Lesson 2.

Limit Laws (Properties).

1) Constant Law

If $f(x) = a$, then $\lim_{x \to c} f(x) = \lim_{x \to c} a = a$

2) Power Law

If $f(x) = x^n$, where $n \in \mathbb{R}$.

then $\lim_{x \to c} x^n = c^n$, if exists.

3) Constant Multiplication Law.

If $y = af(x)$, then $\lim_{x \to c} [af(x)] = a \lim_{x \to c} f(x)$

i.e. $\lim_{x \to 3} [\sqrt{2x}] = \sqrt{2} \lim_{x \to 3} \sqrt{x} = \sqrt{2}\sqrt{3} = \sqrt{6}$

4) Sum Law.

If $y = f(x) + g(x)$, then $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$.

5) Difference Law

If $y = f(x) - g(x)$. then $\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$.

Using Law 1 through Law 5, we could evaluate the limit of any polynomial function.

For example.

$$\lim_{x\to -2} [x^4 - 3x^2 + x - 8]$$

$$= \lim_{x\to -2} x^4 - \lim_{x\to -2} (3x^2) + \lim_{x\to -2} x - \lim_{x\to -2} 8$$

$$= (-2)^4 - 3(-2)^2 + (-2) - 8$$

$$= 16 - 12 - 10 = -6.$$

6) Product Law

If $y = f(x)g(x)$. then $\lim_{x\to c} [f(x)g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$.

7) Quotient Law

If $y = \frac{f(x)}{g(x)}$, then $\lim_{x\to c} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, if exists.

When we apply Limit Laws. some indeterminate form, such as $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, 0^0, 1^∞, ∞^0, $0 \cdot \infty$ may occur. Further investigation is needed.

For example, $\lim_{x\to 4} \frac{\sqrt{x} - 2}{x^2 - 16}$, this is "$\frac{0}{0}$"

Sol. $\lim_{x\to 4} \frac{\sqrt{x} - 2}{(x-4)(x+4)} = \lim_{x\to 4} \frac{\cancel{\sqrt{x} - 2}^{\,1}}{\cancel{(\sqrt{x}-2)}_{1}(\sqrt{x}+2)(x+4)} = \frac{1}{(\sqrt{4}+2)(4+4)} = \frac{1}{32}$

In this question, $\sqrt{x}-2$ or $x-4$ is "zero factor". that causes "0". we need to factor them and cancel them.

8) Ratical Root Law

$$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$$, if exists.

For example. $\lim_{x\to 3} \sqrt[4]{1-x} = \sqrt[4]{\lim_{x\to 3}(1-x)}$

$= \sqrt[4]{1-3} = \sqrt[4]{-2}$

$= (\sqrt{-2})^{\frac{1}{2}} = (\sqrt{2}\, i)^{\frac{1}{2}} = \sqrt[4]{2}\sqrt{i}$

But D.N.E.

Other basic Limits.

1. Squeeze or Sandwich Theorem for Limit.

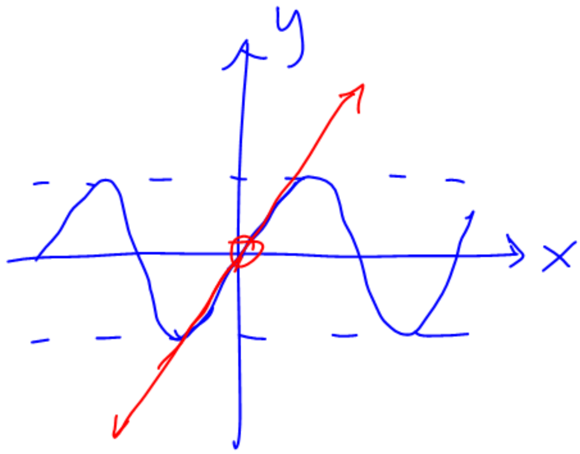
For $x \in [a,b]$, except possibly at $x=c$.

if ① $f(x) \le h(x) \le g(x)$; ② $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$

then $\lim_{x\to c} h(x) = L$

2. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ "$\frac{0}{0}$".

$y = \sin x \approx y = x$, when $x \to 0$.



Generally.

$$\lim_{x\to 0} \frac{\sin ax}{ax} = 1.$$

where $a \in \mathbb{R}$ but $a \neq 0$.

i.e. $$\lim_{x\to 0} \frac{\sin(3x)}{3x} = 1$$

$y = \sin(3x) \approx y = 3x$, as $x \to 0$.

i.e. $$\lim_{x\to 0} \frac{\sin(\tan^2(3x))}{2x^2} = \lim_{x\to 0} \frac{\sin((3x)^2)}{2x^2}$$

$$= \lim_{x\to 0} \frac{(3x)^2}{2x^2} = \frac{9}{2}$$

If $x \to 0$, $\sin(ax) \approx ax \approx \tan(ax)$

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3) $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e \approx 2.718281828$

or $\lim_{x \to \infty} (1+\frac{1}{x})^{x} = e$. $x = 10000000000$ "1^{∞}"

$\lim_{x \to 0^-} (1+x)^{\frac{1}{x}} = e$, $\lim_{x \to 0^+} (1+x)^{\frac{1}{x}} = e$

$x \approx -0.0000000001$ $x \approx 0.0000000001$

$\lim_{x \to -\infty} (1+x)^{\frac{1}{x}} = e$.

$x = -10000000000$

4) $\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$

are polynomial functions.

$$\lim_{x \to \pm\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0} = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}$$

where $n, m \in R$.

For example.

$$\lim_{x\to-\infty} \frac{\sqrt{3x^2-x+6}}{4x^{4/3}+x-7}$$

Sol. $$\lim_{x\to-\infty} \frac{\sqrt{3x^2-x+6}}{4x^{4/3}+x-7} = \lim_{x\to-\infty} \frac{\sqrt{3x^2}}{4x^{4/3}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{3}\,|x|}{4\,x^{4/3}} = \frac{\sqrt{3}}{4} \lim_{x\to-\infty} \frac{-x}{x^{4/3}}$$

$$= -\frac{\sqrt{3}}{4} \lim_{x\to-\infty} \frac{1}{x^{1/3}} = -\frac{\sqrt{3}}{4} \frac{1}{(-\infty)^{1/3}}$$

$$= -\frac{\sqrt{3}}{4} \frac{1}{-\infty} = -\frac{\sqrt{3}}{4}(0^-) = 0$$

Asymptotes by Limit.

vertical asymptotes. (V.A.s)

If $\lim_{x\to a} f(x) = \pm\infty$, then $x=a$ is a V.A.

Horizontal Asymptote (H.A.s)
and Oblique linear Asymptotes (O.A.s)

If $\lim_{x \to \pm\infty} [f(x) - (ax+b)] = 0$

then $y = ax+b$ is an O.A.

If $a=0$. $y=b$ is a H.A.

Theorems related to Continuous functions

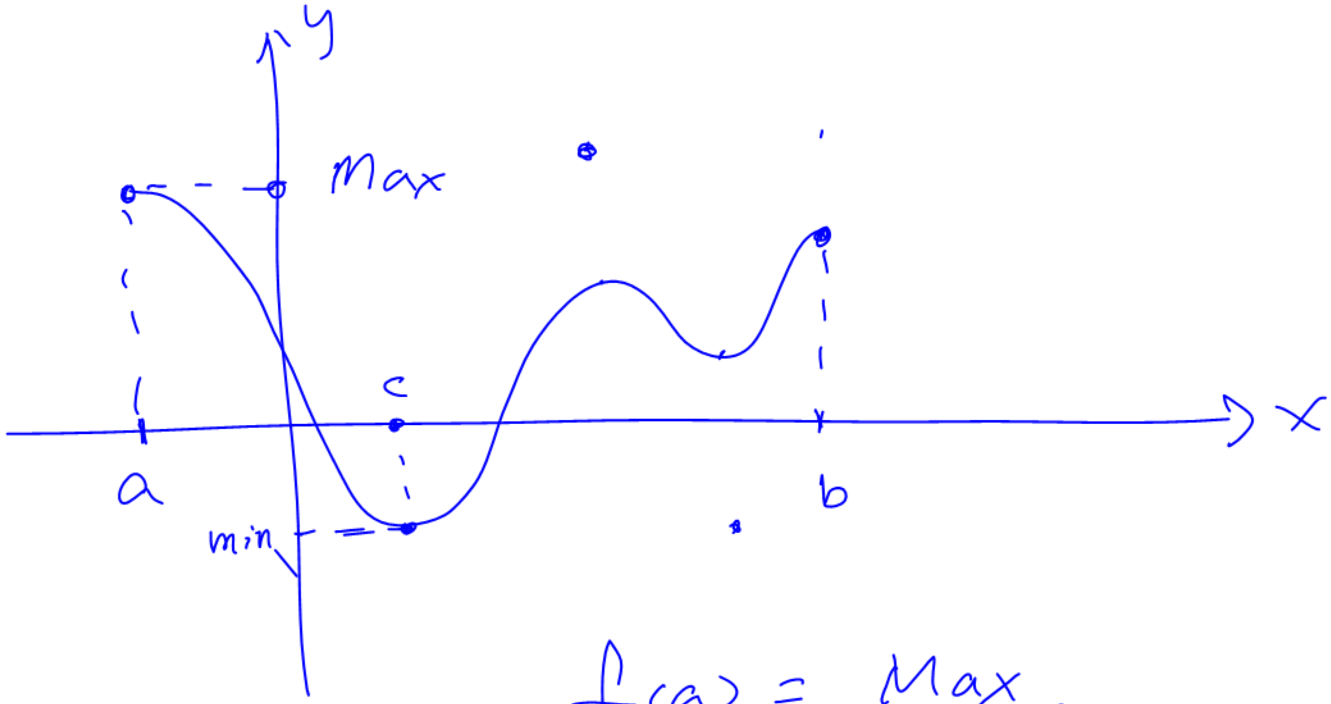
1) The Extreme Value theorem.

If $y=f(x)$ is continuous over $x \in [a,b]$

then $f(x)$ attains a minimum value.

an a maximum value. somewhere in

the interval.



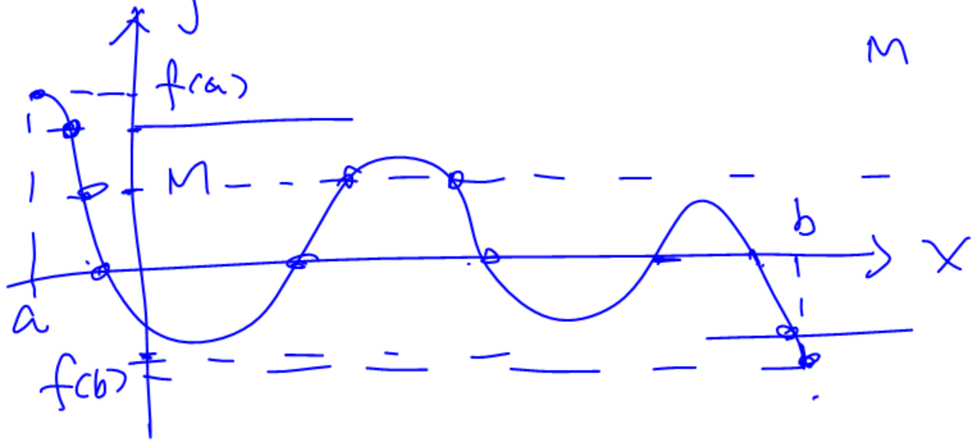
$f(a) = Max.$
$f(c) = Min.$

2) The Intermediate Value Theorem

If $y=f(x)$ is continuous over $x \in [a,b]$.
and $f(a) \le M \le f(b)$ or $f(a) \ge M \ge f(b)$.
then there is at least one x value $c \in [a,b]$
such that $f(c) = M$.

$M=0.$

M



3) Properties of continuous functions

If $f(x)$ and $g(x)$ are continuous at $x = c$. Then the following functions

i) $y = kf(x)$

ii) $y = f(x) + g(x)$

iii) $y = f(x) - g(x)$

iv) $y = f(x)g(x)$

v) $y = \frac{f(x)}{g(x)}$. where $g(c) \neq 0$.

are also continuous at $x = c$