Counting and Calculations

1. Adding and Subtracting Polynomials

1) Term
A term is a number, variable or the product of a number and variable(s).
Examples of terms are 3x, $5y^3$, 2ad, z.

2) Coefficient
Here are the coefficients of the terms listed above:

Term	Coefficient
3s	3
$5y^3$	5
2ab	2
z	1

3) Constant Term
A constant term is a term that contains only a number.
In other words, there is no variable in a constant term. Examples of constant terms are 4, 100, and -5.

Standard Form of a Polynomial
$c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$
where *n* is a non-negative integer.
c_n is called the leading coefficient. c_0 is a constant.

In other words, a polynomial is a finite sum of terms where the exponents on the variables are non-negative integers. Note that the terms are separated by +'s and -'s.

An example of a polynomial expression is $3x^5 - 5x^3 + x - 10$.

4) Degree of a Term
The degree of a term is the sum of the exponents on the variables contained in the term.

5) Degree of the Polynomial
The degree of the polynomial is the largest degree of all its terms.

6) Descending Order
This means that the term that has the highest degree is written first, the term with the next highest degree is written next, and so forth.

7) Some Types of Polynomials

Type	**Definition**	**Example**
Monomial	A polynomial with **one** term	$5x$
Binomial	A polynomial with **two** terms	$5x - 10$
Trinomial	A polynomial with **three** terms	$7x^4 - 6x^2 + 5$

8) Combining Like Terms
Recall that like terms are terms that have the exact same variables raised to the exact same exponents.

One example of like terms is $3x^2, -5x^2$. Another example is $5ab^2$, $4ab^2$.

2. Adding Polynomials

Step 1: Remove the ( ).
If there is only a + sign in front of ( ), then the terms inside of ( ) remain the same when you remove the ( ).

Step 2: Combine like terms.
Example: Perform the indicated operation and simplify: $(7x^5 + 4x^3 - 2x) + (-8x^5 + 4x)$
$(7x^5 + 4x^3 - 2x) + (-8x^5 + 4x) = 7x^5 + 4x^3 - 2x - 8x^5 + 4x = -x^5 + 4x^3 + 2x$

3. Subtracting Polynomials

Step 1: Remove the ( ) .
If there is a - in front of the ( ) then distribute it by multiplying every term in the ( ) by a -1. Or you can think of it as negating every term in the ( ).

Step 2: Combine like terms.
Example: Perform the indicated operation and simplify:
Subtract $9x^3 - 6$ from $3x^3 - 7x + 5$.
$(3x^3 - 7x + 5) - (9x^3 - 6) = 3x^3 - 7x + 5 - 9x^3 + 6 = -6x^3 - 7x + 11$

4. Multiplying Polynomials

1) (Monomial)(Monomial)
In this case, there is only one term in each polynomial. You simply multiply the two terms together.

Example: Find the following product $(-7x^5)(5x^3)$.
$(-7x^5)(5x^3) = (-7)(5)(x^5 \cdot x^3) = -35x^8$

2) (Monomial)(Polynomial)
In this case, there is only one term in one polynomial and more than one term in the other. You need to distribute the monomial to EVERY term of the other polynomial.

Example:
Find the following product $-2a(5ab+3a^2b^2+7a^3b^3)$.

$$-2a(5ab+3a^2b^2+7a^3b^3)=$$
$$-2a(5ab)-2a(3a^2b^2)-2a(7a^3b^3)=$$
$$-10a^2b-6a^3b^2-14a^4b^3$$

3) (Binomial)(Binomial)
In this case, both polynomials have two terms. You need to distribute both terms of one polynomial times both terms of the other polynomial.

One way to keep track of your distributive property is to use the FOIL method. Note that this method only works on (Binomial)(Binomial).

F	**First terms**
O	**Outside terms**
I	**Inside terms**
L	**Last terms**

This is a fancy way of saying take every term of the first binomial times every term of the second binomial. In other words, do the distributive property for every term in the first binomial.

Example:
Find the following product $(3y-2)^2$.

$$(3y-2)^2=(3y-2)(3y-2)$$
$$\underbrace{(3y)(3y)}_{F}+\underbrace{(3y)(-2)}_{O}+\underbrace{(-2)(3y)}_{I}+\underbrace{(-2)(-2)}_{L}=$$
$$9y^2-6y-6y+4=9y^2-12y+4$$

As mentioned above, use the distributive property until every term of one polynomial is multiplied times every term of the other polynomial. Make sure that you simplify your answer by combining any like terms.

Example:
Find the following product $(3y-1)(2y^2+5y-8)$.

$$(3y-1)(2y^2+5y-8)=(3y)(2y^2)+(3y)(5y)+(3y)(-8)-(1)(2y^2)-(1)(5y)-1(-8)=$$
$$6y^3+15y^2-24y-2y^2-5y+8=6y^3+13y^2-29y+8$$

5. Exponent

1) What Is an Exponent?

There's nothing mysterious! An exponent is simply shorthand for multiplying that number of identical factors. So 4^3 is the same as (4)(4)(4), three identical factors of 4.

Such as: $4^3 = (4)(4)(4)$.

And x^3 is just three factors of x, (x)(x)(x).

One warning: Remember the order of operations.

Exponents are the first operation (in the absence of grouping symbols like parentheses), so the exponent applies only to what it's directly attached to. $3x^3$ is 3(x)(x)(x), not (3x)(3x)(3x).

If we wanted (3x)(3x)(3x), we'd need to use grouping: $(3x)^3$.

2) Negative Exponents

$$x^{-n} = \frac{1}{x^n} \quad \text{or} \quad \frac{1}{x^{-n}} = x^n$$

A negative exponent means to divide by that number of factors instead of multiplying.

So 4^{-3} is the same as $1/(4^3)$, $4^{-3} = \frac{1}{4^3} = \frac{1}{(4)(4)(4)} = \frac{1}{64}$, and $x^{-3} = \frac{1}{x^3}$.

As you know, you can't divide by zero. So there's a restriction that $x^{-n} = \frac{1}{x^n}$, only when x is not zero. When x = 0, x^{-n} is undefined.

Be careful with negative exponents. The temptation is to negate the base, which would not be a correct thing to do.

Example:

Simplify $\frac{1}{3^{-3}}$.

$$\frac{1}{3^{-3}} = 3^3 = 27$$

3) Simplifying an Exponential Expression

When simplifying an exponential expression, write it so that each base is written one time with one POSITIVE exponent.

Example:

Simplify. Write answer with positive exponents. $\frac{(15a^4)(a^5)}{(3a^{11})}$

$$\frac{(15a^4)(a^5)}{(3a^{11})} = \frac{15a^{4+5}}{3a^{11}} = \frac{15a^{4+5}}{3a^{11}} \quad , \; 5a^{9-11} = 5a^{-2} \, , \quad \frac{5}{a^2}$$

4) Scientific Notation

A positive number is written in scientific notation if it is written in the form:
$a \times 10^r$, where $1 \leq a < 10$ and r is an integer power of 10.

5) Review of Exponent Rules

Product Rule:	$x^m \cdot x^n = x^{m+n}$
Power Rule for Exponents:	$(x^m)^n = x^{mn}$
Power of a Product:	$(xy)^n = x^n y^n$
Power of a Quotient:	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$
Quotient Rule for Exponents:	$\frac{x^m}{x^n} = x^{m-n}$
Zero Exponent:	$x^0 = 1$
Negative Exponent:	$x^{-n} = \frac{1}{x^n}$

6. Radicals

1) Radical Notation

The exponent l/n and the radical sign $\sqrt[n]{}$ are both used to indicate the nth root.

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

The number a is called the **radicand** and n is the **index** of the radical.
If n = 2, $\sqrt[2]{a} = \sqrt{a}$

The denominator of a fractional exponent indicates the root.
In general,

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

Example 1:

Rewrite in exponential form.

$$\sqrt{x}\,\sqrt[3]{x} = x^{1/2} \cdot x^{1/3}$$
$$= x^{1/2 + 1/3}$$
$$= x^{5/6}$$

2) Simplify a Radical

WE SAY THAT A SQUARE ROOT RADICAL is simplified, or in its simplest form, when the radicand has no square factors.

Example 1:

33, for example, has no square factors. Its factors are $3 \cdot 11$, neither of which is a square number. Therefore, $\sqrt{33}$ is in its simplest form.

Example 2:

Extracting the square root. 18 has the square factor 9.

$$18 = 9 \cdot 2.$$

Therefore, $\sqrt{18}$ is not in its simplest form. We have,

$$\sqrt{18} = \sqrt{9 \cdot 2}$$

We may now extract, or take out, the square root of 9:

$$\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}.$$

$\sqrt{18}$ is now simplified. The radicand no longer has any square factors.

The justification for taking out the square root of 9, is this theorem:

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

The square root of a product
is equal to the product of the square roots
of each factor.

Example 3:

Since the *square* of any power produces an even exponent --

$$(a^3)^2 = a^6$$

-- then the square root of an even power will be *half* the exponent.

$$\sqrt{a^6} = a^3.$$

As for an odd power, such as a^7, it is composed of an even power times a:

$$a^7 = a^6 a.$$

Therefore,

$$\sqrt{a^7} = \sqrt{a^6 a} = a^3\sqrt{a}.$$

(These results hold only for $a \geq 0$.)

3) Similar Radicals

Similar radicals have the same radicand. We add them as like terms.

$$7 + 2\sqrt{3} + 5\sqrt{2} + 6\sqrt{3} - \sqrt{2} = 7 + 2\sqrt{3} + 6\sqrt{3} + 5\sqrt{2} - \sqrt{2}$$

$$= 7 + 8\sqrt{3} + 4\sqrt{2}.$$

4) Conjugate Pairs

The conjugate of $a + \sqrt{b}$ is $a - \sqrt{b}$. They are a conjugate pair.

Example 1:

Multiply $6 - \sqrt{2}$ with its conjugate.

The product of a conjugate pair -- $(6 - \sqrt{2})(6 + \sqrt{2})$ -- is the difference of two squares.

Therefore,

$$(6 - \sqrt{2})(6 + \sqrt{2}) = 36 - 2 = 34.$$

When we multiply a conjugate pair, the radical vanishes and we obtain a rational number.

Example 2:

Simplify the fraction by rationalize the denominator:

$$\frac{1}{3 + \sqrt{2}}$$

Multiply both the denominator and the numerator by the conjugate of the denominator; that is, multiply them by $3 - \sqrt{2}$.

$$\frac{1}{3 + \sqrt{2}} = \frac{3 - \sqrt{2}}{9 - 2} = \frac{3 - \sqrt{2}}{7}$$

The numerator becomes $3 - \sqrt{2}$. The denominator becomes the difference of the two squares.

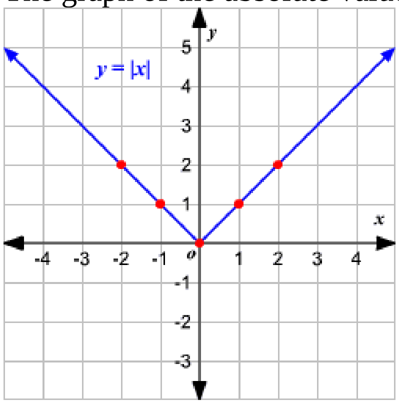
7. The Absolute Value Function and its Properties

One of the most used functions in mathematics is the absolute value function. Its definition and some of its properties are given below.

The absolute value of a real number x, |x|, is

|x| = x if $x \geq 0$
−x if $x < 0$

The graph of the absolute value function is shown below



Example 1:
|2| = 2, |−2| = − (−2) = 2
The absolute value function is used to measure the distance between two numbers. Thus, the distance between x and 0 is |x − 0| = |x|, and the distance between x and y is |x − y|. Thus, the distance from −2 to −4 is |−2 − (−4)| = |−2 + 4| = |2| = 2, and the distance from −2 to 5 is |−2 − 5| = |−7| = 7.

Example 2:
What is the value of $\sqrt{7^2}$?

$$\sqrt{7^2} = \sqrt{49} = 7$$

Example 3:
What is the value of $\sqrt{(-7)^2}$?

$$\sqrt{(-7)^2} = \sqrt{49} = 7$$

Notice that the result is always positive. Another way to represent this is by using the absolute value notation, namely:

$$\sqrt{x^2} = |x|$$

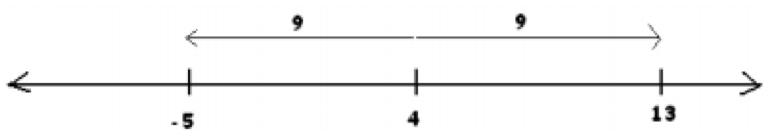
Example 4:

Find all values x that satisfy: | x – 4 | = 9

Approach 1:

(GEOMETRY) | x – 4 | = 9 reads: “The distance between x and 4 is 9.”

That is, if | x – 4 | = 9, then x is a number whose distance from 4 is 9.That is, x is either 9 units up or 9 units down from 4.



Thus x = 13 or x = −5

Approach 2:

(ARITHMETIC) If | x – 4 | = 9, then (x – 4) is a quantity, which, when made positive, equals 9. So either x – 4 = 9 or – (x – 4) = 9.

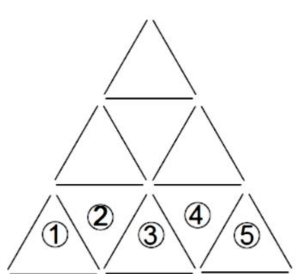
Now solve the equation to obtain: x = 13 or x = −5.

In-class questions

1. A set of tiles numbered 1 through 100 is modified repeatedly by the following operation: remove all tiles numbered with a perfect square, and renumber the remaining tiles consecutively starting with 1. How many times must the operation be performed to reduce the number of tiles in the set to one?

2. Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

3. A large equilateral triangle is constructed by using toothpicks to create rows of small equilateral triangles. For example, in the figure we have 3 rows of small congruent equilateral triangles, with 5 small triangles in the base row. How many toothpicks would be needed to construct a large equilateral triangle if the base row of the triangle consists of 2003 small equilateral triangles?



4. How many distinct four-digit numbers are divisible by 3 and have 23 as their last two digits?

5. How many four-digit positive integers have at least one digit that is a 2 or a 3?

6. How many ordered pairs (m, n) of positive integers, with $m > n$, have the property that their squares differ by 96?

7. A set of 25 square blocks is arranged into a 5×5 square. How many different combinations of 3 blocks can be selected from that set so that no two are in the same row or column?

8. Seven distinct pieces of candy are to be distributed among three bags. The red bag and the blue bag must each receive at least one piece of candy, the white bag may remain empty. How many arrangements are possible?

Extra problem

9. In base 10, the number 2013 ends in the digit 3. In base 9, on the other hand, the same number is written as $(2676)_9$ and ends in the digit 6. For how many positive integers b does the base-b representation of 2013 end in the digit 3?