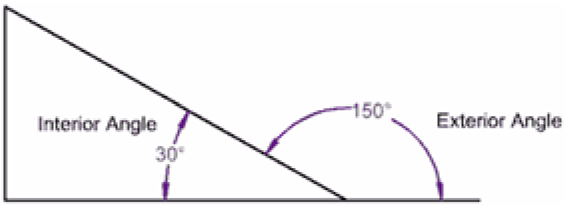
Notice:
Part of the notes are the same as before, more concepts on circles are added. The questions in class and homework are different from before. Please review the notes and do the questions in class and homework.

Geometry 3

1. Interior and Exterior Angle

An Interior Angle is an angle inside a shape.

The Exterior Angle is the angle between any side of a shape, and a line extended from the next side.



Note: If you add up the Interior Angle and Exterior Angle you get a straight line, 180°.

1) Interior Angles sum of Polygons

The sum of the measures of the interior angles of a polygon with *n* sides is (n-2)180.

For examples:

- Triangle or ('3 - gon')
    - sum of interior angles: (3-2) 180 = 180°
- Quadrilateral which has four sides (' 4 - gon')
    - sum of interior angles: (4-2)180 = 360°
- Hexagon which has six sides ('6 - gon')
    - sum of interior angles: (6-2)180 = 720°

An interior angle of a regular polygon with n sides is $\frac{(n-2)\times 180}{n}$.

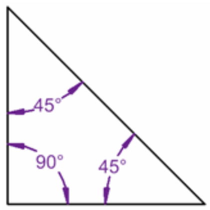
Example:

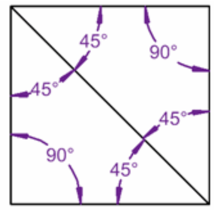
To find the measure of an interior angle of a regular octagon, which has 8 sides, apply the formula above as follows:
((8-2) × 180) /8 = 135°

There are Two Triangles in a Square

The internal angles in this triangle add up to 180°

(90°+45°+45°=180°)





... and for this square they add up to **360°**

... because the square can be made from two triangles!

2) Exterior Angles sum of Polygons

The Exterior Angle is the angle between any side of a shape, and a line extended from the next side.

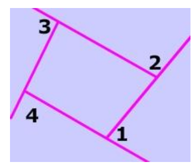
The sum of the measures of the exterior angles of a polygon, one at each vertex, is 360°.

For example:

A Polygon is any flat shape with straight sides.

The Exterior Angles of a Polygon add up to 360°.

$\angle 1+\angle 2+\angle 3+\angle 4=360^0$



2. Pythagorean Triples

A "Pythagorean Triple" is a set of positive integers, **a**, **b** and **c** that fits the rule:

$a^2 + b^2 = c^2$

1) Endless

The set of Pythagorean Triples is endless.

It is easy to prove this with the help of the first Pythagorean Triple, (3, 4, and 5):
Let n be any integer greater than 1, then 3n, 4n and 5n would also be a set of Pythagorean Triple.

This is true because: $(3n)^2 + (4n)^2 = (5n)^2$

Examples:

n	**(3n, 4n, 5n)**
2	(6,8,10)
3	(9,12,15)
...	... etc ...

So, you can make infinite triples just using the (3,4,5) triple.

2) Euclid's Proof that there are Infinitely Many Pythagorean Triples

However, Euclid used a different reasoning to prove the set of Pythagorean Triples is unending.

The proof was based on the fact that the difference of the squares of any two consecutive numbers is always an odd number.

Examples:

$2^2 - 1^2 = 4 - 1 = \mathbf{3}$ (an odd number),
$15^2 - 14^2 = 225\text{-}196 = \mathbf{29}$ (an odd number)

3) Properties

It can be observed that a Pythagorean Triple always consists of:

- all even numbers, or
- two odd numbers and an even number.

A Pythagorean Triple can never be made up of all odd numbers or two even numbers and one odd number. This is true because:

- The square of an odd number is an odd number and the square of an even number is an even number.
- The sum of two even numbers is an even number and the sum of an odd number and an even number is in odd number.

Therefore, if one of a and b is odd and the other is even, c would have to be odd. Similarly, if both a and b are even, c would be an even number too!

4) Constructing Pythagorean Triples

It is easy to construct sets of Pythagorean Triples.

When **m** and **n** are any two positive integers ($m < n$):
$a = n^2 - m^2$
$b = 2nm$
$c = n^2 + m^2$
Then, a, b, and c form a Pythagorean Triple.

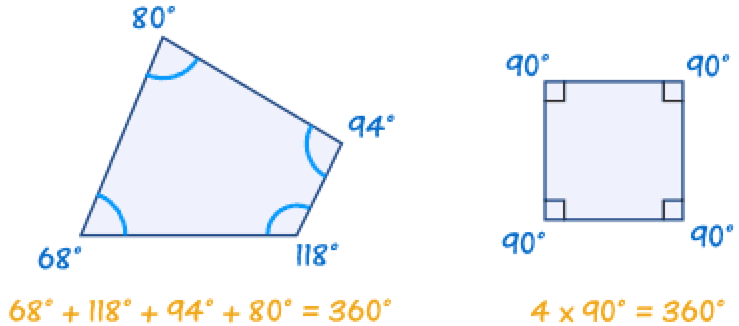
Example:
m=1 and n=2
$a = 2^2 - 1^2 = 4 - 1 = \mathbf{3}$
$b = 2 \times 2 \times 1 = \mathbf{4}$
$c = 2^2 + 1^2 = \mathbf{5}$
Thus, we obtain the first Pythagorean Triple (3,4,5).

Similarly, when m=2 and n=3 we get the next Pythagorean Triple (5,12,13).

3. Four sides (or edges)

1) Properties

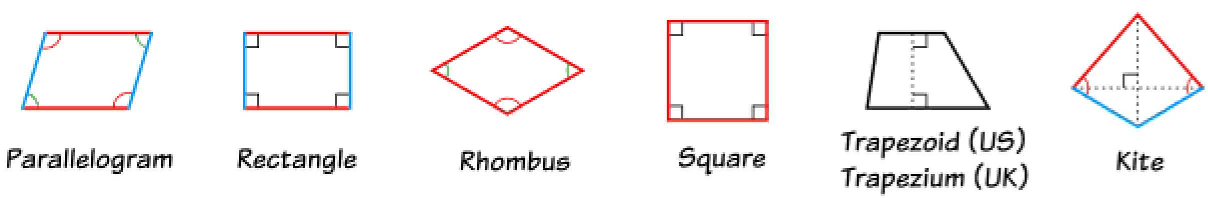
Four vertices (or corners). The interior angles add up to **360 degrees**:



Try drawing a quadrilateral, and measure the angles. They should add to **360°**

2) Types of Quadrilaterals

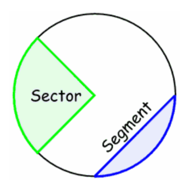
There are special types of quadrilateral:



Some types are also included in the definition of other types! For example a **square**, **rhombus** and **rectangle** are also ***parallelograms***.

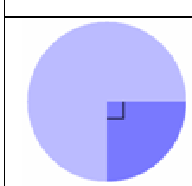
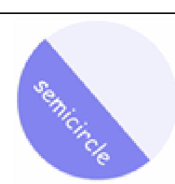
4. Circle Sector and Segment

Slices
There are two main "slices" of a circle:
The "pizza" slice is called a **Sector**.
And the slice made by a chord is called a **Segment**.



1) Common Sectors

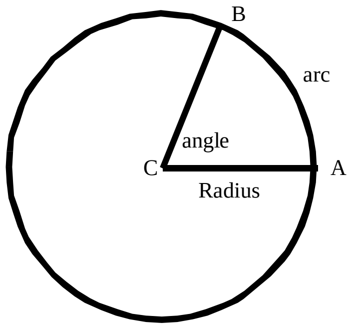
The Quadrant and Semicircle are two special types of Sector:

Quarter of a circle is called a **Quadrant**	Half a circle is called a **Semicircle.**
	

2) Area of a Sector

Radian Measure

- We can measure angles in several ways - one of which is degrees
- Another way to measure an angle is by means of radians
- One radian is defined as the measure of the angle subtended at the center of a circle by an arc equal in length to the radius of the circle

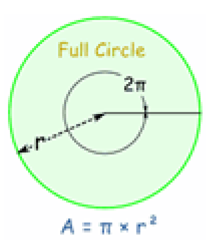


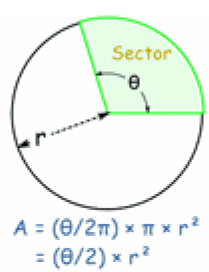
So in terms of radians, the formula is θ = arc length/radius

θ for one revolution = Circumference/r = $2\pi r/r = 2\pi$ radians

So then an angle of **360° = 2 π radians** or more easily, an angle of **180° = π radians**

You can work out the Area of a Sector by comparing its angle to the angle of a full circle.
Note: I am using radians for the angles.





This is the reasoning:

- A circle has an angle of 2π and an Area of: πr^2
- So a Sector with an angle of θ (instead of 2π) must have an area of: $(\theta/2\pi) \times \pi r^2$
- Which can be simplified to: $(\theta/2) \times r^2$

Area of Sector = $\frac{1}{2} \times \theta \times r^2$ ***(when θ is in radians)***

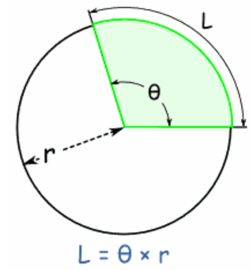
Area of Sector = $\frac{1}{2} \times (\theta \times \pi/180) \times r^2$ ***(when θ is in degrees)***

3) Arc Length of Sector or Segment

By the same reasoning, the arc length (of a Sector or Segment) is:

Arc Length "L" = $\theta \times r$ ***(when θ is in radians)***

Arc Length "L" = $(\theta \times \pi/180) \times r$ ***(when θ is in degrees)***



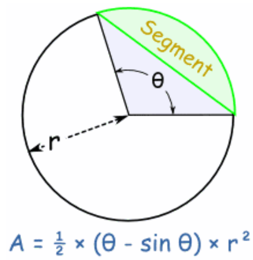
4) Area of Segment

The Area of a Segment is the area of a sector minus the triangular piece (shown in light blue here).

There is a lengthy derivation, but the result is a slight modification of the Sector formula:

Area of Segment = $\frac{1}{2} \times (\theta - \sin\theta) \times r^2$ ***(when θ is in radians)***

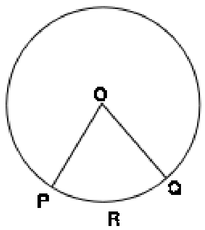
Area of Segment = $\frac{1}{2} \times ((\theta \times \pi/180) - \sin\theta) \times r^2$ ***(when θ is in degrees)***



5 Angles in a Circle

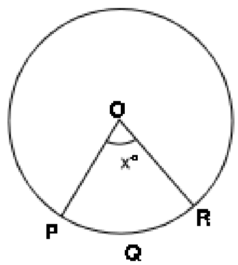
Central Angle - The angle made at the centre of a circle by the radii at the end points of an arc (or a chord) is called the **central angle** or angle subtended by an arc (or chord) at the centre.

In Figure, ∠POQ is the central angle made by arc PRQ.



The length of an arc is closely associated with the central angle subtended by the arc. Let us define the "degree measure" of an arc in terms of the central angle.

The degree measure of a minor arc of a circle is the measure of its corresponding central angle.
In Figure 19.2, Degree measure of PQR = x°



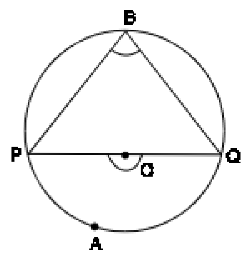
Relationship between length of an arc and its degree measure.

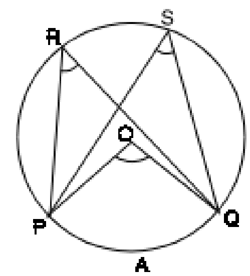
$$\text{Length of an arc} = \text{circumference} \times \frac{\text{degree measure of the arc}}{360^\circ}$$

Inscribed angle: The angle subtended by an arc (or chord) on any point on the remaining part of the circle is called an inscribed angle.

Theorem: The angle subtended at the centre of a circle by an arc is double the angle subtended by it on any point on the remaining part of the circle.

In figure, $\angle POQ = 2\angle PRQ$





Theorem: Angle in a semicircle is a right angle.

$\angle PBQ = 90^\circ$ since PQ is the diameter.

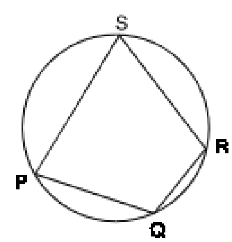
Theorem: Angles in the same segment of a circle are equal.

Given: A circle with centre O and the angles $\angle PRQ$ and $\angle PSQ$ in the same segment formed by the chord PQ (or arc PAQ)
To prove: $\angle PRQ = \angle PSQ$

Construction: Join OP and OQ.
Proof: As the angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle, therefore we have
$\angle POQ = 2\ \angle PRQ$ ...(i) and $\angle POQ = 2\angle PSQ$ ...(ii)
From (i) and (ii), we get $2\angle PRQ = 2\angle PSQ$
$\therefore\ \angle PRQ = \angle PSQ$
We take some examples using the above results

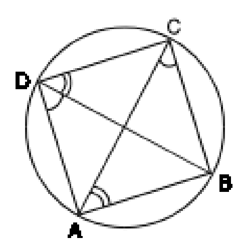
6. Cyclic Quadrilateral

A quadrilateral is said to be a **cyclic quadrilateral** if there is a circle passing through all its four vertices. For example, figure to the right shows a cyclic quadrilateral PQRS.



Theorem. Sum of the opposite angles of a cyclic quadrilateral is 180°.
Given: A cyclic quadrilateral ABCD.
To prove: $\angle BAD + \angle BCD = \angle ABC + \angle ADC = 180^\circ$

Construction: Draw AC and DB



Proof: $\angle ACB = \angle ADB$ and $\angle BAC = \angle BDC$
[Angles in the same segment]
$\therefore\ \angle ACB + \angle BAC = \angle ADB + \angle BDC = \angle ADC$
Adding $\angle ABC$ on both the sides, we get
$\angle ACB + \angle BAC + \angle ABC = \angle ADC + \angle ABC$
But $\angle ACB + \angle BAC + \angle ABC = 180^\circ$ [Sum of the angles of a triangle]
$\therefore\ \angle ADC + \angle ABC = 180^\circ$
$\therefore\ \angle BAD + \angle BCD = 360^\circ - (\angle ADC + \angle ABC) = 180^\circ$.
Hence proved.
Converse of this theorem is also true.

If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.

◆ **Things you should know:**

***SAS* Postulate**
If two triangles have two sides and the included angle of one triangle congruent respectively to two sides and the included angle of the other triangle, then the triangles are congruent.

ASA Postulate
If two triangles have two angles and the included of one triangle congruent respectively to two angles and the included side of the other triangle, then the triangle are congruent

SSS Postulate
If two triangles have three sides of one triangle congruent respectively to the three sides of the other triangle, then the triangles are congruent

HA Postulate
If two right triangles have the hypotenuse and an acute angle of one triangle congruent respectively to the hypotenuse and an acute angle of the other, then the triangles are congruent

SAA Corollary
If two triangles have a side and two angles of one congruent respectively to a side and two angles of the other, then the triangles are congruent.

Angle-Sum Theorem for Polygons
The formula for the sum, S, of the measures of the angles of a polygon of n sides is
$S = (n - 2)180$.
Exterior Angle-Sum Corollary for triangles
The measure of one exterior angle of a triangle equals the sum of the measures of its remote Interior angles

Theorem The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.

Theorem If two sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

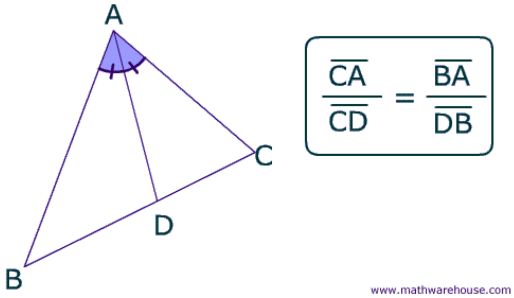
Theorem If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram

Theorem The diagonals of a rectangle are congruent.

Theorem The diagonals of a rhombus are perpendicular to each other

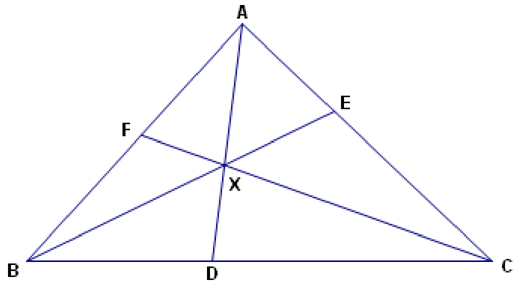
Theorem Each diagonal of a rhombus bisects a pair of opposite angles

Angle Bisector Theorem – An angle bisector of an angle of a triangle divides the opposite side in two segments that are proportional to the other two sides of the triangle.



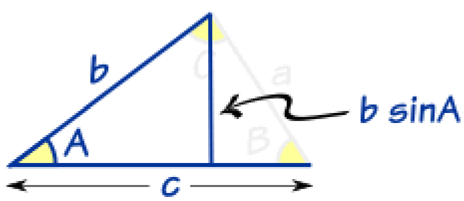
Ceva's Theorem Let ABC be a triangle, and let D, E, F be points on lines BC, CA, AB, respectively. Lines AD, BE, CF are concurrent (they intersect at a point) iff (if and only if)

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$



Area of Triangle

Area = ½ × base × height



In this triangle:

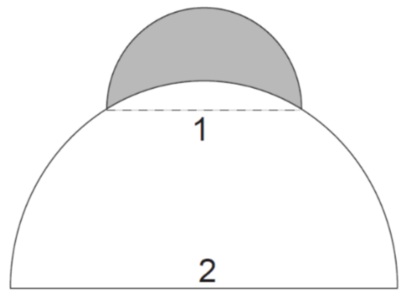
- the base is: **c**
- the height is: **b × sin A**

Putting that together gets us: Area = ½ × (c) × (b × sin A)

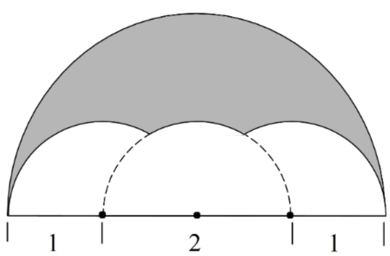
Which is (more simply): **Area = ½bc sin A**

In-class questions

1. A semicircle of diameter 1 sits at the top of a semicircle of diameter 2, as shown. The shaded area inside the smaller semicircle and outside the larger semicircle is called a lune. Determine the area of this lune.

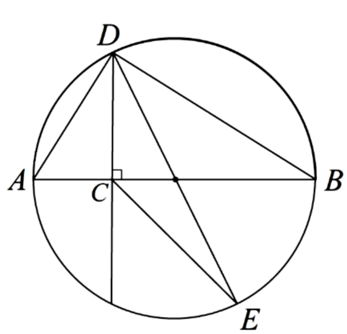


2. Three semicircles of radius 1 are constructed on diameter AB of a semicircle of radius 2. The centers of the small semicircles divide AB into four line segments of equal length, as shown. What is the area of the shaded region that lies within the large semicircle but outside the smaller semicircles?



3. In ΔABC we have $AB = 7,\ AC = 8,\ and\ BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects $\angle BAC$. What is the value of AD / CD?

4. Let AB be a diameter of a circle and C be a point on AB with $2 \cdot AC = BC$. Let D and E be points on the circle such that $DC \perp AB$ and DE is a second diameter. What is the ratio of the area of ΔDCE to the area of ΔABD?



5. A circle with center 0 has area 156π. Triangle *ABC* is equilateral, $\overline{BC}$ is a chord on the circle, $OA = 4\sqrt{3}$, and point O is outside ΔABC. *W*hat is the side length of ΔABC?

6. A circle of radius 2 is centered at *O*. Square *OABC* has side length 1. Sides *AB* and *CB* are extended past *B* to meet the circle at *D* and *E*, respectively. What is the area of the shaded region in the figure, which is bounded by *BD*, *BE*, and the minor arc connecting *D* and *E?*