AP Calculus Class 13

Homework 12.

$v(t) = \sin \omega t \cos^2 \omega t$

$s = f(t) \qquad f(0) = 0. \qquad\qquad s = \int v\,dt.$

$s = f(t) = \int \sin \omega t \cos^2 \omega t \, dt.$

let $u = \cos \omega t. \qquad du = -\sin(\omega t)(\omega)\,dt$

$\Rightarrow -\frac{1}{\omega} du = \sin \omega t \, dt.$

$s = -\frac{1}{\omega} \int u^2 du = -\frac{1}{\omega} \frac{u^3}{3} + C = -\frac{1}{3\omega} \cos^3 \omega t + C.$

$f(0) = 0 = -\frac{1}{3\omega} \cos^3(\omega \cdot 0) + C$

$= -\frac{1}{3\omega} (1)^3 + C$

$\Rightarrow C = \frac{1}{3\omega}$

$\Rightarrow f(t) = -\frac{1}{3\omega} \cos^3 \omega t + \frac{1}{3\omega} = \frac{1 - \cos^3 \omega t}{3\omega}$

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4. $\int \frac{1}{x^2 - 6x + 8} dx$

$\frac{1}{x^2 - 6x + 8} = \frac{1}{(x-4)(x-2)} = \frac{A}{x-4} + \frac{B}{x-2}$

$$\frac{1}{(x-4)(x-2)} = \frac{A(x-2)+B(x-4)}{(x-4)(x-2)}$$

$$\Rightarrow \quad A(x-2)+B(x-4)=1$$

$$Ax-2A+Bx-4B=1$$

$$(A+B)x-2(A+2B)=1 \qquad (A+B)x-2(A+2B)=\boxed{0}\,x+\boxed{1}$$

$$A+B=0 \qquad A+2B=-\frac{1}{2}$$

$$A=\frac{1}{2} \qquad B=-\frac{1}{2}$$

$$\Rightarrow \frac{1}{x^2-6x+8} = \frac{1}{2}\left(\frac{1}{x-4}-\frac{1}{x-2}\right)$$

$$\int \frac{1}{x^2-6x+8}\,dx = \frac{1}{2}\left(\ln|x-4|-\ln|x-2|\right)+C.$$

$$= \frac{1}{2}\ln\left|\frac{x-4}{x-2}\right|+C \qquad \boxed{A}$$

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5. $\displaystyle\int_2^3 \frac{3}{(x-1)(x+2)}\,dx$

$$= \ln\left(\frac{8}{5}\right) \qquad \boxed{D}$$

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1. b) $\displaystyle\int x\cos^2 x\,dx \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x).$

$$= \int \frac{1}{2}x(1+\cos 2x)\,dx.$$

$$= \int \frac{1}{2}x + \frac{1}{2}x\cos 2x\,dx$$

$$= \frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos 2x\,dx$$

$$= \frac{1}{4}x^2 + \frac{1}{2}\int x\cos 2x\,dx$$

let $f = x$ $\qquad g' = \cos 2x$

$f' = 1$ $\qquad g = \frac{1}{2}\sin 2x$

$$\frac{1}{2}\int x\cos 2x\,dx = \frac{1}{2}\left(\frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx\right).$$

$$= \frac{1}{2}\left(\frac{x}{2}\sin 2x - \frac{1}{2}\left(-\frac{1}{2}\cos 2x + C\right)\right).$$

$$= \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C.$$

$$\int x\cos^2 x\,dx = \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C.$$

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1. c). $\int \cos^2 x\tan^3 x\,dx.$ $\qquad \tan x = \frac{\sin x}{\cos x}$

$$= \int \cos^2 x \cdot \frac{\sin^3 x}{\cos^3 x}\,dx$$

$$= \int \frac{\sin^3 x}{\cos x}\,dx = \int \frac{\sin^2 x\,\sin x}{\cos x}\,dx.$$ $\qquad \sin^2 x = 1-\cos^2 x$

$$= \int \frac{(1-\cos^2 x)\sin x}{\cos x}\,dx$$

let $u = \cos x$

$du = -\sin x\,dx$

$$= -\int \frac{1-u^2}{u}\,du = -\int \frac{1}{u}\,du + \int \frac{u^2}{u}\,du.$$

$= -\ln u + \frac{1}{2}u^2 + C.$

$= \frac{1}{2}\cos^2 x - \ln|\cos x| + C$

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1. d) $\int \cos^2 x \sin 2x \, dx$ $\quad\quad$ $\sin 2x = 2\sin x \cos x.$

$= 2\int \cos^2 x \,(2\sin x \cos x)\, dx.$

$= 2\int \cos^3 x \sin x \, dx$

let $u = \cos x$ $\quad\quad$ $du = -\sin x \, dx$

$= -2\int u^3 \, du = -2\left(\frac{1}{4}u^4\right) + C.$

$= -\frac{1}{2}\cos^4 x + C.$

Integration by Partial Fractions

$$f(x) = \frac{P(x)}{Q(x)}$$

$$Q(x) = (ax+b)(cx+d) \quad \text{or} \quad (ax+b)^n$$

Example $\displaystyle\int \frac{3x+5}{(x^2+1)(x+2)}\,dx$

$$\frac{Ax+B}{ax^2+bx+c}$$

$$\frac{3x+5}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2}$$

$$= \frac{(Ax+B)(x+2) + C(x^2+1)}{(x^2+1)(x+2)}$$

$$\Rightarrow (Ax+B)(x+2) + C(x^2+1) = 3x+5$$

$$\Rightarrow (A+C)x^2 + (2A+B)x + (2B+C) = 3x+5.$$

$$\Rightarrow A+C=0 \qquad 2A+B=3 \qquad 2B+C=5.$$

$$A = \frac{1}{5} \qquad B = \frac{13}{5} \qquad C = -\frac{1}{5}$$

$$\int \frac{3x+5}{(x^2+1)(x+2)}\,dx = \int \frac{\frac{1}{5}x + \frac{13}{5}}{x^2+1} - \frac{\frac{1}{5}}{x+2}\,dx.$$

$$= \frac{1}{5}\int \frac{x}{x^2+1}\,dx + \frac{13}{5}\int \frac{1}{x^2+1}\,dx - \frac{1}{5}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{5}\left(\frac{1}{2}\ln|x^2+1|\right) + \frac{13}{5}\tan^{-1}x - \frac{1}{5}\ln|x+2| + C.$$

$$a_1x^3 + a_2x^2 + a_3x + a_4 = (b_1x^2 + b_2x + b_3)(c_1x + c_2)\,d$$

$$f(x) = \frac{P(x)}{Q(x)} \qquad \deg P < \deg Q.$$

what happens if $\deg P \geq \deg Q$?

Example: $\displaystyle\int \frac{x^3+x}{x-1}\,dx$

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}, \qquad \text{where } \deg R < \deg Q.$$

$$\begin{array}{r|l}
 & x^2 + x + 2 \longrightarrow S(x) \\
\hline
x-1 & x^3 + 0x^2 + x \\
 & x^3 - x^2 \\
\hline
 & \quad x^2 + x \\
 & \quad x^2 - x \\
\hline
 & \qquad 2x + 0 \\
 & \qquad 2x - 2 \\
\hline
 & \qquad\quad 2 \longrightarrow R(x)
\end{array}$$

$$\begin{array}{c|cccc}
1 & 1 & 0 & 1 & 0 \\
 & & 1 & 1 & \\
\hline
 & 1 & 1 & & 2 \\
 & x-1 & & 2 &
\end{array}$$

$$\int \frac{x^3-x}{x-1}\,dx = \int \underbrace{x^2+x+2}_{S(x)} + \underbrace{\frac{2}{x-1}}_{R(x)}\,dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

Summary: 4 Cases for IBPF

① Distinct linear factors

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

② Repeated linear factors

$$\frac{P(x)}{Q(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

③ Distinct, irreducible quadratic factors

$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \cdots + \frac{A_nx+B_n}{a_nx^2+b_nx+c_n}$$

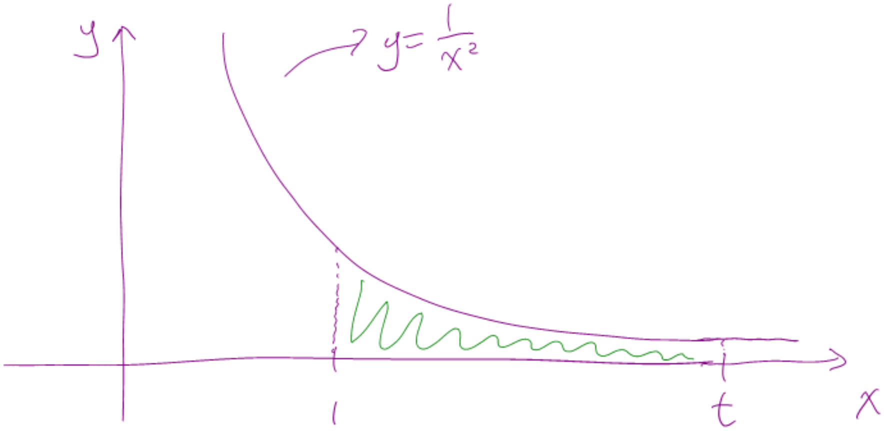
④ Repeated, irreducible quadratic factors

$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Improper Integrals.

Two types:

① Infinite intervals ② Discontinuous integrand.



$$A(t) = \int_1^t \frac{1}{x^2}\,dx$$
$$= -\frac{1}{x}\Big]_1^t$$
$$= 1 - \frac{1}{t}$$

$A(t) < 1$

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = 1.$$

The area of the shaded region approaches 1 as $t \to \infty$.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = 1.$$

Def^n: Type 1 Improper Integral.

a) If $\int_a^t f(x)\,dx$ exists $\forall$ numbers $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

b) If $\int_t^b f(x)\,dx$ exists $\forall$ numbers $t \le b$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

The improper integral integrals in a) and b) are called **convergent** if the corresponding limits exist and **divergent** if the limits do not.

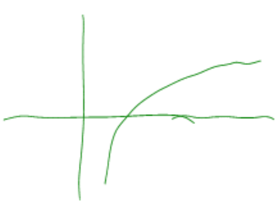
c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^\infty f(x)\,dx$$

Example $\int_1^\infty \frac{1}{x}\,dx$ convergent or divergent?

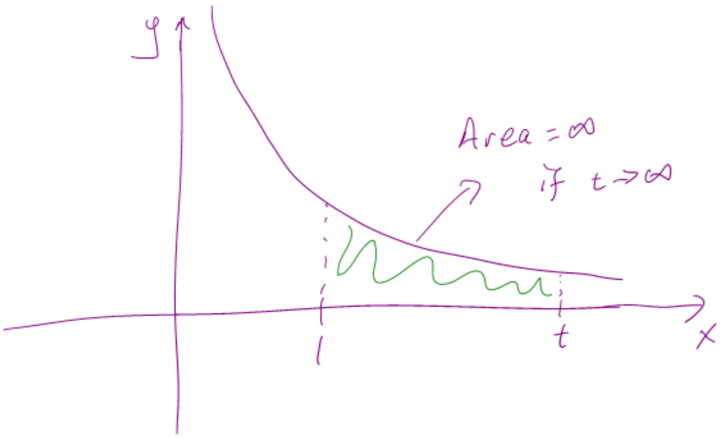
$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \left[\ln|x| \right]_1^t$$

$$= \lim_{t\to\infty} \left[\ln t - \ln 1 \right] = \lim_{t\to\infty} \ln t = \infty$$

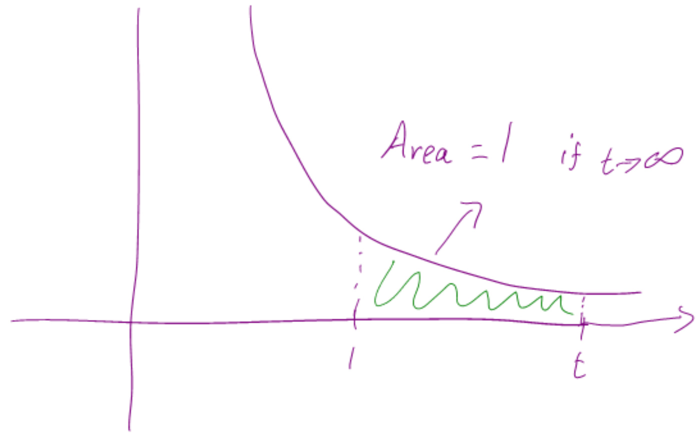


$\Rightarrow$ Divergent.

$y = \frac{1}{x}$



$y = \frac{1}{x^2}$



$\int_1^\infty \frac{1}{x^p}\,dx$ For what value of p does the integral converge or diverge.

We know that if $p=1$, the integral diverges.

Let's assume $p \neq 1$.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx$$

$$= \lim_{t\to\infty} \left. \frac{x^{-p+1}}{-p+1} \right]_1^t$$

$$= \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right].$$

$$\frac{x^{-p+1}}{-p+1} = \frac{x^{-p+1}}{1-p} = \left(\frac{1}{1-p}\right)\left(x^{-p+1}\right) = \left(\frac{1}{1-p}\right)\left(x^{-(p-1)}\right)$$

$$= \left(\frac{1}{1-p}\right)\left(\frac{1}{x^{(p-1)}}\right)$$

① If $p > 1$, then $p - 1 > 0$

$\Rightarrow$ as $t \to \infty$, $t^{p-1} \to \infty$

$$\Rightarrow \frac{1}{t^{p-1}} \to 0$$

$$\Rightarrow \int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} \to \text{Finite number.}$$

$\Rightarrow$ Convergent.

② If $p < 1$, then $p - 1 < 0$.

$$\Rightarrow \frac{1}{t^{p-1}} = t^{1-p} \to \infty \quad \text{as } t \to \infty.$$

$\Rightarrow$ Divergent

$\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Example: $\int_{-\infty}^0 x e^x \, dx$.

By def^n: $\int_{-\infty}^0 x e^x \, dx = \lim_{t \to -\infty} \int_t^0 x e^x \, dx$.

Apply IBP.

let $f = x$ $\quad g' = e^x$

$f' = 1$ $\quad g = e^x$

$$\int_t^0 xe^x dx = xe^x\Big]_t^0 - \int_t^0 e^x dx.$$

$$= -te^t - e^x\Big]_t^0 = -te^t - [e^0 - e^t]$$

$$= -te^t - 1 + e^t.$$

Apply l'Hospital's rule to

$$\lim_{t\to\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} \frac{1}{-e^{-t}}$$

$$= \lim_{t\to-\infty} (-e^t) = 0.$$

$$\Rightarrow \int_{-\infty}^0 xe^x dx = \lim_{t\to-\infty} (-te^t - 1 + e^t)$$

$$= -0 - 1 + 0 = -1.$$

Example: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$

$\displaystyle\int_{-\infty}^a f(x) + \int_a^\infty f(x)$ $\qquad$ let $a = 0$.

$$= \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^\infty \frac{1}{1+x^2} dx$$

For $\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \tan^{-1} x \Big]_{t}^{0}$

$$= \lim_{t\to-\infty} (\tan^{-1} 0 - \tan^{-1} t)$$

$$= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \tan^{-1} x \Big]_{0}^{t}$$

$$= \lim_{t\to\infty} (\tan^{-1} t - \tan^{-1} 0)$$

$$= \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi .$$