AP Calculus Class 4

Example: An object is attached to the end of a string and it's stretched 4 cm beyond its rest position, and the release time is at $t=0$.

$s(t) = 4\cos t$

Find the velocity.

Sol^n: $s'(t) = (4\cos t)'$

$v(t) = -4\sin t.$

$x(t) = 3t^2 + 2t + 1.$ position fun^n.

$x'(t) = 6t + 2 = v(t)$ velocity fun^n.

$v'(t) = 6 = a(t) = x''(t)$ acceleration fun^n.

Example: a) $f(x) = \dfrac{\sec x}{1+\tan x}$. Find $f'(x)$

b) For what value of x does the graph f have a horizontal tangent?

Sol^n: $f'(x) = \dfrac{(\sec x)'(1+\tan x) - (1+\tan x)'(\sec x)}{(1+\tan x)^2}$

$= \dfrac{(\sec x\tan x)(1+\tan x) - (\sec^2 x)(\sec x)}{(1+\tan x)^2}$

$$= \frac{\sec x \tan x + \sec x \tan^2 x - (\sec x)(\sec^2 x)}{(1+\tan x)^2}$$

$\tan^2 x + 1 = \sec^2 x$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

b) Horizontal tangent $\Rightarrow f'(x) = 0$.

$\Rightarrow \cancel{\sec x = 0}$ or $\tan x - 1 = 0$

$\Downarrow$

$\frac{1}{\cos x} = 0$ $\quad$ $\tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}$ $\quad n \in \mathbb{Z}$.

The Chain Rule

Let's say we have the fun^n $\quad F(x) = (x^2+1)^3$

$y = f(u) = u^3 \qquad u = g(x) = x^2 + 1$

$f(g(x)) = (f \circ g)(x)$

The Chain Rule

If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F = f(g(x))$, then F is differentiable and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

If $y=f(u)$ and $y=g(x)$

then $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Example: $F(x)=\sqrt{x^2+1}$ Find $F'(x)$

$F(x)=f(g(x))$ let $f=\sqrt{u}$, $g=x^2+1$

$f'(u)=(u^{\frac{1}{2}})'=\frac{1}{2}u^{-\frac{1}{2}}=\frac{1}{2\sqrt{u}}$

$g'(x)=(x^2+1)'=2x$

$F'(x)=f'(g(x))\,g'(x)=\frac{1}{2\sqrt{x^2+1}}\cdot 2x=\frac{x}{\sqrt{x^2+1}}$

Example:

① $y=\sin(x^2)$ ② $y=\sin^2 x$

Find y'

①. $y=f(g(x)) \rightarrow f=\sin u$ $u=g(x)=x^2$

$y'=f'(g(x))\,g'(x)$ $f'(u)=\cos u$. $g'(x)=2x$

$=\cos(x^2)\,2x=2x\cos x^2$

② $y=f(g(x))$ $f=u^2$ $g=\sin x$, $(\sin x)^2$

$=(u^2)'(\sin x)'=2u(\cos x)$

$=2\sin x\cos x$

when applying the chain rule,

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

outter fun^n inner fun^n derivative of the outter fun^n evaluated at the inner fun^n. derivative of the inner fun^n.

The Power Rule Combined with the Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Example: $y = (x^3 - 1)^{100}$ $\qquad u = x^3 - 1$

$$y' = (100u^{99})u' = 100(x^3-1)^{99}(3x^2)$$

$$= 300x^2(x^3-1)^{99}$$

Example: $g(t) = \Big(\underbrace{\frac{t-2}{2t+1}}_{u}\Big)^9$ $\qquad (u^9)' = 9u^8$

sol^n: $g'(t) = (9u^8)u'$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \left(\frac{t-2}{2t+1}\right)'$$

$$\left(\frac{t-2}{2t+1}\right)' = \frac{(t-2)'(2t+1) - (t-2)(2t+1)'}{(2t+1)^2}$$

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \left(\frac{(2t+1)\cdot 1 - 2(t-2)}{(2t+1)^2}\right) = \frac{45(t-2)^8}{(2t+1)^{10}}$$

Example: $y = e^{\sin x}$ find y'.

let $u = \sin x$, then $y = e^u$

$u = \sin x$

$u' = \cos x$

$y' = (e^u)' \cdot u' = e^u \cdot u'$

$= e^{\sin x} \cos x$

$a^x = (e^{\ln a})^x = e^{\ln a x}$

$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{\ln a x}) = e^{\ln a x} \cdot \frac{d}{dx}(\ln a x)$

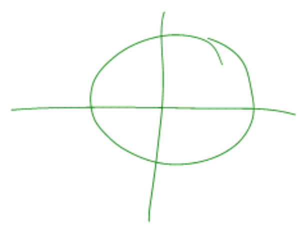
$= e^{\ln a x} \cdot \ln a = a^x \ln a$.

$\Rightarrow \boxed{\frac{d}{dx}(a^x) = a^x \ln a}$

Implicit Differentiation

$y = 2x^2 + 3$ $\quad y' = 4x \rightarrow y(x) = 2x^2 + 3$.

① $x^2 + y^2 = 25 \Rightarrow y^2 = 25 - x^2 = y = \pm\sqrt{25 - x^2}$



② $x^3 + y^3 = 6xy$ $\quad y = f(x)$

$\Rightarrow x^3 + (f(x))^3 = 6x f(x)$.

Example. $x^2+y^2=25$. Find $\frac{dy}{dx}$ $\quad x^2+y^2(x)=25$

$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(25)$ $\longrightarrow$ Differentiating both sides of the equ^n w.r.t. x.

$\frac{d}{dx}x^2+\frac{d}{dx}y^2=0$ $\quad \frac{d}{dx}(y(x))^2$

$\Rightarrow 2x+2y\frac{dy}{dx}=0$ $\Rightarrow$ $2x=-2y\frac{dy}{dx}$ $\quad 2y\frac{dy(x)}{dx}$

(chain rule).

$\Rightarrow \frac{dy}{dx}=-\frac{x}{y}$

At the point $(3,4)$, the slope is $\frac{dy}{dx}=-\frac{3}{4}$

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(2) $x^3+y^3=6xy$.

$\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(6xy)$. $\nearrow (\frac{d}{dx}6x)\cdot y+6x(\frac{d}{dx}y)$

$3x^2+3y^2\cdot\frac{dy}{dx}=6y+6x\cdot\frac{dy}{dx}$ $\longrightarrow$ $\frac{dy}{dx}=y'$

$3x^2+3y^2\cdot y'=6y+6x\cdot y'$

Solve for y'

$\Rightarrow x^2+y^2\cdot y'=2y+2x\cdot y'$

$\Rightarrow y^2\cdot y^2-2xy^2=2y-x^2$ $\Rightarrow$ $(y^2-2x)y'=2y-x^2$

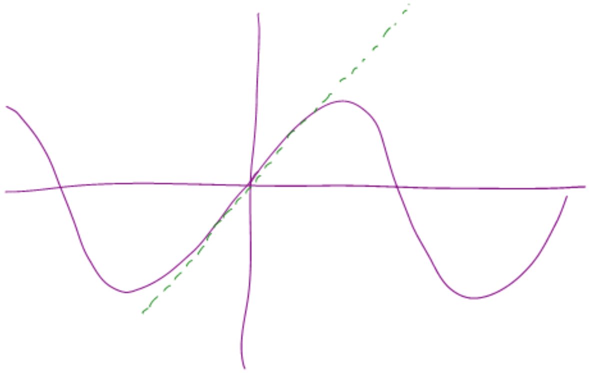
$\Rightarrow y'=\frac{2y-x^2}{y^2-2x}$

Derivatives of Inverse Trigonometric Functions

$\not\Rightarrow (\sin x)^{-1} = \frac{1}{\sin x}$

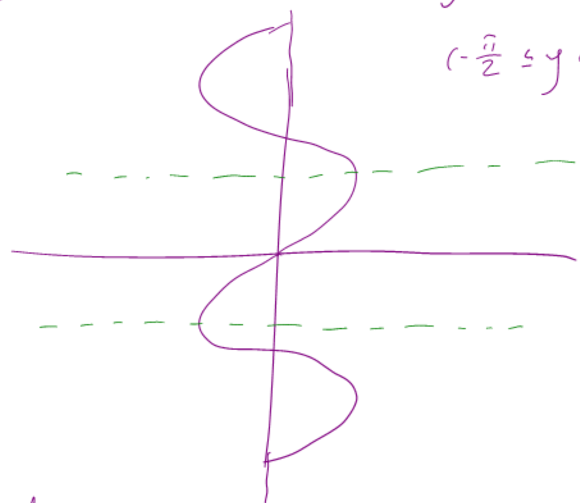
The inverse for $\sin x$ is $\sin^{-1} x$ or $\arcsin x$.

$y = \sin x$



$y = \sin^{-1} x \Rightarrow \sin y = x$

$\left(-\frac{\pi}{2} \le y \le \frac{\pi}{2}\right)$



$$(\sin y)' = x' \Rightarrow \cos y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\Rightarrow \cos x = \sqrt{1 - \sin^2 x}$$

$$= \frac{1}{\sqrt{1 - \sin^2 y}}$$

sub $\sin y = x$ into the result.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

$$\Rightarrow \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{x\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

Higher Order Derivatives

Suppose f is a differentiable function, then we can differentiate f to get f'. If f' is also differentiable, then we can differentiate it again to get $(f')' = f''$. The function f'' is the second derivative of f.

order.

$$f'' = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2f}{dx^2} \qquad \frac{d^3f}{dx^3}$$

Example $f(x) = x\cos x$ Find $f''(x)$.

Sol^n: $f'(x) = \cos x + x(-\sin x) = \cos x - x\sin x$

$f''(x) = -\sin x - (\sin x + x\cos x)$

$= -x\cos x - 2\sin x$.

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f'', f''', $f^{(4)}$, $f^{(n)}$

Example:

Find the $f^{(27)}(x)$ if $f(x) = \cos x$.

Sol^n: $f'(x) = -\sin x$ $\quad f''(x) = -\cos x$

$f'''(x) = \sin x$ $\quad f^{(4)} = \cos x$.

$f^{(5)} = -\sin x$

$f^{(26)} = \cos x$ $\quad f^{(27)} = \sin x$.

$8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$.

$8^{\frac{2}{3}} = (64)^{\frac{1}{3}} = 4$.

$\sqrt{-1} = i$ $\quad i^2 = -1$

$(-1)^{\frac{3}{2}} = (i)^3 = i \cdot i \cdot i = -i$

$(-1)^{\frac{3}{2}} = (-1)^{\frac{1}{2}} = i$

$\frac{z}{w} = \frac{a}{b}$

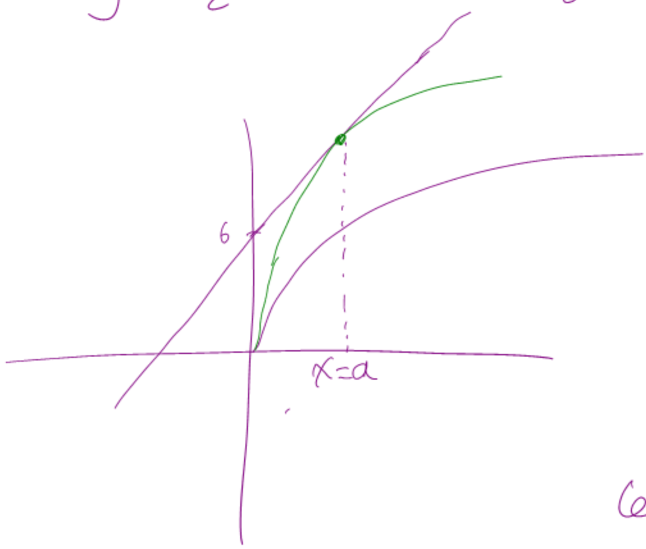
$z = a+bi$

$w = c+di$

$\frac{z}{w} = \frac{z\overline{w}}{|w|^2}$

Homework 3.

5. $y = \frac{3}{2}x + 6$ tangent to the curve $y = c\sqrt{x}$



$y = c\sqrt{x}$

$y' = \frac{c}{2\sqrt{x}}$

$y' = \frac{3}{2}$

$\Rightarrow \frac{3}{2} = \frac{c}{2\sqrt{x}}$

Let's assume the tangent occurs at $x = a$

$\Rightarrow \frac{3}{2} = \frac{c}{2\sqrt{x}} \Rightarrow 3\sqrt{x} = c \Rightarrow 3\sqrt{a} = c.$

$y = c\sqrt{a} = \frac{3}{2}a + 6$

sub $c = \sqrt{a}$ into the above eqn

$\Rightarrow 3\sqrt{a} \cdot \sqrt{a} = \frac{3}{2}a + 6. \Rightarrow 3a = \frac{3}{2}a + 6.$

$\Rightarrow 6a = 3a + 12 \Rightarrow a = 4.$

$\Rightarrow c = 3\sqrt{4} = 6.$

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1. a) $\lim_{x \to 5} \frac{2^x - 32}{x - 5}$. $f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to a} \frac{2^x - 2^5}{x - 5}$ $f(x) = 2^x$

8.a) $y = x^2 f(x)$.

$$y' = (x^2)' f(x) + x^2 f'(x)$$
$$= 2x f(x) + x^2 f'(x)$$