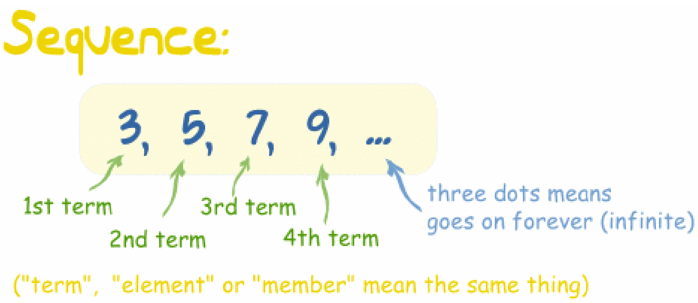
Sequences

1. Definition of Sequence

A Sequence is a set of things (usually numbers) that are in order.

Each number in the sequence is called a term (or sometimes "element" or "member"):



2. Finding Missing Numbers

To find a missing number you need to first find a Rule behind the Sequence.
Sometimes it is just a matter of looking at the numbers and seeing a pattern.

The rule can be expressed in many ways. The following two ways are used most often.

a) **Explicit Formula** of a Sequence - A formula that allows direct computation of any term for a sequence a1, a2, a3, . . . , an, ...

Example: $a_n = 4n - 7$

b) **Recursive Formula** - For a sequence a1, a2, a3, . . . , an, . . . a recursive formula is a formula that requires the computation of all previous terms in order to find the value of an .

Example: $\begin{cases} a_1 = 3 \\ a_n = 2a_{n-1} + 5 \end{cases}$

Example: 1, 4, 9, 16,?
Answer: they are Squares ($1^2=1$, $2^2=4$, $3^2=9$, $4^2=16$, ...)
Rule: $x_n = n^2$
Sequence: 1, 4, 9, 16, 25, 36, 49, ...

Did you see how we wrote down that rule with "x" and "n" ?
x_n means "term number n", so term 3 would be written x_3

And we also used "n" in the formula, so the formula for term 3 is $3^2 = 9$. This could be written $x_3 = 3^2 = 9$

Once we have a Rule we can use it to find any term, for example, the 25th term can be found by "plugging in" 25 wherever n is.
$x_{25} = 25^2 = 625$

Example: 3, 5, 8, 13, 21, ?

After 3 and 5 all the rest are the sum of the two numbers before, that is 3 + 5 = 8, 5 + 8 = 13 and so on (it is actually part of the Fibonacci Sequence):
Rule: $x_n = x_{n-1} + x_{n-2}$
Sequence: 3, 5, 8, 13, 21, 34, 55, 89, ...

Now what does x_{n-1} mean? Well that just means "the previous term" because the term number (n) is 1 less (n-1).
So, if n was 6, then $x_n = x_6$ (the 6th term) and $x_{n-1} = x_{6-1} = x_5$ (the 5th term)
So, let's apply that Rule to the 6th term:
$x_6 = x_{6-1} + x_{6-2}$
$x_6 = x_5 + x_4$

We already know the 4th term is 13, and the 5th is 21, so the answer is:
$x_6 = 21 + 13 = 34$

3. Many Rules

One of the troubles with finding "the next number" in a sequence is that mathematics is so powerful you can find more than one Rule that works.

Example: What is the next number in the sequence 1, 2, 4, 7,?
Here are three solutions (there can be more!):

Solution 1: Add 1, then add 2, 3, 4, ...
So, 1+1=2, 2+2=4, 4+3=7, 7+4=11, etc...

Rule: $x_n = n(n-1)/2 + 1$
Sequence: 1, 2, 4, 7, 11, 16, 22, ...
(That rule looks a bit complicated, but it works)

Solution 2: After 1 and 2, add the two previous numbers, plus 1:
Rule: $x_n = x_{n-1} + x_{n-2} + 1$
Sequence: 1, 2, 4, 7, 12, 20, 33, ...

Solution 3: After 1, 2 and 4, add the three previous numbers
Rule: $x_n = x_{n-1} + x_{n-2} + x_{n-3}$
Sequence: 1, 2, 4, 7, 13, 24, 44, ...

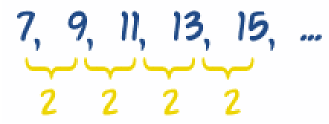
So, we had three perfectly reasonable solutions, and they created totally different sequences.

4. Simplest Rule

When in doubt choose the simplest rule that makes sense, but also mention that there are other solutions.

5. Finding Differences

Sometimes it helps to find the differences between each pair of numbers ... this can often reveal an underlying pattern.
Here is a simple case:



The differences are always 2, so we can guess that "2n" is part of the answer.
Let us try 2n:

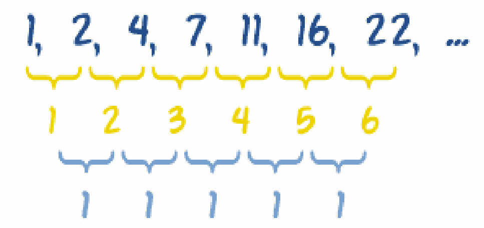
n:	1	2	3	4	5
Terms (x_n):	7	9	11	13	15
2n:	2	4	6	8	10
Wrong by:	5	5	5	5	5

The last row shows that we are always wrong by 5, so just add 5 and we are done:
Rule: $x_n = 2n + 5$

OK, you could have worked out "2n+5" by just playing around with the numbers a bit, but we want a systematic way to do it, for when the sequences get more complicated.

6. Second Differences

In the sequence {1, 2, 4, 7, 11, 16, 22, ...} we need to find the differences and then find the differences of those (called *second differences*), like this:



The second differences in this case are 1. If the second difference is the same while the first is not, the function must be quadratic.
With second differences you multiply by "$n^2 / 2$".

In our case the difference is 1, so let us try $n^2 / 2$:

n:	1	2	3	4	5
Terms (x_n):	1	2	4	7	11
n^2:	1	4	9	16	25

$n^2 / 2$:	0.5	2	4.5	8	12.5
Wrong by:	0.5	0	-0.5	-1	-1.5

We are close, but seem to be drifting by 0.5, so let us try: $n^2 / 2 - n/2$

$n^2 / 2 - n/2$:	0	1	3	6	10
Wrong by:	1	1	1	1	1

Wrong by 1 now, so let us add 1:

$n^2 / 2 - n/2 + 1$:	1	2	4	7	11
Wrong by:	0	0	0	0	0

The formula $n^2 / 2 - n/2 + 1$ can be simplified to $n(n-1)/2 + 1$
So, by "trial-and-error" we were able to discover a rule that works:
Rule: $x_n = n(n-1)/2 + 1$
Sequence: 1, 2, 4, 7, 11, 16, 22, 29, 37, ...

7. Arithmetic Sequence

An **arithmetic sequence** or arithmetic progression $a_1, a_2, a_3, \cdots, a_n, \cdots$ is a sequence of numbers such that the difference of any two successive members of the sequence is a constant.

For instance, the sequence 3, 5, 7, 9, 11, 13... is an arithmetic sequence with common difference 2. So we have

$a_1, a_1+d, a_1+2d, a_1+3d, \cdots$

If the initial term of an arithmetic progression is a_1 and the common difference of successive members is d, then the nth term of the sequence is given by:

$$a_n = a_1 + (n-1)d$$

One of the properties of an arithmetic sequence is $a_k = \dfrac{a_{k-1} + a_{k+1}}{2} \quad (k > 1)$

8. Geometric Sequence

A **geometric sequence**, or geometric progression, is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed non-zero number called the common ratio.

For example, the sequence 2, 6, 18, 54, ... is a geometric progression with common ratio 3.

Similarly 10, 5, 2.5, 1.25, … is a geometric sequence with common ratio 1/2. So we have: $a_1, a_1r, a_1r^2, a_1r^3, \cdots$

Thus, the general form of a geometric sequence is

$a_n = a_1 r^{n-1}$

One of the properties of a geometric sequence is

$a_k = \pm\sqrt{a_{k-1}a_{k+1}} \qquad (a_{k-1}a_{k+1} > 0)$

►In-class questions

1. When the mean, median, and mode of the list 10, 2, 5, 2, 4, 2, x are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x?

2. Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$, for all positive integers m and n. Then a_{12} is

3. The first four terms in an arithmetic sequence are $x+y, x-y, xy$, and x/y, in that order. What is the fifth term?

4. Let $a_1, a_2, ...,$ be a sequence with the following properties.

 1) $a_1 = 1$, and

 2) $a_{2n} = n \cdot a_n$ for any positive integer n.

What is the value of $a_{2^{100}}$?

5. Let 1, 4, … and 9, 16, … be two arithmetic progressions. The set S is the union of the first 2004 terms of each sequence. How many distinct numbers are in S?

6. Let $a_1, a_2, ...$ be a sequence for which $a_1 = 2, a_2 = 3$, and $a_n = \dfrac{a_{n-1}}{a_{n-2}}$ for each positive integer $n \ge 3$. What is a_{2006}?

7. A finite sequence of three-digit integers has the property that the tens and units digits of each term are, respectively, the hundreds and tens digits of the next term, and the tens and units digits of the last term are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with terms 247, 475, and 756 and end with the term 824. Let S be the sum of all the terms in the sequence. What is the largest prime number that always divides S?

8. Two non-decreasing sequences of nonnegative integers have different first terms. Each sequence has the property that each term beginning with the third is the sum of the previous two terms, and the seventh term of each sequence is N. What is the smallest possible value of N?