Lesson 1.

Definition of Limit : Continuity;

Limit Laws (Properties) :

In a Day school.

G 12. MHF4U — Advanced Functions

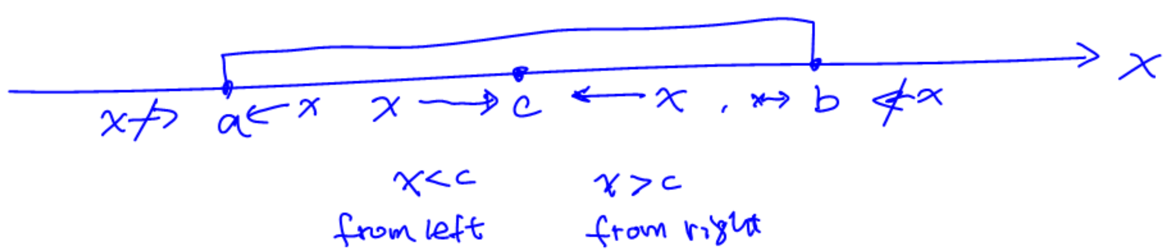
MCV4U — Calculus and Vectors.

MDM4U — Math of Data Management.

Definition of Limit.

For a function $y=f(x)$ that is defined over an interval $x \in [a, b]$, containing a x-value c.

However, $y=f(x)$ may not be defined at $x=c$, or $f(c)$ does not exist. but we could still discuss the Limit of $f(x)$ at $x=c$. Graphically



For example,

$$f(x) = \frac{x^2-1}{x-1}$$

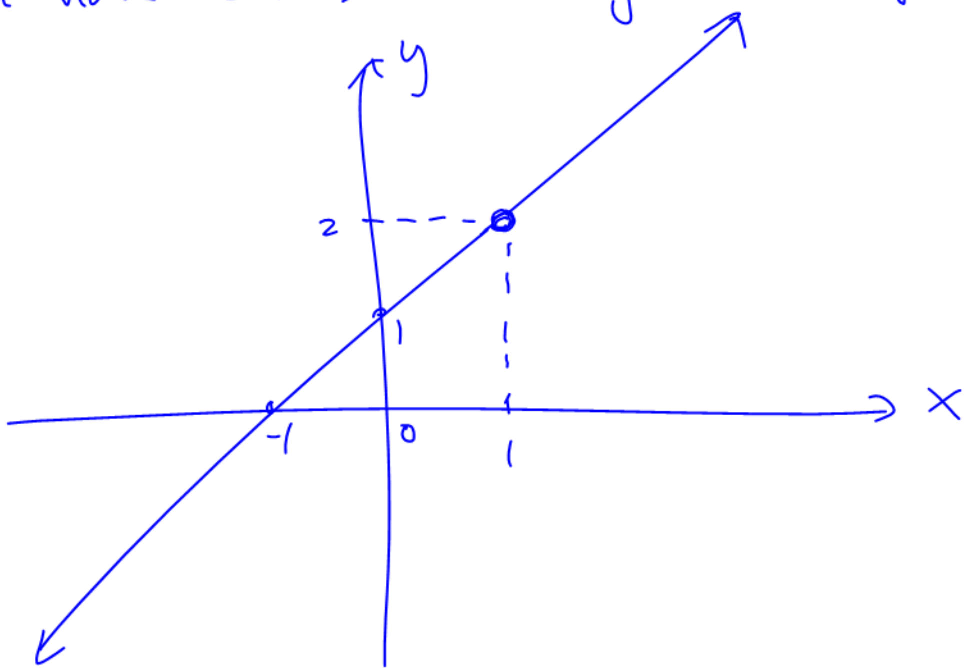
is defined over $x \in (-\infty, \infty)$ except $x=1$.

or we could make $x \in [-2, 2]$. except $x=1$.

Since $f(x) = \frac{x^2-1}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1$

so $f(x) = \frac{x^2-1}{x-1}$ is the same as $y = x+1$ except at $x=1$.

there is a hole $(1, 2)$ on the graph of $f(x) = \frac{x^2-1}{x-1}$.



$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x \to 1} (x+1)$$

$$= 1+1 = 2.$$

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots, \left(\frac{1}{2}\right)^n, \cdots$

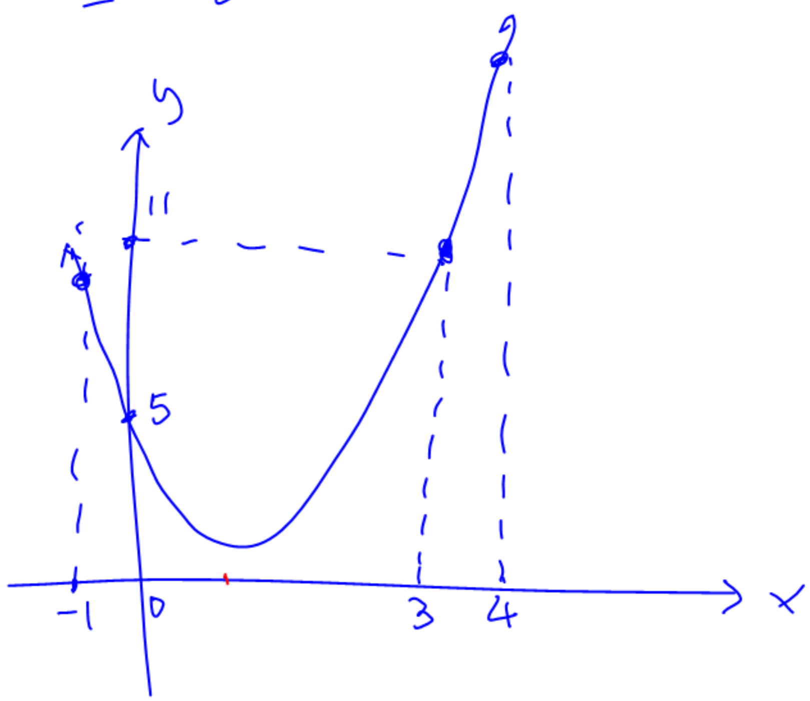
$$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$$

$$f(n) = \left(\frac{1}{2}\right)^n$$

If $f(x) = x^2 - x + 5$, where $x \in [-1, 4]$

let $c = 3$.

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} [x^2 - x + 5]$$
$$= 3^2 - 3 + 5 = 9 - 3 + 5 = 11 = f(3)$$



Definition of Limit.

For a function $y=f(x)$ defined over $x \in [a,b]$ that contains $x=c$, and $f(c)$ may not exist.

$\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$.

where L is a real number constant.

$\lim_{x\to c^-} f(x)$ is the "left-hand side limit".

$x\to c^- \iff x\to c$ and $x<c$.

$\lim_{x\to c^+} f(x)$ is the "right-hand side limit".

$x\to c^+ \iff x\to c$ and $x>c$.

Both $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ are also called "two one-sided limits".

$\lim_{x\to 0^-} \frac{1}{x} = \frac{1}{0^-} = -\infty$;

if $x\to 0^-$. may consider $x \approx -0.000\cdots01$

$\frac{1}{x} \approx \frac{1}{-0.000\cdots01} = -1000\cdots0$

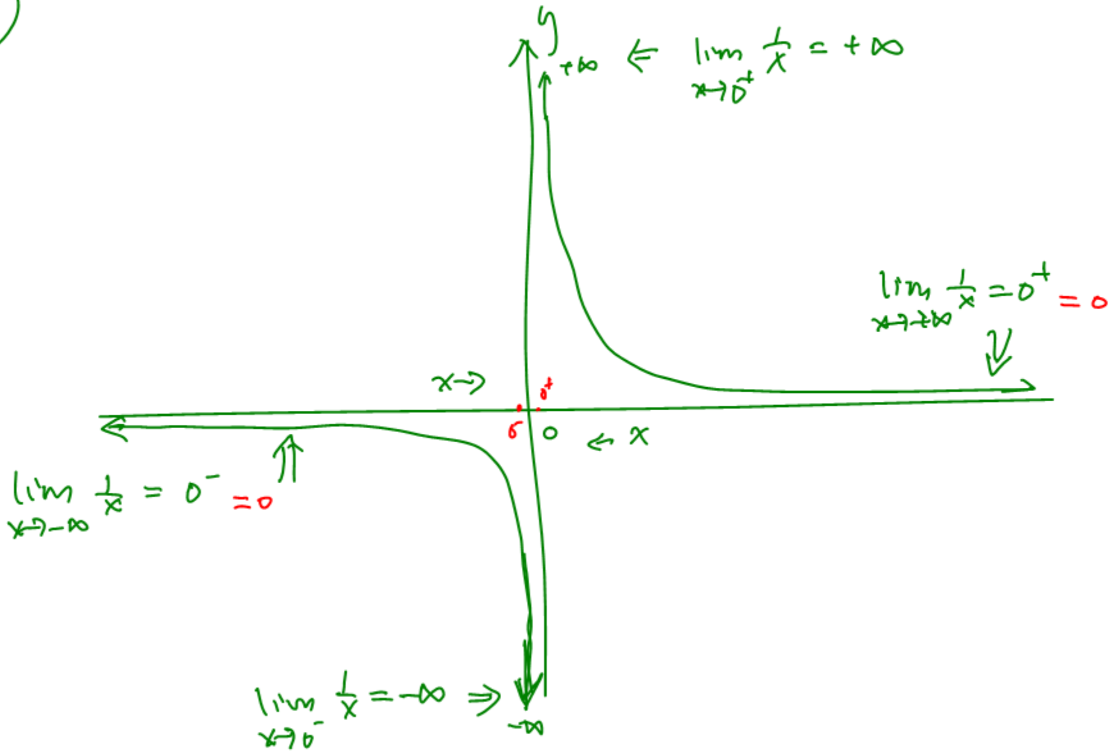
$\lim_{x\to 0^+} \frac{1}{x} = \frac{1}{0^+} = +\infty$;

if $x\to 0^+$, may consider $x \approx +0.000\cdots01$

$\frac{1}{x} \approx \frac{1}{+0.000\cdots01} = +1000\cdots0$

$y = \frac{1}{x}$

$+\infty$



Definition of Continuity

A function $y=f(x)$ is cotinuous at $x=c$ if $\boxed{\lim_{x\to c} f(x) = f(c)}$. Otherwise, $y=f(x)$ is discontinuous at $x=c$.

So possible steps to check continuity:

i) Check whether $f(c)$ exists or not.
   if $f(c)$ DNE, then $y=f(x)$ is discontinuous at $x=c$.

ii) Check whether $\lim_{x\to c} f(x)$ exists or not.
   if $\lim_{x\to c} f(x)$ DNE, then $y=f(x)$ is discontinuous at $x=c$.

iii) If $\lim_{x\to c} f(x) = L$ exists, then check whether $L = f(c)$ or not. if not, $y=f(x)$ is discontinuous at $x=c$.

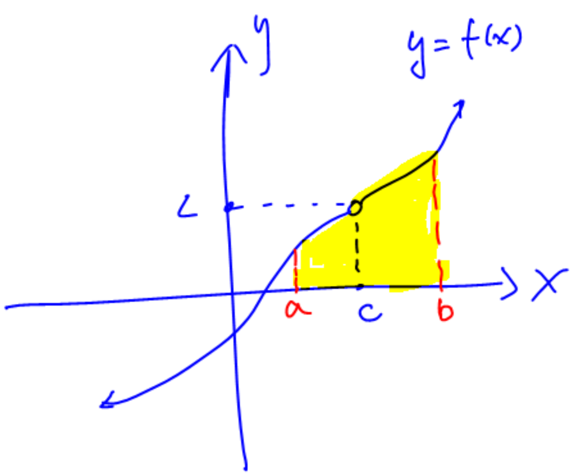
Usually, we need to identify type of discontinuity:

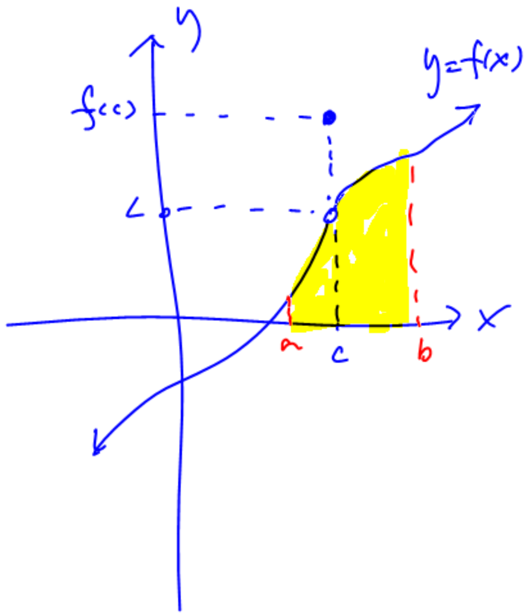
Three types of discontinuity:

① Removable discontinuity.

If $\lim_{x \to c} f(x) = L$ but $L \neq f(c)$.

Two possible examples:





f(c) DNE.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

converges

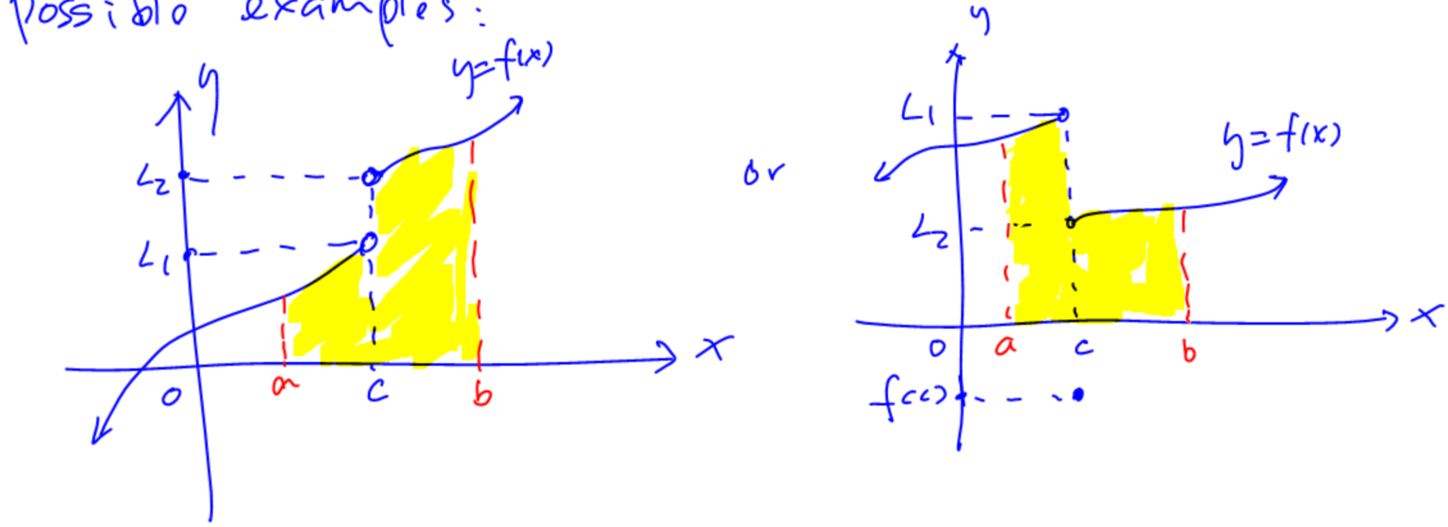
② Jump discontinuity.

If $\lim_{x \to c^-} f(x) = L_1$ and $\lim_{x \to c^+} f(x) = L_2$

but $L_1 \neq L_2$ so $\lim_{x \to c} f(x)$ DNE.

No matter what value $f(c)$ is.

possible examples:



$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

converges.

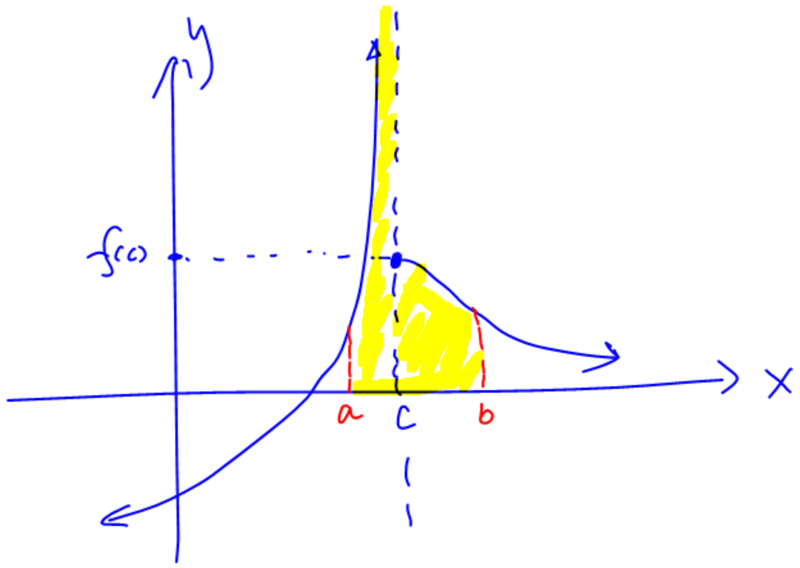
improper integrals

③ Infinite discontinuity

If one of the two one-sided limits (or both) is (are) infinite, so $\lim_{x \to c} f(x)$ DNE.

then no matter what $f(c)$ is.

Possible examples:



In this case, $\lim_{x \to c^-} f(x) = +\infty$,

$\lim_{x \to c^+} f(x) = f(c)$.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$\Downarrow$ diverges or converges

$\Downarrow$ converges

diverges or converges