AP Calculus Class 11.

The substitution Rule.

Definite Integrals.

$$\int_1^2 \frac{1}{(3-5x)^2}\,dx.$$

Let $u = 3-5x$

$du = -5\,dx$

$-\frac{1}{5}\,du = dx$

$u(1) = 3-(5)(1) = -2$

$u(2) = 3-5(2) = -7$

$$= \int_{-2}^{-7} \frac{1}{u^2}\left(-\frac{1}{5}\right)du$$

$$= -\frac{1}{5}\int_{-2}^{-7} \frac{1}{u^2}\,du$$

$$= -\frac{1}{5}\int_{-2}^{-7} u^{-2}\,du = -\frac{1}{5}\left[-u^{-1}\right]_{-2}^{-7}$$

$$= -\frac{1}{5}\left[-\frac{1}{(-7)} - \left(-\frac{1}{(-2)}\right)\right] = \frac{1}{5}\left[\frac{1}{2}-\frac{1}{7}\right]$$

$$= \frac{1}{14}$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1}$$

$$\int u^{-2}\,du = \frac{u^{-2+1}}{-2+1}$$

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

Example: $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

Let $u = \ln x$

$du = \frac{1}{x}\,dx$

$u(1) = \ln 1 = 0$

$u(e) = \ln e = 1$

$$= \int_0^1 u\,du = \left.\frac{u^2}{2}\right]_0^1$$

$$= \frac{1}{2} - \frac{0}{2} = \frac{1}{2}$$

Integration Techniques

Integration by Parts.

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x).$$

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x) + g'(x)f(x)\,dx$$

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int g'(x)f(x)\,dx.$$

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

The formula for <u>integration by parts</u>

$$\int fg' = fg - \int gf'$$

$$\int f(x)\underbrace{g'(x)dx} = f(x)g(x) - \int g(x)\underbrace{f'(x)dx}$$

$$f'(x)dx = df$$

$$g'(x)dx = dg$$

$$\frac{d}{dx}g(x)\,dx$$

let $f = f(x)$
$g = g(x)$

$$\int f\,dg = fg - \int g\,df. \longrightarrow \int u\,dv = uv - \int v\,du.$$

Example: $\int x \cdot \sin x \, dx$

x, $\sin x$.

$f = x \qquad g' = \sin x$.

$f' = 1 \qquad g = -\cos x$

$$\int x \sin x \, dx = x \cdot (-\cos x) - \int -\cos x \cdot 1 \, dx$$

$$= -x\cos x + \underbrace{\int \cos x \, dx}_{\sin x}$$

$$= -x\cos x + \sin x + C$$

$\int x \sin x \, dx$.

let $f = \sin x \qquad g' = x$

$f' = \cos x \qquad g = \frac{x^2}{2}$

$$\int x \sin x \, dx = \sin x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \cos x \, dx$$

Example: $\int \ln x \, dx \rightarrow \int \ln x \cdot 1 \, dx$

let $f = \ln x \qquad g' = 1$

$f' = \frac{1}{x} \qquad g = x$

$$\int f g' \, dx = fg - \int g f' \, dx$$

$$\int \ln x \, dx = x \ln x - \int \frac{1}{x} \cdot x \, dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

Example: $\int t^2 e^t \, dt$

let $f = t^2$ $\qquad g' = e^t$

$f' = 2t$ $\qquad g = e^t$

$$\int t^2 e^t dt = t^2 e^t - \int 2t e^t \, dt$$

$$= t^2 e^t - 2 \int t e^t \, dt$$

$f = t \qquad g' = e^t$

$f' = 1 \qquad g = e^t$

$$= t^2 e^t - 2 \left[t e^t - \int 1 \cdot e^t \, dt \right].$$

$$= t^2 e^t - 2 \left[t e^t - e^t + C \right]$$

$$= t^2 e^t - 2t e^t + 2e^t + C.$$

Example: $\int e^x \sin x \, dx$.

let $f = e^x$ $\qquad g' = \sin x$

$f' = e^x$ $\qquad g = -\cos x.$

$$\int e^x \sin x\,dx = -e^x\cos x + \int \cos x \cdot e^x\,dx \longrightarrow \text{Let } f = e^x$$

$$= -e^x \cos x + \left[e^x \sin x - \int e^x \sin x\,dx\right] \qquad g' = \cos x$$

$$= -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx. \qquad f' = e^x$$

$$g = \sin x.$$

$$2\int e^x \sin x\,dx = -e^x\cos x + e^x \sin x$$

$$\Rightarrow \int e^x \sin x\,dx = \frac{1}{2} e^x(\cos x + \sin x) + C$$

IBP for Definite Integrals.

$$\int_a^b f(x)\,g'(x)\,dx = f(x)\,g(x)\Big]_a^b - \int_a^b g'(x)\,f(x)\,dx.$$

Example: $\int_0^1 \tan^{-1} x\,dx$

Let $f = \tan^{-1} x \qquad g' = 1.$

$f' = \frac{1}{1+x^2} \qquad g = x$

$$\int_0^1 \tan^{-1}x\,dx = x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx.$$

$$= \left[1\cdot\tan^{-1}(1) - 0\cdot\tan^{-1}(0)\right] - \int_0^1 \frac{x}{1+x^2}\,dx.$$

$$= \frac{\pi}{4} - 0 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$\int_0^1 \frac{x}{1+x^2}\,dx$$

let $u = 1+x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx.$

$u(0) = 1$

$u(1) = 2.$

$$\Rightarrow \int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u}\,du$$

$$\Rightarrow \int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \left(\frac{1}{2}\int_1^2 \frac{1}{u}\,du\right)$$

$$= \frac{\pi}{4} - \frac{1}{2}\Big[\ln u\Big]_1^2$$

$$= \frac{\pi}{4} - \frac{1}{2}\left[\ln 2 - \ln 1\right].$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \ln\sqrt{2}$$

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'g - gf'}{g^2}$$

$$\frac{d}{dx}\left[f\cdot\frac{1}{g}\right]$$

Integration Techniques.

- *substitution rule
- Int. by parts.
- Trig. int.
- Int. by partial fraction.
- Improper int.

Homework 10.

8. $\int_3^5 [f(x)+g(x)]dx = \int_3^5 [2g(x)+7]\,dx$

$= \int_3^5 2g(x)\,dx + \int_3^5 7\,dx$

$= 2\int_3^5 g(x)\,dx + 7[5-3] = 2\int_3^5 g(x)\,dx + 14.$

$\boxed{B}$.

9. $\int_1^e \frac{x^2-1}{x}\,dx = \int_1^e x - \frac{1}{x}\,dx$

$= \left[\frac{1}{2}x^2 - \ln x\right]_1^e = \left[\left(\frac{1}{2}e^2-1\right) - \left(\frac{1}{2}-0\right)\right]$

$= \frac{1}{2}e^2 - \frac{3}{2}$

$\boxed{E}$.

10. $\boxed{\text{II} \checkmark}$ $\qquad$ III ✗ $\qquad$ I ✗

$f(x) = 1 \qquad g(x) = x^2 - 9.$

$\boxed{B}$.

11. $\int (x^2+1)^2\,dx = \int x^4 + 2x^2 + 1\,dx$

$= \frac{x^5}{5} + \frac{2}{3}x^3 + x + C.$

$\boxed{E}$.

12. $\int \frac{3x^2}{\sqrt{x^3+1}}\,dx$

Let $u = x^3+1$

$du = 3x^2\,dx$

$\int \frac{1}{\sqrt{u}}\,du = \int u^{-\frac{1}{2}}\,du$

$= \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2u^{\frac{1}{2}} + C$

$= 2\sqrt{x^3+1} + C$

$\boxed{A}$

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7. $\int_0^9 f(x)\,dx = 4.$ $\int_0^3 x f(x^2)\,dx.$

Let $u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$u(0) = 0^2 = 0$

$u(3) = 3^2 = 9.$

$\Rightarrow \int_0^3 x f(x^2)\,dx = \frac{1}{2}\int_0^9 f(u)\,du.$

$\frac{1}{2}\int_0^9 f(x)\,dx = \frac{1}{2}(4) = 2.$

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6. e) $\int_0^{\frac{1}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx.$

Let $u = \sin^{-1}x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$

$$\int_0^{\frac{\pi}{6}} u\,du = \frac{1}{2}u^2\Big]_0^{\frac{\pi}{6}}$$

$u(0) = 0$

$u(\frac{1}{2}) = \frac{\pi}{6}$

$$= \frac{1}{2}\left[\frac{\pi}{6}\right]^2 = \frac{\pi^2}{72}$$

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2. g) $\displaystyle\int_{-1}^{2} (x - 2|x|)\,dx.$

$$\int_{-b}^{a} |x|\,dx = \int_0^a x\,dx + \int_{-b}^{0} \overset{-(-x)}{(-x)}\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

$$\int_0^2 x - 2x\,dx + \int_{-1}^{0} (x - 2(-x))\,dx$$

$$= \int_0^2 (-x)\,dx + \int_{-1}^{0} (3x)\,dx.$$

$$= \left[-\frac{1}{2}x^2\right]_0^2 + \left[\frac{3}{2}x^2\right]_{-1}^{0} = -2 - \frac{3}{2} = -\frac{7}{2}$$

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3. a) We want displacement.

$$\int v(t) = \int t^2 - 2t - 8$$

$$\Rightarrow \int_1^6 v(t)\,dt = \int_1^6 t^2 - 2t - 8\,dt.$$

$$= \left[\frac{t^3}{3} - t^2 - 8t \right]_1^6 = -\frac{10}{3} \text{ meters.}$$

b) $v(t) = t^2 - 2t - 8 = (t-4)(t+2) = 0.$

$$\downarrow$$

$t = 4$

$[1, 4]$
$[4, 6]$.

$$\int_1^6 |v(t)|\,dt = \int_1^4 -v(t)\,dt + \int_4^6 v(t)\,dt.$$

$$= -\int_1^4 t^2 - 2t - 8\,dt + \int_4^6 t^2 - 2t - 8\,dt.$$

$$= -\left[\frac{1}{3}t^3 - t^2 - 8t \right]_1^4 + \left[\frac{1}{3}t^3 - t^2 - 8t \right]_4^6$$

$$= \frac{54}{3} + \frac{44}{3} = \frac{98}{3}.$$