Notice:

The following notes are the same as the notes that you studied before, but the questions in class and homework are different from before.

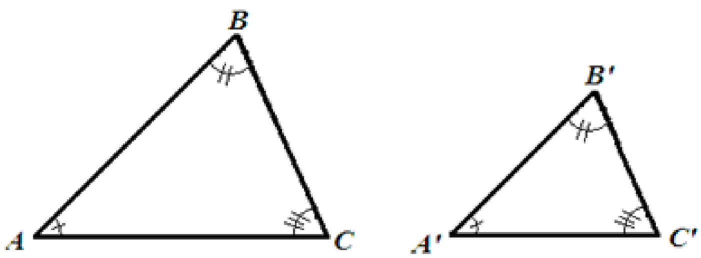
Geometry 2

1. Similar Triangles

1) Definition

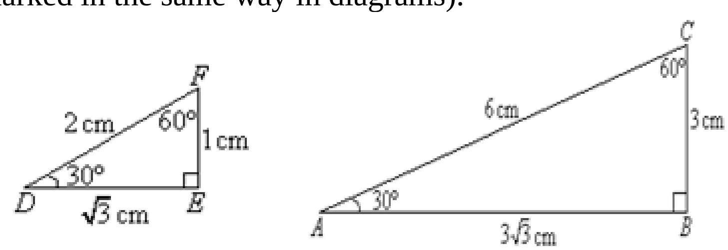
Two triangles ABC and A'B'C' are similar if the three angles of the first triangle are congruent to the corresponding three angles of the second triangle and the lengths of their corresponding sides are proportional as follows.

$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$ or AB: A'B'= BC: B'C' = CA: C'A'



Equiangular triangles have the same shape but may have different sizes. So, equiangular triangles are also called **similar triangles**.

For example, triangle *DEF* is similar to triangle *ABC* as their three angles are equal (equal angles are marked in the same way in diagrams).



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The length of each side in triangle *DEF* is multiplied by the same number, 3, to give the sides of triangle *ABC*.

$\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = 3$

Generally speaking, if two triangles are similar, then the corresponding sides are in the same ratio.

2) Angle-Angle (AA) Similarity

Theorem

If two angles in a triangle are congruent to the two corresponding angles in a second triangle, then the two triangles are similar. This is because sum of three angles of a triangle equals 180^{o} which assures the third pair of corresponding angles must be equal.

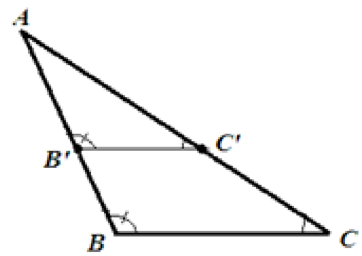
Example 1: Let ABC be a triangle and B'C' a segment parallel to BC. Prove triangles ABC is similar triangle A'BC'.

Solution

Since B'C' is parallel to BC, angles AB'C' and ABC are congruent (corresponding angles).

Also angles AC'B' and ACB are congruent (corresponding angles).

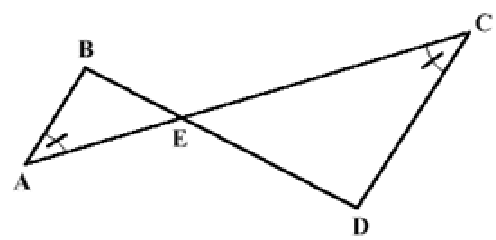
Since the two triangles have two corresponding congruent angles, they are similar by **Angle-Angle**.



Example 2: If AB//CD, Prove triangle ABE is similar to triangle CDE.

Solution:

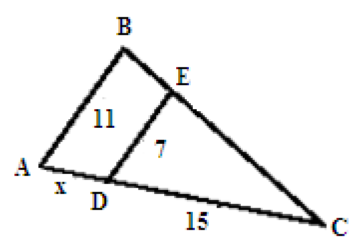
Since AB is parallel to CD, and line AC acts as a transversal across the parallel lines AB and DE, *angle A* and *angle C* are congruent (alternate angles).
Also, a*ngle AEB* and *angle CED* are congruent (opposite angles)



Since the two triangles have two corresponding congruent angles, t*riangle ABE* and *triangle CDE* are similar by **Angle-Angle**.

Example 3

Given that lines DE and AB are parallel in the figure to the right, determine the value of x, the distance between points A and D.



Solution:

First, we can demonstrate that $\Delta CDE \backsim \Delta CAB$, because C = C (by identity).

And $\angle CDE = \angle CAB$ (corresponding angles)

Since two pairs of corresponding angles are equal for the two triangles, we have demonstrated that they are similar triangles.

Hence we have $\frac{AC}{DC} = \frac{BC}{EC} = \frac{AB}{DE}$

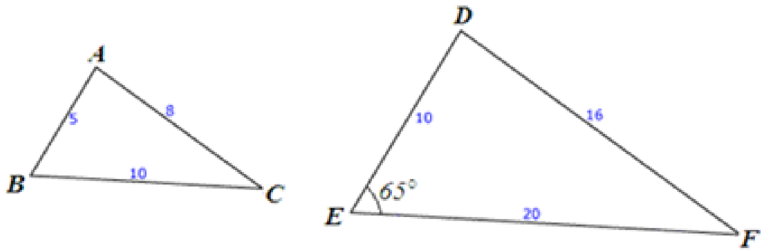
This gives $\frac{15+x}{15}=\frac{11}{7}$. Solving the equation,

$15 + x = \frac{11}{7}(15)$, $x = 8.57$

3) Side-Side-Side (SSS) Similarity

Theorem
If the three sides of a triangle are proportional to the corresponding sides of a second triangle, then the triangles are similar.
Or, the lengths of the corresponding sides are proportional and therefore the two triangles are similar.

Example 4
Prove the two triangles shown below are similar triangles and determine the angle of B.



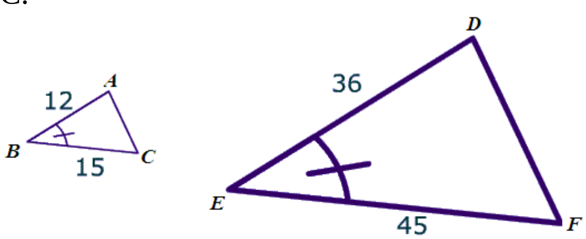
Solution
Since AB : DE =5:10= 1: 2, and AC: DF = 8:16 =1:2, and BC: EF = 10:20 = 1:2
Therefore triangle ABC and triangle DEF are similar triangles.

Since corresponding angles are congruent for similar triangles, and angle E in triangle DEF is 65^{o}, then angle B = angle E = 65^{o}

4) Side-Angle-Side (SAS) Similarity

Theorem
If an angle of a triangle is congruent to the corresponding angle of a second triangle, and the lengths of the two sides including the angle in one triangle are proportional to the lengths of the corresponding two sides in the second triangle, then the two triangles are similar.

Example 5: Prove the triangles shown below are similar. If DF equals 27, determine the length of AC.



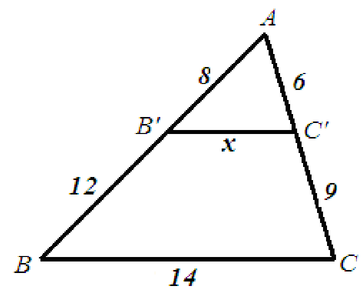
Solution

Since angle B equals angle E, and BC : EF = 15 : 45 =1:3, and BA : ED=12: 36=1:3.
The two triangles have two sides whose lengths are proportional and a congruent angle included between the two sides. Therefore the two triangles are similar(SAS). We may calculate the ratios of the lengths of the corresponding sides.

Then AB : DE = AC : DF, substitute the given value in written proportion, we have 12 : 36 =AC : 27, we get AC =9

Example 6: In the figure shown below, if B'C'// BC, determine the value of x.

Solution:
Since angles BAC and B'AC' are congruent, the lengths of the sides including the congruent angles are given in figure,



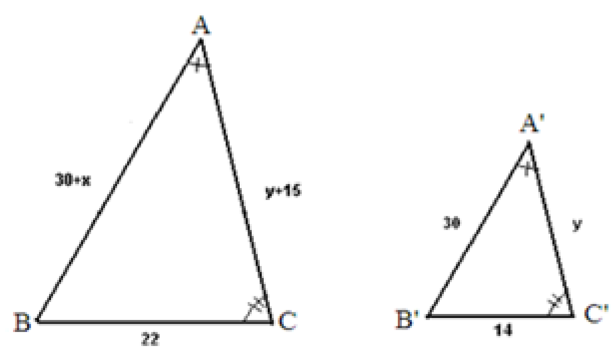
and AB : AB' = 12 : 8 = 3: 2
and AC : AC' = 9 : 6 =3:2

The two triangles have two sides whose lengths are proportional and a congruent angle included between the two sides. Therefore the two triangles are similar. We may calculate the ratios of the lengths of the corresponding sides.

Then we have AB' : AB = B'C' : BC, substitute the value shown in the figure in proportion,
8 : 20 = x : 14, we get x = 5.6

Example 7: In the figure shown below, if angle A = angle A' and angle C=angle C' and some sides are given specific value, determine x and y.

Solution
Since the two triangles have two corresponding congruent angles, they are similar by **Angle-Angle**. Now we use the proportionality of the lengths of the side to write equations that help in solving for x and y.
(30 + x) / 30 = 22 / 14 = (y + 15) / y
First, an equation in x may be written as follows.
(30 + x) / 30 = 22 / 14



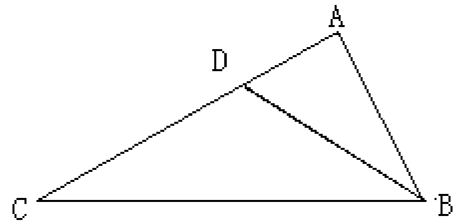
Solve the above for x.
420 + 14 x = 660
x = 17.1 (rounded to one decimal place).

Second, an equation in y may be written as follows.
22 / 14 = (y + 15) / y
Solve the above for y to obtain.
y = 26.25

Example 8: In the diagram, $\Delta ABC \sim \Delta ADB$, and $\angle ABD = \angle C$. List all corresponding sides and angles.

Solution:
Corresponding sides: AB and AC, AD and AB, DB and AC

Corresponding angles: $\angle A$ and $\angle A$, $\angle ADB$ and $\angle ABC$, $\angle ABD$ and $\angle C$



Example 9: Find the value of the pronumeral in the following diagram.

Solution:
ΔADE and ΔABC are similar as they are equiangular.

$$\therefore \frac{AD}{AB} = \frac{DE}{BC}$$

$$\frac{x+4}{x} = \frac{6}{3}$$

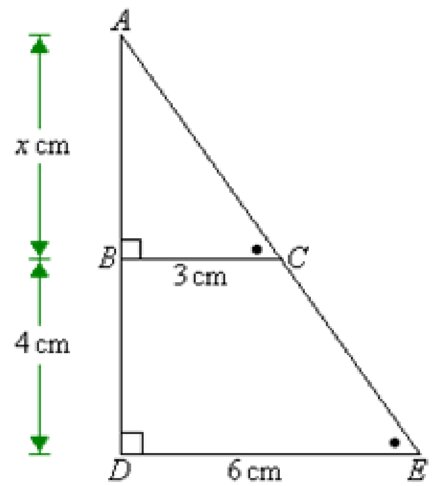
$$\frac{x+4}{x} = 2 \qquad \{\text{Multiply both sides by } x\}$$

$$x\left(\frac{x+4}{x}\right) = x \times 2$$

$$x+4 = 2x \qquad \{\text{Subtract } x \text{ from both sides}\}$$

$$x+4-x = 2x-x$$

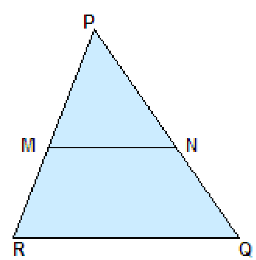
$$4 = x \qquad x = 4$$



5) Triangle Proportionality Theorem

If a line is parallel to one side of a triangle and intersects the other two sides in two distinct points, then it separates these sides into segments of proportional lengths.

If MN// RQ, then $\frac{PM}{RM} = \frac{PN}{QN}$

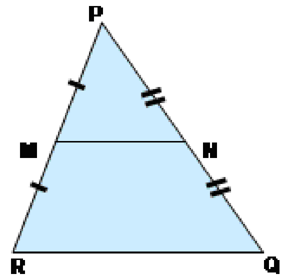


6) Triangle Proportionality Theorem Converse

If a line intersects two sides of a triangle in two distinct points and separates these sides into segments of proportional lengths, then it is parallel to the third side.

If $\frac{PM}{RM} = \frac{PN}{QN}$, then MN// RQ

7) Triangle Mid-segment Theorem



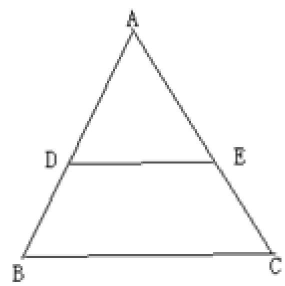
A midsegment of a triangle is parallel to one side of the triangle, and it's length is one-half the length of that side.

If M and N are midpoints of PR and PQ respectively, then MN//RQ and MN = ½ RQ.

Midsegment is a segment whose endpoints are midpoints of two sides of a triangle.

Example 10: In the diagram, if $\frac{AD}{AB}=\frac{AE}{AC}$, and AD = 3.2cm, DB = 2.4cm, AE = 2cm, find the length of EC.

Solution:



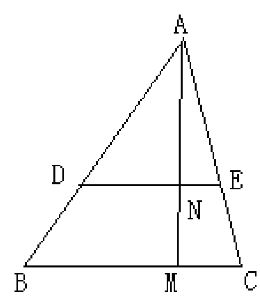
In ΔADE and ΔABC, $\because \frac{AD}{AB}=\frac{AE}{AC}$, $\angle$A is the common angle, so ΔADE∽ΔABC (SAS)

$$\therefore \frac{AD}{AD+DB}=\frac{AE}{AE+EC}$$

$$\therefore \frac{3.2}{3.2+2.4}=\frac{2}{2+EC} \qquad \therefore EC=\frac{2\times 5.6}{3.2}-2=1.5$$

Example 11: In ΔABC, DE // BC. Height AM intersects DE at N. If DE : BC = 4 : 5 and AM = 15, then what is the length of AN?

Solution:



$$\because BC//DE \qquad \therefore \triangle ADE \sim \triangle ABC,\ \triangle ADN \sim \triangle ABM$$

$$\therefore \frac{AD}{AB}=\frac{DE}{BC}=\frac{4}{5}$$

$$\because \triangle ADN \sim \triangle ABM \qquad \therefore \frac{AN}{AM}=\frac{AD}{AB}=\frac{4}{5}$$

$$\therefore AN=\frac{4}{5}AM=12$$

Example 12: In right triangle ABC, $\angle ABC=90°$. E is a point on AB extended and F is on AC such that EF $\perp AC$. EF intersects BC at D. G is on AC such that $BG \perp AC$. How many triangles are similar to ΔEBD (not including itself)?

Solution:

$\because \angle EDB = \angle CDF$ and $\angle EBD = \angle CFD = 90°$

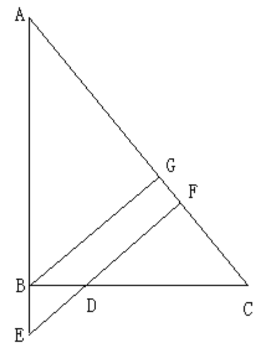
$\therefore \Delta EBD \sim \Delta CFD$

$\because DF // BG \quad \therefore \Delta CDF \sim \Delta CBG$

$\because \angle C = 90° - \angle CBG = \angle ABG$

$\therefore \triangle CGB \sim \triangle BGA \sim \triangle ABC$

$\because BG // EF \quad \therefore \Delta ABG \sim \Delta AEF$

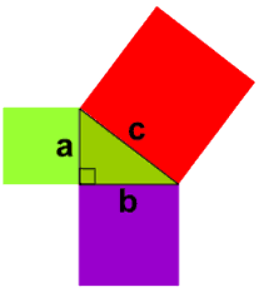


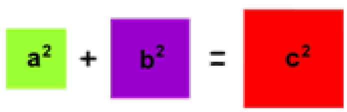
2. Pythagorean Triples

A "Pythagorean Triple" is a set of positive integers, **a**, **b** and **c** that fits the rule: $a^2 + b^2 = c^2$

Triangles

And when you make a triangle with sides **a**, **b** and **c** it will be a right angled triangle:





Note:

c is the **longest side** of the triangle, called the "hypotenuse"

a and **b** are the other two sides

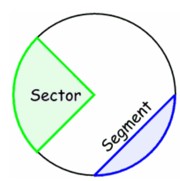
3. Circle Sector and Segment

Slices

There are two main "slices" of a circle:

The "pizza" slice is called a **Sector**.

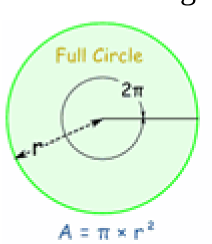
And the slice made by a chord is called a **Segment**.

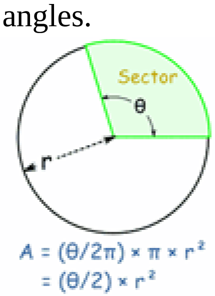


1) Area of a Sector

You can work out the Area of a Sector by comparing its angle to the angle of a full circle.

Note: I am using radians for the angles.





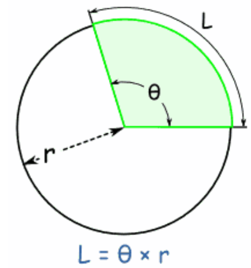
This is the reasoning:

- A circle has an angle of 2π and an Area of: πr^2

- So a Sector with an angle of θ (instead of 2π) must have an area of: $(\theta/2\pi) \times \pi r^2$
- Which can be simplified to: $(\theta/2) \times r^2$

Area of Sector = $\frac{1}{2} \times \theta \times r^2$ ***(when θ is in radians)***
Area of Sector = $\frac{1}{2} \times (\theta \times \pi/180) \times r^2$ ***(when θ is in degrees)***

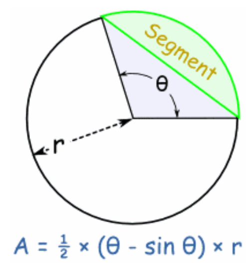
2) Arc Length of Sector or Segment



By the same reasoning, the arc length (of a Sector or Segment) is:

Arc Length "L" = $\theta \times r$ ***(when θ is in radians)***
Arc Length "L" = $(\theta \times \pi/180) \times r$ ***(when θ is in degrees)***

3) Area of Segment

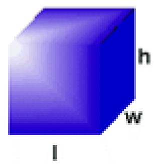
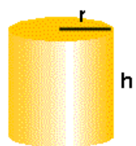
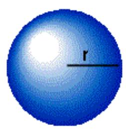
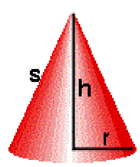
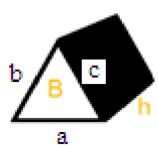
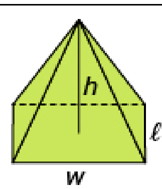
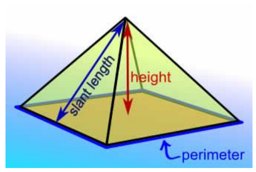


The Area of a Segment is the area of a sector minus the triangular piece (shown in light blue here).

There is a lengthy derivation, but the result is a slight modification of the Sector formula:

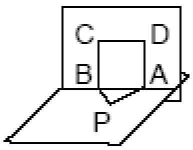
Area of Segment = $\frac{1}{2} \times (\theta - \sin\theta) \times r^2$ ***(when θ is in radians)***
Area of Segment = $\frac{1}{2} \times ((\theta \times \pi/180) - \sin\theta) \times r^2$ ***(when θ is in egrees)***

►Volume (V) and Surface Area (SA) Formulas

Name	Shapes	Formula
Rectangular Solid		$Volume = Length \cdot Width \cdot Height$ $V = lwh$ SA: Surface Area $SA = 2lh + 2hw + 2lw$
Cylinder		$Volume = \pi r^2 \cdot height$ $V = \pi r^2 h$ SA: Surface Area $SA = 2\pi rh + 2\pi r^2$
Sphere		$V = \frac{4}{3}\pi r^3$ SA: Surface Area $SA = 4\pi r^2 = \pi d^2$
Cone		$V = \frac{1}{3}\pi r^2 h$ SA: Surface Area $SA = s\pi r + \pi r^2$, $s = \sqrt{r^2 + h^2}$
Prism		$V = \frac{1}{2}Bh$ (B: Area) SA: Surface Area $SA = 2B + Ph$ $SA = 2B + (a + b + c) \cdot h$
Pyramid		$V = \frac{1}{3}Bh = \frac{1}{3}wlh$, where B is the area of the base.
Pyramid		SA: Surface Area When all side faces are the same: [Base Area] + $^1/_2$ × Perimeter × [Slant Length] When side faces are different: [Base Area] + [Lateral Area]

► In-class questions

1. Triangle PAB and square ABCD are in perpendicular planes. Given that PA =3, PB = 4, and AB = 5, what is PD?



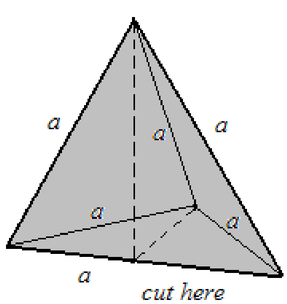
2. Survivors on a desert island find a piece of plywood (ABC) in the shape of an equilateral triangle with sides of length 2 m. To shelter their goat from the sun, they place edge BC on the ground, lift corner A, and put in a vertical post PA which is h m long above ground.

When the sun is directly overhead, the shaded region (ΔPBC) on the ground directly underneath the plywood is an isosceles triangle with largest angle ($\angle$BPC) equal to 120^o. Determine the value of h, to the nearest centimetre.

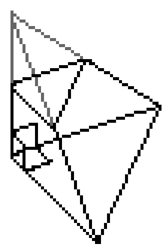
3. A cone, a cylinder and a sphere all have radius r. The height of the cylinder is H and the height of the cone is h. The cylinder and the sphere have the same volume. The cone and the cylinder have the same total surface area. Prove that h and H cannot both be integers.

4. The 6 edges of a regular tetrahedron are of length a. the tetrahedron is sliced along one of its edges to form two identical solids.

a) find the perimeter of the slice.   b) find the area of the slice



5. A tetrahedron, a solid with four triangular faces, is cut from the corner of a unit cube. It has three faces that are mutually perpendicular isosceles right triangles and the fourth face an equilateral triangle. The three perpendicular edges are edges of the cube and so have unit length. A plane parallel to one of the isosceles triangles bisects the third perpendicular edge to form the solid shown. What is the total surface area of the solid?



6. A tetrahedron is a three dimensional solid in the shape of a pyramid with a triangular base. a tetrahedron can be constructed by first drawing a triangle ABC in the plane, then placing a point D in space so that it is not in the same plane as ABC, and finally connecting D to each of A, B, and C. If the corners of a tetrahedron are cut off so that a triangle is formed at each corner, then what is the maximum number of edges in the resulting solid?

7. A pyramid made from a solid block of wood rests with its square base on a table top. Each edge of the square base has length 10 cm. The lateral sides of the pyramid are equilateral triangles. An ant is walking on the sides of the pyramid from the midpoint *A* of one of the edges of the base to the midpoint *B* of the opposite edge. See the diagram. Find the length of the shortest path the ant has to walk.