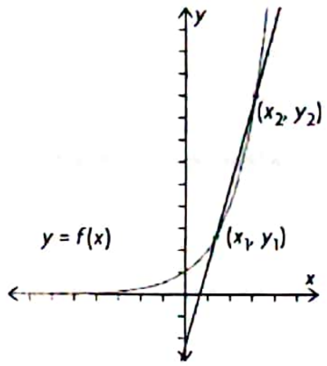
Determining Average Rate of Change

- The average rate of change is the change in the quantity represented by the dependent variable (Δy) divided by the corresponding change in the quantity represented by the independent variable (Δx) over an interval. Algebraically, the average rate of change for any function $y = f(x)$ over the interval $x_1 \le x \le x_2$ can be determined by

$$\text{Average rate of change } = \frac{change\ \ in\ \ y}{change\ \ in\ \ x} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- Graphically, the average rate of change for any function $y = f(x)$ over the interval $x_1 \le x \le x_2$ is equivalent to the slope of the secant line passing through two points (x_1, y_1) and (x_2, y_2).



Average rate of change

$$= m_{secant} = \frac{change\ \ in\ \ y}{change\ \ in\ \ x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

- Average rate of change is expressed using the units of the two quantities that are related to each other.
- A ***positive average rate of change*** indicates that the quantity represented by the ***dependent variable is increasing on the specified interval***, compared with the quantity represented by the independent variable. Graphically, this is indicated by a ***secant line that has a positive slope*** (the secant line rises from left to right).
- A ***negative average rate of change*** indicates that the quantity represented by the ***dependent variable is decreasing on the specified interval,*** compared with the quantity represented by the independent variable. Graphically, this is indicated by ***a secant line that has a negative slope*** (the secant line falls from left to right).
- ***All linear relationships* have *a constant rate of change***. Average rate of change calculations over different intervals of the independent variable give the same result.
- ***Nonlinear relationships do not have a constant rate of change***. Average rate of change calculations over different intervals of the independent variable give different results.

Example 1

Which of the following are examples of average rates of change?

a) The average height of the players on the basketball team is 2.1 m.
b) A child grows 8 cm in 6 months.
c) A plane travelled 650 km in 3 h.
d) The snowboarder raced across the finish line at 60 km/h

Example 2

The table shows the percent of Canadian households that used e-mail from 1999 to 2003.

Year	Households (%)
1999	26.3
2000	37.4
2001	46.1
2002	48.9
2003	52.1

Determine the average rate of change of the percent of households using e-mail for each pair of consecutive years from 1999 to 2003.

Example 3

If a ball is dropped from the top of a 120-m cliff, its height, h, in metres, after t seconds can be modelled by $h(t) = 120 - 4.9t^2$.

a) Find the average rate of change of the height of the ball with respect to time over the intervals

i. 1 s to 4 s

ii. 4 s to 6 s

iii. 6 s to 7 s

b) What does the average rate of change represent in this situation?

c) Interpret the significance of your answers in part (a).

Estimating Instantaneous Rates of Change from Tables of Values and Equations

The ***instantaneous rate of change*** of the dependent variable is the ***rate at which the dependent variable changes at a specific value of the independent variable,*** $x = a$.

- The instantaneous rate of change of the dependent variable, in a table of values or an equation of the relationship, can be estimated using the following methods:
  - Using **a series of preceding** $(a - h \le x \le a)$ **and following** $(a \le x \le a + h)$ **intervals**:
    Calculate the average rate of change by keeping one endpoint of each interval fixed. (This is $x = a$, the location where the instantaneous rate of change occurs.) Move the other endpoint of the interval closer and closer to the fixed point from either side by making h smaller and smaller. Based on the trend for the average rates of change, make an estimate for the instantaneous rate of change at the specific value.

  - Using **a series of centred intervals** $(a - h \le x \le a + h)$:
    Calculate the average rate of change by picking endpoints for each interval on either side of $x = a$, where the instantaneous rate of change occurs. Choose these endpoints so that the value where the instantaneous rate of change occurs is the midpoint of the interval. Continue to calculate the average rate of change by moving both endpoints closer and closer to where the instantaneous rate of change occurs. Based on the trend, make an estimate for the instantaneous rate of change.

  - Using **the difference quotient and a general point:**
    Calculate the average rate of change using the location where the instantaneous rate of change occurs $(a, f(a))$ and a general point $(a + h, f(a + h))$, i.e., $\dfrac{f(a+h) - f(a)}{h}$. Choose values for h that are very small, such as ± 0.01 or ± 0.001. The smaller the value used for h, the better the estimate will be.

Example 1

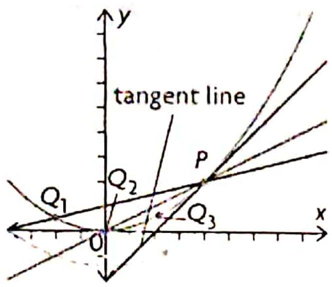
The population of a small town appears to be growing exponentially. Town planners think that the equation $P(t) = 35\,000(1.05)^t$, where $P(t)$ is the number of people in the town and t is the number of years after 2000, models the size of the population. Estimate the instantaneous rate of change in the population in 2020.

Example 2 In the table below, the distance travelled by a parachutist in the first 4 seconds after jumping out of an airplane is recorded at 0.5-s intervals. Estimate the parachutist's velocity at 2 s.

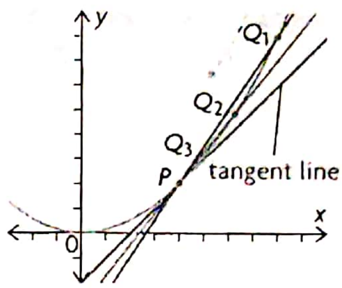
Time (s)	Distance (m)
0	0
0.5	1.25
1.0	5.00
1.5	11.25
2.0	20.00
2.5	31.25
3.0	45.00
3.5	61.25
4.0	80.00

Exploring Instantaneous Rates of Change Using Graphs

- The slope of a tangent at a point on a graph is equivalent to the instantaneous rate of change of a function at this point. The slope of a tangent cannot be calculated directly using the slope formula because the coordinates of only one point are known. The slope can be estimated, however, by calculating the slopes of a series of secant lines that go through the fixed point of tangency P and points that get closer and closer to this fixed point.



As Q approaches P from the left, the slope of QP increases and approaches the slope of the tangent line.



As Q approaches P from the right, the slope of QP decreases and approaches the slope of the tangent line.

Example 1 Create a table to estimate the slope of the tangent to $f(x)=4^{-x}+3$ at $x=-1$. Be sure to approach P from both directions.

Example 2

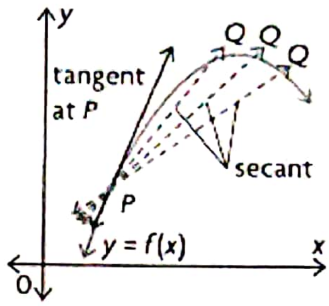
A thermometer is taken from a room temperature of 20°C to an outdoor temperature of 5°C. Temperature readings (T) are taken every 5.0 min as shown in the table.

t (min)	0.0	5.0	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0
T (°C)	20	15	12	9.8	8.3	7.2	6.5	6.0	5.7	5.5	5.3

a) Sketch the graph of T as a function of t. Draw a smooth curve through the points. Use your graph to estimate the instantaneous rate of change of the temperature with respect to time when $t = 20$ min.

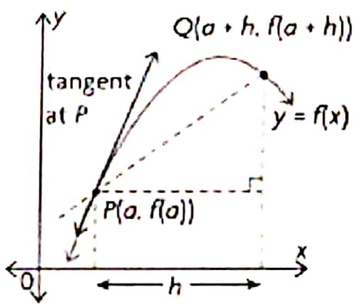
b) Use the table to estimate the instantaneous rate of change of the temperature with respect to time when $t = 20$ min.

The Slope of a Tangent



- The slope of the tangent to a curve at a point P is the limit of the slopes of the secants PQ as Q moves closer to P.

  $m_{tangent} = \lim\limits_{Q \to P}$ (slope of secant PQ)



- The slope of the tangent to the graph of $y = f(x)$ at $P(a, f(a))$ is given by

  $m_{tangent} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

- To find the slope of the tangent at a point $P(a, f(a))$,
  - find the value of $f(a)$
  - find the value of $f(a+h)$
  - evaluate $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Example 1 a) Copy and complete the table for $f(x) = 3x - x^2$ and a tangent at the point where $x = 4$.

Tangent Point $(a, f(a))$	**Increment** h	**Second Point** $(a+h, f(a+h))$	**Slope of Secant** $\dfrac{f(a+h)-f(a)}{h}$
	1		
	0.1		
	0.01		
	0.001		
	0.0001		

a) What do the values in the last column indicate about the slope of the tangent?

Example 2 Determine the slope of the tangent to each curve at the given value of x.

a) $f(x) = x^2$, $x = 2$

b) $g(x) = \sqrt{x - 4}$, $x = 5$

c) $k(x) = \frac{1}{x+3}$, $x = 1$