AP Calculus Lesson Four Notes

Chapter Two – Differentiation

2.3 Implicit Differentiation
2.4 Estimating a Derivative
2.5 Derivative of the Inverse of a Function

2.1 Implicit Differentiation

When a functional relationship between *x* and *y* is defined by an equation of the form $F(x,y)=0$, we say that the equation define *y* implicitly as a function of *x*. Some examples are $xy = 1$, $x^2 + y^2 = 4$, $y^2 - 3x = 0$, and $cos(xy) = y^2 - 5$.

If $y = f(x)$, then $\frac{d(y^n)}{dx} = ny^{n-1}\frac{dy}{dx}$, or $\left(y^n\right)' = ny^{n-1}y'$. This formula is very useful in implicit differentiation.

Example 2.3-1 Find $xy=1$

Solution

There are actually two solution methods for this problem.

<u>*Solution 1 :*</u>

This is the simple way of doing the problem. Just solve for *y* to get the function in the form that we're used to dealing with and then differentiate.

$$y=\frac{1}{x} \qquad \Rightarrow \qquad y' = -\frac{1}{x^2}$$

So, that's easy enough to do. However, there are some functions for which this can't be done. That's where the second solution technique comes into play.

<u>*Solution 2 :*</u>

In this case we're going to leave the function in the form that we were given and work with it in that form. However, let's recall from the first part of this solution that if we could solve for *y* then we will get *y* as a function of *x*. In other words, if we could solve for y (as we could in this case, but won't always be able to do) we get y = y(x). Let's rewrite the equation to note this.

$$xy = x\,y(x) = 1$$

Be careful here and note that when we write $y\,(x)$ we don't mean y times x. What we are noting here is that y is some (probably unknown) function of x. This is important to recall when doing this solution technique.

The next step in this solution is to differentiate both sides with respect to x as follows,

$$\frac{d}{dx}\left(x\,y(x)\right) = \frac{d}{dx}(1)$$

The right side is easy. It's just the derivative of a constant. The left side is also easy, but we've got to recognize that we've actually got a product here, the x and the $y(x)$. So to do the derivative of the left side we'll need to do the product rule. Doing this gives,

$$(1)\,y(x) + x\frac{d}{dx}\left(y(x)\right) = 0$$

Now, recall that we have the following notational way of writing the derivative.

$$\frac{d}{dx}\left(y(x)\right) = \frac{dy}{dx} = y'$$

Using this we get the following,

$$y + xy' = 0$$

Note that we dropped the (x) on the y as it was only there to remind us that the y was a function of x and now that we've taken the derivative it's no longer really needed. We just wanted it in the equation to recognize the product rule when we took the derivative.

So, let's now recall just what were we after. We were after the derivative, y', and notice that there is now a y' in the equation. So, to get the derivative all that we need to do is solve the equation for y'

$$y' = -\frac{y}{x}$$

There it is. Using the second solution technique this is our answer. This is not what we got from the first solution however. Or at least it doesn't look like the same derivative that we got from the first solution. Recall however, that we really do know what y is in terms of x and if we plug that in we will get,

$$y' = -\frac{1/\cancel{x}}{x} = -\frac{1}{x^2}$$

which is what we got from the first solution. Regardless of the solution technique used we should get the same derivative.

Example 2.3-2 Differentiate each of the following.

(a) $\left(5x^3 - 7x + 1\right)^5$, $\left[f(x)\right]^5$, $\left[y(x)\right]^5$

(b) $\sin\left(3 - 6x\right)$, $\sin\left(y(x)\right)$

(c) $\mathbf{e}^{x^2 - 9x}$, $\mathbf{e}^{y(x)}$

Solution

These are written a little differently from what we're used to seeing here. This is because we want to match up these problems with what we'll be doing in this section. Also, each of these parts has several functions to differentiate starting with a specific function followed by a general function. This again, is to help us with some specific parts of the implicit differentiation process that we'll be doing.

(a) $\left(5x^3 - 7x + 1\right)^5$, $\left[f(x)\right]^5$, $\left[y(x)\right]^5$

With the first function here we're being asked to do the following,

$$\frac{d}{dx}\left[\left(5x^3 - 7x + 1\right)^5\right] = 5\left(5x^3 - 7x + 1\right)^4\left(15x^2 - 7\right)$$

and this is just the chain rule. We differentiated the outside function (the exponent of 5) and then multiplied that by the derivative of the inside function (the stuff inside the parenthesis).

For the second function we're going to do basically the same thing. We're going to need to use the chain rule. The outside function is still the exponent of 5 while the inside

function this time is simply $f(x)$. We don't have a specific function here, but that doesn't mean that we can't at least write down the chain rule for this function. Here is the derivative for this function,

$$\frac{d}{dx}\left[f(x)\right]^5 = 5\left[f(x)\right]^4 f'(x)$$

We don't actually know what $f(x)$ is so when we do the derivative of the inside function all we can do is write down notation for the derivative, *i.e.* $f'(x)$.

With the final function here we simply replaced the f in the second function with a y since most of our work in this section will involve y's instead of f's. Outside of that this function is identical to the second. So, the derivative is,

$$\frac{d}{dx}\left[y(x)\right]^5 = 5\left[y(x)\right]^4 y'(x)$$

(b) $\sin(3-6x)$, $\sin(y(x))$

The first function to differentiate here is just a quick chain rule problem again so here is it's derivative,

$$\frac{d}{dx}\left[\sin(3-6x)\right] = -6\cos(3-6x)$$

For the second function we didn't bother this time with using $f(x)$ and just jumped straight to $y(x)$ for the general version. This is still just a general version of what we did for the first function. The outside function is still the sine and the inside is give by $y(x)$ and while we don't have a formula for $y(x)$ and so we can't actually take its derivative we do have a notation for its derivative. Here is the derivative for this function,

$$\frac{d}{dx}\left[\sin(y(x))\right] = y'(x)\cos(y(x))$$

(c) $\mathbf{e}^{x^2-9x}$, $\mathbf{e}^{y(x)}$

In this part we'll just give the answers for each and leave out the explanation that we had in the first two parts.

$$\frac{d}{dx}\left(\mathbf{e}^{x^2-9x}\right) = (2x-9)\mathbf{e}^{x^2-9x} \qquad \frac{d}{dx}\left(\mathbf{e}^{y(x)}\right) = y'(x)\mathbf{e}^{y(x)}$$

Example 2.3-3 Find y' for the following function.

$$x^2 + y^2 = 9$$

Solution

Now, this is just a circle and we can solve for y which would give,

$$y = \pm\sqrt{9 - x^2}$$

Prior to starting this problem we stated that we had to do implicit differentiation here because we couldn't just solve for y and yet that's what we just did. So, why can't we use "normal" differentiation here? The problem is the " ±". With this in the "solution" for y we see that y is in fact two different functions. Which should we use? Should we use both? We only want a single function for the derivative and at best we have two functions here.

So, in this example we really are going to need to do implicit differentiation so we can avoid this. In this example we'll do the same thing we did in the first example and remind ourselves that y is really a function of x and write y as $y\,(x)$. Once we've done this all we need to do is differentiate each term with respect to x.

$$\frac{d}{dx}\left(x^2 + \left[y(x)\right]^2\right) = \frac{d}{dx}(9)$$

As with the first example the right side is easy. The left side is also pretty easy since all we need to do is take the derivative of each term and note that the second term will be similar the part (a) of the second example. All we need to do for the second term is use the chain rule.

After taking the derivative we have,

$$2x + 2\left[y(x)\right]^1 y'(x) = 0$$

At this point we can drop the (x) part as it was only in the problem to help with the differentiation process. The final step is to simply solve the resulting equation for y'.

$$\begin{aligned} 2x + 2yy' &= 0 \\ y' &= -\frac{x}{y} \end{aligned}$$

Unlike the first example we can't just plug in for y since we wouldn't know which of the two functions to use. Most answers from implicit differentiation will involve both x and y so don't get excited about that when it happens.

Example 2.3-4 Find the equation of the tangent line to

$$x^2 + y^2 = 9$$

at the point $(2, \sqrt{5})$.

Solution

First note that unlike all the other tangent line problems we've done in previous sections we need to be given both the *x* and the *y* values of the point. Notice as well that this point does lie on the graph of the circle (you can check by plugging the points into the equation) and so it's okay to talk about the tangent line at this point.

Recall that to write down the tangent line we need is slope of the tangent line and this is nothing more than the derivative evaluated at the given point. We've got the derivative from the previous example so as we need to do is plug in the given point.

$$m = y'\Big|_{x=2,\, y=\sqrt{5}} = -\frac{2}{\sqrt{5}}$$

The tangent line is then.

$$y = \sqrt{5} - \frac{2}{\sqrt{5}}(x-2)$$

Example 2.3-5 Find y' for each of the following.

(a) $x^3 y^5 + 3x = 8y^3 + 1$

(b) $x^2 \tan(y) + y^{10} \sec(x) = 2x$

(c) $\mathbf{e}^{2x+3y} = x^2 - \ln(xy^3)$

Solution

(a) $x^3 y^5 + 3x = 8y^3 + 1$

First differentiate both sides with respect to *x* and remember that each *y* is really *y* (*x*) we just aren't going to write it that way anymore. This means that the first term on the left will be a product rule.

We differentiated these kinds of functions involving *y*'s to a power with the chain rule in the Example 2 above. Also, recall the discussion prior to the start of this problem. When doing this kind of chain rule problem all that we need to do is differentiate the *y*'s as normal and then add on a y', which is nothing more than the derivative of the "inside function".

Here is the differentiation of each side for this function.

$$3x^2y^5 + 5x^3y^4y' + 3 = 24y^2y'$$

Now all that we need to do is solve for the derivative, y'. This is just basic solving algebra that you are capable of doing. The main problem is that it's liable to be messier than what you're used to doing. All we need to do is get all the terms with y' in them on one side and all the terms without y' in them on the other. Then factor y' out of all the terms containing it and divide both sides by the "coefficient" of the y'. Here is the solving work for this one,

$$\begin{aligned} 3x^2y^5 + 3 &= 24y^2y' - 5x^3y^4y' \\ 3x^2y^5 + 3 &= \left(24y^2 - 5x^3y^4\right)y' \\ y' &= \frac{3x^2y^5 + 3}{24y^2 - 5x^3y^4} \end{aligned}$$

The algebra in these problems can be quite messy so be careful with that.

(b) $x^2\tan\left(y\right) + y^{10}\sec\left(x\right) = 2x$

We've got two product rules to deal with this time. Here is the derivative of this function.

$$2x\tan\left(y\right) + x^2\sec^2\left(y\right)y' + 10y^9y'\sec\left(x\right) + y^{10}\sec\left(x\right)\tan\left(x\right) = 2$$

Notice the derivative tacked onto the secant! Again, this is just a chain rule problem similar to the second part of Example 2 above.

Now, solve for the derivative.

$$\begin{aligned} \left(x^2\sec^2\left(y\right) + 10y^9\sec\left(x\right)\right)y' &= 2 - y^{10}\sec\left(x\right)\tan\left(x\right) - 2x\tan\left(y\right) \\ y' &= \frac{2 - y^{10}\sec\left(x\right)\tan\left(x\right) - 2x\tan\left(y\right)}{x^2\sec^2\left(y\right) + 10y^9\sec\left(x\right)} \end{aligned}$$

(c) $\mathbf{e}^{2x+3y} = x^2 - \ln\left(xy^3\right)$

We're going to need to be careful with this problem. We've got a couple chain rules that we're going to need to deal with here that are a little different from those that we've dealt with prior to this problem.

In both the exponential and the logarithm we've got a "standard" chain rule in that there is something other than just an *x* or *y* inside the exponential and logarithm. So, this means we'll do the chain rule as usual here and then when we do the derivative of the inside function for each term we'll have to deal with differentiating *y*'s.

Here is the derivative of this equation.

$$\mathbf{e}^{2x+3y}\left(2+3y'\right)=2x-\frac{y^3+3xy^2y'}{xy^3}$$

In both of the chain rules note that the y' didn't get tacked on until we actually differentiated the *y*'s in that term.

Now we need to solve for the derivative and this is liable to be somewhat messy. In order to get the y' on one side we'll need to multiply the exponential through the parenthesis and break up the quotient.

$$\begin{aligned}
2\mathbf{e}^{2x+3y}+3y'\mathbf{e}^{2x+3y} &= 2x-\frac{y^3}{xy^3}-\frac{3xy^2y'}{xy^3}\\
2\mathbf{e}^{2x+3y}+3y'\mathbf{e}^{2x+3y} &= 2x-\frac{1}{x}-\frac{3y'}{y}\\
\left(3\mathbf{e}^{2x+3y}+3y^{-1}\right)y' &= 2x-x^{-1}-2\mathbf{e}^{2x+3y}\\
y' &= \frac{2x-x^{-1}-2\mathbf{e}^{2x+3y}}{3\mathbf{e}^{2x+3y}+3y^{-1}}
\end{aligned}$$

Note that to make the derivative at least look a little nicer we converted all the fractions to negative exponents.

2.4 Estimating a Derivative

Numerically

Example 2.4-1

The table shown gives the temperatures of a polar bear on a very cold arctic day (*t* = minutes; *T* = degrees Fahrenheit):

t	0	1	2	3	4	5	6	7	8
T	98	94.95	93.06	91.90	91.17	90.73	90.45	90.28	90.17

Our task is to estimate the derivative of T numerically at various times.

Solution

Use the difference quotient $\dfrac{T(t+h)-T(t)}{h}$ with h equal to 1, we see that

$$T'(0)=\frac{T(1)-T(0)}{1-0}=\frac{94.95-98}{1}=-3.05°/\text{min}$$

Also,

$$T'(1)=\frac{T(2)-T(1)}{2-1}=\frac{93.06-94.95}{1}=-1.89°/\text{min}$$

$$T'(2)=\frac{T(3)-T(2)}{3-2}=\frac{91.90-93.06}{1}=-1.16°/\text{min}$$

And so on.

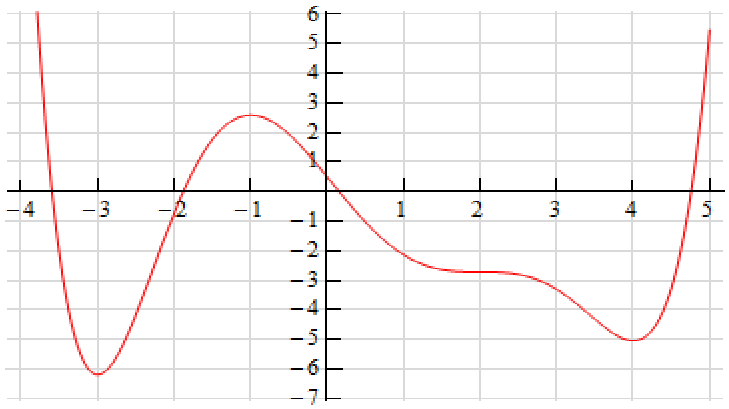
The following table shows the approximate values of $T'(t)$ obtained from the difference quotients above.

t	0	1	2	3	4	5	6	7	8
$T'(t)$	-3.05	-1.89	-1.16	-0.73	-0.47	-0.28	-0.17	-0.11	n/a

Graphically

If we have the graph of a function $f(x)$, we can use it to graph $f'(x)$. We accomplish this by estimating the slope of the graph of $f(x)$ at enough points to assure a smooth curve for $f'(x)$.

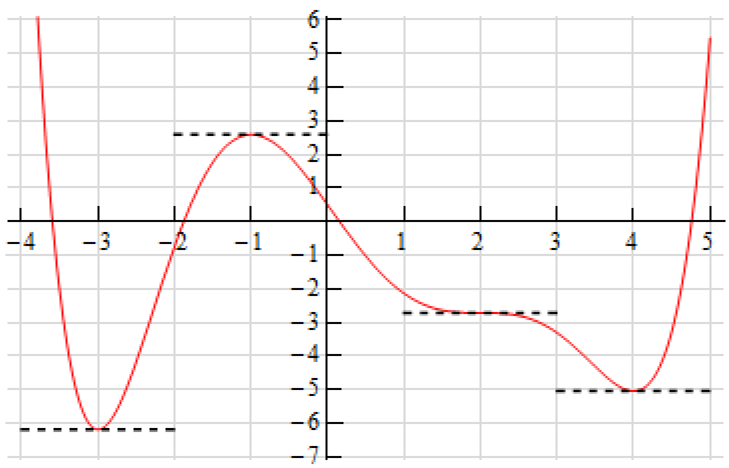
Example 2.4-2 Below is the sketch of a function $f(x)$. Sketch the graph of the derivative of this function, $f'(x)$.



Solution

At first glance this seems to an all but impossible task. However, if you have some basic knowledge of the interpretations of the derivative you can get a sketch of the derivative. It will not be a perfect sketch for the most part, but you should be able to get most of the basic features of the derivative in the sketch.

Let's start off with the following sketch of the function with a couple of additions.



Notice that at $x = -3$, $x = -1$, $x = 2$, and $x = 4$ the tangent line to the function is horizontal. This means that the slope of the tangent line must be zero. Now, we know that the slope of the tangent line at a particular point is also the value of the derivative of the function at that point. Therefore, we now know that,

$$f'(-3) = 0 \qquad f'(-1) = 0 \qquad f'(2) = 0 \qquad f'(4) = 0$$

This is a good starting point for us. It gives us a few points on the graph of the derivative. It also breaks the domain of the function up into regions where the function is increasing and decreasing. We know, from our discussions above, that if the function is increasing at a point then the derivative must be positive at that point. Likewise, we know that if the function is decreasing at a point then the derivative must be negative at that point.

We can now give the following information about the derivative.

$x < -3$	$f'(x) < 0$
$-3 < x < -1$	$f'(x) > 0$
$-1 < x < 2$	$f'(x) < 0$
$2 < x < 4$	$f'(x) < 0$
$x > 4$	$f'(x) > 0$

Remember that we are giving the signs of the derivatives here and these are solely a function of whether the function is increasing or decreasing. The sign of the function itself is completely immaterial here and will not in any way effect the sign of the derivative.

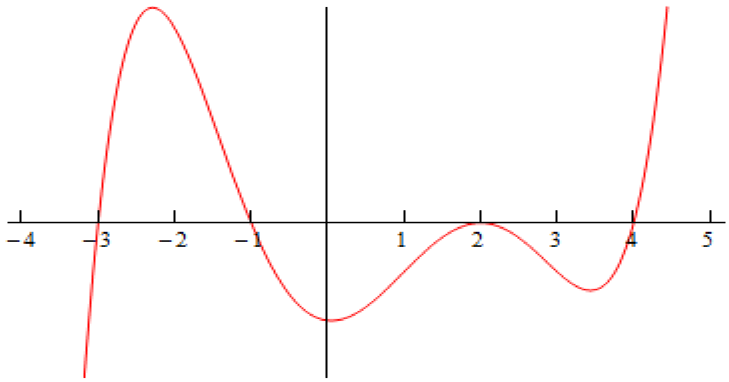
This may still seem like we don't have enough information to get a sketch, but we can get a little bit more information about the derivative from the graph of the function. In the range $x < -3$ we know that the derivative must be negative, however we can also see that the derivative needs to be increasing in this range. It is negative here until we reach $x = -3$ and at this point the derivative must be zero. The only way for the derivative to be negative to the left of $x = -3$ and zero at $x = -3$ is for the derivative to increase as we increase x towards $x = -3$.

Now, in the range $-3 < x < -1$ we know that the derivative must be zero at the endpoints and positive in between the two endpoints. Directly to the right of $x = -3$ the derivative must also be increasing (because it starts at zero and the goes positive therefore it must be increasing). So, the derivative in this range must start out increasing and must eventually get back to zero at $x = -1$. So, at some point in this interval the derivative must start decreasing before it reaches $x = -1$. Now, we have to be careful here because this is just general behavior here at the two endpoints. We won't know where the derivative goes from increasing to decreasing and it may well change between increasing and decreasing several times before we reach $x = -1$. All we can really say is that immediately to the right of $x = -3$ the derivative will be increasing and immediately to the left of $x = -1$ the derivative will be decreasing.

Next, for the ranges $-1 < x < 2$ and $2 < x < 4$ we know the derivative will be zero at the endpoints and negative in between. Also, following the type of reasoning given above we can see in each of these ranges that the derivative will be decreasing just to the right of the left hand endpoint and increasing just to the left of the right hand endpoint.

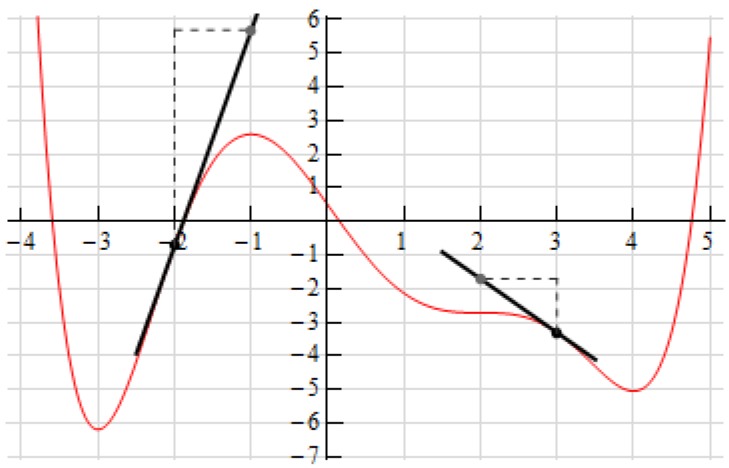
Finally, in the last region $x > 4$ we know that the derivative is zero at $x = 4$ and positive to the right of $x = 4$. Once again, following the reasoning above, the derivative must also be increasing in this range.

Putting all of this material together (and always taking the simplest choices for increasing and/or decreasing information) gives us the following sketch for the derivative.



Note that this was done with the actual derivative and so is in fact accurate. Any sketch you do will probably not look quite the same. The "humps" in each of the regions may be at different places and/or different heights for example. Also, note that we left off the vertical scale because given the information that we've got at this point there was no real way to know this information.

That doesn't mean however that we can't get some ideas of specific points on the derivative other than where we know the derivative to be zero. To see this let's check out the following graph of the function (not the derivative, but the function).

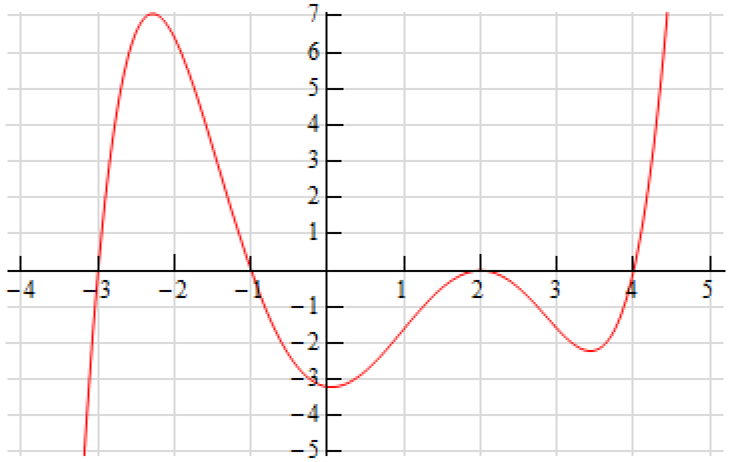


At $x = -2$ and $x = 3$ we've sketched in a couple of tangent lines. We can use the basic rise/run slope concept to estimate the value of the derivative at these points.

Let's start at $x = 3$. We've got two points on the line here. We can see that each seem to be about one-quarter of the way off the grid line. So, taking that into account and the fact that we go through one complete grid we can see that the slope of the tangent line, and hence the derivative, is approximately -1.5.

At $x = -2$ it looks like (with some heavy estimation) that the second point is about 6.5 grids above the first point and so the slope of the tangent line here, and hence the derivative, is approximately 6.5.

Here is the sketch of the derivative with the vertical scale included and from this we can see that in fact our estimates are pretty close to reality.



Note that this idea of estimating values of derivatives can be a tricky process and does require a fair amount of (possible bad) approximations so while it can be used, you need to be careful with it.

Linear Approximations

A fruitful way to think about the derivative $f'(x)$ is that it is the best linear approximation to $f(x)$ at the point x. For a differentiable function $f(x)$, if we think of Δx as a very small in change in x, then $\Delta f = f(a + \Delta x) - f(a)$ is also small.

Since $f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x} \approx \frac{f(a+\Delta x)-f(a)}{\Delta x}$, it follows that when Δx is small enough, we have $\Delta f\ \Delta x \approx f'(a)$, so $\Delta f \approx f'(a)\Delta x$, or $f(a+\Delta x) \approx f(a) + f'(a)\Delta x$.

Example 2.4-3 Use a linear approximation to estimate $\sqrt[3]{8.002}$.

Solution

We want a linear approximation to $f(x) = \sqrt[3]{x}$ for $a = 8$ and $\Delta x = 0.002$.

Since $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$, and $f(a + \Delta x) \approx f(a) + f'(a)\Delta x$

we have $\sqrt[3]{8.002} \approx \sqrt[3]{8} + f'(8)(0.002) = 2 + \frac{1}{3\sqrt[3]{8^2}}(0.002) = 2 + \frac{0.002}{12} \approx 2.00017.$

2.5 Derivative of the Inverse of a Function

Two functions $f(x)$ and $f^{-1}(x)$ are inverses if $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

If $y = f(x)$ and for $x = a$, $y = f(a) = b$, so (a, b) is a point of the graph of $f(x)$ and (b, a) is the corresponding point of the graph of $f^{-1}(x)$. Now, we can take derivatives of both sides of $f^{-1}(f(x)) = x$ with respect to x, we have,

$$\frac{d\left[f^{-1}(f(x))\right]}{dx} = \frac{dx}{dx}, \text{ or } \frac{d\left(f^{-1}(y)\right)}{dy}\frac{dy}{dx} = 1, \text{ or } \frac{d\left(f^{-1}(y)\right)}{dy} = \frac{1}{\frac{dy}{dx}}.$$

We have, therefore, the derivative of $f^{-1}(x)$ at the point (b, a) of its graph,

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}$$

Or generally,

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}$$

Example 2.5-1

Let $y = f(x) = 3x^2 + x - 2$ and let g be the inverse function. Evaluate $g'(0)$.

Solution

Since $f'(x) = 6x + 1$ and $g'(y) = \dfrac{1}{6x+1}$. To find *x* when *y* = 0, we must solve the equation $3x^2 + x - 2 = 0$. Note by inspection that *x* = −1, or *x* = 2/3 so $g'(0) = \dfrac{1}{6(-1)+1} = \dfrac{1}{-5}$, or $g'(0) = \dfrac{1}{6(2/3)+1} = \dfrac{1}{5}$.

Example 2.5-2

Find where the tangent to the curve $4x^2 + 9y^2 = 36$ is vertical.

Solution

We differentiate the equation implicitly to get $\dfrac{dy}{dx}$:

$$8x + 18y\frac{dy}{dx} = 0$$

So $\dfrac{dy}{dx} = -\dfrac{4x}{9y}$.

Since the tangent line to a curve is vertical when $\dfrac{dx}{dy} = 0$, we conclude that

$$-\frac{9y}{4x} = 0$$

So *y* = 0. When we substitute *y* = 0 in the original equation, we get *x* = ± 3. The points $(\pm 3, 0)$ are the ends of the major axis of the ellipse, where the tangents are indeed vertical.

Derivatives of Parametrically Defined Functions

If *x* = *f* (*t*) and *y* = *g*(*t*) are differentiable functions of *t*, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d\left(\frac{dy}{dx}\right)}{dt}}{\frac{dx}{dt}}$$

Example 2.5-3 Find the second derivative for the following set of parametric equations.

$x = t^5 - 4t^3$ $\qquad\qquad$ $y = t^2$

Solution

This is the set of parametric equations and we have the following computations.

$$\frac{dy}{dt} = 2t \qquad \frac{dx}{dt} = 5t^4 - 12t^2 \qquad \frac{dy}{dx} = \frac{2}{5t^3 - 12t}$$

We will first need the following,

$$\frac{d}{dt}\left(\frac{2}{5t^3 - 12t}\right) = \frac{-2\left(15t^2 - 12\right)}{\left(5t^3 - 12t\right)^2} = \frac{24 - 30t^2}{\left(5t^3 - 12t\right)^2}$$

The second derivative is then,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \\ &= \frac{\frac{24 - 30t^2}{\left(5t^3 - 12t\right)^2}}{5t^4 - 12t^2} \\ &= \frac{24 - 30t^2}{\left(5t^4 - 12t^2\right)\left(5t^3 - 12t\right)^2} \\ &= \frac{24 - 30t^2}{t\left(5t^3 - 12t\right)^3} \end{aligned}$$

Example 2.5-4 Find the tangent line(s) to the parametric curve given by

$$x = t^5 - 4t^3 \qquad y = t^2$$

at (0,4).

Solution

Note that there is apparently the potential for more than one tangent line here! We will look into this more after we're done with the example.

The first thing that we should do is find the derivative so we can get the slope of the tangent line.

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2t}{5t^4-12t^2}=\frac{2}{5t^3-12t}$$

At this point we've got a small problem. The derivative is in terms of *t* and all we've got is an *x*-*y* coordinate pair. The next step then is to determine that value(s) of *t* which will give this point. We find these by plugging the *x* and *y* values into the parametric equations and solving for *t*.

$$0=t^5-4t^3=t^3\left(t^2-4\right) \qquad \Rightarrow \qquad t=0,\pm 2$$
$$4=t^2 \qquad \Rightarrow \qquad t=\pm 2$$

Any value of *t* which appears in both lists will give the point. So, since there are two values of *t* that give the point we will in fact get two tangent lines. That's definitely not something that happened back in Calculus I and we're going to need to look into this a little more. However, before we do that let's actually get the tangent lines.

Since we already know the *x* and *y*-coordinates of the point all that we need to do is find the slope of the tangent line.

$$m=\left.\frac{dy}{dx}\right|_{t=-2}=-\frac{1}{8}$$

The tangent line (at t = - 2) is then,

$$y=4-\frac{1}{8}x$$

Again, all we need is the slope.

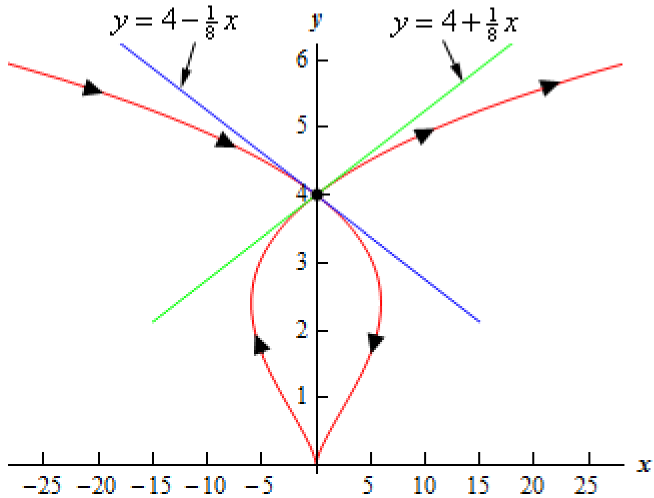
$$m=\left.\frac{dy}{dx}\right|_{t=2}=\frac{1}{8}$$

The tangent line (at $t = 2$) is then,

$$y=4+\frac{1}{8}x$$

Now, let's take a look at just how we could possibly get two tangents lines at a point. This was definitely not possible back in Calculus I where we first ran across tangent lines.

A quick graph of the parametric curve will explain what is going on here.



So, the parametric curve crosses itself! That explains how there can be more than one tangent line. There is one tangent line for each instance that the curve goes through the point.

The Mean Value Theorem

Before we get to the Mean Value Theorem we need to cover the following theorem.

Rolle's Theorem

Suppose $f(x)$ is a function that satisfies all of the following.

1. $f(x)$ is continuous on the closed interval $[a,b]$.
2. $f(x)$ is differentiable on the open interval (a,b).
3. $f(a) = f(b)$.

Then there is a number c such that $a < c < b$ and $f'(x) = 0$. Or, in other words $f(x)$ has a critical point in (a,b).

Example 2.5-5 Show that the following polynomial function has exactly one real root.

$f(x) = 4x^5 + x^3 + 7x - 2$

Solution

From basic Algebra principles we know that since $f(x)$ is a 5th degree polynomial there it will have five roots. What we're being asked to prove here is that only one of those 5 is a real number and the other 4 must be complex roots.

First, we should show that it does have at least one real root. To do this note that $f(0) = -2$ and that $f(1) = 10$ and so we can see that $f(0) < 0 < f(1)$. Now, because $f(x)$ is a polynomial we know that it is continuous everywhere and so by the Intermediate Value Theorem there is a number c such that $0 < c < 1$ and $f(c) = 0$. In other words $f(x)$ has at least one real root.

We now need to show that this is in fact the only real root. To do this we'll use an argument that is called contradiction proof. What we'll do is assume that $f(x)$ has at least two real roots. This means that we can find real numbers a and b (there might be more, but all we need for this particular argument is two) such that $f(a) = f(b) = 0$. But if we do this then we know from Rolle's Theorem that there must then be another number c such that $f'(x) = 0$.

This is a problem however. The derivative of this function is,

$$f'(x) = 20x^4 + 3x^2 + 7$$

Because the exponents on the first two terms are even we know that the first two terms will always be greater than or equal to zero and we are then going to add a positive number onto that and so we can see that the smallest the derivative will ever be is 7 and this contradicts the statement above that says we MUST have a number c such that $f'(c) = 0$.

We reached these contradictory statements by assuming that $f(x)$ has at least two roots. Since this assumption leads to a contradiction the assumption must be false and so we can only have a single real root.

Mean Value Theorem

Suppose $f(x)$ is a function that satisfies both of the following.

1. $f(x)$ is continuous on the closed interval $[a,b]$.
2. $f(x)$ is differentiable on the open interval (a,b).

Then there is a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or,

$$f(b)-f(a)=f'(c)(b-a)$$

Note that the Mean Value Theorem doesn't tell us what c is. It only tells us that there is at least one number c that will satisfy the conclusion of the theorem.

Also note that if it weren't for the fact that we needed Rolle's Theorem to prove this we could think of Rolle's Theorem as a special case of the Mean Value Theorem. To see that just assume that $f(a) = f(b)$ and then the result of the Mean Value Theorem gives the result of Rolle's Theorem.

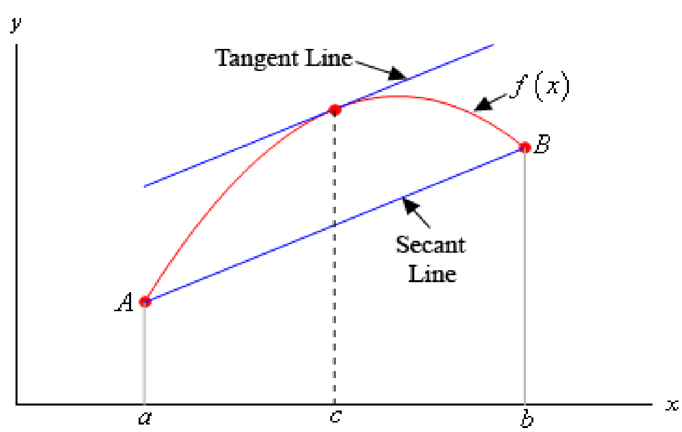
Before we take a look at a couple of examples let's think about a geometric interpretation of the Mean Value Theorem. First define $A = (a, f(a))$ and $B = (b, f(b))$ and then we know from the Mean Value theorem that there is a c such that $a < c < b$ and that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Now, if we draw in the secant line connecting A and B then we can know that the slope of the secant line is,

$$\frac{f(b)-f(a)}{b-a}$$

Likewise, if we draw in the tangent line to $f(x)$ at $x = c$ we know that its slope is $f'(c)$. What the Mean Value Theorem tells us is that these two slopes must be equal or in other words the secant line connecting A and B and the tangent line at $x = c$ must be parallel. We can see this in the following sketch.



Let's now take a look at a couple of examples using the Mean Value Theorem.

Example 2.5-6 Determine all the numbers *c* which satisfy the conclusions of the Mean Value Theorem for the following function.

$$f(x) = x^3 + 2x^2 - x \quad \text{on} \quad [-1,2]$$

Solution

There isn't really a whole lot to this problem other than to notice that since $f(x)$ is a polynomial it is both continuous and differentiable (*i.e.* the derivative exists) on the interval given.

First let's find the derivative.

$$f'(x) = 3x^2 + 4x - 1$$

Now, to find the numbers that satisfy the conclusions of the Mean Value Theorem all we need to do is plug this into the formula given by the Mean Value Theorem.

$$f'(c) = \frac{f(2) - f(-1)}{2-(-1)}$$
$$3c^2 + 4c - 1 = \frac{14-2}{3} = \frac{12}{3} = 4$$

Now, this is just a quadratic equation,

$$3c^2 + 4c - 1 = 4$$
$$3c^2 + 4c - 5 = 0$$

Using the quadratic formula on this we get,

$$c = \frac{-4 \pm \sqrt{16 - 4(3)(-5)}}{6} = \frac{-4 \pm \sqrt{76}}{6}$$

So, solving gives two values of *c*.

$$c = \frac{-4+\sqrt{76}}{6} = 0.7863 \qquad\qquad c = \frac{-4-\sqrt{76}}{6} = -2.1196$$

Notice that only one of these is actually in the interval given in the problem. That means that we will exclude the second one (since it isn't in the interval). The number that we're after in this problem is,

$$c = 0.7863$$

Be careful to not assume that only one of the numbers will work. It is possible for both of them to work.

Example 2.5-7 Suppose that we know that $f(x)$ is continuous and differentiable on $[6,\ 15]$. Let's also suppose that we know that $f(6) = -2$ and that we know that $f'(x) \le 10$. What is the largest possible value for $f(15)$?

Solution

Let's start with the conclusion of the Mean Value Theorem.

$$f(15) - f(6) = f'(c)(15-6)$$

Plugging in for the known quantities and rewriting this a little gives,

$$f(15) = f(6) + f'(c)(15-6) = -2 + 9f'(c)$$

Now we know that $f'(x) \le 10$ so in particular we know that $f'(c) \le 10$. This gives us the following,

$$\begin{aligned} f(15) &= -2 + 9f'(c) \\ &\le -2 + (9)10 \\ &= 88 \end{aligned}$$

All we did was replace $f'(c)$ with its largest possible value.

This means that the largest possible value for $f(15)$ is 88.