AP Calculus Class 12

Trigonometric Integrals.

Example: $\int \cos^3 x \, dx \longrightarrow$ substitution.

$\cos^2 x \cdot \cos x$

$\cos^3 x = \underbrace{\cos^2 x}_{(1-\sin^2 x)} \cos x$

$\sin^2 x + \cos^2 x = 1.$

$= (1-\sin^2 x)\cos x.$

$\int (1-\sin^2 x)\cos x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$= \int (1-u^2)\, du = \int 1 \, du - \int u^2 \, du$

$= u - \frac{1}{3}u^3 + C$

$= \sin x - \frac{1}{3}\sin^3 x + C.$

Example: $\int \sin^5 x \cos^2 x \, dx$

$\sin^5 x \cos^2 x = \sin^4 x \cos^2 x \sin x$

$= (\sin^2 x)^2 \cos^2 x \sin x$

$= (1-\cos^2 x)^2 \cos^2 x \sin x$

let $u = \cos x$ $\quad du = -\sin x \, dx$

$$\int \sin^5 x \cos^2 x \, dx = \int (1-\cos^2 x)^2 \cos^2 x \sin x \, dx.$$

$$\Rightarrow \int (1-u^2)^2 u^2 (-du)$$

$$= -\int (1-u^2)^2 u^2 \, du$$

$$= -\int (u^2 - 2u^4 + u^6) \, du$$

$$= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C.$$

$$= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C.$$

Example: $\int \sin^4 x \, dx$

$\sin^4 x = (\sin^2 x)^2$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x).$

$$\Rightarrow \int \sin^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1-\cos 2x)^2 dx.$$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x \, dx.$$

$\cos^2 2x = \frac{1}{2}(1+\cos 4x)$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right) dx.$$

$$= \frac{1}{4}\int \frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x \, dx.$$

$$= \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x \right] + C.$$

$\int \sin^m x \cos^n x \, dx.$

a) Power of $\cos x$ is odd $(n = 2k+1)$

- Save a cos factor, then use $\cos^2 x = 1 - \sin^2 x$
- Express the remaining factors in sine.

$$\int \sin^m x \cos^n x \, dx = \int \sin^m x \cos^{(2k+1)} x \, dx.$$

$$= \int \sin^m x \cos^{2k} x \cos x \, dx$$

$$= \int \sin^m x (\cos^2 x)^k \cos x \, dx.$$

$$= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx$$

Then $u = \sin x$

b) If the power of sine is odd, $(m = 2k+1)$

Apply the same idea from before and pull out a sine factor.

Use the identity $\sin^2 x = 1 - \cos^2 x.$

let $u = \cos x$.

c) If both m and n are even, then use the following trig identities.

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$.

Example: $\int \tan^6 x \sec^4 x \, dx$.

$\sec^2 x = 1 + \tan^2 x$

let $u = \tan x$ $\quad\quad du = \sec^2 x \, dx$.

$\tan^6 x (\sec^2 x)(\sec^2 x) = \tan^6 x (1+\tan^2 x) \sec^2 x$

$$\int \tan^6 x (\sec^2 x)(\sec^2 x) \, dx = \int \tan^6 x (1+\tan^2 x) \sec^2 x \, dx.$$

$$= \int u^6 (1+u^2) \, du$$

$$= \int u^6 + u^8 \, du = \frac{u^7}{7} + \frac{u^9}{9} + C.$$

$$= \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C.$$

Example: $\int \tan^5\theta \sec^7\theta \, d\theta$

Separate $\tan\theta \sec\theta$ factor out.

$$\int \tan^5\theta \sec^7\theta \, d\theta = \int \tan^4\theta \sec^6\theta \tan\theta \sec\theta \, d\theta$$

let $u = \sec\theta$ $\quad du = \sec\theta \tan\theta \, d\theta$

$$= \int (\sec^2\theta - 1)^2 \sec^6\theta \sec\theta \tan\theta \, d\theta$$

$$= \int (u^2-1)^2 u^6 \, du$$

$$= \int u^{10} - 2u^8 + u^6 \, du$$

$$= \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C.$$

$$= \frac{1}{11}\sec^{11}\theta - \frac{2}{9}\sec^9\theta + \frac{1}{7}\sec^7\theta + C.$$

$\int \tan^m x \sec^n x \, dx$

a) If the power of sec x is even. $(n=2k)$

- Save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$.

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx.$$

$$= \int \tan^m x (1+\tan^2 x)^{k-1} \sec^2 x \, dx.$$

$u = \tan x$, $\quad du = \sec^2 x \, dx$

b) The power of $\tan x$ is odd $(m = 2k+1)$.

- Save a factor of $\sec x \tan x$.

  Use $\tan^2 x = \sec^2 x - 1$.

  Let $u = \sec x$.

$$\int \sec^3 x \, dx = \int \sec^2 x \sec x \, dx.$$

$\sec^3 x = \sec^2 x \sec x.$

Let $f = \sec x$ $\qquad g' = \sec^2 x.$

$f' = \sec x \tan x$ $\qquad g = \tan x.$

$$\begin{aligned}
\int \sec^2 x \sec x \, dx &= \sec x \tan x - \int \sec x \tan^2 x \, dx. \\
&= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx \\
&= \sec x \tan x - \left[\int \sec^3 x \, dx - \int \sec x \, dx \right]
\end{aligned}$$

$$\Rightarrow 2 \int \sec^2 x \sec x \, dx = \sec x \tan x + \int \sec x \, dx.$$

$\int \sec x \, dx =$
$\ln|\sec x + \tan x| + C.$

$$= \int \sec^3 x \, dx = \frac{1}{2} (\sec x \tan x + \ln|\sec x + \tan x| + c).$$

$$\int \sin 4x \cos 5x \, dx.$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right].$$

$$\int fg' = fg - \int gf' \qquad \int e^x \sin x \, dx$$

Integration by Partial Fraction Decomposition.

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx.$$

$$= 2\int \frac{1}{x-1}\,dx - \int \frac{1}{x+2}\,dx.$$

$$= 2\ln|x-1| - \ln|x+2| + C.$$

$$f(x) = \frac{P(x)}{Q(x)}$$ polynomial funⁿ.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x_1 + a_0 \qquad a_n \neq 0$$

$\deg(P) = n.$

$\deg(P) < \deg(Q)$, this rational funⁿ is proper.

$\deg(P) \geq \deg(Q)$, this " " is improper.

Example: $\int \frac{x+5}{x^2+x-2}\,dx.$ $\qquad x^2+x-2=(x+2)(x-1)$

$$\frac{x+5}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$\Rightarrow \frac{x+5}{x^2+x-2} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$$\Rightarrow A(x-1)+B(x+2) = x+5.$$

$$Ax-A+Bx+2B = x+5$$

$$(A+B)x+(2B-A) = x+5 \longrightarrow \text{Compare coefficients.}$$

$$\Rightarrow A+B=1, \quad 2B-A=5,$$

$$\Rightarrow B=2, \quad A=-1$$

$$\Rightarrow \frac{x+5}{x^2+x-2} = -\frac{1}{x+2} + \frac{2}{x-1}$$

Example: Evaluate $\int \frac{x+18}{3x^2+17x+20}\,dx.$

$$3x^2+17x+20 = (3x+5)(x+4)$$

$$\frac{x+18}{3x^2+17x+20} = \frac{A}{3x+5} + \frac{B}{x+4}$$

$$= \frac{A(x+4)+B(3x+5)}{(3x+5)(x+4)}$$

$\Rightarrow A(x+4) + B(3x+5) = x+18.$

$Ax + 4A + 3Bx + 5B = x + 18.$

$(A+3B)x + (4A+5B) = x+18.$

$\Rightarrow A+3B=1, \quad 4A+5B=18.$

$A = 7 \qquad B = -2.$

$$\int \frac{x+18}{(3x+5)(x+4)}\,dx = \int \frac{7}{3x+5} - \frac{2}{x+4}\,dx.$$

$$= 7\int \frac{1}{3x+5}\,dx - 2\int \frac{1}{x+4}\,dx.$$

$\downarrow$ let $u = 3x+5$, $du = 3dx$, $\frac{1}{3}du = dx$

$\downarrow$ $v = x+4$, $dv = dx$

$$= 7 \cdot \int \frac{1}{u} \cdot \frac{1}{3}\,du - 2\int \frac{1}{v}\,dv$$

$$= \frac{7}{3}\ln|u| - 2\ln|v| + C.$$

$$= \frac{7}{3}\ln|3x+5| - 2\ln|x+4| + C$$

Example: $\int \frac{2x+4}{(x-1)^2}\,dx.$ $\qquad (x-1)^2 = (x-1)(x-1)$

$$\frac{2x+4}{(x-1)^2} \neq \frac{A}{x-1} + \frac{B}{x-1} = \frac{A+B}{x-1}$$

$$\neq \frac{A}{(x-1)^2}$$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$= \frac{A(x-1)+B}{(x-1)^2}$$

$$\Rightarrow A(x-1)+B = 2x+4$$

$$\Rightarrow Ax - A + B = 2x+4.$$

$$A=2, \quad B=6.$$

$$\Rightarrow \int \frac{2x+4}{(x-1)^2}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{6}{(x-1)^2}\,dx.$$

$$= 2\ln|x-1| + \frac{6}{x-1} + C.$$

$$\frac{2x+4}{x^2(x-1)^3} \qquad x^2(x-1)^3 = x^2 \cdot (x-1)^3$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

Homework 11.

4. $\int x \sec^2 x \, dx$.

let $f = x \quad g' = \sec^2 x$

$f' = 1 \quad g = \tan x$.

$$\int x \sec^2 x \, dx = x \tan x - \int 1 \cdot \tan x \, dx.$$

$$= x \tan x + \ln|\cos x| + C$$

E.

5. $\int x \sin 2x \, dx$

let $f = x \quad g' = \sin 2x$

$f' = 1 \quad g = -\frac{1}{2} \cos 2x$.

$$\int x \sin 2x \, dx = -\frac{1}{2} x \cos 2x + \int \frac{1}{2} \cos 2x \, dx$$

$$= -\frac{1}{2} x \cos 2x + \frac{1}{2} \sin 2x + C.$$

A.

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1. j) $\int_0^t e^s \sin(t-s) \, ds$.

let $f = \sin(t-s) \quad g' = e^s$

$f' = -\cos(t-s) \quad g = e^s$.

$$= e^s \sin(t-s)\Big]_0^t + \int_0^t e^s \cos(t-s)\,ds.$$

$$= [e^t \sin(\cancel{t-t})^{0} - e^0(\sin t)] + \int_0^t e^s \cos(t-s)\,ds.$$

$$= -\sin t + \underbrace{\int_0^t e^s \cos(t-s)\,ds.}$$

$$f = \cos(t-s) \qquad g' = e^s$$

$$f' = +\sin(t-s) \qquad g = e^s$$

$$\Rightarrow \int_0^t e^s(\cos(t-s))\,ds = [e^s \cos(t-s)]_0^t - \int_0^t e^s \sin(t-s)\,ds$$

$$= e^t - \cos t - \int_0^t e^s \sin(t-s)\,ds.$$

$$\Rightarrow \int_0^t e^s \sin(t-s)\,ds = -\sin t + e^t - \cos t - \int_0^t e^s \sin(t-s)\,ds.$$

$$\Rightarrow 2\int_0^t e^s \sin(t-s)\,ds = -\sin t + e^t - \cos t.$$

$$\Rightarrow \int_0^t e^s \sin(t-s)\,ds = \frac{1}{2}(e^t - \sin t - \cos t).$$

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1. i) $\displaystyle\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr.$

$$f = r^2 \qquad g' = \frac{r}{\sqrt{4+r^2}}$$

$$f' = 2r \qquad g = \sqrt{4+r^2}$$

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \left[r^2\sqrt{4+r^2} \right]_0^1 - \int_0^1 2r\sqrt{4+r^2}\,dr.$$

$$= \sqrt{5} - 2\underbrace{\int_0^1 r\sqrt{4+r^2}\,dr}.$$

let $u = 4+r^2$ $\quad u(0) = 4$

$du = 2r\,dr$ $\quad u(1) = 5,$

$\frac{1}{2}du = r\,dr.$

$$\frac{1}{2}\int_4^5 \sqrt{u}\,du = \frac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^5 = \frac{1}{3}\left[u^{\frac{3}{2}}\right]_4^5 = \frac{1}{3}(5\sqrt{5}-8)$$

$$\Rightarrow \int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \sqrt{5} - \frac{2(5\sqrt{5}-8)}{3}$$

$$= \sqrt{5} + \frac{16-10\sqrt{5}}{3} = \frac{16-7\sqrt{5}}{3}$$