Lesson 3.

Definition of Derivative

Defferentiation Rules

1. Definition of Derivative

For a function $y=f(x)$ is defined over interval $[a, x]$.

Average rate of change, $ARC = \dfrac{f(x)-f(a)}{x-a}$

is the slope of the secant line that passes through two points $(a, f(a))$ and $(x, f(x))$.

Instantaneous rate of change, $IRC = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$

is the slope of the tangent line that touches the curve of $y=f(x)$ at the point $(a, f(a))$,

Now, we call IRC as the derivative of $f(x)$ with respect to x at $x=a$, and using the following notations:

$f'(x) = \lim\limits_{x\to x} \dfrac{f(x)-f(x)}{x-x}$?

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \quad --- \quad \text{Formula 1}$$

or $\left.\dfrac{df}{dx}\right|_{x=a}$ or $\left.\dfrac{dy}{dx}\right|_{x=a}$ or $\dfrac{df(a)}{dx}$ $= f'(a)$.

In Formula 1. if let $h = x - a$, then $x = a + h$.

and $x \to a \iff h \to 0$.

thus, $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \cdots \text{ Formula 2.}$$

or $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Both Formula 1 and Formula 2 are called

" the first principles of definition of derivative "

or just " the first principles ".

For example, given $f(x) = \sqrt{x+7}$, find the derivative of $f(x)$ using the first principles.

Sol. If using Formula 1.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\sqrt{x+7} - \sqrt{a+7}}{x - a} \quad "\frac{0}{0}"$$

$$= \lim_{x \to a} \frac{(\sqrt{x+7} - \sqrt{a+7})(\sqrt{x+7} + \sqrt{a+7})}{(x-a)(\sqrt{x+7} + \sqrt{a+7})} = \lim_{x \to a} \frac{x + \cancel{7} - (a + \cancel{7})}{(x-a)(\sqrt{x+7} + \sqrt{a+7})}$$

$$= \lim_{x \to a} \frac{\cancel{x-a}}{\cancel{(x-a)}(\sqrt{x+7} + \sqrt{a+7})} = \frac{1}{\sqrt{a+7} + \sqrt{a+7}} = \frac{1}{2\sqrt{a+7}} = \frac{\sqrt{a+7}}{2(a+7)}$$

$$\boxed{\therefore f'(x) = \frac{\sqrt{x+7}}{2(x+7)}}$$

If using Formula 2. $\quad f(x)=\sqrt{x+7}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h+7}-\sqrt{x+7}}{h}, \quad \text{" } \frac{0}{0} \text{ "}$$

$$=\lim_{h\to 0}\frac{(\sqrt{x+h+7}-\sqrt{x+7})(\sqrt{x+h+7}+\sqrt{x+7})}{h(\sqrt{x+h+7}+\sqrt{x+7})}$$

$$=\lim_{h\to 0}\frac{\cancel{x}+h+\cancel{7}-(\cancel{x+7})}{h(\sqrt{x+h+7}+\sqrt{x+7})}=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+7}+\sqrt{x+7})}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h+7}+\sqrt{x+7}}=\frac{1}{\sqrt{x+0+7}+\sqrt{x+7}}=\frac{1}{2\sqrt{x+7}}=\frac{\sqrt{x+7}}{2(x+7)}$$

By the way, if we want to find $f'(2)$.

then $f'(2)=\dfrac{\sqrt{2+7}}{2(2+7)}=\dfrac{3}{2(9)}=\dfrac{1}{6}$

or

$$f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{\sqrt{x+7}-\sqrt{2+7}}{x-2}=\cdots=\frac{1}{6}.$$

or $$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{\sqrt{2+h+7}-\sqrt{2+7}}{h}=\cdots=\frac{1}{6}.$$

Example 2. Given $f(x)=\dfrac{1}{3-\sqrt{x}}$.

① find $f'(x)$. using the first principles.

② find $f'(4)$.

Sol. ①

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\frac{1}{3-\sqrt{x}}-\frac{1}{3-\sqrt{a}}}{x-a}$$

$$= \lim_{x \to a} \frac{\dfrac{3-\sqrt{a} - (3-\sqrt{x})}{(3-\sqrt{x})(3-\sqrt{a})}}{x-a} = \lim_{x \to a} \frac{\sqrt{x}-\sqrt{a}}{(x-a)(3-\sqrt{x})(3-\sqrt{a})}$$

$$= \lim_{x \to a} \frac{\cancel{\sqrt{x}-\sqrt{a}}}{\cancel{(\sqrt{x}-\sqrt{a})}(\sqrt{x}+\sqrt{a})(3-\sqrt{x})(3-\sqrt{a})} = \frac{1}{(\sqrt{a}+\sqrt{a})(3-\sqrt{a})(3-\sqrt{a})}$$

$\because\ x-a = (\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})$

$$= \frac{1}{2\sqrt{a}(3-\sqrt{a})^2} = \frac{\sqrt{a}(3+\sqrt{a})^2}{2a(9-a)^2}$$

$$\therefore\ f'(x) = \frac{\sqrt{x}(3+\sqrt{x})^2}{2x(9-x)^2}$$

② $f'(4) = \dfrac{\sqrt{4}(3+\sqrt{4})^2}{3(4)(9-4)^2} = \dfrac{\cancel{2}(\cancel{5})^2}{3(\cancel{4}_2)(\cancel{5})^2} = \dfrac{1}{6}$

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2. Differentiation Rules.

It is a long process to use the first principles to get the derivative of a function.

So a set of rules is developed for taking derivatives.

① Constant Rule

If $f(x) = C$, a constant function.

then $f'(x) = [C]' = 0$

For example. $f(x) = \sqrt{3}$

$f'(x) = (\sqrt{3})' = 0$

② Power Rule.

If $f(x) = x^n$. where $n \in R$

then $f'(x) = (x^n)' = n x^{n-1}$

For example $f(x) = x$

$$f'(x) = (x)' = (1) x^{1-1} = x^0 = 1$$

$$g(x) = \frac{1}{x}$$

$$g'(x) = \left(\frac{1}{x}\right)' = (x^{-1})' = (-1) x^{-1-1} = -x^{-2}$$

$$h(x) = \sqrt[3]{x^2}$$

$$h'(x) = (\sqrt[3]{x^2})' = (x^{\frac{2}{3}})' = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3} x^{-\frac{1}{3}}$$

$$u(x) = x^e$$

$$u'(x) = (x^e)' = e x^{e-1}$$

③ If $y = af(x)$. Constant Multiple Rule

then $y' = [af(x)]' = \underline{\underline{a}}[f(x)]' = af'(x)$

For example. $y = \sqrt{3x}$

$$y' = (\sqrt{3x})'$$

$$= (\sqrt{3}\sqrt{x})' = \sqrt{3}(\sqrt{x})'$$

$$= \sqrt{3}\left(\tfrac{1}{2}x^{-\frac{1}{2}}\right)$$

$$= \frac{\sqrt{3}}{2}x^{-\frac{1}{2}}$$

(4) Sum Rule.

If $y = f(x) + g(x)$

then $y' = [f(x) + g(x)]' = f'(x) + g'(x)$

(5) Difference Rule

If $y = f(x) - g(x)$

then $y' = [f(x) - g(x)]' = f'(x) - g'(x)$

Using Rule 1 through Rule 5.

We could take derivative of any polynomial function

For example.

$$y = 2x^5 - x^4 + 5x^2 - x + \underline{10}$$

$$y' = 2(x^5)' - (x^4)' + 5(x^2)' - (x)' + (10)'$$

$= 2(5x^4) - 4x^3 + 5(2x) - 1 + 0$

$= 10x^4 - 4x^3 + 10x - 1$

(6) Product Rule

If $y = f(x)g(x)$

then $y' = [f(x)g(x)]'$

$= f'(x)g(x) + f(x)g'(x)$

(7) Quotient Rule

If $y = \frac{f(x)}{g(x)}$.

then $y' = \left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

(8) Chain Rule for composite function

If $y = f(g(x)) = f \circ g(x)$

then $\frac{dy}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

$y'_x = f'(g) \cdot g'(x)$

For example, find $\frac{dy}{dx}$ or y'

a) $y = (x^2 - x + 3)(\sqrt{5} - x^3)$

If using Product Rule,

$$y' = (x^2 - x + 3)'(\sqrt{5} - x^3) + (x^2 - x + 3)(\sqrt{5} - x^3)'$$

$$= (2x - 1 + 0)(\sqrt{5} - x^3) + (x^2 - x + 3)(0 - 3x^2)$$

$$= 2\sqrt{5}x - 2x^4 - \sqrt{5} + x^3 - 3x^4 + 3x^3 - 9x^2$$

$$= -5x^4 + 4x^3 - 9x^2 + 2\sqrt{5}x - \sqrt{5}$$

If not using Product Rule

$$y = (x^2 - x + 3)(\sqrt{5} - x^3) = \sqrt{5}x^2 - x^5 - \sqrt{5}x + x^4 + 3\sqrt{5} - 3x^3$$

$$y' = \sqrt{5}(x^2)' - (x^5)' - \sqrt{5}(x)' + (x^4)' + (3\sqrt{5})' - 3(x^3)'$$

$$= 2\sqrt{5}x - 5x^4 - \sqrt{5} + 4x^3 - 9x^2$$

b) $y = \dfrac{4 - x^2}{2x + 3}$ ← rational function

$$y' = \frac{(4 - x^2)'(2x + 3) - (4 - x^2)(2x + 3)'}{(2x + 3)^2}$$

$$= \frac{(0 - 2x)(2x + 3) - (4 - x^2)(2 + 0)}{(2x + 3)^2}$$

$$= \frac{-4x^2 - 6x - 8 + 2x^2}{(2x + 3)^2} = \frac{-2x^2 - 6x - 8}{(2x + 3)^2}$$

c) $y = \sqrt[3]{x^2 - x + 5}$

$(\sqrt[3]{x})'$

$(x^2 - x + 5)'$

$y' = (\sqrt[3]{x^2 - x + 5})'$

$\begin{cases} f(g) = \sqrt[3]{g} \rightarrow f'(g) = \frac{1}{3} g^{-\frac{2}{3}} \\ g(x) = x^2 - x + 5 \rightarrow g'(x) = 2x - 1 \end{cases}$

$= f'(g) \cdot g'(x)$

$= \frac{1}{3} g^{-2/3} \cdot (2x-1)$

$= \frac{1}{3} (x^2 - x + 5)^{-2/3} \cdot (2x - 1)$

d) $y = (3x^4 - x + 5)^{100}$

$y' = 100 (3x^4 - x + 5)^{100-1} (3x^4 - x + 5)'$

$= 100 (3x^4 - x + 5)^{99} (12x^3 - 1)$

e) $y = \dfrac{1}{x^5 + 2x^3 + x - 8}$

Find $\frac{dy}{dx}$ using Chain Rule or using Quotient Rule.

Sol, If using Chain Rule.

$y = [x^5 + 2x^3 + x - 8]^{-1}$

$y' = (-1)(x^5 + 2x^3 + x - 8)^{-1-1} (x^5 + 2x^3 + x - 8)'$

$= -(x^5 + 2x^3 + x - 8)^{-2} (5x^4 + 6x^2 + 1)$

$$= \frac{-(5x^4+6x^2+1)}{(x^5+2x^3+x-8)^2}$$

If using Quotient Rule.

$$y' = \left(\frac{1}{x^5+2x^3+x-8}\right)'$$

$$= \frac{(1)'(x^5+2x^3+x-8) - (1)(x^5+2x^3+x-8)'}{(x^5+2x^3+x-8)^2}$$

$$= \frac{(0) - (5x^4+6x+1)}{(x^5+2x^3+x-8)^2}$$