AP Calculus Lesson Twenty Notes

Chapter 9 Infinite Series

9.3 Positive Term Series and Alternating Series

Tests for Positive Term Series

9.3.1 Integral Test

The last topic that we discussed in the previous section was the harmonic series. In that discussion we stated that the harmonic series was a divergent series. It is now time to prove that statement. This proof will also get us started on the way to our next test for convergence that we'll be looking at.

So, we will be trying to prove that the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

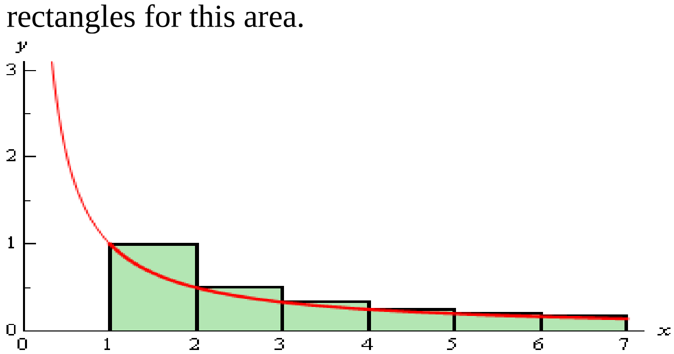
We'll start this off by looking at an apparently unrelated problem. Let's start off by asking what the area under $f(x) = \frac{1}{x}$ is on the interval $[1, \infty)$. From the section on Improper Integrals we know that this is,

$$\int_1^{\infty} \frac{1}{x}\,dx = \infty$$

and so we called this integral divergent (yes, that's the same term we're using here with series….).

So, just how does that help us to prove that the harmonic series diverges? Well, recall that we can always estimate the area by breaking up the interval into segments and then sketching in rectangles and using the sum of the area all of the rectangles as an estimate of the actual area. Let's do that for this problem as well and see what we get.

We will break up the interval into subintervals of width 1 and we'll take the function value at the left endpoint as the height of the rectangle. The image below shows the first few rectangles for this area.



So, the area under the curve is approximately,

$$A \approx \left(\frac{1}{1}\right)(1)+\left(\frac{1}{2}\right)(1)+\left(\frac{1}{3}\right)(1)+\left(\frac{1}{4}\right)(1)+\left(\frac{1}{5}\right)(1)+\cdots$$
$$=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$
$$=\sum_{n=1}^{\infty}\frac{1}{n}$$

Now note a couple of things about this approximation. First, each of the rectangles overestimates the actual area and secondly the formula for the area is exactly the harmonic series!
Putting these two facts together gives the following,

$$A \approx \sum_{n=1}^{\infty}\frac{1}{n} > \int_1^{\infty}\frac{1}{x}dx = \infty$$

Notice that this tells us that we must have,

$$\sum_{n=1}^{\infty}\frac{1}{n} > \infty \qquad \Rightarrow \qquad \sum_{n=1}^{\infty}\frac{1}{n} = \infty$$

Since we can’t really be larger than infinity the harmonic series must also be infinite in value. In other words, the harmonic series is in fact divergent.

So, we’ve managed to relate a series to an improper integral that we could compute and it turns out that the improper integral and the series have exactly the same convergence.
Let’s see if this will also be true for a series that converges. When discussing the Divergence Test we made the claim that

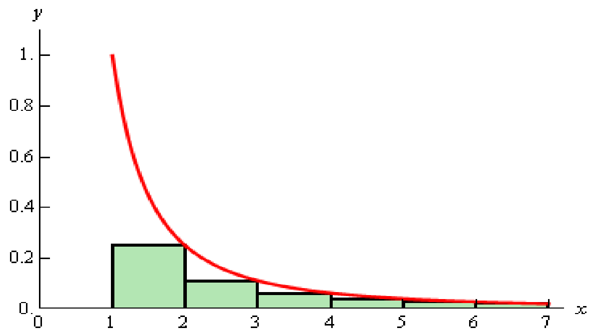
$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

converges. Let’s see if we can do something similar to the above process to prove this.

We will try to relate this to the area under $f(x)=\frac{1}{x^2}$ is on the interval [1, ∞). Again, from the Improper Integral section we know that,

$$\int_1^{\infty}\frac{1}{x^2}dx = 1$$

and so this integral converges. We will once again try to estimate the area under this curve. We will do this in an almost identical manner as the previous part with the exception that we will instead of using the left end points for the height of our rectangles we will use the right end points. Here is a sketch of this case,



In this case the area estimation is,

$$A \approx \left(\frac{1}{2^2}\right)(1)+\left(\frac{1}{3^2}\right)(1)+\left(\frac{1}{4^2}\right)(1)+\left(\frac{1}{5^2}\right)(1)+\cdots$$
$$=\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots$$

This time, unlike the first case, the area will be an underestimation of the actual area and the estimation is not quite the series that we are working with. Notice however that the only difference is that we're missing the first term. This means we can do the following,

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\underbrace{\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots}_{\text{Area Estimation}}<1+\int_1^{\infty}\frac{1}{x^2}\,dx=1+1=2$$

Or, putting all this together we see that,

$$\sum_{n=1}^{\infty}\frac{1}{n^2}<2$$

With the harmonic series this was all that we needed to say that the series was divergent. With this series however, this isn't quite enough. For instance $-\infty < 2$, and if the series did have a value of $-\infty$, then it would be divergent (when we want convergent). So, let's do a little more work.

First, let's notice that all the series terms are positive (that's important) and that the partial sums are,

$$s_n=\sum_{i=1}^{n}\frac{1}{i^2}$$

Because the terms are all positive we know that the partial sums must be an increasing sequence. In other words,

$$s_n=\sum_{i=1}^{n}\frac{1}{i^2}<\sum_{i=1}^{n+1}\frac{1}{i^2}=s_{n+1}$$

In s_{n+1} we are adding a single positive term onto s_n and so must get larger. Therefore, the partial sums form an increasing (and hence monotonic) sequence.

Also note that, since the terms are all positive, we can say,

$$s_n = \sum_{i=1}^{n} \frac{1}{i^2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < 2 \qquad \Rightarrow \qquad s_n < 2$$

and so the sequence of partial sums is a bounded sequence.

In the second section on Sequences we gave a theorem that stated that a bounded and monotonic sequence was guaranteed to be convergent. This means that the sequence of partial sums is a convergent sequence. So, who cares right? Well recall that this means that the series must then also be convergent!

So, once again we were able to relate a series to an improper integral (that we could compute) and the series and the integral had the same convergence.

We went through a fair amount of work in both of these examples to determine the convergence of the two series. Luckily for us we don't need to do all this work every time. The ideas in these two examples can be summarized in the following test.

Integral Test

Suppose that $f(x)$ is a continuous, positive and decreasing function on the interval $[k, \infty)$ and that $f(n) = a_n$ then,

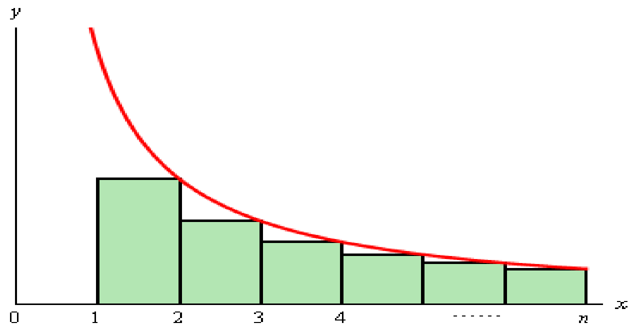
1. If $\int_k^{\infty} f(x)dx$ is convergent so is $\sum_{n=k}^{\infty} a_n$.
2. If $\int_k^{\infty} f(x)dx$ is divergent so is $\sum_{n=k}^{\infty} a_n$.

Proof of Integral Test

First, for the sake of the proof we'll be working with the series $\sum_{n=1}^{\infty} a_n$

The original test statement was for a series that started at a general $n = k$ and while the proof can be done for that it will be easier if we assume that the series starts at $n = 1$.

Another way of dealing with the $n = k$ is we could do an index shift and start the series at $n = 1$ and then do the Integral Test. Either way proving the test for $n = 1$ will be sufficient. Let's start off and estimate the area under the curve on the interval $[1, n]$ and we'll underestimate the area by taking rectangles of width one and whose height is the right endpoint. This gives the following figure.



Now, note that,

$$f(2) = a_2 \qquad f(3) = a_3 \qquad \cdots \qquad f(n) = a_n$$

The approximate area is then,

$$A \approx (1)f(2) + (1)f(3) + \cdots + (1)f(n) = a_2 + a_3 + \cdots a_n$$

and we know that this underestimates the actual area so,

$$\sum_{i=2}^{n} a_i = a_2 + a_3 + \cdots a_n < \int_1^n f(x)\,dx$$

Now, let's suppose that $\int_1^\infty f(x)dx$ is convergent and so must have a finite value. Also, because $f(x)$ is positive we know that,

$$\int_1^n f(x)\,dx < \int_1^\infty f(x)\,dx$$

This in turn means that,

$$\sum_{i=2}^{n} a_i < \int_1^n f(x)\,dx < \int_1^\infty f(x)dx$$

Our series starts at $n = 1$ so this isn't quite what we need. However, that's easy enough to deal with.

$$\sum_{i=1}^{n} a_i = a_1 + \sum_{i=2}^{n} a_i < a_1 + \int_1^\infty f(x)\,dx = M$$

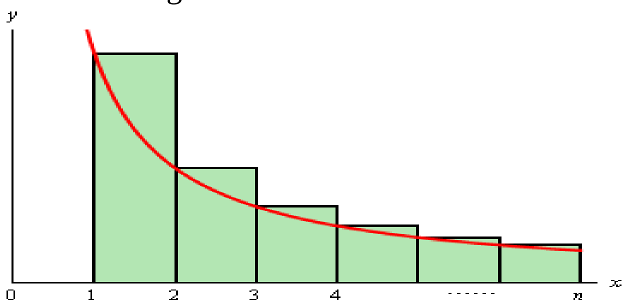
So, just what has this told us? Well we now know that the sequence of partial sums, $s_n = \sum_{i=1}^{n} a_i$ are bounded above by M.

Next, because the terms are positive we also know that,

$$s_n \le s_n + a_{n+1} = \sum_{i=1}^{n} a_i + a_{n+1} = \sum_{i=1}^{n+1} a_i = s_{n+1} \qquad \Rightarrow \qquad s_n \le s_{n+1}$$

and so the sequence $\{s_n\}_{n=1}^{\infty}$ is also an increasing sequence. So, we now know that the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ converges and hence our series $\sum_{n=1}^{\infty} a_n$ is convergent.

So, the first part of the test is proven. The second part is somewhat easier. This time let's over estimate the area under the curve by using the left endpoints of interval for the height of the rectangles as shown below.



In this case the area is approximately,

$$A \approx (1)f(1) + (1)f(2) + \cdots + (1)f(n-1) = a_1 + a_2 + \cdots a_{n-1}$$

Since we know this overestimates the area we also then know that,

$$s_{n-1} = \sum_{i=1}^{n-1} a_i = a_1 + a_2 + \cdots a_{n-1} > \int_1^{n-1} f(x)\,dx$$

Now, suppose that $\int_1^{\infty} f(x)dx$ is divergent. In this case this means that $\int_1^{\infty} f(x)dx \to \infty$ as $n \to \infty$ because $f(x) \geq 0$. However, because $n-1 \to \infty$ as $n \to \infty$ we also know that $\int_1^{n-1} f(x)dx \to \infty$ Therefore, since $s_{n-1} > \int_1^{n-1} f(x)dx$ we know that as $n \to \infty$ we must have $s_{n-1} \to \infty$. This in turn tells us that $s_{n-1} \to \infty$ as $n \to \infty$.

So, we now know that the sequence of partial sums, $\{s_n\}_{n=1}^{\infty}$, is a divergent sequence and so $\sum_{n=1}^{\infty} a_n$ is a divergent series.

Example 9.3-1 Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Solution

In this case the function we'll use is,

$$f(x) = \frac{1}{x \ln x}$$

This function is clearly positive and if we make *x* larger the denominator will get larger and so the function is also decreasing. Therefore, all we need to do is determine the convergence of the following integral.

$$\begin{aligned}\int_2^\infty \frac{1}{x\ln x}\,dx &= \lim_{t\to\infty}\int_2^t \frac{1}{x\ln x}\,dx \qquad u=\ln x\\ &= \lim_{t\to\infty}\left(\ln(\ln x)\right)\Big|_2^t\\ &= \lim_{t\to\infty}\left(\ln(\ln t)-\ln(\ln 2)\right)\\ &= \infty\end{aligned}$$

The integral is divergent and so the series is also divergent by the Integral Test.

Example 9.3-2 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} n\,\mathbf{e}^{-n^2}$$

Solution

The function that we'll use in this example is,

$$f(x) = x\mathbf{e}^{-x^2}$$

This function is always positive on the interval that we're looking at. Now we need to check that the function is decreasing. It is not clear that this function will always be decreasing on the interval given. We can use our Calculus I knowledge to help us however. The derivative of this function is,

$$f'(x) = \mathbf{e}^{-x^2}\left(1-2x^2\right)$$

This function has two critical points (which will tell us where the derivative changes sign) at $x = \pm\frac{1}{\sqrt{2}}$. Since we are starting at $n = 0$, we can ignore the negative critical point. Picking a couple of test points we can see that the function in increasing on the interval $\left[0, \frac{1}{\sqrt{2}}\right]$ and it is decreasing on $\left[\frac{1}{\sqrt{2}}, \infty\right)$, Therefore, eventually the function will be decreasing and that's all that's required for us to use the Integral Test.

$$\begin{aligned}\int_0^\infty x\mathbf{e}^{-x^2}\,dx &= \lim_{t\to\infty}\int_0^t x\mathbf{e}^{-x^2}\,dx \qquad u=-x^2\\ &= \lim_{t\to\infty}\left(-\frac{1}{2}\mathbf{e}^{-x^2}\right)\Bigg|_0^t\\ &= \lim_{t\to\infty}\left(\frac{1}{2}-\frac{1}{2}\mathbf{e}^{-t^2}\right)=\frac{1}{2}\end{aligned}$$

The integral is convergent and so the series must also be convergent by the Integral Test.

9.3.2 The *p* - series Test

If $k > 0$ then $\sum_{n=k}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Example 9.3-3 Determine if the following series are convergent or divergent.

(a) $\sum_{n=} \frac{1}{n^7}$ (b) $\frac{1}{\sqrt{n}}$

Solution

(a) In this case $p = 7$ and so by this fact the series is convergent.

(b) For this series $p = \frac{1}{2} \le 1$ and so the series is divergent by the fact.

9.3.3 Comparison Test and Limit Comparison Test

Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \ge 0$ for all *n* and $a_n \le b_n$ for all *n*. then

1. If $\sum b_n$ is convergent then so is $\sum a_n$.
2. If $\sum a_n$ is divergent then so is $\sum b_n$.

In other words, we have two series of positive terms and the terms of one of the series is always larger than the terms of the other series. Then if the larger series is convergent the smaller series must also be convergent. Likewise, if the smaller series is divergent then the larger series must also be divergent. Note as well that in order to apply this test we need both series to start at the same place.

Proof of Comparison Test

The test statement did not specify where each series should start. We only need to require that they start at the same place so to help with the proof we'll assume that the series start at $n = 1$. If the series don't start at $n = 1$ the proof can be redone in exactly the same manner or you could use an index shift to start the series at $n = 1$ and then this proof will apply.
We'll start off with the partial sums of each series.

$$s_n = \sum_{i=1}^{n} a_i \qquad\qquad t_n = \sum_{i=1}^{n} b_i$$

Let's notice a couple of nice facts about these two partial sums. First, because $a_n, b_n \ge 0$, we know that,

$$s_n \le s_n + a_{n+1} = \sum_{i=1}^{n} a_i + a_{n+1} = \sum_{i=1}^{n+1} a_i = s_{n+1} \qquad \Rightarrow \qquad s_n \le s_{n+1}$$

$$t_n \le t_n + b_{n+1} = \sum_{i=1}^{n} b_i + b_{n+1} = \sum_{i=1}^{n+1} b_i = t_{n+1} \qquad \Rightarrow \qquad t_n \le t_{n+1}$$

So, both partial sums form increasing sequences.
Also, because $a_n \le b_n$ for all *n* we know that we must have $s_n \le t_n$ for all *n*.
With these preliminary facts out of the way we can proceed with the proof of the test itself.

Let's start out by assuming that $\sum_{n=1}^{\infty} b_n$ is a convergent series. Since $b_n \ge 0$ we know that,

$$t_n = \sum_{i=1}^{n} b_i \le \sum_{i=1}^{\infty} b_i$$

However, we also have established that $s_n \le t_n$ for all *n* and so for all *n* we also have,

$$s_n \le \sum_{i=1}^{\infty} b_i$$

Finally since $\sum_{n=1}^{\infty} b_n$ is a convergent series it must have a finite value and so the partial sums, s_n are bounded above. Therefore, from the second section on sequences we know that a monotonic and bounded sequence is also convergent and so $\{s_n\}_{n=1}^{\infty}$ is convergent and so $\sum_{n=1}^{\infty} a_n$ is convergent.

Next, let's assume that $\sum_{n=1}^{\infty} a_n$ is divergent. Because $a_n \ge 0$ we then know that we must have $s_n \to \infty$ as $n \to \infty$. However, we also know that for all *n* we have $s_n \le t_n$ and therefore we also know that $t_n \to \infty$ as $n \to \infty$. So, $\{t_n\}_{n=1}^{\infty}$ is a divergent sequence as so $\sum_{n=1}^{\infty} b_n$ is divergent.

Example 9.3-4 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2(n)}$$

Solution
Since the cosine term in the denominator doesn't get too large we can assume that the series terms will behave like,

$$\frac{n}{n^2} = \frac{1}{n}$$

which, as a series, will diverge. So, from this we can guess that the series will probably diverge and so we'll need to find a smaller series that will also diverge.

Recall that from the comparison test with improper integrals that we determined that we can make a fraction smaller by either making the numerator smaller or the denominator larger. In this case the two terms in the denominator are both positive. So, if we drop the cosine term we will in fact be making the denominator larger since we will no longer be subtracting off a positive quantity. Therefore,

$$\frac{n}{n^2 - \cos^2(n)} > \frac{n}{n^2} = \frac{1}{n}$$

Then, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (it's harmonic or the *p*-series test) by the Comparison Test our original series must also diverge.

Example 9.3-5 Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^2 + 2}{n^4 + 5}$$

Solution

In this case the "+2" and the "+5" don't really add anything to the series and so the series terms should behave pretty much like

$$\frac{n^2}{n^4} = \frac{1}{n^2}$$

which will converge as a series. Therefore, we can guess that the original series will converge and we will need to find a larger series which also converges.

This means that we'll either have to make the numerator larger or the denominator smaller. We can make the denominator smaller by dropping the "+5". Doing this gives,

$$\frac{n^2 + 2}{n^4 + 5} < \frac{n^2 + 2}{n^4}$$

At this point, notice that we can't drop the "+2" from the numerator since this would make the term smaller and that's not what we want. However, this is actually all the further that we need to go. Let's take a look at the following series.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^2 + 2}{n^4} &= \sum_{n=1}^{\infty} \frac{n^2}{n^4} + \sum_{n=1}^{\infty} \frac{2}{n^4} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{2}{n^4} \end{aligned}$$

As shown, we can write the series as a sum of two series and both of these series are convergent by the *p*-series test. Therefore, since each of these series are convergent we know that the sum,

$$\sum_{n=1}^{\infty} \frac{n^2 + 2}{n^4}$$

is also a convergent series. Recall that the sum of two convergent series will also be convergent.

Now, since the terms of this series are larger than the terms of the original series we know that the original series must also be convergent by the Comparison Test.

The comparison test is a nice test that allows us to do problems that either we couldn't have done with the integral test or at the best would have been very difficult to do with the integral test. That doesn't mean that it doesn't have problems of its own.
Consider the following series.

$$\sum_{n=0}^{\infty} \frac{1}{3^n - n}$$

This is not much different from the first series that we looked at. The original series converged because the 3^n gets very large very fast and will be significantly larger than the n. Therefore, the n doesn't really affect the convergence of the series in that case. The fact that we are now subtracting the n off now instead of adding the n on really shouldn't change the convergence. We can say this because the 3^n gets very large very fast and the fact that we're subtracting n off won't really change the size of this term for all sufficiently large value of n.

So, we would expect this series to converge. However, the comparison test won't work with this series. To use the comparison test on this series we would need to find a larger series that we could easily determine the convergence of. In this case we can't do what we did with the original series. If we drop the n we will make the denominator larger (since the n was subtracted off) and so the fraction will get smaller and just like when we looked at the comparison test for improper integrals knowing that the smaller of two series converges does not mean that the larger of the two will also converge.

So, we will need something else to do help us determine the convergence of this series. The following variant of the comparison test will allow us to determine the convergence of this series.

Limit Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n. Define,

$$c = \lim_{n \to \infty} \frac{a_n}{b_n}$$

If c is positive (*i.e.* $c > 0$) and is finite (*i.e.* $c < \infty$) then either both series converge or both series diverge.

Proof of Limit Comparison Test

Because $0 < c < \infty$ we can find two positive and finite numbers, m and M, such that $m < c < M$.

Now, because $c = \lim_{n \to \infty} \frac{a_n}{b_n}$, we know that for large enough n the quotient $\frac{a_n}{b_n}$ must be close to c and so there must be a positive integer N such that if $n > N$, we also have,

$$m < \frac{a_n}{b_n} < M$$

Multiplying through by b_n gives,

$$mb_n < a_n < Mb_n$$

provided $n > N$. Now, if $\sum b_n$ diverges then so does $\sum mb_n$ and so since $mb_n < a_n$ for all sufficiently large n by the Comparison Test $\sum a_n$ also diverges.

Likewise, if $\sum b_n$ converges then so does $\sum Mb_n$ and since $a_n < Mb_n$ for all sufficiently large n by the Comparison Test $\sum a_n$ also converges.

Example 9.3-6 Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{1}{3^n - n}$$

Solution

To use the limit comparison test we need to find a second series that we can determine the convergence of easily and has what we assume is the same convergence as the given series. On top of that we will need to choose the new series in such a way as to give us an easy limit to compute for c.

We've already guessed that this series converges and since it's vaguely geometric let's use $\sum_{n=0}^{\infty} \frac{1}{3^n}$ as the second series. We know that this series converges and there is a chance that since both series have the 3^n in it the limit won't be too bad.

Here's the limit.

$$c = \lim_{n\to\infty} \frac{1}{3^n} \frac{3^n - n}{1}$$

$$= \lim_{n\to\infty} 1 - \frac{n}{3^n}$$

Now, we'll need to use L'Hospital's Rule on the second term in order to actually evaluate this limit.

$$c = 1 - \lim_{n\to\infty} \frac{1}{3^n \ln(3)}$$

$$= 1$$

So, c is positive and finite so by the Comparison Test both series must converge since $\sum_{n=0}^{\infty} \frac{1}{3^n}$ converges.

Example 9.3-7 Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{4n^2 + n}{\sqrt[3]{n^7 + n^3}}$$

Solution

Fractions involving only polynomials or polynomials under radicals will behave in the same way as the largest power of *n* will behave in the limit. So, the terms in this series should behave as,

$$\frac{n^2}{\sqrt[3]{n^7}} = \frac{n^2}{n^{\frac{7}{3}}} = \frac{1}{n^{\frac{1}{3}}}$$

and as a series this will diverge by the *p*-series test. In fact, this would make a nice choice for our second series in the limit comparison test so let's use it.

$$\begin{aligned}
\lim_{n\to\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}} \frac{n^{\frac{1}{3}}}{1} &= \lim_{n\to\infty} \frac{4n^{\frac{7}{3}}+n^{\frac{4}{3}}}{\sqrt[3]{n^7\left(1+\frac{1}{n^4}\right)}} \\
&= \lim_{n\to\infty} \frac{n^{\frac{7}{3}}\left(4+\frac{1}{n}\right)}{n^{\frac{7}{3}}\sqrt[3]{1+\frac{1}{n^4}}} \\
&= \frac{4}{\sqrt[3]{1}} = 4 = c
\end{aligned}$$

So, c is positive and finite and so both limits will diverge since $\sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{3}}}$ diverges.

9.3.4 Alternating Series Test

Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all *n*. Then if,

1. $\lim_{n\to\infty} b_n = 0$ and,
2. $\{b_n\}$ is a decreasing sequence

the series $\sum a_n$ is convergent.

Proof of Alternating Series Test

Without loss of generality we can assume that the series starts at $n = 1$. If not we could modify the proof below to meet the new starting place or we could do an index shift to get the series to start at $n = 1$.

First, notice that because the terms of the sequence are decreasing for any two successive terms we can say, $b_n - b_{n+1} \geq 0$. Now, let's take a look at the even partial sums.

$$s_2 = b_1 - b_2 \ge 0$$
$$s_4 = b_1 - b_2 + b_3 - b_4 = s_2 + b_3 - b_4 \ge s_2 \qquad \text{because } b_3 - b_4 \ge 0$$
$$s_6 = s_4 + b_5 - b_6 \ge s_4 \qquad \text{because } b_5 - b_6 \ge 0$$
$$\vdots$$
$$s_{2n} = s_{2n-2} + b_{2n-1} - b_{2n} \ge s_{2n-2} \qquad \text{because } b_{2n-1} - b_{2n} \ge 0$$

So, $\{s_{2n}\}$ is an increasing sequence. Next, we can also write the general term as,

$$s_{2n} = b_1 - b_2 + b_3 - b_4 + b_5 + \cdots - b_{2n-2} + b_{2n-1} - b_{2n}$$
$$= b_1 - (b_2 - b_3) - (b_4 - b_5) + \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

Each of the quantities in parenthesis are positive and by assumption we know that b_{2n} is also positive. So, this tells us that $s_{2n} \le b_1$ for all n.

We now know that $\{s_{2n}\}$ is an increasing sequence that is bounded above and so we know that it must also converge. So, let's assume that its limit is s or,

$$\lim_{n\to\infty} s_{2n} = s$$

Next, we can quickly determine the limit of the sequence of odd partial sums, $\{s_{2n+1}\}$, as follows,

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty}(s_{2n} + b_{2n+1}) = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} = s + 0 = s$$

So, we now know that both $\{s_{2n}\}$ and $\{s_{2n+1}\}$ are convergent sequences and they both have the same limit and so we also know that $\{s_n\}$ is a convergent sequence with a limit of s. This in turn tells us that $\sum a_n$ is convergent.

Example 9.3-8 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Solution

First, identify the b_n for the test.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} \qquad b_n = \frac{1}{n}$$

Now, all that we need to do is run through the two conditions in the test.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$
$$b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$$

Both conditions are met and so by the Alternating Series Test the series must converge.

Example 9.3-9 Determine if the following series is convergent or divergent.
$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$$

Solution
First, identify the b_n for the test.
$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5} = \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+5} \qquad \Rightarrow \qquad b_n = \frac{n^2}{n^2+5}$$

Let's check the conditions.
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^2+5} = 1 \neq 0$$

So, the first condition isn't met and so there is no reason to check the second. Since this condition isn't met we'll need to use another test to check convergence. In these cases where the first condition isn't met it is usually best to use the divergence test.
$$\begin{aligned}\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5} &= \left(\lim_{n\to\infty} (-1)^n\right)\left(\lim_{n\to\infty} \frac{n^2}{n^2+5}\right) \\ &= \left(\lim_{n\to\infty} (-1)^n\right)(1) \\ &= \lim_{n\to\infty} (-1)^n \qquad \text{doesn't exist}\end{aligned}$$
This limit doesn't exist and so by the Divergence Test this series diverges.

Example 9.3-10 Determine if the following series is convergent or divergent.
$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

Solution
The point of this problem is really just to acknowledge that it is in fact an alternating series. To see this we need to acknowledge that,
$$\cos(n\pi) = (-1)^n$$

and so the series is really,
$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} \qquad \Rightarrow \qquad b_n = \frac{1}{\sqrt{n}}$$

Checking the two condition gives,
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$
$$b_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = b_{n+1}$$
The two conditions of the test are met and so by the Alternating Series Test the series is convergent.