Algebra 1

►Basic Knowledge

1. Laws of Exponents

The exponent of a number says how many times to use the number in a multiplication.
In this example: $8^2 = 8 \times 8 = 64$
So an Exponent just saves you writing out lots of multiplies!

Example: a^7

$$a^7 = a \times a \times a \times a \times a \times a \times a = aaaaaaa$$

Notice how I just wrote the letters together to mean multiply? We will do that a lot here.

Example: $y^6 = y \times y \times y \times y \times y \times y = yyyyyy$

The Key to the Laws
Writing all the letters down is the key to understanding the Laws

Example: $x^2 x^3 = (xx)(xxx) = xxxxx = x^5$

Which shows that $x^2 x^3 = x^{2+3} = x^5$, but more on that later!

A fractional exponent like 1/n means to take the nth root: $x^{\frac{1}{n}} = \sqrt[n]{x}$

2. Using Exponents in Algebra

You might like to read the page on Exponents first.

1) Whole Number Exponents

The exponent "n" in a^n says how many times to use a in a multiplication:

$$a^n = a \times a ... \times a = n \times a$$

2) Negative Exponents

Example: $5^{-3} = 1/5^3 = 1/125 = 0.008$

A positive exponent a^n is equal to $1/a^{-n}$ (1 divided by the negative exponent) $a^n = \dfrac{1}{a^{-n}}$

3) Variables with Exponents

What is a Variable with an Exponent?
A Variable is a symbol for a number we don't know yet. It is usually a letter like x or y.

An exponent (such as the 2 in x^2) says how many times to use the variable in a multiplication.
Example: $y^2 = yy$
(yy means y *multipled by* y, because in Algebra putting two letters next to each other means to multiply them)
Likewise $z^3 = zzz$ *and* $x^5 = xxxxx$

4) Exponents of 1 and 0
a) Exponent of 1
If the exponent is 1, then you just have the variable itself (example $x^1 = x$)
We usually don't write the "1", but it sometimes helps to remember that x is also x^1

b) Exponent of 0
If the exponent is 0, then you are not multiplying by anything and the answer is just "1"
(example $y^0 = 1$)

5) Multiplying Variables with Exponents
So, how do you multiply this:
$(y^2)(y^3)$
We know that $y^2 = yy$, and $y^3 = yyy$ so let us write out all the multiplies:
$y^2\ y^3 = yyyyy$
That is 5 "y"s multiplied together, so the new exponent must be 5:
$y^2\ y^3 = y^5$
But why count the "y"s when the exponents already tell us how many?
The exponents tell us that there are two "y"s multiplied by 3 "y"s for a total of 5 "y"s:
$y^2\ y^3 = y^{2+3} = y^5$
So, the simplest method is to just add the exponents! (Note: this is one of the Laws of Exponents)

6) Negative Exponents
Negative Exponents Mean Dividing!

$$x^{-3} = \frac{1}{x}, \quad x^{-2} = \frac{1}{x^2}, \quad x^{-3} = \frac{1}{x^3}$$

Get familiar with this idea, it is very important and useful!

7) Dividing

$$\frac{y^3}{y^2} = \frac{yyy}{yy} = y^{3-2} = y^1 = y$$

OR, you could have done it like this:

$$\frac{y^3}{y^2} = y^3 y^{-2} = y^{3-2} = y^1 = y$$

So, it just subtract the exponents of the variables you are dividing by!

You can see what is going on if you write down all the multiplies, then "cross out" the variables that are both top and bottom:

$$\frac{x^3yz^2}{xy^2z^2} = \frac{xxxyzz}{xyyzz} = \frac{xx}{y} = \frac{x^2}{y}$$

3. Complex Fractions

A complex fraction is a fraction where the numerator, denominator, or both contain a fraction.

Example 1: $\frac{3}{1/2}$ is a complex fraction. The numerator is 3 and the denominator is 1/2.

Example 2: $\frac{3/7}{9}$ is a complex fraction. The numerator is 3/7 and the denominator is 9.

Example 3: $\frac{3/4}{9/10}$ is a complex fraction. The numerator is 3/4 and the denominator is 9/10.

Rule: To multiply two complex fractions, convert the fractions to simple fractions and follow the steps you use to multiply two simple fractions.

Example: Calculate $\frac{3/4}{1/2} \times \frac{8}{3/16}$.

Solution 1: Convert the numerator $\frac{3/4}{1/2}$ to a simple fraction. $\frac{3/4}{1/2}$ can be written

$$\frac{3/4}{1/2} = \frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \times \frac{2}{1} = \frac{3\times2}{2\times2} = \frac{3}{2}\times1 = \frac{3}{2}.$$

Convert the denominator to a simple fraction. $\frac{8}{3/16}$ can be written

$$\frac{8}{3/16} = \frac{8}{1} \div \frac{3}{16} = \frac{8}{1} \times \frac{16}{3} = \frac{8\times16}{1\times3} = \frac{128}{3}.$$

The problem $\frac{3/4}{1/2} \times \frac{8}{3/16}$ can now be written $\frac{3}{2} \times \frac{128}{3}$.

Multiply the numerators and multiply the denominators. $\frac{3\times128}{2\times6}$.

The problem is reduced by $\dfrac{3\times 2\times 64}{2\times 3}=\dfrac{2}{2}\times\dfrac{3}{3}\times\dfrac{64}{1}=64$.

Solution 2: Convert the 8 to a fraction, multiply the numerators, and multiply the denominators.

$$\frac{3/4}{1/2}\times\frac{8}{3/16}=\frac{\frac{3\times 8}{4\times 1}}{\frac{1\times 3}{2\times 16}}=\frac{\frac{6}{1}}{\frac{3}{32}}\times\frac{\frac{32}{1}}{\frac{32}{1}}=\frac{6\times 32}{3}=64.$$

4. Remember these Identities

1) $(x+a)(x+b)=x^2+(a+b)x+ab$

2) $(a\pm b)^2=a^2\pm 2ab+b^2$

3) $(a\pm b)^3=a^3\pm 3a^2b+3ab^2\pm b^3$

4) $a^2-b^2=(a-b)(a+b)$

5) $a^3\pm b^3=(a\pm b)(a^2\mp ab+b^2)$

6) $a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+ab^{n-2}+b^{n-1})$ (n is a positive integer)

7) $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$

8) $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

9) $\dfrac{a}{b}\pm\dfrac{c}{b}=\dfrac{a\pm c}{b}$

10) $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad\pm bc}{bd}$

11) $\dfrac{a}{b}\cdot\dfrac{c}{d}=\dfrac{ac}{bd}$

12) $\dfrac{a}{b}\div\dfrac{c}{d}=\dfrac{ad}{bc}$

►Examples

<u>Example 1</u>

1. Find the sum of the following: $\frac{1}{2}(1-\frac{1}{3})+\frac{1}{2}(\frac{1}{3}-\frac{1}{5})+\frac{1}{2}(\frac{1}{5}-\frac{1}{7})+...+\frac{1}{2}(\frac{1}{51}-\frac{1}{53})$

Solution:

This sequence "telescopes" if you multiply it our and rearrange the terms. It will look like:

$$\left(\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{3}\right)+\left(\frac{1}{2}\cdot\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{5}\right)+\left(\frac{1}{2}\cdot\frac{1}{5}-\frac{1}{2}\cdot\frac{1}{7}\right)+\cdots+\left(\frac{1}{2}\cdot\frac{1}{49}-\frac{1}{2}\cdot\frac{1}{51}\right)+\left(\frac{1}{2}\cdot\frac{1}{51}-\frac{1}{2}\cdot\frac{1}{53}\right)$$

$$=\frac{1}{2}+\left(-\frac{1}{2}\cdot\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{3}\right)+\left(-\frac{1}{2}\cdot\frac{1}{5}+\frac{1}{2}\cdot\frac{1}{5}\right)+\left(-\frac{1}{2}\cdot\frac{1}{7}+\frac{1}{2}\cdot\frac{1}{7}\right)+\cdots+\left(-\frac{1}{2}\cdot\frac{1}{51}+\frac{1}{2}\cdot\frac{1}{51}\right)-\frac{1}{2}\cdot\frac{1}{53}$$

$$=\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{53}=\frac{1}{2}\left(1-\frac{1}{53}\right)=\frac{1}{2}\left(\frac{52}{53}\right)=\frac{26}{53}$$

Example 2

$$\frac{\frac{a}{c}+\frac{a}{b}+1}{\frac{b}{a}+\frac{b}{b}+1}=11,$$

where a, b, and c are positive integers, find the number of different ordered triples (a, b, c) such that $a+2b+c\le 40$.

Solution:

$$\frac{\frac{\text{ab +ac+ bc}}{bc}}{\frac{\text{bc+ab +ac}}{ac}}=11 \qquad \frac{ac}{bc}=11\text{, or a=11b}$$

By substitution, the condition $a+2b+c=40$ becomes $13b+c=40$.

Since b and c are positive integers, then b can only take on the values 1, 2, or 3. The values of a correspond directly to the values of b, since $a = 11b$.

If b=3, there is one corresponding value of c. When b = 2, there are 14 possible values of c. Finally if b=1, there are 27 possible values of c.

Therefore, the number of different ordered triples satisfying the given conditions is 1+14+27=42.

Example 3

There are four unequal, positive integers a, b, c, and N such that N=5a+3b+5c. It is also true that N=4a+5b+4c and N is between 131 and 150. What is the value of a+b+c?

Solution:

We are told that $N=5a+3b+5c$ (1) and $N=4a+5b+4c$ (2). Multiply equation (1) by 4 to get $4N=20a+12b+20c$ (3). Similarly, multiply equation (2) by 5 to get $5N=20a+25b+20c$ (4). Subtract equation (3) from equation (4) to get $N=13b$.

Since N and b are both positive integers with $131<N<150$, N must be a multiple of 13. The only possible value for N is 143, when $b=11$.

Substitute $N = 143$ and $b=11$ into equation (1) to get $143=5a+3(11)+5c$

110=5a+5c

Thus, the value of $a+b+c$ is $22+11=33$.

► **In-class questions**

1. The number 2005 can be written in the form a^2-b^2, where a and b are positive integers less than 1000, in exactly one way. What is the value of a^2+b^2?

2. Given that $z^2+z-3=5$, what is the numerical value of $z^4+2z^3-5z^2-6z+5$?

3. Using only odd digits, all possible two-digit numbers are formed. What is the sum of all such numbers?

4. Given that $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}+\cdots=1$, determine $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}+\cdots$.

5. For $n = 1, 2, 3, \ldots$, the *n*th *harmonic number* H_n is defined by $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$.

Find the sum $H_1 + H_2 + \ldots + H_n$?

6. The range of a list of numbers is the difference between the largest number and the smallest number in the list. For example, the range of the list 1, 5, 1, 6, 3 is 6-1 = 5.

a) Determine the range of the list 7, 13, 4, 9, 6.
b) The list 11, 5, a, 13, 10 has a range of 12. Determine the two possible values of a.
c) The list $6+2x^2,\ 6+4x^2,\ 6,\ 6+5x^2$ has a range of 80. Determine the two possible values of *x*.
d) The list $5x+3y,\ 0,\ x+y,\ 3x+y$ has a range of 19. If x and y are integers with $x > 0$ *and* $y > 0$, determine the values of x *and* y.