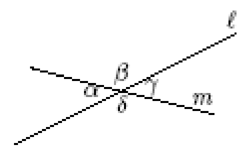
Geometry 3

Concepts

1. Lines and angles

We take as an axiom that the angle at a point on a straight line is a constant regardless of the point or line. Such an angle is called a straight angle and its measure is 180^{o} .

When two lines intersect, four angles are formed. Two such angles are called vertically opposite (or just opposite) if they are not formed on the same side of one of the lines. The straight angle axiom (postulate) implies the following theorem.

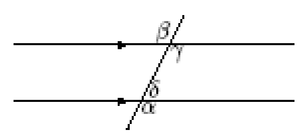


Theorem 1.
The opposite angles formed by intersecting straight lines are equal. In the diagram, $\alpha = \gamma$ and $\beta = \delta$.

Parallel lines

In the diagram the two horizontal lines are parallel. The line cutting the parallel lines is called a transversal.

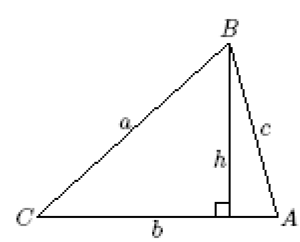
Theorem 2
Angles α and β are called alternate angles, α and γ are corresponding angles, and angles γ and δ are supplementary angles. Alternate angles and corresponding angles are equal, and pairs of supplementary angles sum to 180^{o}.



2. Areas and perimeters

Theorem 1.
The area of a parallelogram is equal to bh where b is the length of its base and h is its height (the perpendicular distance from the base to the parallel side opposite).

Theorem 2.
The area of a triangle is equal to $\frac{1}{2}$bh where b is the length of its base and h is its height (the perpendicular distance from the base to the vertex opposite).



Theorem 3.

(Heron's Theorem). If the lengths of the sides of a triangle are a, b and c, so that the semi perimeter $s = (a+b+c) = 2$ then the area of the triangle is $\sqrt{s(s-a)(s-b)(s-c)}$

3. Congruence of triangles

Any triangle has a property that an exterior angle of a triangle equals the sum of the two non-adjacent interior angles.

Two polygons are called congruent if their corresponding sides and corresponding angles are equal. Triangles may be determined to be congruent by any of the following rules.

- **SAS Rule** If two sides and the included angle of one triangle are equal to the two sides and the included angle of another, then the triangles are congruent.

- **SSS Rule** If three sides of one triangle are equal to the three sides of another, then the triangles are congruent.

- **ASA Rule** If two angles and the included side of one triangle are equal to the two angles and the included side of another, then the triangles are congruent.

- **RHS Rule** If the hypotenuse and one other side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle, then the triangles are congruent.

Note: that when we say two triangles ABC and XY Z are congruent we mean that the correspondence of vertex A to X, B to Y and C to Z determines the congruence. We denote that two triangles ABC and XY Z are congruent by writing $\Delta ABC \cong \Delta XYZ$.

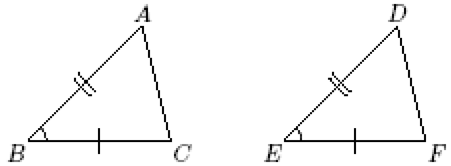
A triangle is isosceles if two of its sides are equal. By convention, the common vertex of the two equal sides of an isosceles triangle is written between the other two vertices, i.e. to say ΔXYZ is isosceles we imply that $YX = YZ$.

Proof of SAS Rule.

In the triangles ABC and DEF we have side $AB = DE$, included angle $\angle ABC = \angle DEF$, and side $BC = EF$. We must show $\Delta ABC \cong \Delta DEF$.

Place triangle ABC over triangle DEF so that B falls on E and edge BC runs along line EF. Since $BC = EF$, C falls on F. Since $\angle ABC = \angle DEF$, line BA falls on ED, and since $AB = DE$, A fall on D. Since A falls on D and C falls on F, line segment AC falls on DF. Hence $\Delta ABC \cong \Delta DEF$.

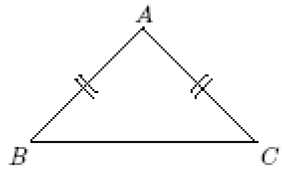
Theorem 1.
If a triangle is isosceles then the angles opposite the equal sides are equal. Conversely, if two angles of a triangle are equal then the two sides opposite the equal angles are equal, so that the triangle is isosceles.



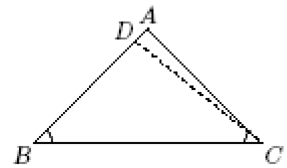
Proof of Theorem 1.

Assume in triangle ABC that AB = AC. Then

AB = AC,	given (1)
∠BAC = ∠CAB,	same angle
AC = AB,	equivalent to (1)
ΔABC ≅ ΔACB,	by SAS Rule
∠ABC = ∠ACB	



So we have shown that an isosceles triangle has the angles opposite its equal sides equal.



Now assume in ΔABC that ∠ABC = ∠ACB.

Along the ray BA, construct (by compass) the point D such that DB = AC. Now we have

DB = AC,	by construction
∠ABC = ∠DBC = ∠ACB,	given
BC = CB,	same line segment
ΔDBC ≅ ΔACB,	by SAS Rule

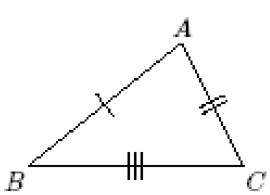
∴ ∠DCB = ∠ABC = ∠ACB

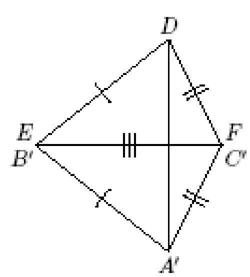
Thus line DC coincides with line AC. Hence D = A, and AB = DB = AC. So we have shown that a triangle with two angles equal has the sides opposite the equal angles equal.

Proof of SSS Rule.

Assume in triangles ABC and DEF that AB = DE, BC = EF and CA = FD. Transport triangle ABC so that B falls on E and line BC runs along EF. Since BC = EF, C falls on F.

Now let triangle ABC fall on the opposite side of line EF to triangle DEF so that A falls on A0. The transported copy of ΔABC is ΔA'B'C' in the diagram.



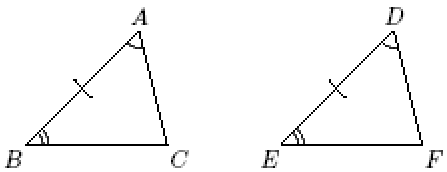


By construction, ΔABC ≅ ΔA'B'C', where E = B' and F = C'. In particular, A'E = AB = DE and A'F = AC = DF, so that triangles DEA' and DFA' are isosceles. So by Theorem 1, we have ∠EDA' = ∠EA'D and ∠FDA' = ∠FA'D.

Hence, ∠EDF = ∠EDA' + ∠FDA' = ∠EA'D + ∠FA'D = ∠EA'F = ∠BAC.

So now ΔABC ≅ ΔDEF by the SAS Rule.
Proof of ASA Rule.

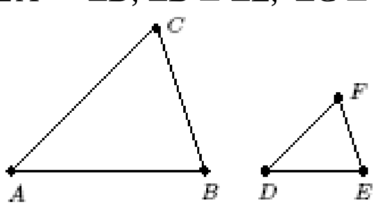
In the triangles ABC and DEF we have ∠ABC = ∠DEF, included side AB = DE and ∠BAC = ∠EDF. Place ΔABC over ΔDEF so that A falls on D and AB runs along DE.
Since AB = DE, B falls on E. Also AC runs along DF because ∠BAC = ∠EDF. Similarly,
BC runs along EF because ∠ABC = ∠DEF.
Thus the intersection point C of AC and BC must fall on the intersection point F of EF and DF. So ΔABC is exactly superimposed over ΔDEF, and hence ΔABC ≅ ΔDEF.



4. Similarity of triangles

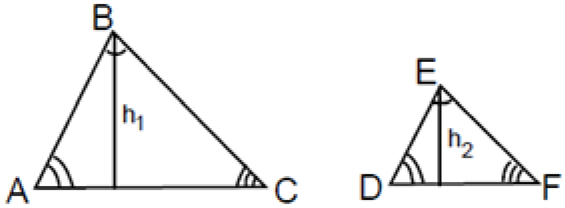
1) Definition

Two triangles are said to be similar if there is a correspondence between their vertices such that corresponding angles are congruent. The notation ΔABC ∽ ΔDEF means that ΔABC is similar to ΔDEF under the correspondence A ↔ D, B ↔ E, and C ↔ F, or more specifically that
∠A ≌ ∠D, ∠B ≅ ∠E, ∠C ≅ ∠F.



The first thing to notice is that in Euclidean geometry, it is only necessary to check that two of the corresponding angles are congruent. Each of the congruence rules has a corresponding similarity rule, by replacing side-length equality by proportionality. Thus, triangles may be determined to be similar by any of the following rules.

Similar triangles have proportional sides, that is, $\frac{ED}{AB}=\frac{EF}{BC}=\frac{DF}{AC}=\frac{h_1}{h_2}$



• SAS Rule
If two sides of one triangle are in the same proportion as the two sides of another, and the included angles of the sides that correspond are equal then the triangles are similar.

• SSS Rule
If three sides of one triangle are in the same proportion as the three sides of another, then the triangles are similar.

• AA Rule (or AAA Rule)
If two angles of one triangle are equal to two angles of another, then the triangles are similar. (The equality of the two remaining corresponding angles are then necessarily equal.)

• RHS Rule
If the hypotenuse and one other side of a right-angled triangle are in the same proportion as the hypotenuse and one side of another right-angled triangle, then the triangles are similar.

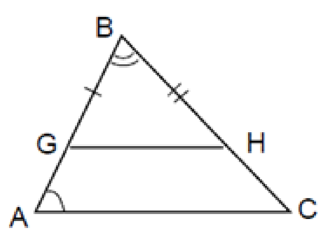
As with congruence, when we say two triangles ABC and XY Z are similar we mean that the correspondence of vertex A to X, B to Y and C to Z determines the similarity. We denote that two triangles ABC and XY Z are similar by writing ΔABC ∽ ΔXY Z.

Theorem 1.
If a line joins the midpoints of two sides of a triangle then that line is parallel to the third side and its length is equal to one half of the length of the third side.

Theorem 2.
A line parallel to one side of a triangle divides the other two sides in the same proportion. That is

If $GH//AC$ then

$$\frac{BG}{BH}=\frac{BA}{BC}$$

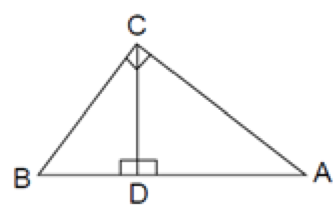


Theorem 3.
The bisector of one side of a triangle divides the opposite side in the same ratio as the other two sides.

2) Right triangles

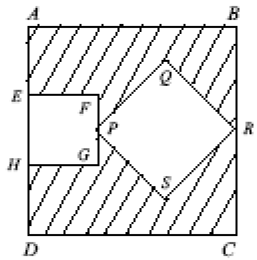
Theorem 4. Right Angle Similarity Theorem: In the right ΔABC. drawing the perpendicular from the right angle at C to the side AB gives the three similar right triangles; ΔABC ΔACD *and* ΔBDC, and $\frac{BD}{BC}=\frac{BC}{BA}$.

This implies that the BC is the ***geometric mean*** of BD and BA

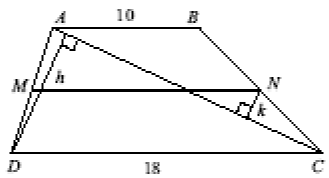


► Questions in class

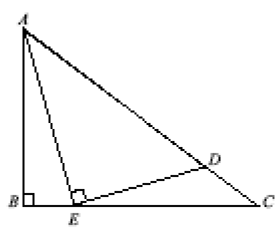
1. $ABCD$ is a square with $AB = 9$, $EFGH$ is a square in which E and H are trisection points of AD, and $PQRS$ is a square in which R is the midpoint of BC and P is the midpoint of FG, as shown. If X is the area of $ABCD$ and Y is the area of the shaded region, determine Y/X.



2. Trapezoid $ABCD$ with AB parallel to DC and median MN. If $AB = 10$, $DC = 18$, and h and k are the lengths of perpendiculars to AC drawn from D and N, determine h/k.



3. In right triangle ΔABC, $AB = 3$ and $BC = 4$. ΔDEA is inscribed in ΔABC such that $\Delta DEA \sim \Delta ABC$. If the perimeter of ΔDEA is m and the perimeter of ΔABC is n, determine m/n.



4. $ABCD$ is a rectangle, E and F are midpoints of AD and AB respectively. If $EF = 2\sqrt{3}$ and $FC = \sqrt{13}$, compute AB/AD