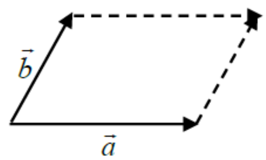
Unit: Applications of vectors (2)

Area of Parallelogram

If $\vec{a}$ and $\vec{b}$ are vectors represented by 2 non-parallel sides of a parallelogram, then area of the parallelogram is:

Area of ▱ $= \left|\vec{a} \times \vec{b}\right|$



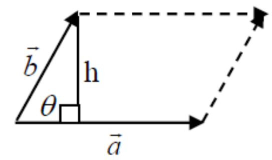
Proof:

Area of ▱ = base x height

Area $= \left|\vec{a}\right| h$

$\sin\theta = \dfrac{h}{\left|\vec{b}\right|}$ $\quad h = \left|\vec{b}\right| \sin\theta$

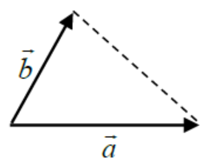
Area $= \left|\vec{a}\right|\left|\vec{b}\right| \sin\theta = \left|\vec{a} \times \vec{b}\right|$

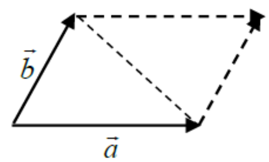


Area of Triangle

Similarly, area of triangle is given by

Area of $\Delta = \dfrac{1}{2}\left|\vec{a} \times \vec{b}\right|$





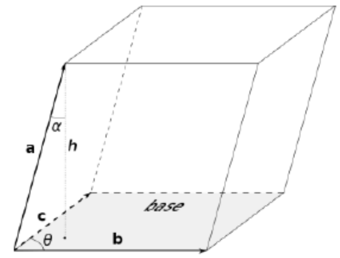
Example 1: Area of triangle using vectors

Find the area of the triangle with the given vertices. $A(7, 3, 4)$, $B(1, 0, 6)$ and $C(4, 5, -2)$

Volume of Parallelepiped

If $\vec{a}, \vec{b}, \& \vec{c}$ are vectors represented by the 3 non-parallel sides of a parallelepiped as shown in the figure, the volume of it is given by: $V = \left|\vec{a} \bullet (\vec{b} \times \vec{c})\right|$

The volume of a parallelepiped is the product of the area of its base A and its height h. The base is any of the six faces of the parallelepiped. The height is the perpendicular distance between the base and the opposite face. $V = \left|\vec{b} \times \vec{c}\right| h$.



$\cos\alpha = \dfrac{h}{|\vec{a}|}$ $\quad h = |\vec{a}|\cos\alpha$

$$\therefore V = \left|\vec{b} \times \vec{c}\right|\left|\vec{a}\right|\cos\alpha$$
$$= \vec{a} \cdot \vec{b} \times \vec{c} = \left|\vec{a} \cdot \vec{b} \times \vec{c}\right| \quad \because V > 0$$

Example 2: Volume of Parallelepiped

Find the volume of the parallelepiped determined by the vectors $\vec{a}$ = (2,-5, -1), $\vec{b}$ = (4, 0, 1), $\vec{c}$ = (3, -1, -1).

Example. 3 Prove that the vectors $\vec{a}$ = (−1,2,−7), $\vec{b}$ = (2,0,1), and $\vec{c}$ = (−7,6,0) are not coplanar.

Important!

Coplanar Test:

Recall: Coplanar - vectors on the same plane

If $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=0$ (Coplanar)

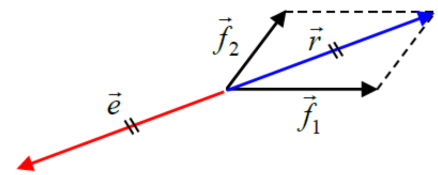
If $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\neq 0$ (Non - coplanar)

Vectors as Forces

What is force?

Force is a vector quantity with magnitude and direction. e.g. A force of 10N moving N30°E.

Resultant and Equilibrant Forces (by Parallelogram Law)



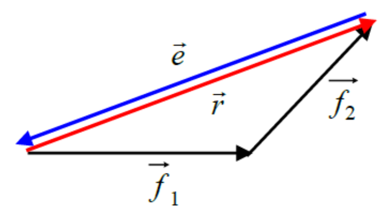
In the figure, sketch $\vec{r} = \overrightarrow{f_1} + \overrightarrow{f_2}$

$\boxed{\vec{e} = -\vec{r} \quad |\vec{e}| = |\vec{r}|}$

$\vec{r}$ is the resultant vector of $\overrightarrow{f_1}$ and $\overrightarrow{f_2}$.

$\vec{e}$ is the equilibrant vector which is equal in magnitude to $\vec{r}$ but opposite in direction.

Resultant and Equilibrant Forces (by Triangular Law)



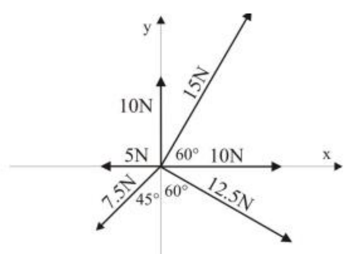
In the figure, sketch $\vec{r} = \overrightarrow{f_1} + \overrightarrow{f_2}$

$\vec{r} = \overrightarrow{f_1} + \overrightarrow{f_2}$

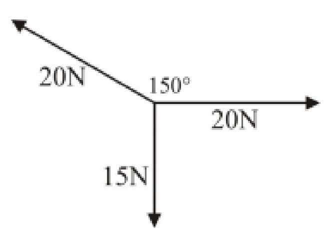
$\Rightarrow -\vec{e} = \overrightarrow{f_1} + \overrightarrow{f_2}$

$\Rightarrow \vec{0} = \overrightarrow{f_1} + \overrightarrow{f_2} + \vec{e}$

Example 4. Find the resultant of the following system of forces (magnitude and direction).



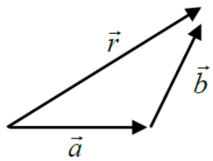
Example 5. Find an equilibrant for the following system of forces.



Velocity & Speed

Velocity is vector quantity with magnitude and direction. e.g. Velocity of 60 km/h due N60ºE.

Speed is a scalar quantity with magnitude only. e.g. Speed of 60 km/h.



Ground velocity & Resultant velocity

In flying planes, ground velocity $\Leftrightarrow$ resultant velocity.

$\vec{r} = \vec{a} + \vec{b}$ represents sum of two vectors, any 2 of the vectors are given, the third vector can be found.

Example 6. A car is traveling at $\vec{v_c}$ = 100km/ h[E], a motorcycle is traveling at $\vec{v_m}$ = 80km/h[W], a truck is traveling at $\vec{v_t}$ =120km/ h[N] and an SUV is traveling at $\vec{v_s}$ = 100km/ h[SW]. Find the relative velocity of the car relative to:

a) motorcycle

b) truck

c) SUV

Example 7. Thieves are fleeing in a stolen boat travelling at 30 km/h due west. A police boat is sent to catch them. When the stolen boat is 3 km due north of the police, the police set out at a speed of 40 km/h.

a. In what direction must the police head in order to intercept the thieves?
b. When will the interception occur?

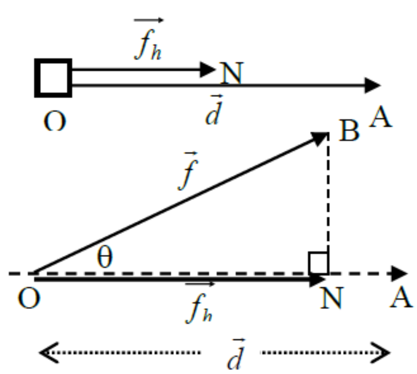
Physical Application of the Dot Product - Work

- **Work** is applied to any form of activity that requires physical exertion or mental effort.
- In physics, **work** is done when a force acting on an object causes a displacement of the object from one position to another.
- In math, **work** is defined as the **product** of the force and the parallel distance moved by it, shown in figure.
- Work is a **scalar quantity**. The unit of work is a *joule* (J)

1) $W = |\vec{f_h}||\vec{d}|$

2) $W = |\vec{f}||\vec{d}|\cos\theta$ (θ is the angle between $\vec{f}$ & $\vec{d}$)

3) $W = \vec{f} \bullet \vec{d}$



$\overrightarrow{OA} = \vec{d}$ $\quad$ $\overrightarrow{ON} = proj_{\overrightarrow{OA}}\overrightarrow{OB} = proj_{\vec{d}}\vec{f} = \vec{f_h}$

$$J = \frac{kg \cdot m^2}{s^2} = N \cdot m = Pa \cdot m^3 = W \cdot s = C \cdot V$$

where kg is the kilogram, m is the metre, s is the second, N is the newton, Pa is the pascal, W is the watt, C is the coulomb, and V is the volt.

$1\ N = 1\ kg(m/s^2)$ -- (Mass x Acceleration)

Example 8: Calculating the Work

Calculate the work done by a force $\vec{F}$ that causes a displacement $\vec{d}$, if the angle between the force and the displacement is θ. $|\vec{F}| = 14$ N, $|\vec{d}| = 6$ m, i) $\theta = 50°$ ii) $\theta = 110°$

Physical Application of the Cross Product - Magnitude of Torque - (Moment)

- When a force causes an object to turn; that is, the force causes an angular rather than a linear displacement. This turning effect of a force is called a **torque.**
- Torque is a **vector quantity**. It is measured in units of **newton metres (N-m)**
- The torque caused by a force is defined as the cross product

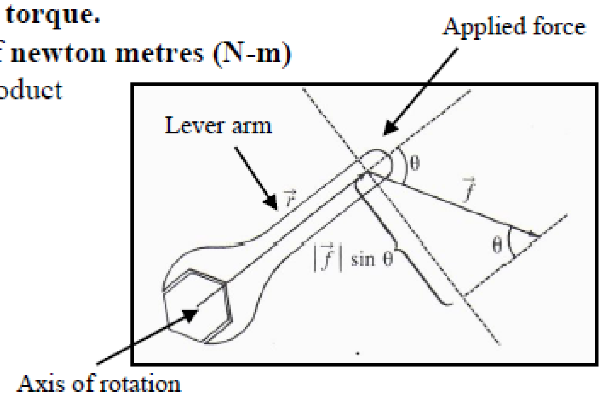
$$\vec{T} = \vec{r}\times\vec{f} = |\vec{r}||\vec{f}|\sin\theta(\hat{n})$$

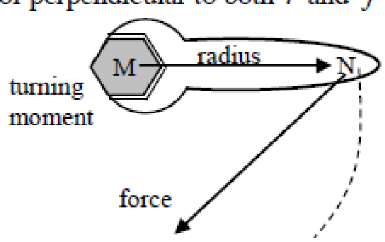
$\vec{f}$ - applied force $|\vec{f}|$ is in Newtons (N)

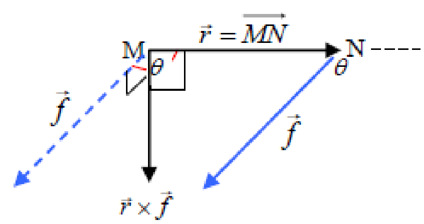
$\vec{r}$ - the vector determined by the lever arm acting from the axis of rotation. $|\vec{r}|$ is in m.

θ - the angle between the force and the level arm.

$\hat{n}$ - unit vector perpendicular to both $\vec{r}$ and $\vec{f}$.







Example 9: Calculating torque

A 50-N force is applied to the end of a 20-cm wrench and makes an angle of 30° with the handle of the wrench.

a) What is the torque on a bolt at the other end of the wrench?

b) What is the maximum torque that can be exerted by a 50-N force on the wrench and how can it be achieved?