Ecuaciones de Friedmann.

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A partir de las ecuaciones de campo de Einstein y considerando que el Universo es homegeneo e isotrópico se utilizan la métrica de Friedmann Robertson Walker Lemetre FRWL para resolver las ecuaciones de campo y obtener una descripción de la evolución de materia en el universo temprano.

Utilizamos que el elemento de línea tiene la siguiente forma

$$ds^2 = dt^2 - a(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sen^2\theta d\phi^2\right) \tag{1}$$

con $c = 1$, donde $a(t)$ es el factor de escala, k una constante ($k = -1, 0, 1$), con r tal que $0 < r < 1$, r, θ, ϕ coordenadas comoviles espaciales.

La métrica tiene la siguiente forma

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2r^2 & 0 \\ 0 & 0 & 0 & -a^2r^2sin^2\theta \end{pmatrix}$$

donde $g_{00} = 1$, $g_{rr} = \frac{-a^2}{1-kr^2}$, $g_{\theta\theta} = -a^2r^2$, $g_{\phi\phi} = -a^2r^2sin^2\theta$

Dado que las ecuaciones de campo de Einstein son:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{2}$$

con $R_{\mu\nu} = g^{\lambda k}R_{\lambda\mu k\nu}$ el tensor de Ricci, y $R_{\lambda\mu k\nu}$ el tensor de Riemann

$$R^{\mu}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha} \tag{3}$$

donde

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta}) \tag{4}$$

además $R = g^{\mu k}R_{\mu k}$ es el escalar de Ricci

Utilizando los coeficientes de la métrica FRWL se calculan los símbolos de Cristoffel $\Gamma^{\lambda}_{\mu\nu}$ para obtener el tensor y el escalar de Ricci:

Tenemos $\Gamma^{\lambda}_{\mu\nu}$ como

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} (g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}) \tag{5}$$

calculando para cada caso, tenemos

1) $\lambda = 0$

$$\Gamma^{0}_{\mu\nu} = \frac{1}{2} g^{0\alpha} (g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}) = \frac{1}{2} g^{00} (g_{0\mu,\nu} + g_{0\nu,\mu} - g_{\mu\nu,0}) \tag{6}$$

$$\Gamma^{0}_{\mu\nu} = -\frac{1}{2} g_{\mu\nu,0} \tag{7}$$

$\mu = 0$

$$\Gamma^{0}_{0\nu} = -\frac{1}{2} g_{0\nu,0} = 0 \tag{8}$$

$$\Gamma^{0}_{\nu 0} = 0 \tag{9}$$

$\mu = i$

$$\Gamma^{0}_{i\nu} = -\frac{1}{2} g_{i\nu,0}, \nu = 0, \Gamma^{0}_{i0} = 0 \tag{10}$$

$$\nu = i, \quad \Gamma^{0}_{ij} = -\frac{1}{2} g_{ij,0} = -\frac{\dot{a}}{a} g_{ij} \tag{11}$$

$$\frac{d}{dt} a^2(t) = 2a\dot{a} = \left(2\frac{\dot{a}}{a}\right) a^2 \tag{12}$$

$$g_{rr} = -\frac{a^2}{1 - kr^2}, \quad g_{rr,0} = 2\frac{\dot{a}}{a} g_{rr}, \quad g_{ij,0} = 2\frac{\dot{a}}{a} g_{ij} \tag{13}$$

2) $\lambda = i$

$$\Gamma^{i}_{\mu\nu} = \frac{1}{2} g^{i\alpha} \left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right) \tag{14}$$

$$\Gamma^{i}_{\mu\nu} = \frac{1}{2} g^{ii} \left(g_{i\mu,\nu} + g_{i\nu,\mu} - g_{\mu\nu,i}\right) \tag{15}$$

$\mu = 0$

$$\Gamma^{i}_{0\nu} = \frac{1}{2} g^{ii} (\cancel{g}_{io,\nu} + g_{i\nu,0} - \cancel{g}_{0\nu,i}) = \frac{1}{2} g^{ii} g_{i\nu,0} \tag{16}$$

$\nu = 0$

$$\Gamma^{i}_{00} = 0 \tag{17}$$

$\nu = j$

$$\Gamma^{i}_{0j} = \frac{1}{2} g^{ii} g_{ij,0} = \frac{\dot{a}}{a} g^{ii} g_{ij} = \frac{\dot{a}}{a} \delta^{i}_{j} \tag{18}$$

Finalmente

$$\Gamma^{0}_{0\nu} = \Gamma^{0}_{\nu 0} = 0, \quad \Gamma^{0}_{i0} = \Gamma^{0}_{0i} = 0, \quad \Gamma^{i}_{00} = 0 \tag{19}$$

$$\Gamma^{0}_{ij} = -\frac{\dot{a}}{a} g_{ij}, \quad \Gamma^{i}_{0j} = \Gamma^{i}_{j0} = \frac{\dot{a}}{a} \delta^{i}_{j} \tag{20}$$

Calculamos el tensor de Ricci $R_{\nu\beta} = R^{\mu}_{\nu\mu\beta}$, con $\nu = \beta = 0$

$$R_{00} = \cancel{\Gamma}^{\mu}_{00,\mu} - \Gamma^{\mu}_{0\mu,0} + \cancel{\Gamma}^{\mu}_{\sigma\mu}\Gamma^{\sigma}_{00} - \Gamma^{\mu}_{\sigma 0}\Gamma^{\sigma}_{\mu 0} \tag{21}$$

$$\sum \Gamma^{\mu}_{0\mu,0} = \cancel{\Gamma}^{0}_{00,0} + \sum \Gamma^{i}_{0i,0} = \sum \Gamma^{i}_{0i,0} = \frac{\ddot{a}}{a}\sum_{i=j}^{3}\delta^{i}_{j} = \left(3\frac{\dot{a}}{a}\right)_{0} = 3\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) \tag{22}$$

$$\Gamma^{\mu}_{\sigma 0}\Gamma^{\sigma}_{\mu 0} = \cancel{\Gamma}^{\mu}_{00}\Gamma^{0}_{\mu 0} + \Gamma^{\mu}_{i0}\Gamma^{i}_{\mu 0} = \Gamma^{0}_{0i}\Gamma^{i}_{00} + \Gamma^{\mu}_{i0}\Gamma^{i}_{\mu 0} = \left(\frac{\dot{a}}{a}\right)^2 \delta^{i}_{j}\delta^{i}_{j} = 3\left(\frac{\dot{a}}{a}\right)^2 \tag{23}$$

Obtenemos

$$R_{00} = -3\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) - \left(\frac{\dot{a}}{a}\right)^2 = -3\frac{\ddot{a}}{a}, \qquad R_{00} = -3\frac{\ddot{a}}{a} \tag{24}$$

con $\nu = i$, $\beta = j$, $R_{\nu\beta} = R_{ij} = R^{\mu}_{i\mu j}$

$$R_{\nu\beta} = \Gamma^{\mu}_{\nu\beta,\mu} - \Gamma^{\mu}_{\nu\mu,\beta} + \Gamma^{\mu}_{\sigma\mu}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\mu\nu} \tag{25}$$

$$R_{ij} = \Gamma^{\mu}_{ij,\mu} - \Gamma^{\mu}_{i\mu,\beta} + \Gamma^{\mu}_{\sigma\mu}\Gamma^{\sigma}_{ij} - \Gamma^{\mu}_{\sigma j}\Gamma^{\sigma}_{\mu i} \tag{26}$$

$$\Gamma^{\mu}_{\sigma j}\Gamma^{\sigma}_{\mu i} = \Gamma^{\mu}_{0j}\Gamma^{0}_{\mu i} + \Gamma^{\mu}_{ij}\Gamma^{i}_{\mu i} \tag{27}$$

obtenemos

$$R_{ij} = -\left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2k}{a^2}\right) g_{ij} \tag{28}$$

Calculamos el escalar de Ricci, dado por $R = R_{\nu\beta}g^{\nu\beta}$

$$R = R_{\nu\beta}g^{\nu\beta} = R_{00}g^{00} + R_{ij}R^{ij} = -3\frac{\dot{a}^2}{a} - 3\left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2k}{a^2}\right) \tag{29}$$

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) \tag{30}$$

Obtenemos ahora el tensor de energía-momento

$$T^{\nu}_{\mu} = diag(+\rho, -P, -P, -P) \tag{31}$$

$$T^{\nu}_{\mu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix}$$

$$T_{\mu\alpha} = g_{\alpha\nu}T^{\nu}_{\mu} \quad \Rightarrow \quad T_{ij} = -g_{ij}P, \quad T_{00} = \rho \tag{32}$$

Obtenidos todos los tensores basta sustituir en las ecuaciones de campo de Einstein

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{33}$$

De los resultados obtenidos para el tensor y escalar de Ricci, y el tensor de energía-momento, sustituimos primero la parte temporal y luego la espacial

Hacemos $R_{00} - \frac{1}{2} g_{00} R = 8\pi G T_{00}, \quad$ con $g_{00} = 1$

$$-3\frac{\ddot{a}}{a} + \frac{1}{2}6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G\rho \tag{34}$$

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{35}$$

Luego $R_{ij} - \frac{1}{2} g_{ij} R = 8\pi G T_{ij}$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G P \tag{36}$$

Reescribiendo el factor de escala y su derivada como $H = \dot{a}/a$, tenemos

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{37}$$

$$2\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2} = -8\pi G P \tag{38}$$

Las ecuaciones 37 y 38 son las llamadas ecuaciones de Friedmann.