AP Calculus Lesson Twelve Notes

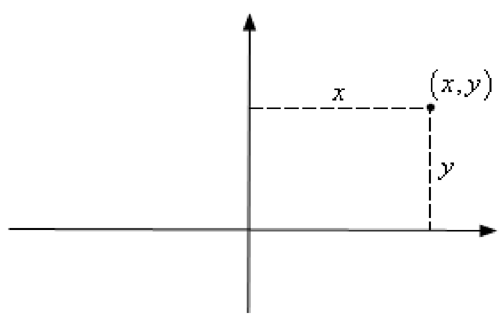
Chapter 5 Applications of the Definite Integral and Polar Coordinates

5.4 Polar Coordinates

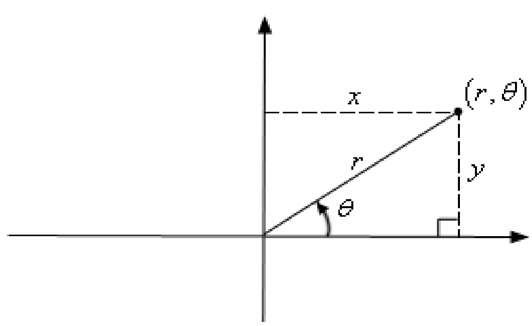
Polar Coordinates

Up to this point we've dealt exclusively with the Cartesian (or Rectangular, or x-y) coordinate system. However, as we will see, this is not always the easiest coordinate system to work in. So, in this section we will start looking at the polar coordinate system.

Coordinate systems are really nothing more than a way to define a point in space. For instance in the Cartesian coordinate system at point is given the coordinates (x, y) and we use this to define the point by starting at the origin and then moving x units horizontally followed by y units vertically. This is shown in the sketch below.

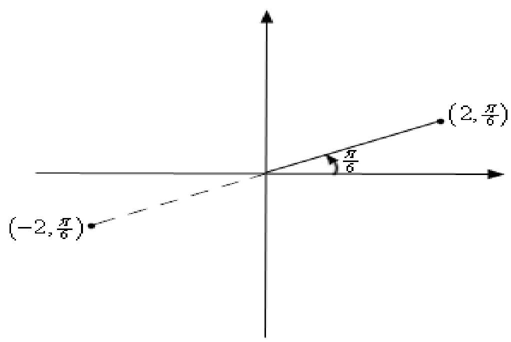


This is not, however, the only way to define a point in two dimensional space. Instead of moving vertically and horizontally from the origin to get to the point we could instead go straight out of the origin until we hit the point and then determine the angle this line makes with the positive x-axis. We could then use the distance of the point from the origin and the amount we needed to rotate from the positive x-axis as the coordinates of the point. This is shown in the sketch below.



Coordinates in this form are called **polar coordinates**.

The above discussion may lead one to think that r must be a positive number. However, we also allow r to be negative. Below is a sketch of the two points $(2, \pi/6)$ and $(-2, \pi/6)$.

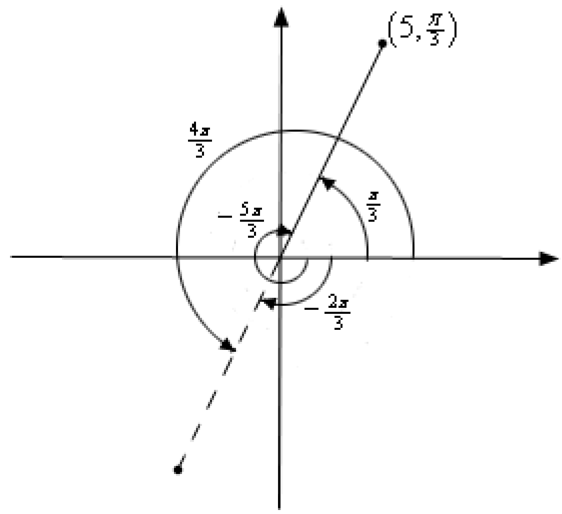


From this sketch we can see that if r is positive the point will be in the same quadrant as θ. On the other hand if r is negative the point will end up in the quadrant exactly opposite θ. Notice as well that the coordinates $(-2, \pi/6)$ describe the same point as the coordinates $(2, 7\pi/6)$ do. The coordinates $(2, 7\pi/6)$ tells us to rotate an angle of $7\pi/6$ from the positive x-axis, this would put us on the dashed line in the sketch above, and then move out a distance of 2.

This leads to an important difference between Cartesian coordinates and polar coordinates. In Cartesian coordinates there is exactly one set of coordinates for any given point. With polar coordinates this isn't true. In polar coordinates there is literally an infinite number of coordinates for a given point. For instance, the following four points are all coordinates for the same point.

$$\left(5, \frac{\pi}{3}\right) = \left(5, -\frac{5\pi}{3}\right) = \left(-5, \frac{4\pi}{3}\right) = \left(-5, -\frac{2\pi}{3}\right)$$

Here is a sketch of the angles used in these four sets of coordinates.



In the second coordinate pair we rotated in a clock-wise direction to get to the point. We shouldn't forget about rotating in the clock-wise direction. Sometimes it's what we have to do.

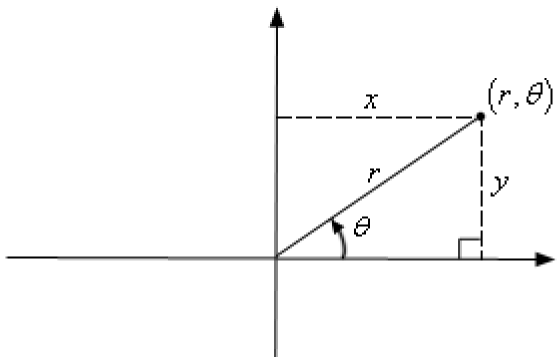
The last two coordinate pairs use the fact that if we end up in the opposite quadrant from the point we can use a negative r to get back to the point and of course there is both a counter clock-wise and a clock-wise rotation to get to the angle.

These four points only represent the coordinates of the point without rotating around the system more than once. If we allow the angle to make as many complete rotation about the axis system as we want then there are an infinite number of coordinates for the same point. In fact the point (r, θ) can be represented by any of the following coordinate pairs.

$(r, \theta + 2n\pi)$ or $(-r, \theta + (2n+1)\pi)$, where n is any integer.

Next we should talk about the origin of the coordinate system. In polar coordinates the origin is often called the **pole**. Because we aren't actually moving away from the origin/pole we know that $r = 0$. However, we can still rotate around the system by any angle we want and so the coordinates of the origin/pole are $(0, \theta)$.

Now that we've got a grasp on polar coordinates we need to think about converting between the two coordinate systems. Well start out with the following sketch reminding us how both coordinate systems work.



Note that we've got a right triangle above and with that we can get the following equations that will convert polar coordinates into Cartesian coordinates.

Polar to Cartesian Conversion Formulas

$x = r\cos\theta$ and $y = r\sin\theta$

Cartesian to Polar Conversion Formulas

$r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$

Some Polar Equations and Their Graphs

LEMNISCATE

Equation in polar coordinates:

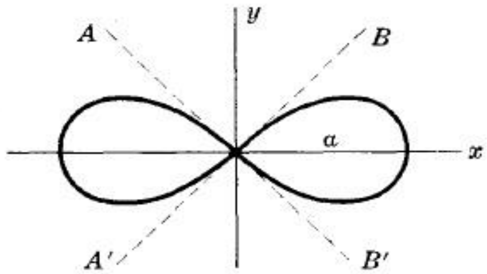
$r^2 = a^2\cos 2\theta$

Equation in rectanular coordinates:

$(x^2 + y^2)^2 = a^2(x^2 - y^2)$

Angle between AB' or A'B and x axis = 45º

Area of one loop = $a^2/2$



CYCLOID

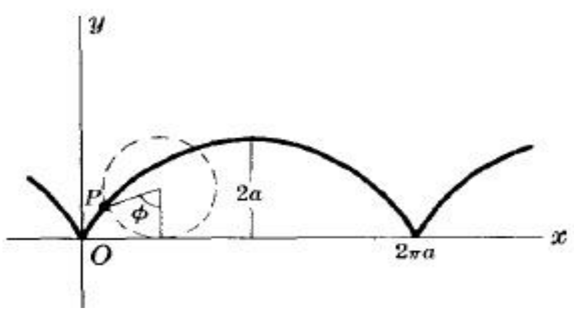
Equations in parametric form:

$$\begin{cases} x = a(\phi - \sin\phi) \\ y = a(1 - \cos\phi) \end{cases}$$

Area of one arch = $3\pi a^2$

Arc length of one arch = $8a$

This is a curve described by a point P on a circle of radius a rolling along x axis.



HYPOCYCLOID WITH FOUR CUSPS

Equation in rectangular coordinates:

$x^{2/3} + y^{2/3} = a^{2/3}$

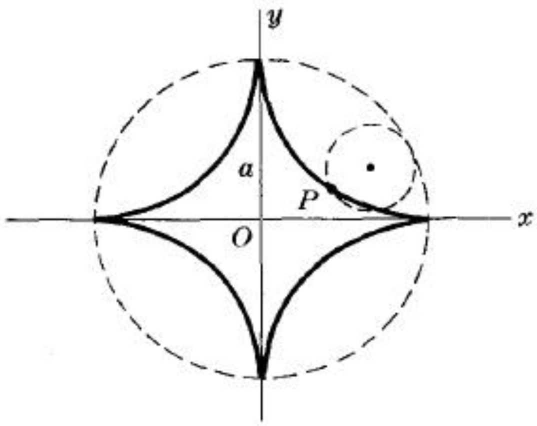
Equations in parametric form:

$$\begin{cases} x = a\cos^3\theta \\ y = a\sin^3\theta \end{cases}$$

Area bounded by curve = $3\pi a^2/8$

Arc length of entire curve = 6a

This is a curve described by a point P on a circle of radius a/4 as it rolls on the inside of a circle of radius a.



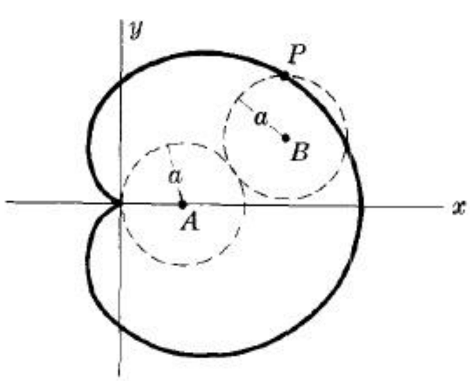
CARDIOID

Equation: $r = a(1 + \cos\theta)$

Area bounded by a curve = $3\pi a^2/2$

Arc length of a curve = $8a$

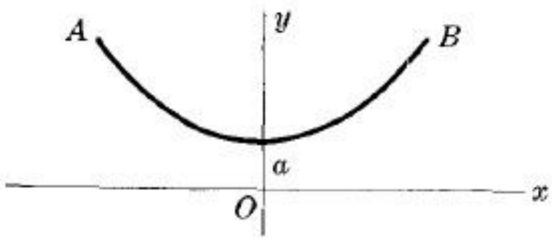
This is the curve described by a point P of a circle of radius a as it rolls on the outside of a fixed circle of radius *a*.The curve is also a special case of the limacon of Pascal.



CATENARY

Equation: $y = a(e^{x/a} + e^{-x/a})/2 = a\cosh(x/a)$

This is a curve in which a heavy uniform chain would hang if suspended vertically from fixed points A and B.

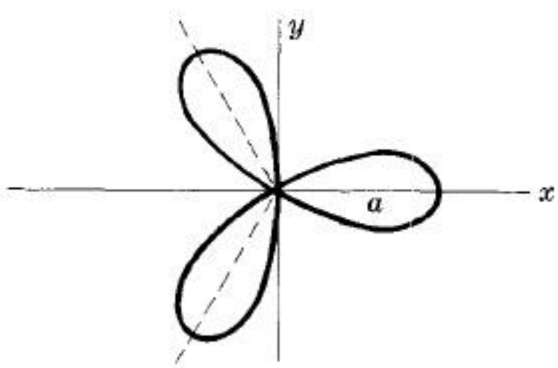


THREE-LEAVED ROSE

Equation: $r = a\cos 3\theta$

The equation $r = a\cos 3\theta$ is a similar curve obtained by rotating the curve counterclockwise 30º or $\pi/6$ radians.

In general $r = a\cos n\theta$ or $r = a\sin n\theta$ has n leaves if n is odd.

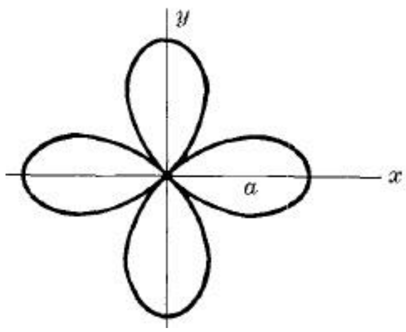


FOUR-LEAVED ROSE

Equation: $r = a\cos 2\theta$

The equation $r = a\sin 2\theta$ is a similar curve obtained by rotating the curve counterclockwise through 45º or $\pi/4$ radians.

In general $r = a\cos n\theta$ or $r = a\sin n\theta$ has 2n leaves if n is even.

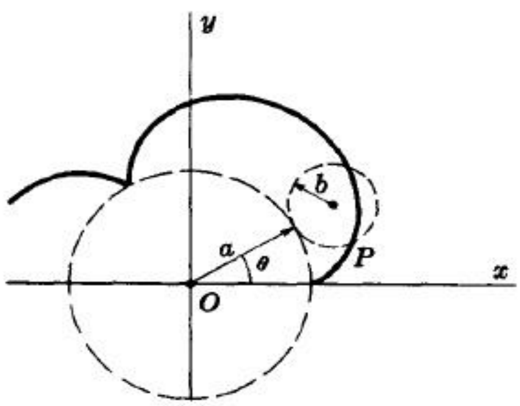


EPICYCLOID

Parametric equations:

$$\begin{cases} x = (a+b)\cos\theta - b\cos\left(\dfrac{a+b}{b}\right)\theta \\ y = (a+b)\sin\theta - b\sin\left(\dfrac{a+b}{b}\right)\theta \end{cases}$$

This is a curve described by a point P on a circle of a radius b as it rolls on the outside of a circle of radius a.The cardioid is a special case of an epicycloid.



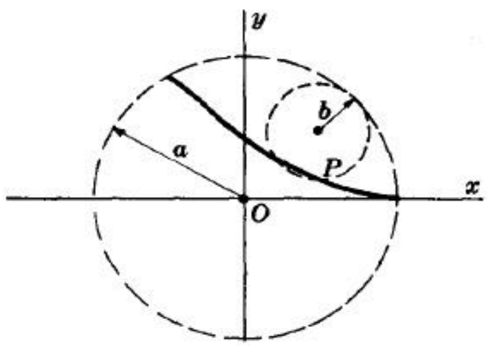
GENERAL HYPOCYCLOID

Parametric equations:

$$\begin{cases} x = (a-b)\cos\phi + b\cos\left(\dfrac{a-b}{b}\right)\phi \\ y = (a-b)\sin\phi - b\sin\left(\dfrac{a-b}{b}\right)\phi \end{cases}$$

This is a curve described by a point P on a circle of a radius b as it rolls on the outside of a circle of radius a.

If b = a/4, the curve is hypocycloid with four cusps.



TROCHOID

Parametric equations:

$$\begin{cases} x = a\phi - b\sin\phi \\ y = a - b\cos\phi \end{cases}$$

This is a curve described by a pint P at distance b from the center of a circle of radius a as the circle rolls on the x axis.

If b < a, the curve is as shown on Fig.11-10 and is called *curtate cycloid.*

If b > a, the curve is as shown on Fig.11-11 and is called a *prolate cycloid.*

If b = a, the curve is a cycloid.

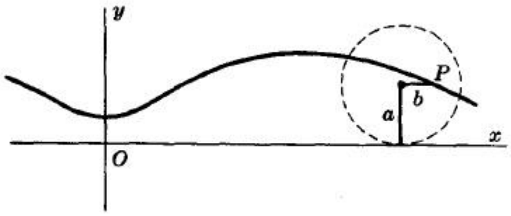


Fig. 11-10

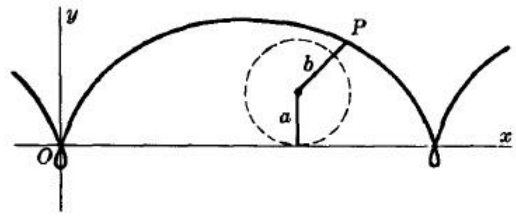


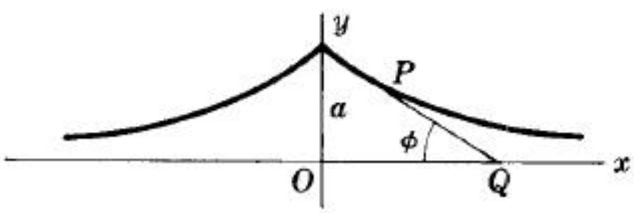
Fig. 11-11

TRACTRIX

Parametric equations:

$$\begin{cases} x = a(\ln \cot \tfrac{1}{2}\phi - \cos \phi) \\ y = a \sin \phi \end{cases}$$

This is a curve described by endpoint P of a taut string PQ of length a as the other end Q is moved along the x axis.



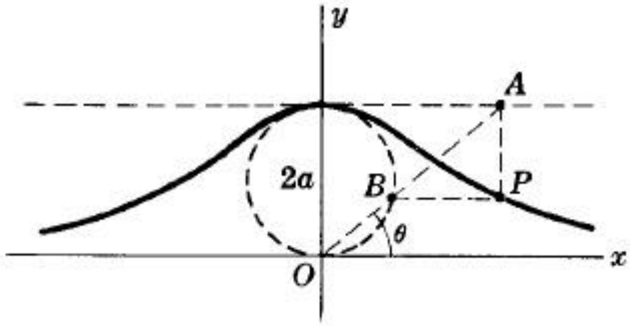
WHITCH OF AGNESI

Equation in rectangular coordinates: $y = 8a^3/(x^2 + 4a^2)$

Parametric equations:

$$\begin{cases} x = 2a \cot \theta \\ y = a(1 - \cos 2\theta) \end{cases}$$

In the figure the variable line OA intersects y = 2a and the circle of radius a with center (0,a) at A and B respectively. Any point P on the "which" is located by constructing lines parallel to the x and y axes through B and A respectively and determining the point P of intersection.



FOLIUM OF DESCARTES

Equation in rectangular coordinates:

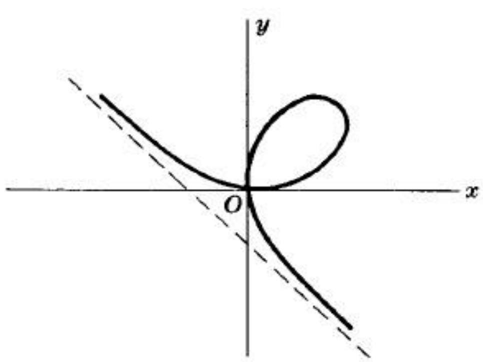
$x^3 + y^3 = 3axy$

Parametric equations:

$$\begin{cases} x = \dfrac{3at}{1 + t^3} \\ y = \dfrac{3at^2}{1 + t^3} \end{cases}$$

Area of loop $3a^2/2$

Equation of asymptote: $x + y + a = 0$.

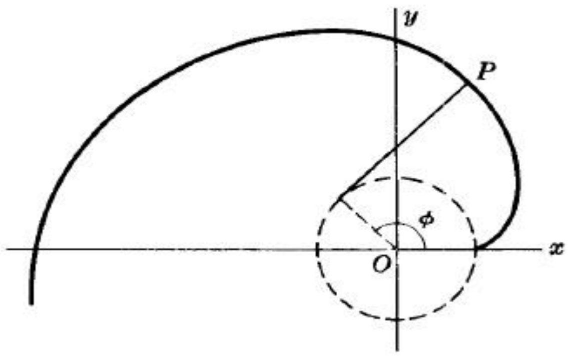


INVOLUTE OF A CIRCLE

Parametric equations:

$$\begin{cases} x = a(\cos\phi + \phi\sin\phi) \\ y = a(\sin\phi - \phi\cos\phi) \end{cases}$$

This is a curve described by the endpoint P of a string as it unwinds from a circle of radius a while held taut.



EVOLUTE OF AN ELLIPSE

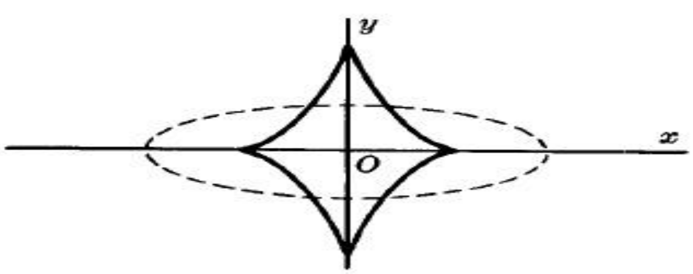
Equation in ractangular coordinates:

$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

Parametric equations:

$$\begin{cases} ax = (a^2 - b^2)\cos^3\theta \\ by = (a^2 - b^2)\sin^3\theta \end{cases}$$

This curve is the envelope of the normals to the ellipse $x^2/a^2 + y^2/b^2 = 1$.



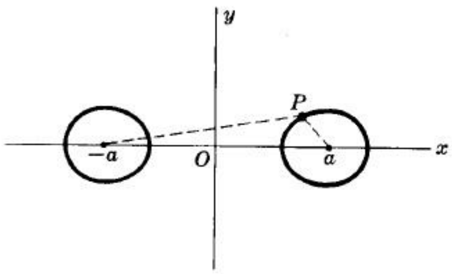
OVALS OF CASSINI

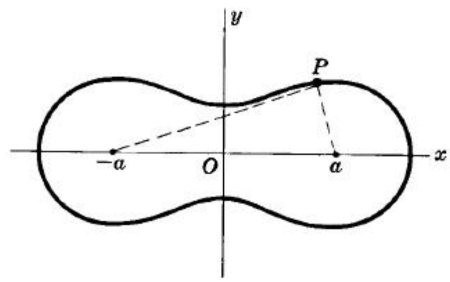
Polar equation: $r^4 + a^4 - 2a^2r^2\cos 2\theta = b^4$.

This is the curve described by point P such that the product of its distances from two fixed points[distance 2a apart] is a constant b^2.

The curve is as in the figures according as b < a or b > a respectively.

If b = a, the curve is a *lemniscate*



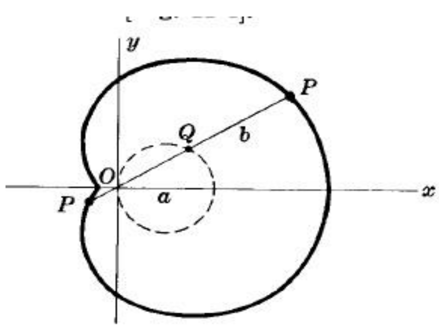


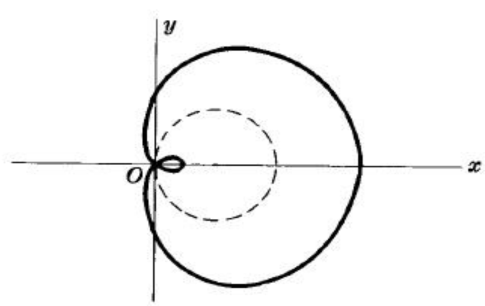
LIMASCON OF PASCAL

Polar equation: $r = b + a\cos\theta$

Let OQ be a line joining origin O to any point Q on a circle of diameter a passing through O.Then the curve is the locus of all points P such that PQ = b.

The curve is as in the figures below according as b > a or b < a respectively. If b = a, the curve is a cardioid.





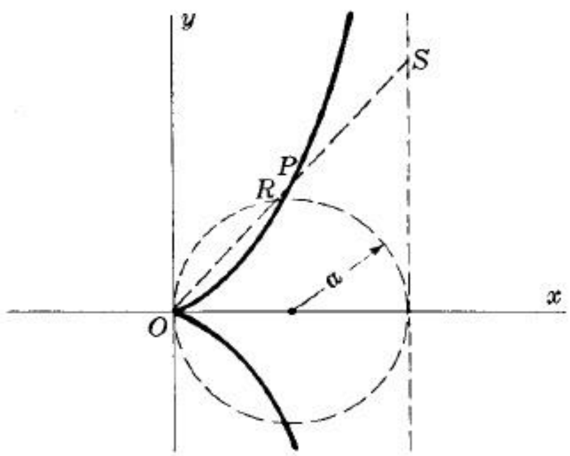
CISSOID OF DIOCLES

Equation in rectangular coordinates: $y^2 = x^3/(2a - x)$

Parametric equations:

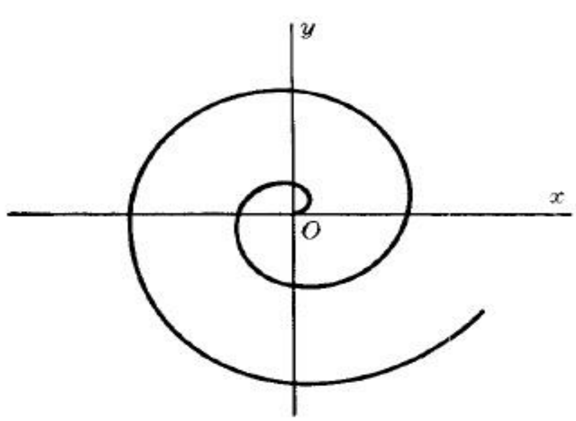
$$\begin{cases} x = 2a\sin^2\theta \\ y = \dfrac{2a\sin^3\theta}{\cos\theta} \end{cases}$$

This is the curve described by a point P such that the distance OP = distance RS. It is used in the problem if *duplication of a cube*, i.e. finding the side of a cube which has twice the volume of a given cube.



SPIRAL OF ARCHIMEDES

Polar equation: r = aθ

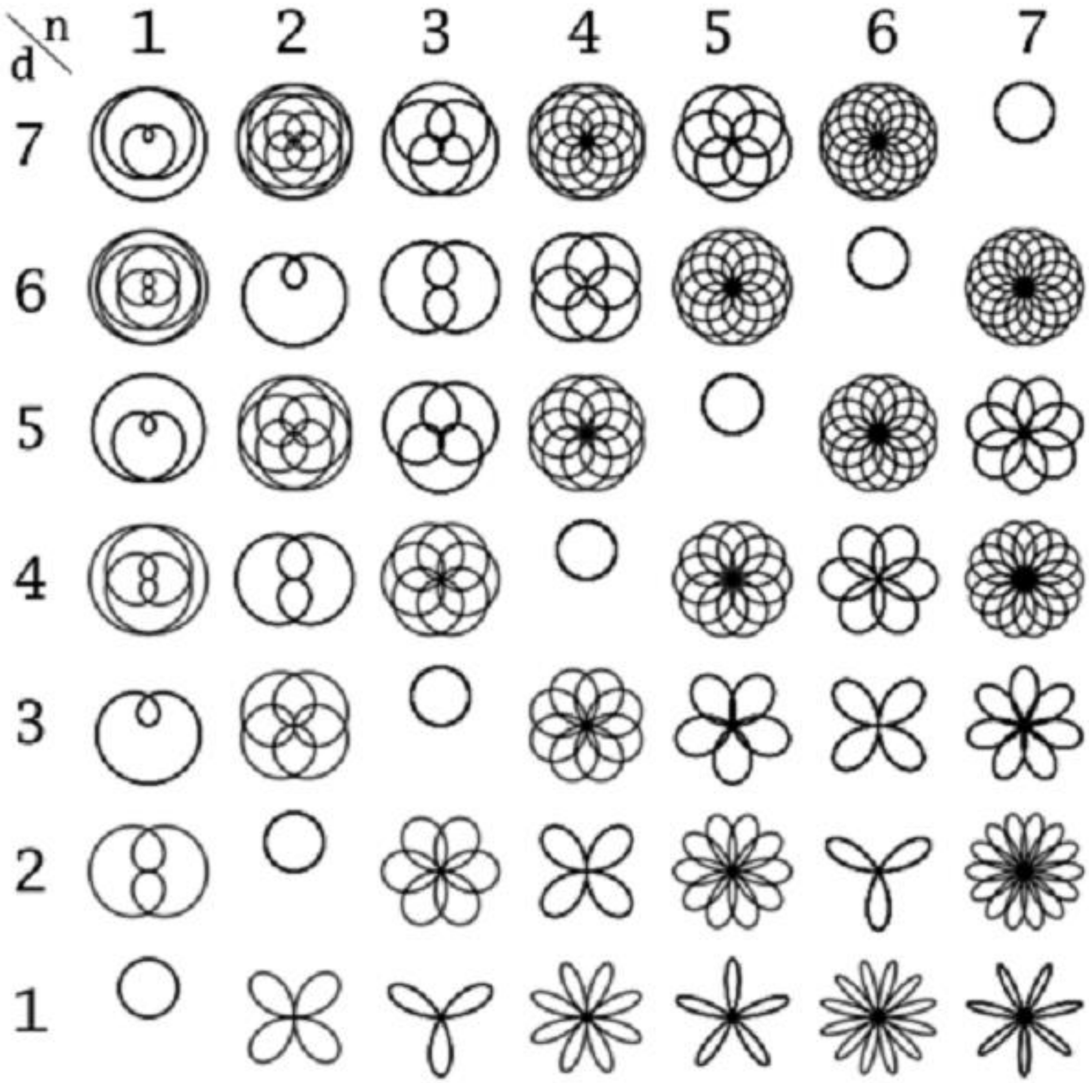


POLAR ROSES

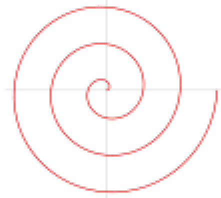
Perhaps some of the prettiest graphs in polar coordinates are ***polar roses***, also known as ***rhodonea curves***. The equations of these fascinating figures have the simple form

$r = \cos k\theta$

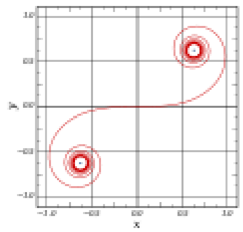
where k is an arbitrary rational number. (If k is irrational, the curve never closes.) It turns out that if k is an integer, then the resulting polar rose looks like a flower with k petals if k is odd or 2k petals if k is even. If k is not an integer, then the curve intersects itself. The following figure shows all with k = n/d for n and d both ranging from 1 to 7.



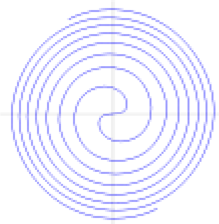
- The Archimedean spiral: $r = a + b \cdot \theta$(see also:Involute)
- The Euler spiral, Cornu spiral or *clothoid*
- Fermat's spiral: $r = \theta^{1/2}$
- The Fibonacci spiral and golden spiral: special cases of the logarithmic spiral
- The hyperbolic spiral: $r = a / \theta$
- The lituus: $r = \theta^{-1/2}$
- The logarithmic spiral: $r = a \cdot e^{b\theta}$; approximations of this are found in nature
- The Spiral of Theodorus: an approximation of the Archimedean spiral composed of contiguous right triangles



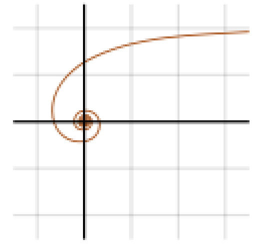
Archimedean spiral



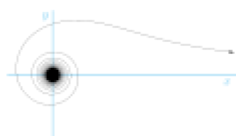
Cornu spiral



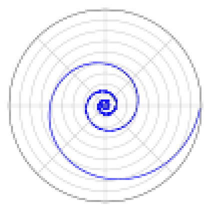
Fermat's spiral



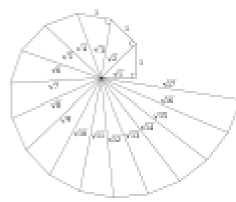
hyperbolic spiral



lituus



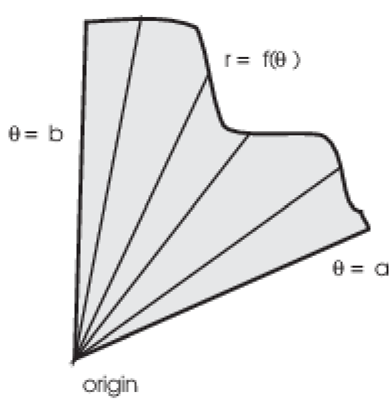
logarithmic spiral



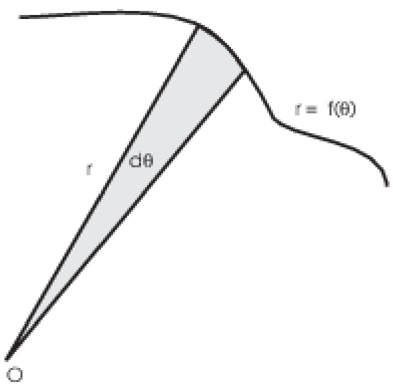
spiral of Theodorus

Area in Polar Coordinates

As the picture shows, a region in polar is "swept out" as if by a revolving searchlight beam.



Look at a small wedge-shaped piece of the region. It subtends an angle $d\theta$ and the radius is r.



If angle $d\theta$ is small, the wedge is approximately a circular wedge. The area of a circular wedge of radius r and angle $d\theta$ is $\frac{1}{2}r^2 d\theta$, so this is a good approximation to the area of the wedge-shaped piece above.

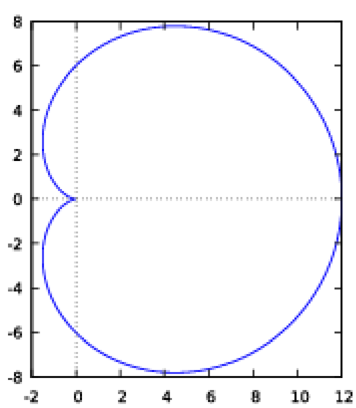
As usual, we obtain the total area by integrating to add up the areas of the little pieces:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

Example 1. Find the area of the region inside the cardioid $r = 6 + 6cos\theta$ that is shown below.

Solution

$r = 6 + 6\cos\theta$

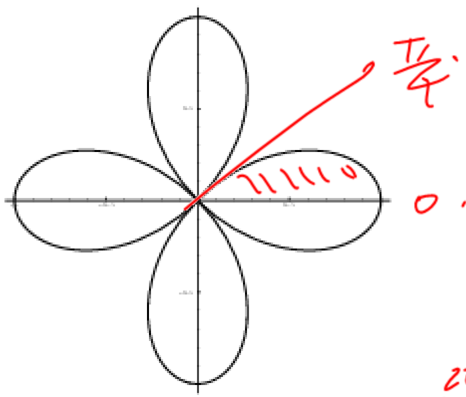


The entire cardioid is traced out once as angle θ goes from 0 to 2π. The area is

$\int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$

$$A = \int_0^{2\pi} \frac{1}{2}(6 + 6cos\theta)^2 d\theta = \int_0^{2\pi} (27 + 36cos\theta + 9cos2\theta) d\theta = 54\pi$$

Example 2. Find the area of the region inside $r = cos2\theta$ whose graph is shown below.



Solution

$\int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$

The 4-leaved rose is traced out once as θ goes from 0 to 2π.

$$A = \int_0^{2\pi} \frac{1}{2}(cos2\theta)^2 d\theta = \int_0^{2\pi} \frac{1}{4}(1 + cos4\theta) d\theta = \frac{\pi}{2}$$

$\cos^2(2\theta) = \frac{1}{2}(1+\cos 4\theta)$

$= 8\int_0^{\pi/4} \frac{1}{4}(1+\cos 4\theta)\, d\theta = \frac{\pi}{2}$

You could also use symmetry. The half of right-hand leaf is traced out as angle θ goes from 0 to π/4 , so you could find the area of that half-leaf and multiply by 8.

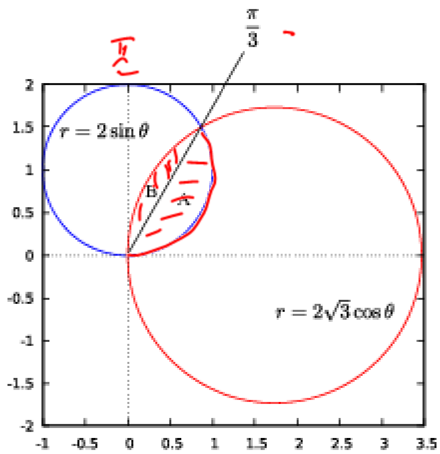
Example 3. Find the area of the region which is inside both $r = 2sin\theta$ and $r = 2\sqrt{3}cos\theta$ simultaneously.

(That is, find the area of the intersection of the interiors of the curves.)

Solution

Find where the curves intersect:

Let $2sin\theta = 2\sqrt{3}cos\theta$, then we get $tan\theta = \sqrt{3}$, this gives $\theta = \frac{\pi}{3}$.



The region is shaped like an orange slice, that can be broken up into two pieces. Piece A is the area inside $r = 2sin\theta$ (the circle that goes up the y-axis) from θ goes from 0 to π/3. Piece B is the area inside $r = 2\sqrt{3}cos\theta$ (the circle that goes along the x-axis) from θ goes from π/3 to π/2. The area in this example is the sum of the areas of A and B, so it is

$$A = \int_0^{\pi/3} \frac{1}{2}(2sin\theta)^2 d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(2\sqrt{3}cos\theta)^2 d\theta = \frac{5\pi}{6} - \sqrt{3}$$

Arc Length in Polar Coordinates

If a curve is given in polar coordinates $r = f(\theta)$, an integral for the length of the curve can be derived using the arc length formula for a parametric curve. Regard θ as the parameter. The parametric arc length formula becomes

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

Since $x = rcos\theta$ and $y = rsin\theta$, so

$$\frac{dx}{d\theta} = -r\sin\theta + \left(\frac{dr}{d\theta}\right)\cos\theta,$$

$$\frac{dy}{d\theta} = r\cos\theta + \left(\frac{dr}{d\theta}\right)\sin\theta,$$

Square and add, using $sin^2\theta + cos^2\theta = 1$

$$\begin{aligned}
\left(\frac{dx}{d\theta}\right)^2 &= r^2(\sin\theta)^2 - 2r\left(\frac{dr}{d\theta}\right)\sin\theta\cos\theta + \left(\frac{dr}{d\theta}\right)^2(\cos\theta)^2 \\
\left(\frac{dy}{d\theta}\right)^2 &= r^2(\cos\theta)^2 + 2r\left(\frac{dr}{d\theta}\right)\cos\theta\sin\theta + \left(\frac{dr}{d\theta}\right)^2(\sin\theta)^2 \\
\hline
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= r^2 + \left(\frac{dr}{d\theta}\right)^2
\end{aligned}$$

Hence,

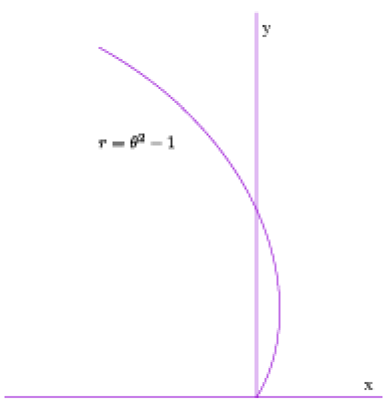
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

$$\int_{\theta_1}^{\theta_2} \sqrt{(r'(\theta))^2 + (r(\theta))^2}\, d\theta$$

Note: As with other arc length computations, it's pretty easy to come up with polar curves which lead to integrals with non-elementary antiderivatives. In that case, the best you might be able to do is to approximate the integral using a calculator or a computer.

Example 4. Find the length of the curve $r = \theta^2 - 1$ from θ = 1 to θ = 2.

Solution



Since $r = \theta^2 - 1$, so

$$\frac{dr}{d\theta} = 2\theta.$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (\theta^2 - 1)^2 + 4\theta^2 = \theta^4 - 2\theta^2 + 1 + 4\theta^2 = \theta^4 + 2\theta^2 + 1 = (\theta^2 + 1)^2.$$

$\theta^2 + 1$

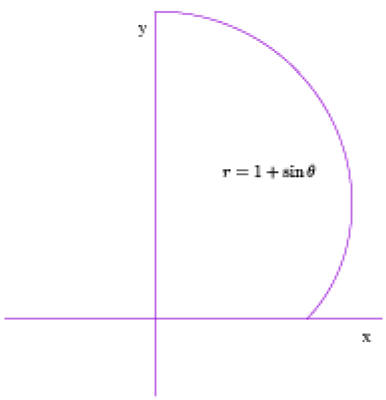
$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \theta^2 + 1.$$

The length is

$$\int_1^2 (\theta^2 + 1)\, d\theta = \left[\frac{1}{3}\theta^3 + \theta\right]_1^2 = \frac{10}{3} = 3.33333\ldots. \quad \square$$

Example 5. Find the length of the cardioid $r = 1 + sin\theta$ from θ = 0 to θ = π/2.

Solution



Since $r = 1 + sin\theta$, so

$$\frac{dr}{d\theta} = \cos\theta.$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 + \sin\theta)^2 + (\cos\theta)^2 = 1 + 2\sin\theta + (\sin\theta)^2 + (\cos\theta)^2 = 2 + 2\sin\theta.$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{2}\sqrt{1 + \sin\theta}.$$

We could do the antiderivative separately:

$$\int \sqrt{2}\sqrt{1+\sin\theta}\,d\theta = \sqrt{2}\int \frac{\sqrt{1+\sin\theta}\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\,d\theta = \sqrt{2}\int \frac{\sqrt{1-(\sin\theta)^2}}{\sqrt{1-\sin\theta}}\,d\theta =$$

$$\sqrt{2}\int \frac{\sqrt{(\cos\theta)^2}}{\sqrt{1-\sin\theta}}\,d\theta = \sqrt{2}\int \frac{\cos\theta}{\sqrt{1-\sin\theta}}\,d\theta =$$

$$\left[u = 1-\sin\theta, \quad du = -\cos\theta\,d\theta, \quad d\theta = \frac{du}{-\cos\theta}\right]$$

$$\sqrt{2}\int \frac{\cos\theta}{\sqrt{u}}\cdot\frac{du}{-\cos\theta} = -\sqrt{2}\int \frac{1}{\sqrt{u}}\,du = -\sqrt{2}\cdot 2\sqrt{u}+c = -2\sqrt{2}\sqrt{1-\sin\theta}+c.$$

The length is

$$\int_0^{\pi/2} \sqrt{2}\sqrt{1+\sin\theta}\,d\theta = \left[-2\sqrt{2}\sqrt{1-\sin\theta}\right]_0^{\pi/2} = 2\sqrt{2} = 2.82842\ldots. \quad \square$$