AP Calculus Class 22

Infinite Sequences and Series

Def^n: A sequence is a set of numbers in a defined order.

$a_1, a_2, a_3, \ldots, a_n, \ldots$

For the rest of this section, all sequences and series have infinite number of terms.

$\{a_n\}$, $\{a_n\}_{n=1}^{\infty}$

Example:

a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$, $\quad a_n = \frac{n}{n+1}$, $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\right\}$

b) $\left\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \ldots, \underline{\qquad\qquad}, \ldots\right\}$

$\left\{\frac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$, $\quad a_n = \frac{(-1)^n(n+1)}{3^n}$

Example: Find the general formula for the term a_n

$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \ldots\right\}$

$a_1 = \frac{3}{5}$ $\qquad$ $a_n = (-1)^{n-1}\frac{n+2}{5^n}$

Example: Fibonacci sequence.

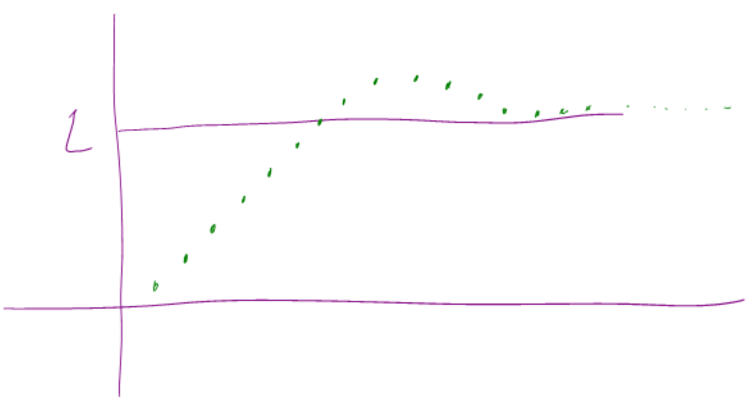
$1, 1, 2, 3, 5, \ldots \longrightarrow \{a_n\}. \quad a_n = a_{n-1} + a_{n-2}$

Def^n: A sequence $\{a_n\}$ has the limit L and we write

$\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n\to\infty$.

if we can make the term a_n as close to L as we like by taking n sufficiently large.

If the $\lim_{n\to\infty} a_n$ exists, we say the sequence converges (or is convergent). Otherwise we say the sequence diverges (or divergent)



Thm: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, when n is an integer, then

$\lim_{n\to\infty} a_n = L$.

Def^n: $\lim_{n\to\infty} a_n = \infty$ means that $\forall$ positive integer M, $\exists$ an integer N s.t.

if $n > N$, then $a_n > M$.

Limit Laws.

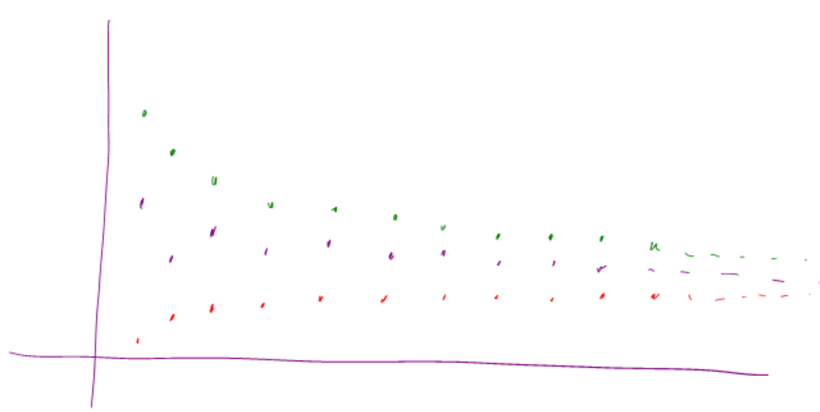
$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (c\, a_n) = c \lim_{n\to\infty} a_n.$$

$\vdots$

Sandwich/Squeeze Thm.

If $a_n \le b_n \le c_n$, for some $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



Thm: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example: $\lim_{n\to\infty} \frac{n}{n+1}$

$$\lim_{n\to\infty} \frac{n/n}{n/n + 1/n} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$$

Example: $\lim_{n\to\infty} \frac{\ln(n)}{n}$

- We can't apply l'Hospital's Rule directly b/c it doesn't apply to sequences.

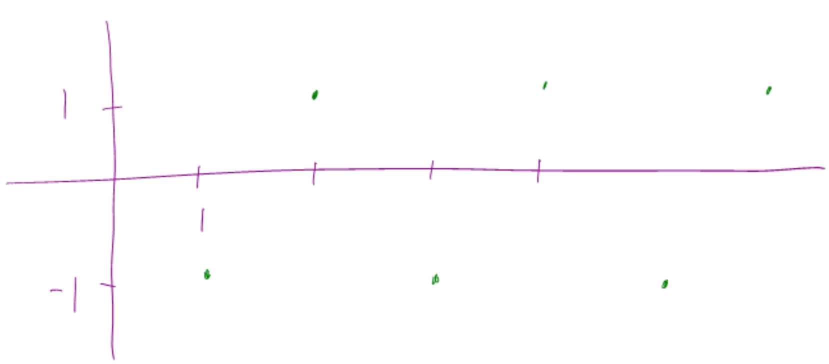
- change $\frac{\ln(n)}{n}$ to $f(x) = \frac{\ln(x)}{x}$

$$\Rightarrow \lim_{x\to\infty} \frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

By the previous thm,

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$$

Example: Determine if $a_n = (-1)^n$ is convergent or divergent.



Sol^n: Divergent.

Example: $a_n = \frac{n!}{n^n}$, $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots$

$a_1 = 1$ $\quad a_2 = \frac{1 \cdot 2}{2 \cdot 2}$ $\quad a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n}$$

$$\Rightarrow a_n = \left(\frac{1}{n}\right)\left(\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}\right)$$

$$a_n = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}\right) \leq \frac{1}{n}$$

$$\Rightarrow \quad 0 < a_n \leq \frac{1}{n}$$

$\Rightarrow$ We know $\frac{1}{n} \to 0$ as $n \to \infty$.

$\Rightarrow$ $a_n \to 0$ as $n \to \infty$ by the Squeeze thm.

Example: For what value of r is the sequence $\{r^n\}$ convergent?

$\lim_{x \to \infty} a^x = \infty$ if $a > 1$.

$\lim_{x \to \infty} a^x = 1$ if $a = 1$

$\lim_{x \to \infty} a^x = 0$ if $0 < a < 1$.

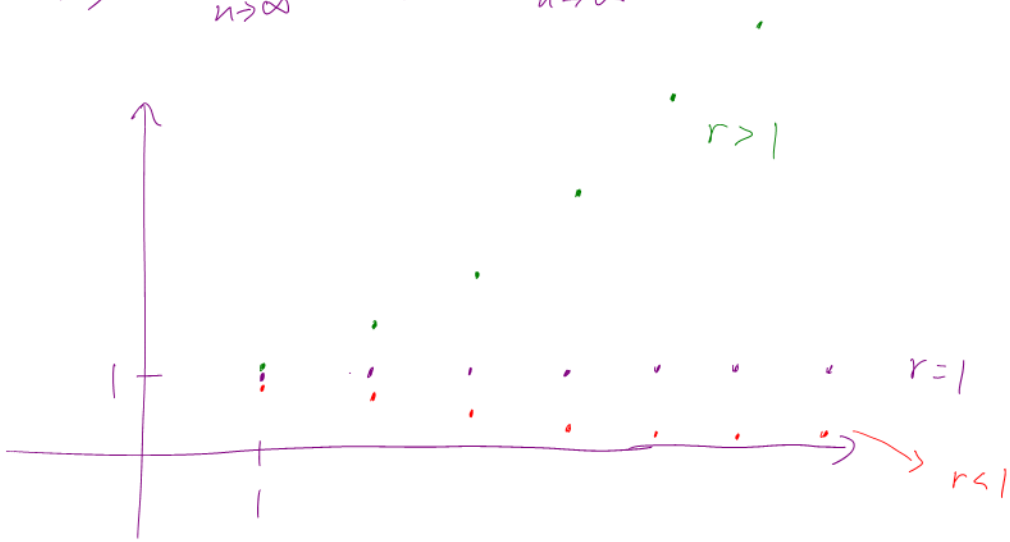
$$\Rightarrow \lim_{n\to\infty} r^n = 0 \qquad \text{if} \quad 0<r<1$$

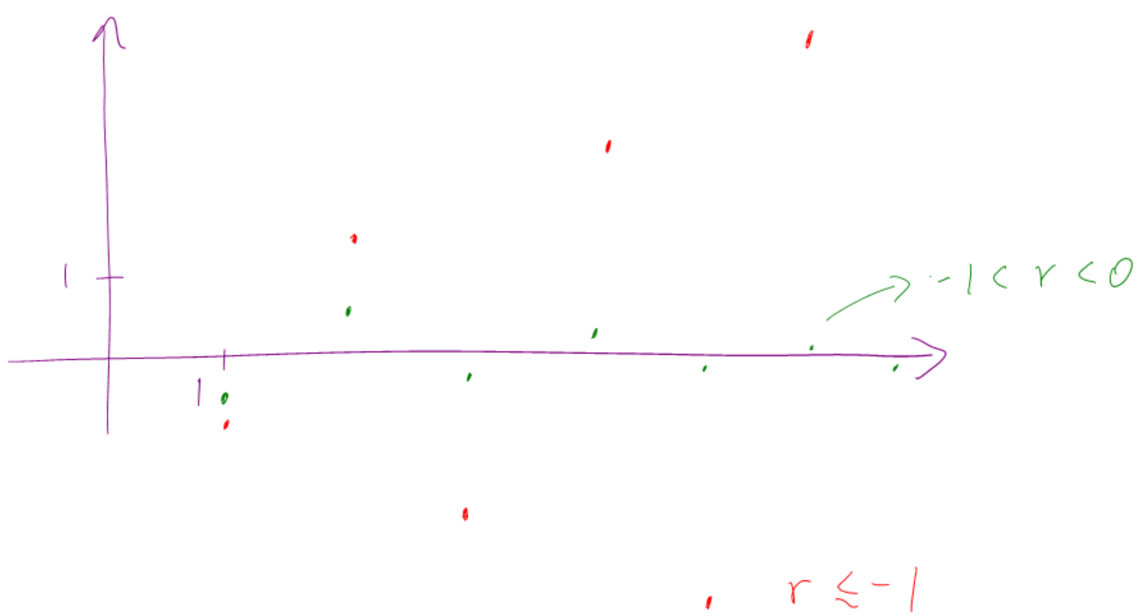
$$\lim_{n\to\infty} r^n = 1 \qquad \text{if} \quad r=1 \qquad \underline{\underline{0 \leq r \leq 1}}$$

$$\lim_{n\to\infty} r^n = 0 \qquad \text{if} \quad r=0$$

If $\quad -1 < r < 0 \quad \Rightarrow \quad 0 < |r| < 1$

$$\Rightarrow \lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0.$$





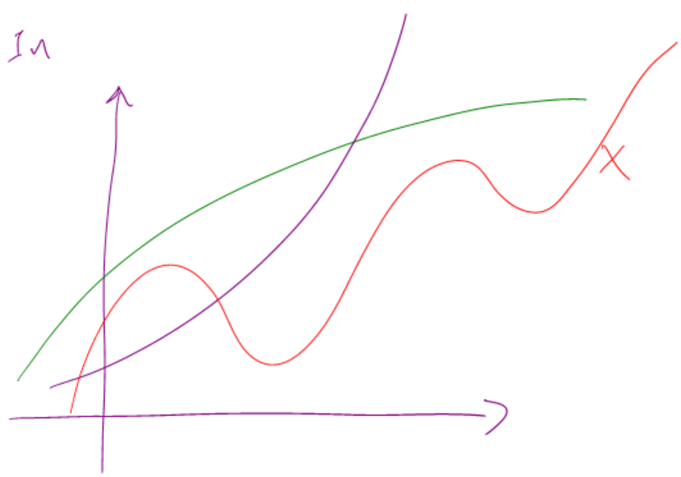
The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1. \end{cases}$$

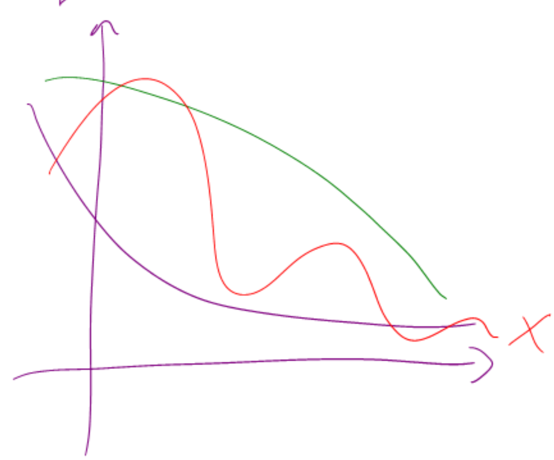
Def^n: A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ $\forall\ n \geq 1$. It's called decreasing if $a_n > a_{n+1}$ $\forall\ n \geq 1$.

It's called monotonic if it's either increasing or decreasing.

In



Dec



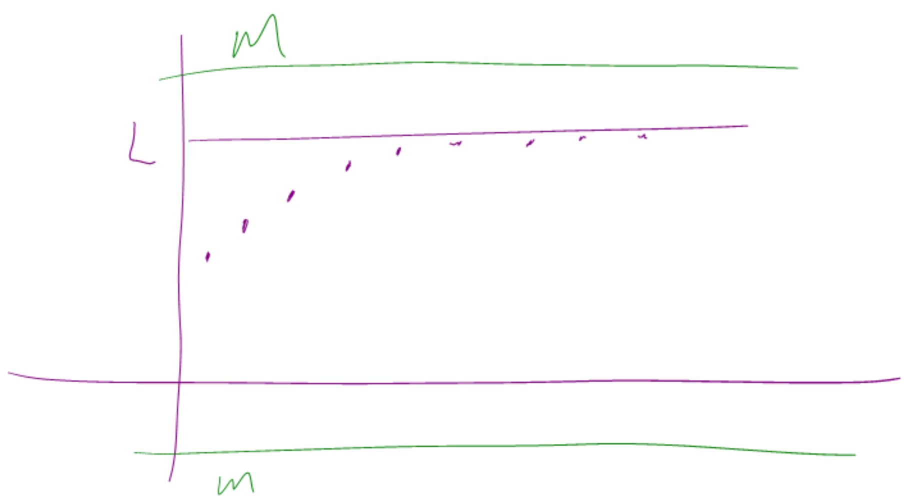
Def^n: A sequence $\{a_n\}$ is bounded above if there is a number M s.t.

$$a_n \leq M \qquad \forall\ n \geq 1.$$

It's bounded below if there is a number m s.t.

$m \le a_n \quad \forall n \ge 1.$

If it's bounded from above and below, then $\{a_n\}$ is a bounded sequence.



Monotonic Sequence Theorem.

Every bounded, monotonic sequence is convergent.

Series.

$\{a_n\} \rightarrow \quad a_1, a_2, \ldots, a_n, \ldots$

Series: $a_1 + a_2 + \cdots + a_n + \cdots$

Infinte series (series)

$\sum_{n=1}^{\infty} a_n$ or $\sum a_n$

$1 + 2 + \cdots + n + \cdots$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$

$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

Partial Sum (Important!)

$a_1 + a_2 + \cdots + a_n + \cdots$

partial sum s_n

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$\vdots$

$s_n = a_1 + a_2 + \cdots + a_n$

$\Rightarrow \quad s_n = \sum_{i=1}^{n} a_i$

We want to define the convergence and divergence of series based on the sequence $\{s_n\}$.

Def^n: Given a series $\sum_{n=1}^{\infty} a_n = a_1 + \cdots + a_n$, let s_n denote the n^{th} partial sum,

$$s_n = \sum_{i=1}^{\infty} a_i = a_1 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n \to \infty} s_n = S$ exists as a real number, then the series $\sum_{n=1}^{\infty} a_n$ is called convergent.

$$a_1 + \cdots + a_n + \cdots = S \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = S.$$

The number s is called the sum of the series.

Otherwise, the series is divergent.

Example

$0.999 \ldots\ldots$

$0.\overline{9} = 0.9 + 0.09 + 0.009 + \cdots$

$s_1 = 0.9$

$s_2 = 0.9 + 0.09 = 0.99$

$s_3 = 0.9 + 0.09 + 0.009 = 0.999$

$\vdots$

$s_n = 0.\underbrace{99\ldots9}_{n}$

$\{s_n\} = \{0.9, 0.99, 0.999, \ldots\}$

Example: Geometric Series

$$a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

If $r=1$, then $s_n = a + a + \cdots + a = na$.

If $n \to \infty$, then $s_n = \pm\infty$

If $r \neq 1$, we have

$$s_n = a + ar + \cdots + ar^{n-1}$$

$$\Rightarrow \quad rs_n = ar + ar^2 + \cdots + ar^n$$

$$\Rightarrow \quad s_n - rs_n = a - ar^n$$

$$\Rightarrow \quad s_n(1-r) = a(1-r^n)$$

$$\Rightarrow \quad s_n = \frac{a(1-r^n)}{1-r}$$

If $-1 < r < 1$, then $r^n \to 0$ when $n \to \infty$.

$$\Rightarrow \quad s_n = \frac{a}{1-r} \qquad \text{when } n \to \infty$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n = \frac{a}{1-r}$$

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + \cdots$$

is <u>convergent</u> if $|r|<1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n} = \frac{a}{1-r} \qquad |r|<1.$$

If $|r| \geq 1$, then the geometric series is <u>divergent</u>.

Example: Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$

convergent or divergent?

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum (2^2)^n \, 3^{-(n-1)} = \sum \frac{4^n}{3^{n-1}}$$

$$= \sum \frac{4 \cdot 4^{n-1}}{3^{n-1}} = \sum 4\left(\frac{4}{3}\right)^{n-1}$$

$$3^n \cdot \frac{1}{3}$$

$$= 3\left(\frac{4}{3}\right)^n$$

$\Rightarrow a = 4$ and $r = \frac{4}{3}$

Since $r > 1$, the series diverge

Example: Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)}$$

Simplify using partial fraction decomp

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

$$\Rightarrow \quad s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \cdots$$

$$\cancel{\frac{1}{n}}\Big) - \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1.$$