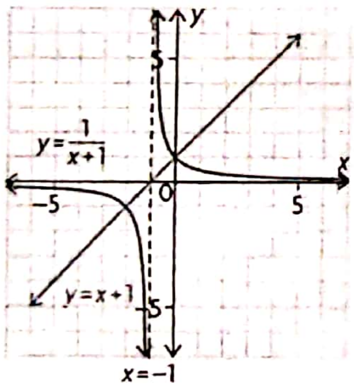
Lesson # 5

Graphs of Reciprocal Functions

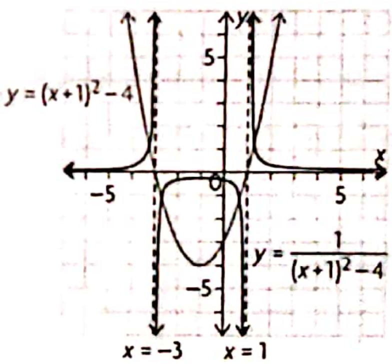
- All the y-coordinates of a reciprocal function are the reciprocals of the y-coordinates of the original function.
- The graph of a reciprocal function has a vertical asymptote at each zero of the original function.
- A reciprocal function will always have $y = 0$ as a horizontal asymptote if the original function is linear or quadratic.
- A reciprocal function has the same positive/negative intervals as the original function.
- Intervals of increase on the original function are intervals of decrease on the reciprocal function. Intervals of decrease on the original function are intervals of increase on the reciprocal function.
- If the range of the original function includes 1 and/or -1, the reciprocal function will intersect the original function at a point (or points) where the y-coordinate is 1 or -1.
- If the original function has a local minimum point, the reciprocal function will have a local maximum point at the same x-value (and vice versa).

A linear function and its reciprocal



Both functions are negative when $x \in (-\infty, -1)$ and positive when $x \in (-1, \infty)$. The original function is increasing when $x \in (-\infty, \infty)$. The reciprocal function is decreasing when $x \in (-\infty, -1)$ or $(-1, \infty)$.

A quadratic function and its reciprocal



Both functions are negative when $x \in (-3, 1)$ and positive when $x \in (-\infty, -3)$ or $(1, \infty)$. The original function is decreasing when $x \in (-\infty, -1)$ and increasing when $x \in (-1, \infty)$. The reciprocal function is increasing when $x \in (-\infty, -3)$ or $(-3, -1)$ and decreasing when $x \in (-1, 1)$ or $(1, \infty)$.

Example 1 State the equation of the reciprocal of each function, and determine the equations of the vertical asymptotes of the reciprocal.

a) $f(x) = x - 6$

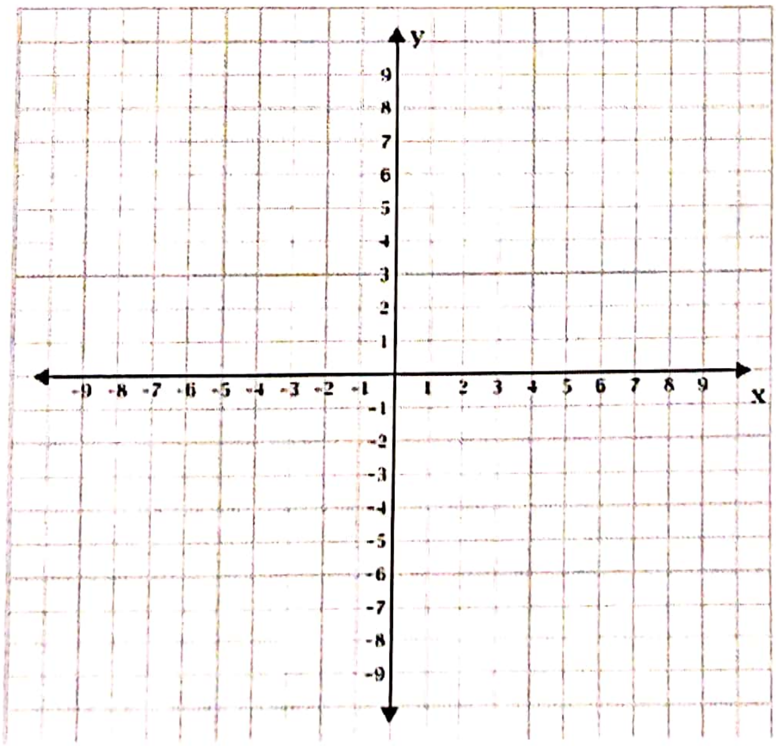
c) $f(x) = -x^2 + 25$

b) $f(x) = -3x + 1$

d) $f(x) = 2x^2 + 5x - 12$

Example 2 Given the function $f(x) = 7 - 2x$,

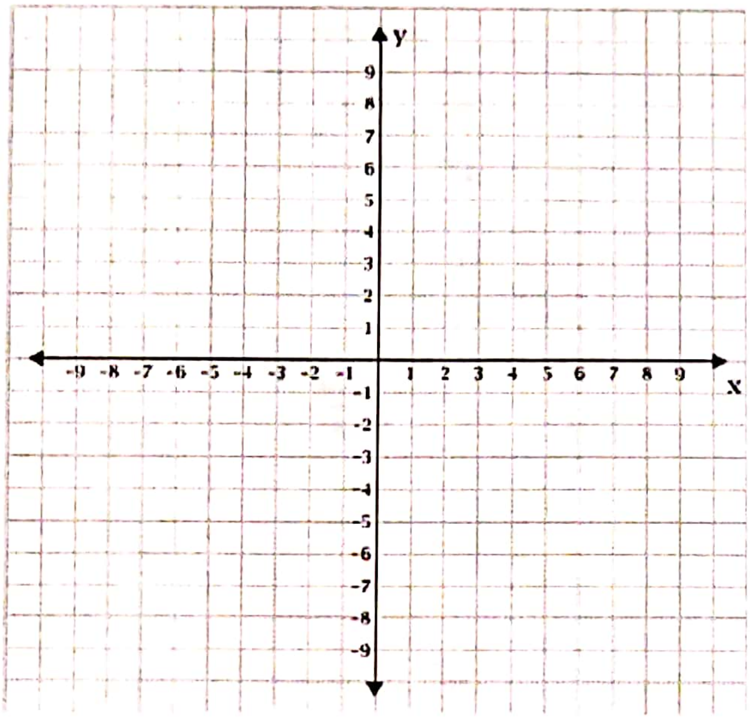
a) determine the domain and range, intercepts, positive/negative intervals, and increasing/decreasing intervals
b) use your answers from part (a) to sketch the graph of the reciprocal function



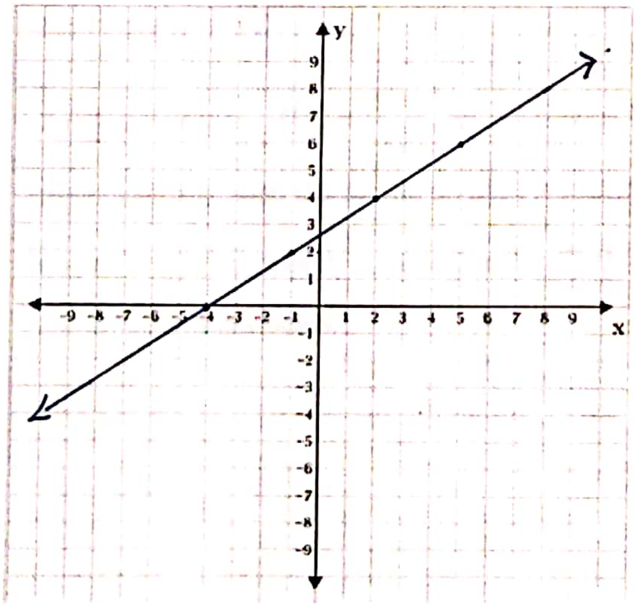
Example 3 Given the function $f(x) = -x^2 - 6x + 16$,

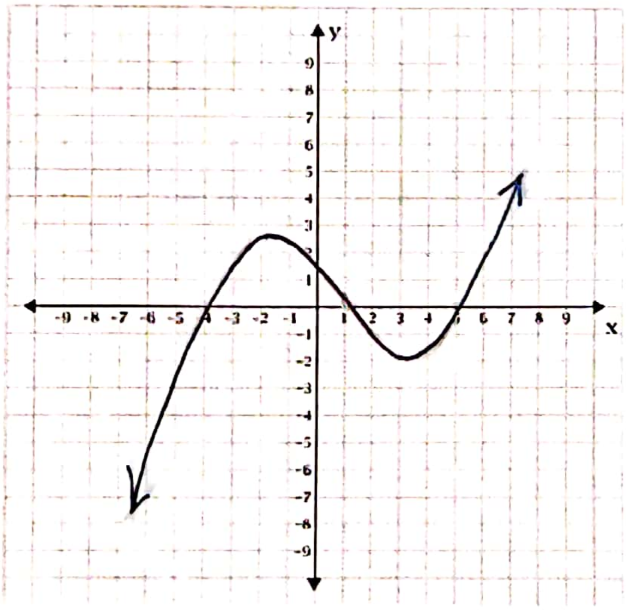
a) determine the domain and range, intercepts, positive/negative intervals, and increasing/decreasing intervals

b) use your answers from part (a) to sketch the graph of the reciprocal function



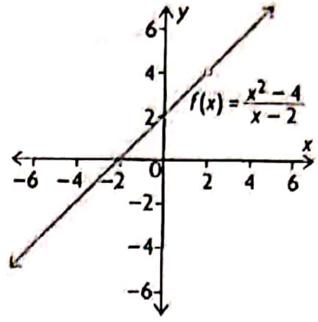
The graph is given. Sketch its reciprocal.



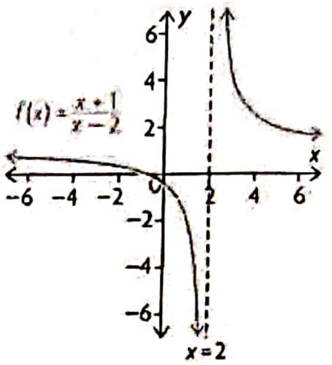


Exploring Quotients of Polynomial Functions

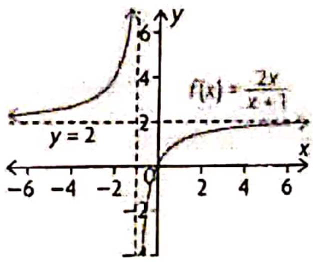
- A rational function, $f(x)=\dfrac{p(x)}{q(x)}$, has a hole at $x=a$ if $\dfrac{p(a)}{q(a)}=\dfrac{0}{0}$. This occurs when $p(x)$ and $q(x)$ contain a common factor of $(x-a)$. For example, $f(x)=\dfrac{x^2-4}{x-2}$ has the common factor of $(x-2)$ in the numerator and the denominator. This results in a hole in the graph of $f(x)$ at $x=2$.



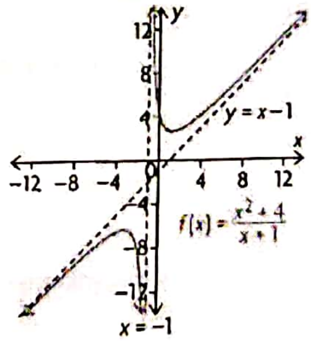
- A rational function, $f(x)=\dfrac{p(x)}{q(x)}$, has a vertical asymptote at $x=a$ if $\dfrac{p(a)}{q(a)}=\dfrac{p(a)}{0}$. For example, $f(x)=\dfrac{x+1}{x-2}$ has a vertical asymptote at $x=2$.



- A rational function, $f(x)=\dfrac{p(x)}{q(x)}$, has a horizontal asymptote only when the degree of $p(x)$ is less than or equal to the degree of $q(x)$. For example, $f(x)=\dfrac{2x}{x+1}$ has a horizontal asymptote at $y=2$.



- A rational function, $f(x)=\dfrac{p(x)}{q(x)}$, has an oblique (slant) asymptote only when the degree of $p(x)$ is greater than the degree of $q(x)$ by exactly 1. For example, $f(x)=\dfrac{x^2+4}{x+1}$ has an oblique asymptote.



Example 1 For each function, determine the equations of any vertical asymptotes, the locations of any holes, and the existence of any horizontal or oblique asymptotes.

a) $f(x)=\dfrac{x^2-5x+4}{x-5}$

c) $h(x)=\dfrac{3x}{2x-1}$

b) $p(x)=\dfrac{2x+5}{2x^2+x-10}$

d) $m(x)=\dfrac{x^2-64}{2x+16}$

Example 2 Write an equation for a rational function with the properties as given.

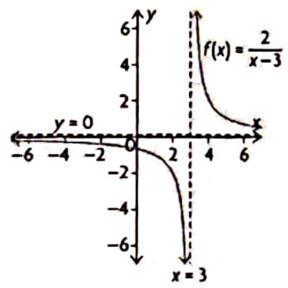
a) a vertical asymptote at $x=-5$ and a horizontal asymptote at $y=\dfrac{1}{2}$

b) a hole at $x=3$ and a vertical asymptote at $x=-1$

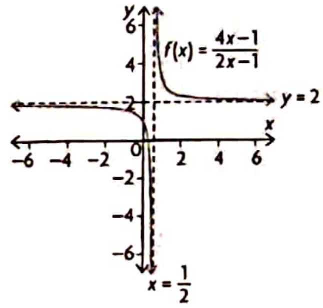
c) a vertical asymptote at $x=-\dfrac{5}{4}$ and an oblique asymptote

Graphs of Rational Functions of the Form $f(x)=\dfrac{ax+b}{cx+d}$

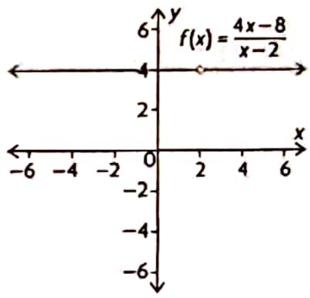
- Rational functions of the form $f(x)=\dfrac{b}{cx+d}$ have a vertical asymptote defined by $x=-\dfrac{d}{c}$ and a horizontal asymptote defined by $y=0$. For example, see the graph of $f(x)=\dfrac{2}{x-3}$.



- Most rational functions of the form $f(x)=\dfrac{ax+b}{cx+d}$ have a vertical asymptote defined by $x=-\dfrac{d}{c}$ and a horizontal asymptote defined by $y=\dfrac{a}{c}$. For example, see the graph of $f(x)=\dfrac{4x-1}{2x-1}$.

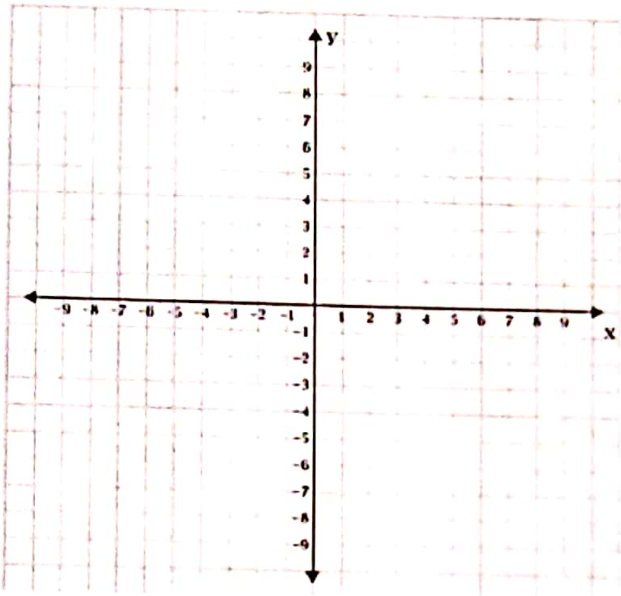


The exception occurs when the numerator and the denominator both contain a common linear factor. This results in a graph of a horizontal line that has a hole where the zero of the common factor occurs. As a result, the graph has no asymptotes. For example, see the graph of $f(x)=\dfrac{4x-8}{x-2}=\dfrac{4(x-2)}{x-2}$.

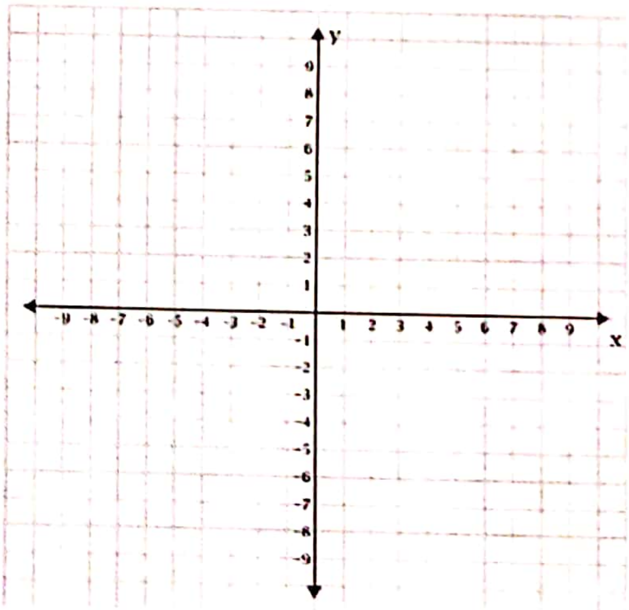


Example 1 For each function, determine the domain, intercepts, asymptotes, and positive/negative intervals. Use these characteristics to sketch the graph of the function. Then describe where the function is increasing or decreasing.

a) $f(x)=\dfrac{6}{x-2}$



b) $g(x)=\dfrac{4x-1}{3x+5}$



Sketch the following rational functions.

1. $y = \frac{x}{9-x^2}$

2. $y = \frac{x^3-27}{x^2-9}$

Sketch the following rational functions.

3. $y = \frac{x^3-3x^2-4x}{x^2+3x}$

4. $y = \frac{2x^3-x^2-2x+1}{x^2-3x+2}$