AP Calculus Class 14

Homework 13.

5. $\int_0^\infty x^2 e^{-x^3} = \lim_{t\to\infty} \int_0^t x^2 e^{-x^3}\,dx.$

let $u = -x^3$

$du = -3x^2\,dx$

$-\frac{1}{3}\,du = x^2\,dx$

$= \lim_{t\to\infty} \int_0^{-t^3} -\frac{1}{3} e^u\,du$

$= \lim_{t\to\infty} -\frac{1}{3} e^{-x^3}\Big|_0^t$

$= \lim_{t\to\infty} \left[-\frac{1}{3} e^{-t^3} - (-\frac{1}{3} e^{-0})\right]$

$= \frac{1}{3}$

$\boxed{C}$

3. $\int_2^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x^2}\,dx$

$= \lim_{t\to\infty} \left[-\frac{1}{x}\right]_2^t$

$= \lim_{t\to\infty} \left[\frac{1}{2} - \frac{1}{t}\right] = \frac{1}{2}$

$\boxed{A}$

4. $\int_1^\infty \frac{x}{(1+x^2)^2}\,dx = \lim_{t\to\infty} \int_0^t x(1+x^2)^{-2}\,dx$

$= \lim_{t\to\infty} -\frac{1}{2}(1+x^2)^{-1}\Big]_1^t$

$= \frac{1}{4}$

$\boxed{C}$

1. e) $\int \frac{1}{x^3-1}\,dx$

$x^3-1^3=(x-1)(x^2+x+1)$ $\quad$ $a^3+b^3=(a+b)(a^2-ab+b^2)$

$$\Rightarrow \frac{1}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

$$\Rightarrow A(x^2+x+1)+(Bx+C)(x-1)=1$$

$$\Rightarrow (A+B)x^2+(A-B+C)x+(A-C)=1$$

$$\Rightarrow A+B=0 \qquad A-B+C=0 \qquad A=C+1$$

$$\Rightarrow A=\frac{1}{3} \qquad B=-\frac{1}{3} \qquad C=-\frac{2}{3}$$

$$\Rightarrow \frac{1}{x^3-1}=\frac{\frac{1}{3}}{x-1}+\frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}=\frac{1}{3(x-1)}-\frac{1}{3}\,\frac{x+2}{x^2+x+1}$$

$$\int\frac{1}{x^3-1}\,dx=\underbrace{\frac{1}{3}\int\frac{1}{x-1}\,dx}_{I_1}-\underbrace{\frac{1}{3}\int\frac{x+2}{x^2+x+1}\,dx}_{I_2}$$

$I_1 \Rightarrow$ let $u=x-1$

$$I_1=\frac{1}{3}\int\frac{1}{u}\,du=\frac{1}{3}\ln u+C=\frac{1}{3}\ln|x-1|+C$$

$$I_2=\int\frac{x+2}{x^2+x+1}\,dx$$

We will split the integrand into two parts.

let $u = x^2 + x + 1$ $\quad du = 2x+1\, dx$ $\quad \frac{1}{2} du = x + \frac{1}{2} dx$

$$\Rightarrow x + 2 = x + \frac{1}{2} + \frac{3}{2}$$

$$I_2 = \int \frac{x+\frac{1}{2}}{x^2+x+1} dx + \int \frac{\frac{3}{2}}{x^2+x+1} dx$$

$$\downarrow \qquad\qquad \downarrow$$

$$I_{2a} \qquad\qquad I_{2b}$$

$$I_{2a} = \int \frac{\frac{1}{2}}{u} du = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln u + C$$

$$= \frac{1}{2} \ln |x^2 + x + 1| + C$$

$$I_{2b} = \frac{3}{2} \int \frac{1}{x^2+x+1} dx$$ (complete the square).

$$= \frac{3}{2} \int \frac{1}{x^2 + x + (\frac{1}{2})^2 - (\frac{1}{2})^2 + 1} dx$$

$$= \frac{3}{2} \int \frac{1}{(x^2+\frac{1}{2})^2 + \frac{3}{4}} dx$$ $\quad \int \frac{1}{u^2+1} du = \tan^{-1} u + C$

$$\int \frac{1}{u^2+a^2} du = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C.$$

$$= \frac{3}{2} \int \frac{1}{u^2 + \left(\frac{\sqrt{3}}{2}\right)^2} du = \frac{3}{2} \left(\frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{u}{\frac{\sqrt{3}}{2}} \right) + C \right)$$

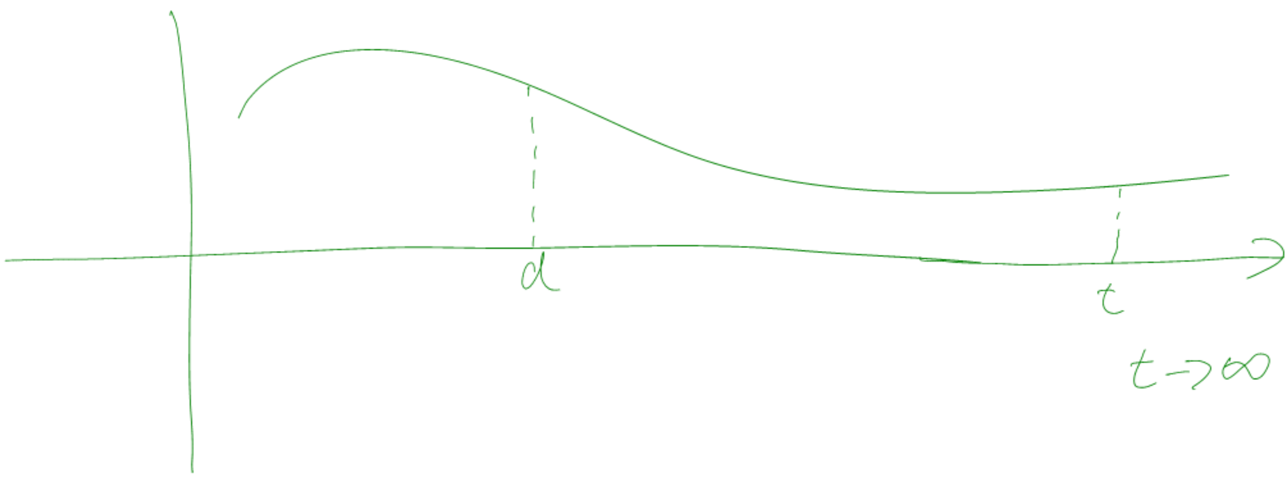
$$= \frac{3}{2} \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2(x+\frac{1}{2})}{\sqrt{3}} \right) + C.$$

$$= \sqrt{3} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}} + C\right)$$

$$\int \frac{1}{x^3-1}\,dx = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1|$$

$$- \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Improper Integrals.



$$\int_1^{\infty} \frac{1}{x}\,dx$$

Type 2: Discontinuous Integrands

Definition of an Improper Integral of Type 2

a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

if this limit exists (as a finite number).

b) If f is continuous on $(a, b]$ and is discontinuous at $\overset{a}{\cancel{b}}$, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

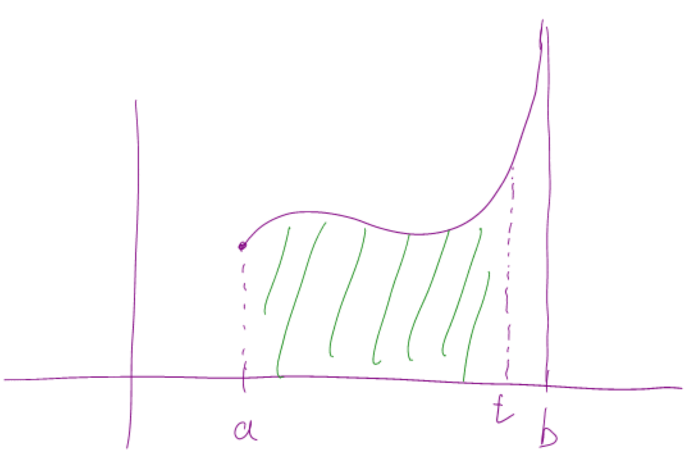
provided this limit exists (as a finite number).

The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

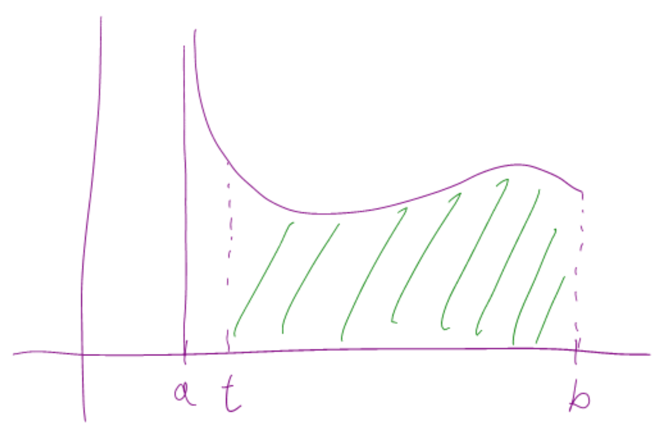
c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

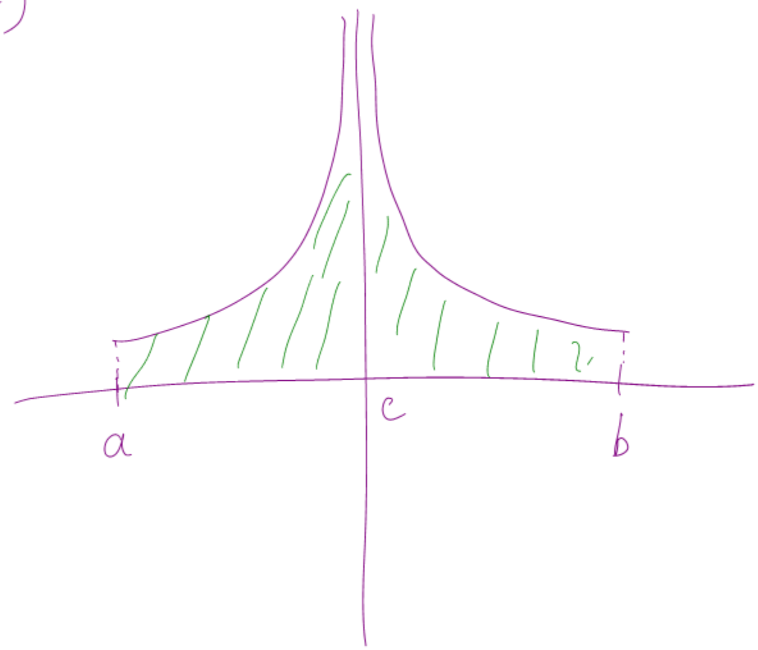
a)

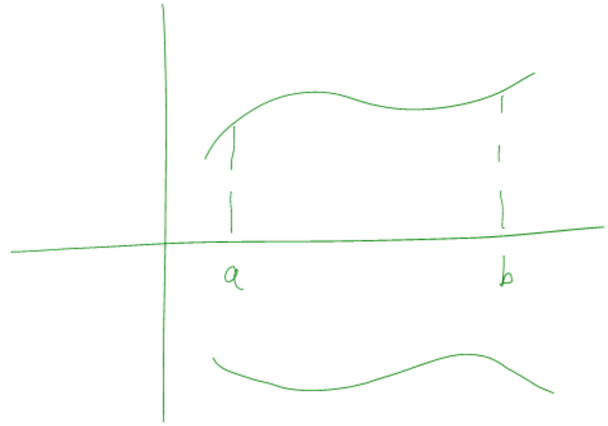


b)



c)





$$\int_a^b f(x)\,dx$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)$$

Example: $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

The V.A. $x=2$. This is on the left end of $(2,5]$.

Use part b) of the def^n.

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx.$$

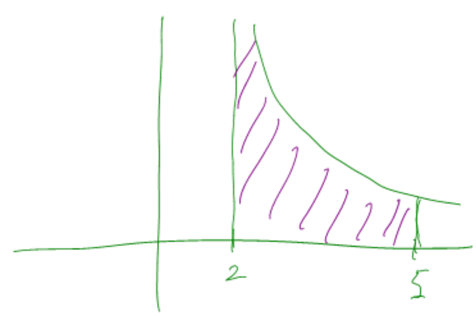
Let $u = x-2$.

$$\to \int \frac{1}{\sqrt{u}}\,du$$

$$= \lim_{t\to 2^+} 2\sqrt{x-2}\,\Big]_t^5$$

$$= \lim_{t\to 2^+} 2\left(\sqrt{3}-\sqrt{t-2}\right) = 2\sqrt{3}$$

$\Rightarrow$ converges to $2\sqrt{3}$



Example: $\int_0^3 \frac{1}{x-1}\,dx$

$$\int_0^3 \frac{1}{x-1}\,dx = \ln|x-1|\Big]_0^3 = \ln 2 - \ln 1 = \ln 2.$$

V.A. at $x=1$.

Since the V.A. is in between 0 and 3, use part c) of the def^n.

$$\int_0^3 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx.$$

$$\int_0^1 \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx$$

$$= \lim_{t\to 1^-} \Big[\ln|x-1|\Big]_0^t$$

$$= \lim_{t\to 1^-} \ln|t-1| - \ln|-1|$$

$$= \lim_{t\to 1^-} \ln(1-t) = -\infty$$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$|t-1| = \begin{cases} t-1 & \text{if } t \geq 1 \\ -(t-1) & \text{if } t < 1 \end{cases} = \begin{cases} t-1 & \text{if } t \geq 1 \\ 1-t & \text{if } t < 1 \end{cases}$$

$\Rightarrow \int_0^1 \frac{1}{x-1}\,dx$ diverges.

$\Rightarrow \int_0^3 \frac{1}{x-1}\,dx$ diverges b/c it does not satisfy part c) of the def^n.

$\int_0^1 f(x)\,dx = -\infty$ $\int_1^3 f(x)\,dx = +\infty$.

$-\infty + \infty \neq 0.$ $\neq a$ $+, -, \times, \div$

∞ , $\infty + 1 = \infty$

Example: $\int_0^1 \ln x\,dx$ V.A. is at $x=0$

Apply IBP.

Let $f = \ln x$ $g' = 1$

$f' = \frac{1}{x}$ $g = x$

$$\Rightarrow \int_0^1 \ln x\,dx = x\ln x\Big]_t^1 - \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\cdot x\,dx.$$

$$= x\ln x\Big]_t^1 - \lim_{t\to 0^+} x\Big]_t^1$$

$$= 1\ln 1 - t\ln t - (1-t).$$

$$= -t\ln t - 1 + t$$

$$= \lim_{t\to 0^+} \underbrace{-t\ln t}_{} - 1 + t$$

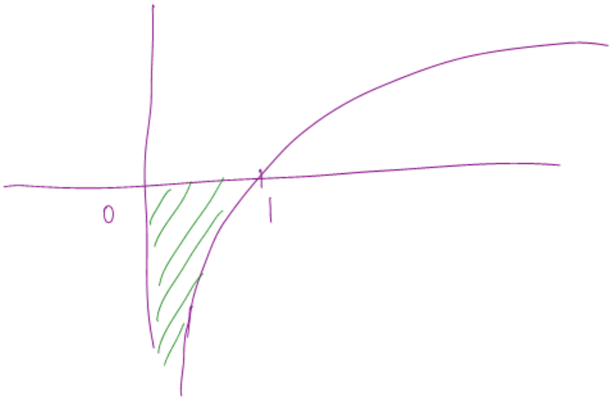
Apply l'Hospital's Rule

$0\cdot(-\infty)$

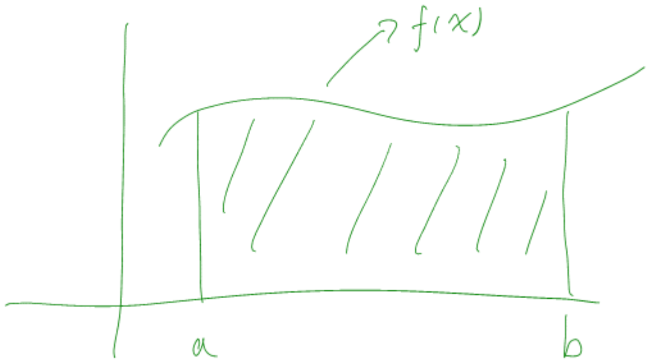
$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} \overset{H}{=} \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = \lim_{t\to 0^+} (-t) = 0$$

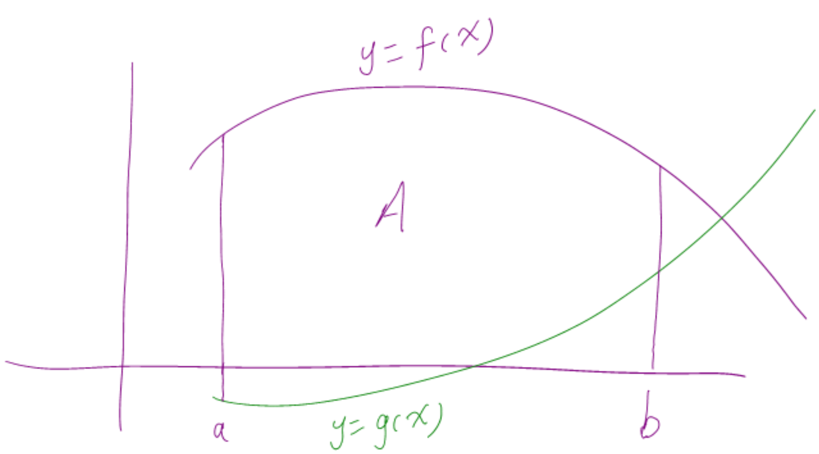
$\frac{\infty}{\infty}$

$$\Rightarrow \int_0^1 \ln x \, dx = \lim_{t \to 0^+} (-t \ln t - 1 + t) = 0 - 1 + 0 = -1.$$



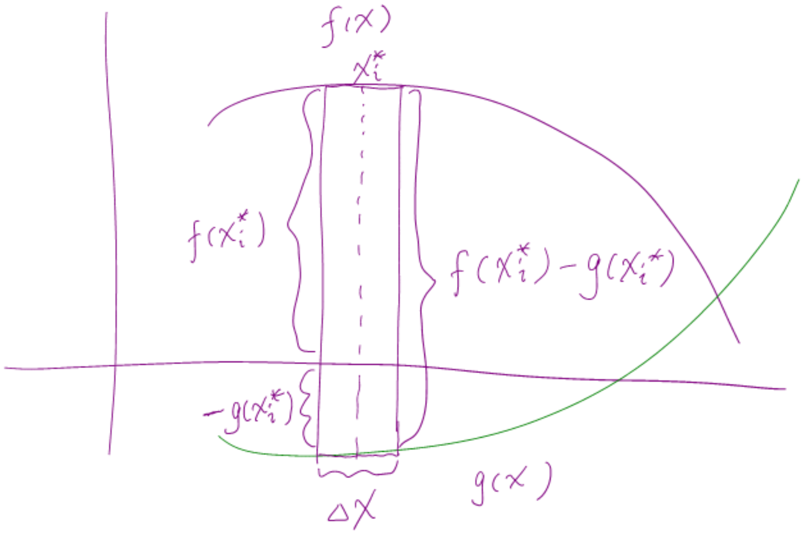
Applications of Integration





f and g are continuous

$f \geq g \quad \forall\ x \in [a, b].$



The area for one rectangular strip is $[f(x_i^*)-g(x_i^*)]\Delta x$.

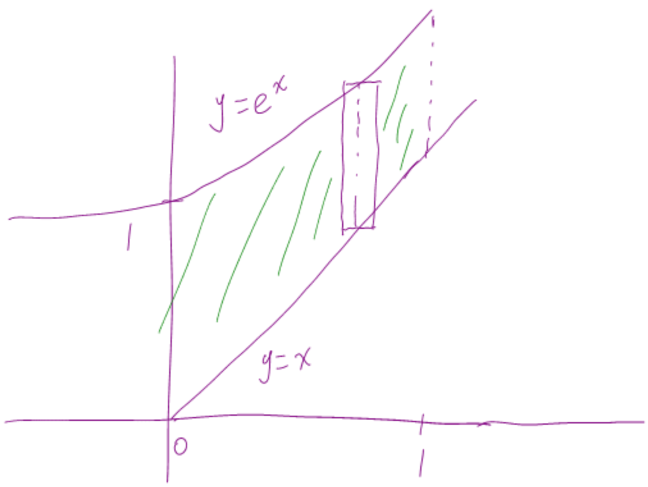
$$A=\lim_{n\to\infty}\sum_{i=1}^{n}\left[f(x_i^*)-g(x_i^*)\right]\Delta x$$

Def^n: The area A of the region bounded by the curves $y=f(x)$, $y=g(x)$ and the lines $x=a$, $x=b$. where f and g are continuous $f(x)\geq g(x)$ $\forall$ $x\in[a,b]$, is

$$A=\int_a^b [f(x)-g(x)]\,dx.$$

Example: Find the region bounded by

$y = e^x$, $y = x$, $x = 0$, and $x = 1$.



$y_T = e^x$ $\quad$ $y_B = x$

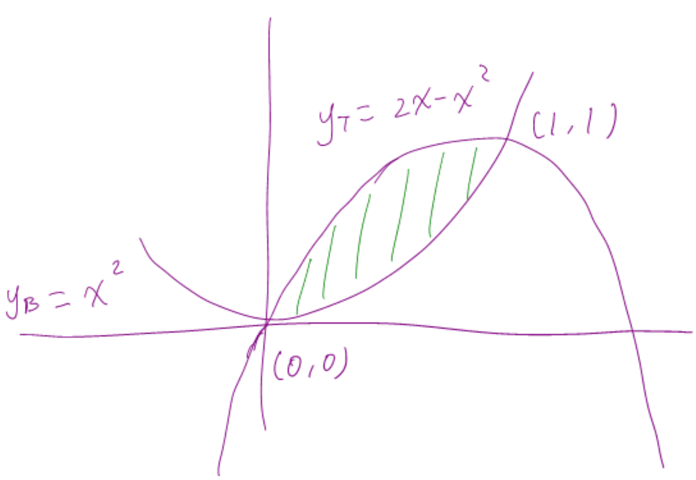
$a = 0$ $\quad$ $b = 1$

$$A = \int_0^1 e^x - x \, dx$$

$$= e^x - \frac{1}{2}x^2 \Big]_0^1$$

$$= e - \frac{1}{2} - 1 = e - \frac{3}{2}$$

Example: Find the area enclosed by the curves $y = x^2$ and $y = 2x - x^2$.



$$x^2 = 2x - x^2$$

$$\Rightarrow 2x^2 - 2x = 0$$

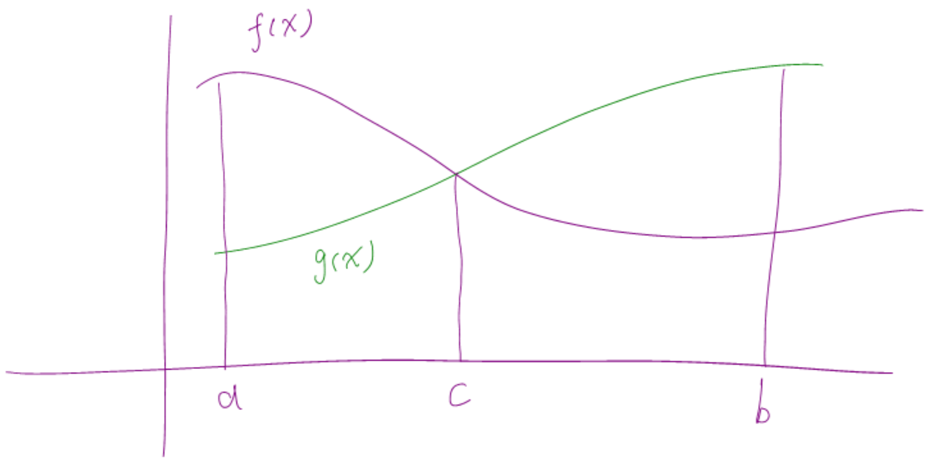
$$\Rightarrow 2x(x-1) = 0$$

$$\Rightarrow x = 0 \text{ and } x = 1.$$

$$A = \int_a^b \underset{\substack{\downarrow \\ y_T}}{f(x)} - \underset{\substack{\downarrow \\ y_B}}{g(x)} \, dx = \int_0^1 2x - x^2 - x^2 \, dx$$

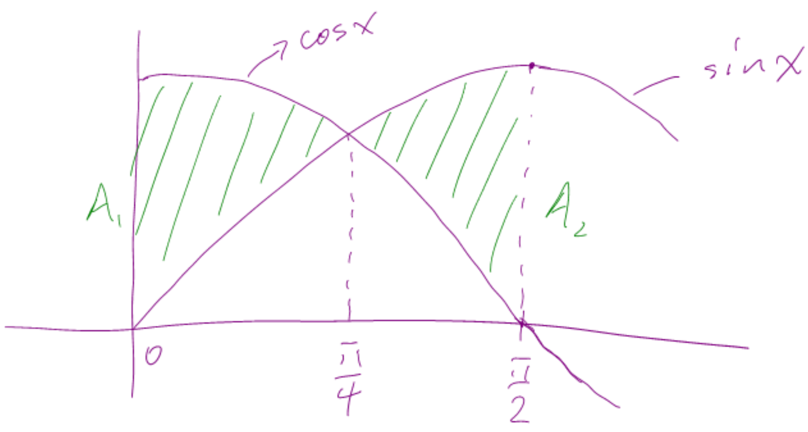
$$= \int_0^1 2x - 2x^2 \, dx = 2 \int_0^1 x - x^2 \, dx.$$

$$= 2 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$



$$A = \int_a^b |f(x) - g(x)| \, dx$$

Example: Find the area bounded by $y = \sin x$
$y = \cos x$, $x = 0$, $x = \frac{\pi}{2}$



$$A = \int_0^{\frac{\pi}{2}} |\cos x - \sin x|\, dx = A_1 + A_2$$

$$= \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \underbrace{[-\cos x + \sin x]}_{\sin x - \cos x} dx$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= 2\sqrt{2} - 2.$$