AP Calculus Class 5

Homework 4.

12. $f(x) = x\sqrt{2x-3}$, $f'(x) = ?$

$$f'(x) = \sqrt{2x-3} + x\left(\sqrt{2x-3}\right)'$$

$$= (2x-3)^{\frac{1}{2}} + x\left((2x-3)^{\frac{1}{2}}\right)'$$ let $u = 2x-3$

$$= (2x-3)^{\frac{1}{2}} + x\left(\tfrac{1}{2}(2x-3)^{-\frac{1}{2}} \cdot 2\right)$$

$$= (2x-3)^{\frac{1}{2}} + x(2x-3)^{-\frac{1}{2}}$$

$$= (2x-3)\cdot(2x-3)^{-\frac{1}{2}} + x(2x-3)^{-\frac{1}{2}}$$

$$= (2x-3)^{-\frac{1}{2}}(2x-3+x)$$

$$= \frac{3x-3}{\sqrt{2x-3}}$$ $\boxed{A}$

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13. $\frac{d}{dx}\left(x\, e^{\ln x^2}\right)$

$$e^{\ln x^2} = x^2 \Rightarrow \frac{d}{dx}\big(\overbrace{x \cdot x^2}^{x^3}\big) = 3x^2$$

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14. $y^2 + (xy+1)^3 = 0$, at $(2, -1)$.

Differentiate implicitly.

$$\underset{\uparrow}{\overset{\frac{dy}{dx}}{}}$$

$$2y\cdot y' + 3(xy+1)^2(xy'+y) = 0,$$

$$\Rightarrow 2(-1)\cdot y' + 3(2\cdot(-1)+1)^2(2y'+(-1)) = 0,$$

$$\Rightarrow -2y' + 6y' - 3 = 0 \quad \Rightarrow \quad y' = \frac{3}{4}$$

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15. $\frac{dy}{dx} = \sqrt{1-y^2}$, find $\frac{d^2y}{dx^2}$

let $u = 1-y^2$

$$\frac{d}{dx}\left((1-y^2)^{\frac{1}{2}}\right) = \frac{d}{dx}(u^{\frac{1}{2}})$$

$$= \frac{1}{2}u^{-\frac{1}{2}}\left(\frac{du}{dy}\right) = \frac{1}{2}(1-y^2)^{-\frac{1}{2}}(-2y)\frac{dy}{dx}$$

$$= \frac{1}{2}(1-y^2)^{-\frac{1}{2}}(-2y)(1-y^2)^{\frac{1}{2}} = -y$$

$$\boxed{B}$$

The Derivative of Logarithmic Functions

$y = \log_a x$ Differentiate Implicitly.

$a^y = x$

$\Rightarrow a^y \ln a \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{a^y \ln a}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x \ln a}$ (since $a^y = x$).

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}}$$

sub e for a.

$\Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x \ln e} = \frac{1}{x}$

$$\Rightarrow \boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$$

Example: $y = \ln(x^3+1)$ Find y'.

Let $u = x^3+1$

$y = \ln u \quad \Rightarrow \quad y' = \frac{1}{u} u' = \frac{1}{x^3+1}(3x^2)$

$$= \frac{3x^2}{x^3+1}$$

Example: $f(x) = \log_{10}(2+\sin x)$. $f'(x) = ?$

$$(\log_a x)' = \frac{1}{x \ln a}$$

$u = 2+\sin x$.

$$f'(x) = \frac{1}{u \ln a} u'$$

$$= \frac{1}{(2+\sin x)\ln a}(\cos x)$$

$$= \frac{\cos x}{(2+\sin x)\ln 10}$$

Example: $y = \ln\left(\frac{x+1}{\sqrt{x-2}}\right)$. Find y'.

$u = \frac{x+1}{\sqrt{x-2}}$

$$y' = \frac{1}{u} u'$$

$$= \frac{1}{\frac{x+1}{\sqrt{x-2}}}\left(\frac{x+1}{\sqrt{x-2}}\right)'$$

$$= \frac{\sqrt{x-2}}{x+1}\left(\frac{\sqrt{x-2} - (x+1)\frac{1}{2}(x-2)^{-\frac{1}{2}}}{x-2}\right)$$

$$= \frac{(x-2) - \frac{1}{2}(x+1)}{(x+1)(x-2)} = \frac{x-5}{2(x+1)(x-2)}$$

Logarithmic Differentiation.

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

Take the ln on both sides of the eqn.

$$\ln y = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right) = \ln\left(x^{3/4}\sqrt{x^2+1}\right) - \ln(3x+2)^5$$

$$= \ln x^{3/4} + \ln(x^2+1)^{\frac{1}{2}} - \ln(3x+2)^5$$

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiate Implicitly

$$\frac{1}{y}y' = \frac{3}{4}\frac{1}{x} + \frac{1}{2}\frac{1}{x^2+1}(2x) - 5\frac{1}{3x+2}(3).$$

$$y' = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

$$y' = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right).$$

Example: $y = x^{\sqrt{x}}$, $y' = ?$

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$$

Differentiate implicitly.

$$\frac{1}{y}y' = \sqrt{x}'\ln x + \sqrt{x}(\ln x)'$$

$$= \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\,\frac{1}{x}$$

$$\Rightarrow y' = y\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$$

$$\Rightarrow y' = x^{\sqrt{x}}\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$$

Example: $y = (\sin x)^{\ln x}$ Find y'.

$$\ln y = \ln(\sin x)^{\ln x}$$

$$= \ln x \cdot \ln(\sin x)$$

Differentiate implicitly.

$$\frac{1}{y}\cdot y' = \frac{1}{x}\ln(\sin x) + \ln x \cdot \frac{1}{\sin x}\cos x.$$

$$= y\left(\frac{\ln \sin x}{x} + \frac{\ln x \cos x}{\sin x}\right)$$

$$= (\sin x)^{\ln x}\left(\frac{\ln \sin x}{x} + \ln x \cot x\right).$$

Related Rates.

Parametric Equations.

$y = 7x^2 + x$.

Sometime, x and y are represented by a third variable, denoted by t.

t is called a parameter.

$x(t) = t$

$y(t) = 7t^2 + t$

$x(t) = t+2$

$y(t) = 7(t+2)^2 + (t+2)$

If we want to differentiate, we have to do so w.r.t. the third variable t.

Differentiate $y = 7x^2 + x$.

$$\frac{dy}{dt} = 14x \cdot \frac{dx}{dt} + \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = (14x + 1)\frac{dx}{dt}$$

Example: $y = x^3 + 2x$ and $\frac{dx}{dt} = 5$. Find $\frac{dy}{dt}$ when $x = 2$.

Diffe. both w.r.t. t.

$$(1)\ \frac{dy}{dt} = 3x^2\left(\frac{dx}{dt}\right) + 2\left(\frac{dx}{dt}\right)$$

$$\Rightarrow \frac{dy}{dt} = 3(4)(5) + 2(5) = 60 + 10 = 70.$$

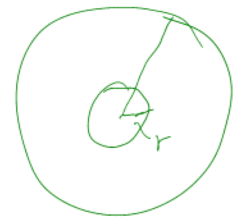
Example

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

V = volume, $\rightarrow$ $\frac{dV}{dt}$ = increase in volume. $= 100\ \text{cm}^3/\text{s}$.

$r = 25$ cm.

$\frac{dr}{dt} = ?$



The volume of a sphere is

$$V = \frac{4}{3}\pi r^3.$$

Need the volume equ^n in order to connect $\frac{dV}{dt}$ to $\frac{dr}{dt}$.

$$V = \frac{4}{3}\pi r^3 \quad \rightarrow \quad V(r) = \frac{4}{3}\pi r^3.$$

$$\Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi(3r^2)$$

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \underbrace{\frac{4}{3}\pi(3r^2)}_{\frac{dV}{dr}}\frac{dr}{dt}$$

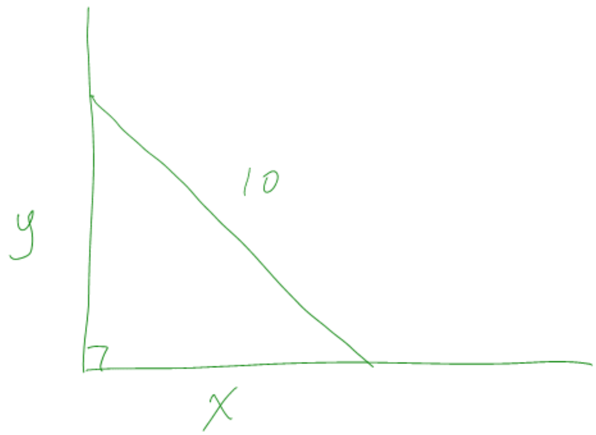
Solve the equ^n.

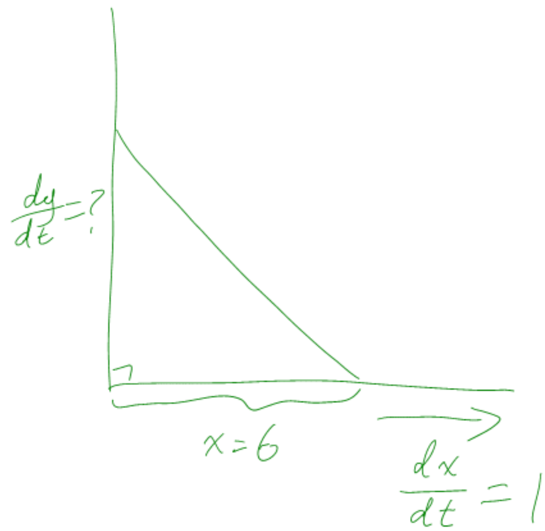
$\frac{dV}{dt} = 100$ $\quad r = 25 \quad$ Find $\frac{dr}{dt}$

$$\Rightarrow \frac{dr}{dt} = \frac{\frac{dV}{dt}}{\frac{4}{3}\pi(3r^2)} = \frac{\cancel{100}}{\cancel{\frac{4}{3}}\pi(\cancel{3}\cdot 25^{\cancel{2}})} = \frac{1}{25\pi} \approx 0.0127\ \text{cm/s}.$$

Example

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?





Given: $\frac{dx}{dt} = 1$ Find $\frac{dy}{dt}$ when $x = 6$.

By the Pythagorean thm,

$$x^2 + y^2 = 100. \Rightarrow x^2(t) + y^2(t) = 100$$

Differentiate implicitly.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

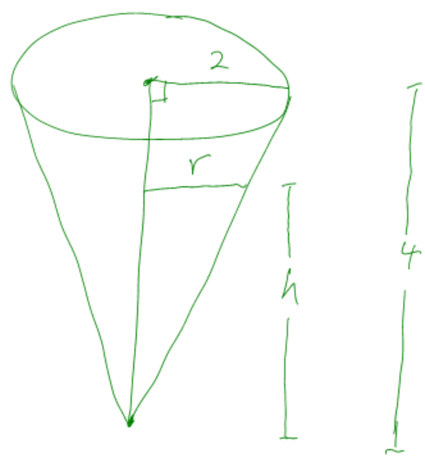
$$\Rightarrow 2x\frac{dx}{dt} = -2y\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When $x = 6$, then $y = 8$ by the Pyth.. thm.

$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4} \text{ ft/s}.$$

Example

A water tank has the shape of inverted circular cone with base radius 2 m and height 4 m. If the water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.



Let V = volume of the cone.

h = height of the water.

r = radius of the water

$\frac{dV}{dt} = 2 \text{ m}^3/\text{min}$

Find $\frac{dh}{dt}$ when $h = 3$.

$V = \frac{1}{3}(\pi r^2) h$

By similar triangles, we have

$\frac{r}{h} = \frac{2}{4} \Rightarrow r = \frac{1}{2}h.$

$\Rightarrow V = \frac{1}{3}(\pi(\frac{1}{2}h)^2) \cdot h = \frac{1}{3}\frac{\pi}{4}h^3 = \frac{\pi}{12}h^3.$

Differentiate w.r.t t.

$\frac{dV}{dt} = \frac{\pi}{12} 3h^2 \frac{dh}{dt}$

$= \frac{\pi}{4} h^2 \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{4}{\pi h^2}$

$= 2\left(\frac{4}{\pi 3^2}\right) = \frac{8}{9\pi}$ m/min.

- Write down everything you know and what to find.
- Find an equ^n that relates the variables.
- Differentiate implicitly w.r.t t.
- sub in all the numbers you know.

Homework.

1. c). $f(x) = (xe^x)(\csc x.)$

$$f'(x) = (xe^x)'(\csc x) + (xe^x)(\csc x)'$$

$$= (e^x + xe^x)(\csc x) + (xe^x)(-\csc x \cot x).$$

$$= e^x(1+x)\csc x - (xe^x)(\csc x \cot x)$$

$$= e^x \csc x ((1+x) - x\cot x).$$

6. $$f(x) = \frac{1}{x - \frac{2}{x+\sin x}}.$$

$$f(x) = \left(x - \frac{2}{x+\sin x}\right)^{-1}$$

$$f'(x) = -\left(x - \frac{2}{x+\sin x}\right)^{-2}\left(x - 2(x+\sin x)^{-1}\right)'$$

$$= -\left(x - \frac{2}{x+\sin x}\right)^{-2}\left(1 - 2\left(-(x+\sin x)^{-2}(1+\cos x)\right)\right)$$

$$= -\frac{1}{\left(x - \frac{2}{x+\sin x}\right)^2}\left(1 + \frac{2+2\cos x}{(x+\sin x)^2}\right)$$

$$= -\frac{1 + \frac{2+2\cos x}{(x+\sin x)^2}}{\left(x - \frac{2}{(x+\sin x)}\right)^2}$$