An introduction to calculus (2)

Continuity

Continuity

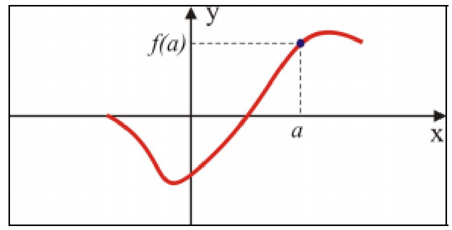
A function $y = f(x)$ is *continuous* at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

Note: A function is continuous at a if the following three conditions are met:

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $f(a)$ and $\lim_{x \to a} f(x)$ are equal.

Note: A function is continuous if the graph can be drawn *without lifting* the pen from paper.



Discontinuity

If $y = f(x)$ is not continuous at a then we say:

- $y = f(x)$ is *discontinuous* at a or
- $y = f(x)$ has a discontinuity at a

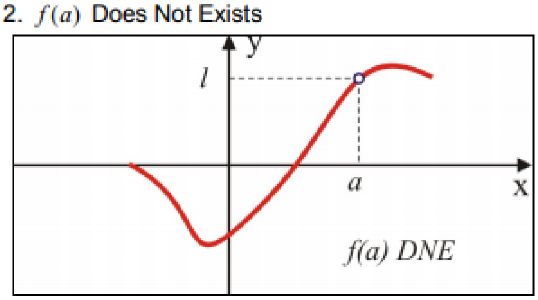
Note: There are three types of discontinuity:

a) *removable* or *point* discontinuity
b) *jump* discontinuity
c) *infinite* discontinuity (break)

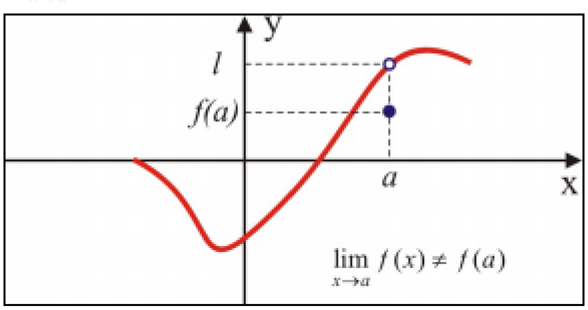
Removable or Point Discontinuity

A function $y = f(x)$ has a *removable* or *point discontinuity* at a if:

1. $\lim_{x \to a} f(x)$ exists
2. $f(a)$ Does Not Exists



or $\lim_{x \to a} f(x) \neq f(a)$



Note: A removable or point discontinuity *can be removed* by redefining the function at a as

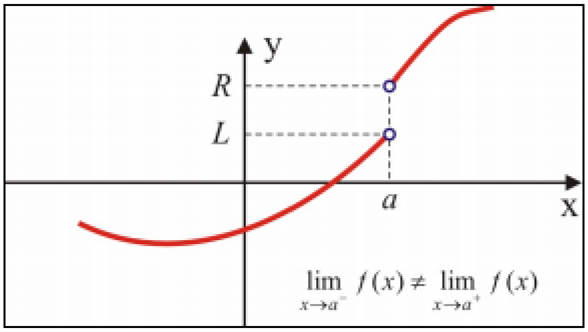
$$f(a) \overset{def}{=} \lim_{x \to a} f(x).$$

Ex 1. Redefine $y = f(x) = \dfrac{x^2 - 4}{x - 2}$ such that $y = f(x)$ is to be continuous everywhere (at any number). Graph the old and the new function.

Jump Discontinuity

A function $y = f(x)$ has a *jump discontinuity* at a if the left-side and the right-side limits exist but they are not equal:

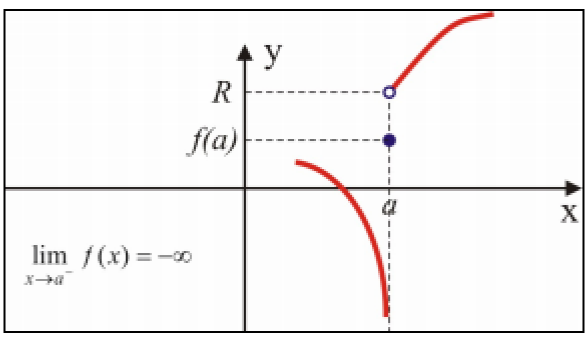
$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

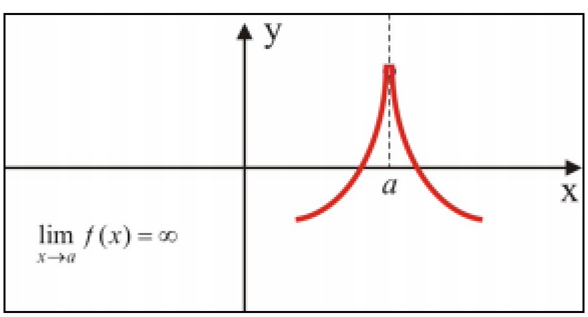


Ex 2. Analyze the continuity of the function $y = f(x) = \dfrac{|x-3|}{x-3}$ at $x = 3$. Graph the function to illustrate the situation.

Infinite Discontinuity

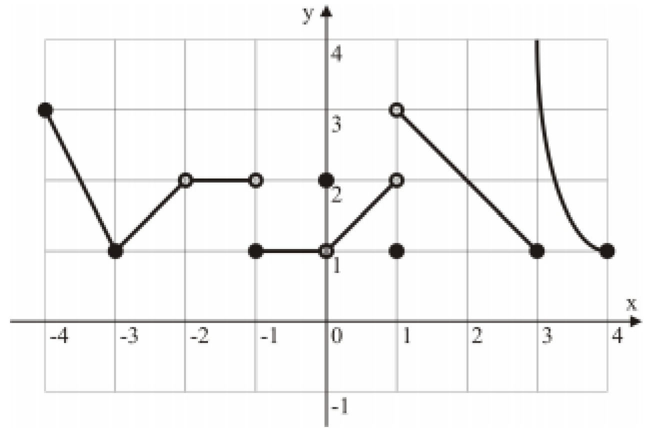
A function $y = f(x)$ has an *infinite discontinuity* at a if at least one of the left-side or the right-side limits is *unbounded* (approaches to ∞ or $-\infty$).





To write $\lim_{x \to a} f(x) = \infty$ is better (more information is included) than to write $\lim_{x \to a} f(x)$ DNE.

Ex 3. Analyze the continuity of the function $f(x) = \dfrac{1}{x}$ at $x = 0$.

Ex 4. The function $y = f(x)$ is represented graphically in the figure on the right side. Analyze the continuity of this function at: a) $x = -3$ b) $x = -2$ c) $x = -1$ d) $x = 0$ e) $x = 3$	
Elementary Functions Elementary functions (polynomial, power, rational, trigonometric, exponential, and logarithmic) are continuous over their domain.	

Ex 5. Analyze the continuity of the function: $f(x)=\begin{cases} x, & x<0 \\ x^2, & 0\le x\le 1 \\ x^3+1, & x>1 \end{cases}$	Ex 6. For what value of the constant c is the function $f(x)=\begin{cases} x+c, & x<2 \\ cx^2+1, & x\ge 2 \end{cases}$ continuous at any number (everywhere)?

Limits at Infinity

When we take the limit of a function at infinity, we are interested in the end-behavior of a graph. We can write the analysis of each end-behavior of a function f(x) using the following notations:

$$\lim_{x \to \infty} f(x) \quad \text{or} \quad \lim_{x \to -\infty} f(x)$$

Assuming a function exists on either end, for infinitely large and infinitely small values of x, it is limited in what it can do. We will now analyze these possibilities.

Ex 1: Evaluate the following limits without:

a. $\lim_{x \to \infty} \frac{1}{x}$

b. $\lim_{x \to -\infty} \frac{1}{x}$

c. $\lim_{x \to \infty} (x^3 - 2x)$

d. $\lim_{x \to \infty} \frac{-6x^2-3x+1}{5x-1}$

e. $\lim_{x \to \infty} \frac{4x^2-x+2}{6x^2+5x+1}$

f. $\lim_{x \to -\infty} \frac{x}{x^2-x-6}$

More practice on LIMITS

1. $\lim\limits_{x\to 0}\dfrac{5x^3+8x^2}{3x^4-16x^2}=$

2. $\lim\limits_{x\to 5}\dfrac{\dfrac{2}{x+3}-\dfrac{1}{4}}{x-5}=$

3. $\lim\limits_{t\to 2}\dfrac{t^3+2t^2-13t+10}{t^3+4t^2-4t-16}=$

4. $\lim\limits_{x\to 0}\dfrac{(2+x)^3-8}{x}=$

5. $\lim\limits_{h\to 0}\dfrac{4(x+h)^2-3(x+h)+5-\left(4x^2-3x+5\right)}{h}=$

6. $\lim_{x \to 3} \frac{\sqrt{x+6}-3}{x-3} =$

7. $\lim_{x \to 1} \frac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+x-3} =$