Number Theory

1. Factoring

Factoring is sort of the opposite of multiplying, but more creative than division.

To factor a number what we want to do is find numbers to multiply and get it, for example we can factor 15 as 3 x 5, and we can factor 12 as 2 x 6 or 3 x 4.

Some numbers can be factored many different ways, but others can't be factored very many ways at all.

2. Primes

Natural numbers bigger than 1 that can only be factored as themselves times 1 are called primes.

A prime number is only divisible by 1 and itself, so to check whether a number is prime, you need to test each number between 2 and the number minus 1 to make sure none of them divide evenly into the number.

3. Composites

The opposite of prime is composite. Numbers that can be factored in more interesting ways are called composites.

1 is neither prime nor composite, it isn't in the game, probably partly because in earlier times it wasn't regarded as a number.

4. Short Division

A shortcut for long division for use with small divisors call short division.

In short division you write the long division sign upside down and you do the multiplying and the subtraction in one step and express the remainder kind of like a carried number. Here's an example.

$$876543 \div 7$$

To do this by short division we make the long division symbol upside down and carry the remainders like this.

$$\begin{array}{r|l} 7 & 8\;{}^{1}7\;{}^{3}6\;{}^{1}5\;{}^{1}4\;3 \\ \hline & 1\;2\;5\;2\;2\;0 \quad R\,3 \end{array}$$

7 goes into 8 once, 1 times 7 is 7, 8-7=1, so that is 1 left over, so we carry that 1 to the next place, writing it little next to the 7. Then we continue saying 7 goes into 17 twice with 3 leftover, carry the 3, 7 goes into 36 five times with 1 leftover carry the 1 to the 5, 7 goes into 15 twice with 1 leftover carry the 1 to the 4, 7 goes into 14 twice with nothing leftover. Then in the final step 7 goes into 3 zero times with a remainder of 3.

5. Prime Factoring

The idea behind prime factoring, as opposed to just any old factoring, is to factor and then keep factoring the factors until you can no longer factor anything anymore, namely until you have it written as one big product of prime numbers, because the numbers that you can no longer factor will be by definition prime numbers.

For a good example let's take a number that can be factored in many ways, 60. There are several different ways that you can factor 60.

60 = 1 x 60
60 = 2 x 30
60 = 3 x 20
60 = 4 x 15
60 = 5 x 12
60 = 6 x 10

The idea is to just start with one of these, it doesn't matter which one, except that 1 x 60 won't get you anywhere, and then factor each factor, and then the factors of the factors, and so on until you can do no more.

Let's try 6 x 10, since that would probably be the easiest one to see. We can factor 6 into 2 x 3, and 10 into 2 x 5. Replacing 6 with 2 x 3 and 10 with 2 x 5, this becomes

60 = 2 x 3 x 2 x 5.

2, 3, and 5 are all prime numbers, so we have prime factored 60. All we have to do now is neaten our answer up a bit. It is customary to write prime factorizations in increasing order, that is with the smallest numbers first, so by that custom it is better to write our answer like this.

2 x 2 x 3 x 5

It is also customary to use exponents as a shorthand when we have repeated multiplication. See the whole numbers part of my article Integer Exponents if you need some review with exponents. By using exponents to write the repeat multiplication we would write the final answer for the prime factorization of 60 as

2^2 x 3 x 5.

One interesting thing about prime factoring is that no matter how you start, you will in the end get the same factorization. If instead of starting with 6 x 10, we had started with 4 x 15, we would have factored 4 into 2 x 2, and 15 into 3 x 5, and gotten

2 x 2 x 3 x 5 = 2^2 x 3 x 5.

If we had started with 5 x 12, we would have left 5 as is and factored 12 into 2 x 6 or 3 x 4. If we had factored it into 2 x 6, we would have factored 6 into 2 x 3 and gotten

5 x 2 x 2 x 3 = 2^2 x 3 x 5,

and if we had factored it into 3 x 4, we would have factored 4 into 2 x 2 and gotten

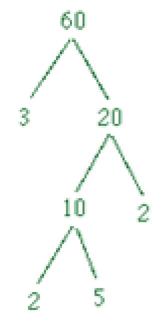
5 x 3 x 2 x 2 = 2^2 x 3 x 5.

And I'll leave the checking of the rest of the possibilities as exercises for the students.

There are a couple of good ways to keep track of things when you are prime factoring. One of them involves making trees and the other involves short divisions one on top of the other.

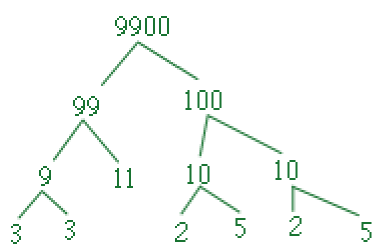
The Tree Method
The method involves drawing branches for a tree every time you factor, and then at the end you put together the ends of the branches for you prime factorization. So here is how the prime factorization of 60 would go using this method.

For variety I am starting with a different factorization this time and first factoring 60 into 3 x 20, drawing branches for the 3 and the 20. Then I factor the 20 into 10 times 2, drawing 10 and 2 branches. Since 3 is prime it doesn't branch. The 2 is also prime, so it doesn't branch either, but I factor the 10 into 2 and 5, making branches for them, and now I am down to only primes. Then for the factorization all I have to do is put together all of the ends of the branches in numerical order and use exponents to indicate repeated multiplication, so I again get an answer of 2^2 x 3 x 5.



For examples

Tree:



It is very easy to factor out a factor of 100 and get it down to much smaller numbers right from the beginning. Then both 99 and 100 are easy to factor. The only disadvantage is that we have to do some sorting in the end and make sure to include all of the ends of the branches, including the 11. However we do it, we have two 2's, two 5's, two 3's, and one 11, so that's 2^2 x 3^2 x 5^2 x 11.

6. GCFs and LCMs

1) Factors and Multiples
Factors and multiples are sort of opposites. If one number is a factor of another number, then the other number is a multiple of it and vice versa.

'A is a factor of B' means the same thing as 'B is a multiple of A'.

For example
2 is a factor of 12 and 12 is a multiple of 2. 125 is a multiple of 5 and 5 is a factor of 125.
9 is a factor of 900 and 900 is a multiple of 9.
70 is a multiple of both 7 and 10, and both 7 and 10 are factors of 70.

2) Common
Common in this setting means something that two or more numbers share.

For example
12 is a common multiple of 4 and 6, because 12 is the result of multiplying 3 times 4 and also the result of multiplying 2 times 6.
2 is a common factor of 4 and 6, because 2 divides evenly into 4, and it also divides evenly into 6, 2 x 2 = 4 and 2 x 3 = 6.

3) Relative Primes
Two numbers are called relative prime if their only common factor is 1.

If the numbers themselves are prime they will always be relative prime, so 3 and 5 are relative prime, but they don't have to be prime to be relative prime.

Example:
8 and 27 are relative prime, but definitely not primes themselves. Even a number with lots of factors like 60 can be relatively prime with respect to another number,
60 and 121 are relatively prime. Perhaps you can think of some other interesting examples of relative primes that are not primes.

4) Finding GCFs and LCMs by Guessing
The LCM of a collection of numbers is the least or smallest number that is a common multiple of them. So if we took all of the common multiples of the numbers and lined them up and asked which one was the smallest, then that would be the least common multiple.

Similarly the GCF of a collection of numbers is the greatest or largest number that is a common factor of the numbers. If we took all of the common factors of those numbers and lined them up and then took that largest one that would be the greatest common factor.

To find GCFs this way, if you don't see a large common factor right away, try looking for primes that divide the numbers evenly, and then if you find more than one of them try multiplying them together and see if that is still a common factor.

For example if you have 12 and 18, you could see that both 2 and 3 are common factors, so maybe 6 will also be a common factor, and it is.

For LCMs there are a couple of techniques you can use. You can always find a common multiple by multiplying the numbers.

For example
15 is a common multiple of 3 and 5, because it is 5 x 3 and 3 x 5.
216 is a common multiple of 12 and 18, since it is 18 x 12 and 12 x 18.

If you are dealing with only two numbers, this will be the smallest one, so the LCM, whenever the two numbers are relatively prime, but if the numbers have a common factor larger than 1, there should be a smaller number that will work.

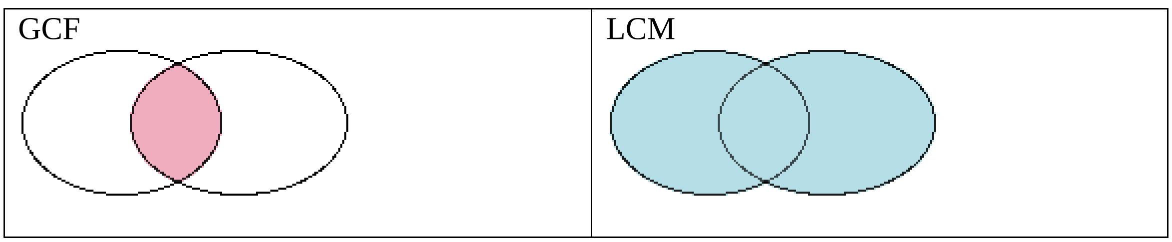
15 is the LCM of 3 and 5, but 216 is not the LCM of 12 and 18, but at least you know from this that the LCM for them can be no larger than 216.

5) Finding GCFs and LCMs by Prime Factoring

Now we are ready to use prime factorizations to find GCFs and LCMs. Using the condition that we just obtained, what are trying to do for the GCF is put together the largest collection of factors in a prime factorization so that it is contained in both of the prime factorizations, and for the LCM we want the smallest prime factorization that both of them are contained in.

If you have heard of intersections and unions, it is sort of like the intersection for the GCF and the union for the LCM.

If you haven't heard of such things, it looks sort of like this for the LCM, that is we want to put together everything they have in common for the GCF and throw everything together, but without duplication for the LCM.



Example: Find the GCF and the LCM of 2 x 3 x 5 and 3 x 5 x 7 x 11.

Solution:

To write down the GCF we just write down all of the factors that they have in common. That would be the 3 and the 5. So the GCF is **3 x 5**.

For the LCM we throw everything together, but without duplication. Instead of using just the numbers that they have in common, we use every number than occurs in at least one of them. So that's 2, 3, 5, 7, and 11. So the LCM of the numbers is **2 x 3 x 5 x 7 x 11**.

Summary

For the GCF use all the primes that occur in all factorizations, and raise them to the lowest powers.

For the LCM use all the primes that occur in any of the factorizations, and raise them to the highest powers.

7. Table of Properties

Let ***a***, ***b***, and ***c*** be real numbers, variables, or algebraic expressions.
(These are properties you need to know.)

	Property	Example
1.	Commutative Property of Addition $a + b = b + a$	$2 + 3 = 3 + 2$
2.	Commutative Property of Multiplication $a \bullet b = b \bullet a$	$2 \bullet (3) = 3 \bullet (2)$
3.	Associative Property of Addition $a + (b + c) = (a + b) + c$	$2 + (3 + 4) = (2 + 3) + 4$
4.	Associative Property of Multiplication $a \bullet (b \bullet c) = (a \bullet b) \bullet c$	$2 \bullet (3 \bullet 4) = (2 \bullet 3) \bullet 4$
5.	Distributive Property $a \bullet (b + c) = a \bullet b + a \bullet c$	$2 \bullet (3 + 4) = 2 \bullet 3 + 2 \bullet 4$
6.	Additive Identity Property $a + 0 = a$	$3 + 0 = 3$
7.	Multiplicative Identity Property $a \bullet 1 = a$	$3 \bullet 1 = 3$
8.	Additive Inverse Property $a + (- a) = 0$	$3 + (-3) = 0$
9.	Multiplicative Inverse Property $a \cdot \left(\frac{1}{a}\right) = 1$ **Note:** ***a*** cannot = 0	$3 \cdot \left(\frac{1}{3}\right) = 1$
10.	Zero Property $a \bullet 0 = 0$	$5 \bullet 0 = 0$

8. Factorial

The factorial function (symbol: **!**) means to multiply a series of descending natural numbers.
Examples:
4! = 4 x 3 x 2 x 1 = 24
1! = 1

➔ **n! = n (n – 1)(n – 2)…1**

n! = n (n-1)! **Example:** 5! = 5 x 4!

n! = (n+1)! / n+1 **Example:** 4! = 5!/5

Using the above definition we have 0! = 1!/1 = 1 → **0! = 1**

In-class questions

1. How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5?

2. Let n be a 5-digit number, and let q and r be the quotient and remainder, respectively, when n is divided by 100. For how many values of n is $q+r$ divisible by 11?

3. What is the largest integer that is a divisor of $(n+1)(n+3)(n+5)(n+7)(n+9)$ for all positive even integers n?

4. How many distinct four-digit numbers are divisible by 3 and have 23 as their last two digits?

5. For each positive integer $m>1$, let $P(m)$ denote the greatest prime factor of m. For how many positive integers n is it true that both $P(n)=\sqrt{n}$ and $P(n+48)=\sqrt{n+48}$?

6. For how many positive integers n less than or equal to 24 is $n!$ evenly divisible by $1+2+\cdots+\mathrm{n}$?

7. Let n denote the smallest positive integer that is divisible by both 4 and 9, and whose base-10 representation consists of only 4's and 9's, with at least one of each. What are the last four digits of n?

8. Let $k=2008^2+2^{2008}$. What is the units digit of k^2+2^k ?