# AP Calculus Class 8

Guidelines for sketching a Curve.

① Domain: check the domain of f.

② Intercepts: x and y-intercepts.

Let $x=0$ and find $f(0)$ – y-intercept

Let $y=0$ and find $x$ – x-intercept.

③ Symmetry: Determine whether f is an even or an odd funⁿ.

E: $f(-x)=f(x)$ $\forall x$ in D.

O: $f(-x)=-f(x)$ $\forall x$ in D.

④ Asymptotes

Horizontal Asymptote: Let $x \to \pm\infty$

For $\lim_{x\to\pm\infty} f(x) = L$.

Find $y=L$.

Vertical Asymptote: Let $f(x) \to \pm\infty$.

For $\lim_{x\to a^{\pm}} f(x) = \pm\infty$

Find $x=a^{\pm}$

– check oblique asymptote.

5. Intervals of Increasing or Decreasing.

Use the I/D Test to find intervals for which $f$ is inc or dec.

6. Local Max or Min Values.

Find the critical numbers.

7. Concavity and Points of Inflection.

Find $f''(x)$ and the concavity of $f$. and the points of Inflections based on concavity intervals.

8. sketch the entire curve.

Example: sketch the curve $f(x) = \frac{2x^2}{x^2-1}$.

① Domain: Not the entire real line.

$\Rightarrow x^2 - 1 \neq 0 \quad \Rightarrow \quad x \neq \pm 1.$

$\Rightarrow \quad x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$


② Intercepts:

let $x = 0$ and $y = 0$.

$\Rightarrow$ there is a point on the origin of the graph.

③ Symmetry: $f(-x) = f(x) \quad \Rightarrow \quad$ Even fun$^n$.

④ Asymptotes:

H.A. let $x \to \pm\infty$.

$$\lim_{x\to\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\infty} \frac{\frac{2x^2}{x^2}}{\frac{x^2-1}{x^2}} = \lim_{x\to\infty} \frac{2}{1 - \frac{1}{x^2}}$$

$$= \frac{2}{1-0} = 2.$$

The same goes for $x \to -\infty$. $\to \lim_{x\to-\infty} \frac{2x^2}{x^2-1} = 2$.

$\Rightarrow$ The H.A. is 2.

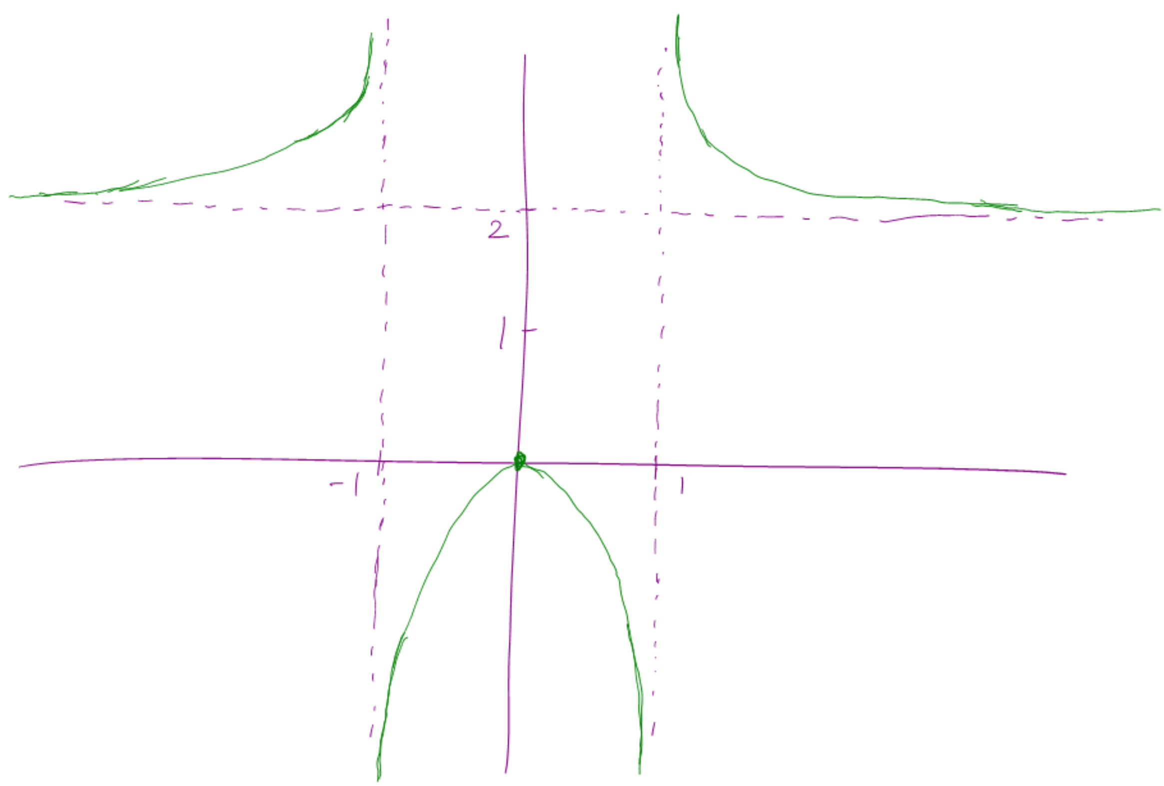
V.A. $f(x) = \dfrac{2x^2}{x^2-1}$

$\Rightarrow x^2 - 1 = 0 \quad \Rightarrow x = \pm 1$

Need to find the following limits.

$\lim\limits_{x \to 1^+} \dfrac{2x^2}{x^2-1} = \infty \qquad \lim\limits_{x \to -1^+} \dfrac{2x^2}{x^2-1} = -\infty$

$\lim\limits_{x \to 1^-} \dfrac{2x^2}{x^2-1} = -\infty \qquad \lim\limits_{x \to -1^-} \dfrac{2x^2}{x^2-1} = \infty$



⑤ Intervals. $f(x) = \dfrac{2x^2}{x^2-1}$

$$f'(x) = \frac{4x(x^2-1) - 2x^2(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

crit numbers → $x=0$, $x=\pm 1$.

| Intervals | $f'(x)$ | $f(x)$ |
|---|---|---|
| $x<-1$ | + | inc |
| $-1<x<0$ | + | inc |
| $0<x<1$ | − | dec |
| $x>1$ | − | dec |

⑥ Local extremum.

check crit point : $x=0$.

Since $f'$ changes from + to − at 0,

then $f(0)=0$ is a local maximum.

⑦ Concavity : $f'(x) = \dfrac{-4x}{(x^2-1)^2}$

$$f''(x) = \frac{-4(x^2-1)^2 + 4x\left(2(x^2-1)\right)2x}{(x^2-1)^4}$$

$$= \frac{12x^2+4}{(x^2-1)^3}$$

Then $f''(x) > 0, \Rightarrow x^2 - 1 > 0. \Rightarrow x^2 > 1$

$f''(x) < 0, \Rightarrow x^2 < 1.$

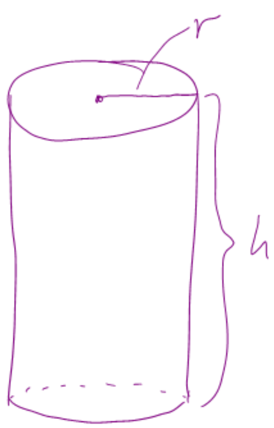
$\Rightarrow$ The curve is concave upward on intervals $(-\infty, -1)$ and $(1, \infty)$

concave downward on $(-1, 1)$.

This curve has no inflection points b/c 1 and $-1$ aren't in the domain of $f$.

# Optimization Problems.

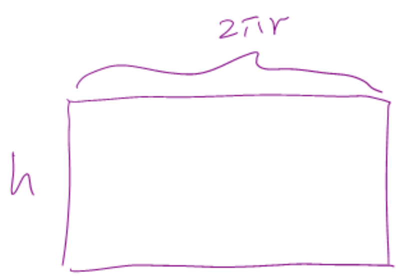
Example: A cylindrical can is to be made to hold 1 L of oil. Find the dimension that will minimize the cost of metal to manufacture the can.




r: radius of the circular base

h: height of the can.

A: area of the circles + the area of the rectangular side.




V: Volume of the can

$1 \text{ L} = 1000 \text{ cm}^3$.

$$A = 2(\pi r^2) + 2\pi r h$$

To eliminate h, use $\pi r^2 h = 1000 \text{ cm}^3$.

$$\Rightarrow h = \frac{1000}{\pi r^2}$$

$$\Rightarrow A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right).$$

$$= 2\pi r^2 + \frac{2000}{r}$$

The fun$^n$ we want to minimize is

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad r>0.$$

Find the crit numbers.

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

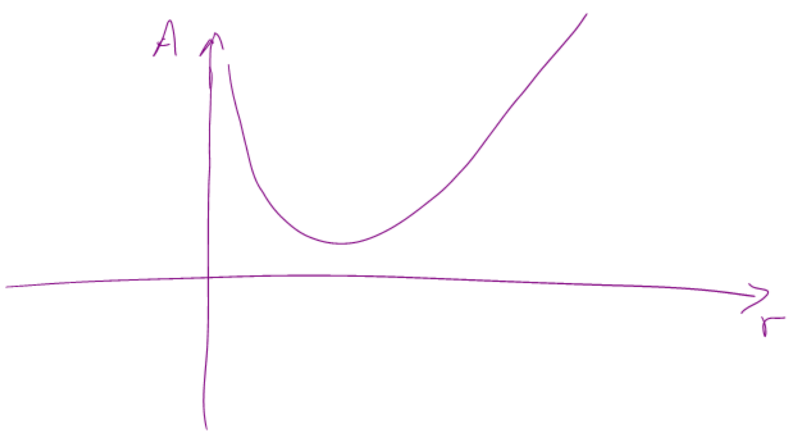
let $A'(r) = 0 \quad \Rightarrow \pi r^3 = 500$

$\Rightarrow$ the crit number $r = \sqrt[3]{\frac{500}{\pi}}$

The domain of $A$ is in $(0, \infty)$.

We see that $A(r) \to \infty$ as $r \to \infty$.

$A(r) \to \infty$ as $r \to 0^+$




There must be a minimum value, and this min value is also the absolute min value.

The value of $h$ would be

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi \left(\frac{500}{\pi}\right)^{\frac{2}{3}}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

$\Rightarrow$ To minimize the cost, the $r$ should be $\sqrt[3]{\frac{500}{\pi}}$ cm and $h$ should be $2r$.

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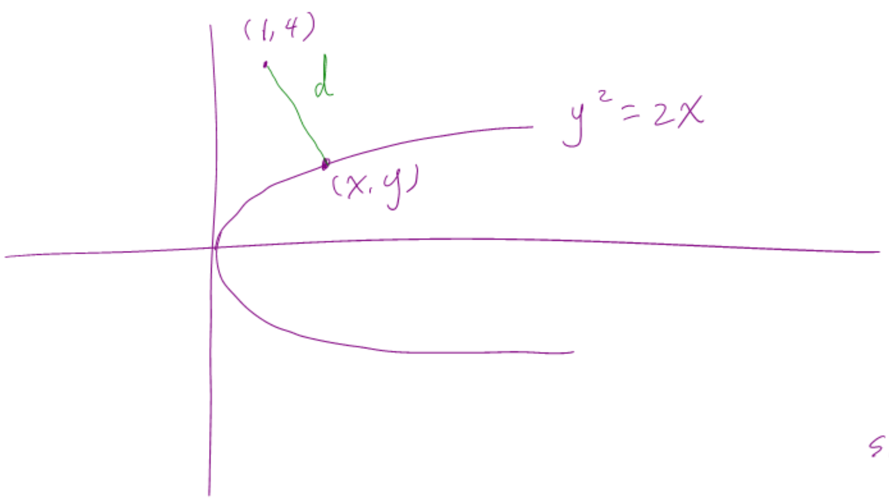
Example: Find the point on the parabola $y^2 = 2x$ that's closest to the point $(1, 4)$.




since $y^2 = 2x$

$\Rightarrow x = \frac{1}{2}y^2$

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

$$= \sqrt{\left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2}$$

$$\Rightarrow d^2 = \left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2$$

let $d^2 = f(y)$

$\Rightarrow \quad f(y) = (\frac{1}{2}y^2 - 1)^2 + (y-4)^2$

$$f'(y) = 2(\frac{1}{2}y^2 - 1)(y) + 2(y-4).$$

$$= y^3 - 2y + 2y - 8 \quad = y^3 - 8.$$

So $f'(y) = 0$ when $y = 2$.

We see that $f'(y) < 0$ when $y < 2$.

$f'(y) > 0$ when $y > 2$.

$f$ goes from dec to inc at $y = 2$.

$\Rightarrow$ The abs min is at $y = 2$.

$\Rightarrow \quad x = \frac{1}{2}(2)^2 = 2,$

The point on $y^2 = 2x$ that's closest to

$(1, 4)$ is $\underline{(2, 2)}$.

Homework 7.

10. $s = x + \frac{1}{x}$

$s' = 1 - \frac{1}{x^2}$     let $s' = 0$.

$\Rightarrow 1 - \frac{1}{x^2} = 0. \quad \Rightarrow x^2 = 1 \quad \Rightarrow x = \pm 1.$

$1 + \frac{1}{1} = 2. \qquad -1 - 1 = -2.$

$\Rightarrow x = 1.$

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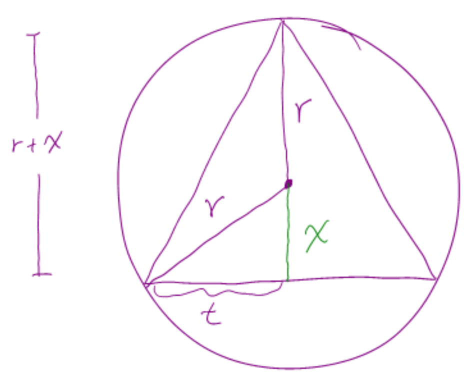
13.




A: The area of the isosceles triangle.

r: the radius of the circle.

r+x: The height of the triangle.

2t: The base of the triangle.

$A = \frac{1}{2}(\text{base})(\text{height})$

$= \frac{1}{2}(2t)(r+x). = t(r+x).$

$t = \sqrt{r^2 - x^2}$

$\Rightarrow A = \sqrt{r^2 - x^2}\,(r+x).$

Then $A'(x) = r\dfrac{-2x}{2\sqrt{r^2-x^2}} + \sqrt{r^2-x^2} + x\dfrac{-2x}{2\sqrt{r^2-x^2}}$

$$= -\frac{x^2+rx}{\sqrt{r^2-x^2}} + \sqrt{r^2-x^2}$$

$A'(x) = 0$

$\Rightarrow \sqrt{r^2-x^2} = \dfrac{x^2+rx}{\sqrt{r^2-x^2}}$

$\Rightarrow r^2 - x^2 = x^2 + rx \quad \Rightarrow 2x^2 + rx - r^2 = 0.$

$\Rightarrow (2x-r)(x+r) = 0$

$\Rightarrow x = \frac{1}{2}r \quad$ or $\quad x = -r.$

Now $A(r) = 0 = A(-r). \to$ reject.

$\Rightarrow$ Max value is at $x = \frac{1}{2}r.$

So the triangle has height

$r + \frac{1}{2}r = \underline{\frac{3}{2}r.}$ and base

$2\sqrt{r^2 - (\frac{1}{2}r)^2} = 2\sqrt{\frac{3}{4}r^2} \;\underline{= \sqrt{3}\,r.}$