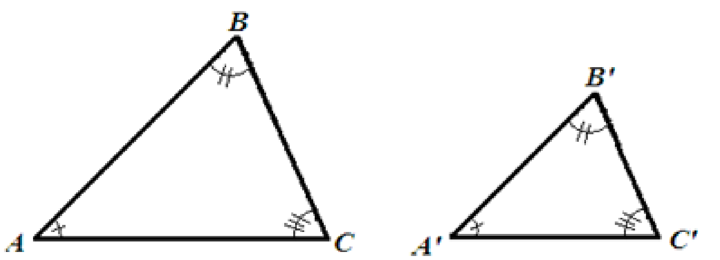
Geometry 1

1. Similar triangles

1) Definition

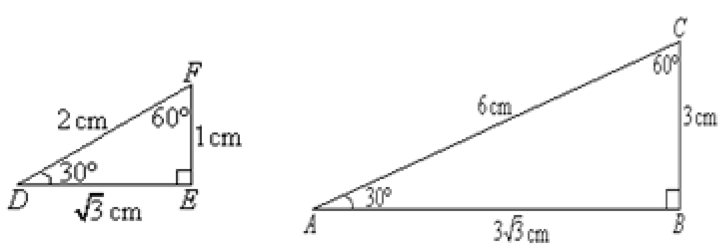
Two triangles ABC and A'B'C' are similar if the three angles of the first triangle are congruent to the corresponding three angles of the second triangle and the lengths of their corresponding sides are proportional as follows.

$\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CA}{C'A'}$ or AB: A'B'= BC: B'C' = CA: C'A'



Equiangular triangles have the same shape but may have different sizes. So, equiangular triangles are also called **similar triangles**.

For example, triangle *DEF* is similar to triangle *ABC* as their three angles are equal (equal angles are marked in the same way in diagrams).



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The length of each side in triangle *DEF* is multiplied by the same number, 3, to give the sides of triangle *ABC*.

$\therefore \dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD} = 3$

Generally speaking, if two triangles are similar, then the corresponding sides are in the same ratio.

2) Angle-Angle (AA) Similarity

Theorem

If two angles in a triangle are congruent to the two corresponding angles in a second triangle, then the two triangles are similar. This is because sum of three angles of a triangle equals 180º which assures the third pair of corresponding angles must be equal.

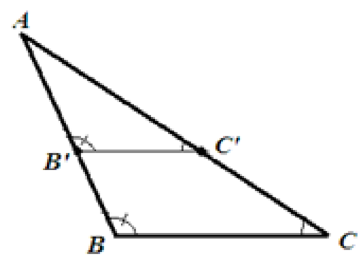
Example 1: Let ABC be a triangle and B'C' a segment parallel to BC. Prove triangles ABC is similar triangle A'BC'.

Solution:

Since B'C' is parallel to BC, angles AB'C' and ABC are congruent (alternate angles).

Also angles AC'B' and ACB are congruent (alternate angles).

Since the two triangles have two corresponding congruent angles, they are similar by **Angle-Angle**.

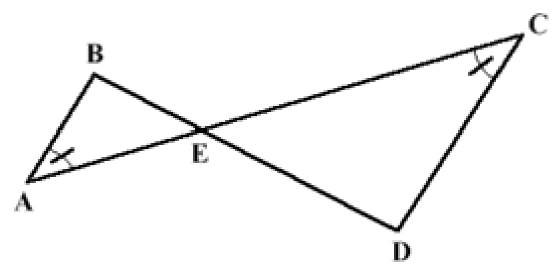


Example 2: If AB//CD, Prove triangle ABE is similar to triangle CDE.

Solution:

Since AB is parallel to CD, and line AC acts as a transversal across the parallel lines AB and DE, *angle A* and *angle C* are congruent (alternate angles).

Also, a*ngle AEB* and *angle CED* are congruent (opposite angles)



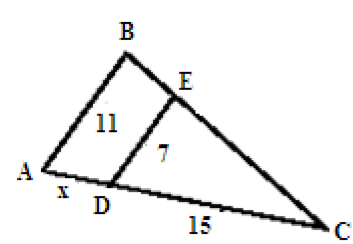
Since the two triangles have two corresponding congruent angles, t*riangle ABE* and *triangle CDE* are similar by **Angle-Angle**.

Example 3: Given that lines DE and AB are parallel in the figure to the right, determine the value of x, the distance between points A and D.

Solution:

First, we can demonstrate that $\Delta CDE \backsim \Delta CAB$, because C = C (by identity).

And $\angle CDE = \angle CAB$ (corresponding angles)



Since two pairs of corresponding angles are equal for the two triangles, we have demonstrated that they are similar triangles.

Hence we have $\frac{AC}{DC} = \frac{BC}{EC} = \frac{AB}{DE}$

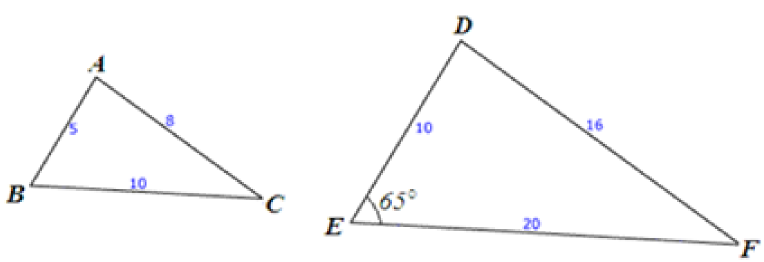
This gives $\frac{15+x}{15} = \frac{11}{7}$. Solving the equation,

$15 + x = \frac{11}{7}(15)$, $x = 8.57$

3) Side-Side-Side (SSS) Similarity

Theorem
If the three sides of a triangle are proportional to the corresponding sides of a second triangle, then the triangles are similar.
Or, the lengths of the corresponding sides are proportional and therefore the two triangles are similar.

Example 4: Prove the two triangles shown below are similar triangles and determine the angle of B.



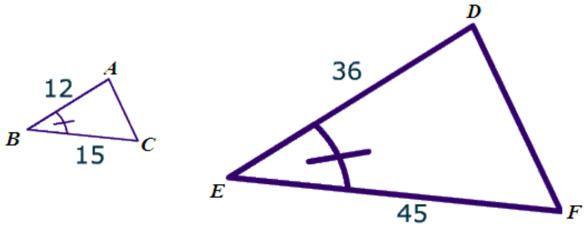
Solution
Since AB : DE = 5:10 = 1: 2, and AC: DF = 8:16 = 1:2, and BC: EF = 10:20 = 1:2
Therefore triangle ABC and triangle DEF are similar triangles.

Since corresponding angles are congruent for similar triangles, and angle E in triangle DEF is 65°, then angle B = angle E = 65°

4) Side-Angle-Side (SAS) Similarity

Theorem
If an angle of a triangle is congruent to the corresponding angle of a second triangle, and the lengths of the two sides including the angle in one triangle are proportional to the lengths of the corresponding two sides in the second triangle, then the two triangles are similar.

Example 5: Prove the triangles shown below are similar. If DF equals 27, determine the length of AC.



Solution

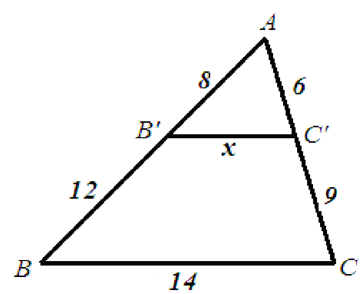
Since angle B equals angle E, and BC : EF = 15 : 45 = 1 : 3, and BA : ED = 12 : 36 = 1 : 3.
The two triangles have two sides whose lengths are proportional and a congruent angle included between the two sides. Therefore the two triangles are similar (SAS). We may calculate the ratios of the lengths of the corresponding sides.
Then AB : DE = AC : DF, substitute the given value in written proportion, we have 12 : 36 =AC : 27, we get AC = 9.

Example 6: In the figure shown below, if B'C'// BC, determine the value of x.

Solution:

Since angles BAC and B'AC' are congruent, the lengths of the sides including the congruent angles are given in figure,

and AB : AB' = 12 : 8 = 3: 2
and AC : AC' = 9 : 6 =3:2



The two triangles have two sides whose lengths are proportional and a congruent angle included between the two sides. Therefore the two triangles are similar. We may calculate the ratios of the lengths of the corresponding sides.

Then we have AB' : AB = B'C' : BC, substitute the value shown in the figure in proportion,
8 : 20 = x : 14, we get x = 5.6

Example 7: In the figure shown below, if angle A = angle A' and angle C=angle C' and some sides are given specific value, determine x and y.

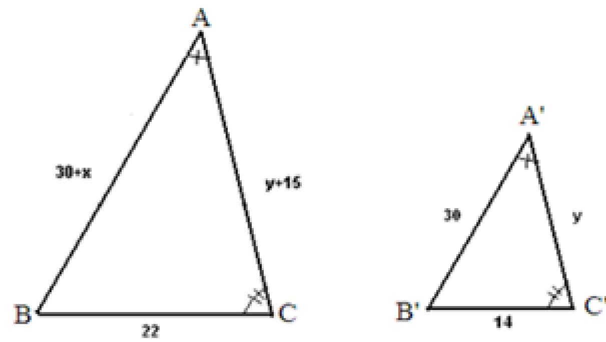
Solution

Since the two triangles have two corresponding congruent angles, they are similar by **Angle-Angle**. Now we use the proportionality of the lengths of the side to write equations that help in solving for x and y.

(30 + x) / 30 = 22 / 14 = (y + 15) / y
First, an equation in x may be written as follows.
(30 + x) / 30 = 22 / 14

Solve the above for x.
420 + 14 x = 660
x = 17.1 (rounded to one decimal place).

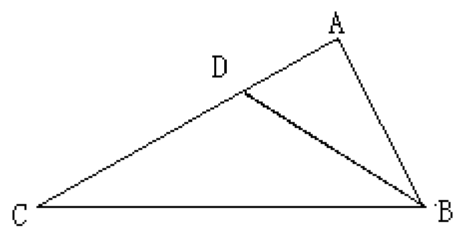
Second, an equation in y may be written as follows.
22 / 14 = (y + 15) / y
Solve the above for y to obtain.
y = 26.25



Example 8: In the diagram, $\Delta ABC \sim \Delta ADB$, and $\angle ABD = \angle C$. List all corresponding sides and angles.

Solution:
Corresponding sides: AB and AC, AD and AB, DB and AC

Corresponding angles: $\angle A$ and $\angle A$, $\angle ADB$ and $\angle ABC$, $\angle ABD$ and $\angle C$



Example 9: Find the value of the pronumeral in the following diagram.

Solution:
ΔADE and ΔABC are similar as they are equiangular.

$$\therefore \frac{AD}{AB} = \frac{DE}{BC}$$

$$\frac{x+4}{x} = \frac{6}{3}$$

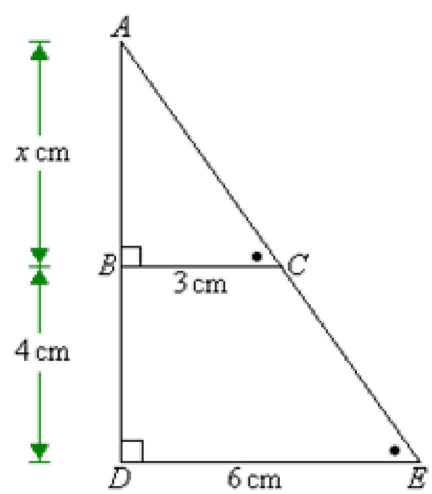
$$\frac{x+4}{x} = 2 \qquad \text{(Multiply both sides by } x\text{)}$$

$$x\left(\frac{x+4}{x}\right) = x \times 2$$

$$x+4 = 2x \qquad \text{(Subtract } x \text{ from both sides)}$$

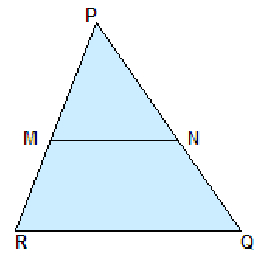
$$x+4-x = 2x-x$$

$$4 = x \qquad x = 4$$



5) Triangle proportionality Theorem

If a line is parallel to one side of a triangle and intersects the other two sides in two distinct points, then it separates these sides into segments of proportional lengths.



If MN// RQ, then $\frac{PM}{RM} = \frac{PN}{QN}$

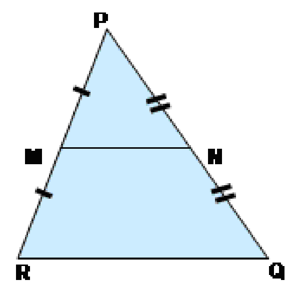
6) Triangle proportionality Theorem Converse

If a line intersects two sides of a triangle in two distinct points and separates these sides into segments of proportional lengths, then it is parallel to the third side.

If $\frac{PM}{RM} = \frac{PN}{QN}$, then MN// RQ

7) Triangle Mid-segment Theorem

A midsegment of a triangle is parallel to one side of the triangle, and its length is one-half the length of that side.



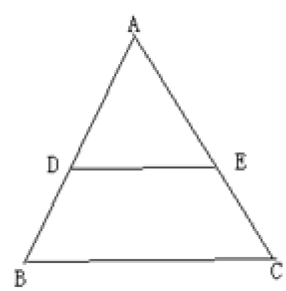
If M and N are midpoints of PR and PQ respectively, then MN//RQ and MN = ½ RQ.

Midsegment is a segment whose endpoints are midpoints of two sides of a triangle.

Example 10: In the diagram, if $\frac{AD}{AB} = \frac{AE}{AC}$, and AD = 3.2cm, DB = 2.4cm, AE = 2cm, find the length of EC.

Solution:

In ΔADE and ΔABC, $\because \frac{AD}{AB} = \frac{AE}{AC}$, $\angle$A is the common angle, so ΔADE∽ΔABC (SAS)



$$\therefore \frac{AD}{AD+DB} = \frac{AE}{AE+EC}$$

$$\therefore \frac{3.2}{3.2+2.4} = \frac{2}{2+EC} \qquad \therefore EC = \frac{2\times 5.6}{3.2} - 2 = 1.5$$

Example 11: In ΔABC, DE // BC. Height AM intersects DE at N. If DE : BC = 4 : 5 and AM = 15, then what is the length of AN?

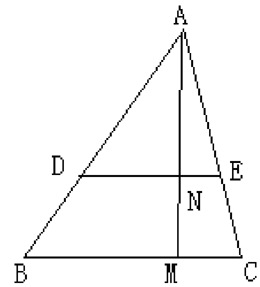
Solution:

$\because BC // DE \quad \therefore \triangle ADE \sim \triangle ABC$, $\triangle ADN \sim \triangle ABM$

$\therefore \frac{AD}{AB} = \frac{DE}{BC} = \frac{4}{5}$

$\because \triangle ADN \sim \triangle ABM \qquad \therefore \frac{AN}{AM} = \frac{AD}{AB} = \frac{4}{5}$

$\therefore AN = \frac{4}{5} AM = 12$



Example 12: In right triangle ABC, $\angle ABC = 90°$. E is a point on AB extended and F is on AC such that EF $\perp AC$. EF intersects BC at D. G is on AC such that $BG \perp AC$. How many triangles are similar to ΔEBD (not including itself)?

Solution:

$\because \angle EDB = \angle CDF$ and $\angle EBD = \angle CFD = 90°$

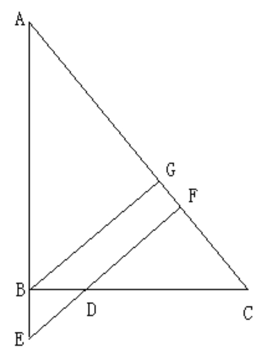
$\therefore \Delta EBD \sim \Delta CFD$

$\because DF // BG \quad \therefore \Delta CDF \sim \Delta CBG$

$\because \angle C = 90° - \angle CBG = \angle ABG$

$\therefore \triangle CGB \sim \triangle BGA \sim \triangle ABC$

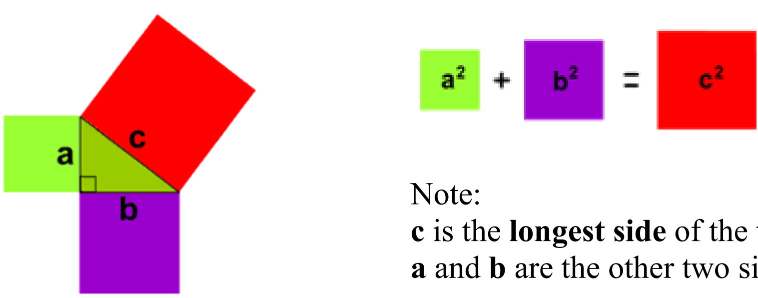
$\because BG // EF \quad \therefore \Delta ABG \sim \Delta AEF$



2. Pythagorean Triples

A "Pythagorean Triple" is a set of positive integers, **a**, **b** and **c** that fits the rule: ***$a^2 + b^2 = c^2$***

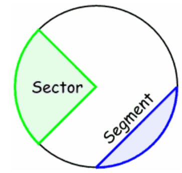
When you make a triangle with sides **a**, **b** and **c** it will be a right angled triangle:



Note:
c is the **longest side** of the triangle, called the "hypotenuse"
a and **b** are the other two sides

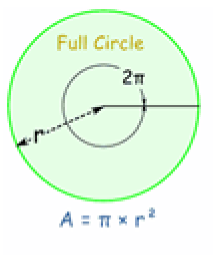
3. Circle Sector and Segment

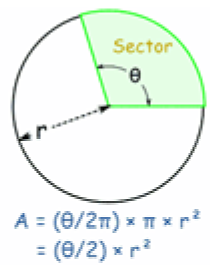
Slices
There are two main "slices" of a circle:
The "pizza" slice is called a **Sector**.
And the slice made by a chord is called a **Segment**.



1) Area of a Sector

You can work out the Area of a Sector by comparing its angle to the angle of a full circle.
Note: I am using radians for the angles.





This is the reasoning:

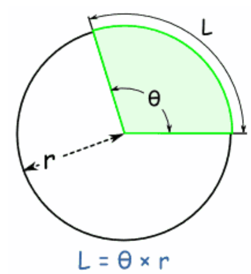
- A circle has an angle of 2π and an Area of: πr^2
- So a Sector with an angle of θ (instead of 2π) must have an area of: $(\theta/2\pi) \times \pi r^2$
- Which can be simplified to: $(\theta/2) \times r^2$

Area of Sector = $\frac{1}{2} \times \theta \times r^2$ ***(when θ is in radians)***
Area of Sector = $\frac{1}{2} \times (\theta \times \pi/180) \times r^2$ ***(when θ is in degrees)***

2) Arc Length of Sector or Segment

By the same reasoning, the arc length (of a Sector or Segment) is:

Arc Length "L" = $\theta \times r$ ***(when θ is in radians)***
Arc Length "L" = $(\theta \times \pi/180) \times r$ ***(when θ is in degrees)***

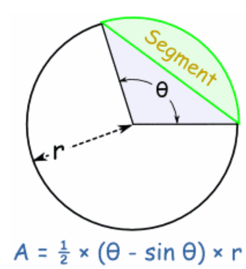


3) Area of Segment

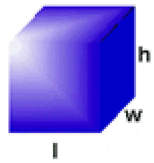
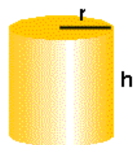
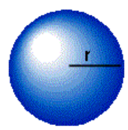
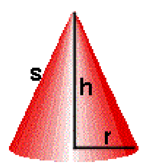
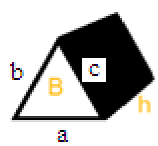
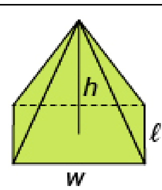
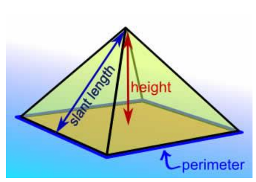
The Area of a Segment is the area of a sector minus the triangular piece (shown in light blue here).

There is a lengthy derivation, but the result is a slight modification of the Sector formula:

Area of Segment = $\frac{1}{2} \times (\theta - \sin\theta) \times r^2$ ***(when θ is in radians)***
Area of Segment = $\frac{1}{2} \times ((\theta \times \pi/180) - \sin\theta) \times r^2$ ***(when θ is in egrees)***



►Volume (V) and Surface Area (SA) Formulas

<table>
<tr><th>Name</th><th>Shapes</th><th>Formula</th></tr>
<tr><td>Rectangular Solid</td><td>
</td><td>$Volume = Length \cdot Width \cdot Height$
$V = lwh$
SA: Surface Area
$SA = 2lh + 2hw + 2lw$</td></tr>
<tr><td>Cylinder</td><td>
</td><td>$Volume = \pi r^2 \cdot height$
$V = \pi r^2 h$
SA: Surface Area
$SA = 2\pi rh + 2\pi r^2$</td></tr>
<tr><td>Sphere</td><td>
</td><td>$V = \frac{4}{3}\pi r^3$
SA: Surface Area
$SA = 4\pi r^2 = \pi d^2$</td></tr>
<tr><td>Cone</td><td>
</td><td>$V = \frac{1}{3}\pi r^2 h$
SA: Surface Area
$SA = s\pi r + \pi r^2$, $s = \sqrt{r^2 + h^2}$</td></tr>
<tr><td>Prism</td><td>
</td><td>$V = \frac{1}{2}Bh$ (B: Area)
SA: Surface Area
$SA = 2B + Ph$
$SA = 2B + (a + b + c) \cdot h$</td></tr>
<tr><td>Pyramid</td><td>
</td><td>$V = \frac{1}{3}Bh = \frac{1}{3}wlh$,
where B is the area of the base.</td></tr>
<tr><td>Pyramid</td><td>
</td><td>SA: Surface Area

When all side faces are the same:
[Base Area] + $^1/_2$ × Perimeter × [Slant Length]

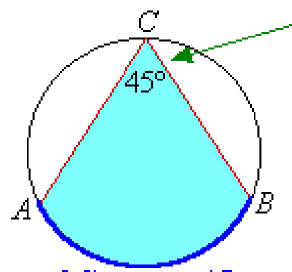
When side faces are different:
[Base Area] + [Lateral Area]</td></tr>
</table>

4. Angle at the Circumference

If the end points of an arc are joined to a third point on the circumference of a circle, then an angle is formed. It is called the **inscribed angle**.

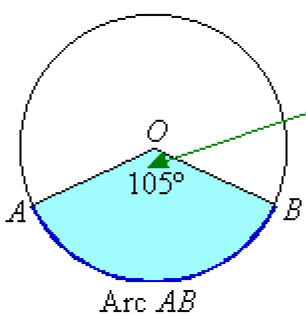
For example, the minor arc *AB* subtends an angle of 45º at *C*. The angle *ACB* is said to be the angle subtended by the minor arc *AB* (or simply arc *AB*) at *C*.

The angle *ACB* is an angle at the circumference standing on the arc *AB*.



5. Angle at the Centre

If the end points of an arc are joined to the centre of a circle, then an angle is formed.
For example, the minor arc *AB* subtends an angle of 105º at *O*. The angle *AOB* is said to be the angle subtended by the minor arc *AB* (or simply arc *AB*) at the centre *O*.

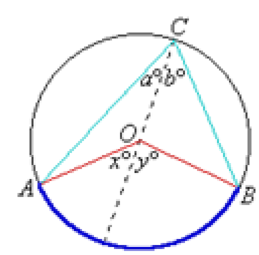


The angle *AOB* is an angle at the centre *O* standing on the arc *AB,* it is also called the **central angle**.

6. Angle at Centre Theorem

Theorem

Use the information given in the diagram to prove that the angle at the centre of a circle is twice the angle at the circumference if both angles stand on the same arc.



Given: $\angle AOB$ and $\angle ACB$ stand on the same arc; and O is the centre of the circle.

To prove: $\angle AOB = 2\angle ACB$

Proof:

From ΔOAC, $x = a + a$ {Exterior angle of a triangle}

$\therefore x = 2a$ ... (1)

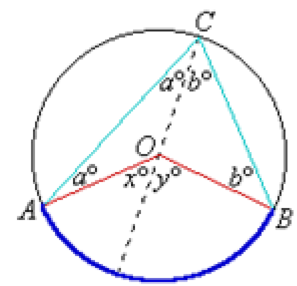
From ΔOBC, $y = b + b$ {Exterior angle of a triangle}

$\therefore y = 2b$ ... (2)

Adding (1) and (2) gives:

$x + y = 2a + 2b = 2(a + b)$ $\therefore \angle AOB = 2\angle ACB$

As required.



In general: The angle at the centre of a circle is twice the angle at the circumference if both angles stand on the same arc. This is called the **Angle at Centre Theorem**.

We also call this the **basic property**, as the other angle properties of a circle can be derived from it.

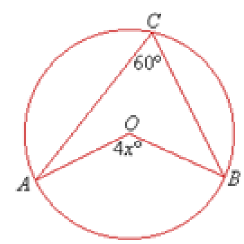
Example 1: Find the value of unknown in the following circle centred at *O*.

Solution:

$4x = 2 \times 60$ {Angle at Centre Theorem}

$4x = 120$

$\frac{4x}{4} = \frac{120}{4}$ $x = 30$



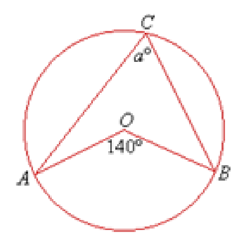
Example 2: Find the value of the unknown in the following circle centred at *O*.

Solution:

$2a = 140$ {Angle at Centre Theorem}

$\frac{2a}{2} = \frac{140}{2}$

$a = 70$



7. Angle in a Semi-Circle

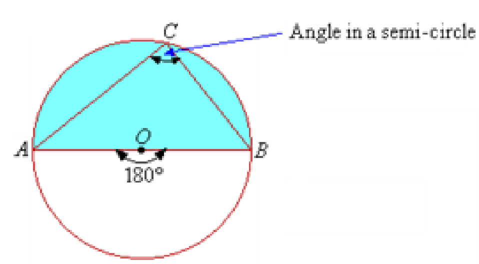
Let $\angle AOB = 180°$.

Then $\angle ACB$ is in a semi-circle.

By the Angle at Centre Theorem, we have:

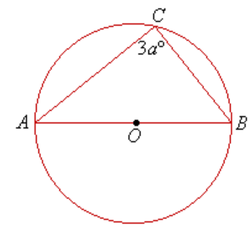
$\angle AOB = 2\angle C$ $2\angle C = 180°$

$\therefore \angle C = 90°$



In general: The angle in a semi-circle is a right angle.

Example 1: Find the value of the unknown in the following circle centred at *O*.



Solution:

$3a = 90$ {Angle in a semi-circle}

$\frac{3a}{3} = \frac{90}{3}$

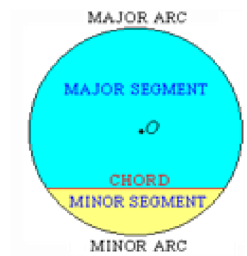
$a = 30$

8. Segments of a Circle

A **chord** of a circle divides the circle into two regions, which are called the **segments** of the circle.

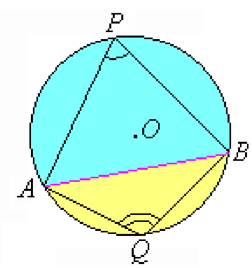
The **minor segment** is the region bounded by the chord and the minor arc intercepted by the chord.

The **major segment** is the region bounded by the chord and the major arc intercepted by the chord.



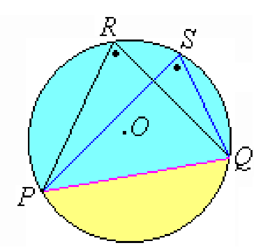
9. Angles in Different Segments

In the left diagram, $\angle APB$ is in the major segment and $\angle AQB$ is in the minor segment. So, we say that angle *APB* and angle *AQB* are in different segments.

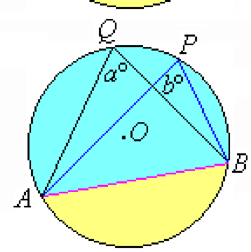


10. Angles in the Same Segment

In the left diagram, $\angle PRQ$ and $\angle PSQ$ are in the major segment. So, we say that angle *PRQ* and angle *PSQ* are in same segments.



Theorem: Use the information given in the diagram to prove that the angles in the same segment of a circle are equal. That is, *a* = *b*.

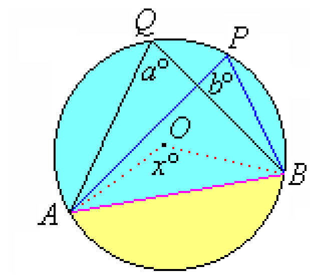


Given: $\angle APB$ and $\angle AQB$ are in the same segment; and O is the center of the circle.

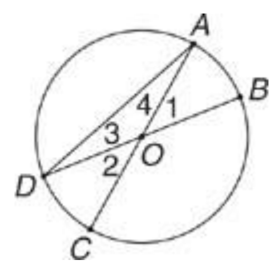
To prove: $\angle APB = \angle AQB$.

Construction: Join O to A and B.
Proof:
Let $\angle AOB = x°$.

Clearly,	$x = 2a$	{Angle at Centre Theorem}
	$x = 2b$	{Angle at Centre Theorem}
	$\therefore\ 2a = 2b$	{Transitive}
	$\therefore\ a = b$	As required.



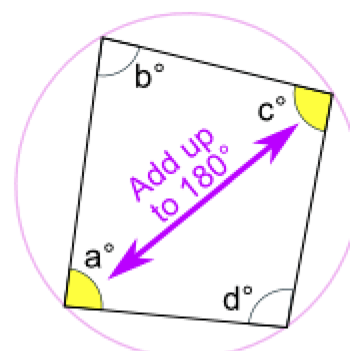
In general: Angles in the same segment of a circle are equal.
Also, it is true that for all equal inscribed angles in the same circle, subtending arcs must be equal.



Given angle 3 = angle 4, then we can conclude that arc AB = arc CD

11. Cyclic Quadrilateral

A "Cyclic" Quadrilateral has every vertex on a circle's circumference.



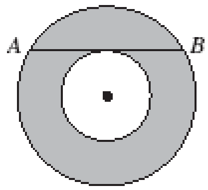
A Cyclic Quadrilateral's **opposite angles add to 180°**:

- a + c = 180°
- b + d = 180°

► In-class questions

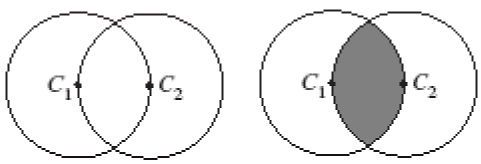
1. In the diagram at right, 2 concentric circles are drawn and the area of the ring trapped between them is shaded. A chord AB is drawn in the larger circle that is tangent to (just touches) the smaller of the 2 circles.

1) If the radii of the 2 circles are 8 and 17, calculate the area of the ring.
2) If the radii of the 2 circles are 8 and 17, calculate the length of the chord AB.
3) Show that for any 2 such circles, with the length of the chord AB equal to x , the area of the ring is $A = \pi x^2 / 4$.

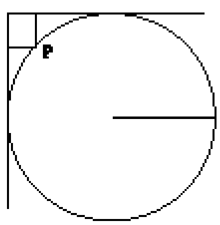


2. If 3 lead spheres of radii 3, 4, and 5 are melted and recast as a single larger sphere, what is the radius of the new sphere, assuming no lead is lost in the process?

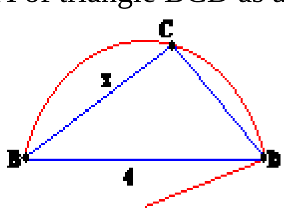
3. Two circles of radius 6 are drawn with centres $C_1$and C_2 as shown. If each of the 2 circles has its centre on the other circle, calculate the area covered by the 2 circles.



4. A circular table is pushed into the corner of a square room so that a point P on the edge of the table is 8 inches from one wall and 9 inches from the other wall as shown. Find the radius of the table in inches.

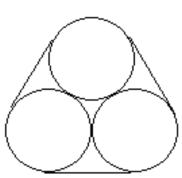


5. Triangle BCD is inscribed in a semi-circle of diameter 4 as shown in the figure. What is the area A of triangle BCD as a function of *x?*

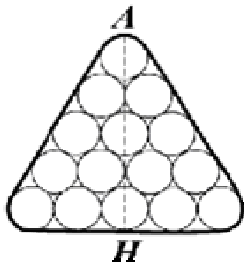


6. A square and a circle have equal perimeters. What is the ratio of the side of the square to the radius of the circle?

7. Three cylindrical drums of 2-foot diameters are to be securely fastened in the form of a triangle by a steel band. What length of band will be required?



8. When people play pool, they first arrange the balls inside a frame, as shown in the figure. The diameters of the balls are all the same. Given that AH=33 cm, find the diameter of a ball.



9. Find the difference between the area of the circle inscribed (the inner circle) in a regular hexagon of side length $\sqrt{3}$ and the circle in which the hexagon is inscribed (the outer circle)?