AP Calculus Class 25

Representation of Functions as Power series.

The geometric series.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \qquad |x|<1$$

$a = 1 \qquad r = x.$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad f(x) = \frac{1}{1-x}$$

Example: Express $\dfrac{1}{1+x^2}$ as a power series and find the interval of convergence.

Observe
$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$
$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots$$

For interval of convergence,

$$|-x^2| < 1 \quad \Rightarrow \quad x^2 < 1 \quad \Rightarrow \quad |x| < 1$$

$\Rightarrow R = 1 \qquad$ So $\quad -1 < x < 1.$

Example: Find the power series representation of $\dfrac{1}{2+x}$.

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\left(\frac{1}{1-\left(-\frac{x}{2}\right)}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}}$$

Example: Find the power series representation of $\dfrac{x^3}{x+2}$

$$\frac{x^3}{x+2} = x^3 \frac{1}{x+2} = x^3\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+3}}{2^{n+1}}$$

Differentiating and Integrating Power Series.

Theorem

If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(1) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \displaystyle\sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$

(2) $\displaystyle\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (1) and (2) are both R.

Example: Express $\frac{1}{(1-x)^2}$ as a power series.

What is the radius of convergence?

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

$$\left(\frac{1}{1-x}\right)' = \left((1-x)^{-1}\right)' = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=0}^{\infty} n x^{n-1}$$

$\Rightarrow$ By part 1 of the above theorem, $R=1$.

Example: $\ln(1-x)$ find the radius of convergence.

$$-\ln(1-x) = \int \frac{1}{1-x}\,dx = \int (1+x+x^2+\cdots)\,dx$$

$$= C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \qquad \left(C + \sum_{n=1}^{\infty} \frac{x^n}{n}\right)$$

$$\Rightarrow \ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

$\Rightarrow$ The radius of convergence is 1.

The Taylor Series.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \qquad |x-a| < R$$

$$f(a) = c_0 + c_1(0) + c_2(0)^2 + \cdots$$
$$= c_0$$

$$f'(x) = 0 + c_1 + c_2 2(x-a) + c_3 3(x-a)^2 + \cdots$$

$$f'(a) = c_1 + c_2 2(0) + c_3 3(0)^2 + \cdots$$
$$= c_1$$

$$f''(x) = 2c_2 + 2\cdot 3\cdot c_3(x-a) + 3\cdot 4\cdot c_4(x-a)^2 + \cdots$$

$$f''(a) = 2c_2 = 2!\, c_2$$

$$f'''(x) = 2\cdot 3\cdot c_3 + 2\cdot 3\cdot 4\cdot c_4(x-a) + \cdots$$

$$f'''(a) = 2\cdot 3\cdot c_3 = 3!\, c_3$$

$$\vdots$$

$$f^{(n)}(a) = 1\cdot 2\cdot 3\cdot \cdots \cdot n\cdot c_n = n!\, c_n$$

$$\Rightarrow \quad c_n = \frac{f^{(n)}(a)}{n!}$$

Theorem

If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \longrightarrow 0! = 1$$

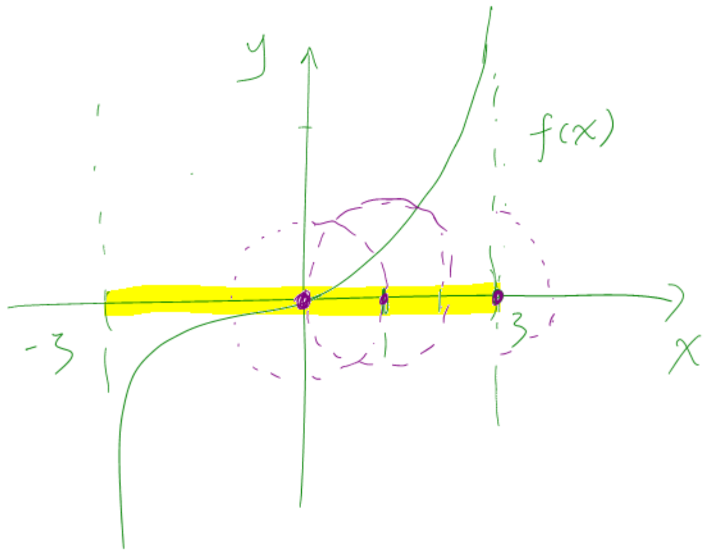
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

The above series is called Taylor Series of the fun^n f at a (or about a)

For the Taylor series when $a=0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

The above series when $a=0$ is called the Maclaurin series.



Example: Find the Maclaurin series of the fun^n $f(x) = e^x$ and its radius of convergence.

$f^{(n)}(x) = e^x$ $\qquad f^{(n)}(0) = e^0 = 1 \qquad \forall n.$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

For $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\qquad$ let $a_n = \frac{x^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1}$$

$$\lim_{n\to\infty} \frac{1}{n+1}|x| < 1.$$

$\Rightarrow$ By the Ratio Test, the series converges $\forall x$.

$\Rightarrow R = \infty$.

$$\boxed{f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

Example: Find the Taylor series for $f(x) = e^x$ at $a = 2$.

$f^{(n)}(2) = e^2$ Plug $a = 2$ into the Taylor series.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n$$

Example: Find the Maclaurin series for $\sin x$.

$f(x) = \sin x$ $\qquad f(0) = 0$

$f'(x) = \cos x$ $\qquad f'(0) = 1$

$f''(x) = -\sin x$ $\qquad f''(0) = 0$

$f'''(x) = -\cos x$ $\qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x$ $\qquad f^{(4)}(0) = 0.$

$$f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$

$$= 0 + x + \left(-\frac{1}{3!} x^3\right) + 0 + \frac{1}{5!} x^5 + 0 - \frac{1}{7!} x^7 + \frac{1}{9!} x^9 + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\Rightarrow \boxed{\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned}}$$

Example: Find the Maclaurin series for $\cos x$.

$$\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$$

Important Maclaurin Series.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$$