AP Calculus Class 23

Quick Sequence and Series Review.

Sequence : $\{a_n\} \to \{a_1, a_2, \dots a_n, \dots\}$

Series : $\sum_{n=1}^{\infty} a_n \to a_1 + a_2 + \cdots + a_n + \cdots$

$\hookrightarrow$ Partial sums:

$s_n = \sum_{i=1}^{n} a_n$

$s_1 = a_1 \qquad s_2 = a_1 + a_2 \quad \cdots \quad s_n = a_1 + \cdots + a_n$

Partial sum sequence $\{s_n\} \to \{s_1, s_2, \dots s_n, --\}$.

Homework 22.

5. $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\}$

$a_1 = \sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{2-1}{2}} = 2^{1-\frac{1}{2}}$

$a_2 = \sqrt{2\sqrt{2}} = (2 \cdot 2^{\frac{1}{2}})^{\frac{1}{2}} = (2^{\frac{3}{2}})^{\frac{1}{2}} = 2^{\frac{3}{4}} = 2^{\frac{2^2-1}{2^2}} = 2^{1-\frac{1}{2^2}}$

$a_3 = \sqrt{2\sqrt{2\sqrt{2}}} = (2 \cdot 2^{\frac{3}{4}})^{\frac{1}{2}} = (2^{\frac{7}{4}})^{\frac{1}{2}} = 2^{\frac{7}{8}} = 2^{\frac{2^3-1}{2^3}} = 2^{1-\frac{1}{2^3}}$

$a_4 = 2^{\frac{15}{16}}$

$\vdots$

$a_n = 2^{1-\frac{1}{2^n}}$

$\Rightarrow \lim_{n\to\infty} a_n = 2^{1-\frac{1}{2^\infty}} = 2^1 = 2.$

$\lim_{n\to\infty} a_n = L. \longrightarrow L = \sqrt{2\sqrt{2\sqrt{2}\ldots\ldots}}$

$L = \sqrt{2 \cdot L} \Rightarrow L^2 = 2 \cdot L$

$\Rightarrow L^2 - 2L = 0 \Rightarrow L(L-2) = 0.$

$\Rightarrow \cancel{L=0}$ or $\underline{\underline{L=2}}$

Example: Harmonic Series.

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \longrightarrow$ Divergent.

$s_1 = 1$

$s_2 = 1 + \frac{1}{2}$

$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2} = 2$

$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2} = 2.5$$

$$S_{2^n} > 1 + \frac{n}{2}$$

$\Rightarrow$ The harmonic series is divergent.

Thm: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then

$$\lim_{n\to\infty} a_n = 0.$$

NB! The converse of this statement is not necessarily true.

The Test for divergence.

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series is divergent.

Example: show that the series

$\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{5n^2}{n^2}+\frac{4}{n^2}} = \frac{1}{5} \neq 0.$$

Thm

① $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

② $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$

Example: Find the sum for $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

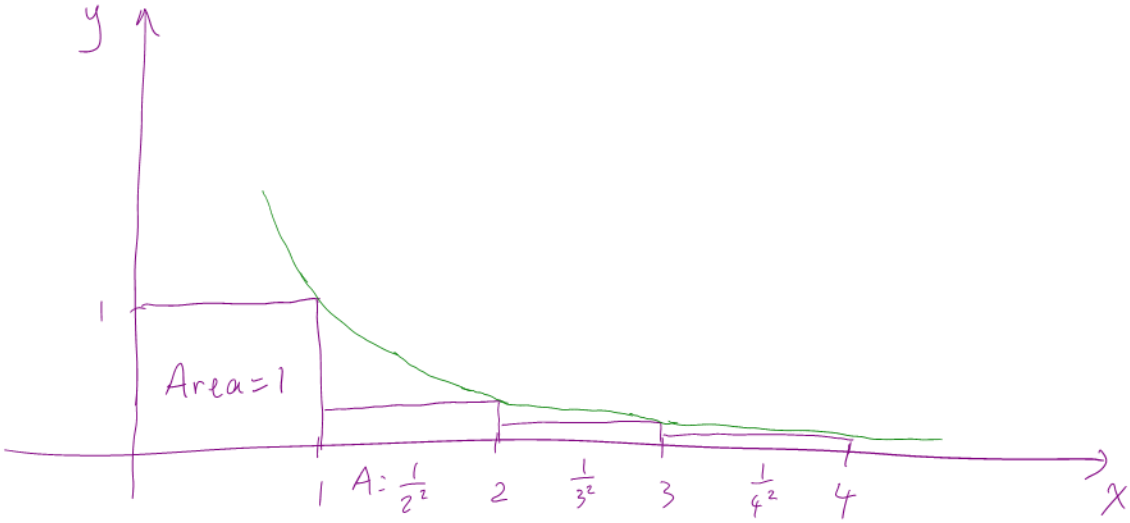
For $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$ $\quad a = \frac{1}{2} \quad r = \frac{1}{2}$

For $\sum_{n=1}^{\infty} \frac{3}{n(n+1)} = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 3 \cdot 1 = 3.$

$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 3 + 1 = 4.$

The Integral Test.

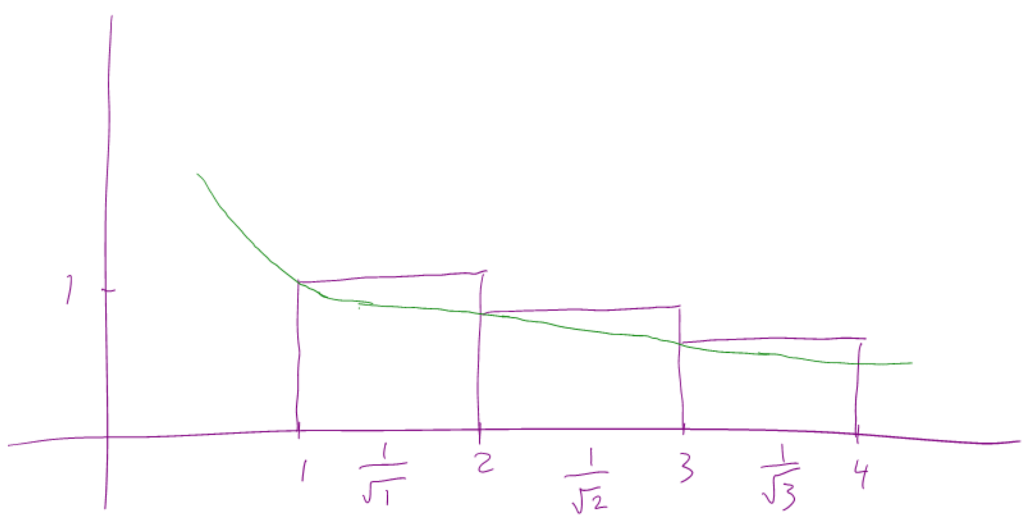
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$



$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$



$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx \longrightarrow \text{diverges}$$

The Integral Test.

Suppose f is a continuous, positive, decreasing fun^n on $[1, \infty)$, and let $a_n = f(n)$. Then the series $\sum a_n$ is convergent iff the improper integral $\int_1^\infty f(x)\,dx$ is convergent.

(i) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

Example: $\sum_{n=1}^\infty \frac{1}{n^2+1}$. Test for conv or div.

$$\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx.$$

$$= \lim_{t\to\infty} \tan^{-1} x \Big]_1^t = \lim_{t\to\infty} \left(\tan^{-1} t - \frac{\pi}{4}\right).$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

By the integral test, since the integral converges, the series $\sum_{n=1}^\infty \frac{1}{n^2+1}$ converges.

The p-series test.

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p>1$,

and divergent if $p \leq 1$.

$$\sum \frac{1}{n^3} = \frac{1}{1} + \frac{1}{8} + \frac{1}{27} + \cdots$$

$$\sum = \frac{1}{n^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{1}} + \frac{1}{\sqrt[3]{2}} + \cdots$$

Example: $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges?

$$f(x) = \frac{\ln x}{x}$$

$$f'(x) = \frac{(\frac{1}{x})x - \ln x}{x^2} = \frac{1-\ln x}{x^2}$$

$$f'(x) < 0 \quad \Rightarrow \quad \frac{1-\ln x}{x^2} < 0.$$

$$\ln x > 1 \quad \Rightarrow x > e.$$

since the fun^n is decreasing when $x>0$,
apply the integral test.

$$\int_1^{\infty} \frac{\ln x}{x}\, dx = \lim_{t \to \infty} \int_1^t \frac{\ln x}{x}\, dx.$$

$$= \lim_{t\to\infty} \frac{(\ln x)^2}{2} \Big]_1^t = \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty.$$

$\Rightarrow$ The series is divergent based on the integral test.

The Comparison Test.

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms,

① If $\sum b_n$ is convergent and $a_n \le b_n$ $\forall n$, then $\sum a_n$ is also convergent.

② If $\sum b_n$ is divergent and $a_n \ge b_n$ $\forall n$, then $\sum a_n$ is also divergent.

Example: Determine if $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

For $\frac{5}{2n^2} \longrightarrow \sum \frac{5}{2}\frac{1}{n^2} = \frac{5}{2}\sum \frac{1}{n^2}$

since $\frac{5}{2}\sum \frac{1}{n^2}$ converges by the p-series test, and $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} < \sum_{n=1}^{\infty} \frac{5}{2n^2}$,

then $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges by the comparison test.

The Limit Comparison Test.

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$

where c is a finite number and $c>0$, then either both series converge or both series diverge.

Example: Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.

$a_n = \frac{1}{2^n-1}$, let $b_n = \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n-1}$$

$$= \lim_{n\to\infty} \frac{\frac{2^n}{2^n}}{\frac{2^n}{2^n} - \frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} = 1 > 0$$

$\Rightarrow$ By the limit comp test, $\sum \frac{1}{2^n-1}$ is convergent.

Alternating Series.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \cdots = \sum (-1)^n \frac{n}{n+1}$$

Alternating Series Test.

If the alternating series

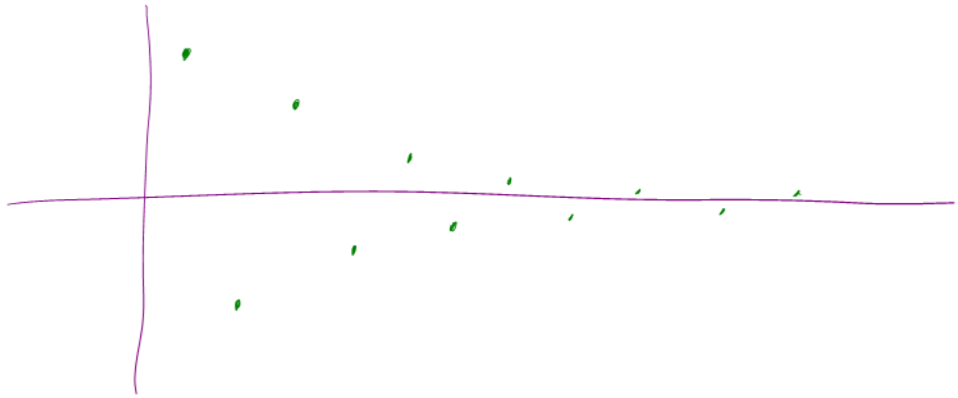
$$\sum_{n=1}^{\infty} (-1)^n b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \cdots \qquad b_n > 0$$

satisfies

1. $b_{n+1} \le b_n \quad \forall n.$

2. $\lim_{n\to\infty} b_n = 0$

then, the alternating series is convergent.



Example: Is the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ conv or div?

let $b_n = \frac{(-1)^n 3n}{4n-1}$

$b_1 = -\frac{3}{3} = -1$

$b_2 = \frac{6}{7} < 1$

$\Rightarrow b_{n+1} < b_n$

$$\lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{\frac{3n}{n}}{\frac{4n}{n}-\frac{1}{n}} = \lim_{n\to\infty} \frac{3}{4-\frac{1}{n}} = \frac{3}{4} \neq 0.$$

$\Rightarrow$ The alternating test is not satisfied.

$\Rightarrow$ The limit doesn't exist.

By the test for divergence, $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is divergent.