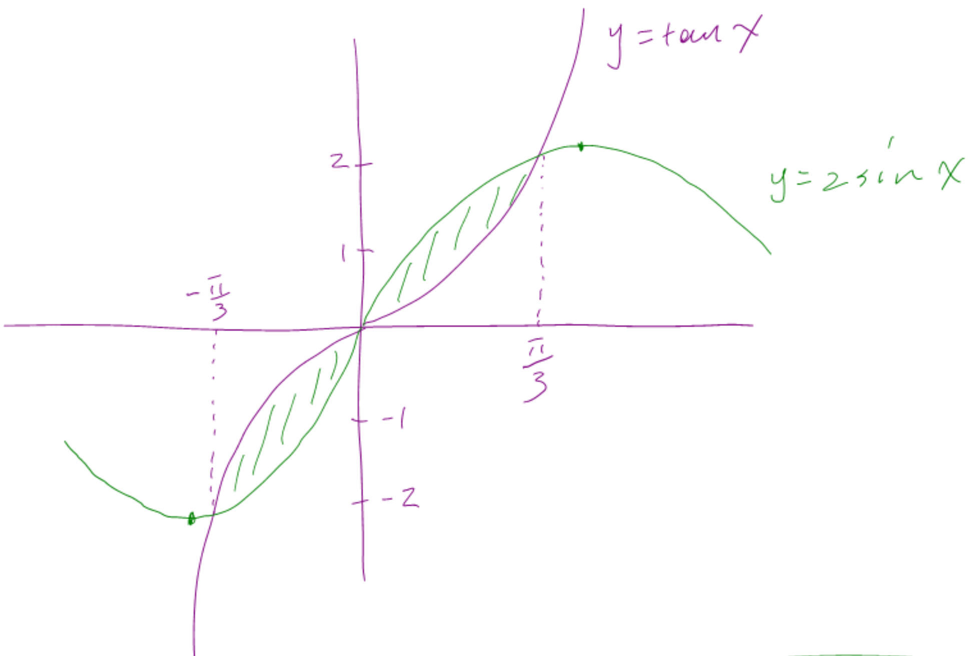
AP Calculus Class 15

2. d) $y = \tan x$, $y = 2\sin x$, $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$



$$A = 2\int_0^{\frac{\pi}{3}} 2\sin x - \tan x \, dx$$

$$= 2\left[2(-\cos x) - (-\ln|\cos x|)\right]_0^{\frac{\pi}{3}}$$

$$= 2\left[-1 + \ln\left(\tfrac{1}{2}\right)\right] - 2\left[-2(1) + \ln 1\right]$$

$$= \underline{2 - 2\ln 2.}$$

$$\int \frac{\sin x}{\cos x} \, dx$$

let $u = \cos x$

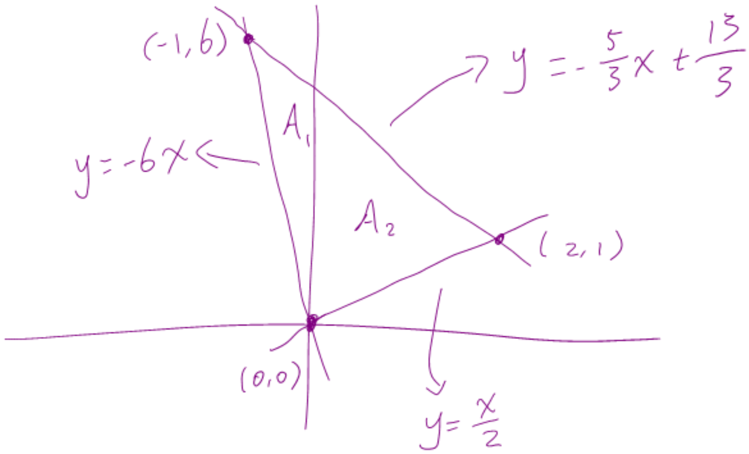
$du = -\sin x \, dx$

$-du = \sin x \, dx$

$$-\int \frac{1}{u} \, du = -\ln|u|$$

$$\int \tan x = -\ln|\cos x| + C$$

3.



$A = A_1 + A_2$

$$= \int_{-1}^{0} \left(-\frac{5}{3}x+\frac{13}{3}\right) - \underbrace{(-6x)}_{\frac{18}{3}x} dx + \int_{0}^{2} \left(\underbrace{-\frac{5}{3}x}_{-\frac{10}{6}x}+\frac{13}{3}\right) - \underbrace{\frac{1}{2}x}_{\frac{3}{6}x} dx$$

$$= \int_{-1}^{0} \frac{13}{3}x + \frac{13}{x} dx + \int_{0}^{2} -\frac{13}{6}x + \frac{13}{3} dx$$

$$= \frac{13}{3}\left[\frac{1}{2}x^2 + x\right]_{-1}^{0} + \frac{13}{3}\left[-\frac{1}{4}x^2 + x\right]_{0}^{2}$$

$$= \frac{13}{3}\cdot\frac{1}{2} + \frac{13}{3} = \frac{13}{2}$$

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1. g) $\int_0^1 \frac{\ln x}{\sqrt{x}} dx$ V.A.: $x=0$.

$$\int_0^1 \frac{\ln x}{\sqrt{x}} dx = \lim_{t\to 0^+} \int_t^1 \frac{\ln x}{\sqrt{x}} dx.$$

Apply IBP. $\int_t^1 \frac{\ln x}{\sqrt{x}} dx$ let $f=\ln x$ $g'=\frac{1}{\sqrt{x}}$

$f'=\frac{1}{x}$ $g=2\sqrt{x}$

$$\int_t^1 \frac{\ln x}{\sqrt{x}}\,dx = \left[2\sqrt{x}\ln x\right]_t^1 - 2\int_t^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \left[2\sqrt{x}\ln x\right]_t^1 - 4\left[\sqrt{x}\right]_t^1$$

$$= -2\sqrt{t}\ln t - 4 + 4\sqrt{t}$$

$$\Rightarrow \int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left[-2\sqrt{t}\ln t - 4 + 4\sqrt{t}\right].$$

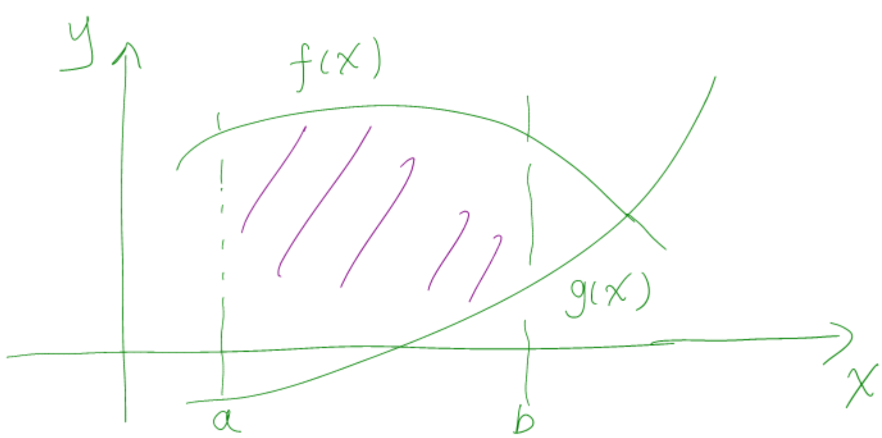
$$= 0 - 4 = -4.$$

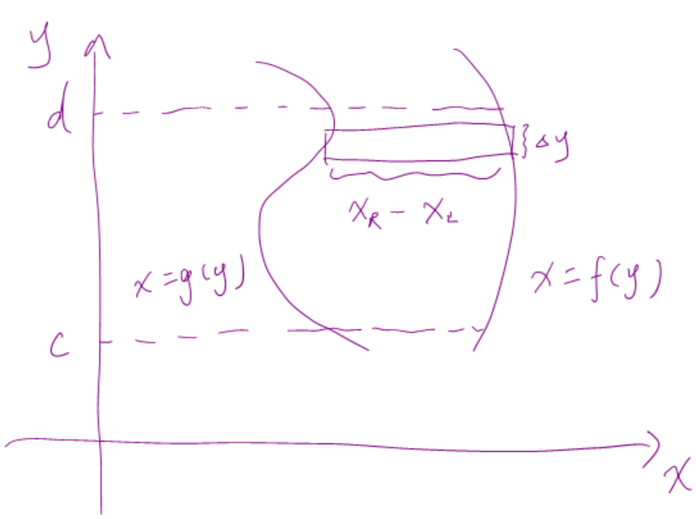
$$\sqrt{t}\ln t \;\to\; \frac{\ln t}{\frac{1}{\sqrt{t}}} \quad \begin{matrix}\to\infty \\ \to\infty\end{matrix}$$

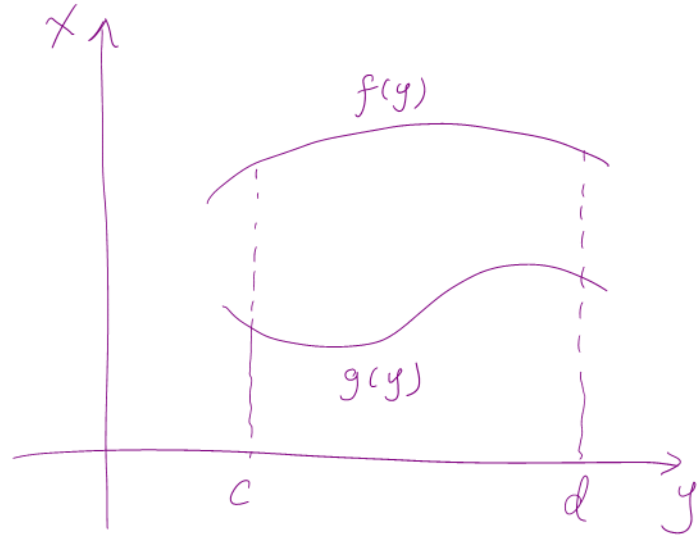
$$-2t^{\frac{1}{2}} \leftarrow \frac{\frac{1}{t}}{\frac{1}{-2t^{\frac{3}{2}}}}$$

$\Rightarrow$ Convergent.

Area Between Curves.

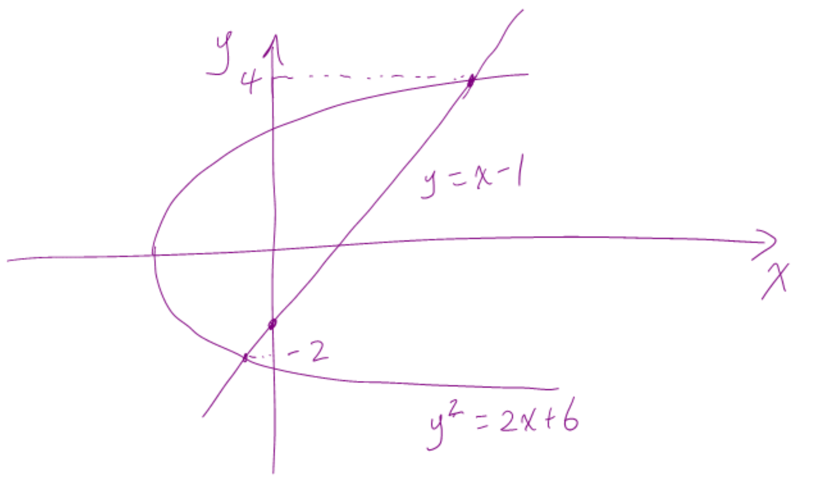






$$A = \int_c^d (x_R - x_L)\, dy$$

Example: Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



$y = x - 1 \Rightarrow x = y + 1$

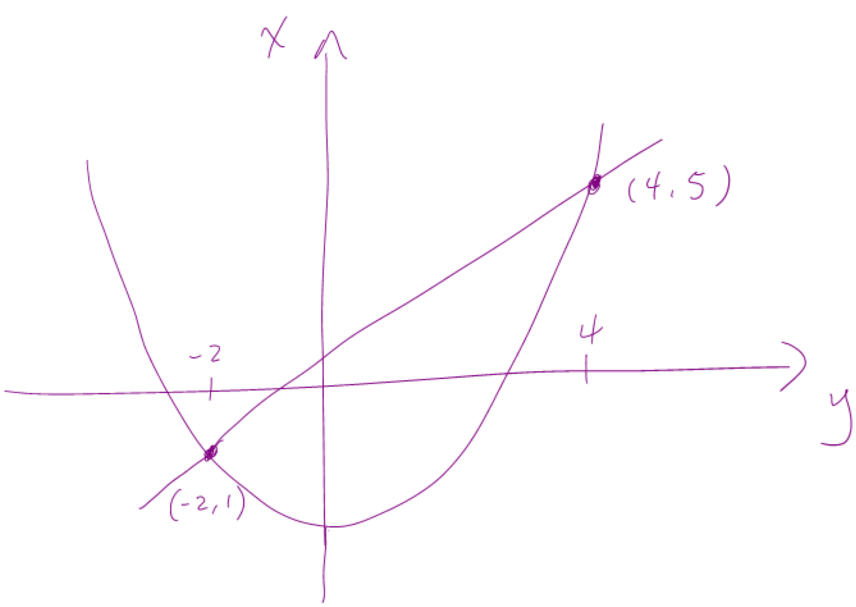
$y^2 = 2(y+1) + 6$ $\qquad x = \frac{1}{2}y^2 - 3$

$y^2 = 2y + 8.$

$\Rightarrow y^2 - 2y - 8 = 0.$

$(y+2)(y-4) = 0.$

$\Rightarrow y = -2 \qquad y = 4.$



$$A = \int_{-2}^{4} (x_R - x_L)\, dy$$

$$= \int_{-2}^{4} \left((y+1) - \left(\tfrac{1}{2}y^2 - 3\right) \right) dy$$

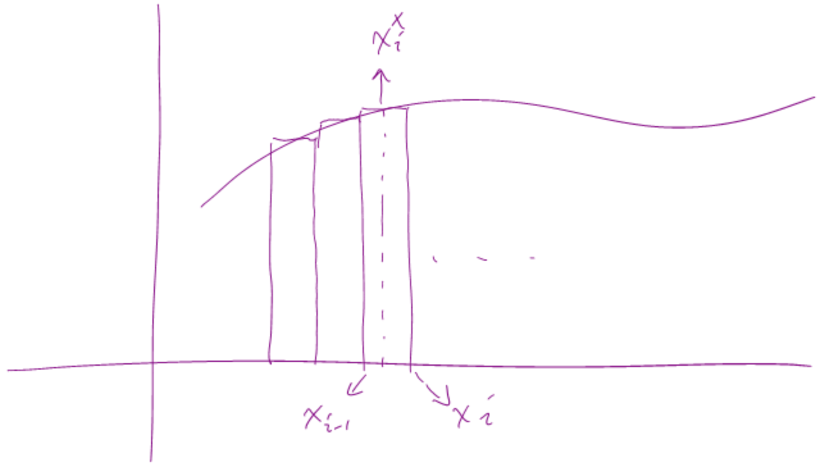
$$= \int_{-2}^{4} \left(-\tfrac{1}{2}y^2 + y + 4\right) dy$$

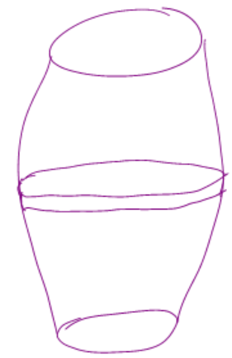
$$= \tfrac{1}{2}\left(\frac{y^3}{3} + \frac{y^2}{2} + 4y\right)\Bigg]_{-2}^{4}$$

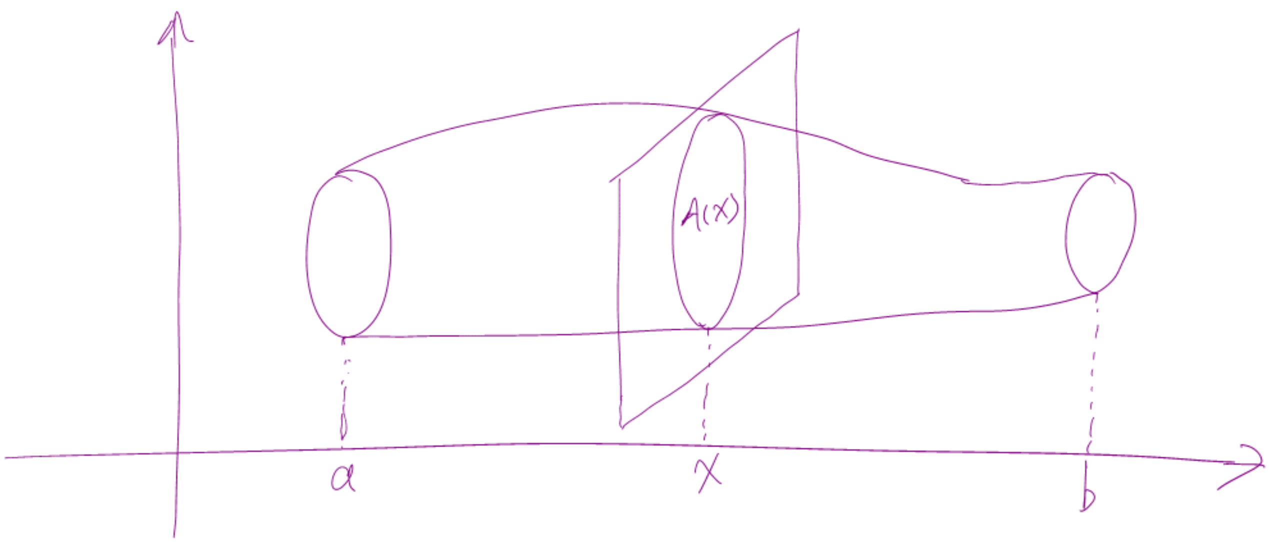
$$= -\tfrac{1}{6}(64) + 8 + 16 - \left(\tfrac{4}{3} + 2 - 8\right)$$

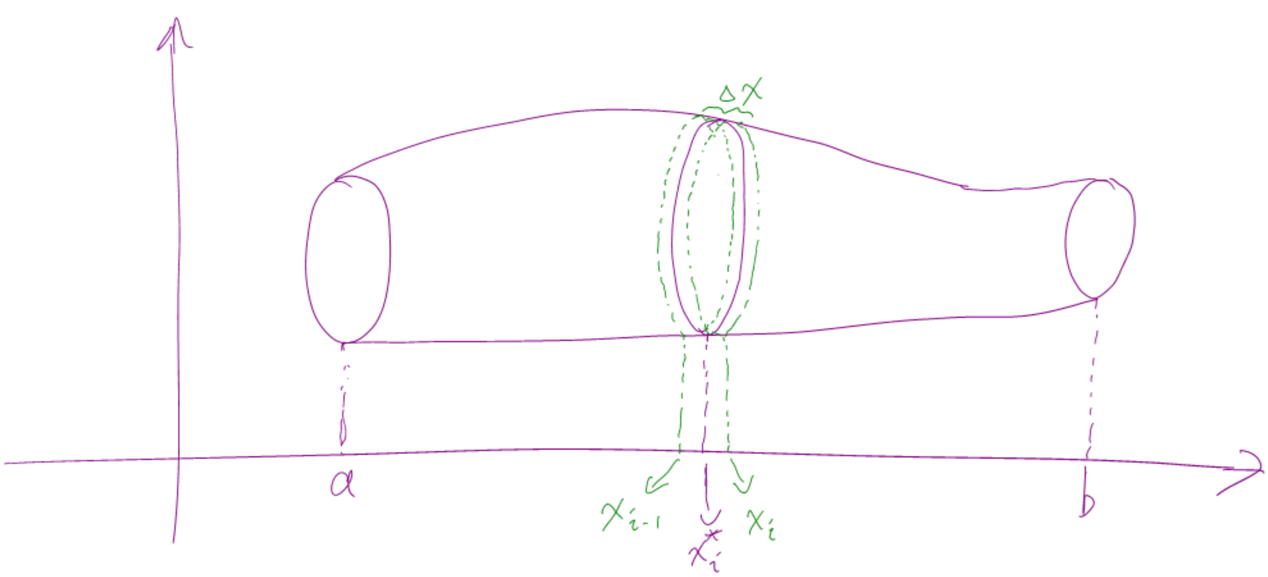
$$= 18$$

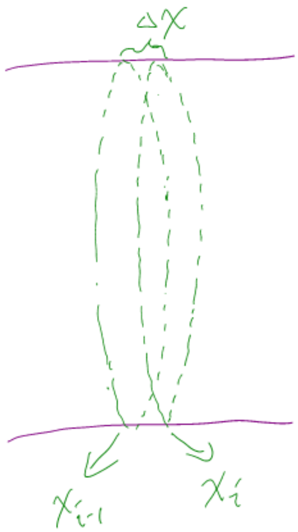
Volume.











slice the volume into pieces of equal width Δx.

choose a sample point x_i^* between $[x_{i-1}, x_i]$.

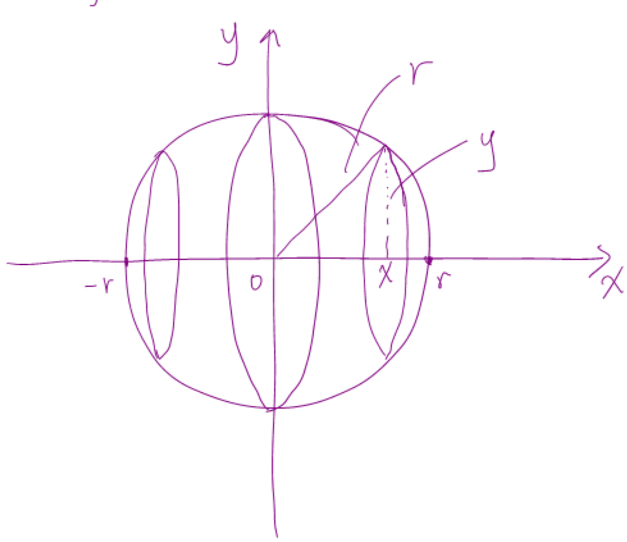
$$V \approx A(x_i^*)\Delta x$$

The total volume, $V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x \qquad n \to \infty$.

Def^n of volume.

$$\boxed{V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx}$$

Example: show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.



$$x^2 + y^2 = r^2 \rightarrow y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

$$y = \sqrt{r^2 - x^2}$$

$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

$$a = -r \qquad b = r.$$

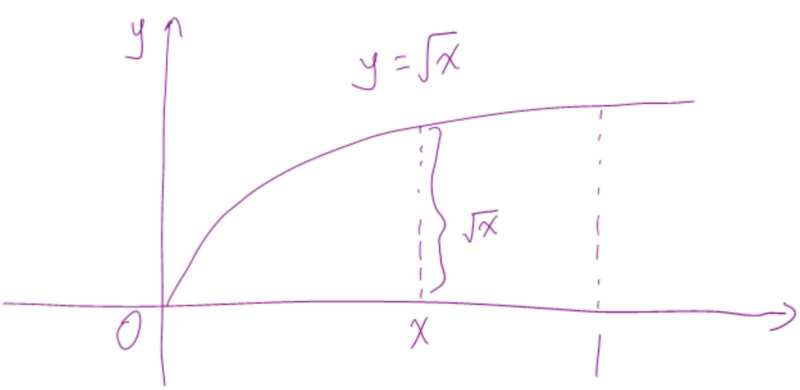
Apply the definition

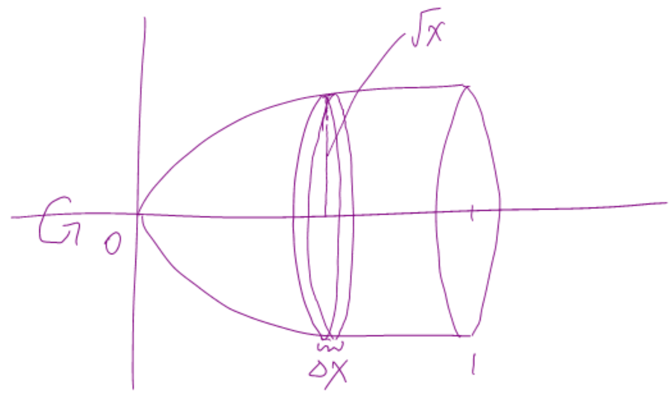
$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx$$

$$= 2\pi \int_0^r r^2 - x^2\,dx = 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r$$

$$= 2\pi \left(r^3 - \frac{r^3}{3} \right) = \frac{4}{3}\pi r^3$$

Example: Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.





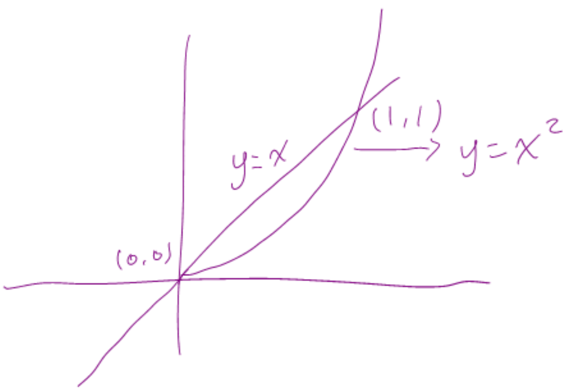
Cross-sectional area

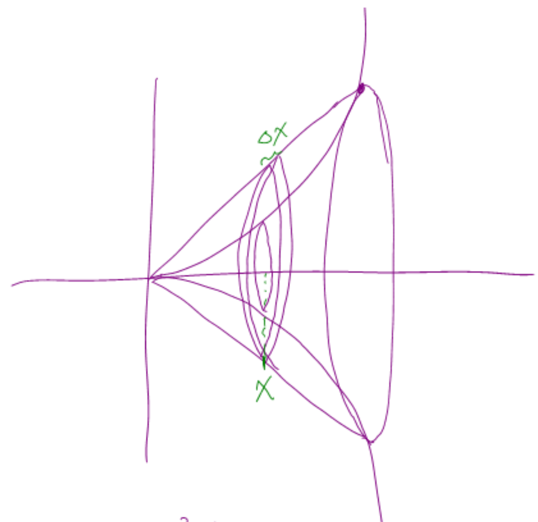
$$A(x) = \pi \sqrt{x}^2 = \pi x$$

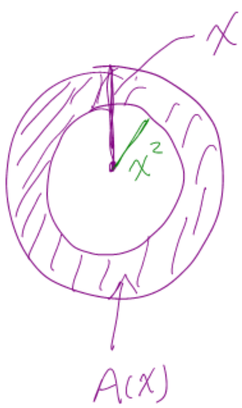
The volume for one cylinder is $A(x)\Delta x = \pi x \Delta x$.

$$\Rightarrow V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \pi \frac{x^2}{2} \Big]_0^1 = \frac{\pi}{2}.$$

Example: the region R is enclosed by the curves $y=x$ and $y=x^2$ is rotated around the x-axis. Find the volume of the resulting solid.







$$A(x) = \pi\left(x^2 - (x^2)^2\right)$$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx$$

$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2\pi}{15}$$

The above three examples show solids that are called <u>solids of revolution</u>

$$V = \int_a^b A(x)\,dx$$

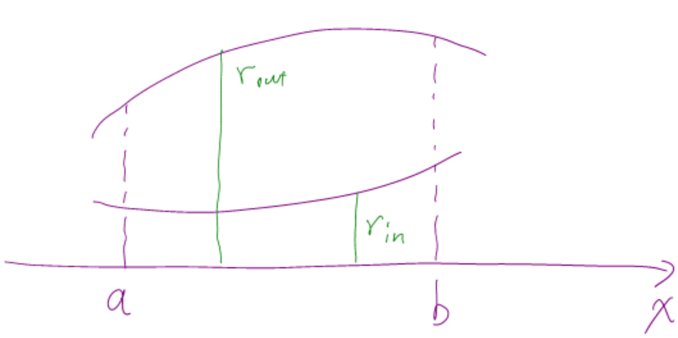
① For the 2nd example with a solid cylinder,

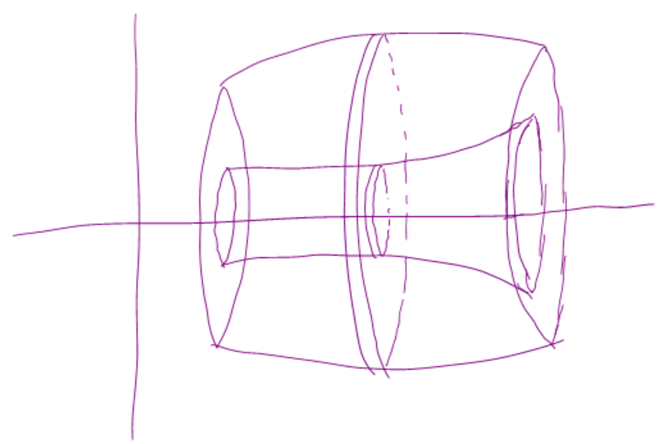
$A = \pi (\text{radius})^2$.

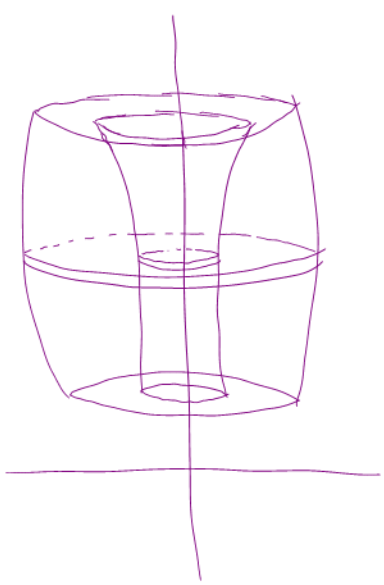
② For the 3rd example with a hollow cylinder,

$A = \pi (\text{radius})^2$

$= \pi (\text{outter radius})^2 - \pi (\text{inner radius})^2$.







Def^n of Volume.

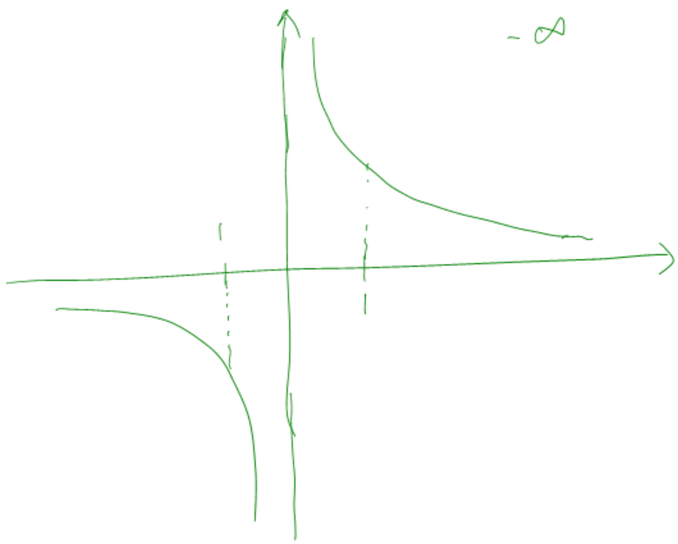
Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis is $A(x)$, where A is a continuous fun^n, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*) \Delta x = \int_a^b A(x)\,dx.$$

$$\int_{-\infty}^{\infty} \frac{1}{x}\,dx = \underbrace{\int_{-\infty}^{0} \frac{1}{x}\,dx}_{\downarrow\ -\infty} + \underbrace{\int_{0}^{\infty} \frac{1}{x}\,dx}_{\downarrow\ \infty}$$

$\infty + \infty \neq 2\infty$

$-\infty + \infty \neq 0.$



Divergent.

$$\int_{-1}^{1} \frac{1}{x}\,dx = \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx.$$

$$= \int_{1}^{0} \frac{1}{-x}(-dx) + \int_{0}^{1} \frac{1}{x}\,dx$$

$$= \int_{1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$$

$$= -\int_{0}^{1} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$$

$$= 0.$$

$\int_{1}^{0} \frac{1}{-x}(-dx)$

Let $u = -x$

$du = -dx$

$u(0) = 0$

$u(1) = -1$

$$\int_{-1}^{1} \frac{1}{x}\,dx = -\int_0^1 \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx$$

$$\int_0^2 \frac{\frac{1}{2}}{\frac{1}{2}x}\,dx$$

$$= -\int_0^1 \frac{1}{x}\,dx + \int_0^2 \frac{\frac{1}{2}}{\frac{1}{2}x}\,dx$$

let $u = \frac{1}{2}x$

$du = \frac{1}{2}dx$

$u(0) = 0$

$u(2) = 1$

$$\int_0^1 \frac{1}{u}\,du$$

$$= -\int_0^1 \frac{1}{x}\,dx + \int_0^2 \frac{1}{x}\,dx$$

$$= -\int_0^1 \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx + \int_1^2 \frac{1}{x}\,dx$$

$$= \int_1^2 \frac{1}{x}\,dx$$

$$= \ln 2 - \ln 1 = \ln 2.$$

Cauchy principle value.

singularity: "special discontinuities"

$$\int_{-\infty}^{\infty} \frac{1}{x}\,dx = \lim_{a\to\infty} \lim_{\varepsilon\to 0} \left[\int_{-a}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{a} \frac{1}{x}\,dx\right] = 0.$$