AP Calculus Homework Two – Limit and Continuity

1.4 Other Basic Limits; 1.5 Asymptotes

1. Use the Sandwich Theorem and the fact that $\lim_{x\to 0}(|x|+1)=1$ to prove that $\lim_{x\to 0}(x^2+1)=1$.

If $x\in(-\frac{1}{2},\frac{1}{2})$, let $f(x)=x^2+1$, $g(x)=1$, $h(x)=|x|+1$; then $g(x)\le f(x)\le h(x)$ for $x\in(-\frac{1}{2},\frac{1}{2})$, and $\lim_{x\to 0}g(x)=\lim_{x\to 0}(1)=1$, $\lim_{x\to 0}h(x)=\lim_{x\to 0}(|x|+1)=1$; By the Sandwich theorem. $\lim_{x\to 0}f(x)=\lim_{x\to 0}(x^2+1)=1$.

2. Find limits.

(a) $\lim_{x\to-\infty}\dfrac{5x^3+27}{20x^2+10x+9}$

$=\lim_{x\to-\infty}\dfrac{5x^3}{20x^2}=\frac{1}{4}\lim_{x\to-\infty}(x)=\frac{1}{4}(-\infty)=-\infty$

(b) $\lim_{x\to\infty}\dfrac{2^{-x}}{2^x}$

$=\lim_{x\to\infty}\dfrac{1}{2^x\cdot 2^x}=\lim_{x\to\infty}\dfrac{1}{4^x}=\dfrac{1}{4^\infty}=\dfrac{1}{\infty}=0$

(c) $\lim_{x\to 0}\dfrac{4x^2+3x\sin x}{x^2}$ $=4\lim_{x\to 0}\dfrac{x^2}{x^2}+3\lim_{x\to 0}\dfrac{x\sin x}{x^2}$

$=4\lim_{x\to 0}(1)+3\lim_{x\to 0}\dfrac{\sin x}{x}=4+3(1)=7$

(d) $\lim_{t\to 0}\dfrac{1-\cos t}{t^{2/3}}$ $=\lim_{t\to 0}\dfrac{(1-\cos t)(1+\cos t)}{t^{2/3}(1+\cos t)}=\lim_{t\to 0}\dfrac{\sin^2 t}{t^{2/3}(1+\cos t)}$

$=\lim_{t\to 0}\sin t\ \lim_{t\to 0}\dfrac{t^{1/3}\sin t}{t^{1/3}t^{2/3}}\ \lim_{t\to 0}\dfrac{1}{1+\cos t}=\lim_{t\to 0}\dfrac{\sin t}{1+\cos t}\ \lim_{t\to 0}t^{1/3}\ \lim_{t\to 0}\dfrac{\sin t}{t}$

$=\dfrac{0}{1+1}(0)(1)=0$

(e) $\lim_{x\to+\infty}\left(1+\dfrac{2}{x}\right)^x$

let $\dfrac{1}{t}=\dfrac{2}{x}$, then $x=2t$, $x\to\infty \Leftrightarrow t\to\infty$

$=\lim_{t\to\infty}(1+\frac{1}{t})^{2t}$

$=\left[\lim_{t\to\infty}(1+\frac{1}{t})^t\right]^2$

$=e^2$

$\because \lim_{x\to\infty}(1+\frac{1}{x})^x=e$

or

$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

$$" \frac{\infty}{\infty} "$$

(f) $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}$ $= \lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{\sqrt{x}} = 1$

or $\lim\limits_{x\to\infty} \dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}} = \lim\limits_{x\to\infty} \dfrac{\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x}}}{\frac{\sqrt{x+1}}{\sqrt{x}}} = \lim\limits_{x\to\infty} \dfrac{\sqrt{\frac{x}{x}+\sqrt{\frac{x}{x^2}+\sqrt{\frac{x}{x^4}}}}}{\sqrt{\frac{x}{x}+\frac{1}{x}}} = \lim\limits_{x\to\infty}\dfrac{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}}{\sqrt{1+\frac{1}{x}}}$

$= \dfrac{\sqrt{1+\sqrt{0+\sqrt{0}}}}{\sqrt{1+0}} = \dfrac{1}{1} = 1$

$\infty - \infty$

(g) $\lim\limits_{x\to+\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$

$= \lim\limits_{x\to\infty} \dfrac{(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x})(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x})}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}} = \lim\limits_{x\to\infty}\dfrac{\cancel{x}+\sqrt{x+\sqrt{x}}-\cancel{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}} = \lim\limits_{x\to\infty}\dfrac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$

$= \lim\limits_{x\to\infty}\dfrac{\sqrt{x}}{\sqrt{x}+\sqrt{x}} = \dfrac{1}{2}$

3. Find a value of k such that $g(x)$ is continuous at $x = 0$.

$$g(x) = \begin{cases} \ln(x+k), & \text{if } 0 < x < 3 \\ \cos(kx), & \text{if } x \le 0 \end{cases}$$

$\therefore g(0) = \lim\limits_{x\to 0} g(x)$

$g(0) = \cos(k(0)) = \cos(0) = 1.$

$\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+}\cos(kx) = \cos(0) = 1$; $\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-}\ln(x+k) = \ln(0+k) = \ln k$

$\therefore$ let $\ln k = 1 \Rightarrow k = e$

4. Find all asymptotes for the graph of $f(x) = \dfrac{2x^2+4}{2+7x-4x^2}$.

let $2+7x-4x^2 = 0$, $\begin{matrix} 2 & & -x \\ & \times & \\ 1 & & 4x \end{matrix}$

$(2-x)(1+4x) = 0$

$\therefore$ two vertical asymptotes: $x = -\frac{1}{4}$ and $x = 2$.

$\because \lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty}\dfrac{2x^2}{-4x^2} = -\frac{1}{2}$

$\therefore y = -\frac{1}{2}$ is a horizontal asymptote.

$f(x)$ has no oblique linear asymptote.

5. Find all vertical and horizontal asymptotes for the graph of $h(x) = \dfrac{e^{-x}}{x}$.

$\because \lim\limits_{x\to 0} h(x) = \lim\limits_{x\to 0}\dfrac{e^{-x}}{x} = \dfrac{e^0}{0^+}$ or $\dfrac{e^0}{0^-} = +\infty$ or $-\infty$, $\therefore x = 0$ is a vertical asymptote

$\because \lim\limits_{x\to+\infty}\dfrac{e^{-x}}{x} = \dfrac{1}{\infty(e^{\infty})} = 0$, $\therefore y = 0$ is a horizontal asymptote.

$h(x)$ has no O.A.

6. For what vales of k will $\lim\limits_{x\to 3}\dfrac{x-3}{x^2-6x+k}$ exist?

$\because \lim\limits_{x\to 3}(x-3) = 0$, $\lim\limits_{x\to 3}(x^2-6x+k) = 9-18+k = k-9.$

If $k\neq 9$. $\lim\limits_{x\to 3}\dfrac{x-3}{x^2-6x+k} = \dfrac{0}{k-9} = 0$; If $k = 9$, $\lim\limits_{x\to 3}\dfrac{x-3}{x^2-6x+9} = \lim\limits_{x\to 3}\dfrac{x-3}{(x-3)^2} = \lim\limits_{x\to 3}\dfrac{1}{x-3} = \dfrac{1}{0^-}$ or $\dfrac{1}{0^+}$

$= -\infty$ or ∞.

DNE.

$\therefore k\in\mathbb{R}$ but $k\neq 9$

7. Show that $f(x) = \dfrac{x^2-5}{x+1}$ has a root between $x = 2$ and $x = 3$.

$\because f(2) = \dfrac{2^2-5}{2+1} = -\frac{1}{3} < 0$ and $f(3) = \dfrac{3^2-5}{3+1} = 1 > 0$

and $f(x)$ is continuous over $x\in[2, 3]$. By the intermediate value theorem. there exists at least x_0 between 2 and 3 so that $f(x_0) = 0$. x_0 is a root. Actually let $f(x) = 0 \Rightarrow x^2-5=0 \Rightarrow x = \pm\sqrt{5}$.

$\therefore x_0 = \sqrt{5}$