AP Calculus Lesson Nineteen Notes

Chapter 9 Infinite Series

9.1 Infinite Sequences
9.2 Convergent or Divergent Infinite Series

9.1 Infinite Sequences

Let's start off this section with a discussion of just what a sequence is. A sequence is nothing more than a list of numbers written in a specific order. The list may or may not have an infinite number of terms in them although we will be dealing exclusively with infinite sequences in this class. General sequence terms are denoted as follows,

$$\begin{aligned}
&a_1 - \text{first term} \\
&a_2 - \text{second term} \\
&\quad\vdots \\
&a_n - n^{th} \text{ term} \\
&a_{n+1} - (n+1)^{\text{st}} \text{ term} \\
&\quad\vdots
\end{aligned}$$

Because we will be dealing with infinite sequences each term in the sequence will be followed by another term as noted above. In the notation above we need to be very careful with the subscripts. The subscript of $n + 1$ denotes the next term in the sequence and NOT one plus the n^{th} term! In other words, $a_{n+1} \neq a_n + 1$.

So be very careful when writing subscripts to make sure that the "+1" doesn't migrate out of the subscript! This is an easy mistake to make when you first start dealing with this kind of thing.
There is a variety of ways of denoting a sequence. Each of the following are equivalent ways of denoting a sequence.

$$\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \qquad \{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

In the second and third notations above a_n is usually given by a formula.

A couple of notes are now in order about these notations. First, note the difference between the second and third notations above. If the starting point is not important or is implied in some way by the problem it is often not written down as we did in the third notation. Next, we used a starting point of $n = 1$ in the third notation only so we could write one down. There is absolutely no reason to believe that a sequence will start at $n = 1$. A sequence will start wherever it needs to start.

Let's take a look at a couple of sequences.

Example 9.1-1 Write down the first few terms of each of the following sequences.

(a) $\left\{\dfrac{n+1}{n^2}\right\}_{n=1}^{\infty}$, (b) $\left\{\dfrac{(-1)^{n+1}}{2^n}\right\}_{n=1}^{\infty}$, (c) $\{b_n\}_{n=1}^{\infty}$, where $b_n = n^{th}$ digit of π

Solution

(a) $\left\{\dfrac{n+1}{n^2}\right\}_{n=1}^{\infty}$,

To get the first few sequence terms here all we need to do is plug in values of n into the formula given and we'll get the sequence terms.

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{\underbrace{2}_{n=1}, \underbrace{\frac{3}{4}}_{n=2}, \underbrace{\frac{4}{9}}_{n=3}, \underbrace{\frac{5}{16}}_{n=4}, \underbrace{\frac{6}{25}}_{n=5}, \ldots\right\}$$

Note the inclusion of the "…" at the end! This is an important piece of notation as it is the only thing that tells us that the sequence continues on and doesn't terminate at the last term.

(b) $\left\{\dfrac{(-1)^{n+1}}{2^n}\right\}_{n=1}^{\infty}$,

This one is similar to the first one. The main difference is that this sequence doesn't start at $n = 1$.

$$\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty} = \left\{-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots\right\}$$

Note that the terms in this sequence alternate in signs. Sequences of this kind are sometimes called alternating sequences.

(c) $\{b_n\}_{n=1}^{\infty}$, where $b_n = n^{th}$ digit of π

This sequence is different from the first two in the sense that it doesn't have a specific formula for each term. However, it does tell us what each term should be. Each term should be the n^{th} digit of π. So we know that π = 3.1415926535...

The sequence is then, {3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, ...}.

In the first two parts of the previous example note that we were really treating the formulas as functions that can only have integers plugged into them. Or,

$$f(n) = \frac{n+1}{n^2}, \qquad g(n) = \frac{(-1)^{n+1}}{2^n}$$

This is an important idea in the study of sequences (and series). Treating the sequence terms as function evaluations will allow us to do many things with sequences that couldn't do otherwise. Before delving further into this idea however we need to get a couple more ideas out of the way.

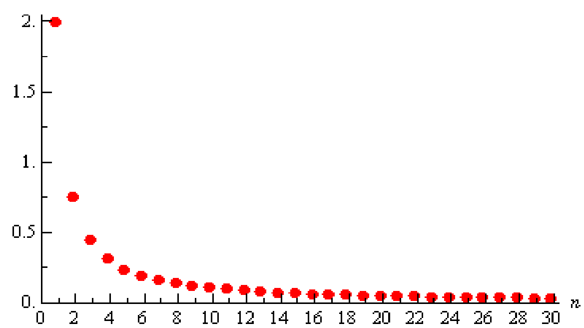
First we want to think about "graphing" a sequence. To graph the sequence $\{a_n\}$ we plot the points (n, a_n) as n ranges over all possible values on a graph. For instance, let's graph the sequence

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$$

The first few points on the graph are,

$$(1,2),\ \left(2,\frac{3}{4}\right),\ \left(3,\frac{4}{9}\right),\ \left(4,\frac{5}{16}\right),\ \left(5,\frac{6}{25}\right),\ \ldots$$

The graph, for the first 30 terms of the sequence, is then,



This graph leads us to an important idea about sequences. Notice that as n increases the sequence terms from our sequence terms, in this case, get closer and closer to zero. We then say that zero is the **limit** (or sometimes the **limiting value**) of the sequence and write,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n+1}{n^2} = 0$$

This notation should look familiar to you. It is the same notation we used when we talked about the limit of a function. In fact, if you recall, we said earlier that we could think of sequences as functions in some way and so this notation shouldn't be too surprising.

Using the ideas that we developed for limits of functions we can write down the following *working definition* for limits of sequences

Working **Definition of Limit**

1. We say that
   $$\lim_{n\to\infty} a_n = L$$
   if we can make a_n as close to L as we want for all sufficiently large n. In other words, the value of the a_n's approach L as n approaches infinity.

2. We say that

   $$\lim_{n\to\infty} a_n = \infty$$

   if we can make a_n as large as we want for all sufficiently large n. Again, in other words, the value of the a_n's get larger and larger without bound as n approaches infinity.

3. We say that

   $$\lim_{n\to\infty} a_n = -\infty$$

   if we can make a_n as large and negative as we want for all sufficiently large n. Again, in other words, the value of the a_n's are negative and get larger and larger without bound as n approaches infinity.

The working definitions of the various sequence limits are nice in that they help us to visualize what the limit actually is. Just like with limits of functions however, there is also a precise definition for each of these limits. Let's give those before proceeding

Precise **Definition of Limit**

1. We say that $\lim_{n\to\infty} a_n = L$ if for every number $\varepsilon > 0$ there is an integer N such that

   $|a_n - L| < \varepsilon$ whenever $n > N$.

2. We say that $\lim_{n\to\infty} a_n = \infty$ if for every number $M > 0$ there is an integer N such that

   $a_n > M$ whenever $n > N$.

3. We say that $\lim_{n\to\infty} a_n = -\infty$ if for every number $M < 0$ there is an integer N such that

   $a_n < M$ whenever $n > N$.

Now that we have the definitions of the limit of sequences out of the way we have a bit of terminology that we need to look at. If $\lim_{n\to\infty} a_n$ exists and is finite we say that the sequence is **convergent**. If $\lim_{n\to\infty} a_n$ doesn't exist or is infinite we say the sequence **diverges**. Note that sometimes we will say the sequence **diverges to** ∞ if $\lim_{n\to\infty} a_n = \infty$ and if $\lim_{n\to\infty} a_n = -\infty$ we will sometimes say that the sequence **diverges to** $-\infty$.

Get used to the terms "convergent" and "divergent" as we'll be seeing them quite a bit throughout this chapter. So just how do we find the limits of sequences? Most limits of most sequences can be found using one of the following theorems.

Theorem 9.1

Given the sequence $\{a_n\}$ if we have a function $f(x)$ such that $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$

This theorem is basically telling us that we take the limits of sequences much like we take the limit of functions. So, now that we know that taking the limit of a sequence is nearly identical to taking the limit of a function we also know that all the properties from the limits of functions will also hold.

Properties

$$\lim_{n\to\infty}\left(a_n \pm b_n\right) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}\left(a_n\, b_n\right) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \quad \text{provided } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p, \text{ provided } a_n \geq 0$$

These properties can be proved using Theorem 9.1 above and the function limit properties we saw in Calculus before or we can prove them directly using the precise definition of a limit using nearly identical proofs of the function limit properties.

Next, just as we had a Squeeze Theorem for function limits we also have one for sequences and it is pretty much identical to the function limit version.

Squeeze Theorem for Sequences 9.2

If $a_n \leq c_n \leq b_n$ for all $n > N$ for some N and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, then $\lim_{n\to\infty} c_n = L$.

Note that in this theorem the "for all n > N for some *N*" is really just telling us that we need to have $a_n \leq c_n \leq b_n$ for all sufficiently large *n*, but if it isn't true for the first few *n* that won't invalidate the theorem.

As we'll see not all sequences can be written as functions that we can actually take the limit of. This will be especially true for sequences that alternate in signs. While we can always write these sequence terms as a function we simply don't know how to take the limit of a function like that. The following theorem will help with some of these sequences.

Theorem 9.3

If $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Note that in order for this theorem to hold the limit MUST be zero and it won't work for a sequence whose limit is not zero. This theorem is easy enough to prove so let's do that.

Proof of Theorem 9.3

The main thing to this proof is to note that,

$-|a_n| \le a_n \le |a_n|$

Then note that, $\lim_{n\to\infty}(-|a_n|) = -\lim_{n\to\infty}|a_n| = 0$.

We then have $\lim_{n\to\infty}(-|a_n|) = \lim_{n\to\infty}|a_n| = 0$. So by the Squeeze Theorem we must also have,

$\lim_{n\to\infty} a_n = 0$.

The next theorem is a useful theorem giving the convergence/divergence and value (for when it's convergent) of a sequence that arises on occasion.

Theorem 9.4

The sequence $\{r^n\}_{n=0}^{\infty}$ converges if $-1 < r \le 1$ and diverges for all other values of *r*. Also,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Here is a quick (well not so quick, but definitely simple) partial proof of this theorem.

Partial Proof of Theorem 9.4

We'll do this by a series of cases although the last case will not be completely proven.

Case 1 :$r > 1$. We know from Calculus that $\lim_{x\to\infty} r^x = \infty$ if $r > 1$ and so by Theorem 9.1 above we also know that $\lim_{n\to\infty} r^n = \infty$ and so the sequence diverges if $r > 1$.

Case 2 : $r = 1$. In this case we have,

$$\lim_{n\to\infty} r^n = \lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1$$

So, the sequence converges for $r = 1$ and in this case its limit is 1.
Case 3 : $0 < r < 1$.We know from Calculus that $\lim_{x\to\infty} r^x = 0$ if $0 < r < 1$ and so by Theorem 9.1 above we also know that $\lim_{n\to\infty} r^n = 0$ and so the sequence converges if $0 < r < 1$ and in this case its limit is zero.

Case 4 : $r = 0$ In this case we have,

$$\lim_{n\to\infty} r^n = \lim_{n\to\infty} 0^n = \lim_{n\to\infty} 0 = 0$$

So, the sequence converges for $r = 0$ and in this case its limit is zero.

Case 5 : $-1 < r < 0$. First let's note that if $-1 < r < 0$ then $0 < |r| < 1$, then by Case 3 above we have,

$$\lim_{n\to\infty} \left|r^n\right| = \lim_{n\to\infty} |r|^n = 0$$

Theorem 9.3 above now tells us that we must also have, $\lim_{n\to\infty} r^n = 0$ and so if $-1 < r < 0$ he sequence converges and has a limit of 0.

Case 6 : $r = -1$. In this case the sequence is,

$$\left\{r^n\right\}_{n=0}^{\infty} = \left\{(-1)^n\right\}_{n=0}^{\infty} = \{1,-1,1,-1,1,-1,1,-1,\ldots\}_{n=0}^{\infty}$$

and hopefully it is clear that $\lim_{n\to\infty}(-1)^n$ doesn't exist. Recall that in order of this limit to exist the terms must be approaching a single value as n increases. In this case however the terms just alternate between 1 and −1 and so the limit does not exist.
So, the sequence diverges for $r = -1$.

Case 7 : $r < -1$. In this case we're not going to go through a complete proof. Let's just see what happens if we let $r = -2$ for instance. If we do that the sequence becomes,

$$\left\{r^n\right\}_{n=0}^{\infty} = \left\{(-2)^n\right\}_{n=0}^{\infty} = \{1,-2,4,-8,16,-32,\ldots\}_{n=0}^{\infty}$$

So, if $r = -2$ we get a sequence of terms whose values alternate in sign and get larger and larger and so $\lim_{n\to\infty}(-2)^n$ doesn't exist. It does not settle down to a single value as n increases nor do the terms ALL approach infinity. So, the sequence diverges for $r = -2$.
We could do something similar for any value of r such that $r < -1$ and so the sequence diverges for $r < -1$.

Before moving onto the next section we need to give one more theorem that we'll need for a proof down the road.

Theorem 9.5

For the sequence $\{a_n\}$ if both $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$ then $\{a_n\}$ then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

Proof of Theorem 9.5

For every given $\varepsilon > 0$, since $\lim_{n\to\infty} a_{2n} = L$ there is an $N_1 > 0$ such that if $n > N_1$ we know that, $|a_{2n} - L| < \varepsilon$

Likewise, because $\lim_{n\to\infty} a_{2n+1} = L$ there is an $N_2 > 0$ such that if $n > N_2$ we know that, $|a_{2n+1} - L| < \varepsilon$

Now, let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. Then either $a_n = a_{2k}$ for some $k > N_1$ or $a_n = a_{2k+1}$ for some $k > N_2$ and so in either case we have that, $|a_2 - L| < \varepsilon$

Therefore, $\lim_{n\to\infty} a_n = L$ and so $\{a_n\}$ is convergent.

Next, we want to take a quick look at some ideas involving sequences.
Given any sequence $\{a_n\}$. We have the following.

1. We call the sequence **increasing** if $a_n < a_{n+1}$ for every n.
2. We call the sequence **decreasing** if $a_n > a_{n+1}$ for every n.
3. If $\{a_n\}$ is an increasing sequence or $\{a_n\}$ is a decreasing sequence we call it **monotonic**.
4. If there exists a number m such that $m \le a_n$ for every n we say the sequence is **bounded below**. The number m is sometimes called a **lower bound** for the sequence.
5. If there exists a number M such that $a_n \le M$ for every n we say the sequence is **bounded above**. The number M is sometimes called an **upper bound** for the sequence.
6. If the sequence is both bounded below and bounded above we call the sequence **bounded**.

Example 9.1-2 Determine if the following sequences are monotonic and/or bounded.

(a) $\{-n^2\}_{n=0}^{\infty}$ (b) $\{(-1)^{n+1}\}_{n=1}^{\infty}$ (c) $\left\{\frac{2}{n^2}\right\}_{n=5}^{\infty}$

Solution

(a) $\left\{-n^2\right\}_{n=0}^{\infty}$

This sequence is a decreasing sequence (and hence monotonic) because, $-n^2 > -(n+1)^2$ for every *n*. Also, since the sequence terms will be either zero or negative this sequence is bounded above. We can use any positive number or zero as the bound, *M*, however, it's standard to choose the smallest possible bound if we can and it's a nice number. So, we'll choose $M = 0$, since $-n^2 \leq 0$ for every *n*.

This sequence is not bounded below however since we can always get below any potential bound by taking *n* large enough. Therefore, while the sequence is bounded above it is not bounded. As a side note we can also note that this sequence diverges (to $-\infty$ if we want to be specific).

(b) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$

The sequence terms in this sequence alternate between 1 and −1 and so the sequence is neither an increasing sequence or a decreasing sequence. Since the sequence is neither an increasing nor decreasing sequence it is not a monotonic sequence. The sequence is bounded however since it is bounded above by 1 and bounded below by −1.
Again, we can note that this sequence is also divergent.

(c) $\left\{\frac{2}{n^2}\right\}_{n=5}^{\infty}$

This sequence is a decreasing sequence (and hence monotonic) since, $\frac{2}{n^2} > \frac{2}{(n+1)^2}$. The terms in this sequence are all positive and so it is bounded below by zero. Also, since the sequence is a decreasing sequence the first sequence term will be the largest and so we can see that the sequence will also be bounded above by 2/25. Therefore, this sequence is bounded. We can also take a quick limit and note that this sequence converges and its limit is zero.

Theorem 9.6

If $\{a_n\}$ is bounded and monotonic then $\{a_n\}$. is convergent.

Be careful to not misuse this theorem. It does not say that if a sequence is not bounded and/or not monotonic that it is divergent. we can easily find a sequence that is not monotonic but does converge.

9.2 Convergent or Divergent Infinite Series

In this section we will introduce the topic that we will be discussing for the rest of this chapter. That topic is infinite series. So just what is a infinite series? Well, let's start with

a sequence $\{a_n\}_{n=1}^{\infty}$ (note the $n = 1$ is for convenience, it can be anything) and define the following,
$s_1 = a_1$
$s_2 = a_1 + a_2$
$s_3 = a_1 + a_2 + a_3$

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

They are called **partial sums** and notice that they will form a sequence, $\{s_n\}_{n=1}^{\infty}$. Also recall that the $\sum$ is used to represent this summation and called a variety of names. The most common names are : **series notation**, **summation notation**, and **sigma notation**.

Now back to series. We want to take a look at the limit of the sequence of partial sums, $\{s_n\}_{n=1}^{\infty}$. We define,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i$$

We call $\sum_{i=1}^{\infty} a_i$ an **infinite series** and note that the series "starts" at $i = 1$ because that is where our original sequence, $\{a_n\}_{n=1}^{\infty}$, started. Had our original sequence started at 2 then our infinite series would also have started at 2. The infinite series will start at the same value that the sequence of terms (as opposed to the sequence of partial sums) starts.
If the sequence of partial sums, $\{s_n\}_{n=1}^{\infty}$, is convergent and its limit is finite then we also call the infinite series, $\sum_{i=1}^{\infty} a_i$ **convergent** and if the sequence of partial sums is divergent then the infinite series is also called **divergent**.

Note that sometimes it is convenient to write the infinite series as,

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

We do have to be careful with this however. This implies that an infinite series is just an infinite sum of terms and as well see in the next section this is not really true.
In the next section we're going to be discussing in greater detail the value of an infinite series, provided it has one of course as well as the ideas of convergence and divergence.
This section is going to be devoted mostly to notational issues as well as making sure we can do some basic manipulations with infinite series so we are ready for them when we need to be able to deal with them in later sections.
First, we should note that in most of this chapter we will refer to infinite series as simply series. If we ever need to work with both infinite and finite series we'll be more careful with terminology, but in most sections we'll be dealing exclusively with infinite series and so we'll just call them series.

Now, in $\sum_{i=1}^{\infty} a_i$ the i is called the **index of summation** or just **index** for short and note that the letter we use to represent the index does not matter. So for example the following series are all the same. The only difference is the letter we've used for the index.

$$\sum_{i=0}^{\infty} \frac{3}{i^2+1} = \sum_{k=0}^{\infty} \frac{3}{k^2+1} = \sum_{n=0}^{\infty} \frac{3}{n^2+1} \quad etc.$$

It is important to again note that the index will start at whatever value the sequence of series terms starts at and this can literally be anything. So far we've used $n = 0$ and $n = 1$ but the index could have started anywhere. In fact, we will usually use $\sum a_n$ to represent an infinite series in which the starting point for the index is not important. When we drop the initial value of the index we'll also drop the infinity from the top so don't forget that it is still technically there.

We will be dropping the initial value of the index in quite a few facts and theorems that we'll be seeing throughout this chapter. In these facts/theorems the starting point of the series will not affect the result and so to simplify the notation and to avoid giving the impression that the starting point is important we will drop the index from the notation. Do not forget however, that there is a starting point and that this will be an infinite series.
Note however, that if we do put an initial value of the index on a series in a fact/theorem it is there because it really does need to be there.

Now that some of the notational issues are out of the way we need to start thinking about various ways that we can manipulate series. We'll start this off with basic arithmetic with infinite series as we'll need to be able to do that on occasion. We have the following properties.

Properties of Infinite Series

If $\sum a_n$ and $\sum b_n$ are both convergent series then,

1. $\sum ca_n$, where c is any number, is also convergent and
   $\sum ca_n = c\sum a_n$.

2. $\sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n$ is also convergent and,
   $\sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n = \sum_{n=k}^{\infty} \left(a_n \pm b_n\right)$

Example 9.2-1 Perform the following index shifts.

(a) Write $\sum_{n=1}^{\infty} ar^{n-1}$ as a series that starts at $n = 0$.

(b) Write $\sum_{n=1}^{\infty} \frac{n^2}{1-3^{n+1}}$ as a series that starts at $n = 3$.

Solution

(a) In this case we need to decrease the initial value by 1 and so the n's (okay the single n) in the term must increase by 1 as well.

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{(n+1)-1} = \sum_{n=0}^{\infty} ar^{n}$$

(b) For this problem we want to increase the initial value by 2 and so all the n's in the series term must decrease by 2.

$$\sum_{n=1}^{\infty} \frac{n^2}{1-3^{n+1}} = \sum_{n=3}^{\infty} \frac{(n-2)^2}{1-3^{(n-2)+1}} = \sum_{n=3}^{\infty} \frac{(n-2)^2}{1-3^{n-1}}$$

Notice that if we ignore the first term the remaining terms will also be a series that will start at n = 2 instead of n = 1. So, we can rewrite the original series as follows,

$$\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n$$

In this example we say that we've stripped out the first term.

We could have stripped out more terms if we wanted to. In the following series we've stripped out the first two terms and the first four terms respectively.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \sum_{n=3}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \sum_{n=5}^{\infty} a_n$$

Being able to strip out terms will, on occasion, simplify our work or allow us to reuse a prior result so it's an important idea to remember.

Notice that in the second example above we could have also denoted the four terms that we stripped out as a finite series as follows,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \sum_{n=5}^{\infty} a_n = \sum_{n=1}^{4} a_n + \sum_{n=5}^{\infty} a_n$$

This is a convenient notation when we are stripping out a large number of terms or if we need to strip out an undetermined number of terms. In general, we can write a series as follows,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

We'll leave this section with an important warning about terminology. Don't get sequences and series confused! A sequence is a list of numbers written in a specific order while an infinite series is a limit of a sequence of finite series and hence, if it exists will be a single value.

So, once again, a sequence is a list of numbers while a series is a single number, provided it makes sense to even compute the series. Students will often confuse the two and try to use facts pertaining to one on the other. However, since they are different beasts this just won't work. There will be problems where we are using both sequences and series so we'll always have to remember that they are different.

If the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence (*i.e.* its limit exists and is finite) then the series is also called **convergent** and in this case if $\lim_{n\to\infty} s_n = s$ then, $\sum_{i=1}^{\infty} a_i = s$. Likewise, if the sequence of partial sums is a divergent sequence (*i.e.* its limit doesn't exist or is plus or minus infinity) then the series is also called **divergent**.

Let's take a look at some series and see if we can determine if they are convergent or divergent and see if we can determine the value of any convergent series we find.

Example 9.2-2 Determine if the series $\sum_{n=1}^{\infty} n$ is convergent or divergent. If it converges determine its value.

Solution

To determine if the series is convergent we first need to get our hands on a formula for the general term in the sequence of partial sums.

$$s_n = \sum_{i=1}^{n} i$$

This is a known series and its value can be shown to be,

$$s_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Don't worry if you didn't know this formula (I'd be surprised if anyone knew it…) as you won't be required to know it in my course.

So, to determine if the series is convergent we will first need to see if the sequence of partial sums,

$$\left\{\frac{n(n+1)}{2}\right\}_{n=1}^{\infty}$$

is convergent or divergent. That's not terribly difficult in this case. The limit of the sequence terms is,

$$\lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

Therefore, the sequence of partial sums diverges to ∞ and so the series also diverges.

Example 9.2-3 Determine if the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges or diverges. If it converges determine its sum.

Solution
This is actually one of the few series in which we are able to determine a formula for the general term in the sequence of partial fractions. However, in this section we are more interested in the general idea of convergence and divergence and so we'll put off discussing the process for finding the formula until the next section.
The general formula for the partial sums is,

$$s_n = \sum_{i=2}^{n} \frac{1}{i^2 - 1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}$$

and in this case we have,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}\right) = \frac{3}{4}$$

The sequence of partial sums converges and so the series converges also and its value is,

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{3}{4}$$

Example 9.2-4 Determine if the series $\sum_{n=0}^{\infty} (-1)^n$ converges or diverges. If it converges determine its sum.

Solution
In this case we really don't need a general formula for the partial sums to determine the convergence of this series. Let's just write down the first few partial sums.

$$s_0 = 1$$
$$s_1 = 1 - 1 = 0$$
$$s_2 = 1 - 1 + 1 = 1$$
$$s_3 = 1 - 1 + 1 - 1 = 0$$

etc.
So, it looks like the sequence of partial sums is,

$$\{s_n\}_{n=0}^{\infty} = \{1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots\}$$

and this sequence diverges since $\lim_{n\to\infty} s_n$ doesn't exist. Therefore, the series also diverges.

Theorem 9.7

If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof

First let's suppose that the series starts at $n = 1$. If it doesn't then we can modify things as appropriate below. Then the partial sums are,

$$s_{n-1} = \sum_{i=1}^{n-1} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_{n-1} \qquad s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_{n-1} + a_n$$

Next, we can use these two partial sums to write,
$a_n = s_n - s_{n-1}$
Now because we know that $\sum a_n$ is convergent we also know that the sequence $\{s_n\}_{n=1}^{\infty}$ is also convergent and that $\lim\limits_{n\to\infty} s_n = s$ for some finite value *s*. However, since $n-1 \to \infty$ as $n \to \infty$, we also have $\lim\limits_{n\to\infty} s_{n-1} = s$. We now have,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(s_n - s_{n-1}\right) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

Be careful to not misuse this theorem! This theorem gives us a requirement for convergence but not a guarantee of convergence. In other words, the converse is NOT true. If $\lim\limits_{n\to\infty} a_n = 0$, the series may actually diverge!

Divergence Test

If $\lim\limits_{n\to\infty} a_n \neq 0$ then $\sum a_n$ will diverge.

Again, do NOT misuse this test. This test only says that a series is guaranteed to diverge if the series terms don't go to zero in the limit. If the series terms do happen to go to zero the series may or may not converge! Again, recall the following two series,

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{converges}$$

One of the more common mistakes that students make when they first get into series is to assume that if $\lim\limits_{n\to\infty} a_n = 0$ then $\sum a_n$ will converge. There is just no way to guarantee this so be careful!

Let's take a quick look at an example of how this test can be used.

Example 9.2-5 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{4n^2 - n^3}{10 + 2n^3}$$

Solution
With almost every series we'll be looking at in this chapter the first thing that we should do is take a look at the series terms and see if they go to zero of not. If it's clear that the terms don't go to zero use the Divergence Test and be done with the problem.
That's what we'll do here.

$$\lim_{n\to\infty}\frac{4n^2-n^3}{10+2n^3}=-\frac{1}{2}\neq 0$$

The limit of the series terms isn't zero and so by the Divergence Test the series diverges.

Now, since the main topic of this section is the convergence of a series we should mention a stronger type of convergence. A series $\sum a_n$ is said to **converge absolutely** if $\sum |a_n|$ also converges. Absolute convergence is *stronger* than convergence in the sense that a series that is absolutely convergent will also be convergent, but a series that is convergent may or may not be absolutely convergent.

In fact if $\sum a_n$ converges and $\sum |a_n|$ diverges, the series $\sum a_n$ is called **conditionally convergent**.

At this point we don't really have the tools at hand to properly investigate this topic in detail nor do we have the tools in hand to determine if a series is absolutely convergent or not. So we'll not say anything more about this subject for a while. When we finally have the tools in hand to discuss this topic in more detail we will revisit it. Until then don't worry about it. The idea is mentioned here only because we were already discussing convergence in this section and it ties into the last topic that we want to discuss in this section.

In the previous section after we'd introduced the idea of an infinite series we commented on the fact that we shouldn't think of an infinite series as an infinite sum despite the fact that the notation we use for infinite series seems to imply that it is an infinite sum. It's now time to briefly discuss this.

First, we need to introduce the idea of a **rearrangement**. A rearrangement of a series is exactly what it might sound like, it is the same series with the terms rearranged into a different order.

For example, consider the following the infinite series.

$$\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+a_4+a_5+a_6+a_7+\cdots$$

A rearrangement of this series is,

$$\sum_{n=1}^{\infty} a_n = a_2+a_1+a_3+a_{14}+a_5+a_9+a_4+\cdots$$

The issue we need to discuss here is that for some series each of these arrangements of terms can have a different values despite the fact that they are using exactly the same terms. Here is an example of this. It can be shown that,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots=\ln 2 \qquad (1)$$

Since this series converges we know that if we multiply it by a constant c its value will also be multiplied by c. So, let's multiply this by $1/2$ to get,

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\frac{1}{16}+\cdots=\frac{1}{2}\ln 2 \qquad (2)$$

Now, let's add in a zero between each term as follows.

$$0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+0+\frac{1}{10}+0-\frac{1}{12}+0+\cdots=\frac{1}{2}\ln 2 \qquad (3)$$

Note that this won't change the value of the series because the partial sums for this series will be the partial sums for the (2) except that each term will be repeated. Repeating terms in a series will not affect its limit however and so both (2) and (3) will be the same.

We know that if two series converge we can add them by adding term by term and so add (1) and (3) to get,

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots=\frac{3}{2}\ln 2 \qquad (4)$$

Now, notice that the terms of (4) are simply the terms of (1) rearranged so that each negative term comes after two positive terms. The values however are definitely different despite the fact that the terms are the same.

Note as well that this is not one of those "tricks" that you see occasionally where you get a contradictory result because of a hard to spot math/logic error. This is a very real result and we've not made any logic mistakes/errors.

Here is a nice set of facts that govern this idea of when a rearrangement will lead to a different value of a series.

Facts

Given the series $\sum a_n$,

1. If $\sum a_n$ is absolutely convergent and its value is *s* then any rearrangement of $\sum a_n$ will also have a value of *s*.

2. If $\sum a_n$ is conditionally convergent and *r* is any real number then there is a rearrangement of $\sum a_n$ whose value will be *r*.

In this section we are going to take a brief look at three special series. Actually, special may not be the correct term. All three have been named which makes them special in some way, however the main reason that we're going to look at two of them in this section is that they are the only types of series that we'll be looking at for which we will be able to get actual values for the series. The third type is divergent and so won't have a value to worry about.

In general, determining the value of a series is very difficult and outside of these two kinds of series that we'll look at in this section we will not be determining the value of series in this chapter.

Geometric Series

A geometric series is any series that can be written in the form,

$$\sum_{n=1}^{\infty} ar^{n-1}$$

or, with an index shift the geometric series will often be written as,

$$\sum_{n=0}^{\infty} ar^n$$

These are identical series and will have identical values, provided they converge of course. If we start with the first form it can be shown that the partial sums are,

$$s_n = \frac{a\left(1-r^n\right)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

The series will converge provided the partial sums form a convergent sequence, so let's take the limit of the partial sums.

$$\begin{aligned}\lim_{n\to\infty} s_n &= \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right)\\ &= \lim_{n\to\infty}\frac{a}{1-r} - \lim_{n\to\infty}\frac{ar^n}{1-r}\\ &= \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n\end{aligned}$$

Now, from Theorem 3 from the Sequences section we know that the limit above will exist and be finite provided $-1 < r \le 1$. However, note that we can't let $r = 1$, since this will give division by zero. Therefore, this will exist and be finite provided $-1 < r < 1$ and in this case the limit is zero and so we get,

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}$$

Therefore, a geometric series will converge if $-1 < r < 1$, its value is,

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Note that in using this formula we'll need to make sure that we are in the correct form. In other words, if the series starts at $n = 0$ then the exponent on the r must be n. Likewise if the series starts at $n = 1$ then the exponent on the r must be $n - 1$.

Example 9.2-6 Determine if the following series converge or diverge. If they converge give the value of the series.

(a) $\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$ (b) $\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

Solution

(a) $\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$

This series doesn't really look like a geometric series. However, notice that both parts of the series term are numbers raised to a power. This means that it can be put into the form of a geometric series. We will just need to decide which form is the correct form. Since the series starts at $n = 1$, we will want the exponents on the numbers to be $n - 1$.

It will be fairly easy to get this into the correct form. Let's first rewrite things slightly. One of the *n*'s in the exponent has a negative in front of it and that can't be there in the geometric form. So, let's first get rid of that.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 9^{-(n-2)}4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}}$$

Now let's get the correct exponent on each of the numbers. This can be done using simple exponent properties.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}} = \sum_{n=1}^{\infty} \frac{4^{n-1}4^{2}}{9^{n-1}9^{-1}}$$

Now, rewrite the term a little.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 16(9)\frac{4^{n-1}}{9^{n-1}} = \sum_{n=1}^{\infty} 144\left(\frac{4}{9}\right)^{n-1}$$

So, this is a geometric series with $a = 144$ and r = 4/9 < 1.

Therefore, since $|r| < 1$, we know the series will converge and its value will be,

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \frac{144}{1-\frac{4}{9}} = \frac{9}{5}(144) = \frac{1296}{5}$$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

Again, this doesn't look like a geometric series, but it can be put into the correct form. In this case the series starts at n = 0, so we'll need the exponents to be *n* on the terms. Note that this means we're going to need to rewrite the exponent on the numerator a little

$$\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}} = \sum_{n=0}^{\infty} \frac{\left((-4)^{3}\right)^{n}}{5^{n}5^{-1}} = \sum_{n=0}^{\infty} 5\frac{(-64)^{n}}{5^{n}} = \sum_{n=0}^{\infty} 5\left(\frac{-64}{5}\right)^{n}$$

So, we've got it into the correct form and we can see that $a = 5$ and $r = -64/5$. Also note that $|r| \geq 1$ and so this series diverges.

Telescoping Series

It's now time to look at the second of the three series in this section. In this portion we are going to look at a series that is called a telescoping series. The name in this case comes from what happens with the partial sums and is best shown in an example.

Example 9.2-7 Determine if the following series converges or diverges. If it converges find its value.

$$\sum_{n=0}^{\infty}\frac{1}{n^2+3n+2}$$

Solution

We first need the partial sums for this series.

$$s_n=\sum_{i=0}^{n}\frac{1}{i^2+3i+2}$$

Now, let's notice that we can use partial fractions on the series term to get,

$$\frac{1}{i^2+3i+2}=\frac{1}{(i+2)(i+1)}=\frac{1}{i+1}-\frac{1}{i+2}$$

I'll leave the details of the partial fractions to you. By now you should be fairly adept at this since we spent a fair amount of time doing partial fractions back in the Integration Techniques chapter. If you need a refresher you should go back and review that section.
So, what does this do for us? Well, let's start writing out the terms of the general partial sum for this series using the partial fraction form.

$$\begin{aligned}s_n&=\sum_{i=0}^{n}\left(\frac{1}{i+1}-\frac{1}{i+2}\right)\\&=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots+\left(\cancel{\frac{1}{n}}-\cancel{\frac{1}{n+1}}\right)+\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+2}\right)\\&=1-\frac{1}{n+2}\end{aligned}$$

Notice that every term except the first and last term cancelled out. This is the origin of the name **telescoping series**.

This also means that we can determine the convergence of this series by taking the limit of the partial sums.

$$\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(1-\frac{1}{n+2}\right)=1$$

The sequence of partial sums is convergent and so the series is convergent and has a value of

$$\sum_{n=0}^{\infty}\frac{1}{n^2+3n+2}=1$$

Harmonic Series

This is the third and final series that we're going to look at in this section. Here is the harmonic series.

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

The harmonic series is divergent and we'll need to wait until the next section to show that. This series is here because it's got a name and so I wanted to put it here with the other two named series that we looked at in this section. We're also going to use the harmonic series to illustrate a couple of ideas about divergent series that we've already discussed for convergent series. We'll do that with the following example.

Example 9.2-8
Show that each of the following series are divergent.

(a) $\sum_{n=1}^{\infty} \frac{5}{n}$ (b) $\sum_{n=4}^{\infty} \frac{1}{n}$

Solution

(a) $\sum_{n=1}^{\infty} \frac{5}{n}$

To see that this series is divergent all we need to do is use the fact that we can factor a constant out of a series as follows,

$$\sum_{n=1}^{\infty} \frac{5}{n} = 5\sum_{n=1}^{\infty} \frac{1}{n}$$

Now, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and so five times this will still not be a finite number and so the series has to be divergent. In other words, if we multiply a divergent series by a constant it will still be divergent.

(b) $\sum_{n=4}^{\infty} \frac{1}{n}$

In this case we'll start with the harmonic series and strip out the first three terms.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \sum_{n=4}^{\infty} \frac{1}{n} \qquad \Rightarrow \qquad \sum_{n=4}^{\infty} \frac{1}{n} = \left(\sum_{n=1}^{\infty} \frac{1}{n}\right) - \frac{11}{6}$$

In this case we are subtracting a finite number from a divergent series. This subtraction will not change the divergence of the series. We will either have infinity minus a finite number, which is still infinity, or a series with no value minus a finite number, which will still have no value.
Therefore, this series is divergent. Just like with convergent series, adding/subtracting a finite number from a divergent series is not going to change the fact the convergence of the series.

So, some general rules about the convergence/divergence of a series are now in order. Multiplying a series by a constant will not change the convergence/divergence of the series and adding or subtracting a constant from a series will not change the convergence/divergence of the series. These are nice ideas to keep in mind.