AP Calculus Class 19

Homework 18.

7. 5 c), 4 b,

7. $f'(x) = f(x)(1-f(x))$ and $f(0) = \frac{1}{2}$

$y' = y(1-y)$ $y_0 = \frac{1}{2}$

$\Rightarrow \frac{dy}{dx} = y(1-y)$

$\Rightarrow \frac{1}{y(1-y)}\,dy = dx.$

$\Rightarrow \int \frac{1}{y(1-y)}\,dy = \int dx$

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$$

$\Rightarrow A(1-y) + By = 1$

$A - Ay + By = 1$

$\Rightarrow A - B = 0$ and $A = 1$

$\Rightarrow B = 1$

$$\int \frac{1}{y(1-y)}\,dy = \int \frac{1}{y}\,dy + \int \frac{1}{1-y}\,dy$$

$$= \ln|y| - \ln|1-y|$$

So $\displaystyle\int \frac{1}{y(1-y)}\,dy = \int dx$

$\Rightarrow \ln|y| - \ln|1-y| = x + c$ $\qquad \ln a - \ln b = \ln\left(\frac{a}{b}\right)$

$\Rightarrow \ln\left|\frac{y}{1-y}\right| = x + c.$

$\Rightarrow \left|\frac{y}{1-y}\right| = e^{x+c} = e^c e^x$ $\qquad$ Let $e^c = C.$

$\Rightarrow \frac{y}{1-y} = Ce^x$

$\Rightarrow y = Ce^x(1-y) = Ce^x - Ce^x y$

$\Rightarrow y + Ce^x y = Ce^x$

$\Rightarrow y(1+Ce^x) = Ce^x$

$\Rightarrow y = \dfrac{Ce^x}{1+Ce^x}$

$f(x) = \dfrac{Ce^x}{1+Ce^x}$ $\qquad f(0) = \frac{1}{2}$

$\Rightarrow f(0) = \dfrac{Ce^0}{1+Ce^0} = \dfrac{1}{2}$ $\qquad \Rightarrow \dfrac{C}{1+C} = \dfrac{1}{2}$

$\Rightarrow \quad 2C = 1 + C \qquad \Rightarrow C = 1.$

$$\Rightarrow \quad f(x) = \frac{e^x}{1+e^x}$$

5.c) $\quad xy' + y = y^2 \qquad y(1) = -1$

$$x\frac{dy}{dx} = y^2 - y$$

$$\Rightarrow \frac{1}{y^2-y}\,dy = \frac{1}{x}\,dx.$$

$$\Rightarrow \int \frac{1}{y^2-y}\,dy = \int \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1}{y(y-1)}\,dy = \ln|x| + C.$$

$A = -1 \qquad B = 1$

$$\Rightarrow \int \frac{1}{y(y-1)}\,dy = \int \frac{-1}{y}\,dy + \int \frac{1}{y-1}\,dy$$

$$= -\ln|y| + \ln|y-1|$$

$\Rightarrow -\ln|y| + \ln|y-1| = \ln|x| + C. \qquad y(1) = -1$

$\Rightarrow -\ln|-1| + \ln|-2| = \ln 1 + C.$

$\Rightarrow \quad C = \ln 2$

$$\Rightarrow -\ln|y| + \ln|y-1| = \ln|x| + \ln 2.$$

$$\Rightarrow \ln\left|\frac{y-1}{y}\right| = \ln|2x|$$

$$\Rightarrow e^{\ln\left|\frac{y-1}{y}\right|} = e^{\ln|2x|} \quad \Rightarrow \left|\frac{y-1}{y}\right| = |2x|$$

$$\Rightarrow \frac{y-1}{y} = 2x \quad \Rightarrow 1 - \frac{1}{y} = 2x$$

$$\Rightarrow 1 - 2x = \frac{1}{y}$$

$$\Rightarrow y = \frac{1}{1-2x}$$

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4. b) $(x^2+1)\, y' = xy$

$$(x^2+1)\frac{dy}{dx} = xy$$

$$\Rightarrow \frac{1}{y}\,dy = \frac{x}{x^2+1}\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{x}{x^2+1}\,dx \longrightarrow$$

let $u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$$\Rightarrow \ln|y| + C_1 = \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + C_2$$

$$= \frac{1}{2}\ln|x^2+1| + C_2$$

$\Rightarrow \ln|y| = \frac{1}{2}\ln|x^2+1| + C$

$\Rightarrow |y| = e^{\frac{1}{2}\ln|x^2+1|+C}$

$= e^c \, e^{\frac{1}{2}\ln|x^2+1|}$

$= e^c \left(e^{\ln(x^2+1)}\right)^{\frac{1}{2}}$

let $e^c = A$.

$= e^c \sqrt{x^2+1}$

$|y| = A\sqrt{x^2+1}$

$\cancel{A}$ $y = A\sqrt{x^2+1}$

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5. b) $x\cos x = (2y + e^3 y)\, y'$

$x\cos x = (2y + e^3 y)\,\frac{dy}{dx}$

$x\cos x \, dx = (2y + e^3 y)\, dy$

$\Rightarrow \int x\cos x \, dx = \int (2y + e^3 y)\, dy$

$\int x\cos x \, dx$

let $f = x \qquad g' = \cos x$

$f' = 1 \qquad g = \sin x.$

$\Rightarrow \int x\cos x\,dx = x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + C.$

$$\int 2y+e^{3y}\,dy = \int 2y\,dy + \int e^{3y}\,dy$$

$$= y^2 + \frac{1}{3}e^{3y} + C$$

$y(0)=0$

$\frac{1}{3}e^0 = 0+1+C \qquad \Rightarrow C = -\frac{2}{3}$

$\Rightarrow y^2+\frac{1}{3}e^{3y} = x\sin x+\cos x - \frac{2}{3}$

$\Rightarrow$ There is no explicit solution.

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3. d)

4. c) $\frac{du}{dr} = \frac{1+\sqrt{r}}{1+\sqrt{u}}$

$(1+\sqrt{u})du = (1+\sqrt{r})\,dr$

$\int 1+\sqrt{u}\,du = \int 1+\sqrt{r}\,dr$

$u+\frac{2}{3}u^{\frac{3}{2}} = r+\frac{2}{3}r^{\frac{3}{2}}+C.$

$\Rightarrow$ No explicit solution.

Population Growth

$$\frac{dP}{dt} = kP \longrightarrow \text{law of natural growth.}$$

Example: P as the population of bacteria.

$P = 1000$. $\quad P'$ or $\frac{dP}{dt} = 300$ bac/hour.

k: proportionality constant.

we guessed that $P = Ce^{kt}$

$$\frac{dP}{dt} = kP \quad \Rightarrow \quad \frac{1}{P}\,dP = k\,dt$$

$$\Rightarrow \int \frac{1}{P}\,dP = k\int dt.$$

$$\Rightarrow \ln|P| = kt + C.$$

$$\Rightarrow |P| = e^{kt+c} = e^{c}e^{kt} \qquad e^{c} = A.$$

$$\Rightarrow \quad P = Ae^{kt}$$

$$\frac{dP}{dt} = kP$$

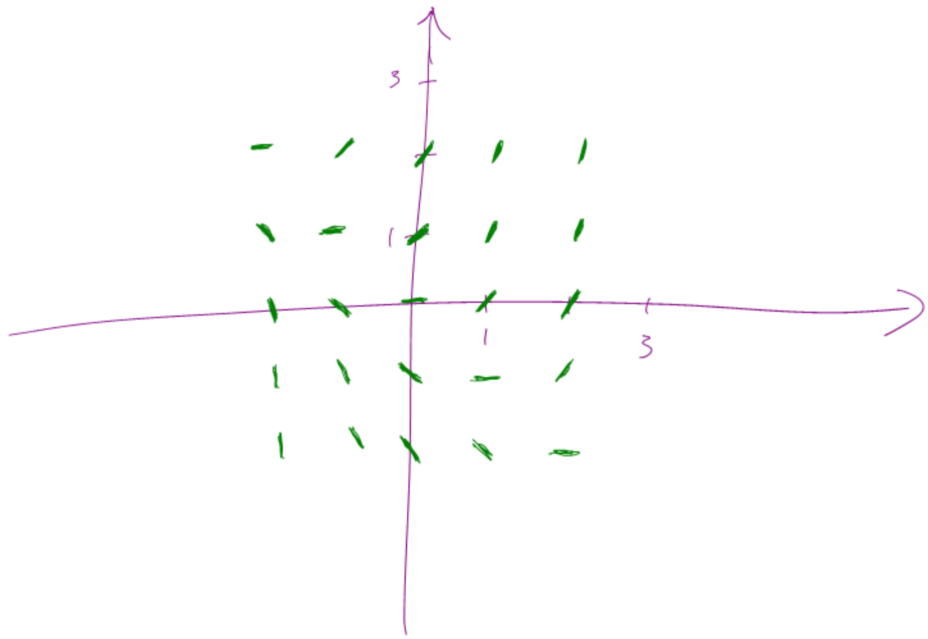
The sol^n is always $P(t) = Ae^{kt} \rightarrow$ General sol^n

If $P(0) = P_0$, then $P(t) = P_0 e^{kt} \rightarrow$ Particular sol^n.

Direction / Slope Field.

Example $y' = x + y$. $\qquad y(0) = 1$.

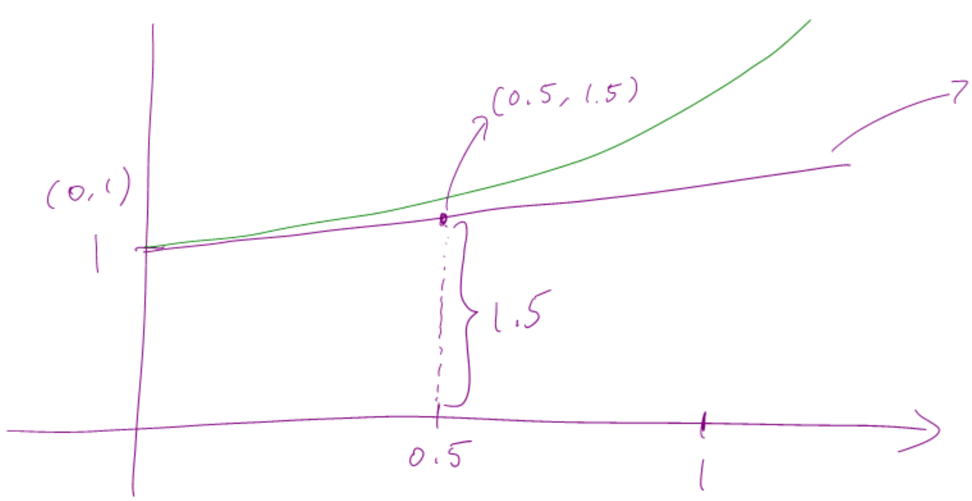
y'	x	y
0	0	0
1	0	1
2	0	2
-1	0	-1
-2	0	-2
1	1	0
2	2	0



$$y' = x^2 + y^2 - 1$$

Euler's Method.

$y' = x + y \qquad y(0) = 1$.

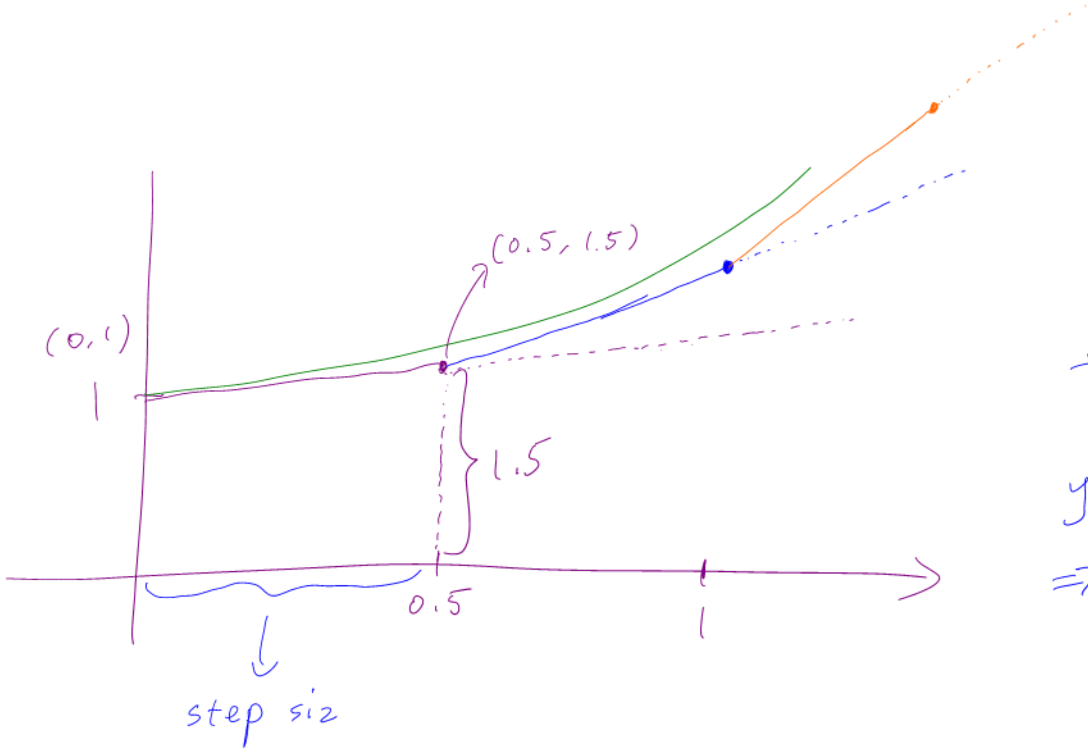


$y = L(x) = x + 1$

$L(0.5) = 0.5 + 1$

$= 1.5$

$$\Rightarrow y'(0.5) = 0.5 + 1.5 = 2$$

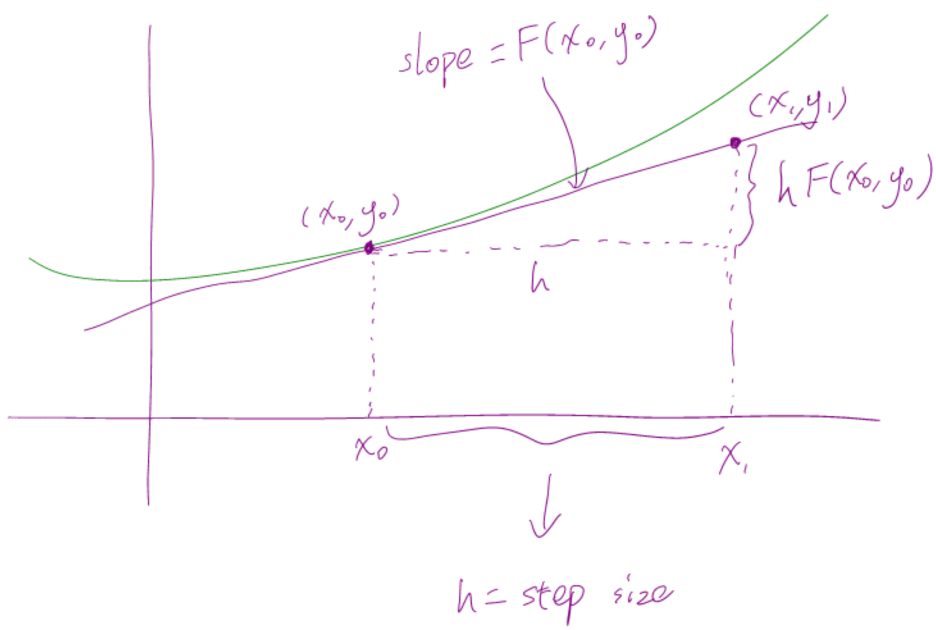


$$\frac{y - 1.5}{x - 0.5} = 2$$

$$y - 1.5 = 2(x - 0.5)$$

$$\Rightarrow y = 1.5 + 2(x - 0.5)$$

$$y = 2x + 0.5,$$



$$F = \frac{y_1 - y_0}{x_1 - x_0}$$

$$F = \frac{y_1 - y_0}{h}$$

$$\Rightarrow y_1 - y_0 = h F(x_0, y_0)$$

$$\boxed{\begin{aligned} \Rightarrow \quad y_1 &= y_0 + h F(x_0, y_0) \\ y_2 &= y_1 + h F(x_1, y_1) \\ &\vdots \\ y_n &= y_{n-1} + h F(x_{n-1}, y_{n-1}) \end{aligned}}$$

Example: Use Euler's Method with step size 0.1 to construct a table of approximate values for the solⁿ of the IVP

$$y' = x + y \qquad y(0) = 1.$$

Solⁿ: $h = 0.1$, $(x_0, y_0) = (0, 1)$

$F(x_0, y_0) = \text{slope} \Rightarrow y' = 0 + 1 = 1 = F(x_0, y_0)$

$$\Rightarrow y_1 = y_0 + h F(x_0, y_0)$$
$$= 1 + 0.1(1) = 1 + 0.1 = 1.1 \longrightarrow (0.1, 1.1)$$

$$\Rightarrow y_2 = y_1 + h F(x_1, y_1)$$
$$= 1.1 + 0.1(0.1 + 1.1) = 1.22 \longrightarrow (0.2, 1.22)$$

$$\Rightarrow \quad y_3 = y_2 + h\,F(x_2, y_2)$$

$$= 1.22 + 0.1\,(0.2 + 1.22) = 1.362. \rightarrow (0.3, 1.362)$$

n	x_n	y_n
1	0.1	0.11
2	0.2	1.22
3	0.3	1.362
4	0.4	1.5282
5	0.5	1.721