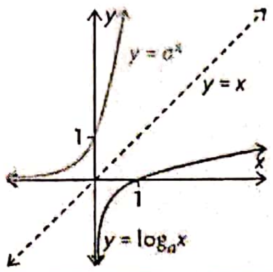
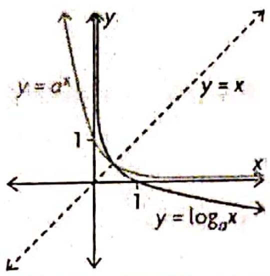
Exploring the Logarithmic Function

- The inverse of the exponential function $y = a^x$ is the logarithmic function $y = \log_a x$ (it is read as "the logarithm of x to the base a").

Properties of the Exponential Function $y = a^x$	Properties of the Logarithmic Function $y = \log_a x$
$a > 0$ and $a \neq 1$	$a > 0$ and $a \neq 1$
if $a > 1$, then the exponential function is an increasing function	if $a > 1$, then the logarithmic function is an increasing function
	
if $0 < a < 1$, then the exponential function is a decreasing function	if $0 < a < 1$, then the logarithmic function is a decreasing function
	
the x-axis is a horizontal asymptote	the y-axis is a vertical asymptote
the y-intercept is 1	the x-intercept is 1
the domain is the set of all real numbers $D : \{x \in \Re\}$	the domain is the set of positive real numbers $D : \{x \in \Re \mid x > 0\}$
the range is the set of positive real numbers $R : \{y \in \Re \mid y > 0\}$	the range is the set of all real numbers $R : \{y \in \Re\}$

- Note that the inverse of the exponential function $y = a^x$ can be written as $x = a^y$.
  Hence, $y = \log_a x$ is equivalent to $x = a^y$.

Example 1 Rewrite each equation in logarithmic form.

a) $49 = 7^2$

b) $3^{-4} = \frac{1}{81}$

c) $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$

Example 2 Rewrite each equation in exponential form.

a) $\log_2 32 = 5$

b) $\log_{\frac{1}{5}} 125 = -3$

c) $\log_{27} 9 = \frac{2}{3}$

Transformations of Logarithmic Functions

- A logarithmic function of the form $f(x) = a\log_{10}[k(x-d)]+c$ can be graphed by applying the appropriate transformations to the parent function $f(x) = \log_{10} x$.

- To graph a transformed logarithmic function, apply the stretches/compressions/reflections given by the parameters a and k first. Then apply the vertical and horizontal translations given by the parameters c and d.

- Consider a logarithmic function of the form $f(x) = a\log_{10}[k(x-d)]+c$.

Transformations of the Parent Function
$\lvert a\rvert$ gives the vertical stretch/compression factor. If $a<0$, there is also a reflection in the x-axis.
$\left\lvert\frac{1}{k}\right\rvert$ gives the horizontal stretch/compression factor. If $k<0$, there is also a reflection in the y-axis.
d gives the horizontal translation
c gives the vertical translation

- The vertical asymptote changes when a horizontal translation is applied. The domain of a transformed logarithmic function depends on where the vertical asymptote is located and whether the function is to the left or the right of the vertical asymptote. If the function is to the left of the asymptote $x=d$, the domain is $x<d$. If it is to the right of the asymptote, the domain is $x>d$.

- The range of a transformed logarithmic function is always $\{y \in \Re\}$.

Example 1 Each of the following functions is a transformation of $f(x)=\log_{10} x$. Describe the transformations that must be applied to $f(x)$ to graph $g(x)$.

a) $g(x)=6\log_{10}[4(x-1)]+2$

b) $g(x)=\frac{3}{5}\log_{10}[-(x+4)]-9$

c) $g(x)=-\log_{10}\left(\frac{1}{2}x-3\right)+7$

Example 2 For each sequence of transformations of the parent function $y=\log_{10} x$, write the equation of the resulting function.

a) vertical stretch by a factor of 8, horizontal stretch by a factor of 3, horizontal translation 2 units left

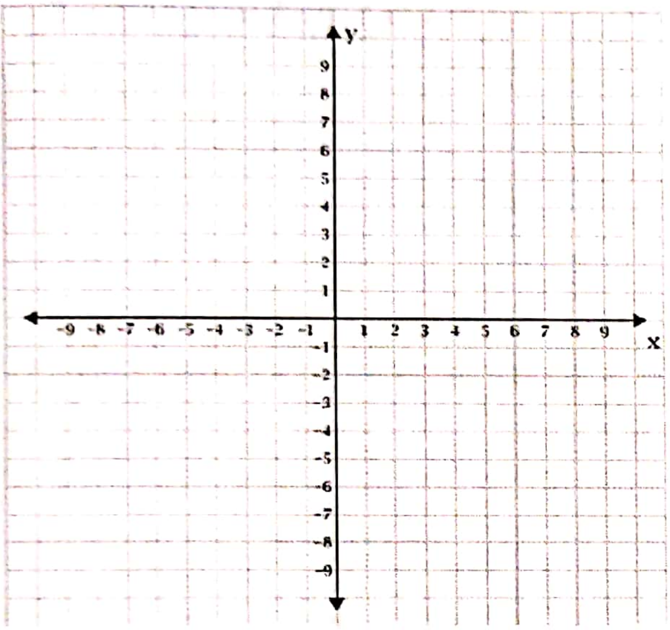
b) horizontal compression by a factor of $\frac{4}{5}$, reflection in the y-axis, vertical translation 10 units down

c) vertical compression by a factor of $\frac{2}{3}$, reflection in the x-axis, horizontal translation 6 units right, vertical translation 8 units up

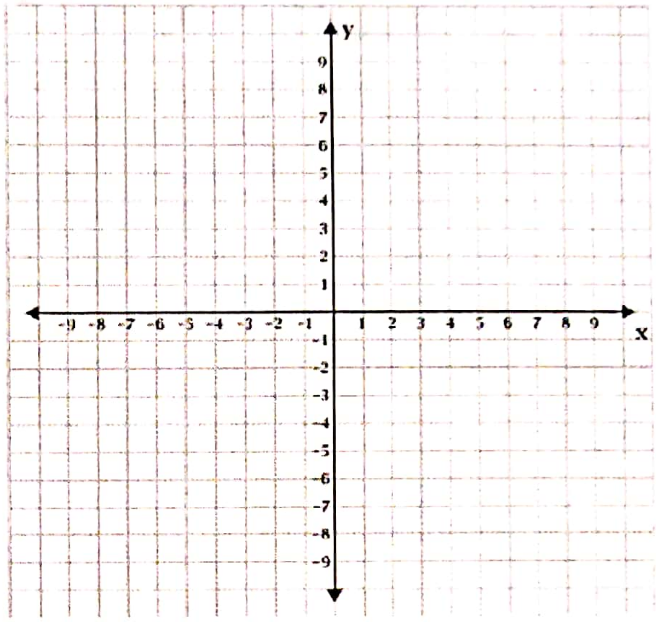
Sketching Exponential and Logarithmic Functions

$y = b^x$ and $y = log_b\, x \quad (x = b^y)$

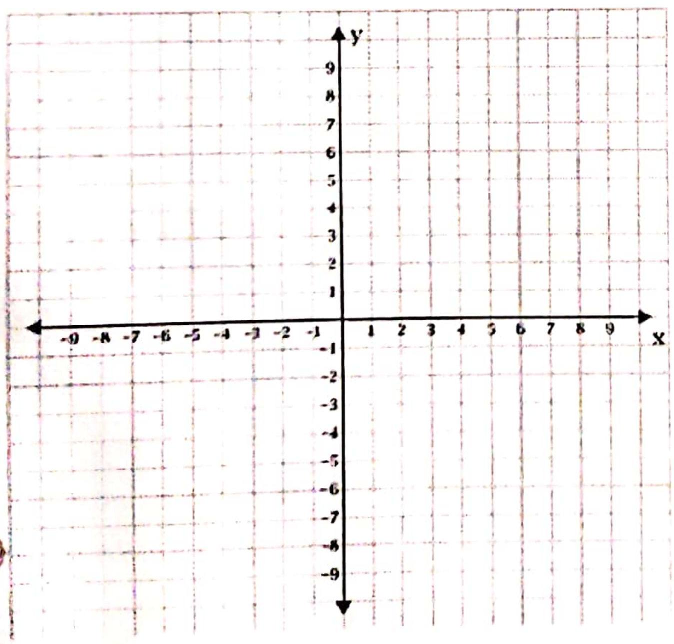
1. $y = 2^x$ and $y = log_2\, x \quad (x = 2^y)$



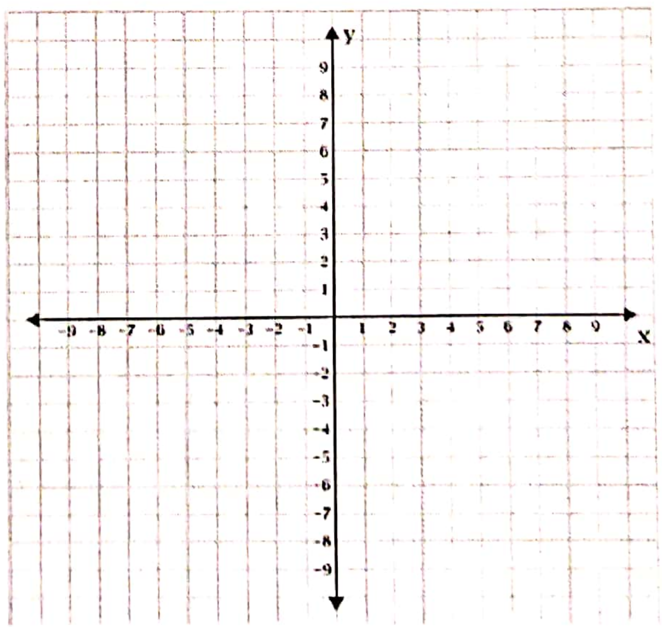
2. $y = 3^x$ and $y = log_3\, x$



3. $y = \left(\frac{1}{2}\right)^x$ and $y = log_{\frac{1}{2}}\, x$



4. $y = \left(\frac{1}{3}\right)^x$ and $y = log_{\frac{1}{3}}\, x$

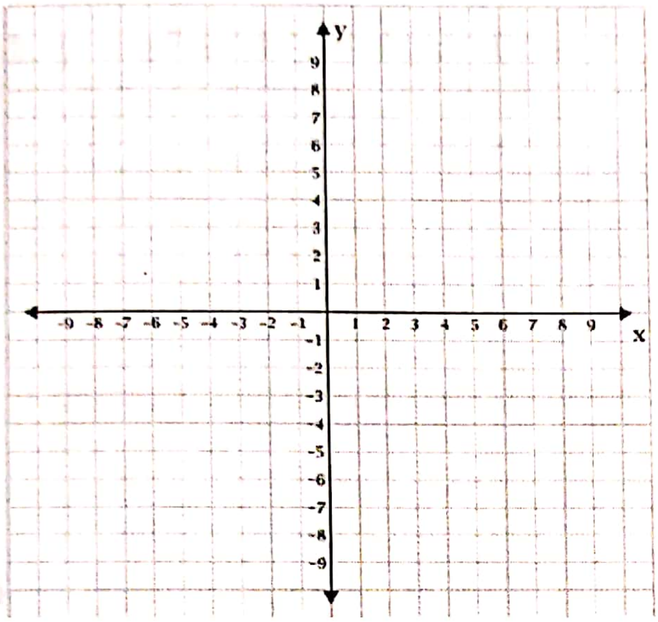


Transformations of Logarithmic Functions

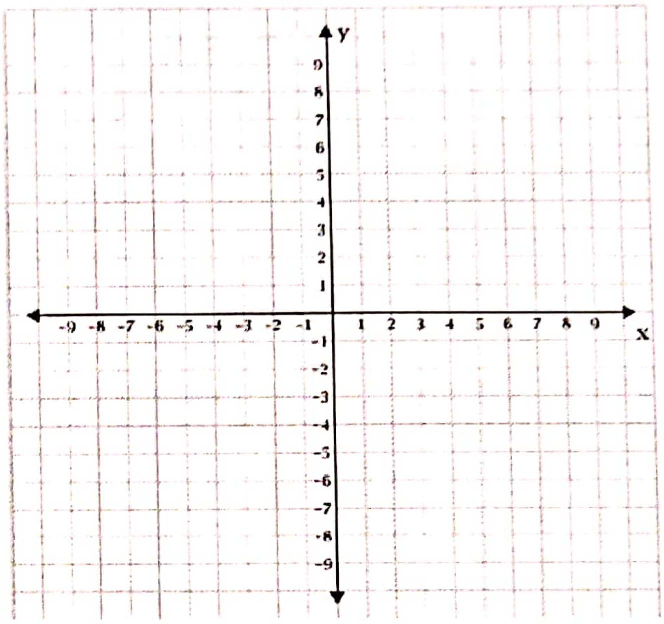
$$y = \log_b x \rightarrow y = a\log_b[k(x-d)] + c$$

$$(x, y) \rightarrow \left(\frac{1}{k}x + d, ay + c\right)$$

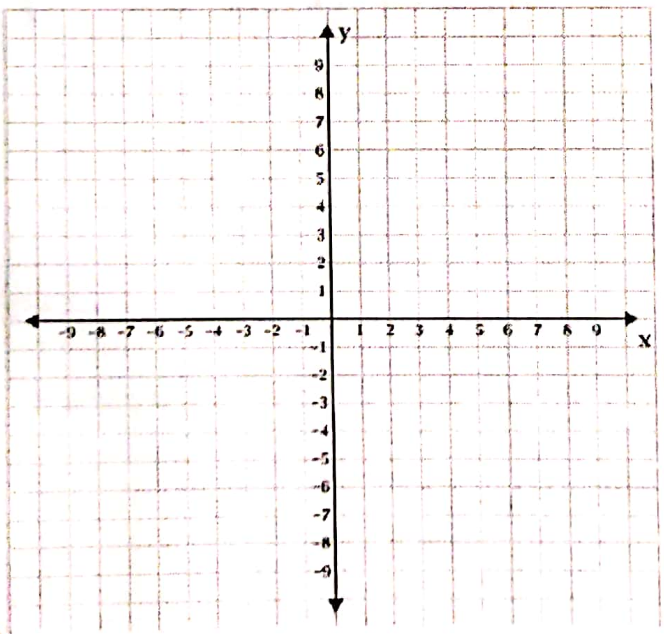
1. $y = log_2(x+4)$



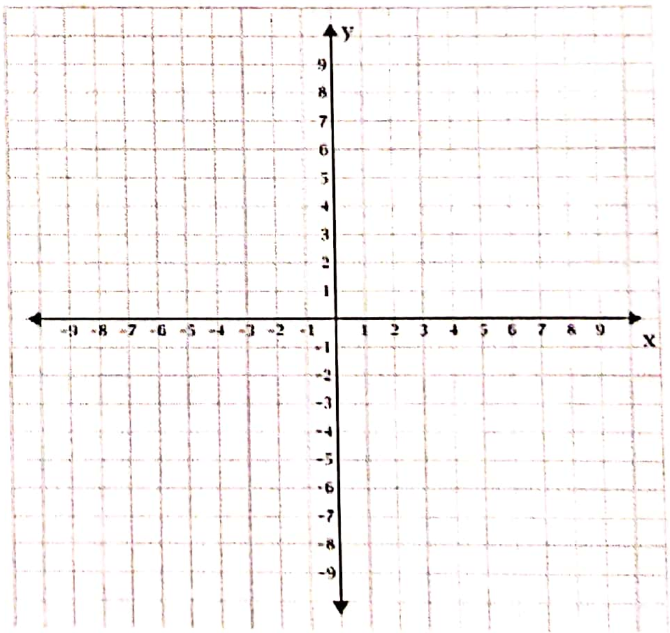
2. $y = -2\,log_3\,x$



3. $y = 3\,log_{\frac{1}{2}}[-(x-3)]$



4. $y = 5\,log(x+6) + 2$

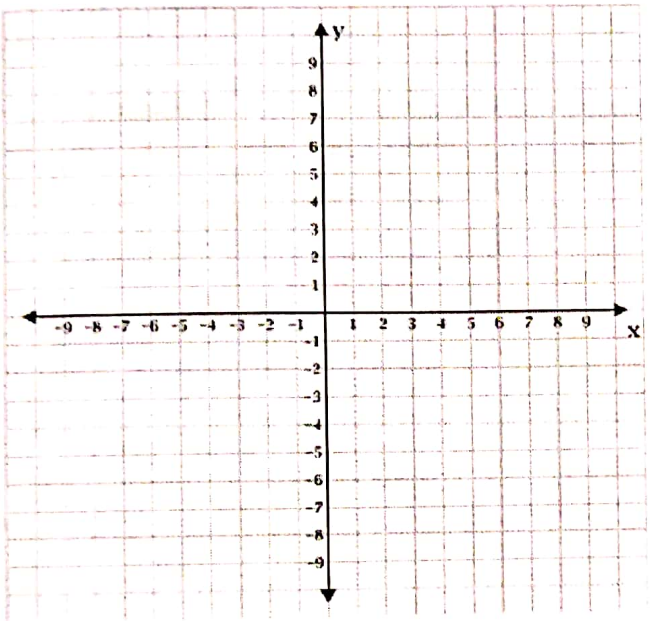


Sketching Natural Logarithmic Functions

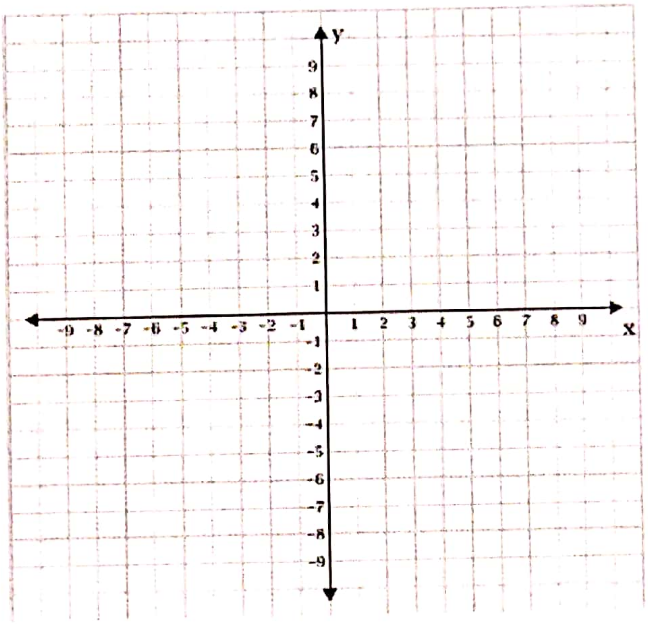
$$y = \ln x \rightarrow y = a\ln[k(x-d)] + c$$

$$(x, y) \rightarrow \left(\frac{1}{k}x + d, ay + c\right)$$

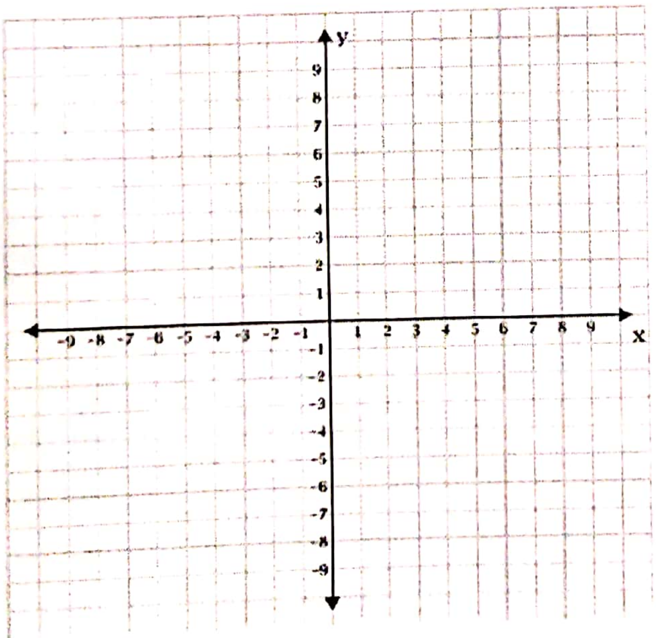
1. $y = e^x$ and $y = \ln x \quad (x = e^y)$



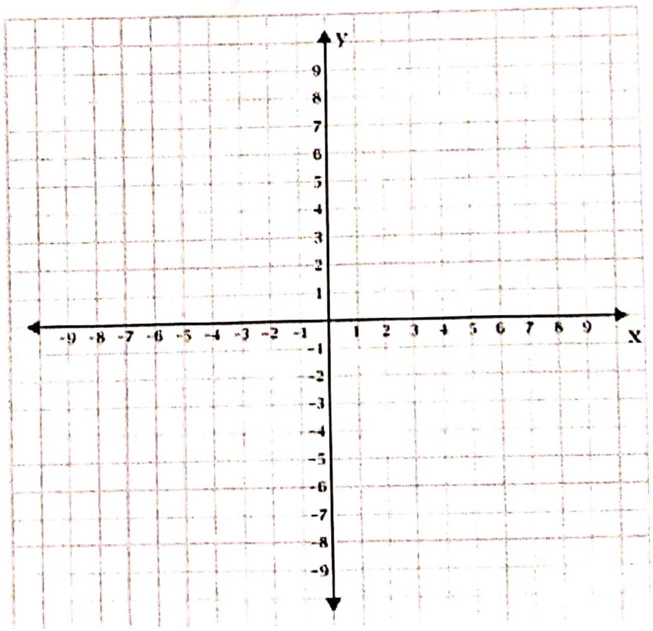
2. $y = \ln(x-1) + 3$



3. $y = \ln\left[-{}^{1}\!/_{2}\,(x+1)\right]$



4. $y = 2\ln(2x-6)$



Evaluating Logarithms

- The following are some properties of logarithms, where $a > 0$ and $a \neq 1$:
  - $\log_a 1 = 0$
  - $\log_a a^x = x$
  - $a^{\log_a x} = x$

- The expression $\log x$ is called a common logarithm. It means $\log_{10} x$, and it can be evaluated using the log key on a calculator.

Example 1 Evaluate each logarithm.

a) $5^{\log_5 29}$

b) $\log_{12} 1$

c) $\log_2 \sqrt[4]{32}$

d) $\log_6 216$

Example 2 Evaluate each common logarithm.

a) $\log 1$

b) $\log 1000$

c) $\log \sqrt[3]{100\,000}$

d) $\log \dfrac{1}{10\,000}$

Laws of Logarithms

The laws of logarithms are as follows, where $x > 0$, $y > 0$, $a > 0$, and $a \neq 1$:

- **product law of logarithms:** $\log_a xy = \log_a x + \log_a y$
- **quotient law of logarithms:** $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
- **power law of logarithms:** $\log_a x^r = r\log_a x$

Example 1 Evaluate each of the following.

a) $\log 1.25 + \log 80$

b) $\log_4 192 - \log_4 3$

c) $\log \sqrt[3]{100}$

d) $\log_2 40 - \log_2 \left(\frac{5}{2}\right)$

e) $\log_8 6 - \log_8 3 + 2\log_8 2$

f) $\log_{25}\left(\sqrt{5} \times 125\right)$

Example 2 Express as a single logarithm.

a) $\log\left(\sqrt[3]{p}\right)+\log\left(\sqrt{p}\right)+\log\sqrt[6]{p}$

b) $\log_3\left(x^2-1\right)-\log_3\left(x+1\right)$

c) $\frac{1}{3}\log_a x+\frac{1}{4}\log_a y-\frac{2}{5}\log_a w$

Example 4 Rewrite each expression with no logarithms of products, quotients, or powers

a) $\log_a\sqrt[4]{\frac{x^3y^2}{w}}$

b) $\log_a\frac{x^3y^4}{\sqrt{x^{\frac{1}{4}}y^{\frac{2}{3}}}}$

Solving Exponential Equations

- If $M = N$, then $\log_a M = \log_a N$, where $a, M, N > 0$ and $a \neq 1$.
  If we cannot rewrite the powers in an exponential equation so that they have the same base, the equation can be solved by taking the logarithm of each side of the equation, then applying the laws of logarithms to solve for the variable.

Example 1 Solve the following exponential equations.

a) $3 = 2^x$

b) $8^{3d+5} = 5^{d+2}$

c) $2 \cdot 3^{2x} = 6^x$

Applications of Exponential Functions

Exponential growth occurs when quantities increase or decrease at a rate proportional to the quantity present. This growth or decay occurs in savings accounts, the size of populations, and the quantity of decay that occurs in radioactive chemicals. These situations can be modelled by exponential functions of the form $f(x) = ab^x$. Note that

- $f(x)$ is the final amount or number
- a is the initial amount or number
- for exponential growth, $b = 1 + growth\ rate$
- for exponential decay, $b = 1 - growth\ rate$
- x is the number of growth or decay periods

Example 1 An investment pays 12% interest, compounded weekly. Determine how long it will take for this investment to triple in value.

Example 2 The value of a particular model of car depreciates by 18% per year. If it sells for $35 000, how long will it take for the car to depreciate to half its original value?

Example 4 A bacteria culture starts with 3000 bacteria. After 3 h, the estimated count is 48 000. What is the doubling period?

Example 5 After 90 days, a sample of silver-110, Ag^{110}, has decayed to about 80% of its original size. Determine the half-life of Ag^{110}.

Solving Logarithmic Equations

- If $\log_a M = \log_a N$, then $M = N$, where $a, M, N > 0$ and $a \neq 1$.
- A logarithmic equation can be solved by simplifying it using the laws of logarithms.
- A logarithmic equation can be solved by expressing it in exponential form and solving the resulting exponential equation.
- When solving logarithmic equations, be sure to check for inadmissible solutions. A solution is inadmissible if its substitution in the original equation results in an undefined value. Remember that the argument and the base of a logarithm must both be positive.

Example 1 Solve the following logarithmic equations.

a) $\log_{11} x + \log_{11}(x+1) = \log_{11} 6$

b) $\log_3(n^2 - 2n - 6) = 2$

c) $\log(3w+6) = 1 + \log w$

d) $\log_7(c+1) + \log_7(c-5) = 1$

Solving Problems of Logarithmic Functions

When a range of values can vary greatly, using a logarithmic scale with powers of 10 makes comparisons between the large and small values more manageable. Scales that measure a wide range of values, such as the pH scale, Richter scale, and decibel scale, are logarithmic scales.

- *Logarithms and Chemistry*

Chemists define the acidity of a liquid on a pH scale,

$pH = -\log\left[H^+\right]$ where $[H^+]$ is the concentration of the hydrogen ion in moles per litre.

A liquid with a pH lower than 7 is called an acid. A substance with a pH greater than 7 is called a base. Chemists calculate the pH of a substance to an accuracy of two decimal places.

Example 1 A soft drink has a pH of approximately 3. What is the concentration of hydronium ions in a soft drink?

Logarithms and Earthquakes

The formula Richter used to define the magnitude of an earthquake is

$M = \log\left(\frac{I}{I_0}\right)$ where I is the intensity of the earthquake being measured,

I_0 is the intensity of a reference earthquake,

and M is the Richter number used to measure the intensity of earthquakes.

Earthquakes below magnitude 4 usually cause no damage, and quakes below 2 cannot be felt. A magnitude 6 earthquake is strong, while one of magnitude 7 or higher causes major damage.

Example 2 An Alaskan earthquake was 4 times more intense than a San Francisco earthquake that had a magnitude of 3.4 on the Richter scale. What was the magnitude of the Alaskan earthquake on the same scale?

Logarithms and Sound

The formula used to compare sound is

$$L = 10\log\left(\frac{I}{I_0}\right)$$

where I is the intensity of the sound being measured,

I_0 is the intensity of a sound at the threshold of hearing,

and L is the loudness measured in decibels.

The following table shows the loudness of a selection of sounds.

30 dB	Soft whisper
60 dB	Normal conversation
80 dB	Shouting
90 dB	Subway
100 dB	Screaming child
120 dB	Rock concert
140 dB	Jet engine
180 dB	Space-shuttle launch

At the threshold of hearing, the loudness of sound is zero decibels (0 dB).

Example 3 A power mower makes a noise that is measured at 106 dB. Ordinary traffic registers about 70 dB. How many times louder is the mower than the traffic?

Solve Exponential and Logarithmic Equations

1. $ln(2x - 1) = 1$

2. $ln(ln\, x) = 2$

3. $e^{e^x} = 5$

4. $e^x - 6e^{-x} = 1$

5. $(\ln x)^2 + (\ln x) - 2 = 0$

6. $e^{3x+6} = 10$

7. $e^x + 4e^{-x} - 5 = 0$

8. $e^{3+x} = \pi^x$

9. $\log_2(\log_3(\log_4(3^{4x-10} - 11))) = 0$

Change of Base

- $log_b\, a = \frac{log_c\, a}{log_c\, b}$
- $\log_b a = \frac{1}{\log_a b}$

1. $2^{x+1} = 5^{1-2x}$

2. $3 \cdot 4^{3x} = 8^x$

3. *Prove*

   a) $\frac{1}{log_5\, a} + \frac{1}{log_3\, a} = \frac{1}{log_{15}\, a}$

   b) $\frac{2}{log_8\, a} - \frac{4}{log_2\, a} = \frac{1}{\log_4 a}$