AP Calculus Lesson Thirteen Notes

Chapter 6 Additional Techniques of Integration

6.1 Integration by Parts
6.2 Trigonometric Substitutions

6.1 Integration by Parts

The following result is useful for simplifying certain types of integrals

Integration by parts Formula (6.1.1)

If $u = f(x)$ and $v = g(x)$, where f' and g' are continuous, then $$\int u\,dv = uv - \int v\,du.$$

Proof
By the product rule,

$$D_x[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Or, equivalently,

$$f(x)g'(x) = D_x[f(x)g(x)] - g(x)f'(x).$$

Integrating both sides of the previous equation gives us

$$\int f(x)g'(x)\,dx = \int D_x[f(x)g(x)]\,dx - \int g(x)f'(x)\,dx.$$

By Theorem (5.29) the first integral on the right side equals $f(x)g(x)+C$. since we obtain another constant of integration from the second integral, it is unnecessary to include C in the formula; that is,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

Since $du = f'(x)dx$ and $dv = g'(x)$, the preceding formula may be written as in (6.1).

When applying formula (6.1) to a given integral, we begin by letting one part of the integrand correspond to *dv*. The expression chosen for *dv* must include the differential *dx*. After selecting *dv*, the remaining part of the integrand is designated by *u*. since this process involves splitting the integrand into two parts, the use of (6.1) is referred to as **integrating by parts.** A proper choice for *dv* is crucial. A good rule of thumb is to choose the most complicated part of the integrand that can be readily integrated. The following examples illustrate this important method of integration.

Example 1

Find $\int xe^{2x}dx$

Solution

There are four possible choices for *dv*, namely *dx*, *xdx*, $e^{2x}dx$, or $xe^{2x}dx$. If we let $dv = e^{2x}dx$, then the remaining part of the integrand is *u*; that is, *u*=*x*. to find *v* we integrate *dv*, obtaining $v = \frac{1}{2}e^{2x}$. Note that a constant of integration is not added at this stage of the solution. (in Exercise 51 you are asked to prove that if a constant *is* added to *v*, the same result is obtained). Since *u*=*x* we see that *du*=*dx*. For ease of reference it is convenient to display these expressions as follows:

$$dv = e^{2x}\,dx \qquad u = x$$
$$v = \tfrac{1}{2}e^{2x} \qquad du = dx.$$

Substituting these expressions in Formula (6.1), that is, *integrating by parts,* we obtain

$$\int xe^{2x}\,dx = x(\tfrac{1}{2}e^{2x}) - \int \tfrac{1}{2}e^{2x}\,dx.$$

The integral on the right side may be found by means of Theorem (7.21). this gives us

$$\int xe^{2x}\,dx = \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C.$$

It takes considerable practice to become proficient in making a suitable choice for *dv*. To illustrate, if we had chosen *dv*=*x dx* in Example 1, then it would have been necessary to let $u = e^{2x}$, giving us

$$dv = x\,dx \qquad u = e^{2x}$$
$$v = \tfrac{1}{2}x^2 \qquad du = 2e^{2x}\,dx.$$

Integrating by parts, we obtain

$$\int xe^{2x}\,dx = \tfrac{1}{2}x^2e^{2x} - \int x^2e^{2x}\,dx.$$

Since the exponent associated with *x* has increased, the integral on the right is more complicated than the given integral. This indicates an incorrect choice for *dv.*

Example 2

Evaluate $\int x\sec^2 x dx$

Solution

Since $\sec^2 x$ can be integrated readily, we let $dv = \sec^2 x dx$. The remaining part of the integrand is *x* and hence we must let *u*=*x*. thus

$$dv = \sec^2 x\,dx \qquad u = x$$
$$v = \tan x \qquad du = dx$$

And integration by parts gives us

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$$
$$= x\tan x - \ln|\sec x| + C.$$

In this event integration by parts leads to

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx.$$

If we now substitute in (a), we obtain

$$\int e^x \cos x\, dx = e^x \sin x - \left[e^x \sin x - \int e^x \cos x\, dx\right]$$

Which reduces to

$$\int e^x \cos x\, dx = \int e^x \cos x\, dx.$$

Although this is a true statement, it is not a solution to the problem! Incidentally, the integral in Example 5 *can* be evaluated by using $dv = e^x dx$ for *both* the first and second applications of the integration by parts formula.

Integration by parts may sometimes by employ to obtain **reduction formulas** for integrals. Such formulas can be used to write an integral involving powers of an expression in terms of integrals that involve lower powers of the expression.

Example 3

Find a reduction formula for $\int \sin^n x dx$

Solution

Let

$$dv = \sin x\, dx \qquad u = \sin^{n-1} x$$

$$v = -\cos x \qquad du = (n-1)\sin^{n-2} x \cos x\, dx.$$

Integrating by parts,

$$\int \sin^n x\, dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx.$$

Since $\cos^2 x = 1 - \sin^2 x$, we may write

$$\int \sin^n x\, dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx.$$

Consequently,

$$\int \sin^n x\, dx + (n-1)\int \sin^n x\, dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\, dx.$$

The left side of the last equation reduces to $n\int \sin^n x dx$. Dividing both sides by *n*, we obtain

$$\int \sin^n x\, dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{(n-1)}{n}\int \sin^{n-2} x\, dx.$$

Example 4

Evaluate $\int \sin^4 x dx$

Solution

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx$$
$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$
$$= \tfrac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx.$$

We apply a half-angle formula again and write

$$\cos^2 2x = \tfrac{1}{2}(1 + \cos 4x) = \tfrac{1}{2} + \tfrac{1}{2}\cos 4x.$$

Substituting in the last integral and simplifying gives us

$$\int \sin^4 x\,dx = \tfrac{1}{4}\int (\tfrac{3}{2} - 2\cos 2x + \tfrac{1}{2}\cos 4x)\,dx$$
$$= \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C.$$

Integrals involving only products of sin*x* and cos*x* may be evaluated using the following guidelines.

(6.1.2) guidelines for evaluating integrals of the form $\int \sin^m x \cos^n x dx$

1. if both *m* and *n* are even integers, use half-angle formulas for $\sin^2 x$ and $\cos^2 x$ to reduce the exponents by one-half.
2. if *n* is an odd integer, write the integral as
$$\int \sin^m x \cos^n x\,dx = \int \sin^m x \cos^{n-1} x \cos x\,dx$$
3. if *m* is an odd integer, write the integral as
$$\int \sin^m x \cos^n x\,dx = \int \sin^{m-1} x \cos^n x \sin x\,dx$$
And express $\sin^{m-1}$ in terms of $\cos x$ by using the trigonometric identity $\sin^2 x = 1 - \cos^2 x$. Use the substation $u = \cos x$ to evaluate the resulting integral/

Example 5

Evaluate $\int \cos^3 x \sin^4 x dx$

Solution

Using Guideline 2 of (6.2),

$$\int \cos^3 x \sin^4 x\,dx = \int \cos^2 x \sin^4 x \cos x\,dx$$
$$= \int (1 - \sin^2 x)\sin^4 x \cos x\,dx.$$

If we let *u*=sin*x,* then *du*=cos*x* and the integral may be written

$$\int \cos^3 x \sin^4 x\,dx = \int (1-u^2)u^4\,du = \int (u^4 - u^6)\,du$$
$$= \tfrac{1}{5}u^5 - \tfrac{1}{7}u^7 + C$$
$$= \tfrac{1}{5}\sin^5 x - \tfrac{1}{7}\sin^7 x + C.$$

(6.1.3) Guidelines for evaluating integrals of the form $\int \tan^m x \sec^n x\,dx$

1. if n is an even integer, write the integral as

$$\int \tan^m x \sec^{n-2} x \sec^2 x\,dx$$

And express $\sec^{n-2} x$ in terms of tan x by using the trigonometric identity $\sec^2 x = 1 + \tan^2 x$. The substitution u=tan$x$ leads to a simple integral.

2. if m is an odd integer, write the integral as

$$\int \tan^{m-1} x \sec^{n-1} x \sec x \tan x\,dx.$$

since m-1 is even, $\tan^{m-1}$ may be expressed in terms of $\sec x$ by means of the identity $\tan^2 x = \sec^2 x - 1$. The substitution $u = \sec x$ then leads to a form that is readily integrable.

3. if n is odd and m is even, then another method such as integration by parts should be used.

Example 5

Evaluate $\int \tan^2 x \sec^4 x\,dx$

Solution

Using Guideline 1 of (6.1.3)

$$\int \tan^2 x \sec^4 x\,dx = \int \tan^2 x \sec^2 x \sec^2 x\,dx$$
$$= \int \tan^2 x(\tan^2 x + 1)\sec^2 x\,dx.$$

If we let $u = \tan x$, then $du = \sec^2 x dx$ and

$$\int \tan^2 x \sec^4 x\,dx = \int u^2(u^2+1)\,du$$
$$= \int (u^4 + u^2)\,du$$
$$= \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 + C$$
$$= \tfrac{1}{5}\tan^5 x + \tfrac{1}{3}\tan^3 x + C.$$

Example 6

Evaluate $\int \tan^3 x \sec^5 x\,dx$

Solution
Using Guideline 2 of (6.1.3),

$$\int \tan^3 x \sec^5 x\,dx = \int \tan^2 x \sec^4 x(\sec x \tan x)\,dx$$

$$= \int (\sec^2 x - 1)\sec^4 x(\sec x \tan x)\,dx.$$

Substituting $u = \sec x$ and $du = \sec x \tan x dx$, we obtain

$$\int \tan^3 x \sec^5 x\,dx = \int (u^2 - 1)u^4\,du$$

$$= \int (u^6 - u^4)\,du$$

$$= \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C$$

$$= \tfrac{1}{7}\sec^7 x - \tfrac{1}{5}\sec^5 x + C.$$

Integrals of the form $\int \cot^m x \csc^n x dx$ may be evaluated in similar fashion.

Finally, integrals of the form $\int \sin mx \cos nx dx$ may be evaluated by means of the product formulas (see(1.36)), as illustrated in the next example.

6.2 Trigonometric Substitutions

If an integrand contains one of the expressions $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, where $a > 0$, the radical sign can be eliminated by using the trigonometric substitution listed in the following table.

Given expression	Trigonometric substitution
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$

When making a trigonometric substitution we shall assume that θ is in the range of the corresponding inverse trigonometric function. Thus, for the substitution $x = a\sin\theta$ we have $-\pi/2 \le \theta \le \pi/2$. In this event, $\cos\theta \ge 0$ and

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta}$$

$$= \sqrt{a^2(1 - \sin^2 \theta)}$$

$$= \sqrt{a^2 \cos^2 \theta}$$

$$= a \cos \theta.$$

If $\sqrt{a^2 - x^2}$ occurs in a denominator, we add the restriction $|x| \ne a$, or equivalently, $-\pi/2 < \theta < \pi/2$.

Example 1

Evaluate $\int \dfrac{1}{x^2\sqrt{16 - x^2}}\,dx$

Solution

Let $x = 4\sin\theta$, where $-\pi/2 < \theta < \pi/2$. It follows that

$$\sqrt{16 - x^2} = \sqrt{16 - 16\sin^2\theta} = 4\sqrt{1 - \sin^2\theta} = 4\cos\theta.$$

Since $x = 4\sin\theta$, we have $dx = 4\cos\theta d\theta$. Substituting in the given integral,

$$\begin{aligned}\int \frac{1}{x^2\sqrt{16 - x^2}}\,dx &= \int \frac{1}{(16\sin^2\theta)4\cos\theta}\,4\cos\theta\,d\theta \\ &= \frac{1}{16}\int \frac{1}{\sin^2\theta}\,d\theta \\ &= \frac{1}{16}\int \csc^2\theta\,d\theta \\ &= -\frac{1}{16}\cot\theta + C.\end{aligned}$$

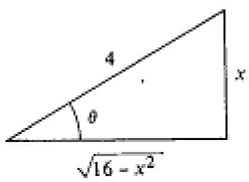

FIGURE 9.1 $\frac{x}{4} = \sin\theta$

It is now necessary to return to the original variable of integration x. since $\theta = \arcsin(x/4)$, we could express the result as $-\frac{1}{16}\cot\arcsin(x/4)$.

However, since the given integral involves $\sqrt{16 - x^2}$, it is desirable to have the evaluated form also contain this radical. A method for accomplishing this is to use the following geometric device. If $0 < \theta < \pi/2$, then since $\sin\theta = x/4$, we may interpret θ as an acute angle of a right triangle having opposite side and hypotenuse of lengths x and 4, respectively (see Figure 9.1). the length $\sqrt{16 - x^2}$ of the adjacent side is calculated by means of the Pythagorean Theorem. Referring to the triangle we see that

$$\cot\theta = \frac{\sqrt{16 - x^2}}{x}.$$

It can be shown that the last formula is also true if $-\pi/2 < \theta < 0$. Thus, Figure 9.1 may be used whether θ is positive or negative.

Substituting $\sqrt{16 - x^2}\,/\,x$ for $\cos\theta$ in our integral evaluation gives us

$$\begin{aligned}\int \frac{1}{x^2\sqrt{16 - x^2}}\,dx &= -\frac{1}{16}\cdot\frac{\sqrt{16 - x^2}}{x} + C \\ &= -\frac{\sqrt{16 - x^2}}{16x} + C.\end{aligned}$$

If an integrand contains $\sqrt{a^2 + x^2}$, where $a > 0$, then the substitution $x = a\tan\theta$ will eliminate the radical sign. When using this substitution it will be assumed that θ is in the range of the inverse tangent function; that is, $-\pi/2 < \theta < \pi/2$. In this event $\sec\theta > 0$ and

$$\begin{aligned}\sqrt{a^2 + x^2} &= \sqrt{a^2 + a^2\tan^2\theta} \\ &= \sqrt{a^2(1 + \tan^2\theta)} \\ &= \sqrt{a^2\sec^2\theta} \\ &= a\sec\theta.\end{aligned}$$

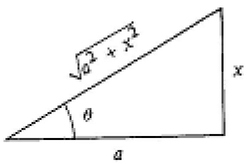

FIGURE 9.2 $\frac{x}{a} = \tan\theta$

After making this substitution and evaluating the resulting trigonometric integral, it is necessary to return to the variable *x*. the preceding formulas show that

$$\tan\theta = \frac{x}{a} \quad \text{and} \quad \sec\theta = \frac{\sqrt{a^2+x^2}}{a}.$$

As in the solution of Example 1, the trigonometric functions of θ can be found by referring to the triangle in Figure 9.2, whether θ is positive or negative.

Example 2

Evaluate $\int \frac{1}{\sqrt{4+x^2}}\,dx$

Solution

Let us substitute as follows:

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta.$$

Consequently,

$$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta} = 2\sqrt{1+\tan^2\theta} = 2\sqrt{\sec^2\theta} = 2\sec\theta$$

and

$$\begin{aligned}\int \frac{1}{\sqrt{4+x^2}}\,dx &= \int \frac{1}{2\sec\theta}\, 2\sec^2\theta\, d\theta \\ &= \int \sec\theta\, d\theta \\ &= \ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

Since $\tan\theta = x/2$ we see from the triangle in Figure 9.3 that

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}$$

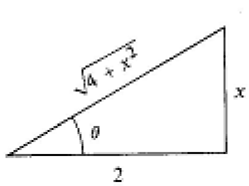

FIGURE 9.3 $\frac{x}{2} = \tan\theta$

And hence

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$$

The expression on the right may be written

$$\ln\left|\frac{\sqrt{4+x^2}+x}{2}\right| + C = \ln|\sqrt{4+x^2}+x| - \ln 2 + C.$$

Since $\sqrt{4+x^2} + x > 0$ for all *x*, the absolute value sign is unnecessary. If we also let $D = -\ln 2 + C$, then

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \ln(\sqrt{4+x^2} + x) + D.$$

For integrands containing $\sqrt{x^2 - a^2}$ we substitute $x = a\sec\theta$, where θ is chosen in the range of the inverse secant function; that is, either $0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$. In this case, $\tan\theta \ge 0$ and

$$\begin{aligned}\sqrt{x^2 - a^2} &= \sqrt{a^2\sec^2\theta - a^2}\\ &= \sqrt{a^2(\sec^2\theta - 1)}\\ &= \sqrt{a^2\tan^2\theta}\\ &= a\tan\theta.\end{aligned}$$

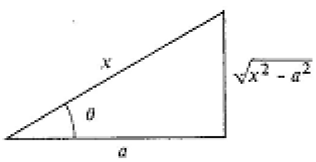

FIGURE 9.4 $\frac{x}{a} = \sec\theta$

Since $$\sec\theta = \frac{x}{a} \quad \text{and} \quad \tan\theta = \frac{\sqrt{x^2 - a^2}}{a}$$

It follows that we may refer to the triangle in Figure 9.4 when changing from the variable θ to the variable *x*.

Example 3

Evaluate $\int \frac{\sqrt{x^2 - 9}}{x}dx$

Solution

Let us substitute as follows:

$$x = 3\sec\theta, \qquad dx = 3\sec\theta\tan\theta\, d\theta.$$

Consequently,

$$\sqrt{x^2 - 9} = \sqrt{9\sec^2\theta - 9} = 3\sqrt{\sec^2\theta - 1} = 3\sqrt{\tan^2\theta} = 3\tan\theta$$

And, therefore,

$$\begin{aligned}\int\frac{\sqrt{x^2 - 9}}{x}dx &= \int\frac{3\tan\theta}{3\sec\theta}3\sec\theta\tan\theta\, d\theta\\ &= 3\int\tan^2\theta\, d\theta\\ &= 3\int(\sec^2\theta - 1)\, d\theta\\ &= 3(\tan\theta - \theta) + C.\end{aligned}$$

Since $\sec\theta = x/3$ we may refer to the triangle to Figure 9.5 and write

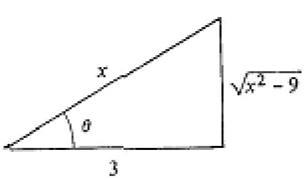

FIGURE 9.5 $\frac{x}{3} = \sec\theta$

$$\begin{aligned}\int\frac{\sqrt{x^2 - 9}}{x}dx &= 3\left[\frac{\sqrt{x^2 - 9}}{3} - \sec^{-1}\left(\frac{x}{3}\right)\right] + C\\ &= \sqrt{x^2 - 9} - 3\sec^{-1}\left(\frac{x}{3}\right) + C.\end{aligned}$$

Hyperbolic functions may also be used to simplify certain integrations. For example, $\cosh^2 u = 1 + \sinh^2 u$ and hence, if an integrand contains the expression $\sqrt{a^2 + x^2}$, the substitution $x = a\sinh u$ leads to *a*coshu.