AP Calculus Lesson Twenty-One Notes

Chapter 9 Infinite Series

9.4 Absolute Convergence
9.5 Power Series and Power Series Representations of Functions

9.4. Absolute Convergence

9.4.1 Definition

A series $\sum a_n$ is called **absolutely convergent** if $\sum |a_n|$ is convergent. If $\sum a_n$ is convergent and $\sum |a_n|$ is divergent we call the series **conditionally convergent**.

Fact

If $\sum a_n$ is absolutely convergent then it is also convergent.

Proof

First notice that $\sum |a_n|$ is either a_n or it is $-a_n$ depending on its sign. This means that we can then say,
$0 \le a_n + |a_n| \le 2|a_n|$
Now, since we are assuming that $\sum |a_n|$ is convergent then $\sum 2|a_n|$ is also convergent since we can just factor the 2 out of the series and 2 times a finite value will still be finite. This however allows us to use the Comparison Test to say that $\sum a_n + |a_n|$ is also a convergent series. Finally, we can write, $\sum a_n = \sum a_n + |a_n| - \sum |a_n|$
and so $\sum a_n$ is the difference of two convergent series and so is also convergent.

Example 9.4-1 Determine if each of the following series are absolute convergent, conditionally convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n^2}$ (c) $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

Solution

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

This is the alternating harmonic series and we saw in the last section that it is a convergent series so we don't need to check that here. So, let's see if it is an absolutely convergent series. To do this we'll need to check the convergence of.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

This is the harmonic series and we know from the integral test section that it is divergent. Therefore, this series is not absolutely convergent. It is however conditionally convergent since the series itself does converge.

(b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+2}}{n^2}$

In this case let's just check absolute convergence first since if it's absolutely convergent we won't need to bother checking convergence as we will get that for free.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+2}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

This series is convergent by the *p*-series test and so the series is absolute convergent. Note that this does say as well that it's a convergent series.

(c) $\sum_{n=1}^{\infty}\frac{\sin n}{n^3}$

In this part we need to be a little careful. First, this is NOT an alternating series and so we can't use any tools from that section.

What we'll do here is check for absolute convergence first again since that will also give convergence. This means that we need to check the convergence of the following series.

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^3}\right| = \sum_{n=1}^{\infty}\frac{|\sin n|}{n^3}$$

To do this we'll need to note that

$$-1 \le \sin n \le 1 \qquad \Rightarrow \qquad |\sin n| \le 1$$

and so we have,

$$\frac{|\sin n|}{n^3} \le \frac{1}{n^3}$$

Now we know that $\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges by the *p*-series test and so by the Comparison Test we also know that $\sum_{n=1}^{\infty}\frac{\sin n}{n^3}$ converges.

Therefore the original series is absolutely convergent (and hence convergent).

Fact

If $\sum a_n$ is absolutely convergent and its value is *s* then any rearrangement of $\sum a_n$ will also have a value of *s*.

9.4.2 Ratio Test

Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

Then,

1. if $L < 1$, the series is absolutely convergent (and hence convergent).
2. If $L > 1$, the series is divergent.
3. if $L = 1$, the series may be divergent, conditionally convergent, or absolutely convergent.

Proof of Ratio Test

First note that we can assume without loss of generality that the series will start at $n = 1$ as we've done for all our series test proofs.

Let's start off the proof here by assuming that $L < 1$, and we'll need to show that $\sum a_n$ is absolutely convergent. To do this let's first note that because $L < 1$, there is some number r such that $L < r < 1$.

Now, recall that,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

and because we also have chosen r such that $L < r$ there is some N such that if $n \geq N$ we will have,

$$\left|\frac{a_{n+1}}{a_n}\right| < r \qquad \Rightarrow \qquad |a_{n+1}| < r|a_n|$$

Next, consider the following,

$$|a_{N+1}| < r\,|a_N|$$
$$|a_{N+2}| < r\,|a_{N+1}| < r^2\,|a_N|$$
$$|a_{N+3}| < r\,|a_{N+2}| < r^3\,|a_N|$$
$$\vdots$$
$$|a_{N+k}| < r\,|a_{N+k-1}| < r^k\,|a_N|$$

So, for $k = 1, 2, 3, \ldots$ we have $|a_{N+K}| < r^K|a_N|$. Just why is this important? Well we can now look at the following series.

$$\sum_{k=0}^{\infty}|a_N|\,r^k$$

This is a geometric series and because $0 < r < 1$, we in fact know that it is a convergent series. Also because $|a_{N+K}| < r^K|a_N|$ by the Comparison test the series

$$\sum_{n=N+1}^{\infty}|a_n| = \sum_{k=1}^{\infty}|a_{N+k}|$$

is convergent. However since,

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N}|a_n| + \sum_{n=N+1}^{\infty}|a_n|$$

we know that $\sum_{n=1}^{\infty} |a_n|$ is also convergent since the first term on the right is a finite sum of finite terms and hence finite. Therefore $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Next, we need to assume that $L > 1$, and we'll need to show that $\sum a_n$ is divergent. Recalling that,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and because $L > 1$, we know that there must be some N such that if $n \geq N$ we will have,

$$\left| \frac{a_{n+1}}{a_n} \right| > 1 \qquad \Rightarrow \qquad |a_{n+1}| > |a_n|$$

However, if $|a_{n+1}| > |a_n|$ for all $n \geq N$ then we know that, $\lim_{n\to\infty} |a_n| \neq 0$ because the terms are getting larger and guaranteed to not be negative. This in turn means that, $\lim_{n\to\infty} a_n \neq 0$

Therefore, by the Divergence Test $\sum a_n$ is divergent.

Finally, we need to assume that $L = 1$, and show that we could get a series that has any of the three possibilities. To do this we just need a series for each case. We'll leave the details of checking to you but all three of the following series have $L = 1$, and each one exhibits one of the possibilities.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{absolutely convergent}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad \text{conditionally convergent}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{divergent}$$

Example 9.4-2 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$$

Solution

With this first example let's be a little careful and make sure that we have everything down correctly. Here are the series terms a_n.

$$a_n = \frac{(-10)^n}{4^{2n+1}(n+1)}$$

Recall that to compute a_{n+1} all that we need to do is substitute $n+1$ for all the n's in a_n.

$$a_{n+1} = \frac{(-10)^{n+1}}{4^{2(n+1)+1}((n+1)+1)} = \frac{(-10)^{n+1}}{4^{2n+3}(n+2)}$$

Now, to define L we will use,

$$L = \lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right|$$

since this will be a little easier when dealing with fractions as we've got here. So,

$$L = \lim_{n\to\infty} \left| \frac{(-10)^{n+1}}{4^{2n+3}(n+2)} \frac{4^{2n+1}(n+1)}{(-10)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{-10(n+1)}{4^2(n+2)} \right|$$

$$= \frac{10}{16} \lim_{n\to\infty} \frac{n+1}{n+2}$$

$$= \frac{10}{16} < 1$$

So, $L < 1$ and so by the Ratio Test the series converges absolutely and hence will converge.

Example 9.4-3 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{n!}{5^n}$$

Solution

Now that we've worked one in detail we won't go into quite the detail with the rest of these. Here is the limit.

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!}{5^{n+1}} \frac{5^n}{n!} \right| = \lim_{n\to\infty} \frac{(n+1)!}{5\ n!}$$

In order to do this limit we will need to eliminate the factorials. We simply can't do the limit with the factorials in it. To eliminate the factorials we will recall from our discussion on factorials above that we can always "strip out" terms from a factorial. If we do that with the numerator (in this case because it's the larger of the two) we get,

$$L = \lim_{n\to\infty} \frac{(n+1)\ n!}{5\ n!}$$

at which point we can cancel the n! for the numerator an denominator to get,

$$L = \lim_{n\to\infty} \frac{(n+1)}{5} = \infty > 1$$

So, by the Ratio Test this series diverges.

9.4.3 Root Test

Suppose that we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.

3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

Proof of Root Test

First note that we can assume without loss of generality that the series will start at $n = 1$ as we've done for all our series test proofs. Also note that this proof is very similar to the proof of the Ratio Test.

Let's start off the proof here by assuming that $L < 1$ and we'll need to show that $\sum a_n$ is absolutely convergent. To do this let's first note that because $L < 1$ there is some number r such that $L < r < 1$. Now, recall that,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

and because we also have chosen r such that L < r there is some N such that if $n \geq N$ we will have,

$$|a_n|^{\frac{1}{n}} < r \qquad \Rightarrow \qquad |a_n| < r^n$$

Now the series $\sum_{n=0}^{\infty} r^n$ is a geometric series and because $0 < r < 1$ we in fact know that it is a convergent series. Also because $|a_n| < r^n$, $n \geq N$ by the Comparison test the series $\sum_{n=N}^{\infty} |a_n|$ is convergent. However since,

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|$$

we know that $\sum_{n=1}^{\infty} |a_n|$ is also convergent since the first term on the right is a finite sum of finite terms and hence finite. Therefore $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Next, we need to assume that L > 1 and we'll need to show that $\sum a_n$ is divergent. Recalling that,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

and because $L > 1$ we know that there must be some N such that if $n \geq N$ we will have,

$$|a_n|^{\frac{1}{n}} > 1 \qquad \Rightarrow \qquad |a_n| > 1^n = 1$$

However, if $|a_n| > 1$ for all $n \geq N$ then we know that,

$$\lim_{n\to\infty} |a_n| \neq 0$$

This in turn means that,

$$\lim_{n\to\infty} a_n \neq 0$$

Therefore, by the Divergence Test $\sum a_n$ is divergent.

Finally, we need to assume that L = 1 and show that we could get a series that has any of the three possibilities. To do this we just need a series for each case. We'll leave the details of checking to you but all three of the following series have L = 1 and each one exhibits one of the possibilities.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{absolutely convergent}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad \text{conditionally convergent}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{divergent}$$

Example 9.4-4 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \left(\frac{5n-3n^3}{7n^3+2} \right)^n$$

Solution

Again, there isn't too much to this series.

$$L = \lim_{n\to\infty} \left| \left(\frac{5n-3n^3}{7n^3+2} \right)^n \right|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{5n-3n^3}{7n^3+2} \right| = \left| \frac{-3}{7} \right| = \frac{3}{7} < 1$$

Therefore, by the Root Test this series converges absolutely and hence converges.
Note that we had to keep the absolute value bars on the fraction until we'd taken the limit to get the sign correct.

Example 9.4-5 Determine if the following series is convergent or divergent.

$$\sum_{n=3}^{\infty} \frac{(-12)^n}{n}$$

Solution

Here's the limit for this series.

$$L = \lim_{n\to\infty} \left| \frac{(-12)^n}{n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{12}{n^{\frac{1}{n}}} = \frac{12}{1} = 12 > 1$$

After using the fact from above we can see that the Root Test tells us that this series is divergent.

9.4.4 Strategy for Series

Now that we've got all of our tests out of the way it's time to think about organizing all of them into a general set of guidelines to help us determine the convergence of a series.

Note that these are a general set of guidelines and because some series can have more than one test applied to them we will get a different result depending on the path that we take through this set of guidelines. In fact, because more than one test may apply, you should always go completely through the guidelines and identify all possible tests that can be used on a given series. Once this has been done you can identify the test that you feel will be the easiest for you to use.
With that said here is the set of guidelines for determining the convergence of a series.

1. With a quick glance does it look like the series terms don't converge to zero in the limit, *i.e.* does $\lim_{n\to\infty} a_n \neq 0$? If so, use the Divergence Test. Note that you should only do the divergence test if a quick glance suggests that the series terms may not converge to zero in the limit.
2. Is the series a *p*-series ($\sum \frac{1}{n^p}$) or a geometric series ($\sum_{n=1}^{\infty} ar^n$)? If so use the fact that *p*-series will only converge if p > 1 and a geometric series will only converge if $|r| < 1$. Remember as well that often some algebraic manipulation is required to get a geometric series into the correct form.

3. Is the series similar to a *p*-series or a geometric series? If so, try the Comparison Test.

4. Is the series a rational expression involving only polynomials or polynomials under radicals (*i.e.* a fraction involving only polynomials or polynomials under radicals)? If so, try the Comparison Test and/or the Limit Comparison Test. Remember however, that in order to use the Comparison Test and the Limit Comparison Test the series terms all need to be positive.

5. Does the series contain factorials or constants raised to powers involving *n*? If so, then the Ratio Test may work. Note that if the series term contains a factorial then the only test that we've got that will work is the Ratio Test.

6. Can the series terms be written in the form $a_n = (-1)^n b_n$? If so, then the Alternating Series Test may work.

7. Can the series terms be written in the form $a_n = (b_n)^n$? If so, then the Root Test may work.
8. If $a_n = f(n)$ for some positive, decreasing function and $\int_a^{\infty} f(x)dx$ is easy to evaluate then the Integral Test may work.

9.4.5. Estimating the Value of an Alternating Series

Once again we will start off with a convergent series
$\sum a_n = \sum (-1)^n b_n$

which in this case happens to be an alternating series, so we know that $b_n \geq 0$ for all *n*. Also note that we could have any power on the "−1" we just used *n* for the sake of convenience. We want to know how good of an estimation of the actual series value will the partial sum, s_n, be. As with the prior cases we know that the remainder, R_n, will be the error in the estimation and so if we can get a handle on that we'll know approximately how good the estimation is.

From the proof of the Alternating Series Test we can see that *s* will lie between s_n and s_{n+1} for any *n* and so,

$$|s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}$$

Therefore,

$$|R_n| = |s - s_n| \leq b_{n+1}$$

We needed absolute value bars because we won't know ahead of time if the estimation is larger or smaller than the actual value and we know that the b_n's are positive.
Let's take a look at an example.

Example 9.4-6 Using $n = 15$ to estimate the value of

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Solution

This is an alternating series and is does converge. In this case the exact value is known and so for comparison purposes,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} = -0.8224670336$$

Now, the estimation is,

$$s_{15} = \sum_{n=1}^{15} \frac{(-1)^n}{n^2} = -0.8245417574$$

From the fact above we know that

$$|R_{15}| = |s - s_{15}| \leq b_{16} = \frac{1}{16^2} = 0.00390625$$

So, our estimation will have an error of no more than 0.00390625. In this case the exact value is known and so the actual error is,

$$|R_{15}| = |s - s_{15}| = 0.0020747238$$

9.5 Power Series and Power Series Representations of Functions

9.5.1 Power Series

In this section we are going to start talking about power series. A **power series about a**, or just **power series**, is any series that can be written in the form,

$$\sum_{n=0}^{\infty} c_n \left(x-a\right)^n$$

where a and c_n are numbers. The c_n's are often called the **coefficients** of the series. The first thing to notice about a power series is that it is a function of x. That is different from any other kind of series that we've looked at to this point. In all the prior sections we've only allowed numbers in the series and now we are allowing variables to be in the series as well. This will not change how things work however. Everything that we know about series still holds.

In the discussion of power series convergence is still a major question that we'll be dealing with. The difference is that the convergence of the series will now depend upon the value of x that we put into the series. A power series may converge for some values of x and not for other values of x.

Before we get too far into power series there is some terminology that we need to get out of the way.

First, as we will see in our examples, we will be able to show that there is a number R so that the power series will converge for, $|x-a| < R$ and will diverge for $|x-a| > R$. This number is called the **radius of convergence** for the series. Note that the series may or may not converge if $|x-a| = R$. What happens at these points will not change the radius of convergence.

Secondly, the interval of all x's, including the end points if need be, for which the power series converges is called the **interval of convergence** of the series.

These two concepts are fairly closely tied together. If we know that the radius of convergence of a power series is R then we have the following.

$$a-R < x < a+R \qquad \text{power series converges}$$

$$x < a-R \text{ and } x > a+R \qquad \text{power series diverges}$$

The interval of convergence must then contain the interval $a - R < x < a + R$ since we know that the power series will converge for these values. We also know that the interval of convergence can't contain x's in the ranges $x < a - R$ and $x > a + R$ since we know the power series diverges for these value of x. Therefore, to completely identify the interval of convergence all that we have to do is determine if the power series will converge for $x = a - R$ or $x = a + R$.

If the power series converges for one or both of these values then we'll need to include those in the interval of convergence.

Before getting into some examples let's take a quick look at the convergence of a power series for the case of $x = a$. In this case the power series becomes,

$$\sum_{n=0}^{\infty} c_n (a-a)^n = \sum_{n=0}^{\infty} c_n (0)^n = c_0 (0)^0 + \sum_{n=1}^{\infty} c_n (0)^n = c_0 + \sum_{n=1}^{\infty} 0 = c_0 + 0 = c_0$$

and so the power series converges. Note that we had to strip out the first term since it was the only non-zero term in the series.
It is important to note that no matter what else is happening in the power series we are guaranteed to get convergence for $x = a$. The series may not converge for any other value of x, but it will always converge for $x = a$.

Let's work some examples. We'll put quite a bit of detail into the first example and then not put quite as much detail in the remaining examples.

Example 9.5-1 Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$$

Solution
Let's jump right into the ratio test.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{2^{n+1}(4x-8)^{n+1}}{n+1} \; \frac{n}{2^n (4x-8)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{2n(4x-8)}{n+1} \right| \\ &= |4x-8| \lim_{n\to\infty} \frac{2n}{n+1} \\ &= 2|4x-8| \end{aligned}$$

So we will get the following convergence/divergence information from this.

$$\begin{aligned} &2|4x-8| < 1 && \text{series converges} \\ &2|4x-8| > 1 && \text{series diverges} \end{aligned}$$

We need to be careful here in determining the interval of convergence. The interval of convergence requires $|x-a| < R$ and $|x-a| > R$. In other words, we need to factor a 4 out of the absolute value bars in order to get the correct radius of convergence. Doing this gives,

$$\begin{aligned} &8|x-2| < 1 && \Rightarrow && |x-2| < \frac{1}{8} && \text{series converges} \\ &8|x-2| > 1 && \Rightarrow && |x-2| > \frac{1}{8} && \text{series diverges} \end{aligned}$$

So, the radius of convergence for this power series is R = 1/8.

Now, let's find the interval of convergence. Again, we'll first solve the inequality that gives convergence above.

$$-\frac{1}{8} < x - 2 < \frac{1}{8}$$
$$\frac{15}{8} < x < \frac{17}{8}$$

Now check the end points.

If x = 15/8
The series here is,

$$\sum_{n=1}^{\infty}\frac{2^n}{n}\left(\frac{15}{2}-8\right)^n = \sum_{n=1}^{\infty}\frac{2^n}{n}\left(-\frac{1}{2}\right)^n$$
$$= \sum_{n=1}^{\infty}\frac{2^n}{n}\frac{(-1)^n}{2^n}$$
$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

This is the alternating harmonic series and we know that it converges.

if x = 17/8
The series here is,

$$\sum_{n=1}^{\infty}\frac{2^n}{n}\left(\frac{17}{2}-8\right)^n = \sum_{n=1}^{\infty}\frac{2^n}{n}\left(\frac{1}{2}\right)^n$$
$$= \sum_{n=1}^{\infty}\frac{2^n}{n}\frac{1}{2^n}$$
$$= \sum_{n=1}^{\infty}\frac{1}{n}$$

This is the harmonic series and we know that it diverges.

So, the power series converges for one of the end points, but not the other. This will often happen so don't get excited about it when it does. The interval of convergence for this power series is then,

$$\frac{15}{8} \le x < \frac{17}{8}$$

Example 9.5-2 Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=0}^{\infty} n!(2x+1)^n$$

Solution
We'll start this example with the ratio test as we have for the previous ones.

$$L = \lim_{n \to \infty} \left| \frac{(n+1)!(2x+1)^{n+1}}{n!(2x+1)^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(n+1)n!\ (2x+1)}{n!} \right|$$

$$= |2x+1| \lim_{n \to \infty} (n+1)$$

At this point we need to be careful. The limit is infinite, but there is that term with the x's in front of the limit. We'll have $L = \infty > 1$ provided $x \neq -1/2$ So, this power series will only converge if $x = -1/2$. If you think about it we actually already knew that however. From our initial discussion we know that every power series will converge for $x = a$ and in this case $x = -1/2$. Remember that we get a from $(x-a)^n$ and notice the coefficient of the x must be a one!.

In this case we say the radius of convergence is $R = 0$ and the interval of convergence is $x = -1/2$ and yes we really did mean interval of convergence even though it's only a point.

Example 9.5-3 Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(x-6)^n}{n^n}$$

Solution

In this example the root test seems more appropriate. So,

$$L = \lim_{n \to \infty} \left| \frac{(x-6)^n}{n^n} \right|^{\frac{1}{n}}$$

$$= \lim_{n \to \infty} \left| \frac{x-6}{n} \right|$$

$$= |x-6| \lim_{n \to \infty} \frac{1}{n}$$

$$= 0$$

So, since $L = 0 < 1$ regardless of the value of x this power series will converge for every x.

In these cases we say that the radius of convergence is $R = \infty$ and interval of convergence is $-\infty < x < \infty$.

9.5.2 Power Series and Functions

With this section we will start talking about how to represent functions with power series. Let's start off with one that we already know how to do, although when we first ran across this series we didn't think of it as a power series nor did we acknowledge that it represented a function.

Recall that the geometric series is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \qquad \text{provided } |r| < 1$$

Don't forget as well that if $|r| \geq 1$ the series diverges.

Now, if we take a = 1 and r = x this becomes,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \qquad \text{provided } |r| < 1 \tag{1}$$

Turning this around we can see that we can represent the function

$$f(x) = \frac{1}{1-x} \tag{2}$$

with the power series

$$\sum_{n=0}^{\infty} x^n \qquad \text{provided } |x| < 1 \tag{3}$$

Example 9.5-4 Find a power series representation for the following function and determine its interval of convergence.

$$g(x) = \frac{1}{1+x^3}$$

Solution

What we need to do here is to relate this function back to (2). This is actually easier than it might look. Recall that the *x* in (2) is simply a variable and can represent anything. So, a quick rewrite of $g(x)$ gives,

$$g(x) = \frac{1}{1-(-x^3)}$$

and so the $-x^3$ in $g(x)$ holds the same place as the *x* in (2). Therefore, all we need to do is replace the *x* in (3) and we've got a power series representation for $g(x)$.

$$g(x) = \sum_{n=0}^{\infty} \left(-x^3\right)^n \qquad \text{provided } \left|-x^3\right| < 1$$

Notice that we replaced both the *x* in the power series and in the interval of convergence. All we need to do now is a little simplification.

$$g(x) = \sum_{n=0}^{\infty} (-1)^n x^{3n} \qquad \text{provided } |x|^3 < 1 \qquad \Rightarrow \qquad |x| < 1$$

So, in this case the interval of convergence is the same as the original power series. This usually won't happen. More often than not the new interval of convergence will be different from the original interval of convergence.

Example 9.5-5 Find a power series representation for the following function and determine its interval of convergence.

$$h(x) = \frac{2x^2}{1+x^3}$$

Solution

This function is similar to the previous function. The difference is the numerator and at first glance that looks to be an important difference. Since (2) doesn't have an *x* in the numerator it appears that we can't relate this function back to that.

However, now that we've worked the first example this one is actually very simple since we can use the result of the answer from that example. To see how to do this let's first rewrite the function a little.

$$h(x) = 2x^2 \frac{1}{1+x^3}$$

Now, from the first example we've already got a power series for the second term so let's use that to write the function as,

$$h(x) = 2x^2 \sum_{n=0}^{\infty} (-1)^n x^{3n} \qquad \text{provided } |x| < 1$$

Notice that the presence of *x*'s outside of the series will NOT affect its convergence and so the interval of convergence remains the same.

The last step is to bring the coefficient into the series and we'll be done. When we do this make sure and combine the *x*'s as well. We typically only want a single *x* in a power series.

$$h(x) = \sum_{n=0}^{\infty} 2(-1)^n x^{3n+2} \qquad \text{provided } |x| < 1$$

Example 9.5-6 Find a power series representation for the following function and determine its interval of convergence.

$$f(x) = \frac{x}{5-x}$$

Solution

So, again, we've got an *x* in the numerator. So, as with the last example let's factor that out and see what we've got left.

$$f(x) = x \frac{1}{5-x}$$

If we had a power series representation for

$$g(x) = \frac{1}{5-x}$$

we could get a power series representation for *f*(*x*). So, let's find one. We'll first notice that in order to use (4) we'll need the number in the denominator to be a one. That's easy enough to get.

$$g(x) = \frac{1}{5}\frac{1}{1-\frac{x}{5}}$$

Now all we need to do to get a power series representation is to replace the *x* in (3) with *x/*5. Doing this gives,

$$g(x) = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n \qquad \text{provided} \quad \left|\frac{x}{5}\right| < 1$$

Now let's do a little simplification on the series.

$$\begin{aligned} g(x) &= \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^n}{5^n} \\ &= \sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} \end{aligned}$$

The interval of convergence for this series is,

$$\left|\frac{x}{5}\right| < 1 \qquad \Rightarrow \qquad \frac{1}{5}|x| < 1 \qquad \Rightarrow \qquad |x| < 5$$

Okay, this was the work for the power series representation for *g*(*x*), let's now find a power series representation for the original function. All we need to do for this is to multiply the power series representation for *g*(*x*) by *x* and we'll have it.

$$\begin{aligned} f(x) &= x\,\frac{1}{5-x} \\ &= x\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}} \end{aligned}$$

The interval of convergence doesn't change and so it will be $|x| < 5$.

9.5.3. Differentiation and Integration of Power Series

We now need to look at some further manipulation of power series that we will need to do on occasion. We need to discuss differentiation and integration of power series. Let's start with differentiation of the power series,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Now, we know that if we differentiate a finite sum of terms all we need to do is differentiate each of the terms and then add them back up. With infinite sums there are some subtleties involved that we need to be careful with, but are somewhat beyond the scope of this course.

Nicely enough for us however, it is known that if the power series representation of $f(x)$ has a radius of convergence of $R = 0$ then the term by term differentiation of the power series will also have a radius of convergence of R and (more importantly) will in fact be the power series representation of $f'(x)$ provided we stay within the radius of convergence.

Again, we should make the point that if we aren't dealing with a power series then we may or may not be able to differentiate each term of the series to get the derivative of the series. So, what all this means for is that,

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n (x-a)^n = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$$

Note the initial value of this series. It has been changed from $n = 0$ to $n = 1$. This is an acknowledgement of the fact that the derivative of the first term is zero and hence isn't in the derivative. Notice however, that since the $n=0$ term of the above series is also zero, we could start the series at $n = 0$ if it was required for a particular problem. In general however, this won't be done in this class.

We can now find formulas for higher order derivatives as well now.

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n (x-a)^{n-2}$$

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)c_n (x-a)^{n-3}$$

$$etc.$$

Once again, notice that the initial value of n changes with each differentiation in order to acknowledge that a term from the original series differentiated to zero.

Let's now briefly talk about integration. Just as with the differentiation, when we've got an infinite series we need to be careful about just integration term by term. Much like with derivatives it turns out that as long as we're working with power series we can just integrate the terms of the series to get the integral of the series itself. In other words,

$$\begin{aligned}\int f(x)\,dx &= \int \sum_{n=0}^{\infty} c_n (x-a)^n\,dx \\ &= \sum_{n=0}^{\infty} \int c_n (x-a)^n\,dx \\ &= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}\end{aligned}$$

Notice that we pick up a constant of integration, C, that is outside the series here.

Let's summarize the differentiation and integration ideas as follows.

If $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence of $R > 0$ then,

$$f'(x)=\sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \qquad \int f(x)\,dx = C+\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

and both of these also have a radius of convergence of R.

Example 9.5-7 Find a power series representation for the following function and determine its interval of convergence.

$$g(x)=\frac{1}{(1-x)^2}$$

Solution

To do this problem let's notice that

$$\frac{1}{(1-x)^2}=\frac{d}{dx}\left(\frac{1}{1-x}\right)$$

Then since we've got a power series representation for $\frac{1}{1-x}$, all that we'll need to do is differentiate that power series to get a power series representation for $g(x)$.

$$\begin{aligned} g(x) &= \frac{1}{(1-x)^2} \\ &= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\ &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) \\ &= \sum_{n=1}^{\infty} n x^{n-1} \end{aligned}$$

Then since the original power series had a radius of convergence of $R = 1$, the derivative, and hence $g(x)$, will also have a radius of convergence of $R = 1$.

Example 9.5-8 Find a power series representation for the following function and determine its interval of convergence.

$$h(x)=\ln(5-x)$$

Solution

In this case we need to notice that

$$\int \frac{1}{5-x}\,dx = -\ln(5-x)$$

and the recall that we have a power series representation for $\frac{1}{5-x}$.

Remember we found a representation for this in Example 3. So,

$$\begin{aligned}\ln(5-x) &= -\int \frac{1}{5-x}\,dx \\ &= -\int \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}\,dx \\ &= C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,5^{n+1}}\end{aligned}$$

We can find the constant of integration, C, by plugging in a value of x. A good choice is $x = 0$ since that will make the series easy to evaluate.

$$\begin{aligned}\ln(5-0) &= C - \sum_{n=0}^{\infty} \frac{0^{n+1}}{(n+1)\,5^{n+1}} \\ \ln(5) &= C\end{aligned}$$

So, the final answer is,

$$\ln(5-x) = \ln(5) - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,5^{n+1}}$$

Note that it is okay to have the constant sitting outside of the series like this. In fact, there is no way to bring it into the series so don't get excited about it.

9.5.4 Taylor Series

In the previous section we started looking at writing down a power series representation of a function. The problem with the approach in that section is that everything came down to needing to be able to relate the function in some way to $\frac{1}{1-x}$, and while there are many functions out there that can be related to this function there are many more that simply can't be related to this.

So, without taking anything away from the process we looked at in the previous section, we need to do is come up with a more general method for writing a power series representation for a function.

So, for the time being, let's make two assumptions. First, let's assume that the function $f(x)$ does in fact have a power series representation about $x = a$,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

Next, we will need to assume that the function, $f(x)$, has derivatives of every order and that we can in fact find them all.

Now that we've assumed that a power series representation exists we need to determine what the coefficients, c_n, are. This is easier than it might at first appear to be. Let's first just evaluate everything at $x = a$. This gives,

$$f(a) = c_0$$

So, all the terms except the first are zero and we now know what c_0 is. Unfortunately, there isn't any other value of x that we can plug into the function that will allow us to quickly find any of the other coefficients. However, if we take the derivative of the function (and its power series) then plug in $x = a$ we get,

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$
$$f'(a) = c_1$$

and we now know c_1.

Lets' continue with this idea and find the second derivative.

$$f''(x) = 2c_2 + 3(2)c_3(x-a) + 4(3)c_4(x-a)^2 + \cdots$$
$$f''(a) = 2c_2$$

So, it looks like,

$$c_2 = \frac{f''(a)}{2}$$

Using the third derivative gives,

$$f'''(x) = 3(2)c_3 + 4(3)(2)c_4(x-a) + \cdots$$
$$f'''(a) = 3(2)c_3 \qquad \Rightarrow \qquad c_3 = \frac{f'''(a)}{3(2)}$$

Using the fourth derivative gives,

$$f^{(4)}(x) = 4(3)(2)c_4 + 5(4)(3)(2)c_5(x-a)\cdots$$
$$f^{(4)}(a) = 4(3)(2)c_4 \qquad \Rightarrow \qquad c_4 = \frac{f^{(4)}(a)}{4(3)(2)}$$

Hopefully by this time you've seen the pattern here. It looks like, in general, we've got the following formula for the coefficients.

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This even works for $n=0$ if you recall that $0! = 1$ and define

$$f^{(0)}(x) = f(x)$$

So, provided a power series representation for the function $f(x)$ about $x = a$ exists.

The **Taylor Series for *f*(*x*) about *x* = *a*** is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

If we use $a = 0$, so we talking about the Taylor Series about $x = 0$, we call the series a

Maclaurin Series for $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Before working any examples of Taylor Series we first need to address the assumption that a Taylor Series will in fact exist for a given function. Let's start out with some notation and definitions that we'll need.

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To determine a condition that must be true in order for a Taylor series to exist for a function let's first define the **n^{th} degree Taylor polynomial of *f*(*x*) a**s,

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Note that this really is a polynomial of degree at most *n*! If we were to write out the sum without the summation notation this would clearly be an n^{th} degree polynomial. We'll see a nice application of Taylor polynomials in the next section. Notice as well that for the full Taylor Series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the n^{th} degree Taylor polynomial is just the partial sum for the series.

Next, the **remainder** is defined to be,

$$R_n(x) = f(x) - T_n(x)$$

So, the remainder is really just the *error* between the function $f(x)$ and the n^{th} degree Taylor polynomial for a given *n*. With this definition note that we can then write the function as,

$$f(x) = T_n(x) + R_n(x)$$

We now have the following Theorem.

Theorem

Suppose that $f(x) = T_n(x) + R_n(x)$. Then if $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$ then,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

on $|x-a| < R$.

In general showing that $\lim_{n\to\infty} R_n(x) = 0$ is a somewhat difficult process and so we will be assuming that this can be done for some R in all of the examples that we'll be looking at.

Now let's look at some examples.

Example 9.5-9 Find the Taylor Series for $f(x) = e^x$ about $x = 0$.

Solution

This is actually one of the easier Taylor Series that we'll be asked to compute. To find the Taylor Series for a function we will need to determine a general formula for $f^{(n)}(a)$. This is one of the few functions where this is easy to do right from the start.

To get a formula for $f^{(n)}(0)$ all we need to do is recognize that,

$$f^{(n)}(x) = \mathbf{e}^x \qquad n = 0,1,2,3,\ldots$$

and so,

$$f^{(n)}(0) = \mathbf{e}^0 = 1 \qquad n = 0,1,2,3,\ldots$$

Therefore, the Taylor series for $f(x) = e^x$ about $x=0$ is,

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example 9.5-10 Find the Taylor Series for $f(x) = \cos(x)$ about $x = 0$.

Solution

First we'll need to take some derivatives of the function and evaluate them at $x=0$.

$$\begin{array}{ll}
f^{(0)}(x) = \cos x & f^{(0)}(0) = 1 \\
f^{(1)}(x) = -\sin x & f^{(1)}(0) = 0 \\
f^{(2)}(x) = -\cos x & f^{(2)}(0) = -1 \\
f^{(3)}(x) = \sin x & f^{(3)}(0) = 0 \\
f^{(4)}(x) = \cos x & f^{(4)}(0) = 1 \\
f^{(5)}(x) = -\sin x & f^{(5)}(0) = 0 \\
f^{(6)}(x) = -\cos x & f^{(6)}(0) = -1 \\
\vdots & \vdots
\end{array}$$

In this example, unlike the previous ones, there is not an easy formula for either then general derivative or the evaluation of the derivative. However, there is a clear pattern to the evaluations. So, let's plug what we've got into the Taylor series and see what we get,

$$\begin{aligned}\cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots \\ &= \underbrace{1}_{n=0} + \underbrace{0}_{n=1} \underbrace{- \frac{1}{2!}x^2}_{n=2} + \underbrace{0}_{n=3} \underbrace{+ \frac{1}{4!}x^4}_{n=4} + \underbrace{0}_{n=5} \underbrace{- \frac{1}{6!}x^6}_{n=6} + \cdots\end{aligned}$$

So, we only pick up terms with even powers on the *x*'s. This doesn't really help us to get a general formula for the Taylor Series. However, let's drop the zeroes and "renumber" the terms as follows to see what we can get.

$$\cos x = \underbrace{1}_{n=0} \underbrace{- \frac{1}{2!}x^2}_{n=1} \underbrace{+ \frac{1}{4!}x^4}_{n=2} \underbrace{- \frac{1}{6!}x^6}_{n=3} + \cdots$$

By renumbering the terms as we did we can actually come up with a general formula for the Taylor Series and here it is,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Example 9.5-11 Find the Taylor Series $f(x) = \sin(x)$ about $x = 0$.

Solution

As with the last example we'll start off in the same manner.

$$\begin{aligned}
&f^{(0)}(x) = \sin x && f^{(0)}(0) = 0 \\
&f^{(1)}(x) = \cos x && f^{(1)}(0) = 1 \\
&f^{(2)}(x) = -\sin x && f^{(2)}(0) = 0 \\
&f^{(3)}(x) = -\cos x && f^{(3)}(0) = -1 \\
&f^{(4)}(x) = \sin x && f^{(4)}(0) = 0 \\
&f^{(5)}(x) = \cos x && f^{(5)}(0) = 1 \\
&f^{(6)}(x) = -\sin x && f^{(6)}(0) = 0 \\
&\quad\vdots && \quad\vdots
\end{aligned}$$

So, we get a similar pattern for this one. Let's plug the numbers into the Taylor Series.

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\end{aligned}$$

In this case we only get terms that have an odd exponent on x and as with the last problem once we ignore the zero terms there is a clear pattern and formula. So renumbering the terms as we did in the previous example we get the following Taylor Series.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Example 9.5-12 Find the Taylor Series $f(x) = \ln(x)$ about $x = 2$.

Solution

Here are the first few derivatives and the evaluations.

$$\begin{array}{ll}
f^{(0)}(x) = \ln(x) & f^{(0)}(2) = \ln 2 \\
f^{(1)}(x) = \dfrac{1}{x} & f^{(1)}(2) = \dfrac{1}{2} \\
f^{(2)}(x) = -\dfrac{1}{x^2} & f^{(2)}(2) = -\dfrac{1}{2^2} \\
f^{(3)}(x) = \dfrac{2}{x^3} & f^{(3)}(2) = \dfrac{2}{2^3} \\
f^{(4)}(x) = -\dfrac{2(3)}{x^4} & f^{(4)}(2) = -\dfrac{2(3)}{2^4} \\
f^{(5)}(x) = \dfrac{2(3)(4)}{x^5} & f^{(5)}(2) = \dfrac{2(3)(4)}{2^5} \\
\vdots & \vdots \\
f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{x^n} & f^{(n)}(2) = \dfrac{(-1)^{n+1}(n-1)!}{2^n} \qquad n = 1, 2, 3, \ldots
\end{array}$$

Note that while we got a general formula here it doesn't work for $n = 0$. This will happen on occasion so don't worry about it when it does.

In order to plug this into the Taylor Series formula we'll need to strip out the $n = 0$ term first.

$$\begin{aligned}
\ln(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= f(2) + \sum_{n=1}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!\ 2^n}(x-2)^n \\
&= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\, 2^n}(x-2)^n
\end{aligned}$$

Notice that we simplified the factorials in this case. You should always simplify them if there are more than one and it's possible to simplify them.

Also, do not get excited about the term sitting in front of the series. Sometimes we need to do that when we can't get a general formula that will hold for all values of n.

Before leaving this section there are three important Taylor Series that we've derived in this section that we should summarize up in one place. In my class I will assume that you know these formulas from this point on.

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Example 9.5-13 Determine a Taylor Series about $x = 0$ for the following integral.

$$\int \frac{\sin x}{x}\,dx$$

Solution

To do this we will first need to find a Taylor Series about $x = 0$ for the integrand. This however isn't terribly difficult. We already have a Taylor Series for sine about $x = 0$ so we'll just use that as follows,

$$\frac{\sin x}{x} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

We can now do the problem.

$$\int \frac{\sin x}{x}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\,dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,(2n+1)!}$$

So, while we can't integrate this function in terms of known functions we can come up with a series representation for the integral.

9.5.5. Power Series over Complex Numbers

A complex number is one of the form $a + bi$, where a and b are real and $i^2 = -1$. If we allow complex numbers as replacements for x in power series, we obtain some interesting results.

Consider, for instance, the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \qquad (1)$$

when $x = yi$, then (1) becomes

$$\begin{aligned} e^{yi} &= 1 + yi + \frac{(yi)^2}{2!} + \frac{(yi)^3}{3!} + \cdots + \frac{(yi)^n}{n!} + \cdots \\ &= 1 + yi - \frac{y^2}{2!} - \frac{y^3}{3!}i + \frac{y^4}{4!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \end{aligned}$$

Then, $e^{yi} = \cos y + i \sin y$ (2)

Equation (2) is called "Euler's formula". It follows form (2) that

$e^{\pi i} = -1$

sometimes referred to as Euler's magic Formula.