AP Calculus Lesson One Notes

Chapter One - Limits and Continuity

1.1 Definitions of Limits

We write $\lim_{x \to a} f(x) = L$ and say "the limit of *f*(*x*), as *x* approaches *a*, equals *L*" if we can make the value of *f*(*x*) arbitrarily close to *L* (as close to *L* as we like) by taking *x* to be sufficiently close to *a*, but not equal to *a*.

Given a function *f*(*x*) if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$, then the normal limit will exist and $\lim_{x \to a} f(x) = L$.

Likewise, if $\lim_{x \to a} f(x) = L$ then, $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.

This fact can be turned around to also say that if the two one-sided limits have different values, $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, then the normal limit will not exist.

We always use **one-sided limits** to discuss the limit of the function. As the name implies, with one-sided limits we will only be looking at one side of the point in question. Here are the definitions for the two one sided limits.

Right-handed limit

We say that $\lim_{x \to a^+} f(x) = L$, provided we can make *f*(*x*) as close to *L* as we want for all *x* sufficiently close to *a* and *x* > *a* without actually letting *x* be *a*.

Left-handed limit

We say that $\lim_{x \to a^-} f(x) = L$, provided we can make *f*(*x*) as close to *L* as we want for all *x* sufficiently close to *a* and *x* < *a* without actually letting *x* be *a*.

Note that the change in notation is very minor and in fact might be missed if you aren't paying attention. The only difference is the bit that is under the "lim" part of the limit. For the right-handed limit we now have $x \to a^+$ (note the "+") which means that we know will only look at *x*>*a*. Likewise for the left-handed limit we have $x \to a^-$(note the "-") which means that we will only be looking at *x*<*a*.

Also, note that as with the "normal" limit (*i.e.* the limits from the previous section) we still need the function to settle down to a single number in order for the limit to exist. The only difference this time is that the function only needs to settle down to a single number on either the right side of *x* = *a* or the left side of *x* = *a*depending on the one-sided limit we're dealing with.

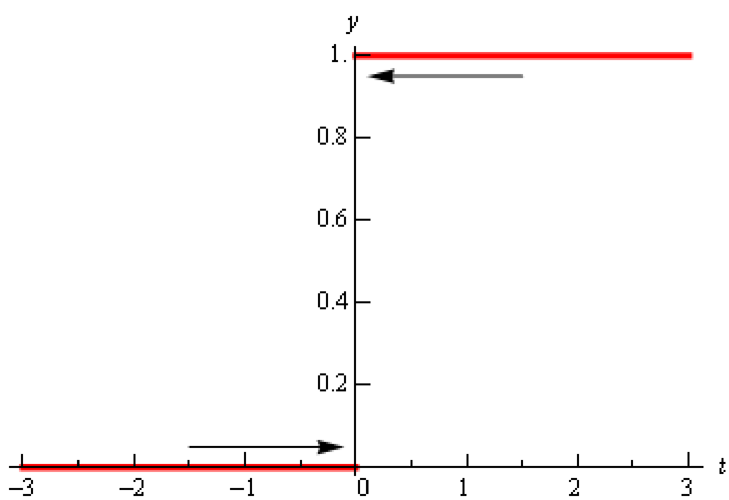
So when we are looking at limits it's now important to pay very close attention to see whether we are doing a normal limit or one of the one-sided limits. Let's now take a look at the some of the problems from the last section and look at one-sided limits instead of the normal limit.

Example 1.1-1 Estimate the value of the following limits.

$$\lim_{t\to 0^+} H(t) \quad \text{and} \quad \lim_{t\to 0^-} H(t) \qquad \text{where, } H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Solution

H(*t*) is a piecewise function. To remind us what this function looks like here's the graph.



So, we can see that if we stay to the right of *t* = 0 (i.e. *t* > 0) then the function is moving in towards a value of 1 as we get closer and closer to *t* = 0, but staying to the right. We can therefore say that the right-handed limit is

$$\lim_{t\to 0^+} H(t) = 1$$

Likewise, if we stay to the left of *t* = 0 (i.e. *t* < 0) the function is moving in towards a value of 0 as we get closer and closer to *t* = 0, but staying to the left. Therefore the left-handed limit is

$$\lim_{t\to 0^-} H(t) = 0$$

In this example we do get one-sided limits even though the normal limit itself doesn't exist.

Example 1.1-2 Estimate the value of the following limit.

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$$

Solution

Notice that I did say estimate the value of the limit. Again, we are not going to directly compute limits in this section. The point of this section is to give us a better idea of how limits work and what they can tell us about the function.

So, with that in mind we are going to work this in pretty much the same way that we did in the last section. We will choose values of *x* that get closer and closer to *x=2* and plug these values into the function. Doing this gives the following table of values.

x	f(x)	x	f(x)
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000

Note that we made sure and picked values of *x* that were on both sides of $x = 2$ and that we moved in very close to $x = 2$ to make sure that any trends that we might be seeing are in fact correct.

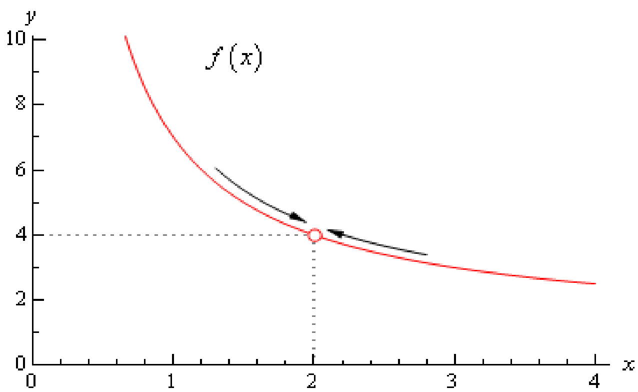
Also notice that we can't actually plug in $x = 2$ into the function as this would give us a division by zero error. This is not a problem since the limit doesn't care what is happening at the point in question.

From this table it appears that the function is going to 4 as *x* approaches 2, so $\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = 4$.

Let's think a little bit more about what's going on here. Let's graph the function from the last example. The graph of the function in the range of *x*'s that were interested in is shown below.

First, notice that there is a rather large open dot at $x = 2$. This is there to remind us that the function (and hence the graph) doesn't exist at $x = 2$.

As we were plugging in values of x into the function we are in effect moving along the graph in towards the point as $x = 2$. This is shown in the graph by the two arrows on the graph that are moving in towards the point.



When we are computing limits the question that we are really asking is what y value is our graph approaching as we move in towards $x = a$ on our graph. We are **NOT** asking what y value the graph takes at the point in question. In other words, we are asking what the graph is doing **around** the point $x = a$. In our case we can see that as x moves in towards 2 (from both sides) the function is approaching $y = 4$, even though the function itself doesn't even exist at $x = 2$. Therefore we can say that the limit is in fact 4.

So what have we learned about limits? Limits are asking what the function is doing **around** $x = a$ and are **not** concerned with what the function is actually doing at $x = a$, This is a good thing as many of the functions that we'll be looking at won't even exist at $x = a$ as we saw in our last example.

Let's work another example to drive this point home.

Example 1.1-3 Estimate the value of the following limit.

$$\lim_{x\to 2} g(x) \qquad \text{where,} \qquad g(x) = \begin{cases} \dfrac{x^2+4x-12}{x^2-2x} & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases}$$

Solution

The first thing to note here is that this is exactly the same function as the first example with the exception that we've now given it a value for $x = 2$. So, let's first note that

$g(2) = 6$

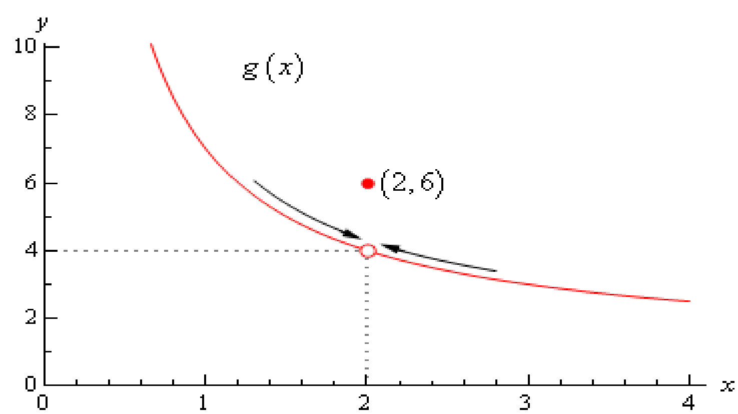
As far as estimating the value of this limit goes, nothing has changed in comparison to the first example. We could build up a table of values as we did in the first example or we could take a quick look at the graph of the function. Either method will give us the value of the limit.

Lets' first take a look at a table of values and see what that tells us. Notice that the presence of the value for the function at $x = 2$ will not change our choices for x. We only choose values of x that are getting closer to $x = 2$ but we never take $x = 2$. In other words the table of values that we used in the first example will be exactly the same table that we'll use here. So, since we've already got it down once there is no reason to redo it here.

$$\lim_{x \to 2} g(x) = 4$$

The limit is **NOT** 6! Remember from the discussion after the first example that limits do not care what the function is actually doing at the point in question. Limits are only concerned with what is going on **around** the point. Since the only thing about the function that we actually changed was its behavior at $x = 2$ this will not change the limit.

Let's also take a quick look at this functions graph to see if this says the same thing.



Again, we can see that as we move in towards $x = 2$ on our graph the function is still approaching a y value of 4. Remember that we are only asking what the function is doing **around** $x = 2$ and we don't care what the function is actually doing at $x = 2$. The graph then also supports the conclusion that the limit is,

$$\lim_{x \to 2} g(x) = 4$$

Let's make the point one more time just to make sure we've got it. Limits are not concerned with what is going on at . Limits are only concerned with what is going on around . We keep saying this, but it is a very important concept about limits that we must always keep in mind. So, we will take every opportunity to remind ourselves of this idea.

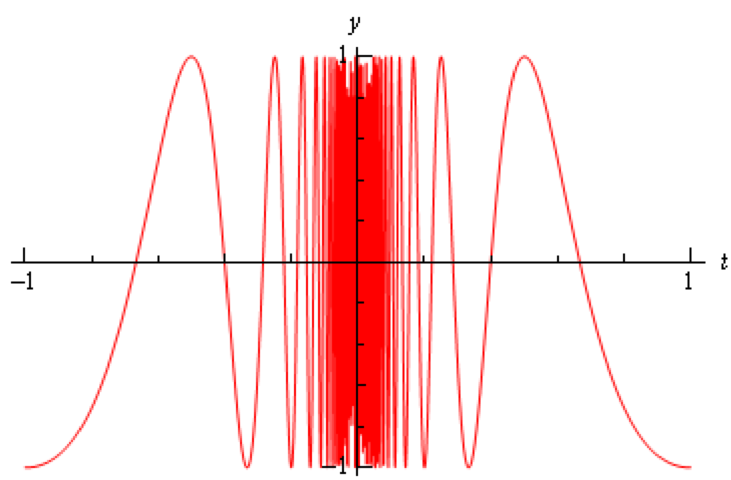
Since limits aren't concerned with what is actually happening at we will, on occasion, see situations like the previous example where the limit at a point and the function value at a point are different. This won't always happen of course. There are times where the function value and the limit at a point are the same and we will eventually see some examples of those. It is important however, to not get excited about things when the function and the limit do not take the same value at a point. It happens sometimes and so we will need to be able to deal with those cases when they arise.

Example 1.1-4 Estimate the value of the following limits.

$$\lim_{t \to 0^+} \cos\left(\frac{\pi}{t}\right) \qquad \lim_{t \to 0^-} \cos\left(\frac{\pi}{t}\right)$$

Solution

From the graph of this function shown below,



we can see that both of the one-sided limits suffer the same problem that the normal limit did in the previous section. The function does not settle down to a single number on either side of *t* = 0. Therefore, neither the left-handed nor the right-handed limit will exist in this case.

So, one-sided limits don't have to exist just as normal limits aren't guaranteed to exist.

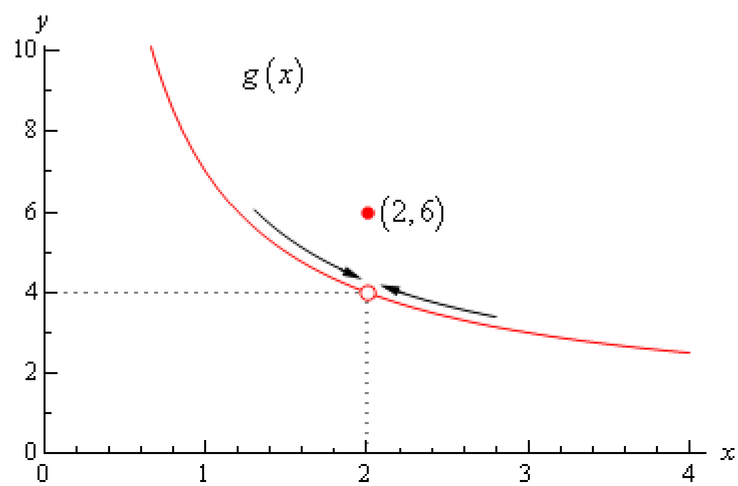
Let's take a look at another example from the previous section.

Example 1.1-5 Estimate the value of the following limits.

$$\lim_{x\to 2^+} g(x) \quad \text{and} \quad \lim_{x\to 2^-} g(x) \qquad \text{where, } g(x)=\begin{cases}\dfrac{x^2+4x-12}{x^2-2x} & \text{if } x\neq 2\\ 6 & \text{if } x=2\end{cases}$$

Solution

So as we've done with the previous two examples, let's remind ourselves of the graph of this function.



In this case regardless of which side of *x* = 2 we are on the function is always approaching a value of 4 and so we get,

$$\lim_{x\to 2^+} g(x)=4 \qquad\qquad \lim_{x\to 2^-} g(x)=4$$

Note that one-sided limits do not care about what's happening at the point any more than normal limits do. They are still only concerned with what is going on around the point. The only real difference between one-sided limits and normal limits is the range of *x*'s that we look at when determining the value of the limit.

Now let's take a look at the first and last example in this section to get a very nice fact about the relationship between one-sided limits and normal limits. In the last example the one-sided limits as well as the normal limit existed and all three had a value of 4. In the first example the two one-sided limits both existed, but did not have the same value and the normal limit did not exist.

The relationship between one-sided limits and normal limits can be summarized by the following fact.

Given a function *f*(*x*) if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$, then the normal limit will exist and $\lim_{x\to a} f(x) = L$.

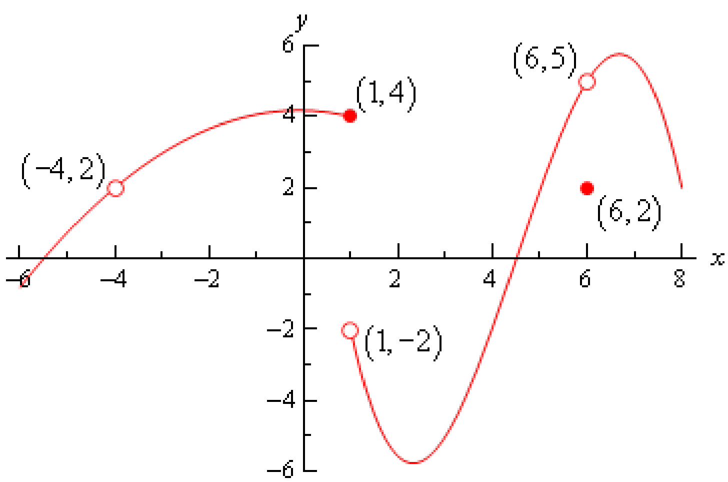
Likewise, if $\lim_{x\to a} f(x) = L$ then, $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.

This fact can be turned around to also say that if the two one-sided limits have different values, $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, then the normal limit will not exist.

This should make some sense. If the normal limit did exist then by the fact the two one-sided limits would have to exist and have the same value by the above fact. So, if the two one-sided limits have different values (or don't even exist) then the normal limit simply can't exist.

Let's take a look at one more example to make sure that we've got all the ideas about limits down that we've looked at in the last couple of sections.

Example 1.1-6 Given the following graph,



compute each of the following.

(a) $f(-4)$ (b) $\lim_{x\to -4^-} f(x)$ (c) $\lim_{x\to -4^+} f(x)$ (d) $\lim_{x\to -4} f(x)$

(e) $f(1)$ (f) $\lim_{x\to 1^-} f(x)$ (g) $\lim_{x\to 1^+} f(x)$ (h) $\lim_{x\to 1} f(x)$

(i) $f(6)$ (j) $\lim_{x\to 6^-} f(x)$ (k) $\lim_{x\to 6^+} f(x)$ (l) $\lim_{x\to 6} f(x)$

Solution

(a) $f(-4)$ doesn't exist. There is no closed dot for this value of *x* and so the function doesn't exist at this point.

(b) $\lim\limits_{x\to -4^-} f(x) = 2$. The function is approaching a value of 2 as *x* moves in towards -4 from the left.

(c) $\lim\limits_{x\to -4^+} f(x) = 2$. The function is approaching a value of 2 as *x* moves in towards -4 from the right.

(d) $\lim\limits_{x\to -4} f(x) = 2$. We can do this one of two ways. Either we can use the fact here and notice that the two one-sided limits are the same and so the normal limit must exist and have the same value as the one-sided limits or just get the answer from the graph. Also recall that a limit can exist at a point even if the function doesn't exist at that point.

(e) $f(1) = 4$. The function will take on the *y* value where the closed dot is.

(f) $\lim\limits_{x\to 1^-} f(x) = 4$.

(g) $\lim\limits_{x\to 1^+} f(x) = -2$. The function is approaching a value of -2 as *x* moves in towards 1 from the right. Remember that the limit does NOT care about what the function is actually doing at the point, it only cares about what the function is doing around the point. In this case, always staying to the right of *x* = 1, the function is approaching a value of -2 and so the limit is -2. The limit is not 4, as that is value of the function at the point and again the limit doesn't care about that!

(h) $\lim\limits_{x\to 1} f(x)$ doesn't exist. The two one-sided limits both exist, however they are different and so the normal limit doesn't exist.

(i) $f(6) = 2$. The function will take on the *y* value where the closed dot is.

(j) $\lim\limits_{x\to 6^-} f(x) = 5$. The function is approaching a value of 5 as *x* moves in towards 6 from the left.

(k) $\lim\limits_{x\to 6^+} f(x) = 5$. The function is approaching a value of 5 as *x* moves in towards 6 from the right.

(l) $\lim\limits_{x\to 6} f(x) = 5$. Again, we can use either the graph or the fact to get this. Also, once more remember that the limit doesn't care what is happening at the point and so it's possible for the limit to have a different value than the function at a point. When dealing with limits we've always got to remember that limits simply do not care about what the function is doing at the point in question. Limits are only concerned with what the function is doing around the point.

Hopefully over the last couple of sections you've gotten an idea on how limits work and what they can tell us about functions. Some of these ideas will be important in later sections so it's important that you have a good grasp on them.

1.2 Continuity

Over the last few sections we've been using the term "nice enough" to define those functions that we could evaluate limits by just evaluating the function at the point in question. It's now time to formally define what we mean by "nice enough".

Definition

A function $f(x)$ is said to be continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

A function is said to be continuous on the interval [a, b] if it is continuous at each point in the interval.

This definition can be turned around into the following fact.

Fact 1

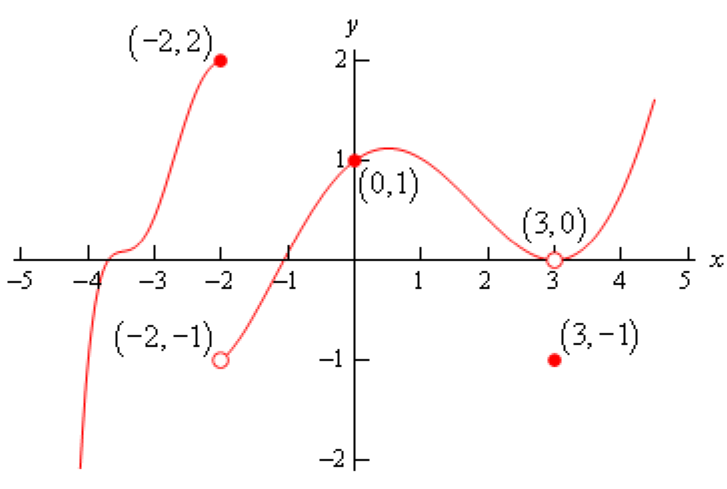
If $f(x)$ is continuous at x = a then, $\lim_{x \to a^-} f(x) = f(a)$ and $\lim_{x \to a^+} f(x) = f(a)$

This is exactly the same fact that we first put down back when we started looking at limits with the exception that we have replaced the phrase "nice enough" with continuous.

It's nice to finally know what we mean by "nice enough", however, the definition doesn't really tell us just what it means for a function to be continuous. Let's take a look at an example to help us understand just what it means for a function to be continuous.

Example 1.2-1 Given the graph of $f(x)$, shown below, determine if $f(x)$ is continuous at

$x = -2$, $x = 0$, and $x = 3$.



Solution

To answer the question for each point we'll need to get both the limit at that point and the function value at that point. If they are equal the function is continuous at that point and if they aren't equal the function isn't continuous at that point.

First at $x = -2$, $f(-2) = 2$, but $\lim_{x\to -2} f(x)$ does not exist.

The function value and the limit aren't the same and so the function is not continuous at this point. This kind of discontinuity in a graph is called a **jump discontinuity**. Jump discontinuities occur where the graph has a break in it is as this graph does.

Now at $x = 0$, $f(0) = 1$ and $\lim_{x\to -0} f(x) = 1$

The function is continuous at this point since the function and limit have the same value.

Finally at $x = 3$, $f(3) = -1$ and $\lim_{x\to -3} f(x) = 0$.

The function is not continuous at this point. This kind of discontinuity is called a **removable discontinuity**. Removable discontinuities are those where there is a hole in the graph as there is in this case.

From this example we can get a quick "working" definition of continuity. A function is continuous on an interval if we can draw the graph from start to finish without ever once picking up our pencil. The graph in the last example has only two discontinuities since there are only two places where we would have to pick up our pencil in sketching it.

In other words, a function is continuous if its graph has no holes or breaks in it.

For many functions it's easy to determine where it won't be continuous. Functions won't be continuous where we have things like division by zero or logarithms of zero. Let's take a quick look at an example of determining where a function is not continuous.

Example 1.2-2 Determine where the function below is not continuous.

$$h(t) = \frac{4t+10}{t^2 - 2t - 15}$$

Solution

Rational functions are continuous everywhere except where we have division by zero. So all that we need to is determine where the denominator is zero. That's easy enough to determine by setting the denominator equal to zero and solving.

$t^2 - 2t - 15 = (t-5)(t+3) = 0$

So, the function will not be continuous at *t=-3* and *t=5*.

A nice consequence of continuity is the following fact.

Fact 2

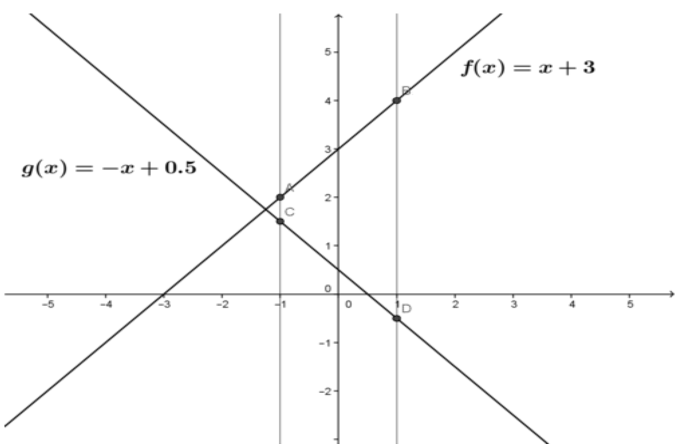
If $f(x)$ is continuous at x = b and $\lim_{x \to a} g(x) = b$ then, $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$.

Properties of Continuous Functions

The previous concept identified the characteristics of a function that is continuous at a point, and over an interval. Let's explore Given two functions f(x) and g(x) that are continuous over a closed interval [a,b], would you expect that arithmetic operations on these two functions would also yield functions continuous over [a,b]?

Example 1.2-3 Given the functions f(x)=x+3 and g(x)=−x+0.5 in the closed interval [−1,1], determine if f(x) and g(x) are continuous in the interval.

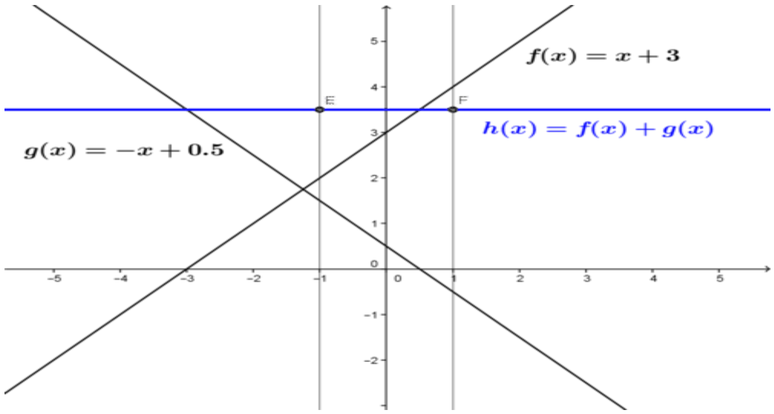
The functions f(x) and g(x) are shown in the graph. Inspection of each function graph and its equation, shows that they are each defined over the closed interval and the function limit at each point in the interval equals the function value at the point. Both functions are continuous in the interval.



Using the same functions and interval as above, determine if h(x)=f(x)+g(x) is continuous in the interval.

Solution

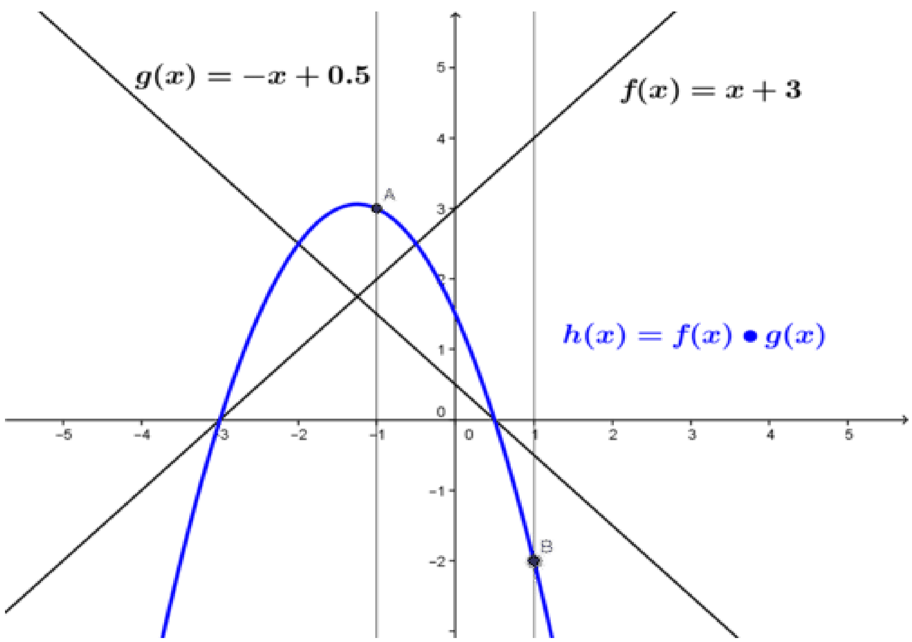
The sum of the two functions is given by h(x) = 3.5, and is shown in the figure. The sum function, a constant, is defined over the closed interval and the function limit at each point in the interval equals the constant function value at each point. The sum function is continuous in the interval.



Example 1.2-4 Still using the interval and functions as above, determine if h(x)=f(x)g(x) is continuous in the interval.

Solution

The product of the two functions is given by h(x)=(x+3)(−x+0.5)=−x^2+2.5x−1.5, and is shown in the figure. The product function, a parabola, is defined over the closed interval and the function limit at each point in the interval equals the product function value at each point. The product function is continuous in the interval.

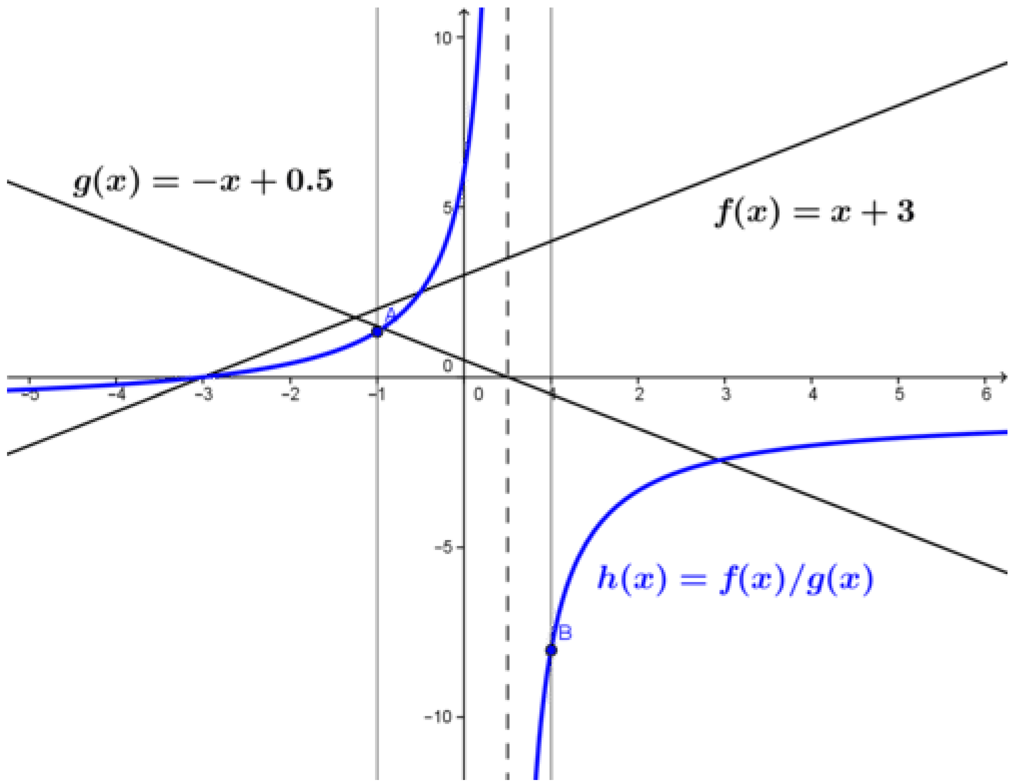


What about the quotient of two continuous functions?

Example 1.2-5 Given the functions f(x)=x+3 and g(x)=−x+0.5 in the closed interval [−1,1], determine if f(x)/g(x) is continuous in the interval.

Solution

The quotient of the two functions is given by h(x)=(x+3)/(−x+0.5), and is shown in the figure.



In the closed interval [−1,1], x=0.5 is the only place where the function h(x) is undefined, and limx→0.5h(x) does not exist. The function h(x) is not continuous at x=0.5, or in the closed interval.

The findings in the above simple functions can be generalized in the following properties.

If f(x) and g(x) are continuous at any real value c over the closed interval [a,b], then the following are also continuous at any real value c over the closed interval [a,b]:

f(x)+g(x)

f(x)−g(x)

f(x)g(x)

f(x)/g(x), as long as g(c)≠0.

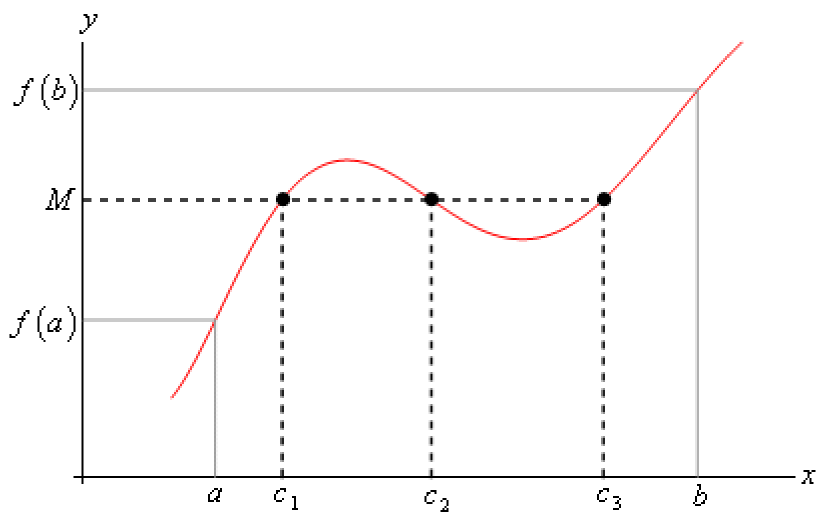
Intermediate Value Theorem and the Extreme Value (Min-Max) Theorem are two other properties of a function that is continuous over a closed interval.

Intermediate Value Theorem

Suppose that *f(x)* is continuous on [*a, b*] and let *M* be any number between *f* (*a*) and *f* (*b*). Then there exists a number *c* such that, $a < c < b$ and

$$f(c) = M$$

All the Intermediate Value Theorem is really saying is that a continuous function will take on all values between *f* (*a*) and *f* (*b*). Below is a graph of a continuous function that illustrates the Intermediate Value Theorem.



As we can see from this image if we pick any value, *M*, that is between the value of *f* (*a*) and the value of *f* (*b*) and draw a line straight out from this point the line will hit the graph in at least one point. In other words somewhere between *a* and *b* the function will take on the value of *M*. Also, as the figure shows the function may take on the value at more than one place.

It's also important to note that the Intermediate Value Theorem only says that the function will take on the value of *M* somewhere between *a* and *b*. It doesn't say just what that value will be. It only says that it exists.

So, the Intermediate Value Theorem tells us that a function will take the value of *M* somewhere between *a* and *b* but it doesn't tell us where it will take the value nor does it tell us how many times it will take the value. There are important idea to remember about the Intermediate Value Theorem.

A nice use of the Intermediate Value Theorem is to prove the existence of roots of equations as the following example shows.

Example 1.2-6 Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere in the interval [-1, 2].

Solution

What we're really asking here is whether or not the function will take on the value

$p(x) = 0$.

Somewhere between -1 and 2. In other words, we want to show that there is a number c such that $-1 < c < 2$ and $p(c) = 0$. However if we define $M = 0$ and acknowledge that $a = -1$ and $b = 2$, we can see that these two condition on c are exactly the conclusions of the Intermediate Value Theorem.

So, this problem is set up to use the Intermediate Value Theorem and in fact, all we need to do is to show that the function is continuous and that $M = 0$.

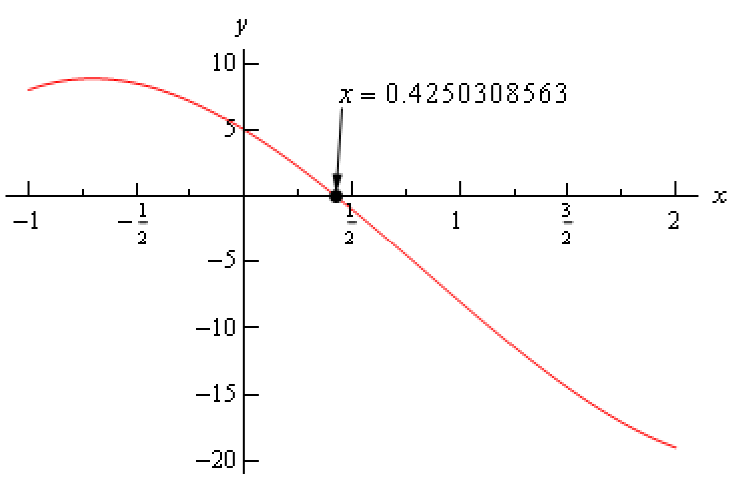
is between $p(-1)$ and $p(2)$ (i. e. $p(-1) < p(2)$ or $p(2) < p(-1)$, and we'll be done.

To do this all we need to do is compute, $p(-1) = 8$ and $p(2) = -19$. So we have,

$-19 = p(2) < 0 < p(-1) = 8$. Therefore $M = 0$ is between $p(-1)$ and $p(2)$, and since $p(x)$ is a polynomial it's continuous everywhere and so in particular it's continuous on the interval [-1,2]. So by the Intermediate Value Theorem there must be a number

$-1 < c < 2$ so that $p(c) = 0$. Therefore the polynomial does have a root between -1 and 2.

For the sake of completeness here is a graph showing the root that we just proved existed. Note that we used a computer program to actually find the root and that the Intermediate Value Theorem did not tell us what this value was.



The Extreme Value (Min-Max Theorem)

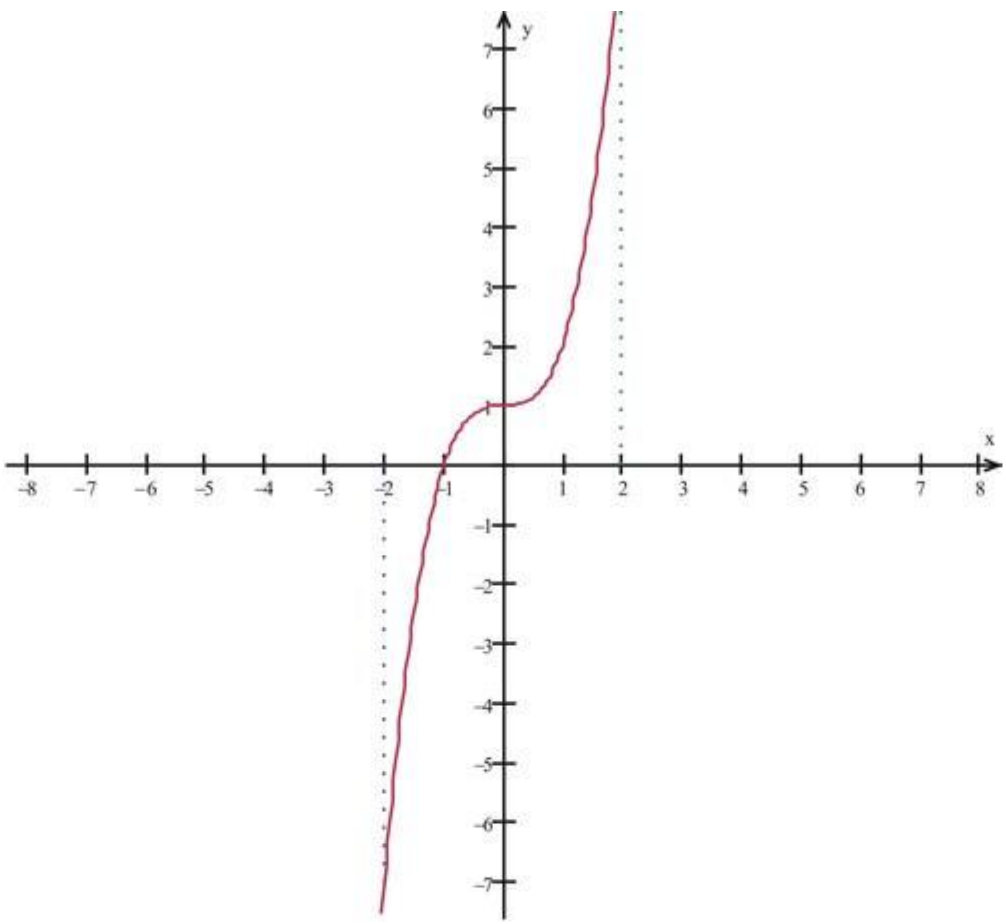
The Extreme Value (Min-Max Theorem) is a consequence of the Intermediate Value Theorem.

The Extreme Value (Min-Max) Theorem states that if a function f(x) is continuous in a closed interval I, then f(x) has both a maximum value and a minimum value in I.

Example 1.2-7 Consider $f(x)=x^3+1$ and interval $I=[-2,2]$. Determine minimum and maximum values.

Solution

Since the function is continuous on the closed interval I, this function has a minimum and a maximum on the interval. The function graph shows that at $x=-2$ the function has a minimum value $f(-2)=-7$; and at $x=2$, a maximum value $f(2)=9$.



1.3 Limit Properties

First we will assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and that *c* is any constant. Then,

1. $\lim_{x \to a}[cf(x)] = c\lim_{x \to a} f(x)$

In other words we can "factor" a multiplicative constant out of a limit.

2. $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

So to take the limit of a sum or difference all we need to do is take the limit of the individual parts and then put them back together with the appropriate sign. This is also not limited to two functions. This fact will work no matter how many functions we've got separated by "+" or "-".

3. $\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$

We take the limits of products in the same way that we can take the limit of sums or differences. Just take the limit of the pieces and then put them back together. Also, as with sums or differences, this fact is not limited to just two functions.

4. $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$

As noted in the statement we only need to worry about the limit in the denominator being zero when we do the limit of a quotient. If it were zero we would end up with a division by zero error and we need to avoid that.

5. $\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$, where n is any real number

In this property *n* can be any real number (positive, negative, integer, fraction, irrational, zero, etc.). In the case that *n* is an integer this rule can be thought of as an extended case of 3.

6. $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$, where n is any real number

This is just a special case of Property5.

7. $\lim\limits_{x \to a} c = c$

In other words, the limit of a constant is just the constant. You should be able to convince yourself of this by drawing the graph of $f(x) = c$.

8. $\lim\limits_{x \to a} x = a$

As with the last one you should be able to convince yourself of this by drawing the graph of $f(x) = x$.

9. $\lim\limits_{x \to a} x^n = a^n$

This is really just a special case of Property 5 using $f(x) = x$.

Note that all these properties also hold for the two one-sided limits as well we just didn't write them down with one sided limits to save on space.

Let's compute a limit or two using these properties. The next couple of examples will lead us to some truly useful facts about limits that we will use on a continual basis.

Example 1.3-1 Compute the value of the following limit

$$\lim_{x \to -2} \left(3x^2 + 5x - 9\right)$$

Solution

First we will use property **2** to break up the limit into three separate limits. We will then use property **1** to bring the constants out of the first two limits. Doing this gives us,

$$\begin{aligned} \lim_{x \to -2} \left(3x^2 + 5x - 9\right) &= \lim_{x \to -2} 3x^2 + \lim_{x \to -2} 5x - \lim_{x \to -2} 9 \\ &= 3\lim_{x \to -2} x^2 + 5\lim_{x \to -2} x - \lim_{x \to -2} 9 \end{aligned}$$

We can now use properties **7** through **9** to actually compute the limit.

$$\begin{aligned}\lim_{x\to-2}\left(3x^2+5x-9\right) &= 3\lim_{x\to-2}x^2+5\lim_{x\to-2}x-\lim_{x\to-2}9\\ &= 3\left(-2\right)^2+5\left(-2\right)-9\\ &= -7\end{aligned}$$

Example 1.3-2 Evaluate the following limit.

$$\lim_{z\to1}\frac{6-3z+10z^2}{-2z^4+7z^3+1}$$

Solution

First notice that we can use property **4)** to write the limit as,

$$\lim_{z\to1}\frac{6-3z+10z^2}{-2z^4+7z^3+1}=\frac{\lim\limits_{z\to1}6-3z+10z^2}{\lim\limits_{z\to1}-2z^4+7z^3+1}$$

Well, actually we should be a little careful. We can do that provided the limit of the denominator isn't zero. As we will see however, it isn't in this case so we're okay.

Now, both the numerator and denominator are polynomials so we can use the fact above to compute the limits of the numerator and the denominator and hence the limit itself.

$$\begin{aligned}\lim_{z\to1}\frac{6-3z+10z^2}{-2z^4+7z^3+1} &= \frac{6-3\left(1\right)+10\left(1\right)^2}{-2\left(1\right)^4+7\left(1\right)^3+1}\\ &= \frac{13}{6}\end{aligned}$$

Notice that the limit of the denominator wasn't zero and so our use of property **4** was legitimate.

Example 1.3-3 Evaluate the following limit.

$$\lim_{x\to3}\left(-\sqrt[5]{x}+\frac{\mathbf{e}^x}{1+\ln\left(x\right)}+\sin\left(x\right)\cos\left(x\right)\right),$$

where *e* is an irrational number and *e* = 2.71828182845905…,we will discuss more about *e* in the following section.

Solution

This is a combination of several of the functions listed above and none of the restrictions are violated so all we need to do is plug in $x = 3$ into the function to get the limit.

$$\begin{aligned}\lim_{x\to 3}\left(-\sqrt[5]{x}+\frac{\mathbf{e}^{x}}{1+\ln(x)}+\sin(x)\cos(x)\right) &= -\sqrt[5]{3}+\frac{\mathbf{e}^{3}}{1+\ln(3)}+\sin(3)\cos(3)\\ &= 8.185427271\end{aligned}$$

Not a very pretty answer, but we can now do the limit.