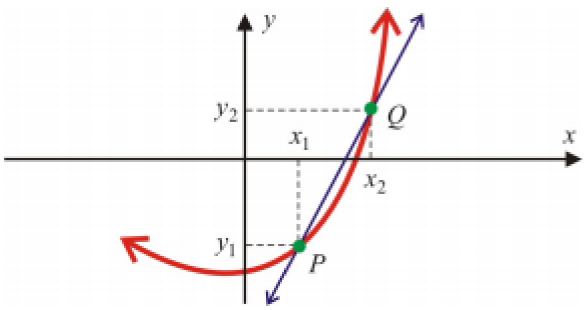
An introduction to calculus (1)

The Slope of the Tangent

Secant Line

Let $y = f(x)$ be a function and $P(x_1, y_1)$ and $Q(x_2, y_2)$ two points on its graph.
The *slope* of the *secant line* that passes through the points P and Q is given by:

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



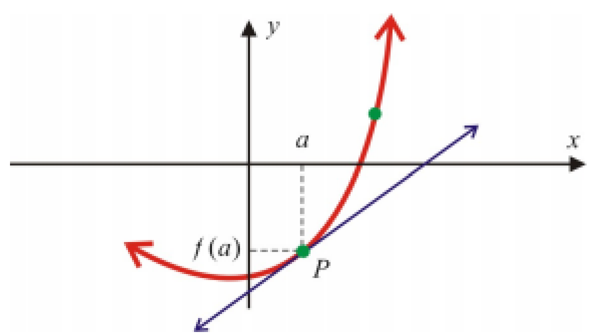
If $P(a, f(a))$ and $Q(a+h, f(a+h))$ then the *slope* of the *secant line* is given by:

$$m = \frac{f(a+h) - f(a)}{h}$$

Tangent Line

As the point Q approaches the point P, the secant line approaches the *tangent line* at P. See the diagram on the right side.
The *slope* of the *tangent line* at $P(a, f(a))$ is:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad (1)$$



Algebraic Computation 1. Use the formula $m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$. 2. Do not substitute h by 0 because you will get the indeterminate case $\frac{0}{0}$. 3. Compute algebraic the *difference quotient* $DQ = \frac{f(a+h)-f(a)}{h}$ until you succeed to cancel out the factor h. 4. *Substitute* in the remaining expression h by 0.	**Ex 1.** Consider $y = f(x) = x^2 - 2x$. a) Find the slope of the tangent line at the generic point $P(a, f(a))$. b) Find the point where the tangent line is horizontal.
c) Find the point P such that $m_P = 2$.	d) Find the point P such that the tangent line at P is perpendicular to the line $L_2 : x - 3y = 3$.

Rate of Change

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Average Rate of Change

$y = f(x), \quad y_1 = f(x_1), \quad y_2 = f(x_2)$
$\Delta x = x_2 - x_1$ (change in variable x)
$\Delta y = y_2 - y_1$ (change in variable y)
The *Average Rate of Change* (ARC) in y variable over the interval $[x_1, x_2]$ is given by:

$$ARC = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Note: The *Average Rate of Change* is the same as the *slope of the secant line* passing through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.
If $x_1 = a$ and $x_2 = a + h$ then:

$$ARC = \frac{f(a+h) - f(a)}{h}$$

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Instantaneous Rate of Change

As $h \to 0$ the Average Rate of Change approaches to the *Instantaneous Rate of Change* (IRC):

$$IRC = RC = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Note: The *Instantaneous Rate of Change* (IRC) is the same as the *slope of the tangent line* at the point $P(a, f(a))$.

Similarly, the *Average Velocity* (AV) approaches *Instantaneous Velocity* (IV):

$$IV = v = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

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Ex 2. Consider the following position function:
$s(t) = t^2 - 4t.$

a) Find the instantaneous velocity at $t = 3s$.

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b) Find the instantaneous velocity at the generic moment $t = a$

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	c) Use the formula at part b) to compute the velocity at time t = 5s.
d) Find the moment(s) of time at which the velocity is zero.	**Ex 3.** Consider $y = f(x) = (x+1)^2$. Find the rate of change in the y variable over the interval $[-1, 2]$.

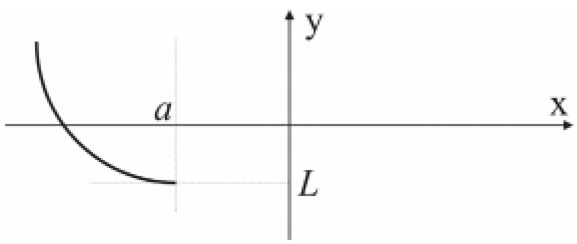
Limit of a Function

Left-Hand Limit

If the values of $y = f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a with $x < a$, then:

$$\lim_{x \to a^-} f(x) = L$$

Read: The limit of the function $f(x)$ as x approaches a from the left is L.



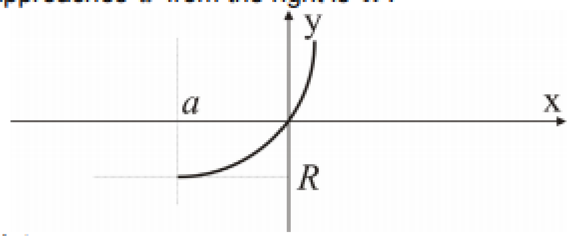
Notes:
1. The function *may be or not defined* at a.
2. DNE stands for *Does Not Exist*.
3. L must be a number.
4. ∞ is not a number.

Right-Hand Limit

If the values of $y = f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a with $x > a$, then:

$$\lim_{x \to a^+} f(x) = R$$

Read: The limit of the function $f(x)$ as x approaches a from the right is R.



Notes:
1. R must be a number. ∞ is not a number.
2. The function *may be or not defined* at a.

Limit

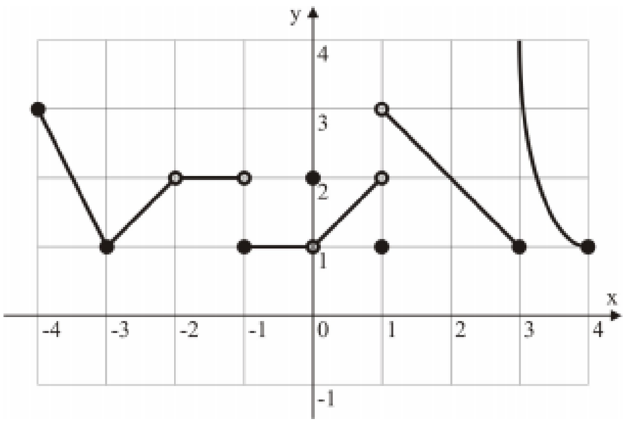
If the values of $y = f(x)$ can be made arbitrarily close to l by taking x sufficiently close to a (*from both* sides), then:

$$\lim_{x \to a} f(x) = l$$

Read: The limit of the function $f(x)$ as x approaches a is l.

Notes:
1. If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$ then $\lim_{x \to a} f(x)$ does exist and $L = R = l$.
2. If $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$ then $\lim_{x \to a} f(x)$ Does Not Exist (DNE).
3. l must be a number. ∞ is not a number.
4. The function *may be or not defined* at a.

Ex 1. Use the function $y = f(x)$ defined by the following graph to find each limit.



a) $\lim\limits_{x\to -4^-} f(x)$

b) $\lim\limits_{x\to -2^-} f(x)$

c) $\lim\limits_{x\to -1^-} f(x)$

d) $\lim\limits_{x\to 3^-} f(x)$

Ex 2. Use the function $y = f(x)$ defined at Ex 1. to find each limit.

a) $\lim\limits_{x\to -1^+} f(x)$

b) $\lim\limits_{x\to 3^+} f(x)$

c) $\lim\limits_{x\to 1^+} f(x)$

d) $\lim\limits_{x\to -2^+} f(x)$

Ex 3. Use the function $y = f(x)$ defined at Ex 1. to find each limit.

a) $\lim\limits_{x\to -4} f(x)$

b) $\lim\limits_{x\to -1} f(x)$

c) $\lim\limits_{x\to 3} f(x)$

d) $\lim\limits_{x\to -3} f(x)$

e) $\lim\limits_{x\to -2} f(x)$

Substitution

If the function is defined by a *formula* (algebraic expression) then the limit of the function at a point a may be determined by *substitution*:

$$\lim_{x\to a} f(x) = f(a)$$

Notes:

1. In order to use substitution, the function must be defined *on both sides* of the number a .
2. Substitution does not work if you get one of the following 7 *indeterminate cases*:

$$\infty - \infty \quad 0\times\infty \quad \frac{0}{0} \quad \frac{\infty}{\infty} \quad 1^{\infty} \quad \infty^0 \quad 0^0$$

Ex 4. Compute each limit.

a) $\lim\limits_{x\to 1^-} \dfrac{x^2}{x+1}$

b) $\lim\limits_{x\to 1^+} \dfrac{x^2}{x+1}$

c) $\lim\limits_{x\to 1} \dfrac{x^2}{x+1}$

d) $\lim\limits_{x\to 2^-} \sqrt{x-2}$

e) $\lim\limits_{x\to 2^+} \sqrt{x-2}$

f) $\lim\limits_{x\to 2} \sqrt{x-2}$

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Piece-wise defined functions

If the function changes formula at a then:
1. Use the appropriate formula to find first the *left-side* and the *right-side* limits.
2. Compare the left-side and the right-side limits to conclude about the limit of the function at a.
Example:

$$f(x)=\begin{cases} f_1(x), & x<a \\ f_2(x), & x>a \end{cases}$$

$L=f_1(a),\quad R=f_2(a)$ (if exist)

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Ex 5. Consider $f(x)=\begin{cases} 2x-3, & x<2 \\ 0, & x=2 \\ x^2-1, & x>2 \end{cases}$

a) Find $\lim_{x\to 2} f(x)$.

b) Find $\lim_{x\to 0} f(x)$.

c) Draw a diagram to illustrate the situation.

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Properties of Limits

Limits Properties

We assume that $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist. Then:

1. $\lim\limits_{x\to a} k = k$
2. $\lim\limits_{x\to a} x = a$
3. $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$
4. $\lim\limits_{x\to a}[cf(x)] = c\lim\limits_{x\to a} f(x)$
5. $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}, \quad \lim\limits_{x\to a} g(x) \neq 0$
6. $\lim\limits_{x\to a}[f(x)g(x)] = [\lim\limits_{x\to a} f(x)][\lim\limits_{x\to a} g(x)]$
7. $\lim\limits_{x\to a}[f(x)]^n = [\lim\limits_{x\to a} f(x]^n$
8. If $P(x)$ is a polynomial function, then

$$\lim_{x\to a} P(x) = P(a)$$

Ex 1. Given $\lim\limits_{x\to 3} f(x) = -2$ and $\lim\limits_{x\to 3} g(x) = 1$, use the limits properties to find $\lim\limits_{x\to 3}\dfrac{2f(x)+g(x)}{-4\sqrt{g(x)}}$.

Substitution

Substitution is the best strategy to find a limit.
Note: Substitution does not work if (by substitution) you get one of the following 7 *indeterminate cases*:

$$\infty - \infty \quad 0\times\infty \quad \frac{0}{0} \quad \frac{\infty}{\infty} \quad 1^\infty \quad \infty^0 \quad 0^0$$

Specific strategies are available to avoid an indeterminate case to appear.

Ex 2. Compute $\lim\limits_{x\to 2}\dfrac{x^2-2x+1}{x-1}$.

Factoring

The indeterminate form $\frac{0}{0}$ may be eliminated by factoring and *canceling out the common factor* that generates zeros:

$$\lim_{x\to a}\frac{F(x)}{G(x)} = \lim_{x\to a}\frac{(x-a)f(x)}{(x-a)g(x)} = \lim_{x\to a}\frac{f(x)}{g(x)}$$

Note: Canceling out the common factor $(x-a)$ is a correct operation because $\lim\limits_{x\to a}$ means that x approaches a but is not equal to a.

Ex 3. Compute $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x-1}$.

Conjugate Radicals When dealing with the indeterminate form $\frac{0}{0}$ you may use the *conjugate radicals* to cancel out the common factor that generates zeros.	Ex 4. Compute $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$.
Change of Variables By changing the variable, the process of canceling the common factor may be simplified. Note. If the change of variable is $u=g(x)$ then: as $x\to a$, $u\to g(a)$	Ex 5. Change the variable to compute $\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$.

Absolute Value Function	
When dealing with an absolute value function, rewrite it as piece-wise defined function according to: $\lvert f(x)\rvert=\begin{cases} f(x), & f(x)\ge 0 \\ -f(x), & f(x)<0 \end{cases}$	Ex 6. Compute each limit. Draw a diagram to illustrate. a) $\lim\limits_{x\to 0}\dfrac{\lvert x\rvert}{x}$
b) $\lim\limits_{x\to 0} x\lvert x\rvert$	c) $\lim\limits_{x\to 0}\dfrac{x^2}{\lvert x\rvert}$