Notice:
The notes are the same as Class 7. The questions in class and homework are different from before. Please review the notes and do the questions in class and homework.

Patterns 2

1. Patterns

Patterns are recognizable as repetitive sequences and can be found in nature, shapes, events, sets of numbers and almost everywhere you care to look. For example, seeds in a sunflower, snowflakes, geometric designs on quilts or tiles, the number sequence 0, 4, 8, 12, 16, ...

Investigation: Patterns

Can you spot any patterns in the following lists of numbers?

- 2; 4; 6; 8; 10; ...
- 1; 2; 4; 7; 11; ...
- 1; 4; 9; 16; 25; ...
- 5; 10; 20; 40; 80; ...

2. Common Number Patterns

Numbers can have interesting patterns. Here we list the most common patterns and how they are made.

1) Arithmetic Sequences

An Arithmetic Sequence is made by adding some value each time.

Examples:

1, 4, 7, 10, 13, 16, 19, 22, 25, ...

This sequence has a difference of 3 between each number.
The pattern is continued by adding 3 to the last number each time.

3, 8, 13, 18, 23, 28, 33, 38, ...

This sequence has a difference of 5 between each number.
The pattern is continued by adding 5 to the last number each time.

The value added each time is called the "common difference"

What is the common difference in this example?

19, 27, 35, 43, ...

Answer: The common difference is 8

The common difference could also be negative, like this:

25, 23, 21, 19, 17, 15, ...

This common difference is -2
The pattern is continued by subtracting 2 each time.

2) Geometric Sequences

A Geometric Sequence is made by multiplying by some value each time.

Examples:
2, 4, 8, 16, 32, 64, 128, 256, ...

This sequence has a factor of 2 between each number.
The pattern is continued by multiplying the last number by 2 each time.

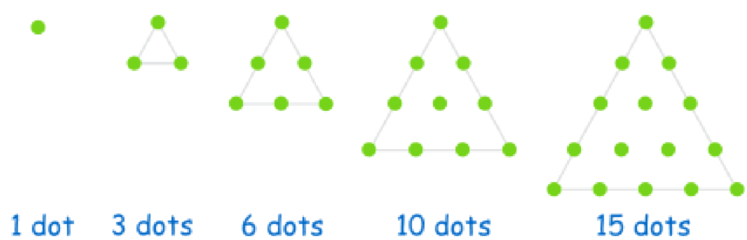
3, 9, 27, 81, 243, 729, 2187, ...

This sequence has a factor of 3 between each number.
The pattern is continued by multiplying the last number by 3 each time.

3) Special Sequences

a) Triangular Numbers
1, 3, 6, 10, 15, 21, 28, 36, 45, ...
This sequence is generated from a pattern of dots which form a triangle.

By adding another row of dots and counting all the dots we can find the next number of the sequence:



b) Square Numbers

1, 4, 9, 16, 25, 36, 49, 64, 81, ...
The next number is made by squaring where it is in the pattern.
The second number is 2 squared (2^2 or 2×2)
The seventh number is 7 squared (7^2 or 7×7) etc

c) Cube Numbers

1, 8, 27, 64, 125, 216, 343, 512, 729, ...

The next number is made by cubing where it is in the pattern.
The second number is 2 cubed (2^3 or $2 \times 2 \times 2$)
The seventh number is 7 cubed (7^3 or $7 \times 7 \times 7$) etc

4) Fibonacci Sequence

The Fibonacci Sequence is the series of numbers:
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

The next number is found by adding up the two numbers before it.

- The 2 is found by adding the two numbers before it (1+1)
- Similarly, the 3 is just (1+2),
- And the 5 is just (2+3),
- The 21 is found by adding the two numbers in front of it (8+13)
- The next number in the sequence above would be 55 = (21+34)
- and so on!

5. Make your own Number Patterns

You can make your own number patterns using coins or matchsticks. Here is an example using dots:

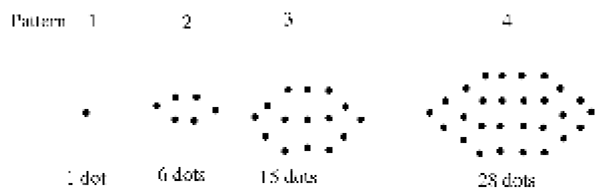


Figure 1

How many dots would you need for pattern 5? Can you make a formula that will tell you how many coins are needed for any size pattern? For example the pattern 20? The formula may look something like:

Dots = pattern × pattern +...

Exercise 1: Study Table

Say you and 3 friends decide to study for Maths, and you are seated at a square table. A few minutes later, 2 other friends join you and would like to sit at your table and help you study. Naturally, you move another table and add it to the existing one. Now 6 of you sit at the table. Another 2 of your friends join your table, and you take a third table and add it to the existing tables. Now 8 of you can sit comfortably.

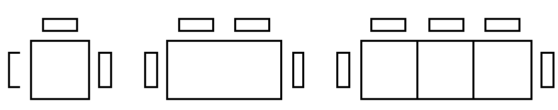


Figure 2: Two more people can be seated for each table added.

Examine how the number of people sitting is related to the number of tables.

Solution

Step 1. Tabulate a few terms to see if there is a pattern:

Number of Tables, *n*	**Number of people seated**
1	4 = 4
2	4 + 2 = 6
3	4 + 2 + 2 = 8
4	4 + 2 + 2 + 2 = 10
⋮	⋮
n	4 + 2 + 2 + 2 + ... + 2
Table 1	

Step 2. Describe the pattern:
We can see that for 3 tables we can seat 8 people, for 4 tables we can seat 10 people and so on. We started out with 4 people and added two each time. Thus, for each table added, the number of persons increased by 2.

6. Notation

A sequence does not have to follow a pattern but when it does we can often write down a formula to calculate the n^{th}-term, a_n. In the sequence

1; 4; 9; 16; 25; ...

where the sequence consists of the squares of integers, the formula for the n^{th}-term is

$$a_n = n^2$$

You can check this by looking at:
$a_1 = 1^2 = 1$, $a_2 = 2^2 = 4$, $a_3 = 3^2 = 9$, $a_4 = 4^2 = 16$, $a_5 = 5^2 = 25$, ...

Therefore, using Equation 3, we can generate a pattern, namely squares of integers.

Exercise 2: Study Table continued ....
As before, you and 3 friends are studying for Maths, and you are seated at a square table. A few minutes later, 2 other friends join you move another table and add it to the existing one. Now 6 of you sit at the table. Another 2 of your friends join your table, and you take a third table and add it to the existing tables. Now 8 of you sit comfortably as illustrated:

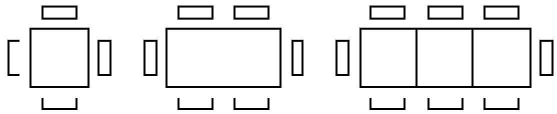


Figure 3: Two more people can be seated for each table added.

Find the expression for the number of people seated at *n* tables. Then, use the general formula to determine how many people can sit around 12 tables and how many tables are needed for 20 people.

Solution

Step 1. Tabulate a few terms to see if there is a pattern :

Number of Tables, *n*	**Number of people seated**	**Formula**
1	4 = 4	$= 4 + 2\cdot(0)$
2	4 + 2 = 6	$= 4 + 2\cdot(1)$
3	4 + 2 + 2 = 8	$= 4 + 2\cdot(2)$
4	4 + 2 + 2 + 2 = 10	$= 4 + 2\cdot(3)$
⋮	⋮	⋮
n	4 + 2 + 2 + 2 + ... + 2	$= 4+ 2\cdot(n- 1)$

Table 2

Step 2. Describe the pattern:

The number of people seated at *n* tables is:
$a_n = 4 + 2 \cdot (n - 1)$

Step 3. Calculate the 12th term :

Considering the example from the previous section, how many people can sit around, say, 12 tables? We are looking for a_{12}, that is, where $n = 12$:

$$\begin{aligned} a_n &= a_1 + d \cdot (n-1) \\ a_{12} &= 4 + 2 \cdot (12-1) \\ &= 4 + 2(11) \\ &= 4 + 22 \\ &= 26 \end{aligned}$$

Step 4. Calculate the number of terms if $a_n = 20$:

$$\begin{aligned} a_n &= a_1 + d \cdot (n - 1) \\ 20 &= 4 + 2 \cdot (n - 1) \\ 20 - 4 &= 2 \cdot (n - 1) \\ 16 \div 2 &= n - 1 \\ 8 + 1 &= n \\ n &= 9 \end{aligned}$$

Step 5. Final Answer:
26 people can be seated at 12 tables and 9 tables are needed to seat 20 people
It is also important to note the difference between n and a_n. n can be compared to a place holder, while a_n is the value at the place "held" by n. Like our "Study Table" example above, the first table (Table 1) holds 4 people. Thus, at place $n = 1$, the value of $a_1 = 4$, and so on:

n	1	2	3	4	...
a_n	4	6	8	10	...

7. Investigation: General Formula

Find the general formula for the following sequences and then find a_{10}, a_{50} and a_{100}:

- 2,5,8,11,14,...
- 0,4,8,12,16,...
- 2,−1,−4,−7,−10,...

The general term has been given for each sequence below. Work out the missing terms.

- 0; 3; ... ;15; 24 $n^2 - 1$
- 3; 2; 1;0; ...; − 2 $-n + 4$
- −11; ...; −7; ...; −3 $-13 + 2n$

8. Patterns and Conjecture

In mathematics, a conjecture is a mathematical statement which appears to be true, but has not been formally proven to be true. Other words that have a similar in meaning to conjecture are: hypothesis, theory, assumption and premise.

For example: Make a conjecture about the next number based on the pattern 2; 6; 11; 17:...
The numbers increase by 4, 5, and 6.

Conjecture: The next number will increase by 7. So, it will be 17 + 7 or 24.

Exercise 3: Number patterns

Consider the following pattern.

$1^2+1 = 2^2-2$

$2^2+2 = 3^2-3$

$3^2+3 = 4^2-4$

$4^2+4 = 5^2-5$

Add another two rows to the end of the pattern.
Make a conjecture about this pattern. Write your conjecture in words.
Generalize your conjecture for this pattern (in other words, write your conjecture algebraically).
Prove that your conjecture is true.

9. Counting Principles

To determine the number of ways that one action can be performed after another, use the **Fundamental Counting Principle**.

1) Fundamental Counting Principle (FCP)
If one action can be performed in n ways, then another in m ways, then both actions can be performed, in order, in n x m ways.

Example: In how many ways can a Canadian postal code be made? A postal code has the format A9A 9A9, where A is any letter and 9 is any number.

Think of this problem as selecting a letter, then selecting a number, then a letter, and so on.
There are 26 possible letters, and 10 possible numbers, for each position.
According to the FCP, the total number of postal codes is
26 x 10 x 26 x 10 x 26 x 10 = 17 576 000

Example: A cafeteria offers lunch specials consisting of one item from each category.

Entree	**Beverage**	**Dessert**
Hamburger	Soft Drink	Ice Cream
Sandwich	Milk	Fruit Cup
Wrap	Juice	
Pasta		
Chicken Salad		

Determine the number of possible lunch specials.

Using the FCP, there are 5 x 3 x 2 = 30 lunch specials.

2) Additive Counting Principle (ACP)
If one action can be performed in n ways, and another in m ways, and both actions cannot be performed together, then either action can be performed in n + m ways.

Example: Determine the number of ways of drawing a red face card or a spade from a standard deck of 52 cards.

There are 6 red face cards (J Heart, J Diamond, Q Heart, Q Diamond, K Heart, K Diamond) and 13 spades. Thus, according to the ACP, there are 6 + 13 = 19 ways of drawing a red face card or a spade.

3) Factorial

Question: How many different ways can 8 girls be arranged in line.

1^{st} girl could be anyone of 8 of them, there are 8 choices.
If the 1^{st} girl is decided, then the 2^{nd} girl has 7 choices, and so on.

Therefore by FCP we have 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 40320 ways.

To work more easily with calculations like 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1,
we can use factorial notation (!).

Example:
(a) 5! ....."five factorial"
5! = 5 x 4 x 3 x 2 x 1 = 120
(b) 1! = 1
(c) 0! = 1

For a natural number "n" ,

n! = n x (n - 1) x (n - 2) x ....x 3 x 2 x 1

and is read "n factorial"

4) Permutation

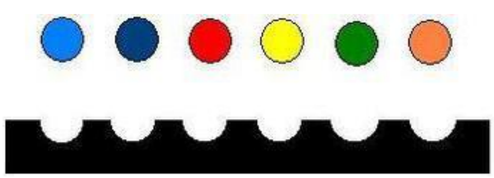
A permutation is an ordered selection (or arrangement) of elements from a given set.
A permutation of "n" distinct objects taken "r" at a time is an arrangement of "r" of the "n" object in a definite order. The total number of such permutations (arrangements) is denoted by:

$$nPr = P(n,r) = \frac{n!}{(n-r)!}$$

There are three different type of permutations:

a) Permutations of all elements; $P(n,\ n) = n!$

Example: What are the total number of arrangements for the following six balls if all balls must be used?



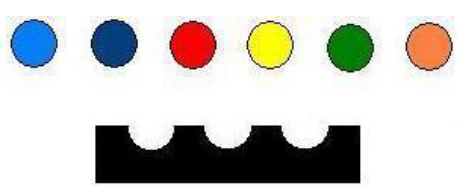
$$\underline{6} \times \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 6!$$

Or

$${}_6P_6 = P(6,6) = 720$$

b) Permutations involving some elements from the population; $P(n,\ r) = \dfrac{n!}{(n-r)!}$

Example: What are the total number of arrangements for the following six balls if only three balls must be used?



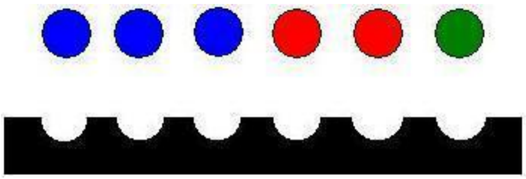
$$\underline{6} \times \underline{5} \times \underline{4} = \frac{6!}{(6-3)!}$$

Or

$${}_6P_3 = P(6,3) = 120$$

c) Permutations involving some identical elements; $\frac{n!}{a!b!c!...}$

Example: What are the total number of arrangements for the following: 3 blue balls, 2 red balls, and 1 green ball?



$$\text{\# of distinct permutations} = \frac{6!}{3!2!} = 60$$

<u>**Explanation**</u>**:**

The total number of permutations for the arrangement of the six balls is 6!, but these are not distinct permutations. The distinction between the permutations within three blue balls and also between the two red balls is not possible, resulting in repetition which includes non-distinct permutations. In order to eliminate this repetition, the total number of permutations are divided by the permutations of the repeated elements.

Therefore, resulting in:

$$\frac{6!}{3!2!} = 60$$

<u>**Example 1**</u>**:** How many distinct permutations can be made from the name AJAY?

The total number of permutations for the name AJAY are 4!.
These are not distinct permutations, the permutations of the identical letters (elements), “A”, do not affect the total number of distinct permutations possible.

Therefore, the number of distinct permutations for AJAY does not equal 4!

To eliminate the non-distinct permutations, the total number of permutations are divided by the permutations of the repeated identical elements.

Therefore, the total number of distinct permutations for AJAY are:

$$\frac{4!}{2!} = 12$$

<u>**Example 2**</u>**:** How many 7-digit even numbers less than 3,000,000 can be formed using the following digits: 1, 2, 2, 3, 5, 5, 6?

This questions involves cases:

Case 1: First digit is 1 and last digit is 6.

$$\underline{1}\times\underline{5}\times\underline{4}\times\underline{3}\times\underline{2}\times\underline{1}\times\underline{1} = \frac{5!\times1\times1}{2!2!} = 30$$

Case 2: First digit is 2 and last digit is 6.

$$\underline{2}\times\underline{5}\times\underline{4}\times\underline{3}\times\underline{2}\times\underline{1}\times\underline{1} = \frac{5!\times2\times1}{2!2!} = 60$$

Case 3: First digit is 2 and last digit is 2.

$$\underline{2}\times\underline{5}\times\underline{4}\times\underline{3}\times\underline{2}\times\underline{1}\times\underline{1} = \frac{5!\times2\times1}{2!2!} = 60$$

Case 4: First digit is 1 and last digit is 2.

$$\underline{1}\times\underline{5}\times\underline{4}\times\underline{3}\times\underline{2}\times\underline{1}\times\underline{2} = \frac{5!\times1\times2}{2!2!} = 60$$

Now use the **Additive Counting Principle (ACP)**:

$$\begin{aligned}\text{Total \# of permutations} &= \text{Case 1} + \text{Case 2} + \text{Case 3} + \text{Case 4}\\ &= 30 + 60 + 60 + 60\\ &= 210\end{aligned}$$

Therefore, there are a total 210 distinct permutations.

5) Combination

A combination of "n" distinct objects, taken "r" at a time is a selection of "r" of the "n" objects without regard for order.

The total # of such combinations is:

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

Example: For the school dance a clean up crew of 3 people is needed.
If there are 8 members of students' council available to help, how many different clean up crew groups could be formed?

Order doesn't matter → Combination → ${}_8C_3 = 56$ groups

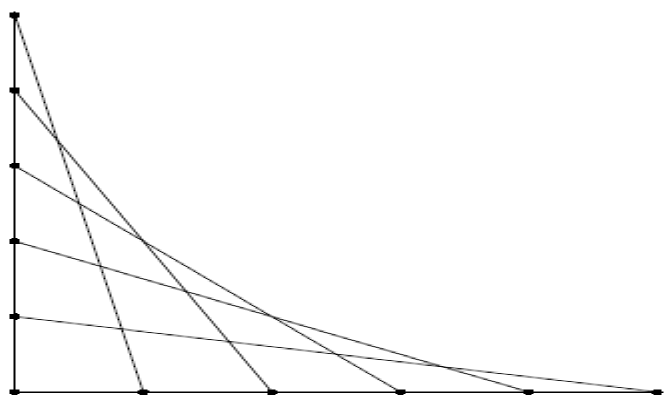
In-class questions

1. Two hockey teams play to a score of 3, i.e. the game is over as soon as one team gets 3 goals. You make a list showing how the scoreboard changes over time. For example, if Team 1 scores, scores again, then Team 2 scores, then Team 1 scores again, then your list is:

$$(0-0),\ (1-0),\ (2-0),\ (2-1),\ (3-1).$$

What is the number of different lists that can be made in this way?

2. What is the number of triangles formed by the intersection points of the lines in the diagram?



3. Consider a regular hexagon *ABCDEF*. Triangles are formed by choosing three of the vertices *A*, *B*, *C*, *D*, *E*, and *F*. Two such triangles are distinct if they have at least one different vertex, even though they may be congruent. What is the number of distinct such triangles that have at least two equal sides?

4. Nine people attend a dinner where there are three choices for the type of meal. Three people order Combo A, three order Combo B, and three order Combo C. The server distributes the nine meals in random order. In how many different ways can exactly one person receive the correct meal?

5. A particle moves through the *xy*-plane in the following spiral formation: it starts at (0, 0), then it moves 1 unit parallel to an axis in each second. The first several moves are
$(0,\ 1),\ (1,\ 1),\ (1,\ 0),$
$(1,\ -1),(0,\ -1),\ (-1,\ -1),\ (-1,\ 0),\ (-1,\ 1),\ (-1,\ 2),\ (0,\ 2),\ (1,\ 2),\ (2,\ 2),\ (2,\ 1)$, and so on. At which point will the particle be after exactly 2017 seconds?

6. The first three figures in a pattern are shown below.

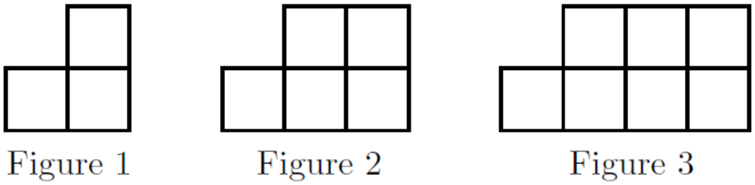


Figure 1 is formed by three identical squares of side length 1 cm arranged in two rows. The perimeter of Figure 1 is 8 cm. Given a figure in the pattern, the next figure is obtained by adding a square of side length 1 cm on the right-hand end of each of the two rows.

a) How many squares of side length 1 cm are used to form Figure 8?
b) Determine the perimeter of Figure 12.
c) Determine the positive integer C for which the perimeter of Figure C is 38 cm.