Lesson 3-*Unit 2 – Derivatives(1)*

The Derivative

In our study of limits and rates of change, we saw that the slope of the tangent line to the graph of $y=f(x)$ at the point P(a, $f(a)$) is calculated by finding the limit of the **difference quotient**, and this is defined as

$$\lim_{h\to 0}\frac{\Delta y}{\Delta x} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

provided this limit exists.

The slope of the tangent line is used to find the **instantaneous rate of change** of y with respect to x at $x=a$.

In calculus, this limit is called the **derivative** of $f(x)$ at $x=a$. The process of finding the derivative is called **differentiation**.

By definition, the derivative, $f'(x)$, of function $f(x)$ for any value x in the domain of f is thus

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \text{ or } f'(x) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

if the limit exists.

In function notation (Joseph-Louis Lagrange, 1736-1813), the derivative of the function f with respect to x is symbolized as $f'(x)$ and pronounced "*f* prime of *x*."

In Leibniz notation (Gottfried Wilhelm Leibniz, 1684), the derivative of $y(x)$ is symbolized as $\frac{dy}{dx}$ and pronounced "dee *y* by dee *x*." We can also use the short form y', pronounced "*y* prime."

The domain of the derivative function depends on whether the value of the limit exists for all values within the domain of the original function.

The domain of the derivative function may be smaller than the domain of the original function.

Differentiation is the process to find the derivative function for a given function.

First Principles is the process of differentiation by computing the limit:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Ex. Use first principles to differentiate each function.

a. $f(x) = 3x-x^3$
b. $f(x) = \frac{5}{x^2}$

Solution

a.

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)-(x+h)^3-(3x-x^3)}{h}$
$= \lim_{h \to 0} \frac{3(x+h-x)-(x+h-x)((x+h)^2+(x+h)x+x^2)}{h}$
$= \lim_{h \to 0} \frac{3h-h(3x^2+3xh+h^2)}{h} = -3x^2+3$

b. $-\frac{10}{x^3}$

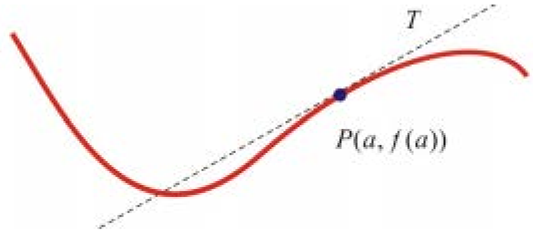
A function is not differentiable at x = a if f '(a) does not exist.

Notes: 1. If a function f is not continuous at x = a then the function f is not differentiable at x = a.

2. If a function f is continuous at x = a then the function f may be or not differentiable at x = a.

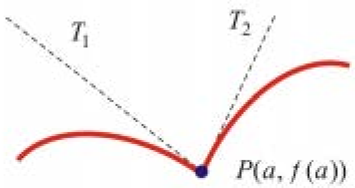
Differentiability Point

If the function y = f (x) is differentiable at x = a then the tangent line at P(a, f (a)) is unique and not vertical (the slope of the tangent line is not ∞ or −∞).



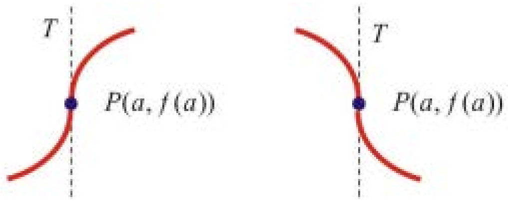
Corner Point

P(a, f (a)) is a corner point if there are two distinct tangent lines at P , one for the left-hand branch and one for the right-hand branch. For example:



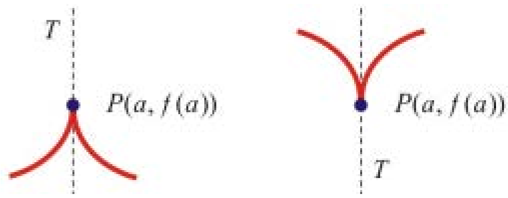
Infinite Slope Point

P(a, f (a))is a infinite slope point if the tangent line at P is vertical and the function is increasing or decreasing in the neighborhood at the of the point P . f '(a) = ∞ or f '(a) = −∞

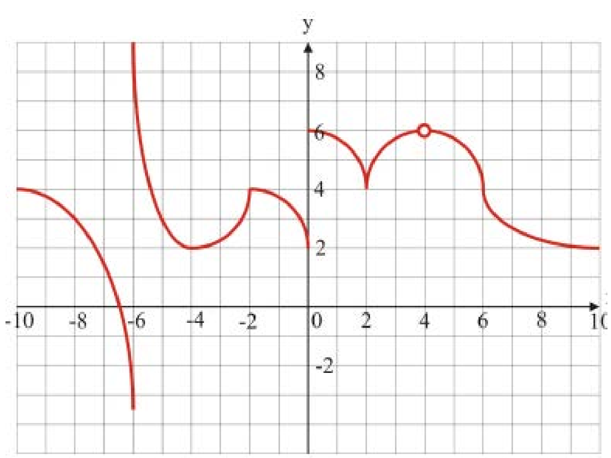


Cusp Point

P(a, f (a)) is a cusp point if the tangent line at P is vertical and the function is increasing on one side of the point P and decreasing on the other side. f '(a) = DNE.



Ex. Find the numbers x where the function y = f (x) (see the graph below) is not differentiable and explain why.



The function y = f (x) is not differentiable at:

x = −6 (infinite break)

x = −2 (P(−2,4) is a corner point)

x = 0 (jump discontinuity)

x = 2 (P(2,4) is a cusp point)

x = 4 (removable discontinuity)

x = 6 (P(6,4) is an infinite slope point)

Next we will uncover patterns of differentiation, by differentiating general statements, to uncover a rule for finding the derivative without needing first principles.

f(x) = c (constant)

f'(x) = $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ = $\lim_{h\to 0}\frac{c-c}{h}$ = 0

$f(x)$ = x^2

f'(x) = $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ = $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$ = $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$ = 2x

f(x) = x^3

f'(x) = 3x^2

Power Rule

Consider the power function: y =x^n,   x, n ∈ R. Then:

$(x^n)'$ = nx^{n-1} or $\frac{d}{dx}x^n$= nx^{n-1}

Ex. For each case, differentiate.

1. $(x^{10})'$=10x^9
2. $(\frac{1}{x^3})'$= $(x^{-3})'$= $\frac{-3}{x^4}$
3. $(\sqrt[7]{x^2})'$= $(x^{\frac{2}{7}})'$= $\frac{2}{7\sqrt[7]{x^5}}$

Some useful specific cases:

$(1)'$=0

$(x)'$= 1

$(\sqrt{x})'$= $\frac{1}{2\sqrt{x}}$

Constant Multiple Rule

Let consider g(x) = cf (x) . Then:

$[cf(x)]'$ = cf'(x)

Sum and Difference Rules

$[f(x) \pm g(x)]'$ = f'(x) $\pm$ g'(x)

Ex. Differentiate.

1. f(x) = -2x^3 + 4x^2 – x + 100
   f'(x) = -6x^2 + 8x -1
2. f(x) = $\frac{2}{9}x^9$ - 6$\sqrt{x}$ + 3x - $\frac{1}{4x^4} + 2^9$
   f'(x) = 2x^8 - $\frac{3}{\sqrt{x}}$ + 3 + $\frac{1}{x^5}$
3. f(x) = $\frac{x^4+2x^3-x+1}{x^2}$ = $\frac{x^4}{x^2}$ + $\frac{2x^3}{x^2}$ - $\frac{x}{x^2}$ + $\frac{1}{x^2}$
   f'(x) = 2x + 2 + $\frac{1}{x^2}$ - $\frac{2}{x^3}$
4. f(x) = $\frac{3x^5+\sqrt{x}}{x}$
   f'(x) = 12x^3 - $\frac{1}{2x\sqrt{x}}$

Tangent Line

To find the equation of the tangent line at the point P(a. f (a)) :

1. Find derivative function f '(x) .

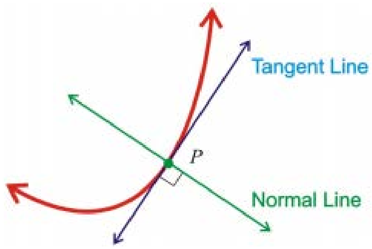
2. Find the slope of the tangent line using: m = f '(a)

3. Use the slope-point formula to get the equation of the tangent line: y − f (a) = m(x − a)

Normal Line

If m_T is the slope of the tangent line, then slope of the normal line m_N is given by:

m_N = - $\frac{1}{m_T}$



Ex. Find the equation of the normal line to the curve y = $x^2 - 3$ at (2, 1).

Solution

f'(x) = 2x

m_T = $f'(2)$ = 4 ⇨ m_N = - $\frac{1}{4}$

y – 1 = - $\frac{1}{4}$(x – 2)

x + 4y -6 = 0

Differentiability for piece-wise defined functions

Let consider the piece-wise defined function: *f(x)*= $\begin{cases} f_1(x) & if\ x < a \\ c & if\ x = a \\ f_2(x) & if\ x > a \end{cases}$

The function f is differentiable at x = a if:

(a) the function is continuous at x = a

(b) $f_1(a)'$ = $f_2(a)'$ (the slope of the tangent line for the left branch is equal to the slope of the tangent line for the right branch).

Ex. Analyze the differentiability of the function

y = f(x) = $|x - 3|$

Solution

f(x) = $\begin{cases} x - 3 & if\ x \geq 3 \\ -x + 3 & if\ x < 3 \end{cases}$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3) \Rightarrow$ f is continuous for all x∈R.

$f'(x) = \begin{cases} 1 & if\ x \geq 3 \\ -1 & if\ x < 3 \end{cases}$

$\lim_{x \to 3^-} f'(x) \neq \lim_{x \to 3^+} f'(x) \Rightarrow$ f is NOT differentiable at x=3 $\Rightarrow D_{f'} = \{x: x \in R, x \neq 3\}$

(3, 0) is a corner point.

Product Rule

If f and g are differentiable at x then so is fg and:

$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

Proof

We know that f and g are differentiable at x then:

$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$ and $\lim_{h \to 0} \frac{g(x+h)-g(x)}{h} = g'(x)$.

$$\lim_{h \to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = \lim_{h \to 0} \frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{g(x)(f(x+h)-f(x))}{h} + \lim_{h \to 0} \frac{f(x+h)(g(x+h)-g(x))}{h}$$

$$= (\lim_{h \to 0} g(x))(\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}) + (\lim_{h \to 0} f(x+h))(\lim_{h \to 0} \frac{g(x+h)-g(x)}{h})$$

$$= f'(x)g(x) + f(x)g'(x)$$

Particular case : $(cf(x))' = cf'(x)$

Ex. Find the equation of the tangent line to the curve $y = (x+\sqrt{x})(x^2+\frac{1}{x})$ at the point P(1,4).

Solution

If x =1then y = 4 . So P(1,4) is on the graph.

$\frac{dy}{dx}$ = (1+$\frac{1}{2\sqrt{x}}$) (x^2+$\frac{1}{x}$) + (x+$\sqrt{x}$)(2x - $\frac{1}{x^2}$)

m =$\frac{dy}{dx}$ (1) = 5

y = 5x-1