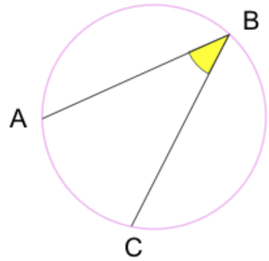
Geometry 3 - Circle 2

1. Inscribed Angle

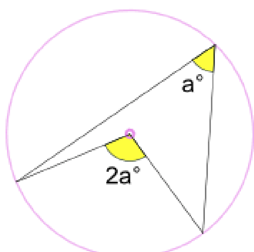
Inscribed Angle: an angle made from points sitting on the circle's circumference.



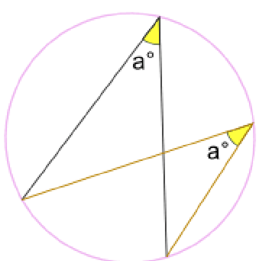
A and C are "end points", B is the "apex point".

2. Inscribed Angle Theorems

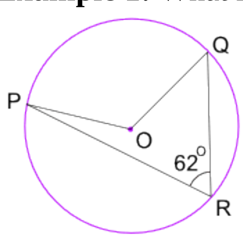
An inscribed angle a° is half of the central angle 2a° (Called the Angle at the Center Theorem).



The angle a° is always the same, no matter where it is on the circumference: Angle a° is the same. (Called the Angles Subtended by Same Arc Theorem)



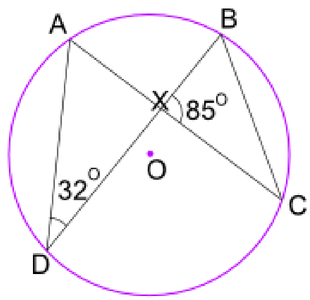
Example 1: What is the size of Angle POQ? (O is circle's center)



Solution:

Angle POQ = 2 × Angle PRQ = 2 × 62° = 124°

Example 2: What is the size of Angle CBX?

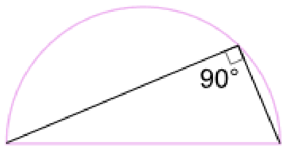


Solution:
Angle ADB = 32° equals Angle ACB.
And Angle ACB equals Angle XCB.
So in triangle BXC we know Angle BXC = 85°, and Angle XCB = 32°
Now use angles of a triangle add to 180° :

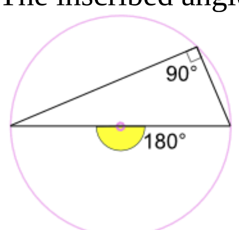
- Angle CBX + Angle BXC + Angle XCB = 180°
- Angle CBX + 85° + 32° = 180°
- Angle CBX = 63°

3. Angle in a Semicircle

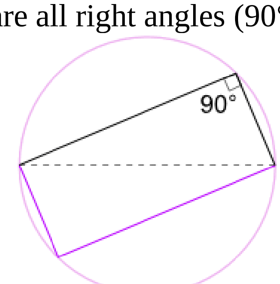
An angle inscribed in a semicircle is always a right angle. *(The end points are either end of a circle's diameter, the apex point can be anywhere on the circumference.)*

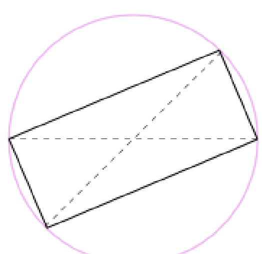


The inscribed angle 90° is half of the central angle 180°

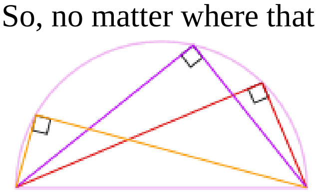


We could also rotate the shape around 180° to make a rectangle!
It is a rectangle, because all sides are parallel, and both diagonals are equal. And so its internal angles are all right angles (90°).

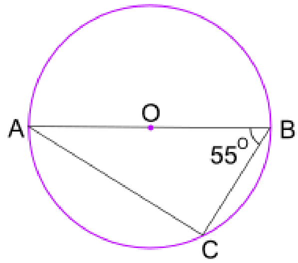




So, no matter where that angle is on the circumference, it is always 90°



Example 3: What is the size of Angle BAC?

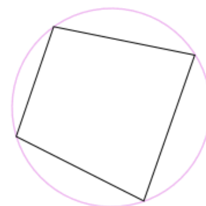


Solution:
The Angle in the Semicircle Theorem tells us that Angle ACB = 90°
Now use angles of a triangle add to 180° to find Angle BAC:

- Angle BAC + 55° + 90° = 180°
- Angle BAC = 35°

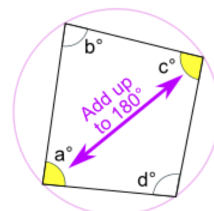
4. Cyclic Quadrilateral

A "Cyclic" Quadrilateral has every vertex on a circle's circumference.

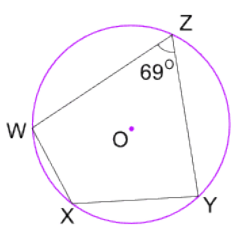


A Cyclic Quadrilateral's opposite angles add to 180°:

- a + c = 180°
- b + d = 180°



Example 4: What is the size of Angle WXY?



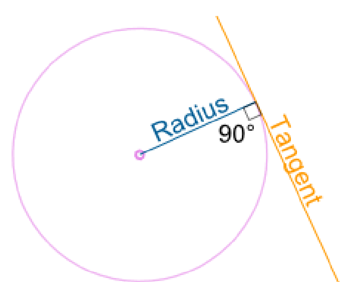
Solution:
Opposite angles of a cyclic quadrilateral add to 180°

- Angle WZY + Angle WXY = 180°
- 69° + Angle WXY = 180°
- Angle WXY = 111°

5. Tangent Angle

A tangent is a line that just touches a circle at one point.

It always forms a right angle with the circle's radius as shown here.



►Formulas for Working with Angles in Circles

1. Central Angle:

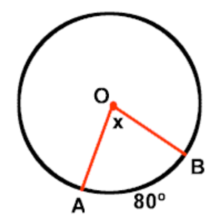
A central angle is an angle formed by two intersecting radii such that its vertex is at the center of the circle.

Central Angle = Intercepted Arc

$\angle AOB = AB$, $\angle AOB$ is a central angle.

Its *intercepted arc* is the minor arc from *A* to *B*.

$\angle AOB$ = 80°



Theorem involving central angles:

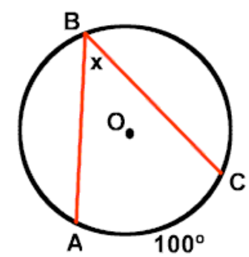
In a circle, or congruent circles, congruent central angles have congruent arcs.

2. Inscribed Angle:

An inscribed angle is an angle with its vertex "on" the circle, formed by two intersecting chords.

Inscribed Angle = $\frac{1}{2}$ Intercepted Arc, $\angle ABC = \frac{1}{2} AC$

<*ABC* is an inscribed angle. Its *intercepted arc* is the minor arc from *A* to *C*. $\angle ABC = 50^0$



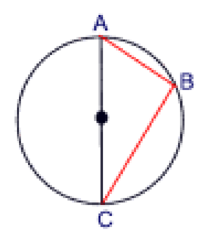
Special situations involving inscribed angles:

A quadrilateral inscribed in a circle is called a cyclic quadrilateral.

An angle inscribed in a semi-circle is a right angle.

$\angle ABC = \frac{1}{2}(AC) = \frac{1}{2}(180^0) = 90^0$

In a circle, inscribed circles that intercept the same arc are congruent.

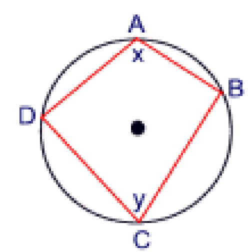


The opposite angles in a cyclic quadrilateral are supplementary.

$\angle x = \frac{1}{2}(DCB)$; $\angle y = \frac{1}{2}(DAB)$

$\angle x + \angle y = \frac{1}{2}(DCB + DAB)$

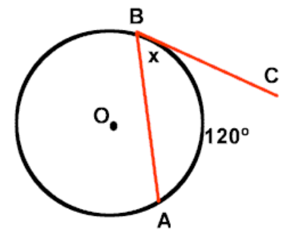
$\angle x + \angle y = \frac{1}{2}(360^0) = 180^0$



3. Tangent Chord Angle:

An angle formed by an intersecting tangent and chord has its vertex "on" the circle.

Inscribed Angle = $\frac{1}{2}$ Intercepted Arc; $\angle ABC = \frac{1}{2} AB$



$\angle ABC$ is an angle formed by a tangent and chord.

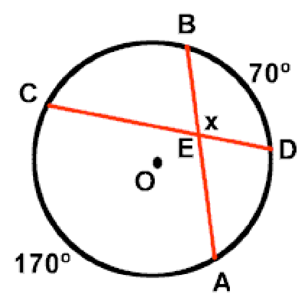
Its *intercepted arc* is the minor arc from A to B. $\angle ABC = 60^0$

4. Angle Formed Inside of a Circle by Two Intersecting Chords

When two chords intersect "inside" a circle, four angles are formed. At the point of intersection, two sets of vertical angles can be seen in the corners of the X that is formed on the picture. Remember: vertical angles are equal.

Angle Formed Inside by Two Chords = $\frac{1}{2}$ Sum of Intercepted Arcs

$\angle BED = \frac{1}{2}(AB + BD)$. $\angle BED$ is formed by two intersecting chords.



Its *intercepted arcs* are BD and CA.

[Note: the intercepted arcs belong to the set of vertical angles.]

$\angle BED = \frac{1}{2}(70 + 170) = \frac{1}{2}(240) = 120^0$

also, $\angle CEA = 120^0$ (vertical angle).

<*BEC* and <*DEA* = 60° by straight line.

5. Angle Formed Outside of a Circle by the Intersection

Angle Formed Outside of a Circle by the Intersection of "Two Tangents" or "Two Secants" or "a Tangent and a Secant".

The formulas for all THREE of these situations are the same:

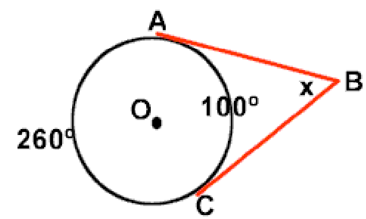
Angle Formed Outside = $\frac{1}{2}$ Difference of Intercepted Arcs

(When subtracting, start with the larger arc.)

1) Two Tangents:

<ABC is formed by two tangents intersecting outside of circle *O*.

The *intercepted arcs* are minor arc AC and major arc AC. These two arcs together comprise the entire circle.



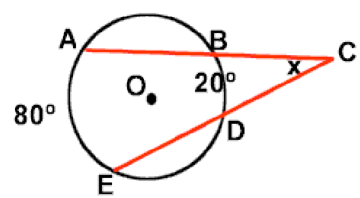
$$\angle ABC=\frac{1}{2}\left(\overset{Major}{AC}-\overset{Minor}{AC}\right);\quad \angle ABC=\frac{1}{2}(260-100)=80^0$$

Special situation for this set up
It can be proven that $\angle ABC$ and central $\angle AOC$ are supplementary. Thus the angle formed by the two tangents and its first intercepted arc also add to 180^0.

2) Two Secants:

$\angle ACE$ is formed by two secants intersecting outside of circle *O*.

The *intercepted arcs* are minor arcs BD and AE.

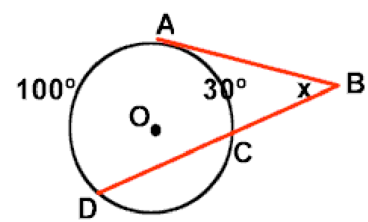


$$\angle ACE=\frac{1}{2}(AE-BD);\quad \angle ACE=\frac{1}{2}(80-20)=30^0$$

3) a Tangent and a Secant:

<ABD is formed by a tangent and a secant intersecting outside of circle *O*.

The *intercepted arcs* are minor arcs AC and AD.



$$\angle ABD=\frac{1}{2}(AD-AC);\quad \angle ABD=\frac{1}{2}(100-30)=35^0$$

►Circle Theorems

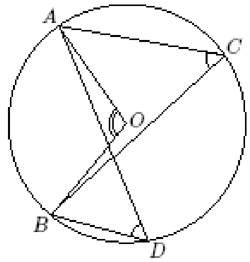
Theorem 1.
There is a unique circle through any triple of non-collinear points.

Proof
Let the three non-collinear points be A, B, C. The points are distinct since two distinct points are sufficient to define a line, so that if any points are coincident then the points would be collinear, contradicting their non-collinearity. Form the perpendicular bisector of each of AB and BC. (Recall that the perpendicular bisector of two points is the locus of points that are equidistant from two given points.) These bisectors are non-parallel, since A, B, C are non-collinear. Hence the bisectors intersect.

Let the point of intersection be O. Then OA = OB since O lies on the bisector of AB. Similarly, OB = OC since O lies on the bisector of BC.

Thus OA = OB = OC so that O is equidistant from A, B and C. Hence we A, B and C lie on a circle with centre O and radius OA. Since O is the unique intersection of the perpendicular bisectors of AB and BC the circle through A, B and C is unique.

Theorem 2.
If AB is an arc of a circle then angles subtended at the circumference opposite AB are equal and are equal to half the angle subtended at the centre, i.e. in the diagram



$\angle ACB = \angle ADB = \frac{1}{2}\angle AOB$.

Proof. Construct OC, forming isosceles triangles AOC and BOC. Let the equal base angles of ΔAOC be x and the equal base angles of ΔBOC be y. Then
$\angle ACB = x + y$
$\angle AOB = 360^{o} - (180^{o} - 2x) - (180^{o} - 2y) = 2x + 2y = 2\angle ACB$
$\therefore \angle ACB = \frac{1}{2}\angle AOB$
Suppose that C is in fact at D, and x and y are defined as before. Then $\angle ADB = y - x$ and
$\angle AOB = 180^{o} - 2x - (180^{o} - 2y) = 2(y - x) = 2\angle ACB$ with the same conclusion as before.

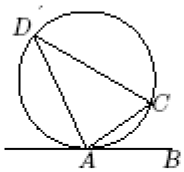
Theorem 3.
If AB is a semicircular arc of a circle and C is any point on the circumference of the circle then $\angle ACB$ is a right angle.

Theorem 4.
If A and B are points on the circumference of a circle with centre O and C is an exterior point of the circle such that BC is a tangent to the circle then

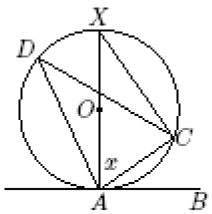
$\angle ABC = \frac{1}{2}\angle AOB$.

Theorem 5 (Tangent-chord Theorem or Alternate Segment Theorem).
Let AC be a chord in a circle and let AB be a line meeting the circle at A.

Then AB is tangent to the circle if and only if $\angle CAB$ is equal to the angle subtended by the chord at points on the circumference on the opposite of AC from B.



Proof. Draw a diameter from A through the centre O (to X) and let x = ∠XAC.



Then ∠XAB = 90^o , by Theorem 23, OA ? AB
∴ ∠CAB = ∠XAB - ∠XAC= 90^o - x

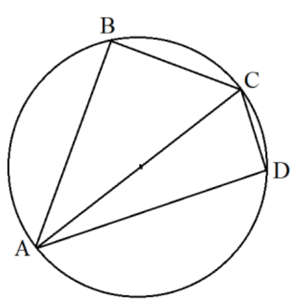
∠ACX = 90^o , by Theorem 20, since AX is a diameter
∴ ∠AXC = 180^o - ∠ACX - ∠XAC = 90^o - x = ∠CAB

∠ADC = ∠AXC, by Theorem 19, common arc: AC
∴ ∠ADC = ∠CAB:

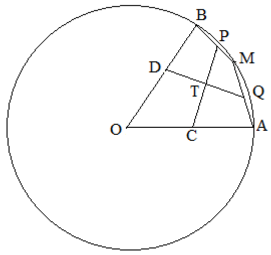
Remark:
In the diagram of the proof above, if we think of X moving around the circumference of the circle, we always have ∠AXC = ∠ADC, by Theorem 19. As X moves around toward A, ∠BAC can be thought of as the limit of ∠AXC as X ⇒A, since the chord XA (extended) becomes the tangent AB when X and A become the one point.

In-class questions

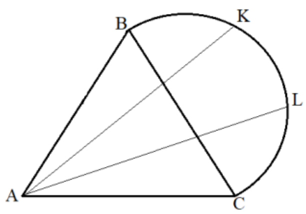
1. One of the diagonals of a quadrilateral inscribed in a circle is a diameter of the circle. Prove that (the lengths of) the projections of the opposite sides of the quadrilateral on the other diagonal are equal.



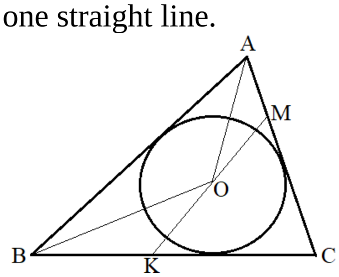
2. On a circle centered at O, points A and B single out an arc of 60^0. Point M belongs to this arc. Prove that the straight line passing through the midpoints of MA and OB is perpendicular to that passing through the midpoints of MB and OA.



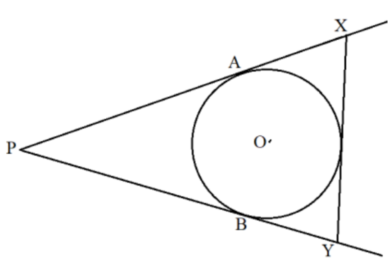
3. A semicircle is constructed outwards on side BC of an equilateral triangle ABC as on the diameter. Given points K and L that divide the semicircle into three equal arcs, prove that lines AK and AL divide BC into three equal parts.



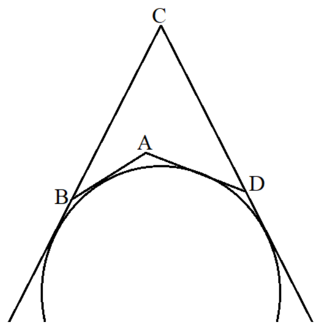
4. Point O is the center of the circle inscribed in △ABC. On sides AC and BC points M and K, respectively, are selected so that $BK \cdot AB = BO^2$ *and* $AM \cdot AB = AO^2$. Prove that M, O and K lie on one straight line.



5. Lines PA and PB are tangent to a circle centered at O, let A and B be the tangent points. A third tangent to the circle is drawn, it intersects with segments PA and PB at points X and Y, respectively. Prove that the value of angle XOY does not depend on the choice of the third tangent.



6. Quadrilateral ABCD is such that there exists a circle inscribed into angle $\angle BAD$ and tangent to the extensions of sides BC and CD. Prove that $AB+BC=AD+DC$.



7. Common outer tangents AB and CD are drawn to two circles of distinct radii. Prove that quadrilateral ABCD is a circumscribed one if and only if the circles are tangent to each other.