Transformations of Exponential Functions

In functions of the form $g(x)=a\,f[k(x-d)]+c$, the constants d and c change the location of the graph of $f(x)$. The shape is dependent on the value of the base function $f(x)=b^x$, as well as on the values of a and k.

- Functions of the form $g(x)=a\,f[k(x-d)]+c$ can be graphed by applying the appropriate transformations to the key points of the base function $f(x)=b^x$, following the order of operations.
- In exponential functions of the form $g(x)=ab^{k(x-d)}+c$:
  - If $|a|>1$, a vertical stretch by a factor of $|a|$ occurs. If $0<|a|<1$, a vertical compression by a factor of $|a|$ occurs. If a is also negative, then the function is reflected in the x-axis.
  - If $|k|>1$, a horizontal compression by a factor of $\left|\frac{1}{k}\right|$ occurs. If $0<|k|<1$, a horizontal stretch by a factor of $\left|\frac{1}{k}\right|$ occurs. If k is also negative, then the function is reflected in the y-axis.
  - If $d>0$, a horizontal translation of d units to the right occurs. If $d<0$, a horizontal translation to the left occurs.
  - If $c>0$, a vertical translation of c units up occurs. If $c<0$, a vertical translation down occurs.
  - You might have to factor the exponent to see what the transformations are. For example, if the exponent is $2x+2$, it is easier to see that there was a horizontal compression of $\frac{1}{2}$ and a horizontal translation of 1 to the left if you factor to $2(x+1)$.
  - When transforming functions, consider the order. You might perform stretches and reflections followed by translations, but if the stretch involves a different coordinate than the translation, the order doesn't matter.
  - The domain is always $\{x\in\Re\}$. Transformations do not change the domain.
  - The range depends on the location of the horizontal asymptote and whether the function is above or below the asymptote. The horizontal asymptote changes when vertical translations are applied. If the function is above the asymptote, its range is $y>c$. If the function is below the asymptote, its range is $y<c$.

Example 1 Write the equation for the function that results when each set of transformations is applied to the base function $y=5^x$.

a) stretch horizontally by a factor of 4, reflect in the y-axis, translate right 16 units, and shift up 8 units

b) compress vertically by a factor of $\frac{1}{7}$, reflect in the x-axis, shift left 2 units, and translate down 12 units

Example 3 For each transformation, state the base function and describe the transformations in the order in which they could be applied. Use transformations to sketch the graph of each of the following functions. Also state the y-intercept, the equation of the asymptote, and the domain and range for each function.

a) $f(x) = 2\left[3^{-(x-5)}\right] + 1$

b) $g(x) = -\left(\frac{1}{2}\right)^{2x+12} - 4$

Solving Exponential Equations.

1. $49^{x-1} = 7\sqrt{7}$

2. $2^{3x-4} = 0.25$

3. $\left(\frac{1}{4}\right)^{x+4} = \sqrt{8}$

4. $36^{2x+4} = \sqrt{1296}^{x}$

5. $2^{2x+1} + 7 = 71$

6. $9^{2x+1} = 81(27)^{x}$

7. $4^{x+1} + 4^{x} = 80$

8. $2^{x+2} + 2^{x} = 320$

9. $2^{x+2} - 2^{x} = 96$

10. $10^{x+1} - 10^{x} = 9000$

11. $3^{x+2} + 3^{x} = 30$

12. $4^{x+3} - 4^{x} = 63$

13. $2^{x^2} = 32(2^{4x})$

14. $3^{x^2+20} = \left(\frac{1}{27}\right)^{3x}$

15. $\left(\sqrt{8}\right)^{x} = 2^{x-2}$

16. $9^{k-1} = \left(\sqrt{27}\right)^{2k}$

17. $2^{2x} - 9(2^{x}) + 8 = 0$

18. $3^{2x} - 12(3^x) = -27$

19. $4^{2x} = -16 + 17(4^x)$

20. $4^{2x} - 16 = 0$

21. $2^{2x} - 18(2^x) = -32$

22. $3^{2x} - 6(3^x) + 9 = 0$

23. $2(2^x) = 8 + 2^{2x}$

Answers

1. $x = \frac{7}{4}$
2. $x = \frac{2}{3}$
3. $x = -\frac{19}{4}$
4. $x = -4$
5. $x = \frac{5}{2}$
6. $x = 2$
7. $x = 2$
8. $x = 6$
9. $x = 5$
10. $x = 3$
11. $x = 1$
12. $x = 0$
13. $x = 5, x = -1$
14. $x = -4, x = -5$
15. $x = -4$
16. $k = -2$
17. $x = 3, x = 0$
18. $x = 2, x = 1$
19. $x = 2, x = 0$
20. $x = 1$
21. $x = 4, x = 1$
22. $x = 1$
23. $x = 2$

EXPONENTIAL FUNCTIONS

1. Solve each equation.

a) $6^{4y-7}=1$

b) $8^{k-2}=\left(\sqrt{128}\right)^{2k}$

c) $3^{b+1}+3^{b+2}-108=0$

d) $12\left(4^{2x}\right)=-5\left(4^{x}\right)+2$

Applications Involving Exponential Functions

- The exponential function $f(x) = ab^x$ and its graph can be used as a model to solve problems involving exponential growth and decay. Note that
  - $f(x)$ is the final amount or number
  - a is the initial amount or number
  - for exponential growth, $b = 1 + growth\ rate$; for exponential decay, $b = 1 - decay\ rate$
  - x is the number of growth or decay periods
- For situations that can be modelled by an exponential function:
  - If the *growth rate* (as a percent) is given, then the base of the power in the equation can be obtained by *adding* the rate, as a decimal to 1. For example, a growth rate of 8% involves multiplying repeatedly by 1.08.
  - If the *decay rate* (as a percent) is given, then the base of the power in the equation is obtained by *subtracting* the rate, as a decimal, from 1. For example, a decay rate of 8% involves multiplying repeatedly by 0.92.
  - One way to tell the difference between growth and decay is to consider whether the quantity in question (e.g., light intensity, population, dollar value) has increased or decreased.
  - The units for the growth/decay rate and for the number of growth/decay periods must be the same. For example, if light intensity decreases "per metre", then the number of decay periods in the equation is measured in metres, too.

Example 1 A very convenient measure of population growth is a doubling period. The population of the world was 6 billion in 1999. This population is growing exponentially and doubles every 35 years. Estimate the world population in 2050, to the nearest half billion.

Example 2 Carbon-14 is a radioactive substance with a half-life of 5730 years. It is used to determine the age of artifacts. An archaeologist discovers that the burial cloth on an Egyptian mummy has 12.5% of the carbon-14 that it contained originally. How long ago was the mummy buried?

Example 3 A used-car dealer sells a five year-old car for $4200. What was the original value of the car if the depreciation is 15% a year?

Example 4 For a biology experiment, there are 50 cells present. After 2 hours, there are 1800 bacteria. How many bacteria would there be in 6 hours?

Example 5 A population of mould doubles every 3 hours. If the initial population was 600, how long will it take to reach a population of 153600?