Homework 4 Solutions.

1. a) $y' = e^u(-\sin u + c) + e^u(\cos u + cu)$

$= e^u(\cos u - \sin u + c(1+u))$.

b) $y' = \dfrac{x^2\cos x - 2x\sin x}{x^4} = \dfrac{x\cos x - 2\sin x}{x^3}$

c) $f'(x) = (xe^x)'(\csc x) + (xe^x)(\csc x)'$

$= (e^x + xe^x)(\csc x) + (xe^x)(-\csc x \cot x)$.

$= e^x(1+x)\csc x - (xe^x)(\csc x \cot x)$

$= e^x \csc x\,((1+x) - x\cot x)$.

d) $y' = \dfrac{[(1+x^2)\tan^{-1}x - x]'(2) - 0}{4}$

$= \dfrac{(1+x^2)'\tan^{-1}x + (1+x^2)(\tan^{-1}x)' - 1}{2}$

$= \frac{1}{2}\left(2x\tan^{-1}x + (1+x^2)\dfrac{1}{1+x^2} - 1\right)$

$= \frac{1}{2}(2x\tan^{-1}x + 1 - 1) = x\tan^{-1}x$

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2. $y' = \sec x \tan x + 2\sin x$

$y'(\frac{\pi}{3}) = \sec\frac{\pi}{3}\tan\frac{\pi}{3} + 2\sin\frac{\pi}{3}$

$$= \frac{1}{\cos\frac{\pi}{3}} \quad \tan\frac{\pi}{3} + 2\sin\frac{\pi}{3}$$

$$= 2\sqrt{3} + 2\frac{\sqrt{3}}{2} = 3\sqrt{3}$$

$$y - 1 = 3\sqrt{3}\left(x - \frac{\pi}{3}\right)$$

$$\Rightarrow \quad y = 3\sqrt{3}\,x - \pi\sqrt{3} + 1.$$

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3. a) $g(t) = (t^4+1)^{-3}$

$$g'(t) = -3(t^4+1)^{-4}(t^4+1)'$$

$$= -3(t^4+1)^{-4}(4t^3) = \frac{-12t^3}{(t^4+1)^4}$$

b) $y' = x'e^{-kx} + x(e^{-kx})'$

$$= e^{-kx} + x(e^{-kx}(-kx)')$$

$$= e^{-kx} - kx(e^{-kx}) = e^{-kx}(1-kx)$$

c) $y = (x^2+1)(x^2+2)^{\frac{1}{3}}$

$$y' = (x^2+1)'(x^2+2)^{\frac{1}{3}} + (x^2+1)\left[(x^2+2)^{\frac{1}{3}}\right]'$$

$$= 2x\sqrt[3]{x^2+2} + (x^2+1)\left(\frac{1}{3}(x^2+2)^{-\frac{2}{3}}(2x)\right)$$

$$= 2x\sqrt[3]{x^2+2} + \frac{(x^2+1)(2x)}{3(x^2+2)^{2/3}}$$

Next few steps are optional.

$$= \frac{2x(x^2+2)^{\frac{1}{3}} \cdot 3(x^2+2)^{2/3} + (x^2+1)\,2x}{3(x^2+2)^{2/3}}$$

$$= \frac{6x(x^2+2) + 2x(x^2+1)}{3(x^2+2)^{2/3}}$$

$$= \frac{2x(3x^2+6) + 2x(x^2+1)}{(\qquad)}$$

$$= \frac{2x(3x^2+6x+x^2+1)}{(\qquad)} = \frac{2x(4x^2+7)}{3(x^2+2)^{2/3}}$$

d) $y' = (e^{-5x})'(\cos 3x) + (e^{-5x})(\cos 3x)'$

$= e^{-5x}(-5x)'(\cos 3x) + (e^{-5x})(-\sin 3x)(3x)'$

$= -5e^{-5x}\cos 3x - e^{-5x}(\sin 3x)(3)$

$= -e^{-5x}(5\cos 3x + 3\sin 3x)$

e) $y' = \cos(\tan(2x))\,(\tan(2x))'$

$= \cos(\tan(2x))(\sec^2(2x))(2x)'$

$= 2\cos(\tan 2x)(\sec^2 2x)$

f) $y' = 2^{3^{x^2}} \ln 2 (3^{x^2})'$

$= 2^{(\ )} \ln 2 (3^{x^2} \ln 3 (x^2)')$

$= 2^{(\ )} \ln 2 (3^{x^2} \ln 3 (2x))$

$$= 2x \cdot 2^{3^{x^2}} \cdot 3^{x^2} \ln 2 \ln 3$$

9) $y = [x + (x + \sin^2 x)^3]^4$

$$y' = 4 [\quad]^3 (x + (x + \sin^2 x)^3)'$$

$$= 4 [\quad]^3 (1 + 3(x + \sin^2 x)^2 (x + \sin^2 x)')$$

$$= 4 [\quad]^3 \cdot (1 + 3(x + \sin^2 x)^2 (1 + 2 \sin x \cos x))$$

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4. $y' = 10(1+2x)^9 (2) = 20 (1+2x)^9$

$$y'(0) = 20(1)^9 = 20$$

$$y - 1 = 20x \quad \Rightarrow \quad y = 20x + 1.$$

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5. $F(x) = f(x f(x f(x)))$

$$F'(x) = f'(x f(x f(x))) \, (x f(x f(x)))'$$

$$(x f(x f(x)))' = f(x f(x)) + x \, (f(x f(x)))'$$

$$= f(x f(x)) + x f'(x f(x)) \, (x f(x))'$$

$$= \text{''} \qquad\qquad \text{''} \, (f(x) + x f'(x))$$

$$= f(x f(x)) + x f'(x f(x)) \, (f(x) + x f'(x))$$

Sub 1 into x.

$$(1f(1f(1)))' = f(1\cdot 2) + 1f'(1\cdot 2)(2+1\cdot 4)$$
$$= f(2) + 1\cdot f'(2)(6)$$
$$= 3 + 5(6) = 33$$

$$f'(xf(xf(x))) = f'(1f(1\cdot 2)) = f'(1\cdot 3) = f'(3) = 6$$

$$F'(1) = 6\cdot 33 = 198$$

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6. $$f(x) = \frac{1}{x - \frac{2}{x+\sin x}} = \left(x - \frac{2}{x+\sin x}\right)^{-1}$$

$$f'(x) = -\left(x - \frac{2}{x+\sin x}\right)^{-2}\left(1 - 2\left[(x+\sin x)^{-1}\right]'\right)$$

$$= -\frac{1}{\left(x - \frac{2}{x+\sin x}\right)^2}\left(1 - 2\left[-(x+\sin x)^{-2}(1+\cos x)\right]\right)$$

$$= -\frac{1}{\left(x - \frac{2}{x+\sin x}\right)^2}\left(1 + \frac{2+2\cos x}{(x+\sin x)^2}\right)$$

$$= -\frac{1 + \frac{2+2\cos x}{(x+\sin x)^2}}{\left(x - \frac{2}{x+\sin x}\right)^2}$$

7, a) $5y^4 \frac{dy}{dx} + 2xy^3 + x^2(3y^2)\frac{dy}{dx} = \frac{dy}{dx} e^{x^2} + y(e^{x^2})(2x)$

$\Rightarrow 5y^4 \frac{dy}{dx} + 3x^2y^2 \frac{dy}{dx} - \frac{dy}{dx} e^{x^2} = 2xy e^{x^2} - 2xy^3$

$\Rightarrow \frac{dy}{dx}(5y^4 + 3x^2y - e^{x^2}) = 2xye^{x^2} - 2xy^3$

$\Rightarrow \frac{dy}{dx} = \frac{2xye^{x^2} - 2xy^3}{5y^4 + 3x^2y - e^{x^2}}$

b) $\frac{dy}{dx}\sin(x^2) + y\cos(x^2)\,2x = \sin(y^2) + x\cos(y^2)\,2y\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}\sin(x^2) - 2xy\cos(y^2)\frac{dy}{dx} = \sin(y^2) - 2xy\cos(x^2)$

$\Rightarrow \frac{dy}{dx}(\sin(x^2) - 2xy\cos(y^2)) =$ " "

$\Rightarrow \frac{dy}{dx} = \frac{\sin(y^2) - 2xy\cos(x^2)}{\sin(x^2) - 2xy\cos(y^2)}$

c) $\sqrt{xy} = 1 + x^2y \quad \Rightarrow (xy)^{\frac{1}{2}} = 1 + x^2y$

$\frac{1}{2}(xy)^{-\frac{1}{2}}(y + x\frac{dy}{dx}) = 2xy + x^2\frac{dy}{dx}$

$\Rightarrow \frac{1}{2\sqrt{xy}}(y) + \frac{1}{2\sqrt{xy}} x \frac{dy}{dx} = 2xy + x^2\frac{dy}{dx}$

$\Rightarrow \frac{x}{2\sqrt{xy}}\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy - \frac{y}{2\sqrt{xy}}$

$\Rightarrow \frac{dy}{dx}\left(\frac{x}{2\sqrt{xy}} - x^2\right) = 2xy - \frac{y}{2\sqrt{xy}}$

$\Rightarrow \frac{dy}{dx}(x - x^2 2\sqrt{xy}) = 2xy\,2\sqrt{xy} - y$

$$\Rightarrow \frac{dy}{dx} = \frac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$$

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8. $$2x + 2y\frac{dy}{dx} = 2(2x^2 + 2y^2 - x)(4x + 4y\frac{dy}{dx} - 1)$$

Plug in the numbers $(0, \frac{1}{2})$.

$$\Rightarrow 1\frac{dy}{dx} = 2(0 + \tfrac{1}{2} - 0)(0 + 2\frac{dy}{dx} - 1)$$

$$\Rightarrow \frac{dy}{dx} = 1(2\frac{dy}{dx} - 1) \Rightarrow \frac{dy}{dx} = 2\frac{dy}{dx} - 1$$

$$\Rightarrow \frac{dy}{dx} = 1.$$

$$\Rightarrow y - \tfrac{1}{2} = 1(x) \Rightarrow y = x + \tfrac{1}{2}$$

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9. a) $$g'(x) = \frac{1}{2}(x^2-1)^{-\frac{1}{2}}(2x)(\sec^{-1}x) + (x^2-1)^{\frac{1}{2}}\frac{1}{x\sqrt{x^2-1}}$$

$$= \frac{x\sec^{-1}x}{\sqrt{x^2-1}} + \frac{\sqrt{x^2-1}}{x\sqrt{x^2-1}}$$

$$= \frac{x\sec^{-1}x}{\sqrt{x^2-1}} + \frac{1}{x}$$

b) $$h'(t) = -\frac{1}{1+t^2} - \frac{1}{1+(\frac{1}{t})^2}(\tfrac{1}{t})'$$

$$= -\frac{1}{1+t^2} - \frac{1}{1+\frac{1}{t^2}}(-\tfrac{1}{t^2})$$

$$= -\frac{1}{1+t^2} + \frac{1}{\frac{t^2+1}{t^2}}(\tfrac{1}{t^2}) = 0$$

c) $F'(\theta) = \dfrac{1}{\sqrt{1-\sin\theta}} \left((\sin\theta)^{\frac{1}{2}} \right)'$

$= \dfrac{1}{\sqrt{1-\sin\theta}} \; \dfrac{1}{2} (\sin\theta)^{-\frac{1}{2}} \cos\theta$

$= \dfrac{1}{2} \; \dfrac{\cos\theta}{\sqrt{\sin\theta - \sin^2\theta}}$

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10. $y = xe^{cx}$

$y' = e^{cx} + xe^{cx}(c)$

$y'' = ce^{cx} + ce^{cx} + c^2xe^{cx} = 2ce^{cx} + c^2xe^{cx}$

$= ce^{cx}(2+cx).$

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11. $f(x) = xg(x^2)$

$f'(x) = g(x^2) + x(g(x^2))'$

$= g(x^2) + xg'(x^2)(2x)$

$= g(x^2) + 2x^2g'(x^2)$

$f''(x) = g'(x^2)(2x) + 4xg'(x) + 2x^2g''(x^2)(2x)$

$= 6xg'(x^2) + 4x^3g''(x^2).$

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12. A 13. C 14. D 15. B