AP Calculus Lesson Five Notes

Chapter Three – Applications of Differential Calculus

3.1 Slope; Critical Points, Tangents and Normals
3.2 Increasing and Decreasing Functions
3.3 Maximum, Minimum, and Inflection Points

3.1 Slope; Critical Numbers (or Points), Tangents and Normals

If the derivative of $y = f(x)$ exists at $P(x_1, y_1)$, then the slope of the curve at P (which is defined to be the slope of the tangent to the curve at P) is $f'(x_1)$, the derivative of $f(x)$ at $x = x_1$.

Any c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ is undefined is called a critical number, $f(c)$ is a critical value of f, and $(c, f(c))$ is a critical point. However, some textbooks also directly call critical numbers as critical points. If f has a derivative everywhere, we find the critical numbers by solving the equation $f'(x) = 0$.

Example 3.1-1 Determine all the critical numbers (or points) for the function.

$$f(x) = 6x^5 + 33x^4 - 30x^3 + 100$$

Solution

We first need the derivative of the function in order to find the critical numbers (or points) and so let's get that and notice that we'll factor it as much as possible to make our life easier when we go to find the critical numbers (or points).

$$\begin{aligned} f'(x) &= 30x^4 + 132x^3 - 90x^2 \\ &= 6x^2\left(5x^2 + 22x - 15\right) \\ &= 6x^2(5x-3)(x+5) \end{aligned}$$

Now, our derivative is a polynomial and so will exist everywhere. Therefore the only critical points will be those values of x which make the derivative zero. So, we must solve.

$$6x^2(5x-3)(x+5) = 0$$

Because this is the factored form of the derivative it's pretty easy to identify the three critical numbers (or points). They are,

$$x = -5, \quad x = 0, \quad x = \frac{3}{5}$$

Example 3.1-2 Determine all the critical numbers (or points) for the function.

$$R(w) = \frac{w^2 + 1}{w^2 - w - 6}$$

Solution

We'll leave it to you to verify that using the quotient rule we get that the derivative is,

$$R'(w) = \frac{-w^2 - 14w + 1}{\left(w^2 - w - 6\right)^2} = -\frac{w^2 + 14w - 1}{\left(w^2 - w - 6\right)^2}$$

Notice that we factored a "-1" out of the numerator to help a little with finding the critical points. This negative out in front will not affect the derivative whether or not the derivative is zero or not exist but will make our work a little easier.

Now, we have two issues to deal with. First the derivative will not exist if there is division by zero in the denominator. So we need to solve,

$$w^2 - w - 6 = (w - 3)(w + 2) = 0$$

We didn't bother squaring this since if this is zero, then zero squared is still zero and if it isn't zero then squaring it won't make it zero.

So, we can see from this that the derivative will not exist at $w = 3$ and $w = -2$. However, these are NOT critical points since the function will also not exist at these points. Recall that in order for a point to be a critical point the function must actually exist at that point.

At this point we need to be careful. The numerator doesn't factor, but that doesn't mean that there aren't any critical points where the derivative is zero. We can use the quadratic formula on the numerator to determine if the fraction as a whole is ever zero.

$$w = \frac{-14 \pm \sqrt{(14)^2 - 4(1)(-1)}}{2(1)} = \frac{-14 \pm \sqrt{200}}{2} = \frac{-14 + 10\sqrt{2}}{2} = -7 \pm 5\sqrt{2}$$

So, we get two critical points. Also, these are not "nice" integers or fractions. This will happen on occasion. Don't get too locked into answers always being "nice". Often they aren't.

Note as well that we only use real numbers for critical points. So, if upon solving the quadratic in the numerator, we had gotten complex number these would not have been considered critical points.

Summarizing, we have two critical numbers (or points). They are,

$$w = -7 + 5\sqrt{2}, \quad w = -7 - 5\sqrt{2}$$

Again, remember that while the derivative doesn't exist at $w = 3$ and -2 neither does the function and so these two points are not critical points for this function.

Example 3.1-3 Determine all the critical numbers (points) for the function.

$$y = 6x - 4\cos(3x)$$

Solution

First get the derivative and don't forget to use the chain rule on the second term.

$$y' = 6 + 12\sin(3x)$$

Now, this will exist everywhere and so there won't be any critical points for which the derivative doesn't exist. The only critical numbers (or points) will come from numbers (or points) that make the derivative zero. We will need to solve,

$$\begin{aligned} 6 + 12\sin(3x) &= 0 \\ \sin(3x) &= -\frac{1}{2} \end{aligned}$$

Solving this equation gives the following.

$$\begin{aligned} 3x &= 3.6652 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots \\ 3x &= 5.7596 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Don't forget the $2\pi\, n$ on these! There will be problems down the road in which we will miss solutions without this! Also make sure that it gets put on at this stage! Now divide by 3 to get all the critical points for this function.

$$x = 1.2217 + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$x = 1.9199 + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

Example 3.1-4 Determine all the critical numbers (or points) for the function.

$$h(t) = 10t\mathbf{e}^{3-t^2}$$

Solution

Here's the derivative for this function.

$$h'(t) = 10\mathbf{e}^{3-t^2} + 10t\mathbf{e}^{3-t^2}(-2t) = 10\mathbf{e}^{3-t^2} - 20t^2\mathbf{e}^{3-t^2}$$

Now, this looks unpleasant, however with a little factoring we can clean things up a little as follows,

$$h'(t) = 10\mathbf{e}^{3-t^2}\left(1 - 2t^2\right)$$

This function will exist everywhere and so no critical points will come from that. Determining where this is zero is easier than it looks. We know that exponentials are never zero and so the only way the derivative will be zero is if,

$$\begin{aligned} 1 - 2t^2 &= 0 \\ 1 &= 2t^2 \\ \frac{1}{2} &= t^2 \end{aligned}$$

We will have two critical points for this function.

$$t = \pm\frac{1}{\sqrt{2}}$$

Example3.1-5 Determine all the critical numbers (or points) for the function.

$$f(x) = x^2 \ln(3x) + 6$$

Solution

Before getting the derivative let's notice that since we can't take the log of a negative number or zero we will only be able to look at $x > 0$.

The derivative is then,

$$\begin{aligned} f'(x) &= 2x\ln(3x) + x^2\left(\frac{3}{3x}\right) \\ &= 2x\ln(3x) + x \\ &= x\left(2\ln(3x) + 1\right) \end{aligned}$$

Now, this derivative will not exist if x is a negative number or if $x = 0$, but then again neither will the function and so these are not critical points. Remember that the function will only exist if $x > 0$ and nicely enough the derivative will also only exist if $x > 0$ and so the only thing we need to worry about is where the derivative is zero.

First note that, despite appearances, the derivative will not be zero for $x = 0$. As noted above the derivative doesn't exist at $x = 0$, because of the natural logarithm and so the derivative can't be zero there!

So, the derivative will only be zero if,

$$\begin{aligned} 2\ln(3x) + 1 &= 0 \\ \ln(3x) &= -\frac{1}{2} \end{aligned}$$

Recall that we can solve this by exponentiating both sides.

$$\begin{aligned} \mathbf{e}^{\ln(3x)} &= \mathbf{e}^{-\frac{1}{2}} \\ 3x &= \mathbf{e}^{-\frac{1}{2}} \\ x &= \frac{1}{3}\mathbf{e}^{-\frac{1}{2}} = \frac{1}{3\sqrt{\mathbf{e}}} \end{aligned}$$

There is a single critical number (or point) for this function.

Tangents and Normals

The equation of the tangent to the curve $y = f(x)$ at point $P(x_1, y_1)$ is
$y - y_1 = f'(x)(x - x_1)$

The line through *P* that is perpendicular to the tangent, called the normal to the curve at *P*, has slope $-\dfrac{1}{f'(x_1)}$. Its equation is

$$y - y_1 = -\frac{1}{f'(x_1)}(x - x_1)$$

If the tangent to a curve is horizontal at a point, then the derivative at the point is 0. If the tangent is vertical at a point, then the derivative does not exist at the point.

Example 3.1-6 Find the equations of the tangent and normal to the curve of $f(x) = x^3 - 3x^2$ at the point (1, –2).

Solution

Since $f'(x) = 3x^2 - 6x$ and $f'(1) = 3(1)^2 - 6(1) = -3$, the equation of the tangent is

$y = 2 = -3(x - 1)$ or $y = -3x + 1$

And the equation of the normal is

$y = 2 = (1/3)(x - 1)$ or $y = (1/3)x - (7/3)$

Example 3.1-7 Find the equations of the tangent to the curve of $x^2y - x = y^3 - 8$ at the point where $x = 0$.

Solution

Here we differentiate implicitly to get $\dfrac{dy}{dx} = \dfrac{1 - 2xy}{x^2 - 3y^2}$. Since $y = 2$ when $x = 0$ and the slope at this point is $\dfrac{1-0}{0-12} = -\dfrac{1}{12}$, the equation of the tangent is

$y - 2 = -(1/12)x$ or $x + 12y = 24$

Example 3.1-8 Determine the *x*-*y* coordinates of the points where the following parametric equations will have horizontal or vertical tangents.

$x = t^3 - 3t$ and $y = 3t^2 - 9$

Solution

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{dt} = 3t^2 - 3 \quad \text{and} \quad \frac{dy}{dt} = 6t$$

Horizontal Tangents

We'll have horizontal tangents where,

$$6t = 0 \qquad \text{or} \quad t = 0$$

Now, this is the value of *t* which gives the horizontal tangents and we were asked to find the *x*-*y* coordinates of the point. To get these we just need to plug *t* into the parametric equations. Therefore, the only horizontal tangent will occur at the point (0,-9).

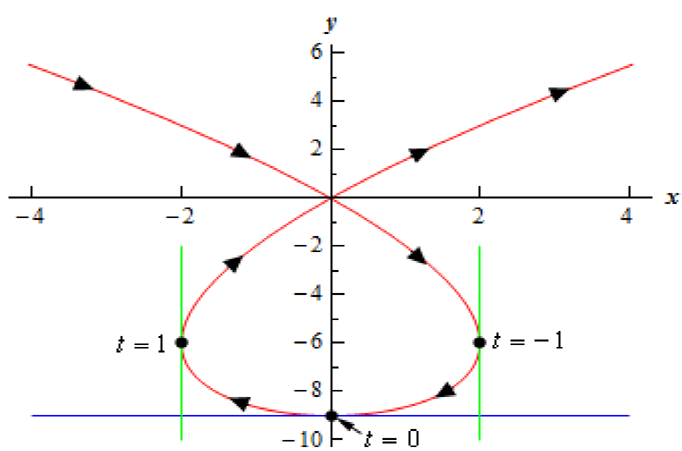
Vertical Tangents

In this case we need to solve,

$$3(t^2 - 1) = 0 \quad \text{so} \quad t = \pm 1$$

The two vertical tangents will occur at the points (2,-6) and (-2,-6).

For the sake of completeness and at least partial verification here is the sketch of the parametric curve.



3.2 Increasing and Decreasing Functions

Definition

1. Given any x_1 and x_2 from an interval I with $x_1 < x_2$ if $f(x_1) < f(x_2)$, then $f(x)$ is ***increasing*** on I.

2. Given any x_1 and x_2 from an interval I with $x_1 < x_2$ if $f(x_1) > f(x_2)$, then $f(x)$ is ***decreasing*** on I.

Fact

1. If $f'(x) > 0$ for every x on some interval I, then $f(x)$ is increasing on the interval.

2. If $f'(x) < 0$ for every x on some interval I, then $f(x)$ is decreasing on the interval.

3. If $f'(x) = 0$ for every x on some interval I, then $f(x)$ is constant on the interval.

Example 3.2-1 Determine all intervals where the following function is increasing or decreasing.

$$f(x) = -x^5 + \frac{5}{2}x^4 + \frac{40}{3}x^3 + 5$$

Solution

To determine if the function is increasing or decreasing we will need the derivative.

$$f'(x) = -5x^4 + 10x^3 + 40x^2 = -5x^2(x-4)(x+2)$$

Note that when we factored the derivative we first factored a "-1" out to make the rest of the factoring a little easier. From the factored form of the derivative we see that we have three critical numbers $x -2$, $x = 0$ and $x = 4$. We'll need these in a bit.

We now need to determine where the derivative is positive and where it's negative. We've done this several times now in both the Review chapter and the previous chapter. Since the derivative is a polynomial it is continuous and so we know that the only way for it to change signs is to first go through zero.

In other words, the only place that the derivative *may* change signs is at the critical points of the function. We've now got another use for critical points. So, we'll build a number line, graph the critical points and pick test points from each region to see if the derivative is positive or negative in each region.

Here is the chart and the test values for the derivative.

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 4)$	$(4, +\infty)$
Test Value	-3	-1	2	5
-5	–	–	–	–
$x-4$	–	–	–	+
x + 2	–	+	+	+
$f'(x)$	–	+	+	–
$f(x)$	decreasing	increasing	increasing	decreasing

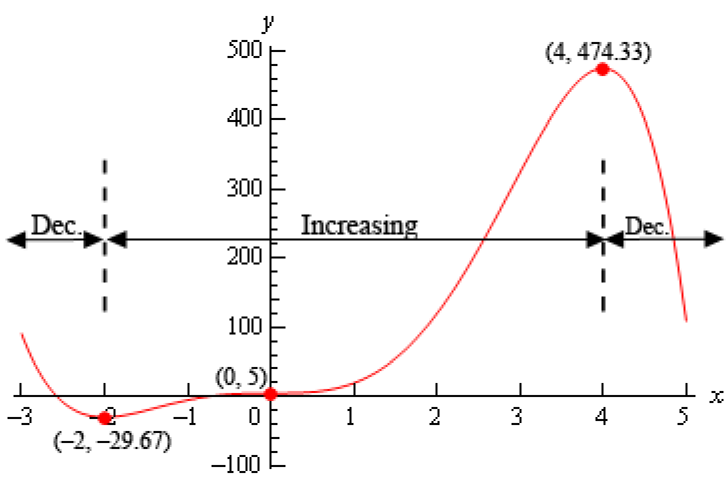
Make sure that you test your points in the derivative. One of the more common mistakes here is to test the points in the function instead! Recall that we know that the derivative will be the same sign in each region. The only place that the derivative can change signs is at the critical points and we've marked the only critical points on the number line.

So, it looks we've got the following intervals of increase and decrease.

$(-2, 0)$ and $(0, 4)$ are increasing intervals, $(-\infty, -2)$ and $(4, +\infty)$ are decreasing intervals.

To get this sketch we start at the very left of the graph and knowing that the graph must be decreasing and will continue to decrease until we get to $x = -2$. At this point the function will continue to increase until it gets to $x = 4$. However, note that during the increasing phase it does need to go through the point at $x = 0$ and at this point we also know that the derivative is zero here and so the graph goes through $x = 0$ horizontally. Finally, once we hit $x = 4$, the graph starts, and continues, to decrease. Also, note that just like at $x = 0$ the graph will need to be horizontal when it goes through the other two critical points as well.

Here is the graph of the function.



3.3 Maximum, Minimum, and Inflection Points

Definition

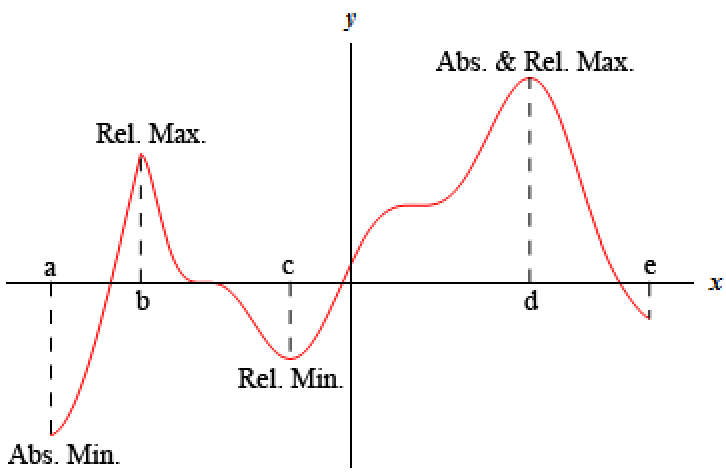
1. We say that *f(x)* has an **absolute (or global) maximum** at $x = c$ if $f(x) \le f(c)$ for every x in the domain we are working on.

2. We say that *f(x)* has a **relative (or local) maximum** at $x = c$ if $f(x) \le f(c)$ for every x in some open interval around $x = c$.

3. We say that *f(x)* has an **absolute (or global) minimum** at $x = c$ if $f(x) \ge f(c)$ for every x in the domain we are working on.

4. We say that *f(x)* has a **relative (or local) minimum** at $x = c$ if $f(x) \ge f(c)$ for every x in some open interval around $x = c$.

It's usually easier to get a feel for the definitions by taking a quick look at a graph.



For the function shown in this graph we have relative maximums at $x = b$ and $x = d$. Both of these point are relative maximums since they are interior to the domain shown and are the largest point on the graph in some interval around the point. We also have a relative minimum at $x = c$ since this point is interior to the domain and is the lowest point on the graph in an interval around it. The far right end point, $x = e$, will not be a relative minimum since it is an end point.

The function will have an absolute maximum at $x = d$ and an absolute minimum at $x = a$. These two points are the largest and smallest that the function will ever be. We can also notice that the absolute extrema for a function will occur at either the endpoints of the domain or at relative extrema. We will use this idea in later sections so it's more important than it might seem at the present time.

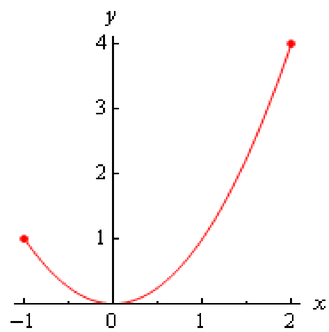
Let's take a quick look at some examples to make sure that we have the definitions of absolute extrema and relative extrema straight.

Example 3.3-1 Identify the absolute extrema and relative extrema for the following function.

$f(x) = x^2$ on $[-1, 2]$

Solution

Since this function is easy enough to graph let's do that. However, we only want the graph on the interval [-1,2]. Here is the graph,



Note that we used dots at the end of the graph to remind us that the graph ends at these points.

We can now identify the extrema from the graph. It looks like we've got a relative and absolute minimum of zero at $x = 0$ and an absolute maximum of four at $x = 2$. Note that $x = -1$ is not a relative maximum since it is at the end point of the interval.

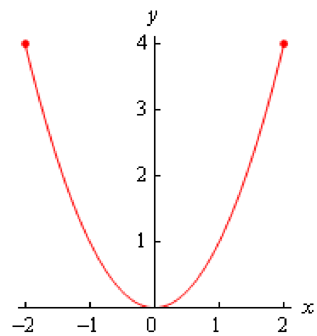
This function doesn't have any relative maximums.

Example 3.3-2 Identify the absolute extrema and relative extrema for the following function.

$f(x) = x^2$ on $[-2, 2]$

Solution

Here is the graph for this function.



In this case we still have a relative and absolute minimum of zero at $x = 0$. We also still have an absolute maximum of four. However, unlike the first example this will occur at two points, $x = -2$ and $x = 2$.

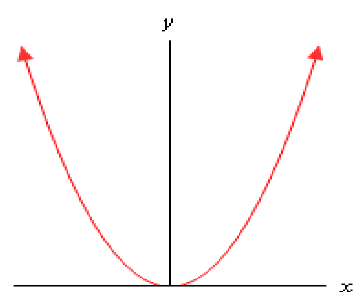
Again, the function doesn't have any relative maximums.

Example 3.3-3 Identify the absolute extrema and relative extrema for the following function.

$$f(x) = x^2$$

Solution

In this case we've given no domain and so the assumption is that we will take the largest possible domain. For this function that means all the real numbers. Here is the graph.



In this case the graph doesn't stop increasing at either end and so there are no maximums of any kind for this function. No matter which point we pick on the graph there will be points both larger and smaller than it on either side so we can't have any maximums (or any kind, relative or absolute) in a graph.

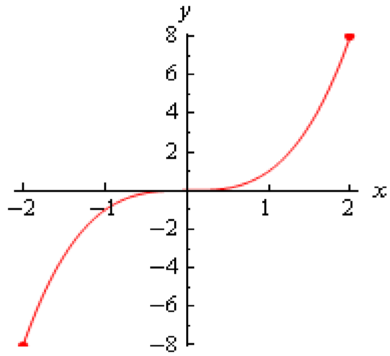
We still have a relative and absolute minimum value of zero at $x = 0$.

Example 3.3-4 Identify the absolute extrema and relative extrema for the following function.

$f(x) = x^3$ on $[-2, 2]$

Solution

Here is the graph for this function.



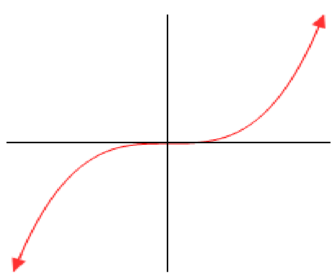
This function has an absolute maximum of eight at $x = 2$ and an absolute minimum of negative eight at $x = -2$. This function has no relative extrema.

Example 3.3-5 Identify the absolute extrema and relative extrema for the following function.

$f(x) = x^3$

Solution

Again, we aren't restricting the domain this time so here's the graph.

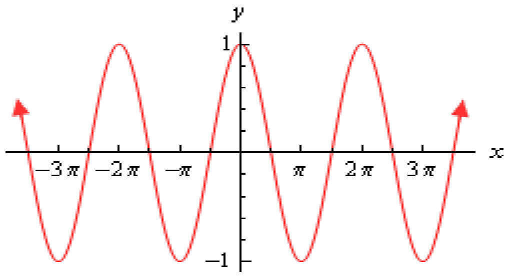


In this case the function has no relative extrema and no absolute extrema.

Example 3.3-6 Identify the absolute extrema and relative extrema for the function $f(x) = \cos x$.

Solution

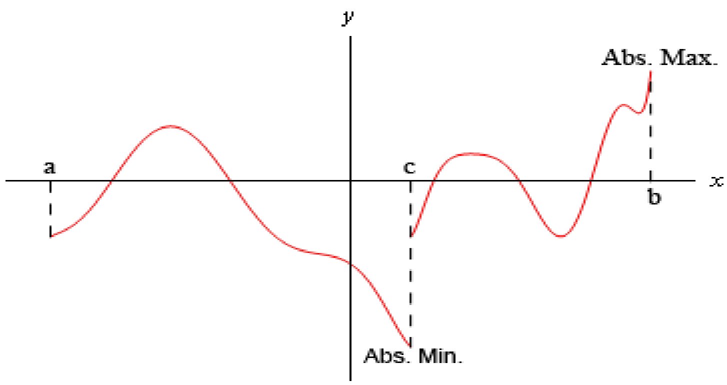
We've not restricted the domain for this function. Here is the graph.



Cosine has extrema (relative and absolute) that occur at many points. Cosine has both relative and absolute maximums of 1 at $x = \ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$

Cosine also has both relative and absolute minimums of -1 at $x = \ldots, -3\pi, -\pi, 0, \pi, 3\pi, \ldots$

We should also point out that just because a function is not continuous at a point that doesn't mean that it won't have both absolute extrema in an interval that contains that point. Below is the graph of a function that is not continuous at a point in the given interval and yet has both absolute extrema.



This graph is not continuous at $x = c$, yet it does have both an absolute maximum ($x = b$) and an absolute minimum ($x = c$). Also note that, in this case one of the absolute extrema occurred at the point of discontinuity, but it doesn't need to. The absolute minimum could just have easily been at the other end point or at some other point interior to the region. The point here is that this graph is not continuous and yet does have both absolute extrema.

First Derivative Test

Suppose that $x = c$ is a critical point of $f(x)$ then,

1. If $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right $x = c$ then $f(c)$ is a relative maximum.

2. If $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right $x = c$ then $f(c)$ is a relative minimum.

3. If $f'(x)$ is the same sign on both sides $x = c$ then $f(c)$ is neither a relative maximum nor a relative minimum.

Example 3.3-7 Find and classify all the critical points of the following function. Give the intervals where the function is increasing and decreasing.

$$g(t) = t\sqrt[3]{t^2 - 4}$$

Solution

First we'll need the derivative so we can get our hands on the critical points. Note as well that we'll do some simplification on the derivative to help us find the critical points.

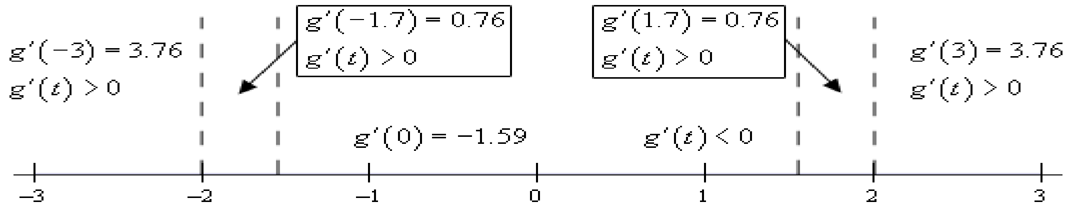
$$\begin{aligned} g'(t) &= \left(t^2 - 4\right)^{\frac{1}{3}} + \frac{2}{3}t^2\left(t^2 - 4\right)^{-\frac{2}{3}} \\ &= \left(t^2 - 4\right)^{\frac{1}{3}} + \frac{2t^2}{3\left(t^2 - 4\right)^{\frac{2}{3}}} \\ &= \frac{3\left(t^2 - 4\right) + 2t^2}{3\left(t^2 - 4\right)^{\frac{2}{3}}} \\ &= \frac{5t^2 - 12}{3\left(t^2 - 4\right)^{\frac{2}{3}}} \end{aligned}$$

So, it looks like we'll have four critical points here. They are,

$t = \pm 2$ The derivative doesn't exist here.

$t = \pm\sqrt{\frac{12}{5}} = \pm 1.549$ The derivative is zero here.

Finding the intervals of increasing and decreasing will also give the classification of the critical points so let's get those first. Here is a number line with the critical points graphed and test points.



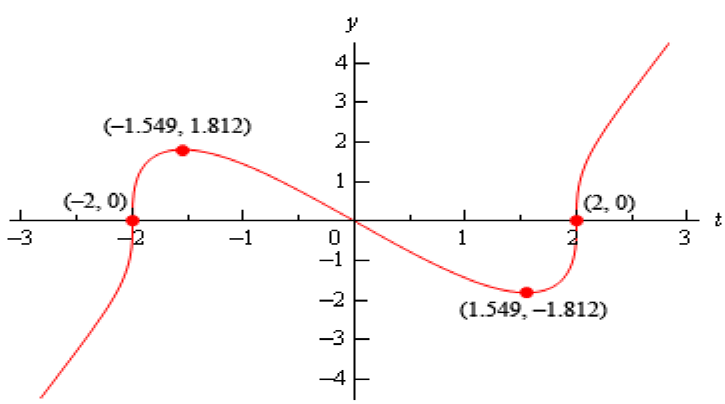
So, it looks like we've got the following intervals of increasing and decreasing.

$$\text{Increase}: \quad -\infty < x - 2, \;\; -2 < x < -\sqrt{\frac{12}{5}}, \;\; \sqrt{\frac{12}{5}} < x < 2, \;\; \& \;\; 2 < x < \infty$$

$$\text{Decrease}: \; -\sqrt{\frac{12}{5}} < x < \sqrt{\frac{12}{5}}$$

From this it looks like the function has neither relative minimum or relative maximums at $t = -2$ and $t = 2$ since the function is increasing on both side of them. On the other hand, $f\left(-\sqrt{\frac{12}{5}}\right)$ is a relative maximum and $f\left(\sqrt{\frac{12}{5}}\right)$ is a relative minimum.

For completeness sake here is the graph of the function.



Example 3.3-8 Suppose that the elevation above sea level of a road is given by the following function.

$$E(x) = 500 + \cos\left(\frac{x}{4}\right) + \sqrt{3}\sin\left(\frac{x}{4}\right)$$

where *x* is in miles. Assume that if *x* is positive we are to the east of the initial point of measurement and if *x* is negative we are to the west of the initial point of measurement.

If we start 25 miles to the west of the initial point of measurement and drive until we are 25 miles east of the initial point how many miles of our drive were we driving up an incline?

Solution

Okay, this is just a really fancy way of asking what the intervals of increasing and decreasing are for the function on the interval [-25,25]. So, we first need the derivative of the function.

$$E'(x) = -\frac{1}{4}\sin\left(\frac{x}{4}\right) + \frac{\sqrt{3}}{4}\cos\left(\frac{x}{4}\right)$$

Setting this equal to zero gives,

$$\begin{aligned} -\frac{1}{4}\sin\left(\frac{x}{4}\right) + \frac{\sqrt{3}}{4}\cos\left(\frac{x}{4}\right) &= 0 \\ \tan\left(\frac{x}{4}\right) &= \sqrt{3} \end{aligned}$$

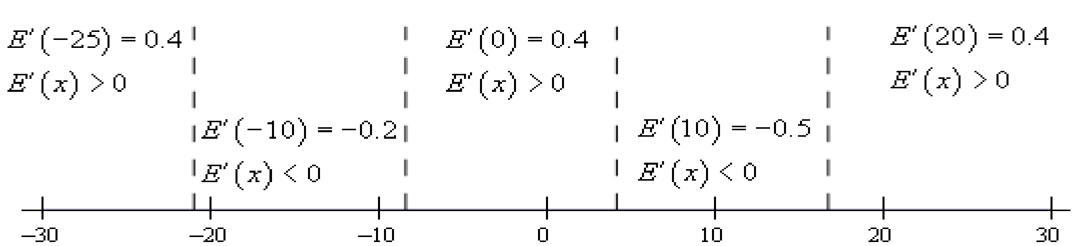
The solutions to this and hence the critical points are,

$$\begin{aligned} \frac{x}{4} &= 1.0472 + 2\pi n, \; n = 0, \pm 1, \pm 2, \ldots \\ \frac{x}{4} &= 4.1888 + 2\pi n, \; n = 0, \pm 1, \pm 2, \ldots \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 4.1888 + 8\pi n, \; n = 0, \pm 1, \pm 2, \ldots \\ x &= 16.7552 + 8\pi n, \; n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

I'll leave it to you to check that the critical points that fall in the interval that we're after are,

$$-20.9439, \; -8.3775, \; 4.1888, \; 16.7552$$

Here is the number line with the critical points and test points.



So, it looks like the intervals of increasing and decreasing are,

$$\text{Increase}: \quad -25 < x < -20.9439,\ -8.3775 < x < 4.1888 \text{ and } 16.7552 < x < 25$$
$$\text{Decrease}: \ -20.9439 < x < -8.3775 \ \text{ and } \ 4.1888 < x < 16.7552$$

Notice that we had to end our intervals at -25 and 25 since we've done no work outside of these points and so we can't really say anything about the function outside of the interval [-25, 25].

From the intervals we can actually answer the question. We were driving on an incline during the intervals of increasing and so the total number of miles is

$$\begin{aligned}\text{Distance } &= \left(-20.9439-\left(-25\right)\right)+\left(4.1888-\left(-8.3775\right)\right)+\left(25-16.7552\right)\\ &= 24.8652 \text{ miles}\end{aligned}$$

Even though the problem didn't ask for it we can also classify the critical points that are in the interval [-25, 25].

$$\text{Relative Maximums}: \ -20.9439, 4.1888$$
$$\text{Relative Minimums}: \ -8.3775, 16.7552$$

Example 3.3-9 The population of rabbits (in hundreds) after t years in a certain area is given by the following function,

$$P\left(t\right) = t^2 \ln\left(3t\right) + 6$$

Determine if the population ever decreases in the first two years.

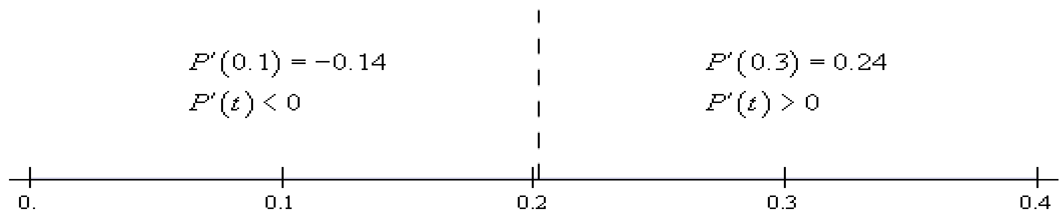
Solution

So, again we are really after the intervals and increasing and decreasing in the interval [0,2].

We found the only critical point to this function back in the Critical Points section to be,

$$x = \frac{1}{3\sqrt{\mathbf{e}}} = 0.202$$

Here is a number line for the intervals of increasing and decreasing.



So, it looks like the population will decrease for a short period and then continue to increase forever.

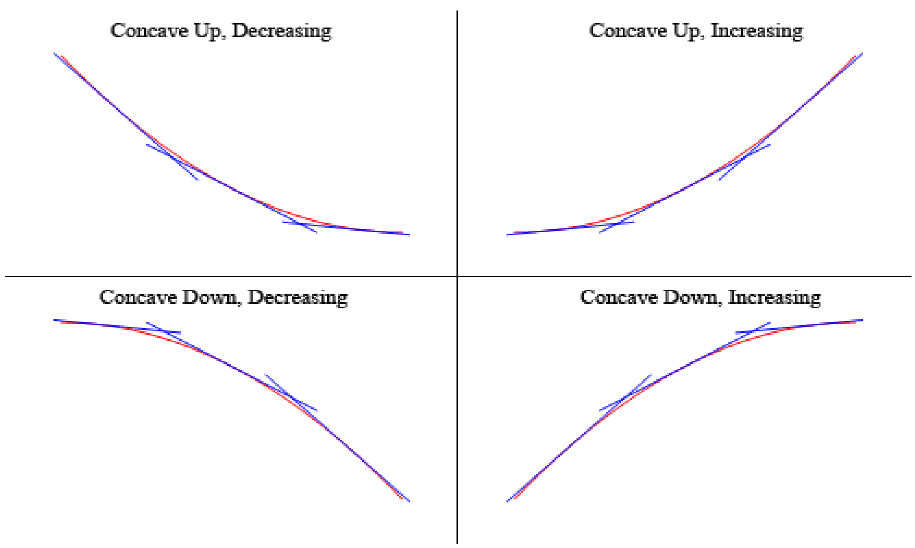
Also, while the problem didn't ask for it we can see that the single critical point is a relative minimum.

Definition of the Concavity of a Function

Given the function $f(x)$ then

1. $f(x)$ is **concave up** on an interval I if all of the tangents to the curve on I are below the graph of $f(x)$.
2. $f(x)$ is **concave down** on an interval I if all of the tangents to the curve on I are above the graph of $f(x)$.

To show that the graphs above do in fact have concavity claimed above here is the graph again (blown up a little to make things clearer).



So, as you can see, in the two upper graphs all of the tangent lines sketched in are all below the graph of the function and these are concave up. In the lower two graphs all the tangent lines are above the graph of the function and these are concave down.

Again, notice that concavity and the increasing/decreasing aspect of the function is completely separate and do not have anything to do with the other. This is important to note because students often mix these two up and use information about one to get information about the other.

There's one more definition that we need to get out of the way.

Definition of Inflection Point

A point $x = c$ is called an **inflection point** if the function is continuous at the point and the concavity of the graph changes at that point.

Fact

Given the function $f(x)$ then,

1. If $f''(x) > 0$ for all x in some interval I then $f(x)$ is concave up on I.
2. If $f''(x) < 0$ for all x in some interval I then $f(x)$ is concave down on I.

Second Derivative Test

Suppose that $x = c$ is a critical point of $f'(c)$ such that $f'(c) = 0$ and that $f''(c)$ is continuous in a region around $x = c$. Then,

1. If $f''(c) < 0$ then $f(c)$ is a relative maximum.
2. If $f''(c) > 0$ then $f(c)$ is a relative minimum.
3. If $f''(c) = 0$ then $f(c)$ can be a relative maximum, relative minimum or neither.

Example 3.3-8 Use the second derivative test to classify the critical points of the function,

$$h(x) = 3x^5 - 5x^3 + 3$$

Solution

Note that all we're doing here is verifying the results from the first example. The second derivative is,

$h''(x) = 60x^3 - 30x$

The three critical points ($x = -1$, $x = 0$, and $x = 1$) of this function are all critical points where the first derivative is zero so we know that we at least have a chance that the Second Derivative Test will work. The value of the second derivative for each of these are,

$h''(-1) = -30 \qquad h''(0) = 0 \qquad h''(1) = 30$

The second derivative at $x = -1$ is negative so by the Second Derivative Test this critical point this is a relative maximum as we saw in the first example. The second derivative at $x = 1$is positive and so we have a relative minimum here by the Second Derivative Test as we also saw in the first example.

In the case of $x = 0$ the second derivative is zero and so we can't use the Second Derivative Test to classify this critical point. Note however, that we do know from the First Derivative Test we used in the first example that *in this case* the critical point is not a relative extrema.

Example 3.3-9 For the following function find the inflection points and use the second derivative test, if possible, to classify the critical points. Also, determine the intervals of increase/decrease and the intervals of concave up/concave down and sketch the graph of the function.

$$f(t) = t(6-t)^{\frac{2}{3}}$$

Solution

We'll need the first and second derivatives to get us started.

$$f'(t) = \frac{18-5t}{3(6-t)^{\frac{1}{3}}} \qquad f''(t) = \frac{10t-72}{9(6-t)^{\frac{4}{3}}}$$

The critical points are,

$$t = \frac{18}{5} = 3.6 \qquad t = 6$$

Notice as well that we won't be able to use the second derivative test on t = 6 to classify this critical point since the derivative doesn't exist at this point. To classify this we'll need the increasing/decreasing information that we'll get to sketch the graph.

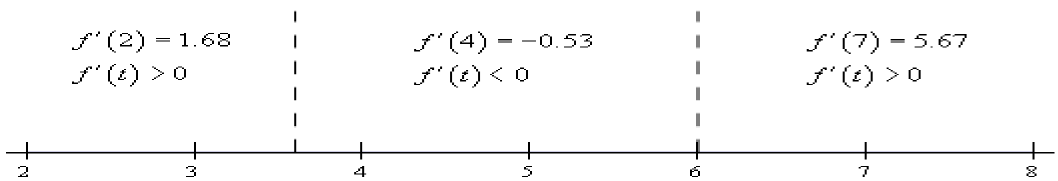
We can however, use the Second Derivative Test to classify the other critical point so let's do that before we proceed with the sketching work. Here is the value of the second derivative at $t = 3.6$

$$f''(3.6) = -1.245 < 0$$

So, according to the second derivative test $f(3.6)$ is a relative maximum.

Now let's proceed with the work to get the sketch of the graph and notice that once we have the increasing/decreasing information we'll be able to classify $t = 6$.

Here is the number line for the first derivative.



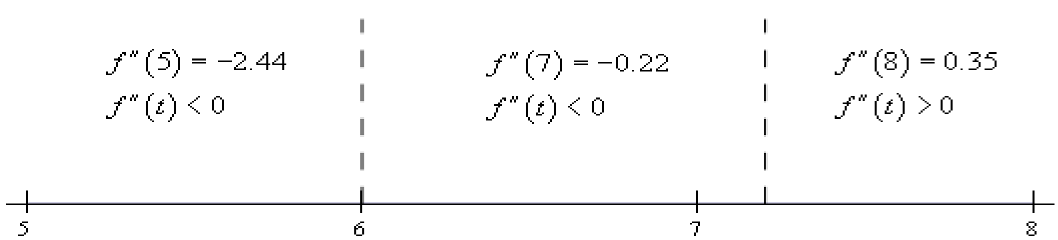
So, according to the first derivative test we can verify that $f(3.6)$ is in fact a relative maximum. We can also see that $f(6)$ is a relative minimum.

Be careful not to assume that a critical point that can't be used in the second derivative test won't be a relative extrema. We've clearly seen now both with this example and in the discussion after we have the test that just because we can't use the Second Derivative Test or the Test doesn't tell us anything about a critical point doesn't mean that the critical point will not be a relative extrema. This is a common mistake that many students make so be careful when using the Second Derivative Test.

Okay, let's finish the problem out. We will need the list of possible inflection points. These are,

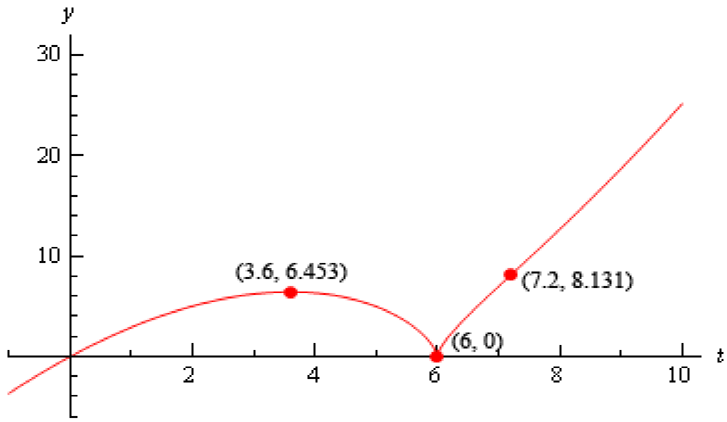
$$t = 6 \qquad\qquad t = \frac{72}{10} = 7.2$$

Here is the number line for the second derivative. Note that we will need this to see if the two points above are in fact inflection points.



So, the concavity only changes at $t = 7.2$ and so this is the only inflection point for this function.

Here is the sketch of the graph.



The change of concavity at $t = 7.2$ is hard to see, but it is there it's just a very subtle change in concavity.