AP Calculus Class 3

Homework, 1.

10. b) $\lim\limits_{x\to-\infty}(x+\sqrt{x^2+2x})\cdot\dfrac{x-\sqrt{x^2+2x}}{x-\sqrt{x^2+2x}}$

$= \lim\limits_{x\to-\infty}\dfrac{x^2-(x^2+2x)}{x-\sqrt{x^2+2x}} = \lim\limits_{x\to-\infty}\dfrac{-2x}{x-\sqrt{x^2+2x}}\cdot\dfrac{\frac{1}{x}}{\frac{1}{x}}$

$= \lim\limits_{x\to-\infty}\dfrac{-2\cancel{\frac{x}{x}}}{\frac{x}{x}-\frac{\sqrt{x^2+2x}}{\sqrt{x^2}}} = \lim\limits_{x\to-\infty}\dfrac{-2}{1-\left(-\sqrt{\frac{x^2+2x}{x^2}}\right)}$

$= \dfrac{-2}{1+\sqrt{1+0}} = \dfrac{-2}{1+1} = -1$

Homework 2.

1. a) $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}\cdot\dfrac{\sqrt{x+1}}{\sqrt{x+1}}$

$= \lim\limits_{x\to+\infty}\dfrac{\sqrt{(x+\sqrt{x+\sqrt{x}})\cdot(x+1)}}{x+1} = \lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2+x\sqrt{x+\sqrt{x}}+x+\sqrt{x+\sqrt{x}}}}{x+1}$

$= \lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2+x\sqrt{x+\sqrt{x}}+x+\sqrt{x+\sqrt{x}}}}{x+1}\cdot\dfrac{\frac{1}{x}}{\frac{1}{x}}$

$$= \lim_{x \to +\infty} \frac{\sqrt{\frac{x^2}{x^2} + \frac{x}{x^2}\sqrt{x+\sqrt{x}} + \frac{x}{x^2} + \frac{\sqrt{x+\sqrt{x}}}{x^2}}}{1 + \frac{1}{x}}$$

$$= \lim_{x \to +\infty} \frac{\sqrt{1 + \frac{\sqrt{x+\sqrt{x}}}{x} + \frac{1}{x} + \frac{\sqrt{x+\sqrt{x}}}{x^2}}}{1 + \frac{1}{x}}$$

$$\frac{\sqrt{x+\sqrt{x}}}{x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \frac{\sqrt{1+0+0+0}}{1+0} = 1$$

$$= \frac{\sqrt{\frac{1}{x} + \frac{1}{\sqrt{x}}}}{1}$$

$$= \sqrt{\frac{1}{x} + \frac{1}{\sqrt{x}}}$$

1\. b) $\lim_{x \to +\infty} \sqrt{x + \sqrt{x+\sqrt{x}}} - \sqrt{x}$

$\lim_{x \to \infty} x^2 - x$

$$= \lim_{x \to +\infty} \left(\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}\right) \cdot \frac{\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x}} \qquad \left(\frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}}\right)$$

$$= \frac{1}{2}.$$

When solving complicated limits, try the following:

- Rationalize the denominator.
- simplifying by multiplying by 1 in the form of 1

① $\frac{a \pm b}{a \pm b}$ ② $\frac{\frac{1}{x}}{\frac{1}{x}}$

6. a) $\lim_{x \to 4} \dfrac{5+\sqrt{x}}{\sqrt{5+x}} = \dfrac{5+\sqrt{4}}{\sqrt{5+4}} = \dfrac{5+2}{\sqrt{9}} = \dfrac{7}{3}$

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8. a) $f(x) = \dfrac{x^4 - 1}{x-1} = \dfrac{(x^2+1)(x+1)(x-1)}{x-1}$

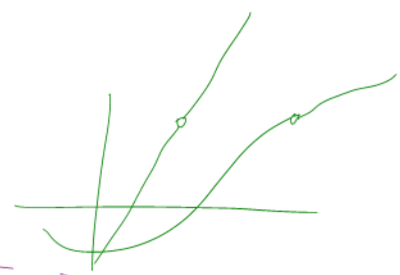
$\lim_{x \to 1} \dfrac{(x^2+1)(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x \to 1} (x^2+1)(x+1)$

$= (1+1)(1+1) = 4$

since the limit exists at $x=1$,

$\Rightarrow$ $x=1$ is a removable discontinuity.

$\Rightarrow g(x) = \begin{cases} f(x) & \text{if } x \neq 1 \\ 4 & \text{if } x = 1 \end{cases}$



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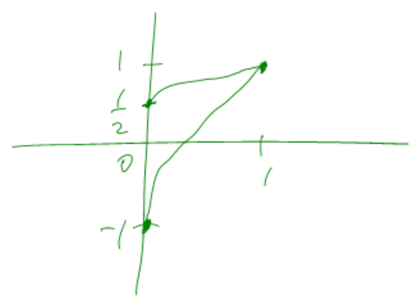
9. $\sqrt[3]{x} = 1 - x$, $(0,1)$



$f(x) = \sqrt[3]{x} - 1 + x = 0.$

$f(0) = -1$ and $f(1) = 1$

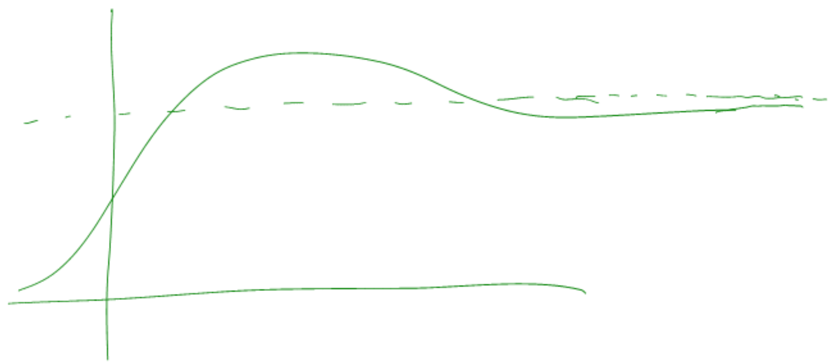
since $-1 < 0 < 1$, there is a number $c \in (0,1)$

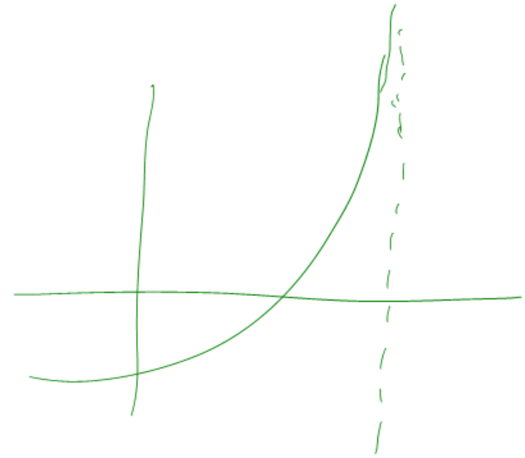
s.t. $f(c) = 0$ by the IVT.

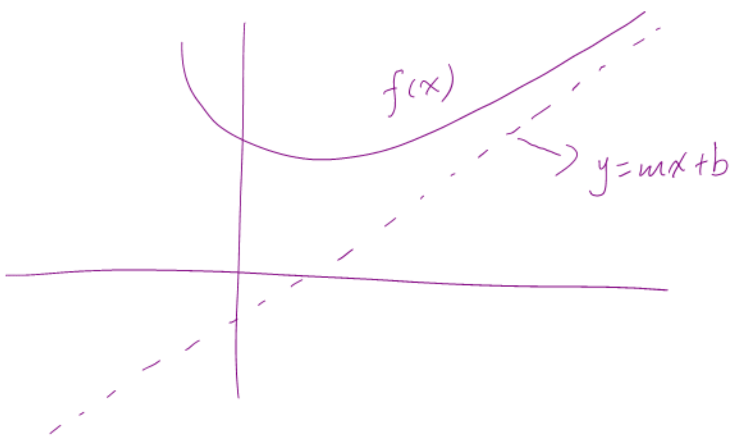
$\Rightarrow$ There is a root of the equ^n $\sqrt[3]{x} = 1 - x$

in the interval $(0,1)$.

Slant/oblique asymptote.







$$\lim_{x\to\infty}[f(x)-(mx+b)]=0$$

Homework 2.

2. a) $y=\dfrac{x^2+1}{x+1} = x-1+\dfrac{2}{x+1}$

$$\frac{7}{3}=2+\frac{1}{3}$$

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2+0x+1 \\ & x^2+x \\ \hline & -x+1 \\ & -x-1 \\ \hline & 2 \end{array}$$

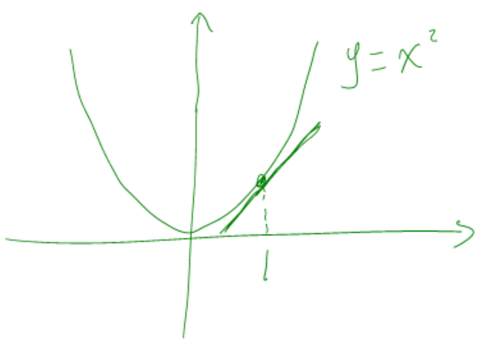
$$\Rightarrow \quad y = f(x) = x - 1 + \frac{2}{x+1}$$

$$\Rightarrow \quad f(x) - (x-1) = \frac{2}{x+1}$$

$$\lim_{x\to\infty} f(x) - (x-1) = \lim_{x\to\infty} \frac{2}{x+1} = 0.$$

$\Rightarrow$ $y = x - 1$ is the slant asymptote.

Derivative at a point $x = 1$.



$f'(1) = 2$

$f'(x) = 2x$

Derivatives of Functions

Derivative of a constant function

$$\frac{d}{dx}(c) = 0$$

$\frac{d}{dx}(5) = 0$ $\frac{d}{dx}(\pi) = 0.$

Derivative of a power function

If n is a real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$f(x) = x^5$ $f'(x) = \frac{d}{dx} f(x) = 5x^{5-1} = 5x^4$

$y = t^{88}$ $\frac{dy}{dt} = 88t^{88-1} = 88t^{87}$

Differentiate

a) $f(x) = \frac{1}{x^2}$

$f(x) = x^{-2}$

$f'(x) = -2x^{-2-1}$

$= -2x^{-3}$

b) $y = \sqrt[3]{x}$

$= x^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3} x^{\frac{1}{3}-1} = \frac{1}{3} x^{-\frac{2}{3}}$

The Constant Multiple Rule

If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

$\frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4) = 3(4x^3) = 12x^3$

$\frac{d}{dx}(-x) = -1\frac{d}{dx}x = -1 \cdot 1(x^{1-1}) = -1$

The Sum Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Difference Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Find $f'(x)$ if $f(x) = x^4 - 5x^5 + 9x^2 + 6$.

$f'(x) = \frac{d}{dx}(x^4 - 5x^5 + 9x^2 + 6)$

$= 4x^3 - 25x^4 + 18x$

Example: Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

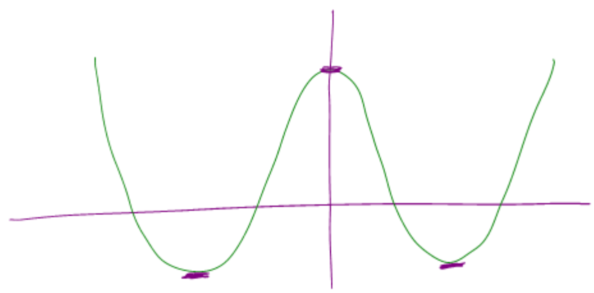
Horizontal tangent $\Rightarrow$ slope is 0. $\Rightarrow$ derivative is 0.

$y = x^4 - 6x^2 + 4$ $\quad$ $y' = 4x^3 - 12x$

let $y' = 0$ $\quad$ $y' = 4x(x^2 - 3) = 0.$ $\Rightarrow$ $x = 0$ or $x = \pm\sqrt{3}$

$f(0) = 4$ $\quad$ $f(-\sqrt{3}) = -5$ $\quad$ $f(\sqrt{3}) = -5.$

$(0,4), (-\sqrt{3}, -5), (\sqrt{3}, -5)$



Definition of the Number e

e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

$f(x) = e^x - x$ $\quad$ $f'(x) = e^x - 1.$

The Product and Quotient Rules

The Product Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

$$(fg)' = fg' + gf'$$

Example: $f(x) = \underbrace{x}_{f}\,\underbrace{e^x}_{g}$, find $f'(x)$.

$$f'(x) = x(e^x)' + e^x(x')$$

$$f'(x) = xe^x + e^x = e^x(x+1)$$

Example: $f(t) = \sqrt{t}\,(1-t)$

$$f'(t) = (\sqrt{t}\,(1-t))' = \sqrt{t}\,(1-t)' + (1-t)(\sqrt{t})'$$

$$= \sqrt{t}\,(-1) + (1-t)(t^{\frac{1}{2}})' = -\sqrt{t} + (1-t)\left(\tfrac{1}{2}t^{-\frac{1}{2}}\right)$$

$$= -t^{\frac{1}{2}} + \frac{1-t}{2t^{\frac{1}{2}}} = \frac{1-3t}{2\sqrt{t}}$$

The Quotient Rule

If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Example: $$y = \frac{x^2 + x - 2 \;\to f}{x^3 + 6 \;\to g}, \quad \text{Find } y'.$$

$$f' = 2x - 1 \qquad\qquad g' = 3x^2$$

$$y' = \frac{gf' - fg'}{g^2} = \frac{(x^3+6)(2x-1) - (x^2+x+2)(3x^2)}{(x^3+6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3+6)^2}$$

$$f(x) = \frac{3x^2 + 2\sqrt{x}}{x} = 3x + 2x^{-\frac{1}{2}}$$

Table of Differentiation Formulae

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f \pm g)' = f' \pm g'$$

$$(fg)' = gf' + fg' \qquad \left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$$

Derivative of Trigonometric Functions

$f(x) = \sin x.$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$\sin(a+b) = \sin a \cos b + \cos a \sin b.$

$\cosh.$

$$f'(x) = \lim_{h \to 0} \frac{\sin x \cos(h) + \cos x \sin(h) - \sin x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin x \cos(h) - \sin x}{h} + \frac{\cos x \sin(h)}{h} \right].$$

$$= \lim_{h \to 0} \left[\frac{\sin x (\cos(h) - 1)}{h} \right] + \lim_{h \to 0} \frac{\cos x \sin(h)}{h}$$

$$= \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= \sin x \cdot 0 + \cos x \cdot 1$$

$$= \cos x$$

$$\Rightarrow \frac{d}{dx} \sin x = \cos x$$

$$\Rightarrow \frac{d}{dx} \cos x = -\sin x$$

$f(x) = \tan x.$

$$f'(x) = \left(\frac{\overset{f}{\sin x}}{\underset{g}{\cos x}}\right)' = \frac{\cos x (\sin x)' - \sin x (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \csc x = -\csc x \cot x$

$\frac{d}{dx} \sec x = \sec x \tan x$

$\frac{d}{dt} \cot x = -\csc^2 x.$