# AP Calculus Lesson Sixteen Notes

**Chapter Seven - Differential Equations**

**7.4 Solving First-order Differential Equations**
**7.5 Exponential Growth and Decay**

## 7.5 Solving First-Order Differential Equations Analytically

A differential equation has variables separable if it is of the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ or $g(y)dy - f(x)dx = 0$. The general solution is:

$$\int g(y)dy - \int f(x)dx = C$$

***Example 7.5-1***

Solve the differential equation $\frac{dy}{dx} = -\frac{x}{y}$, given the initial condition $y(0) = 2$.

***Solution***

We rewrite the given differential equation as $ydy = xdx$, then integrate both sides

$$\int ydy = -\int xdx$$

or

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

Since the initial condition $y(0) = 2$, so $\frac{1}{2}(2)^2 = -\frac{1}{2}(0)^2 + C$, that gives $C$ = 2.

Therefore, we have the solution as follows

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + 2 \quad \text{or} \quad y^2 = -x^2 + 4$$

We can express the solution in an explicit form $y = \sqrt{4 - x^2}$ since $y(0) = 2 > 0$.

***Example 7.5-2***

If $\frac{dS}{dt} = \sqrt{St}$ and $t = 0$ where $S = 1$, find $S$ when $t = 9$.

***Solution***

The original equation can be written as $\frac{dS}{\sqrt{S}} = \sqrt{t}dt$, then by taking integration both sides, we have

$$2S^{1/2} = \frac{2}{3}t^{3/2} + C$$

Substituting the given initial condition of $t = 0$ and $S = 1$, we have

$$2(1)^{1/2} = \frac{2}{3}(0)^{3/2} + C$$

So $C = 2$, and $S^{1/2} = \frac{1}{3}t^{3/2} + 1$

When $t = 9$, $S^{1/2} = \frac{1}{3}(9)^{3/2} + 1 = 10$ or $S = 100$.

## 7.6 Exponential Growth and Decay

### Case I : Exponential Growth

An interesting special differential equation with wide applications is defined as " A positive quantity $y$ increases (or decreases) at a rate at any time $t$ is proportional to the amount present". It follows that the quantity $y$ satisfies the differential equation

$$\frac{dy}{dt} = ky \qquad (7.6.1)$$

Where $k > 0$ if $y$ increases and $k < 0$ if $y$ decreases.

From (7.6.1) it follows that

$$\frac{dy}{y} = kdt$$

$$\int \frac{1}{y} dy = \int k dt$$

$$\ln y = kt + C$$

$$y = e^{kt+C} = Ce^{kt}$$

If $y(0) = y_0$, then $C = y_0$, so $y = y_0 e^{kt}$

The length of time required for a quantity that is decaying exponentially to be reduced by half is called its half-life.

***Example 7.6-1***
The population of a country is growing at a rate proportional to its population. If the growth rate per year is 4% of the current population. how long will it take for the population to double?

***Solution***

Since $\frac{dP}{dt} = 0.04P$, so $k = 0.04$

$P = P_0 e^{0.04t}$, Let $P = 2P_0$, then we have

$2P_0 = P_0 e^{0.04t}$, by taking ln both sides, the solution is

$t = \frac{\ln 2}{0.04} = 17.33$ years

Therefore, it will take about 17 years and 4 months for the population to double.

***Example 7.6-2***
Radium-226 decays at a rate proportional to the quantity present. Its half-life is 1612 years. How long will it take for one quarter of a given quantity of radium-226 to decay?

***Solution***

Let $Q(t)$ be the amount present at time $t$, so

$$Q(t) = Q_0 e^{kt}$$

then by the half-time is 1612, we have

$$\frac{1}{2}Q_0 = Q_0 e^{k(1612)}$$

by taking ln both sides,

$$k = \frac{\ln(1/2)}{1612} \approx -0.00043$$

So

$$Q(t) = Q_0 e^{-0.00043t}$$

Let $Q(t) = 0.75Q_0$, then solve for $t$,

$$0.75Q_0 = Q_0 e^{-0.00043t}$$

$$t = \frac{\ln 0.75}{-0.00043} \approx 669 \text{ years}$$

So it will take about 669 years for one quarter of a given quantity of radium-226 to decay.

**Case II. Restricted Growth.**

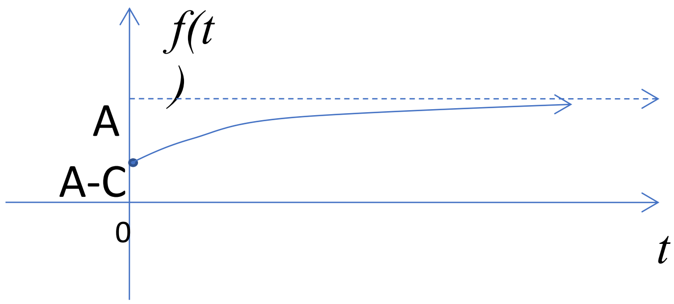
The rate of change of a quantity $y = f(t)$ may be proportional not to the amount present, but to a difference between that amount and a fixed constant. Two situations are to be distinguished:

(a) $f'(t) = k[A - f(t)]$ or (b) $f'(t) = -k[f(t) - A]$

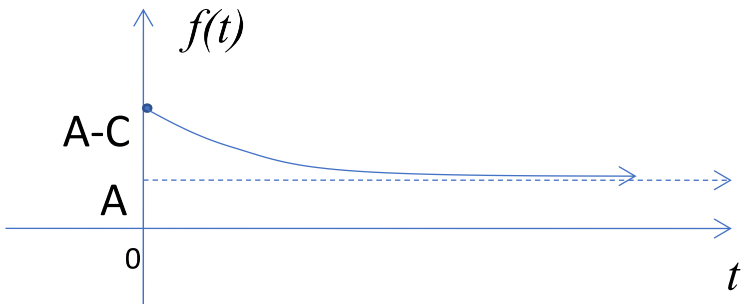
where $f(t)$ is the amount at time $t$, $k$ and $A$ are both positive.

In Model (a), $f(t) = A - Ce^{-kt}$ is increasing for some C > 0.
In Model (b), $f(t) = A + Ce^{-kt}$ is decreasing for some C > 0.

Graphically,




Model (a), $f(t) = A - Ce^{-kt}$ for some C > 0.




Model (b), $f(t) = A + Ce^{-kt}$ for some C > 0.

For Model (a), $\dfrac{dy}{dt} = k(A - y)$, or $\dfrac{dy}{A - y} = kdt$, thus $-\ln(A - y) = kt + C$, that implies $y = A - Ce^{-kt}$

***Example 7.6-3***

According to Newton's Law of cooling, a hot object cools at a rate proportional to the difference between its own temperature and that of its environment. If a roast at room temperature 68˚F is put into a 20˚F freezer, and if after 2 hr, the temperature of roast is 40˚F.

(a) What is its temperature after 5 hr?

(b) How long will it take for the temperature of the roast to fall to 21˚F?

***Solution***

(a) $\dfrac{dR}{dt} = -k[R(t) - 20]$ or $\dfrac{dR}{k} = [20 - R(t)]dt$

Then, we can have the solution,

$$R(t) = 20 + Ce^{-kt}$$

Using the initial condition $R(0) = 68$, we have $C = 48$. Using $R(2) = 40$, we can have,

$$e^{-k} \approx 0.65$$

Therefore,

$$R(t) = 20 + 48(0.65)^t$$

and

$$R(5) = 20 + 48(0.65)^5 \approx 26°F$$

(b) Let $R(t) = 21$, and solve for $t$:

$$21 = 20 + 48(0.65)^t$$

By solving this exponential equation, we have $t \approx 9$ hours

**Case III: Logistic Growth - Another Restricted Growth**

The rate of change of a quantity may be proportional both to the amount of the quantity and to the difference between a fixed constant $A$ and its amount. If $y = f(t)$ is the amount, then

$$\frac{dy}{dt} = ky(A - y) \qquad (7.6.2)$$

where $k$ and $A$ are both positive constants

Equation (7.6.2) is called the logistic differential equation, that is used to model logistic growth.

The solution of the differential equation (7.6.2) is

$$y = \frac{A}{1 + Ce^{-Akt}} \qquad (7.6.3)$$

for some positive constant $C$. In most applications, $C > 1$, so $\frac{A}{1+C} < \frac{A}{2}$.

When $t \to \infty$, we have,

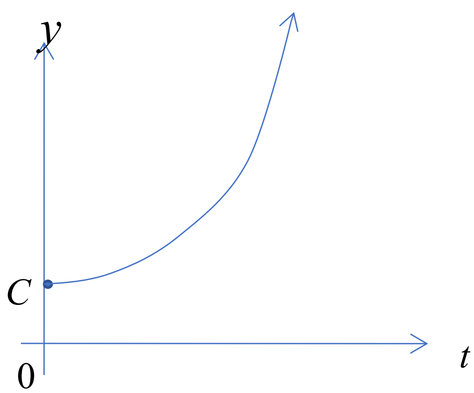
(1) $Ce^{-Akt} \to 0$

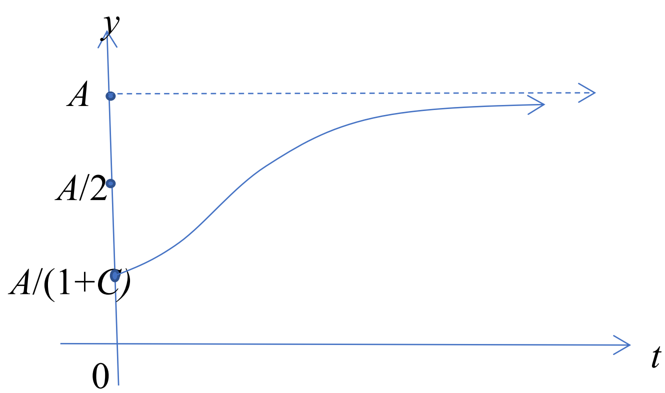
(2) $1 + Ce^{-Akt} \to 1$

(3) $y = \dfrac{A}{1 + Ce^{-Akt}} \to A$

Thus, $A$ is an upper limit of $f(t)$ in this growth model. When applied to population, $A$ is called " the carrying capacity " or "the maximum sustainable population".

**Unrestricted versus Restricted Growth**




Case I: $y = Ce^{kt}$

Case III: $y = \dfrac{A}{1 + Ce^{-Akt}}$

It is clear, for example, that human populations cannot continue endlessly to grow exponentially. Not only is Earth's land area fixed, but also there are limited supplies of food, energy and other natural resources. The growth function in Case III allows for such factors.

The two graphs are quite similar close to 0. This similarity implies that logistic growth is exponential at the start -- a reasonable conclusion, since populations are small at the outset.

The $S$ - shaped cure in Case III is often called a logistic curve, It shows that the rate of growth $y'$:

(1) increases slowly for a while; i.e. $y'' > 0$.

(2) attains a maximum when $y = \dfrac{A}{2}$, at the point of inflection.

(3) then decreases ($y'' < 0$), approaching 0 as $y$ approaches its upper limit $A$.

***Example 7.6-4***

Suppose a flu-like virus is spreading through a population of 50000 at a rate proportional both to the number of people already infected and to the number still uninfected. If 100 people were infected yesterday and 130 are infected today.

(a) Write a expression for the number of people $N(t)$ infected after $t$ days.
(b) Determine how many will be infected a week from today.

***Solution***

(a) We are told that $N'(t) = kN(50000 - N)$, and $N(0)$ = 100, $N(1)$ = 130, then the differential equation describing logistic growth leads to

$$N(t) = \frac{50000}{1 + Ce^{-50000kt}}$$

From $N(0)$ = 100, we get $100 = \frac{50000}{1 + C}$, so $C$ = 499.

From $N(1)$ = 130, we get $130 = \frac{50000}{1 + 499e^{-50000k}}$, so $e^{-50000k} \approx 0.77$. Thus,

$$N(t) = \frac{50000}{1 + 499(0.77)^t}.$$

(b) We are going to find $N(8)$, since $t$ = 0 represents yesterday.

$$N(8) = \frac{50000}{1 + 499(0.77)^8} \approx 798$$

So there are about 798 people will be infected a week from today.