Compound Angle Formulas

Addition Formulas

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

Subtraction Formulas

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$
$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

Example 1

Simplify each expression.

a) $\sin\frac{7x}{5}\cos\frac{2x}{5} - \cos\frac{7x}{5}\sin\frac{2x}{5}$

b) $\tan\left(x + \frac{\pi}{4}\right)$

Example 2

Evaluate each expression.

a) $\cos 20^\circ \cos 190^\circ - \sin 20^\circ \sin 190^\circ$

b) $\dfrac{\tan\frac{8\pi}{9} - \tan\frac{5\pi}{9}}{1 + \tan\frac{8\pi}{9}\tan\frac{5\pi}{9}}$

Example 3

Determine the exact value of

a) $\sin\left(\frac{7\pi}{12}\right)$

b) $\tan\left(-195^\circ\right)$

Example 4

The tangent of the acute angle α is 0.75, and the tangent of the acute angle β is 2.4. Without using a calculator, determine the value of $\sin(\alpha + \beta)$ and $\cos(\alpha - \beta)$.

Proving Trigonometric Identities

The following trigonometric identities are important for you to remember:

Reciprocal Identities

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Addition and Subtraction Formulas

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Double Angle Formulas

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 2\cos^2 x - 1$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

To prove that a given equation is an identity, the two sides of the equation must be shown to be equivalent. This can be accomplished using a variety of strategies, such as

- simplifying the more complicated side until it is identical to the other side, or manipulating both sides to get the same expression
- rewriting expressions using any of the identities stated above
- using a common denominator or factoring, where possible

Example 3

Prove each identity.

a) $(1-\cos\beta)^2+\sin^2\beta=2(1-\cos\beta)$

b) $\cos^4\theta-\sin^4\theta=1-2\sin^2\theta$

c) $\dfrac{1}{1-\sin x}-\dfrac{1}{1+\sin x}=\dfrac{2\tan x}{\cos x}$

d) $\dfrac{\sin\alpha+\tan\alpha}{1+\sec\alpha}=\sin\alpha$

Example 4

Prove each identity.

a) $\sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$

b) $\sin(\alpha+\beta)\times\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta$

c) $\tan 2x=\dfrac{2}{\cot x-\tan x}$

d) $\dfrac{\sin 2\theta}{1-\cos 2\theta}=2\csc 2\theta-\tan\theta$

Related & Corelated Angles

$\frac{\pi}{2}$

$\sin(\pi - \theta) =$

$\cos(\pi - \theta) =$

$\tan(\pi - \theta) =$

$\sin\theta$

$\cos\theta$

$\tan\theta$

π

$0^r / 2\pi$

$\sin(\pi + \theta) =$

$\cos(\pi + \theta) =$

$\tan(\pi + \theta) =$

$\sin(2\pi - \theta) =$

$\cos(2\pi - \theta) =$

$\tan(2\pi - \theta) =$

$\sin(-\theta) =$

$\cos(-\theta) =$

$\tan(-\theta) =$

$\frac{3\pi}{2}$

$\frac{\pi}{2}$

$\sin\left(\frac{\pi}{2} + \theta\right)$

$\cos\left(\frac{\pi}{2} + \theta\right)$

$\tan\left(\frac{\pi}{2} + \theta\right)$

$\sin\left(\frac{\pi}{2} - \theta\right)$

$\cos\left(\frac{\pi}{2} - \theta\right)$

$\tan\left(\frac{\pi}{2} - \theta\right)$

π

$0^r / 2\pi$

$\sin\left(\frac{3\pi}{2} - \theta\right)$

$\cos\left(\frac{3\pi}{2} - \theta\right)$

$\tan\left(\frac{3\pi}{2} - \theta\right)$

$\sin\left(\frac{3\pi}{2} + \theta\right)$

$\cos\left(\frac{3\pi}{2} + \theta\right)$

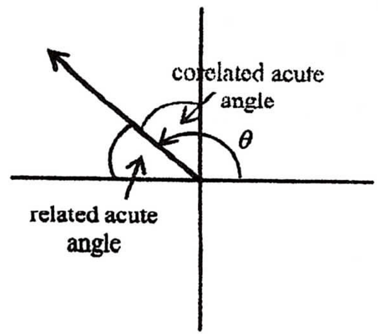
$\tan\left(\frac{3\pi}{2} + \theta\right)$

$\frac{3\pi}{2}$

Related and Corelated Angle Identities

Recall:

- If an angle is in standard position, the acute angle between the terminal arm and the x-axis is the ***related acute angle.***
- If an angle is in standard position, the acute angle between the terminal arm and the y-axis is the ***correlated acute angle.***



*Any angle can be defined in terms of its related (or correlated acute angle.

Related Angle Identities:

Eg. 1 Simplfiy.

a) $\sin(\pi - x)$

$= \sin\pi\cos x - \cos\pi\sin x$

$= (0)\cos x - (-1)\sin x$

$= \sin x$

b) $\cos(\pi - x)$

$= \cos\pi\cos x + \sin\pi\sin x$

$= (-1)\cos x + (0)\sin x$

$= -\cos x$

c) $\tan(\pi - x)$

$= \dfrac{\sin(\pi - x)}{\cos(\pi - x)}$

$= \dfrac{\sin x}{-\cos x}$

$= -\tan x$

Summary:

$\sin(\pi - x) = \sin x$	$\sin(\pi + x) = -\sin x$	$\sin(2\pi - x) = -\sin x$	$\sin(-x) = -\sin x$
$\cos(\pi - x) = -\cos x$	$\cos(\pi + x) = -\cos x$	$\cos(2\pi - x) = \cos x$	$\cos(-x) = \cos x$
$\tan(\pi - x) = -\tan x$	$\tan(\pi + x) = \tan x$	$\tan(2\pi - x) = -\tan x$	$\tan(-x) = -\tan x$

Correlated Angle Identities:

Eg. 2 Simplfiy.

a) $\sin(\frac{\pi}{2} - x)$

$= \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$

$= (1)\cos x - (0)\sin x$

$= \cos x$

b) $\cos(\frac{\pi}{2} - x)$

$= \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$

$= (0)\cos x + (1)\sin x$

$= \sin x$

c) $\tan(\frac{\pi}{2} - x)$

$= \dfrac{\sin(\frac{\pi}{2} - x)}{\cos(\frac{\pi}{2} - x)}$

$= \dfrac{\cos x}{\sin x}$

$= \cot x$

Summary:

$\sin(\frac{\pi}{2} + x) = \cos x$	$\sin(\frac{\pi}{2} - x) = \cos x$	$\sin(\frac{3\pi}{2} - x) = -\cos x$	$\sin(\frac{3\pi}{2} + x) = -\cos x$
$\cos(\frac{\pi}{2} + x) = -\sin x$	$\cos(\frac{\pi}{2} - x) = \sin x$	$\cos(\frac{3\pi}{2} - x) = -\sin x$	$\cos(\frac{3\pi}{2} + x) = \sin x$
$\tan(\frac{\pi}{2} + x) = -\cot x$	$\tan(\frac{\pi}{2} - x) = \cot x$	$\tan(\frac{3\pi}{2} - x) = \cot x$	$\tan(\frac{3\pi}{2} + x) = -\cot x$

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Related Angle, Co – Related Angle Identities

1. Write each in a simpler form.

   a) $\sin(\pi - x)$ b) $\sec(\pi + x)$ c) $\tan(-x)$ d) $\csc(\pi + x)$ e) $\sec(\pi - x)$

   f) $\cot(\pi + x)$ g) $\cos(\pi - x)$ h) $\sin(-x)$ i) $\csc(\pi - x)$ j) $\sec(-x)$

2. Find the exact value of the following.

   a) $\sin(\pi + \frac{\pi}{6})$ b) $\cot(\pi - \frac{\pi}{3})$ c) $\sec(\pi + \frac{\pi}{4})$ d) $\csc(\pi - \frac{\pi}{6})$

3. Find the exact value of the following.

   a) $\sec\left(\pi - \frac{\pi}{6}\right)$ b) $\cos\left(\pi - \frac{\pi}{3}\right)$ c) $\cot\left(\pi + \frac{\pi}{4}\right)$ d) $\sin\left(\pi + \frac{\pi}{6}\right)$

4. Use the relationship $\cos(\pi - x) = -\cos x$ to evaluate the following.

   a) $\cos\frac{5\pi}{6}$ b) $\cos\frac{2\pi}{3}$ c) $\cos\frac{3\pi}{4}$

5. Use the relationship $\tan(-x) = -\tan x$ to evaluate the following.

   a) $\tan(-\frac{\pi}{3})$ b) $\tan(-\frac{\pi}{6})$ c) $\tan(-\frac{\pi}{4})$

6. Write each in a simpler form.

   a) $\sin(\frac{3\pi}{2} - x)$ b) $\sec(\frac{\pi}{2} + x)$ c) $\tan(\frac{3\pi}{2} + x)$ d) $\csc(\frac{3\pi}{2} + x)$ e) $\sec(\frac{\pi}{2} - x)$

7. Write each in a simpler form.

   a) $\sin\left(\frac{\pi}{2} - x\right)$ b) $\tan\left(\frac{3\pi}{2} - x\right)$ c) $\sec\left(\frac{3\pi}{2} + x\right)$ d) $\csc\left(\frac{\pi}{2} + x\right)$ e) $\cot\left(\frac{\pi}{2} + x\right)$

8. Find the exact value of the following.

   a) $\sin(\frac{\pi}{2} + \frac{\pi}{6})$ b) $\cot(\frac{3\pi}{2} - \frac{\pi}{3})$ c) $\sec(\frac{3\pi}{2} + \frac{\pi}{4})$ d) $\csc(\frac{\pi}{2} - \frac{\pi}{6})$

9. Find the exact value of the following.

   a) $\sec\left(\frac{\pi}{2} + \frac{\pi}{6}\right)$ b) $\cos\left(\frac{3\pi}{2} - \frac{\pi}{3}\right)$ c) $\cot\left(\frac{3\pi}{2} + \frac{\pi}{4}\right)$ d) $\sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$

ANSWERS

1. a) $\sin x$ b) $-\sec x$ c) $-\tan x$ d) $-\csc x$ e) $-\sec x$ f) $\cot x$ g) $-\cos x$ h) $-\sin x$ i) $\csc x$ j) sec 2. a) $\frac{-1}{2}$ b) $-\frac{1}{\sqrt{3}}$ c) $-\sqrt{2}$ d) 2
3. a) $-\frac{2\sqrt{3}}{3}$ or $-\frac{2}{\sqrt{3}}$ b) $\frac{-1}{2}$ c) 1 d) $\frac{-1}{2}$ 4. a) $\frac{-\sqrt{3}}{2}$ b) $\frac{-1}{2}$ c) $-\frac{1}{\sqrt{2}}$ 5. a) $-\sqrt{3}$ b) $-\frac{1}{\sqrt{3}}$ c) -1 6. a) $-\cos x$ b) $-\csc x$ c) $-\cot x$ d) $-\sec x$
e) $\csc x$ 7. a) $\cos x$ b) $\cot x$ c) $\csc x$ d) $\sec x$ e) $-\tan x$ 8. a) $\frac{\sqrt{3}}{2}$ b) $\sqrt{3}$ c) $\sqrt{2}$ d) $\frac{2}{\sqrt{3}}$ 9. a) -2 b) $-\frac{\sqrt{3}}{2}$ c) -1 d) $\frac{\sqrt{3}}{2}$

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Proving Trigonometric Identities

Strategies

- begin with the more complex side and continue until it is the same as the simpler side
- sometimes it is easier to work with both sides and manipulate them until they are the same
- express all functions in terms of sine and/or cosine (if all terms are in terms of the same trig function, it is often easier to work with those instead of changing to sin and cos)
- look for factoring opportunities
- often just doing the operations in the identity will help immensely
- look for squares and use the Pythagorean identities
- express all functions with the same argument

A: Prove that following identities using the reciprocal, quotient, and Pythagorean identities.

1. $\sin x \tan x = \sec x - \cos x$
2. $\csc^2 x + \sec^2 x = \csc^2 x \sec^2 x$
3. $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$
4. $\sec x - \cos x = \sin x \tan x$
5. $\dfrac{1-\sin x}{\cos x} = \dfrac{\cos x}{1+\sin x}$
6. $\dfrac{\sec x - 1}{1-\cos x} = \sec x$
7. $\dfrac{1+\tan x}{\sin x} - \sec x = \csc x$
8. $\tan x + \cot x = \dfrac{1}{\csc x \sin x}$
9. $\dfrac{\sin x - \cos x}{\cos x} + \dfrac{\sin x + \cos x}{\sin x} = \sec x \csc x$
10. $\dfrac{1+\tan^2 x}{1+\cot^2 x} = \dfrac{1-\cos^2 x}{\cos^2 x}$
11. $\tan x + \cot x = \sec^2 x \cot x$
12. $\dfrac{1}{1+\sec x} + \dfrac{1}{1-\sec x} = -2\cot^2 x$
13. $\dfrac{1-\sin^2 x\cos^2 x}{\cos^4 x} = \tan^4 x + \tan^2 x + 1$

B: Prove that following identities using related, co-related angles, Add/Sub and double angle formulas,

1. $\cos(x+y)\cos y + \sin(x+y)\sin y = \cos x$
2. $\sin x + \tan y \cos x = \dfrac{\sin(x+y)}{\cos y}$
3. $\cos\left(\dfrac{3\pi}{4} + x\right) + \sin\left(\dfrac{3\pi}{4} - x\right) = 0$
4. $\sin(x+y)\sin(x-y) = \cos^2 y - \cos^2 x$
5. $\dfrac{\tan(x-y)+\tan y}{1-\tan(x-y)\tan y} = \tan x$
6. $\dfrac{\sin(\pi - x)}{\tan(\pi + x)} \dfrac{\cot\left(\dfrac{\pi}{2} - x\right)}{\tan\left(\dfrac{\pi}{2} + x\right)} \dfrac{\cos(2\pi - x)}{\sin(-x)} = \sin x$

7. $$\frac{\csc(\pi - x)}{\sec(\pi + x)} \cdot \frac{\cos(-x)}{\cos\left(\frac{\pi}{2} + x\right)} = \cot^2 x$$

8. $$\frac{\cos\left(\frac{\pi}{2} + x\right)\sec(-x)\tan(\pi - x)}{\sec(2\pi + x)\sin(\pi + x)\cot\left(\frac{\pi}{2} - x\right)} = -1$$

9. $$\frac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi - x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{3\pi}{2} + x\right)} = \sin^4 x - \sin^2 x$$

C: Prove that following identities using a variety of formulas and identities.

1. $\sin^2 x + \cos^4 x = \cos^2 x + \sin^4 x$

2. $\cos x = \sin x \tan^2 x \cot^3 x$

3. $(\sin x + \cos x)(\tan x + \cot x) = \sec x + \csc x$

4. $\sin^3 x + \cos^3 x = (1 - \sin x \cos x)(\sin x + \cos x)$

6. $\sin(x + y) + \sin(x - y) = 2\sin x \cos y$

7. $\tan x + \tan(\pi - x) + \cot\left(\frac{\pi}{2} + x\right) = \tan(2\pi - x)$

8. $\sin\left(\frac{\pi}{2} + x\right)\cos(\pi - x)\cot\left(\frac{3\pi}{2} + x\right) = \sin\left(\frac{\pi}{2} - x\right)\sin\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} + x\right)$

9. $\tan\left(\frac{\pi}{2} - x\right) - \cot\left(\frac{3\pi}{2} - x\right) + \tan(2\pi - x) - \cot(\pi - x) = \dfrac{4 - 2\sec^2 x}{\tan x}$

10. $\csc^2\left(\frac{\pi}{2} - x\right) = 1 + \sin^2 x \csc^2\left(\frac{\pi}{2} - x\right)$

11. $\tan\left(\frac{\pi}{4} + x\right) + \tan\left(\frac{\pi}{4} - x\right) = 2\sec 2x$