Algebra 1 (Equations 1)

1. Algebra

Algebra is a simple language, used to create mathematical models of real-world situations and to handle problems that we can't solve using just arithmetic.

Algebraic Expressions

An *algebraic expression* is one or more algebraic terms in a phrase. It can include variables, constants, and operating symbols, such as plus and minus signs. It's only a phrase, not the whole sentence, so it doesn't include an equal sign.

Algebraic expression: $3x^2 + 2y + 7xy + 5$

In an algebraic expression, terms are the elements separated by the plus or minus signs. This example has four terms, $\mathbf{3x^2}$, **2y**, **7xy**, and **5**. Terms may consist of variables and coefficients, or constants.

2. Equation

An equation says that two things are equal. It will have an equals sign "=" like this:

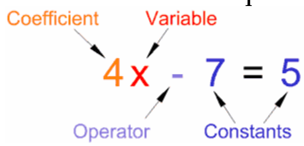
$$x \quad + \quad 2 \quad = \quad 6$$

That equation says: what is on the left (x + 2) is equal to what is on the right (6)
So an equation is like a statement "*this* equals *that*"

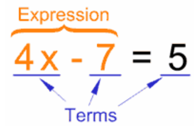
Parts of an Equation

So that people can discuss equations, there are names for different parts (better than saying "that thingy there"!)

Here we have an equation that says 4x - 7 equals 5, and all its parts:



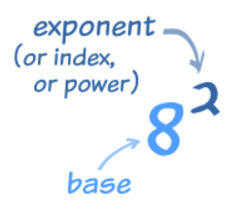
A **Variable** is a symbol for a number we don't know yet. It is usually a letter like x or y.
A number on its own is called a **Constant**.
A **Coefficient** is a number used to multiply a variable (4*x* means 4 times x, so 4 is a coefficient)
An **Operator** is a symbol (such as +, ×, etc) that represents an operation.



A **Term** is either a single number or a variable, or numbers and variables multiplied together.
An **Expression** is a group of terms (the terms are separated by + or - signs).

So, now we can say things like "that expression has only two terms", or "the second term is a constant", or even "are you sure the coefficient is really 4?"

1) Exponents



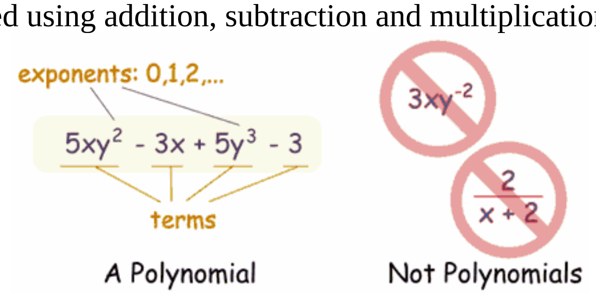
The exponent (such as the 2 in x^2) says how many times to use the value in a multiplication.
Examples:
$8^2 = 8 \times 8 = 64$
$y^3 = y \times y \times y$
$y^2z = y \times y \times z$

Exponents make it easier to write and use many multiplications

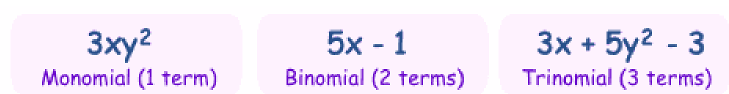
Example: $\mathbf{y^4z^2}$ is easier than $\mathbf{y \times y \times y \times y \times z \times z}$, or even **yyyyzz**

2) Polynomial
Example of a Polynomial: $3x^2 + x - 2$

A polynomial can have constants, variables and the exponents 0,1,2,3,...
And they can be combined using addition, subtraction and multiplication, but not division!



3) Monomial, Binomial, Trinomial
There are special names for polynomials with 1, 2 or 3 terms:



Like Terms
Like Terms are terms whose variables (and their exponents such as the 2 in x^2) are the same.
In other words, terms that are "like" each other. (Note: the coefficients can be different)

Example:
$(1/3)xy^2$ $\quad$ $-2xy^2$ $\quad$ $6xy^2$
Are all like terms because the variables are all xy^2.

3. Quadratic equation

A **quadratic equation** is a trinomial of the form $ax^2 + bx + c = 0$. There are three main ways of solving quadratic equations that are covered below.

1) Factoring

If a quadratic equation can be factored, then it can be written as a product of two binomials. Since the trinomial is equal to 0, one of the two binomial factors must also be equal to zero. By factoring the quadratic equation, we can equate each binomial expression to zero and solve each for x.

Example: Solve $x^2 + 2x = 15$ by factoring.

$$x^2 + 2x = 15 : x^2 + 2x = 15$$
$$: x^2 + 2x - 15 = 0$$
$$: (x+5)(x-3) = 0$$

$$x + 5 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = -5 \qquad\qquad x = 3$$

$\therefore$ the solution to the quadratic equation is $x = -5$ or $x = 3$.

2) Completing the Square

Quadratic equations cannot always be solved by factoring. They can always be solved by the method of completing the squares. To complete the square means to convert a quadratic to its standard form.

We want to convert $ax^2 + bx + c = 0$ to a statement of the form $a(x - h)^2 + k = 0$. To do this, we would perform the following steps:

a) Group together the ax^2 and bx terms in parentheses and factor out the coefficient a.

$$a(x^2 + \frac{b}{a}x) + c = 0$$

b) In the parentheses, add and subtract $(b/2a)^2$, which is half of the x coefficient, squared.

$$a(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2) + c = 0$$

c) Remove the term $-(b/2a)^2$ from the parentheses. Don't forget to multiply the term by a, when removing from parentheses.

$$a(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2) - a\left(\frac{b}{2a}\right)^2 + c = 0$$

d) Factor the trinomial in parentheses to its perfect square form, $(x + b/2a)^2$.

$$a\left(x+\frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c = 0$$

e) Transpose (or shift) all other terms to the other side of the equation and divide each side by the constant a.

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

f) Take the square root of each side of the equation.

$$\left(x+\frac{b}{2a}\right) = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

g) Transpose the term -b/2a to the other side of the equation, isolating x.

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

The quadratic equation is now solved for x. The method of completing the square seems complicated since we are using variables a, b and c.

Example: Solve $x^2 + 4x - 13 = 0$ by completing the square.

$$\begin{aligned}
x^2 + 4x - 13 = 0 \;&:\; (x^2 + 4x) - 13 = 0 \\
&:\; (x^2 + 4x + 4 - 4) - 13 = 0 \\
&:\; (x^2 + 4x + 4) - 4 - 13 = 0 \\
&:\; (x^2 + 4x + 4) - 17 = 0 \\
&:\; (x+2)^2 - 17 = 0 \\
&:\; (x+2)^2 = 17 \\
&:\; (x+2) = \pm\sqrt{17} \\
&:\; x = -2 \pm \sqrt{17}
\end{aligned}$$

$$x = -2 + \sqrt{17} \text{ or } x = -2 - \sqrt{17}$$

Note: For more examples of solving a quadratic equation by completing the square, see questions #1 and #2 in the Additional Examples section at the bottom of the page.

3) The Quadratic Formula

The method of completing the square can often involve some very complicated calculations involving fractions. To make calculations simpler, a general formula for solving quadratic equations, known as the quadratic formula, was derived. To solve quadratic equations of the form $ax^2 + bx + c = 0$, substitute the coefficients a, b, and c into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The value contained in the square root of the quadratic formula is called the **discriminant**. If,

$b^2 - 4ac > 0$ – There are 2 real roots, $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

$b^2 - 4ac = 0$ – There is 1 real root, $x = \dfrac{-b}{2a}$

$b^2 - 4ac < 0$ – There are no real roots.

Example: Solve $4x^2 - 5x + 1 = 0$ using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(4)(1)}}{2(4)} = \frac{5 \pm \sqrt{25-16}}{8} = \frac{5 \pm \sqrt{9}}{8} = \frac{5 \pm 3}{8}$$

$$x = \frac{5+3}{8} = 1 \quad \text{or} \quad x = \frac{5-3}{8} = \frac{1}{4}$$

3. Solving Inequalities

Sometimes we need to solve Inequalities like these:

Symbol	Words	Example
>	greater than	$x + 3 > 2$
<	less than	$7x < 28$
≥	greater than or equal to	$5 \geq x - 1$
≤	less than or equal to	$2y + 1 \leq 7$

1) Solving
Our aim is to have x (or whatever the variable is) on its own on the left of the inequality sign: Something like: $x < 5$ or: $y \geq 11$.

2) How to Solve
Some things you do will change the direction!
< would become >
> would become <
≤ would become ≥
≥ would become ≤

Safe Things To Do
These are things you can do without affecting the direction of the inequality:

- Add (or subtract) a number from both sides
- Multiply (or divide) both sides by a **positive** number
- Simplify a side

Example: $3x < 7 + 3$
You can simplify $7 + 3$ without affecting the inequality: $3x < 10$

But these will change the direction of the inequality ("<" becomes ">" for example):
- Multiply (or divide) both sides by a **negative** number
- Swapping left and right hand sides

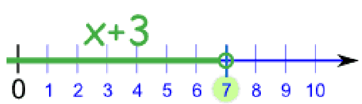
Example: $2y + 7 < 12$

When you swap the left and right hand sides, you must also change the direction of the inequality: $12 > 2y + 7$

3) Adding or Subtracting a Value
We can often solve inequalities by adding (or subtracting) a number from both sides, like this:
Solve: $x + 3 < 7$

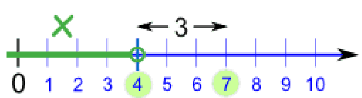
If we subtract 3 from both sides, we get:
$x + 3 \mathbf{- 3} < 7 \mathbf{- 3}$
$x < 4$
And that is our solution: $\mathbf{x < 4}$

4) What did we do?
We went from this:



$x+3 < 7$

To this:



$x < 4$

And that works well for **adding** and **subtracting**, because if you add (or subtract) the same amount from both sides, it does not affect the inequality

Example: Alex has more coins than Billy. If both Alex and Billy get three more coins each, Alex will still have more coins than Billy.

What If I Solve It, But "x" Is On The Right?
No matter, just swap sides, but reverse the sign so it still "points at" the correct value!

Example: $12 < x + 5$

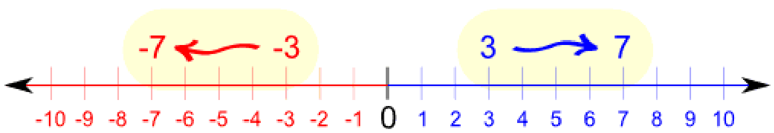
If we subtract 5 from both sides, we get:
$12 \mathbf{- 5} < x + 5 \mathbf{- 5}$
$7 < x$
But it is normal to put "x" on the left hand side, so let us flip sides: $x > 7$

5) Positive Values
Everything is fine if you want to multiply or divide by a positive number:
Solve: 3y < 15

If we divide both sides by 3 we get:
3y/3 < 15/3
y < 5
And that is our solution: y < 5

6) Negative Values
When you multiply or divide by a negative number you have to reverse the inequality.

For example, from 3 to 7 is an increase, but from -3 to -7 is a decrease.



-7 < -3 7 > 3

Solve: - 2y < - 8

Let us divide both sides by -2 and **reverse the inequality**!
-2y < -8
-2y/**-2** > -8/**-2**
y > 4
And that is the correct solution: **y > 4**

Remember:
When multiplying or dividing by a negative number, **reverse** the inequality.

7) Multiplying or Dividing by Variables

Solve: bx < 3b
It seems easy just to divide both sides by **b**, which would give us: **x < 3**
But if **b** is **negative** we need to reverse the inequality like this: **x > 3**
But we don't know if b is positive or negative, so **we can't answer this one**!

To help you understand, imagine replacing **b** with **1** or **-1** in that example:

- if **b is 1**, then the answer is simply **x < 3**
- but if **b is -1**, then you would be solving **-x < -3**, and the answer would be **x > 3**

Do not try dividing by a variable to solve an inequality (unless you know the variable is always positive, or always negative).

A Bigger Example: Solve: (x-3)/2 < -5
First, let us clear out the "/2" by multiplying both sides by 2.
Because you are multiplying by a positive number, the inequalities will not change.
(x-3)/2 × 2 < -5 × 2
(x-3) < -10

Now add 3 to both sides:
x - 3 **+ 3** < -10 + **3**
x < -7
And that is our solution: **x < -7**

Two Inequalities At Once!
Solve: - 2 < (6 - 2x)/3 < 4

First, let us clear out the "/3" by multiplying each part by 3:
Because you are multiplying by a positive number, the inequalities will not change.
-6 < 6 - 2x < 12

Now subtract 6 from each part:
-12 < -2x < 6

Now multiply each part by - (1/2).
Because you are multiplying by a negative number, the inequalities change direction.
6 > x > -3

But to be neat it is better to have the smaller number on the left, larger on the right. So let us swap them over (and make sure the inequalities point correctly): -3 < x < 6

4. Basic Formulae

$$(x+a)(x+b) = x^2 + (a+b)x + ab$$
$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$
$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$
$$a^2 - b^2 = (a-b)(a+b)$$
$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$
$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}) \quad (n \text{ is a positive integer})$$
$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$
$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

In-class questions

1. Let A, M, and C be nonnegative integers such that $A+M+C=10$. What is the maximum value of $A\cdot M\cdot C+A\cdot M+M\cdot C+C\cdot A$?

2. The mean, median, unique mode, and range of a collection of eight integers are all equal to 8. What is the largest integer that can be an element of this collection?

3. Let a, b, and c be real numbers such that $a-7b+8c=4$ *and* $8a+4b-c=7$. Then what is $a^2-b^2+c^2$?

4. Let *S* be the set of the 2005 smallest positive multiples of 4, and let *T* be the set of the 2005 smallest positive multiples of 6. How many elements are common to *S* and *T*?

5. What is the average (mean) of all 5-digit numbers that can be formed by using each of the digits 1, 3, 5, 7, and 8 exactly once?

6. Suppose that the number *a* satisfies the equation $4=a+a^{-1}$. What is the value of a^4+a^{-4}?

7. How many ordered pairs (*m*, *n*) of positive integers, with $m>n$, have the property that their squares differ by 96?

8. Brian writes down four integers $w>x>y>z$ whose sum is 44. The pairwise positive differences of these numbers are 1, 3, 4, 5, 6, and 9. What is the sum of the possible values for *w*?