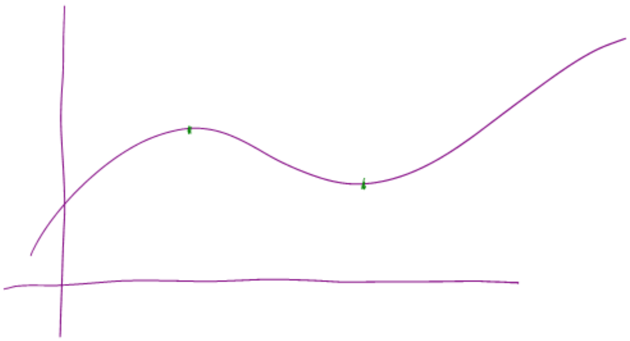
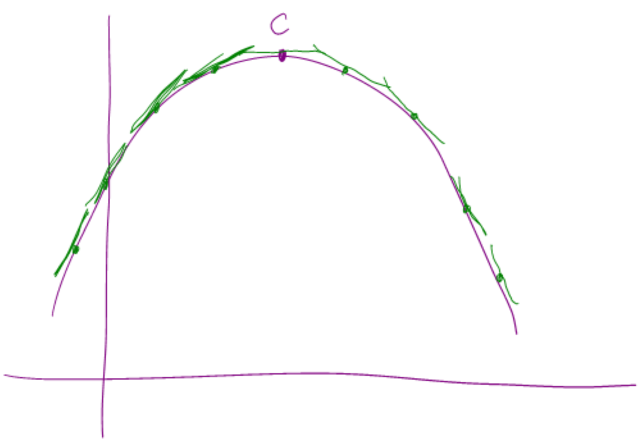
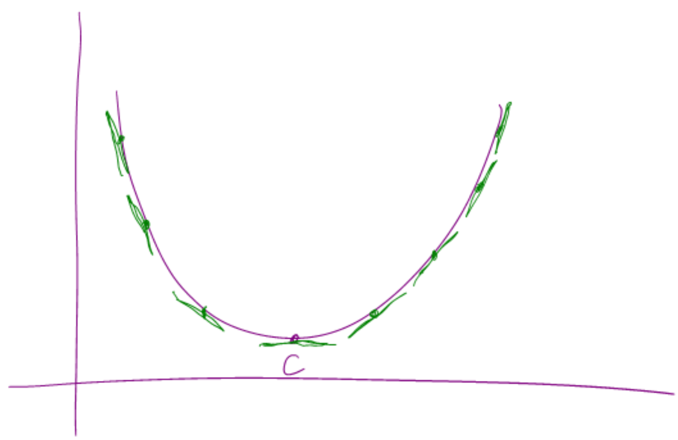
AP Calculus Class 7

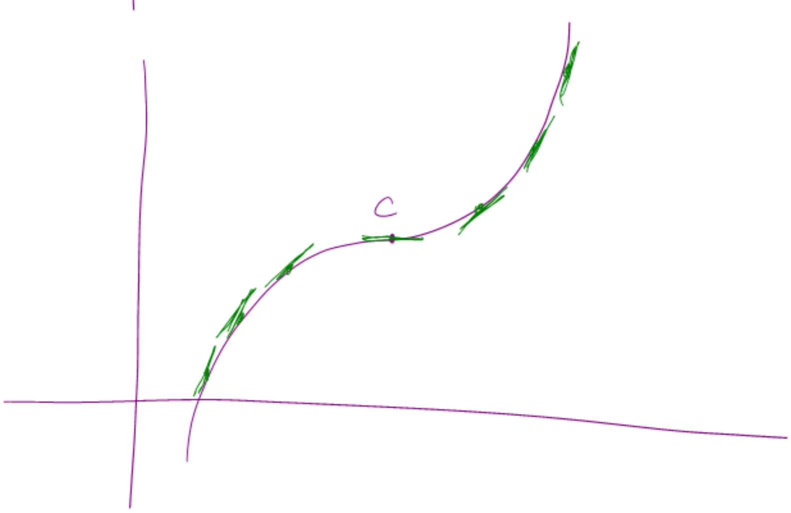


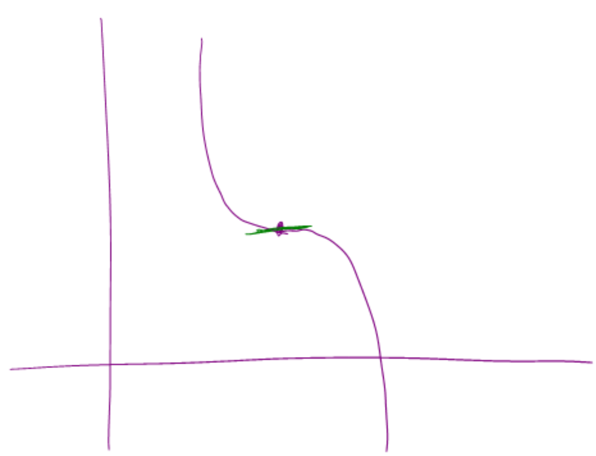
First Derivative Test

> Suppose that c is a critical number of a continuous function f.
> (1) If f' changes from positive to negative at c, then f has a local maximum at c.
> (2) If f' changes from negative to positive at c, then f has a local minimum at c.
> (3) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.



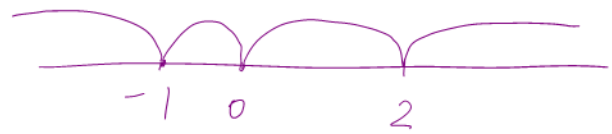






Example: $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Interval	$f'(x)$	$f(x)$
$x < -1$	$-$	dec
$-1 < x < 0$	$+$	inc
$0 < x < 2$	$-$	dec
$x > 2$	$+$	inc.



Crit numbers: $-1, 0, 2$.

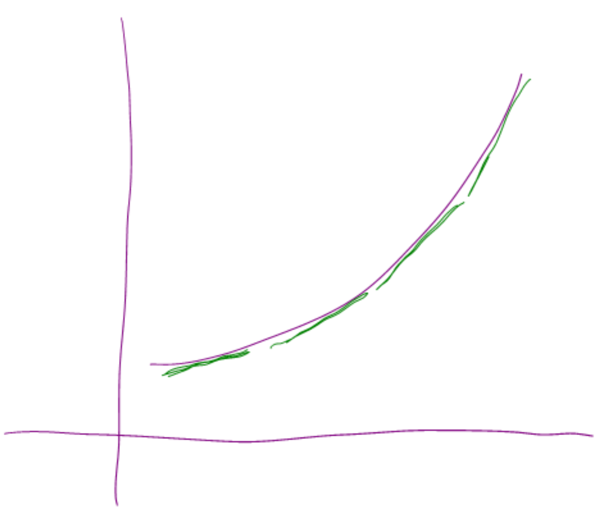
At $f(-1) = 0$, this is a local min.

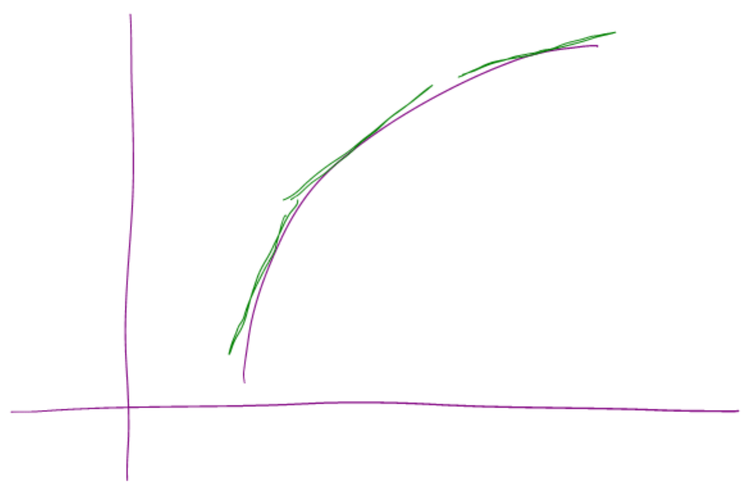
At $f(0) = 5$, this is a local max.

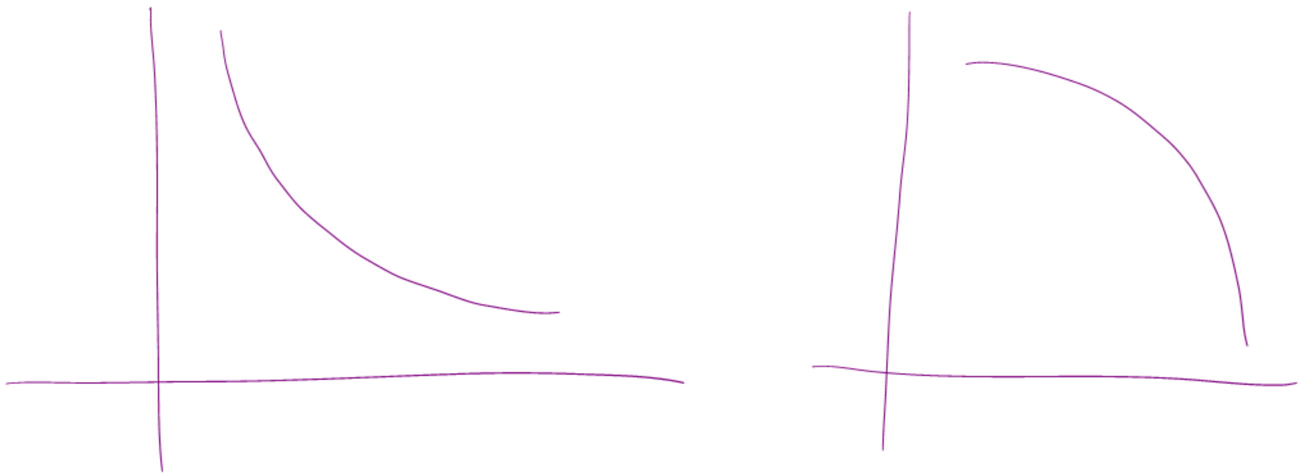
At $f(2) = -27$, this is a local min.

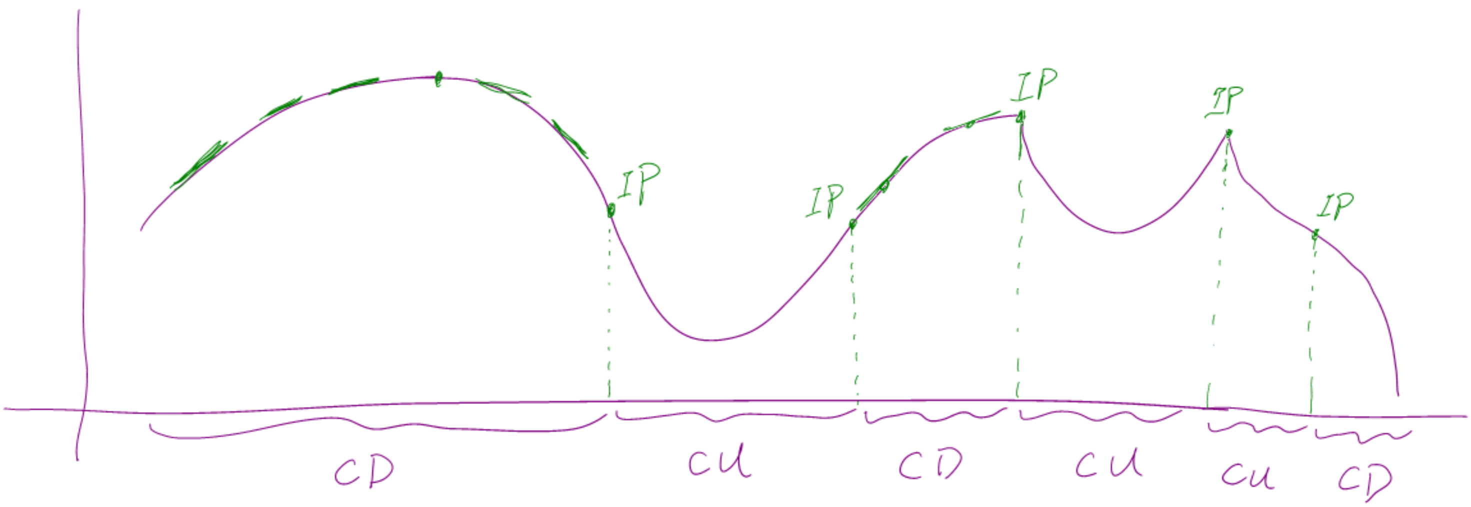
Definition

> If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on an interval I, then it is called **concave downward** on I.









CD: The f' goes from $+$ to $-$.

$\Rightarrow$ f' is a decreasing fun^n

CU: The f' goes from $-$ to $+$.

$\Rightarrow$ f' is an increasing fun^n.

Concavity Test

(1) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
(2) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test

Suppose f'' is continuous near c.
(1) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
(2) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example: Analyze the curve $y = x^4 - 4x^3$ w.r.t concavity, inflection points, local max and local min.

$f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$.

$f'(x) = 0 \Rightarrow 4x^2(x-3) = 0$.

$\Rightarrow x = 0$ and $x = 3$ are the crit. numbers.

Apply the 2nd Derivative Test.

$f''(x) = 12x^2 - 24x = 12x(x-2)$

$f''(0) = 0.$ $\quad f''(3) = 36 > 0.$

$\Rightarrow f'(3) = 0$ and $f''(3) > 0$

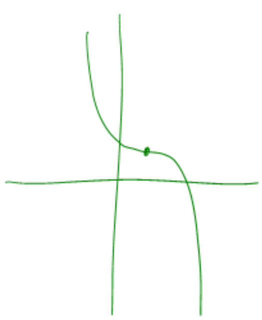
$\Rightarrow f(3) = -27$ is a local min.

For $f''(0)$,

since $f'(x) < 0$ for $x < 0$, and also for $0 < x < 3$,

the 1st Derivative Test tells us that f

does not have a max or min at $x = 0$.

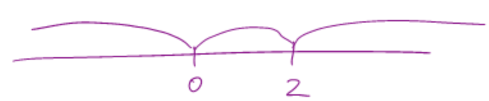


To find the inflection points,

check for $f''(x) = 0$.

$\Rightarrow f''(x) = 0$, when $x = 0$ or $x = 2$

Interval	$f''(x) = 12x(x-2)$	Concavity
$(-\infty, 0)$	+	up
$(0, 2)$	−	down
$(2, \infty)$	+	up.



The inflection points are $(0, 0)$ and $(2, -16)$.

Optimization Problems

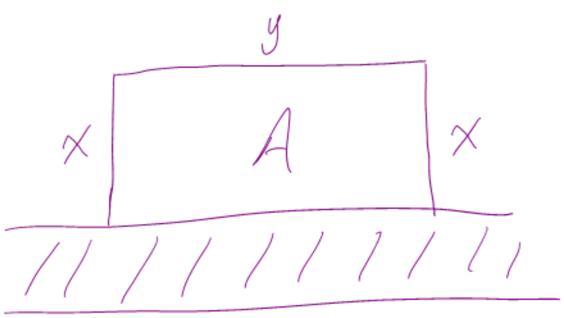
Example

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

① Understand the problem.

2400 ft of fence, maximize the area.

② Draw the diagram.



③ Introduce Notations.

Let A = the area of the field.

y = the length of the rectangular fence.

x = the width of the fence.

④ Express "A" in terms of the other quantities (x, y).

$$A = xy$$

(5) If "A" is represented by more than one variable, use the given info to find a relationship between the variables.

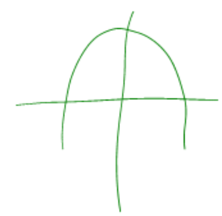
$$2x + y = 2400 \quad \Rightarrow \quad y = 2400 - 2x$$

$$\Rightarrow \quad A = x(2400 - 2x) = 2400x - 2x^2$$

(6) Use the methods we developed before to find the max or min values of the fun^n.

$A(x) = 2400x - 2x^2$.

Note that $x \geq 0$ and $x \leq 1200$.



We want to maximize $A(x)$ $\quad 0 \leq x \leq 1200$.

To find the max value, we differentiate

$A'(x) = 2400 - 4x$.

The crit. number is.

$A'(x) = 0 \quad \Rightarrow \quad 2400 - 4x = 0. \quad \underline{\underline{x = 600.}}$

$A(0) = 0. \qquad A(1200) = 0.$

$A(600) = 720\,000$

Indeterminate Forms and L'Hospital's Rule

$$F(x) = \frac{\ln x}{x-1} \qquad \lim_{x \to 1} \frac{\ln x}{x-1} = \frac{0}{0}$$

Indeterminate form of type $\frac{0}{0}$.

$$\lim_{x \to \infty} \frac{x^2}{x-1} \qquad \frac{\infty}{\infty}.$$

L'Hospital's Rule.

Suppose f and g are differentiable, and $g'(x) \neq 0$ near a.

Suppose that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$.

or $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

Then $\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$ if the limit on either side exists.

Example $\lim_{x \to 1} \frac{\ln x}{x-1}$ $\frac{\to 0}{\to 0.}$, apply l'Hospital's rule.

$$\lim_{x \to 1} \frac{(\ln x)'}{(x-1)'} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = \lim_{x \to 1} \frac{1}{x} = 1.$$

Example: $\lim_{x\to\infty} \dfrac{e^x}{x^2}$ $\begin{matrix}\to \infty \\ \to \infty\end{matrix}$

Apply l'Hospital's rule.

$$\lim_{x\to\infty} \frac{(e^x)'}{(x^2)'} = \lim_{x\to\infty} \frac{e^x}{2x} \to \text{Apply l'H's rule again.}$$

$$\lim_{x\to\infty} \frac{e^x}{2} = \infty$$

Indeterminate form of $0\cdot\infty$.

$$f\cdot g \to \frac{f}{1/g} \quad \text{or} \quad \frac{g}{1/f}$$

Example: $\lim_{x\to 0^+} x\ln x$.

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{(\ln x)'}{(1/x)'} = \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} (-x) = 0.$$

Indeterminant form of $\infty - \infty$.

Example: $\lim_{x\to(\frac{\pi}{2})^-} \sec x - \tan x$.

$$\lim_{x\to(\frac{\pi}{2})^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to(\frac{\pi}{2})^-} \frac{1-\sin x}{\cos x} = \cdots = 0$$

Indeterminate Powers.

0^0, ∞^0, 1^∞

Example: $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} (e^{\ln x})^x$

$= \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1.$

Homework 6.

7, 5, 3e, 10.

7. show $1+2x+x^3+4x^5=0$. has exactly one root.

$f(-1) = -6 < 0$, $f(0) = 1 > 0$.

since f is a polynomial, it's continuous.

so the MVT says $\exists$ a number $-1 < c < 0$

s.t. $f(c)=0$.

$\Rightarrow$ The equation has a real root.

Suppose that f has distinct real roots a and b.

where $a<b$.

Then $f(a) = f(b) = 0$.

Since f is a polynomial, then the 1st two conditions of Rolle's Thm are satisfied.

$\Rightarrow$ By Rolle's thm, $\exists$ a number r in (a,b)

s.t. $f'(r)=0$.

But, $f'(x) = 2+3x^2+20x^4 \geq 2$. for all x.

$\Rightarrow f'(x) \neq 0$.

$\Rightarrow$ Contradiction.

$\Rightarrow$ The eqn has only one root.