Algebra

1. Basics of Algebra

Algebra is a division of mathematics designed to help solve certain types of problems quicker and easier.

Algebra is based on the concept of unknown values called **variables**, unlike arithmetic which is based entirely on known number values.

This lesson introduces an important algebraic concept known as the **Equation**. The idea is that an equation represents a scale such as the one shown on the right.

Instead of keeping the scale balanced with weights, numbers, or constants are used. These numbers are called **constants** because they constantly have the same value.

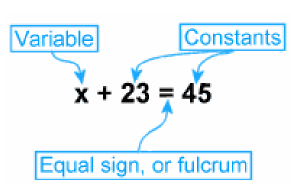
For example the number 47 always represents 47 units or 47 multiplied by an unknown number. It never represents another value.

The equation may also be balanced by a device called a **variable**. A variable is an unknown number represented by any letter in the alphabet (often x).

Several symbols are used to relate all of the variables and constants together. These symbols are listed and explained below.

x	Multiply
*	Multiply
/	Divide
+	Add or Positive
-	Subtract or Negative
( )	Calculate what is inside of the parentheses first. (also called grouping symbols)

2. Basics of the Equation



The diagram on the right shows a basic equation. This equation is similar to problems which you may have done in ordinary mathematics such as:

__ + 16 = 30

You could easily guess that __ equals 14 or do 30 - 16 to find that __ equals 14.

In this problem __ stood for an unknown number; in an equation we use variables, or any letter in the alphabet.

When written algebraically the problem would be: $x + 16 = 30$
and the answer should be written: $x = 14$

3. Solving Equations

These equations can be solved relatively easy and without any formal method. But, as you use equations to solve more complex problems, you will want an easier way to solve them.

Pretend you have a scale like the one shown. On the right side there are 45 pennies and on the left side are 23 pennies and an unknown amount of pennies.

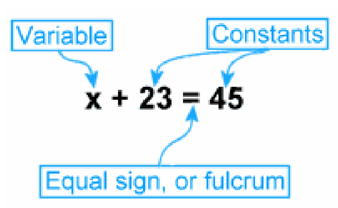
The scale is balanced, therefore, we know that there must be an equal amount of weight on each side.

As long as the same operation (addition, subtraction, multiplication, etc.) is done to both sides of the scale, it will remain balanced.

To find the unknown amount of pennies of the left side, remove 23 pennies from each side of the scale.

This action keeps the scale balanced and isolates the unknown amount.

Since the weight(amount of pennies) on both sides of the scale are still equal and the unknown amount is alone, we now know that the unknown amount of pennies on the left side is the same as the remaining amount (22 pennies) on the right side.



Because an equation represents a scale, it can also be manipulated like one. The diagram below shows a simple equation and the steps to solving it.

Initial Equation / Problem	$x + 23 = 45$
Subtract 23 from each side	$x + 23 - 23 = 45 - 23$
Result / Answer	$x = 22$

Again, to solve the next question you must keep both sides of the equation equal; perform the same operation on each side to get the variable "x" alone. The steps to solving the equation are shown below.

Initial Equation / Problem:	x + 23	= 2x + 45
Subtract x from each side	x - x + 23	= 2x - x + 45
Result	23	= x + 45
Subtract 45 from each side	23 - 45	= x + 45 - 45
Result	-22	= x
Answer	x	= -22

Take a look at the equation below. As you can see, after the variable is subtracted from the left and the constants are subtracted from the right, you are still left with 2x on one side.

Initial Equation / Problem	x + 23	= 3x + 45
Subtract x from each side	x - x + 23	= 3x - x + 45
Result	23	= 2x + 45
Subtract 45 from each side	23 - 45	= 2x + 45 - 45
Result	-22	= 2x
Switch the left and right sides of the equation	2x	= -22

This means that the unknown number multiplied by two, equals -22.

But our end goal is to determine what x is, not what 2x is! We need to divide by the coefficient of the variable. Since we determined that the coefficient of x is 2, we divide each side of the equation by 2:
x = -11

4. Identifying and Using Coefficients

The coefficient of a variable is the number which the variable is being multiplied by.

In the previous equation 2x = -22, 2 is the coefficient of x because 2x is present in the equation. Some additional examples of coefficients:

Term	Coefficient of x
2x	2
0.24x	0.24
x	1
-x	-1

Note that in the last two examples, the following rules are applied

- If the variable has no visible coefficient, then it has an implied coefficient of 1.
- If the variable only has a negative sign, then it has an implied coefficient of -1.

5. Law of Exponents

The exponent of a number says how many times to use the number in a multiplication.
In this example: $8^2 = 8 \times 8 = 64$
2 is the **exponent**. 8 is the **base.**

The **laws of exponents** help us to evaluate or simplify expressions with exponents.

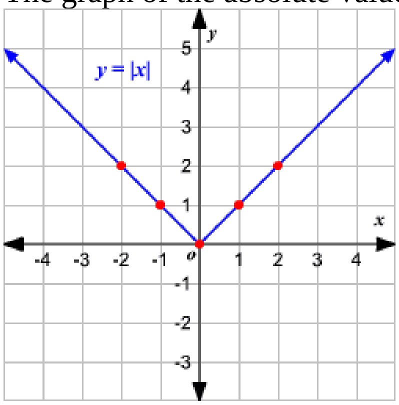
Law	Example
$x^1 = x$	$6^1 = 6$
$x^0 = 1$	$7^0 = 1$
$x^{-1} = 1/x$	$4^{-1} = 1/4$
$x^m x^n = x^{m+n}$	$x^2x^3 = x^{2+3} = x^5$
$x^m/x^n = x^{m-n}$	$x^6/x^2 = x^{6-2} = x^4$
$(x^m)^n = x^{mn}$	$(x^2)^3 = x^{2\times3} = x^6$
$(xy)^n = x^n y^n$	$(xy)^3 = x^3y^3$
$(x/y)^n = x^n/y^n$	$(x/y)^2 = x^2 / y^2$
$x^{-n} = 1/x^n$	$x^{-3} = 1/x^3$

6. Absolute Value

The absolute value of a real number x, |x|, is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The graph of the absolute value function is shown below



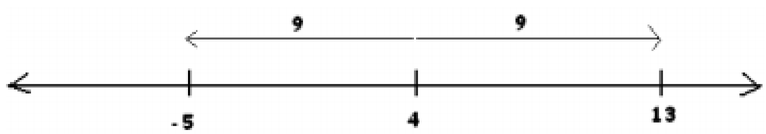
Example 1: $|2| = 2$, $|-2| = -(-2) = 2$

The absolute value function is used to measure the distance between two numbers. Thus, the distance between x and 0 is $|x - 0| = |x|$, and the distance between x and y is $|x - y|$. Thus, the distance from -2 to -4 is $|-2 - (-4)| = |-2 + 4| = |2| = 2$, and the distance from -2 to 5 is $|-2 - 5| = |-7| = 7$.

Example 2: Find all values x that satisfy: $| x - 4 | = 9$

Approach 1: (GEOMETRY) $| x - 4 | = 9$ reads: "The distance between x and 4 is 9."

That is, if $| x - 4 | = 9$, then x is a number whose distance from 4 is 9.That is, x is either 9 units up or 9 units down from 4.



Thus $x = 13$ or $x = -5$

Approach 2: (ARITHMETIC) If $| x - 4 | = 9$, then $(x - 4)$ is a quantity, which, when made positive, equals 9. So either $x - 4 = 9$ or $x - 4 = -9$ Now add 4 throughout to obtain: $x = 13$ or $x = -5$.

7. Inequality

A linear inequality has this standard form: $ax + b < c$.

When a is positive, then solving it is identical to solving an equation:

$$ax < c - b$$

$$x < \frac{c - b}{a}$$

As with equations, the inequality is "solved" when positive x is isolated on the left. The only difference between solving an inequality and solving an equation, is the following:

When we multiply or divide by a negative number, the <u>sign</u> must change.

Example: Solve for x.

$$-2x + 5 < 11$$
$$-2x < 6$$
$$x > -3.$$

The signs changed, of course, because we divided both sides by *negative* 2.

Alternatively, we could immediately make $2x$ positive -- by changing all the signs on both sides. But then we must also change the sense.

$-2x + 5 < 11$ implies $2x - 5 > -11$, and so on.

Questions in class

1. After 15 liters of gasoline was added to a partially filled fuel tank, the tank was 75% full. If the tank's capacity is 28 liters, then what was the number of liters in the tank before adding the gas?

2. The perimeter of a rectangle is 56 meters. The ratio of its length to width is 4: 3. What is the length, in meters, of a diagonal of the rectangle?

3. If x, y, and z solve the system of equations, then what is $x + y + z$?

$$\begin{cases} 2x + 3y - z = -8 \\ 5x - y - z = 3 \\ x - 6y - 3z = 1 \end{cases}$$

4. A three digit decimal number abc may be expressed as 100a +10 b + c where each of the digits is multiplied by its respective place value and subsequently summed. If a = b = c and a > 0, which of the following numbers must be a factor of the three digit number abc?
a) 7 b) 11 c) 17 d) 23 e) 37

5. The sides of a triangle with positive area have lengths 4, 6, and x. The sides of a second triangle with positive area have lengths 4, 6, and y. What is the smallest positive number that is not a possible value of | x - y |?
(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

6. How many solutions does the equation $|x+2| = 2x$ have?

7. The ratio of the length to the width of a rectangle is 4: 3. If the rectangle has diagonal of length d, then the area may be expressed as kd^2 for some constant k. What is k?

8. Points ($\sqrt{\pi}$, a) and ($\sqrt{\pi}$, b) are distinct points on the graph of $y^2 + x^4 = 2x^2y + 1$. What is the value of $|a - b|$?

9. Suppose a, b, c are nonnegative numbers, and $3a + 2b + c = 5$, $2a + b - 3c = 1$. Find the Maximum value of $S = 3a + b - 7c$.