AP Calculus Lesson Nine Notes

Chapter 4 Antiderivatives and the Definite Integral

4.3 Definition of Definite Integral and Properties of Definite Integral
4.4 The Mean Value Theorem for Definite Integral
4.5 The Fundamental Theorem of Calculus

4.3 Definition of Definite Integral

Limiting processes similar to the one for areas used in the preceding section arise frequently in mathematics and its applications. The situations that occur often lead to limits of the form

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(w_i)\Delta x$$

A special instance of this limit appeared in Definition (4.2.4); however, in the general case, there are several major differences from our work with areas. In our discussion of area in the previous section we made the following assumptions:

1. the function f is continuous on a closed interval $[a,b]$
2. $f(x)$ is nonnegative for all x in $[a,b]$
3. All the subintervals $[x_{i-1},x_i]$ determined by the subdivision of $[a,b]$have the same length Δx
4. the number w_i are chosen such that $f(w_i)$ is always the minimum (or maximum) value of f on $[x_{i-1},x_i]$.

These four conditions are not always present in applied problems. For this reason it is necessary to allow the following changes in 1-4.

1’ the function f may be discontinuous at some number in $[a,b]$
2’ $f(x)$ may be negative for some x in $[a,b]$
3’ The lengths of the subintervals $[x_{i-1},x_i]$ may be different
4’ The number w_i is *any* number in $[x_{i-1},x_i]$

We shall begin by introducing some new terminology and notations. A **partition** P of a closed interval $[a,b]$ is any decomposition $[a,b]$ into subintervals of the form

$$[x_0,x_1],[x_1,x_2],[x_2,x_3],...,[x_{n-1},x_n]$$

Where n is a positive integer and the x_i are numbers such that

$$a = x_0 < x_1 < x_2 < x_3 < ... < x_{n-1} < x_n = b$$

The length of the *i*th subinterval $[x_{i-1}, x_i]$will be denoted by Δx, that is,

$$\Delta x_i = x_i - x_{i-1}$$

A typical partition of [*a*,*b*] is illustrated in Figure 5.6. the largest of the numbers $\Delta x_1, \Delta x_2, ..., \Delta x_n$ is called the **norm** of the partition *P* and is denoted by ||*P*||.

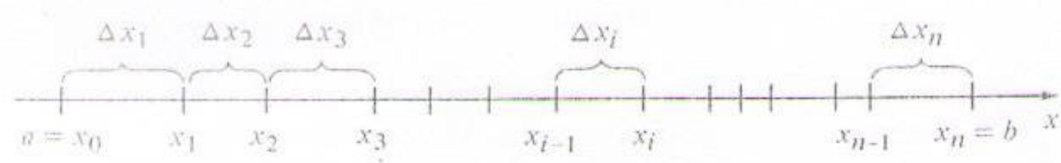


FIGURE 5.6 A partition of [*a*, *b*]

The following concept, named after the mathematician G.F.B Riemann (1826-1866), is fundamental for the definition of the definite integral.

Definition (4.3.1)

Let *f* be a function that is defined on a closed interval [*a.b*]and let *P* be a partition of [*a*,*b*]. A **riemann sum** of *f* for *P* is any expression *Rp* of the form

$$R_P = \sum_{i=1}^{n} f(w_i)\Delta x_i$$

Where w_i is some number in $[x_{i-1}, x_i]$ for i=1,2,…,*n*

The sum in Definition (4.2.4), which represents a sum of areas of certain inscribed rectangles, is a special type of Riemann sum. In Definition (4.3.1). $f(w_i)$ is not necessarily a maximum or minimum value of *f* on $[x_{i-1}, x_i]$. thus, if we construct a rectangle of length $f(w_i)$ and width Δx_i as illustrated in Figure 5.7, the rectangle may be neither inscribed nor circumscribed.

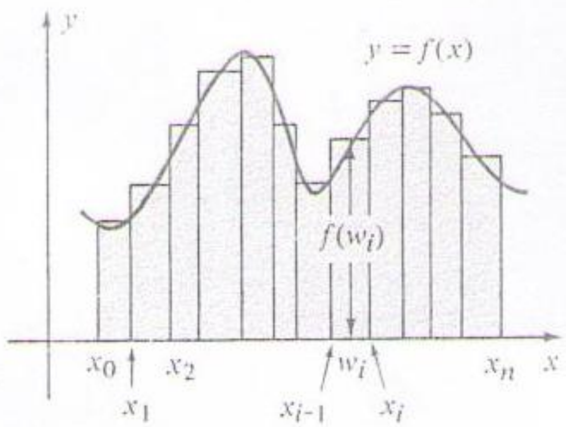
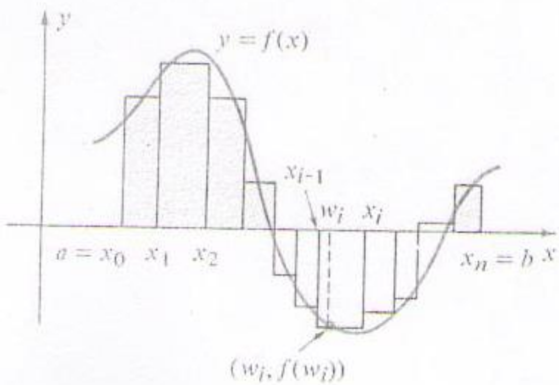


FIGURE 5.7



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In general, since *f(x)* may be negative for some *x* in [*a*,*b*], some terms of R_p in definition (4.3.1) may be negative. Consequently, a Riemann sum does not always represent a sum of areas of rectangles. It is possible to interpret a Riemann sum geometrically as follows. If R_P is defined as in (4.3.1), then for each subinterval $[x_{i-1}, x_i]$let us construct a horizontal line segment through the point $(w_i, f(w_i))$, thereby obtaining a collection of rectangles. If $f(w_i)\Delta x_i$ is the area of this rectangle. If $f(w_i)$is negative, then the rectangle lies below the x-axis as illustrated by the unshaded rectangles in Figure 5.8. In this case the product $f(w_i)$ Δx_i is the *negative* of the area of a rectangle. It follows that R_P is the sum of the areas of the rectangles that lie above the *x*-axis and the *negatives* of the areas of the rectangles that lie below the *x*-axis.

Example

Suppose $f(x)=8-(x^2/2)$ and *P* is the partition of [0,6]into the five subintervals determined by

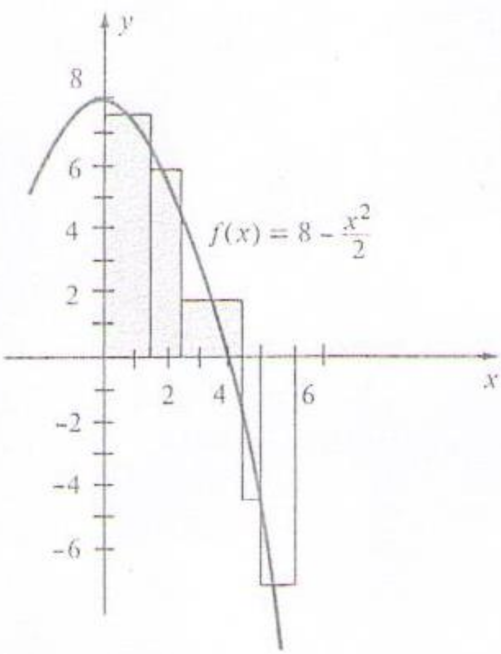

FIGURE 5.9

$x_0 = 0, x_1 = 1.5, x_2 = 2.5, x_3 = 4.5, x_4 = 5, x_5 = 6$. Find (a) the norm of the partition and (b) the Riemann sum R_P if $w_1 = 1, w_2 = 2, w_3 = 3.5, w_4 = 5, x_5 = 5.5$.

Solution

The graph of *f* is sketched in Figure 5.9. Also shown in the figure are the points on the x-axis that correspond to x_i and the rectangles of length $|f(w_i)|$ for *i*=1,2,3,4, and 5.

Thus, $\Delta x_1 = 1.5, \Delta x_2 = 1, \Delta x_3 = 2, \Delta x_4 = 0.5, \Delta x_5 = 1$
And hence, the norm ||P|| of the partition is Δx_3, or 2. By definition (4.3.1),

$$\begin{aligned}R_P &= f(w_1)\Delta x_1 + f(w_2)\Delta x_2 + f(w_3)\Delta x_3 + f(w_4)\Delta x_4 + f(w_5)\Delta x_5\\ &= f(1)(1.5) + f(2)(1) + f(3.5)(2) + f(5)(0.5) + f(5.5)(1)\\ &= (7.5)(1.5) + (6)(1) + (1.875)(2) + (-4.5)(0.5) + (-7.125)(1)\end{aligned}$$

Which reduces to R_P=11.625.
In the future we shall not always specify the number *n* of subintervals in a partition P of $[a,b]$. In this event a Riemann sum (4.3.1) will be written

$$R_P = \sum_i f(w_i)\Delta x_i$$

Where it is understood that terms of the form $f(w_i)\,\Delta x_i$ are to be summed over all subintervals $[x_{i-1},x_i]$ of partition P.
In a manner similar to that used in formulating Definition (4.2.4), we may define what is meant by

$$\lim_{\|P\|\to 0} \sum_i f(w_i)\Delta x_i = I$$

Where I is a real number. Intuitively, it will mean that if the norm ||P|| of the partition P is close to 0, then every Riemann sum for F is close to I.

Definition (4.3.2)

Let f be a function that is defined on a closed interval $[a,b]$, and let I be a real number. The statement

$$\lim_{\|P\|\to 0} \sum_i f(w_i)\Delta x_i = I$$

Means that for every $\varepsilon > 0$ there exists a $\delta > 0$, such that if P is a partition $[a,b]$with $\|P\| < \delta$, then

$$\left|\sum_i f(w_i)\Delta x_i - I\right| < \varepsilon$$

For any choice of numbers w_i in the subintervals $[x_{i-1},x_i]$of P. the number I is called a **limit of a sum.**

Note that for every $\delta > 0$ there are infinitely many partitions P such that $\|P\| < \delta$. Moreover, for each such partition P there are infinitely many ways of choosing the numbers w_i in $[x_{i-1},x_i]$. Consequently, there may be an infinite number of different Riemann sums associated with *each* partition P. however, if the limit I exists, then for any ε units of I, provided a small enough norm is chosen. Although similar to that given in Appendix II may be used to show that if the limit I exists, then it is unique.

We next define the definite integral as a limit of a sum.

Definition (4.3.3)

Let *f* be a function that is defined on a closed interval [*a,b*]. the **definite integral of *f* from *a* to *b*,** denoted by $\int_a^b f(x)dx$, is given by

$$\int_a^b f(x)dx = \lim_{\|p\|\to 0} \sum_i f(w_i)\Delta x_i$$

Provided the limit exists.

If the definite integral of *f* from *a* to *b* exists, then *f* is said to be **integrable** on the closed interval [*a,b*], or we say that the integral $\int_a^b f(x)dx$ **exists**. The process of finding the number represented by the limit is called **evaluating the integral.**

The simple $\int$ in Definition (4.3.3) is called an **integral sign.** It may be thought of as an elongated letter *S* (the first letter of the word *sum*) and is used to indicate the connection between definite integrals and Riemann sums. The numbers *a* and *b* are referred to as the **limits of integration,** *a* being called the **lower limit** and *b* and **upper limit.** Note that the word *limit* in this terminology is used in conjunction with the smallest and largest numbers in the interval [*a,b*] and has no connection with any definitions of limits given earlier. The expression *f(x)* which appears to the right of the integral sign (we sometimes say "*behind* the integral sign") is called the **integrand.** Finally, the symbol *dx* which follows *f(x)* should not be confused with the differential of *x* defined in Section 3.5. at this stage of our work it is merely used to indicate the variable. Later in the text the use of the differential symbol will have certain practical advantages.

Letters other than *x* may be used in the notation for the definite integral. This follows from the fact that when we describe a function, the symbol used for the independent variable is immaterial. Thus, if *f* is integrable on [*a,b*], then

$$\int_a^b f(x)dx = \int_a^b f(s)ds = \int_a^b f(t)dt$$

Etc. for this reason the letter *x* in Definition (4.3.3) is sometimes referred to as **dummy variable**.

Whenever an interval [*a,b*]is employed it is assumed that $a<b$. consequently, Definition (4.3.3) does not take into account the cases in which the lower limit of integration is greater than or equal to the upper limit. The definition may be extended to include the case where the lower limit is greater than the upper limit, as follows

Definition (4.3.4)

If $c>d$, then $\int_c^d f(x)dx = -\int_d^c f(x)dx$

In words, Definition (4.3.4) may be phrased as "interchanging the limits of integration changes the sign of the integral." One reason for the form of Definition (4.3.4) will become apparent later, after we have considered the Fundamental Theorem of Calculus.

The case in which the lower the upper limits of integration are equal is covered by the next definition.

Definition (4.3.5)

If *f(a)* exists, then $\int_a^a f(x)dx = 0$

Not every function *f* is integrable. For example, if *f(x)* becomes positively or negatively infinite at some number in [*a,b*], then the definite integral does not exist. To illustrate, let *f* be defined on [*a,b*] and $\lim_{x \to a^+} f(x) = \infty$. In the first subinterval $[x_0, x_1]$ of any partition *P* of [*a,b*], a number w_1 can be found such that $f(w_1)\Delta x_1$ is larger than any given number *M*. it follows that for any partition *P*, we can form a Riemann sum $\sum_i f(w_i)\Delta x_i$ that is arbitrarily large. Hence if *I* is any real number an *P* any partition of [*a,b*], then there exist Riemann sums $R_P - I$ is arbitrarily large. This implies that *f* is not integrable. A similar argument can be given if *f* becomes infinite at any other number in [*a,b*]. Consequently, *if a function f is integrable on [a,b], then it is bounded on [a,b]*; that is, there is a real number *M* such that $|f(x)| \le M$ for all *x* in [*a,b*].

The reader may be tempted to conjecture that if a function is discontinuous somewhere in [*a,b*], then it is not integrable. This conjecture is false/ definite integrals of discontinuous functions may or may not exist, depending on the nature of the discontinuities. However, according to the next theorem, continuous functions are *always* integrable.

Theorem (4.3.6)

If *f* is continuous on [*a,b*], then *f* is integrable on [*a,b*]

A proof of Theorem 4.3.6 may be found in texts on advanced calculus.

If *f* is integrable, then the limit in Definition (4.3.3) exists for all choices of w_i in $[x_{i-1}, x_i]$. This fact allows us to specialize w_i if we wish to do so. For example, we could always choose w_i as the smallest number x_{i-1} in the subinterval, or as the largest number x_i, or as the midpoint of the subinterval, or as the number that always produces the minimum or maximum value in $[x_{i-1}, x_i]$, etc. in addition, since the limit is independent of th e partitions *P* of [*a,b*] (provided that ||P|| is sufficiently small) we may specialize the partition s to the case in which all the subintervals $[x_{i-1}, x_i]$ have the same length Δx. A partition of this type is called a **regular partition.** We will see in the next chapter that the specializations we have described are often used in applications. As an immediate illustration, we have the following important result.

Theorem (4.3.7)

If f is continuous and $f(x) \geq 0$ for all x in [a,b], then the area A of the region under the graph of f from a to b is given by

$$A = \int_a^b f(x)dx$$

Proof

The area A was defined, using (4.2.4), as the limit of the sum $\sum_i f(u_i)\Delta x$, where $f(u_i)$ is the minimum value of f in $[x_{i-1}, x_i]$. Since this is a special type of Riemann sum, the conclusion follows from Definition (4.3.3).

properties of the Definite Integral

This section contains some fundamental properties of the definite integral. Most of the proofs are rather technical and have been places in Appendix II, where the reader may study them whenever time permits.

Theorem (4.3.8)

$$\int_a^b kdx = k(b-a)$$

Proof

If f is the constant function defined by $f(x)=k$ for all x in [a,b] and P is a partition of [a,b], then for every Riemann sum of f.

$$\sum_i f(w_i)\Delta x_i = \sum_i k\Delta x_i = k\sum_i \Delta x_i = k(b-a)$$

Since the sum $\sum_i \Delta x_i$ is the length of the interval [a,b]. Consequently,

$$\left|\sum_i f(w_i)\Delta x_i - k(b-a)\right| = |k(b-a) - k(b-a)| = 0,$$

Which is less than any positive number ε *regardless* of the size of ||P||. Therefore, by Definition (4.3.2),

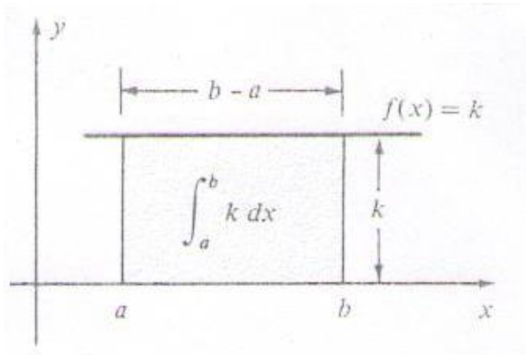

FIGURE 5.10

$$\lim_{\|P\|\to 0}\sum_i f(w_i)\Delta x_i = \lim_{\|P\|\to 0}\sum_i k\Delta x_i = k(b-a)$$

By Definition (5.8), this means that

$$\int_a^b f(x)dx = \int_a^b kdx = k(b-a)$$

Theorem (4.3.8) is in agreement with the discussion of area in Section 4.2, for if *k*>0, then the graph of *f* is a horizontal line *k* units above the *x*-axis, and the region under the graph of *a* to *b* is a rectangle with sides of length *k* and *b*-*a* as illustrated in Figure 5.10. Hence the area $\int_a^b f(x)dx$ of the rectangle is *k(b-a)*

Example 1

Evaluate $\int_{-2}^3 7dx$

Solution

Using (4.3.8),

$$\int_{-2}^3 7dx = 7[3-(-2)] = 7(5) = 35$$

For the special case of (4.3.8) with *k*=1 we shall abbreviate the integrand by writing

$$\int_a^b dx = b-a$$

If a function *f* is integrable on [*a*,*b*] and *k* is a real number, then by (ii) of Theorem (4.2.3) a Riemann sum of the function *kf* may be written

$$\sum_i kf(w_i)\Delta x_i = k\sum_i f(w_i)\Delta x_i$$

It is proved in Appendix II that the limit of the sum on the left is equal to *k* times the limit of the sum on the right. Restating this fact in terms of definite integrals gives us the next theorem.

Theorem (4.3.9) If *f* is integrable on [*a*,*b*] and *k* is any real number, then *kf* is integrable on [*a*,*b*] and

$$\boxed{\int_a^b kf(x)dx = k\int_a^b f(x)dx}$$

The conclusion of Theorem (4.3.9) is sometimes stated "*a constant factor in the integrand may be taken outside the integral sign.*" It is *not* permissible to take expressions involving variables outside the integral sign in this manner.

If two functions *f* and *g* are defined on [*a*,*b*], then by (i) of Theorem (4.2.3) a Riemann sum of *f*+*g* may be written

$$\sum_i [f(w_i)+g(w_i)]\Delta x_i = \sum_i f(w_i)\Delta x_i + \sum_i g(w_i)\Delta x_i$$

It can be shown that if *f* and *g* are integrable, then the limit of the sum on the left may be found by adding the limits of the two sums on the right. This fact is stated in integral form in (i) of the next theorem. A proof may be found in Appendix II. The analogous result for differences is stated in (ii).

Theorem (4.3.10)

if *f* and *g* are integrable on [*a*,*b*], then *f*+*g* and *f*-g are integrable on [*a*,*b*]and

(i) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(ii) $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

Theorem (4.3.10) may be extended to any finite number of functions. Thus, if f_1, $f_{2\ldots}$, f_n are integrable on [*a*,*b*], then so is their sum and

$$\int_a^b [f_1(x)+f_2(x)+\ldots+f_n(x)]dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \ldots + \int_a^b f_n(x)dx$$

Example 2

Given $\int_0^2 x^3 dx = 4$ and $\int_0^2 x dx = 2$, evaluate $\int_0^2 (5x^3 - 3x + 6)dx$

Solution

We may proceed as follows

$$\begin{aligned}\int_0^2 (5x^3-3x+6)dx &= \int_0^2 5x^3 dx - \int_0^2 3x dx + \int_0^2 6dx \\ &= 5\int_0^2 x^3 dx - 3\int_0^2 x dx + 6(2-0) \\ &= 5(4) - 3(2) + 12 = 26\end{aligned}$$

If f is continuous on $[a,b]$ and $f(x) \geq 0$ for all x in $[a,b]$, then by Theorem (4.3.7), the

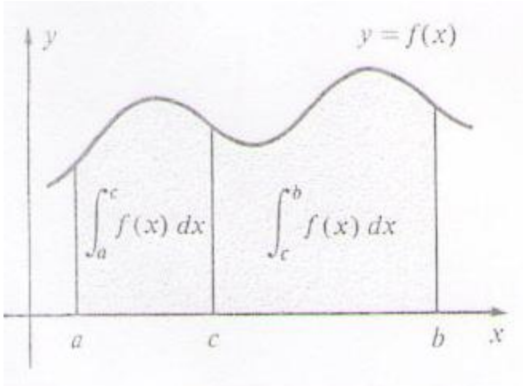

FIGURE 5.11

integral $\int_a^b f(x)dx$ is the area under the graph of f from a to b. in like manner, if $a<c<b$, then the integrals $\int_a^c f(x)dx$ and $\int_a^b f(x)dx$ are the areas under the graph of f from a to c and from c to b, respectively, as illustrated in Figure 11. since the area from a to b is the sum of the two smaller areas, we have

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

The next theorem shows that the last equality is true under a more general hypothesis. The proof is given in Appendix II.

Theorem (4.3.11)

If $a<c<b$, and if f is integrable on both $[a,b]$ and $[c,d]$, then f is integrable on $[a,b]$and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

The following result shows that Theorem (4.3.11) can be generalized to the case where c is not necessarily between a and b.

Theorem (4.3.12)

If f is integrable on a closed interval and if *a,b and c* are any three numbers in the intervals, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Proof

If *a,b and c* are all different, then there are six possible was of ordering these three numbers. The theorem should be verified for each of these cases as well as for the cases in which two or all three, of the numbers are equal. We shall verify one case and leave the remaining parts as exercises. Thus, suppose the number are ordered such that $c<a<b$. using Theorem (4.3.11)

$$\int_c^b f(x)dx = \int_c^a f(x)dx + \int_a^b f(x)dx$$

Which, in turn, may be written

$$A = \int_a^b f(x)dx = -\int_c^a f(x)dx + \int_c^b f(x)dx .$$

The desired conclusion now follows, since interchanging the limits of integration changes the sign of the integral (see Definition (4.3.5)).

If *f* and *g* are continuous on [*a*,*b*] and $f(x) \geq f(g) \geq 0$ for all *x* in [*a*,*b*], then the area under the graph of *f* from *a* to *b* is greater than or equal to the area under the graph of *g* from *a* to *b*. the corollary to the next theorem is a generalization of this fact to arbitrary integrable functions. The proof of the theorem is given in Appendix II.

Theorem (4.3.13)

If *f* is integrable on [*a*,*b*] and if $f(x) \geq 0$ for all *x* in [*a*,*b*], then

$$\int_a^b f(x)dx \geq 0$$

Corollary (4.3.14)

If *f* and *g* are integrable on [*a*,*b*] and $f(x) \geq f(g)$ for all *x* in [*a*,*b*], then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

4.4 The Mean Value Theorem for Definite Integrals

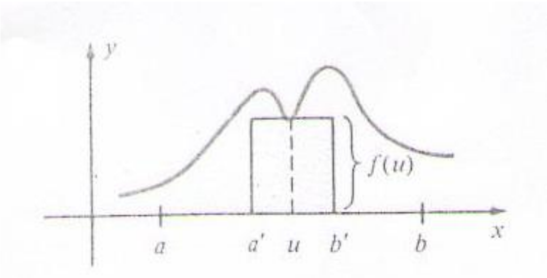

FIGURE 5.12

Suppose *f* is continuous and $f(x) \geq 0$ for all *x* in a closed interval [a,b]. if $f(c) \geq 0$ for some *c* in [a,b], then $\lim_{x \to c} f(x) > 0$ and, by an argument similar to that used in the proof of Theorem (2.7), there is an interval [a',b']contained in [a,b]throughout which *f(x)* is positive. Let *f(u)* be the minimum value of *f* on [a',b'], as illustrated in Figure 5.12. it follows that the area under the graph of *f* from *a* to *b* is at least as large as the area $f(u)(b'-a')$ of the pictured rectangle. Consequently, by Theorem (5.12), $\int_a^b f(x)dx > 0$.

This result can also be proved directly from the definition of the definite integral. Suppose the functions *f* and *g* are continuous on [*a*,*b*]. if $f(x) \geq f(g)$ for all *x* in [*a*,*b*], but $f \neq g$, then $f(x) - g(x) > 0$ for some *x* and, by the previous discussion, $\int_a^b (f(x) - g(x)]dx > 0$, consequently, $\int_a^b f(x)dx > \int_a^b g(x)dx$. This fact will be used in the proof of the next theorem.

The Mean value Theorem for Definite Integrals (4.4.1)

If f is continuous on a closed interval $[a,b]$, then there is a number z in the open interval (a,b) such that $\int_a^b f(x)dx = f(z)(b-a)$

Proof

If f is a constant function, then the result follows trivially from Theorem (4.3.8) where z is *any* number in (a,b). Next assume that f is not a constant function and suppose that m and M are the minimum and maximum values of f, respectively, on $[a,b]$. Let $f(u)=m$ and $f(v)=M$ where u and v are in $[a,b]$. Since f is not a constant function, $m<f(x)<M$ for some x in $[a,b]$ and hence by the remark immediately preceding this theorem.

$$\int_a^b m dx < \int_a^b f(x)dx < \int_a^b M dx$$

Employing Theorem (4.3.8),

$$m(b-a) < \int_a^b f(x)dx < M(b-a)$$

Dividing by b-a and replacing m and M by $f(u)$ and $f(v)$, it follows from the Intermediate Value Theorem (2.29) that there is a number z, strictly between u and v, such that

$$f(z) = \frac{1}{b-a}\int_a^b f(x)dx$$

Multiplying both sides by b-a gives us the conclusion of the theorem.

The number z of Theorem (4.4.1) is not necessarily unique. Indeed, as pointed out in the proof, if f is a constant function then *any* number z can be used. The theorem guarantees that at *least* one number z will produce the desired result.

The Mean Value Theorem has an interesting geometric interpretation if $f(c) \geq 0$ on [*a*,*b*]. in this case $\int_a^b f(x)dx$ is the area under the graph of *f* from *a* to *b* and the number *f(z)* in Theorem (4.4.1) is the y-coordinate of the point P on the graph of *f* having *x*-coordinate *z* (see Figure 5.13). if a horizontal line is drawn through *P*, then the are of the rectangular region bounded by this line, the x-axis, and the lines *x*=*a* and *x*=*b* is *f(z)(b-a)* which, according to Theorem (4.4.1), is the same as the are under the graph of *f* from *a* to *b.*

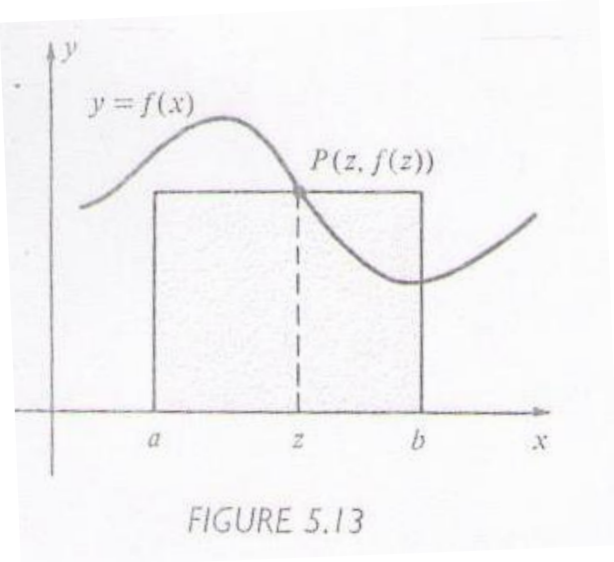

FIGURE 5.13

Example

It can be proved that $\int_0^3 [4-(x^2/4)]dx = \frac{39}{4}$. Find a number that satisfies the conclusion of the Mean Value Theorem for this integral.

Solution

According to the Mean Value Theorem for Definite Integrals, there is a number *z* between 0 and 3 such that

$$\int_0^3 [4-(x^2/4)]dx = \left(4-\frac{z^2}{4}\right)(3-0)$$

Or, equivalently, $$\frac{39}{4} = \left(\frac{16-z^2}{4}\right)(3)$$

Multiplying both sides of the last equation by $\frac{4}{3}$ leads to $13=16-z^2$ and, therefore, $z^2=3$. Consequently, $\sqrt{3}$ satisfies the conclusion of Theorem (4.4.1).

The Mean Value Theorem for Definite Integrals can be used to help prove a number of important theorems. One of the most important is the *Fundamental Theorem of Calculus* given in the next section.

4.5 The Fundamental Theorem of Calculus

The task of evaluating a definite integral by means of Definition (4.3.3) is quite difficult even in the simplest cases. This section contains a theorem that can be used to find the definite integral without using limits of sums. Due to its importance in evaluating definite integrals, and because it exhibits the connection between differentiation and integration, the theorem is aptly called *the Fundamental Theorem of Calculus.* This theorem was discovered independently by Sir Isaac Newton (1642-1727) in England and by Gottfried

Wilhelm Leibniz (1646-1716) in Germany. It is primarily because of this discovery that these outstanding mathematicians are credited with the invention of calculus.

To avoid confusion, in the following discussion we shall use the variable *t* and denote the definite integral of *f* from *a* to *b* by $\int_a^b f(t)dt$. If *f* is continuous on [*a*,*b*] and if *x* is in [*a*,*b*], then *f* is continuous on [*a*,*x*] and hence by Theorem (5.11) *f* is integrable on [*a*,*x*]. Consequently, the equation

$$G(x) = \int_a^x f(t)dt$$

Defines a function *G* with domain [*a.b*], since for each *x* in [*a*,*b*]there corresponds a unique number *G(x)*. the next theorem brings out the remarkable fact that *G* is an antiderivative of *f*. in addition, it shows how any antiderivative may be used to find a definite integral of *f*.

The Fundamental Theorem of Calculus (4.5.1)

Suppose *f* is continuous on a closed interval [*a*,*b*].
Part I if the function *G* is defined by

$$G(x) = \int_a^x f(t)dt$$

For all *x* on [*a*,*b*], then *G* is an antiderivative of *f* on [*a*,*b*].
Part II if *F* is any antiderivative of *f*, then

$$\int_a^b f(t)dt = F(b) - F(a)$$

Proof to establish Part I we must show that if *x* is in [*a*,*b*], then $G'(x) = f(x)$, that is,

$$\lim_{h\to 0}\frac{G(x+h)-G(x)}{h} = f(x)$$

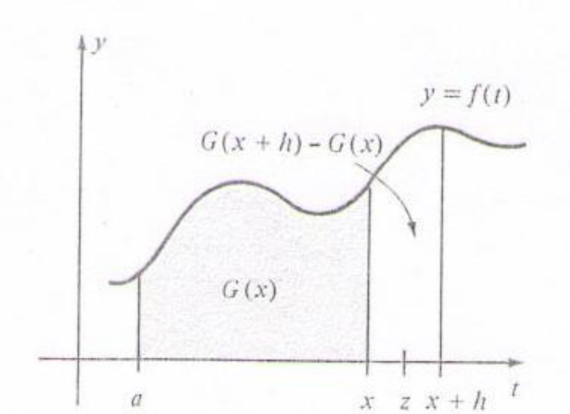

FIGURE 5.14

Before giving a formal proof, it is instructive to consider some geometric aspects of this formula. If $f(x) \geq 0$ throughout [*a*,*b*], then *G(x)* is the area under the graph of *f*from *a* to *x* , as illustrated in Figure 5.14. if *h>0*, then the difference *G(x+h)-G(x)* is the area under the graph of *f* from *x* to *x+h*, the number *h* is the length of the interval [*x*,*x+h*], and *f(x)* is the y-coordinate of the point with x-coordinate *x* on the graph of *f*. we will show that $\frac{G(x+h)-G(x)}{h} = f(z)$, where *z* is between *x* and *x+h*. reasoning intuitively, it appears that if *h*→0, then *z*→*x* and *f(z)*→*f(x)*, which is what we wish to prove.

Let us now give a rigorous proof that $G'(x) = f(x)$. If *x* and *x+h* are in [*a*,*b*], then using the definition of *G*, together with Definition (4.3.4) and Theorem (4.3.11),

$$\begin{aligned} G(x+h) - G(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_a^{x+h} f(t)dt + \int_x^a f(t)dt \\ &= \int_x^{x+h} f(t)dt \end{aligned}$$

Consequently, if $h \neq 0$

$$G(x+h) - G(x) = \frac{1}{h}\int_a^{x+h} f(t)dt$$

If *h>0*, then by the Mean Value Theorem for Integrals (4.4.1), there is a number *z*(depending on *h*) in the open interval (*x*,*x+h*) such that

$$\int_x^{x+h} f(t)dt = f(z)h$$

And, therefore,

$$\frac{G(x+h) - G(x)}{h} = f(z)$$

Since *x<z<x+h* it follows from the continuity of *f* that

$$\lim_{h\to 0^+} f(z) = \lim_{h\to x^+} f(z) = f(x)$$

And hence

$$\lim_{h\to 0} \frac{G(x+h) - G(x)}{h} = \lim_{h\to x^+} f(z) = f(x)$$

If *h<0*, then we may prove in similar fashion that

$$G'(x) = \lim_{h\to 0} \frac{G(x+h) - G(x)}{h} = f(x)$$

Which is what we wished to prove.

To prove Part II, let *F* be any antiderivative of *f* and let *G* be the special antiderivative defined in Part I. it follows from Theorem (4.32) that *F* and *G* differ by a constant; that is, there is a number *C* such that *G(x)-F(x)=C* for all *x* in [*a*.*b*]. hence, from the definition of *G*,

$$\int_a^x f(t)dt - F(x) = C$$

For all *x* in [*a*,*b*]. if we let *x*=*a* and use the fact that $\int_a^a f(t)dt = 0$, we obtain $0 - F(a) = C$. Consequently,

$$\int_a^x f(t)dt - F(x) = -F(a).$$

Since this is an identity for all *x* in [*a*,*b*] we may substitute *b* for *x*. obtaining

$$\int_a^b f(t)dt - F(b) = -F(a)$$

Adding *F(b)* to both sides of this equation and replacing the variable *b* by *x* gives us the desired conclusion.

It is customary to denote the difference *F(b)-F(a)* either by the symbol $F(x)]_a^b$ or by $\left[f(x)\right]_a^b$. Part II of the Fundamental Theorem may then be expressed as follows.

Theorem (4.5.2)

If *f* is continuous on [*a*,*b*] and *F* is any antiderivative of *f*, then

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$$

The formula in Theorem (4.5.2) is also valid if $a \geq b$, thus, if *a*>*b*, then by Definition (5.9),

$$\begin{aligned} \int_a^b f(x)dx &= -\int_b^a f(x)dxF(x) \\ &= -\left[F(a) - F(b)\right] \\ &= F(b) - F(a) \end{aligned}$$

If a=b, *then by Definition (4.3.5),*

$$\int_a^a f(x)dx = 0 = F(a) - F(a)$$

Example 1

Evaluate $\int_{-2}^{3} (6x^2 - 5)dx$

Solution

An antiderivative of $6x^2 - 5$ is given by $F(x) = 2x^2 - 5x$ / Applying Theorem (4.5.2)

$$\int_{-2}^{3}(6x^2-5)dx=\left[2x^3-5x\right]_{-2}^{3}$$
$$=\left[2(3)^3-5(3)\right]-\left[2(-2)^3-5(-2)\right]$$
$$=\left[54-15\right]-\left[-16+10\right]=45$$

Note that if *F(x)+C* is used in place of *F(x)* in Theorem (4.5.2), the same result is obtained, since

$$\left[F(x)+C\right]_a^b=\left\{F(b)+C\right\}-\left\{F(a)+C\right\}$$
$$=F(b)-F(a)=\left[F(x)\right]_a^b$$

This is in keeping with the statement that *any* antiderivative *F* may be employed in the Fundamental Theorem. Also note that for any number *k*,

$$\left[kF(x)\right]_a^b=kF(b)-kF(a)=k\left[F(b)-F(a)\right]=k\left[F(x)\right]_a^b$$

That is, a constant factor can be "taken out" of the bracket when using this notation. This result is analogous to Theorem (4.3.9) on define integrals.

Theorem (4.5.3)

If *k* is a real number, and *r* is a rational number such that $r\neq -1$, then
$$\int_a^b kx^r\,dx=\left[\left(\frac{k}{r+1}\right)x^{r+1}\right]_a^b=\left(\frac{k}{r+1}\right)\left(b^{r+1}-a^{r+1}\right)$$

Proof

If $f(x)=kx^r$, then the function *F* defined by $F(x)=\left[k/(r+1)\right]x^{r+1}$ is an antiderivative of *f* is defined. Applying Theorem (4.5.2) gives us the conclusion.

If an integrand is a sum of terms of the form kx^r where $r\neq -1$, then (4.5.3) may be applied to each term, as illustrated in the next example.

Example 2

Evaluate $\int_{-1}^{2}(x^3+1)^2\,dx$

Solution

Squaring the integrand and then applying Theorem (4.5.3) to each term gives us

$$\begin{aligned}\int_{-1}^{2}(x^3+1)^2\,dx &= \int_{-1}^{2}(x^6+2x^3+1)dx \\ &= \left[\frac{1}{7}x^7+\frac{2}{4}x^4+x\right]_{-1}^{2} \\ &= \left[\frac{1}{7}(2)^7+\frac{1}{2}(2)^4+2\right]-\left[\frac{1}{7}(-1)^7+\frac{1}{2}(-1)^4+(-1)\right] \\ &= \frac{405}{14}\end{aligned}$$

In the preceding example it is very important to note that

$$\int_{-1}^{2}(x^3+1)^2\,dx \neq \left.\frac{(x^3+1)^3}{3}\right]_{-1}^{2}$$

Example 3

Evaluate $\int_{1}^{4}(5x-2\sqrt{x}+\frac{32}{x^3})dx$

Solution

We begin by changing the form of the integrand so that Theorem (4.5.3) may be applied to each term. Thus

$$\begin{aligned}\int_{1}^{4}(5x-2x^{1/2}+32x^{-3})dx &= \left[\frac{5}{2}x^2-\frac{2}{(3/2)}x^{3/2}+\frac{32}{-2}x^{-2}\right]_{1}^{4} \\ &= \left[\frac{5}{2}x^2-\frac{4}{3}x^{3/2}-\frac{16}{x^2}\right]_{1}^{4} \\ &= \left[\frac{5}{2}(4)^2-\frac{4}{3}(4)^{3/2}-\frac{16}{4^2}\right]-\left[\frac{5}{2}-\frac{4}{3}-16\right] \\ &= 259/6\end{aligned}$$

Example 4

Evaluate $\int_{-2}^{3}|x|\,dx$

Solution

By Definition (1.2), $|x| = -x$ if $x < 0$ and $|x| = x$ if $x \geq 0$. This suggests that we use Theorem (4.3.11) to express the given integral as a sum of two definite integrals as follows:

$$\begin{aligned}\int_{-2}^{3} |x|\,dx &= \int_{-2}^{0} |x|\,dx + \int_{0}^{3} |x|\,dx \\ &= \int_{-2}^{0} (-x)dx + \int_{0}^{3} xdx \\ &= -\left[\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{3} \\ &= -\left[0 - \frac{4}{2}\right] + \left[\frac{9}{2} - 0\right] \\ &= 2 + \frac{9}{2} = \frac{13}{2}\end{aligned}$$

Using the antiderivative formulas given in Theorem together with Part II of the Fundamental Theorem (4.5.1) leads to the following definite integral formulas for the sine and cosine functions.

Theorem (4.5.4)

$$\int_a^b \sin kxdx = -\frac{1}{k}\cos kx\Big]_a^b$$

$$\int_a^b \cos kxdx = \frac{1}{k}\sin kx\Big]_a^b$$

Example 5

Find the area *A* under the graph of *y=sin2x* from *x=0* to *x=π/4*

Solution

By Theorems (4.3.7) and (4.5.4)

$$A = \int_0^{\pi/4} \sin 2xdx = -\frac{1}{2}\cos 2x\Big]_0^{\pi/4}$$

$$= -\frac{1}{2}\left[\cos 2\left(\frac{\pi}{4}\right) - \cos 2(0)\right]$$

$$= -\frac{1}{2}\left[\cos\frac{\pi}{2} - \cos 0\right] = -\frac{1}{2}[0-1] = \frac{1}{2}$$

The technique of defining a function by means of a definite integral, as in Part I of the Fundamental Theorem of Calculus (4.3.7), will have a very important application in Chapter 7 (see Definition (7.3)). Recall, from (4.5.1), that if f is continuous on [a,b] and the function G is defined by $G(c)=\int_a^x f(t)dt$ where $a \le x \le b$, then G is an antiderivative of f; that is, $D_x G(x) = f(x)$. This may be stated in integral form as follows:

$$D_x \int_a^x f(t)dt = f(x)$$

The preceding formula is generalized in the next theorem.

Theorem (4.5.5)

Let f be continuous on [a,b]. if $a \le c \le b$, then

$$D_x \int_a^x f(t)dt = f(x)$$

For all x in [a,b]

Proof

If F is an antiderivative of f , then

$$\begin{aligned} D_x \int_a^x f(t)dt &= D_x\left[F(x)-F(c)\right] \\ &= D_x F(x) - D_x F(c) \\ &= f(x) - 0 = f(x) \end{aligned}$$

Example 6

If $G(x)=\int_1^x \frac{1}{t}dt$, where $x>0$, find $G'(x)$.

Solution

Let $f(x)=1/x$ and consider any interval [a,b]where $a>0$ and $b \ge 1$. If $a \le x \le b$, then by Theorem (4.5.5),

$$G'(x) = D_x \int_1^x \frac{1}{t}dt = \frac{1}{x}.$$