AP Calculus Lesson Eight Notes

Chapter 4 Antiderivatives and the Definite Integral

4.1 Antiderivatives
4.2 Area

4.1 Antiderivatives

In chapter 3 certain problems were stated in the form "*give a function g, find the derivative g*', *find the function g*." an equivalent way of stating this converse problem is, "Give a function *f*, find a function *F* such that *F*'=*f*." as a simple illustration, suppose $f(x) = 8x^3$. In this case it is easy to find a function *F* that has *f* as its derivative. We know that differentiating a power of x *reduces* the exponent by 1 and therefore to obtain F we must *increase* the given exponent by 1. Thus $F(x) = ax^4$ for some number *a*. differentiating, we obtain $F'(x) = 4ax^3$ and, in order for this to equal f(x), *a* must equal 2. Consequently, the function F defined by $F(x) = 2x^4$ has the desired property. According to the next definition, *F* is called an *antiderivative of f.*

Definition (4.1.1)

A function *F* is an **antiderivative** of a function *f* if *F*'=*f*

For convenience, we shall use the phrase "*f(x)* is an antiderivative of *f(x)"* synonymously with "*F* is an antiderivative of *f*." the domain of an antiderivative will not usually be specified. It follows from the Fundamental Theorem of Calculus in the next chapter that a suitable domain is any closed interval [*a,b*] on which *f* is continuous.

Antiderivatives are never unique. In deed, since the derivative of a constant is zero, it follows that if *F* is an antiderivative of *f,* so is the function *G* defined by $G(x) = F(x) + C$, for every number *C*. for example, if $f(x)=8x^3$, then functions defined by expressions such as $2x^4+7$, $2x^4-\sqrt{3}$, and $2x^4+\frac{2}{5}$ are antiderivatives of *f*. The next theorem brings out the fact that functions of this type are the only possible antiderivatives of *f.*

Theorem(4.1.2)

If F_1 and F_2 are differentiable functions such that $F'_1=F'_2(x)$ for all *x* in a closed interval [*a,b*], then $F_2(x)=F_1(x)+C$ for some number *C* and all *x* in [*a,b*].

Proof

If we define the function g by $g(x) = F_2(x) - F_1(x)$

Then, $$g'(x) = F'_2(x) - F'_1(x) = 0$$

For all x in [a,b]. if x is any number such that $a < x \le b$, then applying the Mean Value Theorem to the function g and the closed interval [a,b]. there exists a number z in the open interval (a,x) such that

$$g(x) - g(a) = g'(z)(x-a) = 0 \bullet (x-a) = 0$$

Hence $g(x)=g(a)$ for all x in [a,b]. Substitution in the first equation stated in the proof gives us

$$g(a) = F_2(x) - F_1(x)$$

Adding $F_1(x)$ to both sides we obtain the desired conclusion, with $C = g(a)$.

If F_1 and F_2 are antiderivatives of the same function *f*, then $F'_1 = f = F'_2$ and hence, by the theorem just proved, $F_2(x)=F_1(x)+C$ for some *C*. in other words, if *F(x)* is an antiderivative of *f(x)*, then every other antiderivative has the form *F(x) + C* where *C* is an arbitrary constant (that is, an unspecified real number). We shall refer to *F(x) + C* as the **most general antiderivative** of *f(x)*.

Theorem (4.1.3)

If $f'(x)=0$ for all x in [a,b], then f is a constant function on [a,b].

Proof

Denote f by F_2 and let the function F_1 be defined by $F_1(x)=0$ for all *x*. Since $F'_1 = 0$ and $F'_2 = f'(x) = 0$, we see that $F_1'(x) = F_2'(x)$ for all x in [a,b]. Applying Theorem (4.1.2), there is a number *C* such that $F_2(x)=F_1(x)+C$; that is, $f(x)=C$ for all x in [a,b]. This completes the proof.

Rules for derivatives may be used to obtain formulas for antiderivatives as in the proof of the following g important result.

Power Rule for Antidifferentiation (4.1.4)

Let *a* be any real number, *r* any rational number different from -1, and *C* an arbitrary constant.

If $f(x) = ax^r$, then $F(x) = (\frac{a}{r+1})x^{r+1} + C$

Is the most general antiderivative of *f(x)*.

Proof

It is sufficient to show that $F'(x) = f(x)$. This fact follows readily from the Power Rule for Derivatives, since

$$F(x) = (\frac{a}{r+1})(r+1)x^r = ax^r = f(x)$$

Example 1:

Find the most general antiderivative of

(a) $4x^5$ (b) $7/x^3$ (c) $\sqrt[3]{x^2}$

Solution

(a) using the Power Rule with a=4 and r=5 gives us the antiderivative $\frac{4}{6}x^6 + C$, or $\frac{2}{3}x^6 + C$

(b) Writing $7/x^3$ as $7x^{-3}$ and using Rule (4.1.4) with a=7 and r=-3 leads to $\frac{7}{-2}x^{-2} + C$, or $-\frac{7}{2x^2} + C$.

(c) Since $\sqrt[3]{x^2} = x^{2/3}$ we may apply the Power Rule with a=1 and r=2/3, obtaining $\frac{1}{\frac{5}{3}}x^{5/3} + C$, or $\frac{3}{5}x^{5/3} + C$

To avoid algebraic errors, *it is highly recommended to check solutions* of problems involving Antidifferentiation by differentiating the final antiderivative. In each case the given expression should be obtained.

Theorem(4.1.5)

If F(x) and G(x) are antiderivatives of f(x) and g(x), respectively, then
i) F(x) +G (x) is an antiderivative of f(x) + g(x).
ii) cF(x) is an antiderivative of cf(x), where c is any real number.

Proof

By hypothesis, $D_x[F(x)] = f(x)$ and $D_x[G(x)] = g(x)$. Hence,

$$D_x[F(x) + G(x)] = D_x[F(x)] + D_x[G(x)] = f(x) + g(x).$$

This proves (i). to prove (ii) we merely note that

$$D_x[cF(x)] = cD_x[F(x)] = cf(x)$$

Part (i) of the preceding theorem can be extended to any finite sum of functions. This fact may be stated as follow: "*an antiderivative of a sum is the sum of the antiderivatives.*" As usual, when working with several functions we assume that the domain is restricted to the intersection of the domains of the individual functions. A similar result is true for differences.

Example 2

Find the most general antiderivative F(x) of

$$f(x) = 3x^4 - x + 4 + (5/x^3)$$

Solution

Writing the last term of f(x) as $5x^{-3}$ and applying the Power Rule of Antidifferentiation to each term gives us

$$F(x) = \tfrac{3}{5}x^5 - \tfrac{1}{2}x^2 + 4x - \tfrac{5}{2}x^{-2} + C$$

It is unnecessary to introduce an arbitrary constant for each of the four antiderivatives, since they could be added together to produce the one constant C.

The next theorem provides formulas for antiderivatives of the sine and cosine functions.

Theorem(4.1.6)

In each of the following statements, F(x) is the most general antiderivative of f(x), and k is any nonzero real number.

i) if $f(x) = \sin kx$, then $F(x) = -\dfrac{1}{k}\cos kx + C$

ii) if $f(x) = \cos kx$, then $F(x) = \dfrac{1}{k}\sin kx + C$

Proof

It is sufficient to show that F'(x)=f(x). Thus, for part (i) we have

$$F'(x) = D_x[-\frac{1}{k}\cos kx + C]$$
$$= -\frac{1}{k} D_x \cos kx + 0$$
$$= -\frac{1}{k}(-\sin kx)(k) = \sin kx$$

The proof of (ii) is left to the reader.

Example 3

Find the most general antiderivative of

(a) $\sin 5x - \cos x$ (b) $\sqrt{x} + 4\cos 3x$

Solution

(a) Using (4.5) and (4.6) we obtain

$-\frac{1}{5}\cos 5x - \sin x + C$

Where C is an arbitrary constant.

(b) By the Power Rule (4.1.4),

An antiderivative of 4cos3x is $4[\frac{1}{3}\sin 3x] = \frac{4}{3}\sin 3x$.
Thus, the most general antiderivative of $\sqrt{x} + 4\cos 3x$ is

$$\frac{2}{3}x^{3/2} + \frac{4}{3}\sin 3x + C$$

Where C is an arbitrary Constant.

Equations that involve derivatives of an unknown function *f* are very common in mathematical applications. Such equations are called **differential equations.** The function *f* is called a **solution** of the differential equation. To **solve** a differential equation means to find all solutions. Sometimes, in addition to the differential equation we may know certain values of *f*, called **boundary values,** as illustrated in the next example.

Example 4

Solve the differential equation $f'(x)=6x^2+x-5$ with boundary value $f(0)=2$.

Solution

From our discussion of antiderivatives,

$$f(x) = 2x^3 + \tfrac{1}{2}x^2 - 5x + C$$

For some number C. letting x=0 and using the giving boundary value, we obtain

F(0)=0+0-0+C=2

And hence C=2. Consequently, the solution *f* of the differential equation with the given boundary value is $f(x) = 2x^3 + \tfrac{1}{2}x^2 - 5x + 2$

If a point P is moving rectilinearly, then its position function s is an antiderivative of its velocity function, that is, s'(t)=v(t). similarly, since v'(t)=a(t), the velocity function is an antiderivative of the acceleration function. If the velocity or acceleration function is known, then given sufficient boundary conditions it is possible to determine the position function. The particular boundary conditions corresponding to t=0 are sometimes called the **initial conditions.**

Example 5

A motorboat moves away from a dock along a straight line, with an acceleration at time t given by a(t)=12t-4ft/sec^2. if, at time t=0, the boat had a velocity of 8 ft/sec and was 15 ft from the dock, find its distance s(t) from the dock at the end of t seconds.

Solution

From v'(t)=12t-4 we obtain, by Antidifferentiation,

$$v(t) = 6t^2 - 4t + C$$

for some number C. substitution of 0 for t and use of the fact that v(0)=8 gives us 8=0-0+C=C thus

$$v(t) = 6t^2 - 4t + 8$$

Or equivalently, $s'(t) = 6t^2 - 4t + 8$.

The most general antiderivative of s'(t) is $s(t) = 2t^3 - 2t^2 + 8t + D$, where D is some number. Substitution of 0 for t and use the fact that s(0)=15 leads to 15=0-0+0+D=D and, consequently, the desired position function is given by

$$s(t) = 2t^3 - 2t^2 + 8t + 15.$$

An object on or near the surface of the Earth is acted upon by a force called **gravity**, which produces a constant acceleration denoted by g. the approximation to g which is employed for most problems is 32 ft/sec^2 or 980cm/sec^2. Thus use of this important physical constant is illustrated in the following example.

Example 6

A stone is thrown vertically upward from a position 144 ft above the ground with a velocity of 96 ft/sec. if air resistance is neglected, find its distance above the ground after t sec. for what length of time does the stone rise? When, and with what velocity, does it strike the ground?

Solution

The motion of the stone may be represented by a point moving rectilinearly on a vertical line with origin at ground level and position direction upward. The distance above the ground at time t is s(t) and the initial conditions are s(0)=144 and v(0)=96. since the velocity is decreasing, v'(t)<0; that is, the acceleration is negative. Hence, by the remarks preceding this example, the approximation to g which is employed for most problems is 32 ft/sec^2 or 980cm/sec^2.

a(t)= -32.

Since v is an antiderivative of a,

V(t)= -32t +C,

For some number C. substituting 0 for t and using the fact that v(0) = 96 gives us 96=0+C=C and, consequently,

V(t)= -32t+96.

Since s'(t) =v(t) we obtain, by Antidifferentiation,

$$S(t) = -16t^2 + 96t + D$$

For some number D. letting t=0 and using the fact that s(0)=144 leads to 144=0+0+D=D. it follows that the distance from the ground to the stone at time t is given by

$$S(t) = -16t^2 + 96t + 144.$$

The stone will rise until v(t)=0, that is, until -32t+96=0, or t=3. the stone will strike the ground when s(t)=0, that is when $-16t^2 + 96t + 144$=0 or, equivalently, $t^2 - 6t - 9 = 0$. Applying the quadratic formula, $t = 3 \pm 3\sqrt{2}$. the solution $3 - 3\sqrt{2}$ is extraneous (Why?) and hence the stone strikes the ground after $3 + 3\sqrt{2}$ sec. the velocity at that time is

$$v(3+3\sqrt{2}) = -32(3+3\sqrt{2})+96$$
$$= -96\sqrt{2} \approx -135.8\, ft/\sec$$

The Definite Integral

Calculus consists of two main parts, *differential calculus* and *integral calculus.* Differential calculus is based upon the derivative. In this chapter we define the concept which is the basis for integral calculus: the *definite integral.* One of the most important results we shall discuss is the *fundamental Theorem of Calculus.* This theorem demonstrates that differential and integral calculus are very closely related.

4.2 Area

In our development of the definite integral we shall employ sums of many numbers. To express such sums compactly, it is convenient to use **summation notation.** To illustrate, given a collection of numbers {a_1, a_2, …a_n}, the symbol $\sum_{i=1}^{n} a_i$ represents their sum, that is

(4.2.1) $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n$

In (4.2.1), the Greek capital letter $\sum$ (sigma) indicates a sum, and the symbol a_i represents the *i*th term. The letter *I* is called the **index of summation** or the **summation variable.** And the numbers 1 and n indicate the extreme values of the summation variables.

Example 1 Find $\sum_{i=1}^{4} i^2(i-3)$.

Solution in this case, $a_i = i^2(1-3)$. To find the indicated sum we merely substitute, in succession, the integers 1, 2, 3, and 4 for *I* and add the resulting terms. Thus,

$$\sum_{i=1}^{4} i^2(i-3) = 1^2(1-3) + 2^2(2-3) + 3^2(3-3) + 4^2(4-3)$$
$$= (-2) + (-4) + 0 + 16 = 10$$

The letter used for the summation variable is arbitrary. The sum in Example 1 can be written

$$\sum_{i=1}^{4} i^2(i-3) = \sum_{k=1}^{4} k^2(k-3) = \sum_{j=1}^{4} j^2(j-3)$$

Or in many other ways

If $a_i = c$ for each *i* then, for example,

$$\sum_{i=1}^{2} a_i = a_1 + a_2 = c + c = 2c = \sum_{i=1}^{2} c$$

$$\sum_{i=1}^{3} a_i = a_1 + a_2 + a_3 = c + c + c = 3c = \sum_{i=1}^{3} c$$

In general, the following result is true for every positive integer *n*.

(5.2) $$\sum_{i=1}^{n} c = nc$$

For every real number *c*

The domain of the summation variable does not have to begin at 1. For example, the following is self-explanatory:

$$\sum_{i=4}^{8} a_i = a_4 + a_5 + a_6 + a_7 + a_8$$

Example 2

Find $\sum_{i=0}^{3} \frac{2^i}{(i+1)}$

Solution

$$\sum_{i=0}^{3} \frac{2^i}{(i+1)} = \frac{2^0}{(0+1)} + \frac{2^1}{(1+1)} + \frac{2^2}{(2+1)} + \frac{2^3}{(3+1)}$$

$$= 1 + 1 + \frac{4}{3} + 2 = \frac{16}{3}$$

Theorem (4.2.3)

If *n* is any positive integer and $\{a_1, a_2, \ldots a_n\}$and $\{b_1, b_2, \ldots b_n\}$are sets of numbers, then

(i) $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i;$$

(ii) $$\sum_{i=1}^{n} ca_i = c\left(\sum_{i=1}^{n} a_i\right),$$ for any number *c*;

(iii) $$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i;$$

Proof

To prove formula (i) we begin by writing

$$\sum_{i=1}^{n} (a_i + b_i) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \ldots + (a_n + b_n)$$

The terms on the right may be rearranged to produce

$$\sum_{i=1}^{n}(a_i+b_i)=(a_1+a_2+a_3+...+a_n)+(b_1+b_2+b_3+...+b_n)$$

Expressing the right side in summation notation gives us formula (i).

For formula (ii) we have

$$\sum_{i=1}^{n}ca_i = ca_1+ca_2+ca_3+...+ca_n$$

$$=c(a_1+a_2+a_3+...+a_n)=c\left(\sum_{i=1}^{n}a_i\right)$$

The proof of (iii) is left as an exercise.

Before stating the definition of the definite integral, it will be instructive to consider the area of a certain region in a plane. Physical examples could be used; however, we prefer to postpone them until the next chapter. It is important to remember that the discussion of area in this section is *not* to be considered as the definition of the definite integral. It is included only to help motivate the work in Section 5.2 in the same way that slopes of tangent lines and velocities were used to motivate the definition of derivative.

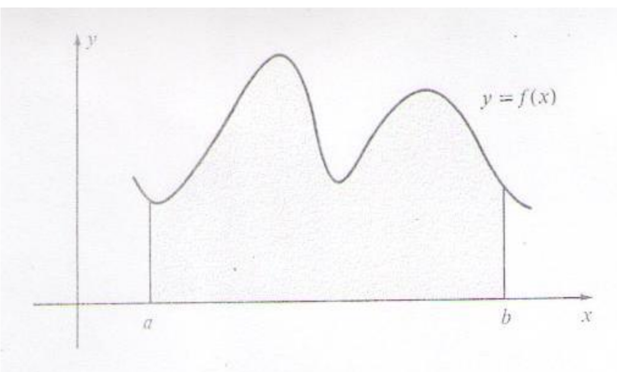

FIGURE 5.1

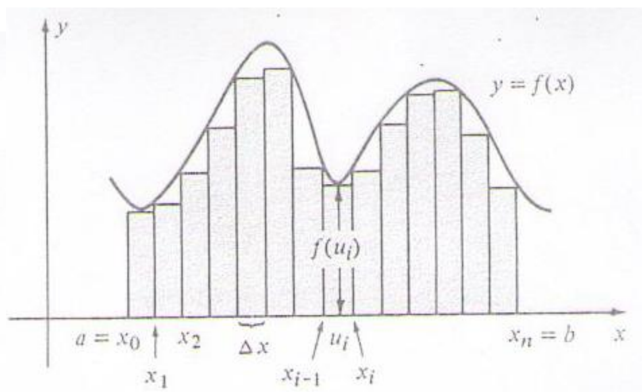

FIGURE 5.2

It is easy to calculate the area of a plane region bounded by straight lines. For example, the area of a rectangle is the product of its length and width. The area of a triangle is one-half the product of the altitude and base. The area of any polygon can be found by subdividing it into triangles. To find areas of more complicated regions, whose boundaries involve graphs of functions, it is necessary to introduce a limiting process and then use methods of calculus. In particular, let us consider a region S in a coordinate plane, bounded by vertical lines with –intercepts *a* and *b*, by the *x*-axis, and by the graph of function *f*, which is continuous and nonnegative on the closed interval [*a*,*b*]. A region of this type of the graph is illustrated in Figure 5.1. Since $f(x)\geq 0$ for every *x* in [*a*,*b*], no part of the graph lies below the *x*-axis. For convenience we shall refer to *S* as **the region under the graph of *f* from *a* to *b*.** our objective is to define the area of *S*.

If n is any positive integer, let us begin by dividing the interval [a,b] into n subintervals, all having the same length(b-a)/n. this can be accomplished by choosing numbers $x_0, x_1, x_2, \ldots x_n$ *where* a=x_0, b=x_x, and

$$x_i - x_{i-1} = \frac{b-a}{n}$$

For i=1,2,…,n. if the length (b-a)/n of each subinterval is denoted by Δx, then for each I we have

$$\Delta x = x_i - x_{i-1}, \text{ and } x_i = x_{i-1} + \Delta x$$

As illustrated in Figure 5.2.

Note that

$$\begin{aligned} &x_0 = a, \qquad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \ldots, \\ &x_i = a + i\Delta x, \quad \ldots, \qquad x_n = a + n\Delta x = b. \end{aligned}$$

Since f is continuous on each subinterval [x_{i-1}, x_i], it follows from Theorem (4.3) that f takes on a minimum value at some number u_i in [x_{i-1}, x_i]. For each I, let us construct a rectangle of width $\Delta x = x_i - x_{i-1}$ and length equal to the minimum distance $f(u_i)$ from the x-axis to the graph of f, as shown in Figure 5.2. The area of the ith rectangle is $f(u_i)\Delta x$. The boundary of the region formed by the totality of these rectangles is called the **inscribed rectangular polygon** associated with the subdivision of [a,b] into n sub-interval. The are of this inscribed polygon is the sum of the areas of the n rectangles, that is,

$$f(u_1)\Delta x + f(u_2)\Delta x + \ldots + f(u_n)\Delta x.$$

Using summation notation we may write:

Area of inscribed rectangular polygon= $\sum_{i=1}^{n} f(u_i)\Delta x$, where $f(u_i)$ is the minimum value of f on [x_{i-1}, x_i].

Referring to Figure 5.2, we see that if n is very large or, equivalently, if Δx is very small, then the sum of the rectangular areas appears to be close to what we wish to consider as the area of the region S. indeed, reasoning intuitively, if there exists a number A that has the property that the sum $\sum_{i=1}^{n} f(u_i)\Delta x$ gets closer and closer to A as Δx gets closer and closer to 0 (but $\Delta x \neq 0$), then we shall call A the **area** of S and write

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(u_i)\Delta x$$

The meaning of this "limit of a sum" is not the same as that for limit of a function, introduced in Chapter 2. to eliminate the hazy phrase "closer and closer" and arrive at a satisfactory definition of *A*, let us take a slightly different point of view. If *A* denotes the are of *S*, then the difference

$$A - \sum_{i=1}^{n} f(u_i)\Delta x$$

Is the area of the unshaded portion in Figure 5.2 that lies under the graph of *f* and over the inscribed rectangular polygon. This number may be thought of as the error involved in using the area of the inscribed rectangular polygon to approximate *A*. if we have the proper notion of area, then we should be able to make this error arbitrarily small by choosing the width Δx of the rectangles sufficiently small. This is the motivation for the following definition of the area *A* of *S*, where the notation is the same as that used in the preceding discussion.

Definition (4.2.4)

Let *f* be continuous and nonnegative on [*a*,*b*]. Let *A* be a real number and let $f(u_i)$ be the minimum value of *f* on $[x_{i-1}, x_i]$. The statement

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(u_i)\Delta x$$

Means that for every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that if $0 < \Delta x < \delta$, then

$$A - \sum_{i=1}^{n} f(u_i)\Delta x < \varepsilon$$

If *A* is the indicated limit and we let $\varepsilon = 10^{-9}$, then Definition (5.4) states that by using sufficiently thin rectangles, the difference between *A* and the area of the inscribed polygon is less than one-billionth of a square unit! Similarly, if $\varepsilon = 10^{-12}$ we can make this difference less than one-trillionth of a square unit. In general, the difference can be made less than *any* preassigned ε.

If *f* is continuous on [*a*,*b*], then, as it is shown in more advanced texts, a number *A* satisfying Definition (5.4) actually exists. We shall call *A* **the area under the graph of *f* from *a* to *b*.**

The area *A* may also be obtained by means of **circumscribed rectangular polygons** of type illustrated in Figure 5.3. in this case we select the number v_i in each interval $[x_{i-1}, x_i]$, such that $f(v_i)$ is the maximum value of *f* on $[x_{i-1}, x_i]$.

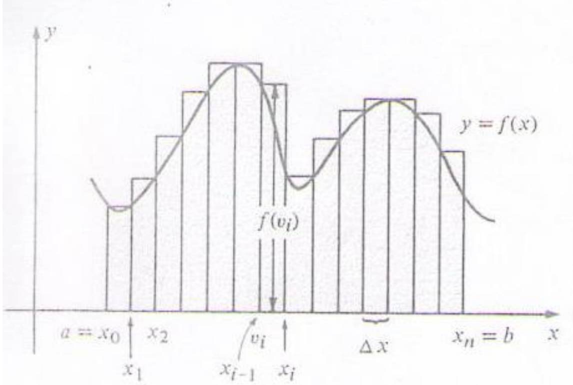

FIGURE 5.3

We may then write:

Area of circumscribed rectangular polygon= $\sum_{i=1}^{n} f(v_i)\Delta x$

The limit of this sum as $\Delta x \to 0$ is defined as in (4.2.4), where the only change is that we use

$$\sum_{i=1}^{n} f(v_i)\Delta x - A < \varepsilon$$

In the definition, since we want this difference to be nonnegative. It can be proved that the same number *A* is obtained using either inscribed or circumscribed rectangles.

The following formulas will be useful in some illustrations of Definition (4.2.4)

$$(i)\sum_{i=1}^{n} i = 1+2+...+n = \frac{n(n+1)}{2}$$

$$(ii)\sum_{i=1}^{n} i^2 = 1^2+2^2+...+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(iii)\sum_{i=1}^{n} i^3 = 1^3+2^3+...+n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

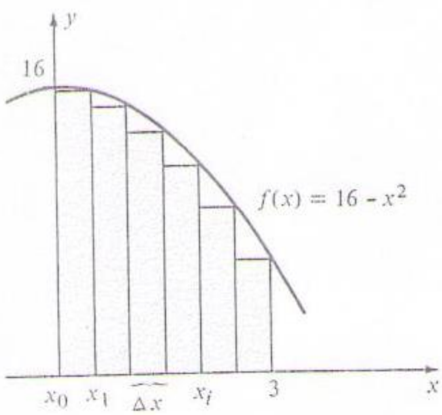

FIGURE 5.4

These may be proved by means of mathematical induction (see Appendix I)

The next two examples provide specific illustrations of how summation properties may be used in conjunction with Definition (4.2.4) to find the areas of certain regions in a coordinate plane.

Example3

If $f(x) = 16 - x^2$, find the area of the region under the graph of *f* from 0 to 3.

Solution

The region is illustrated in Figure 5.4 where for clarity we have used different scales on the *x*- and *y*- axes. If the interval [0,3] is divided into *n* equal subintervals, then the length Δx of a typical subinterval is 3/*n*. employing the notation used in Figure 5.2, with *a=0* and *b=3*.

$$x_0 = 0, x_1 = \Delta x, x_2 = 2(\Delta x), ..., x_i = i(\Delta x), ..., x_n = n(\Delta x) = 3$$

Using the fact that $\Delta x = 3/n$ we may write

$$x_i = i(\Delta x) = i(\frac{3}{n}) = \frac{3i}{n}$$

Since f is decreasing on [0,3], the number u_i in $[x_{i-1}, x_i]$ at which f takes on its minimum value is always the right-hand endpoint x_i of the subinterval, that is, $u_i = x_i = 3i/n$. Since

$$f(u_i) = f\left(\frac{3i}{n}\right) = 16 - \left(\frac{3i}{n}\right)^2 = 16 - \frac{9^2}{n^2},$$

The summation is Definition (5.4) may be written

$$\begin{aligned}\sum_{i=1}^{n} f(u_i)\Delta x &= \sum_{i=1}^{n}\left(16 - \frac{9i^2}{n^2}\right)\left(\frac{3}{n}\right)\\ &= \sum_{i=1}^{n}\left(\frac{48}{n} - \frac{27i^2}{n^3}\right)\end{aligned}$$

Using Theorem (4.2.3) and (4.2.2), the last sum may be simplified as follows:

$$\sum_{i=1}^{n}\frac{48}{n} - \sum_{i=1}^{n}\frac{27i^2}{n^3} = \left(\frac{48}{n}\right)n - \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

Next, applying (ii) of (4.2.5), we obtain

$$\begin{aligned}\sum_{i=1}^{n} f(u_i)\Delta x &= 48 - \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\\ &= 48 - \frac{9}{2n^3}\left[2n^3 + 3n^2 + n\right]\end{aligned}$$

In order to find the area, we must now let Δx approach 0. since $\Delta x = (b-a)/n$, this can be accomplished by letting n increase without bound. Although our discussion of limits involving infinity was concerned with a real variable x, a similar discussion can be given for the integer variable n. assuming that this is true, and that we can replace $\Delta x \to 0$ by $n \to \infty$, we have

$$\begin{aligned}\lim_{\Delta x \to 0}\sum_{i=1}^{n} f(u_i)\Delta x &= \lim_{n\to\infty}\left\{48 - \frac{9}{2n^3}\left[2n^3 + 3n^2 + n\right]\right\}\\ &= \lim_{n\to\infty} 48 - \frac{9}{2}\lim_{n\to\infty}\left[\frac{2n^3 + 3n^2 + n}{n^3}\right]\\ &= 48 - \frac{9}{2}\lim_{n\to\infty}\left[2 + \frac{3}{n} + \frac{1}{n^2}\right]\\ &= 48 - \tfrac{9}{2}\left[2 + 0 + 0\right] = 48 - 9 = 39\end{aligned}$$

Because of the assumptions we made, the preceding solution is not completely rigorous. Indeed, one reason for introducing the definite integral is to enable us to solve problems of this type in a simple and precise manner.

The area in the preceding example may also be found by using circumscribed rectangular polygons. In this case we select, in each subinterval $[x_{i-1},x_i]$, the number $v_i = (i-1)(3/n)$ at which *f* takes on its maximum value.

The next example illustrates the use of circumscribed rectangles in finding an area.

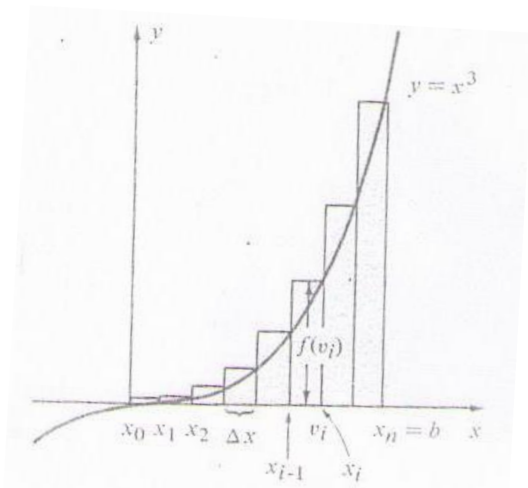

FIGURE 5.5

Example 4

If $f(x) = x^3$, find the area under the graph of *f* from 0 to *b*, where *b*>0.

Solution

If we subdivide the interval [0,b]into *n* equal parts, then as Figure 5.5 illustrates, we obtain a typical circumscribed rectangular polygon where, as in Example 3,

$$\Delta x = \frac{b}{n} \text{ and } x_i = i(\Delta x).$$

For clarity different scales have been used on the *x*- and *y*-axes.

Since *f* is an increasing function, the maximum value $f(v_i)$ in the interval $[x_{i-1},x_i]$occurs at the right-hand endpoint, that is.

$$v_i = x_i = i(\Delta x) = i\left(\frac{b}{n}\right) = \frac{bi}{n}$$

The sum of the areas of the circumscribed rectangles is

$$\sum_{i=1}^{n} f(v_i)\Delta x = \sum_{i=1}^{n}\left(\frac{bi}{n}\right)^3\left(\frac{b}{n}\right) = \sum_{i=1}^{n}\frac{b^4}{n^4}i^3$$
$$= \frac{b^4}{n^4}\sum_{i=1}^{n} i^3 = \frac{b^4}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$