AP Calculus Homework One – Limit and Continuity

1.1 Definitions of Limits; 1.2 Continuity; 1.3 Limits Properties

1. Show that limits do not exist.

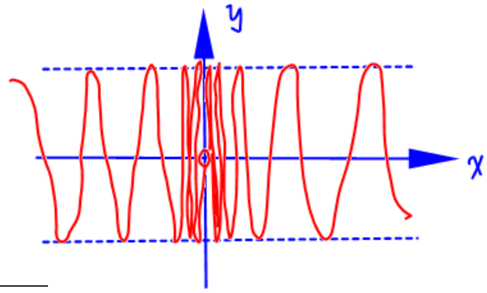
(a) $\lim\limits_{x\to -2}\frac{x+2}{|x+2|}$

$\because \lim\limits_{x\to -2^-}\frac{x+2}{|x+2|} = \lim\limits_{x\to -2^-}\frac{x+2}{-(x+2)} = \lim\limits_{x\to -2^-}(-1) = -1$.

$\lim\limits_{x\to -2^+}\frac{x+2}{|x+2|} = \lim\limits_{x\to -2^+}\frac{x+2}{x+2} = \lim\limits_{x\to -2^+}(1) = 1$.

Since two one-sided Limits are not equal so $\lim\limits_{x\to -2}\frac{x+2}{|x+2|}$ DNE

(b) $\lim\limits_{x\to 0}\sin\frac{1}{x}$



When x is approaching to 0, the value of $\sin\frac{1}{x}$ is oscillating between -1 and 1. Therefore, there is no real number L for the value of $\sin\frac{1}{x}$ to approach to. Hence $\lim\limits_{x\to 0}\sin\frac{1}{x}$ DNE

(c) $\lim\limits_{x\to 0}\sqrt{3+\arctan\frac{1}{x}}$

$\because \lim\limits_{x\to 0^-}\sqrt{3+\arctan\frac{1}{x}} = \sqrt{3+\arctan(\frac{1}{0^-})} = \sqrt{3+\arctan(-\infty)} = \sqrt{3-\frac{\pi}{2}}$

$\lim\limits_{x\to 0^+}\sqrt{3+\arctan\frac{1}{x}} = \sqrt{3+\arctan(\frac{1}{0^+})} = \sqrt{3+\arctan(+\infty)} = \sqrt{3+\frac{\pi}{2}}$.

Since the two one-sided limits are not equal, so $\lim\limits_{x\to 0}\sqrt{3+\arctan\frac{1}{x}}$ DNE

2. Find limits.

(a) $\lim\limits_{x\to 0}\frac{x^2}{2x-1} = \frac{0^2}{2(0)-1} = \frac{0}{-1} = 0$

"$\frac{0}{0}$"

(b) $\lim\limits_{x\to 2}\frac{x^3-8}{x^2-4} = \lim\limits_{x\to 2}\frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+2)} = \frac{2^2+2(2)+4}{2+2} = 3$

(c) $\lim\limits_{x\to -1}\frac{2+2/x}{x^2-4x-5} = 2\lim\limits_{x\to -1}\frac{1+\frac{1}{x}}{(x+1)(x-5)} = 2\lim\limits_{x\to -1}\frac{\frac{\cancel{x+1}}{x}}{\cancel{(x+1)}(x-5)} = 2\,\frac{\frac{1}{-1}}{-1-5} = \frac{1}{3}$

(d) $\lim\limits_{h\to 0}\frac{5(h-1)^2+(h-1)-4}{h} = \lim\limits_{h\to 0}\frac{5(h^2-2h+1)+h-5}{h} = \lim\limits_{h\to 0}\frac{5h^2-9h+\cancel{5}-\cancel{5}}{h}$

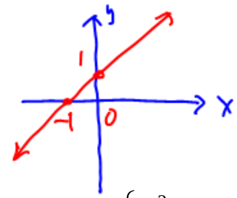
$= \lim\limits_{h\to 0}(5h-9) = 5(0)-9 = -9$

(e) Explain, using examples, when substitution can not be used to solve a limit.

When applying the quotient Law and the radical root Law, pay attention to "$\frac{0}{0}$" and complex result. i.e. $\lim\limits_{x\to 1}\frac{x^2-1}{x-1}$ is "$\frac{0}{0}$", need to cancel out "$x-1$".

$\lim\limits_{x\to 1}\frac{x^2-1}{x-1} = \lim\limits_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim\limits_{x\to 1}(x+1) = 1+1 = 2$; $\lim\limits_{x\to 2}\sqrt{1-x} = \sqrt{\lim\limits_{x\to 2}(1-x)} = \sqrt{1-2} = \sqrt{-1} = i$ DNE.

Checking " $\lim_{x \to c} f(x) = f(c)$ "



Actually. $f(x) = x+1$ for $x \in R$.

3. Discuss the continuity and sketch the graph of $f(x) = \begin{cases} \dfrac{x^2+x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$. $= \begin{cases} x+1, & \text{if } x<0 \\ 1, & \text{if } x=0 \\ x+1, & \text{if } x>0 \end{cases}$

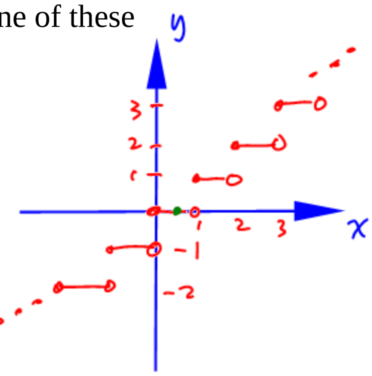
$f(x)$ is a piecewise function. it has three pieces. $f(x)$ is continuous for $x<0$ and $x>0$. So we need to discuss the continuity at $x=0$ only: $f(0)=1$. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x+1) = 0+1 = 1$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1) = 0+1 = 0$. $\therefore \lim_{x\to 0} f(x) = 1 = f(0)$. Hence $f(x)$ is continuous everywhere for $x \in R$.

4. If $[x]$ is the greatest integer not greater than x, then $\lim_{x \to \frac{1}{2}} [x]$ is

(A) 1/2 (B) 1 (C) nonexistent (D) 0 (E) none of these

$\lim_{x\to \frac{1}{2}^-} [x] = [\frac{1}{2}] = 0$ and $\lim_{x\to\frac{1}{2}^+} [x] = [\frac{1}{2}] = 0$

$\therefore \lim_{x\to\frac{1}{2}} [x] = 0$



5. Find a value of k such that $f(x)$ is continuous at $x = 0$.

$$f(x) = \begin{cases} \dfrac{x^2 - x}{2x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

$\because f(0) = K$. and $\lim_{x\to 0^-} \frac{x^2-x}{2x} = \lim_{x\to 0^-} \frac{x-1}{2} = \frac{0-1}{2} = -\frac{1}{2}$; $\lim_{x\to 0^+} \frac{x^2-x}{2x} = \lim_{x\to 0^+} \frac{x-1}{2} = \frac{0-1}{2} = -\frac{1}{2}$

$\therefore \lim_{x\to 0} f(x) = -\frac{1}{2}$. Hence $K = -\frac{1}{2}$

6. The function s(x) is defined as follows. Find a value of k such that $s(x)$ is continuous for all x.

$$s(x) = \begin{cases} 4x - 11, & \text{if } x < 3 \\ kx^2, & \text{if } x \geq 3 \end{cases}$$

$S(x)$ is continuous for all $x \in R$ except $x=3$

At $x=3$. $S(3) = K(3)^2 = 9K$

$\lim_{x\to 3^-} S(x) = \lim_{x\to 3^-} (4x-11) = 4(3)-11 = 1$.

let $9K = 1$, $K = \frac{1}{9}$. with this K value, $S(x)$ is continuous for all $x \in R$.

7. Discuss the continuity of the graph of $y = \dfrac{x^2 - 9}{3x - 9}$, indicating type of discontinuity if there is one.

$\because y = \frac{x^2-9}{3x-9} = \frac{(\cancel{x-3})(x+3)}{3(\cancel{x-3})} = \frac{x+3}{3}$ $\therefore f(3)$ DNE, and

$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{3x-9} = \lim_{x\to 3} \left(\frac{x+3}{3}\right) = \frac{3+3}{3} = 2$. Hence $f(x)$ is continuous everywhere for $x \in R$ except $x=3$. It is a removable discontinuity at $x=3$.