fact that BRL-CAD's usage must go beyond the military sector to break the status quo.

In a bid to increase BRL-CAD's user base from the entertainment industry, a heart-shaped primitive, a symbol of love, was developed to entertain families at weddings, marriage anniversaries, valentine's day celebrations, etc. It can also be used by fashion designers to create magnificent embroidery like on clothing for real-life and cartoon movies.

While adhering to its coding style, the heart-shaped primitive was incorporated into BRL-CAD's ray tracing library using the following steps;

- Room for the heart-shaped primitive was created in BRL-CAD by hooking it unto the raytracing library.
- Wrote callback functions for the heart and tested them using BRL-CAD's inbuilt testing infrastructure in the *regress/* directory.
- Built support for typing in parameters for the heart-shaped primitive in the display interfaces viz *mged* or *archer*.

In order to make room for the heart-shape in BRL-CAD, the heart-shaped primitive was tagged, a data structure designed and a skeletal *hrt.c* file stubbed into the source code repository in *src/librt/primitives/hrt/hrt.c*.

3.3.1 Tagging the heart-shaped primitive

Given that each of the primitives in Figure 2.2 above is uniquely stored in BRL-CAD's database, it was necessary to tag the incoming heart-shaped primitive

with a unique magic number, **0x6872743f**, which is the hexadecimal equivalent of “?hrt?” and increment the maximum number of primitives in *src/libbu/-magic.c*, *include/magic.h* and *include/raytrace.h*.

3.3.2 Data Structure of Heart-shaped primitive

The heart-shape has two lobes which are symmetric about the zaxis and meet at each of its two cusps as the picture in Figure 3.1 shows.

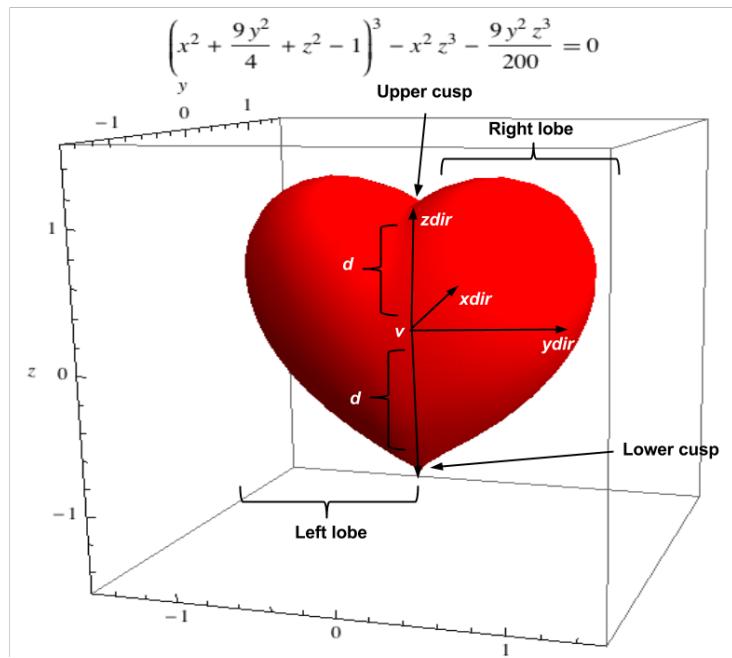


FIGURE 3.1: Diagram of the heart-shaped primitive

The heart-shape was stored in the *include/rtgeom.h* file as an Abstract Data Type (C-like structure) called *rt_hrt_internal* with the following fields representing its parameters.

- A magic number *hrt_magic*
- A center point *v*
- A vector in the direction of the X-axis *xdir*
- A vector in the directions of the Y-axis *ydir*
- A vector in the direction of the Z-axis *zdir*
- Distance from center point to either cusps *d*

These 3 vectors in \mathbb{E}^3 viz $xdir$, $ydir$ and $zdir$ are also called radial vectors because they radiate in the X, Y and Z directions and the aforementioned structure is also known in BRL-CAD's parlance as the heart-shape's internal format.

3.3.3 A bare bones heart-shape

The signatures of the functions (prototypes without return types) which are called to compute the geometric properties for the heart-shaped primitive were casted and enlisted in a function table in *src/librt/primitives/table.c*. The *src/librt/primitives/hrt/* directory was added to the source code repository and the *hrt.c* file to it committed to it. The *hrt.c* file consisted of introductory comments, include directives, structures for raytracing and storage of the heart-shaped primitive in the database as well as stubs for all callback functions which all print a ***rt_hrt_???:Not implemented yet!*** message when called. For surface representation and raytracing reasons, the heart-shape is stored in a more elaborate C-like structure called *hrt_specific* with the following parameters;

- A position vector for the heart-shape's center ***hrt_V***
- A unit vector in the X axis direction ***hrt_X***
- A unit vector in the Y axis direction ***hrt_Y***
- A unit vector in the Z axis direction ***hrt_Z***
- A vector in the Z axis direction ***hrt_d***
- A matrix for scaling and rotating ***hrt_SoR***
- A matrix for scaling and transposing a rotated heart ***hrt_invRSSR***
- A matrix for transposing a rotated heart ***hrt_invR***
- A vector for the inverse of the squared magnitudes of the three (3) aforementioned unit vectors ***hrt_invsq***

The location of the *hrt.c* file in the raytracing library was added to *src/librt/C-MakeLists.txt* for compilation purposes and test driven development.

After hooking the heart-shaped primitive into BRL-CAD's source code repository, callback functions which compute geometrically useful properties for the heart were written such as;

1. Formatted description
2. Database importation and exportation
3. Computing the bounding box of the heart-shaped primitive
4. Plotting the wireframe of the heart-shaped primitive
5. Surface representation and raytracing of the heart-shaped primitive

Now, we explain how the written functions actually compute each of these properties. Each property was tested using BRL-CAD’s commands [?] after having written the requisite function(s) that compute it. It is worth highlighting here that the “*rt_hrt_*” prefix is attached to the heart-shaped primitive’s structure and its functions’ signatures because “*rt*” and “*hrt*” denote the raytracing library and the heart-shape respectively.

3.3.4 Formatted description of the heart-shaped primitive

It often arises that engineers and scientists working on geometric models ask to know the exact values of the objects in question. In order to know a solid’s type and the values of its key parameters, the *rt_hrt_describe()* function was written which simply prints the heart-shape’s parameters in human readable format. The radial vectors in the X, Y and Z directions are printed alongside their magnitudes and the distance from the heart-shape’s center to either of its cusps is also printed.

3.3.5 Database importation and exportation of the heart-shaped primitive

BRL-CAD provides CSG modeling features and its models are usually stored in it’s Geometric Editing database called GED. For the heart-shaped primitive to be used in CSG, functions that import and export data in between the database format and the internal format were written.

The *rt_hrt_import5()* function imports a heart from the database format to the internal format. Firstly, a check that the primitive being imported is the heart-shaped primitive is done by verifying that the primitive’s magic number is the same as the heart-shape’s. Then, this function assigns integers held in an array buffer to the different parameters of the *rt_hrt_internal* structure.

We wrote the `rt_hrt_export5()` function to export the heart-shape's internal format into the database format. The database format for the heart-shape consists rather of only the heart's center and the 3 radial vectors in the X, Y and Z directions. It is not really useful storing value for d in the database as it is equivalent to some components of the radial vectors. After checking that the primitive in question is the heart-shape, the values for v , $xdir$, $ydir$ and $zdir$ are stored in an array buffer holding the integer values for the heart's parameters

3.3.6 Computing the bounding box of the heart-shaped primitive

The bounding box of a geometric model refers to the box with the smallest volume within which the model lies – more like the least upper bound of the set of all enclosing volumes. The spherical equivalent of the bounding box is the bounding sphere which is the smallest sphere within which a model lies. It is a useful property to compute for a model as it is used to appropriately orient an object (within its model coordinates) while positioning the camera (within its world coordinates) and casting lights towards it.

In order to compute the bounding box of the heart-shape, the `rt_hrt_bbox()` function was implemented which had two three-dimensional points as parameters. These points are the minimal and maximal points located at the closest lower left hand corner and the furthest upper right hand corner of the bounding box. Obtaining the coordinates of these two points suffices to compute the bounding as they are located at opposite ends of the box, equidistant from the heart-shape's center and differ from the other 6 edges by a single component.

The X component of the minimal point (closest lower left hand corner point) is obtained by subtracting a factor of $2/3$ of the radial vector $xdir$ from the unit vector of the heart-shape's center v . The X component of the maximal point (furthest upper right hand corner point) is obtained by adding a factor of $2/3$ of the radial vector $xdir$ to the unit vector of the heart-shape's center v .

The Y component of the minimal point (closest lower left hand corner point) is obtained by subtracting the radial vector $ydir$ from the heart-shape's center v . The Y component of the maximal point (furthest upper right hand corner point) is obtained by adding the radial vector $ydir$ to the heart-shape's center v .

The Z component of the minimal point (closest lower left hand corner point) is obtained by subtracting the radial vector $zdir$ from the heart-shape's center v . The Z component of the maximal point (furthest upper right hand

corner point) is obtained by adding a factor of 1.25 of the radial vector $zdir$ to the vector to the heart-shape's center v . This factor of 1.25 represents the distance from the heart's center to either of its cusps (which is 1.0) and the additional 0.25 closely approximates the displacement from the upper cusp to the highest point on either of the heart-shape's lobes.

Hence, with the coordinates of the maximal and minimal points obtained, the bounding box of the heart-shape was obtained.

3.3.7 Plotting the wireframe of the heart-shaped primitive

As earlier discussed, the wireframe of an object enables the sketching and preview of objects before adding colour and texture to them. In order to build the wireframe of the heart-shaped primitive into BRL-CAD's functionality, *rt_hrt_plot()* and *rt_hrt_24pts()* functions were written. The heart-shaped primitive's wireframe is made up of several ellipses aligned along the Z axis each of which consists of 24 edges. With progress in the positive Z - axis direction, 8 ellipses (with decreasing radii) were used to frame the upper portions of the left and right lobes. The 24 required points for each ellipse are computed by the *rt_hrt_24pts()* function which computes 24 points which are 15° apart. Finally, the different ellipses are connected together to enrich the wireframe with more isocontours. Along the XY and XZ planes (when $Z = 0$ and $Y = 0$ respectively), the heart-shape's wireframe appears like an ellipse. Along the YZ plane (when $X = 0$), the heart-shape primitive's wireframe is indeed heart-shaped. The algorithm below was used to develop the wireframe of the heart-shape.

Algorithm 1. Plotting the wireframe of the heart-shape

```

1. for points in the +Z directions starting at the upper cusp
    locate centres of ellipses which constitute the upper half of the left lobe
    determine radial vectors in X, Y and Z directions as need be
    determine the 24 points for each ellipse
    draw each ellipse      // Connect the 24 points
    make connections between ellipses
    end at highest point of left lobe      // the maximum turning point
2. for points in the +Z directions starting at the upper cusp
    locate centres of ellipses which constitute the upper half of the right lobe
    determine radial vectors in X, Y and Z directions as need be
    determine the 24 points for each ellipse
    draw each ellipse      // Connect the 24 points
    make connections between ellipses
    end at highest point of right lobe      // the maximum turning point
3. for chosen levels in the +Z direction starting at the lower cusp

```

```

locate centres of ellipses
determine radial vectors in X, Y and Z directions as need be
determine the 24 points for each ellipse
draw each ellipse      // Connect the 24 points
make connections between ellipses
end at upper cusp level      //the maximum turning point

```

3.3.8 Surface representation and raytracing of the heart-shaped primitive

The heart-shaped primitive was endowed with a surface representation by raytracing it. Raytracing at its very core consists of solving for the intersection point of a line and a surface. Many interesting surfaces have been written as polynomial functions of position and the heart-shape is not left out. The peculiarity of our work is that it showed that raytracing using the Laguerre-based rootfinder works for sextic equations (those with a degree of 6). To do this, *rt_hrt_prep()*, *rt_hrt_norm()*, *rt_hrt_shot()* and *rt_hrt_print()* functions were written.

The *rt_hrt_prep()* function prepares a heart-shape for raytracing by verifying that the object in question is a valid heart-shape. If the primitive is a valid heart, then a specific heart structure called *hrt_specific* is created and stored in memory for use by the *rt_hrt_shot()* function. To check the validity of the heart, we first check that at least one of the three radial vectors has a non-zero magnitude. Secondly, we check that the value for parameter *d* is nonzero. After, we check that the 3 radial vectors are perpendicular to each other. If they are indeed perpendicular to each other, then the heart-shaped primitive is a valid heart-shape. Finally, values for *d*, *v*, *hrt_invRSSR*, *hrt_invsq* and the heart-shape's bounding sphere centered at *v* are computed. After computing these requisite parameters for the heart in *rt_hrt_prep()*, the *rt_hrt_print()* function prints the position vector of the heart-shape's center and the two (2) matrices for transposing, rotating and scaling just to make sure that are correct.

Millions of light rays are shot at the surface of the heart and the pixels where intersections occur are rendered. The *rt_hrt_shot()* function came in handy at this point of our development of the heart-shape. Each point in \mathbb{E}^3 is represented by a treble (x,y,z) and the heart-shaped primitive's surface is implicitly represented as a sextic equation as shown in (2) below.

$$f(x, y, z) = \left(x^2 + \frac{9}{4}y^2 + z^2 - 1 \right)^3 - z^3 \left(x^2 + \frac{9}{80}y^2 \right) \quad (2)$$

Each light ray is modeled as a line in \mathbb{E}^3 written as a linear equation written in the form $W = Dt + P$ where $W = (x, y, z)$, $D = (a, b, c)$, $P = (x_0, y_0, z_0)$ and a, b, c, x_0, y_0

and z_0 are constants in \mathbb{R} . The system of equations for the components of W is given below

$$\begin{aligned} x &= at+x_0 \\ y &= bt+y_0 \\ z &= ct+z_0 \end{aligned} \tag{3}$$

Equation (3) above indicates that each point on the line is represented by a unique value for t . To find the points of intersection of the light ray and the surface of the heart-shaped primitive, x , y and z from (3) above were substituted in equation (2). This yielded a new sextic equation (4) in t shown below

$$S(t) = C_6t^6 + C_5t^5 + C_4t^4 + C_3t^3 + C_2t^2 + C_1t + C_0 = 0 \tag{4}$$

where C_i , $i = 0 \div 6$ are the coefficients of equation (4) which can be viewed in the Appendix.

The real zeroes of (4) indicate an intersection in \mathbb{E}^3 . Even if complex roots are returned by the rootfinder, the roots with imaginary components sufficiently close to zero are considered to be real zeroes and are also used as values for t in intersection points. If there are no real solutions, then the ray does not intersect the heart's surface. The coordinates of the intersection points were determined by substituting the values for real t_i in (3) above.

The Algorithm below shows the intersection of an arbitrary light ray and the heart-shape's surface.

Algorithm 2. Intersect a ray with the heart-shape's surface

```

Normalize distance from the heart-shape
Generate sextic equation
Pass equation through Laguerre-based root finder
if (root finder returns other than 6 roots)
    throw exceptions // root finder did not find roots
else          // Real roots indicate an intersection in real space
    select real roots among the 6 //complex roots with very small imaginary parts
for each real root returned by root finder
    Determine the entry and exit points of the light ray

```

The normal to the surface N at the point of intersection is in the direction of the gradient of $f(x,y,z)$. $N := (f_x, f_y, f_z)$ where f_x , f_y and f_z are the partials derivative

functions of $f(x,y,z)$ with respect to x, y and z respectively. The function `rt.hrt_norm()` computes the surface normal N to the surface of the heart-shaped primitive.

By substituting $w = x^2 + 9/4y^2 + z^2 - 1$, (2) becomes

$$f(x, y, z) = w^3 - z^3(x^2 + 9/80y^2)$$

and the surface normal is given by the system of equations in below.

$$\begin{aligned} f_x(x, y, z) &= 6x(w^2 - z^3/3) \\ f_y(x, y, z) &= 6y(12/27w^2 - 88/3z^3) \\ f_z(x, y, z) &= 6z(w^2 - z/2(x^2 + 9/80y^2)) \end{aligned}$$

Computing the exact algebraic expression for the coefficients of (3) above is cumbersome and error-prone. Instead, several polynomial variables were used to hold the coefficients of (3) and gradually build equation (3). Starting with polynomial variables for x^2 , $9/4y^2$ and $z^2 - 1$, w was obtained. With polynomial variables holding $y^2 + 9/4x^3$ and z^3 , $z^3(x^3 + 9/80x^3)$ was obtained. Hence, the coefficients of (3) were determined.

Once the coefficients of (3) are determined, it is parsed through BRL-CAD's root finder. For polynomials with degrees less than five, there exists exact solutions in radicals. Indeed, linear and quadratic equations can be trivially solved by the substitution method and the quadratic formula respectively. A method for solving cubics was discovered by Cardan and a method for solving quartics was discovered by Ferrari. Although some methods exist to determine the exact solutions of higher order polynomials (with degrees greater than 4) in radicals from Galois theory, the equation (3) does not satisfy these conditions. Indeed, the Galois group of equation (3) is contained neither in the group of order 48 which stabilizes a partition of the set of the roots into three subsets of two roots nor in the group of order 72 which stabilizes a partition of the set of the roots into two subsets of three roots. As a result, only numerical methods can be employed to solve it.

Fortunately, BRL-CAD has a rootsolver which is based on a numerical method called the Laguerre method. Named after the French mathematician Edmund Laguerre, the Laguerre method is a rootfinding algorithm used to find the roots of polynomials. Extensive empirical studies show that this method has come close to being a surefire method because it almost always converges to some root of the polynomial no matter what initial guess is chosen. This is in contrast to the other methods like the Newton-Raphson method which may fail to converge for poorly chosen initial guesses. Algorithm 3 below shows how the Laguerre method works;

Algorithm 3. The Laguerre Root-finding Method

```

Choose an initial guess  $x_0$ 
while ( $k \leq N + 1$  or  $a \approx 0$ )
for k = 0 ÷ N
    Compute G :=  $f'(x_k)/f(x_k)$ 
    Compute H :=  $G^2 - f''(x_k)/f(x_k)$ 
    Compute a :=  $n/G + \sqrt{(n-1)(nH-G^2)f(x)}$  ,n = degree of f(x)
    end for loop
    Set  $x_{k+1} := x_{k-a}$ 
end while loop
If a root is found, then the corresponding linear factor is removed from the polynomial.
This deflation step reduces the degree of the polynomial by one and approximations
for all roots of the polynomial are obtained likewise.

```

BRL-CAD's rootfinder *rt_poly_findroot()* function is found in *src/librt/roots.c*. To test that the root solver solves sextic equations, an arbitrarily chosen equation (5) was passed.

$$x^6 - 8x^5 + 32x^4 - 78x^3 + 121x^2 - 110x + 50 = 0 \quad (5)$$

Equation (5) is parsed through the rootfinder to obtain its roots $x_1 = 1-i$, $x_2 = 1+i$, $x_3 = 2-i$, $x_4 = 2+i$, $x_5 = 1-2i$, $x_6 = 1+2i$. This test is implemented in *src/util/roots_example.c*. Besides this test, the call to the rootfinder by passing equation (2) worked and rendered the heart-shape as our discussion of results in the next chapter will show.

3.3.9 Type in support for the heart-shaped primitive

Finally, appropriate support for typing the parameters of the heart-shaped primitive using the keyboard into the mged or archer interfaces was implemented. To do this, the *p_hrt[]* array was developed and *hrt_in()* routine in *src/libged/typein.c* written to prompt users to input parameters for the heart-shape. Then ,we added the *mk_hrt()* function to the include/wdb.h and *src/libwdb/wdb.c* files to assign values from **mged** or **archer** interfaces to the heart-shape's parameters.

Chapter 4

RESULTS & DISCUSSION

In this chapter, we present and discuss the results obtained from our work. We explain how we tested the different geometric properties of the heart-shaped primitive. Without loss of generality, we used a heart-shaped object centered at the origin (0,0,0), possesses 3 radial vectors (5,0,0), (0,5,0) and (0,0,5) as well as a distance to cusps of 4. Let's suppose this object called *amour* and is stored in the *heart_example.g* database. Given the situation of our work in the field of CAD, we use images to demonstrate that each of the heart-shape's geometric properties work.

4.1 Type-in support for the heart-shaped primitive

During the modeling process, a designer using BRL-CAD creates objects by typing its parameters into either the mged or archer graphical user interfaces. Having built this capacity into the heart-shaped primitive, we used BRL-CAD's *in* [?] command to test its mettle.

The *in* command enables the user to type in the arguments needed to create a shape alongside its name and type. It supports various options and may be invoked with no arguments. The *-s* option invokes the primitive edit mode on a new object immediately it is created. In order to test the type in support of the heart-shaped primitive, we create the *amour* object by typing its name, type and other parameters into the archer interface. After opening the *heart_example.g* database destined to hold the *amour* object, we type “*in amour hrt 0 0 0 5 0 0 0 5 0 0 0 5 4*” into archer’s command line interface. The object’s name, *amour*, is printed on the command line indicating that the object has indeed been created. Figure 4.1 below shows that the *amour* heart-shape object can be created by typing in its parameters through the keyboard.

```
Archer> in
Enter name of solid: amour
Enter solid type: hrt
Enter X, Y, Z of the heart vertex: 0
Enter Y: 0
Enter Z: 0
Enter X, Y, Z of vector xdir: 5
Enter Y: 0
Enter Z: 0
Enter X, Y, Z of vector ydir: 0
Enter Y: 5
Enter Z: 0
Enter X, Y, Z of vector zdir: 0
Enter Y: 0
Enter Z: 5
Enter distance to cusps: 4
amour
Archer> █
```

FIGURE 4.1: Testing type in support for the heart-shaped primitive in archer

4.2 Formatted description of the heart-shaped primitive

After having created objects for modeling, it sometimes becomes necessary to display a description of these objects. For us to test that we can describe the heart-shaped primitive, we used BRL-CAD's *l* [?] command on the *amour* object. The *l* (*listing*) command displays a verbose description of a specific list of objects. If the shape of the object is a primitive, then detailed parameters of that shape are displayed. If the object is a combination of other primitives, then the boolean formula for the combination is listed while indicating any accumulated transformations. If a shader and colour has been assigned to the combination, then all details will be listed. The *-t* (terse) option displays a shorter list of primitive shape parameters.

To describe the *amour* object, we print *amour*'s parameters in both terse and verbose forms by running the “*l -t amour*” and “*l amour*” commands respectively in the archer command prompt. This is shown in Figure 4.2 below.

```
Archer> l -t amour
amour: Heart (HRT)
      V (0, 0, 0)
      Xdir (5, 0, 0) mag=5
      Ydir (0, 5, 0) mag=5
      Zdir (0, 0, 5) mag=5
      d=4

Archer> l amour
amour: Heart (HRT)
      V (0, 0, 0)
      Xdir (5, 0, 0) mag=5
      Ydir (0, 5, 0) mag=5
      Zdir (0, 0, 5) mag=5
      d=4
      Xdir direction cosines=(0, 90, 90)
      Xdir rotation angle=0, fallback angle=0
      Ydir direction cosines=(90, 0, 90)
      Ydir rotation angle=90, fallback angle=0
      Zdir direction cosines=(90, 90, 0)
      Zdir rotation angle=0, fallback angle=90
```

FIGURE 4.2: Testing the formatted description of the heart-shaped primitive

4.3 The bounding box of the heart-shaped primitive

As we earlier stated, the `rt_hrt_bbox()` function was implemented to calculate the bounding box of the heart-shaped primitive. In order to test that the bounding box of the heart-shaped primitive is computed, we use BRL-CAD's ***bb*** [?] command.

The ***bb*** (*bounding box*) command reports dimensional information about objects using bounding boxes. It does this by calculating an axis aligned bounding box for an object and printing the dimensions of that box to the command prompt of archer. The ***bb*** command support various options, most of which control the type of information reported.

- The ***-e*** (*extent*) option reports the extent of the bounding box by printing its minimal and maximal points.
- The ***-d*** (*default*) option reports the length, width and height of the box.
- The ***-v*** (*volume*) option prints the volume of the bounding box by default too.
- The ***-q*** (*quiet*) option prints the properties of an object in quiet mode by disabling the printing of the default header.

Once more we use the amour object and the ***bb*** command to test how effectively the bounding box of the heart-shaped primitive was implemented. To report the extent of the bounding box of the amour object, we print its minimal and maximal points by running the “***bb -qe amour***” command in either the mged or archer command prompts. Then, we ran the “***bb -qv amour***” command in archer so that the volume of the amour object is reported in cubic millimeters. After, we reported the length, width and height of amour’s bounding box by running the “***bb -qd amour***”. Finally, to report the volume and dimensions of the bounding box, we ran the “***bb amour***” command in archer’s command prompt.

The image below in Figure 4.3 shows the results obtained after running the aforementioned commands.

```
Archer> bb -qe amour
min {-3.333333 -5.000000 -5.000000} max {3.333333 5.000000 6.250000}

Archer> bb -qd amour
X Length: 6.7 mm
Y Length: 10.0 mm
Z Length: 11.2 mm

Archer> bb -qv amour
Bounding Box Volume: 750.0 mm^3

Archer> bb amour
Bounding Box Dimensions, Object(s) amour:
X Length: 6.7 mm
Y Length: 10.0 mm
Z Length: 11.2 mm
Bounding Box Volume: 750.0 mm^3

Archer> █
```

FIGURE 4.3: Testing the bounding box of the heart-shape

4.4 Plotting the wireframe of the heart-shaped primitive

The process of modeling sometimes warrants the preview of the skeleton of a specific set of objects. To test that the wireframe of the heart-shaped primitive is working, we use the ***draw*** [?] command in BRL-CAD. The ***draw*** command displays objects in either the mgd or archer interfaces. It is synonymous to BRL-CAD's ***e*** command. The draw command's ***-C (colour)*** option enables the user to specify a colour that overrides all other previous colour specifications.

To draw the wireframe of the amour object using white wires, we run the ***"draw -C 255/255/255 amour"*** command. The image in Figure 4.4 below shows the wireframe of amour with red isocontours.

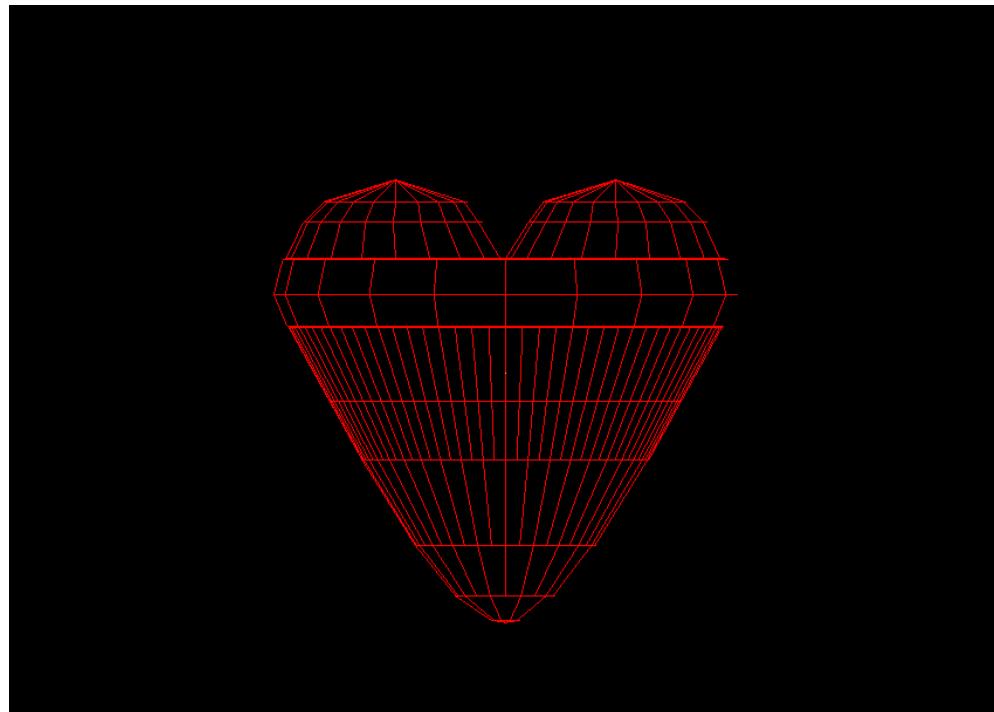


FIGURE 4.4: Testing the wireframe of the heart-shaped primitive

4.5 Ray tracing and surface representation of the heart-shaped primitive

Raytracing the heart-shaped primitive is the most important geometric property because the others lose their relevance without BRL-CAD's capacity to render the heart-shaped primitive. A major difficulty during the raytracing of the heart-shaped primitive was the misunderstanding that the sextic equation could be solved in radicals and only algebraic methods can be employed to solve it.

In order to test that the raytracing property of the heart-shaped primitive works, we use BRL-CAD's ***rt*** [?] command. The ***rt*** command raytraces a set of objects with the default option of ***-s50***. Like other BRL-CAD commands, it has various options.

- The ***-a*** (*azimuth*) option enables the designer to specify the azimuth at which the rendered image will be created.
- The ***-e*** (*elevation*) option enables the designer to specify the elevation at which the rendered image should be created.
- The ***-w*** (*width*) option enables the designer to specify the width of the rendered image.
- The ***-n*** (*height*) option enables the designer to specify the height of the rendered image.
- The ***-s*** (*size*) option enables the designer to specify the length of the rendered image (which is a square).
- The ***-C*** (*colour*) option enables the designer to specify the colour of the rendered image's background.
- The ***-o*** (*output*) option enables the designer to specify the name of the rendered image.

Using the ***rt*** command in BRL-CAD, we produced 360 images each at an azimuth angle of 1 degree from the other and an elevation angle of 35 degrees from our viewpoint. The width and height of these images were 640 and 480 pixels respectively. To do this, we edited the FlyAround Animation script [?] on BRL-CAD's website which is shown below;

```
#!/bin/sh
for i in `loop 000 359 1`; do
    rt -a $i -e 35 -w 640 -n 480 -o image$i.png heart_example.g amour
done
```

Figure 4.5 below shows some images of the heart-shaped primitive obtained from running the script above.

The first image shows the heart-shaped primitive upfront with its two lobes and 2 cusps just as its data structure was designed. The other two images show that the heart-shaped primitive is elliptical on the XY and XZ planes as we earlier highlighted when testing its wireframe.

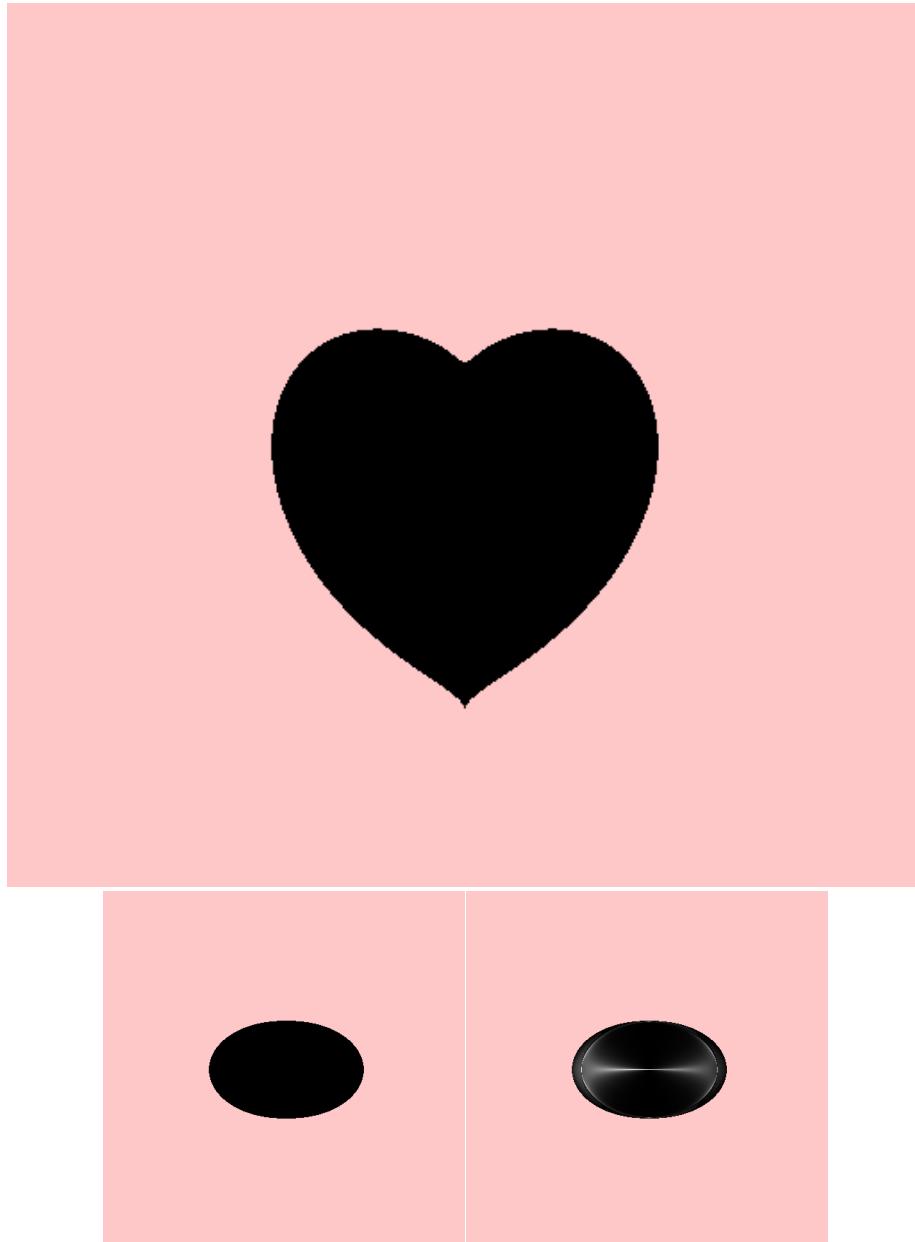


FIGURE 4.5: Images rendered after raytracing the heart-shaped primitive

After appropriately sequencing these images with padded zeroes, we used the ImageMagick ***convert*** command to composite them into a single animated image. Several animations were produced showing the heart-shaped primitive spinning from several viewpoints (above, below and besides) and these can be viewed on my youtube channel [?] [?]. These animations are proof that artists can indeed use the BRL-CAD software to produce cartoon animations, design cards, royal seals and banners, gifts, magnificent embroidery and presents for family and communal celebrations such as weddings, family reunions and Valentine's day for entertainment and fashion.

Chapter 5

CONCLUSIONS

In this chapter, we note the contributions of our work on the heart-shaped primitive to the field of CAD and highlight possible research orientations which can spring up from it. During our work, we first designed the heart-shaped primitive's structure, wrote necessary callback functions and tested them using BRL-CAD's inbuilt testing infrastructure.

5.1 Our Contribution

The contributions of this thesis are outlined below;

- We showed that BRL-CAD's Laguerre-based root solver is indeed a surefire iterative method for finding roots of polynomials and ascertained its stability on sextic equations.
- This work provides a guideline for the development of primitives within open source CAD software by highlighting the implementation of geometrically useful properties for the heart-shaped primitive within BRL-CAD.
- By producing animated videos from resulting rendered images, we showed that BRL-CAD is a software package which artists can use to produce cartoon animations, design cards, royal seals and banners, gifts and presents for family and communal celebrations such as weddings, valentine's day, etc, for entertainment purposes.

5.2 Further Research

In order to invite more artists to use BRL-CAD for the creation of entertainment content and enhance BRL-CAD's functionality so that it becomes a better open source CAD package, we propose the following research ideas;

- Although the heart-shaped primitive has been shown to have possess a wireframe and an implicit surface representation, it would be nice for this primitive to own an explicit surface representation based on its parametric equations.
- BRL-CAD is used within the scientific community for research and computer graphics education, it would be nice to investigate the stability of BRL-CAD's Laguerre-based root solver on septic, octic and nonic equations which are monic polynomials of order 7, 8 and 9 respectively.
- Since BRL-CAD boasts of more than 400 commands within its pocket, it would be worthwhile to invite more artists within the entertainment industry by adding more functionality to its animate command so that several images can be combined into animated videos.

APPENDIX

Coefficients of the terms in equation (4)

$$C_6 = 4320a^2b^2c^2 + 960a^2c^2(a^2 + c^2) + 320(a^6 + c^6) + 2160b^2(a^4 + c^4) + 4860b^4(a^2 + c^2) - 36b^3c^3 + 3645b^6$$

$$C_5 = 8640(ax_0b^2(a^2 + c^2) + by_0a^2c^2 + cz_0b^2(a^2 + c^2)) + 9720b^4(ax_0 + cz_0) + 1920(a^4 + c^4(ax_0 + cz_0)) + 4320(a^4 + c^4by_0) + 3840(a^2 + c^2)(ax_0 + cz_0) + 19440(c^2b^2 + ab^2)by_0 - 108bc(c^2by_0 + cz_0b^2) - 320a^2ba^3 + 21870y_0b^5$$

$$C_4 = 4320(a^2(b^2z_0^2 + c^2y_0^2) + b^2(x_0^2c^2 - a^2 - c^2)) - 960(a^4(1 - z_0^2) + c^4 + b^2(a^2y_0 - b^2x_0^2)) + 4860(b^4(x_0^2 + z_0^2 - 1)) + 4800(a^4x_0^2 + c^4z_0^2) + 2160y_0^2(a^4 + c^4) + 5760a^2c^2(x_0^2 + z_0^2) + 7680ax_0cz_0(a^2 + c^2) + 17280((ax_0by_0 + by_0cz_0)(a^2 + c^2) + ax_0cz_0b^2) + 12960b^2(a^4x_0^2 + c^4z_0^2) + 29160b^2y_0^2(a^2 + c^2) - 108bc(b^2z_0^2 + c^2y_0^2) - 1920a^2c^2 - 54675y_0^2b^4 - 324z_0c^2b^2y_0 - 640ax_0b^3$$

$$C_3 = 3840(ax_0 + cz_0)(a^2z_0^2 + c^2x_0^2 - a^2 + c^2) + 8640(ax_0(z_0^2b^2 + c^2y_0^2 + b^2x_0^2 + a^2y_0^2 - b^2) + by_0(a^2z_0^2 + c^2x_0^2 - a^2 - c^2) + cz_0(a^2y_0^2 + b^2x_0^2 + bz_0^2 + c^2y_0^2 - b^2)) + 6400(a^3x_0^3 + c^3z_0^3) + 19440by_0(b^2(x_0^2 + z_0^2 - 1) + y_0^2(a^2 + c^2)) - 36(b^3z_0^3 + c^3y_0^3) + 11520(ax_0cz_0(ax_0 + cz_0)) - 324by_0cz_0(bz_0 + c + y_0) + 25920by_0(z_0^2c^2 + x_0^2a^2) - 320x_0^2b^3 + 72900y_0^3b^3 + 34560ax_0by_0cz_0 + 58320ax_0b^2y_0^2 - 1920ax_0b^2y_0 + 58320cz_0b^2y_0 - 960a^2by_0^2$$

$$C_2 = 960(a^2(1 + z_0^4) + c^2(1 + x_0^4) - b^2x_0^2y_0) + 2160b^2(1 + x_0^4 + z_0^4) + 4320(y_0^2(a^2 + c^2 + a^2z_0^2 + c^2x_0^2) + b^2(x_0^2z_0^2 - x_0^2 - z_0^2)) + 4800(x_0^3a^2 + z_0^3c^2) + 4860y_0^4(a^2 + c^2) + 5760(z_0^2c^2(1 + x_0^2) + x_0^2a^2(z_0^2 - 1)) + 7680ax_0cz_0(x_0^2 + z_0^2 - 1) - 1920(a^2z_0^2 + c^2x_0^2 + ax_0y_0^2b) + 17280(by_0(ax_0^3 - ax_0 + cz_0^3 - cz_0 + az_0^2) + ax_0cz_0y_0^2) + 38880bz_0^3(ax_0 + cz_0) + 12960y_0^2(a^2x_0^2 - c^2z_0^2) + 29160b^2y_0^2(x_0^2 + z_0^2 - 1) - 180y_0z_0c^2y_0^2 + b^2z_0^2) - 320a^2y_0^3 + 54675y_0^3b^2$$

$$C_1 = 1920((ax_0 + cz_0)(1 + x_0^4 + z_0^4)) + 4320by_0(1 + x_0^4 + z_0^4) + 8640(-by_0z_0^2 + ax_0z_0^2y_0^2 + cz_0x_0^2y_0^2 + by_0^2x_0z_0^2 - ax_0y_0^2 + ay_0^2 + ax_0^3y_0^2 - cz_0y_0^2 + cz_0^3y_0^2 - by_0x_0^2) - 3840(cz_0^3 + ax_0^3 + ax_0z_0^2 - ax_0^2z_0^2 + cz_0x_0^2 - cz_0^3x_0^2) - 19440(by_0^3(1 - x_0^2 - z_0^2)) + 9720(y_0^4(ax_0 + cz_0)) + 21870(by_0^4) - 640(ax_0y_0^3 - by_0^2x_0^2) - 108(cz_0^2y_0^2 + by_0^2y_0^3)$$

$$C_0 = 320(-x_0^6 + z_0^6 - x_0^2y_0^3) + 960(x_0^2 + z_0^4x_0^4 + x_0^2z_0^4 - x_0^4 - z_0^4) + 4320(x_0^2y_0^2z_0^2 - x_0^2y_0^2 - z_0^2y_0^2) - 1920x_0^2z_0^2 - 36x_0^3y_0^3 + 2160(y_0^2(1 + x_0^4 + z_0^4)) + 4860(y_0^4(x_0^2 + z_0^2 + 1)) + 3645y_0^5$$