Algebra and Equations 2

1. Linear Equations

A linear system of two equations with two variables is any system that can be written in the form.

$$\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned}$$

where any of the constants can be zero with the exception that each equation must have at least one variable in it.

Also, the system is called linear if the variables are only to the first power, are only in the numerator and there are no products of variables in any of the equations.

Here is an example of a system with numbers.

$$\begin{aligned} 3x - y &= 7 \\ 2x + 3y &= 1 \end{aligned}$$

Before we discuss how to solve systems we should first talk about just what a solution to a system of equations is. A solution to a system of equations is a value of *x* and a value of *y* that, when substituted into the equations, satisfies both equations at the same time.

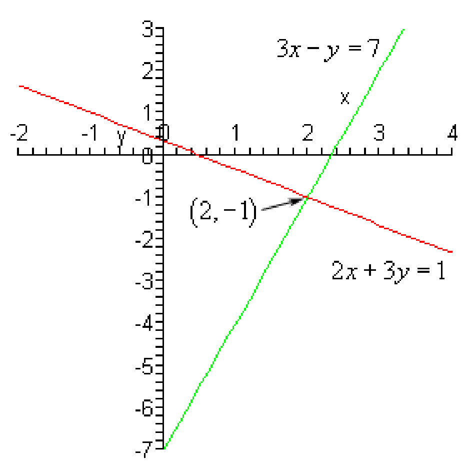
For the example above $x = 2$ and $y = -1$ is a solution to the system. This is easy enough to check.

$$\begin{aligned} 3(2) - (-1) &= 7 \\ 2(2) + 3(-1) &= 1 \end{aligned}$$

So, sure enough that pair of numbers is a solution to the system. Do not worry about how we got these values. This will be the very first system that we solve when we get into examples.

Note that it is important that the pair of numbers satisfy both equations. For instance $x = 1$ and $y = -4$ will satisfy the first equation, but not the second and so isn't a solution to the system. Likewise, $x = -1$ and y = 1 will satisfy the second equation but not the first and so can't be a solution to the system.

Now, just what does a solution to a system of two equations represent? Well if you think about it both of the equations in the system are lines. So, let's graph them and see what we get.



As you can see the solution to the system is the coordinates of the point where the two lines intersect. So, when solving linear systems with two variables we are really asking where the two lines will intersect.

1) Method of substitution

In this method we will solve one of the equations for one of the variables and substitute this into the other equation. This will yield one equation with one variable that we can solve. Once this is solved we substitute this value back into one of the equations to find the value of the remaining variable.

Example 1: Solve the following systems.

$$5x+4y=1$$
$$3x-6y=2$$

Solution

With this system we aren't going to be able to completely avoid fractions. However, it looks like if we solve the second equation for *x* we can minimize them. Here is that work.

$$3x = 6y+2$$
$$x = 2y+\frac{2}{3}$$

Now, substitute this into the first equation and solve the resulting equation for *y*.

$$5\left(2y+\frac{2}{3}\right)+4y = 1$$
$$10y+\frac{10}{3}+4y = 1$$
$$14y = 1-\frac{10}{3} = -\frac{7}{3}$$
$$y = -\left(\frac{7}{3}\right)\left(\frac{1}{14}\right)$$
$$y = -\frac{1}{6}$$

Finally, substitute this into the original substitution to find *x*.

$$x = 2\left(-\frac{1}{6}\right)+\frac{2}{3} = -\frac{1}{3}+\frac{2}{3} = \frac{1}{3}$$

So, the solution to this system is $x=\frac{1}{3}$ and $y=-\frac{1}{6}$.

2) Method of elimination

In this method we multiply one or both of the equations by appropriate numbers (*i.e.* multiply every term in the equation by the number) so that one of the variables will have the same coefficient with opposite signs.

Then next step is to add the two equations together. Because one of the variables had the same coefficient with opposite signs it will be eliminated when we add the two equations.

The result will be a single equation that we can solve for one of the variables. Once this is done substitute this answer back into one of the original equations.

Example 2: Solve the following systems of equations.

$$\begin{aligned} 2x+4y&=-10 \\ 6x+3y&=6 \end{aligned}$$

In this part all the variables are positive so we're going to have to force an opposite sign by multiplying by a negative number somewhere. Let's also notice that in this case if we just multiply the first equation by -3 then the coefficients of the *x* will be -6 and 6.

Sometimes we only need to multiply one of the equations and can leave the other one alone. Here is this work for this part.

$$\begin{array}{lcr} 2x+4y=-10 & \xrightarrow{\times -3} & -6x-12y=30 \\ 6x+3y=6 & \xrightarrow{\text{same}} & \underline{6x+3y=6} \\ & & -9y=36 \\ & & y=-4 \end{array}$$

Finally, plug this into either of the equations and solve for *x*. We will use the first equation this time.

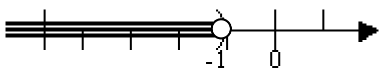
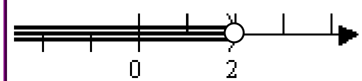
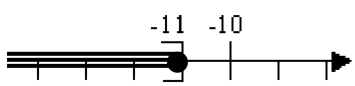
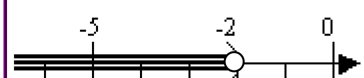
$$\begin{aligned} 2x+4(-4)&=-10 \\ 2x-16&=-10 \\ 2x&=6 \\ x&=3 \end{aligned}$$

So, the solution to this system is $x=3$ and $y=-4$.

1. **Solving Inequalities**

1) Linear Inequalities

Solving linear inequalities is very similar to solving linear equations, except for one small but important detail: **you FLIP the inequality sign whenever you multiply or divide the inequality by a negative.** The easiest way to show this is with some examples:

1) $x + 3 < 2$ $\quad\;\; -3 \quad -3$ $x \quad\;\; < -1$ Graphically, the solution is: 	2) $2 - x > 0$ $\quad + x \quad +x$ $2 \quad > x$ $or \quad x < 2$ Graphically, the solution is: 	Note that the solution to a "less than, but not equal to" inequality is graphed with an **open dot** at the endpoint, indicating that the endpoint is **not included** within the solution.
3) $4x + 6 \le 3x - 5$ $-3x \qquad\quad -3x$ $x + 6 \le -5$ $\quad -6 \quad -6$ $x \quad \le -11$ Graphically, the solution is: 	4) $-2x > 4$ $-2x/-2 < 4/-2$ $x < -2$ Graphically, the solution is: 	(3) Note that the solution to a "less than or equal to" inequality is graphed with a **closed dot** at the endpoint, indicating that the endpoint is **included** within the solution. (4) is the special case noted above. When I divided by the **negative** two, I had to flip the inequality sign.

The rule for example 5 above often seems unreasonable to students the first time they see it. But think about inequalities with numbers in there, instead of variables. You know that the number four is larger than the number two: $4 > 2$. Multiplying through this inequality by -1, we get $-4 < -2$

2) Quadratic Inequalities

When we have an inequality with "x^2" as the highest-degree term, it is called a "quadratic inequality". The method of solution is more complicated.

Example: Solve $x^2 - 3x + 2 > 0$

First, I have to find the x-intercepts of the associated quadratic, because the intercepts are where $y = x^2 - 3x + 2$ is *equal* to zero. Graphically, an inequality like this is asking me to find where the graph is above or below the x-axis. It is simplest to find where it actually *crosses* the x-axis, so I'll start there.

Factoring, I get $x^2 - 3x + 2 = (x - 2)(x - 1) = 0$, so $x = 1$ or $x = 2$. Then the graph crosses the x-axis at 1 and 2, and the number line is divided into the intervals $(-\infty, 1)$, $(1, 2)$, and $(2, \infty)$. Between the x-intercepts, the graph is either above the axis (and thus positive, or greater than zero), or else below the axis (and thus negative, or less than zero).

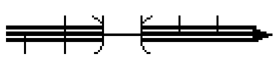
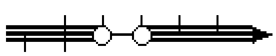
There are two different algebraic ways of checking for this positivity or negativity on the intervals. I'll show both.

1) Test-point method. The intervals between the x-intercepts are $(-\infty, 1)$, $(1, 2)$, and $(2, \infty)$. I will pick a point (any point) inside each interval. I will calculate the value of y at that point. Whatever the sign on that value is, that is the sign for that entire interval.

For $(-\infty, 1)$, let's say I choose $x = 0$; then $y = 0 - 0 + 2 = 2$, which is positive. This says that y is positive on the whole interval of $(-\infty, 1)$, and this interval is thus part of the solution (since I'm looking for a "greater than zero" solution).

For the interval $(1, 2)$, I'll pick, say, $x = 1.5$; then $y = (1.5)^2 - 3(1.5) + 2 = 2.25 - 4.5 + 2 = 4.25 - 4.5 = -0.25$, which is negative. Then y is negative on this entire interval, and this interval is then not part of the solution.

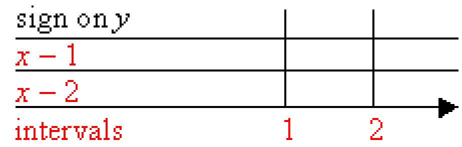
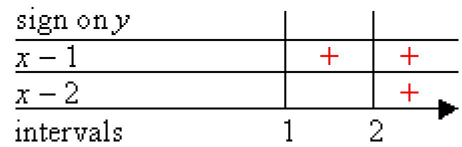
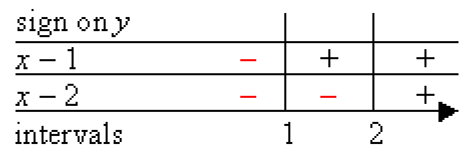
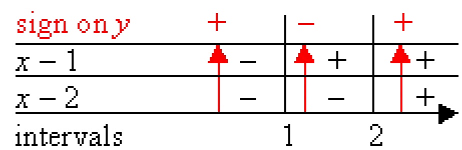
For the interval $(2, \infty)$, I'll pick, say, $x = 3$; then $y = (3)^2 - 3(3) + 2 = 9 - 9 + 2 = 2$, which is positive, and this interval is then part of the solution. Then the complete solution for the inequality is $x < 1$ and $x > 2$. This solution is stated variously as:

$x < 1$ or $x > 2$	**inequality notation**
$x \in (-\infty,1) \cup (2,+\infty)$	**interval, or set, notation**
	**number line with parentheses (brackets are used for closed intervals)**
	**number line with open dots (closed dots are used for closed intervals)**

The particular solution format you use will depend on your text, your teacher, and your taste. Each format is equally valid.

2) Factor method. Factoring, I get $y = x^2 - 3x + 2 = (x - 2)(x - 1)$. Now I will consider each of these factors separately.

The factor $x - 1$ is positive for $x > 1$; similarly, $x - 2$ is positive for $x > 2$. Thinking back to when I first learned about negative numbers, I know that (plus)×(plus) = (plus), (minus)×(minus) = (plus), and (minus)×(plus) = (minus). So, to compute the sign on $y = x^2 - 3x + 2$, I only really need to know the signs on the factors. Then I can apply what I know about multiplying negatives.

First, I set up a grid, showing the factors and the number line.	
Now I mark the intervals where each factor is positive.	
Where the factors aren't positive, they must be negative.	
Now I multiply up the columns, to compute the sign of y on each interval.	

Then the solution of $x^2 - 3x + 2 > 0$ is $(-\infty, 1)$ and $(2, \infty)$.

Example: Solve $-2x^2 + 5x + 12 \leq 0$.

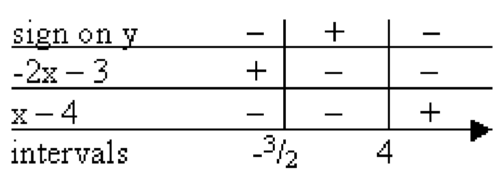
First I find the zeroes, which are the endpoints of the intervals:

$y = -2x^2 + 5x + 12 = (-2x - 3)(x - 4) = 0$ for $x = -{}^3/_2$ and $x = 4$

The intervals are between the endpoints, so the intervals are $(-\infty, {}^{-3}/_2]$, $[{}^{-3}/_2, 4]$, and $[4, \infty)$. (Note that I use *brackets* for the endpoints in "or equal to" inequalities, instead of parentheses, because the endpoints will be included in the final solution.)

To find the intervals where y is negative by the Test-Point Method, I just pick a point in each interval. I can use points such as $x = -2$, $x = 0$, and $x = 5$.

To find the intervals where y is negative by the Factor Method, I just solve each factor: $-2x - 3$ is positive for $-2x - 3 > 0$, $-3 > 2x$, $-3/2 > x$, or $x < -{}^3/_2$; and $x - 4$ is positive for $x - 4 > 0$, $x > 4$. Then I fill out the grid:



Then the solution to this inequality is all x's in

$(-\infty, {}^{-3}/_2]$ and $[4, \infty)$.

In-class questions

1. In base 10, the number 2013 ends in the digit 3. In base 9, on the other hand, the same number is written as $(2676)_9$ and ends in the digit 6. For how many positive integers b does the base-b representation of 2013 end in the digit 3?

2. A lattice point in an xy-coordinate system is any point (x, y) where both x and y are integers. The graph of $y = mx + 2$ passes through no lattice point with $0 < x \leq 100$ for all m such that $1/2 < m < a$. What is the maximum possible value of a?

3. What is the sum of all the solutions of $x = |2x - |60 - 2x||$?

4. The sum of the first m positive odd integers is 212 more than the sum of the first n positive even integers. What is the sum of all possible values of n?

5. For how many integers x is the number $x^4 - 51x^2 + 50$ negative?

6. The real numbers c, b, a form an arithmetic sequence (that is $a - b = b - c$) with $a \geq b \geq c \geq 0$. The quadratic $ax^2 + bx + c$ has exactly one root. What is this root?

7. Let a, b, c, and d be real numbers with $|a - b| = 2$, $|b - c| = 3$, and $|c - d| = 4$. What is the sum of all possible values of $|a - d|$?

8. Let a, b, and c be positive integers with $a \geq b \geq c$ such that $a^2 - b^2 - c^2 + ab = 2011$ and $a^2 + 3b^2 + 3c^2 - 3ab - 2ac - 2bc = -1997$. What is a?