Coordinates (Analytical Geometry)

Introduction

Problems involving the use of coordinates in 2 dimensions are commonplace on many mathematics contests.

For some contests which require full solutions, such problems are generally quite approachable since they lend themselves to step-by-step development of the solutions rather than requiring major insights.

In fact the insightful aspects in the use of a coordinate or analytic approach usually come with the decision of whether or not to attempt a solution using these techniques.

Some useful formulae include:

1. $y = mx + b$: The equation of the line with slope m and y intercept b.

2. The coordinates of the midpoint M of the segment with endpoints A(x_1 , y_1) and B(x_2 , y_2) are: $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$

3. The distance D= $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ between points $A(x1, y1)$ and $B(x2 ,y2)$.

4. $Ax + By + C = 0$: The standard form for a line with slope $-\frac{A}{B}$ and intercepts $-\frac{C}{A}$ *and* $-\frac{C}{B}$

5. $(y- y_0) = m(x- x_0)$: The point-slope equation of the line with slope m through the point P (x_0, y_0).

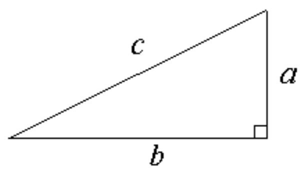
6. $\frac{x}{a}+\frac{y}{b}=1$: The intercept form of the equation of the line with intercepts a and b.

7. The distance: D= $\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2+B^2}}$ between the line $Ax + By + C = 0$ and the point P (x_0, y_0).

8. $(x-h)^2 +(y - k)^2 = r^2$: The equation of the circle centre (h, k) and radius r.

1. Distance Formula

Recall Pythagoras' Theorem:



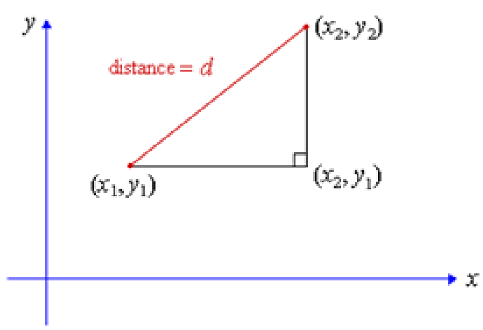
For a right-angled triangle with hypotenuse length *c*,

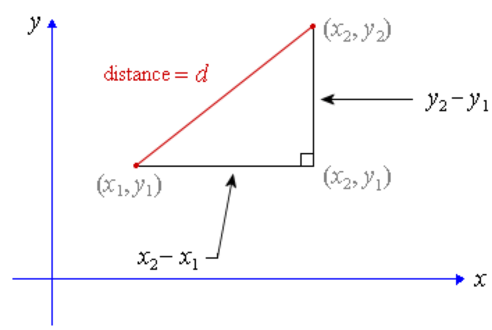
$$c = \sqrt{a^2 + b^2}$$

We use this to find the distance between any two points (x_1, y_1) and (x_2, y_2) on the **Cartesian plane**:

The Cartesian plane was named after Rene Descartes. It is also called the ***x* - *y* plane**.

See more about Descartes in Functions and Graphs.





The point (x_2, y_1) is at the right angle. We can see that:

- The distance between the points (x_1, y_1) and (x_2, y_1) is simply $x_2 - x_1$ and
- The distance between the points (x_2, y_2) and (x_2, y_1) is simply $y_2 - y_1$.

Using Pythagoras' Theorem we have the distance between (x_1, y_1) and (x_2, y_2) given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Find the distance between the points (3, -4) and (5, 7).
Here, $x_1 = 3$ and $y_1 = -4$; $x_2 = 5$ and $y_2 = 7$

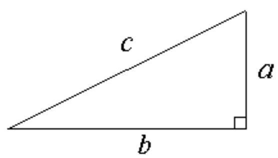
So the distance is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(5-3)^2 + (7-(-4))^2} = \sqrt{4+212} = 11.18$$

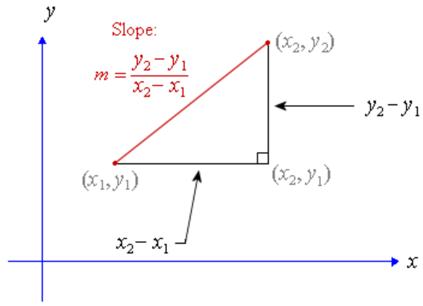
2. Gradient (or slope)

The **gradient** of a line is defined as $\dfrac{\textit{vertical rise}}{\textit{horizontal run}}$



In this triangle, the gradient of the line is given by: $\dfrac{a}{b}$

In general, for the line joining the points (x_1, y_1) and (x_2, y_2):

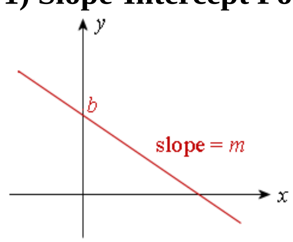


We see from the diagram above, that the **gradient** (usually written *m*) is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

3. The Straight Line

1) Slope-Intercept Form of a Straight Line

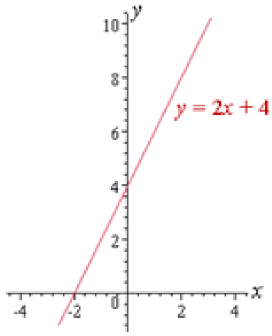


The slope-intercept form (otherwise known as "gradient, *y*-intercept" form) of a line is given by:

$\boldsymbol{y = mx + b}$

For example:

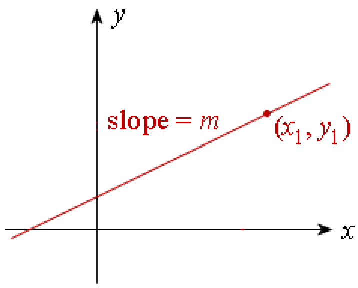
This tells us the slope of the line is *m* and the *y*-intercept of the line is *b*.



The line $y = 2x + 4$ has slope $m = 2$ and y-intercept $b = 4$.

We do not need to set up a table of values to sketch this line. Starting at the y-intercept ($y = 4$), we sketch our line by going up 2 units for each unit we go to the right (since the slope is 2 in this example). To find the x-intercept, we let $y = 0$.
$2x + 4 = 0$
$x = -2$

2) Point-Slope Form of a Straight Line

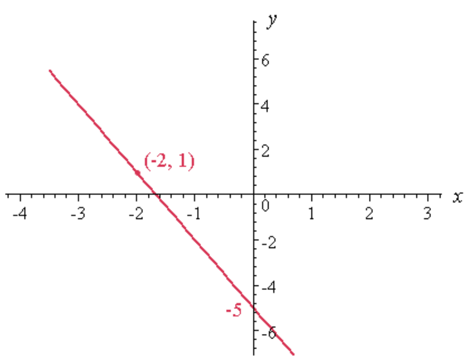


We need other forms of the straight line as well. A useful form is the **point-slope form** (or point - gradient form). We use this form when we need to find the equation of a line passing through a point (x_1, y_1) with slope m:

$$y_2 - y_1 = m(x_2 - x_1)$$

For example:

Find the equation of the line that passes through (-2, 1) with slope of -3.



We use: $y_2 - y_1 = m(x_2 - x_1)$
Here, $x_1 = -2$, $y_1 = 1$, $m = -3$

So the required equation is:
$y - 1 = -3(x - (-2)) = -3x - 6$
$y = -3x - 5$
We have left it in slope-intercept form. We can see the slope is -3 and the y-intercept is -5.

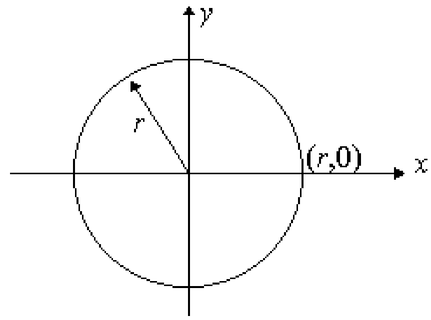
3) General Form

Another form of the straight line which we come across is **general form:**
***Ax* + *By* + *C* = 0**

It can be useful for drawing lines by finding the y-intercept (put $x = 0$) and the x-intercept (put $y = 0$).

4. The Circle

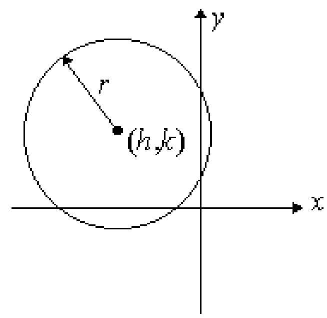
Definition 1:



If A circle is the set of points equidistant from a point C(0, 0) called the center. The fixed distance r from the center to any point on the circle is called the radius.
Than the standard equation of a circle with center C(0, 0) and radius r is as follows:

$$x^2 + y^2 = r^2$$

Definition 2:



If A circle is the set of points equidistant from a point C(h, k) called the center. The fixed distance r from the center to any point on the circle is called the radius.

The standard equation of a circle with center C(h, k) and radius r is as follows:

$$\left(x-h\right)^2 + \left(y-k\right)^2 = r^2$$

Example 1: Find the equation of a circle whose center is at (2, - 4) and radius 5.

Solution:
given (h , k) = (2 , - 4) and r = 5

substitute h, k and r in the standard equation

$(x - 2)^2 + (y - (- 4))^2 = 5^2$
$(x - 2)^2 + (y + 4)^2 = 25$

Example 2: Find the equation of the circle with centre (3/2, -2) and radius 5/2.

Centre (3/2, -2) radius 5/2.

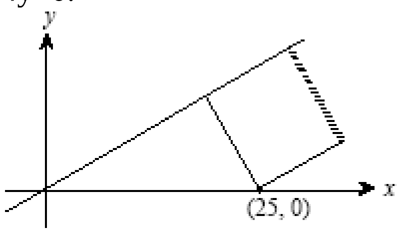
$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-\frac{3}{2})^2 + (y-(-2))^2 = \left(\frac{5}{2}\right)^2$$

$$(x-\frac{3}{2})^2 + (y+2)^2 = \left(\frac{5}{2}\right)^2$$

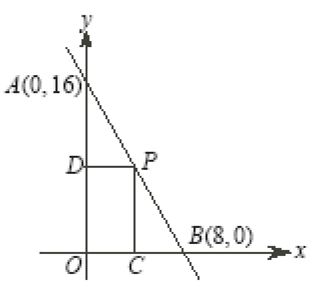
► **In-class questions (Please do the questions again)**

1. Determine the vertices of all squares that have one vertex at (25, 0) and one side along the line 3x-4y=0.



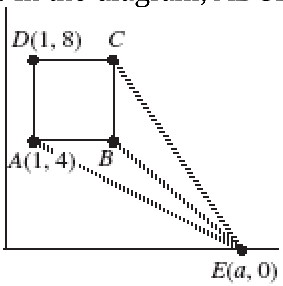
2. Calculate the area of the triangle A(1, 3), B(7, 11), C(9, 8).

3. In the diagram, a line is drawn through the points A(0, 16) and B(8, 0). Point P is chosen in the first quadrant on the line through A and B. Points C and D are then chosen on the x-axis and y-axis, respectively, so that PDOC is a rectangle.



(a) Determine the equation of the line through A and B.
(b) Determine the coordinates of the point P so that PDOC is a square.
(c) Determine the coordinates of all points P that can be chosen so that the area of rectangle PDOC is 30.

4. In the diagram, *ABCD* is a square and the coordinates of *A* and *D* are as shown.



(a) The point *E* (a, 0) is on the x-axis so that the triangles *CBE* and *ABE* lie entirely outside the square *ABCD*. For what value of a is the sum of the areas of triangles *CBE* and *ABE* equal to the area of square *ABCD*?

(b) The point *F* is on the line passing through the points *M* (6, –1) and *N* (12, 2) so that the triangles *CBF* and *ABF* lie entirely outside the square *ABCD*. Determine the coordinates of the point *F* if the sum of the areas of triangle *CBF* and *ABF* equals the area of square *ABCD*.

Extension
Find the set of all points $P(x, y)$ which satisfy the conditions that the triangles *CBP* and *ABP* lie entirely outside the square *ABCD* and the sum of the areas of triangles *CBP* and *ABP* equals the area of square *ABCD*.

5. In the diagram, line segment FCG passes through vertex C of square ABCD, with F lying on AB extended and G lying on AD extended. Prove that $\frac{1}{AB} = \frac{1}{AF} + \frac{1}{AG}$.