AP Calculus Lesson Two Notes

Chapter One - Limits and Continuity

1.4 Other Basic Limits
1.5 Asymptotes

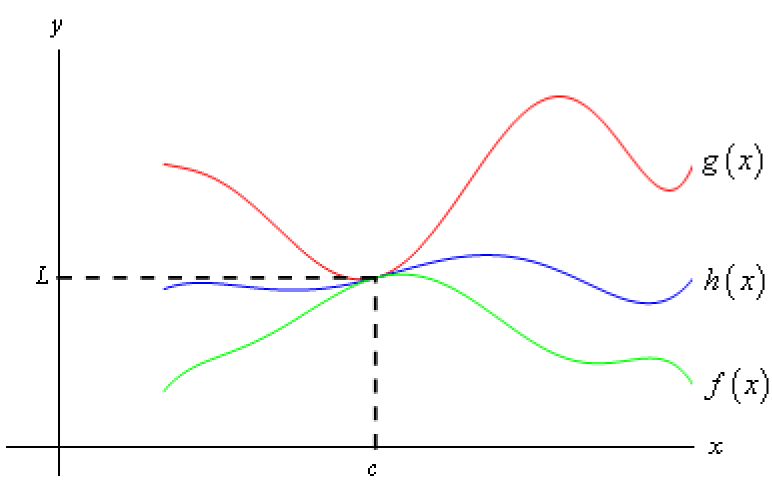
1.4 Other Basic Limits

Squeeze or Sandwich Theorem

Suppose that for all *x* on [*a, b*] (except possibly at *x* = *c*) we have, $f(x) \le h(x) \le g(x)$. Also suppose that, $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$ for some $a \le c \le b$. Then, $\lim_{x \to c} h(x) = L$.

As with the previous fact we only need to know that $f(x) \le h(x) \le g(x)$ is true around *x* = *c* because we are working with limits and they are only concerned with what is going on around *x* = *c* and not what is actually happening at *x* = *c*.

Now, if we again assume that all three functions are nice enough (again this isn't required to make the Squeeze Theorem true, it only helps with the visualization) then we can get a quick sketch of what the Squeeze Theorem is telling us. The following figure illustrates what is happening in this theorem.



From the figure we can see that if the limits of *f(x)* and *g(x)* are equal at *x* = *c*. Then the function values must also be equal at *x* = *c* (this is where we're using the fact that we assumed the functions where "nice enough", which isn't really required for the Theorem). However, because *h(x)* is "squeezed" between *f(x)* and *g(x)* at this point then *h(x)* must have the same value. Therefore, the limit of *h(x)* at this point must also be the same.

Example 1.4-1 Evaluate the following limit.

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$$

Solution

In this example none of the previous examples can help us. There's no factoring or simplifying to do. We can't rationalize and one-sided limits won't work. There's even a question as to whether this limit will exist since we have division by zero inside the cosine at *x=0*.

The first thing to notice is that we know the following fact about cosine.

$$-1 \le \cos(x) \le 1$$

Our function doesn't have just an x in the cosine, but as long as we avoid $x = 0$, we can say the same thing for our cosine.

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1$$

It's okay for us to ignore $x = 0$, here because we are taking a limit and we know that limits don't care about what's actually going on at the point in question, $x = 0$ in this case.

Now if we have the above inequality for our cosine we can just multiply everything by an x^2 and get the following.

$$-x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2$$

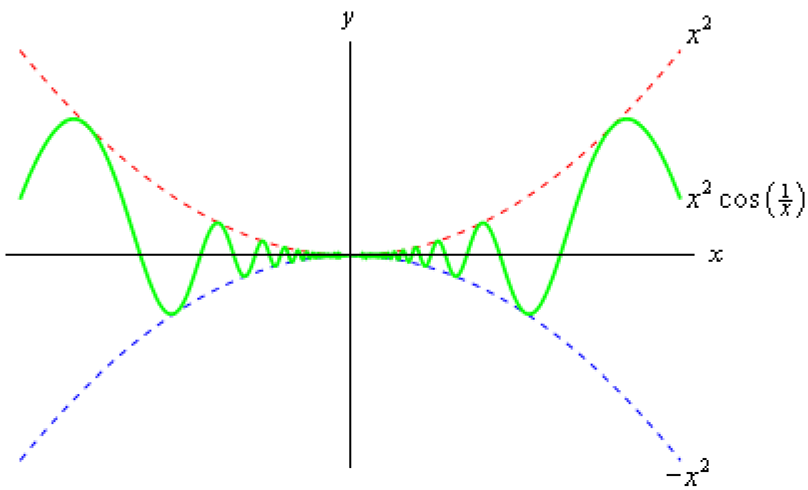
In other words we've managed to squeeze the function that we were interested in between two other functions that are very easy to deal with. So, the limits of the two outer functions are.

$$\lim_{x\to 0} x^2 = 0 \qquad\qquad \lim_{x\to 0}\left(-x^2\right) = 0$$

These are the same and so by the Squeeze theorem we must also have,

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$$

We can verify this with the graph of the three functions. This is shown below.

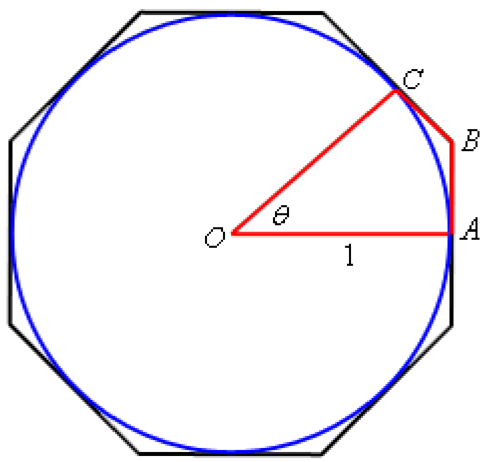


Proof of $\lim_{\theta \leftarrow 0} \dfrac{\sin\theta}{\theta} = 1$

This proof of this limit uses the Squeeze Theorem. However, getting things set up to use the Squeeze Theorem can be a somewhat complex geometric argument that can be difficult to follow so we'll try to take it fairly slow.

Let's start by assuming that $0 \le \theta \le \pi/2$. Since we are proving a limit that has $\theta \to 0$, it's okay to assume that θ is not too large (*i.e.* $0 \le \theta \le \pi/2$). Also, by assuming that θ is positive we're actually going to first prove that the above limit is true if it is the right-hand limit. As you'll see if we can prove this then the proof of the limit will be easy.

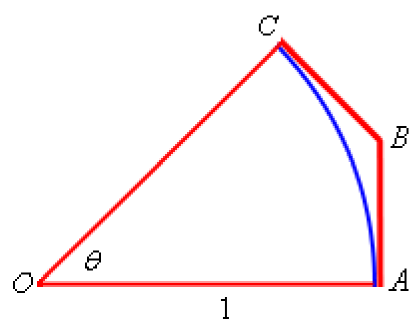
So, now let's start off with the unit circle circumscribed by an octagon with a small slice marked out as shown below.



Points *A* and *C* are the midpoints of their respective sides on the octagon and are in fact tangent to the circle at that point. We'll call the point where these two sides meet *B*.

From this figure we can see that the circumference of the circle is less than the length of the octagon. This also means that if we look at the slice of the figure marked out above then the length of the portion of the circle included in the slice must be less than the length of the portion of the octagon included in the slice.

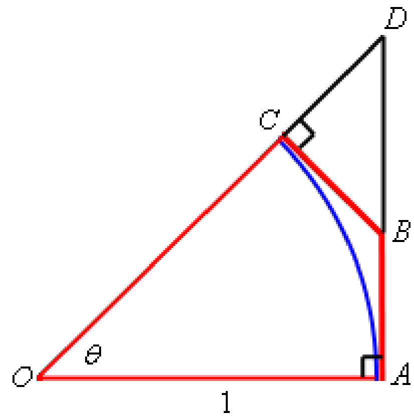
Because we're going to be doing most of our work on just the slice of the figure let's strip that out and look at just it. Here is a sketch of just the slice.



Now denote the portion of the circle by arc *AC* and the lengths of the two portion of the octagon shown by $|AB|$ and $|BC|$. Then by the observation about lengths we made above we must have,

$$arcAc < |AB| + |BC| \qquad (1)$$

Next, extend the lines *AB* and *OC* as shown below and call the point that they meet *D*. The triangle now formed by *AOD* is a right triangle. All this is shown in the figure below.



The triangle *BCD* is a right triangle with hypotenuse *BD* and so we know

$$|BC| < |BD|.$$

Also notice that

$$|AB| + |BD| = |AD|.$$

If we use these two facts in (1) we get,

$$\begin{aligned} \text{arc}\, AC &< |AB| + |BC| \\ &< |AB| + |BD| \\ &= |AD| \end{aligned} \qquad (2)$$

Next, as noted already the triangle *AOD* is a right triangle and so we can use a little right triangle trigonometry to write

$$|AD| = |AO| \tan\theta.$$

Also note that $|AO| = 1$, since it is nothing more than the radius of the unit circle. Using this information in (2) gives,

$$arcAC < |AD| < |AO| \tan\theta = \tan\theta \qquad (3)$$

The next thing that we need to recall is that the length of a portion of a circle is given by the radius of the circle times the angle that traces out the portion of the circle we're trying to measure. For our portion this means that,

$$\text{arc}\,AC = \left|AO\right|\theta = \theta$$

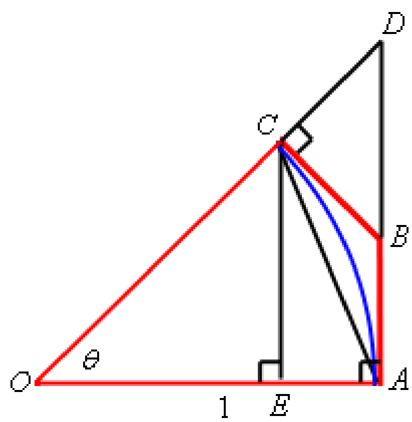
So, putting this into (3) we see that,

$$\theta = \text{arc}\,AC < \tan\theta = \frac{\sin\theta}{\cos\theta}$$

or, if we do a little rearranging we get,

$$\cos\theta < \frac{\sin\theta}{\theta} \qquad (4)$$

We'll be coming back to (4) in a bit. Let's now add in a couple more lines into our figure above. Let's connect *A* and *C* with a line and drop a line straight down from *C* until it intersects *AO* at a right angle and let's call the intersection point *E*. This is all shown in the figure below.



Okay, the first thing to notice here is that,

$$\left|CE\right| < \left|AC\right| < arcAC \qquad (5)$$

Also note that triangle *EOC* is a right triangle with a hypotenuse of $\left|CO\right| = 1$ Using some right triangle trig we can see that,

$$\left|CE\right| = \left|CO\right|\sin\theta = \sin\theta$$

Plugging this into (5) and recalling that $\text{arc}\,AC = \theta$ we get,

$$\sin\theta = \left|CE\right| < \text{arc}\,AC = \theta$$

and with a little rewriting we get,

$$\frac{\sin\theta}{\theta} < 1 \qquad (6)$$

Okay, we're almost done here. Putting (4) and (6) together we see that,

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Provided $0 \le \theta \le \pi/2$. Let's also note that,

$$\lim_{\theta\to 0}\cos\theta = 1 \qquad\qquad \lim_{\theta\to 0} 1 = 1$$

We are now set up to use the Squeeze Theorem. The only issue that we need to worry about is that we are staying to the right of $\theta = 0$ in our assumptions and so the best that the Squeeze Theorem will tell us is,

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$$

So, we know that the limit is true if we are only working with a right-hand limit. However we know that $\sin\theta$ is an odd function and so,

$$\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta}$$

In other words, if we approach zero from the left (*i.e.* negative θs) then we'll get the same values in the function as if we'd approached zero from the right (*i.e.* positive θs) and so,

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1$$

We have now shown that the two one-sided limits are the same and so we must also have,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Example 1.4-2 Evaluate each of the following limits.

(a) $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{6\theta}$ **(b)** $\lim\limits_{x \to 0} \dfrac{\sin(6x)}{x}$ **(c)** $\lim\limits_{x \to 0} \dfrac{x}{\sin(7x)}$

(d) $\lim\limits_{t \to 0} \dfrac{\sin(3t)}{\sin(8t)}$ **(e)** $\lim\limits_{x \to 4} \dfrac{\sin(x-4)}{x-4}$ **(f)** $\lim\limits_{z \to 0} \dfrac{\cos(2z)-1}{z}$

Solution

(a) $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{6\theta}$

There really isn't a whole lot to this limit. In fact, it's only here to contrast with the next example so you can see the difference in how these work. In this case since there is only a 6 in the denominator we'll just factor this out and then use the fact.

$$\lim_{\theta \to 0} \frac{\sin\theta}{6\theta} = \frac{1}{6}\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \frac{1}{6}(1) = \frac{1}{6}$$

(b) $\lim\limits_{x \to 0} \dfrac{\sin(6x)}{x}$

Now, in this case we can't factor the 6 out of the sine so we're stuck with it there and we'll need to figure out a way to deal with it.To do this problem we need to notice that in the fact the argument of the sine is the same as the denominator (*i.e.* both θs). So we need to get both of the argument of the sine and the denominator to be the same. We can do this by multiplying the numerator and the denominator by 6 as follows.

$$\lim_{x \to 0} \frac{\sin(6x)}{x} = \lim_{x \to 0} \frac{6\sin(6x)}{6x} = 6\lim_{x \to 0} \frac{\sin(6x)}{6x}$$

Note that we factored the 6 in the numerator out of the limit. At this point, while it may not look like it, we can use the fact above to finish the limit.

To see that we can use the fact on this limit let's do a **change of variables**. A change of variables is really just a renaming of portions of the problem to make something look more like something we know how to deal with. They can't always be done, but sometimes, such as this case, they can simplify the problem. The change of variables here is to let $\theta = 6x$ and then notice that as $x \to 0$ we also have $\theta \to 6(0) = 0$. When doing a change of variables in a limit we need to change all the x's into θs and that includes the one in the limit.

Doing the change of variables on this limit gives,

$$\lim_{x \to 0} \frac{\sin(6x)}{x} = 6\lim_{x \to 0} \frac{\sin(6x)}{6x} \qquad \text{let } \theta = 6x$$
$$= 6\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$$
$$= 6(1)$$
$$= 6$$

And there we are. Note that we didn't really need to do a change of variables here. All we really need to notice is that the argument of the sine is the same as the denominator and then we can use the fact. A change of variables, in this case, is really only needed to make it clear that the fact does work.

(c) $\lim_{x \to 0} \frac{x}{\sin(7x)}$

In this case we appear to have a small problem in that the function we're taking the limit of here is upside down compared to that in the fact. This is not the problem it appears to be once we notice that,

$$\frac{x}{\sin(7x)} = \frac{1}{\frac{\sin(7x)}{x}}$$

and then all we need to do is recall a nice property of limits that allows us to do ,

$$\lim_{x \to 0} \frac{x}{\sin(7x)} = \lim_{x \to 0} \frac{1}{\frac{\sin(7x)}{x}}$$
$$= \frac{\lim_{x \to 0} 1}{\lim_{x \to 0} \frac{\sin(7x)}{x}}$$
$$= \frac{1}{\lim_{x \to 0} \frac{\sin(7x)}{x}}$$

With a little rewriting we can see that we do in fact end up needing to do a limit like the one we did in the previous part. So, let's do the limit here and this time we won't bother with a change of variable to help us out. All we need to do is multiply the numerator and denominator of the fraction in the denominator by 7 to get things set up to use the fact. Here is the work for this limit.

$$\lim_{x\to 0}\frac{x}{\sin(7x)}=\frac{1}{\lim\limits_{x\to 0}\frac{7\sin(7x)}{7x}}$$
$$=\frac{1}{7\lim\limits_{x\to 0}\frac{\sin(7x)}{7x}}$$
$$=\frac{1}{(7)(1)}$$
$$=\frac{1}{7}$$

(d) $\lim\limits_{t\to 0}\frac{\sin(3t)}{\sin(8t)}$

This limit looks nothing like the limit in the fact, however it can be thought of as a combination of the previous two parts by doing a little rewriting. First, we'll split the fraction up as follows,

$$\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)}=\lim_{t\to 0}\frac{\sin(3t)}{1}\frac{1}{\sin(8t)}$$

Now, the fact wants a t in the denominator of the first and in the numerator of the second. This is easy enough to do if we multiply the whole thing by t/t (which is just one after all and so won't change the problem) and then do a little rearranging as follows,

$$\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)}=\lim_{t\to 0}\frac{\sin(3t)}{1}\frac{1}{\sin(8t)}\frac{t}{t}$$
$$=\lim_{t\to 0}\frac{\sin(3t)}{t}\frac{t}{\sin(8t)}$$
$$=\left(\lim_{t\to 0}\frac{\sin(3t)}{t}\right)\left(\lim_{t\to 0}\frac{t}{\sin(8t)}\right)$$

At this point we can see that this really is two limits that we've seen before. Here is the work for each of these and notice on the second limit that we're going to work it a little differently than we did in the previous part. This time we're going to notice that it doesn't really matter whether the sine is in the numerator or the denominator as long as the argument of the sine is the same as what's in the numerator the limit is still one.

Here is the work for this limit.

$$\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)}=\left(\lim_{t\to 0}\frac{3\sin(3t)}{3t}\right)\left(\lim_{t\to 0}\frac{8t}{8\sin(8t)}\right)$$
$$=\left(3\lim_{t\to 0}\frac{\sin(3t)}{3t}\right)\left(\frac{1}{8}\lim_{t\to 0}\frac{8t}{\sin(8t)}\right)$$
$$=(3)\left(\frac{1}{8}\right)$$
$$=\frac{3}{8}$$

(e) $\lim\limits_{x\to 4}\frac{\sin(x-4)}{x-4}$

This limit almost looks the same as that in the fact in the sense that the argument of the sine is the same as what is in the denominator. However, notice that, in the limit, *x* is going to 4 and not 0 as the fact requires. However, with a change of variables we can see that this limit is in fact set to use the fact above regardless.

So, let θ= *x* - 4 and then notice that as $x\to 4$ we have $\theta\to 0$. Therefore, after doing the change of variable the limit becomes,

$$\lim_{x\to 4}\frac{\sin(x-4)}{x-4}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

(f) $\lim\limits_{z\to 0}\frac{\cos(2z)-1}{z}$

The previous parts of this example all used the sine portion of the fact. However, we could just have easily used the cosine portion so here is a quick example using the cosine portion to illustrate this. We'll not put in much explanation here as this really does work in the same manner as the sine portion.

$$\lim_{z\to 0}\frac{\cos(2z)-1}{z}=\lim_{z\to 0}\frac{2(\cos(2z)-1)}{2z}$$
$$=2\lim_{z\to 0}\frac{\cos(2z)-1}{2z}$$
$$=2(0)$$
$$0$$

All that is required to use the fact is that the argument of the cosine is the same as the denominator.

Limits at Infinity

By limits at infinity we mean one of the following two limit

$$\lim_{x\to\infty} f(x) \qquad\qquad \lim_{x\to-\infty} f(x)$$

In other words, we are going to be looking at what happens to a function if we let *x* get very large in either the positive or negative sense. Also, as we'll soon see, these limits may also have infinity as a value.

First, let's note that the set of Facts from Infinite Limit section also hold if the replace the $\lim\limits_{x\to c}$ with $\lim\limits_{x\to\infty}$ or $\lim\limits_{x\to-\infty}$. In fact, many of the limits that we're going to be looking at we will need the following two facts.

Fact 1

1. If r is a positive rational number and c is any real number then,

$$\lim_{x\to\infty}\frac{c}{x^r}=0$$

2. If r is a positive rational number, c is any real number and x^r is defined for $x<0$ then,

$$\lim_{x\to-\infty}\frac{c}{x^r}=0$$

The first part of this fact should make sense if you think about it. Because we are requiring $r>0$we know that x^r will stay in the denominator. Next as we increase x then x^r will also increase. So, we have a constant divided by an increasingly large number and so the result will be increasingly small. Or, in the limit we will get zero.

The second part is nearly identical except we need to worry about x^r being defined for negative x. This condition is here to avoid cases such as $r=1/2$. If this r were allowed then we'd be taking the square root of negative numbers which would be complex and we want to avoid that at this level.

Note as well that the sign of c will not affect the answer. Regardless of the sign of c we'll still have a constant divided by a very large number which will result in a very small number and the larger x get the smaller the fraction gets. The sign of c will affect which direction the fraction approaches zero (*i.e.* from the positive or negative side) but it still approaches zero.

Example 1.4-3 Evaluate each of the following limits.

(a) $\lim\limits_{x\to\infty}\left(2x^4-x^2-8x\right)$ **(b)** $\lim\limits_{t\to-\infty}\left(\frac{1}{3}t^5+2t^3-t^2+8\right)$

Solution

(a) $\lim_{x \to \infty} \left(2x^4 - x^2 - 8x\right)$

Our first thought here is probably to just "plug" infinity into the polynomial and "evaluate" each term to determine the value of the limit. It is pretty simple to see what each term will do in the limit and so this seems like an obvious step, especially since we've been doing that for other limits in previous sections.

So, let's see what we get if we do that. As *x* approaches infinity, then *x* to a power can only get larger and the coefficient on each term (the first and third) will only make the term even larger. So, if we look at what each term is doing in the limit we get the following,

$$\lim_{x \to \infty} \left(2x^4 - x^2 - 8x\right) = \infty - \infty - \infty$$

Now, we've got a small, but easily fixed, problem to deal with. We are probably tempted to say that the answer is zero (because we have an infinity minus an infinity) or maybe $-\infty$ (because we're subtracting two infinities off of one infinity). However, in both cases we'd be wrong. This is one of those **indeterminate forms** that we will discuss later.

Infinities just don't always behave as real numbers do when it comes to arithmetic. Without more work there is simply no way to know what $\infty - \infty$ will be and so we really need to be careful with this kind of problem.

So, we need a way to get around this problem. What we'll do here is factor the largest power of *x* out of the whole polynomial as follows,

$$\lim_{x \to \infty} \left(2x^4 - x^2 - 8x\right) = \lim_{x \to \infty} \left[x^4 \left(2 - \frac{1}{x^2} - \frac{8}{x^3} \right) \right]$$

If you're not sure you agree with the factoring above (there's a chance you haven't really been asked to do this kind of factoring prior to this) then recall that to check all you need to do is multiply the x^4 back through the parenthesis to verify it was done correctly. Also, an easy way to remember how to do this kind of factoring is to note that the second term is just the original polynomial divided by x^4. This will always work when factoring a power of *x* out of a polynomial.

Now for each of the terms we have,

$$\lim_{x \to \infty} x^4 = \infty \qquad \lim_{x \to \infty} \left(2 - \frac{1}{x^2} - \frac{8}{x^3} \right) = 2$$

The first limit is clearly infinity and for the second limit we'll use the fact above on the last two terms. Therefore using Fact 2 from the previous section we see value of the limit will be,

$$\lim_{x\to\infty}\left(2x^4 - x^2 - 8x\right) = \infty$$

(a) $\displaystyle\lim_{t\to-\infty}\left(\tfrac{1}{3}t^5 + 2t^3 - t^2 + 8\right)$

We'll work this part much quicker than the previous part. All we need to do is factor out the largest power of *t* to get the following,

$$\lim_{t\to-\infty}\left(\tfrac{1}{3}t^5 + 2t^3 - t^2 + 8\right) = \lim_{t\to-\infty}\left[t^5\left(\frac{1}{3}+\frac{2}{t^2}-\frac{1}{t^3}+\frac{8}{t^5}\right)\right]$$

Remember that all you need to do to get the factoring correct is divide the original polynomial by the power of *t* we're factoring out, t^5, in this case.

Now all we need to do is take the limit of the two terms. In the first don't forget that since we're going out towards $-\infty$ and we're raising *t* to the 5th power that the limit will be negative (negative number raised to an odd power is still negative). In the second term well again make heavy use of the fact above to see that is a finite number.

Therefore, using the a modification of the Facts from the previous section the value of the limit is,

$$\lim_{t\to-\infty}\left(\tfrac{1}{3}t^5 + 2t^3 - t^2 + 8\right) = -\infty$$

Fact 2

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree *n* then,

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n \qquad\qquad \lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n$$

What this fact is really saying is that when we go to take a limit at infinity for a polynomial then all we need to really do is look at the term with the largest power and ask what that term is doing in the limit since the polynomial will have the same behavior.

Example 1.4-4 Evaluate both of the following limits.

$$\lim_{x\to\infty}\frac{2x^4 - x^2 + 8x}{-5x^4 + 7} \qquad\qquad \lim_{x\to-\infty}\frac{2x^4 - x^2 + 8x}{-5x^4 + 7}$$

Solution

First, the only difference between these two is that one is going to positive infinity and the other is going to negative infinity. Sometimes this small difference will affect then value of the limit and at other times it won't.

Let's start with the first limit and as with our first set of examples it might be tempting to just "plug" in the infinity. Since both the numerator and denominator are polynomials we can use the above fact to determine the behavior of each. Doing this gives,

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7}=\frac{\infty}{-\infty}$$

This is yet another indeterminate form. In this case we might be tempted to say that the limit is infinity (because of the infinity in the numerator), zero (because of the infinity in the denominator) or -1 (because something divided by itself is one). There are three separate arithmetic "rules" at work here and without work there is no way to know which "rule" will be correct and to make matters worse it's possible that none of them may work and we might get a completely different answer, say -2/5 to pick a number completely at random.

So, when we have a polynomial divided by a polynomial we're going to proceed much as we did with only polynomials. We first identify the largest power of *x* in the denominator (and yes, we only look at the denominator for this) and we then factor this out of both the numerator and denominator. Doing this for the first limit gives,

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7}=\lim_{x\to\infty}\frac{x^4\left(2-\frac{1}{x^2}+\frac{8}{x^3}\right)}{x^4\left(-5+\frac{7}{x^4}\right)}$$

Once we've done this we can cancel the x^4 from both the numerator and the denominator and then use the Fact 1 above to take the limit of all the remaining terms. This gives,

$$\begin{aligned}\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7}&=\lim_{x\to\infty}\frac{2-\frac{1}{x^2}+\frac{8}{x^3}}{-5+\frac{7}{x^4}}\\&=\frac{2+0+0}{-5+0}\\&=-\frac{2}{5}\end{aligned}$$

In this case the indeterminate form was neither of the "obvious" choices of infinity, zero, or -1 so be careful with make these kinds of assumptions with this kind of indeterminate forms.

The second limit is done in a similar fashion. Notice however, that nowhere in the work for the first limit did we actually use the fact that the limit was going to plus infinity. In this case it doesn't matter which infinity we are going towards we will get the same value for the limit.

$$\lim_{x \to -\infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7} = -\frac{2}{5}$$

Example 1.4-5 Evaluate each of the following limits.

$$\lim_{x \to \infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} \qquad \lim_{x \to -\infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x}$$

Solution

The square root in this problem won't change our work, but it will make the work a little messier.

Let's start with the first limit. In this case the largest power of x in the denominator is just an x. So we need to factor an x out of the numerator and the denominator. When we are done factoring the x out we will need an x in both of the numerator and the denominator. To get this in the numerator we will have to factor an x^2 out of the square root so that after we take the square root we will get an x.

This is probably not something you're used to doing, but just remember that when it comes out of the square root it needs to be an x and the only way have an x come out of a square is to take the square root of x^2 and so that is what we'll need to factor out of the term under the radical. Here's the factoring work for this part,

$$\begin{aligned}
\lim_{x \to \infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} &= \lim_{x \to \infty} \frac{\sqrt{x^2\left(3 + \frac{6}{x^2}\right)}}{x\left(\frac{5}{x} - 2\right)} \\
&= \lim_{x \to \infty} \frac{\sqrt{x^2}\sqrt{3 + \frac{6}{x^2}}}{x\left(\frac{5}{x} - 2\right)}
\end{aligned}$$

This is where we need to be really careful with the square root in the problem. Don't forget that $\sqrt{x^2} = |\, x \,|$.

Square roots are ALWAYS positive and so we need the absolute value bars on the *x* to make sure that it will give a positive answer. This is not something that most people every remember seeing in an Algebra class and in fact it's not always given in an Algebra class. However, at this point it becomes absolutely vital that we know and use this fact. Using this fact the limit becomes,

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to\infty}\frac{|x|\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

Now, we can't just cancel the *x*'s. We first will need to get rid of the absolute value bars. To do this let's recall the definition of absolute value.

$$|x|=\begin{cases}x & \text{if } x\ge 0\\ -x & \text{if } x<0\end{cases}$$

In this case we are going out to plus infinity so we can safely assume that the *x* will be positive and so we can just drop the absolute value bars. The limit is then,

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to\infty}\frac{x\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

$$=\lim_{x\to\infty}\frac{\sqrt{3+\frac{6}{x^2}}}{\frac{5}{x}-2}=\frac{\sqrt{3+0}}{0-2}=-\frac{\sqrt{3}}{2}$$

Let's now take a look at the second limit (the one with negative infinity). In this case we will need to pay attention to the limit that we are using. The initial work will be the same up until we reach the following step.

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to-\infty}\frac{|x|\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

In this limit we are going to minus infinity so in this case we can assume that *x* is negative. So, in order to drop the absolute value bars in this case we will need to tack on a minus sign as well. The limit is then,

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to-\infty}\frac{-x\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

$$=\lim_{x\to-\infty}\frac{-\sqrt{3+\frac{6}{x^2}}}{\frac{5}{x}-2}$$

$$=\frac{\sqrt{3}}{2}$$

Defintion of the Number *e*:

e is an irrational number and *e* = 2.71828182845905..., there are in fact a variety of ways to define *e*. Here are a three of them.

1. $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
2. $e=\sum_{n=0}^{\infty}\frac{1}{n\,!}$
3. $\lim_{h\to 0}\frac{e^h-1}{h}=1$

We can verify this limit or the value of *e* with numerical approximation by substituting a large integer *n* into the expression $\left(1+\frac{1}{n}\right)^n$ with a calculator.

Example 1.4-6 Approximate the value of *e*.

Solution

1 We can use a calculator, substituting n = 1 000 000 000 in to $\left(1+\frac{1}{n}\right)^n$, to have

$$\left(1+\frac{1}{1\,000\,000\,000}\right)^{1\,000\,000000}=2.718281827$$

that is a good approximation of the value of *e*.

Example 1.4-7 Evaluate each of the following limits.

$$\lim_{x\to\infty} \mathbf{e}^{x} \qquad \lim_{x\to-\infty} \mathbf{e}^{x} \qquad \lim_{x\to\infty} \mathbf{e}^{-x} \qquad \lim_{x\to-\infty} \mathbf{e}^{-x}$$

Solution

There are really just restatements of facts given in the basic exponential section of the review so we'll leave it to you to go back and verify these.

$$\lim_{x\to\infty} \mathbf{e}^{x} = \infty \qquad \lim_{x\to-\infty} \mathbf{e}^{x} = 0 \qquad \lim_{x\to\infty} \mathbf{e}^{-x} = 0 \qquad \lim_{x\to-\infty} \mathbf{e}^{-x} = \infty$$

Example 1.4-8 Evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{6\mathbf{e}^{4x} - \mathbf{e}^{-2x}}{8\mathbf{e}^{4x} - \mathbf{e}^{2x} + 3\mathbf{e}^{-x}}$ **(b)** $\displaystyle\lim_{t\to-\infty} \frac{\mathbf{e}^{6t} - 4\mathbf{e}^{-6t}}{2\mathbf{e}^{3t} - 5\mathbf{e}^{-9t} + \mathbf{e}^{-3t}}$

Solution

As with the previous example, the only difference between the first two parts is that one of the limits is going to plus infinity and the other is going to minus infinity and just as with the previous example each will need to be worked differently.

(a) $\displaystyle\lim_{x\to\infty} \frac{6\mathbf{e}^{4x} - \mathbf{e}^{-2x}}{8\mathbf{e}^{4x} - \mathbf{e}^{2x} + 3\mathbf{e}^{-x}}$

The basic concept involved in working this problem is the same as with rational expressions in the previous section. We look at the denominator and determine the exponential function with the "largest" exponent which we will then factor out from both numerator and denominator. We will use the same reasoning as we did with the previous example to determine the "largest" exponent. In the case since we are looking at a limit at plus infinity we only look at exponentials with positive exponents.

So, we'll factor an e^{4x} out of both then numerator and denominator. Once that is done we can cancel the e^{4x} and then take the limit of the remaining terms. Here is the work for this limit,

$$\begin{aligned}\lim_{x\to\infty}\frac{6\mathbf{e}^{4x}-\mathbf{e}^{-2x}}{8\mathbf{e}^{4x}-\mathbf{e}^{2x}+3\mathbf{e}^{-x}} &= \lim_{x\to\infty}\frac{\mathbf{e}^{4x}\left(6-\mathbf{e}^{-6x}\right)}{\mathbf{e}^{4x}\left(8-\mathbf{e}^{-2x}+3\mathbf{e}^{-5x}\right)}\\ &= \lim_{x\to\infty}\frac{6-\mathbf{e}^{-6x}}{8-\mathbf{e}^{-2x}+3\mathbf{e}^{-5x}}\\ &= \frac{6-0}{8-0+0}\\ &= \frac{3}{4}\end{aligned}$$

(b) $\lim\limits_{t\to-\infty}\dfrac{\mathbf{e}^{6t}-4\mathbf{e}^{-6t}}{2\mathbf{e}^{3t}-5\mathbf{e}^{-9t}+\mathbf{e}^{-3t}}$

We'll do the work on this part with much less detail.

$$\begin{aligned}\lim_{t\to-\infty}\frac{\mathbf{e}^{6t}-4\mathbf{e}^{-6t}}{2\mathbf{e}^{3t}-5\mathbf{e}^{-9t}+\mathbf{e}^{-3t}} &= \lim_{t\to-\infty}\frac{\mathbf{e}^{-9t}\left(\mathbf{e}^{15t}-4\mathbf{e}^{3t}\right)}{\mathbf{e}^{-9t}\left(2\mathbf{e}^{12t}-5+\mathbf{e}^{6t}\right)}\\ &= \lim_{t\to-\infty}\frac{\mathbf{e}^{15t}-4\mathbf{e}^{3t}}{2\mathbf{e}^{12t}-5+\mathbf{e}^{6t}}\\ &= \frac{0-0}{0-5+0}\\ &= 0\end{aligned}$$

Example 1.4-9 Evaluate each of the following limits.

$\lim\limits_{x\to0^+}\ln x \qquad\qquad \lim\limits_{x\to\infty}\ln x$

Solution

You can verify these restatements from the basic logarithm section.

$\lim\limits_{x\to0^+}\ln x=-\infty \qquad\qquad \lim\limits_{x\to\infty}\ln x=\infty$

Note that we had to do a right-handed limit for the first one since we can't plug negative *x*'s into a logarithm. This means that the normal limit won't exist since we must look at *x*'s from both sides of the point in question and *x*'s to the left of zero are negative.

Example 1.4-10 Evaluate each of the following limits.

(a) $\lim\limits_{x \to \infty} \ln\left(7x^3 - x^2 + 1\right)$ **(b)** $\lim\limits_{t \to -\infty} \ln\left(\dfrac{1}{t^2 - 5t}\right)$

Solution

(a) $\lim\limits_{x \to \infty} \ln\left(7x^3 - x^2 + 1\right)$

So, let's first look to see what the argument of log is doing,

$$\lim_{x \to \infty}\left(7x^3 - x^2 + 1\right) = \infty$$

The argument of the log is going to infinity and so the log must also be going to infinity in the limit. The answer to this part is then,

$$\lim_{x \to \infty} \ln\left(7x^3 - x^2 + 1\right) = \infty$$

(b) $\lim\limits_{t \to -\infty} \ln\left(\dfrac{1}{t^2 - 5t}\right)$

First, note that the limit going to negative infinity here isn't a violation (necessarily) of the fact that we can't plug negative numbers into the logarithm. The real issue is whether or not the argument of the log will be negative or not.

Using the techniques from earlier in this section we can see that,

$$\lim_{t \to -\infty} \frac{1}{t^2 - 5t} = 0$$

and let's also not that for negative numbers (which we can assume we've got since we're going to minus infinity in the limit) the denominator will always be positive and so the quotient will also always be positive. Therefore, not only does the argument go to zero, it goes to zero from the right. This is exactly what we need to do this limit.

So, the answer here is,

$$\lim_{t \to -\infty} \ln\left(\frac{1}{t^2 - 5t}\right) = -\infty$$

Example 1.4-11 Evaluate each of the following limits.

(a) $\lim\limits_{x \to \infty} \tan^{-1} x$

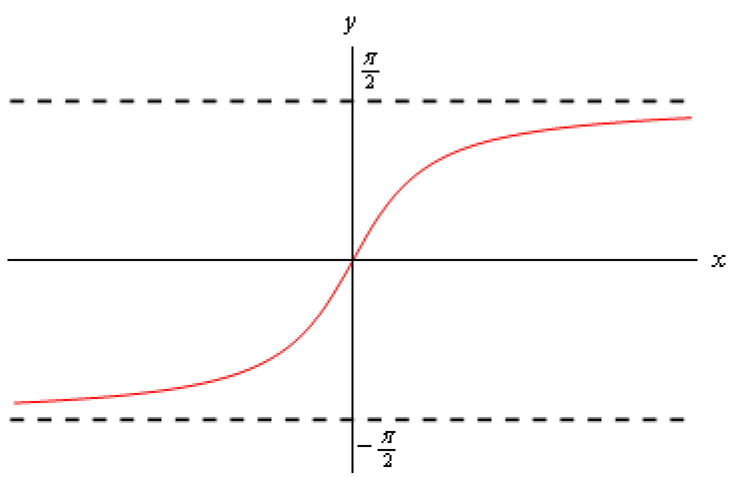
(b) $\lim\limits_{x \to -\infty} \tan^{-1} x$

(c) $\lim\limits_{x \to \infty} \tan^{-1}\left(x^3 - 5x + 6\right)$

(d) $\lim\limits_{x \to 0^-} \tan^{-1}\left(\frac{1}{x}\right)$

Solution

The first two parts here are really just the basic limits involving inverse tangents and can easily be found by examining the following sketch of inverse tangents. The remaining two parts are more involved but as with the exponential and logarithm limits really just refer back to the first two parts as we'll see.



(a) $\lim\limits_{x \to \infty} \tan^{-1} x$

As noted above all we really need to do here is look at the graph of the inverse tangent. Doing this shows us that we have the following value of the limit.

$$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$$

(b) $\lim\limits_{x\to-\infty} \tan^{-1} x$

Again, not much to do here other than examine the graph of the inverse tangent.

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

(c) $\lim\limits_{x\to\infty} \tan^{-1}\left(x^3 - 5x + 6\right)$

Okay, in part (a) above we saw that if the argument of the inverse tangent function (the stuff inside the parenthesis) goes to plus infinity then we know the value of the limit. In this case (using the techniques from the previous section) we have,

$$\lim_{x\to\infty} x^3 - 5x + 6 = \infty$$

So, this limit is,

$$\lim_{x\to\infty} \tan^{-1}\left(x^3 - 5x + 6\right) = \frac{\pi}{2}$$

(d) $\lim\limits_{x\to0^-} \tan^{-1}\left(\dfrac{1}{x}\right)$

Even though this limit is not a limit at infinity we're still looking at the same basic idea here. We'll use part (b) from above as a guide for this limit. We know from the Infinite Limits section that we have the following limit for the argument of this inverse tangent,

$$\lim_{x\to0^-} \frac{1}{x} = -\infty$$

So, since the argument goes to minus infinity in the limit we know that this limit must be,

$$\lim_{x\to0^-} \tan^{-1}\left(\frac{1}{x}\right) = -\frac{\pi}{2}$$

Let's work another couple of examples involving rational expressions.

Example 1.4-12 Evaluate each of the following limits.

$$\lim_{z\to\infty}\frac{4z^2+z^6}{1-5z^3} \qquad\qquad \lim_{z\to-\infty}\frac{4z^2+z^6}{1-5z^3}$$

Solution

Let's do the first limit and in this case it looks like we will factor a z^3 out of both the numerator and denominator. Remember that we only look at the denominator when determining the largest power of z here. There is a larger power of z in the numerator but we ignore it. We ONLY look at the denominator when doing this! So doing the factoring gives,

$$\lim_{z\to\infty}\frac{4z^2+z^6}{1-5z^3}=\lim_{z\to\infty}\frac{z^3\left(\frac{4}{z}+z^3\right)}{z^3\left(\frac{1}{z^3}-5\right)}$$
$$=\lim_{z\to\infty}\frac{\frac{4}{z}+z^3}{\frac{1}{z^3}-5}$$

When we take the limit we'll need to be a little careful. The first term in the numerator and denominator will both be zero. However, the z^3 in the numerator will be going to plus infinity in the limit and so the limit is,

$$\lim_{z\to\infty}\frac{4z^2+z^6}{1-5z^3}=\frac{\infty}{-5}=-\infty$$

The final limit is negative because we have a quotient of positive quantity and a negative quantity.

Now, let's take a look at the second limit. Note that the only different in the work is at the final "evaluation" step and so we'll pick up the work there.

$$\lim_{z\to-\infty}\frac{4z^2+z^6}{1-5z^3}=\lim_{z\to-\infty}\frac{\frac{4}{z}+z^3}{\frac{1}{z^3}-5}=\frac{-\infty}{-5}=\infty$$

In this case the z^3 in the numerator gives negative infinity in the limit since we are going out to minus infinity and the power is odd. The answer is positive since we have a quotient of two negative numbers.

Example 1.4-13 Evaluate the following limit.

$$\lim_{t\to-\infty}\frac{t^2-5t-9}{2t^4+3t^3}$$

Solution

In this case it looks like we will factor a t^4 out of both the numerator and denominator. Doing this gives,

$$\begin{aligned}\lim_{t\to-\infty}\frac{t^2-5t-9}{2t^4+3t^3} &= \lim_{t\to-\infty}\frac{t^4\left(\frac{1}{t^2}-\frac{5}{t^3}-\frac{9}{t^4}\right)}{t^4\left(2+\frac{3}{t}\right)}\\ &= \lim_{t\to-\infty}\frac{\frac{1}{t^2}-\frac{5}{t^3}-\frac{9}{t^4}}{2+\frac{3}{t}}\\ &= \frac{0}{2}\\ &= 0\end{aligned}$$

In this case using Fact 1 we can see that the numerator is zero and so since the denominator is also not zero the fraction, and hence the limit, will be zero.

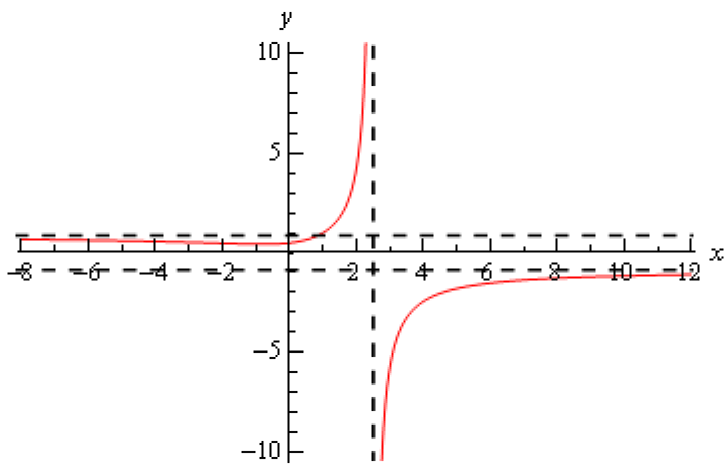
1.5 Asymptotes

Definition of Horizontal Asymptotes

The function *f(x)* will have a horizontal asymptote at *y=L* if either of the following are true.

$$\lim_{x\to\infty} f(x) = L \qquad\qquad \lim_{x\to-\infty} f(x) = L$$

We're not going to be doing much with asymptotes here, but it's an easy fact to give and we can use the previous example to illustrate all the asymptote ideas we've seen in the both this section and the previous section. The function in the last example will have two horizontal asymptotes. It will also have a vertical asymptote. Here is a graph of the function showing these.



In the two examples of the Section 1.4, ***Example 4-4*** and ***Example 4-10***, $x = -2/5$ and $x = \pm\pi/2$ are horizontal asymptotes.

Definition of Vertical Asymptotes

The function *f(x)* will have a vertical asymptote at $x = a$ if we have any of the following limits at $x = a$,

$$\lim_{x \to a^-} f(x) = \pm\infty \qquad \lim_{x \to a^+} f(x) = \pm\infty \qquad \lim_{x \to a} f(x) = \pm\infty$$

Example 1.5-1 Evaluate each of the following limits.

$$\lim_{x \to 4^+} \frac{3}{(4-x)^3} \qquad \lim_{x \to 4^-} \frac{3}{(4-x)^3} \qquad \lim_{x \to 4} \frac{3}{(4-x)^3}$$

Solution

Let's start with the right-hand limit. For this limit we have,

$$x > 4 \qquad \Rightarrow \qquad 4 - x < 0 \qquad \Rightarrow \qquad (4-x)^3 < 0$$

also, $4 - x \to 0$ as $x \to 4$. So, we have a positive constant divided by an increasingly small negative number. The results will be an increasingly large negative number and so it looks like the right-hand limit will be negative infinity.

For the left-handed limit we have,

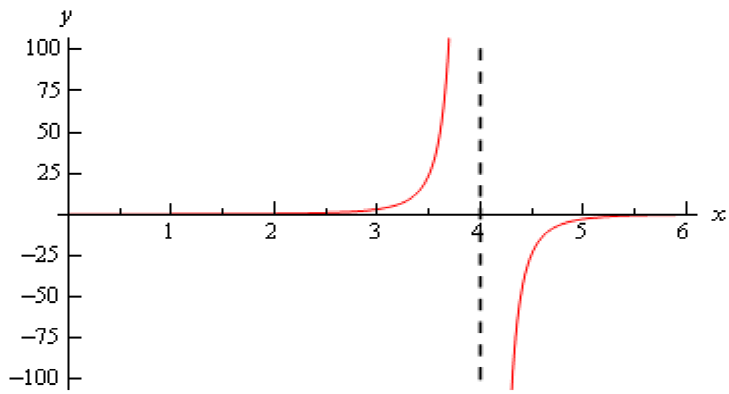
$$x < 4 \qquad \Rightarrow \qquad 4 - x > 0 \qquad \Rightarrow \qquad (4-x)^3 > 0$$

and we still have $4 - x \to 0$ as $x \to 4$. In this case we have a positive constant divided by an increasingly small positive number. The results will be an increasingly large positive number and so it looks like the left-hand limit will be positive infinity.

The normal limit will not exist since the two one-sided limits are not the same. The official answers to this example are then,

$$\lim_{x \to 4^+} \frac{3}{(4-x)^3} = -\infty \qquad \lim_{x \to 4^-} \frac{3}{(4-x)^3} = \infty \qquad \lim_{x \to 4} \frac{3}{(4-x)^3} \text{ doesn't exist}$$

Here is a quick sketch to verify our limits.



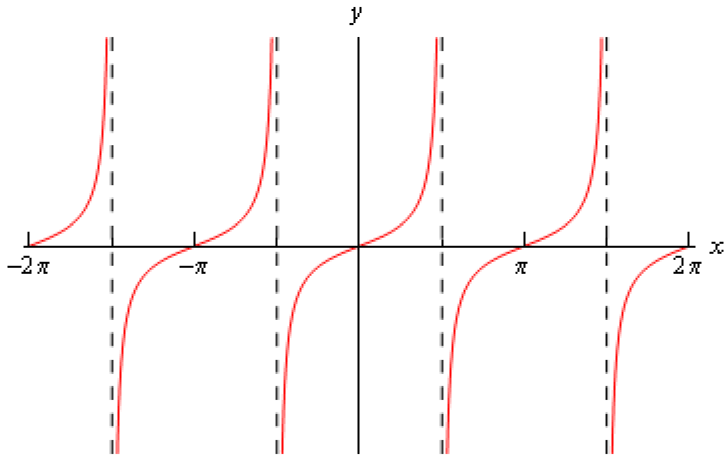
Obviously, $x = 4$ is a vertical asymptote here.

Example 1.5-2 Evaluate both of the following limits.

$$\lim_{x \to \frac{\pi}{2}^+} \tan(x) \qquad \lim_{x \to \frac{\pi}{2}^-} \tan(x)$$

Solution

Here's a quick sketch of the graph of the tangent function.



From this it's easy to see that we have the following values for each of these limits,

$$\lim_{x \to \frac{\pi}{2}^+} \tan(x) = -\infty \qquad \lim_{x \to \frac{\pi}{2}^-} \tan(x) = \infty$$

Note that the normal limit will not exist because the two one-sided limits are not the same, and $x = \pi/2 \pm k\pi$, $k = 0, 1, 2, 3, \ldots$ are vertical asymptotes.

Definition of Slant or Oblique linear Asymptotes

The function $f(x)$ will have a slant or oblique linear asymptote $y = ax + b$ if either of the following are true.

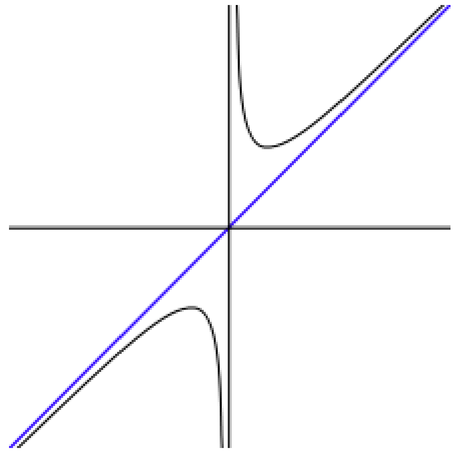
$$\lim_{x \to \infty}\left[f(x) - (ax + b)\right] = 0 \qquad \lim_{x \to -\infty}\left[f(x) - (ax + b)\right] = 0$$

In the first case the line $y = ax + b$ is an oblique asymptote of $f(x)$ when x tends to $+\infty$, and in the second case the line $y = ax + b$ is an oblique asymptote of $f(x)$ when x tends to $-\infty$.

Example 1.5-3 Determine the oblique asymptote for $f(x) = x - 1/x$.

Solution

In the graph of $f(x) = x - 1/x$ below, the y-axis ($x = 0$) is a vertical asymptote and the line $y = x$ is an oblique linear asymptote.



The function has the oblique asymptote *y* = *x* (*a* = 1, *b* = 0) as seen in the following limit,

$$\lim_{x\to\pm\infty}\left[f(x)-x\right]=\lim_{x\to\pm\infty}\left[x-\frac{1}{x}-x\right]=\lim_{x\to\pm\infty}\left(-\frac{1}{x}\right)=0.$$

The following sections are given for reference only, not required by AP or BC Calculus

(ε, δ)-Definition of Limit

2.2 Definition of Limit

Let us discuss $\lim_{x\to 4}\frac{1}{2}(3x-1)=5.5$ in detail, the variation of $f(x)=\frac{1}{2}(3x-1)$ when *x* is close to 4. Using the functional values we arrive at the following statements.

If $3.9 < x < 4.1$, then $5.35 < f(x) < 5.65$
If $3.99 < x < 4.01$, then $5.485 < f(x) < 5.515$
If $3.999 < x < 4.001$, then $5.4985 < f(x) < 5.5015$
If $3.9999 < x 4.0001$, then $5.49985 < f(x) < 5.50015$
If $3.99999 < x < 4.00001$, then $5.499985 < f(x) < 5.500015$

Each of these statements has the following form, where the Great letters ε (epsilon) and δ (delta) are used to denote small positive real numbers:

(i) if $4 - \delta < x < 4 + \delta$, then $5.5 - \varepsilon < f(x) < 5.5 + \varepsilon$

For example, the first statement follow from (i) by letting δ = 0.1 and ε = 0.15; the second is the case δ = 0.01 and ε = 0.015; for the third let δ = 0.001 and ε = 0.0015; and so on.

We may rewrite (i) in terms of intervals as follows:

(ii) If *x* is in the open interval $(4 - \delta, 4 + \delta)$, then $f(x)$ is in the open interval $(5.5 - \varepsilon, 5.5 + \varepsilon)$.

A geometric interpretation of (ii) is given in Figure 2.4, where the curved arrow indicates the correspondence between *x* and *f(x)*.

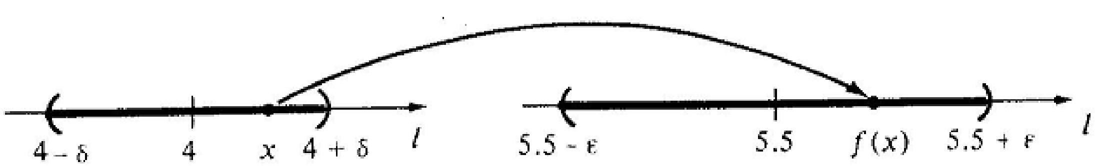


FIGURE 2.4

Evidently (i) is equivalent to following statement.

If $-\delta < x - 4 < \delta$, then $-\varepsilon < f(x) - 5.5 < \varepsilon$

Employing absolute values, this may be written:

If $|x - 4| < \delta$, then $|f(x) - 5.5| < \varepsilon$

If we wish to add the condition $x \neq 4$, then it is necessary to demand that $0 < |x-4|$. This gives us the following extension of (i) and (ii).

If $\quad 0 < |x-4| < \delta$, then $\quad |f(x)-5.5| < \varepsilon$

A statement of type (iii) will appear in the definition of limit; however, it is necessary to change our point of view to some extent. To arrive at (iii), we first considered the domain of *f* and assigned values to *x* that were close to 4. we then noted the closeness of *f(x)* to 5.5. In the definition of limit we shall *reverse* this process by first considering an open interval $(5.5-\varepsilon, 5.5+\varepsilon)$ and then, second, determining whether there is an open interval of the form $(4-\delta, 4+\delta)$ in the domain of *f* such that (iii) is true.

The next definition is patterned after the previous remarks.

Definition of Limit (2.3)

Let *f* be a function that is defined on an open interval containing *a*, except possible at *a* itself, and let *L* be a real number: The statement

$$\lim_{x \to a} f(x) = L$$

Means that for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

If $0 < |x-a| < \delta$, then $|f(x)-L| < \varepsilon$.

If $\lim_{x \to a} f(x) = L$, we say that **the limit of f(x), as x approaches a, is L.** Since ε can be arbitrarily small, the last two inequalities in this definition are sometimes phrased *f(x) can be made arbitrarily close to L by choosing x sufficiently close to a.*

It is sometimes convenient to use the following form of Definition (2.3). Where the two inequalities involving absolute values have been stated in terms of open intervals

Alternative Definition of Limit (2.4)

The statement

$$\lim_{x \to a} f(x) = L$$

Means that for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if *x* is in the open interval $(a-\delta, a+\delta)$, and $x \neq a$, then *f(x)* is in the open interval $(L-\varepsilon, L+\varepsilon)$.

To get a better understanding of the relationship between the positive numbers ε and δ in Definitions (2.3) and (2.4), let us consider a geometric interpretation similar to that in Figure 1.26. Where the domain of *f* is represented by certain points on a coordinate line *l*, and the range by other points on a coordinate line *l'*. The limit process may be outlined by follows.

To prove that $\lim_{x \to a} f(x) = L$

Step1. For any $\varepsilon > 0$ consider the open interval $(L-\varepsilon, L+\varepsilon)$ (see Figure 2.5)

FIGURE 2.5

Step 2. Show that there exists an open interval $(a-\delta, a+\delta)$ in the domain of *f* such that (2.4) is true (see Figure 2.6).

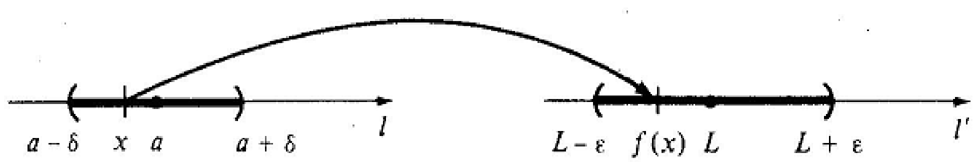


FIGURE 2.6

It is extremely important to remember that *first* we consider the interval $(L-\varepsilon, L+\varepsilon)$ and then, *second*, we show that and interval $(a-\delta, a+\delta)$ of the required type exists in the domain of *f*. one scheme for remembering the proper sequence of events is to think of the function *f* as a cannon that shoots a cannonball from the point on *l* with coordinate *x* to the point on *l'* with coordinate *f(x),* as illustrated by the curved arrow in Figure 2.6. Step 1 may then be regarded as setting up a target of radius ε with bull's eye at *L*. to apply Step 2 we must find an open interval containing *a* in which to place the cannon such that the cannonball hits the target. Incidentally, there is no guarantee that it will hit the bull's eye; however, if $\lim_{x\to a} f(x) = L$ we can make the cannonball land as close as we please to the bull's eye.

It should be clear that the number δ in the limit definition is not unique, for if a specific δ can be found, then any *smaller* positive number δ' will also satisfy the requirements.

Since a function may be described geometrically by means of a graph on a rectangular coordinate system, it is of interest to interpret Definitions (2.3) and (2.4) graphically. Figure 2.7 illustrates the graph of a function *f* where, for any *x* in the domain of *f,* the number *f(x)* is the *y*-coordinate of the point on the graph with *x*-coordinate *x.* given any ε >0, we consider the open interval $(L-\varepsilon, L+\varepsilon)$ on the *y*-axis, and the horizontal lines $y = x \pm \varepsilon$ shown in the figure. If there exists an open interval $(a-\delta, a+\delta)$, with the possible exception of *x=a*, the point $P(x, f(x))$ lies between the horizontal lines, that is, within the shaded rectangle shown in Figure 2.7, then $L-\varepsilon < f(x) < L+\varepsilon$ and hence $\lim_{x\to a} f(x) = L$.

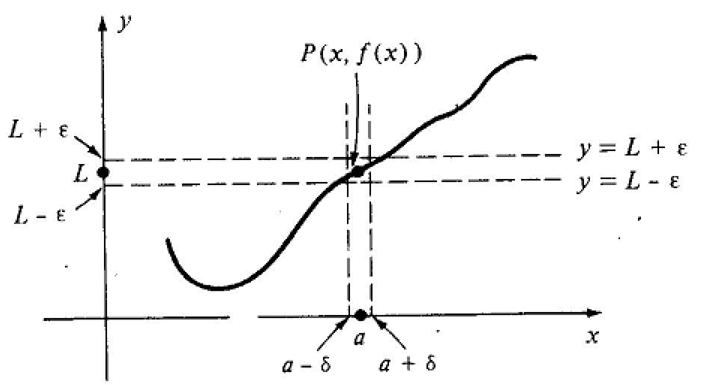


FIGURE 2.7 $\lim_{x\to a} f(x) = L$

The next example illustrates how the geometric process pictured in Figure 2.7 may be applied to a specific function.

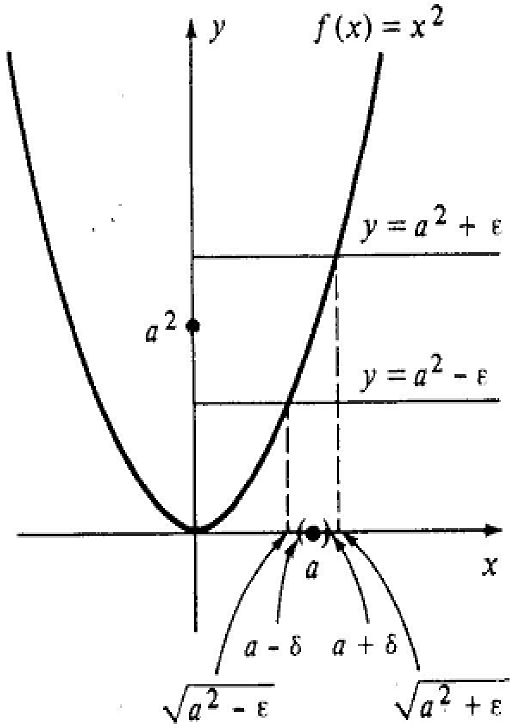


FIGURE 2.8

Example 1

Prove that $\lim_{x\to a} x^2 = a^2$

Solution

Let us consider the case *a*>0, we shall apply (2.4) with $f(x) = x^2$ and $L = a^2$. The graph of *f* is sketched in Figure 2.8, together with typical points on the *x*-and *y*-axes corresponding to *a* and a^2, respectively.

For any positive number ε, consider the horizontal lines $y = a^2 + \varepsilon$ and $y = a^2 - \varepsilon$. These lines intersect the graph of *f* at points with *x*-coordinates $\sqrt{a^2 - \varepsilon}$ and $\sqrt{a^2 + \varepsilon}$, as illustrated in the figure. If *x* is in the open interval $(\sqrt{a^2 - \varepsilon}, \sqrt{a^2 + \varepsilon})$, then

$$\sqrt{a^2 - \varepsilon} < x < \sqrt{a^2 + \varepsilon}$$

Consequently,

$$a^2 - \varepsilon < x^2 < a^2 + \varepsilon .$$

That is, $f(x) = x^2$ is in the open interval $(a^2 - \varepsilon, a^2 + \varepsilon)$. Geometrically, this means that the point (x, x^2) on the graph of *f* lies between the horizontal lines. Let us choose a number δ smaller than both $\sqrt{a^2 + \varepsilon} - a$ and $a - \sqrt{a^2 - \varepsilon}$ as illustrated in Figure 2.8. It follows that if *x* is in the interval $(a - \delta, a + \delta)$, then *x* is also in $(\sqrt{a^2 - \varepsilon}, \sqrt{a^2 + \varepsilon})$, and, therefore, *f(x)* is in $(a^2 - \varepsilon, a^2 + \varepsilon)$. Hence, by Definition (2.4), $\lim_{x\to a} x^2 = a^2$. Although we have considered only *a*>0, a similar argument applies if $a \le 0$.

To shorten explanations, whenever the notation $\lim_{x\to a} f(x) = L$ is used we shall often assume that all the conditions given in Definition (2.3) are satisfied. Thus, it may not always be pointed out that *f* is defined on an open interval containing *a*. Moreover, we shall not always specify *L* but merely write

"$\lim_{x\to a} f(x)$ exists," or "*f(x)* has a limit as *x* approaches *a*." the phrase "find $\lim_{x\to a} f(x)$" means "find a number *L* such that $\lim_{x\to a} f(x)=L$" if no such II exists we write "$\lim_{x\to a} f(x)$ does not exist."

It can be proved that if *f(x)* has a limit as *x* approaches *a,* then that limit is unique.

In the following example we return to the function considered at the beginning of this section and *prove* that the limit exists by means of Definition (2.3).

Example 2

Prove that $\lim\limits_{x\to 4}\dfrac{1}{2}(3x-1)=\dfrac{11}{2}$.

Solution

Let $f(x)=\frac{1}{2}(3x-1), a=4,\ L=\frac{11}{2}$. According to Definition (2.3) we must show that for every $\varepsilon>0$, there exists a $\delta>0$ such that

If $\quad 0<|x-4|<\delta$, then $\quad |\frac{1}{2}(3x-1)-\frac{11}{2}|<\varepsilon$.

A clue to the choice of δ can be found by examining the last inequality involving ε. The following is a list of equivalent inequalities.

$$
\begin{aligned}
|\tfrac{1}{2}(3x-1)-\tfrac{11}{2}|&<\varepsilon\\
\tfrac{1}{2}|(3x-1)-11|&<\varepsilon\\
|3x-1-11|&<2\varepsilon\\
|3x-12|&<2\varepsilon\\
3|x-4|&<2\varepsilon\\
|x-4|&<\tfrac{2}{3}\varepsilon
\end{aligned}
$$

The final inequality gives us the needed clue. If we let $\delta = \frac{2}{3}\varepsilon$, then if $0 < |x-4| < \delta$, the last inequality in the list is true and consequently so is the first. Hence by Definition (2.3), $\lim_{x\to 4} \frac{1}{2}(3x-1) = \frac{11}{2}$.

It is also possible to give a geometric proof similar to that used in Example 1.

It was relatively easy to use the definition of limit in the previous examples because *f(x)* was a simple expression involving *x*. limits of more complicated functions may also be verified by direct applications of the definition however, the task of showing that every $\varepsilon > 0$, there exists a suitable $\delta > 0$ often requires a great deal of ingenuity. In Section 2.3 we shall introduce theorems which can be used to find many limits without resorting to a search for the general number δ that appears in Definition (2.3).

The next two examples indicate how the geometric interpretation illustrated in Figure 2.7 may be used to show that certain limits do not exist.

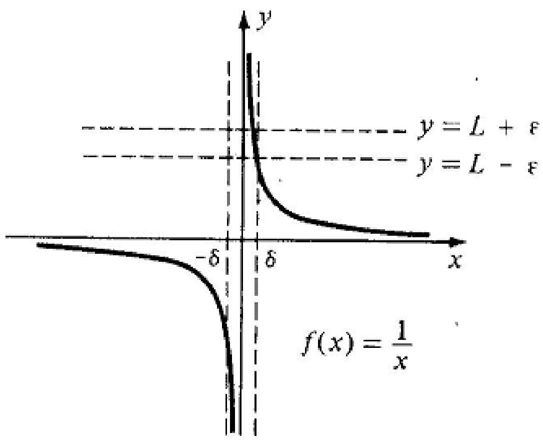


FIGURE 2.9

Example 3

Show that $\lim_{x\to 0} 1/x$ does not exist

Solution

Let us proceed in an indirect manner. Thus, suppose it *were* true that

$$\lim_{x\to 0} 1/x = L$$

For some number *L*. let us consider any pair of horizontal lines $y = L \pm \varepsilon$ as illustrated in Figure 2.9. Since we are assuming that the limit exists, it should be possible to find an open interval $(0-\delta, 0+\delta)$ or equivalently. $(-\delta, \delta)$, containing 0, such that whenever $-\delta < x < \delta$ and $x \neq 0$, the point (*x,1/x*) on the graph lies between the horizontal lines $y = L \pm \varepsilon$. However, since 1/*x* can be made as large as desired by choosing x close to 0, not every point (*x*,1/*x*) with nonzero *x*-coordinate in $(-\delta, \delta)$ has this property. Consequently our supposition is false that is the limit does not exist.

Example 4

If $f(x) = |x|/x$, show that $\lim_{x \to 0} f(x)$ does not exist

Solution

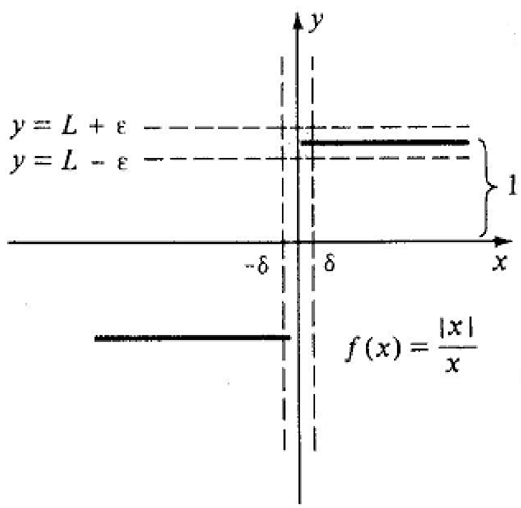


FIGURE 2.10

If *x*>0, then |*x*|/*x*=*x*/*x*=1 and hence, to the right of the *y*-axis, the graph of *f* coincides with the line *y*=1. If *x*<0, then $|x|/x = -x/x = -1$, which means that to the left of the *y*-axis the graph of *f* coincides with the line *y*=-1. if it were true that $\lim_{x \to 0} f(x) = L$ for some *L*, then the preceding remarks imply that $-1 < L < 1$. As shown in Figure 2.10, if we consider any pair of horizontal lines $y = L \pm \varepsilon$, where $0 < \varepsilon < 1$, then there exist points on the graph which are not between these lines for some nonzero *x* in *every* interval $(-\delta, \delta)$ containing 0. It follows that the limit does not exist.

2.3 Theorems on Limits

It would be an excruciating task to solve each problem on limits by means of Definition (2.3). The purpose of this section is to introduce theorems that may be used to simplify the process. To prove the theorems it is necessary to employ the definition of limit; however, once they are established it will be possible to determine many limits without referring to an ε or a δ. Several theorems are proved in this section; the remaining proofs may be found in Appendix II.

The simplest limit to consider involves the constant function defined by $f(x) = c$, where *c* is a real number. If *f* is represented geometrically by means of coordinate lines *l* and *l'*, then *every* arrow from *l* terminated at the same point on *l'*, namely, the point with coordinate *c*, as indicated in Figure 2.11.

FIGURE 2.11

It is easy to prove that for every real number *a*, $\lim_{x\to a} f(x) = c$. Thus, if $\varepsilon > 0$,

Consider the open interval $(c-\varepsilon, c+\varepsilon)$ on *l'* as illustrated in the figure. Since $f(x) = c$ is always in this interval, *any* number δ will satisfy the conditions of Definition (2.3); that is, for *every* $\delta > 0$,

If $\quad 0 < |x-a| < \delta$, then $\quad |f(x) - L| < \varepsilon$.

We have proved the following theorem.

Theorem (2.5)

If *a* and *c* are any real number, then $\lim_{x\to a} c = c$

Sometimes Theorem (2.5) is phrased "the limit of a constant is the constant." To illustrate,

$$\lim_{x\to a} c = c, \lim_{x\to 3} 8 = 8, \lim_{x\to 8} 3 = 3, \lim_{x\to \pi} \sqrt{2} = \sqrt{2} and \lim_{x\to a} 0 = 0$$

The next result tells us that in order to find the limit of a linear function as *x* approaches *a*, we merely substitute *a* for *x*.

Theorem (2.6)

If *a, b,* and *m* are real numbers, then $\lim_{x\to a}(mx+b) = ma+b$

Proof

If *m=0*, then *mx+b=b* and the statement of the theorem reduces to $\lim_{x\to a} b = b$, which was proved in Theorem (2.5).

Next suppose $m \neq 0$. If we let $f(x) = mx + b$ and $L = ma + b$, then according to Definition (2.3) we must show that for every $\delta > 0$ such that

If $0 < |x - a| < \delta$, then $|(mx+b) - (ma+b)| < \varepsilon$.

As in the solution of Example 2 in the previous section, a clue to the choice of δ can be found by examining the inequality involving ε. All inequalities in the following list are equivalent.

$$|(mx+b)-(ma+b)| < \varepsilon$$
$$|mx - ma| < \varepsilon$$
$$|m||x-a| < \varepsilon$$
$$|x-a| < \frac{\varepsilon}{|m|}$$

The last inequality suggests that we choose $\delta = \varepsilon / |m|$. Thus given any $\varepsilon > 0$,

If $0 < |x-a| < \delta$, where $\delta = \varepsilon / |m|$

Then the last inequality in the list is true, and hence, so is the first inequality, which is what we wished to prove.

As special cases of Theorem (2.6), we have

$$\lim_{x\to a} x = a$$
$$\lim_{x\to 4}(3x-5) = 3\cdot 4 - 5 = 7$$
$$\lim_{x\to\sqrt{2}}(13x+\sqrt{2}) = 13\sqrt{2} + \sqrt{2} = 14\sqrt{2}$$

The next theorem states that if a function *f* has a positive limit as *x* approaches *a*, then *(x)* is positive throughout some open interval containing *a*, with the possible exception of *a*.

Theorem (2.7)

If $\lim_{x\to a} f(x) = L$ and $L > 0$, then there exists an open interval $(a-\delta, a+\delta)$ containing *a* such that $f(x) > 0$ for all *x* in $(a-\delta, a+\delta)$, except possibly *x=a*.

Proof

If $\varepsilon = L/2$, then interval $(L-\varepsilon, L+\varepsilon)$, and hence $f(x) > 0$

In like manner, it can be shown that if *f* has a *negative* limit as *x* approaches *a*, then there is an open interval *I,* with the possible exception of *x=a*.

Many functions may be expressed as sums, differences, products, and quotients of other functions. In particular, suppose a function *s* is a sum of two functions *f* and *g*, so that $s(x) = f(x) + g(x)$ for every *x* in the domain of *s*. if *f(x)* and *g(x)* have limits *L* and *M*, respectively, as *x* approaches *a*, it is natural to conclude that *s(x)* has the limit *L+M* as *x* approaches *a*. the fact that this and analogous statements hold for products and quotients are consequences of the next theorem.

Theorem (2.8)

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$(i) \lim_{x\to a}[f(x) + g(x)] = L + M$$
$$(ii) \lim_{x\to a}[f(x) \cdot g(x)] = L \cdot M$$
$$(iii) \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}, M \neq 0$$
$$(iv) \lim_{x\to a}[cf(x)] = cL$$
$$(v) \lim_{x\to a}[f(x) - g(x)] = L - M$$

Although the conclusions of Theorem (2.8) appear to be intuitively evident, the proofs are rather technical and require some ingenuity. Proofs for (i)-(iii) may be found in Appendix II. Part (iv) of the theorem follows readily from part (ii) and Theorem (2.5) as follows:

$$\lim_{x\to a}[cf(x)] = \left[\lim_{x\to a} c\right]\left[\lim_{x\to a} f(x)\right] = cL$$

Finally, to prove (v) we may write

$$f(x) - g(x) = f(x) + (-1)g(x)$$

And then use (i) and (iv) (with c=-1)

The conclusions of Theorem (2.8) are often written as follows:

$$(i)\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$
$$(ii)\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$$
$$(iii)\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \lim_{x\to a} g(x) \neq 0$$
$$(iv)\lim_{x\to a}[cf(x)] = c\left[\lim_{x\to a} f(x)\right]$$
$$(v)\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

2.4 One-Sided Limits

If $\lim_{x\to a}\sqrt{x-2} = \sqrt{\lim_{x\to a} x-2} = \sqrt{a-2}$

The case a=2 is not covered by Definition (2.3) since there is no open interval containing 2 throughout which *f* is defined (note that $\sqrt{x-2}$ is not real if x<2). A natural way to extend the definition of limit to include this exceptional case to restrict *x* to value *greater* than 2. Thus, we replace the condition $2-\delta < x < 2+\delta$, which arises from Definition (2.3), by the condition $\delta < x < 2+\delta$. The corresponding limit is called *the limit of f(x) as x approaches 2 from the right,* or the *right-hand limit* of $\sqrt{x-2}$ as *x* approaches 2.

Definition (2.15)

Let *f* be a function that is defined on an open interval (*a,c*), and let *L* be a real number. The statement

$$\lim_{x \to a^+} f(x) = L$$

Means that for every $\varepsilon > 0$, there exists $\delta > 0$, such that

If $a < x < a + \delta$, then $| f(x) - L | > \varepsilon$.

If $\lim_{x \to a^+} f(x) = L$, we say that **the limit of f(x), as x approaches a from the right, is L.** we also refer to L as the **right-hand limit** of f(x) as x approaches a. the symbol $x \to a^+$ is used to indicate that values of x are always larger than a. Note that the only difference between Definitions (2.15) and (2/3) is that for *right*-hand limits we restrict *x* to the *right* half $(a, a+\delta)$ of the interval $(a-\delta, a+\delta)$. Definition (2.15) is illustrated geometrically in (i) of Figure 2.13. intuitively, we think of *f(x)* getting close to *L* as *x* gets close to *a*, through values *larger* than *a.*

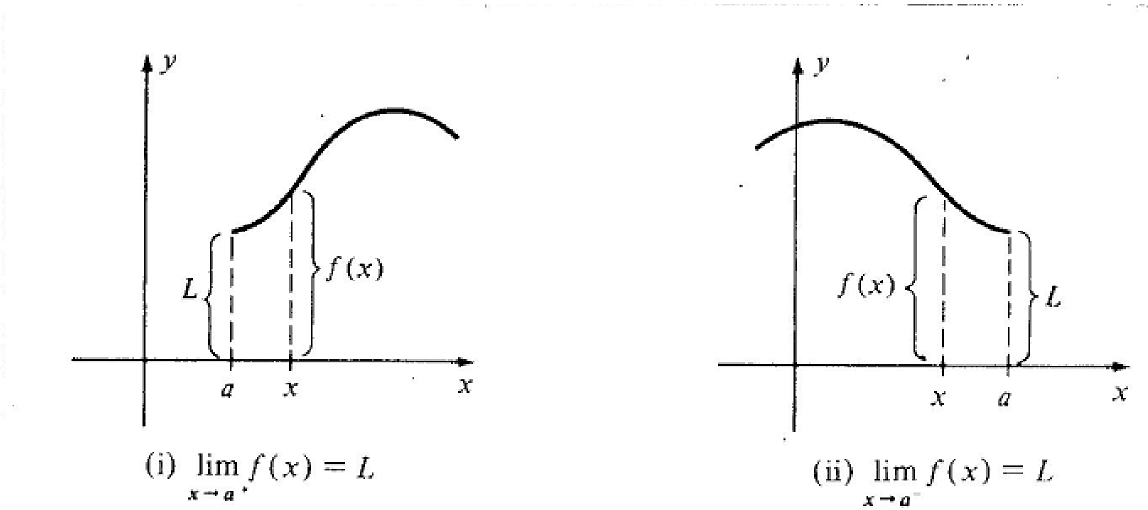


FIGURE 2.13

The notation of *left-hand limit* is defined in similar fashion. For example, if $f(x) = \sqrt{2-x}$, then we restrict *x* to values *less* than 2. The general definition follows.

Definition (2.16)

Let *f* be a function that is defined on an open interval (*c,a*), and let *L* be a real number. The statement

$$\lim_{x \to a^-} f(x) = L$$

Means that for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

If $a - \delta < x < a$, then $|f(x) - L| < \varepsilon$

If $\lim_{x \to a^-} f(x) = L$, we say that **the limit of f(x) as x approaches a from the left, is L;** or that L is the **left-hand limit** of *f(x)* as *x* approaches *a*. the symbol $x \to a^-$ is used to indicate that *x* is restricted to values *less* than *a*. a geometric illustration of Definition (2.16) is given in (ii) of Figure 2.13. Note that for the *left*-hand limit, *x* is the *left* half $(a - \delta, a)$ of the interval $(a - \delta, a + \delta)$.

Sometimes Definitions (2.15) and (2.16) are referred to as **one-sided limits** of *f(x)* as *x* approaches *a*. the relation between one-sided limits and limits is stated in the next theorem. The proof is left as an exercise.

Theorem (2.17)

If *f* is defined throughout an open interval containing *a*, except possibly at *a* itself, then $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

The preceding theorem tells us that the limit of *f(x)* as *x* approaches *a* exists if and only if both the right- and left-hand limits exist and are equal.

Theorems similar to the limit theorems of the previous section can be proved for one-sided limits. For example,

$$\lim_{x \to a^+} [f(x) + g(x)] = \lim_{x \to a^+} f(x) + \lim_{x \to a^+} g(x)$$

and

$$\lim_{x \to a^+} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a^+} f(x)}$$

With the usual restrictions on the existence of limits and *n*th roots. Analogous results are true for left-hand limits.

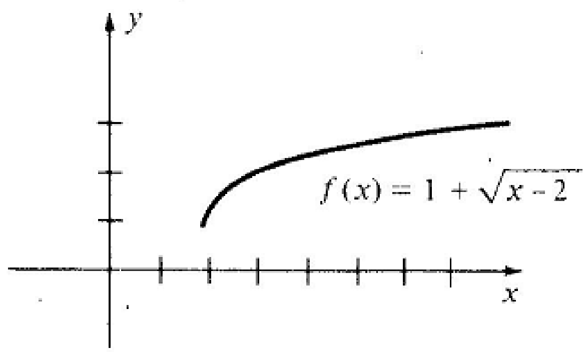


FIGURE 2.14

Example 1

Find $\lim_{x\to 2^+}(1+\sqrt{x-2}) = \lim_{x\to 2^+} 1 + \lim_{x\to 2^+}\sqrt{x-2}$

Solution

Using (one-sided)limit theorems,

$$\begin{aligned}\lim_{x\to 2^+}(1+\sqrt{x-2}) &= \lim_{x\to 2^+} 1 + \lim_{x\to 2^+}\sqrt{x-2}\\ &= 1+\sqrt{\lim_{x\to 2^+} x-2}\\ &= 1+0=1\end{aligned}$$

The graph of $f(x) = 1+\sqrt{x-2}$ is sketched in Figure 2.14. Note that there is no left-hand limit, since $\sqrt{x-2}$ is not real number if x<2.

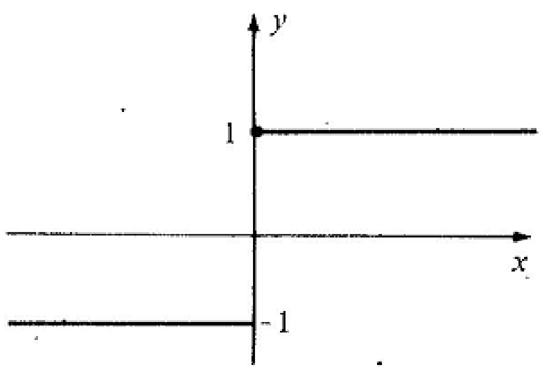


FIGURE 2.15

Example 2

Suppose $f(x) = |x|/x$ if $x \neq 0$ and $f(0) = 1$. Find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$. What is $\lim_{x\to 0} f(x)$?

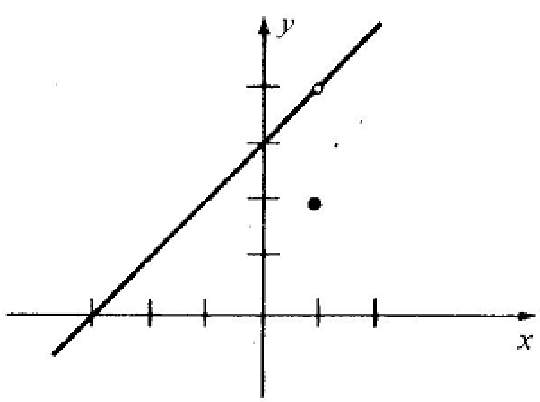


FIGURE 2.16

Solution

If x>0, then |x|=x and f(x) =x/x=1.

Consequently,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1$$

If x<0, then |x|=-x and f(x) =-x/x=-1, therefore,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-1) = -1$$

Since these right- and left-hand limits are unequal, it follows from Theorem (2.17) that $\lim_{x\to 0} f(x)$ does not exist. The graph of *f* is sketched in Figure 2.15.

Example 3

Suppose *f(x) =x+*3 if $x \neq 1$ and *f*(1)=2. Find $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, and $\lim_{x\to 1} f(x)$.

Solution

The graph of *f* consists of the point *P*(1,2) and all points on the line *y=x+*3 except the point with coordinates (1,4) as shown in figure 2.16.

1.1 Evidently, $\lim_{x\to 1^+} f(x) = 4 = \lim_{x\to 1^-} f(x)$. Hence by Theorem (2.17), $\lim_{x\to 1} f(x) = 4$.