AP Calculus Class 20

Homework 19.

3. $h=0.2$, estimate $y(1)$.

$y'=1-xy \qquad y(0)=0 \quad \Rightarrow (x_0,y_0)=(0,0)$.

x-coordinates: 0, 0.2, 0.4, 0.6, 0.8, 1.

$y_1 = y_0 + hF(x_0,y_0)$

$= 0 + 0.2(1-0)$

$= 0 + 0.2 = 0.2. \longrightarrow (0.2, 0.2)$

$y_2 = y_1 + hF(x_1,y_1)$

$= 0.2 + 0.2(1 - 0.2 \cdot 0.2)$

$= 0.2 + 0.2(0.96)$

$= 0.2 + 0.196 = 0.392 \longrightarrow (0.4, 0.392)$

y_3

$\vdots$

$y_5 = 0.782 \longrightarrow (1, 0.782)$.

$y(1) \approx 0.782$.

6. $\frac{dy}{dt} = ky$, $y = .$

$P = P_0 e^{kt} \rightarrow y = y_0 e^{kt}$

$\boxed{B}$ $\frac{dy}{dt} = ky \Rightarrow \frac{1}{y} dy = k\,dt$

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7. $y = y_0 e^{kt}$ Population is $2y_0$ when $t = 10$.

$2y_0 = y_0 e^{k(10)} \Rightarrow 2 = e^{k(10)}$

$\Rightarrow \ln 2 = k(10) \Rightarrow k = \frac{\ln 2}{10} \approx 0.069$

$\boxed{A}$

Parametric Equations.

Parametric equ
Polar coordinates
Vector fun^n

Tangents
Areas
Arc Lengths

$y = f(x)$

Introduce a third variable "t."

let $x = f(t)$ $y = g(t)$

t: parameter.

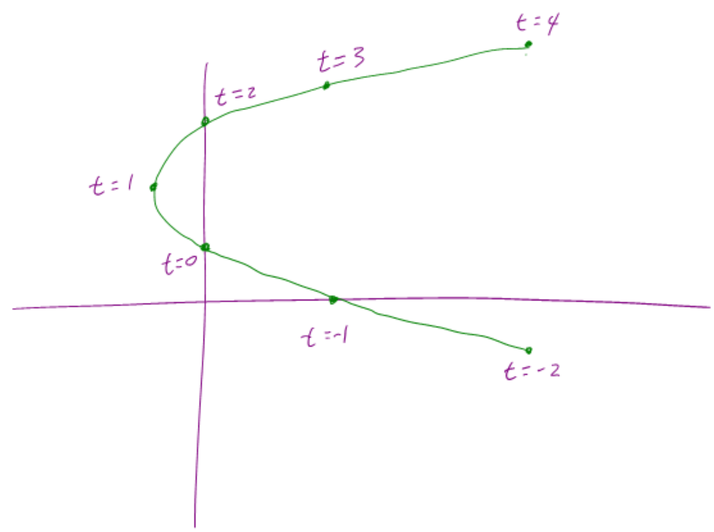
$x = f(t)$ and $y = g(t)$: parametric equations.

The curve C (in red) $(x, y) = (f(t), g(t))$ is called a parametric curve.

Example: Sketch and identify the curve defined by the parametric equ^n

$x = t^2 - 2t$ $y = t + 1$

t	x	y
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5
-1	3	0
-2	8	-1



In general, the curve with parametric equ^n

$x = f(t)$, $y = g(t)$ $\qquad a \le t \le b$.

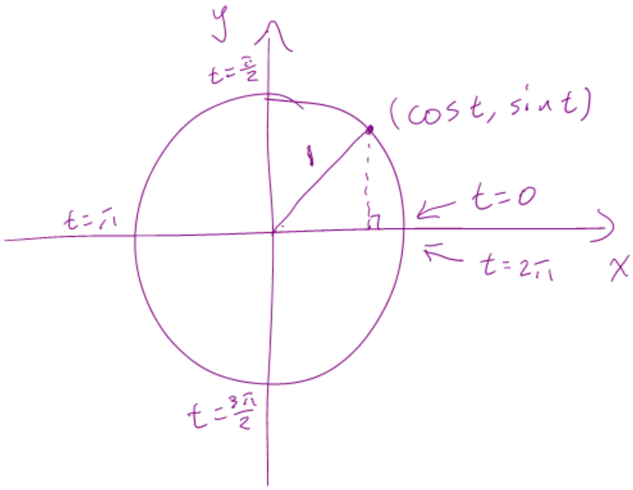
has initial point $(f(a), g(a))$ and terminal point $(f(b), g(b))$

Example: What curve is represented by the following parametric equ^n.

$x = \cos t \qquad y = \sin t. \qquad 0 \le t \le 2\pi$

$\sin^2 t + \cos^2 t = 1$

$\Rightarrow x^2 + y^2 = 1$



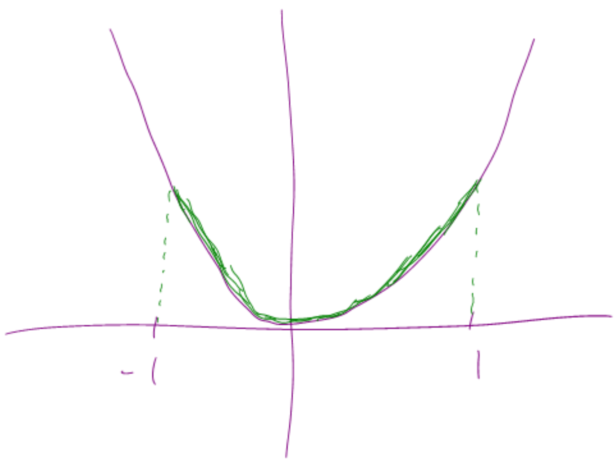
Example: sketch the curve w/ parametric eqn

$x = \sin t \quad y = \sin^2 t$

$\Rightarrow \; y = \sin^2 t = x^2$

since $-1 \leq \sin t \leq 1$

then $x \in [-1, 1]$.



Calculus w/ Parametric Equations.

- Tangents
- Areas
- Arc Lengths.

Parametric Equ^n: $x = f(t)$, $y = g(t)$.

$$y = F(x) \quad \Rightarrow \quad g(t) = F(f(t))$$

$$g'(t) = F'(f(t)) = F'(f(t)) \cdot f'(t) = F'(x) f'(t)$$

$$\Rightarrow F'(x) = \frac{g'(t)}{f'(t)}$$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if } \frac{dx}{dt} \neq 0.} \longrightarrow \frac{d}{dx} y = \frac{\frac{d}{dt} y}{\frac{d}{dt} x}$$

$$\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}} \qquad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\downarrow$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Example: A curve C is defined by the parametric equ^n $x = t^2$, $y = t^3 - 3t$

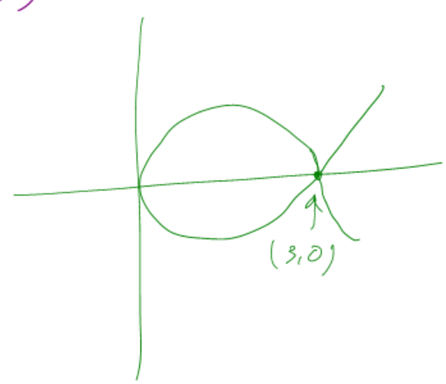
a) Show that C has two tangents at the point $(3,0)$ and find their eqn's.

b) Find the points on C where the tangents are horizontal or vertical.

Sol^n: a) $y = t^3 - 3t = t(t^2-3) = 0$

$t(t^2-3)=0 \Rightarrow t=0$ or $t = \pm\sqrt{3}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$



sub $t = \pm\sqrt{3}$ into $\frac{dy}{dx}$

$$\Rightarrow \pm\frac{3}{2}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = \pm\frac{6}{2\sqrt{3}} = \pm\sqrt{3}$$

$$\frac{dy}{dx} = \pm\sqrt{3}$$

$\Rightarrow y = \sqrt{3}(x-3)$ and $y = -\sqrt{3}(x-3)$.

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b) For horizontal tangents

$\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dt} = 0$ $\quad \frac{dx}{dt} \neq 0$.

$\Rightarrow \frac{dy}{dt} = 0 = 3t^2 - 3 = 0 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$.

sub $t = \pm 1$ into $x = t^2$, $y = t^3 - 3t$.

$\Rightarrow (1,-2)$ and $(1,2)$.

For vertial tangents,

$\frac{dx}{dt} = 0.$ $\quad \frac{dx}{dt} = 2t \quad \Rightarrow t = 0.$

$\Rightarrow$ At $(0,0)$, there is a vertical tangent.

Areas w/ Parametric Eqn

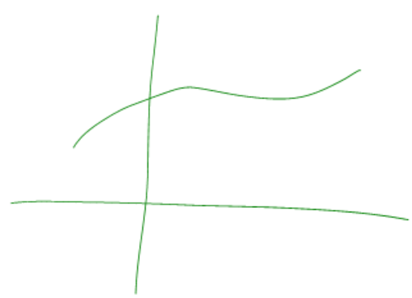
$x = f(t)$, $y = g(t)$, $\quad y = F(x)$.

Assume that the curve is traced out once at t goes from α to β.

$$A = \int_a^b F(x)\,dx$$

For $x = f(t)$, Let $\alpha = f(a)$ and $\beta = f(b)$.

$\frac{dx}{dt} = f'(t) \quad \Rightarrow \quad dx = f'(t)\,dt$



$$\Rightarrow A = \int_\alpha^\beta F(x) f'(t)\,dt$$

$$= \int_\alpha^\beta F(f(t)) f'(t)\,dt$$

Since $y = F(x) = F(f(t)) = g(t)$

$$\boxed{A = \int_{\alpha}^{\beta} g(t) f'(t)\, dt}$$

Example: Determine the area under the parametric curve given by

$x = 6(t - \sin t) \qquad y = 6(1 - \cos t) \qquad 0 \le t \le 2\pi.$

$$A = \int_{\alpha}^{\beta} g(t) f'(t)\, dt. \qquad f'(t) = \frac{dx}{dt} = 6(1-\cos t) = y$$

$$= \int_0^{2\pi} [6(1-\cos t)]^2\, dt$$

$$= 36 \int_0^{2\pi} (1-\cos t)^2\, dt$$

$$= 36 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\, dt.$$

$$= 36 \int_0^{2\pi} (1 - 2\cos t + \tfrac{1}{2}(1 + \cos 2t))\, dt$$

$$= 36 \int_0^{2\pi} (1 - 2\cos t + \tfrac{1}{2} + \tfrac{1}{2}\cos 2t)\, dt$$

$$= 36 \int_0^{2\pi} (\tfrac{3}{2} - 2\cos t + \tfrac{1}{2}\cos 2t)\, dt$$

$$= 36 \left[\tfrac{3}{2}t - 2\sin t + \tfrac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$= 108\pi$$

Arc Lengths w/ Parametric Eqnⁿ

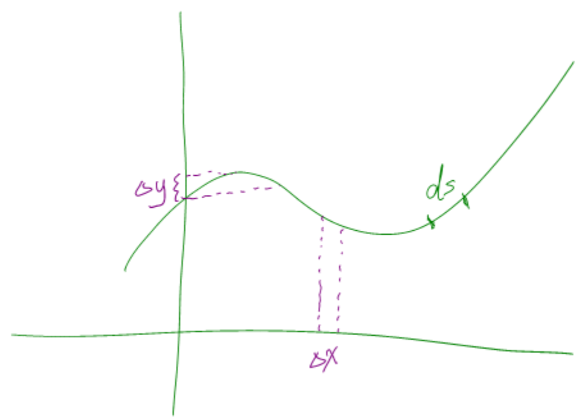
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta.$

Assume that the curve is traced out once from left to right.

$\Rightarrow \frac{dx}{dt} \ge 0.$

$$L = \int_a^b \underbrace{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}_{ds}$$



$$L = \int ds.$$

$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \rightarrow$ if $y = f(x) \qquad a \le x \le b$

$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \rightarrow$ if $x = h(y) \qquad c \le y \le b.$

We know that $x = f(t) \rightarrow \frac{dx}{dt} = f'(x)$

$\Rightarrow dx = f'(t)\, dt$

$\Rightarrow dx = \frac{dx}{dt}\, dt$

$$L = \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2} \; \frac{dx}{dt} \, dt \qquad \frac{dx}{dt} \neq 0.$$

$$= \int_\alpha^\beta \sqrt{1 + \frac{(\frac{dy}{dt})^2}{(\frac{dx}{dt})^2}} \; \frac{dx}{dt} \, dt.$$

$$= \int_\alpha^\beta \sqrt{\frac{(\frac{dx}{dt})^2}{(\frac{dx}{dt})^2} + \frac{(\frac{dy}{dt})^2}{(\frac{dx}{dt})^2}} \; \frac{dx}{dt} \, dt$$

$$= \int_\alpha^\beta \frac{1}{\left|\frac{dx}{dt}\right|} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; \frac{dx}{dt} \, dt$$

Since we assumed that $\frac{dx}{dt} \geq 0$, we can get rid of the abs value sign.

$$\boxed{\Rightarrow L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt}$$

Example: For the following.

$x = \cos t \qquad y = \sin t \qquad 0 \leq t \leq 2\pi$

$\frac{dx}{dt} = -\sin t \qquad \frac{dy}{dt} = \cos t.$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t} \, dt$$

$$= \int_0^{2\pi} 1 \, dt = t \Big]_0^{2\pi} = 2\pi.$$

Homework 19.

4. $h = 0.1$ $\qquad y(0.5) \approx ?$

$F(x,y) = y' = y + xy. \qquad y(0) = 1. \quad \rightarrow \quad (x_0, y_0) = (0, 1).$

$y_1 = y_0 + h F(x_0, y_0)$

$= 1 + 0.1(1 + 0.1)$

$= 1 + 0.1(1) = 1.1 \quad \rightarrow \quad (x_1, y_1) = (0.1, 1.1).$

$y_2 = y_1 + h F(x_1, y_1)$

$= 1.1 + 0.1(1.1 + (0.1)(1.1))$

$= 1.1 + 0.1(1.21) = 1.221 \qquad (x_2, y_2) = (0.2, 1.221).$

$y_3 = 1.368$

$y_4 = 1.545$

$y_5 = 1.761 \quad \longrightarrow \quad (x_5, y_5) = (0.5, 1.761).$