AP Calculus Class 24

Absolute Convergence and Ratio and Root Tests.

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

Def^n: A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Def^n: A series $\sum a_n$ is called conditionally convergent if it's convergent, but not absolutely convergent.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \rightarrow$ conv

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \longrightarrow \text{div.}$$

Thm: If a series is absolutely convergent, then it is convergent.

Proof: Note the following inequality.

$$0 \le a_n + |a_n| \le 2|a_n|$$

If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent.

$\Rightarrow \sum 2|a_n|$ is convergent.

By the comparison test, $\sum (a_n + |a_n|)$ is convergent.

$\Rightarrow \sum (a_n + |a_n|) = \sum a_n + \sum |a_n|$

$\Rightarrow \sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$

$\Rightarrow \sum a_n$ is convergent because it's the difference between two convergent series. $\square$

Example: Determine whether

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \cdots$$

is convergent or divergent.

$$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| = \sum \frac{|\cos n|}{n^2}$$

Notice that $|\cos n| \le 1$

$$\Rightarrow \frac{|\cos n|}{n^2} \le \frac{1}{n^2}$$

Because $\sum \frac{1}{n^2}$ is a p-series with $p = 2 > 1$,

$\Rightarrow \sum \frac{|\cos n|}{n^2}$ is convergent by the comparison test.

$\Rightarrow \sum \left|\frac{\cos n}{n^2}\right|$ is absolutely convergent

$\Rightarrow \sum \frac{\cos n}{n^2}$ is convergent by the absolute convergence theorem.

The Ratio Test

(1) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(2) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(3) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Example: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Let $a_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)^2} \cdot \frac{n^2}{1}\right|$$

$$= \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{n^2+2n+1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}}$$

$$= \frac{1}{1+0+0} = 1$$

$\Rightarrow$ Inconclusive from the Ratio Test.

but $\sum \frac{1}{n^2}$ is convergent (p-series w/ $p=2$).

Observe $\sum \frac{1}{n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1.$$

$\Rightarrow$ By the Ratio Test, it's inconclusive.

but $\sum \frac{1}{n}$ is a harmonic series, $\Rightarrow$ it's divergent.

Example: Test $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}\right|$$

$$= \left|\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n^3}\right|$$

$$= \left| \frac{\cancel{(-1)^n}(-1)(n+1)^3}{\cancel{3^n}\cdot 3} \cdot \frac{\cancel{3^n}}{\cancel{(-1)^n} n^3} \right|$$

$$= \left| (-1) \frac{(n+1)^3}{n^3} \cdot \frac{1}{3} \right|$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1.$$

$\Rightarrow \sum a_n$ is abs convergent by the Ratio Test.

$\Rightarrow \sum a_n$ is convergent.

The Root Test

(1) If $\lim_{n\to\infty} \sqrt[n]{a_n} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(2) If $\lim_{n\to\infty} \sqrt[n]{a_n} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{a_n} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(3) If $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$, the Root Test is inconclusive.

Example: $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

$a_n = \left(\frac{2n+3}{3n+2}\right)^n$

$\Rightarrow \sqrt[n]{a_n} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}}$

$\Rightarrow \lim_{n\to\infty} \frac{2+\frac{3}{n}}{3+\frac{2}{n}} = \frac{2}{3} < 1$

$\Rightarrow$ The series $\sum a_n$ is convergent by the Root Test.

- Divergence Test

$\sum a_n \rightarrow \lim_{n\to\infty} a_n \neq 0.$

If $\lim_{n\to\infty} a_n = 0.$

Cannot say the $\sum a_n$ is convergent.

- Integral Test

$a_n = f(n) \rightarrow$ let n be any number x,

$\rightarrow \int_1^{\infty} f(x)\,dx.$

- Comparison Test.

$a_n \leq b_n \rightarrow$ If b_n is convergent, then a_n is conv.

$a_n \geq b_n \rightarrow$ " " divergent, " " div.

* Limit Comparison

$$\lim_{n\to\infty} \frac{b_n}{a_n} = C.$$

Either both a_n and b_n are convergent or divergent.

- Alternating series test.

$\sum (-1)^n a_n$. If $a_{n+1} \le a_n$ and $\lim_{n\to\infty} a_n = 0$.

then $\sum (-1)^n a_n$ is convergent.

Power Series.

Def^n: A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

$$\sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + \cdots$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

A general version of the power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a power series in $(x-a)$ or a power series centered at a or a power series about a.

For the first term $C_0(x-a)^0$, what happens when $x=a$?

$\to$ We get 0^0.

A: Mathematicians define $0^0 = 1$ in this case.

Example: For what values of x is the series $\sum_{n=0}^{\infty} n!\, x^n$ convergent?

Apply the Ratio Test.

Let's denote $a_n = n!\, x^n$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = \left| \frac{\cancel{n!}(n+1)\cancel{x^n} \cdot x}{\cancel{n!}\,\cancel{x^n}} \right|$$

$\Rightarrow \lim_{n\to\infty} (n+1)|x| = \infty > 1$

$\Rightarrow$ By the Ratio Test, the series diverges when $x \neq 0$.

$\infty \cdot 0 \to$ undefined.

In this case, the series converges only $\boxed{\text{when } x=0}$

Example: For what value of x does $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

let $a_n = \frac{(x-3)^n}{n}$

Apply the Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}}\right| = \left|\frac{\cancel{(x-3)^n}(x-3)}{n+1} \cdot \frac{n}{\cancel{(x-3)^n}}\right|$$

$$= \left|\frac{(x-3)n}{n+1}\right|$$

$$\Rightarrow \lim_{n\to\infty} \frac{n}{n+1}|x-3| = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}|x-3|$$

$$\Rightarrow |x-3| < 1$$

Now $|x-3| < 1 \Rightarrow -1 < x-3 < 1$

$\Rightarrow 2 < x < 4.$

when $|x-3| = 1 \Rightarrow x = 2$ and $\cancel{x=4}$.

For $x = 4$,

$\Rightarrow \sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \frac{1^n}{n} = \frac{1}{n} \longrightarrow$ divergent.

$\longrightarrow$ harmonic series

For $x = 2$.

$\Rightarrow \sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \frac{(-1)^n}{n} \longrightarrow$ convergent.

$\longrightarrow$ alternating harmonic series.

$\Rightarrow 2 \le x < 4$ will make the series convergent.

Theorem

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(1) The series converges only when $x = a$.

(2) The series converges for all x.

(3) There is a $R > 0$ such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$

- The number R in case 3 is called the radius of convergence of the power series.

- The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.

Example: $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ Find the interval of convergence.

let $a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$

Apply the Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right|$$

$$= \left|-3x\sqrt{\frac{n+1}{n+2}}\right|$$

$$= 3\sqrt{\frac{n+1}{n+2}}\,|x|.$$

$$\Rightarrow \lim_{n\to\infty} \sqrt{\frac{n+1}{n+2}}\; 3|x|$$

$$\Rightarrow 3|x| < 1 \qquad \Rightarrow |x| < \frac{1}{3}$$

$$\Rightarrow R = \frac{1}{3} \qquad \Rightarrow -\frac{1}{3} < x < \frac{1}{3}$$

check $x = -\frac{1}{3}$ and $x = \frac{1}{3}$

For $x \cancel{= -\frac{1}{3}}$

$$\sum_{n=0}^{\infty} \frac{(-3)^n\left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$

$\rightarrow$ p-series $p < 1$

$\Rightarrow \sum a_n$ diverges when $x = -\frac{1}{3}$

For $x = \frac{1}{3}$

$$\sum_{n=0}^{\infty} \frac{(-3)^n\left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

By the Alternating series test, $\sum a_n$ is convergent. when $x = \frac{1}{3}$

$\Rightarrow$ The interval of convergence is $-\frac{1}{3} < x \le \frac{1}{3}$