AP Calculus Lesson Eleven Notes

Chapter 5 Applications of the Definite Integral and Polar Coordinates

5.2 Volumes Using Cylindrical Shells and Volumes by Slicing
5.3 Work and Arc Length

5.2 Volumes Using Cylindrical Shells and Volumes by Slicing

Volumes Using Cylindrical Shells

There is another method for finding volumes of solids of revolution which, in certain cases, is simpler to apply than those discussed in Section 6.2. The method employs hollow circular cylinders, that is, thin cylindrical shells of the type illustrated in Figure 6.21.

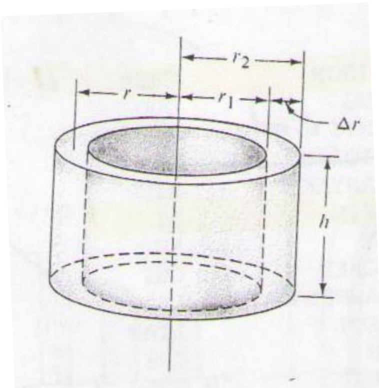

FIGURE 6.21

The volume of a shell having outer radius r_2, inner radius r_1, and altitude h is $\pi r_2^2 h - \pi r_1^2 h$. This expression may also be written

$$\pi(r_2^2 - r_1^2)h = \pi(r_2 + r_1)(r_2 - r_1)h$$
$$= 2\pi\left(\frac{r_2 + r_1}{2}\right)h\left(r_2 - r_1\right)$$

If we let $r = (r_2 + r_1)/2$ (the **average radius** of the shell) and $\Delta r = r_2 - r_1$ (The **thickness** of the shell), then the volume of the shell is given by $2\pi rh\Delta r$, that is,

(5.2.1) Volume of a shell = 2π **(average radius)(altitude)(thickness).**

Let the function f be continuous and $f(x) \geq 0$ for all x in $[a,b]$, where $0 \leq a < b$. Let R be the region bounded by the graph of f, the x-axis, and by the graphs of $x=a$ and $x=b$, as illustrated in (i) of Figure 6.22. the solid generated by revolving R about the y-axis is illustrated in (ii) of the figure. Note that if $a>0$, there is a hole through the solid. Let P be a partition of $[a,b]$and consider a rectangle with base corresponding to the interval $\left[x_{i-1}, x_i\right]$ and altitude $f(w_i)$, where w_i is the mid point of $\left[x_{i-1}, x_i\right]$. If this rectangle is revolved about the y-axis, then as illustrated in (iii) of Figure 6.22 there results a cylindrical shell with average radius w_i, altitude $f(w_i)$, and thickness $\Delta x_i = x_i - x_{i-1}$. By (5.2.1) we may express its volume as

$$2\pi w_i f(w_i)\Delta x_i$$

Doing this for each subinterval in the partition and adding gives us

$$\sum_i 2\pi w_i f(w_i)\Delta x_i$$

Geometrically, this sum represents the volume of a solid of the type illustrated in (iv) of Figure 6.22. Evidently, the smaller the norm $\|P\|$ of the partition, the better the sum approximates the volume *V* of the solid generated by *R*.
In deed, it appears that the volume of the solid illustrated in (ii) of the figure is the limit of the sum. This gives us the following,

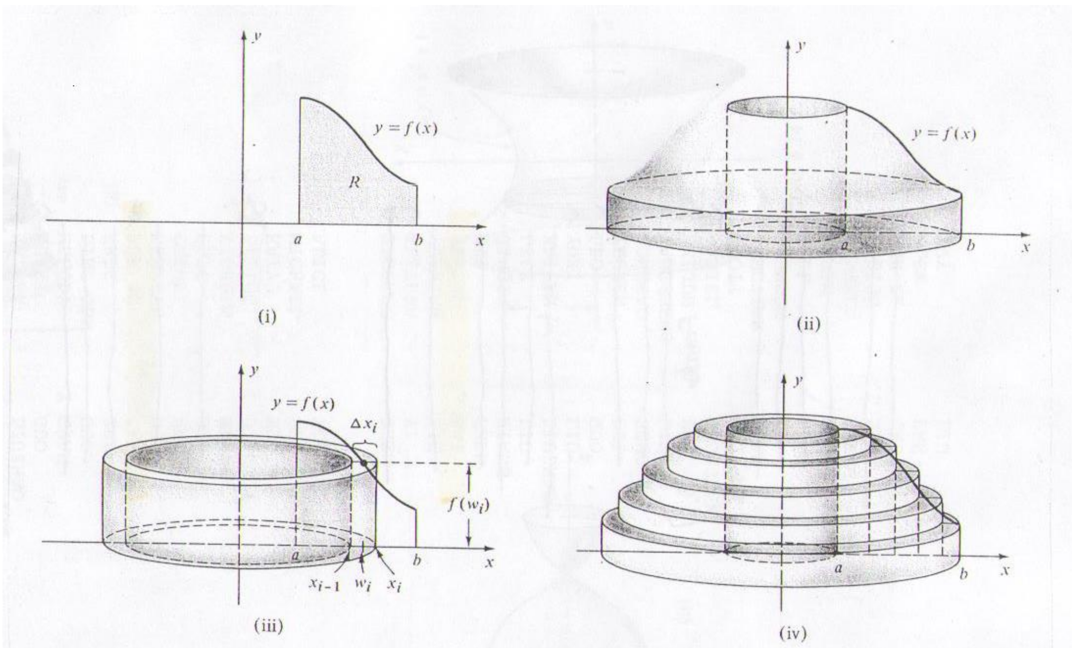

FIGURE 6.22

Definition (5.2.2)

Let *f* be continuous on [*a*,*b*]where $0 \le a < b$. The **volume** *V* of the solid of revolution generated by revolving the region bounded by the graphs of *f*, *x*=*a*, *x*=*b*, and the *x*-axis about the *y*-axis is

$$V = \lim_{\|P\|\to 0} \sum_i 2\pi w_i f(w_i)\Delta x_i = \int_a^b 2\pi x f(x)dx$$

The last equality follows from the definition of the definite integral. It can be proved that if the methods of Section 5.1 are also applicable, then both methods lead to the same answer.

Example 1

The region bounded by the graph of $y = 2x - x^2$ and the *x*-axis is revolved about the *y*-axis. Find the volume of the resulting solid.

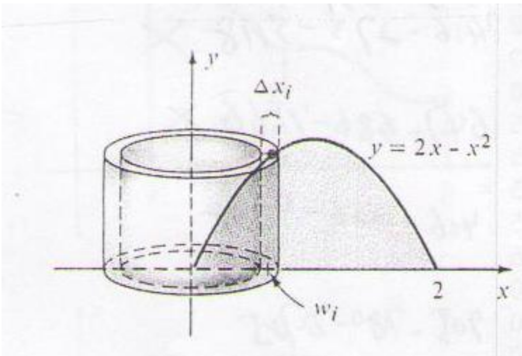

FIGURE 6.23

Solution

The region and a shell generated by a typical rectangle are sketched in Figure 6.23, where w_i is the midpoint of $[x_{i-1}, x_i]$. Since the average radius of the shell is w_i, the altitude is $2w_i - w_i^2$, and the thickness is Δx_i, it follows from (5.2.1) that the volume of the shell is

$$2\pi w_i (2w_i - w_i^2)\Delta x_i$$

Consequently, as in Definition (5.2.2), the volume *V* of the solid is given by

$$V = \lim_{\|P\|\to 0} \sum_i 2\pi w_i (2w_i - w_i^2)\Delta x_i$$

$$= \int_0^2 2\pi x(2x - x^2)dx = 2\pi\int_0^2 (2x^2 - x^3)dx$$

$$= 2\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \frac{8\pi}{3}$$

The volume *V* can also be found using washers; however, the calculations would be more involved since the given equation would have to be solved for *x* in terms of *y*.

Example 2

The region bounded by the graphs of $y = x^2$ and $y = x + 2$ is revolved about the line $x = 3$. Express the volume of the resulting solid as a definite integral.

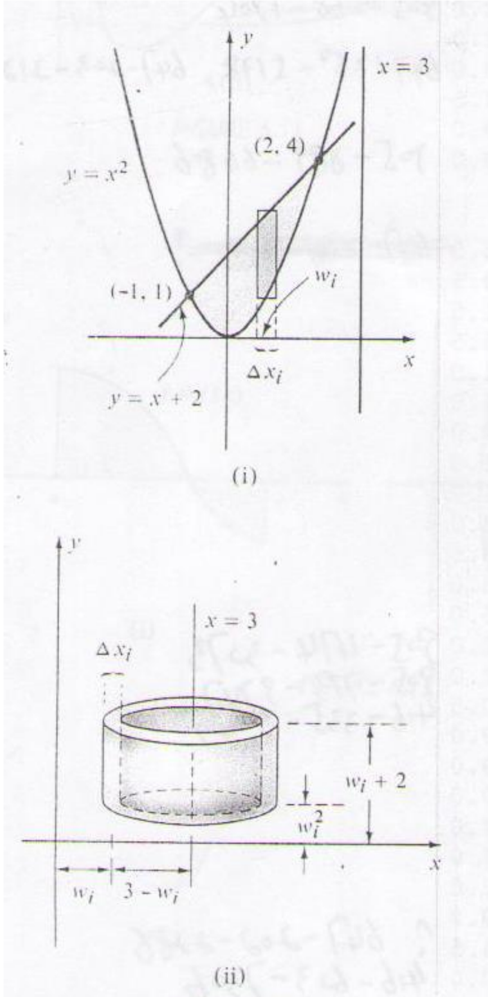

FIGURE 6.24

Solution

The region and a typical rectangle are shown in (i) of Figure 6.24, where w_i represents the midpoint of the ith subinterval $\left[x_{i-1}, x_i\right]$. The cylindrical shell generated by the rectangle is illustrated in (ii) of Figure 6.24. This shell has the following dimensions:

Altitude= $(w_i + 2) - w_i^2$

Average radius= $3 - w_i$

Thickness= Δx_i

Hence, by (5.2.1) its volume is

$$2\pi(3 - w_i)\left[\left(w_i + 2\right) - w_i^2\right]\Delta x_i$$

Taking the limit of a sum of such terms gives us

$$V = \int_{-1}^{2} 2\pi\left(3 - x\right)\left(x + 2 - x^2\right)dx$$

The definite integral in the preceding example could be evaluated by multiplying the factors in the integrand and then integrating each term. Since we have worked a sufficient number of problems of this type, it would not be very instructive to carry out all of these details. For convenience we shall refer to the process of expressing V in terms of an integral as *setting up the integral for V.*

Volumes by Slicing

If a plane intersects a solid, then the region common to the plane and the solid is called a **cross-section** of the solid. In Section 5.1 we encountered circular and washer-shaped cross sections. We shall now consider solids that have the property that for every x in a closed interval $[a,b]$on a coordinate line l, the plane perpendicular to l at the point with coordinate x intersects the solid in a cross section whose area is given by $X(x)$, where A is a continuous function on $[a,b]$. Figures 6.26 and 6.27 illustrate solids of the type we wish to discuss.

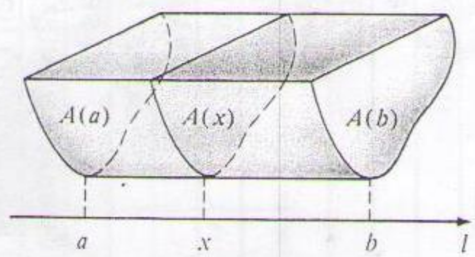

FIGURE 6.26

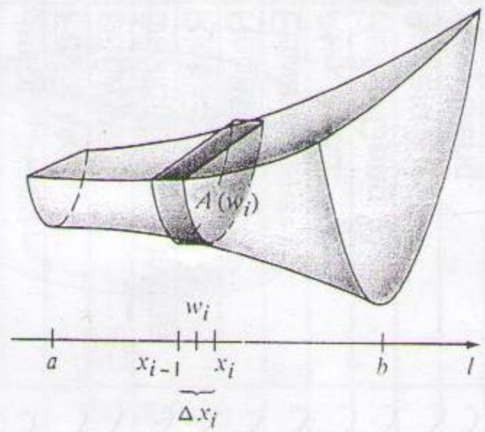

FIGURE 6.27

The solid is called a **cylinder** if, as illustrated in Figure 6.26, all cross sections are the same. If we are only interested in that part of the graph bounded by planes through the points with coordinates *a* and *b*, when the cross sections determined by these planes are called the **bases** of the cylinder and the distance between the bases is called the **altitude.** By definition, the volume of such a cylinder is the area of a base multiplied by the altitude. As special case, a right circular cylinder of base radius *r* and altitude *h* has volume $\pi r^2 h$.

The solid illustrated in Figure 6.27 is not a cylinder since cross sections by planes perpendicular to *l* are not all the same. To find the volume we begin with a partition *P* of [*a*,*b*]by choosing $a = x_0, x_1, x_2, ...x_n = b$. The planes perpendicular to *l* at the points with these coordinates slice the solid into smaller pieces. The *i*th such slice is shown in Figure 6.27. As usual, let $\Delta x_i = x_i - x_{i-1}$ and choose any number w_i in $\left[x_{i-1}, x_i\right]$. It appears that if Δx_i is small, then the volume of the slice can be approximated by the volume of the cylinder of base area $A(w_i)$ and altitude Δx_i, that is, by $A(w_i)\Delta x_i$. Consequently, the total volume of the solid is approximated by the Riemann sum $\sum_{i=1}^{n} A(w_i)\Delta x_i$. Since the approximation improves as $\|P\|$ gets smaller, we define the volume *V* of the solid as the limit of this sum. Our discussion may be summarized as follows, where it is assumed that the conditions on *A(x)* are those we have imposed previously.

(5.2.3) $$V = \lim_{\|P\| \to 0} \sum_i A(w_i)\Delta x_i = \int_a^b A(x)dx$$

Where $A(x)$ is the area of a cross section corresponding to the number x in $[a,b]$.

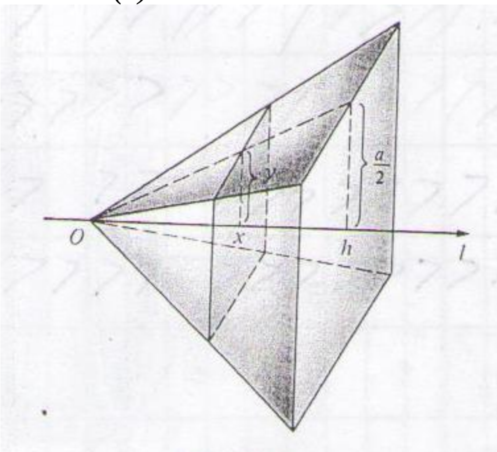

FIGURE 6.28

Example 1

Find the volume of a right pyramid that has altitude h and square base of side a.

Solution

If, as shown in Figure 6.28, we introduce a coordinate line l along the axis of the pyramid, with origin O at the vertex, then cross sections by planes perpendicular to l are squares. If $A(x)$ is the cross-sectional area determined by the plane that intersects the axis x units from O, then

$$A(x) = (2y)^2 = 4y^2$$

Where y is the distance indicated in the figure.

By similar triangles

$$\frac{y}{x} = \frac{a/2}{h}, \text{ or } \quad y = \frac{ax}{2h}$$

And hence $\quad A(x) = 4y^2 = \dfrac{4a^2x^2}{4h^2} = \dfrac{a^2}{h^2}x^2$

Applying (6.8)

$$V = \int_0^h \left(\frac{a^2}{h^2}\right) x^2 dx = \left(\frac{a^2}{h^2}\right)\left[\frac{x^3}{3}\right]_0^h = \frac{a^2h}{3}$$

Example 2

A solid has, as its base, the circular region in the xy-plane bounded by the graph of $x^2 + y^2 = a^2$, where $a>0$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is an equilateral triangle with one side in the base.

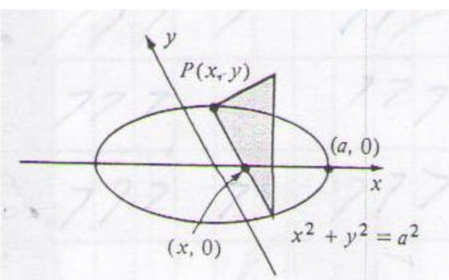

FIGURE 6.29

Solution

A typical cross section by a plane x units from the origin is illustrated in Figure 6.29.

If the point $P(x, y)$ is on the circle, then the length of a side of the triangle is $2y$ and the altitude is $\sqrt{3}y$. Hence the area $A(x)$ of the pictured triangle is

$$A(x) = \tfrac{1}{2}(2y)(\sqrt{3}y) = \sqrt{3}y^2 = \sqrt{3}(a^2 - x^2)$$

Applying (6.8) gives us

$$V = \int_{-a}^{a} \sqrt{3}\left(a^2 - x^2\right) dx = \sqrt{3}\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4\sqrt{3}a^3}{3}$$

5.3 Work and Arc Length

Work

Definition (5.3.1)

Let a constant force ***F*** is appied to an object, moving it a distance d in the direction of the force, then the work ***W*** done on the object is given by ***W*** = ***Fd***.

Let *P* denote the partition of the interval [*a*,*b*] determined by the numbers $a = x_0, x_1, x_2, ...x_n = b$ and let $\Delta x_i = x_i - x_{i-1}$. If w_i is a number in $\left[x_{i-1}, x_i\right]$, the force at w_i is $f(w_i)$. It seems evident that the smaller we choose Δx, the better $f(w_i)\Delta x_i$ approximates the work done in the interval $\left[x_{i-1}, x_i\right]$. If it is also assumed that work is additive, in the sense that the work *W* done as the object moves from *A* to *B* can be found by adding the work done over each subinterval, then

$$W \approx \sum_{i=1}^{n} f(w_i)\Delta x_i$$

Since we expect this approximation to improve as the norm $\|P\|$ of the partition becomes smaller, it is natural to define *W* as the limit of the preceding sum. This limit leads to a definite integral.

Definition (5.3.2)

Let the force at the point with coordinate *x* on a coordinate line *l* be *f(x)*, where *f* is continuous on [*a*,*b*]. The **work** *W* done in moving an object from the point with coordinate *a* to the point with coordinate *b* is

$$W = \lim_{\|P\| \to 0} \sum_i f(w_i)\Delta x_i = \int_a^b f(x)dx$$

The formula in Definition (5.3.2) can be used to find the work done in stretching or compressing a spring. To solve problems of this type it is necessary to use the following law from physics.

Hooke's Law (5.3.3)

The force *f(x)* required to stretch a spring *x* units beyond its natural length is

F(x)=kx

Where *k* is a constant called the **spring constant.**

The formula in (5.3.3) is also used to find the work done in compressing a spring *x* units from its natural length.

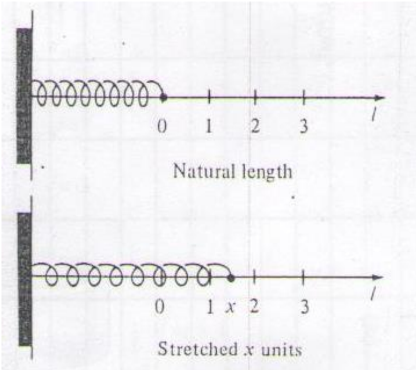

FIGURE 6.32

Example 3

A force of 9 lb is required to stretch a spring from its natural length of 6 in. to a length of 8 in. find the work done in stretching the spring (a) from its natural length to a length of 10 in.; (b) from a length of 7 in. to a length of 9 in.

Solution

(a) Let us introduce a coordinate line *l* as shown in Figure 6.32, where one end of the spring is attached to some point to the left of the origin and the end to be pulled is located at the origin. According to Hooke's Law (5.3.3), the force *f(x)* required to stretch a spring *x* units beyond its natural length is give by.

$$f(x)=kx$$

For some constant *k*. using the given data, $f(2)=9$. Substituting in $f(x)=kx$ we obtain $9=k*2$ and hence the spring constant is $k=\frac{9}{2}$. Consequently, for this spring, Hooke's Law has the form

$$f(x)=\frac{9}{2}x$$

By definition (6.10) the work done in stretching the spring 4 in. is given by

$$W=\int_0^4 \frac{9}{2}x\,dx=\frac{9}{4}x^2\Big]_0^4=36 in.-lb$$

(b) we use the same function *f* but change the interval to [1,3], obtaining

$$W=\int_1^3 \frac{9}{2}x\,dx=\frac{9}{4}x^2\Big]_1^3=\frac{81}{4}-\frac{9}{4}=18 in.-lb$$

Example 4

A right circular conical tank of altitude 20 ft and radius of base 5 ft has its vertex at ground level and axis vertical. If the tank is full of water, find the work done in pumping the water over the top of the tank.

Solution

We begin by introducing a coordinate system as shown in Figure 6.33. the cone intersects the *xy*-plane along the line of slope 4 through the origin. An equation for this line is $y=4x$.

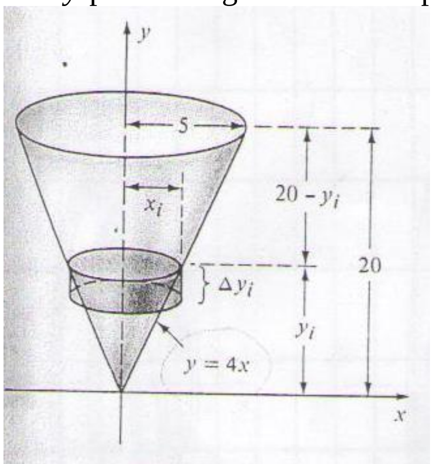

FIGURE 6.33

Let P denote the partition of the interval [0,20]determined by $y = y_0, y_1, y_2, ..., y_n = 20$, let $\Delta y_i = y_i - y_{i-1}$, and let x_i be the *x*-coordinate of the point on $y = 4x$ with *y*-coordinate y_i. If the cone is subdivided by means of planes perpendicular to the *y*-axis at each y_i, then we may think of the water as being sliced into *n* parts. As illustrated in Figure 6.33, the volume of the *i*th slice may be approximated by the volume $\pi x_i^2 \Delta y_i$ of a circular disc or, since $x_i = y_i / 4$, by $\pi(y_i / 4)^2 \Delta y_i$. This leads to the approximation

$$\text{Volume of } i\text{th slice} \approx \pi\left(\frac{y_i^2}{16}\right)\Delta y_i$$

Assuming that water weighs 62.5lb/ft^3, the weight of the disc in Figure 6.33 is approximately $62.5\pi(y_i^2/16)\Delta y_i$. By (5.3.1), the work done in lifting the disc to the top of the tank is the product of the distance 20-y_i and the weight, that is,

$$\text{Work done in lifting } i\text{th slice} \approx (20 - y_i)62.5\pi(y_i^2/16)\Delta y_i$$

The actual work W is obtained by taking the limit of this sum as the norm $\|P\|$ approaches zero. This give us

$$W = \int_0^{20} (20 - y)62.5\pi\left(\frac{y^2}{16}\right)dy$$

$$= \frac{62.5\pi}{16}\left[\frac{20y^3}{3} - \frac{y^4}{4}\right]_0^{20} \approx 163{,}625 \text{ ft} - \text{lb.}$$

Arc Length

Definition (5.3.4)

Let the function $f(x)$ be smooth on a closed interval $[a, b]$. The **arc length of the graph** of $f(x)$ from A(a, $f(a)$) to B(b, $f(b)$) is given by

$$L_a^b = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Example 1

If $f(x) = 3x^{2/3} - 10$, find the are length of the graph of $f(x)$ from the point A (8, 2) to B(27, 17).

Solution The arc length required is

$$L_8^{27} = \int_8^{27} \sqrt{1+(2/x^{1/3})^2}\,dx = \int_8^{27} \sqrt{x^{2/3}+4}\,/\,x^{1/3}dx = 3/2\int_8^{13} \sqrt{u}\,du = u^{3/2}\Big]_8^{13} \approx 24.2$$

where, $u = x^{2/3} + 4,\quad du = 2/(3x^{1/3})dx.$

Definition (5.3.5)

Let the function g be defined by $x = g(y)$, where g is smooth on a closed interval $[c, d]$. The **arc length of the graph** of g from $(g(c), c)$ to $(g(d), d)$ is given by

$$L_c^d = \int_c^d \sqrt{1+[g'(y)]^2}\,dy\ .$$

Example 2

Find the are length, to three decimal places, of the curve $(x-2)^2 = 4y^3$ from $y = 0$ to $y = 1$.

Solution Since $x - 2 = 2y^{3/2}$ and $dx/dy = 3y^{1/2}$. The arc length required is

$$L_0^1 = \int_0^1 \sqrt{1+9y}\,dy \approx 2.268$$