Algebra 2

1. Completing the square

IF WE TRY TO SOLVE this quadratic equation by factoring
$x^2 + 6x + 2 = 0$

We cannot. Therefore, we use a technique called Completing the square. This means to make the quadratic into a perfect square trinomial, for example the form
$a^2 + 2ab + b^2 = (a + b)^2$.

The technique is valid only when 1 is the coefficient of x^2.

1) Transpose the constant term to the right:
$x^2 + 6x = -2$

2) Add a square number to both sides. Add the square of *half* the coefficient of x. In this case, add the square of 3:
$x^2 + 6x + 9 = -2 + 9$

The left-hand side is now the perfect square of $(x + 3)$.
$(x + 3)^2 = 7$

3 is *half* of the coefficient 6.

This equation has the form: $a^2 = b$
which implies $a = \pm\sqrt{b}$

Therefore, $x + 3 = \pm\sqrt{7}$
$x = -3 \pm \sqrt{7}$

That is, the solutions to
$x^2 + 6x + 2 = 0$

are the conjugate pair,
$-3 + \sqrt{7}$, $-3 - \sqrt{7}$.

We can check this. The sum of those roots is -6, which is the *negative* of the coefficient of x. And the product of the roots is
$(-3)^2 - (\sqrt{7})^2 = 9 - 7 = 2$,

which is the constant term. Thus both conditions on the roots are satisfied. These are the two roots of the quadratic.

Example

Solve this quadratic equation by completing the square.
$x^2 - 2x - 2 = 0$

To see the solution, pass your mouse over the colored area. To cover the solution again, click "Refresh" ("Reload").

$$\begin{aligned} x^2 - 2x &= 2 \\ x^2 - 2x + 1 &= 2 + 1 \\ (x-1)^2 &= 3 \\ x - 1 &= \pm\sqrt{3} \\ x &= 1 \pm \sqrt{3} \end{aligned}$$

Before considering the quadratic formula, note that half of any number b is $\frac{b}{2}$. Half of 5 is $\frac{5}{2}$. Half of $\frac{p}{q}$ is $\frac{p}{2q}$.

Quadratic Polynomials: Completing the Square.

If guessing does not work, "completing the square" will do the job.

Let us try to factor $x^2 - 2x - 7$. We will actually consider the equivalent problem of finding the roots, the solutions of the equation: $x^2 - 2x - 7 = 0$

Move the constant term to the other side of the equation: $x^2 - 2x = 7$

The magic trick of this method is to exploit the binomial formula:
$x^2 - 2bx + b^2 = (x+b)^2$

If we look at the left side of the equation we want to solve, we see that it matches the first two terms of the binomial formula if $b = 1$.

Let's write down the binomial formula for $b = 1$:
$x^2 - 2x + 1 = (x+1)^2$

But the third term of the binomial formula does not show up in our equation; we make it show up by force by adding 1 to both sides of our equation:
$x^2 - 2x + 1 = 7 + 1$

This trick is called "completing the square"! Now we use the binomial formula to simplify the left side of our equation (also adding 7 + 1 = 8):
$(x+1)^2 = 8$

Next we take square roots of both sides, but be careful: there are **two** possible cases:

$$(x+1)=\sqrt{8}\text{ , or }(x+1)=-\sqrt{8}$$

In both cases $(x+1)^2=8$. We are done, once we solve the two equations for *x*.

$x=-1+\sqrt{8}$, or $x=-1-\sqrt{8}$

are the two roots of our polynomial. Consequently, our polynomial factors as follows:

$$\begin{aligned} x^2+2x-7 &= (x-(-1+\sqrt{8})^2)\cdot(x-(-1-\sqrt{8})) \\ &= (x+1-2\sqrt{2})\cdot(x+1+2\sqrt{2}) \end{aligned}$$

2. Completing the Square: Solving Quadratic Equations

1) Solve $4x^2 - 2x - 5 = 0$

This is the original problem.	$4x^2 - 2x - 5 = 0$
Move the loose number over to the other side.	$4x^2 - 2x = 5$
Divide through by whatever is multiplied on the squared term. Take half of the coefficient (don't forget the sign!) of the *x*-term, and square it. Add this square to both sides of the equation. Convert the left-hand side to squared form, and simplify the right-hand side. (This is where you use that sign that you kept track of earlier. You plug it into the middle of the parenthetical part.)	$x^2-\frac{1}{2}x=\frac{5}{4}$ $-\frac{1}{4}\rightarrow\frac{1}{16}$ $x^2-\frac{1}{2}x+\frac{1}{16}=\frac{5}{4}+\frac{1}{16}$ $(x-\frac{1}{4})^2=\frac{21}{16}$
Square-root both sides, remembering the "±" on the right-hand side. Simplify as necessary.	$x-\frac{1}{4}=\pm\sqrt{\frac{21}{16}}=\pm\frac{\sqrt{21}}{4}$
Solve for "*x* =".	$x=\frac{1}{4}\pm\frac{\sqrt{21}}{4}$
Remember that the "±" means that you have two values for *x*.	$x=\frac{1}{4}-\frac{\sqrt{21}}{4}$ and $x=\frac{1}{4}+\frac{\sqrt{21}}{4}$

2) Solve $x^2 + 6x - 7 = 0$ by completing the square.

Do the same procedure as above, in exactly the same order. (Study tip: Always working these problems in exactly the same way will help you remember the steps when you're taking your tests.)

This is the original equation.	$x^2 + 6x - 7 = 0$
Move the loose number over to the other side.	$x^2 + 6x = 7$
Take half of the *x*-term (that is, divide it by two) (and don't forget the sign!), and square it. Add this square to both sides of the equation.	$x^2 + 6x = 7$
Convert the left-hand side to squared form. Simplify the right-hand side.	$(x + 3)^2 = 16$
Square-root both sides. Remember to do "±" on the right-hand side.	$x + 3 = \pm 4$
Solve for "*x* =". Remember that the "±" gives you two solutions. Simplify as necessary.	$x = -3 \pm 4$ $= -3 - 4, -3 + 4$ $= -7, +1$

If you are not consistent with remembering to put your plus/minus in as soon as you square-root both sides, then this is an example of the type of exercise where you'll get yourself in trouble. You'll write your answer as "$x = -3 + 4 = 1$", and have no idea how they got "$x = -7$", because you won't have a square root symbol "reminding" you that you "meant" to put the plus/minus in. That is, if you're sloppy, these *easier* problems will embarrass you!

3. Formula for Completing the Square

1) $(x+a)(x+b) = x^2 + (a+b)x + ab$

2) $(a \pm b)^2 = a^2 \pm 2ab + b^2$

3) $a^2 - b^2 = (a-b)(a+b)$

Examples of perfect square trinomials (the red trinomials)

1) $(x+1)^2 = x^2 + 2x + 1$

2) $(x-2)^2 = x^2 - 4x + 4$

3) $x^2 - 1^2 = (x+1)(x-1)$

In-class questions

1. If the only real solution to $x+\frac{9}{x}+y+\frac{25}{y}=4$ is the ordered pair $(a,\ b)$, determine the numerical value of $4a+7b$.

2. Given $2^{\pi}x+(2^{\pi}+5)y=3^{\sqrt{2}}x+(3^{\sqrt{2}}+5)y$, determine the value of x/y.

3. Solve for real *x*: $\sqrt{\frac{1-x}{x}}>\sqrt{\frac{x}{1-x}}$.

4. If r and s are the (complex) roots of the equation $x^2-\sqrt{27}x+13=0$, what is the value of r^2+s^2?

5. The equation $x^2-px+(2p+8)=0$ has two roots, one of which is twice the other. What are the possible values of p?

6. If the reciprocal of $(x-1)$ is $x+\frac{1}{2}$, what is x if $x>0$?

7. Find the sum of all the solutions to the equation $6+\sqrt{x+6}=x$.

8. If *r* and *s* are the (real) roots of the equation $x^2+3x+k=0$, find *k* so that $|r-s|=2$.

9. Solve the system of equations

$$\begin{cases} x+xy+y=2+3\sqrt{2} & (1) \\ x^2+y^2=6 & (2) \end{cases}$$

10. Determine all pairs of integers $(x,\ y)$ which satisfy the equation $6x^2-3xy-13x+5y=-11$.