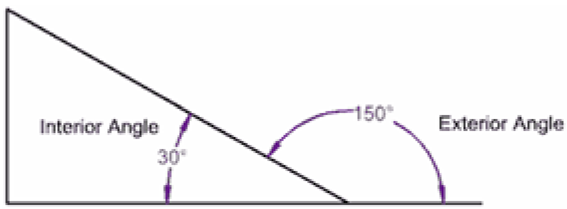
Geometry 2

1. Interior and Exterior Angle

An Interior Angle is an angle inside a shape.

The Exterior Angle is the angle between any side of a shape, and a line extended from the next side.



Note: If you add up the Interior Angle and Exterior Angle you get a straight line, 180°.

1) Interior Angles sum of Polygons

The sum of the measures of the interior angles of a polygon with *n* sides is (n-2)180.

For examples:

- Triangle or ('3 - gon')
  - sum of interior angles: (3-2) 180 = 180°
- Quadrilateral which has four sides (' 4 - gon')
  - sum of interior angles: (4-2)180 = 360°
- Hexagon which has six sides ('6 - gon')
  - sum of interior angles: (6-2)180 = 720°

An interior angle of a regular polygon with n sides is $\frac{(n-2)\times 180}{n}$.

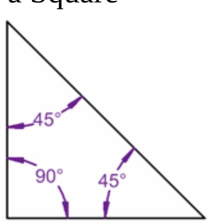
Example:

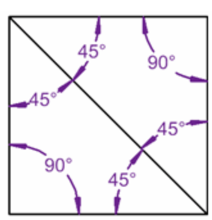
To find the measure of an interior angle of a regular octagon, which has 8 sides, apply the formula above as follows:
((8-2) × 180) /8 = 135°

There are Two Triangles in a Square

The internal angles in this triangle add up to 180°

(90°+45°+45°=180°)





... and for this square they add up to **360°**

... because the square can be made from two triangles!

2) Exterior Angles sum of Polygons

The Exterior Angle is the angle between any side of a shape, and a line extended from the next side.

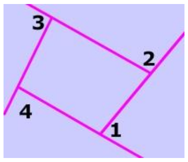
The sum of the measures of the exterior angles of a polygon, one at each vertex, is 360°.

For example:

A Polygon is any flat shape with straight sides.

The Exterior Angles of a Polygon add up to 360°.

$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360^0$



2. Pythagorean Triples

A "Pythagorean Triple" is a set of positive integers, **a**, **b** and **c** that fits the rule:

$a^2 + b^2 = c^2$

1) Endless

The set of Pythagorean Triples is endless.

It is easy to prove this with the help of the first Pythagorean Triple, (3, 4, and 5):
Let n be any integer greater than 1, then 3n, 4n and 5n would also be a set of Pythagorean Triple.

This is true because: $(3n)^2 + (4n)^2 = (5n)^2$

Examples:

n	**(3n, 4n, 5n)**
2	(6,8,10)
3	(9,12,15)
...	... etc ...

So, you can make infinite triples just using the (3,4,5) triple.

2) Euclid's Proof that there are Infinitely Many Pythagorean Triples

However, Euclid used a different reasoning to prove the set of Pythagorean Triples is unending. The proof was based on the fact that the difference of the squares of any two consecutive numbers is always an odd number.

Examples:

$2^2 - 1^2 = 4 - 1 = \mathbf{3}$ (an odd number),
$15^2 - 14^2 = 225-196 = \mathbf{29}$ (an odd number)

3) Properties

It can be observed that a Pythagorean Triple always consists of:

- all even numbers, or
- two odd numbers and an even number.

A Pythagorean Triple can never be made up of all odd numbers or two even numbers and one odd number. This is true because:

- The square of an odd number is an odd number and the square of an even number is an even number.
- The sum of two even numbers is an even number and the sum of an odd number and an even number is in odd number.

Therefore, if one of a and b is odd and the other is even, c would have to be odd. Similarly, if both a and b are even, c would be an even number too!

4) Constructing Pythagorean Triples

It is easy to construct sets of Pythagorean Triples.

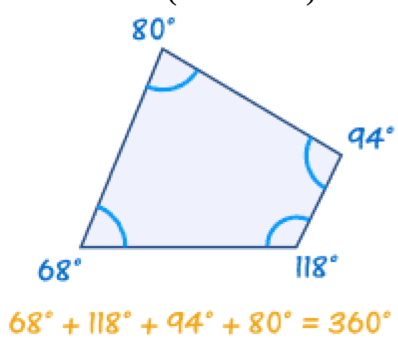
When **m** and **n** are any two positive integers (m < n):
$a = n^2 - m^2, \quad b = 2nm, \quad c = n^2 + m^2$
Then, a, b, and c form a Pythagorean Triple.

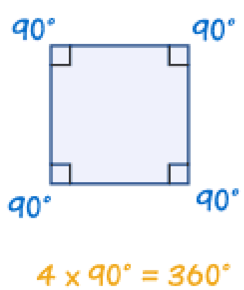
Example:
$m = 1 \text{ and } n = 2, \quad a = 2^2 - 1^2 = 4 - 1 = \mathbf{3}, \quad b = 2 \times 2 \times 1 = \mathbf{4}, \quad c = 2^2 + 1^2 = \mathbf{5}$
Thus, we obtain the first Pythagorean Triple (3,4,5).

Similarly, when m=2 and n=3 we get the next Pythagorean Triple (5,12,13).

3. Four sides (or edges)

1) Properties
Four vertices (or corners). The interior angles add up to **360 degrees**:

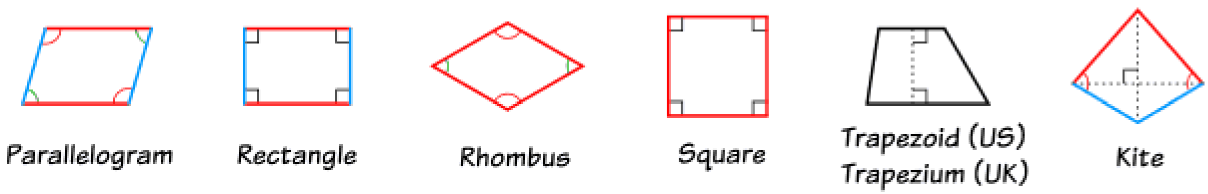




Try drawing a quadrilateral, and measure the angles. They should add to **360°**

2) Types of Quadrilaterals

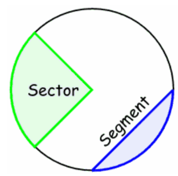
There are special types of quadrilateral:



Some types are also included in the definition of other types! For example a **square**, **rhombus** and **rectangle** are also ***parallelograms***.

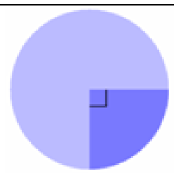
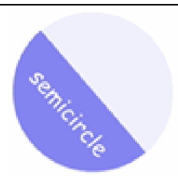
4. Circle Sector and Segment

Slices
There are two main "slices" of a circle:
The "pizza" slice is called a **Sector**.
And the slice made by a chord is called a **Segment**.



1) Common Sectors

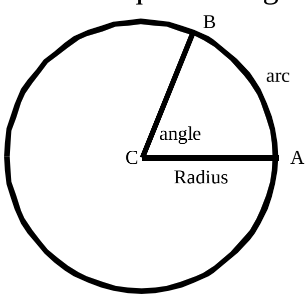
The Quadrant and Semicircle are two special types of Sector:

Quarter of a circle is called a **Quadrant**	Half a circle is called a **Semicircle.**
	

2) Area of a Sector

Radian Measure

- We can measure angles in several ways - one of which is degrees
- Another way to measure an angle is by means of <u>radians</u>
- One radian is defined as the measure of the angle subtended at the center of a circle by an arc equal in length to the radius of the circle

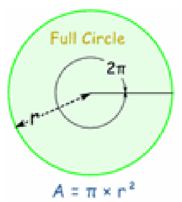


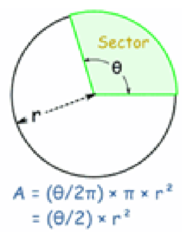
So in terms of radians, the formula is θ = arc length/radius

θ for one revolution = Circumference/r = $2\pi r/r = 2\pi$ radians

So then an angle of **360° = 2 π radians** or more easily, an angle of **180° = π radians**

You can work out the Area of a Sector by comparing its angle to the angle of a full circle.
Note: I am using radians for the angles.





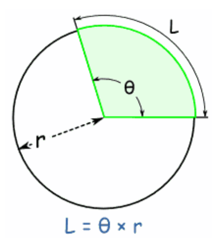
This is the reasoning:

- A circle has an angle of 2π and an Area of: πr^2
- So a Sector with an angle of θ (instead of 2π) must have an area of: $(\theta/2\pi) \times \pi r^2$
- Which can be simplified to: $(\theta/2) \times r^2$

Area of Sector = $\frac{1}{2} \times \theta \times r^2$ ***(when θ is in radians)***

Area of Sector = $\frac{1}{2} \times (\theta \times \pi/180) \times r^2$ ***(when θ is in degrees)***

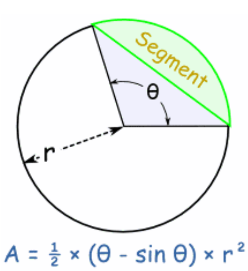
3) Arc Length of Sector or Segment



By the same reasoning, the arc length (of a Sector or Segment) is:
Arc Length "L" = $\theta \times r$ ***(when θ is in radians)***
Arc Length "L" = $(\theta \times \pi/180) \times r$ ***(when θ is in degrees)***

4) Area of Segment

The Area of a Segment is the area of a sector minus the triangular piece (shown in light blue here).



There is a lengthy derivation, but the result is a slight modification of the Sector formula:

Area of Segment = $\frac{1}{2} \times (\theta - \sin\theta) \times r^2$ ***(when θ is in radians)***
Area of Segment = $\frac{1}{2} \times ((\theta \times \pi/180) - \sin\theta) \times r^2$ ***(when θ is in degrees)***

- **Things you should know:**

***SAS* Postulate**
If two triangles have two sides and the included angle of one triangle congruent respectively to two sides and the included angle of the other triangle, then the triangles are congruent.

ASA Postulate
If two triangles have two angles and the included of one triangle congruent respectively to two angles and the included side of the other triangle, then the triangle are congruent

SSS Postulate
If two triangles have three sides of one triangle congruent respectively to the three sides of the other triangle, then the triangles are congruent

HA Postulate
If two right triangles have the hypotenuse and an acute angle of one triangle congruent respectively to the hypotenuse and an acute angle of the other, then the triangles are congruent

SAA Corollary
If two triangles have a side and two angles of one congruent respectively to a side and two angles of the other, then the triangles are congruent.

Angle-Sum Theorem for Polygons
The formula for the sum, S, of the measures of the angles of a polygon of n sides is
$S = (n - 2)180$.
Exterior Angle-Sum Corollary for triangles
The measure of one exterior angle of a triangle equals the sum of the measures of its remote Interior angles

Theorem The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.

Theorem If two sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

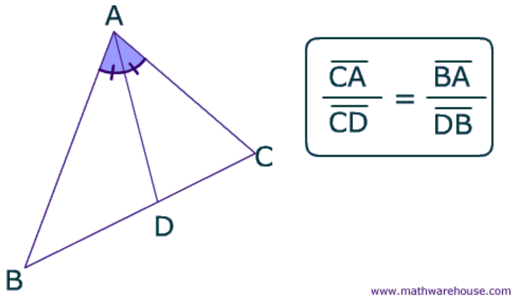
Theorem If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram

Theorem The diagonals of a rectangle are congruent.

Theorem The diagonals of a rhombus are perpendicular to each other

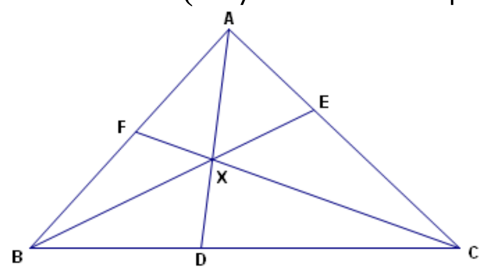
Theorem Each diagonal of a rhombus bisects a pair of opposite angles

Angle Bisector Theorem – An angle bisector of an angle of a triangle divides the opposite side in two segments that are proportional to the other two sides of the triangle.



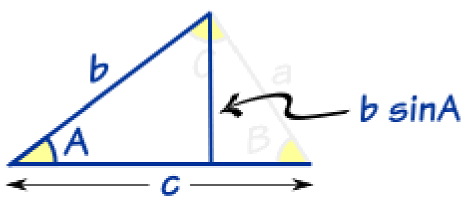
Ceva's Theorem Let ABC be a triangle, and let D, E, F be points on lines BC, CA, AB, respectively. Lines AD, BE, CF are concurrent (they intersect at a point) iff (if and only if)

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$$



Area of Triangle

Area = ½ × base × height



In this triangle:

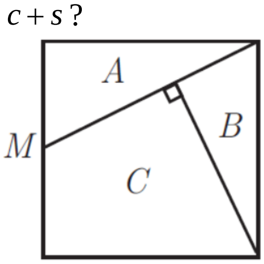
- the base is: **c**
- the height is: **b × sin A**

Putting that together gets us: Area = ½ × (c) × (b × sin A)

Which is (more simply): **Area = ½bc sin A**

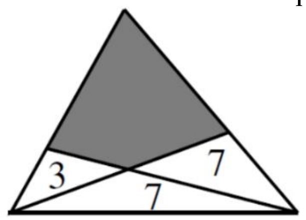
In-class questions

1. A cubical cake with edge length 2 inches is iced on the sides and the top. It is cut vertically into three pieces as shown in this top view, where *M* is the midpoint of a top edge. The piece whose top is triangle *B* contains *c* cubic inches of cake and *s* square inches of icing. What is $c+s$?

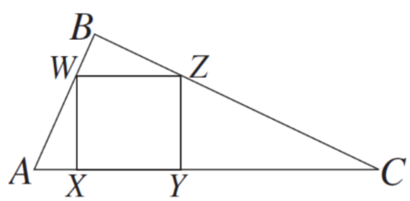


2. In trapezoid $ABCD$ we have AB parallel to DC, E as the midpoint of BC, and F as the midpoint of DA. The area of $ABEF$ is twice the area of $FECD$. What is AB/DC?

3. A triangle is partitioned into three triangles and a quadrilateral by drawing two lines from vertices to their opposite sides. The areas of the three triangles are 3, 7, and 7, as shown. What is the area of the shaded quadrilateral?



4. Right ΔABC has $AB = 3$, $BC = 4$, and $AC = 5$. Square $XYZW$ is inscribed in ΔABC with X and Y on AC, W on AB, and Z on BC. What is the side length of the square?



5. Trapezoid $ABCD$ has bases AB and CD and diagonals intersecting at K. Suppose that $AB = 9$, $DC = 12$, and the area of ΔAKD is 24. What is the area of trapezoid $ABCD$?

6. Convex quadrilateral $ABCD$ has $AB = 9$ *and* $CD = 12$. Diagonals AC and BD intersect at E, $AC = 14$, and ΔAED and ΔBEC have equal areas. What is AE?

7. Rhombus $ABCD$ has side length 2 and $\angle B = 120°$. Region R consists of all points inside the rhombus that are closer to vertex B than any of the other three vertices. What is the area of R?