AP Calculus Lesson Seventeen Notes

Chapter Eight - Indeterminate Forms, Improper Integrals, and Taylor's Formula

8.1 The Indeterminate Form 0/0 and ∞/∞
8.2 Integral with Infinite Limits of Integration

In this chapter we introduce techniques that are useful in the investigation of certain limits. We shall also study definite integrals which have discontinuous integrands or infinite limits of integration. The final section contains a method for approximating functions by means of polynomials. These topics have many mathematical and physical applications. Our most important use for them will occur in the next chapter, when *infinite series* are studies.

8.1 The Indeterminate Forms 0/0 and ∞/∞

in our early work with limits we encountered expressions of the form $\lim_{x\to c} f(x)/g(x)$. Where both *f* and *g* have the limit 0 as *x* approaches *c*. in this event, $f(x)/g(x)$ is said to have the **indeterminate form 0/0** at *x=c*. the word *indeterminate* is used because a further analysis is necessary to conclude whether or not the limit exists. Perhaps the most important examples of the indeterminate form 0/0 occur in the use of the derivative formula

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

If *f* and *g* become positively or negatively as *x* approaches *c*, we say that *f(x)/g(x)* has the **indeterminate form** ∞/∞ at *x=c.*

Indeterminate forms can sometimes be investigated by employing algebraic manipulations/ to illustrate, in Chapter 2 we considered

$$\lim_{x\to 2} \frac{2x^2 - 5x + 2}{5x^2 - 7x - 6}.$$

The indicated quotient has the indeterminate form 0/0; however, the limit may be found as follows:

$$\lim_{x\to 2} \frac{(x-2)(2x-1)}{(x-2)(5x+3)} = \lim_{x\to 2} \frac{2x-1}{5x+3} = \frac{3}{13}.$$

Other indeterminate forms require more complicated techniques. For example, in Chapter 2 a geometric argument was used to show that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

In this section we shall establish **L'Hôpital's Rule** and illustrate how it can be used to investigate many indeterminate forms. The proof makes use of the following formula, which bears the name of the famous French mathematician A. Cauchy (1789-1857)

Cauchy's Formula (8.1.1)

If the functions *f* and *g* are continuous on a closed interval [*a*,*b*], differentiable on the open interval (*a*,*b*), then there is a number *w* in (*a*,*b*) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(w)}{g'(w)}.$$

Proof

We first note that $g(b)-g(a)\neq 0$, for otherwise $g(a)=g(b)$, and by Rolle's Theorem (4.10) there is a number *c* in (*a*,*b*)such that *g'(x)*=0, contrary to hypothesis.

It is convenient to introduce a new function *h* as follows:

$$h(x)=[f(b)-f(a)]g(x)-[g(b)-g(a)]f(x)$$

For all *x* in [*a*,*b*]. it follows that *h* is continuous on [*a*,*b*], differentiable on (*a*,*b*), and that *h(a)*=*h(b)*. by Rolle's Theorem, there is a number *w* in (*a*,*b*)such that *h'(w)*=0, that is

$$[f(b)-f(a)]g'(w)-[g(b)-g(a)]f'(w)=0.$$

The preceding equation may be written in the form stated in the conclusion of the theorem.

As a special case, if we let *g(x)*=*x* in Formula (8.1.1), then the conclusion has the form

$$\frac{f(b)-f(a)}{b-a}=\frac{f'(w)}{1}$$

Which is equivalent to

$$f(b)-f(a)=f'(w)(b-a).$$

This shows that Cauchy's Formula is a generalization of the Mean Value Theorem (4.12).

The next result is the main theorem on indeterminate forms.

L'Hôpital's Rule (8.1.2)

Suppose the function *f* and *g* are differentiable on an open interval (*a*,*b*)containing *c* , except possibly at *c* itself. If $g'(x)\neq 0$ for $x\neq c$, and if *f(x)/g(x)* has the indeterminate form 0/0 and ∞/∞ at *x*=*c*, then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

Provided $f'(x)/g'(x)$ has a limit or becomes infinite as *x* approaches *c*.

Proof

Suppose *f(x)/g(x)* has the indeterminate form 0/0 at *x*=*c* and $\lim_{x\to c} f'(x)/g'(x)=L$ for some number L. We wish to prove that $\lim_{x\to c} f(x)/g(x)=L$. Let us introduce two new functions *F* and *G* where

$$F(x) = f(x) \quad \text{if } x \neq c \text{ and } F(c) = 0,$$
$$G(x) = g(x) \quad \text{if } x \neq c \text{ and } G(c) = 0.$$

Since

$$\lim_{x \to c} F(x) = \lim_{x \to c} f(x) = 0 = F(c).$$

The function *F* is continuous at *c* and hence is continuous *throughout* the interval (*a*,*b*). similarly, *G* is continuous on (*a*,*b*). moreover, at every $x \neq c$. We have $F'(x) = f'(x)$ and $G'(x) = g'(x)$. It follows from Cauchy's Formula, applies either to the interval [*c*,*x*] or to [*x*,*c*], that is a number *w* between *c* and *x* such that

$$\frac{F(x) - F(c)}{G(x) - G(c)} = \frac{F'(w)}{G'(w)} = \frac{f'(w)}{g'(w)}.$$

Using the fact that $F'(x) = f'(x)$, $G'(x) = g'(x)$, and $F(c) = G(c) = 0$ gives us

$$\frac{f(x)}{g(x)} = \frac{f'(w)}{g'(w)}.$$

Since *w* is always between *c* and *x* it follows that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(w)}{g'(w)} = \lim_{w \to c} \frac{f'(w)}{g'(w)} = L$$

Which is what we wished to prove. A similar argument may be given if $f'(x)/g'(x)$ becomes infinite as *x* approaches *c*. the proof for the indeterminate form ∞/∞ is more difficult and may be found in texts on advanced calculus.

Beginning students sometimes use L'Hôpital's Rule incorrectly by applying the quotient rule to $f(x)/g(x)$. Note that (8.1.2) states that the derivatives of *f(x)* and *g(x)* are taken *separately*, after which the limit of the quotient $f'(x)/g'(x)$ is investigated.

Example 1

Find $\lim_{x \to 0} \dfrac{\cos x + 2x - 1}{3x}$

Solution

The quotient has the indeterminate form 0/0 at *x*=0. by L'Hôpital's Rule (8.1.2),

$$\lim_{x \to 0} \frac{\cos x + 2x - 1}{3x} = \lim_{x \to 0} \frac{-\sin x + 2}{3} = \frac{2}{3}.$$

To be completely rigorous in Example 1 we should have determined whether or not $\lim_{x \to 0} (-\sin x + 2)/3$ existed *before* equating it to the given expression; however, to simplify solutions it is customary to proceed as indicated.

Sometimes it is necessary to employ L'Hôpital's Rule several times in the same problem, as illustrated in the next example.

Example 2

Find $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos 2x}$

Solution

The quotient has the indeterminate form 0/0. by L'Hôpital's Rule,

$$\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos 2x}=\lim_{x\to 0}\frac{e^x-e^{-x}}{2\sin 2x}$$

Provided the second limit exists. Since the last quotient has the indeterminate form 0/0, we apply L'Hôpital's Rule a second time, obtaining

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{2\sin 2x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{4\cos 2x}=\frac{1}{2}.$$

It follows that the given limit exists and equals $\frac{1}{2}$.

Example 4

Find $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$

Solution

The indeterminate form is ∞/∞. By L'Hôpital's Rule.

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})}=\lim_{x\to\infty}\frac{2\sqrt{x}}{x}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0.$$

Example 5

Find $\lim_{x\to\infty}\frac{e^{3x}}{x^2}$, if it exists.

Solution

The indeterminate form is ∞/∞. In this case we must apply L'Hôpital's Rule twice, as follows.

$$\lim_{x\to\infty}\frac{e^{3x}}{x^2}=\lim_{x\to\infty}\frac{3e^{3x}}{2x}=\lim_{x\to\infty}\frac{9e^{3x}}{2}=\infty$$

Thus the given quotient increases without bound as x becomes infinite

It is extremely important to verify that a given quotient has the indeterminate form 0/0 or ∞/∞ *before using L'Hôpital's Rul.* Indeed, if the rule is applied to a nonindeterminate form, an incorrect conclusion may be obtained, as illustrated in the next example.

Example 6

Find $\lim_{x\to\infty}\frac{e^x+e^{-x}}{x^2}$, if it exists.

Solution

Suppose we overlook the fact that the quotient does *not* have either of the indeterminate form 0/0 or ∞/∞ at $x=0$. if we (incorrectly) apply L'Hôpital's Rule we obtain

$$\lim_{x\to 0}\frac{e^x+e^{-x}}{x^2}=\lim_{x\to 0}\frac{e^x-e^{-x}}{2x}.$$

Since the last quotient has the indeterminate form 0/0 we may apply L'Hôpital's Rule, obtaining

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{2x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{2}=\frac{1+1}{2}=1.$$

This would lead us to the (wrong) conclusion that the given limit exists and equals 1. A correct method for investigating the limit is to observe that

$$\lim_{x\to 0}\frac{e^x+e^{-x}}{x^2}=\lim_{x\to 0}(e^x+e^{-x})\left(\frac{1}{x^2}\right).$$

Since

$$\lim_{x\to 0}(e^x+e^{-x})=2 \quad \text{and} \quad \lim_{x\to 0}\frac{1}{x^2}=\infty,$$

Other Indeterminate Forms

If $\lim_{x\to c} f(x)=0$ and $\lim_{x\to c}|g(x)|=\infty$, then *f(x)g(x)* is said to have the **indeterminate form 0**. ∞ At *x=c*. the same terminology is used for one-sided limits or if *x* becomes positively or negatively infinite. This form may be changed to one of the indeterminate form 0/0 or ∞/∞ by writing

$$f(x)g(x)=\frac{f(x)}{1/g(x)} \quad \text{or} \quad f(x)g(x)=\frac{g(x)}{1/f(x)}.$$

Example 1

Find $\lim_{x\to 0^+} x^2 \ln x$

Solution

The indeterminate form is 0. ∞. We first write

$$x^2 \ln x=\frac{\ln x}{(1/x^2)}$$

And then apply L'Hôpital's Rule to the resulting indeterminate form ∞/∞. Thus

$$\lim_{x\to 0^+} x^2 \ln x=\lim_{x\to 0^+}\frac{\ln x}{(1/x^2)}=\lim_{x\to 0^+}\frac{(1/x)}{(-2/x^3)}.$$

The last quotient has the indeterminate form ∞/∞; however, further applications of L'Hôpital's Rule would again lead to ∞/∞. In this case we simplify the quotient algebraically and find the limit as follows:

$$\lim_{x\to 0^+}\frac{(1/x)}{(-2/x^3)}=\lim_{x\to 0^+}\frac{x^3}{-2x}=\lim_{x\to 0^+}\frac{x^2}{-2}=0.$$

Example 2

Find $\lim_{x\to \pi/2}(2x-\pi)\sec x$

Solution

The indeterminate form is $0\cdot\infty$. Hence we begin by writing

$$(2x-\pi)\sec x = \frac{2x-\pi}{1/\sec x} = \frac{2x-\pi}{\cos x}.$$

Since the last expression has the indeterminate form 0/0 at $x=\pi/2$, L'Hôpital's Rule may be applied as follows:

$$\lim_{x\to\pi/2}\frac{2x-\pi}{\cos x} = \lim_{x\to\pi/2}\frac{2}{-\sin x} = -2.$$

Indeterminate forms denoted by $\mathbf{0^0}$, ∞^0, and $\mathbf{1}^\infty$ arise from expressions such as $f(x)^{g(x)}$. One method for dealing with these forms is to write

$$y = f(x)^{g(x)}$$

And take the natural logarithm of both sides, obtaining

$$\ln y = \ln f(x)^{g(x)} = g(x)\ln f(x).$$

Note that if the indeterminate form for *y* is 0^0 or ∞^0, then the indeterminate form for $\ln y$ is $0\cdot\infty$, which may be handled using previous methods. Similarly, if the form for *y* is $\mathbf{1}^\infty$, then the indeterminate form for $\ln y$ is $\infty\cdot 0$. It follows that

If $\lim_{x\to c}\ln y = L$, then $\lim_{x\to c} y = \lim_{x\to c} e^{\ln y} = e^L$,

That is, $\lim_{x\to c} f(x)^{g(x)} = e^L$.

This procedure may be summarized as follows.

Guidelines for investigating $\lim_{x\to c} f(x)^{g(x)}$ if the indeterminate for is, $\mathbf{0^0}$, $\mathbf{1}^\infty$, or ∞^0

1. Let $y = f(x)^{g(x)}$.
2. Take logarithms: $\ln y = \ln f(x)^{g(x)} = g(x)\ln f(x)$.
3. Find $\lim_{x\to c}\ln y$, if it exists.
4. If $\lim_{x\to c}\ln y = L$, then $\lim_{x\to c} y = e^L$.

a common error is to stop after showing $\lim_{x\to c}\ln y = L$ and conclude that the given expression has the limit *L*. remember that *we wish to find the limit of y,* and if ln *y* has the limit *L*, the *y* has the limit e^L. The guidelines may also be used if $x\to\infty$, or $x\to-\infty$, or for one-sided limits.

Example 3

Find $\lim_{x\to 0}(x+3x)^{1/2x}$

Solution

The indeterminate form is 1^∞. Following the guidelines we begin by writing

1. $$y = (1+3x)^{1/2x}.$$
2. $$\ln y = \frac{1}{2x}\ln(1+3x) = \frac{\ln(1+3x)}{2x}.$$

The last expression has the indeterminate from 0/0 at *x*=0. by L'Hôpital's Rule,

3. $$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+3x)}{2x} = \lim_{x\to 0}\frac{3/(1+3x)}{2} = \frac{3}{2}.$$

Consequently,

4. $$\lim_{x\to 0}(1+3x)^{1/2x} = \lim_{x\to 0} y = e^{3/2}.$$

If $\lim_{x\to c} f(x) = \lim_x g(c) = \infty$, then *f(x)-g(x)* has the indeterminate form $\infty - \infty$ at $x = c$. In this case the expression should be changed so that one of the fortms we have discussed is obtained.

Example 4

Find $\lim_{x\to 0}\left(\frac{1}{e^x - 1} - \frac{1}{x}\right)$

Solution

the form is $\infty - \infty$; however, if the difference is written as a single fraction, then

$$\lim_{x\to 0}\left(\frac{1}{e^x - 1} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x - e^x + 1}{xe^x - x}.$$

8.2 Integrals with Infinite Limits of Integration

Suppose a function *f* is continuous and nonnegative on an infinite interval $[a,\infty)$ and $\lim_{x\to\infty} f(x) = 0$. If *t>a*, then the area *A(t)* under the graph of *f* from *a* to *t*, as illustrated in (i) of Figure 10.1, is given by

$$A(t) = \int_a^t f(x)\,dx.$$

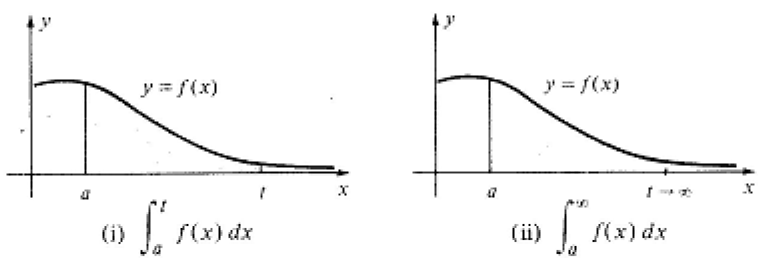

FIGURE 10.1

If $\lim_{t\to\infty} A(t)$ exists, then the limit may be interpreted as the area of the *unbounded* region which lies under the *unbounded* region which lies under the graph of *f*, over the *x*-axis, and to the right of *x=a*, as illustrated in (ii) of the figure. The symbol $\int_a^\infty f(x)dx$ is used to denote this number.

Part (i) of the next definition generalizes the preceding remarks to the case where *f(x)* may be negative in $[a,\infty)$

Definition (8.2.1)

(i) If f is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx.$$

(ii) If f is continuous on $(-\infty, a]$, then

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{t \to -\infty} \int_t^a f(x)\,dx.$$

If $f(x) \geq 0$ for all x, then the limit in (ii) may be regarded as the area under the graph of f, over the x-axis, and to the *left* of x=*a*.

The expressions in definition (8.2.1) are called **improper integrals.** They differ from definite integrals because one of the limits of integration is not a real number. These integrals are said to **converge** if, as $|t|$ increases without bound, the limits on the right side of the equations exist. The limits are called the **values** of the improper integrals. If the limits of integration, as in the following definition

Definition (8.2.2)

Let f be continuous for all x . if a is any real number, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx.$$

The integral on the left in definition (8.2.2) is said to **converge** if the only if *both* of the integrals on the right converge. If one of the integrals diverges, then $\int_{-\infty}^{\infty} f(x)dx$ is said to **diverge.** It can be shown that (10.4) is independent of the real number *a*(see Exercise 40). We may also show that $\int_{-\infty}^{\infty} f(x)dx$ is not necessarily the same as $\lim_{t\to\infty} \int_{-t}^{t} f(x)dx$ (see Exercise 39)

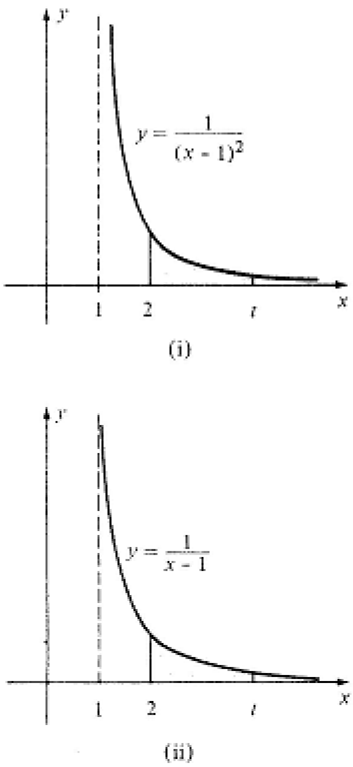

FIGURE 10.2

Example 1

Determine whether the following integrals converge or diverge.

(a) $\int_2^\infty \frac{1}{(x-1)^2}\,dx$ (b) $\int_2^\infty \frac{1}{x-1}\,dx$

Solution

(a) by (i) of Definition (10.3)

$$\int_2^\infty \frac{1}{(x-1)^2}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{(x-1)^2}\,dx = \lim_{t\to\infty}\left.\frac{-1}{x-1}\right]_2^t$$

$$= \lim_{t\to\infty}\left(\frac{-1}{t-1}+\frac{1}{2-1}\right) = 0 + 1 = 1.$$

Thus the integral converges and has the value 1.

(b) by (i) of (10.3)

$$\int_2^\infty \frac{1}{x-1}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x-1}\,dx$$

$$= \lim_{t\to\infty}\left.\ln(x-1)\right]_2^t$$

$$= \lim_{t\to\infty}\left[\ln(t-1) - \ln(2-1)\right]$$

$$= \lim_{t\to\infty}\ln(t-1) = \infty.$$

Consequently, this improper integral diverges.

The graphs of the two functions defined by the integrands in parts (a) and (b) of Example 1 are sketched in Figure 10.2. Note that although the graphs have the same general shape for $x \geq 2$, we may assign an area to the region under the graph shown in (i) of the figure, whereas this is not true for the graph in (ii).

There is an interesting sidelight to the graph in (ii) of Figure 10.2. if the region under the graph of *y=1/(x-1)* from 2 to *t* is revolved about the *x*-axis, then the volume of the resulting solid is

$$\pi\int_2^t \frac{1}{(x-1)^2}\,dx.$$

The improper integral

$$\pi\int_2^\infty \frac{1}{(x-1)^2}\,dx$$

May be regarded as the volume of the *unbounded* solid obtained by revolving about the *x*-axis, the region under the graph of $y = 1/(x-1)$ for $x \geq 2$. By (a) of Example 1, the value of this improper integral is $\pi \bullet 1$ or π. This gives us the rather curious fact that although the area of the region is infinite, the volume of the solid of revolution it generates is finite.

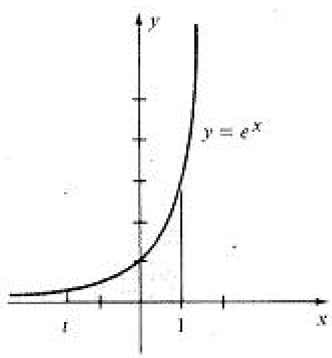

FIGURE 10.3

Example 2

Assign an area to the region which lies under the graph of $y = e^x$, over the *x*-axis, and to the left of *x=1*.

Solution

The region bounded by the graphs of $y = e^x, y = 0, x = 1$ and $x = t$ where *t*<1, is sketched in Figure 10.3.

By our previous remarks, the desired area is given by

$$\int_{-\infty}^{1} e^x\,dx = \lim_{t \to -\infty} \int_t^1 e^x\,dx = \lim_{t \to -\infty} e^x \Big]_t^1$$

$$= \lim_{t \to -\infty} (e - e^t) = e - 0 = e.$$

Example 3

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$. Sketch the graph of $f(x) = \frac{1}{1+x^2}$ and interpret the integral as an area.

Solution

Using Definition (10.4) with *a*=0,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx.$$

Next, applying (i) of Definition (10.3),

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t \to \infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t \to \infty} \arctan x \Big]_0^t$$

$$= \lim_{t \to \infty} (\arctan t - \arctan 0) = \pi/2 - 0 = \pi/2.$$

Similarly, we may show, by using (ii) of (10.3), that

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

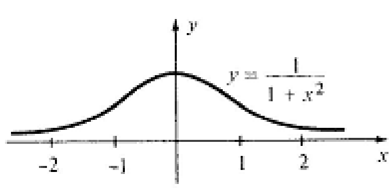

FIGURE 10.4

Consequently, the given improper integral converges and has the value $\pi/2 + \pi/2 = \pi.$

The graph of $y = 1/(1+x^2)$ is sketched in Figure 10.4. as in our previous discussion, the unbounded region that lies under the graph and above the *x*-axis may be assigned an area of π square units.

FIGURE 10.5

Improper integrals with infinite limits of integration have many applications in the physical world. To illustrate, suppose *a* and *b* are the coordinates of two points *A* and *B* on a coordinate line *l*, as shown in Figure 10.5. if *F(x)* is the force acting at the point *P* with coordinate *x*, then by Definition (6.10), the work done as *P* moves from *A* to *B* is given by

$$W = \int_a^b F(x)\,dx.$$

In similar fashion, the improper integral $\int_a^\infty F(x)dx$ is used to define the work done as *P* moves indefinitely to the right (in applications, the terminology "to infinity" is sometimes used). For example, if *F(x)* is the force of attraction between a particle fixed a point *A* and a (movable) particle at *P*, and if $c>a$, then $\int_c^\infty F(x)dx$ represents the work required to move *P* from the point with coordinate *c* to infinity.