Unit: Applications of vectors (1)

Dot Product of two Geometric Vectors

Definition The **<u>dot product</u>** of two vectors $\vec{u}$ and $\vec{v}$ is a scalar given by $\vec{u}\cdot\vec{v}=\lvert\vec{u}\rvert\lvert\vec{v}\rvert\cos(\theta)$ where θ is the angle between the vectors $\vec{u}$ and $\vec{v}$. ***Note:*** *Dot product of two vectors is* ***NOT*** *a vector it is a scalar (number).* ***Note:*** *By convention $0° \le \theta \le 180°$.*	**Ex.** Calculate the dot product, $\vec{u}\cdot\vec{v}$, to one decimal place accuracy, given that $\lvert\vec{u}\rvert=10$, $\lvert\vec{v}\rvert=2$, and the angle between $\vec{u}$ and $\vec{v}$ is 40^0
Properties of Dot Product **1.** $a(\vec{u}\cdot\vec{v})=(a\vec{u})\cdot\vec{v}=\vec{u}\cdot(a\vec{v})$ **2.** $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$ **3.** $\vec{u}\cdot\vec{u}=\lvert\vec{u}\rvert^2$ **4.** $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$ **5.** If $\vec{u}\cdot\vec{v}=0$ where $\vec{u},\vec{v}\neq\vec{0}$, then $\vec{u}\perp\vec{v}$. Conversely, if $\vec{u}\perp\vec{v}$, then $\vec{u}\cdot\vec{v}=0$.	**Ex.** Prove properties 3 and 5.

Ex. The magnitude of the sum of vectors $\vec{a}$ and $\vec{b}$ is equal to the magnitude of their difference. Determine the angle between $\vec{a}$ and $\vec{b}$.	

Dot Product of Algebraic Vectors

The **dot product of the standard unit** vectors is given by: $\vec{i}\cdot\vec{i} = 1$ $\quad\vec{i}\cdot\vec{j} = 0$ $\vec{j}\cdot\vec{j} = 1$ $\quad\vec{j}\cdot\vec{k} = 0$ $\vec{k}\cdot\vec{k} = 1$ $\quad\vec{k}\cdot\vec{i} = 0$	
The dot product of two algebraic vectors $\vec{a} = (a_x, a_y, a_z)$ and $\vec{b} = (b_x, b_y, b)$ is given by $\boldsymbol{\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z}$ *We have now seen two definitions of dot product!* $\vec{u}\cdot\vec{v} = u_x v_x + u_y v_y + u_z v_z = \lvert\vec{u}\rvert\lvert\vec{v}\rvert\cos(\theta)$	**Ex.** Given $\vec{a}$ and $\vec{b}$, determine their dot product. *a.* $\vec{a} = (2,-1)$ and $\vec{b} = (4,3)$ **b.** $\vec{a} = (1,0,3)$ and $\vec{b} = (-2,5,8)$

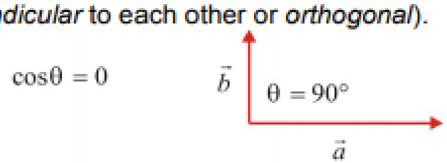
Angle between two Vectors

The angle $\theta = \angle(\vec{a}, \vec{b})$ between two vectors $\vec{a}$ and $\vec{b}$ (when positioned tail to tail) is given by:

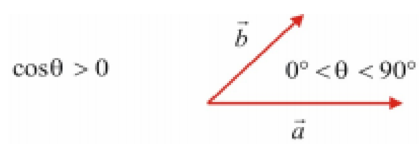
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}}$$

Notes:

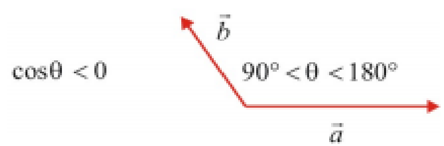
1. If $\cos\theta = 1$ then $\vec{a} \uparrow\uparrow \vec{b}$ (vectors are *parallel* and have *same direction*).
2. If $\cos\theta = -1$ then $\vec{a} \uparrow\downarrow \vec{b}$ (vectors are *parallel* but have *opposite direction*).
3. If $\cos\theta = 0$ then $\vec{a} \perp \vec{b}$ (vectors are *perpendicular* to each other or *orthogonal*).



4. If $\cos\theta > 0$ then $0° < \theta < 90°$ (θ is an *acute* angle).



5. If $\cos\theta < 0$ then $90° < \theta < 180°$ (θ is an *obtuse* angle).



Ex. For what values of k are the vectors $\vec{a} = (k, -2, 3)$ and $\vec{b} = (2, 2k-6, 6)$

a) perpendicular(orthogonal)?

b) parallel (collinear)?

Ex. Find the angle between each pair of vectors:

a. $\vec{u} = 3\hat{\imath} - \hat{\jmath}$ and $\vec{v} = -\hat{\imath} + 2\hat{\jmath}$
b. (2,1,−3) and (1,0,4)

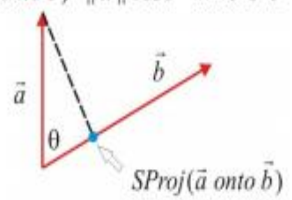
Ex. If the vectors $\overrightarrow{2a} + \vec{b}$ and $\frac{1}{2}\vec{a} - \vec{b}$ are perpendicular to each other and $2|\vec{b}| = 3|\vec{a}|$ find the angle $\theta = \angle(\vec{a}, \vec{b})$.

Scalar and Vector Projections

Scalar Projection

The *scalar projection* of the vector $\vec{a}$ onto the vector $\vec{b}$ is a *scalar* defined as:

$$SProj(\vec{a}\ onto\ \vec{b}) = \|\vec{a}\|\cos\theta \quad \text{where } \theta = \angle(\vec{a},\vec{b})$$



Special Cases

Consider two vectors $\vec{a}$ and $\vec{b}$.

a) If $\vec{a} \uparrow\uparrow \vec{b}$ $(\cos\theta = 1)$, then $SProj(\vec{a}\ onto\ \vec{b}) = \|\vec{a}\|$

b) If $\vec{a} \uparrow\downarrow \vec{b}$ $(\cos\theta = -1)$, then $SProj(\vec{a}\ onto\ \vec{b}) = -\|\vec{a}\|$

c) If $\vec{a} \perp \vec{b}$ then $SProj(\vec{a}\ onto\ \vec{b}) = 0$

Dot Product and Scalar Projection

Recall that the *dot product* of the vectors $\vec{a}$ and $\vec{b}$ is defined as:

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

So, the *scalar projection* of the vector $\vec{a}$ onto the vector $\vec{b}$ can be written as:

$$SProj(\vec{a}\ onto\ \vec{b}) = \|\vec{a}\|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}$$

Note:
For a Cartesian (Rectangular) coordinate system, the *scalar components* a_x, a_y, and a_z of a vector $\vec{a} = (a_x, a_y, a_z)$ are the *scalar projections* of the vector $\vec{a}$ onto the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$.

Proof:

$$SProj(\vec{a}\ onto\ \vec{i}) = \frac{\vec{a}\cdot\vec{i}}{\|\vec{i}\|} = \frac{(a_x, a_y, a_z)\cdot(1,0,0)}{1} = a_x$$

Ex. Given the vector $\vec{a}$= (2,–3,4), find the scalar projection:

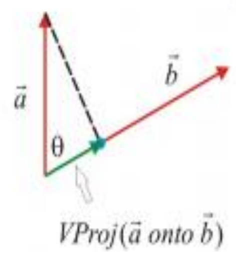
a) of $\vec{a}$ onto the unit vector $\vec{\imath}$

a) of $\vec{a}$ onto the unit vector $\vec{\imath} - \vec{\jmath}$

Vector Projection

The *vector projection* of the vector $\vec{a}$ onto the vector $\vec{b}$ is a vector defined as:

$$VProj(\vec{a}\ onto\ \vec{b}) = \|\vec{a}\|\cos\theta\frac{\vec{b}}{\|\vec{b}\|}$$



Dot Product and Vector Projection

The *vector projection* of the vector $\vec{a}$ onto the vector $\vec{b}$ can be written using the dot product as:

$$VProj(\vec{a}\ onto\ \vec{b}) = \frac{(\vec{a}\cdot\vec{b})\vec{b}}{\|\vec{b}\|^2}$$

Note:
For a Cartesian (Rectangular) coordinate system, the *vector components* $\vec{a}_x = a_x\vec{i}$, $\vec{a}_y = a_y\vec{j}$, and $\vec{a}_z = a_z\vec{k}$ of a vector $\vec{a} = (a_x, a_y, a_z)$ are the *vector projections* of the vector $\vec{a}$ onto the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$.

Ex. Given two vectors $\vec{a}$ = (0,1,−2) and $\vec{b}$ = (−1,0,3), find:

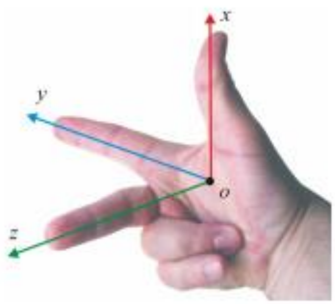
a) the vector projection of the vector $\vec{a}$ onto the vector $\vec{b}$

b) the vector projection of the vector $\vec{\imath}$ onto the vector $\vec{a}$

Cross Product

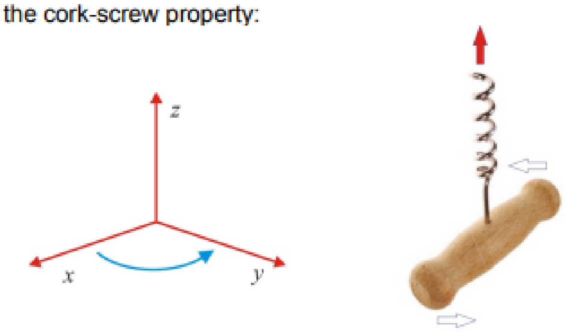
Right Hand System

The *Right Hand System* is based on the position of first three fingers of the right hand as illustrated on the following figure:



Cork-Screw Rule

The *cork-screw rule* describes a *right hand system* based on the cork-screw property:



If you rotate the x-axis towards the y-axis using the shortest path, the screw goes in the positive direction of the z-axis.

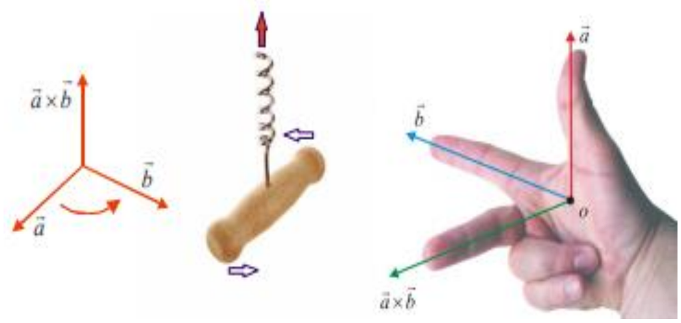
Cross Product

The *cross product* between two vectors $\vec{a}$ and $\vec{b}$ is a *vector* quantity denoted by $\vec{a}\times\vec{b}$ having the following properties:

a) $\|\vec{a}\times\vec{b}\|=\|\vec{a}\|\|\vec{b}\|\sin\alpha$ where $\alpha=\angle(\vec{a},\vec{b})$

b) $\vec{a}\times\vec{b}$ is *perpendicular to both* $\vec{a}$ and $\vec{b}$ (is perpendicular to the plane determined by $\vec{a}$ and $\vec{b}$)

c) the vectors $\vec{a}$, $\vec{b}$, and $\vec{a}\times\vec{b}$ form a *right-handed system*



Specific Cases

1. If $\vec{a}\parallel\vec{b}$ ($\alpha=0$ or $\alpha=\pi=180°$), then $\vec{a}\times\vec{b}=\vec{0}$.
2. If $\vec{a}\perp\vec{b}$ ($\alpha=\pi/2=90°$), then $\|\vec{a}\times\vec{b}\|=\|\vec{a}\|\|\vec{b}\|=maximum$
3. If $\vec{a}\equiv\vec{b}$ then $\vec{a}\times\vec{a}=\vec{0}$.

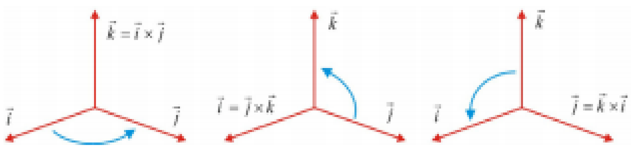
Ex. The magnitudes of two vectors $\vec{a}$ and $\vec{b}$ are $|\vec{a}|$= 2 and $|\vec{b}|$= 3 respectively, and the angle between them is α= 60°. Find the magnitude of the cross product of these vectors.

Cross Product of Unit Vectors

The cross product of the *standard unit vectors* is given by:

$$\vec{i}\times\vec{i}=\vec{0} \qquad \vec{j}\times\vec{j}=\vec{0} \qquad \vec{k}\times\vec{k}=\vec{0}$$

$$\vec{i}\times\vec{j}=\vec{k} \qquad \vec{j}\times\vec{k}=\vec{i} \qquad \vec{k}\times\vec{i}=\vec{j}$$



Cross Product of two Algebraic Vectors

The cross product of two algebraic vectors

$\vec{a}=(a_x,a_y,a_z)=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$ and

$\vec{b}=(b_x,b_y,b_z)=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$ is given by:

$$\vec{a}\times\vec{b}=\vec{i}(a_yb_z-a_zb_y)+\vec{j}(a_zb_x-a_xb_z)+\vec{k}(a_xb_y-a_yb_x)$$

$$=\vec{i}\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}+\vec{j}\begin{vmatrix}a_z & a_x\\ b_z & b_x\end{vmatrix}+\vec{k}\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

$$\vec{a}\times\vec{b}=\begin{array}{ccccc}\vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j}\\ a_x & a_y & a_z & a_x & a_y\\ b_x & b_y & b_z & b_x & b_y\end{array}$$

Ex. Find the cross product $\vec{u}\times\vec{v}$ given that

a. $\vec{u}=3\hat{\imath}-\hat{\jmath}+4\hat{k}$ and $\vec{v}=-\hat{\imath}+2\hat{\jmath}+5\hat{k}$
b. $\vec{u}=(1,2,3)$ and $\vec{v}=(4,-1,5)$
c. $\vec{u}=(-2,1,3)$ and $\vec{v}=(4,-2,-6)$

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Properties of Cross Product

The following properties apply for the cross product:

1. $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$ (anti-commutative property)
2. $\lambda(\vec{a}\times\vec{b})=(\lambda\vec{a})\times\vec{b}=\vec{a}\times(\lambda\vec{b})$
3. $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ (distributive property)
4. $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}=\vec{0}\ or\ \vec{b}=\vec{0}\ or\ \vec{a}\,||\,\vec{b}$
5. $\vec{a}\times\vec{0}=\vec{0}$
6. $\vec{a}\times\vec{a}=\vec{0}$

Note: The dot and cross products have a higher priority in comparison to addition and subtraction operations.

d) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{c}\cdot\vec{a})\vec{b}-(\vec{b}\cdot\vec{a})\vec{c}$ (triple cross product)

$[\vec{a}\times(\vec{b}\times\vec{c})]_x=a_y(\vec{b}\times\vec{c})_z-a_z(\vec{b}\times\vec{c})_y$

$=a_y(b_xc_y-b_yc_x)-a_z(b_zc_x-b_xc_z)$

$=(c_ya_y+c_za_z)b_x-(b_ya_y+b_za_z)c_x+a_xc_xb_x-a_xc_xb_x$

$=(\vec{c}\cdot\vec{a})b_x-(\vec{b}\cdot\vec{a})c_x=RS$

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Ex. Given the vectors $\vec{u}=(-2,1,-1)$ and $\vec{v}=(-1,2,-1)$

a. Find a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$.

b. Find two vectors of magnitude 11 which are perpendicular to both $\vec{u}$ and $\vec{v}$.

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