Derivatives (1)

Derivative Function

Derivative Function	**Differentiability**
Given a function $y = f(x)$, the *derivative function* of f is a *new function* called f' (*f prime*), defined at x by: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$	A function $y = f(x)$ is called *differentiable* at x if $f'(x)$ exists. A function $y = f(x)$ is differentiable over an open interval (a, b) if the function is differentiable at every number in that interval. Note: The domain of derivative function f' is a subset of the domain of the original function f: $D_{f'} \subset D_f$. So a function is defined over D_f but is differentiable over $D_{f'}$.
Interpretations of Derivative Function	**Notations and Reading**
1. The *slope of the tangent line* to the graph of $y = f(x)$ at the point $P(a, f(a))$ is given by $m = f'(a)$. 2. The *instantaneous rate of change* in the variable y with respect to the variable x, where $y = f(x)$, at $x = a$ is given by: $IRC = f'(a)$.	$y' = f'(x)$ [Lagrange Notation] Read: "y prime" or "f prime at x" $\frac{dy}{dx} = \frac{d}{dx} f(x)$ [Leibnitz Notation] Read: "dee y by dee x" $f'(a) = \left.\frac{dy}{dx}\right\rvert_{x=a}$ Read: "f prime at a, dee y by dee x at x equals a"
First Principles	**Ex 1.** Use first principles to differentiate the function
Differentiation is the process to find the derivative function for a given function. *First Principles* is the process of differentiation by computing the limit: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$	$f(x) = \frac{-3}{x^2}$.

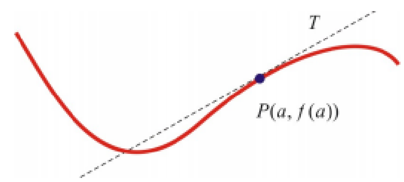
Non-Differentiability

A function is *not differentiable* at $x = a$ if $f'(a)$ *does not exist*.

Notes:
1. If a function f is *not continuous* at $x = a$ then the function f is *not differentiable* at $x = a$.
2. If a function f is *continuous* at $x = a$ then the function f *may be or not* differentiable at $x = a$.

Differentiability Point

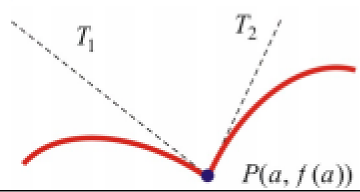
If the function $y = f(x)$ is *differentiable* at $x = a$ then the tangent line at $P(a, f(a))$ is *unique* and *not vertical* (the slope of the tangent line is not ∞ or $-\infty$).



Corner Point

$P(a, f(a))$ is a *corner point* if there are *two* distinct tangent lines at P, one for the left-hand branch and one for the right-hand branch. For example:

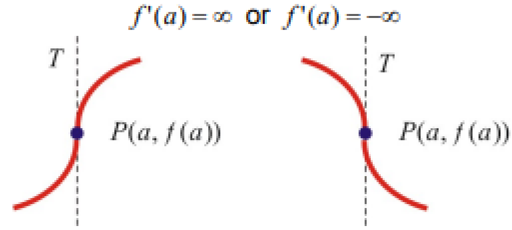
$f(x) = \begin{cases} f_1(x), & x < a \\ f_2(x), & x > a \end{cases}$ and $f_1'(a) \neq f_2'(a)$



Infinite Slope Point

$P(a, f(a))$ is a *infinite slope point* if the tangent line at P is vertical and the function is increasing or decreasing in the neighborhood at the of the point P.

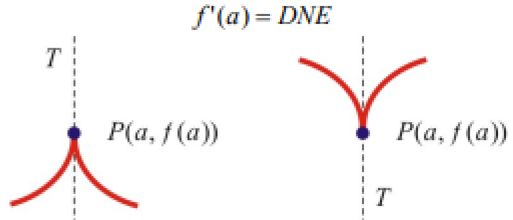
$f'(a) = \infty$ or $f'(a) = -\infty$



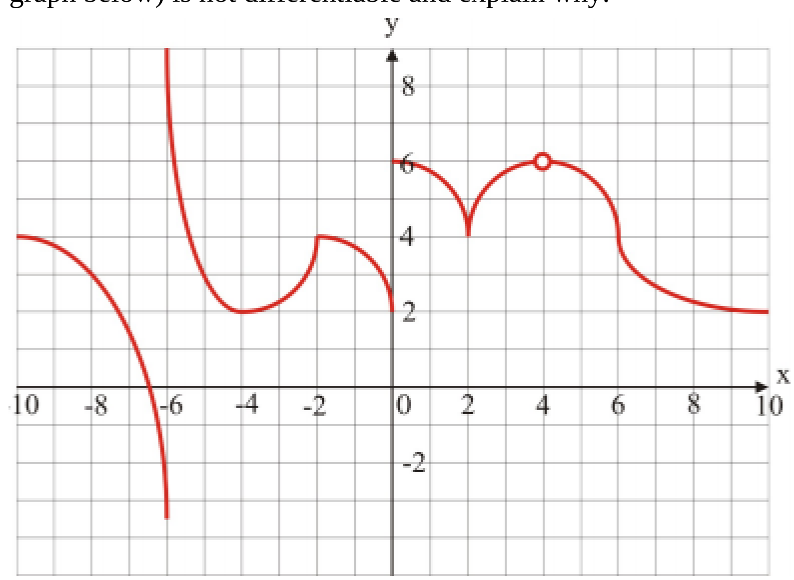
Cusp Point

$P(a, f(a))$ is a *cusp point* if the tangent line at P is vertical and the function is increasing on one side of the point P and decreasing on the other side.

$f'(a) = DNE$



Ex 2. Find the numbers x where the function y = f (x) (see the graph below) is not differentiable and explain why.



Derivative of Polynomial Functions

Computing derivatives from the limit definition is tedious and time-consuming. In this section, we will develop some rules that simplify that process of differentiation.

Constant Function Rule	**Proof**
$\mathrm{f(x) = c, c\ is\ a\ constant, f'(x) = 0}$	

Ex. 1: Constant Rule

Differentiate.

a. f(x) = 2021

b. f(x) = π

Power Rule
$f(x) = x^n, \quad n \in R, \quad f'(x) = nx^{n-1}$

Ex. 2: Power Rule

Differentiate using the power rule.

a. $f(x) = x^7$

b. $f(x) = \frac{1}{x^{20}}$

c. $f(x) = \sqrt{x^3}$

d. $f(x) = x^{2021}$

Constant Multiple Rule

$$f(x) = cg(x), c \text{ is a constant}, f'(x) = cg'(x)$$

Ex. 3: Constant Multiple Rule

Differentiate the following functions.

a. $f(x) = 6x^5$

b. $f(x) = 24x^{\frac{5}{4}}$

c. g(x) = $\frac{-8}{x^{-3}}$

d. $\frac{d}{dx}(15\sqrt[3]{x})$

Sum and Difference Rules

$[f(x) \pm g(x)] = f'(x) \pm g'(x)$

$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

$(f \pm g)' = f' \pm g'$

Ex. 4: Sum and Difference Rules

Differentiate the following functions and simplify into positive exponents.

a. $f(x) = 4x^7 - 6\sqrt{x}$

b. $y = (5x+2)^2$

c. $f(x) = 2x^{-5} + \frac{3}{5}\sqrt[3]{x^2} - \pi x + e$

.

Product Rule

If f and g are differentiable at x then so is fg and:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$
$$(fg)' = f'g + fg'$$
$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$
$$\frac{d}{dx}(uv) = v\frac{d}{dx}u + u\frac{d}{dx}v$$

Ex. 5: Differentiate and simplify.

a. $f(x) = \sqrt{x}(2 - 3x)$

b. $f(x) = (2x^3 + 5)(3x^2 - x)$

Note
Simplifying the derivative means to make the derivative into its factor form with positive exponents.

Note
In some cases, it is easier to expand and simplify the product before differentiating, rather than using the product rule.

c. f(x) = $\frac{3x^7-2x^5+x^3-10x^2+1}{x^2}$, x≠0

Ex 6. Find the equation of the tangent line to the curve y = (x+$\sqrt{x}$)(x^2+$\frac{1}{x}$) at the point P(1,4).

Ex 7. Consider the following piece-wise defined function:

$$f(x) = \begin{cases} -3x - 3x^2 & \text{if } x < -1 \\ 2 + x - x^2 & \text{if } x \geqslant -1 \end{cases}$$

Analyze the differentiability of the function f(x) at x = −1.

Ex 8. Analyze the differentiability of the function $y = f(x) = x^{\frac{2}{3}}$.