Patterns and Sequence

1. Common Number Patterns

Numbers can have interesting patterns. Here we list the most common patterns and how they are made.

1) Arithmetic Sequences

An Arithmetic Sequence is made by adding some value each time.

Example 1:

1, 4, 7, 10, 13, 16, 19, 22, 25, ...

This sequence has a difference of 3 between each number.
The pattern is continued by adding 3 to the last number each time.

Example 2:

3, 8, 13, 18, 23, 28, 33, 38, ...

This sequence has a difference of 5 between each number.
The pattern is continued by adding 5 to the last number each time.
The value added each time is called the "common difference"

What is the common difference in this example?

19, 27, 35, 43, ...

Answer: The common difference is 8

The common difference could also be negative, like this:

25, 23, 21, 19, 17, 15, ...

This common difference is -2
The pattern is continued by subtracting 2 each time.

2) Geometric Sequences

A Geometric Sequence is made by multiplying by some value each time.

Example 1:

2, 4, 8, 16, 32, 64, 128, 256, ...

This sequence has a factor of 2 between each number.
The pattern is continued by multiplying by 2 each time.

Example 2:

3, 9, 27, 81, 243, 729, 2187, ...

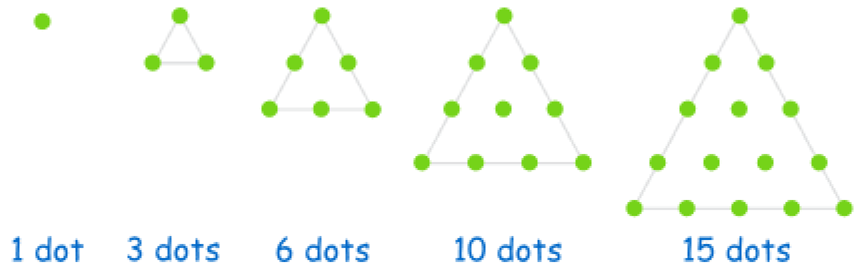
This sequence has a factor of 3 between each number.
The pattern is continued by multiplying by 3 each time.

3) Special Sequences

a) Triangular Numbers

1, 3, 6, 10, 15, 21, 28, 36, 45, ...

This Triangular Number Sequence is generated from a pattern of dots which form a triangle. By adding another row of dots and counting all the dots we can find the next number of the sequence:



b) Square Numbers

1, 4, 9, 16, 25, 36, 49, 64, 81, ...

The next number is made by squaring where it is in the pattern.
The second number is 2 squared (2^2 or 2×2)
The seventh number is 7 squared (7^2 or 7×7) etc

c) Cube Numbers

1, 8, 27, 64, 125, 216, 343, 512, 729, ...

The next number is made by cubing where it is in the pattern.
The second number is 2 cubed (2^3 or 2×2×2)
The seventh number is 7 cubed (7^3 or 7×7×7) etc

d) Fibonacci Numbers

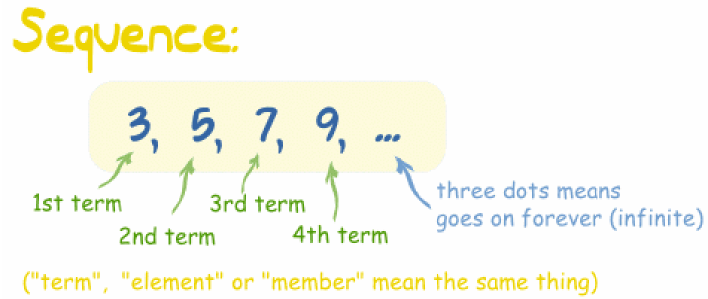
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

The Fibonacci Sequence is found by adding the two numbers before it together.

The 2 is found by adding the two numbers before it (1+1)
The 21 is found by adding the two numbers before it (8+13)
The next number in the sequence above would be 55 (21+34)

2. Definition of Sequence

A Sequence is a set of things (usually numbers) that are in order.
Each number in the sequence is called a term (or sometimes "element" or "member"):



3. Finding Missing Numbers

To find a missing number you need to first find a Rule behind the Sequence.
Sometimes it is just a matter of looking at the numbers and seeing a pattern.

Example: 1, 4, 9, 16,?

Answer: they are Squares (1^2=1, 2^2=4, 3^2=9, 4^2=16, ...)
Rule: $x_n = n^2$
Sequence: 1, 4, 9, 16, 25, 36, 49, ...

Did you see how we wrote down that rule with "x" and "n" ?
x_n means "term number n", so term 3 would be written x_3

And we also used "n" in the formula, so the formula for term 3 is $3^2 = 9$. This could be written $x_3 = 3^2 = 9$

Once we have a Rule we can use it to find any term, for example, the 25th term can be found by "plugging in" 25 wherever n is.
$x_{25} = 25^2 = 625$

Example: 3, 5, 8, 13, 21, ?

After 3 and 5 all the rest are the sum of the two numbers before, that is 3 + 5 = 8, 5 + 8 = 13 and so on (it is actually part of the Fibonacci Sequence):

Rule: $x_n = x_{n-1} + x_{n-2}$
Sequence: 3, 5, 8, 13, 21, 34, 55, 89, ...

Now what does x_{n-1} mean? Well that just means "the previous term" because the term number (n) is 1 less (n-1).

So, if n was 6, then $x_n = x_6$ (the 6th term) and $x_{n-1} = x_{6-1} = x_5$ (the 5th term)

So, let's apply that Rule to the 6th term:

$$x_6 = x_{6-1} + x_{6-2}$$

$$x_6 = x_5 + x_4$$

We already know the 4th term is 13, and the 5th is 21, so the answer is: $x_6 = 21 + 13 = 34$

4. Many Rules

One of the troubles with finding "the next number" in a sequence is that mathematics is so powerful you can find more then one Rule that works.

What is the next number in the sequence 1, 2, 4, 7,?
Here are three solutions (there can be more!):

Solution 1: Add 1, then add 2, 3, 4, ...
So, 1+1=2, 2+2=4, 4+3=7, 7+4=11, etc...

Rule: $x_n = n(n-1)/2 + 1$
Sequence: 1, 2, 4, 7, 11, 16, 22, ...
(That rule looks a bit complicated, but it works)

Solution 2: After 1 and 2, add the two previous numbers, plus 1:
Rule: $x_n = x_{n-1} + x_{n-2} + 1$
Sequence: 1, 2, 4, 7, 12, 20, 33, ...

Solution 3: After 1, 2 and 4, add the three previous numbers
Rule: $x_n = x_{n-1} + x_{n-2} + x_{n-3}$
Sequence: 1, 2, 4, 7, 13, 24, 44, ...

So, we had three perfectly reasonable solutions, and they created totally different sequences.

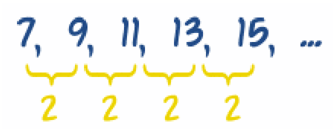
5. Simplest Rule

When in doubt choose the simplest rule that makes sense, but also mention that there are other solutions.

6. Finding Differences

Sometimes it helps to find the differences between each pair of numbers ... this can often reveal an underlying pattern.

Here is a simple case:



The differences are always 2, so we can guess that "2n" is part of the answer.
Let us try 2n:

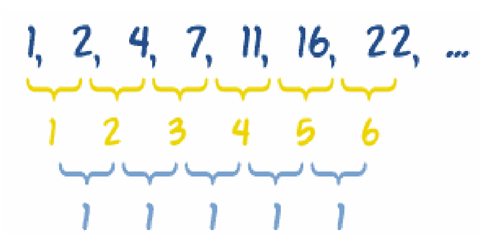
n:	1	2	3	4	5
Terms (x_n):	7	9	11	13	15
2n:	2	4	6	8	10
Wrong by:	5	5	5	5	5

The last row shows that we are always wrong by 5, so just add 5 and we are done:
Rule: $x_n = 2n + 5$

OK, you could have worked out "2n+5" by just playing around with the numbers a bit, but we want a systematic way to do it, for when the sequences get more complicated.

7. Second Differences

In the sequence {1, 2, 4, 7, 11, 16, 22, ...} we need to find the differences and then find the differences of those (called *second differences*), like this:



The second differences in this case are 1.
With second differences you multiply by "$n^2 / 2$".

In our case the difference is 1, so let us try $n^2 / 2$:

n:	1	2	3	4	5
Terms (x_n):	1	2	4	7	11
n^2:	1	4	9	16	25
$n^2 / 2$:	0.5	2	4.5	8	12.5
Wrong by:	0.5	0	-0.5	-1	-1.5

We are close, but seem to be drifting by 0.5, so let us try: $n^2 / 2 - n/2$

$n^2 / 2 - n/2$:	0	1	3	6	10
Wrong by:	1	1	1	1	1

Wrong by 1 now, so let us add 1:

$n^2 / 2 - n/2 + 1$:	1	2	4	7	11
Wrong by:	0	0	0	0	0

The formula $n^2 / 2 - n/2 + 1$ can be simplified to $n(n-1)/2 + 1$

So, by "trial-and-error" we were able to discover a rule that works:

Rule: $x_n = n(n-1)/2 + 1$

Sequence: 1, 2, 4, 7, 11, 16, 22, 29, 37, ...

7. Arithmetic Sequence

An arithmetic sequence or arithmetic progression $a_1, a_2, a_3, \cdots, a_n, \cdots$ is a sequence of numbers such that the difference of any two successive members of the sequence is a constant.

For instance, the sequence 3, 5, 7, 9, 11, 13... is an arithmetic sequence with common difference 2. So we have
$a_1, a_1+d, a_1+2d, a_1+3d, \cdots$

If the initial term of an arithmetic progression is a_1 and the common difference of successive members is d, then the nth term of the sequence is given by:
$$a_n = a_1 + (n-1)d$$

One of the properties of an arithmetic sequence is $a_k = \dfrac{a_{k-1} + a_{k+1}}{2} \qquad (k > 1)$

8. Geometric Sequence

A geometric sequence, or geometric progression, is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed non-zero number called the common ratio.

For example, the sequence 2, 6, 18, 54, ... is a geometric progression with common ratio 3.

Similarly 10, 5, 2.5, 1.25, … is a geometric sequence with common ratio 1/2. So we have: $a_1, a_1q, a_1q_2, a_1q_3, \cdots$

Thus, the general form of a geometric sequence is
$$a_n = a_1 q^{n-1}$$

One of the properties of a geometric sequence is
$$a_k = \pm\sqrt{a_{k-1}a_{k+1}} \qquad (a_{k-1}a_{k+1} > 0)$$

4. Figure pattern

1) Polygonal Numbers

Polygonal Numbers are really just the number of vertexes in a figure formed by a certain polygon. The first number in any group of Polygonal Numbers is always 1, or a point. The second number is equal to the number of vertexes of the polygon.

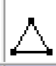
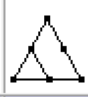
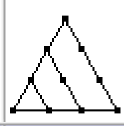
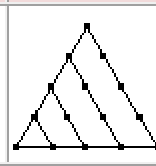
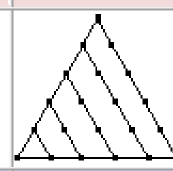
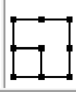
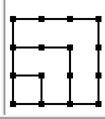
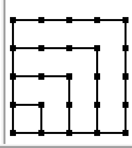
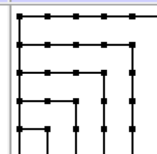
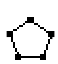
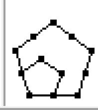
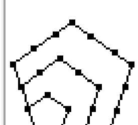
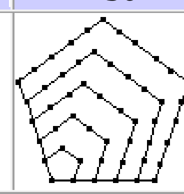
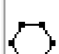
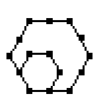
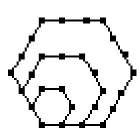
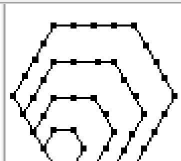
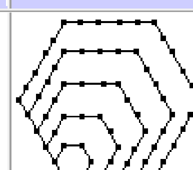
For example, the second Pentagonal Number is 5, since pentagons have 5 vertexes (and sides).

The third Polygonal Number is made by extending two of the sides of the polygon from the second Polygonal Number, completing the larger polygon, and placing vertexes *and other points where necessary*.

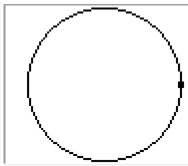
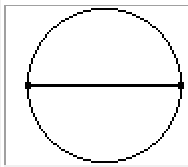
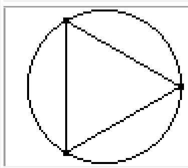
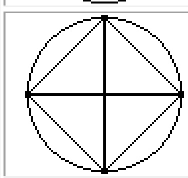
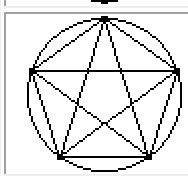
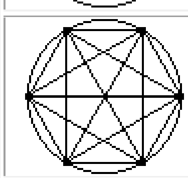
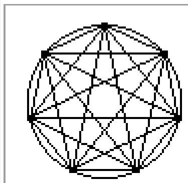
The third Polygonal Number is found by adding all the vertexes *and points* in the resulting figure. (Look at the table below for a clearer explaination).

A formula that will generate the n^{th} x-gonal number (for example: the 2^{nd} 3-gonal, or triangular number) is:

$$\frac{n^2 - n}{2} \times (x - 2) + n$$

Type	**1st**	**2nd**	**3rd**	**4th**	**5th**	**6th**
Triangular						
Value	1	3	6	10	15	21
Square						
Value	1	4	9	16	25	36
Pentagonal						
Value	1	5	12	22	35	51
Hexagonal						
Value	1	6	15	28	45	66

2) Points on a Circle

Image	Points	Segments	Triangles	Quadrilaterals	Pentagons	Hexagons	Heptagons
	1						
	2	1					
	3	3	1				
	4	6	4	1			
	5	10	10	5	1		
	6	15	20	15	6	1	
	7	21	35	35	21	7	1

As you may have noticed, the numbers in the chart above are actually the tip of the right-angled form of Pascal's Triangle, except the preceeding 1's in each row are missing. The circular figures are formed

►In-class questions

1. Peter starts to add up positive integers whose units digit is 5. How many terms of this series, 5 +15 + 25 + 35 + 45 + ... are required to obtain a sum greater than 10 000?

2. What is the sum of all positive integers less than or equal to 2003 that are multiples of either 11 or 13?

3. Given the pattern of numbers below find the sum of the numbers in the nth row.

1
2 3
4 5 6
7 8 9 10
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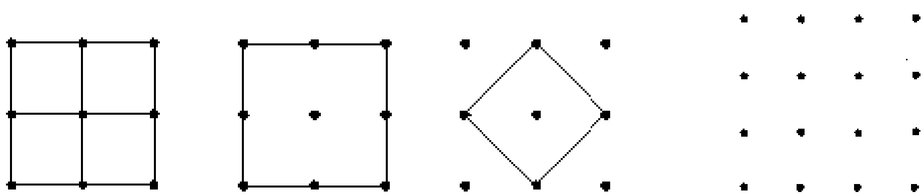
4. A 3 by 3 grid has dots spaced 1 unit apart both horizontally and vertically. Six squares of various side lengths can be formed with corners on the dots, as shown.



(a) Given a similar 4 by 4 grid of dots, there is a total of 20 squares of five different sizes that can be formed with corners on the dots. Draw one example of each size and indicate the number of squares there are of that size.

(b) In a 10 by 10 grid of dots, the number of squares that can be formed with side length $\sqrt{29}$ is two times the number of squares that can be formed with side length 7. Explain why this is true.

(c) Show that the total number of squares that can be formed in a 10 by 10 grid is
$1(9^2) + 2(8^2) + 3(7^2) + 4(6^2) + 5(5^2) + 6(4^2) + 7(3^2) + 8(2^2) + 9(1^2)$.

5. An arithmetic sequence is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, the sequence 2, 11, 20, 29, . . . is an arithmetic sequence. (The “. . .” indicates that this sequence continues without ever ending.)

(a) Find the 11th term in the arithmetic sequence 17, 22, 27, 32, ...

(b) Explain why there is no same number which occurs in each of the following arithmetic sequences:
17, 22, 27, 32, ⋯⋯
13, 28, 43, 58, ⋯⋯

(c) Find a number between 400 and 420 which occurs in both of the following arithmetic sequences:
17, 22, 27, 32, ……
16, 22, 28, 34, ……

6. Jill has a container of small cylindrical rods in six different colours. Each colour of rod has a different length as summarized in the chart.

Colour	Length
Green	3 cm
Pink	4 cm
Yellow	5 cm
Black	7 cm
Violet	8 cm
Red	9 cm

These rods can be attached together to form a pole.

There are 2 ways to choose a set of yellow and green rods that will form a pole 29 cm in length: 8 green rods and 1 yellow rod OR 3 green rods and 4 yellow rods.

(a) How many different sets of yellow and green rods can be chosen that will form a pole 62 cm long? Explain how you obtained your answer.

(b) Among the green, yellow, black and red rods, find, with justification, two colours for which it is impossible to make a pole 62 cm in length using only rods of those two colours.

(c) If at least 81 rods of each of the colours green, pink, violet, and red must be used, how many different sets of rods of these four colours can be chosen that will form a pole 2007 cm in length? Explain how you got your answer.