Probability

1. Mean

Mean: Add up the numbers and divide by how many numbers.

Example: Birthday Activities
Uncle Bob wants to know the average age at the party, to choose an activity.
There will be 6 kids aged 13, and also 5 babies aged 1.

Add up all the ages, and divide by 11 (because there are 11 numbers):
(13+13+13+13+13+13+1+1+1+1+1) / 11 = **7.5**

2. Median

When a set of numbers is arranged in order from smallest to largest, or largest to smallest, the median is the middle number.

Example: Birthday Activities (continued)
List the ages in order:
1, 1, 1, 1, 1, 13, 13, 13, 13, 13, 13

Choose the middle number:
1, 1, 1, 1, 1, **13**, 13, 13, 13, 13, 13

The Median age is **13** ... so let's have a **Disco**!

Sometimes there are **two** middle numbers. Just average them:

Example: What is the Median of 3, 4, 7, 9, 12, 15
There are two numbers in the middle:
3, 4, **7, 9,** 12, 15

So we average them: (7 + 9) / 2 = 16/2 = 8
The Median is **8**

3. Mode

The Mode is the value that occurs most often.

Example: Birthday Activities (continued)
Group the numbers so we can count them:
1, 1, 1, 1, 1, 13, 13, 13, 13, 13, 13

"13" occurs 6 times, "1" occurs only 5 times, so the mode is **13**.
But Mode can be tricky, there can sometimes be more than one Mode.

Example: What is the Mode of 3, 4, 4, 5, 6, 6, 7

Well ... 4 occurs twice but 6 **also** occurs twice.
So **both 4 and 6** are modes.
When there are two modes it is called "bimodal", when there are three or more modes we call it "multimodal".

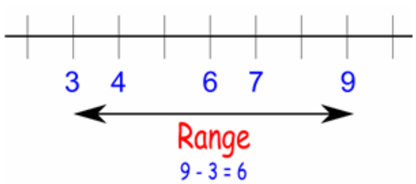
Conclusion
There are other ways of measuring central values, but **Mean, Median and Mode** are the most common.

4. Range (Statistics)

1) What is the Range?
The **Range** is the difference between the lowest and highest values.

Example: In **{4, 6, 9, 3, 7}** the lowest value is 3, and the highest is 9.

So the range is 9-3 = **6**.



2) The Range Can Be Misleading
The range can sometimes be misleading when there are extremely high or low values.

Example: In **{8, 11, 5, 9, 7, 6, 3616}**, the lowest value is 5 and the highest is 3616,
So the range is 3616 - 5 = **3611**.
The single value of 3616 makes the range large, but most values are around 10.

5. Probability

1) Probability

$$Probability\ of\ \ an\ event\ happening = \frac{Number\ of\ \ ways\ it\ can\ happen}{Total\ number\ of\ \ outcomes}$$

Example: there are 5 marbles in a bag: 3 are red, and 2 are blue. What is the probability that a blue marble will be picked?

Number of ways it can happen: 2 (there are 2 blues)

Total number of outcomes: 5 (there are 5 marbles in total)
So the probability is 2/5 = **0.4**

2) Experiment
Experiment: an action where the result is uncertain.
Tossing a coin, throwing dice, seeing what pizza people choose are all examples of experiments

3) Event
Event: a single result of an experiment

Example:
- Getting a Tail when tossing a coin is an event
- Rolling a "5" is an event.
- Rolling an "even number" (2, 4 or 6) is also an event.

6. Probability of an Event

1) Definition of a Probability
Suppose an event E can happen in r ways out of a total of n possible equally likely ways.

Then the probability of occurrence of the event (called its success) is denoted by

$$P(E) = \frac{r}{n}$$

The probability of **non-occurrence** of the event (called its failure) is denoted by

$$P(E) = \frac{n-r}{n} = 1 - \frac{r}{n}.$$

Notice the bar above the E, indicating the event does **not** occur.
Thus, $P(\overline{E}) + P(E) = 1$

In words, this means that the sum of the probabilities in any experiment is 1.

2) Definition of Probability using Sample Spaces
When an experiment is performed, we set up a sample space of all possible outcomes.
In a sample of N equally likely outcomes we assign a chance (or weight) of $\frac{1}{N}$ to each outcome.

We define the **probability** of an event for such a sample as follows:
The probability of an event E is defined as the number of outcomes favourable to E divided by the total number of equally likely outcomes in the sample space S of the experiment.

That is: $P(E) = \dfrac{n(E)}{n(S)}$

where
- $n(E)$ is the number of outcomes favourable to E and
- $n(S)$ is the total number of equally likely outcomes in the sample space S of the experiment.

3) Properties of Probability

(a) $0 \le P(\text{event}) \le 1$
In words, this means that the probability of an event must be a number between 0 and 1 (inclusive).

(b) *P*(impossible event) = 0
In words: The probability of an impossible event is 0.

(c) *P*(certain event) = 1
In words: The probability of an absolutely certain event is 1.

Example 1: What is the probability of...
(a) Getting an ace if I choose a card at random from a standard pack of 52 playing cards.

In a standard pack of 52 playing cards, we have:

♥ 2 3 4 5 6 7 8 9 10 J Q K A
♦ 2 3 4 5 6 7 8 9 10 J Q K A
♣ 2 3 4 5 6 7 8 9 10 J Q K A
♠ 2 3 4 5 6 7 8 9 10 J Q K A

There are 4 aces in a normal pack. So $P(ace) = \frac{4}{52} = \frac{1}{13}$

(b) Getting a 5 if I roll a die.
A die has 6 numbers. There is only one 5 on a die, so $P(5) = \frac{1}{6}$

(c) Getting an even number if I roll a die.
Even numbers are 2, 4, 6. So $P(even) = \frac{3}{6} = \frac{1}{2}$

(d) Having one Tuesday in this week?
Each week has a Tuesday, so probability = 1.

Example 2: There are 15 balls numbered 1 to 15, in a bag. If a person selects one at random, what is the probability that the number printed on the ball will be a prime number greater than 5?

The primes between 5 and 15 are: 7, 11, 13,
So the probability $= \frac{3}{15} = \frac{1}{5}$.

Example 3: The names of four directors of a company will be placed in a hat and a 2-member delegation will be selected at random to represent the company at an international meeting. Let A, B, C and D denote the directors of the company. What is the probability that
(a) A is selected? (b) A or B is selected? (c) A is not selected?

The possible outcomes are: AB, AC, AD, BC, BD, CD.

(a) The probability is $\frac{3}{6}=\frac{1}{2}$

(b) The probability of getting A or B first is $\frac{2}{4}=\frac{1}{2}$.
Now to consider the probability of selecting A or B as the second director. In this case, the first director has to be C or D with probability $\frac{2}{4}$ (2 particular directors out of 4 possible).

Then the probability of the second being A or B is $\frac{2}{3}$ (2 particular directors out of the remaining 3 directors).

We need to multiply the two probabilities.
So the probability of getting A or B for the second director is $\frac{2}{4}\times\frac{2}{3}=\frac{1}{3}$

The total is: $\frac{1}{2}\times\frac{1}{3}=\frac{5}{6}$

(c) Probability that A is not selected is $1-\frac{1}{2}=\frac{1}{2}$

7. Conditional Probability

If E_1 and E_2 are two events, the probability that E_2 occurs given that E_1 has occurred is denoted by $P(E_2|E_1)$.

$P(E_2|E_1)$ is called the **conditional probability** of E_2 given that E_1 has occurred.

Calculating Conditional Probability

Let E_1 and E_2 be any two events defined in a sample space S such that $P(E_1) > 0$.
The **conditional probability** of E_2, assuming E_1 has already occurred, is given by

$$P(E_2 / E_1) = \frac{P(E_2 \ and \ E_1)}{P(E_1)}$$

Example 1: Let *A* denote the event `student is female' and let *B* denote the event `student is French'. In a class of 100 students suppose 60 are French, and suppose that 10 of the French students are females. Find the probability that if I pick a French student, it will be a girl, that is, find $P(A|B)$.

Solution
Since 10 out of 100 students are both French and female, then $P(A \text{ and } B) = 10/100$

Also, 60 out of the 100 students are French, so $P(B) = 60/100$

So the required probability is:

$$P(A/B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{10/100}{60/100} = \frac{1}{6}$$

Example 2: What is the probability that the total of two dice will be greater than 8, given that the first die is a 6?

Solution
Let E_1 = first die is 6;
Let E_2 = total of two dice is > 8

Then "E_1 and E_2" will be given by (6, 3) (6, 4) (6, 5) (6, 6).

There are 36 possible outcomes when we throw 2 dice.

So $P(E_2 \text{ and } E_1) = \frac{4}{36} = \frac{1}{9}$

Therefore

$$P(E_2 / E_1) = \frac{P(E_2 \text{ and } E_1)}{P(E_1)} = \frac{1/9}{1/6} = \frac{2}{3}$$

8. Combinations and Permutations

In English we use the word "combination" loosely, without thinking if the order of things is important. In other words:

"My fruit salad is a combination of apples, grapes and bananas" We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas", it’s the same fruit salad.

"The combination to the safe was 472". Now we do care about the order. "724" would not work, nor would "247". It has to be exactly 4-7-2.

If the order doesn't matter, it is a Combination.

If the order does matter it is a Permutation.

In other words: **A Permutation is an ordered Combination.**

1) Permutations

There are basically two types of permutation:

a) Repetition is Allowed: such as the lock above. It could be "333".
b) No Repetition: for example the first three people in a running race. You can't be first and second.

a) Permutations with Repetition

When you have n things to choose from ... you have n choices each time!

When choosing r of them, the permutations are: **n × n × ... (r times)**

(In other words, there are n possibilities for the first choice, THEN there are n possibilities for the second choice, and so on, multiplying each time.)

Which is easier to write down using an exponent of r: **n × n × ... (r times) =** $\mathbf{n^r}$

Example: in the lock above, there are 10 numbers to choose from (0, 1 … 9) and you choose 3 of them:

10 × 10 × ... (3 times) = 103 = 1,000 permutations

b) Permutations without Repetition

In this case, you have to reduce the number of available choices each time.

For example, what order could 16 pool balls be in?

After choosing, say, number "14" you can't choose it again.

So, your first choice would have 16 possibilities, and your next choice would then have 15 possibilities, then 14, 13, etc. And the total permutations would be:

16 × 15 × 14 × 13 × ... = 20,922,789,888,000

But maybe you don't want to choose them all, just 3 of them, so that would be only:

16 × 15 × 14 = 3,360

In other words, there are 3,360 different ways that 3 pool balls could be selected out of 16 balls.

But how do we write that mathematically? Answer: we use the "factorial function"

!	The factorial function (symbol: !) just means to multiply a series of descending natural numbers. Examples: 4! = 4 × 3 × 2 × 1 = 24 7! = 7 × 6 × 5 × 4 × 3 × 2 × 1 = 5,040 1! = 1

Note: it is generally agreed that 0! = 1. It may seem funny that multiplying no numbers together gets you 1, but it helps simplify a lot of equations.

So, if you wanted to select all of the billiard balls the permutations would be:

16! = 20,922,789,888,000

But if you wanted to select just 3, then you have to stop the multiplying after 14. How do you do that? There is a neat trick ... you divide by 13! ...

$$\frac{16 \times 15 \times 14 \times 13 \times 12 \ldots}{13 \times 12 \ldots} = 16 \times 15 \times 14 = 3{,}360$$

Do you see? 16! / 13! = 16 × 15 × 14

The formula is written:

$$\frac{n!}{(n-r)!}$$

where n is the number of things to choose from, and you choose r of them
(No repetition, order matters)

Examples: Our "order of 3 out of 16 pool balls example" would be:

$$\frac{16!}{(16-3)!} = \frac{16!}{13!} = \frac{20{,}922{,}789{,}888{,}000}{6{,}227{,}020{,}800} = 3{,}360$$

(which is just the same as: 16 × 15 × 14 = 3,360)

How many ways can first and second place be awarded to 10 people?

$$\frac{10!}{(10-2)!} = \frac{10!}{8!} = \frac{3,628,800}{40,320} = 90$$

(which is just the same as: 10 × 9 = 90)

Notation for Permutation

$$P(n,r) = {}^nP_r = {}_nP_r = \frac{n!}{(n-r)!}$$

Example: P(10,2) = 90

c) Permutations with Repeated Elements

Assume that we have an alphabet with k letters and we want to write all possible words containing n_1 times the first letter of the alphabet, n_2 times the second letter,..., n_k times the kth letter. How many words can we write? We call this number $P(n; n_1, n_2, \ldots, n_k)$, where $n = n_1 + n_2 + \cdots + n_k$.

Example: With 3 a's and 2 b's we can write the following 5-letter words: *aaabb, aabab, abaab, baaab, aabba, ababa, baaba, abbaa, babaa, bbaaa.*

We may solve this problem in the following way, as illustrated with the example above. Let us distinguish the different copies of a letter with subscripts: $a_1 a_2 a_3 b_1 b_2$. Next, generate each permutation of this five elements by choosing 1) the position of each kind of letter, then 2) the subscripts to place on the 3 a's, then 3) these subscripts to place on the 2 b's. Task 1) can be performed in $P(5; 3, 2)$ ways, task 2) can be performed in 3! ways, task 3) can be performed in 2!. By the product rule we have $5! = P(5; 3, 2) \times 3! \times 2!$, hence $P(5; 3, 2) = 5!/(3!2!)$.

In general the formula is:

$$P(n; n_1, n_2, \ldots, n_k) = \frac{n!}{n_1!\, n_2! \ldots n_k!}.$$

2) Combinations

There are also two types of combinations (remember the order does not matter now):

a) No Repetition: such as lottery numbers (2,14,15,27,30,33)
b) Repetition is Allowed: such as coins in your pocket (5,5,5,10,10)

a) Combinations without Repetition

This is how lotteries work. The numbers are drawn one at a time, and if you have the lucky numbers (no matter what order) you win!

The easiest way to explain it is to assume that the order does matter (ie permutations), then alter it so the order does not matter.

Going back to our pool ball example, let us say that you just want to know which 3 pool balls were chosen, not the order.

We already know that 3 out of 16 gave us 3,360 permutations.

But many of those will be the same to us now, because we don't care what order!

For example, let us say balls 1, 2 and 3 were chosen. These are the possibilities:

Order does matter	Order doesn't matter
1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1	1 2 3

So, the permutations will have 6 times as many possibilities.

In fact there is an easy way to work out how many ways "1 2 3" could be placed in order, and we have already talked about it. The answer is:

$3! = 3 \times 2 \times 1 = 6$

So, all we need to do is adjust our permutations formula to reduce it by how many ways the objects could be in order (because we aren't interested in the order any more):

$$\frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

That formula is so important it is often just written in big parentheses like this:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

where n is the number of things to choose from, and you choose r of them
(No repetition, order doesn't matter)

It is often called "n choose r" (such as "16 choose 3")

Notation for Combination

$$C(n,r) = {}^{n}C_r = {}_{n}C_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example: Our pool ball example (now without order) is:

$$\frac{16!}{3!(16-3)!} = \frac{16!}{3!\times 13!} = \frac{20{,}922{,}789{,}888{,}000}{6\times 6{,}227{,}020{,}800} = 560$$

Or you could do it this way:

$$\frac{16\times 15\times 14}{3\times 2\times 1} = \frac{3360}{6} = 560$$

It is interesting to also note how this formula is nice and symmetrical:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r} = \binom{n}{n-r}$$

In other words choosing 3 balls out of 16, or choosing 13 balls out of 16 have the same number of combinations.

$$\frac{16!}{3!(16-3)!} = \frac{16!}{13!(16-13)!} = \frac{16!}{3!\times 13!} = 560$$

b) Combinations with Repetition

Assume that we have a set A with n elements. Any selection of r objects from A, where each object can be selected more than once, is called a *combination of n objects taken r at a time with repetition.* For instance, the combinations of the letters a, b, c, d taken 3 at a time with repetition are: *aaa, aab, aac, aad, abb, abc, abd, acc, acd, add, bbb, bbc, bbd, bcc, bcd, bdd, ccc, ccd, cdd, ddd.* Two combinations with repetition are considered identical if they have the same elements repeated the same number of times, regardless of their order.

Example: Assume that we have 3 different (empty) milk containers and 7 quarts of milk that we can measure with a one quart measuring cup. In how many ways can we distribute the milk among the three containers? We solve the problem in the following way. Let x_1, x_2, x_3 be the quarts of milk to put in containers number 1, 2 and 3 respectively. The number of possible distributions of milk equals the number of non negative integer solutions for the equation $x_1 + x_2 + x_3 = 7$. Instead of using numbers for writing the solutions, we will use strokes, so for instance we represent the solution $x_1 = 2, x_2 = 1, x_3 = 4$, or $2 + 1 + 4$, like this: II+I+IIII. Now, each possible solution is an arrangement of 7 strokes and 2 plus signs, so the number of arrangements is $P(9; 7, 2) = 9!/(7!2!)$

The general solution is:

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

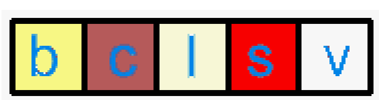
where ***n*** is the number of things to choose from, and you choose ***r*** of them
(Repetition allowed, order doesn't matter)

Example: Let us say there are five flavors of ice cream: **banana, chocolate, lemon, strawberry and vanilla**. You can have three scoops. How many variations will there be?

Let's use letters for the flavors: {b, c, l, s, v}. Example selections would be

- {c, c, c} (3 scoops of chocolate)
- {b, l, v} (one each of banana, lemon and vanilla)
- {b, v, v} (one of banana, two of vanilla)

(And just to be clear: There are **n=5** things to choose from, and you choose **r=3** of them.
Order does not matter, and you **can** repeat!)



Think about the ice cream being in boxes, you could have 5-1 = 4 separators and between separators, you can put 1, 2, or 3 scoops. Therefore, you will have 4 separators + 3 scoops = 7 things to arrange.

This is like the permutation with repeated elements.

$$\frac{7!}{3!\times 4!} = \frac{5040}{6\times 24}$$

$$= 35$$

By using the formula $\binom{n+r-1}{r} = \binom{n+r-1}{n-1} = \frac{(n+r-1)!}{r!(n-1)!}$ we have:

$$\frac{(5+3-1)!}{3!(5-1)!} = \frac{7!}{3!\times 4!} = \frac{5040}{6\times 24} = 35$$

►**More Examples**

Example 1: Two eight-sided dice each have faces numbered 1 through 8. When the dice are rolled, each face has an equal probability of appearing on the top. What is the probability that the product of the two top numbers is greater than their sum?

Solution: There are 8•8 = 64 ordered pairs that can represent the top numbers on the two dice. Let m and n represent the top numbers on the dice. Then $mn > m + n$

Implies that $mn - m - n > 0$, that is,

$$1 < mn - m - n + 1 = (m - 1)(n - 1).$$

This inequality is satisfied except when m = 1, n = 1, or when m = n = 2. There are 16 ordered pairs (m, n) excluded by these conditions, so the probability that the product is greater than the sum is

$$\frac{64-16}{64} = \frac{48}{64} = \frac{3}{4}$$

Example 2: Each face of a cube is painted either red or blue, each with probability 1/2. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

Solution: If the orientation of the cube is fixed, there are $2^6 = 64$ possible arrangements of colors on the faces. There are $2\binom{6}{6} = 2$ arrangements in which all six faces are the same color and $2\binom{6}{5} = 12$ arrangements in which exactly five faces have the same Color. In each of these cases the cube can be placed so that the four vertical faces: have the same color.

The only other suitable arrangements have four faces of one color, with the other color on a pair of opposing faces. Since there are three pairs of opposing faces, there are 2 (3) = 6 such arrangements.

The total number of suitable arrangements is therefore 2 + 12 + 6 = 20, and the probability is 20/64 =5/16.

►In-class questions

1. A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

2. Tina randomly selects two distinct numbers from the set {1, 2, 3, 4, 5}, and Sergio randomly selects a number from the set {1, 2, . . . , 10}. What is the probability that Sergio's number is larger than the sum of the two numbers chosen by Tina?

3. A bag contains two red beads and two green beads. You reach into the bag and pull out a bead, replacing it with a red bead regardless of the color you pulled out. What is the probability that all beads in the bag are red after three such replacements?

4. Each face of a cube is painted either red or blue, each with probability 1/2. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

5. For a particular peculiar pair of dice, the probabilities of rolling 1, 2, 3, 4, 5, and 6 on each die are in the ratio 1 : 2 : 3 : 4 : 5 : 6. What is the probability of rolling a total of 7 on the two dice?

6. Jacob uses the following procedure to write down a sequence of numbers. First he chooses the first term to be *6*. To generate each succeeding term, he flips a fair coin. If it comes up heads, he doubles the previous term and subtracts 1. *H* it comes up tails, he takes half of the previous term and subtracts 1. What is the probability that the fourth term in Jacob's sequence is an integer?

7. Three red beads, two white beads, and one blue bead are placed in a line in random order. What is the probability that no two neighboring beads are the same color?