AP Calculus Lesson Fourteen Notes

Chapter 6 Additional Techniques of Integration

6.3 Partial Fractions and Quadratic Expressions
6.4 Miscellaneous Substitutions

6.3 Partial Fractions and Quadratic Expressions

Partial Fractions

It is easy to verify that

$$\frac{2}{x^2-1} = \frac{1}{x-1} + \frac{-1}{x+1}.$$

The expression on the right side of the equation is called the *partial fraction decomposition* of $2/(x^2-1)$. This decomposition may be used to fin the indefinite integral of $2/(x^2-1)$. We merely integrate each of the fractions that make up the decomposition, obtaining

$$\begin{aligned}\int \frac{2}{x^2-1}\,dx &= \int \frac{1}{x-1}\,dx + \int \frac{-1}{x+1}\,dx \\ &= \ln|x-1| - \ln|x+1| + C \\ &= \ln\left|\frac{x-1}{x+1}\right| + C.\end{aligned}$$

It is theoretically possible to write *any* rational expression *f(x)/g(x)* as a sum of rational expressions whose denominators involve powers of polynomials of degree not greater than two. Specifically, if *f(x)* and *g(x)* are polynomials and *the degree of f(x) is less than the degree of g(x)*, then it follows from a theorem a algebra that

$$\frac{f(x)}{g(x)} = F_1 + F_2 + \cdots + F_k$$

Where each F_i has one of the forms

$$\frac{A}{(px+q)^m} \quad \text{or} \quad \frac{Cx+D}{(ax^2+bx+c)^n}$$

For example, given

$$\frac{x^3-6x^2+5x-3}{x^2-1}$$

We obtain, by long division,

$$\frac{x^3-6x^2+5x-3}{x^2-1} = x-6+\frac{6x-9}{x^2-1}.$$

The partial fraction decomposition is then found for $(6x-9)/(x^2-1)$.

(6.3.1) Guidelines for funding partial fraction decompositions of f(x)/g(x)

A if the degree of *f(x)* is not lower than the degree of *g(x)*, use long division to obtain the proper form.

B express *g(x)* as a product of linear factors *px+q* or irreducible quadratic factors ax^2+bx+c, and collect repeated factors so that *g(x)* is a product of *different* factors of the form $(px+q)^m$ or $(ax^2+bx+c)^n$, where *m* and *n* are nonnegative integers.

C Apply the following rules.

Rule 1. for each factor of the form $(px+q)^m$ where $m \geq 1$, the partial fraction decomposition contains a sum of *m* partial fractions of the form

$$\frac{A_1}{px+q}+\frac{A_2}{(px+q)^2}+\cdots+\frac{A_m}{(px+q)^m}$$

Where each A_i is a real number

Rule 2. for each factor of the form $(ax^2+bx+c)^n$ where $n \geq 1$ and ax^2+bx+c is irreducible, the partial fraction decomposition contains a sum of *n* partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Where each A_i and B_i is a real number.

Example 1

Evaluate $\int \frac{4x^2+13x-9}{x^3+2x^2-3x}dx$

Solution

The denominator of the integrand has the factored form $x(x+3)(x-1)$. Each of the linear factors is handled under Rule 1 of (9.4), with *m=1*. Thus, for the factor *x* there corresponds a partial fraction of the form *A/x*. similarly, for the factors *x*+3 and *x*-1 there correspond partial fraction *B/(x*+3) and *C*/(*x*-1), respectively. Thus the partial fraction decomposition has the form

$$\frac{4x^2+13x-9}{x(x+3)(x-1)}=\frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-1}.$$

Multiplying by the lowest common denominator gives us

$$(*) \qquad 4x^2+13x-9=A(x+3)(x-1)+Bx(x-1)+Cx(x+3).$$

Where we have used the symbol (*) for later reference. In a case such as this, in which the factors are all linear and no repeated, the values for *A, B* and *C* can be found by substituting values for *x* that make the various factors zero. If we let *x*=0 in (*), then

-9=-3*A* or *A*=3.

Letting *x*=1 in (*) gives us

8=4*C* or *C*=2.

Finally, if *x*=-3, then

-12=12*B* or *B*=-1.

The partial fraction decomposition is, therefore,

$$\frac{4x^2+13x-9}{x(x+3)(x-1)} = \frac{3}{x} + \frac{-1}{x+3} + \frac{2}{x-1}.$$

Integrating,

$$\begin{aligned}\int \frac{4x^2+13x-9}{x(x+3)(x-1)}\,dx &= \int \frac{3}{x}\,dx + \int \frac{-1}{x+3}\,dx + \int \frac{2}{x-1}\,dx \\ &= 3\ln|x| - \ln|x+3| + 2\ln|x-1| + D \\ &= \ln|x^3| - \ln|x+3| + \ln|x-1|^2 + D \\ &= \ln\left|\frac{x^3(x-1)^2}{x+3}\right| + D.\end{aligned}$$

Another technique for finding *A*, *B*, and *C* is to compare coefficient of *x*. if the right-hand side of (*) is expanded and like powers of *x* are collected then

$$4x^2+13x-9 = (A+B+C)x^2 + (2A-B+3C)x - 3A.$$

We now use the fact that if two polynomials are equal, then coefficients of like powers are the same. Thus

$$\begin{aligned} A + B + C &= 4 \\ 2A - B + 3C &= 13 \\ -3A \qquad\quad &= -9 \end{aligned}$$

It is left to the reader to show that the solution of this system of equations is $A=3$, $B=-1$, and $C=2$.

Example 2

Evaluate $\int \dfrac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}dx$

Solution

By Rule 1 of (9.4), there is a partial fraction of the form $A/(x+1)$ corresponding to the factor $(x-2)^3$ we apply Rule 1 (with $m=3$), obtaining a sum of three partial fractions $B/(x-2)$, $C/(x-2)^2$, and $D/(x-2)^3$. Consequently, the partial fraction decomposition has the form

$$\frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}.$$

Multiplying both sides by $(x+1)(x-2)^3$ gives us

$$(*) \qquad \begin{aligned} &3x^3-18x^2+29x-4 \\ &\quad = A(x-2)^3 + B(x+1)(x-2)^2 + C(x+1)(x-2) + D(x+1).\end{aligned}$$

Two of the unknown constants may be determined easily. If we let $x=2$ in (*), then

$$24 - 72 + 58 - 4 = 3D, \quad 6 = 3D, \quad \text{and} \quad D = 2.$$

Similarly, letting $x=-1$ in (*)

$$-3 - 18 - 29 - 4 = -27A, \quad -54 = -27A, \quad \text{and} \quad A = 2.$$

The remaining constants may be found by comparing coefficients. If the right side of (*) is expanded and like powers of *x* collected, we see that the coefficient of x^3 is $A+B$. this must equal the coefficient of x^3 on the left, that is,

$$A+B=3$$

Since $A=2$, it follows that $B=3-A=3-2=1$. Finally, we compare, the constant terms in (*) by letting $x=0$. This gives us

$$-4 = -8A + 4B - 2C + D.$$

Substituting the values we have found for *A*, *B*, and *D* leads to

$$-4 = -16 + 4 - 2C + 2$$

Which has the solution $C=-3$. the partial fraction decomposition is. Therefore,

$$\frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3} = \frac{2}{x+1} + \frac{1}{x-2} + \frac{-3}{(x-2)^2} + \frac{2}{(x-2)^3}.$$

To find the given integral we integrate each of the partial fractions on the right side of the last equation. This gives us

$$2\ln|x+1| + \ln|x-2| + \frac{3}{x-2} - \frac{1}{(x-2)^2} + E$$

Which may be written in the more compact form

$$\ln(x+1)^2|x-2| + \frac{3x-7}{(x-2)^2} + E.$$

Example 3

Evaluate $\int \dfrac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4}dx$

Solution

The denominator may be factored by grouping as follows

$$2x^3 - x^2 + 8x - 4 = x^2(2x-1) + 4(2x-1) = (x^2+4)(2x-1).$$

Applying Rule 2 of (9.4) to the irreducible quadratic factor x^2+4 we see that one of the partial fractions has the form $(Ax+B)/(x^2+4)$. By Rule 1, there is also a partial fraction $C/(2x-1)$ corresponding to the factor $2x$-1. Consequently,

$$\frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} = \frac{Ax+B}{x^2+4} + \frac{C}{2x-1}.$$

As in previous examples, this leads to

$$(*) \qquad x^2 - x - 21 = (Ax+B)(2x-1) + C(x^2+4).$$

Substituting $x=1/2$ we obtain $\frac{1}{4} - \frac{1}{2} - 21 = \frac{17}{4}C$, which has the solution $C=-5$. The remaining constants may be found by comparing coefficients. Rearranging the right side of (*) gives us

$$x^2 - x - 21 = (2A+C)x^2 + (-A+2B)x - B + 4C.$$

Comparing the coefficients of x^2 we see that $2A+C=1$. Since $C=-5$ it follows that $2A=6$ or $A=3$. Similarly, comparing the constant terms $-B+4C=-21$ and hence $-B-20=-21$ or $B=1$. Thus the partial fraction decomposition of the integrand is

$$\frac{x^3 - x - 21}{2x^3 - x^2 + 8x - 4} = \frac{3x + 1}{x^2 + 4} + \frac{-5}{2x - 1}$$
$$= \frac{3x}{x^2 + 4} + \frac{1}{x^2 + 4} - \frac{5}{2x - 1}.$$

Quadratic Expressions

Partial fraction decompositions may lead to integrands containing an irreducible quadratic expression ax^2+bx+c. if $b \neq 0$ it is often necessary to complete the square as follows

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$$
$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.$$

The substitution $u=x+(b/2a)$ may then lead to an integrable form.

Example 1

Evaluate $\int \frac{2x-1}{x^2-6x+13}\,dx$

Solution

Note that the quadratic expression $x^2-6x+13$ is irreducible, since $b^2-4ac=36-52=-16<0$. We complete the square as follows.

$$x^2 - 6x + 13 = (x^2 - 6x \qquad) + 13$$
$$= (x^2 - 6x + 9) + 13 - 9 = (x - 3)^2 + 4.$$

If we let $u=x-3$, then $x=u+3$, $dx=du$, and hence

$$\int \frac{2x - 1}{(x - 3)^2 + 4}\,dx = \int \frac{2(u + 3) - 1}{u^2 + 4}\,du$$
$$= \int \frac{2u + 5}{u^2 + 4}\,du$$
$$= \int \frac{2u}{u^2 + 4}\,du + 5\int \frac{1}{u^2 + 4}\,du$$
$$= \ln(u^2 + 4) + \frac{5}{2}\tan^{-1}\frac{u}{2} + C$$
$$= \ln(x^2 - 6x + 13) + \frac{5}{2}\tan^{-1}\frac{x - 3}{2} + C.$$

The technique of completing the square may also be employed if quadratic expressions appear under a radical sign.

Example 2

Evaluate $\int \frac{1}{\sqrt{8+2x-x^2}}\,dx$

Solution

We may complete the square for the quadratic expression $8+2x-x^2$ as follows:

$$8+2x-x^2 = 8-(x^2-2x) = 8+1-(x^2-2x+1)$$
$$= 9-(x-1)^2.$$

Next, letting $u=x-1$ we have $du=dx$, and hence

$$\int \frac{1}{\sqrt{8+2x-x^2}}\,dx = \int \frac{1}{\sqrt{9-u^2}}\,du$$
$$= \sin^{-1}\frac{u}{3} + C$$
$$= \sin^{-1}\frac{x-1}{3} + C.$$

If the next example it is necessary to make a trigonometric substitution after completing the square.

Example 3

Evaluate $\int \frac{1}{\sqrt{x^2+8x+25}}\,dx$

Solution

We complete the square for the quadratic expression as follow

$$x^2+8x+25 = (x^2+8x\qquad)+25$$
$$= (x^2+8x+16)+25-16$$
$$= (x+4)^2+9.$$

Hence
$$\int \frac{1}{\sqrt{x^2+8x+25}}\,dx = \int \frac{1}{\sqrt{(x+4)^2+9}}\,dx.$$

If we next make the trigonometric substitution

$$x+4 = 3\tan\theta$$

Then $dx = 3\sec^2\theta\, d\theta$

And $\sqrt{(x+4)^2+9} = \sqrt{9\tan^2\theta+9} = 3\sqrt{\tan^2\theta+1} = 3\sec\theta.$

6.4 Miscellaneous Substitutions

We have often used a change of variables to aid in the evaluation of a definite or indefinite integral. In this section we shall consider additional substitutions which are sometimes useful. The first example indicates that if an integral contains an expression of the form $\sqrt[n]{f(x)}$, then one of the substitutions $u = \sqrt[n]{f(x)}$ or $u = f(x)$ may simplify the evaluation.

Example 1

Evaluate $\int \frac{x^3}{\sqrt[3]{x^2+4}}\,dx$

Solution 1

The substitution $u = \sqrt[3]{x^2+4}$ leads to the following equivalent equations:

$$u = \sqrt[3]{x^2+4}, \quad u^3 = x^2+4, \quad x^2 = u^3 - 4.$$

Taking the differential of each side of the last equation, we obtain

$$2xdx = 3u^2du, \text{ or } \quad xdx = \tfrac{3}{2}u^2du$$

We now substitute in the given integral as follows:

$$\begin{aligned}
\int \frac{x^3}{\sqrt[3]{x^2+4}}\,dx &= \int \frac{x^2}{\sqrt[3]{x^2+4}} \cdot x\,dx \\
&= \int \frac{u^3-4}{u} \cdot \frac{3}{2}u^2\,du = \frac{3}{2}\int (u^4 - 4u)\,du \\
&= \frac{3}{2}\left(\frac{1}{5}u^5 - 2u^2\right) + C = \frac{3}{10}u^2(u^3 - 10) + C \\
&= \frac{3}{10}(x^2+4)^{2/3}(x^2-6) + C.
\end{aligned}$$

Solution 2

If we substitute *u* for the expression *underneath* the radical, then

$$\begin{aligned}
u &= x^2 + 4 & x^2 &= u - 4 \\
2x\,dx &= du & x\,dx &= \tfrac{1}{2}\,du.
\end{aligned}$$

In this case we may write

$$\begin{aligned}
\int \frac{x^3}{\sqrt[3]{x^2+4}}\,dx &= \int \frac{x^2}{\sqrt[3]{x^2+4}} \cdot x\,dx \\
&= \int \frac{u-4}{u^{1/3}} \cdot \frac{1}{2}\,du = \frac{1}{2}\int (u^{2/3} - 4u^{-1/3})\,du \\
&= \frac{1}{2}\left[\frac{3}{5}u^{5/3} - 6u^{2/3}\right] + C = \frac{3}{10}u^{2/3}[u - 10] + C \\
&= \frac{3}{10}(x^2+4)^{2/3}(x^2-6) + C.
\end{aligned}$$

Example 2

Evaluate $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx$

Solution

If we let $z=\sqrt[6]{x}$, then

$$x=z^6,\qquad \sqrt{x}=z^3,\qquad \sqrt[3]{x}=z^2,\qquad dx=6z^5\,dz$$

And

$$\int\frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx=\int\frac{1}{z^3+z^2}\,6z^5\,dz=6\int\frac{z^3}{z+1}\,dz.$$

By long division,

$$\frac{z^3}{z+1}=z^2-z+1-\frac{1}{z+1}.$$

Consequently,

$$\begin{aligned}\int\frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx&=6\int\left[z^2-z+1-\frac{1}{z+1}\right]dz\\&=6\left(\tfrac{1}{3}z^3-\tfrac{1}{2}z^2+z-\ln|z+1|\right)+C\\&=2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln\left(\sqrt[6]{x}+1\right)+C.\end{aligned}$$

If an integrand is a rational expression in sin x and cos x, then the substitution $z=\tan(x/2)$ where $-\pi<x<\pi$ will transform it into a rational (algebraic) expression in z. to prove this, first note that

$$\cos\frac{x}{2}=\frac{1}{\sec(x/2)}=\frac{1}{\sqrt{1+\tan^2(x/2)}}=\frac{1}{\sqrt{1+z^2}},$$

$$\sin\frac{x}{2}=\tan\frac{x}{2}\cos\frac{x}{2}=z\,\frac{1}{\sqrt{1+z^2}}.$$

Consequently,

$$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}=\frac{2z}{1+z^2},$$

$$\cos x=1-2\sin^2\frac{x}{2}=1-\frac{2z^2}{1+z^2}=\frac{1-z^2}{1+z^2}.$$

Moreover, since $x/2=\tan^{-1}z$, we have $x=2\tan^{-2}z$ and, therefore,

$$dx=\frac{2}{1+z^2}\,dz.$$

The following theorem summarizes this discussion.

Theorem (6.4.1)

If an integrand is a rational expression in sin *x* and cos *x*, the following substitutions will produce a rational expression in *z:*

$$\sin x = \frac{2z}{1+z^2}, \qquad \cos x = \frac{1-z^2}{1+z^2}, \qquad dx = \frac{2}{1+z^2}\,dz.$$

Example 3

Evaluate $\int \frac{1}{4\cos x - 3\cos x}\,dx$

Solution

Applying Theorem (6.4.1) and simplifying the integrand,

$$\int \frac{1}{4\sin x - 3\cos x}\,dx = \int \frac{1}{4\left(\frac{2z}{1+z^2}\right) - 3\left(\frac{1-z^2}{1+z^2}\right)} \cdot \frac{2}{1+z^2}\,dz$$

$$= \int \frac{2}{8z - 3(1-z^2)}\,dz$$

$$= 2\int \frac{1}{3z^2 + 8z - 3}\,dz.$$

Using partial fractions,

$$\frac{1}{3z^2+8z-3} = \frac{1}{10}\left(\frac{3}{3z-1} - \frac{1}{z+3}\right)$$

And hence

$$\int \frac{1}{4\sin x - 3\cos x}\,dx = \frac{1}{5}\int\left(\frac{3}{3z-1} - \frac{1}{z+3}\right)dz$$

$$= \frac{1}{5}(\ln|3z-1| - \ln|z+3|) + C$$

$$= \frac{1}{5}\ln\left|\frac{3z-1}{z+3}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{3\tan(x/2)-1}{\tan(x/2)+3}\right| + C.$$

Other substitutions are sometimes useful; however, it is impossible to state rules that apply to all situations. Whether or not one can express an integrand in a suitable form in often a matter of individual ingenuity.