Games

►Introduction

People have been playing games for the length of recorded history. The uses of these games have ranged from innocently passing the time of day to the settling of disputes in place of war.

There are not very many people that haven't spent time playing tic-tac-toe, checkers or chess. However, it was only during the early 20th century that mathematicians started analyzing games as a separate area of mathematics, which is now known as game theory.

A *game* usually consists of 2 players who are trying to reach some specified goal. In a 2-person game, the two players take turns playing as described by the rules of the game.

For instance, in Tic-tac-toe, the objective of each player is to create either 3 X's or 3 O's in a row, column or diagonal. The *rules* in this game are that the two players alternately draw an X or an O in an unoccupied space of a 3 by 3 grid until either all spaces are used, or one player achieves their goal.

If a player has a means of winning the game regardless of their opponent's moves, then we say that there is a *winning strategy* for that particular player. It shouldn't take too long to convince yourself that neither player has a winning strategy for tic-tac-toe. Thus, you can only win this game if your opponent errs.

In *mathematical* games, there is no element of chance and all of the game's information is available to both players. For instance, the use of dice or cards has some random element, so mathematical games never make use of dice or cards.

For the game's information to be available to both players, both players can examine all things that could affect the game. For example, most games with cards allow a player to view his own cards, but not those of his opponent.

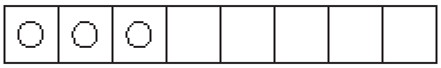
Therefore, these games are not 'mathematical' by our definition. Tic-tac-toe is an example of a mathematical game. (*An interesting result about mathematical games is that if a tie cannot occur, then one of the two players must have a winning strategy*.) If player A cannot *force* a win from a given position, then B must have some strategy to prevent A from winning. Since there is no way to tie, this means B's strategy forces a win for himself!

A typical problem will give both the rules and the goal of the game and ask which player has a winning strategy. This can be stated in many ways such as "What is X's best move from this position?" or "Who has a winning strategy?" or "If it is X's turn, who will win the game?"
For each question, it is not enough to simply state who will win or what move should be made.

A solution must give reasons as to why a certain player can guarantee a win, or why a given move is part of a winning strategy.

► **In-class questions**

Game 1.
Alice and Beth are playing a game with 3 coins on a 1unit by 8 unit strip of paper. The paper is divided up into squares that are the width of the paper as shown. The rules of the game are as follows:



a) On a player's turn, they must move a coin any number of squares to the right.

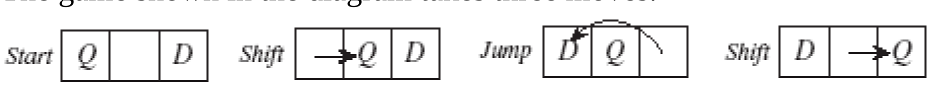
b) The coin may not pass over or land on a square that is occupied by another coin.

c) If it is a player's turn and they have no legal move they lose the game. It is Alice's turn to go first. Describe her first move and then her winning strategy.

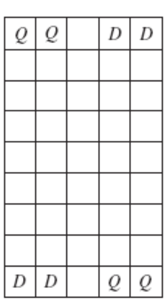
Game 2.
Erin and Fran are playing a game with 4 stacks of cards. On each player's turn, they must remove some number of cards from any one stack. The player who takes the last card wins! It is Erin's turn and the stacks have 11, 12, 14, 15 cards respectively. Show that Fran has a winning strategy.

Game 3.
In the game "Switch", the goal is to make the dimes (D) and quarters (Q) switch spots. The starting position of the game with 1 quarter and 1 dime is shown below. Allowable moves are:
(i) If there is a vacant spot beside a coin then you may *shift* to that space.
(ii) You may *jump* a quarter with a dime or a dime with a quarter if the space on the other side is free.

The game shown in the diagram takes three moves.



(a) Complete the diagram to demonstrate how the game of "Switch" that starts with 2 quarters and 2 dimes can be played in 8 moves.



(b) By considering the number of required *shifts* and *jumps*, explain why the game with 3 quarters and 3 dimes cannot be played in fewer than 15 moves.

Extension to Problem 1:
Explain why the game with n quarters and n dimes cannot be played in fewer than $n(n + 2)$ moves.

Game 4.
Emilia and Omar are playing a game in which they take turns placing numbered tiles on the grid shown.

Emilia starts the game with six tiles:

$\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$, $\boxed{5}$, $\boxed{6}$.

Omar also starts the game with six tiles:

$\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$, $\boxed{5}$, $\boxed{6}$.

Once a tile is placed, it cannot be moved.

After all of the tiles have been placed, Emilia scores one point for each row that has an even sum and one point for each column that has an even sum. Omar scores one point for each row that has an odd sum and one point for each column that has an odd sum. For example, if the game ends with the tiles placed as shown below, then Emilia will score 5 points and Omar 2 points.

3	1	2	4
5	5	2	4
1	3	6	6

(a) In a game, after Omar has placed his second last tile, the grid appears as shown below. Starting with the partially completed game shown, give a final placement of tiles for which Omar scores more points than Emilia. (You do not have to give a strategy, simply fill in the final grid.)

1	3		5
3	1	2	2
	4	4	

(b) Explain why it is impossible for Omar and Emilia to score the same number of points in any game.

(c) In the partially completed game shown below, it is Omar's turn to play and he has a 2 and a 5 still to place. Explain why Omar cannot score more points than Emilia, no matter where he places the 5.

1		3	6
	5		4
3		1	6