AP Calculus Class 1

1 Finding Limits

A rock is dropped from a cliff of 100m. Find the velocity of the rock after 3 seconds.

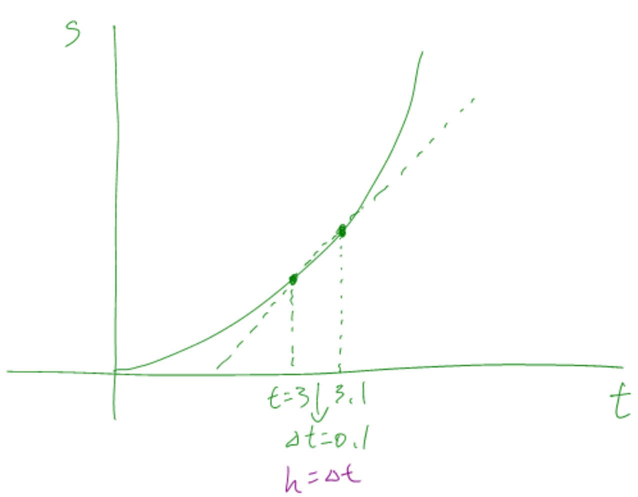
$$V_{avg} = \frac{\Delta s}{\Delta t}$$

$$s(t) = \frac{1}{2}at^2 \rightarrow a = g = 9.8 \, m/s^2$$

$$s(t) = \frac{1}{2}(9.8)t^2 = 4.9t^2$$

Pick an interval of 0.1 s.

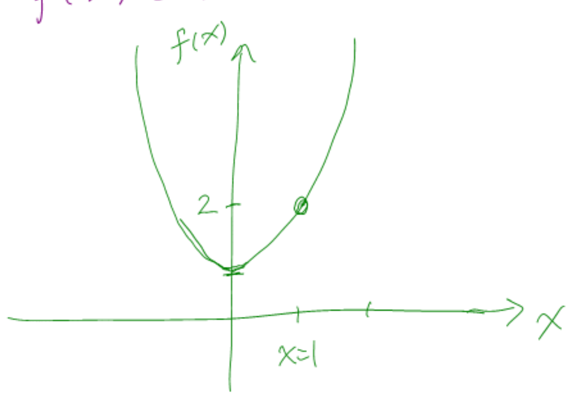
$$V_{avg} = \frac{s(3.1) - s(3)}{3.1 - 3} = \frac{4.9(3.1)^2 - 4.9(3)^2}{0.1} = 29.89 \, m/s$$



$$v = \frac{4.9(t+h)^2 - 4.9(t)^2}{(t+h) - t}$$

$h \rightarrow 0$

$$f(x) = x^2 + 1$$



$$\lim_{x \to 1} f(x) = 2$$

Definition

The limit of $f(x)$ equals to L, written as

$$\lim_{x \to a} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to the number L by taking x sufficiently close to a from both sides, but not equal to a.

Limit Laws

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

1) $\lim_{x \to a} f(x) \pm g(x) = L \pm M$

2) $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$

3) $\lim_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{L}{M}$, provided $M \neq 0$

4) $\lim_{x \to a} cf(x) = cL$, for some constant c.

Theorem 1

If a and c are any real numbers, then

$$\lim_{x \to a} c = c.$$

$\lim_{x \to 2} 6 = 6$

The limit of a constant is the constant.

Theorem 2

If n is a positive integer, then

$$\lim_{x \to a} x^n = a^n$$

Theorem 3

If $a > 0$ and n is a positive integer, or if $a \leq 0$ and n is an odd positive integer, then

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

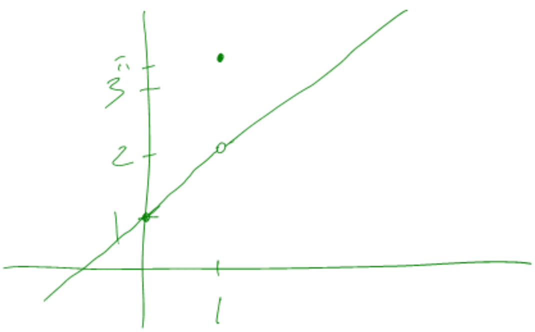
Direct Substitution Property

> If f is a polynomial function and a is a real number, then
>
> $$\lim_{x \to a} f(x) = f(a)$$

Example: Find the $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} x+1 = 2$$



Example: $\displaystyle \lim_{x \to 0} \frac{x}{2 - \sqrt{x+4}}$

$$\lim_{x \to 0} \frac{x}{2 - \sqrt{x+4}} \cdot \frac{2+\sqrt{x+4}}{2+\sqrt{x+4}} = \lim_{x \to 0} \frac{2x + x\sqrt{x+4}}{4 - (x+4)}$$

$$= \lim_{x \to 0} \frac{x(2+\sqrt{x+4})}{-x} = \lim_{x \to 0} -(2+\sqrt{x+4}) = \frac{2+2}{-1} = -4$$

Example: $f(x) = \dfrac{x^2-4x+3}{3x^2-7x-6}$. Find $\lim\limits_{x\to 3} f(x)$

$$\lim_{x\to 3} \frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(3x+2)} = \lim_{x\to 3} \frac{x-1}{3x+2} = \frac{3-1}{9+2} = \frac{2}{11}$$

Example: $\lim\limits_{h\to 0} \dfrac{(3+h)^2-9}{h} \longrightarrow$ Ans: 6.

2 One-Sided Limits

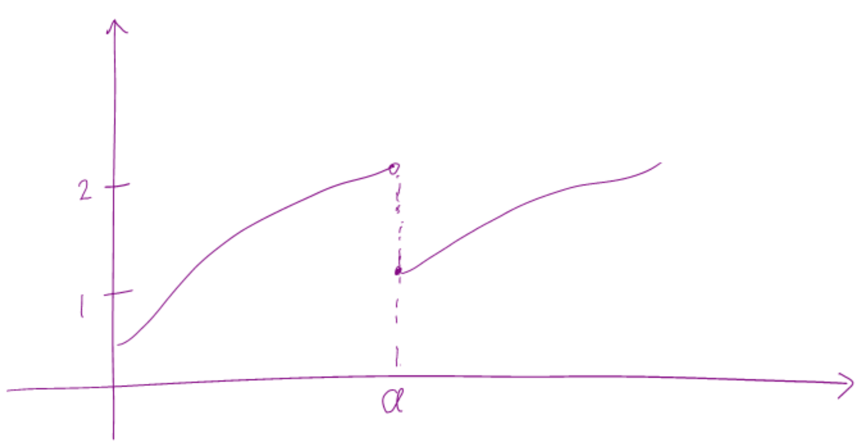
In this section, we will talk about one-sided limits and the condition for the existence of limits.

Definition

> The left-hand limit of $f(x)$ equals to L, written as
>
> $$\lim_{x\to a^-} f(x) = L$$
>
> if we can make the values of $f(x)$ arbitrarily close to the number L by taking x sufficiently close a and x less than a.

Similarly, if we take x greater than a, then we get the **right-hand limit of** $f(x)$. Written as

$$\lim_{x\to a^+} f(x) = L$$



Theorem

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$

Example: show that $\lim_{x\to 0} |x| = 0$.

Recall that $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

For $x \geq 0$, $|x| = x$

$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$

For $x < 0$, $|x| = -x$

$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} -x = 0$

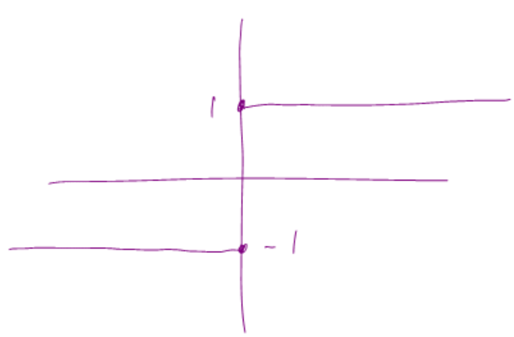
By the previous theorem,

$\lim_{x\to 0} |x| = 0$.

Example: Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1$

$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$



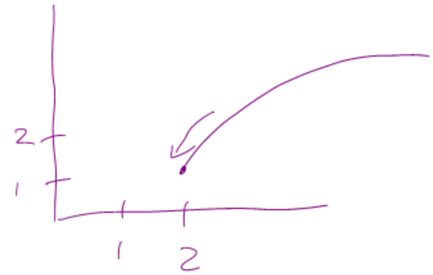
$\Rightarrow \lim_{x\to 0} \frac{|x|}{x}$ DNE

Example: $\lim_{x\to 2^+} (1+\sqrt{x-2})$

$$\lim_{x\to 2^+} (1+\sqrt{x-2})$$

$$= \lim_{x\to 2^+} 1 + \lim_{x\to 2^+} \sqrt{x-2}$$

$$= 1 + \sqrt{2^+ - 2} = 1+0 = 1$$



3 Limits of Trigonometric Functions

Theorem

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

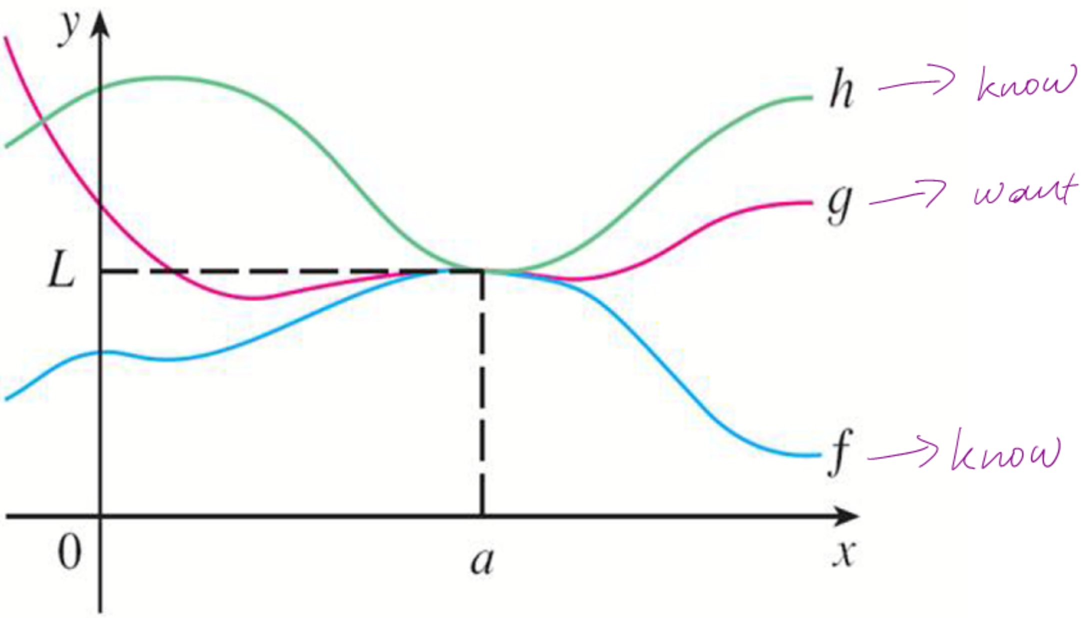
Sandwich/Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

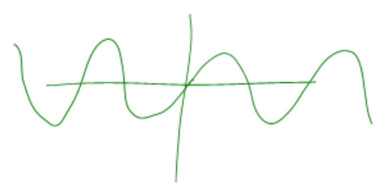
$$\lim_{x\to a} g(x) = L$$



Example: $\lim_{x \to 0} x^2 \sin \frac{1}{x}$

want to use: $\lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

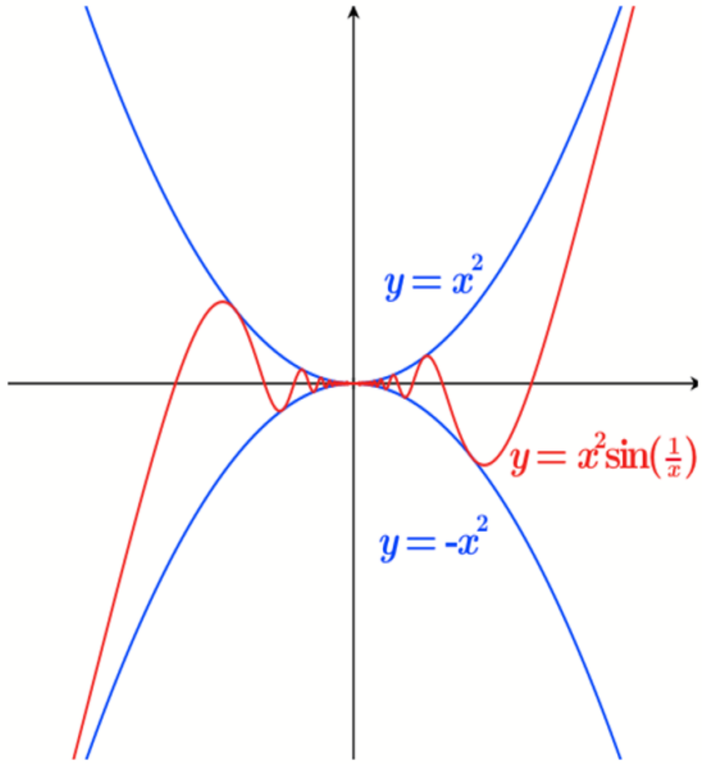


$$\Rightarrow -x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

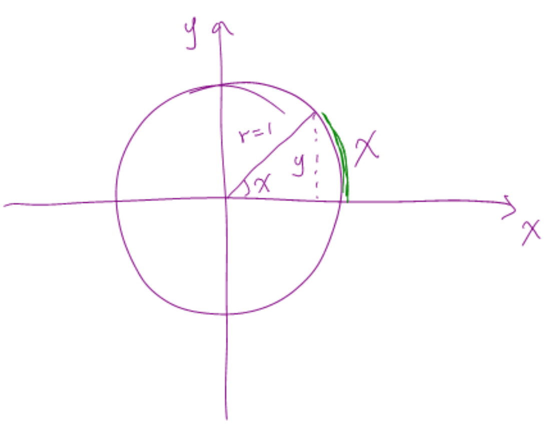
We know that $\lim_{x \to 0} x^2 = 0$ and $\lim_{x \to 0} -x^2 = 0$,

Taking $f(x) = -x^2$ and $g(x) = x^2 \sin \frac{1}{x}$, and $h(x) = x^2$ in the Sandwich Thm, we get

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0.$$



$\lim_{x \to 0} \sin x = 0.$



$r\theta = x$
$\downarrow \ \downarrow$
$1 \ \ x$

$0 < y < x$

$\sin x = \frac{opp}{hyp} = \frac{y}{1} = y$

$\Rightarrow 0 < \sin x < x$

Apply the Squeeze Thm.

Let $x \to 0 \Rightarrow$ squeeze $\sin x \to 0$.

$\Rightarrow \lim_{x \to 0} \sin x = 0.$

$$\lim_{x\to 0} \cos x = 1. \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

Example: Find $\lim_{x\to 0} \frac{1-\cos x}{x}$

$\sin^2 x + \cos^2 x = 1$

$$\lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} = \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$

$$= 1 \cdot \frac{0}{1+1} = 1 \cdot 0 = 0.$$

Example: $\lim_{x\to 0} \frac{\sin 5x}{2x}$

$$\lim_{x\to 0} \frac{\sin 5x}{2x} = \frac{1}{2} \lim_{x\to 0} \frac{\sin 5x}{x} \cdot \frac{5}{5} = \frac{5}{2} \lim_{x\to 0} \frac{\sin 5x}{5x}$$

let $t = 5x$ if $x \to 0$, then $5x \to 0 \Rightarrow t \to 0$.

$$\Rightarrow \frac{5}{2} \cdot \underbrace{\lim_{t\to 0} \frac{\sin t}{t}}_{1} = \frac{5}{2}$$