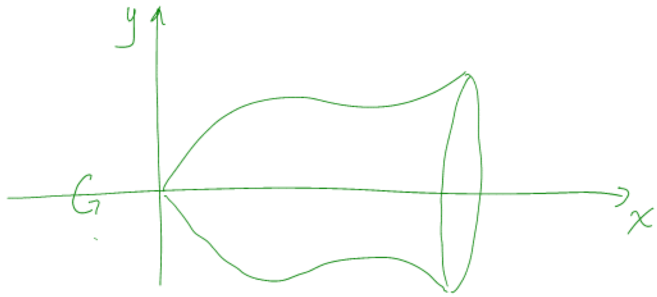
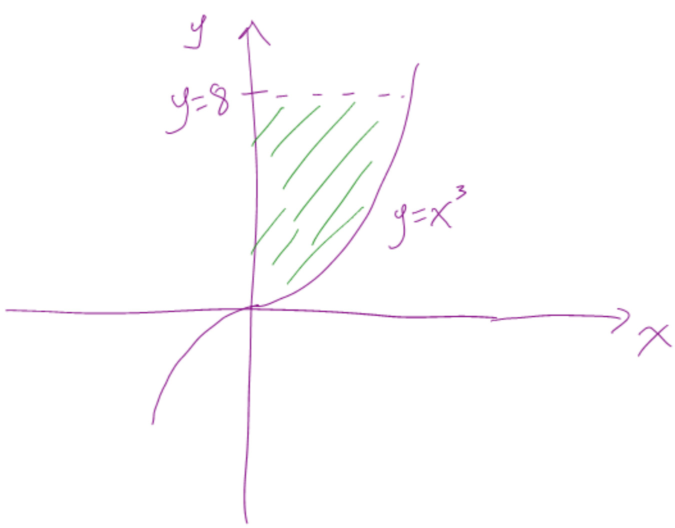
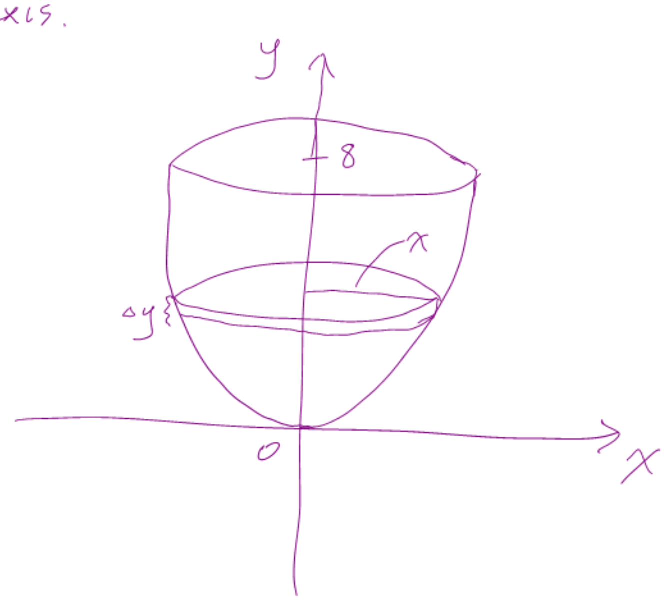
AP Calculus Class 16



Example: Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the y-axis.





$y = x^3 \Rightarrow x = \sqrt[3]{y} \longrightarrow$ radius

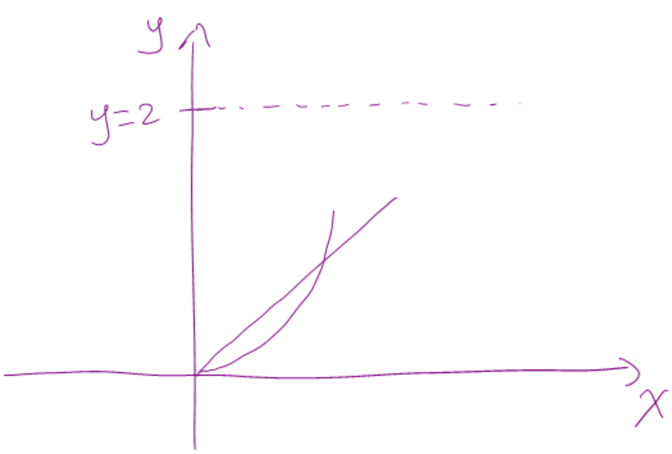
$A(y) = \pi (x)^2 = \pi (\sqrt[3]{y})^2 = \pi y^{\frac{2}{3}}$

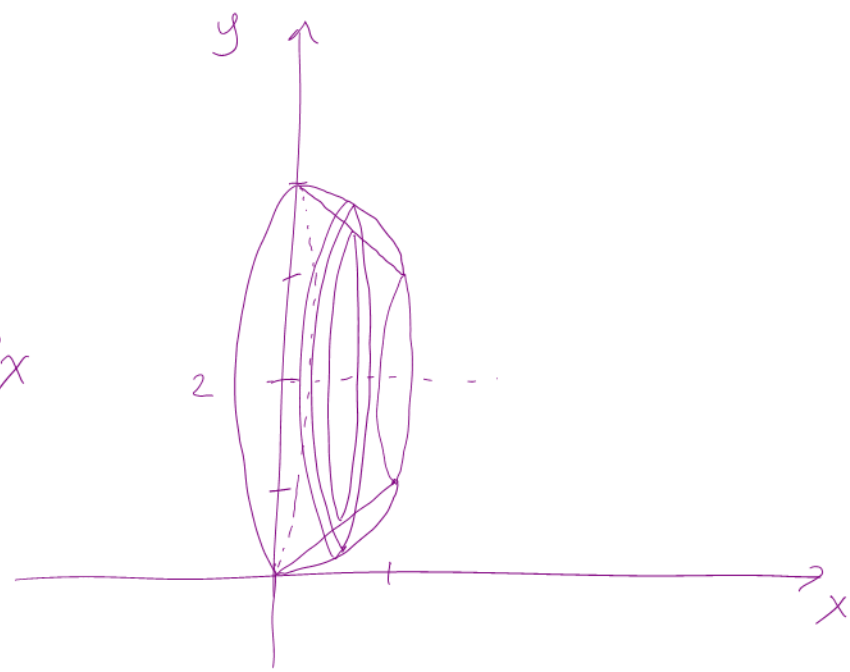
$V_i = A(y)\,\Delta y = \pi y^{\frac{2}{3}} \Delta y$

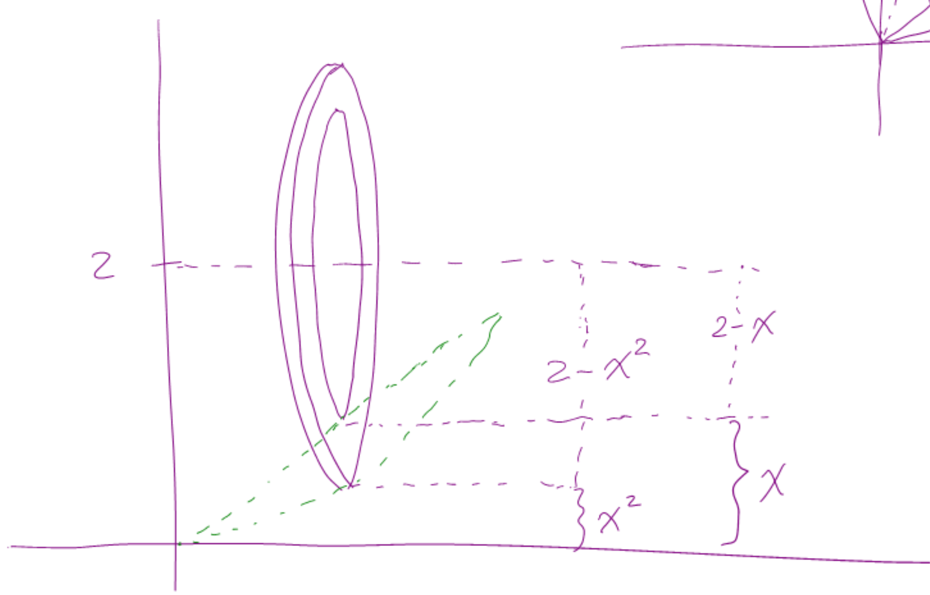
$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{\frac{2}{3}}\,dy$$

$$= \pi \left[\frac{3}{5} y^{\frac{5}{3}} \right]_0^8 = \frac{96\pi}{5}$$

Example: Find the volume of the solid obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2$ about the line $y = 2$.







radius of the disk

outer rad - inner rad

$\Rightarrow (2-x^2) - (2-x)$.

$$A(x) = \pi(2-x^2)^2 - \pi(2-x)^2$$

$$V = \int_0^1 A(x)\,dx$$

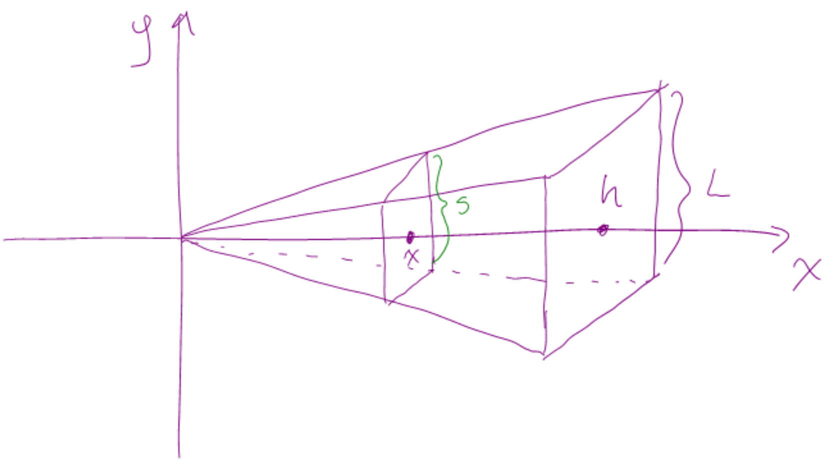
$$= \pi \int_0^1 [(2-x^2)^2 - (2-x)^2]\,dx$$

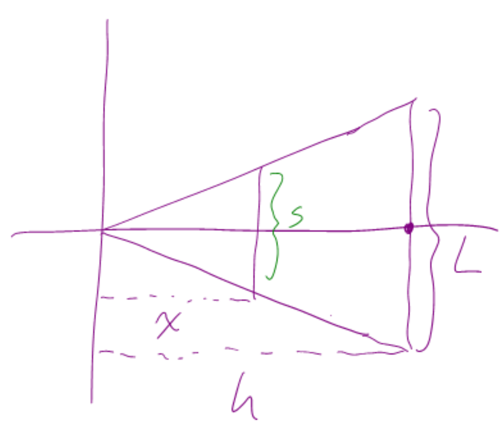
$$= \pi \int_0^1 [4-4x^2+x^4] - [4-2x+x^2]\,dx$$

$$= \pi \int_0^1 [x^4-5x^2+4x]\,dx$$

$$= \pi \left[\frac{x^5}{5} - 5\frac{x^3}{3} + 4\frac{x^2}{2} \right]_0^1 = \frac{8\pi}{15}$$

Example: Find the volume of a pyramid whose base is a square with side length L and height h.





s^2 is the area for the square cross-section.

By similar triangles,

$$\frac{x}{h} = \frac{s/2}{L/2} = \frac{s}{L} \Rightarrow s = \frac{xL}{h}$$

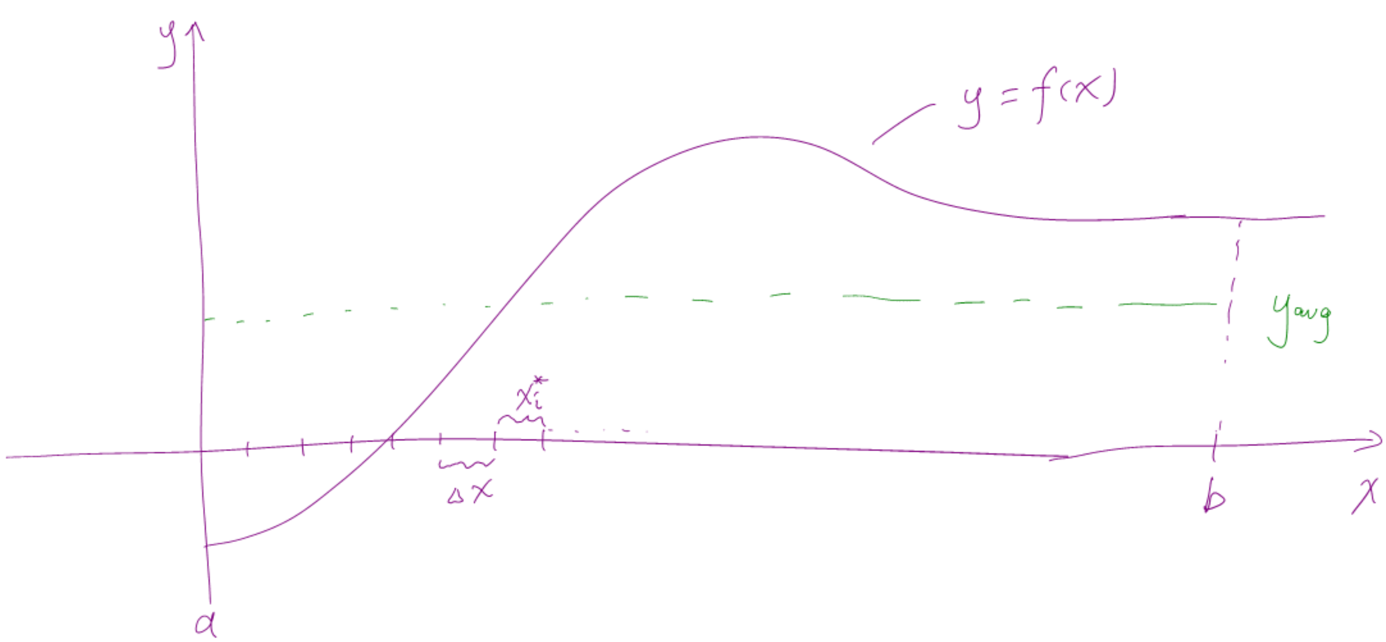
$$A(x) = s^2 = \frac{L^2}{h^2} x^2$$

$$V = \int_0^h A(x)\,dx = \int_0^h \frac{L^2}{h^2} x^2\,dx$$

$$= \frac{L^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{1}{3} L^2 h.$$

The Average Value of a Function

$$y_{avg} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$



A general fun^n $y=f(x)$, $x \in [a,b]$.

Subdivide the fun^n into n equal intervals Δx.

$$\Rightarrow \Delta x = \frac{b-a}{n}$$

Then pick sample points $x_1^*, x_2^*, \ldots, x_n^*$, in each subinterval and calculate the avg of the numbers.

$f(x_1^*), f(x_2^*), \ldots, f(x_n^*)$.

$$\Rightarrow f_{avg} = \frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

since $\Delta x = \dfrac{b-a}{n}$ $\Rightarrow n = \dfrac{b-a}{\Delta x}$

$$\Rightarrow f_{avg} = \frac{f(x_1^*) + \cdots + f(x_n^*)}{\frac{b-a}{\Delta x}}$$

$$= \frac{1}{b-a} \left[f(x_1^*) + \cdots + f(x_n^*) \right] \Delta x$$

$$= \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*) \Delta x.$$

let $n \to \infty$, the limiting value is

$$\lim_{n \to \infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*) \Delta x = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Def^n: The average value of a fun^n f on the interval $[a,b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example: Find the avg value of the fun^n $f(x) = 1+x^2$ on the interval $[-1, 2]$.

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2+1}\int_{-1}^{2} 1+x^2\,dx$$

$$= \frac{1}{3}\left[x+\frac{x^3}{3}\right]_{-1}^{2} = 2.$$

Example: If $f(x) = (x+2)\sin(\sqrt{x+2})$, what's the avg value of f on the closed interval $[0, 6]$.

(A) 2.220 (B) 3.348 (C) 4.757

(D) 20.090 (E) 28.541.

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$$= \frac{1}{6} \underbrace{\int_0^6 \sin(\sqrt{x+2})\,(x+2)\,dx}_{20.090}$$

The Mean Value Theorem for Integrals.

If f is continuous on $[a,b]$, then $\exists$ a number c in $[a,b]$ s.t.

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\Rightarrow \int_a^b f(x)\,dx = f(c)(b-a)$$

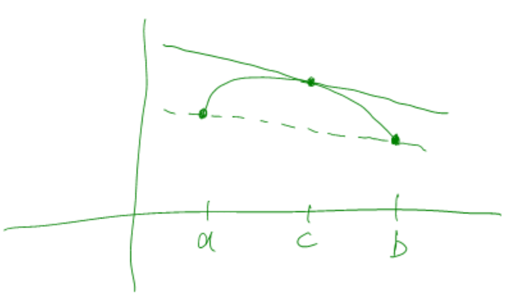
MVT for derivatives.

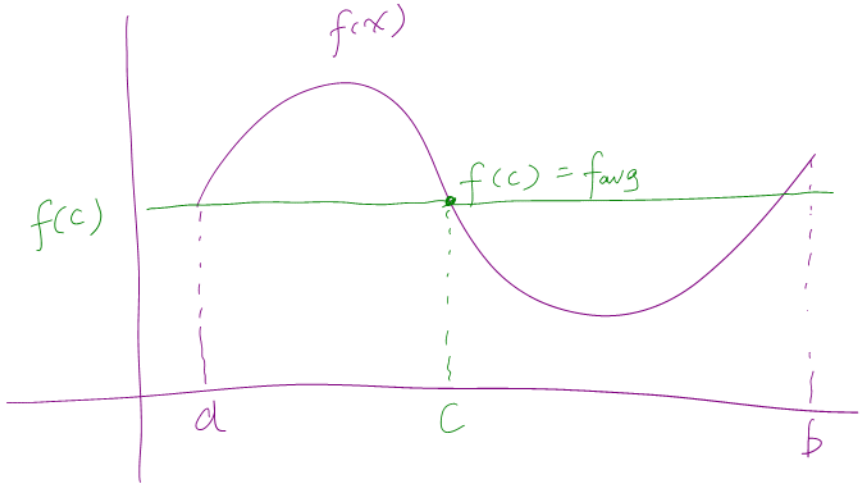
Suppose that f satisfies the following.

1. $f(x)$ is continuous on $[a,b]$.
2. " " differentiable on (a,b),

Then $\exists$ a number c in (a,b) s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$





Example: Since $f(x) = 1 + x^2$ is continuous on $[-1, 2]$.

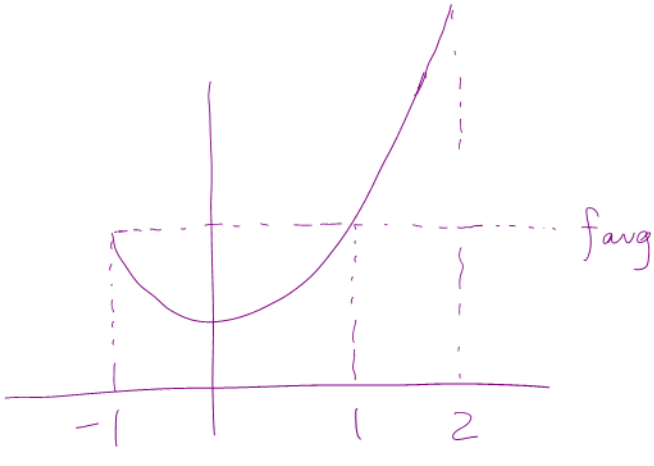
$\Rightarrow$ The MVT for intergal $\Rightarrow \exists$ a number c in $(-1, 2)$ s.t.

$$\int_{-1}^{2} (1+x^2)\,dx = f(c)\,[2-(-1)].$$

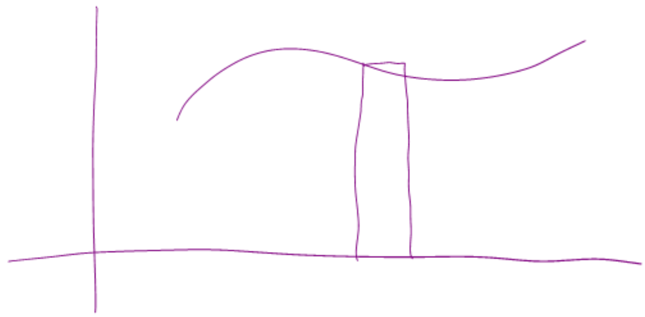
$f_{avg} = f(c) \Rightarrow f(c) = 2.$

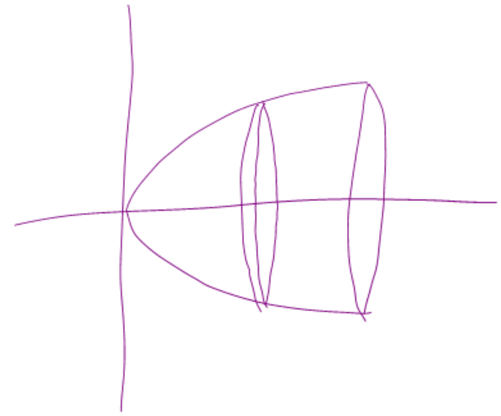
$\Rightarrow f(c) = 1 + c^2 = 2 \qquad \Rightarrow c^2 = 1$

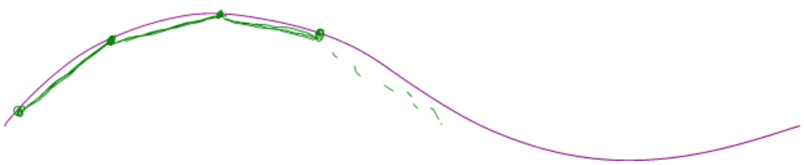
$c = \pm 1.$

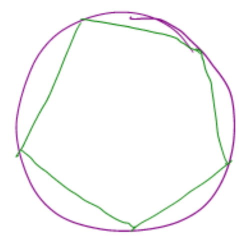


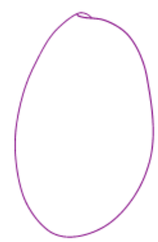
Arc Length.

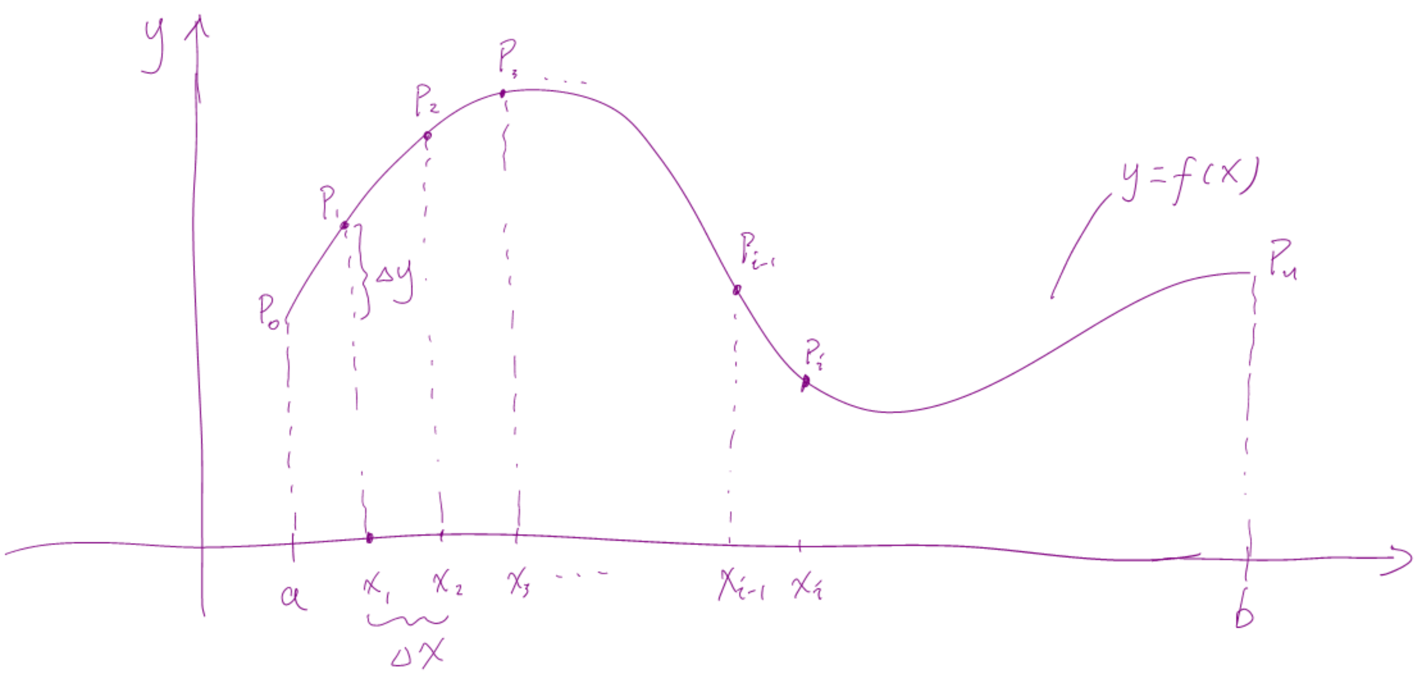










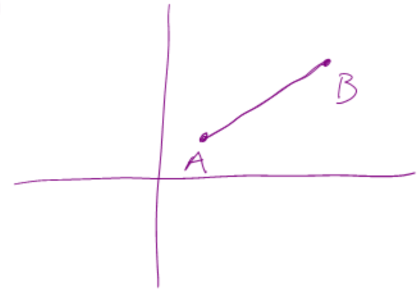


$$\Delta x = x_i - x_{i-1}$$

$$\Delta y_i = y_i - y_{i-1}$$

Length of a curve:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1} P_i| \longrightarrow |AB|$$



$$|P_{i-1} P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

Apply the MVT to the interval $[x_{i-1}, x_i]$.

This means that $\exists$ a number x_i^*

between $[x_{i-1}, x_i]$ s.t.

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

$$\Rightarrow \Delta y_i = f'(x_i^*)(x_i - x_{i-1})$$

$$\Rightarrow |P_{i-1} P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

$$= \sqrt{(\Delta x)^2 + [f'(x_i^*)(x_i - x_{i-1})]^2}$$

$$= \sqrt{(\Delta x)^2 + [f'(x_i^*)\,\Delta x]^2}$$

$$= \sqrt{(\Delta x)^2}\sqrt{1 + (f'(x_i^*))^2}$$

$$= \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x$$

By the previous def^n:

$$\Rightarrow \quad L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+[f'(x_i^*)]^2}\,\Delta x$$

$$\Rightarrow \quad L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

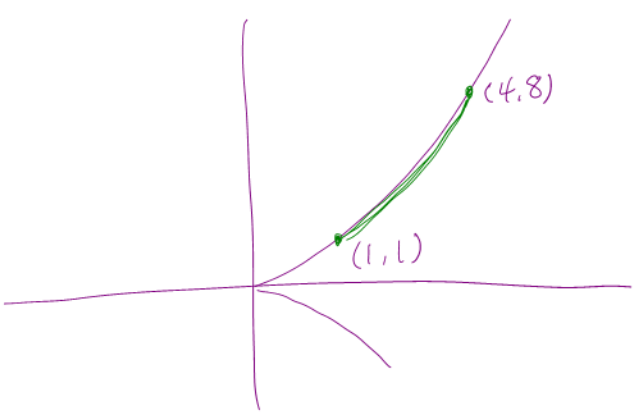
Thm: The arc length formula.

If f' is continuous on $[a,b]$, then the length of the curve $y=f(x)$, $a \le x \le b$, is

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Example: Find the arc length of the curve $y^2 = x^3$ between the points $(1,1)$ and $(4,8)$.



$$y^2 = x^3 \Rightarrow y = x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{3}{2} x^{\frac{1}{2}}.$$

$$L=\int_1^4\sqrt{1+\left(\tfrac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx=\int_1^4\sqrt{1+\tfrac{9}{4}x}\;dx.$$

let $u=1+\frac{9}{4}x$ $\quad du=\frac{9}{4}dx$ $\quad\Rightarrow\ \frac{4}{9}du=dx.$

$u(1)=\frac{13}{4}$ $\quad u(4)=10.$

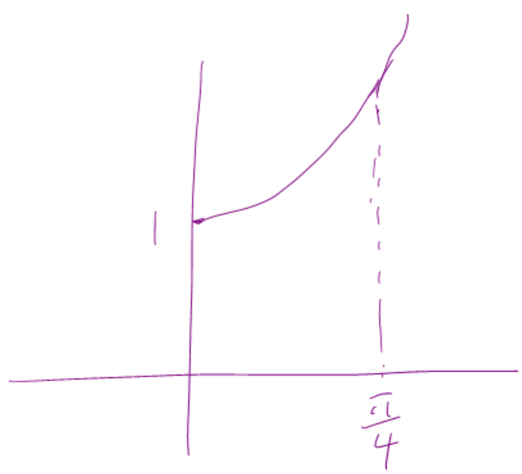
$$\Rightarrow L=\int_{\frac{13}{4}}^{10}\frac{4}{9}\sqrt{u}\,du=\frac{4}{9}\left[\tfrac{2}{3}u^{\frac{3}{2}}\right]_{\frac{13}{4}}^{10}$$

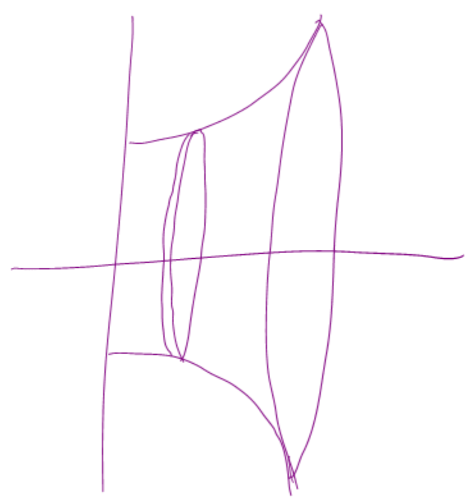
$$=\frac{8}{27}\left[10^{3/2}-\left(\tfrac{13}{4}\right)^{\frac{3}{2}}\right]$$

$$=\frac{1}{27}\left(80\sqrt{10}-13\sqrt{13}\right).$$

Homework 15.

3.





radius is $y=\sec x$.

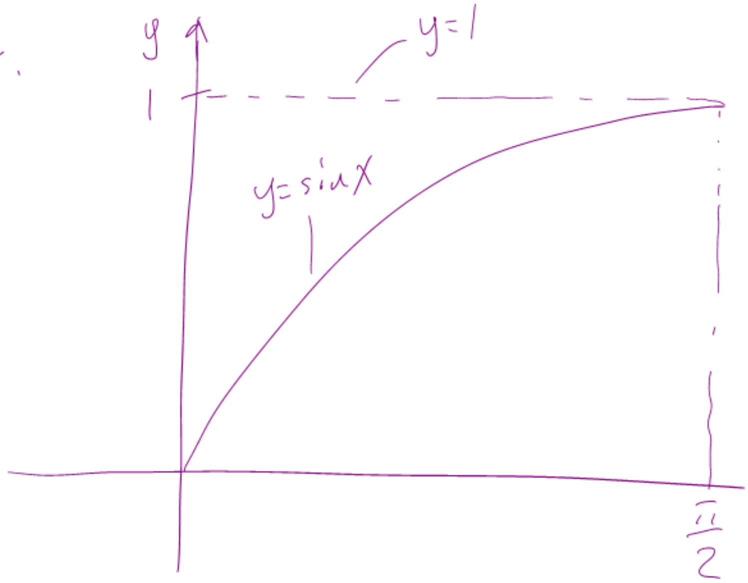
$A(x)=\pi(\sec x)^2$

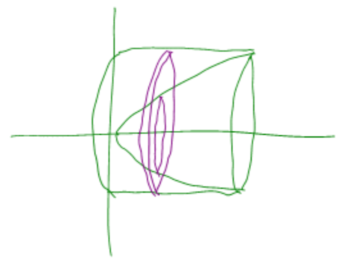
$$V = \int_a^b A(x)\,dx = \int_0^{\frac{\pi}{4}} \pi \sec^2 x\,dx = \pi\left[\tan x\right]_0^{\frac{\pi}{4}}$$

$$= \pi\left[\tan\tfrac{\pi}{4} - \tan 0\right] = \pi \qquad \boxed{C}$$

4. E   5. C.

4.





$$A(x) = \pi(R^2 - r^2)$$

Outer rad is $y=1$

Inner rad is $y=\sin x$

$$V = \int_0^{\frac{\pi}{2}} \pi(1^2 - \sin^2 x)\,dx$$

$$= \pi\int_0^{\frac{\pi}{2}} (1 - \sin^2 x)\,dx$$

$\boxed{E}$

$\int 1 - \sin^2 x\,dx$ E

↑ x

$\int \underline{(1-\sin x)^2}$