Notice:
The notes are the same as before, but the questions in class and homework are different with before.

Algebra

►Basic Knowledge

1. Laws of Exponents

The exponent of a number says how many times to use the number in a multiplication.
In this example: $8^2 = 8 \times 8 = 64$
So an Exponent just saves you writing out lots of multiplies!

Example: a^7
$a^7 = a \times a \times a \times a \times a \times a \times a = aaaaaaa$

Notice how I just wrote the letters together to mean multiply? We will do that a lot here.

Example: $y^6 = y \times y \times y \times y \times y \times y = yyyyyy$
The Key to the Laws
Writing all the letters down is the key to understanding the Laws

Example: $x^2x^3 = (xx)(xxx) = xxxxx = x^5$
Which shows that $x^2x^3 = x^{2+3} = x^5$, but more on that later!

A fractional exponent like 1/n means to take the nth root: $x^{\frac{1}{n}} = \sqrt[n]{x}$

2. Using Exponents in Algebra

You might like to read the page on Exponents first.

1) Whole Number Exponents

The exponent "n" in a^n says how many times to use a in a multiplication:
$a^n = a \times a ... \times a = n \times a$

2) Negative Exponents

Example: $5^{-3} = 1/5^3 = 1/125 = 0.008$

A positive exponent a^n is equal to $1/a^{-n}$ (1 divided by the negative exponent) $a^n = \dfrac{1}{a^{-n}}$

3) Variables with Exponents

What is a Variable with an Exponent?
A Variable is a symbol for a number we don't know yet. It is usually a letter like x or y.
An exponent (such as the 2 in x^2) says how many times to use the variable in a multiplication.

Example: $y^2 = yy$
(yy means y *multipled by* y, because in Algebra putting two letters next to each other means to multiply them)
Likewise $z^3 = zzz$ *and* $x^5 = xxxxx$

4) Exponents of 1 and 0

a) Exponent of 1
If the exponent is 1, then you just have the variable itself (example $x^1 = x$)
We usually don't write the "1", but it sometimes helps to remember that x is also x^1

b) Exponent of 0
If the exponent is 0, then you are not multiplying by anything and the answer is just "1" (example $y^0 = 1$)

5) Multiplying Variables with Exponents
So, how do you multiply this:
$(y^2)(y^3)$

We know that $y^2 = yy$, and $y^3 = yyy$ so let us write out all the multiplies:
$y^2\, y^3 = yyyyy$
That is 5 "y"s multiplied together, so the new exponent must be 5:
$y^2\, y^3 = y^5$

But why count the "y"s when the exponents already tell us how many?
The exponents tell us that there are two "y"s multiplied by 3 "y"s for a total of 5 "y"s:
$y^2\, y^3 = y^{2+3} = y^5$

So, the simplest method is to just add the exponents! (Note: this is one of the Laws of Exponents)

6) Negative Exponents

Negative Exponents Mean Dividing!
$$x^{-3} = \frac{1}{x}, \quad x^{-2} = \frac{1}{x^2}, \quad x^{-3} = \frac{1}{x^3}$$
Get familiar with this idea, it is very important and useful!

7) Dividing
$$\frac{y^3}{y^2} = \frac{yyy}{yy} = y^{3-2} = y^1 = y$$

OR, you could have done it like this:

$$\frac{y^3}{y^2} = y^3 y^{-2} = y^{3-2} = y^1 = y$$

So, it just subtract the exponents of the variables you are dividing by!

You can see what is going on if you write down all the multiplies, then "cross out" the variables that are both top and bottom:

$$\frac{x^3 y z^2}{x y^2 z^2} = \frac{xxxyzz}{xyyzz} = \frac{xx}{y} = \frac{x^2}{y}$$

3. Complex Fractions

A complex fraction is a fraction where the numerator, denominator, or both contain a fraction.

Example 1: $\dfrac{3}{1/2}$ is a complex fraction. The numerator is 3 and the denominator is 1/2.

Example 2: $\dfrac{3/7}{9}$ is a complex fraction. The numerator is 3/7 and the denominator is 9.

Example 3: $\dfrac{3/4}{9/10}$ is a complex fraction. The numerator is 3/4 and the denominator is 9/10.

Rule:
To multiply two complex fractions, convert the fractions to simple fractions and follow the steps you use to multiply two simple fractions.

Example:
Calculate $\dfrac{3/4}{1/2} \times \dfrac{8}{3/16}$.

Solution 1:
Convert the numerator $\dfrac{3/4}{1/2}$ to a simple fraction. $\dfrac{3/4}{1/2}$ can be written

$$\frac{3/4}{1/2} = \frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \times \frac{2}{1} = \frac{3 \times 2}{2 \times 2} = \frac{3}{2} \times 1 = \frac{3}{2}.$$

Convert the denominator to a simple fraction. $\dfrac{8}{3/16}$ can be written

$$\frac{8}{3/16} = \frac{8}{1} \div \frac{3}{16} = \frac{8}{1} \times \frac{16}{3} = \frac{8 \times 16}{1 \times 3} = \frac{128}{3}.$$

The problem $\frac{3/4}{1/2}\times\frac{8}{3/16}$ can now be written $\frac{3}{2}\times\frac{128}{3}$.

Multiply the numerators and multiply the denominators. $\frac{3\times 128}{2\times 6}$.

The problem is reduced by $\frac{3\times 2\times 64}{2\times 3}=\frac{2}{2}\times\frac{3}{3}\times\frac{64}{1}=64$.

Solution 2:

Convert the 8 to a fraction, multiply the numerators, and multiply the denominators.

$$\frac{3/4}{1/2}\times\frac{8}{3/16}=\frac{\frac{3\times 8}{4\times 1}}{\frac{1\times 3}{2\times 16}}=\frac{\frac{6}{1}}{\frac{3}{32}}\times\frac{\frac{32}{1}}{\frac{32}{1}}=\frac{6\times 32}{3}=64.$$

4. Remember these Identities

1) $(x+a)(x+b)=x^2+(a+b)x+ab$

2) $(a\pm b)^2=a^2\pm 2ab+b^2$

3) $(a\pm b)^3=a^3\pm 3a^2b+3ab^2\pm b^3$

4) $a^2-b^2=(a-b)(a+b)$

5) $a^3\pm b^3=(a\pm b)(a^2\mp ab+b^2)$

6) $a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+ab^{n-2}+b^{n-1})$ (n is a positive integer)

7) $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$

8) $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

9) $\frac{a}{b}\pm\frac{c}{b}=\frac{a\pm c}{b}$ 10) $\frac{a}{b}+\frac{c}{d}=\frac{ad\pm bc}{bd}$

11) $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$ 12) $\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$

► In-class questions

1. Given only an 11 litre jug and an 8 litre jug, each with no markings, find the method of using the two jugs to measure out exactly 1 litre of water while using the smallest amount of water. Argue clearly why you think that your procedure uses the smallest amount!

2. Alyssa wishes to spend exactly \$20 buying 49-cent and 53-cent stamps. Show that there is exactly one way to accomplish this! In how many ways could she spend exactly \$100 on such stamps?

3. Consider the equation $5x + 8y = c$. What is the largest value of c for which there are no whole number (non-negative integer) solutions for x and y?

4. Let a and b be two positive integers, where $a \geq b$. Find all pairs a, b such that their sum, their positive difference, their product and their quotient add to 36.

5 (a) When the number 14 has its digits reversed to form the number 41, it is increased by 27. Determine all 2-digit numbers which are increased by 27 when their digits are reversed.

(b) Choose any three-digit integer a b c whose digits are all different.
(When a three-digit integer is written in terms of its digits as a b c, it means the integer is 100a + 10b + c.)

Reverse the order of the digits to get a new three-digit integer c b a.
Subtract the smaller of these integers from the larger to obtain a three-digit integer r s t, where r is allowed to be 0.

Reverse the order of the digits of this integer to get the integer t s r.
Prove that, no matter what three-digit integer a b c you start with, r s t + t s r = 1089.

(c) Suppose that N = a b c d is a four-digit integer with a ≤b ≤ c ≤ d.
When the order of the digits of N is reversed to form the integer M, N is increased by P.
(Again, the first digit of P is allowed to be 0.)
When the order of the digits of P is reversed, an integer Q is formed.
Determine, with justification, all possible values of P + Q.