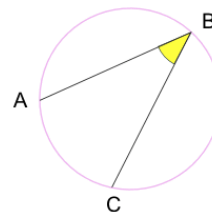


Geometry 4 - Circle

1. Inscribed Angle

Inscribed Angle: an angle made from points sitting on the circle's circumference.

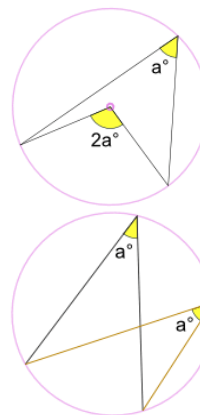
A and C are "end points", B is the "apex point".



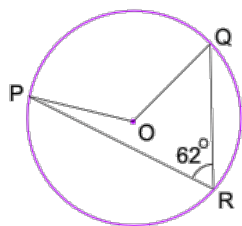
2. Inscribed Angle Theorems

An inscribed angle a° is half of the central angle $2a^\circ$ (Called the Angle at the Center Theorem).

The angle a° is always the same, no matter where it is on the circumference: Angle a° is the same.
(Called the Angles Subtended by Same Arc Theorem)



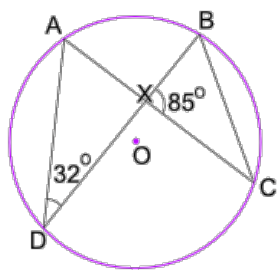
Example 1: What is the size of Angle POQ? (O is circle's center)



Solution:

$$\text{Angle POQ} = 2 \times \text{Angle PRQ} = 2 \times 62^\circ = 124^\circ$$

Example 2: What is the size of Angle CBX?



Solution:

Angle ADB = 32° equals Angle ACB.

And Angle ACB equals Angle XCB.

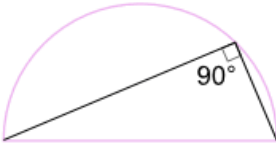
So in triangle BXC we know Angle BXC = 85° , and Angle XCB = 32°

Now use angles of a triangle add to 180° :

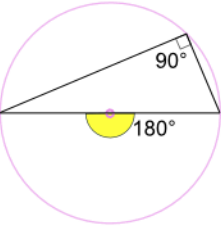
- Angle CBX + Angle BXC + Angle XCB = 180°
- Angle CBX + 85° + 32° = 180°
- Angle CBX = 63°

3. Angle in a Semicircle

An angle inscribed in a semicircle is always a right angle. (The end points are either end of a circle's diameter, the apex point can be anywhere on the circumference.)

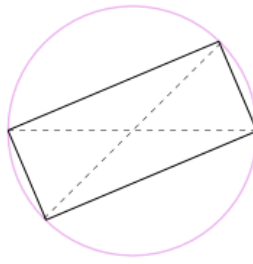
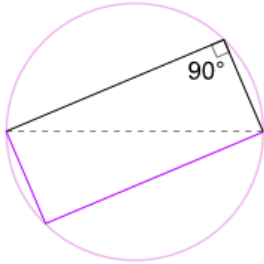


The inscribed angle 90° is half of the central angle 180°

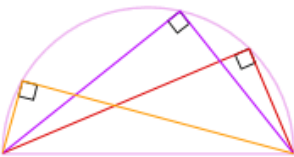


We could also rotate the shape around 180° to make a rectangle!

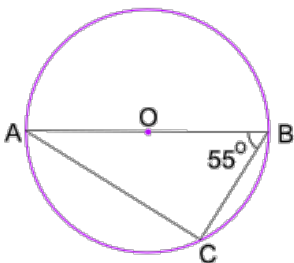
It is a rectangle, because all sides are parallel, and both diagonals are equal. And so its internal angles are all right angles (90°).



So, no matter where that angle is on the circumference, it is always 90°



Example 3: What is the size of Angle BAC?



Solution:

The Angle in the Semicircle Theorem tells us that Angle $ACB = 90^\circ$

Now use angles of a triangle add to 180° to find Angle BAC:

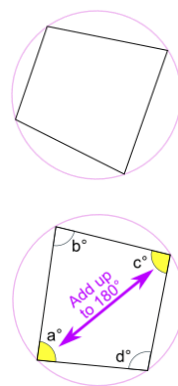
- Angle $BAC + 55^\circ + 90^\circ = 180^\circ$
- Angle $BAC = 35^\circ$

4. Cyclic Quadrilateral

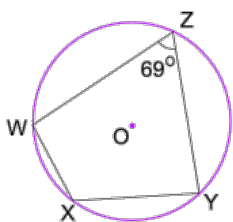
A "Cyclic" Quadrilateral has every vertex on a circle's circumference.

A Cyclic Quadrilateral's opposite angles add to 180° :

- $a + c = 180^\circ$
- $b + d = 180^\circ$



Example 4: What is the size of Angle WXY?



Solution:

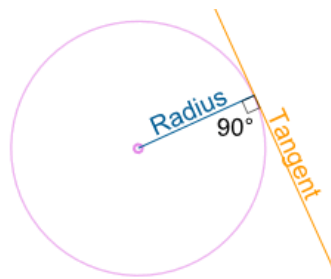
Opposite angles of a cyclic quadrilateral add to 180°

- Angle WZY + Angle WXY = 180°
- $69^\circ + \text{Angle WXY} = 180^\circ$
- Angle WXY = 111°

5. Tangent Angle

A tangent is a line that just touches a circle at one point.

It always forms a right angle with the circle's radius as shown here.



► Formulas for Working with Angles in Circles

1. Central Angle:

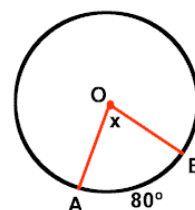
A central angle is an angle formed by two intersecting radii such that its vertex is at the center of the circle.

Central Angle = Intercepted Arc

$\angle AOB = \widehat{AB}$, $\angle AOB$ is a central angle.

Its *intercepted arc* is the minor arc from A to B.

$\angle AOB = 80^\circ$



Theorem involving central angles:

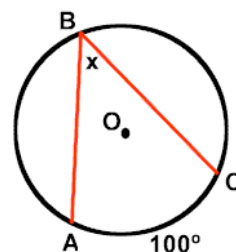
In a circle, or congruent circles, congruent central angles have congruent arcs.

2. Inscribed Angle:

An inscribed angle is an angle with its vertex "on" the circle, formed by two intersecting chords.

$$\text{Inscribed Angle} = \frac{1}{2} \text{ Intercepted Arc, } \angle ABC = \frac{1}{2} \widehat{AC}$$

$\angle ABC$ is an inscribed angle. Its *intercepted arc* is the minor arc from A to C. $\angle ABC = 50^\circ$



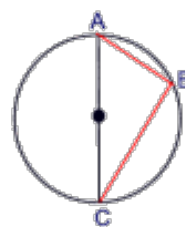
Special situations involving inscribed angles:

A quadrilateral inscribed in a circle is called a cyclic quadrilateral.

An angle inscribed in a semi-circle is a right angle.

$$\angle ABC = \frac{1}{2}(\widehat{AC}) = \frac{1}{2}(180^\circ) = 90^\circ$$

In a circle, inscribed circles that intercept the same arc are congruent.

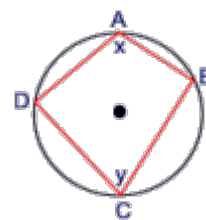


The opposite angles in a cyclic quadrilateral are supplementary.

$$\angle x = \frac{1}{2}(\widehat{DCB}); \angle y = \frac{1}{2}(\widehat{DAB})$$

$$\angle x + \angle y = \frac{1}{2}(\widehat{DCB} + \widehat{DAB})$$

$$\angle x + \angle y = \frac{1}{2}(360^\circ) = 180^\circ$$



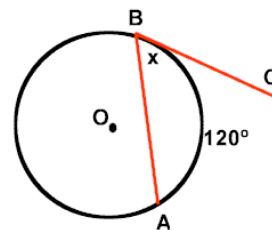
3. Tangent Chord Angle:

An angle formed by an intersecting tangent and chord has its vertex "on" the circle.

$$\text{Inscribed Angle} = \frac{1}{2} \text{ Intercepted Arc; } \angle ABC = \frac{1}{2} \widehat{AB}$$

$\angle ABC$ is an angle formed by a tangent and chord.

Its *intercepted arc* is the minor arc from A to B. $\angle ABC = 60^\circ$



4. Angle Formed Inside of a Circle by Two Intersecting Chords

When two chords intersect "inside" a circle, four angles are formed. At the point of intersection, two sets of vertical angles can be seen in the corners of the X that is formed on the picture. Remember: vertical angles are equal.

Angle Formed Inside by Two Chords = $\frac{1}{2}$ Sum of Intercepted Arcs

$\angle BED = \frac{1}{2}(\widehat{AB} + \widehat{CD})$. $\angle BED$ is formed by two intersecting chords.

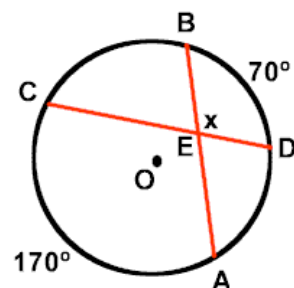
Its *intercepted arcs* are \widehat{BD} and \widehat{CA} .

[Note: the intercepted arcs belong to the set of vertical angles.]

$$\angle BED = \frac{1}{2}(70 + 170) = \frac{1}{2}(240) = 120^\circ$$

also, $\angle CEA = 120^\circ$ (vertical angle).

$\angle BEC$ and $\angle DEA = 60^\circ$ by straight line.



5. Angle Formed Outside of a Circle by the Intersection

Angle Formed Outside of a Circle by the Intersection of "Two Tangents" or "Two Secants" or "a Tangent and a Secant".

The formulas for all THREE of these situations are the same:

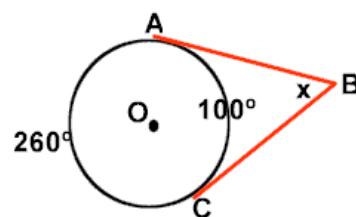
Angle Formed Outside = $\frac{1}{2}$ Difference of Intercepted Arcs (When subtracting, start with the larger arc.)

1) Two Tangents:

$\angle ABC$ is formed by two tangents intersecting outside of circle O .

The *intercepted arcs* are minor arc \widehat{AC} and major arc \widehat{AC} . These two arcs together comprise the entire circle.

$$\angle ABC = \frac{1}{2} \left(\widehat{AC}^{Major} - \widehat{AC}^{Minor} \right); \quad \angle ABC = \frac{1}{2}(260 - 100) = 80^\circ$$



Special situation for this set up

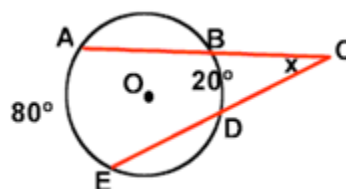
It can be proven that $\angle ABC$ and central $\angle AOC$ are supplementary. Thus the angle formed by the two tangents and its first intercepted arc also add to 180° .

2) Two Secants:

$\angle ACE$ is formed by two secants intersecting outside of circle O .

The *intercepted arcs* are minor arcs \widehat{BD} and \widehat{AE} .

$$\angle ACE = \frac{1}{2}(\widehat{AE} - \widehat{BD}); \quad \angle ACE = \frac{1}{2}(80 - 20) = 30^\circ$$

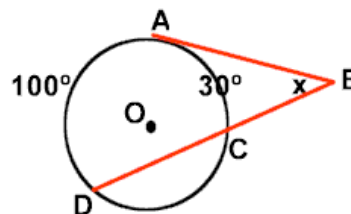


3) a Tangent and a Secant:

$\angle ABD$ is formed by a tangent and a secant intersecting outside of circle O .

The *intercepted arcs* are minor arcs \widehat{AC} and \widehat{AD} .

$$\angle ABD = \frac{1}{2}(\widehat{AD} - \widehat{AC}); \quad \angle ABD = \frac{1}{2}(100 - 30) = 35^\circ$$



► Circle Theorems

Theorem 1.

The area of a circle of radius r is πr^2 and its circumference is $2\pi r$.

Theorem 2.

A line from the centre of a circle perpendicular to a chord bisects the chord and its arc.

Theorem 3.

If AB is a semicircular arc of a circle and C is any point on the circumference of the circle then $\angle ACB$ is a right angle.

Theorem 4.

If A and B are points on the circumference of a circle with centre O and C is an exterior point of the circle such that BC is a tangent to the circle then

$$\angle ABC = \frac{1}{2} \angle AOB.$$

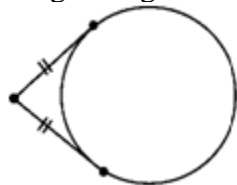
Theorem 5.

A line meeting a circle at a point A is tangent to the circle if and only if the radius to the point of contact with the line at A is perpendicular to the line.



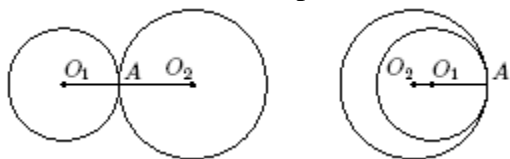
Theorem 6.

Tangent segments from an external point are equal.



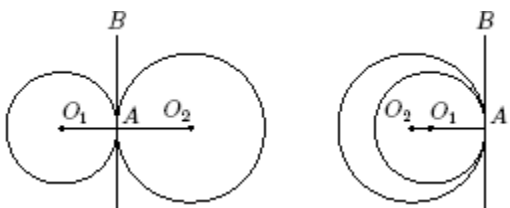
Theorem 7.

If two circles touch at a single point then this point and the centres of the circles are collinear. Below, O_1 , O_2 and the point of contact A of the circles are collinear.



Theorem 8.

If two circles are tangential at a point A then the tangent to one of the circles at A is also a tangent to the other circle.



Theorem 9.

There is a unique circle through any triple of non-collinear points.

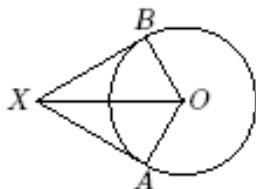
Proof. Let the three non-collinear points be A , B , C . The points are distinct since two distinct points are sufficient to define a line, so that if any points are coincident then the points would be collinear, contradicting their non-collinearity. Form the perpendicular bisector of each of AB and BC . (Recall that the perpendicular bisector of two points is the locus of points that are equidistant from two given points.) These bisectors are non-parallel, since A , B , C are non-collinear. Hence the bisectors intersect. Let the point of intersection be O . Then $OA = OB$ since O lies on the bisector of AB . Similarly, $OB = OC$ since O lies on the bisector of BC .

Thus $OA = OB = OC$ so that O is equidistant from A , B and C . Hence we A , B and C lie on a circle with centre O and radius OA . Since O is the unique intersection of the perpendicular bisectors of AB and BC the circle through A , B and C is unique.

Theorem 10.

The two tangents drawn to a circle from an exterior point of the circle have the same length. In the diagram, $XA = XB$.

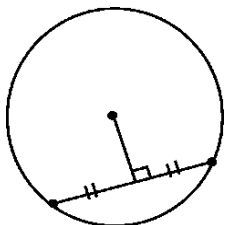
Moreover, the line joining the centre of the circle and the exterior point bisects the angle between the two tangents. In the diagram, OX bisects $\angle AXB$.



Proof. $OA = OB$, (radii of same circle)
 $90^\circ = \angle OAX = \angle OBX$, by Theorem 23
 OX is common
 $\therefore \triangle OAX \cong \triangle OBX$, by the RHS Rule
 $\therefore XA = XB$
 and $\angle OXA = \angle OXB$, i.e. OX bisects $\angle AXB$.

Theorem 11.

A line perpendicular to a chord through the center of the circle bisects the chord.



Proof:

Join OA and OB .

In right triangle OAC and OBC ,

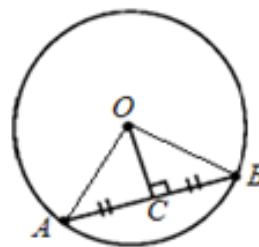
$OC = OC$; common side

$OA = OB$; radius are always equal in some circle

$\angle OCA = \angle OCB = 90^\circ$

given by the question that $OC \perp AB$

$\therefore \triangle OAC \cong \triangle OBC$, then $AC = BC$



Theorem 12.

If AB is an arc of a circle then angles subtended at the circumference opposite AB are equal and are equal to half the angle subtended at the centre, i.e. in the diagram.

$$\angle ACB = \angle ADB = \frac{1}{2} \angle AOB.$$

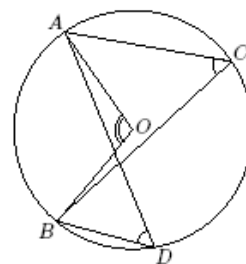
Proof.

Construct OC , forming isosceles triangles AOC and BOC . Let the equal base angles of $\triangle AOC$ be x and the equal base angles of $\triangle BOC$ be y . Then

$$\angle ACB = x + y$$

$$\angle AOB = 360^\circ - (180^\circ - 2x) - (180^\circ - 2y) = 2x + 2y = 2\angle ACB$$

$$\therefore \angle ACB = \frac{1}{2} \angle AOB$$

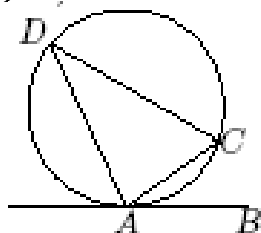


Suppose that C is in fact at D , and x and y are defined as before. Then $\angle ADB = y - x$ and $\angle AOB = 180^\circ - 2x - (180^\circ - 2y) = 2(y - x) = 2\angle ADB$ with the same conclusion as before.

Theorem 13. (Tangent-chord Theorem or Alternate Segment Theorem).

Let AC be a chord in a circle and let AB be a line meeting the circle at A.

Then AB is tangent to the circle if and only if $\angle CAB$ is equal to the angle subtended by the chord at points on the circumference on the opposite of AC from B.



Proof. Draw a diameter from A through the centre O (to X) and let $x = \angle XAC$.

Then $\angle XAB = 90^\circ$, by Theorem 23, $OA \perp AB$

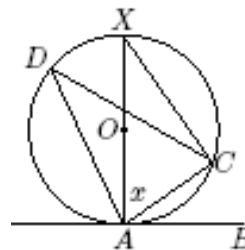
$\therefore \angle CAB = \angle XAB - \angle XAC = 90^\circ - x$

$\angle ACX = 90^\circ$, by Theorem 20, since AX is a diameter

$\therefore \angle AXC = 180^\circ - \angle ACX - \angle XAC = 90^\circ - x = \angle CAB$

$\angle ADC = \angle AXC$, by Theorem 19, common arc: AC

$\therefore \angle ADC = \angle CAB$:



Remark:

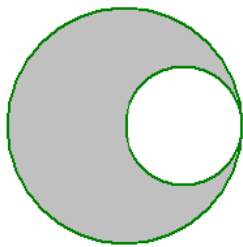
In the diagram of the proof above, if we think of X moving around the circumference of the circle, we always have $\angle AXC = \angle ADC$, by Theorem 19. As X moves around toward A, $\angle CAB$ can be thought of as the limit of $\angle AXC$ as $X \Rightarrow A$, since the chord XA (extended) becomes the tangent AB when X and A become the one point.

► Questions in class

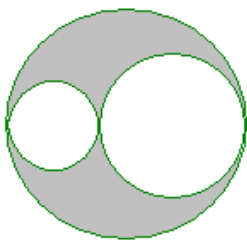
1. A large circle is tangent to each of the sides of a square. A small circle is tangent to the large circle and two sides of the square as shown. If the sides of the square have length 1, find the ratio of the area of the small circle to the area of the shaded region.



2. The smaller of the two circles shown in the figure is tangent to the larger and passes through its center. If the area inside the larger and outside the smaller is 12π , determine the ratio of the area of small circle to the area of large circle.

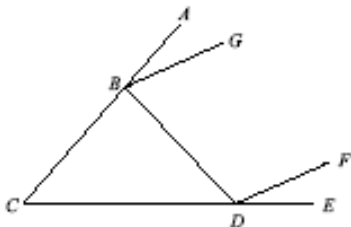


3. Two small circles are tangent to each other externally and tangent to a large circle internally. If the ratio of the radius of the small circle to the radius of medium is 1:2, and the region interior to the largest circle and exterior to the smaller two circles (shaded region) is 36π , determine the ratio of the smallest radius to the largest radius.

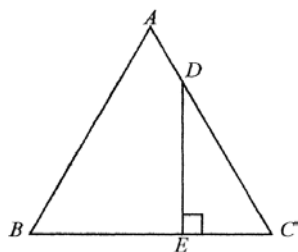


Extra:

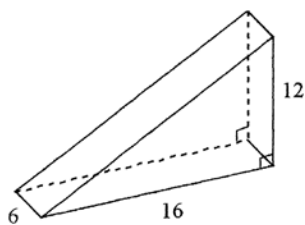
1. If $\angle ABG = \frac{1}{6} \angle ABD$, $\angle FDE = \frac{1}{6} \angle BDE$ and $BG \parallel DF$, find the number of degrees in the measure of $\angle C$.



2. ABC is an equilateral triangle of side 4 cm. If the ratio of the area of DEC to the area of $ADEB$ is 1: 3, find the number of centimeters in the length of DE .



3. I shall seize a city inimical to Marduk. My uncompleted siege ramp has width 6 m, length 16 m and height 12 m. When completed, keeping the width the same, and the length and height in the same ratio, its volume will be 625 m^3 . When it is completed, what will the number of meters in its length be?



4. Starting with a square, a new figure is formed by removing four equilateral triangles. Find the ratio of the perimeter of the new figure to the perimeter of the original square.

