AP Calculus Lesson Three Notes

Chapter Two – Differentiation

2.1 Definition of Derivative
2.2 Differentiation Rules

2.1 Definition of Derivative

The **derivative of *f*(*x*) with respect to *x*** is the function ***f′*(*x*)** and is defined as,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \tag{1}$$

Or the derivative of $y = f(x)$ at $x = a$ is defined as

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \tag{2}$$

Example 2.1-1 Find the derivative of the following function using the definition of the derivative.

$$f(x) = 2x^2 - 16x + 35$$

Solution

All we really need to do is to plug this function into the definition of the derivative, (1), and do some algebra. While, admittedly, the algebra will get somewhat unpleasant at times, but it's just algebra so don't get excited about the fact that we're now computing derivatives.

First plug the function into the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{2(x+h)^2 - 16(x+h) + 35 - \left(2x^2 - 16x + 35\right)}{h} \end{aligned}$$

Be careful and make sure that you properly deal with parenthesis when doing the subtracting.

Now, we know from the previous chapter that we can't just plug in $h = 0$ since this will give us a division by zero error. So we are going to have to do some work. In this case that means multiplying everything out and distributing the minus sign through on the second term. Doing this gives,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 16x - 16h + 35 - 2x^2 + 16x - 35}{h} \\
&= \lim_{h\to 0} \frac{4xh + 2h^2 - 16h}{h}
\end{aligned}
$$

Notice that every term in the numerator that didn't have an *h* in it cancelled out and we can now factor an *h* out of the numerator which will cancel against the *h* in the denominator. After that we can compute the limit.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{h(4x + 2h - 16)}{h} \\
&= \lim_{h\to 0} 4x + 2h - 16 \\
&= 4x - 16
\end{aligned}
$$

So, the derivative is,

$$f'(x) = 4x - 16$$

Example 2.1-2 Find the derivative of the following function using the derivative.

$$R(z) = \sqrt{5z - 8}$$

Solution

First plug into the definition of the derivative as we've done with the previous two examples.

$$
\begin{aligned}
R'(z) &= \lim_{h\to 0} \frac{R(z+h) - R(z)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{5(z+h) - 8} - \sqrt{5z - 8}}{h}
\end{aligned}
$$

In this problem we're going to have to rationalize the numerator. You do remember rationalization from an Algebra class right? In an Algebra class you probably only rationalized the denominator, but you can also rationalize numerators. Remember that in rationalizing the numerator (in this case) we multiply both the numerator and denominator by the numerator except we change the sign between the two terms. Here's the rationalizing work for this problem,

$$
\begin{aligned}
R'(z) &= \lim_{h\to 0}\frac{\left(\sqrt{5(z+h)-8}-\sqrt{5z-8}\right)}{h}\frac{\left(\sqrt{5(z+h)-8}+\sqrt{5z-8}\right)}{\left(\sqrt{5(z+h)-8}+\sqrt{5z-8}\right)}\\
&= \lim_{h\to 0}\frac{5z+5h-8-(5z-8)}{h\left(\sqrt{5(z+h)-8}+\sqrt{5z-8}\right)}\\
&= \lim_{h\to 0}\frac{5h}{h\left(\sqrt{5(z+h)-8}+\sqrt{5z-8}\right)}
\end{aligned}
$$

Again, after the simplification we have only *h*'s left in the numerator. So, cancel the *h* and evaluate the limit.

$$
\begin{aligned}
R'(z) &= \lim_{h\to 0}\frac{5}{\sqrt{5(z+h)-8}+\sqrt{5z-8}}\\
&= \frac{5}{\sqrt{5z-8}+\sqrt{5z-8}}\\
&= \frac{5}{2\sqrt{5z-8}}
\end{aligned}
$$

And so we get a derivative of,

$$R'(z)=\frac{5}{2\sqrt{5z-8}}$$

The two examples above can be also done by using Formula (2) in a similar manner. We'll leave the job to you.

Example 2.1-3 Determine $f'(0)$ for $f(x)=|x|$.

Solution

Since this problem is asking for the derivative at a specific point we'll go ahead and use that in our work. It will make our life easier and that's always a good thing.

So, plug into the definition and simplify.

$$\begin{aligned} f'(0) &= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0}\frac{|0+h|-|0|}{h} \\ &= \lim_{h\to 0}\frac{|h|}{h} \end{aligned}$$

We saw a situation like this back when we were looking at limits at infinity. As in that section we can't just cancel the *h*'s. We will have to look at the two one sided limits and recall that

$$|h| = \begin{cases} h & \text{if } h \geq 0 \\ -h & \text{if } h < 0 \end{cases}$$

$$\begin{aligned} \lim_{h\to 0^-}\frac{|h|}{h} &= \lim_{h\to 0^-}\frac{-h}{h} \qquad \text{because } h<0 \text{ in a left-hand limit.} \\ &= \lim_{h\to 0^-}(-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{h\to 0^+}\frac{|h|}{h} &= \lim_{h\to 0^+}\frac{h}{h} \qquad \text{because } h>0 \text{ in a right-hand limit.} \\ &= \lim_{h\to 0^+}1 \\ &= 1 \end{aligned}$$

The two one-sided limits are different and so

$$\lim_{h\to 0}\frac{|h|}{h}$$

doesn't exist. However, this is the limit that gives us the derivative that we're after.

If the limit doesn't exist then the derivative doesn't exist either.

The preceding discussion leads to the following definition.

Definition

A function $f(x)$ is called **differentiable** at $x = a$ if $f'(x)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval.

Theorem

If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous $x = a$.

Alternate Notation

The typical derivative notation is the "prime" notation. However, there is another notation that is used on occasion so let's cover that.

Given a function $y = f(x)$, all of the following are equivalent and represent the derivative of $f(x)$ with respect to x.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}\big(f(x)\big) = \frac{d}{dx}(y)$$

Because we also need to evaluate derivatives on occasion we also need a notation for evaluating derivatives when using the fractional notation. So if we want to evaluate the derivative at $x=a$ all of the following are equivalent.

$$f'(a) = y'\big|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$$

Note as well that on occasion we will drop the (x) part on the function to simplify the notation somewhat. In these cases the following are equivalent.

$$f'(x) = f'$$

Interpretations of the Derivative

Rate of Change

The first interpretation of a derivative is rate of change. This was not the first problem that we looked at in the limit chapter, but it is the most important interpretation of the derivative. If $f(x)$ represents a quantity at any x then the derivative $f'(a)$ represents the instantaneous rate of change of $f(x)$ at $x=a$.

Slope of Tangent Line

This is the next major interpretation of the derivative. The slope of the tangent line to $f(x)$ at $x=a$ is $f'(a)$. The tangent line then is given by,

$$y = f(a) + f'(a)(x-a)$$

Example 2.1-4 Find the tangent line to the following function at $z = 3$.

$$R(z) = \sqrt{5z-8}$$

Solution

We first need the derivative of the function, which is,

$$R'(z) = \frac{5}{2\sqrt{5z-8}}$$

Now all that we need is the function value and derivative (for the slope) at $z = 3$

$$R(3) = \sqrt{7} \qquad\qquad m = R'(3) = \frac{5}{2\sqrt{7}}$$

The tangent line is then,

$$y = \sqrt{7} + \frac{5}{2\sqrt{7}}(z-3)$$

Velocity

Recall that this can be thought of as a special case of the rate of change interpretation. If the position of an object is given by $f(t)$ after t units of time the velocity of the object at $t = a$ is given by $f'(a)$.

Example 2.1-5 Suppose that the position of an object after t hours is given by,

$$g(t) = \frac{t}{t+1}$$

Answer both of the following about this object.

(a) Is the object moving to the right or the left at t = 10 hours?

(b) Does the object ever stop moving?

Solution

Once again we need the derivative and which is,

$$g'(t) = \frac{1}{(t+1)^2}$$

(a) Is the object moving to the right or the left at *t* = 10 hours?

To determine if the object is moving to the right (velocity is positive) or left (velocity is negative) we need the derivative at $t = 10$.

$$g'(10) = \frac{1}{121}$$

So the velocity at $t = 10$ is positive and so the object is moving to the right at $t = 10$.

(b) Does the object ever stop moving?

The object will stop moving if the velocity is ever zero. However, note that the only way a rational expression will ever be zero is if the numerator is zero. Since the numerator of the derivative (and hence the speed) is a constant it can't be zero. Therefore, the velocity will never stop moving.

In fact, we can say a little more here. The object will always be moving to the right since the velocity is always positive.

2.2 Differentiation Rules

1) The Constant Rule

If $f(x) = c$, then $f'(x) = (c)' = 0$ or $\frac{df(x)}{dx} = \frac{d}{dx}(c) = 0$

The derivative of a constant is zero.

2) The Power Rule

If $f(x) = x^n$, then $f'(x) = (x^n)' = nx^{n-1}$ or $\frac{df(x)}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}$, where n is any real number.

All we are doing here is bringing the original exponent down in front and multiplying and then subtracting one from the original exponent.

Note as well that in order to use this formula *n* must be a number, it can't be a variable. Also note that the base, the *x*, must be a variable, it can't be a number. It will be tempting in some later sections to misuse the Power Rule when we run in some functions where the exponent isn't a number and/or the base isn't a variable.

3) The Sum Rule

If $y = f(x) + g(x)$, then $y' = f'(x) + g'(x)$ or $\frac{dy}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$

4) The Difference Rule

If $y = f(x) - g(x)$, then $y' = f'(x) - g'(x)$ or $\frac{dy}{dx} = \frac{df(x)}{dx} - \frac{dg(x)}{dx}$

5) The Product Rule

If $y = f(x)g(x)$, then $y' = f'(x)g(x) + f(x)g'(x)$ or

$$\frac{dy}{dx} = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$$

6) The Quotient Rule

If $y = \frac{f(x)}{g(x)}$, then $y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ or

$$\frac{dy}{dx} = \frac{\frac{df(x)}{dx}g(x) - f(x)\frac{dg(x)}{dx}}{[g(x)]^2}$$

7) The Chain Rule

If $y = f(g(x))$, then $y' = f'(g)g'(x)$ or $\frac{dy}{dx} = \frac{df(g)}{dg}\frac{dg(x)}{dx}$

Example 2.2-1 Differentiate each of the following functions.

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$

(b) $g(t) = 2t^6 + 7t^{-6}$

(c) $y = 8z^3 - \dfrac{1}{3z^5} + z - 23$

(d) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \dfrac{2}{\sqrt[5]{x^2}}$

(e) $h(x) = x^{\pi} - x^{\sqrt{2}}$

Solution

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$

In this case we have the sum and difference of four terms and so we will differentiate each of the terms using the first property from above and then put them back together with the proper sign. Also, for each term with a multiplicative constant remember that all we need to do is "factor" the constant out (using the second property) and then do the derivative.

$$\begin{aligned} f'(x) &= 15(100)x^{99} - 3(12)x^{11} + 5(1)x^0 - 0 \\ &= 1500x^{99} - 36x^{11} + 5 \end{aligned}$$

Notice that in the third term the exponent was a one and so upon subtracting 1 from the original exponent we get a new exponent of zero. Now recall that $x^0 = 1$. Don't forget to do any basic arithmetic that needs to be done such as any multiplication and/or division in the coefficients.

(b) $g(t) = 2t^6 + 7t^{-6}$

The point of this problem is to make sure that you deal with negative exponents correctly. Here is the derivative.

$$\begin{aligned} g'(t) &= 2(6)t^5 + 7(-6)t^{-7} \\ &= 12t^5 - 42t^{-7} \end{aligned}$$

Make sure that you correctly deal with the exponents in these cases, especially the negative exponents.

(c) $y = 8z^3 - \dfrac{1}{3z^5} + z - 23$

Now in this function the second term is not correctly set up for us to use the power rule. The power rule requires that the term be a variable to a power only and the term must be in the numerator. So, prior to differentiating we first need to rewrite the second term into a form that we can deal with.

$$y = 8z^3 - \frac{1}{3}z^{-5} + z - 23$$

Note that we left the 3 in the denominator and only moved the variable up to the numerator. Remember that the only thing that gets an exponent is the term that is immediately to the left of the exponent. If we'd wanted the three to come up as well we'd have written,

$$\frac{1}{(3z)^5}$$

so be careful with this! It's a very common mistake to bring the 3 up into the numerator as well at this stage.

Now that we've gotten the function rewritten into a proper form that allows us to use the Power Rule we can differentiate the function. Here is the derivative for this part.

$$y' = 24z^2 + \frac{5}{3}z^{-6} + 1$$

(d) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \dfrac{2}{\sqrt[5]{x^2}}$

All of the terms in this function have roots in them. In order to use the power rule we need to first convert all the roots to fractional exponents. Again, remember that the Power Rule requires us to have a variable to a number and that it must be in the numerator of the term. Here is the function written in "proper" form.

$$T(x) = x^{\frac{1}{2}} + 9\left(x^7\right)^{\frac{1}{3}} - \frac{2}{\left(x^2\right)^{\frac{1}{5}}}$$
$$= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - \frac{2}{x^{\frac{2}{5}}}$$
$$= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - 2x^{-\frac{2}{5}}$$

In the last two terms we combined the exponents. You should always do this with this kind of term. In a later section we will learn of a technique that would allow us to differentiate this term without combining exponents, however it will take significantly more work to do. Also don't forget to move the term in the denominator of the third term up to the numerator. We can now differentiate the function.

$$T'(x) = \frac{1}{2}x^{-\frac{1}{2}} + 9\left(\frac{7}{3}\right)x^{\frac{4}{3}} - 2\left(-\frac{2}{5}\right)x^{-\frac{7}{5}}$$
$$= \frac{1}{2}x^{-\frac{1}{2}} + \frac{63}{3}x^{\frac{4}{3}} + \frac{4}{5}x^{-\frac{7}{5}}$$

Make sure that you can deal with fractional exponents. You will see a lot of them in this class.

(e) $h(x) = x^{\pi} - x^{\sqrt{2}}$

In all of the previous examples the exponents have been nice integers or fractions. That is usually what we'll see in this class. However, the exponent only needs to be a number so don't get excited about problems like this one. They work exactly the same.

$$h'(x) = \pi x^{\pi-1} - \sqrt{2}x^{\sqrt{2}-1}$$

The answer is a little messy and we won't reduce the exponents down to decimals. However, this problem is not terribly difficult it just looks that way initially.

Example 2.2-2 Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

(b) $h(t) = \dfrac{2t^5 + t^2 - 5}{t^2}$

Solution

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

In this function we can't just differentiate the first term, differentiate the second term and then multiply the two back together. That just won't work. We will discuss this in detail in the next section so if you're not sure you believe that hold on for a bit and we'll be looking at that soon as well as showing you an example of why it won't work.

It is still possible to do this derivative however. All that we need to do is convert the radical to fractional exponents (as we should anyway) and then multiply this through the parenthesis.

$$y = x^{\frac{2}{3}}\left(2x - x^2\right) = 2x^{\frac{5}{3}} - x^{\frac{8}{3}}$$

Now we can differentiate the function.

$$y' = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

(b) $h(t) = \dfrac{2t^5 + t^2 - 5}{t^2}$

As with the first part we can't just differentiate the numerator and the denominator and the put it back together as a fraction. Again, if you're not sure you believe this hold on until the next section and we'll take a more detailed look at this.

We can simplify this rational expression however as follows.

$$h(t) = \frac{2t^5}{t^2} + \frac{t^2}{t^2} - \frac{5}{t^2} = 2t^3 + 1 - 5t^{-2}$$

This is a function that we can differentiate.

$$h'(t) = 6t^2 + 10t^{-3}$$

Example 2.2-3 Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

(b) $f(x) = (6x^3 - x)(10 - 20x)$

Solution

At this point there really aren't a lot of reasons to use the product rule. As we noted in the previous section all we would need to do for either of these is to just multiply out the product and then differentiate.

With that said we will use the product rule on these so we can see an example or two. As we add more functions to our repertoire and as the functions become more complicated the product rule will become more useful and in many cases required.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

Note that we took the derivative of this function in the previous section and didn't use the product rule at that point. We should however get the same result here as we did then.

Now let's do the problem here. There's not really a lot to do here other than use the product rule. However, before doing that we should convert the radical to a fractional exponent as always.

$$y = x^{\frac{2}{3}}\left(2x - x^2\right)$$

Now let's take the derivative. So we take the derivative of the first function times the second then add on to that the first function times the derivative of the second function.

$$y' = \frac{2}{3}x^{-\frac{1}{3}}\left(2x - x^2\right) + x^{\frac{2}{3}}\left(2 - 2x\right)$$

This is NOT what we got in the previous section for this derivative. However, with some simplification we can arrive at the same answer.

$$y' = \frac{4}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{\frac{5}{3}} + 2x^{\frac{2}{3}} - 2x^{\frac{5}{3}} = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

This is what we got for an answer in the previous section so that is a good check of the product rule.

(b) $f(x) = (6x^3 - x)(10 - 20x)$

This one is actually easier than the previous one. Let's just run it through the product rule.

$$\begin{aligned} f'(x) &= \left(18x^2 - 1\right)\left(10 - 20x\right) + \left(6x^3 - x\right)\left(-20\right) \\ &= -480x^3 + 180x^2 + 40x - 10 \end{aligned}$$

Since it was easy to do we went ahead and simplified the results a little.

Example 2.2-4 Differentiate each of the following functions.

(a) $W(z) = \dfrac{3z+9}{2-z}$ **(b)** $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$

(c) $f(x) = \dfrac{4}{x^6}$ **(d)** $y = \dfrac{w^6}{5}$

Solution

(a) $W(z) = \dfrac{3z+9}{2-z}$

There isn't a lot to do here other than to use the quotient rule. Here is the work for this function.

$$\begin{aligned} W'(z) &= \frac{3(2-z) - (3z+9)(-1)}{(2-z)^2} \\ &= \frac{15}{(2-z)^2} \end{aligned}$$

(b) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$

Again, not much to do here other than use the quotient rule. Don't forget to convert the square root into a fractional exponent.

$$h'(x) = \frac{4\left(\frac{1}{2}\right)x^{-\frac{1}{2}}\left(x^2-2\right) - 4x^{\frac{1}{2}}\left(2x\right)}{\left(x^2-2\right)^2}$$

$$= \frac{2x^{\frac{3}{2}} - 4x^{-\frac{1}{2}} - 8x^{\frac{3}{2}}}{\left(x^2-2\right)^2}$$

$$= \frac{-6x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}}{\left(x^2-2\right)^2}$$

(c) $f(x) = \dfrac{4}{x^6}$

It seems strange to have this one here rather than being the first part of this example given that it definitely appears to be easier than any of the previous two. In fact, it is easier. There is a point to doing it here rather than first. In this case there are two ways to do compute this derivative. There is an easy way and a hard way and in this case the hard way is the quotient rule. That's the point of this example.

Let's do the quotient rule and see what we get.

$$f'(x) = \frac{(0)\left(x^6\right) - 4\left(6x^5\right)}{\left(x^6\right)^2} = \frac{-24x^5}{x^{12}} = -\frac{24}{x^7}$$

Now, that was the "hard" way. So, what was so hard about it? Well actually it wasn't that hard, there is just an easier way to do it that's all. However, having said that, a common mistake here is to do the derivative of the numerator (a constant) incorrectly. For some reason many people will give the derivative of the numerator in these kinds of problems as a 1 instead of 0! Also, there is some simplification that needs to be done in these kinds of problems if you do the quotient rule.

The easy way is to do what we did in the previous section.

$$f'(x) = 4x^{-6} = -24x^{-7} = -\frac{24}{x^7}$$

Either way will work, but I'd rather take the easier route if I had the choice.

(d) $y = \dfrac{w^6}{5}$

This problem also seems a little out of place. However, it is here again to make a point. Do not confuse this with a quotient rule problem. While you can do the quotient rule on this function there is no reason to use the quotient rule on this. Simply rewrite the function as

$$y = \frac{1}{5} w^6$$

and differentiate as always.

$$y' = \frac{6}{5} w^5$$

Example 2.2-5 Suppose that the amount of air in a balloon at any time *t* is given by

$$V(t) = \frac{6\sqrt[3]{t}}{4t+1}$$

Determine if the balloon is being filled with air or being drained of air at $t = 8$.

Solution

If the balloon is being filled with air then the volume is increasing and if it's being drained of air then the volume will be decreasing. In other words, we need to get the derivative so that we can determine the rate of change of the volume at $t = 8$.

This will require the quotient rule.

$$\begin{aligned} V'(t) &= \frac{2t^{-\frac{2}{3}}(4t+1) - 6t^{\frac{1}{3}}(4)}{(4t+1)^2} \\ &= \frac{-16t^{\frac{1}{3}} + 2t^{-\frac{2}{3}}}{(4t+1)^2} \\ &= \frac{-16t^{\frac{1}{3}} + \dfrac{2}{t^{\frac{2}{3}}}}{(4t+1)^2} \end{aligned}$$

Note that we simplified the numerator more than usual here. This was only done to make the derivative easier to evaluate.

The rate of change of the volume at $t = 8$ is then,

$$V'(8) = \frac{-16(2) + \frac{2}{4}}{(33)^2} \qquad (8)^{\frac{1}{3}} = 2 \qquad (8)^{\frac{2}{3}} = \left((8)^{\frac{1}{3}}\right)^2 = (2)^2 = 4$$

$$= -\frac{63}{2178} = -\frac{7}{242}$$

So, the rate of change of the volume $t = 8$ is negative and so the volume must be decreasing. Therefore air is being drained out of the balloon at $t = 8$.

Derivatives of the Six Trigonometric Functions

$$\frac{d}{dx}(\sin(x)) = \cos(x) \qquad \frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x) \qquad \frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x) \qquad \frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

Proof of the Formulas

We'll start with finding the derivative of the sine function. To do this we will need to use the definition of the derivative. It's been a while since we've had to use this, but sometimes there just isn't anything we can do about it. Here is the definition of the derivative for the sine function.

$$\frac{d}{dx}(\sin(x)) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

Since we can't just plug in $h = 0$ to evaluate the limit we will need to use the following trig formula on the first sine in the numerator.

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

Doing this gives us,

$$\begin{aligned}\frac{d}{dx}\left(\sin(x)\right) &= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\ &= \lim_{h\to 0}\frac{\sin(x)\left(\cos(h)-1\right)+\cos(x)\sin(h)}{h}\\ &= \lim_{h\to 0}\sin(x)\frac{\cos(h)-1}{h}+\lim_{h\to 0}\cos(x)\frac{\sin(h)}{h}\end{aligned}$$

As you can see upon using the trig formula we can combine the first and third term and then factor a sine out of that. We can then break up the fraction into two pieces, both of which can be dealt with separately.

Now, both of the limits here are limits as h approaches zero. In the first limit we have a sin*(x)* and in the second limit we have a cos*(x)*. Both of these are only functions of x only and as h moves in towards zero this has no affect on the value of x. Therefore, as far as the limits are concerned, these two functions are constants and can be factored out of their respective limits. Doing this gives,

$$\frac{d}{dx}\left(\sin(x)\right)=\sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

At this point all we need to do is use the limits in the fact above to finish out this problem.

$$\frac{d}{dx}\left(\sin(x)\right)=\sin(x)(0)+\cos(x)(1)=\cos(x)$$

Differentiating cosine is done in a similar fashion. It will require a different trig formula, but other than that is an almost identical proof. The details will be left to you. When done with the proof you should get,

$$\frac{d}{dx}\left(\cos(x)\right)=-\sin(x)$$

With these two out of the way the remaining four are fairly simple to get. All the remaining four trig functions can be defined in terms of sine and cosine and these definitions, along with appropriate derivative rules, can be used to get their derivatives.

Let’s take a look at tangent. Tangent is defined as,

$$\tan(x)=\frac{\sin(x)}{\cos(x)}$$

Now that we have the derivatives of sine and cosine all that we need to do is use the quotient rule on this. Let’s do that.

$$\begin{aligned}\frac{d}{dx}\left(\tan(x)\right) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)\\ &= \frac{\cos(x)\cos(x)-\sin(x)\left(-\sin(x)\right)}{\left(\cos(x)\right)^2}\\ &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}\end{aligned}$$

Now, recall that $\cos^2(x)+\sin^2(x)=1$ and if we also recall the definition of secant in terms of cosine we arrive at,

$$\begin{aligned}\frac{d}{dx}\left(\tan(x)\right) &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)}\\ &= \sec^2(x)\end{aligned}$$

The remaining three trig functions are also quotients involving sine and/or cosine and so can be differentiated in a similar manner. We'll leave the details to you. Here are the derivatives of all six of the trig functions.

Example 2.2-6 Differentiate each of the following functions.

(a) $g(x)=3\sec(x)-10\cot(x)$

(b) $h(w)=3w^{-4}-w^2\tan(w)$

(c) $y=5\sin(x)\cos(x)+4\csc(x)$

Solution

(a) $g(x)=3\sec(x)-10\cot(x)$

There really isn't a whole lot to this problem. We'll just differentiate each term using the formulas from above.

$$\begin{aligned}g'(x) &= 3\sec(x)\tan(x)-10\left(-\csc^2(x)\right)\\ &= 3\sec(x)\tan(x)+10\csc^2(x)\end{aligned}$$

(b) $h(w)=3w^{-4}-w^2\tan(w)$

In this part we will need to use the product rule on the second term and note that we really will need the product rule here. There is no other way to do this derivative unlike what we saw when we first looked at the product rule. When we first looked at the product rule the only functions we knew how to differentiate were polynomials and in those cases all we really needed to do was multiply them out and we could take the derivative without the product rule. We are now getting into the point where we will be forced to do the product rule at times regardless of whether or not we want to.

We will also need to be careful with the minus sign in front of the second term and make sure that it gets dealt with properly. There are two ways to deal with this. One way it to make sure that you use a set of parenthesis as follows,

$$\begin{aligned} h'(w) &= -12w^{-5} - \left(2w\tan(w) + w^2\sec^2(w)\right) \\ &= -12w^{-5} - 2w\tan(w) - w^2\sec^2(w) \end{aligned}$$

Because the second term is being subtracted off of the first term then the whole derivative of the second term must also be subtracted off of the derivative of the first term. The parenthesis make this idea clear.

(c) $y = 5\sin(x)\cos(x) + 4\csc(x)$

As with the previous part we'll need to use the product rule on the first term. We will also think of the 5 as part of the first function in the product to make sure we deal with it correctly. Alternatively, you could make use of a set of parenthesis to make sure the 5 gets dealt with properly. Either way will work, but we'll stick with thinking of the 5 as part of the first term in the product. Here's the derivative of this function.

$$\begin{aligned} y' &= 5\cos(x)\cos(x) + 5\sin(x)\left(-\sin(x)\right) - 4\csc(x)\cot(x) \\ &= 5\cos^2(x) - 5\sin^2(x) - 4\csc(x)\cot(x) \end{aligned}$$

(d) $P(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$

In this part we'll need to use the quotient rule to take the derivative.

$$\begin{aligned} P'(t) &= \frac{\cos(t)\left(3 - 2\cos(t)\right) - \sin(t)\left(2\sin(t)\right)}{\left(3 - 2\cos(t)\right)^2} \\ &= \frac{3\cos(t) - 2\cos^2(t) - 2\sin^2(t)}{\left(3 - 2\cos(t)\right)^2} \end{aligned}$$

Be careful with the signs when differentiating the denominator. The negative sign we get from differentiating the cosine will cancel against the negative sign that is already there.

This appears to be done, but there is actually a fair amount of simplification that can yet be done. To do this we need to factor out a "-2" from the last two terms in the numerator and the make use of the fact that $\cos^2(\theta)+\sin^2(\theta)=1$.

$$\begin{aligned} P'(t) &= \frac{3\cos(t)-2\left(\cos^2(t)+\sin^2(t)\right)}{\left(3-2\cos(t)\right)^2} \\ &= \frac{3\cos(t)-2}{\left(3-2\cos(t)\right)^2} \end{aligned}$$

Derivatives of Exponential and Logarithm Function

We'll start off by looking at the exponential function,

$$f(x)=a^x$$

We want to differentiate this. The power rule that we looked at a couple of sections ago won't work as that required the exponent to be a fixed number and the base to be a variable. That is exactly the opposite from what we've got with this function. So, we're going to have to start with the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x a^h-a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x\left(a^h-1\right)}{h} \end{aligned}$$

Now, the a^x is not affected by the limit since it doesn't have any h's in it and so is a constant as far as the limit is concerned. We can therefore factor this out of the limit. This gives,

$$f'(x) = a^x \lim_{h\to 0}\frac{a^h - 1}{h}$$

For the natural exponential function, $f(x) = e^x$, we have, therefore,

$$f'(x) = e^x \lim_{h\to 0}\frac{e^h - 1}{h} = e^x, \text{ since } \lim_{h\to 0}\frac{e^h - 1}{h} = 1$$

Now, let's discuss the derivatives of $f(x) = \ln x$:

If $y = \ln x$, then $x = e^y$, by implicit differentiation, we have,

$\dfrac{d(x)}{dx} = \dfrac{d(e^y)}{dx}$, that is,

$1 = e^y \dfrac{dy}{dx}$, or $\dfrac{dy}{dx} = \dfrac{1}{e^y} = \dfrac{1}{x}$.

Therefore, $(\ln x)' = \dfrac{1}{x}$

Next, let's discuss the derivatives of $f(x) = a^x$:

If y = a^x, then $\ln y = \ln a^x$, or $\ln y = x\ln a$. Take the derivatives of both sides with respect to x:

$\dfrac{d(\ln y)}{dx} = \dfrac{d(x\ln a)}{dx}$, that leads,

$\dfrac{1}{y}\dfrac{dy}{dx} = \ln a$, so $\dfrac{dy}{dx} = y\ln a = a^x \ln a$.

Therefore, $(a^x)' = a^x \ln a$

Finally, let's derive the derivatives of $f(x) = \log_a x$:

If $y = \log_a x$, then $a^y = x$. Take the derivatives of both sides with respect to *x:*

$\dfrac{d(a^y)}{dx} = \dfrac{dx}{dx} = 1$, or $a^y \ln a \dfrac{dy}{dx} = 1$, or $\dfrac{dy}{dx} = \dfrac{1}{a^y \ln a} = \dfrac{1}{x\ln a}$.

Therefore, $(\log_a x)' = \dfrac{1}{x\ln a}$.

Here is a summary of the derivatives in this section.

1. $\dfrac{d(e^x)}{dx} = e^x$

2. $\dfrac{d(a^x)}{dx} = a^x \ln a$

3. $\dfrac{d(\ln x)}{dx} = \dfrac{1}{x}$

4. $\dfrac{d(\log_a x)}{dx} = \dfrac{1}{x \ln a}$

Example 2.2-7 Differentiate each of the following functions.

(a) $R(w) = 4^w - 5\log_9 w$

(b) $f(x) = 3\mathbf{e}^x + 10x^3 \ln x$

(c) $y = \dfrac{5\mathbf{e}^x}{3\mathbf{e}^x + 1}$

Solution

(a) This will be the only example that doesn't involve the natural exponential and natural logarithm functions.

$$R'(w) = 4^w \ln 4 - \frac{5}{w \ln 9}$$

(b) Not much to this one. Just remember to use the product rule on the second term.

$$\begin{aligned} f'(x) &= 3\mathbf{e}^x + 30x^2 \ln x + 10x^3 \left(\frac{1}{x}\right) \\ &= 3\mathbf{e}^x + 30x^2 \ln x + 10x^2 \end{aligned}$$

(c) We'll need to use the quotient rule on this one.

$$
\begin{aligned}
y &= \frac{5\mathbf{e}^x\left(3\mathbf{e}^x+1\right)-\left(5\mathbf{e}^x\right)\left(3\mathbf{e}^x\right)}{\left(3\mathbf{e}^x+1\right)^2} \\
&= \frac{15\mathbf{e}^{2x}+5\mathbf{e}^x-15\mathbf{e}^{2x}}{\left(3\mathbf{e}^x+1\right)^2} \\
&= \frac{5\mathbf{e}^x}{\left(3\mathbf{e}^x+1\right)^2}
\end{aligned}
$$

The followings are some common formulas for derivatives, in which $u' = \dfrac{du}{dx}$.

Formulas of Derivatives

$$\frac{da}{dx} = 0$$

$$[au]' = au'$$

$$(u^a)' = au^{a-1}u'$$

$$(u+v)' = u' + v'$$

$$(u-v)' = u' - v'$$

$$(uv)' = u'v + uv'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\frac{d[u(v)]}{dx} = \frac{du}{dv}\frac{dv}{dx}$$

$$(\sin u)' = \cos u \cdot u'$$

$$(\cos u)' = -\sin u \cdot u'$$

$$(\tan u)' = \sec^2 u \cdot u'$$

$$(\cot u)' = -\csc^2 u \cdot u'$$

$$(\sec u)' = \sec u \tan u \cdot u'$$

$$(\csc u)' = -\csc u \cot u \cdot u'$$

$$(a^u)' = a^u \ln a \cdot u'$$

$$(e^u)' = e^u \cdot u'$$

$$\left(\log_a^u\right)' = \frac{u'}{u \ln a}$$

$$(\ln u)' = \frac{u'}{u}$$

$$(\sin^{-1} u)' = \frac{1}{\sqrt{1-u^2}} \cdot u'$$

$$(\cos^{-1} u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u'$$

$$(\tan^{-1} u)' = \frac{1}{1+u^2} \cdot u'$$

$$(\cot^{-1} u)' = -\frac{1}{1+u^2} \cdot u'$$

$$(\sec^{-1} u)' = \frac{1}{|u|\sqrt{u^2-1}} \cdot u'$$

$$(\csc^{-1} u)' = -\frac{1}{|u|\sqrt{u^2-1}} \cdot u'$$

$$\sinh u = \frac{e^u - e^{-u}}{2}$$

$$\cosh u = \frac{e^u + e^{-u}}{2}$$

$$(\sinh u)' = \cosh u \cdot u'$$

$$(\cosh u)' = \sinh u \cdot u'$$

$$(\tanh u)' = \sec h^2 u \cdot u'$$

$$(\coth u)' = -\csc h^2 u \cdot u'$$

$$(\sec hu)' = -\sec hu \tanh u \cdot u'$$

$$(\csc hu)' = -\csc hu \coth u \cdot u'$$

$$\sinh^{-1} u = \ln(u + \sqrt{u^2+1})$$

$$\cosh^{-1} u = \ln(u + \sqrt{u^2-1})$$

$$\tanh^{-1} u = \frac{1}{2}\ln\frac{1+u}{1-u}$$

$$\sec h^{-1} u = \ln\frac{1+\sqrt{1-u^2}}{u}$$

$$(\sinh^{-1} u)' = \frac{u'}{\sqrt{u^2+1}}$$

$$(\cosh^{-1} u)' = \frac{u'}{\sqrt{u^2-1}}$$

$$(\tanh^{-1} u)' = \frac{u'}{1-u^2}$$

$$(\sec h^{-1} u)' = -\frac{u'}{u\sqrt{1-u^2}}$$

In the formulas above, $u' = \dfrac{du}{dx}$.