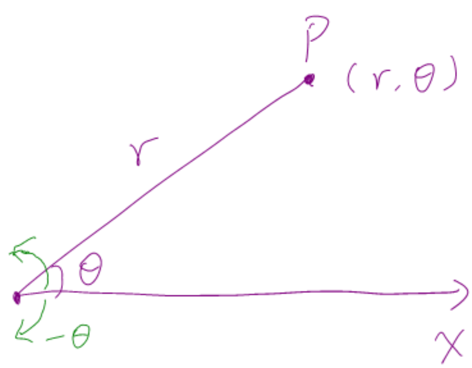
AP Calculus Class 21

Polar Coordinates: $\rightarrow$ Tangents

$\rightarrow$ Areas

Cartesian Coordinates.

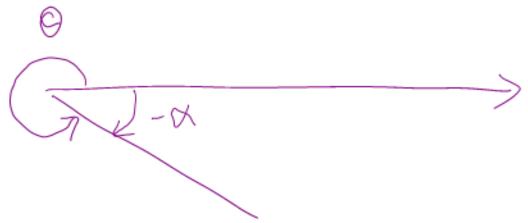
Polar Coordinate.



Pole: origin

Ray: half a line that goes in one direction

Polar axis (x line):
A ray that starts at O.

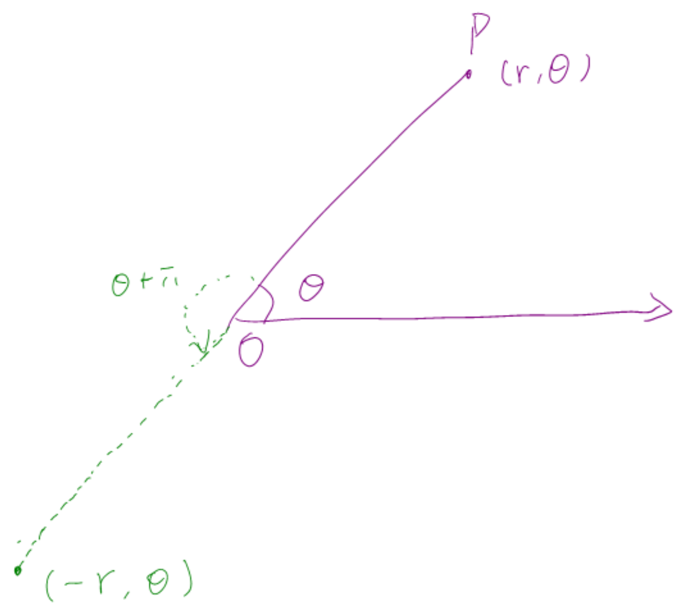


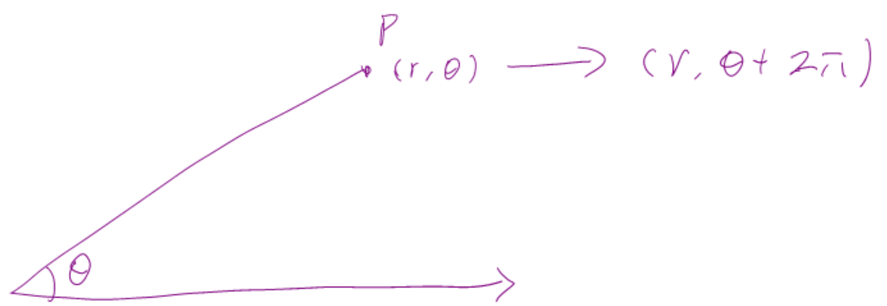
$\sin\theta$

$\sin(-\alpha) = -\sin\alpha$

The coordinate pair (r, θ) is called the polar coordinate of P.

By convention, θ is positive in the counterclockwise direction.





The polar point can have multiple representations.

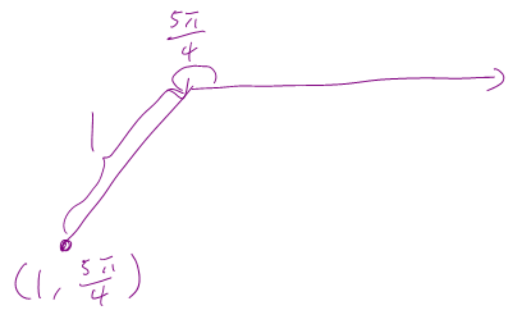
The "first" angle is called the principal angle

$[0, 2\pi]$, $[-\pi, \pi]$

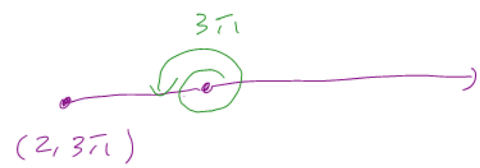
Example: Plot the points whose polar coordinates are given.

a) $\left(1, \frac{5\pi}{4}\right)$ b) $(2, 3\pi)$ c) $\left(2, -\frac{2\pi}{5}\right)$

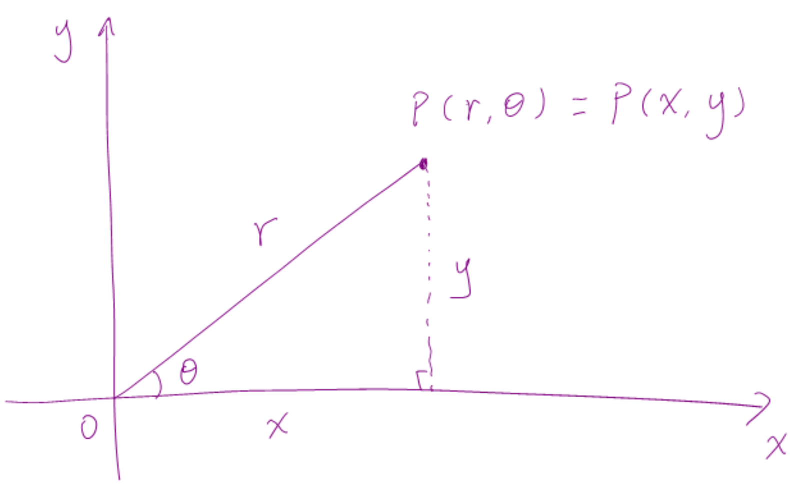
a)



b)



c)



$\cos\theta = \frac{x}{r}$ $\qquad$ $\sin\theta = \frac{y}{r}$

$\Rightarrow$ $\boxed{x = r\cos\theta \qquad y = r\sin\theta}$

$\boxed{x^2+y^2=r^2 \qquad \tan\theta = \frac{y}{x}}$

Example: Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta$$
$$= 2(\cos\tfrac{\pi}{3}) = 2\cdot\tfrac{1}{2} = 1$$

$$y = r\sin\theta$$
$$= 2(\sin\tfrac{\pi}{3}) = 2\cdot\tfrac{\sqrt{3}}{2} = \sqrt{3}$$

$\Rightarrow$ The point is $(1, \sqrt{3})$.

Example: Convert $(1,-1)$ into polar coordinate.

$$r^2 = x^2+y^2$$
$$= 1^2+(-1)^2 = 2 \qquad \Rightarrow r = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1$$

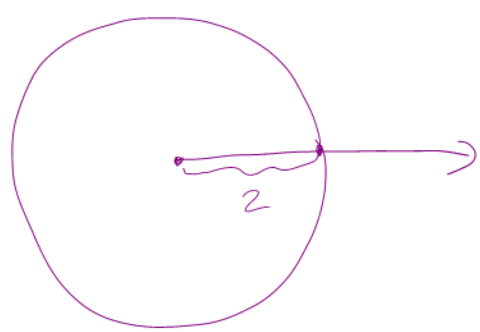
$$\arctan(-1) = -\frac{\pi}{4}$$

$\Rightarrow$ The polar coordinate is $(\sqrt{2}, -\frac{\pi}{4})$.

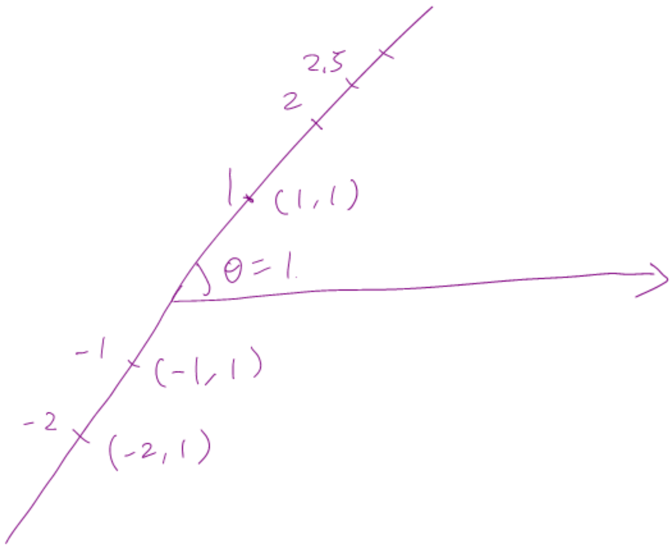
Polar Curve.

Def^n: The graph of a polar equation (polar curve) is written in the form $r = f(\theta)$ or $F(r, \theta) = 0$.

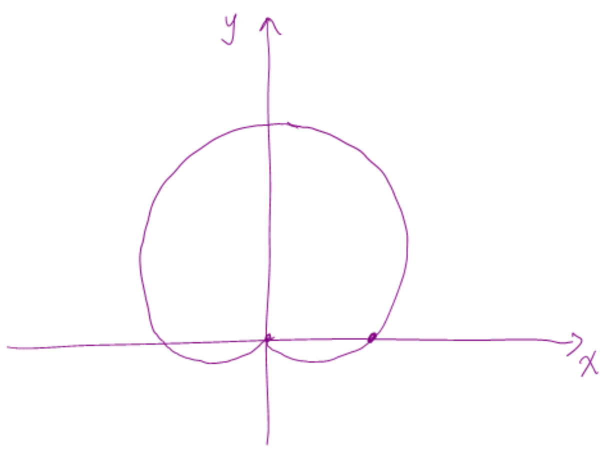
Example: Polare equ^n $r = 2$.



Polar curve with $\theta = 1$.

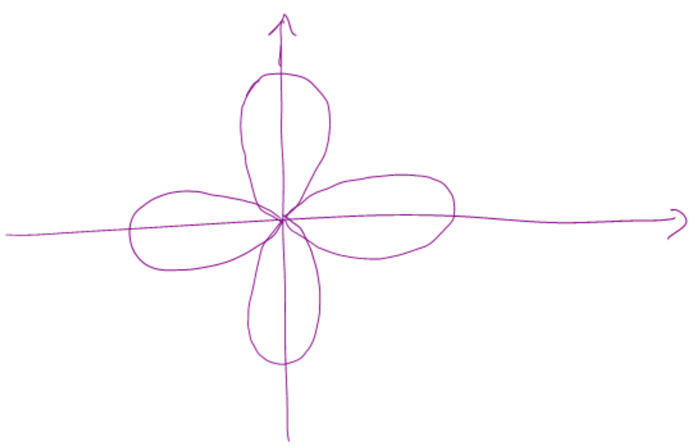


Example: The polar curve $r = 1 + \sin\theta$



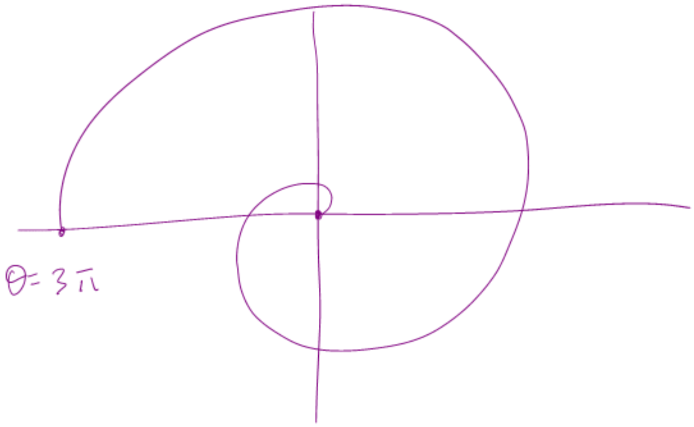
The cardioid.

The polar curve $r = \cos 2\theta$



The four-leaved rose.

The polar curve $r=\theta$



Tangents to Polar Curves.

- The polar curve $r=f(\theta)$.
- Let's regard θ as the parameter

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r'\sin\theta + r(\sin\theta)'}{r'\cos\theta + r(\cos\theta)'}$$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

Example: a) For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$

b) Find the points on the cardiod where the tangent line is horizontal.

a) $r = 1 + \sin\theta$

$$\frac{dr}{d\theta} = \cos\theta.$$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

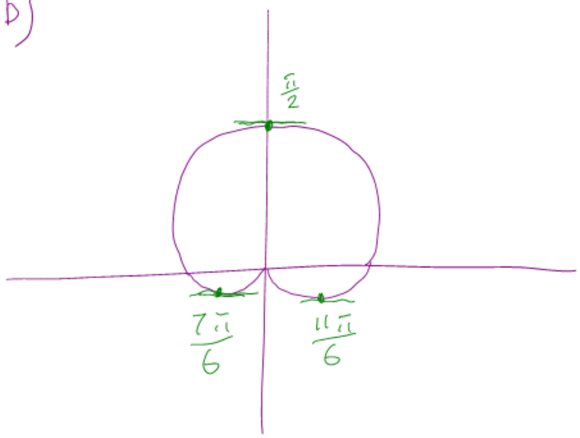
$$= \frac{\cos\theta\,(2\sin\theta + 1)}{\cos^2\theta - \sin\theta - \sin^2\theta}$$

$$= \frac{\cos\theta\,(2\sin\theta + 1)}{(1 - 2\sin^2\theta) - \sin\theta}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{3}} = \frac{\cos(\frac{\pi}{3})\,(2\sin(\frac{\pi}{3}) + 1)}{(1 - 2\sin^2(\frac{\pi}{3})) - \sin(\frac{\pi}{3})}$$

$$= \frac{1+\sqrt{3}}{-(1+\sqrt{3})} = -1$$

b)



We want $\frac{dy}{d\theta} = 0$.

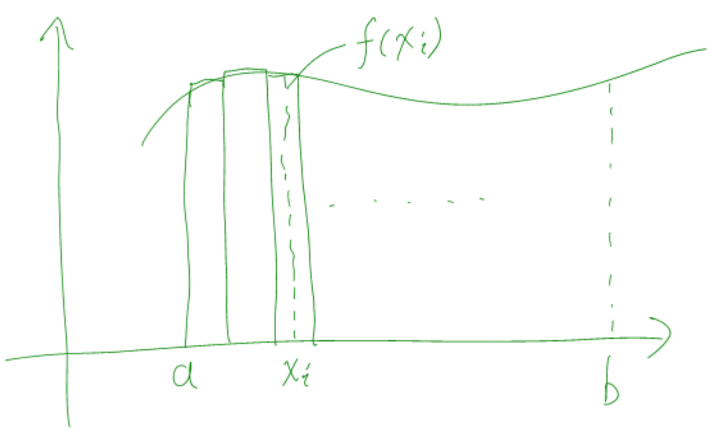
$\Rightarrow \frac{dy}{d\theta} = \cos\theta(2\sin\theta + 1) = 0$.

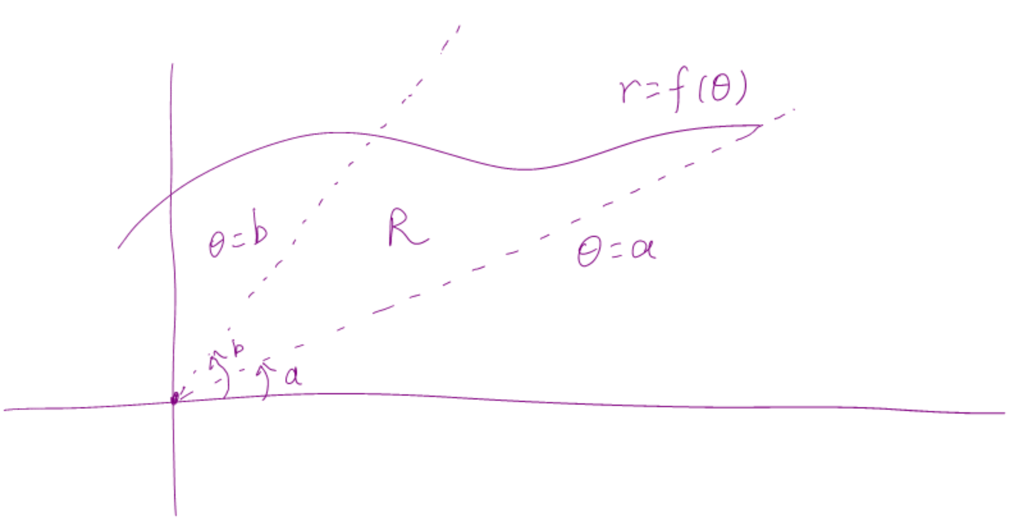
$\Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

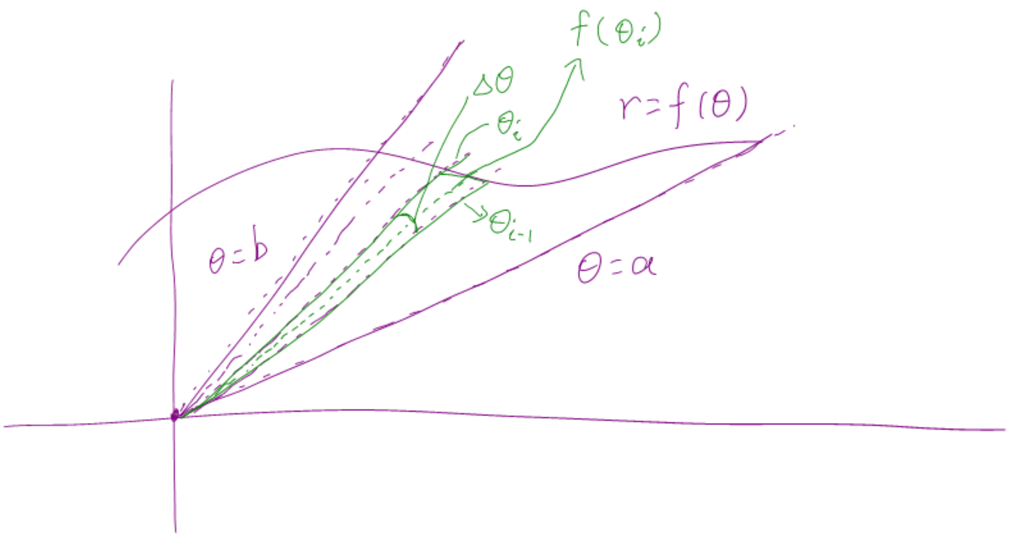
Doesn't work.

$\Rightarrow (2, \frac{\pi}{2}), \quad (\frac{1}{2}, \frac{7\pi}{6}), \quad (\frac{1}{2}, \frac{11\pi}{6})$

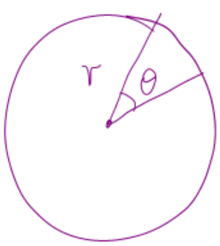
Area for Polar Coordinates.







Area of a sector.



Total area is πr^2

Area of a sector is $A=\frac{\theta}{2\pi}\pi r^2=\frac{1}{2}r^2\theta$

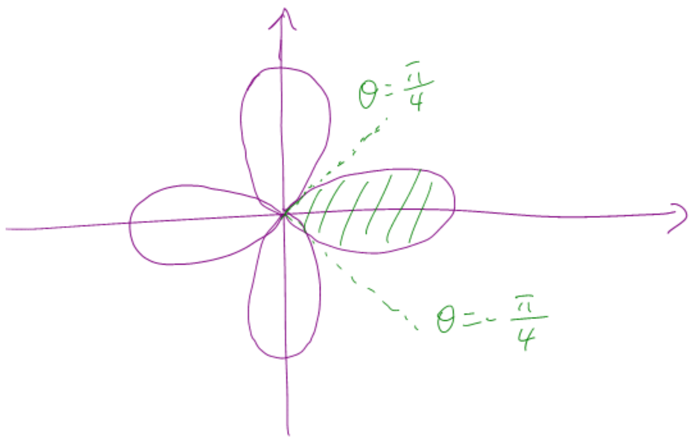
$$\Delta A_i \approx \frac{1}{2}[f(\theta_i)]^2\Delta\theta$$

$$A \approx \sum_{i=1}^{n}\frac{1}{2}[f(\theta_i)]^2\Delta\theta$$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}[f(\theta_i)]^2\Delta\theta=\int_a^b\frac{1}{2}[f(\theta)]^2\,d\theta.$$

$$\boxed{A=\int_a^b\frac{1}{2}[f(\theta)]^2\,d\theta=\int_a^b\frac{1}{2}r^2\,d\theta}$$

Example: Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.



$r = \cos 2\theta$.

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} \cos^2 2\theta \, d\theta.$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1}{2} \cos^2 2\theta \, d\theta = \int_0^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$$

Use trig identity
$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 4\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8}$$

Vector Functions

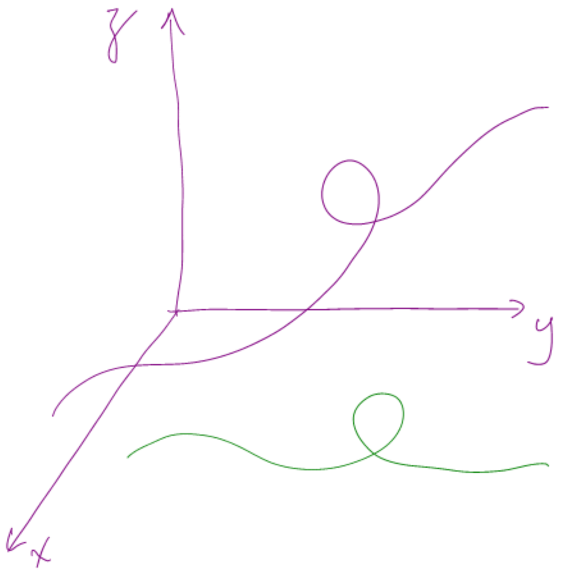
$x, \to f(x) = x^2$

Def^n: A vector fun^n is a fun^n whose domain is a set of real numbers and whose range is a set of vectors.

$\vec{r}(t)$: vector fun^n

$$\vec{r}(t) = (f(t), g(t), h(t)) \longrightarrow \vec{r}(t) = (x, y, z).$$

$$\vec{r}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$$



Multivariable fun^n

$$f(x,y,z) = x^2 + 2z - y.$$

Def^n: If $\vec{r}(t) = (f(t), g(t), h(t))$, then

$$\lim_{t\to a} \vec{r}(t) = \left(\lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t) \right).$$

provided the limits of the components exist.

Example: Find $\lim_{t\to 0} \vec{r}(t)$, where

$$\vec{r}(t) = (1+t^3)\,\hat{i} + te^{-t}\,\hat{j} + \frac{\sin t}{t}\,\hat{k}.$$

$$= \left(1+t^3,\ te^{-t},\ \frac{\sin t}{t} \right).$$

$$\lim_{t\to 0} \vec{r}(t) = \left[\lim_{t\to 0}(1+t^3)\right]\hat{i} + \left[\lim_{t\to 0} te^{-t}\right]\hat{j} + \left[\lim_{t\to 0} \frac{\sin t}{t}\right]\hat{k}.$$

$$= 1\hat{i} + 1\hat{k} = \hat{i} + \hat{k}$$

$$= (1, 0, 1).$$

Derivatives and Integrals of Vector Functions.

Def^n: $\dfrac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{h\to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$

Thm: If $\vec{r}(t) = (f(t), g(t), h(t)) = f(t)i + g(t)j + h(t)k$, where f, g, h are differentiable fun^n, then

$\vec{r}'(t) = (f'(t), g'(t), h'(t)) = f'(t)i + g'(t)j + h'(t)k.$

Example: Differentiate $\vec{r}(t) = (1+t^3, te^{-t}, \sin 2t)$.

$\vec{r}'(t) = (3t^2, e^{-t} - te^{-t}, 2\cos 2t)$.

Diff rules,

$(\vec{f} + \vec{g})' = \vec{f}' + \vec{g}'$

same as single variable fun^n.

Def^n:

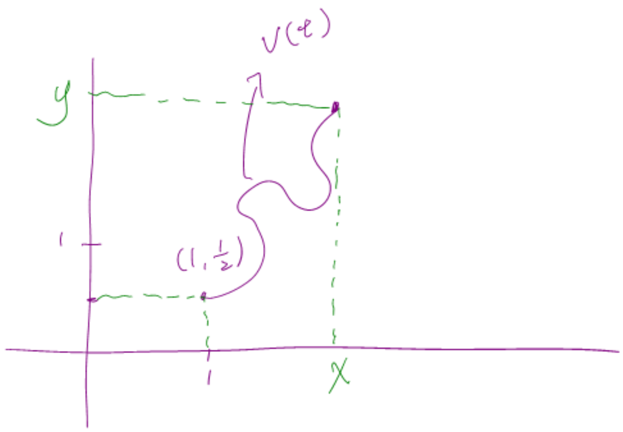
$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)i + \left(\int_a^b g(t)\,dt\right)j + \left(\int_a^b h(t)\,dt\right)k.$$

Example If $\vec{r}(t) = 2\cos t\, i + \sin t\, j + 2t\, k.$

$$\int \vec{r}(t)\,dt = \left(\int 2\cos t\,dt\right)i + \left(\int \sin t\,dt\right)j + \left(\int 2t\,dt\right)k.$$

$$= 2\sin t\, i + (-\cos t)\, j + t^2 k + C.$$

At time $t \geq 0$, a particle moving in the xy-plane has $v(t) = (3, 2^{-t^2})$. If the particle is at $(1, \frac{1}{2})$ at $t=0$, how far is the particle from the origin at time $t=1$?



Evaluate component by component.

$$x = 1 + \int_0^1 v(t)\,dt$$
$$= 1 + \int_0^1 3\,dt$$
$$= 1 + 3 = 4.$$

$$y = \frac{1}{2} + \int_0^1 v(t)\,dt$$

$$= \frac{1}{2} + \int_0^1 2^{-t^2}\,dt$$

$$= \frac{1}{2} + 0.81 = 1.31.$$

Distance from the origin $= \sqrt{4^2 + 1.31^2} = 4.209$