Calculations and Operations

1. Exponent

1) What Is an Exponent?

There's nothing mysterious! An exponent is simply shorthand for multiplying that number of identical factors. So 4^3 is the same as (4)(4)(4), three identical factors of 4.
Such as: $4^3 = (4)(4)(4)$.

And x^3 is just three factors of x, (x)(x)(x).

One warning: Remember the order of operations.

Exponents are the first operation (in the absence of grouping symbols like parentheses), so the exponent applies only to what it's directly attached to.

$3x^3$ is 3(x)(x)(x), suah as $x^{-3} = \dfrac{1}{x^3}$, not (3x)(3x)(3x).

If we wanted (3x)(3x)(3x), we'd need to use grouping: $(3x)^3$.

2) Negative Exponents

$x^{-n} = \dfrac{1}{x^n}$ or $\dfrac{1}{x^{-n}} = x^n$

A negative exponent means to divide by that number of factors instead of multiplying.

So 4^{-3} is the same as $1/(4^3)$, $4^{-3} = \dfrac{1}{4^3} = \dfrac{1}{(4)(4)(4)} = \dfrac{1}{64}$, and $x^{-3} = \dfrac{1}{x^3}$.

As you know, you can't divide by zero. So there's a restriction that $x^{-n} = \dfrac{1}{x^n}$ only when x is not zero. When x = 0, x^{-n} is undefined.

2. Polynomials

1) Term

A term is a number, variable or the product of a number and variable(s).
Examples of terms are 3x, $5y^3$, 2ad, z.

2) Coefficient

Here are the coefficients of the terms listed above:

Term	Coefficient
3s	3
$5y^3$	5
2ab	2
z	1

3) Constant Term

A constant term is a term that contains only a number.
In other words, there is no variable in a constant term. Examples of constant terms are 4, 100, and -5.

Standard Form of a Polynomial
$c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$
where *n* is a non-negative integer.
c_n is called the leading coefficient. c_0 is a constant.

In other words, a polynomial is a finite sum of terms where the exponents on the variables are non-negative integers. Note that the terms are separated by +'s and -'s.

An example of a polynomial expression is $3x^5 - 5x^3 + x - 10$.

4) Degree of a Term
The degree of a term is the sum of the exponents on the variables contained in the term.

5) Degree of the Polynomial
The degree of the polynomial is the largest degree of all its terms.

6) Descending Order
This means that the term that has the highest degree is written first, the term with the next highest degree is written next, and so forth.

7) Some Types of Polynomials

Type	Definition	Example
Monomial	A polynomial with **one** term	$5x$
Binomial	A polynomial with **two** terms	$5x - 10$
Trinomial	A polynomial with **three** terms	$7x^4 - 6x^2 + 5$

8) Combining Like Terms

Recall that like terms are terms that have the exact same variables raised to the exact same exponents.

One example of like terms is $3x^2, -5x^2$. Another example is $5ab^2$, $4ab^2$.

Example: Simplify $15b + 9 + 5b - 2$ by combining like terms.

Are there any like terms that we can combine?
It looks like it. Two terms have the same variable part, *b*. The other pair of terms are constant terms that can be combined together.

$$15b + 9 + 5b - 2 = 15b + 5b + 9 - 2 = (15 + 5)b + 9 - 2 = 20b + 7$$

3. Adding Polynomials

Step 1: Remove the ( ).
If there is only a + sign in front of ( ), then the terms inside of ( ) remain the same when you remove the ( ).

Step 2: Combine like terms.
Example: Perform the indicated operation and simplify:

$$(5x^2 - 4x + 10) + (3x^2 - 2x - 12)$$

$$(5x^2 - 4x + 10) + (3x^2 - 2x - 12) = 5x^2 - 4x + 10 + 3x^2 - 2x - 12 = 8x^2 - 6x - 2$$

4. Subtracting Polynomials

Step 1: Remove the ( ).
If there is a - in front of the ( ) then distribute it by multiplying every term in the ( ) by a -1. Or you can think of it as negating every term in the ( ).

Step 2: Combine like terms.
Example: Perform the indicated operation and simplify:

$$(8y^2 - y - 2) - (7y^2 + 5y - 4)$$

$$(8y^2 - y - 2) - (7y^2 + 5y - 4) = 8y^2 - y - 2 - 7y^2 - 5y + 4 = y^2 - 6y + 2$$

5. Multiplying Polynomials

1) (Monomial)(Monomial)

In this case, there is only one term in each polynomial. You simply multiply the two terms together.

Example: Find the following product $(-2x^3)(3x^2)$.

$(-2x^3)(3x^2) = (-2)(3)(x^3 \cdot x^2) = -6x^5$

2) (Monomial)(Polynomial)

In this case, there is only one term in one polynomial and more than one term in the other. You need to distribute the monomial to EVERY term of the other polynomial.

Example: Find the following product $-5x(7x^5 + 6x^3 - 4x)$.

$-5x(7x^5 + 6x^3 - 4x) =$

$(-5x)(7x^5) - (5x)(6x^3) - (5x)(-4x) =$

$-35x^6 - 30x^4 + 20x^2$

3) (Binomial)(Binomial)

In this case, both polynomials have two terms. You need to distribute both terms of one polynomial times both terms of the other polynomial.

One way to keep track of your distributive property is to use the FOIL method. Note that this method only works on (Binomial)(Binomial).

F	**First terms**
O	**Outside terms**
I	**Inside terms**
L	**Last terms**

This is a fancy way of saying take every term of the first binomial times every term of the second binomial. In other words, do the distributive property for every term in the first binomial.

Example: Find the following product $(3x+5)(2x-7)$.

$(3x+5)(2x-7) =$

$\quad F \qquad\qquad O \qquad\qquad I \qquad\qquad L$

$(3x)(2x) + (3x)(-7) + (5)(2x) + 5(-7) =$

$6x^2 - 21x + 10x - 35 =$

$6x^2 - 11x - 35$

As mentioned above, use the distributive property until every term of one polynomial is multiplied times every term of the other polynomial. Make sure that you simplify your answer by combining any like terms.

4) Special formulas

1) Difference of Squares

When the sum of two numbers multiplies their difference -- $(a + b)(a - b)$ -- then the product is the difference of their squares:

$\boldsymbol{(a + b)(a - b) = a^2 - b^2}$

For, the like terms will cancel: $(a + b)(a - b) = a^2 + ab - ab + b^2 = a^2 - b^2$

Symmetrically, the difference of two squares can be *factored*:

$x^2 - 25 = (x + 5)(x - 5)$

x^2 is the square of x. 25 is the square of 5.

Note that the sum of two squares -- $a^2 + b^2$ -- cannot be factored.

Example 1: Expand.

a) $(x + 9)(x - 9) = \boldsymbol{x^2 - 81}$ b) $(y + z)(y - z) = \boldsymbol{y^2 - z^2}$

Example 2: Factor.

a) $x^2 - 100 = \boldsymbol{(x + 10)(x - 10)}$ b) $y^2 - 1 = \boldsymbol{(y + 1)(y - 1)}$

c) $1 - 4z^2 = \boldsymbol{(1 + 2z)(1 - 2z)}$ d) $25m^2 - 9n^2 = \boldsymbol{(5m + 3n)(5m - 3n)}$

Example 3: Multiply $(x^3 + 2)(x^3 - 2)$.

Recognize the form: $(a + b)(a - b)$. The product will be the difference of two squares:

$(x^3 + 2)(x^3 - 2) = x^6 - 4$.

x^6 is the square of x^3. 4 is the square of 2.

Upon seeing the form $(a + b)(a - b)$, you should *not* do the FOIL method. You should recognize immediately that the product will be $a^2 - b^2$. That is skill in algebra. And the order of factors never matters:

$(a + b)(a - b) = (a - b)(a + b) = a^2 - b^2$.

2) Perfect Square

The square of a binomial has a form of $(a + b)^2$

The square of a binomial comes up so often that you should be able to write the final product immediately. It will turn out to be a very specific trinomial. To see that, let us square $(a + b)$:

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2.$$

For, the outers plus the inners will be $ab + ba = 2ab$.

The order of factors does not matter.

$$(a + b)^2 = a^2 + 2ab + b^2$$

The square of any binomial produces the following three terms:

1. The square of the first term of the binomial: a^2
2. Twice the product of the two terms: $2ab$
3. The square of the second term: b^2

The square of every binomial has that form: $a^2 + 2ab + b^2$.

Example 1: Square the binomial $(x + 6)$.

$(x + 6)^2 = x^2 + 12x + 36$

x^2 is the square of x.
$12x$ is twice the product of x with 6. ($x \cdot 6 = 6x$. Twice that is $12x$.)
36 is the square of 6.

$x^2 + 12x + 36$ is called a perfect square trinomial -- which is the square of a binomial.

Example 2: Square the binomial $(3x - 4)$.

$(3x - 4)^2 = 9x^2 - 24x + 16$

$9x^2$ is the square of $3x$.
$-24x$ is twice the product of $3x \cdot -4$. ($3x \cdot -4 = -12x$. Twice that is $-24x$.)
16 is the square of -4.

Note: If the binomial has a minus sign, then the minus sign appears only in the middle term of the trinomial. Therefore, using the double sign $\pm$ ("plus or minus"), we can state the rule as follows:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

This means: If the binomial is $a + b$, then the middle term will be $+2ab$; but if the binomial is $a - b$, then the middle term will be $-2ab$

The square of $+b$ or $-b$, of course, is always positive. It is always $+b^2$.

Example 3. $(5x^3 - 1)^2 = 25x^6 - 10x^3 + 1$

$25x^6$ is the square of $5x^3$.
$-10x^3$ is twice the product of $5x^3$ and -1. ($5x^3$ times $-1 = -5x^3$. Twice that is $-10x^3$.)
1 is the square of -1.

Note: You should be clear that $(a + b)^2$ is *not* $a^2 + b^2$!

An exponent may not be "distributed" over a sum.

6. Divide

Polynomial ÷ Monomial

Step 1: Use distributive property to write every term of the numerator over the monomial in the denominator.

Distributive Properties:

a(b + c) = ab + ac or *(b + c)a = ba + ca*

Step 2: Simplify the fractions.

Example: Divide $\dfrac{4a^8 - 2a^4 + 10a^2}{2a^2}$.

Step 1: Use distributive property to write every term of the numerator over the monomial in the denominator AND

Step 2: Simplify the fractions.

$$\frac{4a^8 - 2a^4 + 10a^2}{2a^2} = \frac{4a^8}{2a^2} - \frac{2a^4}{2a^2} + \frac{10a^2}{2a^2} = 2a^6 - a^2 + 5$$

7. Inequality

A linear inequality has this standard form: $ax + b < c$.

When a is positive, then solving it is identical to solving an equation:

$$ax < c - b$$

$$x < \frac{c-b}{a}$$

As with equations, the inequality is "solved" when positive x is isolated on the left. The only difference between solving an inequality and solving an equation, is the following:

When we multiply or divide by a negative number, the <u>sign</u> must change.

Example 1: Solve for x.

$$-2x + 5 < 11$$
$$-2x < 6$$
$$x > -3.$$

The signs changed, of course, because we divided both sides by *negative* 2.

Alternatively, we could immediately make $2x$ positive -- by changing all the signs on both sides. But then we must also change the sense.

$-2x + 5 < 11$ implies $2x - 5 > -11$, and so on.

Example 2: Solve this inequality for x.

$$\frac{x-3}{x+5} > 0.$$

Solution. To have a positive quotient, the numerator and denominator must have the same sign. Therefore, either

$$1)\ \ x - 3 > 0 \ \ \textit{and} \ \ x + 5 > 0,$$

or

$$2)\ \ x - 3 < 0 \ \ \textit{and} \ \ x + 5 < 0.$$

Now, 1) implies

$$x > 3 \ \ \textit{and} \ \ x > -5.$$

Which numbers are these that are both greater than 3 *and* greater than −5?
Clearly, any number greater than 3 will also be greater than −5. Therefore, 1) has the solution

$$x > 3.$$

Next, 2) implies

$$x < 3 \ \ \textit{and} \ \ x < -5.$$

Which numbers are these that are both less than 3 *and* less than −5?
Clearly, any number less than −5 will also be less than 3. Therefore, 2) has the solution

$$x < -5.$$

The solution, therefore, is

$$x < -5 \ \text{ or } \ x > 3.$$

► **Questions:**

1. If one minus the reciprocal of (1– x) equals the reciprocal of (1– x), then what is the x?

2. If four times the reciprocal of the circumference of a circle equals the diameter of the circle, then what is the area of the circle?

3. For all non-zero numbers x and y such that x = $\frac{1}{y}$, what is $(x-\frac{1}{x})(y+\frac{1}{y})$ in terms of y?

4. For all non-zero real numbers x and y such that x - y = xy. What is the value of $\frac{1}{x}-\frac{1}{y}$?

5. Find the largest whole number such that seven times the number is less than 100.

6. If $x \neq 0$, $x / 2 = y^2$ and $x / 4 = 4y$, then what is the value of x?

7. What is the value of $\frac{1000^2}{252^2-248^2}$ = ?

8. If x, y and y – 1/x are not 0, then $\frac{x-\frac{1}{y}}{y-\frac{1}{x}}$ equals what?

9. What is the value of $(-1)^{5^2}+1^{2^5}=?$

10. If $\frac{x/4}{2}=\frac{4}{x/2}$, then what is x?

11. If n! = 1 × 2 × 3 ×…× (n - 1) × n, find the value of $\frac{(3!)!}{3!}$

12. Six numbers from a list of nine integers are 7, 8, 3, 5, 9, and 5. What is the largest possible value of the median of all nine numbers in this list?