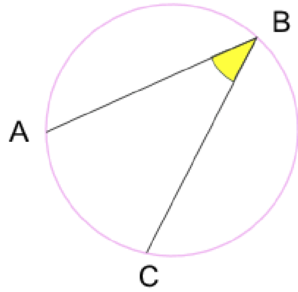
Geometry 3 - Circle 3

1. Inscribed Angle

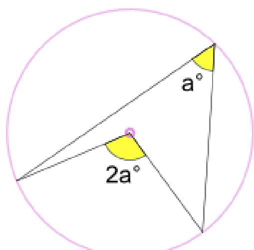
Inscribed Angle: an angle made from points sitting on the circle's circumference.



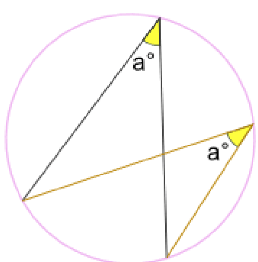
A and C are "end points", B is the "apex point".

2. Inscribed Angle Theorems

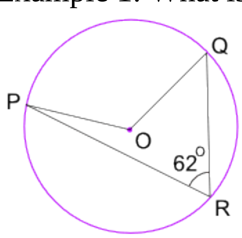
An inscribed angle a° is half of the central angle 2a° (Called the Angle at the Center Theorem).



The angle a° is always the same, no matter where it is on the circumference: Angle a° is the same.
(Called the Angles Subtended by Same Arc Theorem)



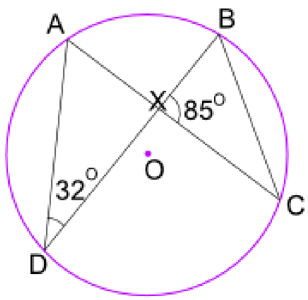
Example 1: What is the size of Angle POQ? (O is circle's center)



Solution:

Angle POQ = 2 × Angle PRQ = 2 × 62° = 124°

Example 2: What is the size of Angle CBX?

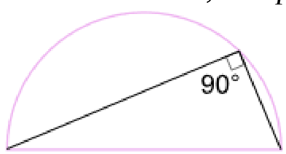


Soultion:
Angle ADB = 32° equals Angle ACB.
And Angle ACB equals Angle XCB.
So in triangle BXC we know Angle BXC = 85°, and Angle XCB = 32°
Now use angles of a triangle add to 180° :

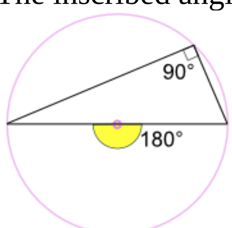
- Angle CBX + Angle BXC + Angle XCB = 180°
- Angle CBX + 85° + 32° = 180°
- Angle CBX = 63°

3. Angle in a Semicircle

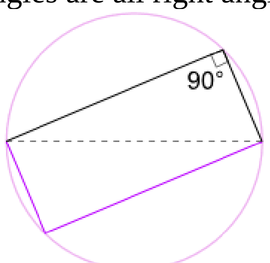
An angle inscribed in a semicircle is always a right angle. *(The end points are either end of a circle's diameter,the apex point can be anywhere on the circumference.)*

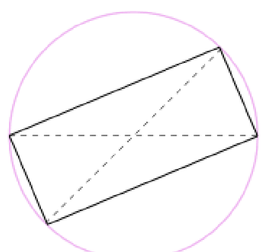


The inscribed angle 90° is half of the central angle 180°

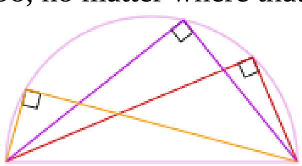


We could also rotate the shape around 180° to make a rectangle!
It is a rectangle, because all sides are parallel, and both diagonals are equal. And so its internal angles are all right angles (90°).

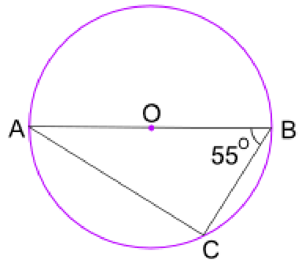




So, no matter where that angle is on the circumference, it is always 90°



Example 3: What is the size of Angle BAC?

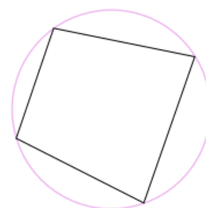


Solution:
The Angle in the Semicircle Theorem tells us that Angle ACB = 90°
Now use angles of a triangle add to 180° to find Angle BAC:

- Angle BAC + 55° + 90° = 180°
- Angle BAC = 35°

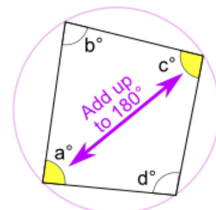
4. Cyclic Quadrilateral

A "Cyclic" Quadrilateral has every vertex on a circle's circumference.

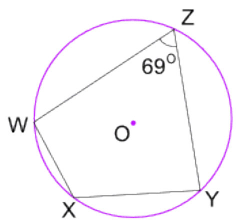


A Cyclic Quadrilateral's opposite angles add to 180°:

- a + c = 180°
- b + d = 180°



Example 4: What is the size of Angle WXY?



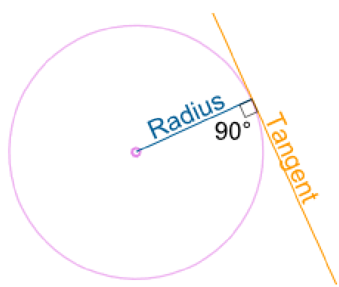
Solution:
Opposite angles of a cyclic quadrilateral add to 180°

- Angle WZY + Angle WXY = 180°
- 69° + Angle WXY = 180°
- Angle WXY = 111°

5. Tangent Angle

A tangent is a line that just touches a circle at one point.

It always forms a right angle with the circle's radius as shown here.



►Formulas for Working with Angles in Circles

1. Central Angle:

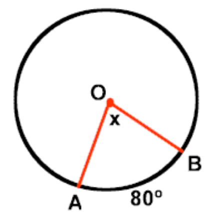
A central angle is an angle formed by two intersecting radii such that its vertex is at the center of the circle.

Central Angle = Intercepted Arc

$\angle AOB = AB$, $\angle AOB$ is a central angle.

Its *intercepted arc* is the minor arc from *A* to *B*.

$\angle AOB$ = 80°



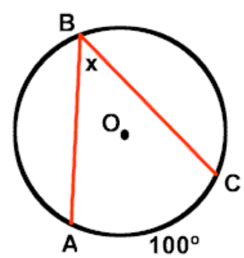
Theorem involving central angles:
In a circle, or congruent circles, congruent central angles have congruent arcs.

2. Inscribed Angle:

An inscribed angle is an angle with its vertex "on" the circle, formed by two intersecting chords.

Inscribed Angle = $\frac{1}{2}$ Intercepted Arc, $\angle ABC = \frac{1}{2}AC$

<*ABC* is an inscribed angle.Its *intercepted arc* is the minor arc from *A* to *C*. $\angle ABC = 50^0$

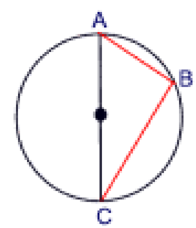


Special situations involving inscribed angles:
A quadrilateral inscribed in a circle is called a cyclic quadrilateral.

An angle inscribed in a semi-circle is a right angle.

$$\angle ABC = \frac{1}{2}(AC) = \frac{1}{2}(180^0) = 90^0$$

In a circle, inscribed circles that intercept the same arc are congruent.

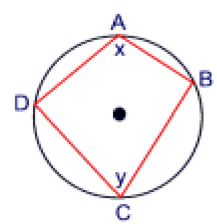


The opposite angles in a cyclic quadrilateral are supplementary.

$$\angle x = \frac{1}{2}(DCB); \ \angle y = \frac{1}{2}(DAB)$$

$$\angle x + \angle y = \frac{1}{2}(DCB + DAB)$$

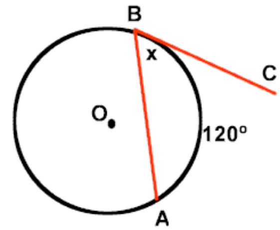
$$\angle x + \angle y = \frac{1}{2}(360^0) = 180^0$$



3. Tangent Chord Angle:

An angle formed by an intersecting tangent and chord has its vertex "on" the circle.

Inscribed Angle = $\frac{1}{2}$ Intercepted Arc; $\angle ABC = \frac{1}{2} AB$



$\angle ABC$ is an angle formed by a tangent and chord.

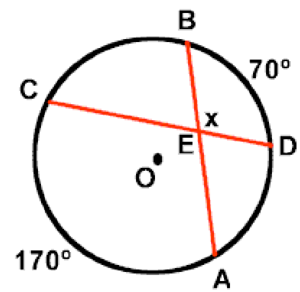
Its *intercepted arc* is the minor arc from *A* to *B*. $\angle ABC = 60^0$

4. Angle Formed Inside of a Circle by Two Intersecting Chords

When two chords intersect "inside" a circle, four angles are formed. At the point of intersection, two sets of vertical angles can be seen in the corners of the X that is formed on the picture. Remember: vertical angles are equal.

Angle Formed Inside by Two Chords = $\frac{1}{2}$ Sum of Intercepted Arcs

$\angle BED = \frac{1}{2}(AB + BD)$. $\angle BED$ is formed by two intersecting chords.



Its *intercepted arcs* are BD and CA.

[Note: the intercepted arcs belong to the set of vertical angles.]

$\angle BED = \frac{1}{2}(70 + 170) = \frac{1}{2}(240) = 120^0$

also, $\angle CEA = 120^0$ (vertical angle).

<*BEC* and <*DEA* = 60° by straight line.

5. Angle Formed Outside of a Circle by the Intersection

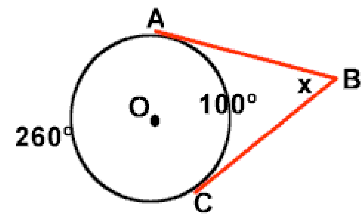
Angle Formed Outside of a Circle by the Intersection of "Two Tangents" or "Two Secants" or "a Tangent and a Secant".

The formulas for all THREE of these situations are the same:

Angle Formed Outside = $\frac{1}{2}$ Difference of Intercepted Arcs (When subtracting, start with the larger arc.)

1) Two Tangents:

<*ABC* is formed by two tangents intersecting outside of circle *O*.



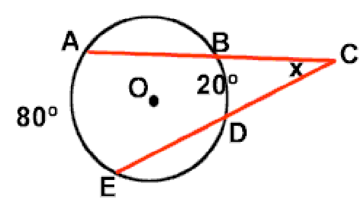
The *intercepted arcs* are minor arc AC and major arc AC. These two arcs together comprise the entire circle.

$$\angle ABC = \frac{1}{2}\left(\overset{Major}{AC} - \overset{Minor}{AC}\right); \quad \angle ABC = \frac{1}{2}(260-100) = 80^0$$

Special situation for this set up
It can be proven that $\angle ABC$ and central $\angle AOC$ are supplementary. Thus the angle formed by the two tangents and its first intercepted arc also add to 180^0.

2) Two Secants:

$\angle ACE$ is formed by two secants intersecting outside of circle *O*.



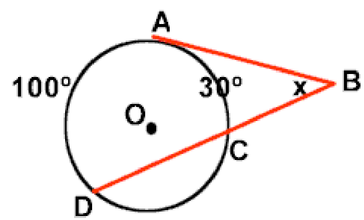
The *intercepted arcs* are minor arcs BD and AE.

$$\angle ACE = \frac{1}{2}(AE - BD); \quad \angle ACE = \frac{1}{2}(80-20) = 30^0$$

3) a Tangent and a Secant:

<*ABD* is formed by a tangent and a secant intersecting outside of circle *O*.

The *intercepted arcs* are minor arcs AC and AD.



$$\angle ABD = \frac{1}{2}(AD - AC); \quad \angle ABD = \frac{1}{2}(100-30) = 35^0$$

►Circle Theorems

Theorem 1.
The area of a circle of radius r is πr^2 and its circumference is $2\pi r$.

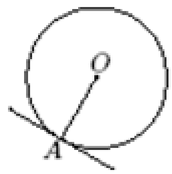
Theorem 2.
A line from the centre of a circle perpendicular to a chord bisects the chord and its arc.

Theorem 3.
If AB is a semicircular arc of a circle and C is any point on the circumference of the circle then ∠ACB is a right angle.

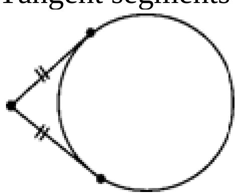
Theorem 4.

If A and B are points on the circumference of a circle with centre O and C is an exterior point of the circle such that BC is a tangent to the circle then
$\angle ABC = \frac{1}{2}\angle AOB$.

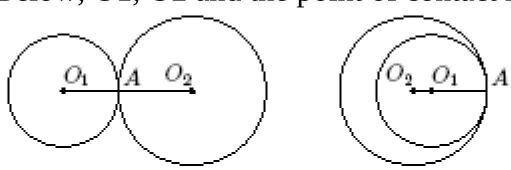
Theorem 5.
A line meeting a circle at a point A is tangent to the circle if and only if the radius to the point of contact with the line at A is perpendicular to the line.



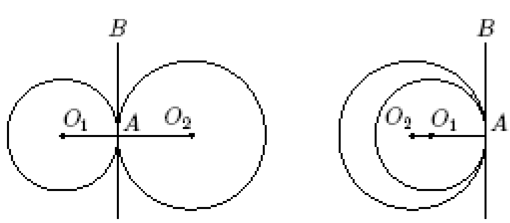
Theorem 6.
Tangent segments from an external point are equal.



Theorem 7.
If two circles touch at a single point then this point and the centres of the circles are collinear. Below, O1, O2 and the point of contact A of the circles are collinear.



Theorem 8.
If two circles are tangential at a point A then the tangent to one of the circles at A is also a tangent to the other circle.



Theorem 9.
There is a unique circle through any triple of non-collinear points.

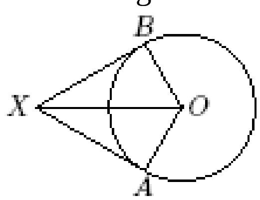
Proof. Let the three non-collinear points be A, B, C. The points are distinct since two distinct points are sufficient to define a line, so that if any points are coincident then the points would be collinear, contradicting their non-collinearity. Form the perpendicular bisector of each of AB and

BC. (Recall that the perpendicular bisector of two points is the locus of points that are equidistant from two given points.) These bisectors are non-parallel, since A, B, C are non-collinear. Hence the bisectors intersect. Let the point of intersection be O. Then OA = OB since O lies on the bisector of AB. Similarly, OB = OC since O lies on the bisector of BC.

Thus OA = OB = OC so that O is equidistant from A, B and C. Hence we A, B and C lie on a circle with centre O and radius OA. Since O is the unique intersection of the perpendicular bisectors of AB and BC the circle through A, B and C is unique.

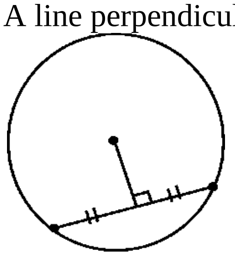
Theorem 10.
The two tangents drawn to a circle from an exterior point of the circle have the same length. In the diagram, XA = XB.

Moreover, the line joining the centre of the circle and the exterior point bisects the angle between the two tangents. In the diagram, OX bisects ∠AXB.

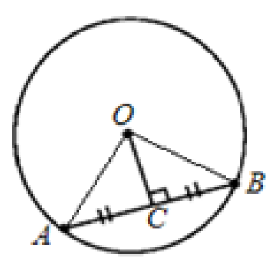


Proof. OA = OB, (radii of same circle)
90^{o} = ∠OAX = ∠OBX, by Theorem 23
OX is common
∴ ΔOAX ≅ ΔOBX, by the RHS Rule
∴ XA = XB
and ∠OXA = ∠OXB, i.e. OX bisects ∠AXB.

Theorem 11.
A line perpendicular to a chord through the center of the circle bisects the chord.



Proof:
Join OA and OB.
In right triangle OAC and OBC,
OC = OC; common side
OA = OB; radius are always equal in some circle
∠OCA=∠OCB=90^{o}
given by the question that OC⊥AB
∴ΔOAC≌ΔOBC, then AC=BC



Theorem 12.
If AB is an arc of a circle then angles subtended at the circumference opposite AB are equal and are equal to half the angle subtended at the centre, i.e. in the diagram.

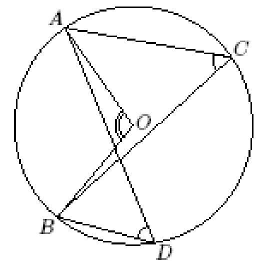
$\angle ACB = \angle ADB = \frac{1}{2}\angle AOB$.

Proof.
Construct OC, forming isosceles triangles AOC and BOC. Let the equal base angles of ΔAOC be x and the equal base angles of ΔBOC be y. Then

$\angle ACB = x + y$

$\angle AOB = 360^{o} - (180^{o} - 2x) - (180^{o} - 2y) = 2x + 2y = 2\angle ACB$

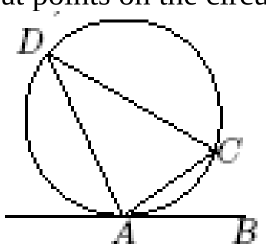
$\therefore \angle ACB = \frac{1}{2}\angle AOB$



Suppose that C is in fact at D, and x and y are defined as before. Then $\angle ADB = y - x$ and $\angle AOB = 180^{o} - 2x - (180^{o} - 2y) = 2(y - x) = 2\angle ACB$ with the same conclusion as before.

Theorem 13. (Tangent-chord Theorem or Alternate Segment Theorem).
Let AC be a chord in a circle and let AB be a line meeting the circle at A.

Then AB is tangent to the circle if and only if $\angle CAB$ is equal to the angle subtended by the chord at points on the circumference on the opposite of AC from B.



Proof. Draw a diameter from A through the centre O (to X) and let $x = \angle XAC$.

Then $\angle XAB = 90^{o}$, by Theorem 23, OA ? AB

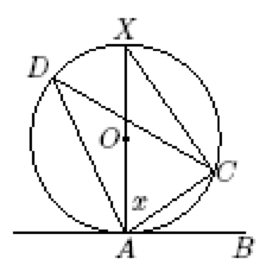
$\therefore \angle CAB = \angle XAB - \angle XAC = 90^{o} - x$

$\angle ACX = 90^{o}$, by Theorem 20, since AX is a diameter

$\therefore \angle AXC = 180^{o} - \angle ACX - \angle XAC = 90^{o} - x = \angle CAB$

$\angle ADC = \angle AXC$, by Theorem 19, common arc: AC

$\therefore \angle ADC = \angle CAB$:

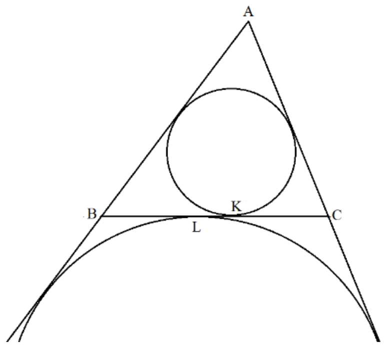


Remark:
In the diagram of the proof above, if we think of X moving around the circumference of the circle, we always have $\angle AXC = \angle ADC$, by Theorem 19. As X moves around toward A, $\angle BAC$

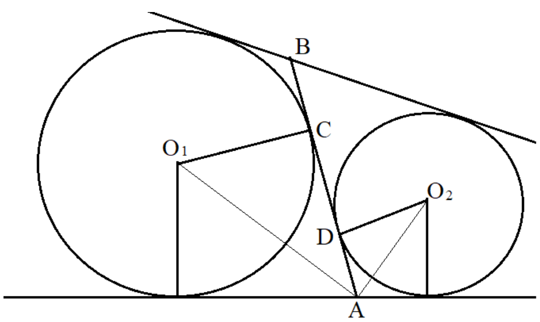
can be thought of as the limit of ∠AXC as X ⇒A, since the chord XA (extended) becomes the tangent AB when X and A become the one point.

In-class questions

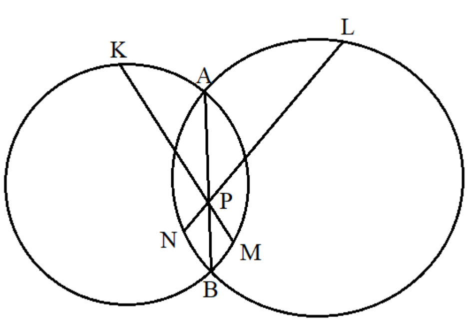
1. The inscribed circle of triangle ABC is tangent to side BC at point K and an escribed circle is tangent at point L. Prove that $CK = BL = \frac{1}{2}(a+b-c)$, where a, b, c are the lengths of the triangle's sides.



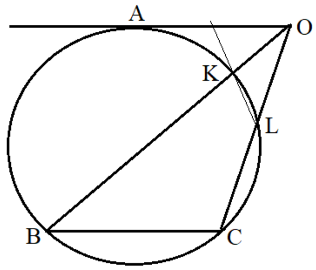
2. The common inner tangent to circles whose radii are R and r intersects their common outer tangents at points A and B and is tangent to one of the circles at point C. Prove that $AC \cdot CB = Rr$.



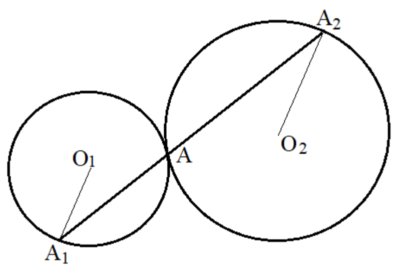
3. Through a point P lying on the common chord AB of two intersecting circles chord KM of the first circle and chord LN of the second circle are drawn. Prove that quadrilateral KLMN is an inscribed one.



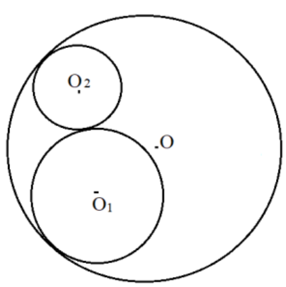
4. Line OA is tangent to a circle at point A and chord BC is parallel to OA. Lines OB and OC intersect the circle for the second time at points K and L, respectively. Prove that line KL divides segment OA in halves.



5. Two circles S_1 *and* S_2 centered at O_1 *and* O_2 are tangent to each other at point A. A line that intersects S_1 at point A_1 and S_2 at point A_2 is drawn through point A. Prove that



6. Two tangent circles centered at O_1 *and* O_2, respectively, are tangent from the inside to the circle of radius R centered at O. Find the perimeter of triangle OO_1O_2.



7. BC is the diameter of semicircle with the center O. CE tangent the semicircle at C. Secant BE intersects the semicircle at D. Point A is on arc BD such that $AD = DC$. If $EC = 3$ *and* $BD = 2.5$, find: 1) $\tan\angle DCE$; 2) the length of AB.