AP Calculus Lesson Eighteen Notes

Chapter Eight - Indeterminate Forms, Improper Integrals, and Taylor's Formula

8.3 Integrals with Discontinuous Integrands
8.4 Taylor's Formula

8.3 Integrals with Discontinuous Integrands

If a function *f* is continuous on a closed interval [*a*,*b*], then by Theorem (4.11) the definite integral $\int_a^b f(x)dx$ exists. If *f* has an infinite discontinuity at some number in the interval it may still be possible to assign a value to the integral. Suppose, for example, that *f* is continuous and nonnegative on the half-open interval [*a*,*b*) and $\lim_{x\to b^-} f(x)=\infty$. If $a<t<b$, then the area *A(t)* under the graph of *f* from *a* to *t* (see(i) of Figure 10.6) is given by

$$A(t) = \int_a^t f(x)\,dx.$$

If $\lim_{t\to b^-} A(t)$ exists, then the limit may be interpreted as the area of the unbounded region that lies under the graph of *f* , over the *x*-axis, and between *x*=*a* and *x*=*b*. it is natural to denote this number by $\int_a^b f(x)dx$

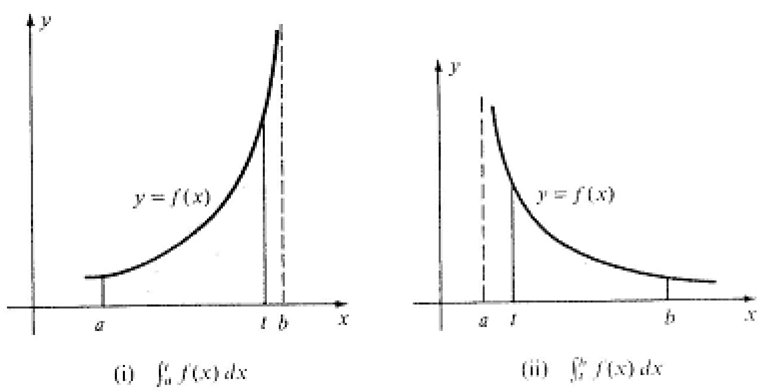

FIGURE 10.6

In like manner, for the situation illustrated in (ii) of the figure, where $\lim_{x\to a^+} f(x)=\infty$, we define $\int_a^b f(x)dx$ as the limit of $\int_t^b f(x)dx$ as $t\to a^+$

These remarks serve as motivation for the following definition.

Definition (8.3.1)

(i) If f is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx.$$

(ii) If f is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx.$$

As in the preceding section, the integrals defined in (8.3.1) are referred to as **improper integrals** and they are said to **converge** if the indicated limits exist. The limits are called the **values** of the improper integrals. If the limits do not exist, the improper integrals are said to **diverge.**

Another type of improper integral is defined as follows

Definition (8.3.2)

If f has a discontinuity at a number c in the open interval (a, b) but is continuous elsewhere in $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

provided *both* of the integrals on the right converge. If both converge, then the value of the integral $\int_a^b f(x)\,dx$ is the sum of the two values.

The graph of a function satisfying the conditions of Definition (8.3.2) is sketched in Figure 10.7.

A definition similar to (8.3.2) is used if *f* has any finite number of discontinuities in (*a*,*b*). for example, if *f* has discontinuities at c_1 and c_2, where $c_1 < c_2$, but is continuous elsewhere in [*a*,*b*], then we choose a number *d* between c_1 and c_2 and express $\int_a^b f(x)dx$ as a sum of four improper integrals over the intervals $[a,c_1]$,$[c_1,d]$,$[d,c_2]$and $[c_2,b]$, respectively/ the given integral converges. Finally, if *f* is continuous on (*a*,*b*) but becomes infinite at *both a* and *b*, then we again define $\int_a^b f(x)dx$ by means of (8.3.2)

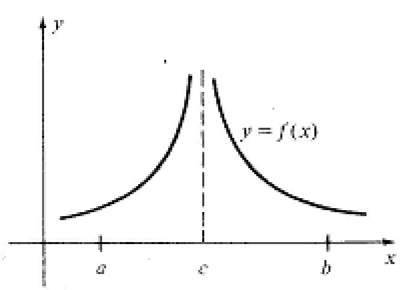

FIGURE 10.7

Example 1

Evaluate

$$\int_0^3 \frac{1}{\sqrt{3-x}}\,dx.$$

Solution

Since the integrand has an infinite discontinuity at *x=3*, we apply (i) of Definition (8.3.1) as follows:

$$\begin{aligned}\int_0^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to 3^-}\int_0^t \frac{1}{\sqrt{3-x}}\,dx\\ &= \lim_{t\to 3^-}\left[-2\sqrt{3-x}\right]_0^t\\ &= \lim_{t\to 3^-}\left[-2\sqrt{3-t}+2\sqrt{3}\right]\\ &= 0+2\sqrt{3}=2\sqrt{3}.\end{aligned}$$

Example 2

Determine whether $\int_0^1 \frac{1}{x}\,dx$ converges or diverges.

Solution

The integrand is undefined at x=0. applying (ii) of (8.3.1),

$$\begin{aligned}\int_0^1 \frac{1}{x}\,dx &= \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\left[\ln x\right]_t^1\\ &= \lim_{t\to 0^+}\left[0-\ln t\right]=\infty.\end{aligned}$$

Since the limit does not exist, the improper integral diverges.

Example 3

Determine whether $\int_0^4 \frac{1}{(x-3)^2}\,dx$ converges or diverges.

Solution

The integrand is undefined at x=3. since this number is in the interior of the interval [0,4], we use Definition (8.3.2) with *c=3*, obtaining

$$\int_0^4 \frac{1}{(x-3)^2}\,dx = \int_0^3 \frac{1}{(x-3)^2}\,dx + \int_3^4 \frac{1}{(x-3)^2}\,dx.$$

For the integral on the left to converge, *both* integrals on the right must converge. Applying (i) of Definition (8.3.1) to the first integral,

$$\begin{aligned}\int_0^3 \frac{1}{(x-3)^2}\,dx &= \lim_{t\to 3^-}\int_0^t \frac{1}{(x-3)^2}\,dx \\ &= \lim_{t\to 3^-} \frac{-1}{x-3}\Big]_0^t \\ &= \lim_{t\to 3^-}\left(\frac{-1}{t-3}-\frac{1}{3}\right) = \infty.\end{aligned}$$

It follows that the given improper integral diverges.

It is important to note that the Fundamental Theorem of Calculus cannot be applied to the integral in Example 3 since the function given by the integrand is not continuous on [0,4]. Indeed, if we had applied the Fundamental Theorem, we would have obtained

$$\begin{aligned}\int_0^4 \frac{1}{(x-3)^2}\,dx &= \frac{-1}{(x-3)}\Big]_0^4 \\ &= -1-\frac{1}{3} = -\frac{4}{3}.\end{aligned}$$

This result is obviously incorrect since the integrand is never negative.

Example 4

Evaluate $\int_{-2}^{7} \frac{1}{(x+1)^{2/3}}\,dx$

Solution

The integrand is undefined at $x=-1$, a number between -2 and 7. consequently, we apply Definition (8.3.2) with $c=-1$, as follows:

$$\int_{-2}^{7} \frac{1}{(x+1)^{2/3}}\,dx = \int_{-2}^{-1} \frac{1}{(x+1)^{2/3}}\,dx + \int_{-1}^{7} \frac{1}{(x+1)^{2/3}}\,dx.$$

We next investigate each integral on the right. Using (i) of (8.3.1) with $b=-1$ gives us

$$\begin{aligned}\int_{-2}^{-1} \frac{1}{(x+1)^{2/3}}\,dx &= \lim_{t\to -1^-}\int_{-2}^{t} \frac{1}{(x+1)^{2/3}}\,dx \\ &= \lim_{t\to -1^-}\Big[3(x+1)^{1/3}\Big]_{-2}^{t} \\ &= \lim_{t\to -1^-}\left[3(t+1)^{1/3} - 3(-1)^{1/3}\right] \\ &= 0+3 = 3.\end{aligned}$$

In similar fashion, using (ii) of (8.3.1) with a=-1,

$$\begin{aligned}\int_{-1}^{7} \frac{1}{(x+1)^{2/3}}\,dx &= \lim_{t\to -1^+} \int_{t}^{7} \frac{1}{(x+1)^{2/3}}\,dx \\ &= \lim_{t\to -1^+} \left[3(x+1)^{1/3}\right]_t^7 \\ &= \lim_{t\to -1^+} \left[3(8)^{1/3} - 3(t+1)^{1/3}\right] \\ &= 6 - 0 = 6.\end{aligned}$$

Since both integrals converge, the given integral converges and has the value 3+6=9.

An improper integral may have both a discontinuity in the integrand and an infinite limit of integration. Integrals of this type may be investigated by expressing them as sums of improper integrals, each of which has one of the forms previously defined. As an illustration, since the integrand of $\int_0^\infty (1/\sqrt{x})dx$ is discontinuous at x = 0, we choose any number (for example, 1) greater than 0 and write

$$\int_0^\infty (1/\sqrt{x})dx = \int_0^1 (1/\sqrt{x})dx + \int_1^\infty (1/\sqrt{x})dx.$$

It is easy to show that the first integral on the right side of the equation converges and the second diverges. Hence, the given integral diverges.

8.4 Taylor's Formula

Recall that *f* is a polynomial function of degree *n* if

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Where each a_i is a real number, $a_n \neq 0$, and the exponents are nonnegative integers. Polynomial functions are the simplest functions to use for calculations, in the sense that their values can be found by employing only additions and multiplications of real number. More complicated operations are needed to calculate values of logarithmic, exponential, or trigonometric functions; however, sometimes it is possible to *approximate* values by using polynomials. For example, since $\lim_{x\to 0}(\sin x)/x = 1$, it follows that if *x* is close to 0, then $\sin x \approx x$; that is, the value of the sign function is almost the same as the value of the polynomial x. we say that sin x *may be approximated by the polynomial x* (provided x is close to 0).

As a second illustration, let f be the natural exponential function, that is, $f(x) = e^x$ for every x. suppose we are interested in calculating values of f when x is close to 0, since $f'(x) = e^x$, the slope of the tangent line at the point (0,1) on the graph of f is $f'(x) = e^0 = 1$. Hence an equation of the tangent line is

$$y - 1 = 1(x - 0) \quad \text{or} \quad y = 1 + x.$$

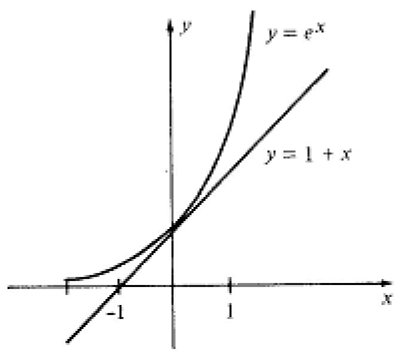

FIGURE 10.9

Referring to Figure 10.9, it is evident that if x is very close to 0, then the point $(x, 1+x)$ on the tangent line is close to the point (x, e^x) on the graph of f and hence we may write $e^x \approx 1 + x$. This formula allows us to approximate e^x by means of a polynomial of degree 1. since the approximation is obviously poor unless x is very close to 0, let us seek a second-degree polynomial

$$g(x) = a + bx + cx^2$$

Such that $e^x \approx g(x)$ when x is numerically small. The first and second derivatives of $g(x)$ are

$$g'(x) = b + 2cx$$
$$g''(x) = 2c.$$

If we want the graph of g (a parabola) to have (i) the same y-intercept, (ii) the same tangent line, and (iii) the same concavity, as the graph of f at the point (0,1), then we must have

$$\text{(i)}\ g(0) = f(0), \quad \text{(ii)}\ g'(0) = f'(0), \quad \text{(iii)}\ g''(0) = f''(0).$$

Since all derivatives of e^x equal e^x, and $e^0 = 1$, these three equations imply that

$$a = 1, \quad b = 1, \quad \text{and} \quad 2c = 1$$

And hence

$$e^x \approx g(x) = 1 + x + \tfrac{1}{2}x^2.$$

The graphs of *f* and *g* are sketched in Figure 10.10. comparing with Figure 10.9 it appears that if *x* is close to 0, then $1+x+\frac{1}{2}x^2$ is closer to e^x than 1+*x*.

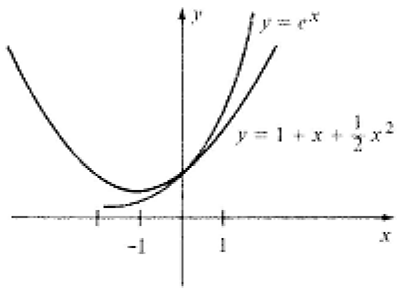

FIGURE 10.10

If we wish to approximate $f(x)=e^x$ by means of a *third*-degree polynomial *h(x)*, it is natural to require that $h(0), h'(0), h''(0),$ and $h'''(0)$ be the same as $f(0), f'(0), f''(0), f'''(0)$, respectively. The graphs of *f* and *h* then have the same tangent line and concavity at (0,1) and, in addition, their *rates of change of concavity* with respect to *x* (that is, the third derivatives) are equal. Using the same technique employed previously would give us

$$e^x \approx h(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3.$$

To get some idea of the accuracy of this approximation, several values of e^x and *h(x)*, approximated to the nearest hundredth, are displayed in the following table.

x	-1.5	-1.0	-0.5	0	0.5	1.0	1.5
e^x	0.22	0.37	0.61	1	1.65	2.72	4.48
$h(x)$	0.06	0.33	0.60	1	1.65	2.67	4.19

Observe that the error in the approximation increases as *x* increases numerically. The graphs of *f* and *h* are sketched in Figure 10.11. notice the improvement in the approximation near *x=0*. we would arrive at

$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$

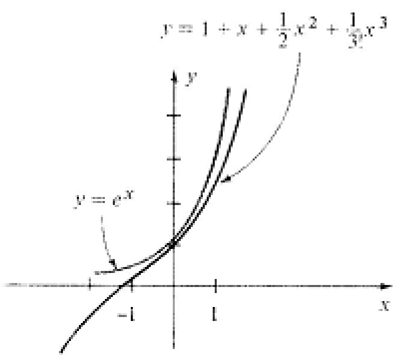

FIGURE 10.11

It will follow from our later work that this remarkably simple formula can be used to approximate e^x to any desired degree of accuracy.

We now ask the following two questions:

1. Does there exist a *general* formula that may be used to obtain polynomial approximations for arbitrary exponential functions, logarithmic functions, trigonometric functions, inverse trigonometric functions, and other transcendental or algebraic functions?
2. can the previous discussion be generalized to the case where *x* is close to an arbitrary number $a \neq 0$?

The answer to both questions is "yes" provided the function under consideration has a sufficient number of derivatives. Specifically, Formula (8.4.2) of this section may be used to obtain polynomial approximations of a wide variety of functions. To see why this is true, suppose *f* is a function that has many derivatives, and consider a number *a* in the domain of *f*. let us proceed as we did for the special case of e^x discussed previously, but with *a* in place of 0. first we note that the only polynomial of degree 0 which coincides with *f* at *a* is constant *f(a)*. to approximate *f(x)* near *a* by a polynomial of degree 1, we choose the polynomial whose graph is the tangent line to the graph of *f* at (*a*,*f(a)*). Since the equation of this tangent line is

$$y - f(a) = f'(a)(x - a), \quad \text{or} \quad y = f(a) + f'(a)(x - a)$$

The desired first-degree polynomial is

$$f(a) + f'(a)(x - a).$$

A better approximation should be obtained by using a polynomial whose graph has the same *concavity* and tangent line as that of *f* at (*a*,*f(a)*). It is left to the reader to verify that these conditions are fulfilled by the second-degree polynomial

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.$$

If we next consider the third-degree polynomial

$$g(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3$$

Then it is easy to show that in addition to $g'(a) = f'(a)$ and $g''(a) = f''(a)$, we also have $g'''(a) = f'''(a)$. It appears that if this pattern is continued, we should get better approximations to *f(x)* when *x* is near *a*. this leads to the first *n+1* terms on the right side of Formula(8.4.2). the form of the last term is a consequence of the next result, which bears the name of the English mathematician Brook Taylor (1685-1731).

Taylor's Formula (8.4.1)

Let *f* be a function and *n* a positive integer such that the derivative $f^{(n+1)}(x)$ exists for every *x* in an interval *I*. if *a* and *b* are distinct numbers in *I*, then there is a number *z* between *a* and *b* such that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b - a)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(b - a)^{n+1}.$$

Proof

There exists a number R_n (depending on *a*,*b*, and *n*) such that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b - a)^n + R_n.$$

Indeed, to find R_n we merely subtract the sum of the first *n*+1 terms on the right side of this equation from *f(b)*. We wish to show that R_n is the same as the last term of the formula given in the statement of the theorem.

Let *g* be the function defined by

$$g(x) = f(b) - f(x) - \frac{f'(x)}{1!}(b - x) - \frac{f''(x)}{2!}(b - x)^2 - \cdots - \frac{f^{(n)}(x)}{n!}(b - x)^n - R_n\frac{(b - x)^{n+1}}{(b - a)^{n+1}}$$

For all x in I. if we differentiate each side of this equation, then many terms on the right cancel. As a matter of fact, it can be shown (see Exercise 41) that

$$g'(x) = -\frac{f^{(n+1)}(x)}{n!}(b-x)^n + R_n(n+1)\frac{(b-x)^n}{(b-a)^{n+1}}.$$

It is easy to see that $g(b)=0$. Moreover, substituting a for x in the expression for $g(x)$ and making use of the first equation of the proof gives us $g(a)=0$. According to Rolle's Theorem, there is a number z between a and b such that $g'(z)=0$. Evaluating $g'(x)$ at z and solving for R_n we see that

$$R_n = \frac{f^{(n+1)}(z)}{(n+1)!}(b-a)^{n+1}$$

Which is what we wished to prove.

Replacement of b by x in Formula (8.4.1) leads to the following.

Talor's Formula with the Remainder (8.4.2)

If f have n+1 derivatives throughout an interval containing a. if x is any number in the interval, then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

Where z is a number between a and x.

For convenience we shall denote the sum of the first $n+1$ terms in Taylor's Formula by $P_n(x)$ and give it a special name, as in (i) of the next definition.

Definition (8.4.3)

(i) the ***n*th-degree Taylor Polynomial** $P_n(x)$ of f at a is

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

(ii) the **Taylor Remainder** $R_n(x)$ of f at a is

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

Where z is between a and x

We may now state the following result

Theorem (8.4.4)

Let f have $n+1$ derivatives throughout an interval containing a. if x is any number in the interval, then

(i) $f(x) = P_n(x) + R_n(x)$

(ii) $f(x) \approx P_n(x)$ if $x \approx a$, where the error involved in using this approximation is less than $|R_n(x)|$

Proof

Part (i) is merely a restatement of Formula (8.4.2) using the notation of (8.4.3). part (ii) follows from the fact that $|f(x) - P_n(x)| = |R_n(x)|$.

Example 1

If $f(x) = \ln x$, find Taylor's Formula with the Remainder for n=3 and a=1.

Solution

If $n=3$ in Taylor's Formula (8.4.2), then we need the first four derivatives of f. it is convenient to arrange our work as follows.

$$\begin{array}{ll} f(x) = \ln x & f(1) = 0 \\ f'(x) = x^{-1} & f'(1) = 1 \\ f''(x) = -x^{-2} & f''(1) = -1 \\ f'''(x) = 2x^{-3} & f'''(1) = 2 \\ f^{(4)}(x) = -3!x^{-4} & f^{(4)}(z) = -6z^{-4} \end{array}$$

By (8.4.2)

$$\ln x = 0 + \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6z^{-4}}{4!}(x-1)^4$$

Where z is between 1 and x. this simplifies to

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4z^4}(x-1)^4.$$

In the next two examples Taylor Polynomials are used to approximate values of functions. To discuss the accuracy of an approximation, it is necessary to agree on what is meant by 1-decimal-place accuracy, 2-decimal-place accuracy, etc. let us adopt the following convention. If E is the error in an approximation, then the approximation will be considered accurate to k decimal places if $|E| < 0.5 \times 10^{-k}$. For example, we have

1-decimal-place accuracy if $|E| < 0.5 \times 10^{-1} = 0.05$
2-decimal-place accuracy if $|E| < 0.5 \times 10^{-2} = 0.005$
3-decimal-place accuracy if $|E| < 0.5 \times 10^{-3} = 0.0005.$

If we are interested in *k*-decimal-place accuracy in the approximation of a sum, we shall approximate each term of the sum to *k*+1 decimal places and then round off the final result to *k* decimal places. In certain cases this may fail to produce the required degree of accuracy; however, it is customary to proceed in this way for elementary approximations. More precise techniques may be found in t4exts on *numerical analysis.*

Example 2

Use the formula obtained in Example 1 to approximate ln (1.1), and estimate the accuracy of this approximation.

Solution

Substituting 1.1 for *x* in the formula of Example 1 gives us

$$\ln(1.1) = 0.1 - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4z^4}(0.1)^4$$

Where 1<*z*<1.1. summing the first three terms we obtain ln(1.1) ≈ 0.0953. Since *z*>1, 1/*z*<1 and, therefore, $1/z^4 < 1$. Consequently,

$$|R_3(1.1)| = \left|\frac{(0.1)^4}{4z^4}\right| < \left|\frac{0.0001}{4}\right| = 0.000025.$$

Since 0.000025<0.00005= 0.5×10^{-4}, it follows from (ii) of Theorem (8.4.4) and our convention concerning accuracy, that the approximation $\ln(1.1) \approx 0.0953$ is accurate to four decimal places.

If we wish to approximate a functional value *f(x)* for some *x*, it is desirable to choose the number *a* in (8.4.2) such that the remainder $R_n(x)$ is very close to 0 when *n* is relatively small (say *n*=3 or *n*=4). This will be true if we choose *a* close to *x*. in addition, *a* should be chosen in such a way that the values of the first *n+1* derivatives of *f* at *a* are easy to calculate. This was done in Example 2, where to approximate ln *x* for x=1.1. we select *a*=1 (see Example 1). The next example provides another illustration of a suitable choice for *a*.

If we let *a*=0 in Formula (8.4.2), we obtain the following important formula, named after the Scottish mathematician Colin Maclaurin (1698-1746)

Maclaurin's Formula (8.4.5)

Let *f* have *n+1* derivatives throughout an interval containing 0. if *x* is any number in the interval, then

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}.$$

Where *z* is between 0 and *x*

Example 4

Find Maclaurin's Formula for $f(x) = e^x$ if *n* is any positive integer.

Solution

For every positive integer *k* we have $f^{(k)}(x) = e^x$, and hence $f^{(k)}(0) = e^0 = 1$.

Substituting in (8.4.5) gives us

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^z}{(n+1)!}x^{n+1}$$

Where *z* is between 0 and *x*. note that the first *n*+1terms are the same as those obtained in our discussion at the beginning of this section.

The formula derived in Example 4 may be used to approximate values of the natural exponential function. Another important application will be discussed in the next chapter in conjunction with representations of functions by means of infinite series.

Example 5

Find Maclaurin's Formula for *f(x)*=sin*x* and *n*=8.

Solution

We need the first nine derivatives of *f(x)*. let us begin as follows

$$\begin{array}{ll} f(x) = \sin x & f(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f'''(x) = -\cos x & f'''(0) = -1 \end{array}$$

Since $f^{(4)}(x) = \sin x$, the remaining derivatives follow the same pattern, and we arrive at

$$f^{(9)}(x) = \cos x \qquad f^{(9)}(z) = \cos z.$$