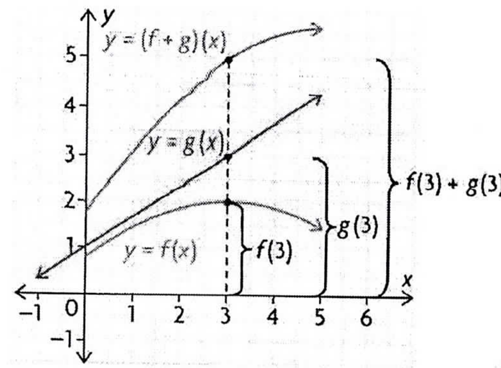
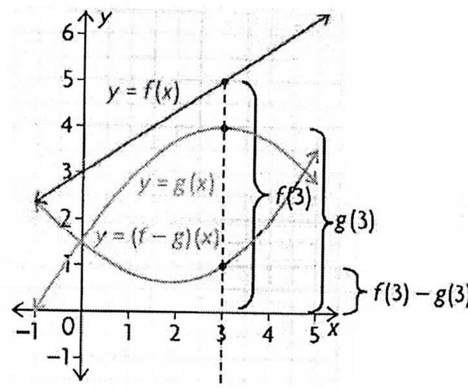
Combining Two Functions: Sums and Differences



- When two functions $f(x)$ and $g(x)$ are combined to form the function $(f+g)(x)$, the new function is called the sum of f and g. For any given value of x, the value of the function is represented by $f(x)+g(x)$. The graph of $f+g$ can be obtained from the graphs of functions f and g by adding the corresponding y-coordinates.

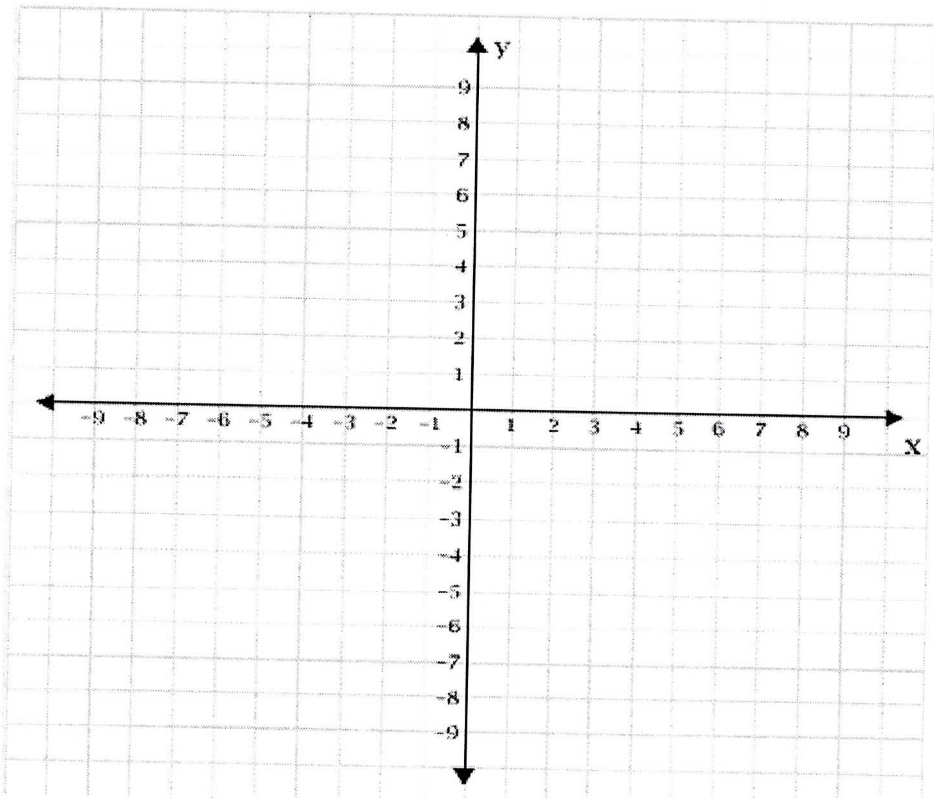


- Similarly, the difference of two functions, $f-g$, is $(f-g)(x)=f(x)-g(x)$. The graph of $f-g$ can be obtained by subtracting the y-coordinate of g from the y-coordinate of f for every pair of corresponding x-values.

- Algebraically, $(f+g)(x)=f(x)+g(x)$ and $(f-g)(x)=f(x)-g(x)$.
- The domain of $f+g$ or $f-g$ is the intersection of the domains of f and g. This means that the functions $f+g$ and $f-g$ are only defined where the domains of both f and g overlap.

Example 1

Given $f(x) = -x^2 + 3$ and $g(x) = -2x$, determine the graphs of $f(x) + g(x)$ and $f(x) - g(x)$. Discuss the key characteristics of the resulting graphs.



Example 2

Given $f = \{(-7, 3), (-4, 0), (-2, -8), (1, -3), (4, 0), (6, 2)\}$ and $g = \{(-7, -5), (-5, -1), (-3, 2), (1, 10), (3, -4), (6, 9)\}$, calculate

a) $f + g$

b) $g - f$

c) $g + g$

d) $f - f$

Example 3

a) If $f(x) = \log(3 - x)$ and $g(x) = \sqrt{x + 3}$, determine the domain of $f + g$.

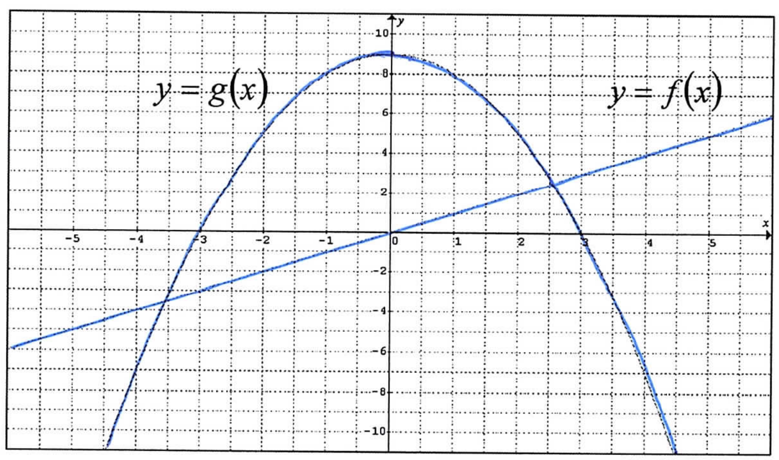
b) If $f(x) = \dfrac{1}{2x + 5}$ and $g(x) = \dfrac{1}{x - 4}$, determine the domain of $f - g$.

Combining Two Functions: Products

When two functions, $f(x)$ and $g(x)$, are combined to form the function $(f \times g)(x)$, the new function is called the product of f and g. For any given value of x, the function value is represented by $f(x)\times g(x)$. The graph of $f \times g$ can be obtained from the graphs of functions f and g by multiplying each y-coordinate of f by the corresponding y-coordinate of g.

- Algebraically, $f \times g$ is defined as $(f \times g)(x) = f(x) \times g(x)$.
- The domain of $f \times g$ is the intersection of the domains of f and g.
- If $f(x) = 0$ or $g(x) = 0$, then $(f \times g)(x) = 0$.
- If $f(x) = \pm 1$, then $(f \times g)(x) = \pm g(x)$. Similarly, if $g(x) = \pm 1$, then $(f \times g)(x) = \pm f(x)$.

Example 1 Determine the graph of $y = (f \times g)(x)$, given the graphs of $f(x) = x$ and $g(x) = -x^2 + 9$.



Example 2

For each of the following pairs of functions, determine $(f \times g)(x)$ and state its domain. Then find $(f \times g)(-2)$, if possible.

a) $f(x) = \{(-6, 1), (-4, -3), (-2, 0), (1, 0), (3, 5), (10, -6)\}$,
$g(x) = \{(-8, 12), (-4, 2), (-2, -7), (0, -6), (5, -9), (10, -1)\}$

b) $f(x) = \sqrt{x+2}$, $g(x) = \dfrac{1}{x+2}$

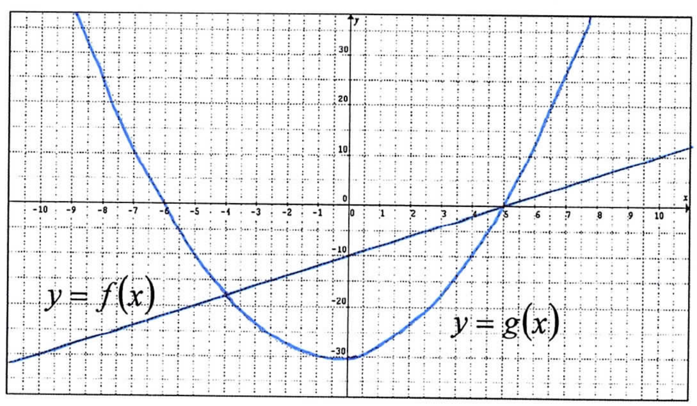
c) $f(x) = \log(x+5)$, $g(x) = 2^x$

d) $f(x) = 5x^2 - 20$, $g(x) = \dfrac{1}{3x+6}$

Exploring Quotients of Functions

- When two functions, $f(x)$ and $g(x)$, are combined to form the function $(f \div g)(x)$, the new function is called the quotient of f and g. For any given value of x, the value of the function is represented by $f(x) \div g(x)$. The graph of $f \div g$ can be obtained from the graphs of functions f and g by dividing each y-coordinate of f by the corresponding y-coordinate of g.
- Algebraically, $(f \div g)(x) = f(x) \div g(x)$.
- $f \div g$ will be defined for all x-values that are in the intersection of the domains of f and g, except in the case where $g(x) = 0$. If the domain of f is A, and the domain of g is B, then the domain of $f \div g$ is $\{x \in \Re \mid x \in A \cap B,\ g(x) \neq 0\}$.
- If $f(x) = 0$ when $g(x) \neq 0$, then $(f \div g)(x) = 0$.
- If $f(x) = \pm 1$, then $(f \div g)(x) = \pm \frac{1}{g(x)}$. Similarly, if $g(x) = \pm 1$, then $(f \div g)(x) = \pm f(x)$. Also, if $f(x) = \pm g(x)$, then $(f \div g)(x) = \pm 1$.

Example 1 Given $f(x) = 2x - 10$ and $g(x) = x^2 + x - 30$, graph the function $y = (f \div g)(x)$.



Example 2

Determine $(f \div g)(x)$ for each of the following pairs of functions, and state its domain.

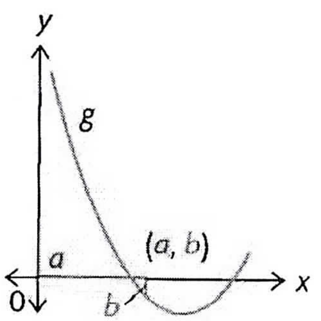
a) $f(x) = \sqrt{1-2x}$, $g(x) = x^2 + 9$

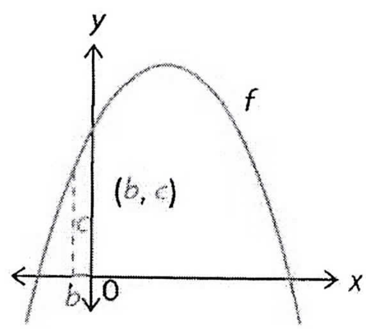
b) $f(x) = 4^x$, $g(x) = \log(x-7)$

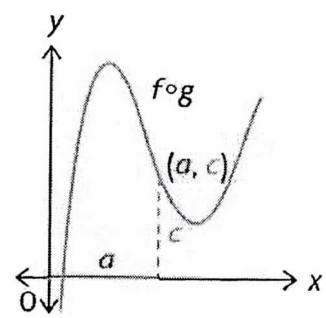
c) $f(x) = \sin x$, $g(x) = \cos x$

Composition of Functions

- Two functions, f and g, can be combined using a process called composition, which can be represented by $f(g(x))$. The output for the inner function g is used as the input for the outer function f. The function $f(g(x))$ can be denoted by $(f \circ g)(x)$.
- Algebraically, the composition of f with g is denoted by $(f \circ g)(x)$, whereas the composition of g with f is denoted by $(g \circ f)(x)$. In most cases, $(f \circ g)(x) \neq (g \circ f)(x)$ because the order in which the functions are composed matters.
- Let $(a, b) \in g$ and $(b, c) \in f$. Then $(a, c) \in f \circ g$. A point in $f \circ g$ exists where an element in the range of g is also in the domain of f. The function $f \circ g$ exists only when the range of g overlaps the domain of f.







- The domain of $(f \circ g)(x)$ is a subset of the domain of g. It is the set of values, x, in the domain of g for which $g(x)$ is in the domain of f.
- If both f and f^{-1} are functions, then $(f^{-1} \circ f)(x) = x$ for all x in the domain of f, and $(f \circ f^{-1})(x) = x$ for all x in the domain of f^{-1}.

Example 1

If $f(x) = x^2 + 2x$ and $g(x) = 10 - 3x$, find the following.

a) $f(g(-4))$

b) $(g \circ f)(6)$

c) $(f \circ f)(1)$

d) $g(g(0))$

Example 2

Let $f(x) = \dfrac{1}{x-1}$ and $g(x) = \sqrt{x+3}$. Determine $f \circ g$, and find its domain.

Example 3

Given the functions $f(x) = x^2$ and $g(x) = \log x$, determine whether $(f \circ g)(x) = (g \circ f)(x)$.

Example 4 Show that if $f(x)=\dfrac{2x-7}{x+9}$, then $(f \circ f^{-1})(x)=(f^{-1} \circ f)(x)=x$.

Example 5 Given $h(x)=\sin(6x^3+5)$, find two functions f and g such that $h=f \circ g$.

Exercises:

1. Consider $f(x) = 4 - x^2$, $g(x) = \sqrt{x+3}$, $h(x) = \frac{1}{2x}$. Evaluate the following.
   (a) $(f \circ g)(1)$
   (b) $(g \circ h)(1)$
   (c) $(f \circ g)(x)$
   (d) $(g \circ h)(x)$
   (e) $(h \circ g)(x)$
   (f) $(f \circ g)(x^2)$
   (g) $(f \circ g \circ h)(x)$

2. Using the functions given in the previous exercise, explain why $(f \circ g)(-4)$ does not exist.

3. Let $s(x) = \sqrt{x}$ and $t(x) = x^2 + 2x + 1$. Evaluate $(s \circ t)(x)$ and state its domain and range.

4. $f(x) = \sqrt{\frac{1}{x^2+2}}$. Write $f(x)$ as the composition of two or more functions.

Answers 1

Section 1

1. (a) 0 (b) $\sqrt{\frac{7}{2}}$ (c) $1 - x$ (d) $\sqrt{\frac{1}{2x} + 3}$ (e) $\frac{1}{2\sqrt{x+3}}$ (f) $1 - x^2$ (g) $1 - \frac{1}{2x}$

2. The number -4 is not in the domain of g.

3. We have $(s \circ t)(x) = \sqrt{x^2 + 2x + 1}$. Its domain is $\mathbb{R}$ and its range is $[0, \infty)$.

4. We have $f(x) = (g \circ h)(x)$ where $g(x) = \sqrt{x}$ and $h(x) = \frac{1}{x^2+2}$. Another possibility is $g(x) = \sqrt{\frac{1}{x+2}}$ and $h(x) = x^2$.

5. For each function h given below, decompose h into the composition of two functions f and g so that $h = f \circ g$.

(a) $h(x) = (x+5)^2$

(b) $h(x) = \sqrt[3]{5x^2+1}$

(c) $h(x) = 2^{\cos x}$

(d) $h(x) = \cos(2^x)$

(e) $h(x) = \dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}$