Algebra 3 (Polynomial Functions and Equations)

1. Fractions

A fraction is expressed as $\frac{a}{b}$, where a is called the numerator and b, the denominator.

The addition or subtraction of fractions can be expressed generally as $\frac{a}{b} \pm \frac{c}{d}$.

In order to carry out the operation indicated it is necessary that the expression have a common denominator, b x d. This is accomplished by multiplying both terms by bd, and dividing both terms by bd. That is, $\frac{a}{b} \pm \frac{c}{d}$ is equal to $\frac{ad \pm bc}{bd}$.

As an example, what is 1/2 + 1/3? Using the previous equation , where now a = 1, b = 2, c = 1 and d = 3, we obtain $\frac{1\times 3 + 1\times 2}{2\times 3} = \frac{5}{6}$.

The product of two fractions is the product of their numerators divided by the product of their denominators. That is $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

2. Exponent

A number raised to a power is written as a^x, where a is the base and x is the power or exponent.

When multiplying a^x by a^y the rule is to add the exponents, thus, a^x times a^y equals a^{x+y}. This is obvious since a^x times a^y is (x + y) a's multiplied together.

When dividing a^x by a^y, the rule is to subtract the exponents. Thus a^x divided by a^y is equal to a^{x+y}.

When raising a number to a power, to a further power, the exponents are multiplied. Thus $(a^x)^y$ is equal to a^{xy}.

How do we handle expressions like a^x times b^y? Now the bases are different and the above rules no longer apply, however, if we can express b as a multiple or a power of a then the problem can be solved, as before.

Suppose b can be expressed as a quantity n times a. Then a^x times b^y equals a^x x $(na)^y$, which is equal to n^y times a^x times a^y, which is equal to n^y times $a^{x + y}$.

Alternatively it could be done this way, suppose that b could be written as a^n. Then we would write a^x times b^y is equal to a^x times $(a^n)^y$ which is equal to a^x times a^{ny}, which is equal to a^{x+ny}. On the other hand, if b is neither a simple multiple nor a simple power of a, then the solution is best handled by resorting to the use of logarithms.

It should be mentioned that the above rules for handling powers are valid for the exponent being ± an integer or a fraction.

Note however that any base raised to the exponent of 0, is 1, and that 0 cannot be used as a base.

3. Proportionality

Now let's look at proportionality. Often in physical problems, one deals with parameters that vary as certain other parameters are varied. These parameters are called variables.

Let's consider two cases. The case where the parameter, x, varies directly with y, or sometimes we say x is proportional to y, $x \propto y$.

And the case where x varies inversely with y, or sometimes we say x is inversely proportional to y, $x \propto \frac{1}{y}$.

These expressions can be written as equalities by inserting constants of proportionality, for example x = ky and $x \propto \frac{k^1}{y}$. Where k and k^1 are unique constants, expressed in the appropriate units, that characterize the particular problem.

4. Linear Equation

We will now examine the solutions of a linear equation in one unknown, remember that an equation is simply a statement of equality.

Let us suppose that we are confronted with an equation of the following form, ad = by + c.

We know values for a, b, c, and d, and wish to solve for the unknown designated y.

First put the expression involving the unknown y on the left hand side of the equation, and everything else on the right hand side.

In other words, transpose terms from one side of the equation to the other, remembering that terms change signs when they change sides.

Thus we obtain, by = ad - c. To obtain y alone on the left hand side, it is necessary to divide the left hand side by b, but to retain the equality, the right hand side must also be divided by b, giving $y = \frac{ad - c}{b}$.

The rule is that you can do anything you want to an equation in terms of adding and subtracting terms, multiplying and dividing by terms etc, as long as you do the same thing to both sides of the equation. One word of caution however, division by 0 is not allowed.

5. Factor

Now how do we deal with common factors? Often the solution of an equation can be simplified by removing factors common to all terms.

For example, suppose we wish to solve the following equation for x, $x^2 - a^2 = 2(x - a)^2$, $x \neq a$. Factor both sides of the equation, giving $(x - a)(x + a) = 2(x - a)(x - a)$. Divide both sides of the equation by $(x - a)$, and we get

$(x + a) = 2(x - a)$. Or finally, $x = 3a$.

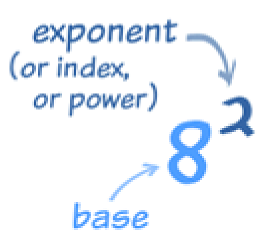
However, we ONLY can do this when we know $x - a \neq 0$, which $x \neq a$. If it is not given in the question, we have to take $x = a$ as another solution into consideration.

i.e. Solve for x: $x^2 - a^2 = 2(x - a)^2$

Solution: $x = a$ or $3a$.

6. Fractional Exponents

Exponents



The exponent of a number says **how many times** to use the number in a **multiplication.**

In this example: $\mathbf{8^2 = 8 \times 8 = 64}$

In words: 8^2 could be called "8 to the second power", "8 to the power 2" or simply "8 squared"

But what if the exponent is a fraction?

Fractional Exponents: ½

Question: What is $x^{½}$?

Answer: $x^{½}$ = the square root of x
(ie $x^{½} = \sqrt{x}$)

Why? Because if you square $x^{½}$ you get: $(x^{½})^2 = x^1 = x$

Similarly, $2^{½} = \sqrt{2}$
Because $\sqrt{2} \times \sqrt{2} = 2$**,** also: $2^{½} \times 2^{½} = 2^1$

To understand that, follow this two-step argument:

1 First, there is the general rule: $\mathbf{(x^m)^n = x^{m \times n}}$
Example: $(x^2)^3 = (xx)^3 = (xx)(xx)(xx) = xxxxxx = x^6$
So $(x^2)^3 = x^{2\times3} = x^6$

2 Now, let's look at what happens when we **square** $x^{½}$:
$(x^{½})^2 = x^{½\times2} = x^1 = x$
When we square $x^{½}$ we get x, so $x^{½}$ must be the square root of x

Let us try that again, but with an exponent of one-quarter (1/4):

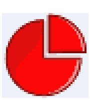
What is $x^{¼}$?
$(x^{¼})^4 = x^{¼\times4} = x^1 = x$

So, what value can be multiplied 4 times to get x?
Answer: ***The fourth root of x.***
So, $x^{¼}$ = The 4th Root of x

General Rule
It worked for **½**, it worked with **¼**, in fact it works generally:

$x^{1/n}$ = The n-th Root of x

So we can come up with this:


A fractional exponent like **1/n** means to **take the n-th root**: $x^{\frac{1}{n}} = \sqrt[n]{x}$

Example: What is $27^{1/3}$?
Answer: $27^{1/3} = \sqrt[3]{27} = 3$

What About More Complicated Fractions?

What about a fractional exponent like $4^{3/2}$? That is really saying to do a **cube** (3) and a **square root** (1/2), in any order.

A fraction (like **m/n**) can be broken into two parts: a whole number part (**m**) , and a fraction (**1/n**) part

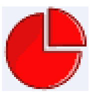
Because **m/n = m × (1/n)** we can do this:

$$x^{\frac{m}{n}} = x^{(m\times\frac{1}{n})} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}$$

The order does not matter, so it also works for **m/n = (1/n) × m**:

$$x^{\frac{m}{n}} = x^{(\frac{1}{n}\times m)} = (x^{\frac{1}{n}})^m = (\sqrt[n]{x})^m$$

And we get this:



A fractional exponent like **m/n** means:
Do the **m-th power**, then take the **n-th root**
OR Take the **n-th root** and then do the **m-th power**

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

Example: What is $4^{3/2}$?

$4^{3/2} = 4^{3\times(1/2)} = \sqrt{4^3} = \sqrt{(4\times4\times4)} = \sqrt{(64)} = \mathbf{8}$

or

$4^{3/2} = 4^{(1/2)\times3} = (\sqrt{4})^3 = (2)^3 = \mathbf{8}$

Either way gets the same result.

Example: What is $27^{4/3}$?

$27^{4/3} = 27^{4\times(1/3)} = \sqrt[3]{(27^4)} = \sqrt[3]{531441} = \mathbf{81}$

or

$27^{4/3} = 27^{(1/3)\times4} = (\sqrt[3]{27})^4 = (3)^4 = \mathbf{81}$

It was certainly easier the 2nd way!

7. Simultaneous Equations

Now let's look at the solutions of two simultaneous equations.
Two equations and two unknowns can be solved in a straight forward fashion using the techniques already developed.

Let us consider the two equations x + y = 3 and 4x - y = 2. We wish to solve for both x and y.

The first step is to eliminate one of the unknowns, this can be accomplished in several ways.

Way 1: By substitution

Solve either one of the equations, for one of the unknowns in terms of the other. Consider the equation x + y = 3, which can be written, x = 3 - y, substitute this into 4x - y = 2 to get 4(3-y) - y = 2.

Solve to get 12 - 4y - y = 2 or 5y = 10 and finally y = 2. Now substitute this value into the previous equation (x = 3 - y) to get x = 3 - 2, which equals 1.

Therefore the solutions to the equation is x = 1 and y = 2.

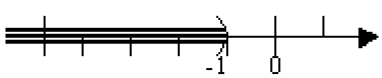
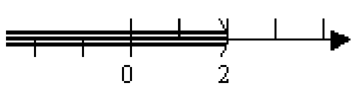
Way 2: By elimination

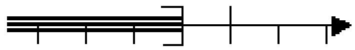
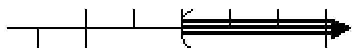
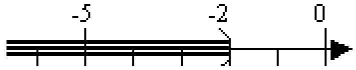
Add both equations, we can eliminate y. We have: 5x = 5 → x = 1, y = 2

Therefore the solutions to the equation is also x = 1 and y = 2.

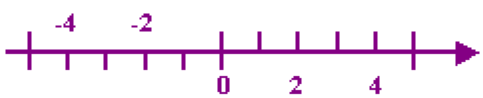
8. Solve Inequality

Solving linear inequalities is very similar to solving linear equations, except for one small but important detail: you flip the inequality sign whenever you multiply or divide the inequality by a negative. The easiest way to show this is with some examples:

1)	$\begin{array}{rcrcr} x & + & 3 & < & 2 \\ & & -3 & & -3 \\ \hline x & & & < & -1 \end{array}$ Graphically, the solution is: 	The only difference between the linear equation "$x + 3 = 2$" and this linear inequality is that I have a "less than" sign, instead of an "equals" sign. The solution method is exactly the same: subtract 3 from either side. Note that the solution to a "less than, but not equal to" inequality is graphed with a parentheses (or else an open dot) at the endpoint, indicating that the endpoint is not included within the solution.
2)	$\begin{array}{rcrcr} 2 & - & x & > & 0 \\ & + & x & & +x \\ \hline 2 & & & > & x \end{array}$ $or \quad x < 2$ Graphically, the solution is: 	The only difference between the linear equation "$2 – x = 0$" and this linear inequality is the "greater than" sign in place of an "equals" sign. Note that "x" in the solution does not "have" to be on the left. However, it is often easier to picture what the solution means with the variable on the left. Don't be afraid to rearrange things to suit your taste.
3)	$\begin{array}{rcrcrcr} 4x & + & 6 & \leq & 3x & - & 5 \\ -3x & & & & -3x & & \\ \hline x & + & 6 & \leq & -5 & & \\ & & -6 & & -6 & & \\ \hline x & & & \leq & -11 & & \end{array}$	The only difference between the linear equation "$4x + 6 = 3x – 5$" and this inequality is the "less than or equal to" sign in place of a plain "equals" sign. The solution method is exactly the same. Note that the solution to a "less than or equal to" inequality is graphed with a square bracket (or else a closed dot) at the

	Graphically, the solution is:  	endpoint, indicating that the endpoint is included within the solution.
4)	$2x > 4$ $x > 2$ Graphically, the solution is:  	The solution method here is to divide both sides by a positive two.
5)	$-2x > 4$ $-2x/-2 < 4/-2$ $x < -2$ Graphically, the solution is:  	This is the special case noted above. When I divided by the **negative** two, I had to flip the inequality sign.

The rule for example 5 above often seems unreasonable to students the first time they see it. But think about inequalities with numbers in there, instead of variables. You know that the number four is larger than the number two: 4 > 2. Multiplying through this inequality by –1, we get –4 < –2, which the number line shows is true:



If we hadn't flipped the inequality, we would have ended up with "–4 > –2", which clearly isn't true.

9. Compound Inequalities

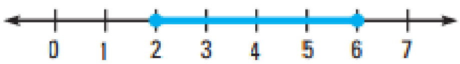
If we join two simple inequalities with the connective "**and**" or the connective "**or**", we get a compound inequality. The solution to two inequalities joined by the word "and" is the **intersection** of their solution sets. The solution to two inequalities joined by the word "or" is the **union** of their solution sets.

Steps to Solve a Compound Inequality
1. Solve and graph each inequality separately.
2. • If the inequalities are joined by the word and, find the intersection of the two solution sets.
• If the inequalities are joined by the word or, find the union of the two solution sets.

Example 1: Solve $-2 \le 3x - 8 \le 10$. Graph the solution.

Isolate the variable x between the two inequality symbols.
$-2 \le 3x - 8 \le 10$ Write original inequality.
$6 \le 3x \le 18$ Add 8 to each expression.
$2 \le x \le 6$ Divide each expression by 3.

The solution is all real numbers that are greater than or equal to 2 and less than or equal to 6.

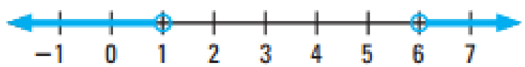


Example 2: Solve $3x + 1 < 4$ or $2x - 5 > 7$. Graph the solution.

A solution of this inequality is a solution of either of its simple parts. You can solve each part separately.

$3x + 1 < 4$	or	$2x - 5 > 7$
$3x < 3$	or	$2x > 12$
$x < 1$	or	$x > 6$

The solution is all real numbers that are less than 1 or greater than 6.



In-class questions (Please do the questions again)

1. Let f be a function for which $f(x/3) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

2. Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. What is the number of possible values of k?

3. What is the sum of the reciprocals of the roots of the equation $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$?

4. The quadratic equation $x^2 + mx + n = 0$ has roots that are twice those of $x^2 + px + m = 0$, and none of m, n, and p is zero. What is the value of $\frac{n}{p}$?

5. Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + (1/b)$ and $b + (1/a)$ are the roots of the equation $x^2 - px + q = 0$. What is q?

6. The polynomial $x^3 - ax^2 + bx - 2010$ has three positive integer zeros. What is the smallest possible value of a?

7. What is the product of all the roots of the equation $\sqrt{5|x|+8} = \sqrt{x^2 - 16}$?

8. The real numbers c, b, a form an arithmetic sequence with $a \geq b \geq c \geq 0$. The quadratic $ax^2 + bx + c$ has exactly one root. What is this root?