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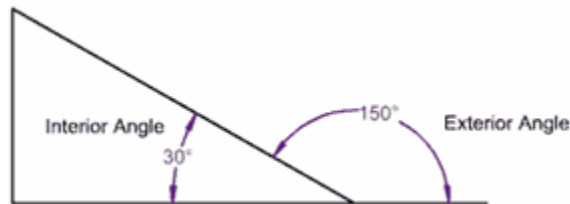
Part of the notes are the same as before, more concepts on circles are added. The questions in class and homework are different from before. Please review the notes and do the questions in class and homework.

Geometry 2

1. Interior and Exterior Angle

An Interior Angle is an angle inside a shape.

The Exterior Angle is the angle between any side of a shape, and a line extended from the next side.



Note: If you add up the Interior Angle and Exterior Angle you get a straight line, 180° .

1) Interior Angles sum of Polygons

The sum of the measures of the interior angles of a polygon with n sides is $(n-2)180$.

For examples:

- Triangle or ('3 - gon')
 - sum of interior angles: $(3-2)180 = 180^\circ$
- Quadrilateral which has four sides ('4 - gon')
 - sum of interior angles: $(4-2)180 = 360^\circ$
- Hexagon which has six sides ('6 - gon')
 - sum of interior angles: $(6-2)180 = 720^\circ$

An interior angle of a regular polygon with n sides is $\frac{(n-2) \times 180}{n}$.

Example:

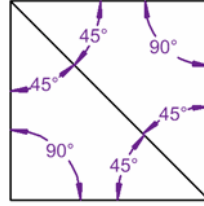
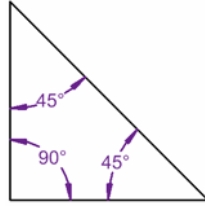
To find the measure of an interior angle of a regular octagon, which has 8 sides, apply the formula above as follows:

$$((8-2) \times 180) / 8 = 135^\circ$$

There are Two Triangles in a Square

The internal angles in this triangle add up to 180°

$$(90^\circ + 45^\circ + 45^\circ = 180^\circ)$$



... and for this square they add up to 360°

... because the square can be made from two triangles!

2) Exterior Angles sum of Polygons

The Exterior Angle is the angle between any side of a shape, and a line extended from the next side.

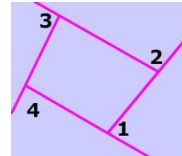
The sum of the measures of the exterior angles of a polygon, one at each vertex, is 360° .

For example:

A Polygon is any flat shape with straight sides.

The Exterior Angles of a Polygon add up to 360° .

$$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360^\circ$$



2. Pythagorean Triples

A "Pythagorean Triple" is a set of positive integers, **a**, **b** and **c** that fits the rule:

$$a^2 + b^2 = c^2$$

1) Endless

The set of Pythagorean Triples is endless.

It is easy to prove this with the help of the first Pythagorean Triple, (3, 4, and 5):

Let n be any integer greater than 1, then $3n$, $4n$ and $5n$ would also be a set of Pythagorean Triple.

This is true because: $(3n)^2 + (4n)^2 = (5n)^2$

Examples:

n	(3n, 4n, 5n)
2	(6,8,10)
3	(9,12,15)
...	... etc ...

So, you can make infinite triples just using the (3,4,5) triple.

2) Euclid's Proof that there are Infinitely Many Pythagorean Triples

However, Euclid used a different reasoning to prove the set of Pythagorean Triples is unending. The proof was based on the fact that the difference of the squares of any two consecutive numbers are always an odd number.

Examples:

$$2^2 - 1^2 = 4 - 1 = \mathbf{3} \text{ (an odd number),}$$

$$15^2 - 14^2 = 225 - 196 = \mathbf{29} \text{ (an odd number)}$$

3) Properties

It can be observed that a Pythagorean Triple always consists of:

- all even numbers, or
- two odd numbers and an even number.

A Pythagorean Triple can never be made up of all odd numbers or two even numbers and one odd number. This is true because:

- The square of an odd number is an odd number and the square of an even number is an even number.
- The sum of two even numbers is an even number and the sum of an odd number and an even number is an odd number.

Therefore, if one of a and b is odd and the other is even, c would have to be odd. Similarly, if both a and b are even, c would be an even number too!

4) Constructing Pythagorean Triples

It is easy to construct sets of Pythagorean Triples.

When **m** and **n** are any two positive integers ($m < n$):

$$a = n^2 - m^2$$

$$b = 2nm$$

$$c = n^2 + m^2$$

Then, a, b, and c form a Pythagorean Triple.

Example:

$$m=1 \text{ and } n=2$$

$$a = 2^2 - 1^2 = 4 - 1 = \mathbf{3}$$

$$b = 2 \times 2 \times 1 = \mathbf{4}$$

$$c = 2^2 + 1^2 = \mathbf{5}$$

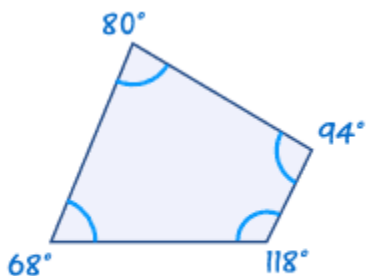
Thus, we obtain the first Pythagorean Triple (3,4,5).

Similarly, when $m=2$ and $n=3$ we get the next Pythagorean Triple (5,12,13).

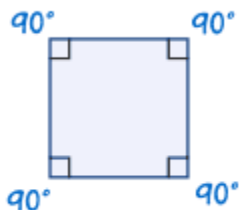
3. Four sides (or edges)

1) Properties

Four vertices (or corners). The interior angles add up to **360 degrees**:



$$68^\circ + 118^\circ + 94^\circ + 80^\circ = 360^\circ$$



$$4 \times 90^\circ = 360^\circ$$

Try drawing a quadrilateral, and measure the angles. They should add to **360°**

2) Types of Quadrilaterals

There are special types of quadrilateral:



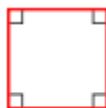
Parallelogram



Rectangle



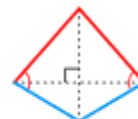
Rhombus



Square



Trapezoid (US)
Trapezium (UK)



Kite

Some types are also included in the definition of other types! For example a **square**, **rhombus** and **rectangle** are also **parallelograms**.

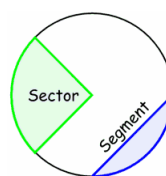
4. Circle Sector and Segment

Slices

There are two main "slices" of a circle:

The "pizza" slice is called a **Sector**.

And the slice made by a chord is called a **Segment**.



1) Common Sectors

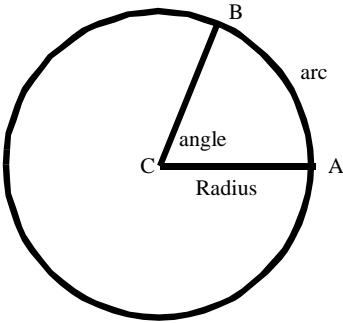
The Quadrant and Semicircle are two special types of Sector:

Quarter of a circle is called a Quadrant	Half a circle is called a Semicircle .

2) Area of a Sector

Radian Measure

- We can measure angles in several ways – one of which is degrees
- Another way to measure an angle is by means of radians
- One radian is defined as the measure of the angle subtended at the center of a circle by an arc equal in length to the radius of the circle



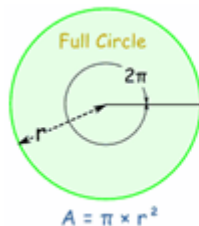
So in terms of radians, the formula is $\theta = \text{arc length}/\text{radius}$

θ for one revolution = Circumference/ $r = 2\pi r/r = 2\pi$ radians

So then an angle of $360^\circ = 2\pi$ **radians** or more easily, an angle of $180^\circ = \pi$ **radians**

You can work out the Area of a Sector by comparing its angle to the angle of a full circle.

Note: I am using radians for the angles.



This is the reasoning:

- A circle has an angle of 2π and an Area of: πr^2
- So a Sector with an angle of θ (instead of 2π) must have an area of: $(\theta/2\pi) \times \pi r^2$
- Which can be simplified to: $(\theta/2) \times r^2$

Area of Sector = $\frac{1}{2} \times \theta \times r^2$ (when θ is in radians)

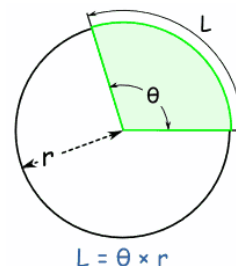
Area of Sector = $\frac{1}{2} \times (\theta \times \pi/180) \times r^2$ (when θ is in degrees)

4) Arc Length of Sector or Segment

By the same reasoning, the arc length (of a Sector or Segment) is:

Arc Length “L” = $\theta \times r$ (when θ is in radians)

Arc Length “L” = $(\theta \times \pi/180) \times r$ (when θ is in degrees)



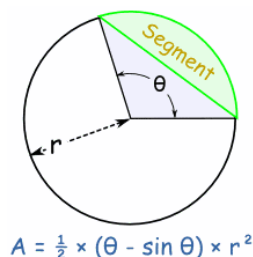
4) Area of Segment

The Area of a Segment is the area of a sector minus the triangular piece (shown in light blue here).

There is a lengthy derivation, but the result is a slight modification of the Sector formula:

Area of Segment = $\frac{1}{2} \times (\theta - \sin \theta) \times r^2$ (when θ is in radians)

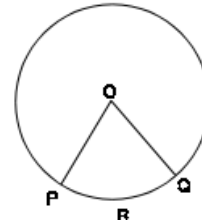
Area of Segment = $\frac{1}{2} \times ((\theta \times \pi/180) - \sin \theta) \times r^2$ (when θ is in degrees)



5 Angles in a Circle

Central Angle - The angle made at the centre of a circle by the radii at the end points of an arc (or a chord) is called the **central angle** or angle subtended by an arc (or chord) at the centre.

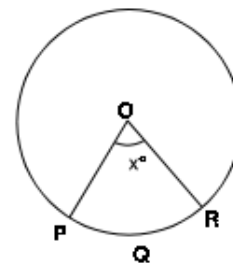
In Figure, $\angle POQ$ is the central angle made by arc PRQ.



The length of an arc is closely associated with the central angle subtended by the arc. Let us define the “degree measure” of an arc in terms of the central angle.

The degree measure of a minor arc of a circle is the measure of its corresponding central angle.

In Figure 19.2, Degree measure of PQR = x°



Relationship between length of an arc and its degree measure.

$$\text{Length of an arc} = \text{circumference} \times \frac{\text{degree measure of the arc}}{360^\circ}$$

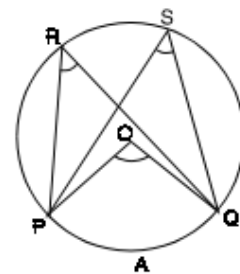
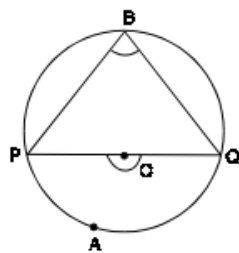
Inscribed angle: The angle subtended by an arc (or chord) on any point on the remaining part of the circle is called an inscribed angle.

Theorem: The angle subtended at the centre of a circle by an arc is double the angle subtended by it on any point on the remaining part of the circle.

In figure, $\angle POQ = 2\angle PRQ$

Theorem: Angle in a semicircle is a right angle.

$\angle PBQ = 90^\circ$ since PQ is the diameter.



Theorem: Angles in the same segment of a circle are equal.

Given: A circle with centre O and the angles $\angle PRQ$ and $\angle PSQ$ in the same segment formed by the chord PQ (or arc PAQ)

To prove: $\angle PRQ = \angle PSQ$

Construction: Join OP and OQ.

Proof: As the angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle, therefore we have

$$\angle POQ = 2\angle PRQ \dots (i) \text{ and } \angle POQ = 2\angle PSQ \dots (ii)$$

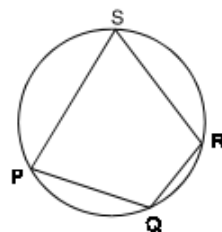
From (i) and (ii), we get $2\angle PRQ = 2\angle PSQ$

$$\therefore \angle PRQ = \angle PSQ$$

We take some examples using the above results

6. Cyclic Quadrilateral

A quadrilateral is said to be a **cyclic quadrilateral** if there is a circle passing through all its four vertices. For example, figure to the right shows a cyclic quadrilateral PQRS.



Theorem. Sum of the opposite angles of a cyclic quadrilateral is 180° .

Given: A cyclic quadrilateral ABCD.

To prove: $\angle BAD + \angle BCD = \angle ABC + \angle ADC = 180^\circ$

Construction: Draw AC and DB

Proof: $\angle ACB = \angle ADB$ and $\angle BAC = \angle BDC$

[Angles in the same segment]

$$\therefore \angle ACB + \angle BAC = \angle ADB + \angle BDC = \angle ADC$$

Adding $\angle ABC$ on both the sides, we get

$$\angle ACB + \angle BAC + \angle ABC = \angle ADC + \angle ABC$$

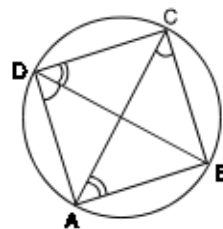
But $\angle ACB + \angle BAC + \angle ABC = 180^\circ$ [Sum of the angles of a triangle]

$$\therefore \angle ADC + \angle ABC = 180^\circ$$

$$\therefore \angle BAD + \angle BCD = 360^\circ - (\angle ADC + \angle ABC) = 180^\circ .$$

Hence proved.

Converse of this theorem is also true.



If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.

◆ **Things you should know:**

SAS Postulate

If two triangles have two sides and the included angle of one triangle congruent respectively to two sides and the included angle of the other triangle, then the triangles are congruent.

ASA Postulate

If two triangles have two angles and the included of one triangle congruent respectively to two angles and the included side of the other triangle, then the triangle are congruent

SSS Postulate

If two triangles have three sides of one triangle congruent respectively to the three sides of the other triangle, then the triangles are congruent

HA Postulate

If two right triangles have the hypotenuse and an acute angle of one triangle congruent respectively to the hypotenuse and an acute angle of the other, then the triangles are congruent

SAA Corollary

If two triangles have a side and two angles of one congruent respectively to a side and two angles of the other, then the triangles are congruent.

Angle-Sum Theorem for Polygons

The formula for the sum, S , of the measures of the angles of a polygon of n sides is
 $S = (n - 2)180$.

Exterior Angle-Sum Corollary for triangles

The measure of one exterior angle of a triangle equals the sum of the measures of its remote Interior angles

Theorem The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.

Theorem If two sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

Theorem If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram

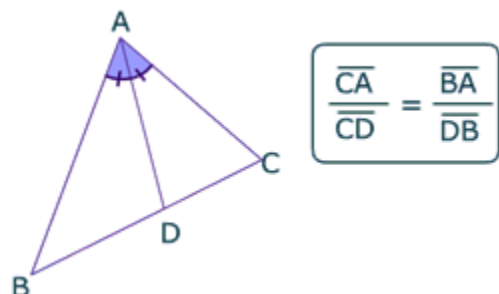
Theorem The diagonals of a rectangle are congruent.

Theorem The diagonals of a rhombus are perpendicular to each other

Theorem Each diagonal of a rhombus bisects a pair of opposite angles

Angle Bisector Theorem

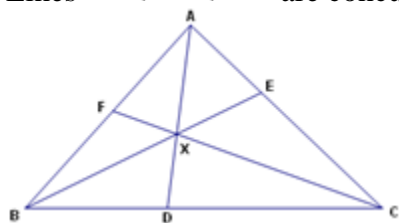
An angle bisector of an angle of a triangle divides the opposite side in two segments that are proportional to the other two sides of the triangle.



Ceva's Theorem

Let ABC be a triangle, and let D, E, F be points on lines BC, CA, AB , respectively.

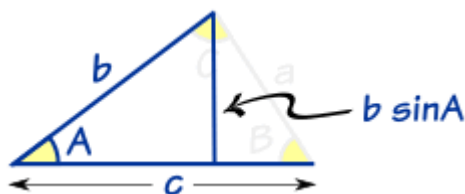
Lines AD, BE, CF are concurrent (they intersect at a point) iff (if and only if)



$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

Area of Triangle

Area = $\frac{1}{2} \times \text{base} \times \text{height}$



In this triangle:

- the base is: c
- the height is: $b \times \sin A$

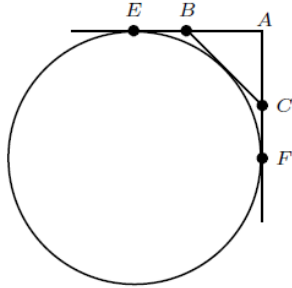
Putting that together gets us: Area = $\frac{1}{2} \times (c) \times (b \times \sin A)$

Which is (more simply): **Area = $\frac{1}{2}bc \sin A$**

In-class questions

1. Three circles with different diameters are mutually tangent externally. Their centres form a triangle with side lengths 5 cm, 6 cm, and 9 cm. What is the total area of the three circles?

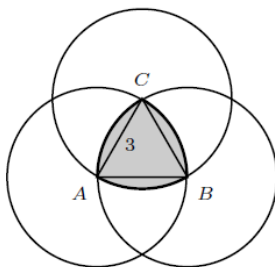
2. In the diagram, a circle is tangent to the hypotenuse of the isosceles right triangle ABC. Side AB is extended and is tangent to the circle at E. Side AC is extended and is tangent to the circle at F. If the area of triangle ABC is 9, then what is the area of the circle?



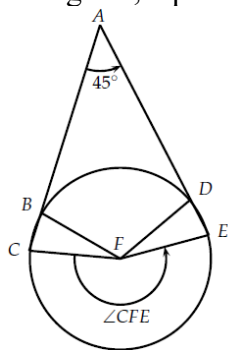
3. If the area of a square inscribed in a circle is 15 cm^2 , then what is the area of the square inscribed in a semicircle of the same circle, in square centimetres?

4. A circle is inscribed in a right triangle with sides a , b , and c , where c is the hypotenuse, as shown in the diagram. What is the radius of the circle?

5. Let ABC be an equilateral triangle with sides of length 3. Let AC be the shorter circular arc with centre B joining A and C, BC be the shorter circular arc with centre A joining B and C, and AB be the shorter circular arc with centre C joining A and B. See the diagram. What is the shaded area contained by the three arcs?



6. In the diagram F is the centre of the circle and AC and AE are secants intersecting the circle at B and D , respectively. Given that $\angle CAE = 45^\circ$, $\angle BFC = 25^\circ$, and $BC = DE$, it follows that $\angle CFE$, measured in degrees, equals what?



7. A rhombus, $ABCD$, has sides of length 1. A circle with centre A passes through C (the opposite vertex.) Likewise, a circle with centre B passes through D . If the two circles are tangent to each other, find the area of the rhombus.

