Algebra 1

1. Quadratic Equation

A quadratic equation is a trinomial of the form $ax^2 + bx + c = 0$. There are three main ways of solving quadratic equations that are covered below.

1) Factoring

If a quadratic equation can be factored, then it can be written as a product of two binomials. Since the trinomial is equal to 0, one of the two binomial factors must also be equal to zero. By factoring the quadratic equation, we can equate each binomial expression to zero and solve each for x.

Examples

I) Solve $x^2 + 2x = 15$ by factoring.

$$x^2 + 2x = 15 : x^2 + 2x = 15$$
$$: x^2 + 2x - 15 = 0$$
$$: (x+5)(x-3) = 0$$

$x + 5 = 0$ or $x - 3 = 0$

$x = -5$ $\quad$ $x = 3$

$\therefore$ the solution to the quadratic equation is $x = -5$ or $x = 3$.

II) Solve $2x^2 - 14x - 13$ using the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-14) \pm \sqrt{(-14)^2 - 4(2)(-13)}}{2(2)} \\ &= \frac{14 \pm \sqrt{196 + 104}}{4} \\ &= \frac{14 \pm \sqrt{300}}{4} \\ &= \frac{14 \pm 10\sqrt{3}}{4} \end{aligned}$$

$$x = \frac{14 \pm 10\sqrt{3}}{4} \quad \text{or} \quad x = \frac{14 - 10\sqrt{3}}{4}$$
$$= \frac{7 + 5\sqrt{3}}{2} \qquad = \frac{7 - 5\sqrt{3}}{2}$$

2) Completing the Square

Quadratic equations cannot always be solved by factoring. They can always be solved by the method of **completing the squares**. To complete the square means to convert a quadratic to its standard form.

We want to convert $ax^2 + bx + c = 0$ to a statement of the form $a(x - h)^2 + k = 0$. To do this, we would perform the following steps:

1) Group together the ax^2 and bx terms in parentheses and factor out the coefficient a.

$$a(x^2 + \frac{b}{a}x) + c = 0$$

2) In the parentheses, add and subtract $(b/2a)^2$, which is half of the x coefficient, squared.

$$a(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2)+c = 0$$

3) Remove the term $-(b/2a)^2$ from the parentheses. Don't forget to multiply the term by a, when removing from parentheses.

$$a(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2)-a\left(\frac{b}{2a}\right)^2+c = 0$$

4) Factor the trinomial in parentheses to its perfect square form, $(x + b/2a)^2$.

$$a\left(x+\frac{b}{2a}\right)^2-a\left(\frac{b}{2a}\right)^2+c = 0$$

Note: If you group the $-(b/2a)^2 + c$ terms together in parentheses, the equation will now be in standard form. The equation is now much simpler to graph as you will see in the Graphing section below. To solve the quadratic equation, continue the following steps.

5) Transpose (or shift) all other terms to the other side of the equation and divide each side by the constant a.

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}-\frac{c}{a}$$

6) Take the square root of each side of the equation.

$$\left(x+\frac{b}{2a}\right) = \pm\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}$$

7) Transpose the term -b/2a to the other side of the equation, isolating x.

$$x = -\frac{b}{2a}\pm\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}$$

The quadratic equation is now solved for x. The method of completing the square seems complicated since we are using variables a, b and c. The examples below show use numerical coefficients and show how easy it can be.

Example

Solve $x^2 + 4x - 13 = 0$ by completing the square.

$$\begin{aligned} x^2 + 4x - 13 = 0 \ &: \ (x^2 + 4x) - 13 = 0 \\ &: \ (x^2 + 4x + 4 - 4) - 13 = 0 \\ &: \ (x^2 + 4x + 4) - 4 - 13 = 0 \\ &: \ (x^2 + 4x + 4) - 17 = 0 \\ &: \ (x + 2)^2 - 17 = 0 \\ &: \ (x + 2)^2 = 17 \\ &: \ (x + 2) = \pm\sqrt{17} \\ &: \ x = -2 \pm \sqrt{17} \end{aligned}$$

$x = -2 + \sqrt{17}$ or $x = -2 - \sqrt{17}$

Note: For more examples of solving a quadratic equation by completing the square, see questions #1 and #2 in the Additional Examples section at the bottom of the page.

3) The Quadratic Formula

The method of completing the square can often involve some very complicated calculations involving fractions. To make calculations simpler, a general formula for solving quadratic equations, known as the **quadratic formula**, was derived. To solve quadratic equations of the form $ax^2 + bx + c = 0$, substitute the coefficients a,b and c into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The value contained in the square root of the quadratic formula is called the **discriminant**. If,

$b^2 - 4ac > 0$ – There are 2 real roots, $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

$b^2 - 4ac = 0$ – There is 1 real root, $x = \dfrac{-b}{2a}$

$b^2 - 4ac < 0$ – There are no real roots.

Examples

I) Solve $4x^2 - 5x + 1 = 0$ using the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(4)(1)}}{2(4)} \\ &= \frac{5 \pm \sqrt{25 - 16}}{8} \\ &= \frac{5 \pm \sqrt{9}}{8} \\ &= \frac{5 \pm 3}{8} \end{aligned}$$

$$x = \frac{5+3}{8} = 1 \quad \text{or} \quad x = \frac{5-3}{8} = \frac{1}{4}$$

II) Solve $2x^2 - 14x - 13 = 0$ using the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-14) \pm \sqrt{(-14)^2 - 4(2)(-13)}}{2(2)} \\ &= \frac{14 \pm \sqrt{196 + 104}}{4} \\ &= \frac{14 \pm \sqrt{300}}{4} \\ &= \frac{14 \pm 10\sqrt{3}}{4} \end{aligned}$$

$$x = \frac{14 \pm 10\sqrt{3}}{4} = \frac{7 + 5\sqrt{3}}{2} \quad \text{or} \quad x = \frac{14 - 10\sqrt{3}}{4} = \frac{7 - 5\sqrt{3}}{2}$$

2. Introduction to Systems of Linear Equations

A **system** has these properties:

- It consists of several parts which interact and affect one another.
- It produces an effect or **output** as a result of some cause or **input**.

One example is a business organization. This system's inputs are its capital, employees, raw materials and factories. Its outputs are its products. Management decides what the interactions among the inputs should be to give the maximum output (for example, how many factories should make which products, etc.)

A **linear system** is a system where the output is proportional to the input. For example, if the business organization is a linear system, then if we double the capital, employees, raw materials and factories (the inputs) then we expect to get double the production (the output).

We can describe mathematically how the parts of a linear system relate to one another and to the input using a **system of linear equations**. If a linear system has *n* parts (where *n* is some number), then we can describe it with a system of *n* linear equations in *n* unknowns or variables. The unknowns in these equations are the values of the inputs. If we did the analysis correctly then there will be a unique solution for the values of the inputs.

Here is an example of a system of two linear equations in the two unknowns *x* and *y*:

$$\begin{cases} 1x + 1y = 4 \\ 2x - 3y = 6 \end{cases}$$

From the input-output point of view, the numbers 4 and 6 on the right-hand-sides are the output, and the unknowns x and y on the left-hand-sides are the input. The numbers 1, 1, 2 and −3 multiplying x and y express the relationships between the parts of the system.

We can verify that $\{x = 3.6, y = 0.4\}$ is the solution of the system by substituting it into the system and getting the pair of equations 4 = 4 and 6 = 6. Now, suppose that we double the output (replace 4 and 6 by 8 and 12):

$$\begin{cases} 1x + 1y = 8 \\ 2x - 3y = 12 \end{cases}$$

Then we can verify that the input is also doubled; the solution now is $\{x = 7.2, y = 0.8\}$. Thus the system is linear. The linearity can be traced back to the fact that the numbers 1, 1, 2 and −3 multiplying x and y are *constants*. If they were replaced by *expressions* involving x and y then the equations would be **non-linear**.

Here is an example of a system of three linear equations in the three unknowns x, y and z:

$$\begin{cases} 4x + 8y + 4z = 80 \\ 2x + 1y - 4z = 7 \\ 3x - 1y + 2z = 22 \end{cases}$$

3. Methods of solving systems of linear equations

1) The substitution method

In the substitution method, we pick one equation and solve it for one of the unknowns, say x. Then we substitute this solution for x into the other $n-1$ equations wherever the variable x appears, and simplify. The result is that the $n-1$ equations contain only $n-1$ unknowns (x no longer appears).

We repeat this process until we get 1 equation in 1 unknown, which is then easily solved. The final step is to back-substitute the solution for x into the previous equations to find the values of all the other unknowns.

Example: Solve the system:

$$\begin{cases} 1x + 1y + 2z = 4 \\ 2x - 3y - 2z = 6 \\ 1x - 2y - 3z = 4 \end{cases}$$

Solution: We arbitrarily choose to solve the first equation for x:
$x = 4 - 1y - 2z$.

Substituting this into the second and third equations we get:

$$\begin{cases} 2(4-1y-2z)-3y-2z=6 \\ 1(4-1y-2z)-2y-3z=4 \end{cases}$$

Simplifying these equations we get:

$$\begin{cases} -5y - 6z = -2 \\ -3y - 5z = 0 \end{cases}$$

Note that we now have 2 equations in 2 unknowns. We repeat the process. We arbitrarily choose to solve the first of these equations for y:
$y = -1.2\,z + 0.4$

Substituting this expression into the second equation gives:
$-3(-1.2\,z + 0.4) - 5\,z = 0.$

Simplifying gives:
$z = -0.8571$

Now that we have the solution for z, we enter the back-substitute phase to find the values of x and y. First substitute the value for z back into either of the previous pair of equations:

$$\begin{cases} -5y - 6z = -2 \\ -3y - 5z = 0 \end{cases}$$

Either choice yields $y = 1.429$. Now substitute the values for both z and y back into any of the original three equations:

$$\begin{cases} 1x + 1y + 2z = 4 \\ 2x - 3y - 2z = 6 \\ 1x - 2y - 3z = 4 \end{cases}$$

Any choice yields $x = 4.286$. Thus the final solution (to 3 significant figures) is:
$\{ x = 4.29, y = 1.43, z = -0.857 \}$.

2) Elimination method

Another method is called the elimination method or the addition or subtraction method. In this way of solving systems of equations, one variable is *eliminated* by *adding* or *subtracting* the equations (understand their names now? = P).

When adding two equations, you basically add all parts of them. Say you had the equations:
4x + 5y = 14
-4x - 3y = -10

Adding them would give 2y = 4

$$\begin{array}{r} 4x + 5y = 14 \\ \underline{+ -4x - 3y = -10} \\ 2y = 4 \end{array}$$

As you can see, the 4x and -4x cancelled out, therefore **eliminating** the variable x, leaving an equation with only one variable (y), able to be solved.

2y = 4
y = 2

Now that you have a value for y, you must find one for x. To do this, just substitute the value for y into either original equation, and solve it for x

4x + 5(2) = 14
4x + 10 = 14
4x = 4
x = 1

Your solution for these two equations is (1, 2).

Notice that only because 4x and -4x, when added, produce 0 (cancel out), the equation can be solved. Their coefficients are opposites of each other. That is why it worked.

4. Formulae

1) $(x+a)(x+b) = x^2 + (a+b)x + ab$

2) $(a \pm b)^2 = a^2 \pm 2ab + b^2$

3) $(a \pm b)^3 = a^3 \pm 3a^2 b + 3ab^2 \pm b^3$

4) $a^2 - b^2 = (a-b)(a+b)$

5) $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

6) $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$ (n is a positive integer)

7) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

8) $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

In-class questions

1. Given the system of equations:

$$\begin{cases} a+b=2m^2 \\ b+c=6m^2 \\ a+c=2 \end{cases}$$

where m is a real number. For what value of m is $a \le b \le c$ 。

2. Solve the system of equations for all possible integral value of a, b, and c.

$$\begin{cases} ab+5=c \\ bc+1=a \\ ca+1=b \end{cases}$$

3. Solve the system of equations for all possible value x and y.

$$\begin{cases} 6x^2-xy+4y^2=24 \\ 4x^2+xy+4y^2=18 \end{cases}$$

4. Solve the system of equations for x。

$$\begin{cases} x+2y-z=5 \\ 3x+2y+z=11 \\ (x+2y)^2-z^2=15 \end{cases}$$

5. If a can be any positive integer and

$$\begin{cases} 2x+a=y \\ a+y=x \\ x+y=z \end{cases}$$

Determine the maximum possible value for $x+y+z$.

6. The graph of $y=mx$ passes through the points (2, 5) and (5, n). What is the value of mn?

7. If x and y are real numbers, determine all solutions x, y of the system of equations

$$\begin{cases} x^2 - xy + 8 = 0 \\ x^2 - 8x + y = 0 \end{cases}$$

8. Solve: $x^3 - 2x^2 y - xy^2 + 2y^3 = 0$.