AP Calculus Lesson Seven Notes

Chapter Three – Applications of Differential Calculus

3.7 Motion along a Curve: Velocity and Acceleration
3.8 Related Rates
3.9 Slope of a Polar Curve

3.7 Motion along a Curve: Velocity and Acceleration Vectors

If a point P moves along a curve defined parametrically by $P(t) = (x(t), y(t))$, where t represents time, then the vector from the origin to P is called the position vector, which may be written as $\vec{R}(t) = (x, y)$ or as $\vec{R} = x\vec{\imath} + y\vec{\jmath}$.

The velocity vector is the derivative of the position vector:

$$\vec{v} = \frac{d\vec{R}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} \text{ or } \vec{v}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(v_x, v_y\right)$$

The slope of $\vec{v}$ is

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx},$$

which is the slope of the curve.

The magnitude of $\vec{v}$ is the vector's length:

$$|\ \vec{v}\ | = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

If the vector $\vec{v}$ is drawn initiating at P, it will be tangent to the curve at P and its magnitude will be the speed of the particale at P.

The acceleration vector $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{R}}{dt}$.

$$\vec{a} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} \text{ or } \vec{a}(t) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right) = \left(a_x, a_y\right)$$

Its magnitude is the vector's length:

$$|\vec{a}| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$$

Example 3.7-1 A particle moves according to the equations $x = 3\cos t$ and $y = 2\sin t$.

(a) Find a single equation in x and y for the path of the particle.
(b) Find the velocity and acceleration vectors at any time *t*, and show that $\vec{a} = -\vec{R}$ at all time.
(c) Find $\vec{R}$,$\vec{v}$, and $\vec{a}$ when $t_1 = \pi/6$ and $t_2 = \pi$.
(d) Find the speed of the particle and the magnitude of its acceleration at each instant in (c).
(e) When is the speed a maximum? A minimum?

Solution

(a) Since $\frac{x^2}{9} = \cos^2 t$ and $\frac{y^2}{4} = \sin^2 t$, therefore

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

And the particale moves in a counterclockwise direction along the ellipse, starting, when $t = 0$, at (3, 0) and returning to this point when $t = 2\pi$.

(b) We have

$$\vec{R} = 3\cos t\,\vec{\imath} + 2\sin t\,\vec{\jmath}$$
$$\vec{v} = -3\sin t\,\vec{\imath} + 2\cos t\,\vec{\jmath}$$
$$\vec{a} = -3\cos t\,\vec{\imath} - 2\sin t\,\vec{\jmath} = -\vec{R}$$

The acceleration, then, is always directed toward the centre of the ellipse.

(c) At $t_1 = \pi/6$,

$$\vec{R}_1 = \frac{3\sqrt{3}}{2}\vec{\imath} + \vec{\jmath}$$
$$\vec{v}_1 = -\frac{3}{2}\vec{\imath} + \vec{\jmath}$$
$$\vec{a}_1 = -\frac{3\sqrt{3}}{2}\vec{\imath} - \vec{\jmath}$$

At $t_2 = \pi$,

$$\vec{R}_2 = -3\vec{\imath}$$
$$\vec{v}_2 = -2\vec{\jmath}$$
$$\vec{a}_2 = 3\vec{\imath}$$

(d) At $t_1 = \pi/6$,

$$|\vec{v}_1| = \sqrt{\frac{9}{4} + 3} = \frac{\sqrt{21}}{2}$$
$$|\vec{a}_1| = \sqrt{\frac{27}{4} + 1} = \frac{\sqrt{31}}{2}$$

At $t_2 = \pi$,

$$|\vec{v}_2| = \sqrt{0 + 4} = 2$$
$$|\vec{a}_2| = \sqrt{9 + 0} = 3$$

(e) For the speed $|\ \vec{v}\ |$ at any time t

$$|\ \vec{v}\ | = \sqrt{9\,sin^2\,t + 4\,cos^2\,t}$$
$$= \sqrt{4 + 5\,sin^2\,t}$$

We see immediately that the speed is a maximum when $t = \pi/2$ or $3\pi/2$, and a minimum when $t = 0$ or π. Generally one can determine maximum or minimum speed by finding $\frac{d|\ \vec{v}\ |}{dt}$, setting it equal to zero, and applying the usual tests to sort out values of t that yield maximum or minimum speeds.

3.8 Related Rates

In this section we are going to look at an application of implicit differentiation. Most of the applications of derivatives are in the next chapter however there are a couple of reasons for placing it in this chapter as opposed to putting it into the next chapter with the other applications. The first reason is that it's an application of implicit differentiation and so putting right after that section means that we won't have forgotten how to do implicit differentiation. The other reason is simply that after doing all these derivatives we need to be reminded that there really are actual applications to derivatives. Sometimes it is easy to forget there really is a reason that we're spending all this time on derivatives.

For these related rates problems it's usually best to just jump right into some problems and see how they work.

Example 3.8-1 Air is being pumped into a spherical balloon at a rate of 5 cm^3/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

Solution

The first thing that we'll need to do here is to identify what information that we've been given and what we want to find. Before we do that let's notice that both the volume of the balloon and the radius of the balloon will vary with time and so are really functions of time, *i.e.* $V(t)$ and $r(t)$.

We know that air is being pumped into the balloon at a rate of 5 cm^3/min. This is the rate at which the volume is increasing. Recall that rates of change are nothing more than derivatives and so we know that,

$$V'(t) = 5$$

We want to determine the rate at which the radius is changing. Again, rates are derivatives and so it looks like we want to determine,

$$r'(t) = ? \qquad \text{when} \quad r(t) = \frac{d}{2} = 10\,\text{cm}$$

Note that we needed to convert the diameter to a radius.

Now that we've identified what we have been given and what we want to find we need to relate these two quantities to each other. In this case we can relate the volume and the radius with the formula for the volume of a sphere.

$$V(t) = \frac{4}{3}\pi\left[r(t)\right]^3$$

As in the previous section when we looked at implicit differentiation, we will typically not use the (t) part of things in the formulas, but since this is the first time through one of these we will do that to remind ourselves that they are really functions of t.

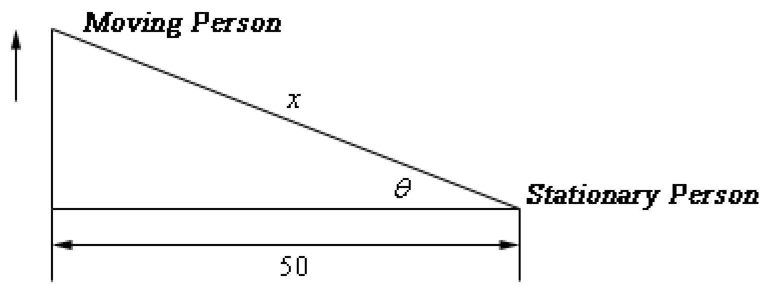
Now we don't really want a relationship between the volume and the radius. What we really want is a relationship between their derivatives. We can do this by differentiating both sides with respect to t. In other words, we will need to do implicit differentiation on the above formula. Doing this gives,

$$V' = 4\pi r^2 r'$$

Note that at this point we went ahead and dropped the (t) from each of the terms. Now all that we need to do is plug in what we know and solve for what we want to find.

$$5 = 4\pi\left(10^2\right)r' \qquad \Rightarrow \qquad r' = \frac{1}{80\pi}\ \text{cm/min}$$

Example 3.8-2 Two people are 50 feet apart. One of them starts walking north at a rate so that the angle shown in the diagram below is changing at a constant rate of 0.01 rad/min. At what rate is distance between the two people changing when θ = 0.5 radians?



Solution

This example is not as tricky as it might at first appear. Let's call the distance between them at any point in time *x* as noted above. We can then relate all the known quantities by one of two trig formulas.

$$\cos\theta = \frac{50}{x} \qquad \sec\theta = \frac{x}{50}$$

We want to find x' and we could find *x* if we wanted to at the point in question using cosine since we also know the angle at that point in time. However, if we use the second formula we won't need to know *x* as you'll see. So, let's differentiate that formula.

$$\sec\theta\tan\theta\ \theta' = \frac{x'}{50}$$

As noted, there are no *x*'s in this formula. We want to determine x' and we know that θ = 0.5 and θ = 0.01 (do you agree with it being positive?). So, just plug in and solve.

$$(50)(0.01)\sec(0.5)\tan(0.5) = x' \qquad \Rightarrow \qquad x' = 0.311254\ \text{ft/min}$$

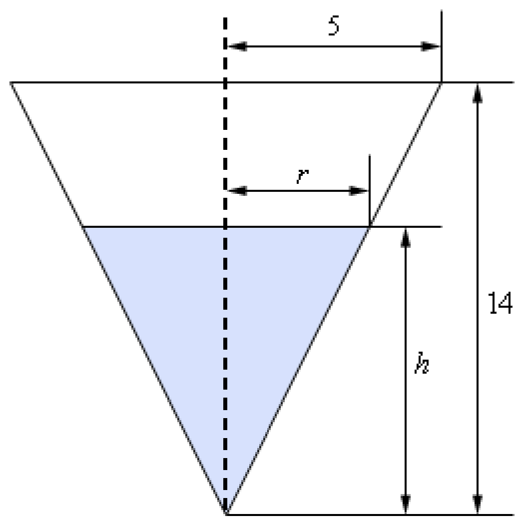
Example 3.8-3 A tank of water in the shape of a cone is leaking water at a constant rate of $2\text{ft}^3/\text{hour}$. The base radius of the tank is 5 ft and the height of the tank is 14 ft.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?

Solution

Okay, we should probably start off with a quick sketch (probably not to scale) of what is going on here.



As we can see, the water in the tank actually forms a smaller cone with the same central angle as the tank itself. The radius of the "water" cone at any time is given by *r* and the height of the "water" cone at any time is given by *h*. The volume of water in the tank at any time *t* is given by,

$$V = \frac{1}{3}\pi r^2 h$$

and we've been given that $V' = -2$.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?

For this part we need to determine h' when $h = 6$ and now we have a problem. The only formula that we've got that will relate the volume to the height also includes the radius and so if we were to differentiate this with respect to t we would get,

$$V' = \frac{2}{3}\pi r r' h + \frac{1}{3}\pi r^2 h'$$

So, in this equation we know V' and h and want to find h', but we don't know r and r'. As we'll see finding r isn't too bad, but we just don't have enough information, at this point, that will allow us to find r' and h' simultaneously.

To fix this we'll need to eliminate the r from the volume formula in some way. This is actually easier than it might at first look. If we go back to our sketch above and look at just the right half of the tank we see that we have two similar triangles and when we say similar we mean similar in the geometric sense. Recall that two triangles are called similar if their angles are identical, which is the case here. When we have two similar triangles then ratios of any two sides will be equal. For our set this means that we have,

$$\frac{r}{h} = \frac{5}{14} \qquad \Rightarrow \qquad r = \frac{5}{14}h$$

If we take this and plug it into our volume formula we have,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5}{14}h\right)^2 h = \frac{25}{588}\pi h^3$$

This gives us a volume formula that only involved the volume and the height of the water. Note however that this volume formula is only valid for our cone, so don't be tempted to use it for other cones! If we now differentiate this we have,

$$V' = \frac{25}{196}\pi h^2 h'$$

At this point all we need to do is plug in what we know and solve for h'.

$$-2 = \frac{25}{196}\pi\left(6^2\right)h' \qquad \Rightarrow \qquad h' = \frac{-98}{225\pi} = -0.1386$$

So, it looks like the height is decreasing at a rate of 0.1386 ft/hr.

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?

In this case we are asking for r' and there is an easy way to do this part and a difficult (well, more difficult than the easy way anyway….) way to do it. The "difficult" way is to redo the work part (a) above only this time use,

$$\frac{h}{r} = \frac{14}{5} \qquad \Rightarrow \qquad h = \frac{14}{5}r$$

to get the volume in terms of *V* and *r* and then proceed as before.

That's not terribly difficult, but it is more work that we need to so. Recall from the first part that we have,

$$r = \frac{5}{14}h \qquad \Rightarrow \qquad r' = \frac{5}{14}h'$$

So, as we can see if we take the relationship that relates *r* and *h* that we used in the first part and differentiate it we get a relationship between r' and h'. At this point all we need to do here is use the result from the first part to get,

$$r' = \frac{5}{14}\left(\frac{-98}{225\pi}\right) = -\frac{7}{45\pi} = -0.04951$$

Much easier that redoing all of the first part. Note however, that we were only able to do this the "easier" way because it was asking for r'at exactly the same time that we asked for h' in the first part. If we hadn't been using the same time then we would have had no choice but to do this the "difficult" way.

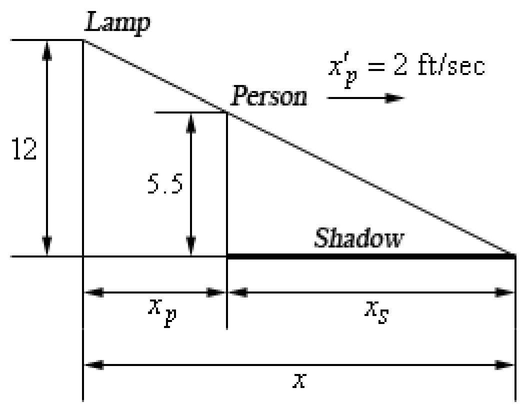
Example 3.8-4 A light is on the top of a 12 ft tall pole and a 5ft 6in tall person is walking away from the pole at a rate of 2 ft/sec.

(a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?

(b) At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?

Solution

We'll definitely need a sketch of this situation to get us started here. The tip of the shadow is defined by the rays of light just getting past the person and so we can form the following set of similar triangles.



Here x is the distance of the tip of the shadow from the pole, x_p is the distance of the person from the pole and x_s is the length of the shadow. Also note that we converted the persons height over to 5.5 feet since all the other measurements are in feet.

(a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?

In this case we want to determine x' when $x_p = 25$ given that $x'_p = 2$.

The equation we'll need here is,

$$x = x_p + x_s$$

but we'll need to eliminate x_s from the equation in order to get an answer. To do this we can again make use of the fact that the two triangles are similar to get,

$$\frac{5.5}{12} = \frac{x_s}{x} = \frac{x_s}{x_p + x_s} \qquad \text{Note : } \frac{5.5}{12} = \frac{\frac{11}{2}}{12} = \frac{11}{24}$$

We'll need to solve this for x_s.

$$\begin{aligned} \frac{11}{24}\left(x_p + x_s\right) &= x_s \\ \frac{11}{24}x_p &= \frac{13}{24}x_s \\ \frac{11}{13}x_p &= x_s \end{aligned}$$

Our equation then becomes,

$$x = x_p + \frac{11}{13}x_p = \frac{24}{13}x_p$$

Now all that we need to do is differentiate this, plug in and solve for x'.

$$x' = \frac{24}{13}x'_p \qquad \Rightarrow \qquad x' = \frac{24}{13}(2) = 3.6923 \text{ ft/sec}$$

The tip of the shadow is then moving away from the pole at a rate of 3.6923 ft/sec. Notice as well that we never actually had to use the fact that $x_p = 25$ for this problem. That will happen on rare occasions.

(b) At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?

This part is actually quite simple if we have the answer from (a) in hand, which we do of course. In this case we know that x_s represents the length of the shadow, or the distance of the tip of the shadow from the person so it looks like we want to determine x'_s when $x_p = 25$.

Again, we can use $x = x_p + x_s$, however unlike the first part we now know that $x'_p = 2$ and $x' = 3.6923$ ft/sec so in this case all we need to do is differentiate the equation and plug in for all the known quantities.

$$x' = x'_p + x'_s$$
$$3.6923 = 2 + x'_s \qquad\qquad x'_s = 1.6923 \text{ ft/sec}$$

The tip of the shadow is then moving away from the person at a rate of 1.6923 ft/sec.

3.9 Slope of a Polar Curve

We now need to discuss some calculus topics in terms of polar coordinates.

We will start with finding tangent lines to polar curves. In this case we are going to assume that the equation is in the form $r = f(x)$ With the equation in this form we can actually use the equation for the derivative $\frac{dy}{dx}$ we derived when we looked at tangent lines with parametric equations. To do this however requires us to come up with a set of parametric equations to represent the curve. This is actually pretty easy to do.

From our work in the previous section we have the following set of conversion equations for going from polar coordinates to Cartesian coordinates.

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

Now, we'll use the fact that we're assuming that the equation is in the form $r = f(x)$. Substituting this into these equations gives the following set of parametric equations (with θ as the parameter) for the curve.

$$x = f(\theta)\cos\theta \qquad\qquad y = f(\theta)\sin\theta$$

Now, we will need the following derivatives.

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \qquad\qquad \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$= \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad\qquad = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

The derivative $\frac{dy}{dx}$ is then,

Derivative with Polar Coordinates

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Note that rather than trying to remember this formula it would probably be easier to remember how we derived it and just remember the formula for parametric equations.

Let's work a quick example with this.

Example 3.9-1 Determine the equation of the tangent line to $r = 3 + 8\sin\theta$ at $\theta = \pi/6$.

Solution

We'll first need the following derivative.

$$\frac{dr}{d\theta} = 8\cos\theta$$

The formula for the derivative $\frac{dy}{dx}$ becomes,

$$\frac{dy}{dx} = \frac{8\cos\theta\sin\theta + (3 + 8\sin\theta)\cos\theta}{8\cos^2\theta - (3 + 8\sin\theta)\sin\theta} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta}$$

The slope of the tangent line is,

$$m = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{4\sqrt{3}+\frac{3\sqrt{3}}{2}}{4-\frac{3}{2}} = \frac{11\sqrt{3}}{5}$$

Now, at $\theta = \pi/6$ we have $r = 7$. We'll need to get the corresponding *x*-*y* coordinates so we can get the tangent line.

$$x = 7\cos\left(\frac{\pi}{6}\right) = \frac{7\sqrt{3}}{2} \qquad y = 7\sin\left(\frac{\pi}{6}\right) = \frac{7}{2}$$

The tangent line is then,

$$y = \frac{7}{2} + \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

For the sake of completeness here is a graph of the curve and the tangent line.