AP Calculus Class 18

Differential Equations.

Mathematical Models : Usually in the form of a funⁿ.

Population growth. → Population models.

Example: human population. animal or insect pop.

bacteria pop.

t : time (independent variable).

P : the number of individuals in the model.

(dependent variable).

$\downarrow$

$P(t)$

$\frac{dP}{dt}$ = the rate of the population growth.

Let's assume that the population growth is proportional to the population size.

$\Rightarrow \frac{dP}{dt} = kP$. $\quad$ k: proportionality constant.

If we are asked to solve a differential equⁿ, we are asked to find the original funⁿ.

$$\frac{dP}{dt} = P \longrightarrow \frac{d}{dt} P = P.$$

$$\frac{d}{dx} e^x = e^x$$

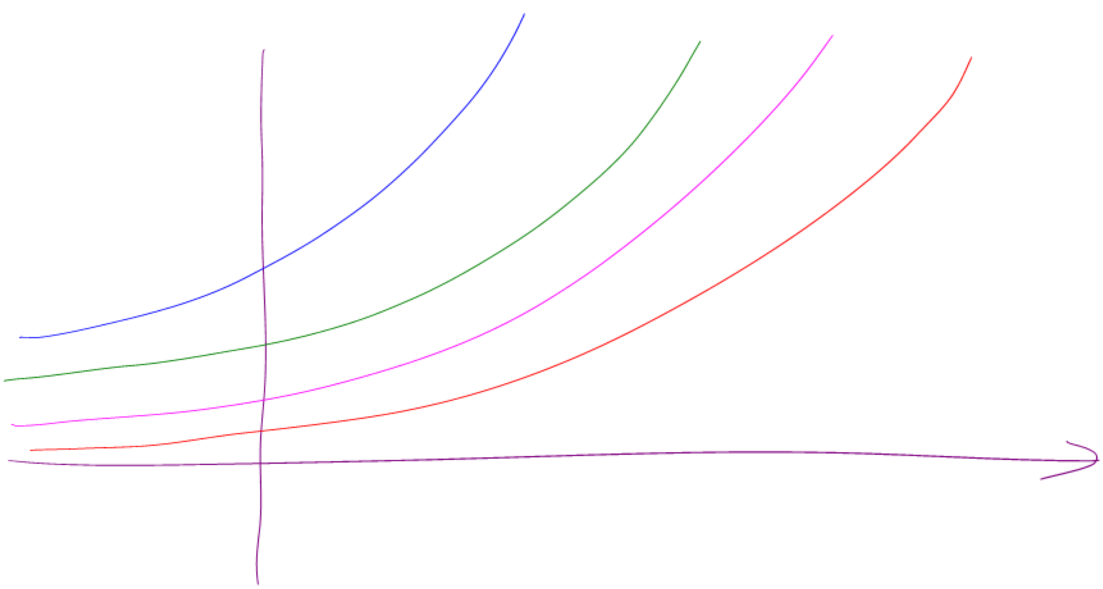
Guess $P(t) = e^t$

$$\frac{dP}{dt} = kP \Rightarrow P(t) = e^{kt}$$

$$\frac{d}{dt} e^{kt} = k e^{kt}$$

let's say if we let $P(t) = C e^{kt}$ $C > 0.$

$$\frac{dP}{dt} = \frac{d}{dt}(Ce^{kt}) = C k e^{kt} = k(Ce^{kt})$$



Def^n: A differential equ^n is an equ^n that contains an unknown fun^n and one or more of its derivatives

$$\frac{dP}{dt} = kP$$

The order of a differential equ^n is the order of the highest derivative in the equ^n.

First order differential equ^n.

- Ordinary Differential Equation (ODE)
  e.g. $f(x)$, $P(t)$, $f(\theta)$.

- Partial Differential Equation (PDE)
  e.g. $f(x, y)$, $f(x_1, x_2, x_3)$.

A fun^n f is called a solution of a differential equ^n if the equ^n is satisfied when $y = f(x)$ and its derivatives are substituded into the equ^n.

For a fun^n y, the derivatives are y' or $\frac{dy}{dx}$

If we have $\frac{dy}{dt}$, y', we can write $\dot{y}$

$\dot{y}$: time derivative. $\frac{dy}{dt}$

Example: show that every number of the family of funⁿ

$$y = \frac{1+ce^t}{1-ce^t}$$

is a solution of the differential equⁿ

$$y' = \frac{1}{2}(y^2-1).$$

$$y' = \frac{(1-ce^t)(ce^t) - (1+ce^t)(-ce^t)}{(1-ce^t)^2}$$

$$= \frac{ce^t - (ce^t)^2 + ce^t + (ce^t)^2}{(1-ce^t)}$$

$$= \frac{2ce^t}{(1-ce^t)^2}$$

$$\frac{1}{2}(y^2-1) = \frac{1}{2}\left[\left(\frac{1+ce^t}{1-ce^t}\right)^2 - 1\right].$$

$$= \frac{1}{2}\left[\left(\frac{1+ce^t}{1-ce^t}\right)^2 - \left(\frac{1-ce^t}{1-ce^t}\right)^2\right].$$

$$= \frac{1}{2}\left[\frac{(1+ce^t)^2 - (1-ce^t)^2}{(1-ce^t)^2}\right]$$

$$= \frac{1}{2}\left[\frac{4ce^t}{(1-ce^t)^2}\right]$$

$$= \frac{2ce^t}{(1-ce^t)^2}$$

$y(t_0) = y_0 \longrightarrow$ Initial Condition. I.C.

$y(0) = 3.$

The problem of finding a sol^n to a differential equ^n given an initial condition is called an initial-value problem (IVP)

Example: Find a sol^n of the differential equ^n $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$.

For $y = \dfrac{1+ce^t}{1-ce^t}$

$\Rightarrow y(0) = \dfrac{1+ce^0}{1-ce^0} = 2. \quad \Rightarrow \quad \dfrac{1+c}{1-c} = 2.$

$\Rightarrow 2 - 2c = 1 + c \qquad \Rightarrow 3c = 1 \qquad \Rightarrow c = \frac{1}{3}$

$\Rightarrow y = \dfrac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \dfrac{3 + e^t}{3 - e^t}$

$y = \dfrac{1+ce^t}{1-ce^t}$, there is a constant $\rightarrow$ General sol^n.

$y = \dfrac{3+e^t}{3-e^t}$, there is a specific constant $\rightarrow$ Specific/particular sol^n.

Separable Equations.

A separable equ^n is a 1st order differential equ^n in the form of

$$\frac{dy}{dx} = g(x)\, f(y) \qquad \text{let } f(y) = \frac{1}{h(y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{g(x)}{h(y)}$$

Solve this type of equ^n through separation of variables

$$\Rightarrow h(y)\, dy = g(x)\, dx$$

Integrate both sides.

$$\Rightarrow \int h(y)\, dy = \int g(x)\, dx.$$

short proof using the chain rule.

$$\frac{d}{dx}\left(\int h(y)\, dy\right) = \frac{d}{dx}\left(\int g(x)\, dx\right). \qquad \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{d}{dy}\left(\int h(y)\, dy\right) \cdot \frac{dy}{dx} = g(x)$$

$$\Rightarrow h(y) \cdot \frac{dy}{dx} = g(x) \quad \Rightarrow \frac{dy}{dx} = \frac{g(x)}{h(y)}$$

Example: a) Solve $\frac{dy}{dx} = \frac{x^2}{y^2}$

b) Find the sol^n to the equ^n that satisfies the I.C. $y(0)=2$

a) $y^2\,dy = x^2\,dx$.

$\Rightarrow \int y^2\,dy = \int x^2\,dx$

$\Rightarrow \frac{1}{3}y^3 = \frac{1}{3}x^3 + c$

$\Rightarrow y^3 = x^3 + c$

$\Rightarrow y = \sqrt[3]{x^3+C}$

b) $y(0)=2 \quad \Rightarrow \sqrt[3]{0+C} = 2$.

$\Rightarrow 2^3 = C \quad \Rightarrow C = 8$.

$\Rightarrow y = \sqrt[3]{x^3+8}$

Example: Solve $\frac{dy}{dx} = \frac{6x^2}{2y+\cos y}$

$\Rightarrow 2y+\cos y\,dy = 6x^2\,dx$.

$\Rightarrow \int 2y+\cos y\,dy = \int 6x^2\,dx$.

$\Rightarrow \quad y^2 + \sin y = 2x^3 + C,$

Impossible to solve y explicity as a fun^n of x.

Example: Solve $y' = x^2 y$.

$$\frac{dy}{dx} = x^2 y \quad \Rightarrow \quad \frac{1}{y} dy = x^2 dx$$

$$\Rightarrow \int \frac{1}{y} dy = \int x^2 dx$$

$$\Rightarrow \ln|y| = \frac{1}{3}x^3 + c.$$

$$\Rightarrow \quad |y| = e^{\frac{x^3}{3} + c} = e^c e^{\frac{x^3}{3}}$$

$$\Rightarrow \quad y = \pm e^c e^{\frac{x^3}{3}} \qquad\qquad C = e^c$$

$$\begin{cases} y = \pm Ce^{\frac{x^3}{3}} \\ y = 0 \end{cases}$$

Example: Solve $y' = x + y$ by making a change of variable $u = x + y$. $y(x)$.

$$\frac{dy}{dx} = x + y \qquad\qquad \text{let } u = x + y.$$

$$\Rightarrow \frac{du}{dx} = 1 + \frac{du}{dy} \cdot \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx} \qquad \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1.$$

$$\frac{du}{dx} = \frac{d}{dx} + \frac{d}{dx} y(x)$$

$$= 1 + \frac{du}{dy}\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} - 1 = \frac{dy}{dx} = x + y = u$$

$$\Rightarrow \frac{du}{dx} - 1 = u$$

$$\Rightarrow \frac{du}{dx} = u + 1$$

$$\Rightarrow \frac{1}{1+u}\, du = dx.$$

$$\Rightarrow \int \frac{1}{1+u}\, du = \int 1\, dx$$

$$\Rightarrow \ln|1+u| = x + c.$$

$$\Rightarrow |1+u| = e^{x+c} = e^c \cdot e^x \qquad \text{let } C = e^c$$

$$\Rightarrow |1+u| = Ce^x$$

Remove the abs sign.

$$1 + u = Ce^x \qquad \Rightarrow u = Ce^x - 1$$

Since we let $u = x + y$.

$$\Rightarrow x + y = Ce^x - 1$$

$$\Rightarrow \boxed{y = Ce^x - x - 1.}$$

$$y' = x + y.$$

$$y' = Ce^x - 1$$

$$x + y = \cancel{x} + (Ce^x - \cancel{x} - 1)$$

Practice Test 2.

5. $\int_1^\infty \frac{1}{x^p}\,dx \qquad \int_0^1 \frac{1}{x^p}\,dx \qquad p = ?$

$\cancel{2}, \quad 1, \quad \frac{1}{2}, \quad 0, \quad -1$

If $p \le 1$, then $\int_1^\infty \frac{1}{x^p}\,dx$ is div.

$\int_0^1 \frac{1}{x^{-1}}\,dx = \int_0^1 x\,dx = \text{conv.}$

$\int_0^1 1\,dx = 1. = \text{conv}$

$\int_0^1 \frac{1}{x^{\frac{1}{2}}}\,dx = \int_0^1 x^{-\frac{1}{2}}\,dx = 2x^{\frac{1}{2}}\Big]_0^1$

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7. $g(x) = \int_{-1}^{x} \frac{t^3 - t^2 - 6t}{\sqrt{t^2+7}}\,dt.$

We want g to decrease.

If $g'(x) < 0$, then $g(x)$ is dec.

Refer to the FTC part 1,

$g(x) = \int_a^x f(t)\,dt \qquad g'(x) = f(x).$

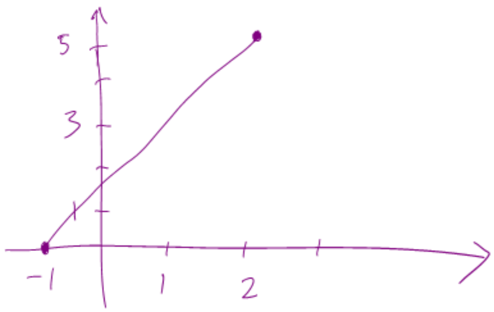
$\Rightarrow g'(x) = \frac{x^3 - x^2 - 6x}{\sqrt{x^2+7}} \quad \Rightarrow \text{check } x^3 - x^2 - 6x < 0.$

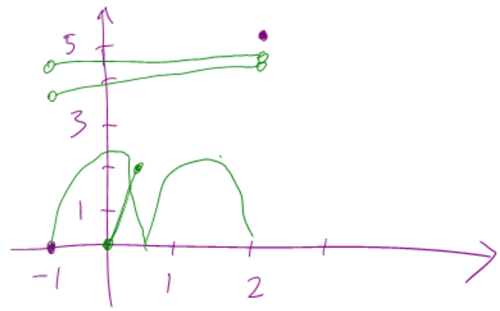
$x(x^2 - x - 6) < 0 \quad \Rightarrow \quad x(x-3)(x+2) < 0.$

$\boxed{A} \qquad x \le -2 \qquad 0 \le x \le 3.$

8. $g \to g(-1) = 0$ $\quad g(2) = 5$.

$x \in (-1, 2)$ for which $g(x) = 3$. $\quad \boxed{B}$

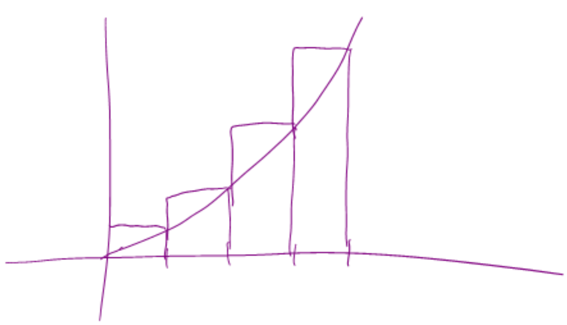




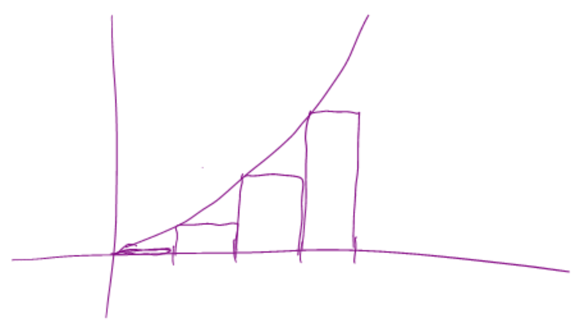
FRQ.

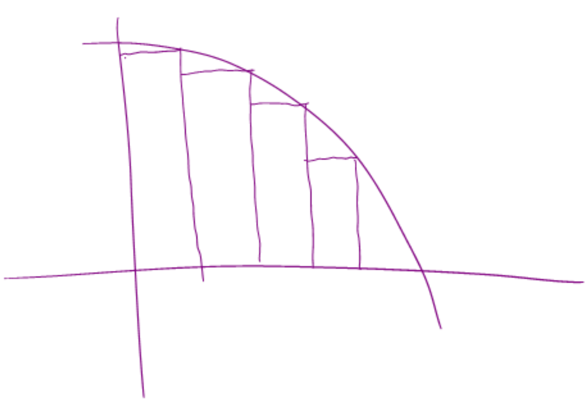
2. b) Use a Right Riemann.

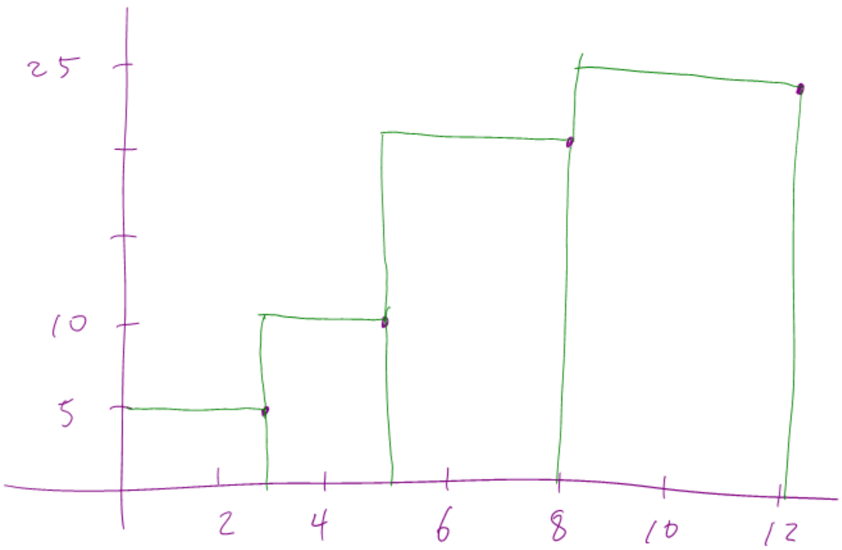
Right Riemann Sum



Left Riemann Sum.







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