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# APMA1360 - Final Project

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## Introduction

Rocketry has been an important motivator for the development of modern control theory. Many of the mathematical techniques for control of dynamical systems, such as the Kalman Filter, were developed in the context of Cold War era missile development and the space race. Historically, rockets were equipped with fins in order to be aerodynamically stable, and controlled using variations on Proportional-Integral-Derivate (PID) controllers. Modern advances in manufacturing and computational power have led to increasingly complex and precise control methods for rockets. A new generation of rocket development, including companies like SpaceX and Blue Origin, eschew rocket fins in favor of activate stabilization by thrust vectoring of the rocket engines. Replacing fins with active thrust vectoring allows the rocket to maintain stability as it falls back to earth, an important step toward reusable rockets.

We investigate a model for controlling such an active thrust rocket system. We begin by modeling the dynamics. We demonstrate that this model, without active thrust, behaves reasonably in simulation and discuss the issues that arise when trying to analyze the stability of the system. We then add control of the rocket engine, allowing a reactive control input to be applied to the rocket to maintain stability.

## Modeling Overview

We model our rocket with motion constrained to a plane. This greatly simplifies the system, while still capturing many of the dynamics that one would expect to observe in a 3D rocket. Certain behaviors, such as the rocket rotating about its longitudinal axis, will not be observable. Nevertheless, much of the interesting system behaviors that we would hope to control will be present in the model.

The state vector consists of six variables: $x, y$ (position), $\dot{x}, \dot{y}$ (velocity), $\theta$ (angular orientation), and $\dot{\theta}$ (angular velocity). In this section we will derive a dynamics model for the vehicle from first principles. The rocket is subject to both thrust and aerodynamic drag forces. We discuss the forces that arise from each of these sources, and then combine them into a unified rocket dynamics equation.

The thrust generated by the rocket motor produces both a force and a torque on the vehicle. Similarly, the aerodynamic drag generated by the linear motion of the rocket produces a force and torque. A second drag force opposes the rotation of the vehicle about its center of mass.

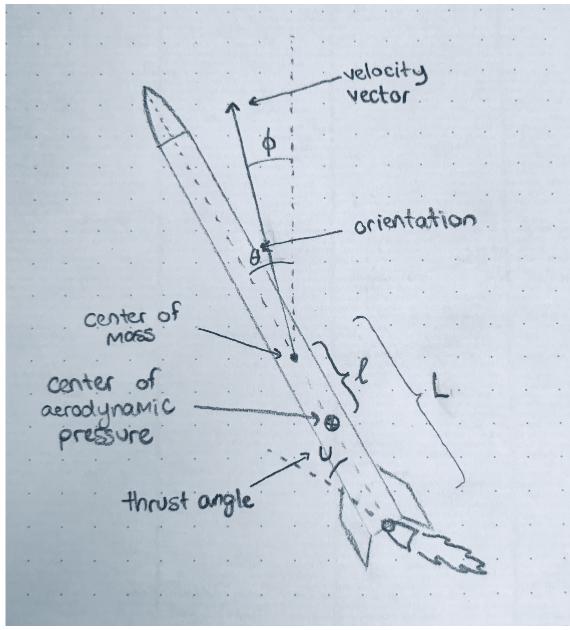


Figure 1: $\theta$ increases as the rocket rotates counter-clockwise, with $\theta = 0$ when the vehicle is straight up. Note that $l$ and $L$ are negative as depicted. $l$ and $L$ are positive when the center of aerodynamic pressure (CP) and the center of thrust are in front of the center of mass (CG).

| Symbol | Definition |
|---|---|
| $x, y$ | Position |
| $\theta$ | Heading |
| $\phi$ | Direction of velocity, $\arctan\left(\frac{\dot{y}}{\dot{x}}\right)$ |
| $v$ | velocity, $\sqrt{\dot{x}^2 + \dot{y}^2}$ |
| $u$ | Thrust angle (relative to rocket vertical) |
| $T$ | Thrust |
| $l$ | Distance from center of gravity to center of pressure (negative for CP behind CG) |
| $L$ | Distance from center of gravity to center of thrust (negative for motor behind CG) |
| $m$ | The mass of the rocket |
| $I$ | Rotation inertia |
| $D_v$ | Drag from forward movement of the rocket |
| $D_\omega$ | Drag from rotational movement of the rocket |
| $g$ | Acceleration due to gravity |

## Forces from thrust

The rocket thrust causes acceleration in both the x and y direction. As shows in Figure 1, the direction of thrust is determined by the rocket's orientation $\theta$ and the amount of tilt of the motor, $u$. The force due to thrust in the $x$ and $y$ directions is thus

$$F_x = -\sin(\theta + u)T \quad (1)$$
$$F_y = \cos(\theta + u)T \quad (2)$$

Applying Newton's Second Law gives

$$\ddot{x} = \frac{-\sin(\theta + u)T}{m} \quad (3)$$
$$\ddot{y} = \frac{\cos(\theta + u)T}{m} \quad (4)$$

Note that angles measure rotation from the positive $y$ axis so the projection on the x axis contains $-\sin(\theta + u)$ instead of $\cos(\theta + u)$.

The thrust may also results in a moment on the rocket. If the motor is aligned with the rocket body (no tilt), there is no moment as the applied force is in the direction of the rocket's center of mass. However, if the motor is tilted, a component of thrust is directed perpendicular to the rocket body (i.e. perpendicular to line segment $L$ in Figure 1), resulting in a rotation.

$$\tau = r \times F \quad (5)$$
$$= LT\sin(u) \quad (6)$$

By Newton's Second Law,

$$\ddot{\theta} = \frac{\tau}{I} \quad (7)$$

Where $I$ is the moment of inertia of the rocket.

Combining equations 5 and 7 yields

$$\ddot{\theta} = \frac{\sin(u)TL}{I} \quad (8)$$

## Forces from drag

We chose to model the rocket as not incurring drag along its longitudinal axis. For example, our model assumes a rocket moving with only vertical velocity that is pointed straight up is not acted upon by drag. When the rocket's heading and orientation are not aligned (i.e. $\theta \neq \phi$) the drag is proportional to $|\sin(\theta - \phi)|$, the cross-sectional area that the rocket presents to the airflow.

The drag force acts in opposition to the direction of motion, $\phi$, and is proportional to the velocity squared.

$$F_{drag} = v^2 D_v \left|\sin(\theta - \phi)\right| \tag{9}$$

It follows that the component of the drag force acting in the $x$ and $y$ direction are

$$F_x = \sin(\phi) v^2 D_v \left|\sin(\theta - \phi)\right| \tag{10}$$
$$F_y = -\cos(\phi) v^2 D_v \left|\sin(\theta - \phi)\right| \tag{11}$$

Finally, it follows from Newton's Second Law that

$$\ddot{x} = \frac{\sin(\phi) v^2 D_v \left|\sin(\theta - \phi)\right|}{m} \tag{12}$$
$$\ddot{y} = \frac{-\cos(\phi) v^2 D_v \left|\sin(\theta - \phi)\right|}{m} \tag{13}$$

We also include the "weather vane" effect of the rocket. When the vehicle is moving, the aerodynamic drag force is applied at the center of pressure (CP). When the center of pressure is not coincident with the center of mass (CG), this force results in a torque, which spins the rocket to align with its direction of motion. The lever arm for this torque is $l$, the distance from CG to CP.

$$\ddot{\theta} = \frac{l v^2 D_v}{I} \sin(\theta - \phi) \tag{14}$$

Finally, we model drag incurred by the rotation rate of the rocket. This damping force is critical because without this force (or some control input) the vehicle would spin indefinitely.

$$\ddot{\theta} = \frac{-|\dot{\theta}|\dot{\theta} D_\omega}{I} \tag{15}$$

## All together now

Summing all of the different forces (including a gravitational acceleration in $y$) we arrive at a dynamics model for the rocket. Specifically, Equations 3, 8, 12, 14, and 15 combine to give the dynamics of the system:

$$\begin{cases} \ddot{x} = \frac{1}{m}\left(-\sin(\theta+u)T + \sin(\phi)\left|v^2\sin(\theta-\phi)D_v\right|\right) \\ \ddot{y} = \frac{1}{m}\left(\cos(\theta+u)T - \cos(\phi)\left|v^2\sin(\theta-\phi)D_v\right|\right) - g \\ \ddot{\theta} = \frac{1}{I}\left(\sin(u)TL - |\dot{\theta}|\dot{\theta}D_\omega + lv^2\sin(\theta-\phi)D_v\right) \end{cases} \tag{16}$$

We can now take our second order ODE and convert it into a system of coupled first order equations by introducing $a = \dot{x},\ b = \dot{y},\ c = \dot{\theta}$

$$\begin{cases} \dot{x} = a \\ \dot{y} = b \\ \dot{\theta} = c \\ \dot{a} = \frac{1}{m}\left(-\sin(\theta+u)T + \sin(\phi)\left|v^2\sin(\theta-\phi)D_v\right|\right) \\ \dot{b} = \frac{1}{m}\left(\cos(\theta+u)T - \cos(\phi)\left|v^2\sin(\theta-\phi)D_v\right|\right) - g \\ \dot{c} = \frac{1}{I}\left(\sin(u)TL - |c|cD_\omega + lv^2\sin(\theta-\phi)D_v\right) \end{cases} \tag{17}$$

## Sanity Check

Let's consider some simple cases to verify our model. If the rocket is flying straight up, and facing straight up, $\dot{y} > 0,\ \dot{x} = 0$. We expect to incur no drag (because our rocket is infinitely thin viewed head on). The only acceleration should be due to thrust and gravity.

$$\begin{cases} \ddot{x} = \frac{1}{m}\left(-\sin(0)T + \sin(0)\left|\dot{y}^2\sin(0)D_v\right|\right) = 0 \\ \ddot{y} = \frac{1}{m}\left(\cos(0)T - \cos(0)\left|\dot{y}^2\sin(0)D_v\right|\right) - g = \frac{T}{m} - g \\ \ddot{\theta} = \frac{1}{I}\left(\sin(0)TL + -|0|0D_\omega + l\dot{y}^2\sin(0)D_v\right) = 0 \end{cases}$$

From this example and some physical intuition, we can see that the only true fixed point is when the rocket is standing straight up and using its thrust to cancel out the acceleration from gravity: $\frac{T}{m} = g$, resulting in $\ddot{y} = 0$.

## The Drag Problem

In order to find stability of fixed points, we must linearize Equation 17 with its Jacobian. Here we see the first problem with the system as modeled. Several terms in the state equation contain absolute values, and the absolute value function has no defined derivative at 0.

In order to make the derivative behave nicely, we can substitute $|a|$ with $\sqrt{a^2+\epsilon} - \sqrt{\epsilon}$. This function is differentiable and approaches the absolute value function as $\epsilon$ goes to 0. However, this replacement just masks a more fundamental problem. The linearization of the drag force

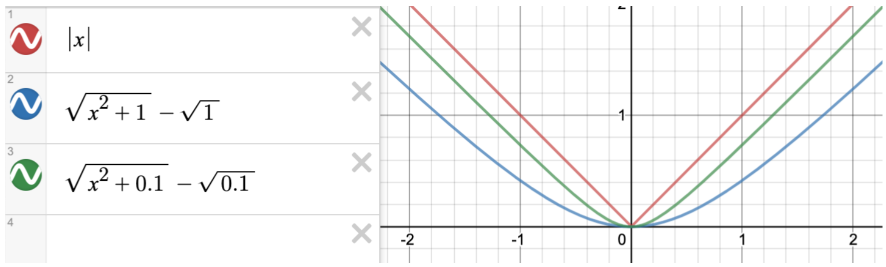


Figure 2: Approximating $|a|$ with $\sqrt{a^2+\epsilon} - \sqrt{\epsilon}$

at a fixed point becomes 0. For example, the force contributed by the rotational damping term $\left|\dot{\theta}\right|\dot{\theta}D_\omega$ changes sign at $\dot{\theta} = 0$, meaning the only reasonable derivative to assign this term at $\dot{\theta} = 0$ is 0. Intuitively this makes sense, as when the rocket rotation rate approaches 0 there is no drag force and the drag force increases as the rocket rotates faster in *either direction*.

## Rocket Dynamics without Drag

The fixed points of our model occur when the rocket is stationary, and therefore not experiencing any drag. As discussed, the derivative of the drag term at these fixed points is zero, and so we do not see any impact from drag in the Jacobian. This means that employing the conventional analytical tool of linearizing about the fixed points essentially ignores drag.

While this is somewhat disappointing, we recall that rockets must be stable both in the atmosphere and in the vacuum of space where there is no drag, so we simplify our dynamics by removing the drag terms:

$$\begin{cases} \ddot{x} = \frac{1}{m}\left(-\sin(\theta+u)T\right) \\ \ddot{y} = \frac{1}{m}\left(\cos(\theta+u)T\right) - g \\ \ddot{\theta} = \frac{1}{I}\left(\sin(u)TL\right) \end{cases} \tag{18}$$

It is at this point that we must consider values for $T$ and $u$, the thrust force and angle respectively — this is how we will control the behavior of the rocket.

Consider $T = mg$ and $u = 0$. With these values the engine is perfectly aligned with the vehicle, and the thrust is just sufficient to cancel out the acceleration due to gravity if the vehicle were pointed straight up.

$$\begin{cases} \ddot{x} = \frac{1}{m}\left(-\sin(\theta+0)mg\right) & = -\sin(\theta)g \\ \ddot{y} = \frac{1}{m}\left(\cos(\theta+0)mg\right) - g & = \cos(\theta)g - g \\ \ddot{\theta} = \frac{1}{I}\left(\sin(0)mgL\right) & = 0 \end{cases} \tag{19}$$

Unsurprisingly, the only fixed point of the system is $\theta = 0$, the rocket is pointed straight up. In order to check stability, we must once again expand the second order ODE

$$\begin{cases} \dot{x} = a \\ \dot{y} = b \\ \dot{\theta} = c \\ \dot{a} = -\sin(\theta)g \\ \dot{b} = \cos(\theta)g - g \\ \dot{c} = 0 \end{cases} \tag{20}$$

We calculate the Jacobian

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g\cos(\theta) & 0 & 0 & 0 \\ 0 & 0 & -g\sin(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

This full Jacobian matrix is large and unwieldy. However, we note that none of the dynamics depend on the $x$ and $y$ state dimensions. If we restrict our stability analysis to the orientation of the rocket, $x$ and $y$ can be removed from the state vector rendering a simpler Jacobian:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ -g\cos(\theta) & 0 & 0 & 0 \\ -g\sin(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

Row 1 shows that $\theta$ depends directly on $\dot{\theta}$, as would be expected. Rows 4 and 5 show that $\dot{x}$ and $\dot{y}$ depend on $\theta$ — the direction of acceleration is set by the orientation of the rocket. In the bottom row we observe that $\dot{\theta}$ is not affected by anything because there are no torques acting on the vehicle when the thrust vector is aligned with the rocket body.

Evaluating at the fixed point, $\theta = 0$ and computing the eigenvalues, we find that the system has all-zero eigenvalues. The degenerate eigenvalues mean that the system is neither stable nor unstable and nothing interesting happens. The state will drift over time, but there are no attractors or repellors. Active control of the thrust direction can counteract this drift and ensure stability of the rocket orientation.

## Better Controllers

With thrust $T = mg$ and motor angle $u = 0$, the thrust and angle are fixed, so the rocket does not self-stabilize. In this section, we evaluate other choices for $u$ and $T$ such that the system actively changes the thrust vector in order to maintain a desired orientation (vertical, in this case).

### P-Control

Consider $u = \theta$ with $T = mg$. In this configuration the rocket once again has constant thrust, however if it tilts over by some amount the engine will gimbal in the same direction, generating a torque which spins the rocket in the opposite direction. In a general control theory context, this is known as "P-controller" or "proportional controller," because the control input is proportional to the difference between the current position and the target position, which in this case is $\theta = 0$.

Taking the eigenvalues of the Jacobian of this new system, we have

$$\left[0, \quad 0, \quad -\sqrt{\frac{Lgm\cos(\theta)}{I}}, \quad \sqrt{\frac{Lgm\cos(\theta)}{I}}\right] \tag{23}$$

At the fixed point, $\theta = 0$, and with $L < 0$ and $0 < g, m, I$, we have two nonzero imaginary eigenvalues. This means that the fixed point is non-hyperbolic, and within the hyperplane spanned by the eigenvectors corresponding to the nonzero eigenvalues, the vehicle orbits the fixed point. This is indeed the behavior one would expect to observe for an undamped, P-controlled system like an undamped mass on a spring or a pendulum without friction.

### D-Control

In a D-controller, the control input is proportional to the derivative of the error. This relationship can be thought of as a damping the system. In our system, this corresponds to letting $u = \dot{\theta}$. Now the motor gimbals to cancel out the rocket's angular velocity.

Taking the eigenvalues of the Jacobian of this new system, we have

$$\left[0, \quad 0, \quad 0, \quad \frac{Lgm\cos(\dot{\theta})}{I}\right] \tag{24}$$

As $L < 0$, the system is stable along some 1-dimensional eigenspace. While this is more asymptotic stability than we found with just the P-control, we would like to find a controller that maintains stability in *all* dimensions in contrast to these controllers that let several state variables drift over time.

### Finding a Stable Controller

The zero-eigenvalues can be thought of as uncontrolled axes. In order to find a fully stable fixed point with all-negative eigenvalues, our controller will need to respond to disturbances along these axes.

We address the $\dot{y}$ control by $T = mg - \dot{y}$. If the rocket has positive $y$-velocity, thrust will decrease to allow gravity to slow the vehicle down. If the rocket has negative $y$-velocity, thrust will increase to stop the rocket's fall.

To control $x$-velocity, the vehicle must change its orientation to direct thrust in opposition to its direction of motion. For example, if the vehicle is moving to the right, then it must tilt over to the left in order to zero its lateral motion.

Additionally, the stable $D$-controller that we introduced for $\theta$ does not actually drive the rocket to $\theta = 0$, it simply stops the rocket, so we introduce a PD-controller. Accordingly, we let $u = \theta + \dot{\theta} - \mu x$ where $\mu > 0$ is a parameter.

The Jacobian for this system is far too complicated to compute by hand, so we used Sympy, a symbolic math package for Python.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ -\frac{2(-b+gm)\cos(-a\mu+c+2\theta)}{m} & \frac{\mu(-b+gm)\cos(-a\mu+c+2\theta)}{m} & \frac{\sin(-a\mu+c+2\theta)}{m} & -\frac{(-b+gm)\cos(-a\mu+c+2\theta)}{m} \\ -\frac{2(-b+gm)\sin(-a\mu+c+2\theta)}{m} & \frac{\mu(-b+gm)\sin(-a\mu+c+2\theta)}{m} & -\frac{\cos(-a\mu+c+2\theta)}{m} & -\frac{(-b+gm)\sin(-a\mu+c+2\theta)}{m} \\ \frac{L(-b+gm)\cos(-a\mu+c+\theta)}{I} & -\frac{L\mu(-b+gm)\cos(-a\mu+c+\theta)}{I} & -\frac{L\sin(-a\mu+c+\theta)}{I} & \frac{L(-b+gm)\cos(-a\mu+c+\theta)}{I} \end{bmatrix}$$

Figure 3: This is the Jacobian for our fully-controlled rocket with $T = mg - \dot{y}$, and $u = \theta + \dot{\theta} - \mu x$ where $\mu > 0$. Recall that $a = \dot{x},\ b = \dot{y},\ c = \dot{\theta}$.

We included the parameter $\mu$, because in our numerical evaluation of the system we observed that two of the real components of the eigenvectors would change sign for different values of $\mu$, indicating a bifurcation. We attempted to use Sympy to find an explicit form for the eigenvalues, which would enable us to solve for the bifurcation value of $\mu$.

The symbolic computation of the eigenvalues proved too intensive to complete within a reasonable time-frame, so we resorted to a numerical search for the bifurcation value. On our second guess, we found that $\mu = 0.5$ results in two eigenvalues with zero real part. Subsequent of testing of values of $\mu$ just above and below 0.5 revealed that this is indeed the bifurcation value.

Somewhat intuitively, another bifurcation occurs when we take let $\mu$ change sign. When $\mu < 0$, the rocket will move it's gimbal in the wrong direction to cancel out the lateral motion. At $\mu = 0$ we find one eigenvalue with zero real part. As $\mu$ becomes negative, the real part of this eigenvalue becomes positive.

Numerical simulation of the system for various values of $\mu$ is shown below in figures 4, 5, and 6.

## Simulation

We wrote a short Python script to solve our system numerically for specific initial conditions using scipy's *odeint* function. We then animated the $x, y$, and $\theta$ components of the result using matplotlib's animation library. See `https://github.com/IzzyBrand/tvc_rocket_model`.

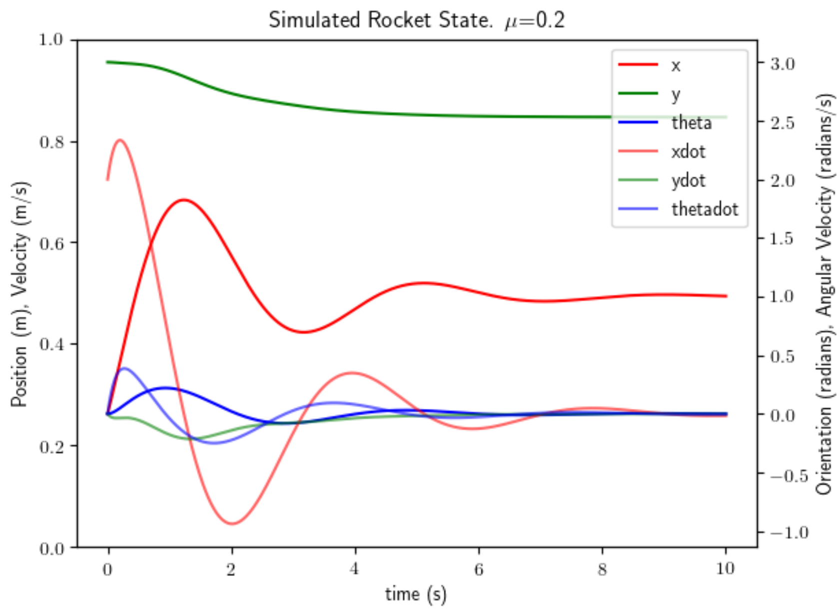


Figure 4: The initial condition is $y = 3$ and $\dot{x} = 1$. All other starting parameters are zero. This is a stable value of $\mu$. Observe that all the velocity terms and $\theta$ are driven to zero. Since the rocket is simply trying to zero it's velocity and not its position, $x$ and $y$ are uncontrolled and nonzero.

# Conclusion

We have shown that a simplified rocket model can successfully be stabilized with a relatively simple controller. We have also seen how the presence of aerodynamic forces complicates both modeling and analysis. As our controllers were analyzed and tested in simulation without these forces, our work would be most applicable when trying to land a rocket on a surface without significant atmosphere such as the moon. Future improvements to our model would consider motion in three dimensions and a more detailed analysis of how mass, inertia, and thrust limitations affect stability and controllability. In spite of the limitations and simplicity of our current model, we have demonstrated that a controlled landing of a rocket on the moon is, in theory, possible.

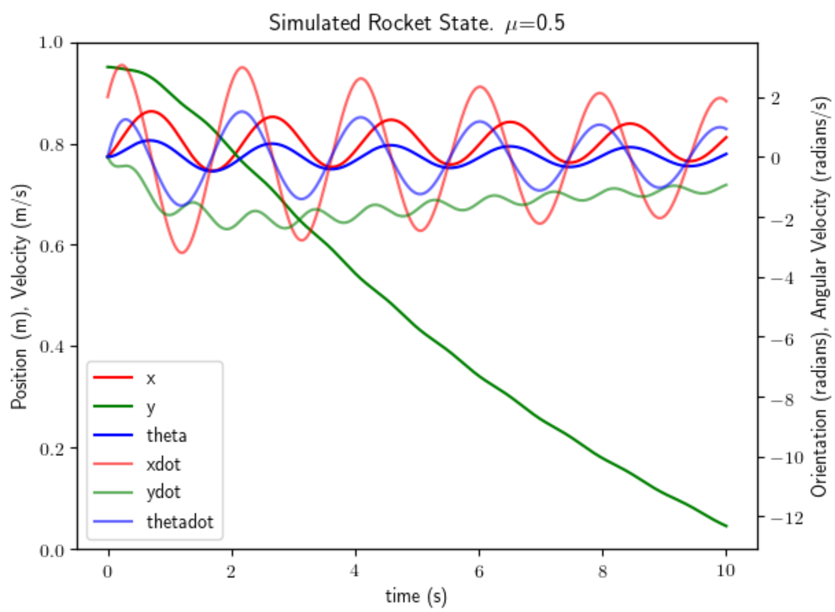


Figure 5: The initial condition is $y = 3$ and $\dot{x} = 1$. All other starting parameters are zero. This is a bifurcation value of $\mu$. The vehicle is oscillating about the fixed point.

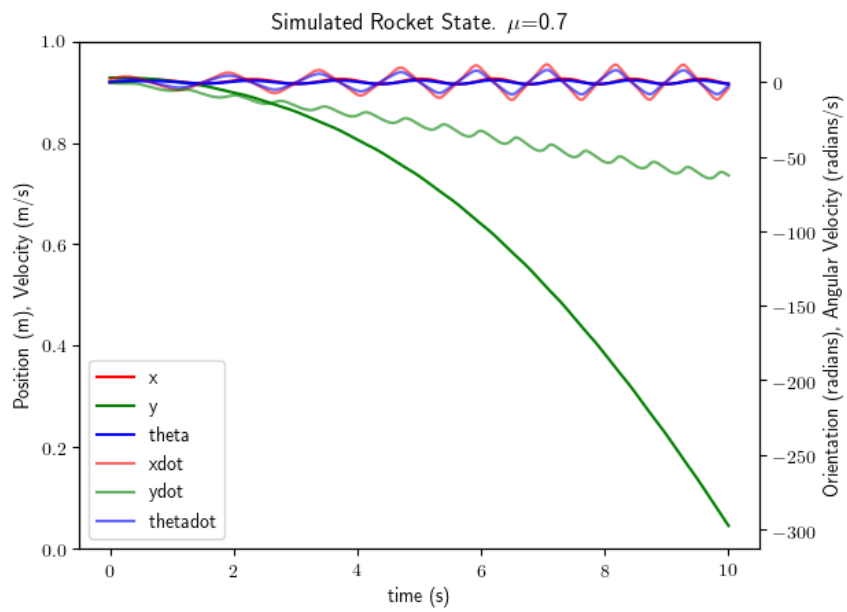


Figure 6: The initial condition is $y = 3$ and $\dot{x} = 1$. All other starting parameters are zero. This is an unstable value of $\mu$. Although the vehicle also oscillates about $\theta = 0$, note that $\dot{y}$ is decreasing, corresponding to the rocket falling with increasing speed.