Unit: Introduction to Vectors

Scalars and Vectors

Scalars (in Mathematics and Physics) are quantities described completely by a number and eventually a measurement unit.	**Vectors** are quantities described by a magnitude (length, intensity or size) and direction.

Ex 1. Classify each quantity as scalar or vector.

a) time　　b) position　　c) temperature

d) electric charge　　e) mass　　f) force

Geometric Vectors

<u>Geometric Vectors</u> are vectors not related to any coordinate system.

To represent vectors we use **<u>rays</u>** (directed line segments).

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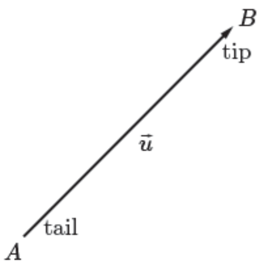
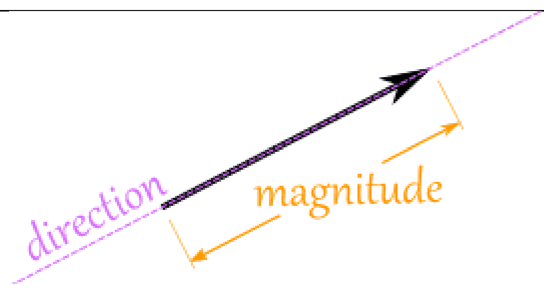
<td>

where A is called the initial (start, tail) point and B is called the final (end, terminal, head or tip) point.</td>
<td>

The length of the ray is a positive real number, which represents the magnitude (size, norm or intensity) of the vector.
The magnitude of the vector $\vec{u}$ is denoted by $|\vec{u}|$, $\|\vec{u}\|$ or u.</td>
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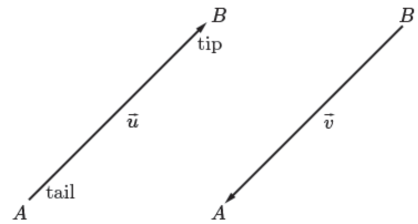
Equivalent or Equal Vectors

Two vectors are **equal (or equivalent)** if and only if they have the same **magnitude** and **direction**. They do not need to be in the same position.

$\overrightarrow{AB} = \overrightarrow{PQ}$

Opposite Vectors

Two vectors are called opposite if they have the same magnitude and opposite direction.



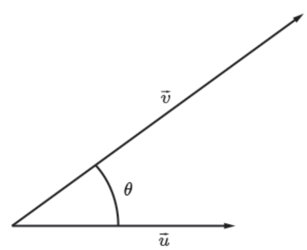
$\vec{u} = -\vec{v}$

Parallel (Collinear) Vectors

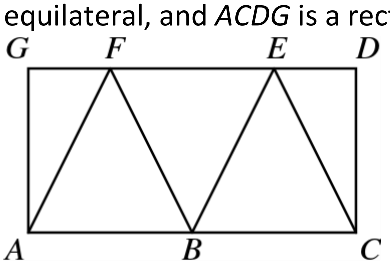
Two vectors are **parallel** if their directions are either the same or opposite.

Angle between Vectors

The angle between two vectors is the angle ≤180° formed when the vectors are placed **tail to tail**, that is, starting at the same point.



Ex 2. In the diagram below, $\triangle AFB$ and $\triangle BEC$ are equilateral, and $ACDG$ is a rectangle.

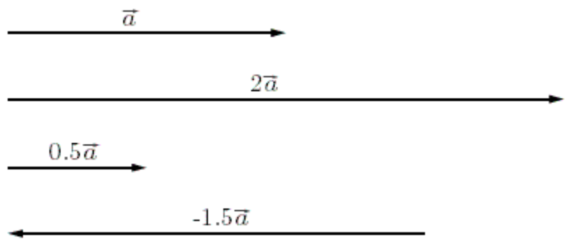


a. Write down two other vectors **equal** to $\overrightarrow{AB}$.
b. Write down three vectors which are **opposite** to $\overrightarrow{FE}$.
c. What vector is the **opposite** of $\overrightarrow{DC}$?
d. Write down 3 vectors which have the same magnitude as $\overrightarrow{BC}$, but different direction.
e. Write down two other vectors **parallel** to $\overrightarrow{AB}$.

Scalar Multiplication

In general, given some real number k, $\boldsymbol{k\vec{a}}$ is a **vector** with the following attributes:

a. Its *magnitude* is $\boldsymbol{|k\vec{a}|=|k||\vec{a}|}$.
ii. Its *direction* is the same as $\vec{a}$ if $k>0$ and opposite to $\vec{a}$ if $k<0$ (if $k=0$, then $k\vec{a}=\vec{0}$).



Important: *When two vectors are parallel, one of the vectors can be expressed in terms of the other using scalar multiplication.*

Unit Vector

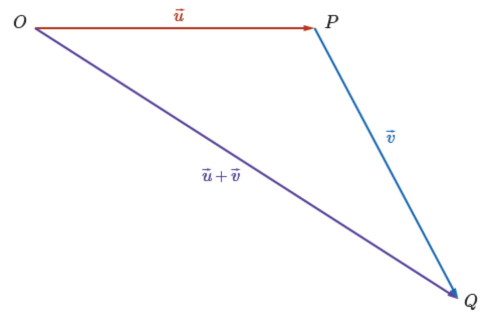
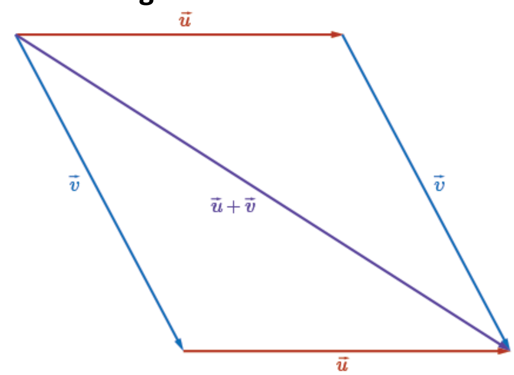
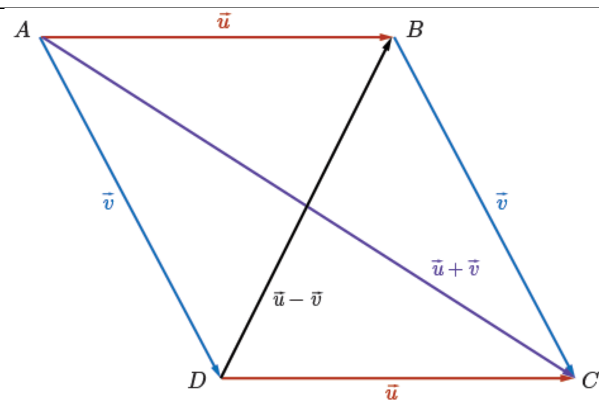
An **unit vector** is a vector having a magnitude of 1.

$$\vec{\mathbf{u}} = \frac{1}{|\vec{\mathbf{v}}|}\vec{\mathbf{v}}, \quad |\vec{\mathbf{u}}| = 1$$

Addition of two Vectors

The vector addition $\vec{s}$ of two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}+\vec{b}$ and is called the sum or resultant of the two vectors.

In order to find the sum or resultant of to vectors we can use two rules:

a. The Triangle Rule(law)  In order to find the sum (resultant) of two geometric vectors: i) Arrange the two vectors **<u>tip to tail.</u>** ii) The **<u>resultant vector</u>** is the third side of the triangle. Its direction is from the start (tail) of the first vector to the tip of the second vector. $\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{PQ}$.	**b. Parallelogram Rule**  i) Arrange the two vectors so they share a common vertex (**<u>tail to tail</u>**). ii) Complete the parallelogram. The **<u>resultant vector</u>**, $\vec{u}+\vec{v}$, is the diagonal drawn from the common vertex. Notice also that $\vec{u}+\vec{v}=\vec{v}+\vec{u}$.
To **subtract** vectors, we add the opposite. Think $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$.	

Ex 3. Given the magnitude of two vectors $|\vec{a}| = 4$ and $|\vec{b}| = 7$, and the angle between them when placed tail to tail as being θ = 60°, find the magnitude of the vector sum $\vec{s} = \vec{a} + \vec{b}$ and the direction (the angles between the vector sum and each vector). Draw a diagram.

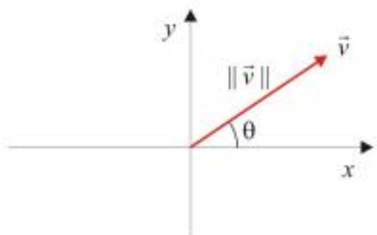
Properties of Vectors

For all $\vec{a}, \vec{b}, \vec{c}$ and $m \in R$, the following properties hold: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ $\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$ $\vec{a}+(-\vec{a})=(-\vec{a})+\vec{a}=\vec{0}$ $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ $\|\| k\vec{a} \|\| = \|k\| \|\| \vec{a} \|\|$ $k(\vec{a}+\vec{b})=k\vec{a}+k\vec{b}$ $(kl)\vec{a}=k(l\vec{a})=l(k\vec{a})$ $(k+l)\vec{a}=k\vec{a}+l\vec{a}$ $1\vec{a}=\vec{a}$ $(-1)\vec{a}=-\vec{a}$ $0\vec{a}=\vec{0}$ $\|\| \vec{0} \|\| = 0$	**Ex**. Given that $\vec{m} = \vec{a} + 3\vec{b}$ and $\vec{n} = 2\vec{a} - 4\vec{b}$, simplify $2\vec{m} - \vec{n}$.

We can see $\vec{0}$ as a vector with magnitude 0 and arbitrary direction.	
Ex 4. If point P is the midpoint of the segment AB then for any point O, we have $\overrightarrow{OP} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$	**Ex 5.** In $\triangle ABC$, AM, BN, and CP are medians. Prove that $\overrightarrow{AM} + \overrightarrow{BN} + \overrightarrow{CP} = \vec{0}$.

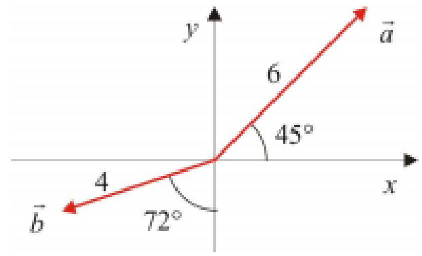
Vectors in R^2 and R^3

A Polar Coordinates
Given a *Cartesian system of coordinates*, a 2D vector $\vec{v}$ may be defined by its *magnitude* $\|\vec{v}\|$ and the counter-clockwise *angle* θ between the positive direction of the x-axis and the vector.

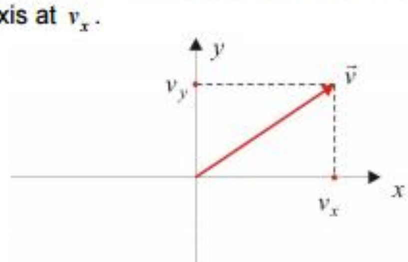


The pair $(\|\vec{v}\|, \theta)$ determines the *polar coordinates* of the 2D vector and $\vec{v} = (\|\vec{v}\|, \theta)$.

Ex 1. Express each vector in polar coordinates in the form $\vec{v} = (\|\vec{v}\|, \theta)$.

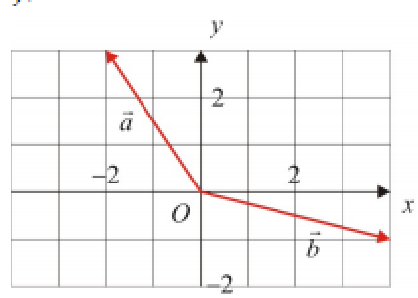


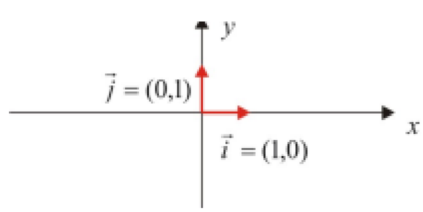
B Scalar Components for a 2D Vector
Let consider a 2D vector with the tail in the origin of the Cartesian system. Parallels through its tip to the coordinates axes intersect the x-axis at v_x and the y-axis at v_x.

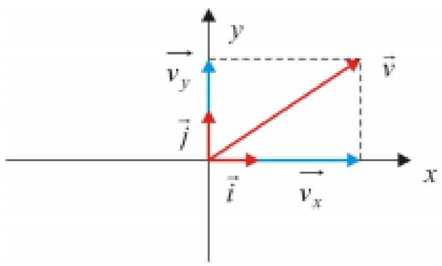


The pair (v_x, v_y) determines the *scalar coordinates* of the 2D vector and $\vec{v} = (v_x, v_y)$.

Ex 2. Express each vector in scalar coordinates in the form $\vec{v} = (v_x, v_y)$.



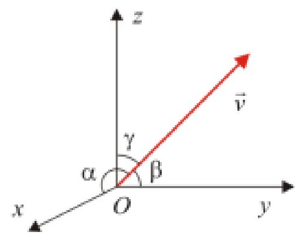
C Link between the Polar Coordinates and Scalar Components To convert a vector from the *polar coordinates* $\vec{v}=(\lVert\vec{v}\rVert,\theta)$ to the *scalar components* $\vec{v}=(v_x,v_y)$ use the formulas: $v_x=\lVert\vec{v}\rVert\cos\theta$ $v_y=\lVert\vec{v}\rVert\sin\theta$ To convert a vector from the *scalar components* $\vec{v}=(v_x,v_y)$ to the *polar coordinates* $\vec{v}=(\lVert\vec{v}\rVert,\theta)$, use the formulas: $\lVert\vec{v}\rVert=\sqrt{{v_x}^2+{v_y}^2}$ (to get the magnitude) $\tan\theta=\frac{v_y}{v_x}$ (to get the direction)	Ex 3. Do the required conversions. a) Convert $\vec{a}=(10,120°)$ to the scalar coordinates. b) Convert $\vec{b}=(-4,-7)$ to the polar coordinates.
D Magnitude of a 2D Algebraic Vector The *magnitude* of a 2D algebraic vector $\vec{v}=(v_x,v_y)$ is given by: $\lVert\vec{v}\rVert=\sqrt{{v_x}^2+{v_y}^2}$	Ex 4. Find the magnitude of the following 2D vector: $\vec{v}=(4,-3)$.
E Standard Unit Vectors The unit vectors $\vec{i}=(1,0)$ and $\vec{j}=(0,1)$ are called the *standard unit vectors* in 2D space. See the figure to the right.	

F Vector Components for a 2D Vector Any vector $\vec{v}$ may be decomposed into two perpendicular *vector components* $\overrightarrow{v_x}$ and $\overrightarrow{v_y}$, parallel to each of the standard unit vectors. $\vec{v} = \overrightarrow{v_x} + \overrightarrow{v_y}$  The link between the *scalar components* and the *vector components* is given by: $\overrightarrow{v_x} = v_x\vec{i}$ $\overrightarrow{v_y} = v_y\vec{j}$ A 2D vector may be written in *algebraic form* as: $\vec{v} = \overrightarrow{v_x} + \overrightarrow{v_y} = v_x\vec{i} + v_y\vec{j} = (v_x, v_y)$	Ex 5. Convert the vector $\vec{v} = -2\vec{i} + 5\vec{j}$ into the form $\vec{v} = (v_x, v_y)$. Ex 6. Convert the vector $\vec{v} = (4, -6)$ into the form $\vec{v} = v_x\vec{i} + v_y\vec{j}$. Ex 7. Find the vector components for $\vec{a} = (-3, -5)$.
G Position 2D Vector The *directed line segment* $\overrightarrow{OP}$, from the origin O to a generic point $P(x, y)$ determines a vector called the *position vector* and: $\overrightarrow{OP} = (x, y) = x\vec{i} + y\vec{j}$	Ex 8. Find the algebraic position vector $\overrightarrow{OA}$, where $A(-2, -3)$.
H Displacement 2D Vector The *directed line segment* $\overrightarrow{AB}$ from the point $A(x_A, y_A)$ to the point $B(x_B, y_B)$ determines a vector called the *displacement vector* and: $\overrightarrow{AB} = (x_B - x_A, y_B - y_A) = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j}$	Ex 9. Find the algebraic displacement vector $\overrightarrow{MN}$, where $M(2, -1)$ and $N(0, 2)$. Draw a diagram.

I Direction Angles

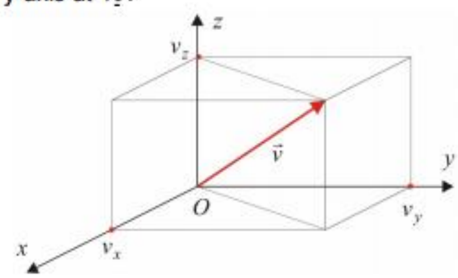
Let consider a 3D coordinate system and a 3D vector $\vec{v}$ with the tail in the origin O.

Direction angles are the angles α, β, and γ between the vector and the positive directions of the coordinates axes:



J Scalar Components of a 3D Vector

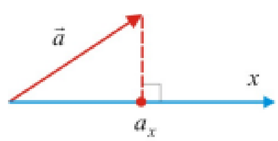
Let consider a 3D coordinate system and a 3D vector $\vec{v}$ with the tail in the origin O. Parallel planes through its tip to the coordinates planes intersect the x-axis at v_x the y-axis at v_y, and the y-axis at v_z.



The triple (v_x, v_y, v_z) determines the *scalar components* of the 3D vector and $\vec{v} = (v_x, v_y, v_z)$.

K Link between the Direction Angles and the 3D Scalar Coordinates

The link between the *direction angles* (α, β, and γ) and the *scalar components* of a vector (v_x, v_y, and v_z) is given by:



$$v_x = \|\vec{v}\| \cos\alpha$$
$$v_y = \|\vec{v}\| \cos\beta$$
$$v_z = \|\vec{v}\| \cos\gamma$$

and by:

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$
$$\cos\alpha = \frac{v_x}{\|\vec{v}\|}$$
$$\cos\beta = \frac{v_y}{\|\vec{v}\|}$$
$$\cos\gamma = \frac{v_z}{\|\vec{v}\|}$$

Note that:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Ex 10. The magnitude of a vector $\vec{a}$ is $\|\vec{a}\| = 20$ and the direction angles are $\alpha = \angle(\vec{a}, Ox) = 60°$, $\beta = \angle(\vec{a}, Oy) = 45°$, and $\gamma = \angle(\vec{a}, Oz) = 60°$. Write the vector $\vec{a}$ in the algebraic form (using the scalar components).

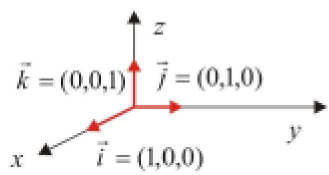
Ex 11. Find the direction angles for the vector $\vec{u} = -2\vec{i} + 3\vec{j} - \vec{k}$.

L Magnitude of a 3D Algebraic Vector
The magnitude of a 3D algebraic vector $\vec{v} = (v_x, v_y, v_z)$ is given by:

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

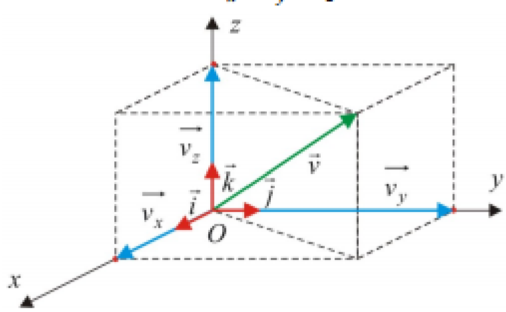
Ex 12. Find the magnitude for the vector $\vec{v} = (2,-3,4)$.

M 3D Standard Unit Vectors
The unit vectors $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$ and are called the *standard unit vectors* in 3D space. See the figure to the right.



N Vector Components for a 3D Vector
Any 3D vector $\vec{v}$ may be decomposed into three perpendicular *vector components* $\overrightarrow{v_x}$, $\overrightarrow{v_y}$ and $\overrightarrow{v_z}$, parallel to each of the 3D standard unit vectors.

$$\vec{v} = \overrightarrow{v_x} + \overrightarrow{v_y} + \overrightarrow{v_z}$$



The link between the *scalar components* and the *vector components* is given by:

$$\overrightarrow{v_x} = v_x\vec{i}, \quad \overrightarrow{v_y} = v_y\vec{j}, \quad \overrightarrow{v_z} = v_z\vec{k}$$

A 3D vector may be written in *algebraic form* as:

$$\vec{v} = \overrightarrow{v_x} + \overrightarrow{v_y} + \overrightarrow{v_z} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k} = (v_x, v_y, v_z)$$

Ex 13. Convert the vector $\vec{v} = -3\vec{i} - 4\vec{j} + 2\vec{k}$ into the form $\vec{v} = (v_x, v_y, v_z)$.

Ex 14. Convert the vector $\vec{v} = (-3,4,-5)$ into the form $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$.

Ex 15. Find the vector components for $\vec{a} = (4,0,-3)$.

O Position 3D Vector
The *directed line segment* $\overrightarrow{OP}$, from the origin O to a generic point $P(x,y,z)$ determines a vector called the position vector and:

$$\overrightarrow{OP} = (x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$$

Ex 16. Find the algebraic position vector $\overrightarrow{OP}$, where $P(3,-2,4)$. Draw a diagram.

P Displacement 3D Vector The *directed line segment* $\overrightarrow{AB}$ from the point $A(x_A, y_A, z_A)$ to the point $B(x_B, y_B, z_B)$ determines a vector called the displacement vector and: $\overrightarrow{AB} = (x_B - x_A, y_B - y_A, z_B - z_A)$ $= (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$	Ex 17. Find the algebraic displacement vector $\overrightarrow{PQ}$, where $P(1,-2,3)$ and $Q(-2,3,-4)$.

Operations with Algebraic Vectors

A 3D Algebraic Vectors A 3D Algebraic Vector may be written in components form as: $\vec{v} = (v_x, v_y, v_z)$ or in terms of unit vectors as: $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$ and has a magnitude given by: $\lVert \vec{v} \rVert = \sqrt{v_x^2 + v_y^2 + v_z^2}$	Ex 1. Consider the vector $\vec{a} = -\vec{i} + 3\vec{j} - 2\vec{k}$. a) Write the vector in components form. b) Find the magnitude of the vector $\vec{a}$.
B Addition of 3D Algebraic Vectors The sum of two 3D algebraic vectors $\vec{a} = (a_x, a_y, a_z) = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ and $\vec{b} = (b_x, b_y, b_z) = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$ is a 3D algebraic vector given by: $\vec{a} + \vec{b} = (a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) + (b_x\vec{i} + b_y\vec{j} + b_z\vec{k})$ $= (a_x + b_x)\vec{i} + (a_y + b_y)\vec{j} + (a_z + b_z)\vec{k}$ $\vec{a} + \vec{b} = (a_x, a_y, a_z) + (b_x, b_y, b_z)$ $= (a_x + b_x, a_y + b_y, a_z + b_z)$	Ex 2. Find the sum of the vector $\vec{a} = -2\vec{i} + 5\vec{j} - \vec{k}$ and $\vec{b} = (2,0,-3)$.

C Subtraction of 3D Algebraic Vectors The difference of two 3D algebraic vectors $\vec{a}=(a_x,a_y,a_z)=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$ and $\vec{b}=(b_x,b_y,b_z)=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$ is a 3D algebraic vector given by: $\vec{a}-\vec{b}=(a_x\vec{i}+a_y\vec{j}+a_z\vec{k})-(b_x\vec{i}+b_y\vec{j}+b_z\vec{k})$ $=(a_x-b_x)\vec{i}+(a_y-b_y)\vec{j}+(a_z-b_z)\vec{k}$ $\vec{a}-\vec{b}=(a_x,a_y,a_z)-(b_x,b_y,b_z)$ $=(a_x-b_x,a_y-b_y,a_z-b_z)$	Ex 3. Find the magnitude of the difference $\vec{a}-\vec{b}$ between the vector $\vec{a}=\vec{i}-\vec{j}$ and $\vec{b}=(1,2,-1)$.
D Multiplication of 3D Algebraic Vector by a Scalar The multiplication of a 3D algebraic vector $\vec{a}=(a_x,a_y,a_z)=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$ by a scalar λ is a 3D algebraic vector given by: $\lambda\vec{a}=\lambda(a_x\vec{i}+a_y\vec{j}+a_z\vec{k})=(\lambda a_x)\vec{i}+(\lambda a_y)\vec{j}+(\lambda a_z)\vec{k}$ $\lambda\vec{a}=\lambda(a_x,a_y,a_z)=(\lambda a_x,\lambda a_y,\lambda a_z)$	Ex 4. Given $\vec{a}=(1,-2,0)$, $\vec{b}=(0,-2,-3)$, and $\vec{c}=(-1,0,2)$, find the vector $\vec{d}=\vec{a}-2\vec{b}+3\vec{c}$.
G Parallelism Two vectors $\vec{a}$ and $\vec{b}$ are parallel ($\vec{a}\parallel\vec{b}$) if there exists λ such that $\vec{a}=\lambda\vec{b}$. Note that parallel vectors may have same direction or opposite direction: 	Ex 8. Prove that the vectors $\vec{a}=(2,4,-6)$ and $b(-1,-2,3)$ are parallel.

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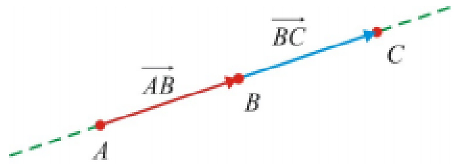
H Co-linearity
Three points A, B, and C are collinear if $\overrightarrow{AB} \parallel \overrightarrow{BC}$.



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Ex 9. Prove that the points $A(2,-1,0)$, $B(-1,0,2)$, and $C(0,1,2)$ are not collinear.

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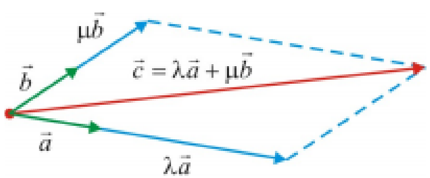
I Linear Dependency
Three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linear dependent if there exist λ and μ such that $\vec{c} = \lambda\vec{a} + \mu\vec{b}$.

Note. In order to be linear dependant the vectors must be coplanar (must belong to the same plan).



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Ex 10. Prove that the vectors $\vec{a} = (-1,2,-3)$, $\vec{b} = (2,0,-1)$, and $\vec{c} = (-7,6,-7)$ are linear dependant.

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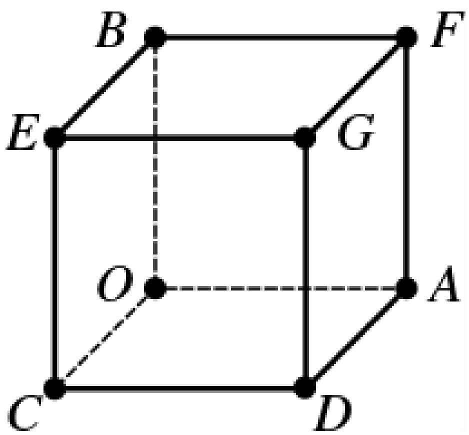
Ex. The drawing below shows a unit cube. Let $\hat{\imath} = \overrightarrow{OA}$, $\hat{\jmath} = \overrightarrow{OB}$, $\hat{k} = \overrightarrow{OC}$.
Write each of the following vectors in terms of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.



a. $\overrightarrow{OF}$ =
b. $\overrightarrow{ED}$ =
c. $\overrightarrow{AG}$ =
d. $\overrightarrow{DF}$ =
e. $\overrightarrow{AC}$ =
f. $\overrightarrow{FC}$ =
g. $\overrightarrow{EA}$ =

Ex. For points $R(-1, 2, -4\sqrt{5})$ and Q(-1, -2, 0) given, find

a. The position vector and the magnitude of the position vector $\overrightarrow{OR}$

b. The displacement vector $\overrightarrow{RQ}$ and its magnitude.

Ex. Find a unit vector parallel to each of the given vectors.

a. $\vec{v}=(2,-5)$
b. $\overrightarrow{OZ}=\hat{\imath}-2\hat{\jmath}+4\hat{k}$
c. $\vec{w}=(-5,12)$