AP Calculus Lesson Ten Notes

Chapter 5 Applications of the Definite Integral and Polar Coordinates

5.1 Area and Solids of Revolution

Applications of the Definite Integral

The definite integral is useful for solving a large variety of applied problems. In this chapter we shall discuss area, volume, work, liquid force, lengths of curves, and some problems from economics and biology. Other applications will be considered later in the text.

5.1 Area and Solids of Revolution

Area

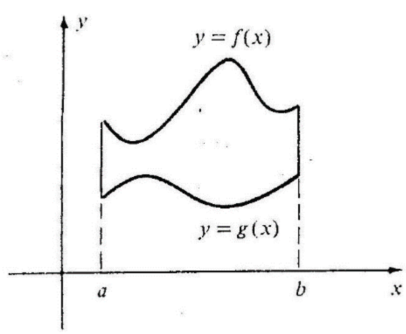

FIGURE 6.1

If a function *f* is continuous on a closed interval [*a*,*b*], and $f(X) \geq 0$ for all *x* in [*a*,*b*], then by Theorem (4.3.7) the area under the graph of *f* from *a* to *b* equals $\int_a^b f(x)dx$. If *g* is another continuous nonnegative valued function on [*a*,*b*], and if $f(x) \geq g(x)$ for all *x* in [*a*,*b*], then the area *A* of the region bounded by the graphs of *f*,*g*,*x*=*a* and *x*=*b* (see Figure 6.1) can be found by subtracting the area under the graph of *g* from the area under the graph of *f*, that is

$$A = \int_a^b f(x)dz - \int_a^b g(x)dx = \int_a^b \left[f(x) - g(x) \right] dx$$

This formula for *A* can be extended to the case in which *f* or *g* is negative for some *x* in [*a*.*b*], as illustrated in (i) of Figure 6.2. to find the area of the indicated region, let us choose a negative number *d* less than the minimum value of *g* on [*a*,*b*] and consider the two functions f_1 and g_1 defined by

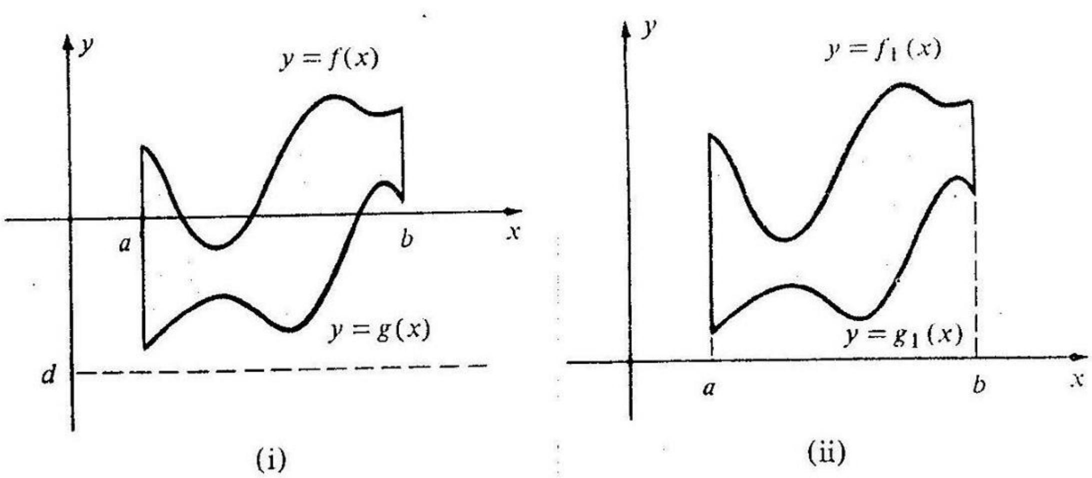

FIGURE 6.2.

$$f_1(x) = f(x) - d\,,\;\; g(x) = g(x) - d$$

For all x in [a,b]. Since d is negative, values of f_1 and g_1 are found by adding the positive number |d| to corresponding values of f and g, respectively. Geometrically, this amounts to raising the graphs of f and g a distance |d| giving us a region having the same shape as the original region. But lying entirely above the x-axis.

If A is the area of the region in (ii) of Figure 6.2, then

$$\begin{aligned} A &= \int_a^b \left[f_1(x) - g_1(x) \right] dx \\ &= \int_a^b \left\{ \left[f(x) - d \right] - \left[g(x) - d \right] \right\} dx \\ &= \int_a^b \left[f(x) - g(x) \right] dx \end{aligned}$$

The preceding discussion may be summarized as follows.

Theorem (5.1.1)

If f and g are continuous and $f(x) \geq g(x)$ for all x in [a,b], then the area A of the region bounded by the graphs of f, g ,x=a, and x=b is

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

The formula for A in Theorem (5.1.1) may be interpreted as a limit of a sum. If we define the function h by $h(x)=f(x)-g(x)$,and if w is in [a,b], then $h(w)$ is the distance from the point on the graph of g with x-coordinate w to the point on the graph of f with x-coordinate w. as in the discussion of Riemann sums in chapter 5, let P denote a partition of [a,b]determined by the numbers $a = x_0, x_1, ..., x_n = b$. For each I, let $\Delta x_i = x_i - x_{i-1}$, and let w_i be an arbitrary number in the ith subinterval [x_{i-1}, x_i] of P. by the definition of h.

$$h(w_i)\Delta x_i = \left[f(w_i) - g(w_i) \right] \Delta x_i$$

Which, in geometric terms, is the area of a rectangle of length $f(w_i) - g(w_i)$ and width Δx_i (see Figure 6.3)

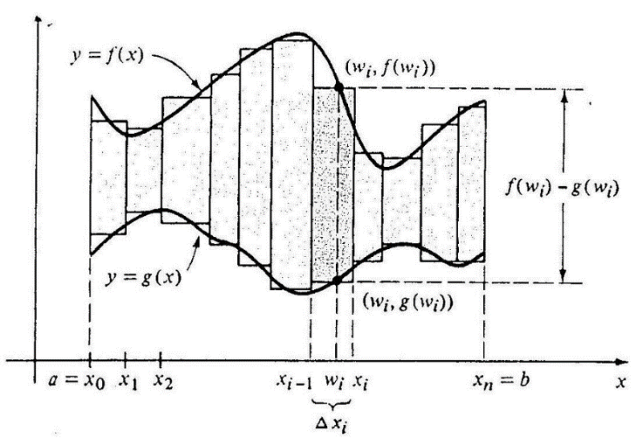

FIGURE 6.3

The Riemann sum

$$\sum_i h(w_i)\Delta x_i = \sum_i \left[f(w_i) - g(w_i) \right] \Delta x_i$$

Is the sum of the areas of all rectangles pictured in Figure 6.3, and may be thought of as an approximation to the area of the region bounded by the graphs of f and g from x=a to x=b. by the definition of the definite integral.

$$\lim_{\|p\| \to 0} \sum_i h(w_i)\Delta x_i = \int_a^b h(x)dx$$

Using the definition of h gives us the following.

Theorem (5.1.2)

If *f* and *g* are continuous and $f(x) \ge g(x)$ for all *x* in [*a*,*b*], then the area *A* of the region bounded by the graphs of *f, g ,x=a,* and *x=b* is

$$A = \lim_{\|p\| \to 0} \sum_i \left[f(w_i) - g(w_i) \right] \Delta x_i = \int_a^b \left[f(x) - g(x) \right] dx$$

When we use Theorem (5.1.2) we initially think of approximating the region by means of rectangles of the type shown in Figure 6.3. After writing a formula for the area of a typical rectangle, we sum all such rectangles and, as in (5.1.2), take the limit of this sum to obtain the area of the region. This technique is illustrated in the following examples.

Example 1

Find the area of the region bounded by the graphs of the equations $y = x^2$ and $y = \sqrt{x}$.

Solution

We shall employ the Riemann sum approach. The region and a typical rectangle are sketched in Figure 6.4.

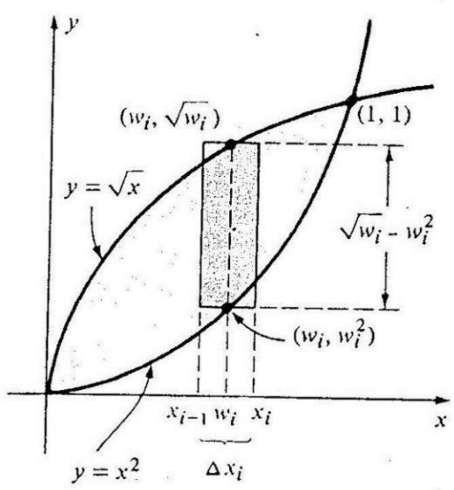

FIGURE 6.4

As indicated in the figure, the length of a typical rectangle is $\sqrt{w_i} - w_i^2$ and its area is $(\sqrt{w_i} - w_i^2)\Delta x_i$. Using Theorem (5.1.2) with *a=0* and *b=1*, we obtain

$$A = \lim_{\|p\| \to 0} \sum_i (\sqrt{w_i} - w_i^2)\Delta x_i$$

$$= \int_0^1 (\sqrt{x} - x^2)dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

The area can also be found by direct substitution in Theorem (5.1.1), with $f(x) = \sqrt{x}$ and $g(x) = x^2$.

As illustrated in Example 1, to find areas by using limits of sums, we always begin by sketching the region with at least one typical rectangle labeling the drawing appropriately. The reason for stressing the summation technique is that similar limiting processes will be employed later for calculating many other mathematical and physical quantities. Treating areas as limits of sums will make it easier to understand those future applications. At the same time it will help solidify the meaning of the definite integral.

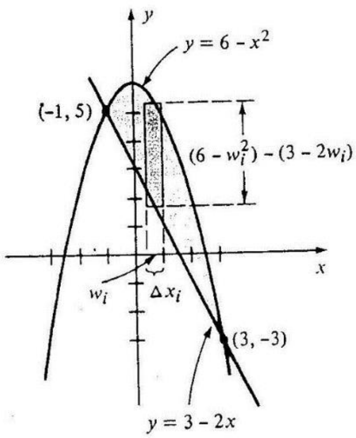

FIGURE 6.5

Example 2

Find the area of the region bounded by the graphs of $y + x^2 = 6$ and $y + 2x - 3 = 0$.

Solution

The region and a typical rectangle are sketched in Figure 6.5. The points of intersection (-1, 5) and (3, -3) of the two graphs may be denotes a number in the *i*th subinterval of a partition. Those who wish to use the nonsubscript approach may (at the discretion of the instructor) proceed immediately to the definite integral which follows a stated limit of a sum.

The following example illustrates that it is sometimes necessary to subdivide a region and use Theorem (5.1.1) or (5.1.2) more than once in order to find the area.

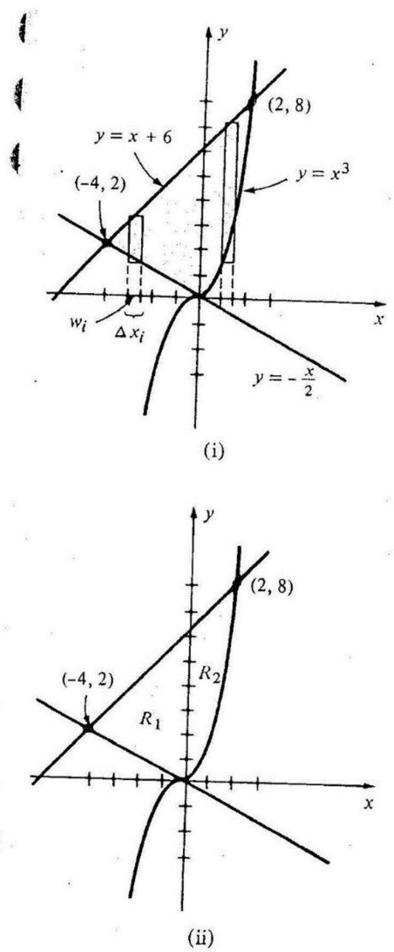

FIGURE 6.6

Example 3

Find the area of the region *R* that is bounded by the graphs of $y - x = 6, y - x^3 = 0,$ and $2y + x = 0$.

Solution

The graphs and region are sketched in (i) of Figure 6.6 where, as indicated, each equation has been solved for *y* in terms of *x* so that appropriate functions may be introduced. Typical rectangles are shown extending from the lower boundary to the upper boundary of *R*. since the lower boundary consists of portions of two different graphs, the area cannot be found by using only one definite integral. However, if *R* is divided into two subregions R_1 and R_2, as shown in (ii) of Figure 6.6, then we can determine the area of each and add them together.

For region R_1 the upper and lower boundaries are the lines $y = x + 6$ and $y = -x/2$, respectively. and hence the length of a typical rectangle is

$$(w_i + 6) - (-w_i / 2)$$

Applying Theorem (5.1.2), the area A_i of R_i is

$$\begin{aligned}
A_1 &= \lim_{\|P\|\to 0} \sum_i \left[(w_i + 6) - \left(-\frac{w_i}{2} \right) \right] \Delta x_i \\
&= \int_{-4}^{0} \left[(x+6) - \left(-\frac{w_i}{2} \right) \right] dx \\
&= \int_{-4}^{0} \left[\frac{3}{2}x + 6 \right] dx = \left[\frac{3}{4}x^2 + 6x \right]_{-4}^{0} \\
&= 0 - (12 - 24) = 12
\end{aligned}$$

Region R_2 has the same upper boundary $y = x^3$. In this case the length of a typical rectangle is

$$(w_i + 6) - w_i^3$$

And the area A_2 of R_2 is

$$\begin{aligned}
A_2 &= \lim_{\|P\|\to 0} \sum_i \left[(w_i + 6) - w_i^3 \right] \Delta x_i \\
&= \int_{0}^{2} \left[(x+6) - x^3 \right] dx \\
&= \left[\frac{x^2}{2} + 6x - \frac{x^4}{4} \right]_{0}^{2} \\
&= (2 + 12 - 4) - 0 = 10
\end{aligned}$$

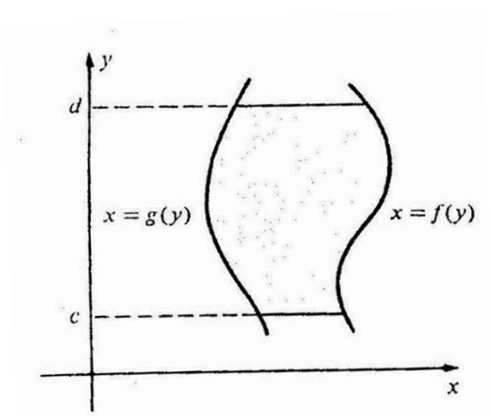

FIGURE 6.7

The area of the entire region *R* is, therefore,

$$A_1 + A_2 = 12 + 10 = 22$$

Sometimes it is necessary to find the area *A* of the region bounded by the graphs of *y=c* and *y=d*, and of two equations of the form *x=f(y)* and *x=g(y)*, where *f* and *g* are continuous functions and $f(y) \geq g(y)$ for all *y* in [*c*,*d*] (see Figure 6.7). in a manner similar to our earlier discussion, but with the roles of *x* and *y* interchanged, we obtain the formula

$$A = \int_c^b \left[f(y) - g(y) \right] dy$$

Where we now regard *y* as the independent variable. We shall refer to this technique as **integration with respect to *y***, whereas Theorem (5.1.1) is called **integration with respect to *x*.**

Summation techniques can also be applied to regions of the type shown in Figure 6.7. In this case we select points on the *y*-axis with *y*-coordinates $y_0, y_1, ..., y_n$ where $y_0 = c$ and $y_n = d$, thereby obtaining a partition of the interval [*c*,*d*]into subintervals of width $\Delta y_i = y_i - y_{i-1}$. For each *I* we choose a number w_i in [y_{i-1}, y_i] and consider rectangles that have areas $\left[f(w_i) - g(w_i) \right] \Delta y_i$, as illustrated in Figure 6.8. This leads to

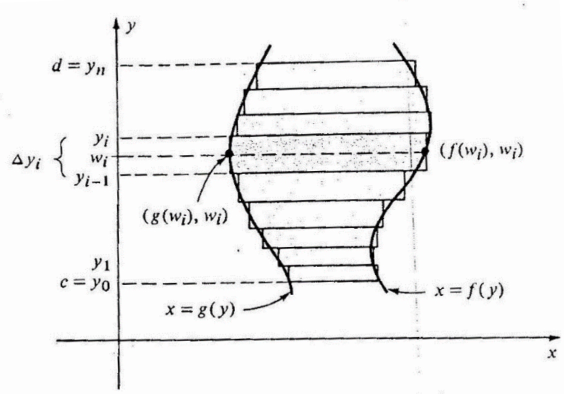

FIGURE 6.8

$$A = \lim_{\|P\| \to 0} \sum_i \left[f(w_i) - g(w_i) \right] \Delta y_i = \int_c^b \left[f(y) - g(y) \right] dy$$

Where the last equality follows form the definition of the definite integral.

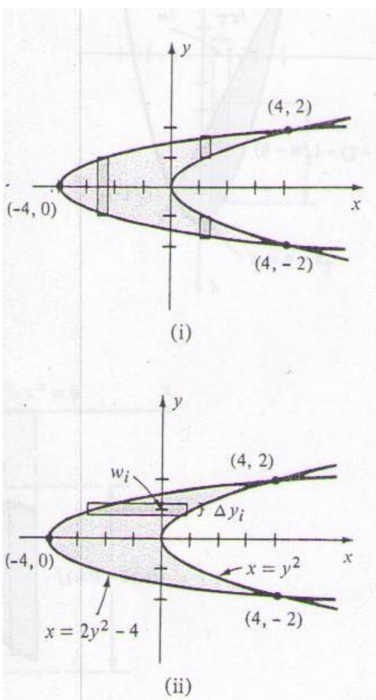

FIGURE 6.9

Example 4
Find the area of the region bounded by the graphs of the equations $2y^2 = x + 4$ and $x = y^2$

Solution
Two sketches of the region are shown in Figure 6.9, where (i) illustrates the situation that occurs if we use vertical rectangles (integration with respect to *x*), and (ii) is the case if we use horizontal rectangles (integration with respect to *y*).

Referring to (i) of the figure we see that several definite integrals are required to find the area. (Why?) However, in (ii) we can use integration with respect to *y* to find the area with only one integration. Letting $f(y) = y^2$, $g(y) = 2y^2 - 4$, and referring to (ii) of Figure 6.9, the length $f(w_i) - g(w_i)$ of a *horizontal* rectangle is

$$w_i^2 - (2w_i^2 - 4).$$

Since the width is Δy_i, the area of the rectangle is

$$\left[w_i^2 - \left(2w_i^2 - 4 \right) \right] \Delta y_i$$

Hence, the area of *R* is

$$A_2 = \lim_{\|P\|\to 0} \sum_i \left[w_i^2 - \left(2w_i^2 - 4\right)\right]\Delta y_i$$
$$= \int_{-2}^{2} \left[y^2 - (2y^2 - 4)\right] dy$$
$$= \int_{-2}^{2} \left(4 - y^2\right) dy$$
$$= \left[4y - \frac{y^3}{3}\right]_{-2}^{2} = \left[8 - \frac{8}{3}\right] - \left[-8 - \left(-\frac{8}{3}\right)\right] = \frac{32}{3}$$

Actually, the integration could have been further simplified. For example, since the *x*-axis bisects the region, it is sufficient to find the area of that part of the region that lies above the *x*-axis and double it, obtaining

$$A = 2\int_{0}^{2} \left[y^2 - (2y^2 - 4)\right] dy .$$

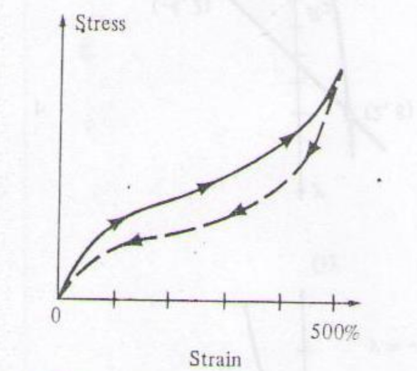

FIGURE 6.10 Stress-strain diagram for an elastic material

Throughout this section we have assumed that the graphs of the functions (or equations) do not cross one another in the interval under discussion. For example, in Theorem (6.1) we demanded that $f(x) \geq g(x)$ for all *x* in [*a*,*b*]. if the graph of *f* and *g* cross at one point $P(c,d)$, where *a*<*c*<*b*, and we wish to find the area bounded by the graphs from *x*=*a* to *x*=*b*, then the theory developed in this section may still be used; however, *two* integrations are required, one corresponding to the interval [*a*,*c*]and the other to [*c*,*b*]. if the graphs cross *several* times, then several integrals may be necessary. Problems whose graphs cross one or more times appear in Exercises 33-36

In scientific investigations, physical interpretations are often attached to areas. One illustration of this occurs in the *theory of elasticity* where, to test the strength of a material, an investigator records values of strain that correspond to different loads (stresses). The ketch in Figure 6.10 is a typical stress-strain diagram for a sample of an elastic material such as vulcanized rubber. Note that it is customary to assign stress values in the vertical direction.

Referring to the figure, we see that as the load applied to the material (the stress) was increased, the strain (indicated by the arrows on the solid part of the graph) increased until the material stretched to five times its original length. As the load was decreased, the elastic material returned to its origin al length; however, the same graph was not retraced. Instead, the graph indicated by the dashes was obtained. This phenomenon is called *elastic hysteresis*.

Solids of Revolution

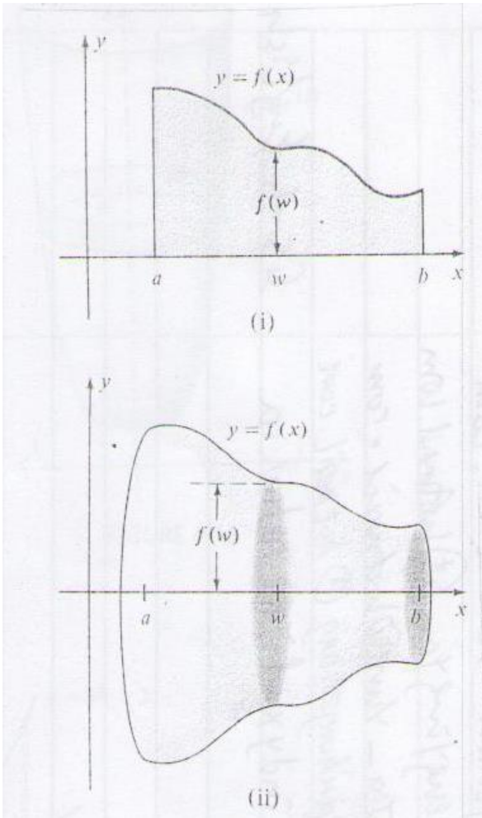

FIGURE 6.11

The volume of an object plays an important role in many problems that arise in the physical sciences. For example, it is essential to know the volume in order to find the center of gravity or moment of inertia of a homogeneous solid. (These concepts will be discussed later in the text.) The task of determining the volume of an irregularly shaped object is often difficult, if not impossible. For this reason, we shall begin with objects that have simple shapes. Included in this category are the solids of revolution discussed here and in the next section.

If a region in a plane is revolved about a line in the plane, the resulting solid is called a **solid of revolution,** and the solid is said to be **generated** by the region. The line about which the revolution takes place is called an **axis of revolution.** If the region bounded by the graph of a continuous nonnegative valued function *f*, the *x*-axis, and the graphs of $x=a$ and $x=b$ (see (i) of Figure 6.11) is revolved about the *x*-axis, a solid of the type shown in (ii) of Figure 6.11 is generated. For example, if *f* is a constant function, then the region is rectangular and the solid generated is a right circular cylinder. If the graph of *f* is a semicircle with endpoint of a diameter at the points $(a,0)$ and $(b,0)$ where $b>a$, then the solid of revolution is a sphere with diameter $b-a$. if the given region is a right triangle with the right angle at one of these points, then a right circular cone is generated.

If a plane perpendicular to the *x*-axis intersects the solid shown in ii of Figure 6.11, a circular cross section is obtained. If, as indicated in the figure, the plane passed through the point on the ix-*axis with ix*-coordinate *w*, then the radius of the circle is *f(w)* and hence its area is $\pi\left[f(w)\right]^2$. We shall arrive at a definition for the volume of such a solid of revolution by using Riemann sums in a manner similar to that used for areas in the previous section.

Suppose *f* is continuous and $f(x) \geq 0$ for all *x* in [*a*,*b*]. Consider a Riemann sum $\sum_i f(w_i)\Delta x_i$, where w_i is any number in the *i*th subinterval $\left[x_{i-1}, x_i\right]$ of a partition *P* of [*a*,*b*]. Geometrically, this gives us a sum of areas of rectangles of the types shown in (i) of Figure 6.12. the solid generated by the polygon formed by these rectangles has the appearance shown in ii of the figure. Observe that the *i*th rectangle generates a circular disc (that is, a “flat” right circular cylinder) of base radius $f(w_i)$ and altitude, or “thickness,” $\Delta x_i = x_i - x_{i-1}$. The volume of this disc is the area of the base times the altitude, that is, $\pi\left[f(w_i)\right]^2 \Delta w_i$. The sum of the volumes of all such discs is the volume of the solid shown in (ii) of Figure 6.12 and is given by $\sum_i \pi\left[f(w_i)\right]^2 \Delta w_i$.

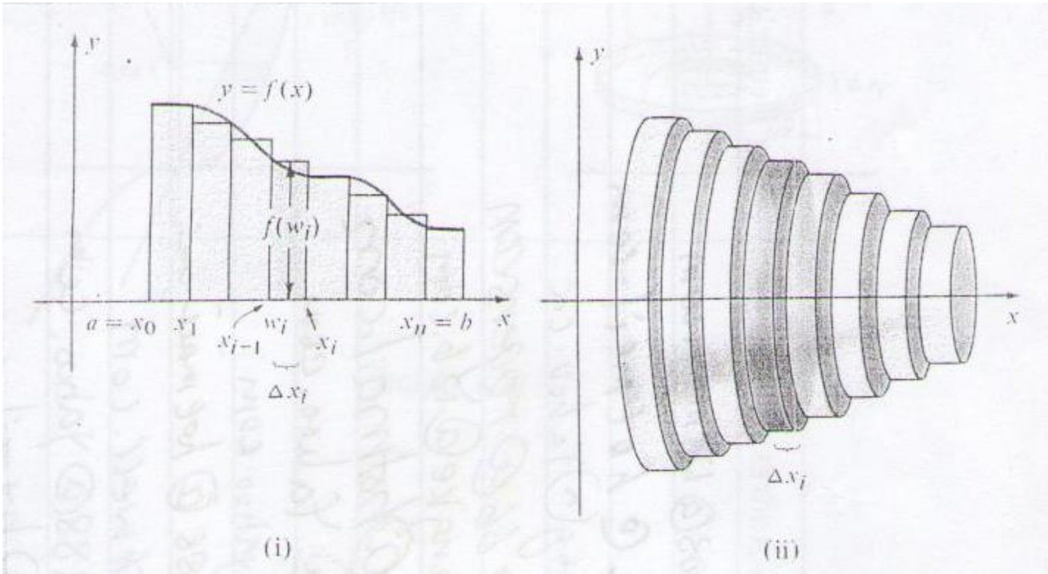

FIGURE 6.12

The sum may be regarded as a Riemann sum for the function *h* defined by $h(x)=\pi\left[f(x)\right]^2$. Intuitively, it appears that if ||P|| is close to zero, then the sum is close to the volume of the solid. It is natural, therefore, to define the volume of revolution as the limit of this sum as follows.

Definition (5.1.3)

Let *f* be continuous on [*a*,*b*]. the **volume** *V* of the solid of revolution generated by revolving the region bounded by the graphs of *f*, *x*=*a*, *x*=*b*, and the *x*-axis about the *x*-axis is

$$V=\lim_{\|P\|\to 0}\sum_i \pi\left[f(w_i)\right]^2\Delta x_i=\int_a^b \pi\left[f(x)\right]^2 dx$$

The fact that the limit of the sum in Definition (5.1.3) equals $\int_a^b \pi\left[f(x)\right]^2 dx$ follows from the definition of the definite integral. Hereafter, when considering limits of Riemann sums, the meanings of all symbols will not be explicitly pointed out. Instead, it will be assumed that the reader is aware of the significance of symbols such as $\|P\|, w_i, and\, \Delta x_i$.

The requirement that $f(x)\geq 0$ for all *x* in [*a*,*b*]was omitted in Definition (5.1.3). if *f* is negative for some *x*, as illustrated in (i) of Figure 6.13, and if the region bounded by the graphs of *f, x*=*a, x*=*b*, and the *x*-axis is revolved about the *x*-axis, a solid of the type shown in (ii) of the figure is obtained. This solid is the same as that generated by revolving the region under the graph of *y*=|*f(x)*| from *a* to *b* about the *x*-axis.

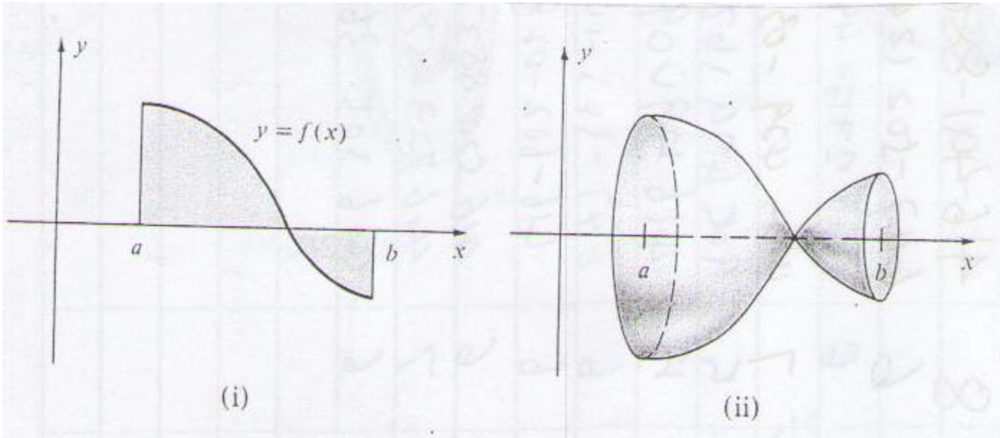

FIGURE 6.13

Since $|f(x)|^2 = [f(x)]^2$, the limit in Definition (5.1.3) is the desired volume.

Example 1

If $f(x) = x^2 + 1$, find the volume of the solid generated by revolving the region under the graph of *f* from -1 to 1 about the *x*-axis.

Solution

The solid is illustrated in Figure 6.14. Included in the sketch is a typical rectangle and the disc that it generates. Since the radius of the disc is $w_i^2 + 1$, its volume is

$$\pi(w_i^2 + 1)^2 \Delta x_i$$

And, as in (5.1.3),

$$\begin{aligned} V &= \lim_{\|P\|\to 0} \sum_i \pi\left(w_i^2 + 1\right)^2 \Delta x_i \\ &= \int_{-1}^{1} \pi(x^2+1)^2\,dx = \pi\int_{-1}^{1}(x^4 + 2x^2 + 1)dx \\ &= \pi\left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_{-1}^{1} \\ &= \pi\left[\left(\frac{1}{5}+\frac{2}{3}+1\right)-\left(-\frac{1}{5}-\frac{2}{3}-1\right)\right] = \frac{56}{15}\pi \end{aligned}$$

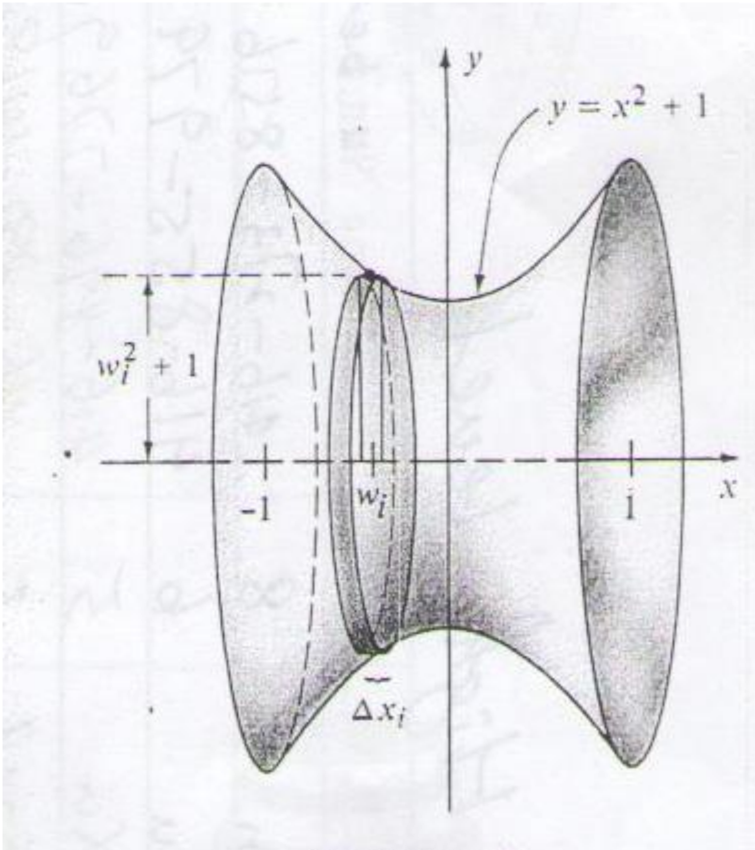

FIGURE 6.14

c If we had used the symmetry of the graph in Example 1, then the volume could have been calculated by integrating from 0 to 1 and doubling the result. Another solution consists of substitution of x^2+1 for $f(x)$ in the formula $V=\int_a^b \pi\left[f(x)\right]^2 dx$. We shall not ordinarily specify the units of measure. If the unit of linear measurement is inches, the volume is expressed in cubic inches. If *x* is in cm, then *V* is in cm^3, etc.

Sometimes it is convenient to find volumes by integrating with respect to *y*. for example, consider a region bounded by horizontal lines with *y*-intercepts *c* and *d*, by the *y*-axis, and by the graph of *x=g(y)*, where the function *g* is continuous for all *y* in [*c*,*d*]. if this region is revolved about the *y*-axis, the volume *V* of the resulting solid may be found by interchanging the roles of *x* and *y* in Definition (5.1.3). Specifically, let *P* be the partition of the interval [*c*,*d*] determined by the numbers $c=y_0, y_1, ..., y_n = d$. Let $\Delta y_i = y_i - y_{i-1}$, let w_i be any number in the ith subinterval, and consider the rectangles of length $g(w_i)$ and width Δy_i as illustrated in (i) of Figure 6.15. The solid generated by revolving these rectangles about the *y*-axis is illustrated in (ii) of the figure.

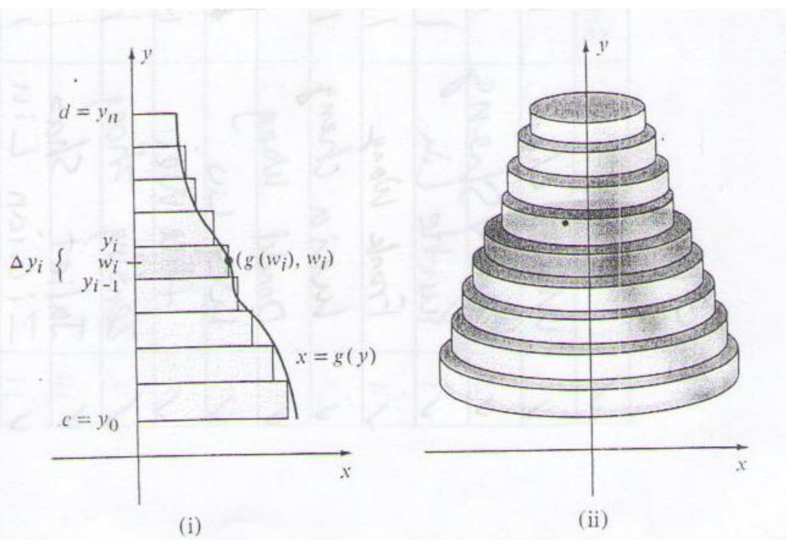

FIGURE 6.15

The volume of the disc generated by the ith rectangle is $\pi\left[g(w_i)\right]^2 \Delta y_i$. Summing and taking the limit gives us the following analogue of Definition (6.3).

Definition (5.1.4)

Let g be continuous on [c,d]. The volume V of the solid of revolution generated by revolving the region bounded by the graphs of $x=g(y)$, $y=c$, $y=d$, and the *y*-axis about the *y*-axis is

$$V = \lim_{\|P\|\to 0} \sum_i \pi \left[g(w_i) \right]^2 \Delta y_i = \int_c^d \pi \left[g(y) \right]^2 dy$$

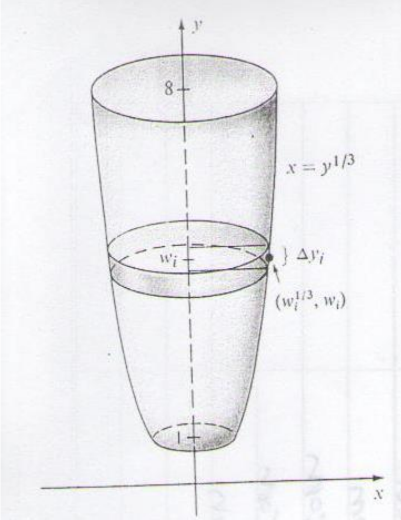

FIGURE 6.16

Example 2

The region bounded by the *y*-axis and the graphs of $y = x^3$, $y=1$, and $y=8$ is revolved about the *y*-axis. Find the volume o the resulting slid.

Solution

The solid is sketched in Figure 6.16 together with a disc generated by a typical horizontal rectangle. Since we plan to integrate with respect to *y*, we solve the equation $y = x^3$ for *x* in terms of *y*, obtaining $x = y^{1/3}$. If we let $x = g(y) = y^{1/3}$, then as shown in Figure 6.16, the radius of a typical disc is $g(w_i) = w_i^{1/3}$ and its volume is $\pi (w_i^{1/3})^2 \Delta y_i$. Applying Definition (6.4) with $g(y) = y^{1/3}$ gives us

$$\begin{aligned} V &= \lim_{\|P\|\to 0} \sum_i \pi \left(w_i^{1/3} \right)^2 \Delta y_i \\ &= \int_1^8 \pi \left(y^{1/3} \right)^2 dy = \pi \int_1^8 y^{2/3} dy \\ &= \pi \left(\frac{3}{5} \right) \left[y^{5/3} \right]_1^8 = \frac{3}{5} \pi \left[8^{5/3} - 1 \right] = \frac{93}{5} \pi \end{aligned}$$

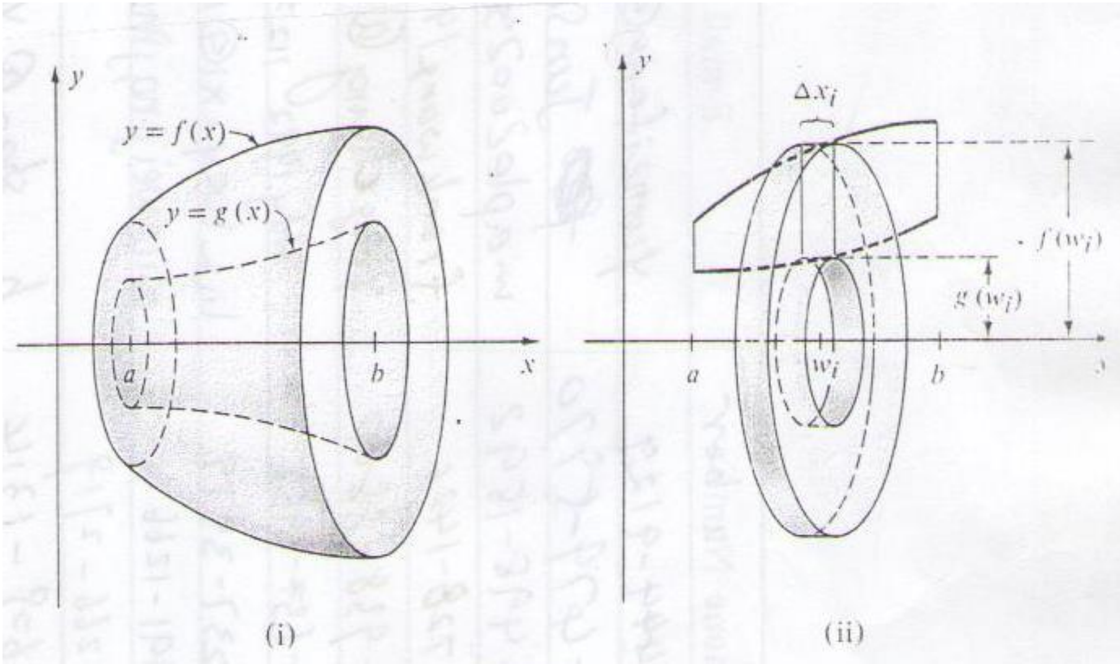

FIGURE 6.17

Let us next consider a region bounded by the graphs of $x=a$, $x=b$ and of two continuous functions f and g where $f(x) \geq g(x) \geq 0$ for all x in $[a,b]$. If this region is revolved about the x-axis, a solid of the type illustrated in (i) of Figure 6.17 may be obtained. Note that if $g(x) > 0$ for all x in $[a,b]$, then there is a hole through the solid.

The volume V may be found by subtracting the volume of the solid generated by the smaller region from the volume of the solid generated by the larger region. Using Definition (6.3) gives us

$$V = \int_a^b \pi \left[f(x) \right]^2 dx - \int_a^b \pi \left[g(x) \right]^2 dx = \int_a^b \pi \left\{ \left[f(x) \right]^2 - \left[g(x) \right]^2 \right\} dx$$

This last integral has an interesting interpretation as a limit of a sum. As illustrated in ii of Figure 6.17, a rectangle extending from the graph of g to the graph of f, through the points with x-coordinate w_i , generates a washer-shaped solid whose volume is

$$\pi \left[f(w_i) \right]^2 \Delta x_i - \pi \left[g(w_i) \right]^2 \Delta x_i = \pi \left\{ \left[f(x_i) \right]^2 - \left[g(x_i) \right]^2 \right\} \Delta x_i$$

Summing the volumes of all such washers and taking the limit gives us the desired integral formula. When working problems of this type it is often convenient to use the following general formula:

(5.1.5) Volume of a washer=π[(outer radius)^2-(inner radius)^2]. (thickness).

In integration problems the thickness will be given by either Δx_i or Δy_i . To find the volume we take a limit of a sum of volumes of these washers.

Example 3
The region bounded by the graphs of the equation $x^2 = y-2$, $2y-x-2=0, x=0$ and $x=1$ is revolved about the *x*-axis. Find the volume of the resulting solid.

Solution
The region and a typical rectangle are sketched in (i) of Figure 6.18. Since we wish to integrate with respect to *x* we solve the first two equations for *y* in terms of *x*, obtaining $y = x^2+2$ and $y=\frac{1}{2}x+1$. The washer generated by the rectangle in (i) is illustrated in (ii) of Figure 6.18.

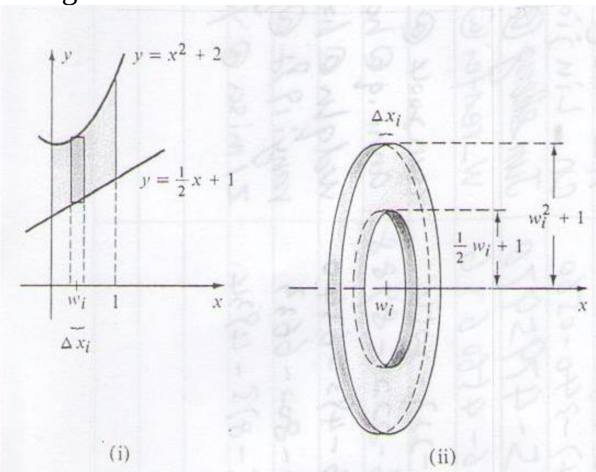

FIGURE 6.18

Since the outer radius of the washer is w_i^2+2 and the inner radius is $\frac{1}{2}w_i+1$, its volume (see (5.1.5)) is

$$\pi\left[\left(w_i^2+2\right)^2-\left(\frac{1}{2}w_i+1\right)^2\right]\Delta x_i$$

Example 5
The region in the first quadrant bounded by the graphs of $y=\frac{1}{8}x^3$ and $y=2x$ is revolved about the *y*-axis. Find the volume of the resulting solid.

Solution
The region and a typical rectangle are shown in (i) of Figure 6.20. since we wish to integrate with respect to *y*, we solve the given equations for *x* in terms of *y*, obtaining

$$x=\frac{1}{2}y \text{ And } x=2y^{1/3}$$

As shown in (ii) of Figure 6.20, the inner and outer radii of the washer generated by the rectangle are $\frac{1}{2}w_i$ and $2w_i^{1/3}$, respectively. Since the thickness is Δy_i it follows form (6.5) that the volume of the washer is

$$\pi\left[\left(2w_i^{1/3}\right)^2-\left(\frac{1}{2}w_i\right)^2\right]\Delta y_i=\pi\left[4w_i^{2/3}-\frac{1}{4}w_i^2\right]\Delta y_i$$

Taking a limit of a sum of such terms gives us

$$V=\int_0^8\pi\left[4y^{2/3}-\frac{1}{4}y^2\right]dy=\pi\left[\frac{12}{5}y^{5/3}-\frac{1}{12}y^3\right]_0^8$$

$$=\pi\left[\frac{12}{5}\left(8^{5/3}\right)-\frac{1}{12}(8^3)\right]=\frac{512}{15}\pi\approx 107.2$$

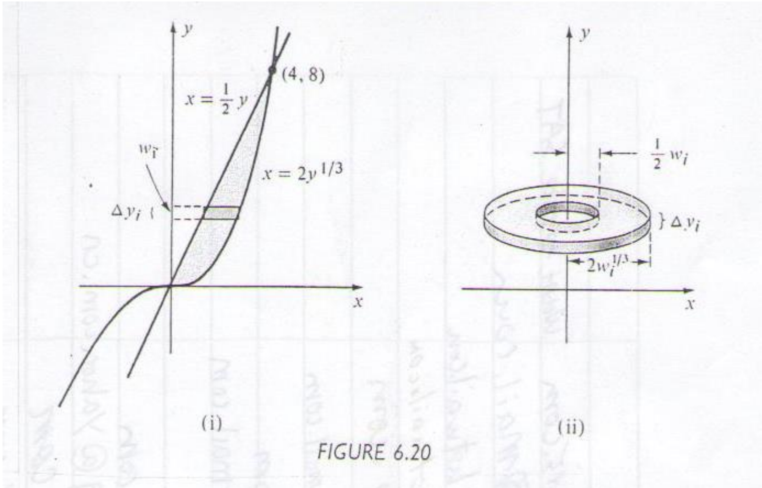
y
(4, 8)
$x = \frac{1}{2}y$
w_i
$x = 2y^{1/3}$
Δy_i
x
(i)
y
$\frac{1}{2}w_i$
Δy_i
$2w_i^{1/3}$
x
(ii)

FIGURE 6.20