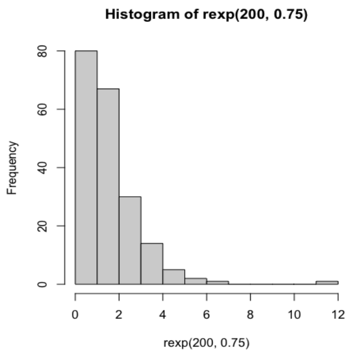
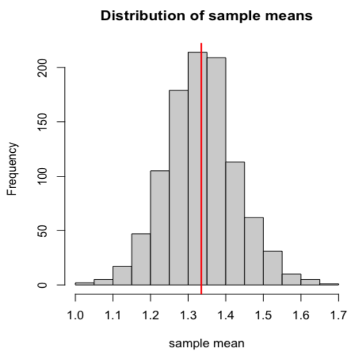
1 Question 1

a)

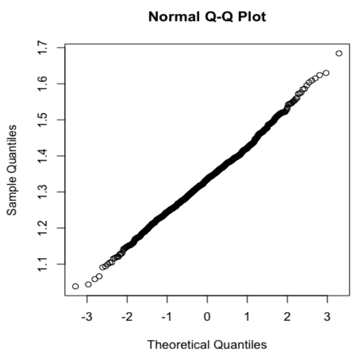
The distribution of the simulated data is right skewed.



b)
The sample means has mean approximately equals to 1.332473.



c)



d)
Sample of size $n = 200$ from the Exponential(3/4) distribution has a right skewed distribution diagram. When there is 1000 samples of size $n = 200$ from Exponential(3/4), the histogram of the sample means is symmetrically bell shaped.

Hence, Central limit theorem ('When independent random variables are added, their properly normalized sum tends toward a normal distribution (informally a bell curve) even if the original variables themselves are not normally distributed.') is verified in this simulation study.

2 Question 2

a)
Given: $f_X(x;\theta) = (\theta+1)x^\theta$, $0 < x < 1$, $\theta > 1$.

$$\begin{aligned}
\mathbb{E}(x) &= \int_0^1 x(\theta+1)x^\theta dx \\
&= \int_0^1 (\theta+1)x^{\theta+1} dx \\
&= \frac{\theta+1}{\theta+2}\left[1^{\theta+2} - 0\right] \\
&= \frac{\theta+1}{\theta+2}.
\end{aligned}$$

$$\begin{aligned}
\overline{X}_n &= \frac{\widetilde{\theta}+1}{\widetilde{\theta}+2} \\
(\widetilde{\theta}+2)\overline{X}_n &= \widetilde{\theta}+1 \\
\widetilde{\theta}\overline{X}_n + 2\overline{X}_n &= \widetilde{\theta}+1 \\
\widetilde{\theta}(\overline{X}_n - 1) &= 1 - 2\overline{X}_n \\
\widetilde{\theta} &= \frac{1-2\overline{X}_n}{\overline{X}_n - 1}.
\end{aligned}$$

Hence, the method of moments estimator $\widetilde{\theta}$ of θ is:

$$\widetilde{\theta} = \frac{1-2\overline{X}_n}{\overline{X}_n - 1}.$$

b)i)

$$L(\theta : x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} (\theta+1)x_i^\theta.$$

$$\begin{aligned}
l(\theta : x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n} ln\left[(\theta+1)x_i^\theta\right] \\
&= \sum_{i=1}^{n} \left[ln(\theta+1) + ln(xi^\theta)\right] \\
&= \sum_{i=1}^{n} ln(\theta+1) + \theta \sum_{i=1}^{n} ln(xi).
\end{aligned}$$

$$\begin{aligned}
l'(\theta) &= \sum_{i=1}^{n} \frac{1}{\theta+1} + \sum_{i=1}^{n} ln(xi) \\
&= \frac{n}{\theta+1} + \sum_{i=1}^{n} ln(xi).
\end{aligned}$$

$$\begin{aligned}
l'' &= \frac{-n}{(\theta+1)^2} \\
l'' &< 0.
\end{aligned}$$

Assume: $l'(\theta) = 0$,

$$\begin{aligned}\frac{n}{\theta+1} + \sum_{i=1}^{n} ln(xi) = 0 \\ \frac{n}{\theta+1} &= -\sum_{i=1}^{n} ln(xi) \\ \theta + 1 &= \Big[\frac{-1}{n}\sum_{i=1}^{n} ln(xi)\Big]^{-1} \\ \theta &= \Big[\frac{-1}{n}\sum_{i=1}^{n} ln(xi)\Big]^{-1} - 1.\end{aligned}$$

Hence, $\widehat{\theta}_{MLE} = \Big[\frac{-1}{n}\sum_{i=1}^{n} ln(xi)\Big]^{-1} - 1$.

b)ii)

Fisher information of $\widehat{\theta}$ is:

$$\begin{aligned}ln(\widehat{\theta}) &= -\mathbb{E}\Big[\frac{-n}{(\theta+1)^2}\Big] \\ &= \frac{n}{(\theta+1)^2}.\end{aligned}$$

b)iii)

$$Se(\widehat{\theta}) = \frac{1}{\sqrt{ln(\widehat{\theta})}} = \frac{1}{\sqrt{\frac{n}{(\theta+1)^2}}} = \frac{\theta+1}{\sqrt{n}}.$$

c)

Given: $y = -ln(x)$. Hence, for $0 < x < 1$, $x = e^{-y}$.

$$\Big|\frac{dx}{dy}\Big| = e^{-y}$$

$$\begin{aligned}f_Y(y) = f_{-ln(Xi)}(y) &= f_X(x;\theta)\Big|\frac{dx}{dy}\Big| \\ &= (\theta+1)x^{\theta}\Big|\frac{dx}{dy}\Big| \qquad \text{for} 0x < 1 \\ &= (\theta+1)x^{\theta}e^{-y} \\ &= (\theta+1)e^{-y\theta}e^{-y} \\ &= (\theta+1)e^{-y(\theta+1)} \qquad for 0 < y < \infty\end{aligned}$$

Hence, probability density function of Y is $(\theta+1)e^{-y(\theta+1)}$, which is same as $Gamma\Big(1, \frac{1}{\theta+1}\Big)$.

3 Question 3

Let X have a geometric distribution with parameter p,
a)

$$X \sim Geometric(p),$$
$$f_X(x;p) = p(1-p)^{x-1}, \text{ when } x = 1, 2, 3, \ldots.$$

$$\begin{aligned}
F_X(x;p) &= \mathbb{P}(X=1) + \mathbb{P}(x=2) + ... + \mathbb{P}(X=x) \\
&= \sum_{x=1}^{\infty} f_X(x;p) \\
&= \sum_{x=1}^{\infty} p(1-p)^{x-1} \\
&= p + p(1-p) + p(1-p)^2 + ... + p*(1-p)^{x-1} \\
&= p[1 + (1-p) + (1-p)^2 + ... + (1-p)^{x-1}] \\
&= p\frac{[(1-p)^x - 1}{(1-p) - 1} \\
&= -[(1-p)^x - 1] \\
&= 1 - (1-p)^x, \quad \text{when} \quad x = 0, 1, 2, \ldots \quad .
\end{aligned}$$

b)
Because, $f_X(x;p) = p(1-p)^{x-1}$, when $x = 1, 2, 3, \ldots$,
Therefore, $\mathbb{P}(X > s) = (1-p)^s$, when $s = 0, 1, 2, \ldots$.

$$\begin{aligned}
\mathbb{P}(X > s+t | X > s) &= \frac{\mathbb{P}(X > s+t, X > t)}{\mathbb{P}(X > t)} \\
&= \frac{\mathbb{P}(x > s+t)}{\mathbb{P}(X > t)} \\
&= \frac{(1-p)^{s+t}}{(1-p)^t} \\
&= (1-p)^s.
\end{aligned}$$

Hence, X has the lack of memory property.

c)
Given $X_1, \ldots, X_n$ be a random sample of size n from a geometric distribution with parameter $p = 0.6$,

$$\begin{aligned}
\mathbb{M}_X(u) &= \mathbb{E}(e^{ux}) \\
&= \sum_{x=1}^{\infty} e^{ux} p(1-p)^{x-1} \\
&= \sum_{x=0}^{\infty} e^{ux} p(1-p)^{x} \\
&= p\sum_{x=0}^{\infty} e^{ux} (1-p)^{x} \\
&= p\sum_{x=0}^{\infty} (e^{u}(1-p))^{x} \\
&= \frac{p}{1-(1-p)e^u} \quad , \quad \text{when} \quad u < -ln(1-p)\ .
\end{aligned}$$

Since, $Y = X_1 + X_2 + \cdots + X_n$,

$$M_y(u) = \Big(\frac{p}{1-(1-p)e^u}\Big)^n \quad .$$

Given, $p = 0.6$,

$$M_Y(u) = \big(\tfrac{0.6}{1-0.4e^u}\big)^n, \text{ for } u < -ln(1-p)\ .$$

4 Question 4

a) The two properties of a density function are as follows:

1. $f(x) \geq 0$, for all x .
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

b)

$$\begin{aligned}
f_Y(y) &= \pi f_{Y_1}(y) + (1-\pi)f_{Y_2}(y) \\
\int_{-\infty}^{\infty} f_Y(y)dy &= \int_{-\infty}^{\infty} (\pi f_{Y_1}(y) + (1-\pi)f_{Y_2}(y))dy \\
\int_{-\infty}^{\infty} f_Y(y)dy &= \pi \int_{-\infty}^{\infty} f_{Y_1}(y)dy + (1-\pi)\int_{-\infty}^{\infty} f_{Y_2}(y)dy \\
\int_{-\infty}^{\infty} f_Y(y)dy &= \pi\times 1 + (1-\pi)\times 1 \\
&= \pi + (1-\pi) = 1.
\end{aligned}$$

This result satisfies both properties of a density function, thus, $f_Y(y)$ is a density function too.

5 Question 5

Given: $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, variance $\sigma^2 < \infty$, mean= μ.
When $n > 3$, $\widetilde{X}_n = \frac{3X_1 + \sum_{i=2}^{n-2} X_i}{n}$.

a)

$$
\begin{aligned}
\mathbb{E}[\widetilde{X}_n] &= \mathbb{E}\left[\frac{3X_1 + X_2 + \cdots + X_{n-2}}{n}\right] \\
&= \frac{1}{n}(3\mu + (n-3)\mu) \\
&= \frac{1}{n}(n\mu) \\
&= \mu,
\end{aligned}
$$

$$
\begin{aligned}
bias(\widetilde{X}_n) &= \mathbb{E}[\widetilde{X}_n] - \mu \\
&= \mu - \mu \\
&= 0.
\end{aligned}
$$

Hence, $\widetilde{X}_n$ is an unbiased estimator of μ.

b)

$$
\begin{aligned}
Var[\widetilde{X}_n] &= Var\left[\frac{3X_1 + X_2 + \cdots + X_{n-2}}{n}\right] \\
&= Var\left[\frac{3X_1}{n} + \frac{X_2 + X_3 + \cdots + X_{n-2}}{n}\right] \\
&= \frac{9\sigma^2}{n^2} + \frac{(n-3)\sigma^2}{n^2} \\
&= \frac{6\sigma^2 + n\sigma^2}{n^2} \\
&= (6+n)\frac{\sigma^2}{n^2}.
\end{aligned}
$$

$$
\begin{aligned}
MSE(\widetilde{X}_n) &= [E]\left[(\widetilde{X}_n - \mu)^2\right] \\
&= \left[bias(\widetilde{X}_n)\right]^2 + \left[Se(\widetilde{X}_n)\right]^2 \\
&= 0 + Var(\widetilde{X}_n) \\
&= (6+n)\frac{\sigma^2}{n^2}.
\end{aligned}
$$

c)

$$
\begin{aligned}
\lim_{n\to\infty}(6+n)\frac{\sigma^2}{n^2} &= \sigma^2 \lim_{n\to\infty}\frac{6+n}{n^2} \\
&= \sigma^2 \lim_{n\to\infty}\left[\frac{6}{n^2} + \frac{n}{n^2}\right] \\
&= \sigma^2\left[\lim_{n\to\infty}\frac{6}{n^2} + \lim_{n\to\infty}\frac{1}{n}\right] \\
&= \sigma^2(0+0) \\
&= 0.
\end{aligned}
$$

Therefore, $\lim_{n\to\infty} MSE(\widetilde{X}_n) = 0$.

d)

Because both $\widetilde{X}_n$ and $\overline{X}_n$ are unbiased estimators for μ with equal sample sizes.

If: $Var(\widetilde{X}_n) < (Var(\overline{X}_n)$, $\widetilde{X}_n$ is a better estimator of μ than $\overline{X}_n$.
Given variable$X$ has variance σ^2.

$$Var(\overline{X}_n) = \frac{\sigma^2}{n}$$
$$Var(\widetilde{X}_n) = \frac{6\sigma^2}{n} + \frac{\sigma^2}{n}.$$

Since $n > 3$ and $\sigma \geq 0$, $\frac{6\sigma^2}{n} \geq 0$.
When $\sigma > 0$, $\frac{6\sigma^2}{n} > 0$,

$$Var(\overline{X}_n) < Var(\widetilde{X}_n)$$

When $\sigma = 0$, $\frac{6\sigma^2}{n} = 0$,

$$Var(\overline{X}_n) = Var(\widetilde{X}_n)$$

Therefore, $Var(\overline{X}_n) \leq Var(\widetilde{X}_n)$.
Hence, $\overline{X}_n$ is uniformly a better estimator of μ than $\widetilde{X}_n$.