AP Calculus Class 9

- Antiderivatives
- Area problems
- Integrals
- Evaluating integrals

Definition

A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

$f(x) = x^2$

$\frac{d}{dx} x^n = n x^{n-1}$

$F(x) = \frac{1}{3}x^3$

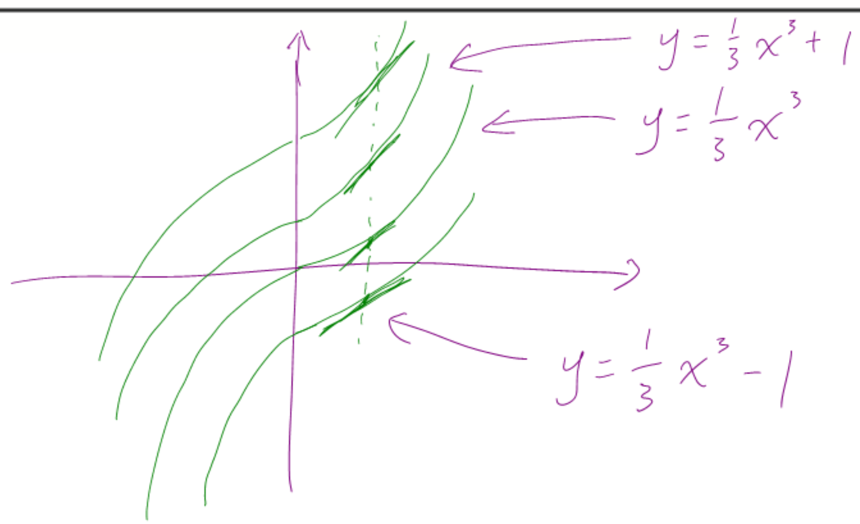
$G(x) = \frac{1}{3}x^3 + 100 \longrightarrow G(x) = \frac{1}{3}x^3 + C$

Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.



Example: Find the most general a.d. of

a) $f(x) = \sin x$ b) $f(x) = \frac{1}{x}$ c) $f(x) = x^n$.

$F' = f$

a) $F'(x) = \sin x$

$\frac{d}{dx} \cos x = -\sin x$

$\Rightarrow -\cos x = F(x)$.

$G(x) = -\cos x + C$.

b) $F'(x) = \frac{1}{x}$ $\Rightarrow F(x) = \ln x$

$\frac{d}{dx} \ln x = \frac{1}{x}$ $\frac{d}{dx} \ln |x| = \frac{1}{x}$

$\Rightarrow G(x) = \ln x + C$.

c) $f(x) = x^n \longrightarrow F'(x) = x^n$

$\frac{d}{dx} x^n = n x^{n-1}$. $F = \frac{x^{n+1}}{n+1}$

$$F' = \frac{\cancel{(n+1)} x^{(n+1)-1}}{\cancel{(n+1)}} = x^n.$$

$$\Rightarrow F = \frac{x^{n+1}}{n+1} + C. \qquad \forall n \neq -1$$

Fun^n	Particular A.D.
$c f(x)$	$c F(x)$
x^n	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$.

Fun^n	Particular A.D.
e^x	e^x
$\sec^2 x$	$\tan x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$

Example: Find all the fun^{n}s g s.t.

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$$

$$= 4\sin x + 2\frac{x^5}{x} - \frac{\sqrt{x}}{x} = 4\sin x + 2x^4 - x^{-\frac{1}{2}}$$

$4\sin x \rightarrow -4\cos x$.

$2x^4 \rightarrow 2\frac{x^5}{5}$

$-x^{-\frac{1}{2}} \rightarrow -\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = -\frac{x^{\frac{1}{2}}}{\frac{1}{2}}$

$\Rightarrow g(x) = -4(\cos x) + \frac{2}{5}x^5 - 2\sqrt{x} + C.$

Example: $f'(x) = e^x + 20(1+x^2)^{-1}$ and $f(0) = -2$

Find f.

$$\underbrace{(1+x^2)^{-1}}_{\frac{1}{1+x^2}}$$

The general antiderivative of $f'(x) = f$ is

$f(x) = e^x + 20(\tan^{-1} x) + C$.

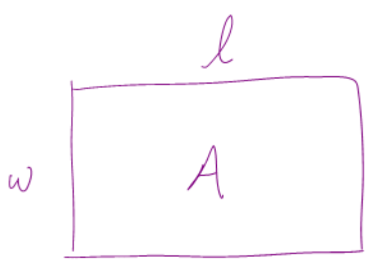
$\Rightarrow f(0) = e^0 + 20(\tan^{-1} 0) + C = -2$.

$\Rightarrow 1 + 20(0) + C = -2$

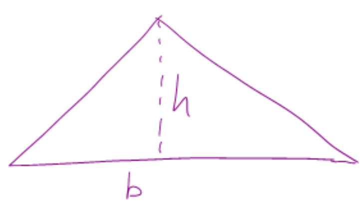
$\Rightarrow C = -3$.

$f(x) = e^x + 20 \tan^{-1} x - 3$.

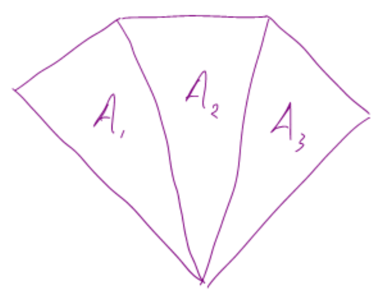
The Area Problem.



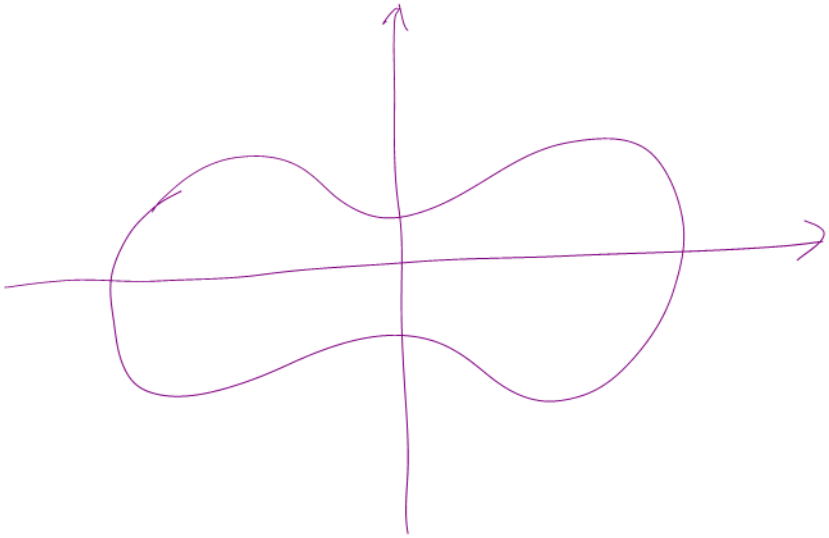
$A = lw$

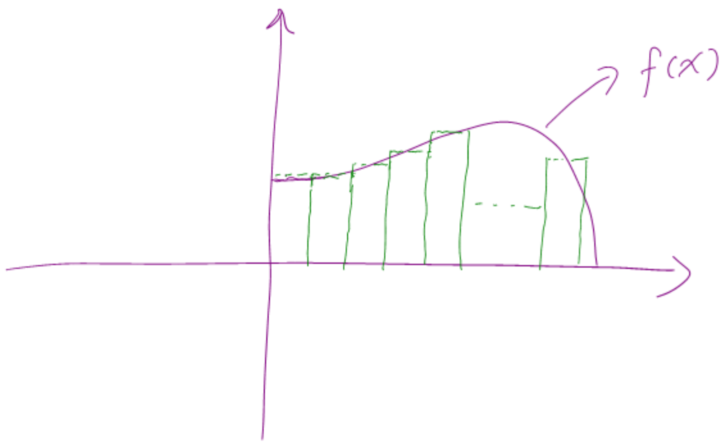


$A = \frac{1}{2}bh$.

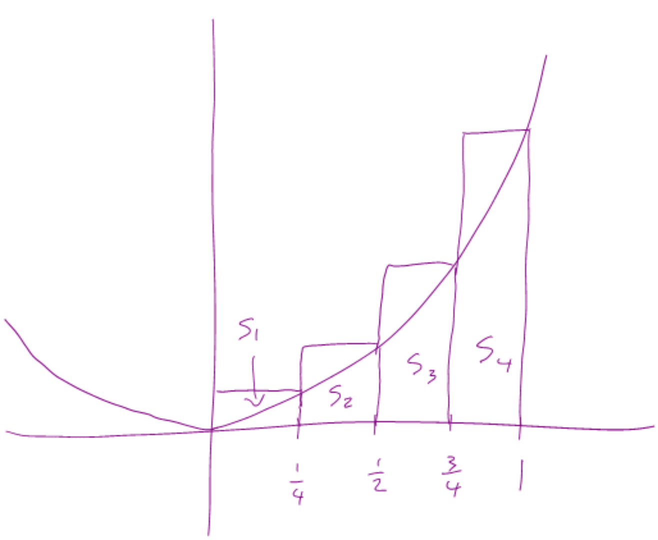


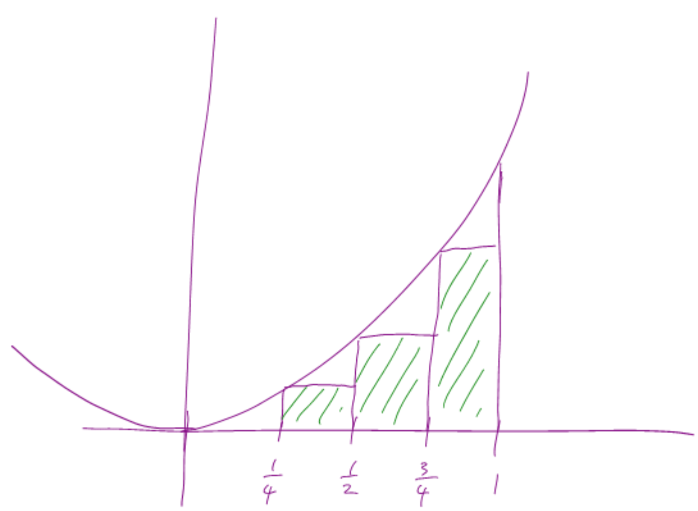
$A = A_1 + A_2 + A_3$

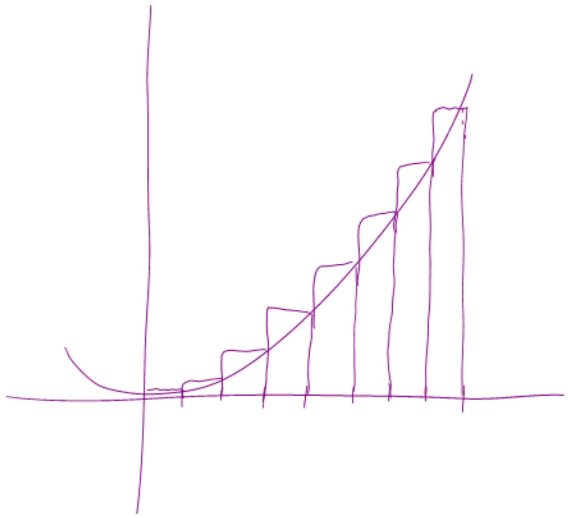




Example: $y = x^2$, approximate the area under this curve between the points $(0, 1)$.







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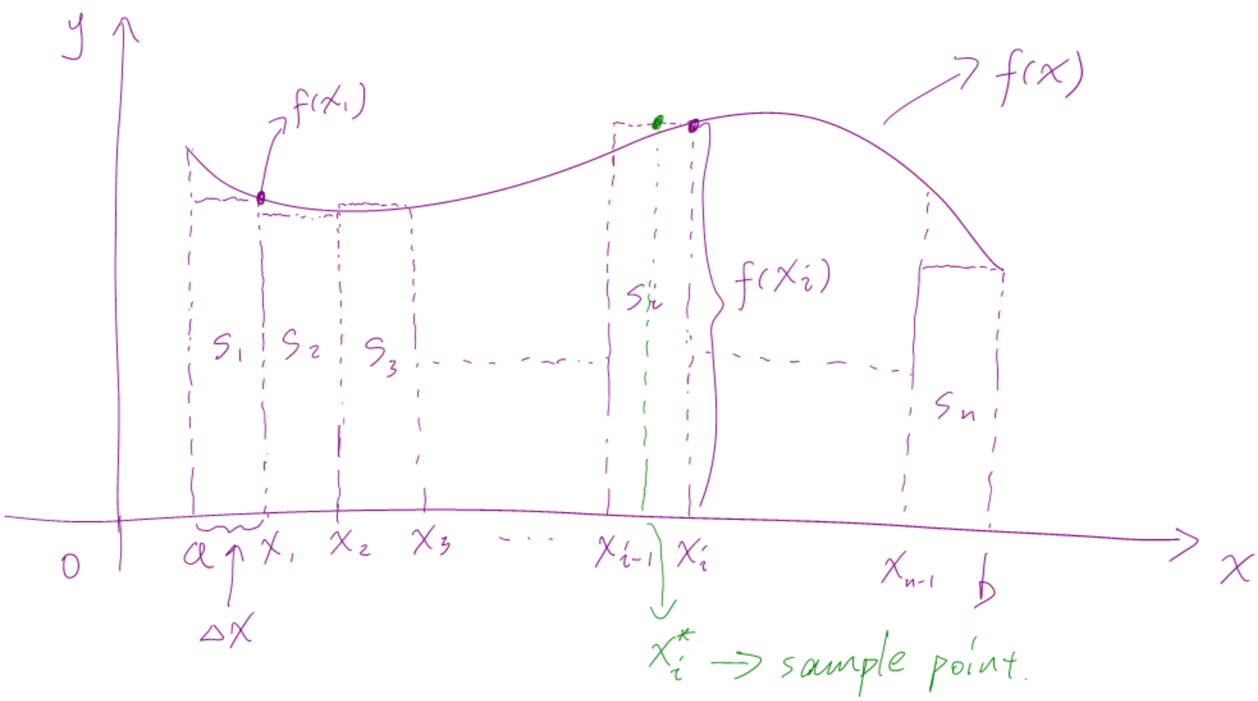
Area $\approx \frac{1}{3}$



$$\Delta x = \frac{b-a}{n}$$

The area for the ith strip is

$$S_i = f(x_i)\,\Delta x$$

If we want to find the sum of the area of all rectangles,

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

Definition

The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

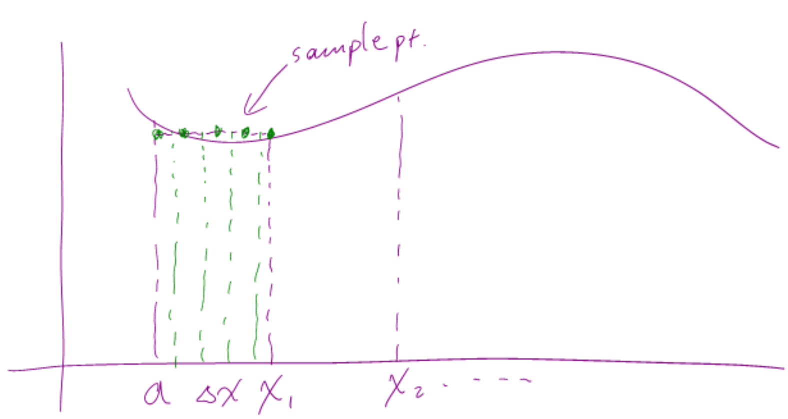
We often use the sigma notation to write sums.

$$\boxed{\sum_{i=1}^{n} f(x_i)\Delta x} = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

$$\prod_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x \cdot f(x_2)\Delta x \cdot \cdots \cdot f(x_n)\Delta x.$$

Don't need to know for this course.

The point within the ith interval $[x_{i-1}, x_i]$ is called a sample point.



For the area above,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \Delta x \qquad \Delta x \to 0.$$

The process of adding rectangles is called the Riemann Sum.

The Definite Integral.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \lim_{n\to\infty} \left(f(x_1^*)\Delta x + \cdots + f(x_n^*) \Delta x \right)$$

Definition of a Definite Integral

If f is a continuous function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0(=a), x_1, x_2, \ldots x_n(=b)$ be the ends points of these subintervals and we let $x_1^*, x_2^*, \ldots x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Sum.

S ʃ ʃ ʃ

$\int$ is called an integral sign.

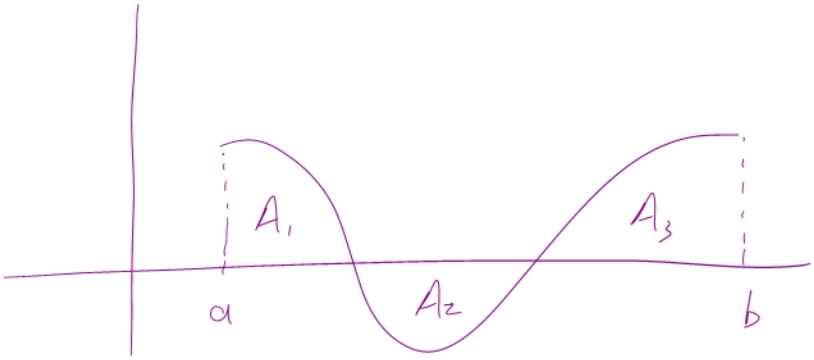
$f(x)$ is called the integrand.

a and b are called the limits of integration.

a is the lower limit and b is called the upper limit.

The procedure of calculating an integral is called integration.

x: dummy variable. (Can use any variable: t, y, θ, r, etc).



Note: the Riemann Sum calculates the <u>net area</u>

The net area:

$A_1 + A_3 - A_2$

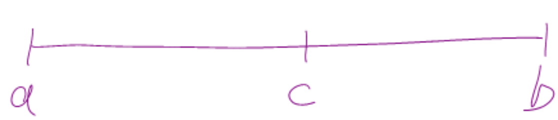
Example: Express $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} (x_i^3 + x_i \sin x_i)\Delta x$ as an integral on the interval $[0, \pi]$.

$\int_0^{\pi} x^3 + x\sin x \, dx$

$\int_0^{\pi}$

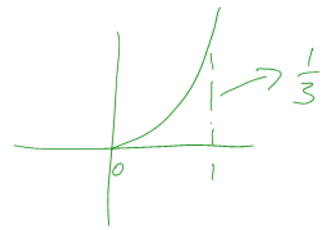
Properties of the Integral

1. $\displaystyle\int_a^b c\,dx = c(b-a)$, where c is an constant
2. $\displaystyle\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
3. $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$
4. $\displaystyle\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
5. $\displaystyle\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$



Example: Evaluate $\int_0^1 (4+3x^2)\,dx$.

$\int_0^1 4\,dx + 3\int_0^1 x^2\,dx$ Prop ②, ③.



From prop ①,

$\int_0^1 4\,dx = 4(1-0) = 4$

From earlier, we know $\int_0^1 x^2\,dx = \frac{1}{3}$

$\Rightarrow \int_0^1 4+3x^2\,dx = 4+3(\frac{1}{3}) = 4+1 = 5$.

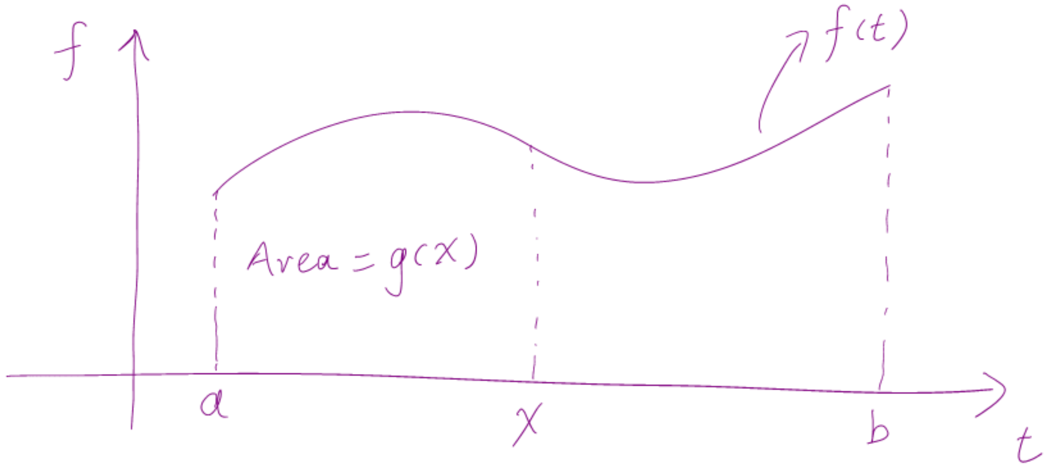
The Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus, Part 1

If f is consinuous on $[a,b]$, then the function f defined by

$$g(x) = \int_a^x f(t)dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.



The Fundamental Theorem of Calculus, Part 2

> If f is consinuous on $[a, b]$, then
>
> $$\int_a^b f(x)dx = F(b) - F(a)$$
>
> where F is any antiderivative of f, that is, a function such that $F' = f$.

Example: $y = x^2$. Find the area under the parabola between 0 and 1.

$$\int_0^1 x^2 dx = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3} - 0 = \frac{1}{3}$$

$x^2 \rightarrow \frac{1}{3}x^3$

For x^2, the antiderivative is $\frac{x^{2+1}}{2+1} = \frac{x^3}{3}$

$$\int_0^1 x^2 dx = F(x)\Big]_a^b = F(b) - F(a) = F(x)\Big|_a^b$$

$$= \frac{x^3}{3}\Big]_a^b = \frac{1}{3} - 0 = \frac{1}{3}.$$