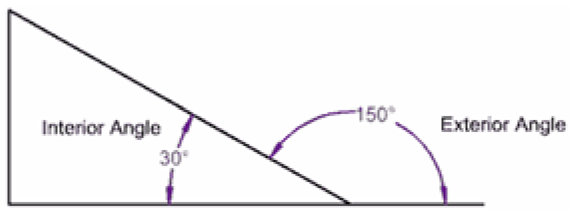
Geometry 1

1. Interior and Exterior Angle

An Interior Angle is an angle inside a shape.

The Exterior Angle is the angle between any side of a shape, and a line extended from the next side.



Note: If you add up the Interior Angle and Exterior Angle you get a straight line, 180°.

1) Interior Angles sum of Polygons

The sum of the measures of the interior angles of a polygon with *n* sides is (n-2)180.

For examples:

- Triangle or ('3 - gon')
  - sum of interior angles: (3-2) 180 = 180°

- Quadrilateral which has four sides (' 4 - gon')
  - sum of interior angles: (4-2)180 = 360°

- Hexagon which has six sides ('6 - gon')
  - sum of interior angles: (6-2)180 = 720°

An interior angle of a regular polygon with n sides is $\frac{(n-2)\times 180}{n}$.

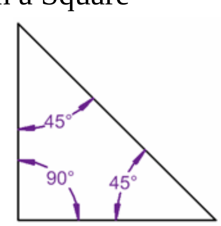
Example:

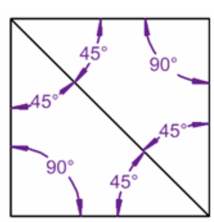
To find the measure of an interior angle of a regular octagon, which has 8 sides, apply the formula above as follows:
((8-2) × 180) /8 = 135°

There are Two Triangles in a Square

The internal angles in this triangle add up to 180°

(90°+45°+45°=180°)





... and for this square they add up to **360°**

... because the square can be made from two triangles!

2) Exterior Angles sum of Polygons

The Exterior Angle is the angle between any side of a shape, and a line extended from the next side.

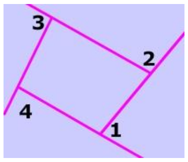
The sum of the measures of the exterior angles of a polygon, one at each vertex, is 360°.

For example:

A Polygon is any flat shape with straight sides.

The Exterior Angles of a Polygon add up to 360°.

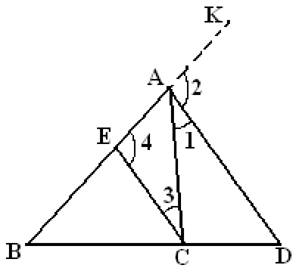
$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360^0$



3) Exterior Angel Bisector Theorem

Exterior angle bisector theorem: The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.

In ΔABC, AD is the bisector of the exterior ∠A and intersects BC produced in D.
Then, BD / CD = AB / AC



Proof:

Statements	Reasons
1) CE ‖ DA	1) By construction
2) ∠1 = ∠3	2) Alternate interior angle
3) ∠2 = ∠4	3) Corresponding angle (CE ‖DA and BK is a transversal
4) AD is a bisector of ∠A	4) Given
5) ∠1 = ∠2	5) Definition of angle bisector
6) ∠3 = ∠4	6) Transitivity (from 2 and 4)
7) AE = AC	7) If angles are equal then side opposite to them are also equal
8) BD / CD = BA/EA	8) By Basic proportionality theorem(EC ‖AD)
9) BD /CD = AB/AE	9) BA = AB and EA = AE
10) BD /CD = AB /AC	10) AE = EC and from(7)

2. Pythagorean Triples

A "Pythagorean Triple" is a set of positive integers, **a**, **b** and **c** that fits the rule:

$a^2 + b^2 = c^2$

1) Endless

The set of Pythagorean Triples is endless.

It is easy to prove this with the help of the first Pythagorean Triple, (3, 4, and 5):
Let n be any integer greater than 1, then 3n, 4n and 5n would also be a set of Pythagorean Triple.

This is true because: $(3n)^2 + (4n)^2 = (5n)^2$

Examples:

n	**(3n, 4n, 5n)**
2	(6,8,10)
3	(9,12,15)
...	... etc ...

So, you can make infinite triples just using the (3, 4, 5) triple.

2) Euclid's Proof that there are Infinitely Many Pythagorean Triples

However, Euclid used a different reasoning to prove the set of Pythagorean Triples is unending. The proof was based on the fact that the difference of the squares of any two consecutive numbers is always an odd number.

Examples:

$2^2 - 1^2 = 4 - 1 = \mathbf{3}$ (an odd number),
$15^2 - 14^2 = 225-196 = \mathbf{29}$ (an odd number)

3) Properties

It can be observed that a Pythagorean Triple always consists of:
- all even numbers, or
- two odd numbers and an even number.

A Pythagorean Triple can never be made up of all odd numbers or two even numbers and one odd number. This is true because:

- The square of an odd number is an odd number and the square of an even number is an even number.
- The sum of two even numbers is an even number and the sum of an odd number and an even number is in odd number.

Therefore, if one of a and b is odd and the other is even, c would have to be odd. Similarly, if both a and b are even, c would be an even number too!

4) Constructing Pythagorean Triples

It is easy to construct sets of Pythagorean Triples.

When **m** and **n** are any two positive integers (m < n):
$a = n^2 - m^2$
$b = 2nm$
$c = n^2 + m^2$
Then, a, b, and c form a Pythagorean Triple.

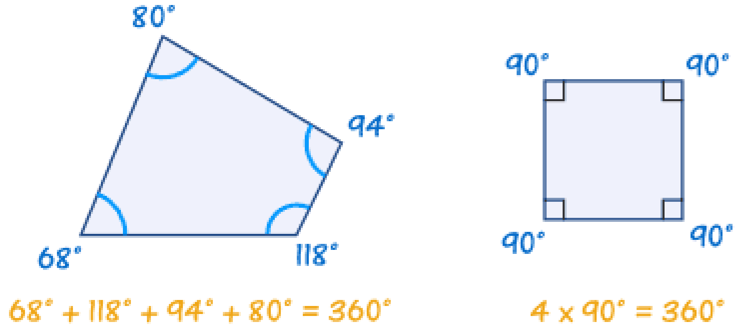
Example:
m=1 and n=2
$a = 2^2 - 1^2 = 4 - 1 = \mathbf{3}$
$b = 2 \times 2 \times 1 = \mathbf{4}$
$c = 2^2 + 1^2 = \mathbf{5}$
Thus, we obtain the first Pythagorean Triple (3, 4, 5).

Similarly, when m=2 and n=3 we get the next Pythagorean Triple (5, 12, 13).

3. Four sides (or edges) Polygon - Quadrilateral

1) Properties

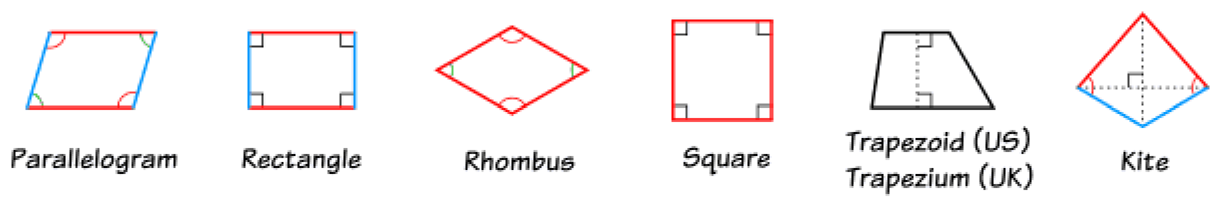
Four vertices (or corners). The interior angles add up to **360 degrees**:



Try drawing a quadrilateral, and measure the angles. They should add to **360°**

2) Types of Quadrilaterals

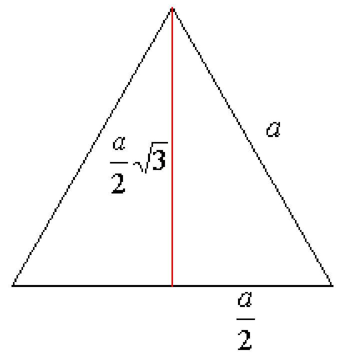
There are special types of quadrilateral:



Some types are also included in the definition of other types! For example a **square**, **rhombus** and **rectangle** are also ***parallelograms***.

4. Area of regular polygon

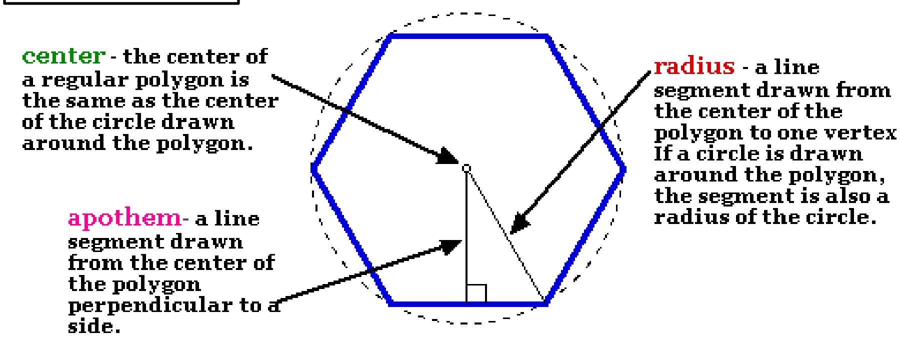
1) Equilateral triangle



Each angle is 60.
It can be divided into two 30°, 60°, 90° triangles.
If the side length / hypotenuse is a, the side that is opposite of 30° is always a/2, and the height / the side that is opposite of 60° is always $\sqrt{3}$ a/2.

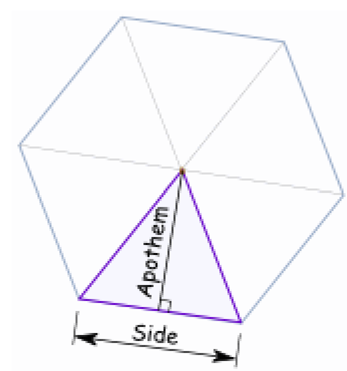
$$\boxed{\textbf{Area} = \frac{\sqrt{3}a^2}{4}}$$

2) Regular n-gon



Breaking into Triangles

We can learn a lot about regular polygons by breaking them into triangles like this:



Notice that: the "base" of the triangle is one side of the polygon.

the "height" of the triangle is the "Apothem" of the polygon

Now, the area of a triangle is half of the base times height, so:

Area of one triangle = base × height / 2 = side × apothem / 2

To get the area of the whole polygon, just add up the areas of all the little triangles ("n" of them):

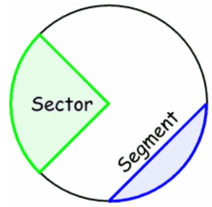
Area of Polygon = n × side × apothem / 2

And since the perimeter is all the sides = n × side, we get:

Area of Polygon = perimeter × apothem / 2 = ap/2

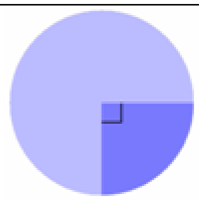
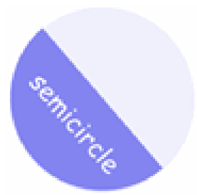
5. Circle Sector and Segment

Slices
There are two main "slices" of a circle:
The "pizza" slice is called a **Sector**.
And the slice made by a chord is called a **Segment**.



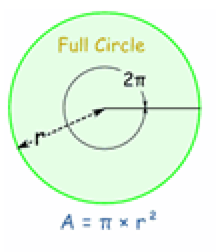
1) Common Sectors

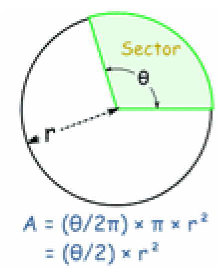
The Quadrant and Semicircle are two special types of Sector:

Quarter of a circle is called a **Quadrant**	Half a circle is called a **Semicircle.**
	

2) Area of a Sector

You can work out the Area of a Sector by comparing its angle to the angle of a full circle. Note: I am using radians for the angles in the diagram. Radian is another unit for measuring angles.





This is the reasoning:

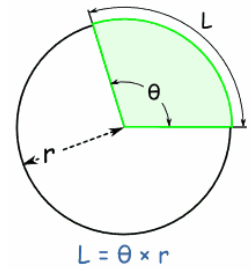
- A circle has an angle of $2\pi = 360°$ and an Area of: πr^2
- So a Sector with an angle of θ (instead of 360°) must have an area of: $(\theta/360) \times \pi r^2$

Area of Sector = $(\theta/360) \times \pi r^2$ ***(when θ is in degrees)***

3) Arc Length of Sector or Segment

By the same reasoning, the arc length (of a Sector or Segment) is:

Arc Length "L" = (θ/360) × 2πr ***(when θ is in degrees)***

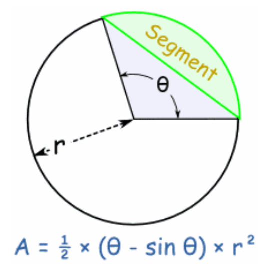


4) Area of Segment

The Area of a Segment is the area of a sector minus the triangular piece (shown in light blue here).

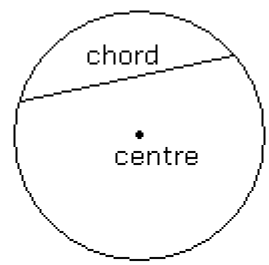
There is a lengthy derivation, but the result is a slight modification of the Sector formula:

Area of Segment = ½ × ((θ × π/180) - sin θ) × r² ***(when θ is in degrees)***

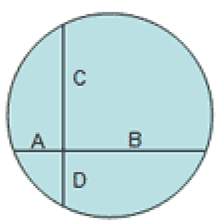


6. Intersecting Chord Theorem

A chord is a straight line joining 2 points on the circumference of a circle.



Intersecting Chord Theorem: When two chords intersect each other inside a circle, the products of their segments are equal.



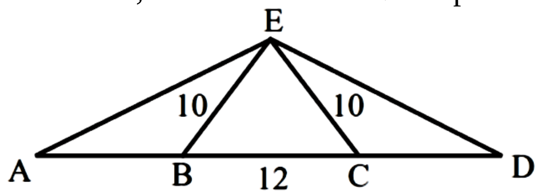
A x B = C x D

In-class questions

1. In rectangle ABCD, we have $AB = 8$, $BC = 9$, H is on BC with $BH = 6$, E is on AD with $DE = 4$, line EC intersects line AH at G, and F is on line AD with $GF \perp AF$. Find the length GF.

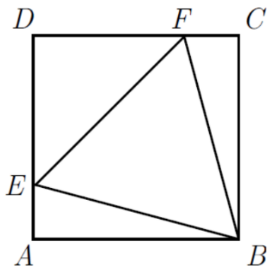
2. In trapezoid *ABCD*, *AB* and *CD* are perpendicular to *AD*, with $AB + CD = BC$, $AB < CD$, and $AD = 7$. What is $AB \cdot CD$?

3. Points $A, B, C,$ *and* D lie on a line, in that order, with $AB = CD$ *and* $BC = 12$. Point E is not on the line, and BE = CE = 10. The perimeter of △AED is twice the perimeter of △BEC. Find AB.

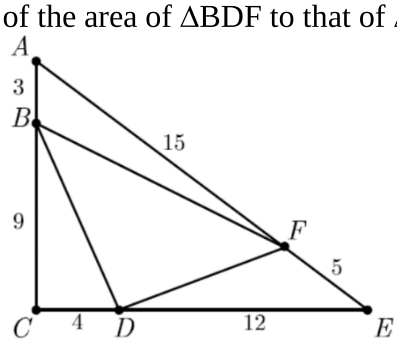


4. Let ΔXOY be a right-angled triangle with $\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ *and* $YM = 22$, find XY.

5. Points E and F are located on square ABCD so that ΔBEF is equilateral. What is the ratio of the area of ΔDEF to that of ΔABE?



6. In right triangle △ACE, we have $AC = 12$, $CE = 16$, *and* $EA = 20$. Points B, D, and F are located on AC, CE, and EA, respectively, so that $AB = 3$, $CD = 4$, *and* $EF = 5$. What is the ratio of the area of ΔBDF to that of ΔACE?



7. In ΔABC points D and E lie on BC and AC, respectively. If AD and BE intersect at T so that AT/DT = 3 and BT/ET = 4, what is CD/BD?