# AP Calculus Lesson Fifteen Notes

## Chapter Seven - Differential Equations



## 7.1 Basic Definitions

### Differential Equation

A differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives. There is one differential equation that everybody probably knows, that is Newton's Second Law of Motion. If an object of mass $m$ is moving with acceleration $a$ and being acted on with force $F$ then Newton's Second Law tells us.

$$F = ma \tag{7.1.1}$$

To see that this is in fact a differential equation we need to rewrite it a little. First, remember that we can rewrite the acceleration, $a$ , in one of two ways.

$$a = \frac{dv}{dt} \qquad \text{or} \qquad a = \frac{d^2u}{dt^2} \tag{7.1.2}$$

Where $v$ is the velocity of the object and $u$ is the position function of the object at any time $t$. We should also remember at this point that the force, $F$ may also be a function of time, velocity, and/or position.
So, with all these things in mind Newton's Second Law can now be written as a differential equation in terms of either the velocity, $v$, or the position, $u$, of the object as follows.

$$m\frac{dv}{dt} = F(t, v) \tag{7.1.3}$$

$$m\frac{d^2u}{dt^2} = F\left(t, u, \frac{du}{dt}\right) \tag{7.1.4}$$

So, here is our first differential equation. We will see both forms of this in later chapters. Here are a few more examples of differential equations.

### Order

The **order** of a differential equation is the largest derivative present in the differential equation. In the differential equations listed above in (7.1.3) is a first order differential equation and (7.1.4) is a second order differential equation. We will be looking almost exclusively at first and second order differential equations. As you will see most of the solution techniques for second order differential equations can be easily (and naturally) extended to higher order differential equations.

**Linear Differential Equations**

A **linear differential equation** is any differential equation that can be written in the following form.

$$a_n(t)\,y^{(n)}(t) + a_{n-1}(t)\,y^{(n-1)}(t) + \cdots + a_1(t)\,y'(t) + a_0(t)\,y(t) = g(t) \qquad (7.1.5)$$

The important thing to note about linear differential equations is that there are no products of the function, $y(t)$, and its derivatives and neither the function or its derivatives occur to any power other than the first power.
The coefficients $a_0(t)$, …, $a_n(t)$ and $g(t)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function, $y(t)$, and its derivatives are used in determining if a differential equation is linear.

If a differential equation cannot be written in the form, (7.1.5) then it is called a **non-linear** differential equation.

***Example 7.1-1***

Show that $y(x) = x^{-3/2}$ is solution to $4x^2 y'' + 12xy' + 3y = 0$ for x > 0.

***Solution*** We'll need the first and second derivative to do this.

$$y'(x) = -\frac{3}{2}x^{-\frac{5}{2}} \qquad\qquad y''(x) = \frac{15}{4}x^{-\frac{7}{2}}$$

Plug these as well as the function into the differential equation.

$$4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right) + 12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right) + 3\left(x^{-\frac{3}{2}}\right) = 0$$

$$15x^{-\frac{3}{2}} - 18x^{-\frac{3}{2}} + 3x^{-\frac{3}{2}} = 0$$

$$0 = 0$$

So, $y(x) = x^{-3/2}$ does satisfy the differential equation and hence is a solution.

**General Solution**

The **general solution** to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account.

**Initial Condition(s)**

Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions (often abbreviated i.c.'s) are of the form,

$$y(t_0) = y_0 \qquad \text{and/or} \qquad y^{(k)}(t_0) = y_k$$

So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points. As we will see eventually, solutions to "nice enough" differential equations are unique and hence only one solution will meet the given conditions.
The number of initial conditions that are required for a given differential equation will depend upon the order of the differential equation.

***Example 7.1-2*** *General solution and particular solution*

It is easy to show that
$x(t) = t^4 - t^3 + C$, where $C$ is a constant, (i)
is a solution to $\dfrac{dx}{dt} = 4t^3 - 3t^2$.
If the initial condition is $x(0) = 3$, then, $x(0) = 0^4 - 0^3 + C = 3$, so $C = 3$, and the solution to the initial-value problem is

$$x(t) = t^4 - t^3 + 3 \qquad \text{(ii)}$$

(i) is the general solution to $\dfrac{dx}{dt} = 4t^3 - 3t^2$.

(ii) is the particular solution to $\dfrac{dx}{dt} = 4t^3 - 3t^2$ with the initial condition $x(0) = 3$.

**7.2 Slope Fields**

We can solve differential equations by obtaining a "slope field" that approximates the general solution of a differential equation. We call the graph of a solution of a differential equation " a solution curve". The slope field of a differential equation is based on the fact that the differential equation can be interpreted as a statement about the slopes of its solution curves.

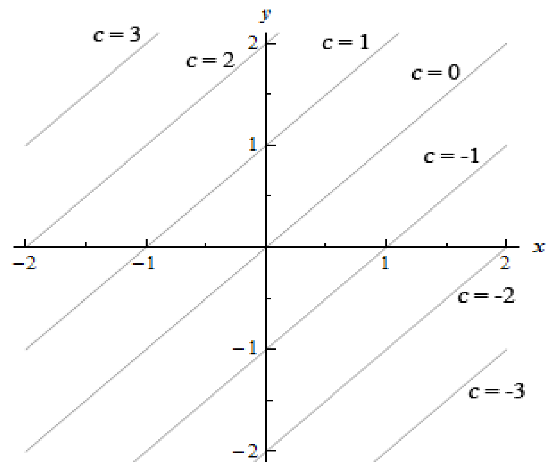
***Example 7.2-1***

Sketch the slope field for the following differential equation. Sketch the set of integral curves for the differential equation $y' = y - x$.

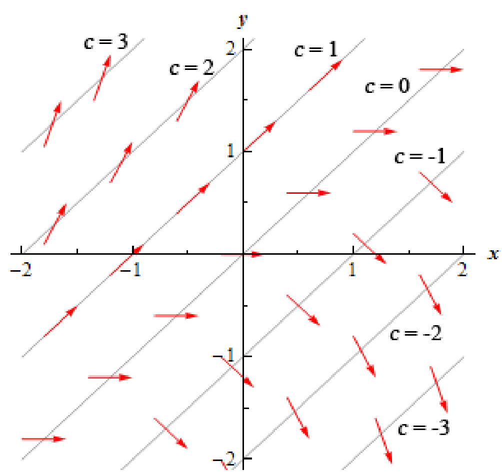
***Solution***

To sketch direction fields for this kind of differential equations we first identify places where the derivative will be constant. To do this we set the derivative in the differential equation equal to a constant, say $c$. This gives us a family of equations, called **isoclines**, that we can plot and on each of these curves the derivative will be a constant value of $c$. Notice that in the previous examples we looked at the isocline for $c = 0$ to get the direction field started. For our case the family of isoclines is $c = y - x$.

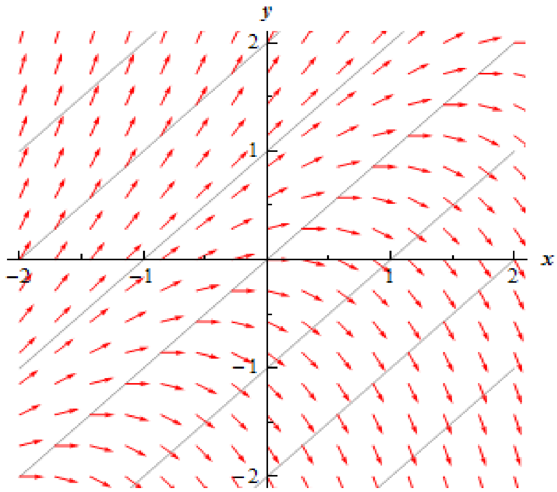
The graph of these curves for several values of $c$ is shown below.



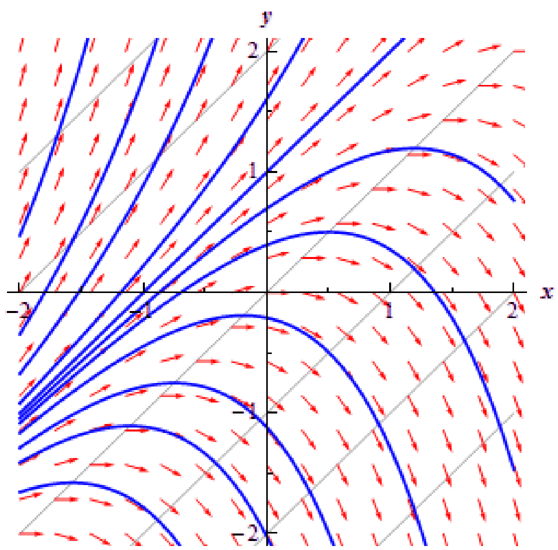

Now, on each of these lines, or isoclines, the derivative will be constant and will have a value of $c$. On the $c = 0$ isocline the derivative will always have a value of zero and hence the tangents will all be horizontal. On the $c = 1$ isocline the tangents will always have a slope of 1 on the $c = -2$ isocline the tangents will always have a slope of $-2$, *etc.* Below is a few tangents put in for each of these isoclines.




To add more arrows for those areas between the isoclines start at say, $c = 0$ and move up to $c = 1$ and as we do that we increase the slope of the arrows (tangents) from 0 to 1. This is shown in the figure below.



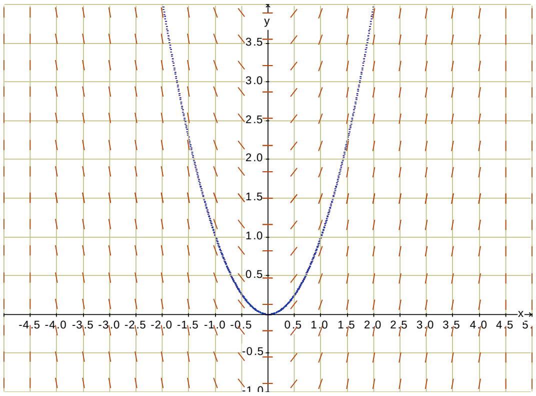

We can then add in integral curves as we did in the previous examples. This is shown in the figure below.

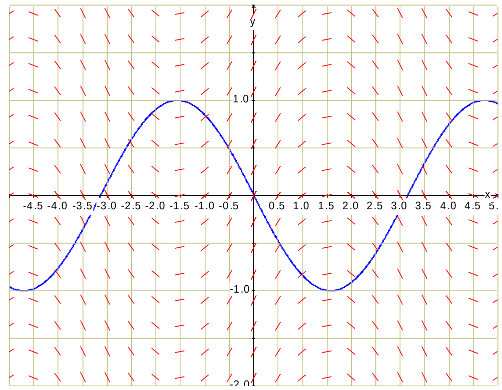


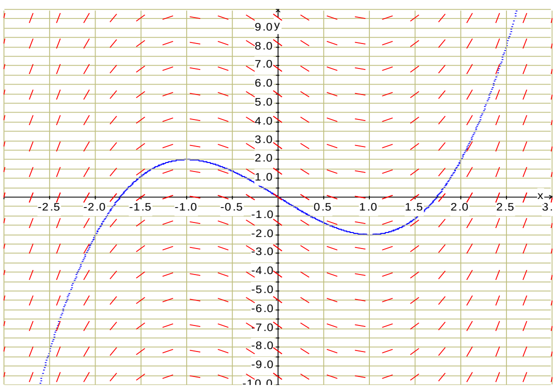

***Example 7.2-2***

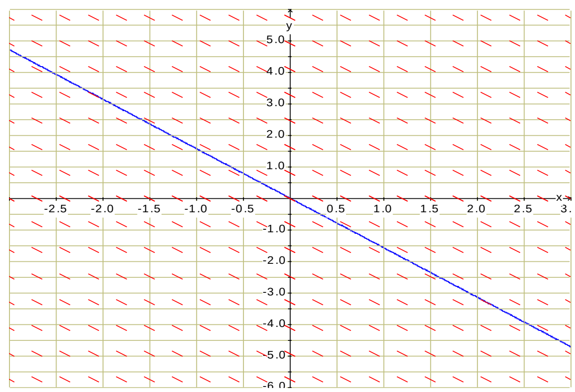
Match each slope field in the following figures with the proper differential equation from the following sets. Find the general solution for each differential equation. The particular solution that goes through (0, 0) has been sketched in.

(A) $y' = \cos x$ (B) $\frac{dy}{dx} = 2x$ (C) $\frac{dy}{dx} = 3x^2 - 3$ (D) $y' = -\pi/2$


(I)


(II)


(III)


(IV)

***Solution***

The general solution of (A) is $y = -\sin x + C$, that matches to (II).
The general solution of (B) is $y = x^2 + C$, that matches to (I).
The general solution of (A) is $y = x^3 - 3x + C$, that matches to (III).
The general solution of (A) is $y = -\dfrac{\pi}{2}x + C$, that matches to (IV).

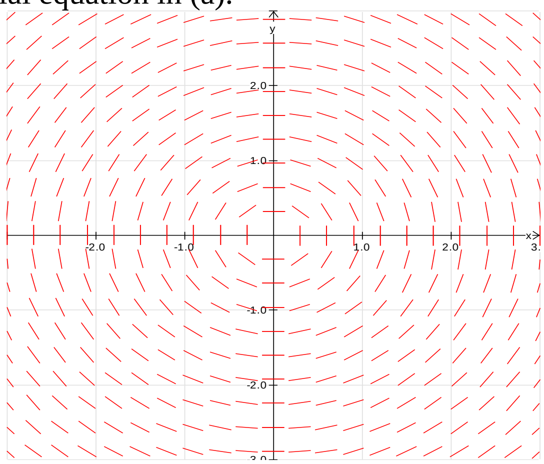
## 7.3 Derivatives of Implicitly Defined Functions

In the previous examples, each differential equation was of the form $\dfrac{dy}{dx} = f(x)$. We were able to find the general solution by finding the antiderivative $y = \int f(x)dx$. Now we consider differential equations of the form $\dfrac{dy}{dx} = f(x, y)$, so $\dfrac{dy}{dx}$ is an implicitly defined function.

***Example 7.3-1***

(a) Verify that a particular solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ is $x^2 + y^2 = 4$.

(b) Using the slope field in the following figure and your answer in (a), find the general solution to the differential equation in (a).




***Solution***

(a) Differentiating the equation $x^2 + y^2 = 4$ implicitly, we get $2x + 2y\frac{dy}{dx} = 0$. From which $\frac{dy}{dx} = -\frac{x}{y}$, which is the given differential equation.

(b) The particular solution in (a) is a circle centred at the origin with radius 2. But every circle with centre at the origin satisfies the differential equation, since if we differentiate $x^2 + y^2 = r^2$ implicitly we again get $\frac{dy}{dx} = -\frac{x}{y}$ for all r. Thus, the general solution of the given differential equation is $x^2 + y^2 = r^2$.

## 7.4 Euler's Method

In the previous section, we found solution curves to the first -order differential equation graphically using slope fields. Now we will find solutions numerically. Using Euler's method, we find points on solution curves.

Let’s start with a general first order initial value problem (IVP)

$$\frac{dy}{dt} = f(t, y) \qquad\qquad y(t_0) = y_0 \tag{7.4.1}$$

where $f(t,y)$ is a known function and the values in the initial condition are also known numbers. Let's assume that everything is nice and continuous so that we know that a solution will in fact exist in some interval surrounding $t_0$.

We want to approximate the solution to (7.4.1) near $t_0$. We'll start with the two pieces of information that we do know about the solution. First, we know the value of the solution at $t = t_0$ from the initial condition. Second, we also know the value of the derivative at $t = t_0$. We can get this by plugging the initial condition into $f(t,y)$ into the differential equation itself. So, the derivative at this point is.

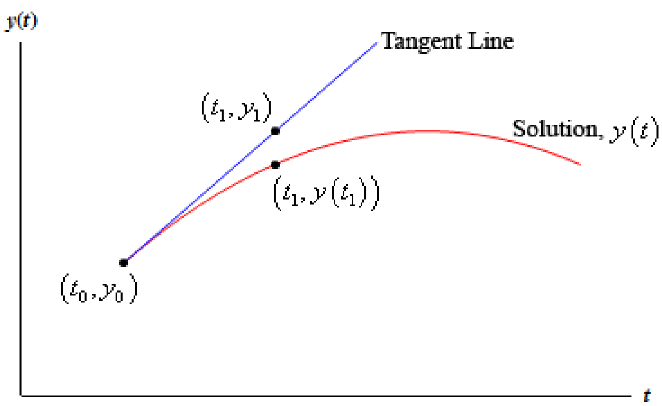
$$\left.\frac{dy}{dt}\right|_{t=t_0} = f\left(t_0, y_0\right)$$

Now, recall from your Calculus I class that these two pieces of information are enough for us to write down the equation of the tangent line to the solution at $t = t_0$.

The tangent line is

$$y = y_0 + f\left(t_0, y_0\right)\left(t - t_0\right)$$

Take a look at the figure below




If $t_1$ is close enough to $t_0$ then the point $y_1$ on the tangent line should be fairly close to the actual value of the solution at $t_1$, or $y(t_1)$. Finding $y_1$ is easy enough. All we need to do is plug $t_1$ in the equation for the tangent line.

$$y_1 = y_0 + f\left(t_0, y_0\right)\left(t_1 - t_0\right)$$

Now, we would like to proceed in a similar manner, but we don't have the value of the solution at $t_1$ and so we won't know the slope of the tangent line to the solution at this point. This is a problem. We can partially solve it however, by recalling that $y_1$ is an approximation to the solution at $t_1$. If $y_1$ is a very good approximation to the actual value of the solution then we can use that to estimate the slope of the tangent line at $t_1$.

So, let's hope that $y_1$ is a good approximation to the solution and construct a line through the point $(t_1, y_1)$ that has slope $f(t_1, y_1)$. This gives

$$y = y_1 + f\left(t_1, y_1\right)\left(t - t_1\right)$$

Now, to get an approximation to the solution at $t = t_2$ we will hope that this new line will be fairly close to the actual solution at $t_2$ and use the value of the line at $t_2$ as an approximation to the actual solution. This gives.

$$y_2 = y_1 + f\left(t_1, y_1\right)\left(t_2 - t_1\right)$$

We can continue in this fashion. Use the previously computed approximation to get the next approximation. So,

$$\begin{gathered} y_3 = y_2 + f\left(t_2, y_2\right)\left(t_3 - t_2\right) \\ y_4 = y_3 + f\left(t_3, y_3\right)\left(t_4 - t_3\right) \\ etc. \end{gathered}$$

In general, if we have $t_n$ and the approximation to the solution at this point, $y_n$, and we want to find the approximation at $t_{n+1}$ all we need to do is use the following.

$$y_{n+1} = y_n + f\left(t_n, y_n\right) \cdot \left(t_{n+1} - t_n\right)$$

If we define

$$f_n = f\left(t_n, y_n\right)$$

we can simplify the formula to

$$y_{n+1} = y_n + f_n \cdot \left(t_{n+1} - t_n\right)$$

Often, we will assume that the step sizes between the points $t_0$ , $t_1$ , $t_2$ , … are of a uniform size of $h$. In other words, we will often assume that

$$t_{n+1} - t_n = h \tag{7.4.2}$$

This doesn't have to be done and there are times when it's best that we not do this. However, if we do the formula for the next approximation becomes.

$$y_{n+1} = y_n + h\, f_n \tag{7.4.3}$$

So, how do we use Euler's Method? It's fairly simple. We start with (7.4.1) and then decide if we want to use a uniform step size or not. Then starting with $(t_0, y_0)$ we repeatedly evaluate (7.4.2) or (7.4.3) depending on whether we chose to use a uniform set size or not. We continue until we've gone the desired number of steps or reached the desired time. This will give us a sequence of numbers $y_1$ , $y_2$ , $y_3$ , … $y_n$ that will approximate the value of the actual solution at $t_1$ , $t_2$ , $t_3$ , … $t_n$.

What do we do if we want a value of the solution at some other point than those used here? One possibility is to go back and redefine our set of points to a new set that will include the points we are after and redo Euler's Method using this new set of points. However this is cumbersome and could take a lot of time especially if we had to make changes to the set of points more than once.

Another possibility is to remember how we arrived at the approximations in the first place. Recall that we used the tangent line

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

to get the value of $y_1$. We could use this tangent line as an approximation for the solution on the interval $[t_0, t_1]$. Likewise, we used the tangent line

$$y = y_1 + f(t_1, y_1)(t - t_1)$$

to get the value of $y_2$. We could use this tangent line as an approximation for the solution on the interval $[t_1, t_2]$. Continuing in this manner we would get a set of lines that, when strung together, should be an approximation to the solution as a whole.

***Example 7.4-1***

For the IVP

$y' + 2y = 2 - \mathbf{e}^{-4t} \qquad y(0) = 1$

Use Euler's Method with a step size of $h = 0.1$ to find approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$, and $0.5$. Compare them to the exact values of the solution as these points.

***Solution***

This is a fairly simple linear differential equation so we'll leave it to you to check that the solution is

$$y(t) = 1 + \frac{1}{2}\mathbf{e}^{-4t} - \frac{1}{2}\mathbf{e}^{-2t}$$

In order to use Euler's Method we first need to rewrite the differential equation into the form given in (7.4.1).

$$y' = 2 - \mathbf{e}^{-4t} - 2y$$

From this we can see that

$$f(t, y) = 2 - \mathbf{e}^{-4t} - 2y$$

Also note that $t_0 = 0$ and $y_0 = 1$. We can now start doing some computations.

$$f_0 = f(0,1) = 2 - \mathbf{e}^{-4(0)} - 2(1) = -1$$
$$y_1 = y_0 + h\,f_0 = 1 + (0.1)(-1) = 0.9$$

So, the approximation to the solution at $t_1 = 0.1$ is $y_1 = 0.9$.
At the next step we have

$$f_1 = f(0.1, 0.9) = 2 - \mathbf{e}^{-4(0.1)} - 2(0.9) = -0.470320046$$
$$y_2 = y_1 + h\, f_1 = 0.9 + (0.1)(-0.470320046) = 0.852967995$$

Therefore, the approximation to the solution at $t_2 = 0.2$ is $y_2 = 0.852967995$.
I'll leave it to you to check the remainder of these computations.

$$f_2 = -0.155264954 \qquad y_3 = 0.837441500$$
$$f_3 = 0.023922788 \qquad y_4 = 0.839833779$$
$$f_4 = 0.1184359245 \qquad y_5 = 0.851677371$$

Here's a quick table that gives the approximations as well as the exact value of the solutions at the given points.

| **Time, $t_n$** | **Approximation** | **Exact** | **Error** |
|---|---|---|---|
| $t_0 = 0$ | $y_0 = 1$ | $y(0) = 1$ | 0 % |
| $t_1 = 0.1$ | $y_1 = 0.9$ | $y(0.1) = 0.925794646$ | 2.79 % |
| $t_2 = 0.2$ | $y_2 = 0.852967995$ | $y(0.2) = 0.889504459$ | 4.11 % |
| $t_3 = 0.3$ | $y_3 = 0.837441500$ | $y(0.3) = 0.876191288$ | 4.42 % |
| $t_4 = 0.4$ | $y_4 = 0.839833779$ | $y(0.4) = 0.876283777$ | 4.16 % |
| $t_5 = 0.5$ | $y_5 = 0.851677371$ | $y(0.5) = 0.883727921$ | 3.63 % |

We've also included the error as a percentage. It's often easier to see how well an approximation does if you look at percentages. The formula for this is,

$$\textit{percent error} = \frac{|\textit{exact} - \textit{approximate}|}{\textit{exact}} \times 100$$

We used absolute value in the numerator because we really don't care at this point if the approximation is larger or smaller than the exact. We're only interested in how close the two are.

In Euler's method, doubling the number of steps will cut the error approximately in half.

Euler's method may not work if the solution curve has discontinuities, cups or asymptotes.

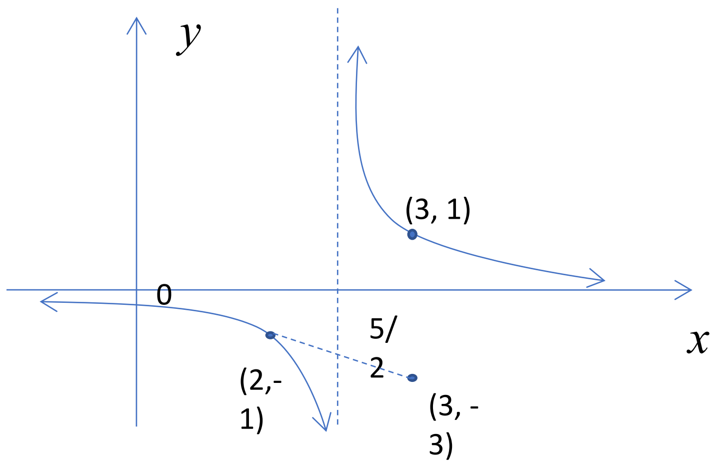
***Example 7.4-2***

Given $y = \dfrac{1}{2x-5}$, for the domain $x < 5/2$. If the corresponding differential equation is $\dfrac{dy}{dx} = -2y^2$ with initial condition $y = -1$ when $x = 2$.

Suppose that we attempt to approximate the solution using Euler's Method with a step size of $\Delta x = 1$, the first step carries us to the point $(3, -3)$ , beyond the discontinuity at $x = 5/2$.

$y_1 = y_0 + \Delta y = -1 + (-2) = -3$, where $\Delta y = slope \times \Delta x = (-2)(-1)^2 \times 1 = -2$
$x_1 = x_0 + \Delta x = 2 + 1 = 3$

The accompanying graph shows that this is nowhere near the solution curve, on which $y = 1$ when $x = 3$.




Here, Euler's method fails because it leaps blindly across the vertical asymptote at $x = 5/2$. So Euler's method may not work if the solution curve has discontinuities, cups or asymptotes.