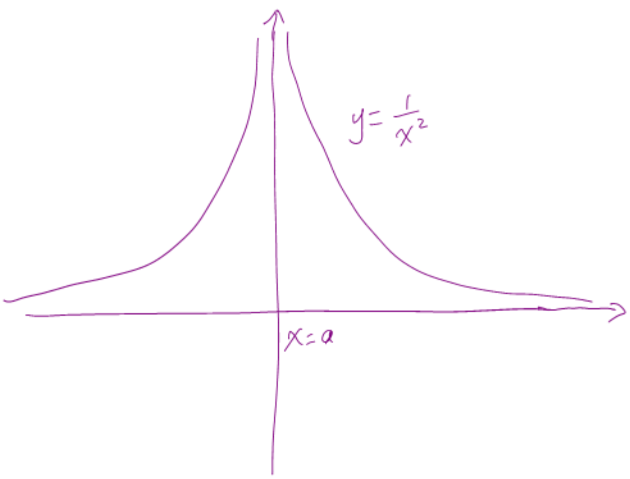
AP Calculus Class 2

Infinite Limits and Vertical Asymptote

Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.



Sol^n: As $x \to 0$, $x^2 \to 0$.

Then $\frac{1}{x^2} \to$ very large.

$\lim_{x \to a} f(x) = \infty$.

$\underbrace{100 \ldots 0}_{1 \text{ billion}}{}^{\overbrace{10 \ldots 0}^{1 \text{ billion}}}$

Because ∞ is not a number, the usual arithmetic operations don't apply.

$3+5$, $3-5$, 3×5, $\frac{3}{5}$.

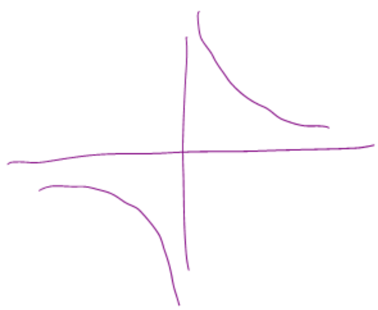
$\infty + \infty \neq 2\infty$ X $\quad \infty - \infty \neq 0$. X

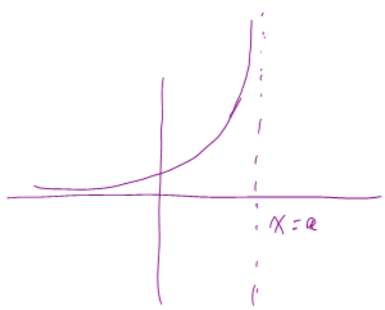
Definition

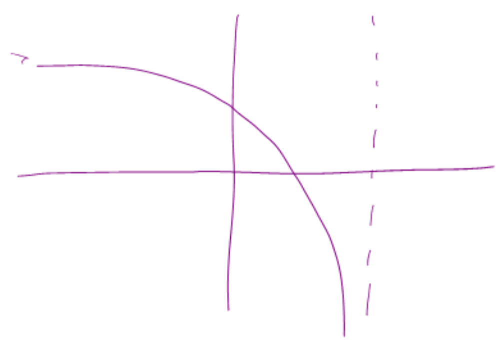
> Let f be a function defined on both sides of a, except possibly at a itself. Then
>
> $$\lim_{x \to a} f(x) = \infty$$
>
> means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close t a, but not equal to a.

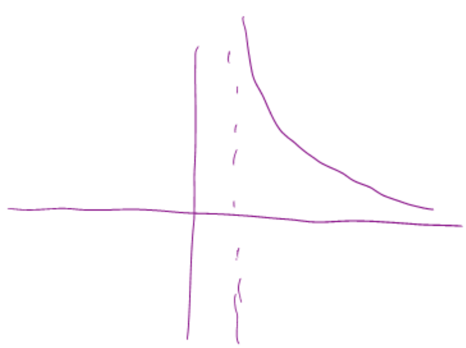
A similar behaviour for fun^n that become infinitely large in the negative direction. then $f(x) \to -\infty$.

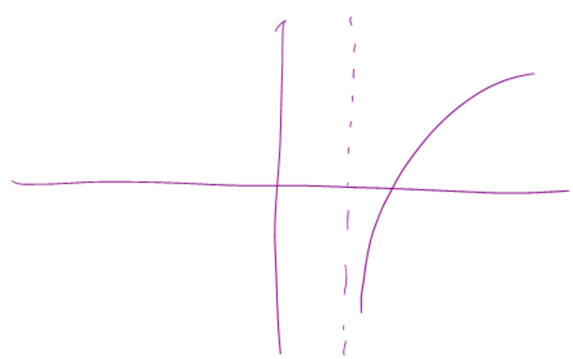
$$\lim_{x\to 0}\left(-\frac{1}{x^2}\right) = -\infty.$$









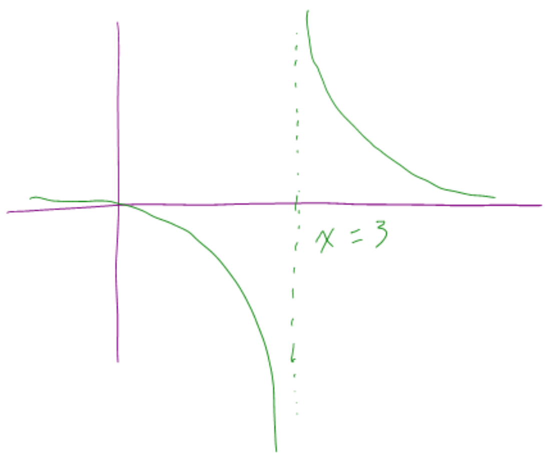


Definition

The line $x = a$ is called a **vertical asymptote** of the function $f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Example: Find the limit $\lim_{x \to 3^+} \frac{2x}{x-3}$ and $\lim_{x \to 3^-} \frac{2x}{x-3}$.



$$\lim_{x \to 3^+} \frac{2(x)}{x-3} = \infty$$

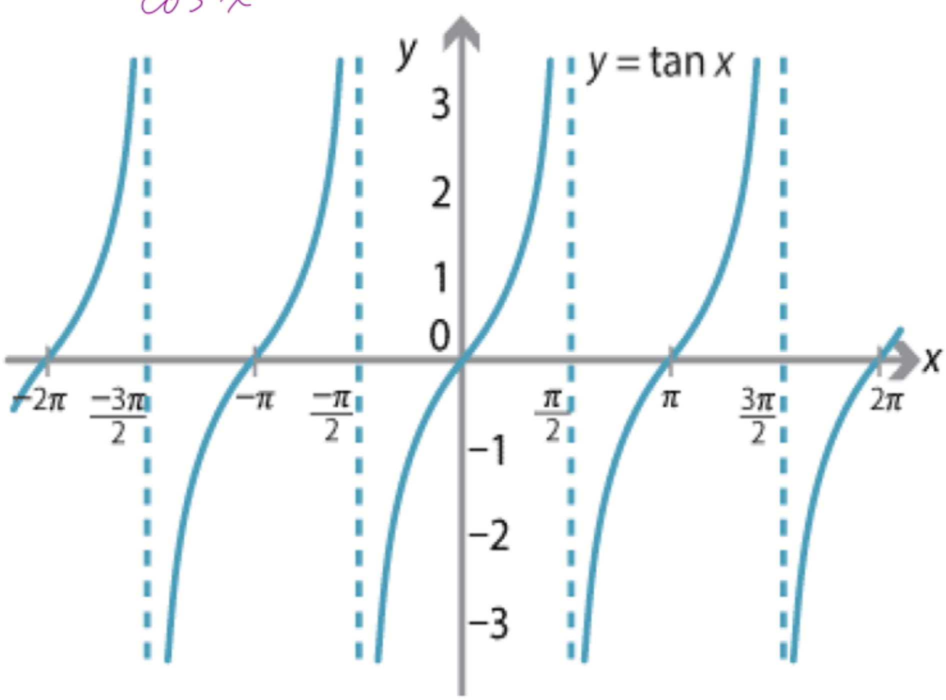
$$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$$

$3^- < 3$

Example

Find the vertical asymptotes of $f(x) = \tan x$.

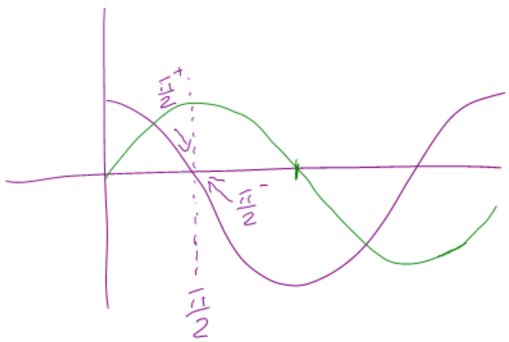
$\tan x = \frac{\sin x}{\cos x}$



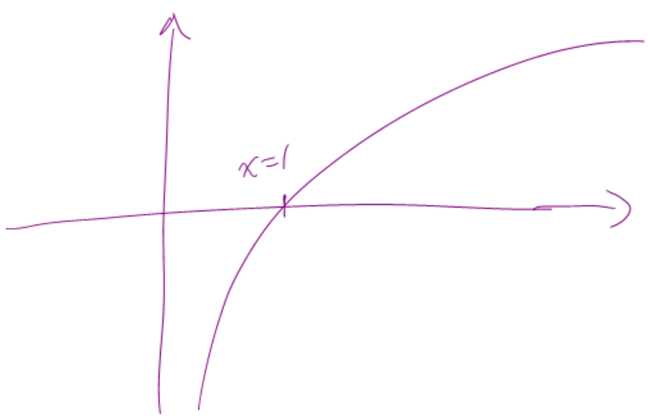
$f(x) = \tan x = \dfrac{\sin x}{\cos x}$

$\cos x \to 0$.

$\cos x = 0 \Rightarrow x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots, \pm \frac{(2n+1)\pi}{2}$



Example: $y = \ln x$



Example: Determine the infinite limit $\lim_{x \to 5^-} \dfrac{e^x}{(x-5)^3}$

$$\lim_{x \to 5^-} \frac{e^x}{(x-5)^3} = \lim_{x \to 5^-} \frac{e^x \ (+)}{(x-5)^3 \ (-)} \to -\infty.$$

$5^- < 5 \Rightarrow (x-5)^3 < 0$

Example $\lim_{x \to \pi^-} \cot x = \lim_{x \to \pi^-} \frac{\cos x}{\sin x} = -\infty$

Limit at Infinity and Horizontal Asymptotes

Definition Limit at ∞

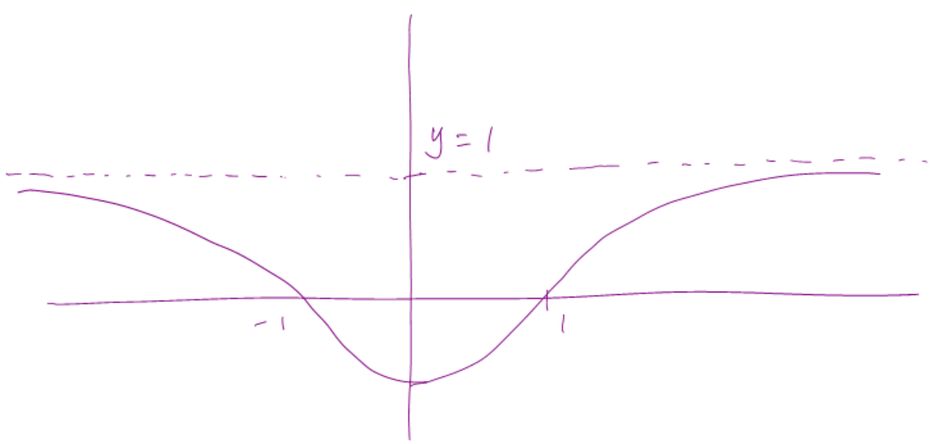
Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

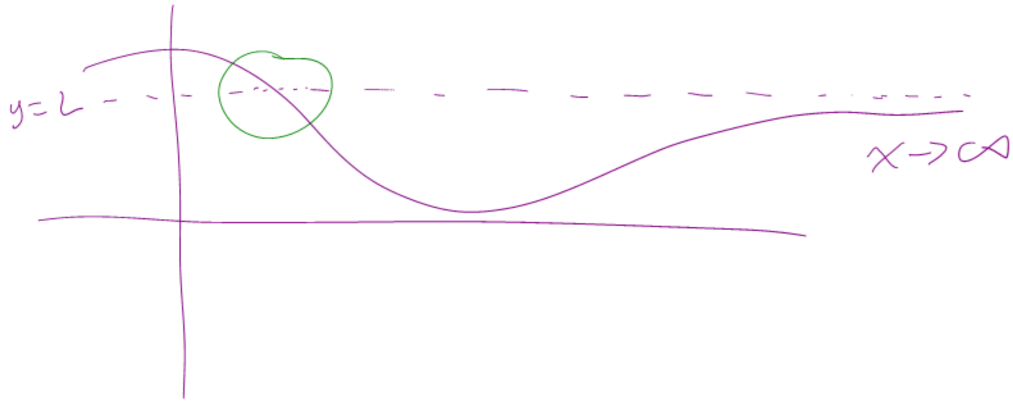
$f(x) = \frac{x^2 - 1}{x^2 + 1}$

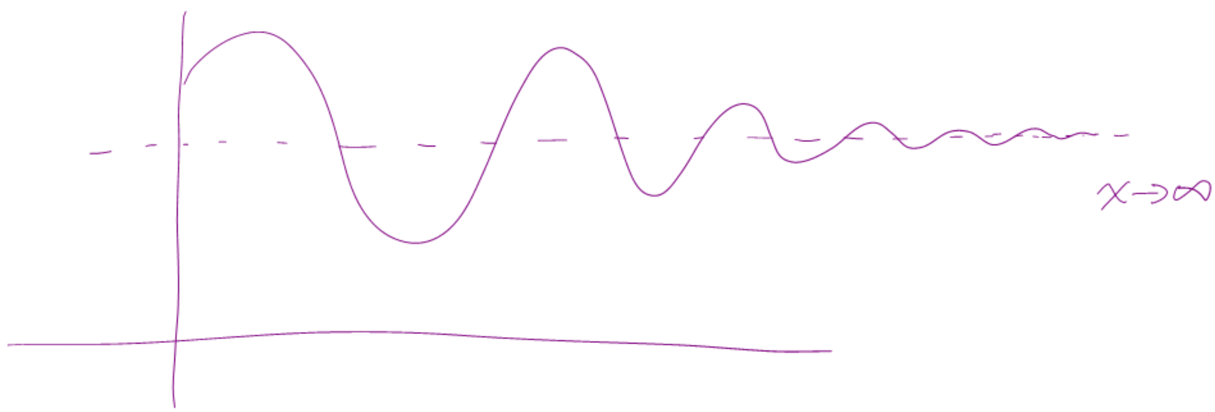
$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$



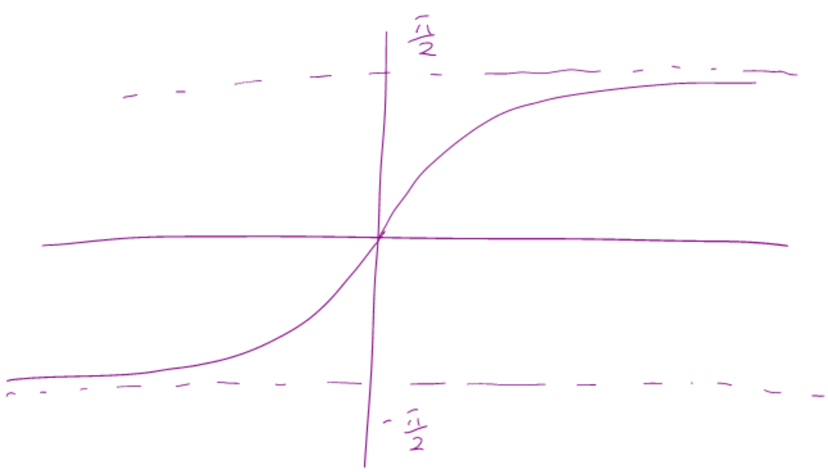
Definition

The line $y = L$ is called a **horizontal asymptote** of the function $f(x)$ if either
$\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$

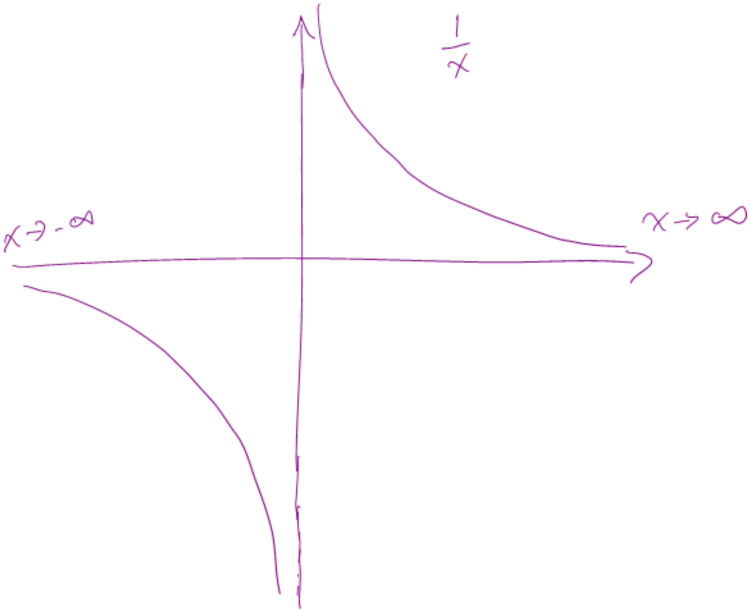




Example: $y = \tan^{-1} x \to \arctan x$.



Example: Find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$



$\lim_{x \to \infty} \frac{1}{x} = 0$

$\lim_{x \to -\infty} \frac{1}{x} = 0$

Example: Find the vertical and horizontal asymptotes of $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$

- For h.a., let $x \to \pm\infty$
- For v.a. set the denominator of a funⁿ to 0. (From the left and the right).

let $x \to \infty$.

$$\lim_{x \to \infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x \to \infty} \frac{\sqrt{2x^2+1} \cdot \frac{1}{x}}{3x-5 \cdot \frac{1}{x}} = \lim_{x \to \infty} \frac{\sqrt{2+\cancel{\frac{1}{x^2}}}}{3-\cancel{\frac{5}{x}}}$$

$$\sqrt{2x^2+1} \cdot \frac{1}{x}$$

$$= \frac{\sqrt{2+0}}{3-0} = \frac{\sqrt{2}}{3}$$

Let $x \to -\infty$

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{-\sqrt{2}}{3}$$

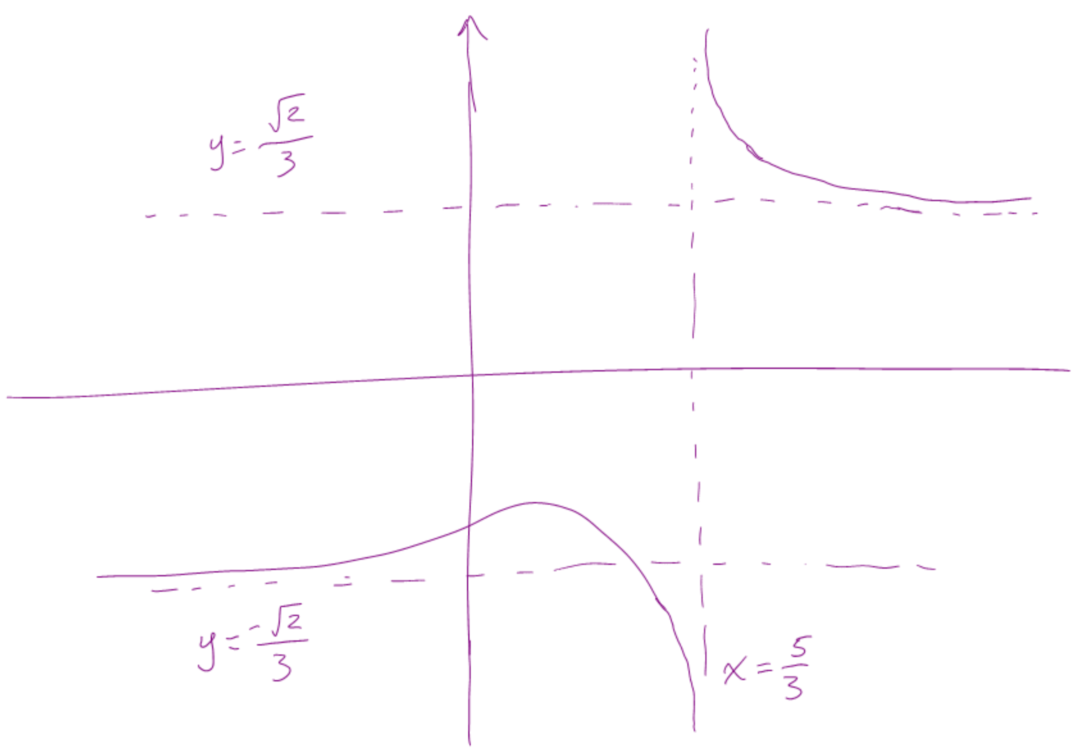
$$\frac{1}{x}\sqrt{2x^2+1} = -\frac{1}{\sqrt{x^2}}\sqrt{2x^2+1} = -\sqrt{2+\frac{1}{x^2}}$$

For the h.a.

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5} \to 0. \quad \Rightarrow \quad 3x-5=0.$$

$$\Rightarrow x = \frac{5}{3} \qquad x \to \frac{5}{3}^- \quad \text{and} \quad x \to \frac{5}{3}^+$$

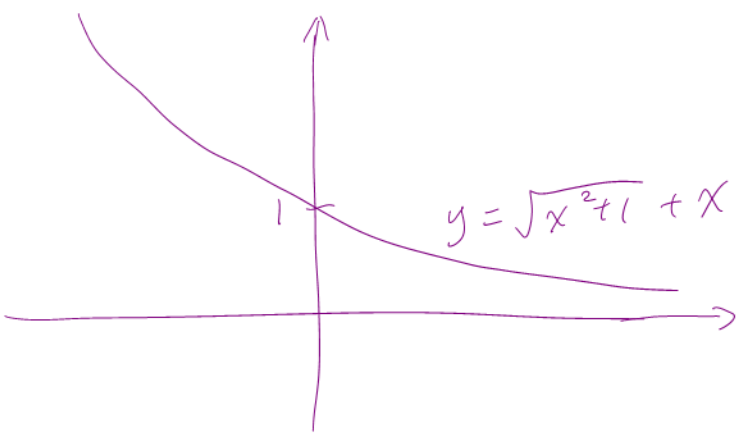
$$\lim_{x\to\frac{5}{3}^+} \frac{\sqrt{2x^2+1}}{\underbrace{3x-5}_{\to +}} = \infty \qquad \lim_{x\to\frac{5}{3}^-} \frac{\sqrt{2x^2+1}}{\underbrace{3x-5}_{\to -}} = -\infty$$



Example: Compute $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$.

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+1}\right)^2-x^2}{\sqrt{x^2+1}+x}$$

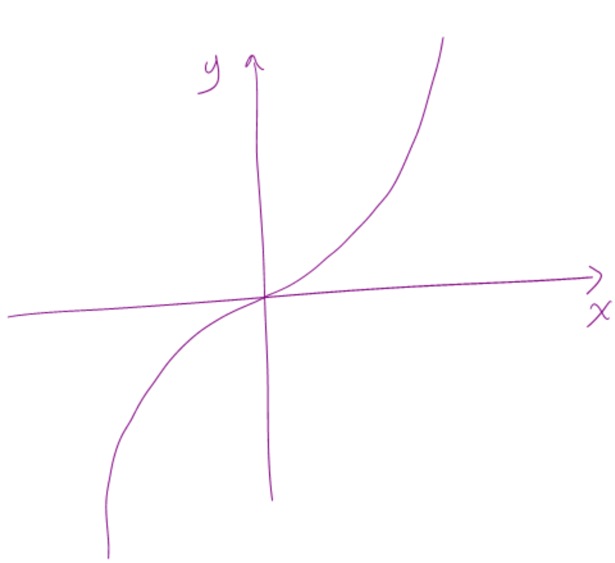
$$=\lim_{x\to\infty}\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=\frac{1}{\infty+\infty}\to 0$$



Infinite Limits at Infinity

$$\lim_{x\to\infty} f(x)=\infty.$$

Example: $f(x)=x^3$



$$\lim_{x\to\infty} x^3=\infty$$

$$\lim_{x\to-\infty} x^3=-\infty$$

Example: Find $\lim_{x \to \infty} (x^2 - x)$.

$$\lim_{x \to \infty} x(x-1) = \infty$$

$$\infty \cdot \infty$$

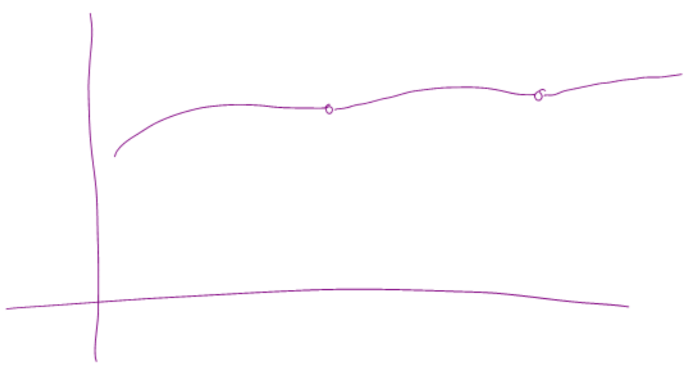
$2\infty + \infty \neq 3\infty$

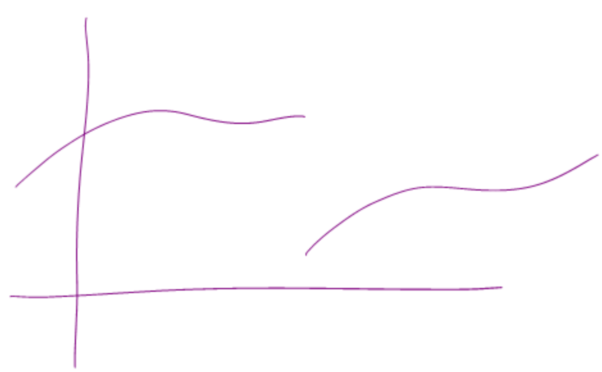
$= \quad \to \infty$

Example: Find the $\lim_{x \to \infty} \frac{x^2+x}{3-x}$

$$\lim_{x \to \infty} \frac{(x^2+x) \cdot \frac{1}{x}}{(3-x) \cdot \frac{1}{x}} = \frac{\infty}{-1} = -\infty$$

Introduction to Continuity





Definition

A function f is **continuous** at a number a if the following three conditions are satisfied:

1) If f is defined on a domain containing a.

2) $\lim_{x \to a} f(x)$ exists.

3) $\lim_{x \to a} f(x) = f(a)$ $\longrightarrow L$

Example: Where are the following fun^n discontinuous?

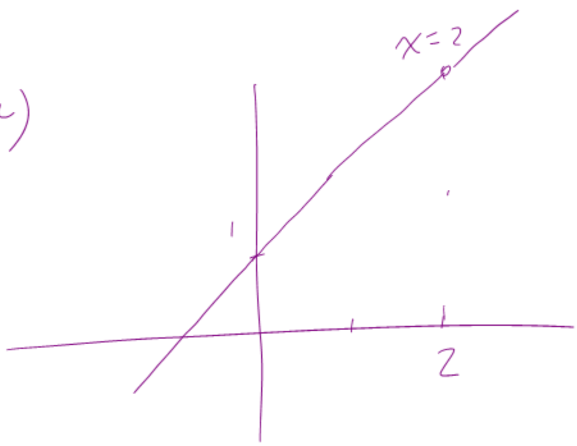
a) $f(x) = \frac{x^2 - x - 2}{x - 2}$

b) $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$

c) $f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$

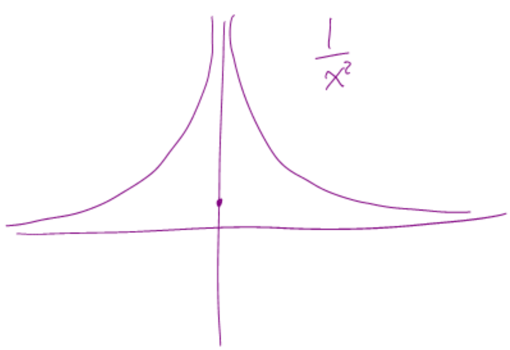
d) $f(x) = \lceil x \rceil$

a)



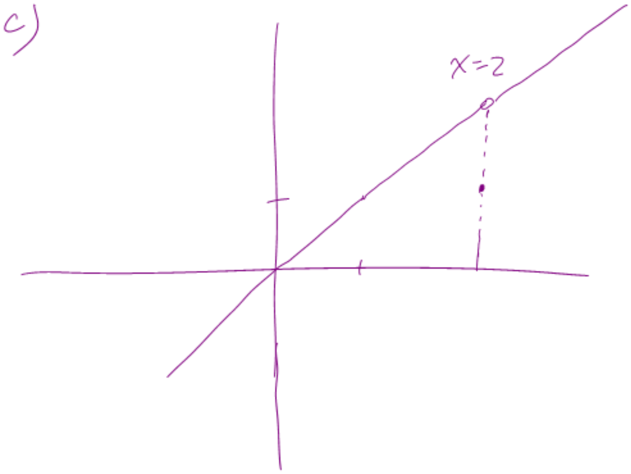
Removable discontinuity.

b)



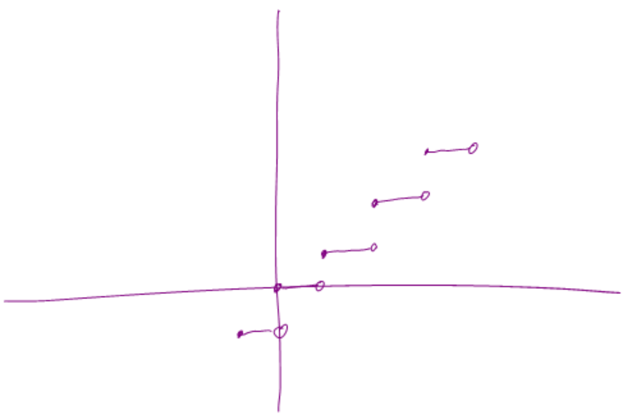
Infinite discontinuity.

c)



Removable discontinuity,

d)



Jump discont. / step discont.

Theorem

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:
1) $f+g$ 2) $f-g$ 3) cf 4) $f \cdot g$ 5) $\frac{f}{g}$ if $\cancel{f(a)}$ $g(a) \neq 0$

Theorem

The following types of functions are continuous at every number in their domain:

polynomial functions	rational functions	trigonometric functions
root functions	exponential functions	logarithmic functions

poly: $a + ax + ax^2 + ax^3 + \cdots$ rat: $\frac{3}{x^2}$ trig: $\sin x^2$

root: $\sqrt[3]{x}$ exp: e^x log: $\ln x$.

Example: show that the fun^n $f(x) = 1 - \sqrt{1-x^2}$ is continuous on the interval $[-1, 1]$.

show: $\lim_{x \to a} 1 - \sqrt{1-x^2} = f(a)$

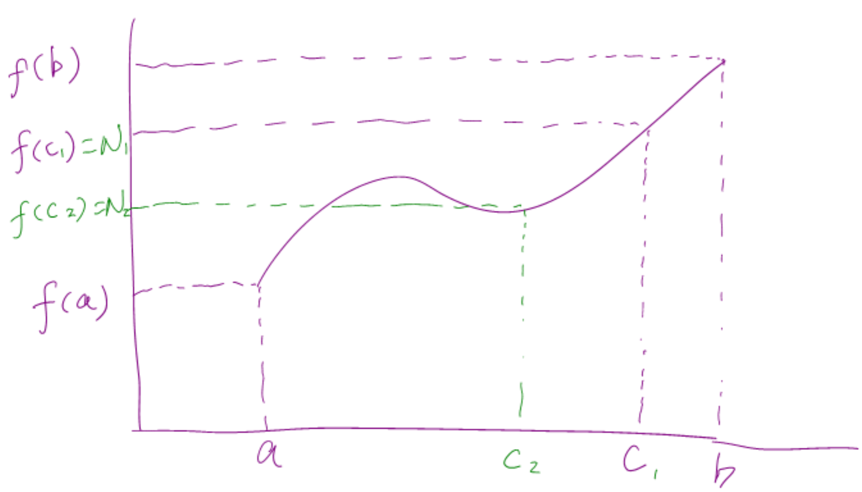
check the endpoints -1, and 1.

Theorem

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$, given by $(f \circ g)(x) = f(g(x))$, is continuous at a.

The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$, and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



Rate of Change and Derivative

Definition

The **tangent line** f at the point $(a, f(a))$ is defined as the following limit, if it exists:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad a = 1 \qquad f(x) = x^2$$

$$= \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to a} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x \to 1} (x+1) = 2$$

$$y = 2x - 1$$

$$h = x - a \quad \rightarrow \quad m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Definition

$$\text{Instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Definition

The **derivative of a function** f **at a number** a, denote by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

Example: Find the derivative of $f(x) = x^2 - 8x + 9$ at the number a.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h}$$

$$= \lim_{h \to 0} 2a + h - 8$$

$$= 2a - 8$$

Example: The position of a particle is given by the expression $s = f(t) = \frac{1}{1+t}$

Find the velocity after 2 seconds.

$$f'(a) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{1}{1+(2+h)} - \frac{1}{1+2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{3-h} - \frac{1}{3}}{h} = \lim_{h \to 0} \frac{3(3+h)}{h}$$

$$= \lim_{h \to 0} \frac{-h}{3(3+h)h} = \lim_{h \to 0} \frac{-1}{3(3+h)} = -\frac{1}{9}$$

Definition

The derivative of a function f is written as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

Example: $f(x) = x^3 - x$, find a formula for $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x} - h - \cancel{x^3} \cancel{+x}}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \lim_{h \to 0} (3x^2 + \cancel{3xh} + \cancel{h^2} - 1) = 3x^2 - 1$$

Definition

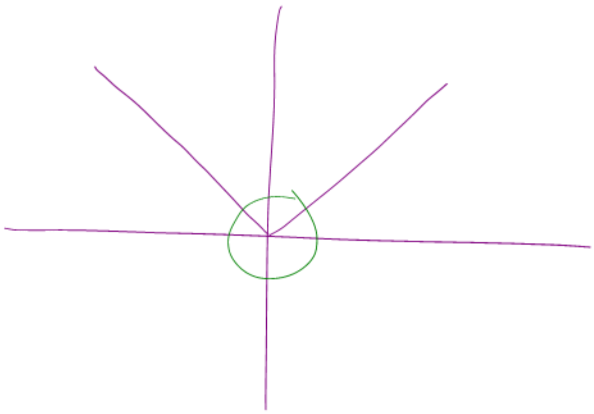
A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) if it is differentiable at every number in the interval.

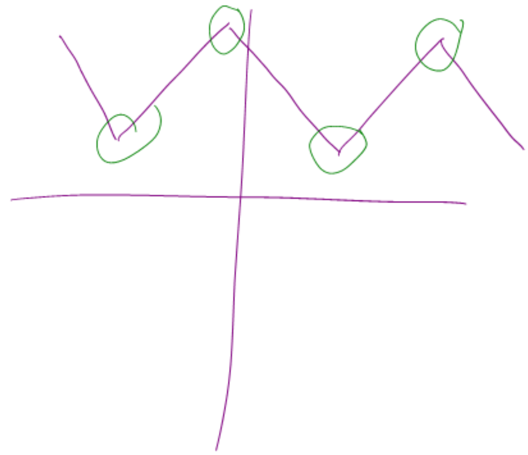
Theorem

If f is differentiable at a, then f is continuous at a.

NB: The converse of this thm is not necessarily true.

$f(x) = |x|$





$f(x)$

Notations:

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}f(x)$$

$\frac{df}{dx}$ → differential.

$f(x,y) = x^2 + 2xy.$

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}$$