Algebra 1 (Number Systems and Operations)

1. Concepts on Number Systems

Number System	Base	Numerals
binary	2	0,1
ternary	3	0,1,2
quaternary	4	0,1,2,3
quinary	5	0,1,2,3,4
senary	6	0,1,2,3,4,5
septenary	7	0,1,2,3,4,5,6
octonary	8	0,1,2,3,4,5,6,7
nonary	9	0,1,2,3,4,5,6,7,8
decimal	10	0,1,2,3,4,5,6,7,8,9
undenary	11	0,1,2,3,4,5,6,7,8,9,A
duodenary	12	0,1,2,3,4,5,6,7,8,9,A,B

2. Positional Number Systems

Our decimal number system is known as a *positional* number system, because the value of the number depends on the position of the digits. For example, the number 123 has a very different value than the number 321, although the same digits are used in both numbers.
(Although we are accustomed to our decimal number system, which is positional, other ancient number systems, such as the Egyptian number system were not positional, but rather used many additional symbols to represent larger values.)

In a positional number system, the value of each digit is determined by which place it appears in the full number. The lowest place value is the rightmost position, and each successive position to the left has a higher place value.

In our decimal number system, the rightmost position represents the "ones" column, the next position represents the "tens" column, the next position represents "hundreds", etc. Therefore, the number 123 represents 1 hundred and 2 tens and 3 ones, whereas the number 321 represents 3 hundreds and 2 tens and 1 one.

The values of each position correspond to powers of the base of the number system. So for our decimal number system, which uses base 10, the place values correspond to powers of 10:

...	1000	100	10	1
...	10^3	10^2	10^1	10^0

3. a) Converting from other number bases to decimal

Other number systems use different bases. The binary number system uses base 2, so the place values of the digits of a binary number correspond to powers of 2. For example, the value of the binary number 10011 is determined by computing the place value of each of the digits of the number:

1	0	0	1	1	the binary number
2^4	2^3	2^2	2^1	2^0	place values

So the binary number 10011 represents the value

$(1 * 2^4) + (0 * 2^3) + (0 * 2^2) + (1 * 2^1) + (1 * 2^0)$
$= 16 + 0 + 0 + 2 + 1$

The same principle applies to any number base. For example, the number 2132 base 5 corresponds to

2	1	3	2	number in base 5
5^3	5^2	5^1	5^0	place values

So the value of the number is

$$
\begin{aligned}
& (2 * 5^3) + (1 * 5^2) + (3 * 5^1) + (2 * 5^0) \\
= \; & (2 * 125) + (1 * 25) + (3 * 5) + (2 * 1) \\
= \; & 250 + 25 + 15 + 2 \\
= \; & 292
\end{aligned}
$$

b) Converting from decimal to other number bases

In order to convert a decimal number into its representation in a different number base, we have to be able to express the number in terms of powers of the other base. For example, if we wish to convert the decimal number 100 to base 4, we must figure out how to express 100 as the sum of powers of 4.

$$
\begin{aligned}
100 &= (1 * 64) + (2 * 16) + (1 * 4) + (0 * 1) \\
&= (1 * 4^3) + (2 * 4^2) + (1 * 4^1) + (0 * 4^0)
\end{aligned}
$$

Then we use the coefficients of the powers of 4 to form the number as represented in base 4:

100 = 1 2 1 0 base 4

One way to do this is to repeatedly divide the decimal number by the base in which it is to be converted, until the quotient becomes zero. As the number is divided, the remainders - in reverse order - form the digits of the number in the other base.

Example: Convert the decimal number 82 to base 6:

82/6 = 13 remainder 4

13/6 = 2 remainder 1

2/6 = 0 remainder 2

The answer is formed by taking the remainders in reverse order: 2 1 4 base 6 .

4. Rules of Exponents

Rules of exponents 1:

To multiply numbers with the same base, add the exponents and keep the base the same.

$$a^m \cdot a^n = a^{m+n}$$

Example:

$$2^2 \times 2^3 = (2\times 2)\times(2\times 2\times 2) = 2\times 2\times 2\times 2\times 2 = 2^5$$

Rules of exponents 2:

When raising a power of a number to a power, multiply the exponents and keep the base the same.

$$(a^m)^n = a^{mn}$$

Example:

$$(a^2)^3 = (a\times a)\times(a\times a)\times(a\times a) = a^6$$

Rules of exponents 3:

When dividing two exponential numbers, subtract the powers.

$$\frac{a^m}{a^n} = a^{m-n}$$

Example:

$$\frac{a^5}{a^2} = \frac{a\times a\times a\times a\times a}{a\times a} = a\times a\times a = a^3$$

Rules of exponents 4:

Any exponential number divided by itself is equal to one.

$$\frac{a^n}{a^n} = 1$$

Example:

$$\frac{a^3}{a^3} = \frac{a \times a \times a}{a \times a \times a} = 1$$

Rules of exponents 5:

To raise a product to a power, raise each factor to that power.

$$(ab)^n = a^n b^n$$

Example:

$$(5 \times 6)^2 = (5 \times 6) \times (5 \times 6) = 5^2 \times 6^2 = 61$$

Rules of exponents 6:

To raise a quotient to a power, raise both the numerator and denominator to that power.

$$(a/b)^n = a^n / b^n$$

Example:

$$\frac{6^3}{4^3} = \frac{6 \times 6 \times 6}{4 \times 4 \times 4} = \frac{27}{8}$$

Any number to the zero power equals one.
The rules for positive exponents apply to negative exponents.
The rules for integer exponents apply to fractional exponents.

5. The Order of Operations

In Mathematics, the order in which mathematical problems are solved is extremely important.

Rules

1) Calculations must be done from left to right.

2) Calculations in brackets (parenthesis) are done first. When you have more than one set of brackets, do the inner brackets first.

3) Exponents (or radicals) must be done next.
4) Multiply and divide in the order the operations occur.
5) Add and subtract in the order the operations occur.

Example 1: Simplify $-\{2x - [3 - (4 - 3x)] + 6x\}$.

$$
\begin{aligned}
&-\{2x - [3 - (4 - 3x)] + 6x\} \\
&\quad = -1\{2x - 1[3 - 1(4 - 3x)] + 6x\} \\
&\quad = -1\{2x - 1[3 - 4 + 3x] + 6x\} \\
&\quad = -1\{2x - 1[-1 + 3x] + 6x\} \\
&\quad = -1\{2x + 1 - 3x + 6x\} \\
&\quad = -1\{2x + 6x - 3x + 1\} \\
&\quad = -1\{5x + 1\} = -5x - 1
\end{aligned}
$$

Example 2: Simplify the following: $6+\frac{16-4}{2^2+2}-2$

First, you must simplify inside the "understood" parentheses. Since 16 – 4 = 12, then:

$$
\begin{aligned}
6+\frac{16-4}{2^2+2}-2 &= 6+\frac{12}{4+2}-2 \\
&= 6+\frac{12}{6}-2 \\
&= 6+2-2 \\
&= 6
\end{aligned}
$$

Do not try to cancel the 4 or 2 into the 12; instead, first add the 4 and 2 to get 6. Only then can you cancel.

Example 3: Calculate $\frac{2^2\times4^2\times6^2}{20\times40\times60}$

First, you must simplify $20\times40\times60=2\times4\times6\times(10)^3$, and then according to rules of exponents 3:

Finally, $\frac{2^2\times4^2\times6^2}{20\times40\times60}=\frac{(2\times4\times6)^2}{2\times4\times6\times(10)^3}=\frac{48}{1000}=0.048$

6. Radicals

Rational number (Q) – can be expressed as a fraction $\frac{m}{n}$, where m and n are integers and n is not zero. Its decimal form is either terminating or repeating.

Ex: $\frac{1}{2}$ 0.68 7

Irrational number (I) – a number that is not rational, a decimal that does not terminate or repeat.

Ex: π $\sqrt{2}$

Radical - denotes a root

$\sqrt{}$ "square root"

$\sqrt[3]{}$ "cube root"

$\sqrt[4]{}$ "fourth root"

$\sqrt[n]{}$ "n^{th} root"

$\sqrt{9} = \sqrt{3^2} = 3$

$\sqrt{x^2} = x$ $\sqrt[3]{x^3} = x$ $\sqrt[4]{x^4} = x$ $\sqrt[n]{x^n} = x$

Rationalizing the Denominator

The process of rewriting a square root radical expression as an equivalent expression in which the denominator no longer contains any radicals is called **rationalizing the denominator.**

- First, simplify any radicals.
- Secondly, multiply the numerator and denominator by the radical factor that remains.

Example 1: Rationalize the denominator of $\frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Example 2: Rationalize the denominator of $\frac{2}{\sqrt{3}}$

$$\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Example 3: Rationalize the denominator of $\frac{3}{2\sqrt{5}}$

$$\frac{3}{2\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{2 \cdot 5} = \frac{3\sqrt{5}}{10}$$

Conjugates

Radical expressions that involve the sum and difference of the same two terms are called **conjugates.** Examples are $\sqrt{2}+\sqrt{5}$ and $\sqrt{2}-\sqrt{5}$ or $3-\sqrt{x}$ and $3+\sqrt{x}$.

The product of two conjugates will contain no radicals!

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)=\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2=a-b$$

In radical expressions with a binomial (two terms) in the denominator, to rationalize the denominator, multiply numerator and denominator by the conjugate of the denominator.

Example 1: Rationalize the denominator of $\dfrac{7}{4-\sqrt{3}}$.

$$\frac{7}{4-\sqrt{3}}\cdot\frac{4+\sqrt{3}}{4+\sqrt{3}}$$

$$=\frac{7(4+\sqrt{3})}{(4-\sqrt{3})(4+\sqrt{3})}$$

$4+\sqrt{3}$ is the conjugate of $4-\sqrt{3}$

$$=\frac{28+7\sqrt{3}}{16-3}$$

$$=\frac{28+7\sqrt{3}}{13}$$

7. Rational Exponents

Definition of $a^{\frac{1}{n}}$ - If $\sqrt[n]{a}$ represents a real number, where $n \geq 2$ is an integer, then $a^{\frac{1}{n}}=\sqrt[n]{a}$.

$$a^{\frac{1}{n}}=\sqrt[n]{a}$$

The denominator of the rational exponent
becomes the index of the radical.

Example 1: Evaluate each, if it exists.

$a)\quad 9^{\frac{1}{2}} = \sqrt{9} = 3$

$b)\quad 125^{\frac{1}{3}} = \sqrt[3]{125} = 5$

$c)\quad \left(\frac{81}{16}\right)^{-\frac{1}{4}} = \left(\frac{16}{81}\right)^{\frac{1}{4}} = \sqrt[4]{\frac{16}{81}} = \frac{2}{3}$

$d)\quad -1^{\frac{1}{8}} = -1$

$e)\quad (-4)^{\frac{1}{2}} = \sqrt{-4}$

Definition of $a^{\frac{m}{n}}$ - If $\sqrt[n]{a}$ represents a real number and $\frac{m}{n}$ is a positive rational number, $n \geq 2$, then $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ *or* $\left(\sqrt[n]{a}\right)^m$. It can be evaluated or simplified by finding the power first, then the root or by finding the root first, then the power.

The numerator is the exponent.

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \; or \; \sqrt[n]{a^m}$$

The denominator is the index.

Example 1: Evaluate, if possible.

$a)\quad 36^{\frac{3}{2}} = \left(\sqrt{36}\right)^3 = 6^3 = 216$

$b)\quad 8^{\frac{4}{3}} = \left(\sqrt[3]{8}\right)^4 = 2^4 = 16$

$c)\quad 16^{\frac{3}{4}} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$

Example 2: Evaluate, if possible.

$a)\quad 4^{-\frac{3}{2}} = \left(\frac{1}{4}\right)^{\frac{3}{2}} = \left(\sqrt[2]{\frac{1}{4}}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$b)\quad (-32)^{-\frac{2}{5}} = \left(\frac{-1}{32}\right)^{\frac{2}{5}} = \left(\sqrt[5]{-\frac{1}{32}}\right)^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

Example 3: Use the properties of exponents to simplify.

a) $\left(\dfrac{-8}{a^3b^6}\right)^{-\frac{2}{3}} = \left(\sqrt[3]{\left(\dfrac{a^3b^6}{-8}\right)}\right)^2 = \left(\dfrac{ab^2}{-2}\right)^2 = \dfrac{a^2b^4}{4}$

b) $\dfrac{9^{\frac{3}{2}}}{(8x^3)^{-\frac{1}{3}}} = \dfrac{3^3}{\left(\dfrac{1}{8x^3}\right)^{\frac{1}{3}}} = \dfrac{27}{\dfrac{1}{2x}} = 54x$

c) $(64x^6y^{-12})^{\frac{5}{6}}(x^6y^{-3})^{\frac{2}{3}} = \left(2xy^{-2}\right)^5\left(x^2y^{-1}\right)^2 = \left(32x^5y^{-10}\right)\left(x^4y^{-2}\right) = 32x^9y^{-12}$

Questions in class

1. If $a \circ b = ab - 1$ and $a \bullet b = a + b^2$, find $5 \circ (4 \bullet 3)$

2. If $\frac{2^2 \cdot 3^3 \cdot 5^5 \cdot 7^7}{20 \cdot 30 \cdot 50 \cdot 70} = 2^a \bullet 3^b \bullet 5^c \bullet 7^d$ for integers a, b, c, and d, find the value of the sum $a + b + c + d$.

3. If n is a positive integer less than 10,000, for how many values of n is $\sqrt[3]{4n}$ an integer?

4. Simplify the expression: $\dfrac{1+\dfrac{1}{1+b}}{1-\dfrac{1}{1+b}}$.

5. Rationalize the denominator and write your answer in lowest terms.

a) $\dfrac{9\sqrt{20}-\sqrt{15}}{\sqrt{5}}$

b) $\dfrac{2\sqrt{6}+\sqrt{3}}{\sqrt{6}-\sqrt{3}}$

6. Simplify $\dfrac{\dfrac{x^2}{y}+\dfrac{y^2}{x}}{y^2-xy+x^2}$

7. Five positive consecutive integers starting with a have average b. What is the average of 5 consecutive integers that start with b, in terms of a?

8. A binary operation $\diamondsuit$ has the properties that $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$ and that $a \diamondsuit a = 1$ for all nonzero real numbers a, b, and c. (Here . represents multiplication). The solution to the equation $2016 \diamondsuit (6 \diamondsuit x) = 100$ can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is p + q?