Math 240A Notes (Professor: Luca Spolaor)

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Lecture 1 Notes: 9/26/2024

Given an indexed family of sets $\{X_\alpha\}_{\alpha \in A}$, we define its <u>Cartesian Product</u> to be:

$$\prod_{\alpha \in A} X_\alpha = \{f : A \longrightarrow \bigcup_{\alpha \in A} X_\alpha \mid f(\alpha \in X_\alpha)\}$$

A projection is a function $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \longrightarrow X_\alpha$ satisfying that $f \mapsto f(\alpha)$.

If X, Y are sets, we define:

- $\mathrm{card}(X) \leq \mathrm{card}(Y)$ if there exists an injection $f : X \longrightarrow Y$.

- $\mathrm{card}(X) \geq \mathrm{card}(Y)$ if there exists a surjection $f : X \longrightarrow Y$.

- $\mathrm{card}(X) = \mathrm{card}(Y)$ if there exists a bijection $f : X \longrightarrow Y$.

> Note that $\mathrm{card}(X) \leq \mathrm{card}(Y) \iff \mathrm{card}(Y) \geq \mathrm{card}(X)$. After all, given an injection in one direction, we can easily make a surjection in the other direction. Or given a surjection in one direction, we can (using A.O.C (axiom of choice)) easily make an injection in the other direction.

> Also, if $\mathrm{card}(X) \leq \mathrm{card}(Y)$ and $\mathrm{card}(Y) \leq \mathrm{card}(X)$, then we know that $\mathrm{card}(Y) = \mathrm{card}(X)$.

>> Proof:
>> We know there exists $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ which are both injective. Hence, $g \circ f$ is an injection from X to $g(Y) \subseteq X$. By an exercise done in my math journal on page 8, we thus there exists a bijection h from X to $g(Y)$. And letting g^{-1} be any left-inverse of g, we then have that $g^{-1} \circ h$ is a bijection from X to Y.

We say X has the <u>cardinality of the continuum</u> if $\mathrm{card}(X) = \mathrm{card}(\mathbb{R})$.
> Proposition: $\mathrm{card}(\mathcal{P}(\mathbb{N})) = \mathrm{card}(\mathbb{R})$.
>> Our textbook goes about proving this by constructing two functions: an injection and a surjection, from $\mathcal{P}(\mathbb{N})$ to $\mathbb{R}$ based on the binary expansion of any real number. That way, we know that $\mathrm{card}(\mathcal{P}(\mathbb{N})) \leq \mathrm{card}(\mathbb{R})$ and $\mathrm{card}(\mathcal{P}(\mathbb{N})) \geq \mathrm{card}(\mathbb{R})$.

Given a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ we know there exists: $\limsup x_n = \inf_{k \geq 1}(\sup_{n \geq k} x_n)$ and $\liminf x_n = \sup_{k \geq 1}(\inf_{n \geq k} x_n)$.

Also, given a function $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, we can define:

$$\limsup_{x \to a} f(x) = \inf_{\delta > 0} \left(\sup_{0 < |x-a| < \delta} f(x) \right).$$

If X is an arbitrary set and $f : X \longrightarrow [0, \infty]$, we define:
$$\sum_{x \in X} f(x) = \sup \left\{ \sum_{x \in F} f(x) \mid F \subseteq X \ s.t. \ F \text{ is finite} \right\}.$$

Cool Proposition from textbook (not covered in lecture):
Let $A = \{x \in X \mid f(x) > 0\}$. If A is uncountable, then $\sum_{x \in X} f(x) = \infty$.

If A is countably infinite and $g : \mathbb{N} \longrightarrow A$ is a bijection, then
$$\sum_{x \in X} f(x) = \sum_{n=1}^{\infty} f(g(n)).$$

Proof of first statement:
$A = \bigcup_{n \in \mathbb{N}} A_n$ where $A_n = \{x \in X \mid f(x) > \frac{1}{n}\}$.

If A is uncountable, we must have that some A_n is uncountable. But then for any finite set $F \subseteq X$, we have that $\sum_{x \in F} f(x) > \frac{\text{card}(F)}{n}$. So $\sum_{x \in X} f(x)$ is unbounded.

A metric space (X, ρ) is a set X equipped with a distance function $\rho : X \times X \longrightarrow [0, \infty)$. We denote the open ball of radius r about x to be $B(r, x) = \{y \in X \mid \rho(x, y) < r\}$. And you remember our definitions from 140A... right?

Proposition 0.21: Every open set in $\mathbb{R}$ is a countable union of disjoint open intervals.
We proved this as part of a homework exercise in Math 140A.

Given a metric space (X, ρ), an element $x \in X$, and sets $F, E \subseteq X$, we can define:

- $\rho(x, E) = \rho_E(x) = \inf\{\rho(x, y) \mid y \in E\}$.

- $\rho(F, E) = \inf\{\rho_E(y) \mid y \in F\}$.

Exercise: $g(x, E) = 0 \iff x \in \overline{E}$.
Proof:
If $\inf\{\rho(x, y) \mid y \in E\} = 0$, then there exists a sequence $\{y_n\}$ in E such that $\rho(x, y_n) \to 0$. This implies $x \in \overline{E}$. Similarly, if $x \in \overline{E}$, we can construct a sequence $\{y_n\}$ such that $\rho(x, y_n) < \frac{1}{n}$ for all n. Then:
$$0 \leq \inf\{\rho(x, y) \mid y \in E\} \leq \inf\{\rho(x, y_n) \mid n \in \mathbb{N}\} = 0.$$

Given a subset E of a metric space (X, ρ), we define:
$$\text{diam}(E) = \sup\{\rho(x, y) \mid x, y \in E\}.$$

If $\text{diam}(E) < \infty$, we say E is <u>bounded</u>. If $\forall \varepsilon > 0$, E can be covered by finitely many balls of radius ε, then we say E is <u>totally bounded</u>.

Exercise: E being totally bounded implies E is bounded.

Pick $\varepsilon > 0$ and let $\{z_1, \ldots, z_n\}$ be the set of points such that $E \subseteq \bigcup\limits_{k=1}^{n} B(\varepsilon, z_n)$.

Then given any $x, y \in E$, we can assume that $x \in B(\varepsilon, z_i)$ and $y \in B(\varepsilon, z_j)$. So, $\rho(x, y) \leq \rho(x, z_i) + \rho(z_i, z_j) + \rho(z_j, y) < 2\varepsilon + \max\{\rho(z_i, z_j) \mid 1 \leq i, j \leq n\}$.

The converse is not generally true. For instance, if you use the discrete metric, then any set with more than one element will have a diameter of 1. But if $0 < \varepsilon < 1$, then it will be impossible to cover an infinite set with finitely many balls.

Lecture 2 Notes: 10/1/2024

Proposition: Suppose E is a subset of a metric space (X, ρ). Then the following are equivalent.

1. E is complete and totally bounded

2. All sequences $(x_n) \subseteq E$, have a convergent subsequence.

3. For all open covers $\{V_\alpha\}_{\alpha \in A}$ of E, there exists $V_{\alpha_1}, \ldots, V_{\alpha_n}$ such that $E \subseteq \bigcup\limits_{i=1}^{n} V_{\alpha_i}$.

Proof:

(1) $\Longrightarrow$ (2):

Lemma:

If E is totally bounded and $F \subseteq E$, then F is totally bounded.

Given any $\varepsilon > 0$, let $\{x_1, \ldots, x_n\}$ be a subset of E such that $E \subseteq \bigcup\limits_{i=1}^{n} B(\varepsilon/2, x_i)$. Then consider the collection of sets:
$$\{F \cap B(\varepsilon/2, x_i)\} - \{\emptyset\}.$$

We know the diameter of each $F \cap B(\varepsilon/2, x_i)$ is at most ε. So in each set, pick $y_i \in F \cap B(\varepsilon/2, x_i)$. Then for some $m \leq n$:
$$F \subseteq \bigcup\limits_{i=1}^{m} B(\varepsilon, y_i)$$

Let $A_1 = E$. Then for $k \geq 2$ we recursively define A_k as follows:

Assuming $A_{k-1} \cap (x_n)_{n \in \mathbb{N}}$ is infinite and A_{k-1} is totally bounded, choose $\{y_1, \ldots, y_m\}$ in A_k such that $A_k \subseteq \bigcup\limits_{i=1}^{m} B(2^{-n}, y_i)$. Importantly, since $(x_n)_{n \in \mathbb{N}} \cap A_{k-1}$ is infinite, we know one of those open balls contains infinitely many points in our sequence. So set A_k equal to that ball intersected with E. Note that by our lemma, A_k is totally bounded.

Now pick any x_{n_1} and then for all $k \geq 2$ pick $x_{n_k} \in A_k$ such that $n_k > n_{k-1}$. That way, $(x_{n_k})_{k \in \mathbb{Z}_+}$ is a subsequence of $(x_n)_{n \in \mathbb{Z}_+}$. Also, we know that $(x_{n_k})_{k \in \mathbb{Z}_+}$ is Cauchy. Hence, since E is complete, we know that it converges to some x in E.

(2) $\implies$ (1):

Firstly, suppose E is not complete. Then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ that is Cauchy but does not converge in E. Importantly, because $(x_n)_{n \in \mathbb{N}}$ is Cauchy, if there was a convergent subsequence, we know the limit of that subsequence would have to be the limit of the whole sequence. But that doesn't exist. So, we know (2) can't be true.

Secondly, suppose E is not totally bounded. Then there exists $\varepsilon > 0$ such that it is impossible to cover E in balls of radius ε. So, we can recursively define a sequence $(x_n)_{n \in \mathbb{N}}$ in E satisfying that:

$$x_n \in E - \bigcup_{i=1}^{n-1} B(\varepsilon, x_i).$$

Importantly, for all natural numbers $n \neq m$, we have that $\rho(x_n, x_m) \geq \varepsilon$. So, it is impossible to find a convergent subsequence of (x_n), meaning (2) is false.

(1) and (2) $\implies$ (3):

Let $\{V_\alpha\}_{\alpha \in A}$ be an open cover of E.

Suppose for the sake of contradiction that for all $n \in \mathbb{N}$, there is a ball B_n of radius 2^{-n} centered in E such that $B_n \cap E \neq \emptyset$ but $B_n \nsubseteq V_\alpha$ for all $\alpha \in A$. Then we can construct a sequence $(x_n)_{n \in \mathbb{N}}$ in E such that $x_n \in B_n \cap E$ for all $n \in \mathbb{N}$. By (2), we know there is a subsequence that converges to some $x \in E$. Importantly, we know $x \in V_\alpha$ for some $\alpha \in A$, and because V_α is open, there is $\varepsilon > 0$ such that $B(\varepsilon, x) \subseteq V_\alpha$. But now we get a contradiction because by picking n such that $2^{-n} < \varepsilon/3$ and $\rho(x, x_n) < \varepsilon/3$, we have for all $y \in B_n$ that:

$$\rho(x, y) \leq \rho(x, x_n) + \rho(x_n, y) < 2^{-n} + 2^{-n+1} < \varepsilon$$

So $B_n \subseteq B(\varepsilon, x) \subseteq V_\alpha$.

We've thus shown that for some $n \in N$, all balls of radius 2^{-n} centered in E are contained by some V_α. And assuming (1), we can cover E with finitely many balls of radius 2^{-n} It follows that by picking a V_α containing a ball for each ball covering E, we've found a finite covering E using the sets in $\{V_\alpha\}_{\alpha \in A}$.

(3) $\implies$ (2):

Suppose $(x_n)_{n \in \mathbb{N}}$ is a sequence in E with no convergent subsequence. Then for each $x \in E$, there must exist $\varepsilon_x > 0$ such that $B(\varepsilon_x, x) \cap (x_n)_{n \in \mathbb{N}}$ is finite. (If ε_x didn't exist, we could construct a Cauchy subsequence converging to x).

But now $\{B(\varepsilon_x, x)\}_{x \in E}$ is an open cover of E with no finite subcover of E because it will take an infinite cover to cover all of $(x_n)_{n \in \mathbb{N}}$.

If E satisfies all three of the above properties, we say E is <u>compact</u>.

Corollary: $K \subseteq \mathbb{R}^n$ is compact iff it's closed and bounded.

Roughly speaking, we want a measure to be a function $\mu : \mathcal{P}(\mathbb{R}^n) \longrightarrow [0, \infty)$ such that $E \mapsto \mu(E) =$ "the area of E". Also, we would like it if:

(i) $\mu([0,1)^n) = 1$

(ii) $\mu(\text{rotation, translation, or reflection of } A) = \mu(A)$

(iii) $\mu(\bigcup\limits_{i=1}^{\infty} A_i) = \sum\limits_{i=1}^{\infty} \mu(A_i)$ if $A_i \cap A_j \neq \emptyset \implies i = j$.

Unfortunately, the properties as written above are inconsistent.

Vitali Sets:
Defining $x \sim y$ iff $x - y \in \mathbb{Q}$, let $N \subseteq [0,1)$ be a set such that $N \cap [x]$ has precisely one element for all $x \in \mathbb{R}$. Next let $R = [0,1) \cap \mathbb{Q}$, and for all $r \in R$ define:
$$N_r = \{x + r \mid x \in N \cap [0, 1-r)\} \cup \{x + r - 1 \mid x \in N \cap [1-r, 1)\}.$$

Importantly, note that $N_r \subseteq [0,1)$. Plus, the two sets being unioned over to make N_r are both disjoint and can be translated around so that they are still disjoint but their union forms N. Hence assuming $\mu : \mathcal{P}(\mathbb{R}^n) \longrightarrow [0, \infty)$ satisfying (ii) and (iii), we know $\mu(N_r) = \mu(N)$.

Also, for all $y \in [0,1)$, if $x \in N \cap [y]$, we know that $y \in N_r$ where $r = x - y$ if $x \geq y$, or $r = x - y + 1$ if $x < y$. Hence, $[0,1) = \bigcup\limits_{r \in R} N_r$.

Also, given any N_r and N_s, if $x \in N_r \cap N_s$, then we'd be able to show that both $x - r$ or $x - r + 1$ and $x - s$ or $x - s + 1$ are distinct elements of N in the same equivalence class, which contradicts how we defined N.

You work through the scratch work of the different cases on your own! :P

So supposing μ satisfies (i) and (iii) and because R is countable, we have that:
$$1 = \sum_{r \in R} \mu(N_r) = \sum_{r \in R} (N) = 0 \text{ or } \infty.$$

This is a contradiction.

Furthermore, the problem is not the countable union property as is demonstrated by the Banach-Tarsky paradox:

Theorem: Let U and V be arbitrary bounded sets in $\mathbb{R}^n$ where $n \geq 3$. Then there exists $E_1, \ldots, E_N, F_1, \ldots, F_N$ in $\mathbb{R}^n$ such that:

- $E_i \cap E_j = \emptyset$ for all $i \neq j$ and $\displaystyle\bigcup_{i=1}^{N} E_i = U$
- $F_i \cap F_j = \emptyset$ for all $i \neq j$ and $\displaystyle\bigcup_{i=1}^{N} F_i = V$
- E_i and F_i are congruent for all $i \in \{1, \ldots, N\}$.

Supposing that $\mu(E_j)$ and $\mu(F_j)$ exists for all j and that μ satisfies (i), (ii), and (iii) except only for finite unions, then that would suggest all sets have the same "area", which we know doesn't make sense.

What we will do to fix this issue is only define μ on a subset of $\mathcal{P}(\mathbb{R}^n)$.

Let $X \neq \emptyset$. An <u>algebra of sets</u> in X is a nonempty collection $\mathcal{A} \subseteq \mathcal{P}(X)$ which is closed under finite unions and complements. If is $\mathcal{A}$ is also closed under countable unions, we say $\mathcal{A}$ is a σ-algebra.

Observations:

1. Algebras of sets are closed under finite intersections and σ-algebras are closed under countable intersection. (This also means algebras of sets are closed under set differences.)

   This is because $\displaystyle\bigcap_{n\in\mathbb{N}} A_n = \left(\bigcup_{n\in\mathbb{N}} A_n^{\mathsf{c}}\right)^{\mathsf{c}}$.

2. If $\mathcal{A}$ is closed under disjoint countable union, then it's closed under arbitrary countable unions.

   This is because $\displaystyle\bigcup_{n\in\mathbb{N}} A_n = A_1 \cup \bigcup_{n\geq 2} \left(A_n \cap \left(\bigcup_{i=1}^{n-1} A_i\right)^{\mathsf{c}}\right)$

3. If $\{\mathcal{E}_\alpha\}_{\alpha\in A}$ is a collection of σ-algebras, then $\displaystyle\bigcap_{\alpha\in A} \mathcal{E}_\alpha$ is a σ-algebra.

   This is pretty trivial to prove. It should remind you of topologies.

Exercise 1.1: A family of sets $\mathcal{R} \subseteq \mathcal{P}(X)$ is called a <u>ring</u> if it is closed under finite unions and difference. If $\mathcal{R}$ is also closed under countable unions, it is called a σ-ring.

(a) Rings are closed under finite intersections and σ-rings are closed under countable intersections.

   If $\mathcal{R}$ is a ring and $A_1, \ldots, A_n \in \mathcal{R}$, then:
   $$\bigcap_{i=1}^{n} A_n = A_1 - \bigcup_{i=2}^{n}(A_1 - A_i) \in \mathcal{R}$$

This is because each $A_1 - A_i \in \mathcal{R}$, meaning $\bigcup_{i=2}^{n}(A_1 - A_i) \in \mathcal{R}$, and so finally $A_1 - \bigcup_{i=2}^{n}(A_1 - A_i) \in \mathcal{R}$.

If $\mathcal{R}$ is a σ-algebra, we can replace the finite intersection and union used in the prior reasoning with a countable intersection and union.

(b) If $\mathcal{R}$ is a ring (or σ-ring), then $\mathcal{R}$ is an algebra (or σ-algebra) iff $X \in \mathbb{R}$.

$(\Longrightarrow)$ Suppose $\mathcal{R}$ is an algebra. Then note that $\emptyset \in \mathcal{R}$ because for any $A \in \mathcal{R}$, $A - A \in \mathcal{R}$. So taking complements, we get that $X \in \mathcal{R}$.

$(\Longleftarrow)$ Suppose $X \in \mathcal{R}$. Then for any $A \in \mathcal{R}$, we know that $A^{\mathsf{C}} = X - A \in \mathcal{R}$. So $\mathcal{R}$ is an algebra (or σ-algebra).

(c) If $\mathcal{R}$ is a σ-ring, then $\mathcal{A} = \{E \subseteq X \mid E \in \mathcal{R} \text{ or } E^{\mathsf{C}} \in \mathcal{R}\}$ is a σ-algebra.

To start, we know that $\mathcal{A}$ is closed under complements because for any $A \in \mathcal{A}$,

$$A \in \mathcal{R} \Longrightarrow (A^{\mathsf{C}})^{\mathsf{C}} \in \mathcal{R} \Longrightarrow A^{\mathsf{C}} \in \mathcal{A}$$
$$A \notin \mathcal{R} \Longrightarrow A^{\mathsf{C}} \in \mathcal{R} \Longrightarrow A^{\mathsf{C}} \in \mathcal{A}$$

Also, let $(E_n)_{n \in \mathbb{N}}$ be a countable collection of sets in $\mathcal{A}$. Then define $A = \{n \in \mathbb{N} \mid E_n^{\mathsf{C}} \notin \mathcal{R}\}$ and $B = \{n \in \mathbb{N} \mid E_n^{\mathsf{C}} \in \mathcal{R}\}$. Clearly, we have that:

$$\bigcup_{n \in \mathbb{N}} E_n = \bigcup_{n \in A} E_n \cup \bigcup_{n \in B} E_n = \bigcup_{n \in A} E_n \cup \bigcup_{n \in B} (E_n^{\mathsf{C}})^{\mathsf{C}}$$

Also $\bigcup_{n \in B} (E_n^{\mathsf{C}})^{\mathsf{C}} = \left(\bigcap_{n \in B} E_n^{\mathsf{C}}\right)^{\mathsf{C}}$, and by part (a), we know that $E_B := \bigcap_{n \in B} E_n^{\mathsf{C}} \in \mathcal{R}$.

Similarly, we know $E_A := \bigcup_{n \in A} E_n \in \mathcal{R}$. So, we've shown that $\bigcup_{n \in \mathbb{N}} E_n = E_A \cup E_B^{\mathsf{C}}$ where $E_A, E_B \in \mathcal{R}$.

Finally, note that $E_A \cup E_B^{\mathsf{C}} = (E_B - E_A)^{\mathsf{C}}$. Since $E_B - E_A \in \mathcal{R}$, we know that $(E_B - E_A)^{\mathsf{C}} \in \mathcal{A}$.

(d) If $\mathcal{R}$ is a σ-ring, then $\mathcal{A} = \{E \subseteq X \mid E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

To start if $E \in \mathcal{A}$, then $E^{\mathsf{C}} \in \mathcal{A}$ because for all $F \in \mathcal{R}$ we have that:

$$E^{\mathsf{C}} \cap F = F - E = F - (E \cap F) \in \mathcal{R}.$$

Also, let $(E_n)_{n \in \mathbb{N}}$ be a countable collection of sets in $\mathcal{A}$. Then for all $F \in \mathcal{R}$, we have that $\left(\bigcup_{n \in \mathbb{N}} E_n\right) \cap F = \bigcup_{n \in \mathbb{N}} (E_n \cap F) \in \mathcal{R}$. So $\mathcal{A}$ is closed under countable union.

Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of sets. Since the intersection of σ-algebras is still a σ-algebra, we define $\mathcal{M}(\mathcal{E})$ to be the smallest σ-algebra that contains $\mathcal{E}$. In other words, $\mathcal{M}(\mathcal{E})$ is the intersection of all σ-algebras that contain $\mathcal{E}$.

We call $\mathcal{M}(\mathcal{E})$ the σ-algebra generated by $\mathcal{E}$.

Lemma: if $\mathcal{E} \in \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$.

Let (X, ρ) be a metric space. We define the <u>Borel σ-algebra</u> on X: $\mathcal{B}_X$, to be the σ-algebra generated by the collection of all open sets, or equivalently the collection of all closed sets.

- A set is G_δ if it is a countable intersection of open sets.

- A set is F_σ if it is a countable union of closed sets.

- A set is $G_{\delta\sigma}$ if it is a countable union of G_δ sets.

- A set is $F_{\sigma\delta}$ if it is a countable intersection of F_σ sets.

You can hopefully see the pattern. Also the professor isn't sure how much we'll use this δ and σ notation in class.

Exercise 1.2: $\mathcal{B}_\mathbb{R}$ is generated by each of the following:

(a) the set of open intervals: $\mathcal{E}_1 = \{(a, b) \mid a < b\}$

(b) the set of closed intervals: $\mathcal{E}_2 = \{[a, b] \mid a < b\}$

(c) the set of half-open intervals:

   (i) $\mathcal{E}_3 = \{(a, b] \mid a < b\}$
   (ii) $\mathcal{E}_4 = \{[a, b) \mid a < b\}$

(c) the set of open rays:

   (i) $\mathcal{E}_5 = \{(a, \infty) \mid a \in \mathbb{R}\}$
   (ii) $\mathcal{E}_6 = \{(-\infty, a) \mid a \in \mathbb{R}\}$

(d) the set of closed rays:

   (i) $\mathcal{E}_7 = \{[a, \infty) \mid a \in \mathbb{R}\}$
   (ii) $\mathcal{E}_8 = \{(-\infty, a] \mid a \in \mathbb{R}\}$

Proof:
We trivially have that $\mathcal{M}(\mathcal{E}_1), \mathcal{M}(\mathcal{E}_2), \mathcal{M}(\mathcal{E}_5), \mathcal{M}(\mathcal{E}_6), \mathcal{M}(\mathcal{E}_7), \mathcal{M}(\mathcal{E}_8) \subseteq \mathcal{B}_\mathbb{R}$ since each of them contain either only open sets or only closed sets. As for the other inclusions, we must do more work.

(a) Note that $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$. Hence, a countable base of $\mathbb{R}$ is the set: $\mathcal{F} = \{(p-q, p+q) \subset \mathbb{R} \mid p, q \in \mathbb{Q} \text{ and } q > 0\}$. In other words, given any open set $E \subseteq \mathbb{R}$, there is a countable subcollection of $\mathcal{F}$ whose union is E.

> To see why, let $x \in E$. Since E is open, there exists $r > 0$ with $B(r, x) \subseteq E$. Next, pick $p \in (x, x + \frac{r}{2}) \cap \mathbb{Q}$, followed by $q \in (p - x, r - p) \cap \mathbb{Q}$. Then $x \in (p - q, p + q) \in \mathcal{F}$ and $(p - q, p + q) \subseteq (x - r, x + r)$.

> With that, we've now shown that for all $x \in E$, there exists $F \in \mathcal{F}$ such that $x \in F \subseteq E$. If we choose such an F_x for all $x \in E$, we then get that $E = \bigcup_{x \in E} F_x$. So E is the union of a subcollection of $\mathcal{F}$. But since $\mathcal{F}$ is countable, the set $\{F_x \in \mathcal{F} \mid x \in E\}$ is also countable.

Importantly, $\mathcal{F} \subset \mathcal{E}_1$. So $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E}_1)$. However as shown above, we must have that $\mathcal{M}(\mathcal{F})$ includes all open sets. So by our lemma on the previous page, $\mathcal{B}_\mathbb{R} \subseteq \mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E}_1)$.

(b) Given any $E = (a, b) \in \mathcal{E}_1$, we can write that $E = \bigcup_{n \in \mathbb{Z}_+} [a + \frac{1}{n}, b - \frac{1}{n}]$. Thus, $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_2)$, meaning $\mathcal{B}_\mathbb{R} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_2)$.

(c) Remember that for these two, we still need to show that $\mathcal{M}(\mathcal{E}_1), \mathcal{M}(\mathcal{E}_2) \in \mathcal{B}_\mathbb{R}$.

(i) Firstly note that if $F = (a, b] \in \mathcal{E}_3$, then $F = \bigcap_{n \in \mathbb{Z}_+} (a, b + \frac{1}{n})$. So $\mathcal{E}_3 \subseteq \mathcal{M}(\mathcal{E}_1)$.

On the other hand, if $E = (a, b) \in \mathcal{E}_1$, we have that $E = \bigcup_{n \in \mathbb{Z}_+} (a, b - \frac{1}{n}]$. So $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_3)$.

By our lemma on the previous page, we thus have that:
$$\mathcal{B}_\mathbb{R} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_1) = \mathcal{B}_\mathbb{R}.$$

(ii) Mostly identical reasoning as with $\mathcal{E}_3$ shows that:
$$\mathcal{B}_\mathbb{R} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_1) = \mathcal{B}_\mathbb{R}$$

(d)

(i) If $E = (a, b) \in \mathcal{E}_1$, then we know that:
$$E = (a, \infty) \cap \left(\bigcap_{n \in \mathbb{Z}_+} (b - \frac{1}{n}, \infty)\right)^\mathsf{C} \in \mathcal{M}(\mathcal{E}_5).$$
So $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_5)$, meaning $\mathcal{B}_\mathbb{R} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_5)$.

(ii) Analogous reasoning to that with $\mathcal{E}_5$ shows that $\mathcal{B}_\mathbb{R} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_6)$.

(e)

(i) If $E = (a, \infty) \in \mathcal{E}_6$, then we have that $E = \bigcup_{n \in \mathbb{Z}_+} [a + \frac{1}{n}, \infty)$. So $\mathcal{E}_5 \subseteq \mathcal{M}(\mathcal{E}_7)$, meaning that $\mathcal{B}_\mathbb{R} = \mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{M}(\mathcal{E}_7)$.

(ii) Analogous reasoning as with $\mathcal{E}_7$ shows that $\mathcal{B}_\mathbb{R} = \mathcal{M}(\mathcal{E}_6) \subseteq \mathcal{M}(\mathcal{E}_8)$.

Exercise 1.3: Let $\mathcal{M}$ be an infinite σ-algebra on X.

(a) $\mathcal{M}$ contains an infinite sequence of disjoint sets.

By the Hausdorff maximum principle, we know there is a subcollection $\mathcal{S}$ of $\mathcal{M}$ which is simply ordered by proper subset and is not contained in any other collection of $\mathcal{M}$ which is simply ordered by proper subset.

We claim $\mathcal{S}$ can't be finite. For suppose $\mathcal{S} = \{E_1, \ldots, E_n\}$ is a sequence of sets in $\mathcal{M}$ simply ordered by proper subset which are indexed such that $E_i \subset E_{i+1}$ for all $i \in \{1, \ldots, n-1\}$.

(Note: if $\mathcal{S}$ is maximal, then we must have $E_1 = \emptyset$ and $E_n = X$.)

We can partition $\mathcal{M}$ into collections $\mathcal{M}_1, \ldots \mathcal{M}_n$ such that $A \in \mathcal{M}_i$ iff i is the least integer for which $A \subseteq E_i$. Importantly, all sets in $\mathcal{M}$ will fall into a partition because all sets from $\mathcal{M}$ are contained in E_n. Also note that while there are infinitely many $A \in \mathcal{M}$, there are only n many partitions. So, there must be a least integer k such that $\mathcal{M}_k$ contains infinitely many $A \in \mathcal{M}$.

And since $\mathcal{M}_1 = \{\emptyset\}$, we know $k \geq 2$.

The fact that $\mathcal{M}_i$ is finite for all $i < k$ means that there are only finitely many sets from $\mathcal{M}$ contained in E_{k-1}. Thus, we can pick a set $B \in \mathcal{A}_k$ such that $B \neq (E_k - E_{k-1}) \cup A$ for any $A \in \mathcal{M}$ that is a subset of E_{k-1}.

Note that since $E_{k-1} \cap B$ is a set in $\mathcal{M}$, we must have that $(E_k - E_{k-1}) \not\subseteq B$ or else B would be the union of $(E_k - E_{k-1})$ and a set from $\mathcal{M}$. Thus, we know E_k contains points that neither B nor E_{k-1} have. At the same time, we know B has points that E_{k-1} doesn't have. It follows that: $E_{k-1} \subset E_{k-1} \cup B \subset E_k$.

Via transitivity, $E_{k-1} \cup B$ is comparable via proper subset with E_i for all $i \in \{1, \ldots, n\}$. Hence, we've shown that $\mathcal{S} \cup \{E_{k-1} \cup B\}$ is a sequence of sets in $\mathcal{M}$ simply ordered by proper subset. But this contradicts that $\mathcal{S}$ is maximal.

Now that we know $\mathcal{S}$ is infinite, let $(E_n)_{n \in \mathbb{Z}_+}$ be a sequence of sets in $\mathcal{S}$ satisfying that $E_n \subset E_{n+1}$. Then we have that $(E_{n+1} - E_n)_{n \in \mathbb{Z}_+}$ is an infinite sequence of nonempty disjoint sets in $\mathcal{M}$.

(b) Show that $\mathrm{card}(\mathcal{M}) \geq \mathfrak{c}$.

Let $(E_n)_{n \in \mathbb{N}}$ be a sequence of nonempty disjoint sets in $\mathcal{M}$. Then if we define the map $f : [0,1]^{\mathbb{N}} \longrightarrow \mathcal{M}$ such that $(a_0, a_1, a_2, \ldots)$ is mapped to the union of all E_n such that $a_n = 1$, we have that f is an injection.

Hence, $\mathrm{card}(\mathcal{M}) \geq \mathrm{card}([0,1]^{\mathbb{N}})$. And since there is a trivial bijection from $[0,1]^{\mathbb{N}}$ and $\mathcal{P}(\mathbb{N})$, plus the fact that we proved early on in the class that $\mathrm{card}(\mathcal{P}(\mathbb{N})) = \mathrm{card}(\mathbb{R})$, we thus know that $\mathrm{card}(\mathcal{M}) \geq \mathfrak{c}$.

Exercise 1.4: An algebra $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is closed under countable increasing unions (meaning $E_1 \subseteq E_2 \subseteq \ldots$).

The rightward implication is true since $\mathcal{A}$ being a σ-algebra means that $\mathcal{A}$ is closed under all countable unions. As for showing the leftward implication, suppose $\{A_n\}_{n \in \mathbb{Z}_+}$ is a countable collection of sets in $\mathcal{A}$. Then for all $n \in \mathbb{Z}_+$, define $E_n = A_1 \cup \ldots \cup A_n$.

Since each E_n are finite unions of sets in $\mathcal{A}$, we know that each E_n is in $\mathcal{A}$. Also, we clearly have that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ In order to make the sets strictly increasing, let $S = \{1\} \cup \{k \in \mathbb{Z} \mid k > 1 \text{ and } E_k - E_{k-1} \neq \emptyset\}$. Then for any $n, m \in S$, we know that $n < m \implies E_n \subset E_m$.

Finally, $\bigcup_{n \in \mathbb{Z}_+} A_n = \bigcup_{n \in \mathbb{Z}_+} E_n = \bigcup_{n \in S} E_n$.

Importantly, S is either finite or countably infinite, and S consists of strictly increasing sets. So by the right hypothesis, we know $\bigcup_{n \in S} E_n \in \mathcal{A}$. Hence, the union over $\{A_n\}_{n \in \mathbb{Z}_+}$ is in $\mathcal{A}$.

Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of nonempty sets, and define $X = \prod_{\alpha \in A} X_\alpha$.

If $\mathcal{M}_\alpha$ is a σ-algebra in X_α for all $\alpha \in A$, then we define the <u>product σ-algebra</u> on X to be: $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \mathcal{M}(\{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{M}_\alpha \text{ and } \alpha \in A\})$.

To get a better geometric intuition for this definition, consider if $A = \{1, 2\}$. Then:
$$\bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \{\pi_1^{-1}(E_1) \mid E_1 \in \mathcal{M}_1\} \cup \{\pi_2^{-1}(E_2) \mid E_2 \in \mathcal{M}_2\}$$
$$= \{E_1 \times X_2 \mid E_1 \in \mathcal{M}_1\} \cup \{X_1 \times E_2 \mid E_2 \in \mathcal{M}_2\}$$

Also, the motivation for this definition is that $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is the smallest σ-algebra where π_α is "measurable" for all α. We'll learn what that means shortly...

Proposition:

(i) A is countable implies $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \mathcal{M}(\{\prod_{\alpha \in A} E_\alpha \mid \forall \alpha \in A, \ E_\alpha \in \mathcal{M}_\alpha\})$

Proof:
If $E_\alpha \in \mathcal{M}_\alpha$, then $\pi_\alpha^{-1}(E_\alpha) = \prod_{\beta \in A} E_\beta$ where $E_\beta = X_\beta$ if $\beta \neq \alpha$ (and $E_\beta = E_\alpha$ if $\beta = \alpha$).

So $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{M}(\{\prod_{\alpha \in A} E_\alpha \mid \forall \alpha \in A, \ E_\alpha \in \mathcal{M}_\alpha\})$

On the other hand, $\prod_{\alpha \in A} E_\alpha = \bigcap_{\alpha \in A} \pi_\alpha^{-1}(E_\alpha)$.

Since A is countable, we thus know that if $E_\alpha \in \mathcal{M}_\alpha$ for all $\alpha \in A$, then $\prod_{\alpha \in A} E_\alpha \in \bigotimes_{\alpha \in A} \mathcal{M}_\alpha$.

(ii) Suppose $\mathcal{M}_\alpha = \mathcal{M}(\mathcal{E}_\alpha)$ for all $\alpha \in A$. Then $\bigotimes\limits_{\alpha \in A} \mathcal{M}_\alpha$ is generated by
$\mathcal{F} = \{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{E}_\alpha \text{ and } \alpha \in A\}$.

Proof:

Since $\mathcal{F} \subseteq \{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{M}_\alpha \text{ and } \alpha \in A\}$, we trivially have that
$\mathcal{M}(\mathcal{F}) \subseteq \bigotimes\limits_{\alpha \in A} \mathcal{M}_\alpha$.

As for showing the other inclusion, define for each $\alpha \in A$:
$$\mathcal{F}_\alpha = \{E \subseteq X_\alpha \mid \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})\}.$$

Note that $\mathcal{F}_\alpha$ is a σ-algebra on X_α that contains $\mathcal{E}_\alpha$.

This is because for any $F \in \mathcal{F}_\alpha$ and $(E_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}_\alpha$, we know that:

- $\left(\pi_\alpha^{-1}(F)\right)^{\mathsf{C}} = \pi_\alpha^{-1}(F^{\mathsf{C}})$

- $\bigcup\limits_{n \in \mathbb{N}} \pi_\alpha^{-1}(E_n) = \left(\pi_\alpha^{-1}(\bigcup\limits_{n \in \mathbb{N}} E_n)\right)$

Also, for any $E \subseteq X_\alpha$, $E \in \mathcal{E}_\alpha \implies \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})$.

By definition, we thus know that $\mathcal{M}_\alpha \subseteq \mathcal{F}_\alpha$. So for all $\alpha \in A$ and $E_\alpha \in \mathcal{M}_\alpha$, we know that $E_\alpha \in \mathcal{F}_\alpha$, which means that $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{M}(\mathcal{F})$. So
$\bigotimes\limits_{\alpha \in A} \mathcal{M}_\alpha \subseteq \mathcal{M}(\mathcal{F})$.

(iii) We can also combine the first two parts of this proposition. If A is countable and $\mathcal{M}_\alpha = \mathcal{M}(\mathcal{E}_\alpha)$ for all $\alpha \in A$, then $\bigotimes\limits_{\alpha \in A} \mathcal{M}_\alpha$ is generated by:
$$\{\prod\limits_{\alpha \in A} E_\alpha \mid \forall \alpha \in A, \ E_\alpha \in \mathcal{E}_\alpha\}$$

Lecture 3 Notes: 10/3/2024

Proposition: Let $X_1, \ldots, X_n$ be metric spaces, and define $X = \prod\limits_{i=1}^{n} X_i$ to be the metric space equipped with the product metric.

The product metric defines the distance between any $x, y \in \prod\limits_{i=1}^{n}$ to be the max distance between a coordinate of x and the corresponding coordinate in y.

- $\bigotimes\limits_{i=1}^{n} \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$.

Proof:

By the previous proposition: $\bigotimes\limits_{i=1}^{n} \mathcal{B}_{X_i}$ is generated by the collection:
$$\{\pi_i^{-1}(U_i) \mid i \in \{1, \ldots, n\} \text{ and } U_i \subseteq X_i \text{ is open}\}.$$

Also, by the definition of a product topology, we know that each $\pi_i^{-1}(U_i)$ is open in X. So by the lemma on page 9, we know that $\bigotimes\limits_{i=1}^{n} \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$.

- If each X_i is separable, then $\displaystyle\bigotimes_{i=1}^{n} \mathcal{B}_{X_i} = \mathcal{B}_X$.

  Proof:
  Let $C_i \subseteq X_i$ be countable with $\overline{C_i} = X_i$ for all $i \in \{1, \ldots, n\}$. Then define $\mathcal{E}_i = \{B(p, x) \mid x \in C_i \text{ and } p \in \mathbb{Q}_+\}$ for each i. Since $\mathcal{E}_i$ is countable and all open sets in X_i are the union of a subcollection of $\mathcal{E}_i$, we know that any open set in X_i is also in $\mathcal{M}(\mathcal{E}_i)$. So, $\mathcal{B}_{X_i} \subseteq \mathcal{M}(\mathcal{E}_i)$. And since $\mathcal{E}_i$ contains only open sets of X_i, the reverse inclusion holds too.

  Also, $C = \displaystyle\prod_{i=1}^{n} C_i$ is a countable dense subset of X.

  Defining $\mathcal{E} = \{B(p, \boldsymbol{x}) \mid \boldsymbol{x} \in C \text{ and } p \in \mathbb{Q}_+\}$, we have that $\mathcal{E}$ is countable and any open set in X is also in $\mathcal{M}(\mathcal{E})$. So, $\mathcal{B}_X \subseteq \mathcal{M}(\mathcal{E})$. And like before since $\mathcal{E}$ contains only open sets of X, the reverse inclusion holds too.

  But now note that given, $B(p, (x_1, \ldots, x_n)) \in \mathcal{E}$, we know that
  $$B(p, (x_1, \ldots, x_n)) = \prod_{i=1}^{n} B(p, x_i) \text{ where } (p, x_i) \in \mathcal{E}_i \text{ for all } i.$$

  So applying part 3 of the previous proposition and the lemma on page 9:
  $$\mathcal{B}_X = \mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}\left(\prod_{i=1}^{n} E_i \mid E_i \in \mathcal{E}_i \text{ for all } i\right) = \bigotimes_{i=1}^{n} \mathcal{M}(\mathcal{E}_i) = \bigotimes_{i=1}^{n} \mathcal{B}_{X_i}$$

Corollary: $\mathcal{B}_{\mathbb{R}^n} = \displaystyle\bigotimes_{i=1}^{n} \mathcal{B}_{\mathbb{R}}$.

  This is because the product metric ρ_1 of $\displaystyle\prod_{i=1}^{n} \mathbb{R}$ is equivalent to the standard metric ρ_2 of $\mathbb{R}^n$, meaning that:
  $$\exists C, C' > 0 \text{ such that } C\rho_1 \leq \rho_2 \leq C'\rho_1.$$

  In the specific case of this corollary, set $C = \sqrt{1/n}$ and $C' = 1$.

  The fact relevant here is that given the metrics ρ_1, ρ_2 on a set X, if ρ_1 is equivalent to ρ_2, then (X, ρ_1) and (X, ρ_2) have the same open sets (this is really trivial to prove).

An underlined elementary family is a collection $\mathcal{E}$ of subsets of a set X such that:

1. $\emptyset \in \mathcal{E}$

2. If $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$.

3. If $E \in \mathcal{E}$, then E^C is a finite disjoint union of members of $\mathcal{E}$.

If $\mathcal{E}$ is an elementary collection, then $\mathcal{A}$ equal to the collection of finite disjoint unions of $\mathcal{E}$ is an algebra.

Proof:
Firstly, given any $A, B \in \mathcal{E}$, we have that $A \cup B = (A - B) \cup B$. Also, by property 3 of elementary families, $(A - B) = (A \cap B^{\mathsf{C}}) = (A \cap \bigcup_{i=1}^{k} C_i)$ where each $C_i \in \mathcal{E}$ and disjoint. By property 2 of elementary families, we thus know $A \cap C_i \in \mathcal{E}$ for all i. So $(A - B)$ is a finite union of disjoint sets in $\mathcal{E}$. In turn, so is $(A - B) \cup B$. Hence, $A \cup B \in \mathcal{A}$.

By induction, we get that for any $A_1, \ldots, A_n \in \mathcal{E}$, $A_1 \cup \ldots \cup A_n$ is a finite union of disjoint sets in $\mathcal{E}$. So $\mathcal{A}$ actually equals the set of all finite unions of $\mathcal{E}$. It follows that $\mathcal{A}$ is closed under finite unions.

I really don't want to write down the proof that $\mathcal{A}$ is closed under complements. It's what you would expect but just heavy on notation.

Exercise 1.5: If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

For the sake of convenience, I will write the union of σ-algebras generated by countable subsets of $\mathcal{E}$ as: $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

To start, since each $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E})$, we trivially know $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E}) = \mathcal{M}$. On the other hand, $\mathcal{E} \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ since each countable $\mathcal{F} \subseteq \mathcal{E}$ is contained in $\mathcal{M}(\mathcal{F}) \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$. So, if we can show that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is a σ-algebra, then we will know that: $\mathcal{M} = \mathcal{M}(\mathcal{E}) \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

Fortunately, it's trivial to show that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is closed under complements. Given any $E \in \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$, we know there exists $\mathcal{M}(\mathcal{F})$ with $E \in \mathcal{M}(\mathcal{F})$. Then $E^{\mathsf{C}} \in \mathcal{M}(\mathcal{F}) \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

Meanwhile, the proof that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is closed under countable unions is more involved:

Suppose $\{E_n\}_{n \in \mathbb{N}}$ is a countable collection of sets in $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$. Then for each $n \in \mathbb{N}$, there exists $\mathcal{F}_n$ such that $E_n \in \mathcal{M}(\mathcal{F}_n)$. Importantly, $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ is still countable. So, setting $\mathcal{F}' = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, we have that:
$$\mathcal{M}(\mathcal{F}') \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$$

Since $\mathcal{F}_n \subseteq \mathcal{F}'$ for all n, we know that $\mathcal{M}(\mathcal{F}_n) \subseteq \mathcal{M}(\mathcal{F}')$ for all n. So, $\{E_n\}_{n \in \mathbb{N}}$ is contained in $\mathcal{M}(\mathcal{F}')$. It follows that $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{M}(\mathcal{F}') \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

Let $X \neq \emptyset$ and $\mathcal{M}$ be a σ-algebra on X. A <u>measure</u> $\mu : \mathcal{M} \longrightarrow [0, \infty]$ is a function satisfying that:

- $\mu(\emptyset) = 0$

- $\mu(\bigcup\limits_{j=1}^{\infty} E_j) = \sum\limits_{j=1}^{\infty} \mu(E_j)$ if $E_j \in \mathcal{M}$ for all j and $E_j \cap E_i = \emptyset$ for all $i \neq j$

$(X, \mathcal{M})$ is called a <u>measurable space</u> and $(X, \mathcal{M}, \mu)$ is called a <u>measure space</u>.

Let $(X, \mathcal{M}, \mu)$ be a measure space.

- μ is called <u>finite</u> if $\mu(X) < \infty$.

  It follows if μ is finite that $\mu(E) < \infty$ for all $E \in \mathcal{M}$ since $E \subseteq X$.
  In probability theory, most measure spaces are finite.

- μ is called <u>σ-finite</u> if $X = \bigcup\limits_{j=1}^{\infty} E_j$, such that $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j.

- μ is called <u>semifinite</u> if $\forall E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $F \subset E$ such that $F \in \mathcal{M}$, and $0 < \mu(F) < \infty$.

  Example: Let $X \neq \emptyset$ and $\mathcal{M} = \mathcal{P}(X)$. Then given a function $\rho : X \longrightarrow [0, \infty]$, $\mu(E) = \sum\limits_{x \in E} \rho(x)$ is a measure.

  - μ is semifinite if and only if $\rho(x) < \infty$ for all $x \in X$.
  - μ is σ-finite if and only if it is semifinite and $\{x \in X \mid \rho(x) > 0\}$ is countable.

  If $\rho(x) = 1$ for all x, then μ is called the <u>counting measure</u>.
  If $\rho(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0, \end{cases}$ then μ is called the <u>Dirac measure</u> at x_0: δ_{x_0}.

  Theorem: Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:
  1. If $E, F \in \mathcal{M}$ with $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
  2. If $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}$, then $\mu(\bigcup\limits_{j=1}^{\infty} E_j) \leq \sum\limits_{j=1}^{\infty} \mu(E_j)$.
  3. If $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}$ with $E_j \subseteq E_{j+1}$ for all $j \in \mathbb{N}$, then $\mu(\bigcup\limits_{j=1}^{\infty} E_j) = \lim\limits_{j \to \infty} \mu(E_j)$.
  4. If $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}$ and $\mu(E_1) < \infty$ and $E_{j+1} \subseteq E_j$ for all $j \in \mathbb{N}$, then $\mu(\bigcap\limits_{j=1}^{\infty} E_j) = \lim\limits_{j \to \infty} \mu(E_j)$.

  Proofs:
  (1) Suppose $E, F \in \mathcal{M}$ with $E \subseteq F$. Then $F = (F - E) \cup E$ is a disjoint union of sets in $\mathcal{M}$, meaning $\mu(F) = \mu(F - E) + \mu(E) \geq \mu(E)$.

(2) Set $F_1 = E_1$ and $F_m = E_m - \bigcup_{i=1}^{m-1} E_i$ for all $m > 1$. Then $(F_i)_{i \in \mathbb{N}}$ is pairwise disjoint and $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$. So $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(F_i)$. On the other hand, $F_i \subseteq E_i$ for all i. So $\sum_{i=1}^{\infty} \mu(F_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$

(3) Setting $E_0 = \emptyset$, we have that $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i - E_{i-1})$. Also, $\mu(E_n) = \sum_{i=1}^{n} \mu(E_i - E_{i-1})$. So:
$$\lim_{n \to \infty} \mu(E_n) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(E_i - E_{i-1}) = \sum_{i=1}^{\infty} \mu(E_i - E_{i-1}) = \mu(\bigcup_{i=1}^{\infty} E_i).$$

(4) Let $F_j = E_1 - E_j$ for all $j \in \mathbb{N}$. Then for all $j \in \mathbb{N}$, $F_j \subseteq F_{j+1}$, $\mu(E_1) = \mu(F_j) + \mu(E_j)$, and $\bigcup_{j=1}^{\infty} F_j = E_1 - \bigcap_{j=1}^{\infty} E_j$. We can thus conclude that:
$$\mu(E_1) = \mu(\bigcap_{j=1}^{\infty} E_j) + \mu(\bigcup_{j=1}^{\infty} F_j)$$
$$= \mu(\bigcap_{j=1}^{\infty} E_j) + \lim_{j \to \infty} (\mu(F_j)) = \mu(\bigcap_{j=1}^{\infty} E_j) + \lim_{j \to \infty} (\mu(E_1) - \mu(E_j))$$

Since $\mu(E_1) < \infty$, we can subtract it out of the expression to get:
$\mu(\bigcap_{j=1}^{\infty} E_j) - \lim_{j \to \infty} (\mu(E_j)) = 0$. Also, we know $\mu(\bigcap_{j=1}^{\infty} E_j) < \infty$ since it's a subset of E_j. So, we can rearrange to get: $\mu(\bigcap_{j=1}^{\infty} E_j) = \lim_{j \to \infty} (\mu(E_j))$.

Exercise 1.9: If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then we have that $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

We know $\mu(E) = \mu(E - f) + \mu(E \cap F)$ and $\mu(F) = \mu(F - E) + \mu(F \cap E)$. Adding those equations together we get that:
$$\mu(E) + \mu(F) = (\mu(E - F) + \mu(E \cap F) + \mu(F - E)) + \mu(E \cap F)$$
$$= \mu(E \cup F) + \mu(E \cap F).$$

Exercise 1.14: If μ is a semifinite measure and $\mu(E) = \infty$, then for any $C > 0$ there exists $F \subset E$ in $\mathcal{M}$ with $C < \mu(F) < \infty$.

Let S be the set of $C > 0$ for which there exists $F \subset E$ in $\mathcal{M}$ with $C < \mu(F) < \infty$. By the definition of semifiniteness, we know S isn't empty. Meanwhile, if for some C we had that there didn't exist a set $F \subset E$ in $\mathcal{M}$ with $C < \mu(F) < \infty$, then we'd know that S is bounded above. Hence, we'd know there exists $\alpha = \sup(S)$.

Now firstly, for all $n \in \mathbb{N}$, choose $G_n \subset E$ in $\mathcal{M}$ such that $\alpha - \frac{1}{n} < \mu(G_n) < \infty$. After that, define $F_n = \bigcup\limits_{i=1}^{n} G_i$ for all $n \in \mathbb{N}$. Since $\mathcal{M}$ is closed under finite unions, we know each F_n is in $\mathcal{M}$. So then observe:

1. $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$
2. For each $n \in \mathbb{N}$, $\alpha - \frac{1}{n} < \mu(F_n) \leq \alpha$

> This is because for each $n \in \mathbb{N}$, $\mu(F_n) < \sum\limits_{i=1}^{n} \mu(G_i)$ which is a finite sum of finite quantities. So $\mu(F_n) < \infty$. At the same time, $F_n \subset E$ since each G_i is a subset of E (we know it is a proper subset because it has a different measure than E). So, if $\mu(F_n) > \alpha$, then $\frac{1}{2}(\mu(F_n) + \alpha)$ would be an element of S greater than α, thus contradicting that $\alpha = \sup(S)$. As for the other inequality, note that $G_n \subseteq F_n$. Thus $\mu(F_n) \geq \mu(G_n) > \alpha - \frac{1}{n}$.

Now $\bigcup\limits_{n=1}^{\infty} F_n \in \mathcal{M}$ due to $\mathcal{M}$ being closed under countable sums. Also, by the two observations above, we know $\mu(\bigcup\limits_{n=1}^{\infty} F_n) = \lim\limits_{n \to \infty} \mu(F_n) = \alpha$. And finally, note that $\bigcup\limits_{n=1}^{\infty} F_n$ is a proper subset of E (we know this because each $F_n \subset E$ and $\bigcup\limits_{n=1}^{\infty} F_n$ can't equal E since their measures are different).

So, we have now proven the existence of a set $F \in \mathcal{M}$ such that $F \subset E$ and $\mu(F) = \alpha$. But now note that $\mu(E - F)$ must be infinite since:
$$\mu(E - F) + \alpha = \mu(E - F) + \mu(F) = \mu(E) = \infty.$$

Because μ is semifinite, there exists $F' \subset E - F$ in $\mathcal{M}$ with $0 < \mu(F') < \infty$. But because F and F' are disjoint subsets of E in $\mathcal{M}$, we know $F \cup F' \in \mathcal{M}$ and $\mu(F \cup F') = \mu(F) + \mu(F') > \alpha$. Plus $F \cup F'$ is a proper subset of E. (It can't equal E because it's measure isn't equal to E. But, both F and F' individually are subsets of E.)

Hence, we have that $\frac{1}{2}(\alpha, \mu(F) + \mu(F)')$ is an element of S greater than α, thus contradicting that α was the supremum of S. We conclude therefore that α does not exist, meaning S is unbounded.

Given a measure space $(X, \mathcal{M}, \mu)$, a set $E \in \mathcal{M}$ satisfying that $\mu(E) = 0$ is called a null set (or μ-null set if we want more precision).

By subadditivity (a.k.a. the fact that for all $(E_j)_{j \in \mathbb{N}} \subset \mathcal{M}$, $\mu(\bigcup\limits_{j=1}^{\infty} E_j) \leq \sum\limits_{j=1}^{\infty} \mu(E_j)$), we know countable unions of null sets are also null sets.

Given a proposition $P(x)$, if there exists a null set $E \in \mathcal{M}$ satisfying that $P(x)$ is true for all $x \in X - E$, then we say P is true <u>almost everywhere</u> (abbreviated as μ-a.e. or just a.e. if the measure being used is clear).

A measure space is <u>complete</u> if given any $E \subseteq X$, we have that $N \in \mathcal{M}$ with $\mu(N) = 0$ and $E \subseteq N$ implies that $E \in \mathcal{M}$.

Proposition: Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Let:

- $\mathcal{N} = \{N \in \mathcal{M} \mid \mu(N) = 0\}$
- $\overline{\mathcal{M}} = \{E \cup F \mid E \in \mathcal{M} \text{ and } F \subseteq N \text{ where } N \in \mathcal{N}\}$.

Then $\overline{\mathcal{M}}$ is a σ-algebra and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

Proof:

Claim 1: $\overline{\mathcal{M}}$ is a σ-algebra.

To see that $\overline{\mathcal{M}}$ is closed under countable union, let $(E_i \cup F_i)_{i \in \mathbb{N}}$ be a sequence of sets in $\overline{\mathcal{M}}$ with each $E_i \in \mathcal{M}$ and $F_i \subseteq N_i$ for some $N_i \in \mathcal{N}$. Then
$$\bigcup_{i \in \mathbb{N}} (E_i \cup F_i) = \bigcup_{i \in \mathbb{N}} E_i \cup \bigcup_{i \in \mathbb{N}} F_i.$$

Importantly, since $\mathcal{M}$ and $\mathcal{N}$ are closed under countable union, we know that $\bigcup_{i \in \mathbb{N}} E_i \in \mathcal{M}$ and $\bigcup_{i \in \mathbb{N}} F_i \subseteq \bigcup_{i \in \mathbb{N}} N_i \in \mathcal{N}$. So, $\bigcup_{i \in \mathbb{N}} (E_i \cup F_i) \in \overline{\mathcal{M}}$.

To show that $\overline{\mathcal{M}}$ is closed under complements, let $E \cup F \in \overline{\mathcal{M}}$ with $E \in \mathcal{M}$ and $F \subseteq N$ for some $N \in \mathcal{N}$. Also note that we can assume $E \cap N = \emptyset$. After all, if $E \cap N \neq \emptyset$, then define $N' = N - E$ and $F' = F - E$. Since $N' \subseteq N$ and $N' \in \mathcal{M}$, we know that $\mu(N') = 0$. Also, $E \cup F = E \cup F'$ with $F' \subseteq N'$. So, E, F', and N' fulfil the same properties we picked E, F, and N for having. But also $E \cap N' = \emptyset$.

Now, $(E \cup F)^{\mathsf{C}} = (E \cup N)^{\mathsf{C}} \cup (N - F)$ where $(E \cup N)^{\mathsf{C}} \in \mathcal{M}$ and $(N - F) \subset N$. So $(E \cup F)^{\mathsf{C}} \in \overline{\mathcal{M}}$.

Now given any $E \cup F \in \overline{\mathcal{M}}$ with $E \in \mathcal{M}$ and $F \subset N$ for some $N \in \mathcal{N}$, define $\overline{\mu}(E \cup F) = \mu(E)$.

Claim 2: $\overline{\mu}$ is well-defined.

Suppose $E_1 \cup F_1 = E_2 \cup F_2$ where for $j \in \{1, 2\}$ we have $E_j \in \mathcal{M}$ and $F_j \subset N_j$ for some $N_j \in \mathcal{N}$. Then $E_1 \subseteq E_2 \cup N_2$, meaning that $\mu(E_1) \leq \mu(E_2) + \mu(N_2) = \mu(E_2)$. By similar reasoning, we can say that $\mu(E_2) \leq \mu(E_1)$. So $\overline{\mu}(E_1 \cup F_1) = \overline{\mu}(E_2 \cup F_2)$.

(The rest is exercise 1.6:)

Claim 3: $\overline{\mu}$ is a complete measure on $\overline{\mathcal{M}}$.

It's easy to show that $\overline{\mu}$ is a measure. After all, $\emptyset \in \mathcal{M} \cap \mathcal{N}$. So, $\overline{\mu}(\emptyset) = \overline{\mu}(\emptyset \cup \emptyset) = \mu(\emptyset) = 0$. Also, suppose $(E_i \cup F_i)_{i \in \mathbb{N}}$ is a sequence of disjoint sets in $\overline{\mathcal{M}}$ with $E_i \in \mathcal{M}$ and $F_i \subset N_i$ for some $N_i \in \mathcal{N}$.

Then $\bigcup_{i\in\mathbb{N}} E_i \in \mathcal{M}$ where each E_i is disjoint and $\bigcup_{i\in\mathbb{N}} F_i \subseteq \bigcup_{i\in\mathbb{N}} N_i \in \mathcal{N}$. So:

$$\overline{\mu}(\bigcup_{i\in\mathbb{N}}(E_i \cup F_i)) = \overline{\mu}(\bigcup_{i\in\mathbb{N}} E_i \cup \bigcup_{i\in\mathbb{N}} F_i)$$

$$= \mu(\bigcup_{i\in\mathbb{N}} E_i) = \sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{\infty} \overline{\mu}(E_i \cup F_i)$$

Finally, to show that $(X, \overline{\mathcal{M}}, \overline{\mu})$ is complete, suppose $A \subseteq X$ and $N_1 \in \overline{\mathcal{M}}$ with $\overline{\mu}(N_1) = 0$ and $A \subseteq N_1$. By definition, we know $N_1 = E \cup F$ where $E \in \mathcal{M}$ and $F \subseteq N_2$ for some $N_2 \in \mathcal{N}$. However, we can also assume $E = \emptyset$. For if $E \neq \emptyset$, then because $\mu(E \cup N_2) \leq \mu(E) + \mu(N_2) \leq 0$, we can define $N_2' = E \cup N_2$ and $F' = E \cup F$. Then $N_1 = \emptyset \cup F'$ and $F' \subseteq N_2'$ where $N_2' \in \mathcal{N}$.

So $A \subset F \subset N_2$ where $N_2 \in \mathcal{N}$. It follows that $A = \emptyset \cup A \in \overline{\mathcal{M}}$.

Claim 4: $\overline{\mu}$ is the unique measure on $\overline{\mathcal{M}}$ that extends μ.

Suppose $\overline{\overline{\mu}}$ is another measure on $\overline{\mathcal{M}}$ such that $\overline{\overline{\mu}}|_{\mathcal{M}} = \mu$. Then consider any $E \cup F \in \overline{M}$ such that $E \in \mathcal{M}$ and $F \subseteq N$ for some $N \in \mathcal{N}$. As shown before, we can assume without loss of generality that $E \cap N = \emptyset$ and thus also $E \cap F = \emptyset$. So, we have that:

$$\overline{\overline{\mu}}(E \cup F) = \overline{\overline{\mu}}(E) + \overline{\overline{\mu}}(F)$$

Next, note that:

$$\mu(E) = \overline{\overline{\mu}}(E) \leq \overline{\overline{\mu}}(E) + \overline{\overline{\mu}}(F) \leq \overline{\overline{\mu}}(E) + \overline{\overline{\mu}}(N) = \mu(E) + \mu(N) = \mu(E)$$

Hence, we know that $\overline{\overline{\mu}}(E \cup F) = \mu(E)$. But also $\overline{\mu}(E \cup F) = \mu(E)$. So $\overline{\overline{\mu}}(E \cup F) = \overline{\mu}(E \cup F)$ for all $E \cup F \in \overline{\mathcal{M}}$.

Note: We call $\overline{\mu}$ the <u>completion</u> of μ and $\overline{\mathcal{M}}$ the <u>completion</u> of $\mathcal{M}$ with respect to μ.

Lecture 4 Notes: 10/8/2024

An <u>outer measure</u> on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \longrightarrow [0, \infty]$ satisfying that:

1. $\mu^*(\emptyset) = 0$.

2. $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$.

3. $\mu^*(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$. (this property is called subadditivity)

Proposition: Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of elementary sets and $\mu : \mathcal{E} \longrightarrow [0, \infty]$ be a function satisfying that $\mu(\emptyset) = 0$.

The textbook only assumes that $\emptyset$ and X are in $\mathcal{E}$.

Then define $\mu^*(A) = \inf \left\{ \sum\limits_{j=1}^{\infty} \mu(E_j) \;\middle|\; E_j \in \mathcal{E} \text{ and } A \subseteq \bigcup\limits_{j=1}^{\infty} E_j \right\}$. This is an outer measure.

Proof:

Given any set $A \subseteq \mathcal{P}(X)$, define $A^* = \left\{ \sum\limits_{j=1}^{\infty} \mu(E_j) \;\middle|\; E_j \in \mathcal{E} \text{ and } A \subseteq \bigcup\limits_{j=1}^{\infty} E_j \right\}$.
That way $\mu^*(A) = \inf(A^*)$.

1. $\mu^*(A)$ is well defined because $\mu(X) \in A^*$ and $\mu(\emptyset) = 0$ is a lower bound of A^*.

2. Since $0 \in \emptyset^*$, we know that $\mu^*(\emptyset) = \inf(\emptyset^*) = 0$.

3. Suppose $A \subseteq B$. Then given any $(E_j)_{j \in \mathbb{N}}$ of sets in $\mathcal{E}$ covering B, we know that they will also cover A. So, $A^* \subseteq B^*$, meaning $\mu^*(A) \leq \mu^*(B)$.

4. Suppose $(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{P}(X)$. Then fix $\varepsilon > 0$. For all $j \in \mathbb{N}$, let $(E_j^{(k)})_{k \in \mathbb{N}}$ be a sequence of sets in $\mathcal{E}$ such that $A_j \subseteq \bigcup\limits_{k \in \mathbb{N}} E_j^{(k)}$ and:

$$\mu^*(A_j) \leq \sum_{k=1}^{\infty} \mu(E_j^{(k)}) \leq \mu^*(A_j) + \varepsilon/2^j.$$

Note that $\bigcup\limits_{j \in \mathbb{N}} A_j \subseteq \bigcup\limits_{j \in \mathbb{N}} \left(\bigcup\limits_{k \in \mathbb{N}} E_j^{(k)} \right)$.

So, $\left(\bigcup\limits_{j \in \mathbb{N}} A_j \right)^* \ni \sum\limits_{j=1}^{\infty} \sum\limits_{k=1}^{\infty} \mu(E_j^{(k)}) \leq \sum\limits_{j=1}^{\infty} (\mu^*(A_j) + \varepsilon/2^j) = \varepsilon + \sum\limits_{j=1}^{\infty} \mu^*(A_j)$.

Since ε was arbitrary, we thus know that:

$$\mu^*\left(\bigcup_{j \in \mathbb{N}} A_j \right) = \inf\left(\bigcup_{j \in \mathbb{N}} A_j \right)^* \leq \sum_{j=1}^{\infty} \mu^*(A_j).$$

Let μ^* be an outer measure on a nonempty set X. Then $A \subseteq X$ is called $\underline{\mu^*\text{-measurable}}$ if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E - A)$ for all $E \subseteq X$.

Note that we trivially have $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E - A)$ for all $E \in \mathcal{P}(X)$. Also, $\mu^*(E \cap A) + \mu^*(E - A) \leq \mu^*(E)$ holds trivially if $\mu^*(E)$. So μ^*-measurability just means that $\mu^*(E \cap A) + \mu^*(E - A) \leq \mu^*(E)$ holds even if $\mu^*(E) < \infty$.

Carathéodory's Theorem: If μ^* is an outer measure on X, then the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

Proof:

Part 1: $\mathcal{M}$ is an algebra and μ^* is additive on $\mathcal{M}$ (meaning $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$ implies that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$).

We know $\mathcal{M}$ is an algebra because:

- $\emptyset \in \mathcal{M}$ because $\mu^*(E) = \mu^*(E \cap \emptyset) + \mu^*(E - \emptyset) = 0 + \mu^*(\emptyset)$ for all $E \subseteq X$.

- Both $A^{\mathsf{C}} \in \mathcal{M}$ and $A \in \mathcal{M}$ are equivalent to us having for all $E \subseteq X$ that $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^{\mathsf{C}})$.

- Suppose A and B are sets in $\mathcal{M}$. Then given $E \subseteq X$, we have:
$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E - A) \\
&= \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) - B) \\
&\quad + \mu^*((E - A) \cap B) + \mu^*((E - A) - B)
\end{aligned}$$

  Now $(E - A) - B = E \cap A^{\mathsf{C}} \cap B^{\mathsf{C}} = E \cap (A \cup B)^{\mathsf{C}}$. Meanwhile, $(E \cap A) - B = E \cap (A - B)$ and $(E - A) \cap B = E \cap (B - A)$. So, by subadditivity, we have that:
$$\begin{aligned}
\mu^*(E \cap (A \cup B)) &+ \mu^*(E - (A \cup B)) \\
&\leq \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) - B) \\
&\quad + \mu^*((E - A) \cap B) + \mu^*((E - A) - B)
\end{aligned}$$

  Hence, $\mu^*(E \cap (A \cup B)) + \mu^*(E - (A \cup B)) \leq \mu^*(E)$. So, $A \cup B \in \mathcal{M}$.

  Next, to show that μ is additive on $\mathcal{M}$, consider any $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$. Then:
$$\mu^*(A \cup B) = \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) - A) = \mu^*(A) + \mu^*(B)$$

Part 2: $\mathcal{M}$ is a σ-algebra and μ^* is σ-additive (think countably additive) on X.

To show that $\mathcal{M}$ is a σ-algebra, it suffices to show that $\mathcal{M}$ is closed under countable disjoint unions. So let $(A_j)_{j \in \mathbb{N}}$ be a sequence of disjoint sets in $\mathcal{M}$. If E is any set in X and $m > 1$, then:
$$\mu^*(E) = \mu^*(E \cap (\bigcup_{j=1}^{m} A_j)) + \mu^*(E - (\bigcup_{j=1}^{m} A_j))$$

But then note that because $\mathcal{M}$ is an algebra, we know $\bigcup_{j=1}^{m} A_j \in \mathcal{M}$. So:
$$\begin{aligned}
\mu^*(E \cap (\bigcup_{j=1}^{m} A_j)) &= \mu^*(E \cap (\bigcup_{j=1}^{m} A_j) \cap A_m) + \mu^*(E \cap (\bigcup_{j=1}^{m} A_j) \cap A_m^{\mathsf{C}}) \\
&= \mu^*(E \cap A_m) + \mu^*(E \cap (\bigcup_{j=1}^{m-1} A_j))
\end{aligned}$$

By induction, we thus have that $\mu^*(E \cap \bigcup_{j=1}^{m} A_j) = \sum_{j=1}^{m} \mu^*(E \cap A_j)$. Also, since $E - (\bigcup_{j=1}^{m} A_j) \supset E - (\bigcup_{j \in \mathbb{N}} A_j)$, we thus know that:
$$\mu^*(E) \geq \sum_{j=1}^{m} \mu^*(E \cap A_j) + \mu^*(E - \bigcup_{j \in \mathbb{N}} A_j)$$

Taking the limit as $m \to \infty$, we thus get that:
$$\begin{aligned}
\mu^*(E) &\geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(E - \bigcup_{j \in \mathbb{N}} A_j) \\
&\geq \mu^*(E \cap (\bigcup_{j \in \mathbb{N}} A_j)) + \mu^*(E - \bigcup_{j \in \mathbb{N}} A_j)
\end{aligned}$$

So, $\bigcup_{j\in\mathbb{N}} A_j$ is μ^*-measurable. Hence, $\mathcal{M}$ is a σ-algebra.

Also, in order to show that $\mu^*(\bigcup_{j\in\mathbb{N}} A_j) = \sum_{j=1}^{\infty} \mu^*(A_j)$, just substitute $E = \bigcup_{j\in\mathbb{N}} A_j$ into the expression at the bottom of the last page.

Part 3: $(X, \mathcal{M}, \mu^*)$ is a complete measure space.

Suppose $\mu^*(A) = 0$. Then given $E \subseteq X$, we have that:
$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E - A) \le \mu^*(A) + \mu^*(E) \le 0 + \mu^*(E)$$

It follows that $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E - A)$ for all E. So $A \in \mathcal{M}$.

Now if $\mu^*(A) = 0$, then $\mu^*(E) = 0$ for all $E \subseteq A$. It follows that all subsets of μ^*-null sets are in $\mathcal{M}$.

The moral of the story is that we'll just call μ^* a measure because it is when restricted to the right σ-algebra.

A <u>premeasure</u> $\mu_0 : \mathcal{A} \longrightarrow [0, \infty]$ is a function on an algebra satisfying that:

- $\mu_0(\emptyset) = 0$

- if $(A_j)_{j\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{A}$ with $\bigcup_{j\in\mathbb{N}} A_j \in \mathcal{A}$, then $\mu_0(\bigcup_{j\in\mathbb{N}} A_j) = \sum_{j=1}^{\infty} \mu_0(A_j)$.

By setting all but finitely many A_j to the emptyset, we can show that μ_0 must be finitely additive. In turn, this is enough to show that $\mu_0(A) \le \mu_0(B)$ if $A \subseteq B$ for any $A, B \in \mathcal{A}$.

We say μ^* is <u>induced by</u> μ_0 if $\mu^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \mu_0(E_j) \,\middle|\, E_j \in \mathcal{A} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$.

Note that μ^* is an outer measure by a previous proposition.

Proposition: In this situation:

1. $\mu^*|_{\mathcal{A}} = \mu_0$

Proof:

Suppose $E \in \mathcal{A}$ and let $(A_j)_{j\in\mathbb{N}}$ be a sequence of sets of in $\mathcal{A}$ covering E. It's trivial that $\mu^*(E) \le \mu_0(E)$ because we could just let $A_1 = E$ and $A_n = \emptyset$ for all $n \ge 2$.

On the other hand, letting $B_1 = E \cap A_1$ and $B_m = E \cap A_m - \bigcup_{j=1}^{m-1} A_j$, we have that $(B_j)_{j\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{A}$ whose union is E. It follows from the second property of a premeasure and the fact that $B_j \subseteq A_j$ for all j that:

$$\mu_0(E) = \sum_{j=1}^{\infty} \mu_0(B_j) \le \sum_{j=1}^{\infty} \mu_0(A_j)$$

Since $(A_j)_{j\in\mathbb{N}}$ was not specified, it follows that $\mu_0(E) \le \mu^*(E)$.

2. Every set in $\mathcal{A}$ is μ^*-measurable.

   Proof:

   Suppose $A \in \mathcal{A}$, $E \subseteq X$, and $\varepsilon > 0$. Then there is a sequence $(B_j)_{j\in\mathbb{N}}$ of sets in $\mathcal{A}$ with $E \subseteq \bigcup_{j=1}^{\infty} B_j$ and $\sum_{j=1}^{\infty} \mu_0(B_j) \le \mu^*(E) + \varepsilon$. Since μ_0 is additive on $\mathcal{A}$, $(B_j \cap A)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ covers $E \cap A$, and $(B_j - A)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ covers $E - A$, we have that:

   $$\mu^*(E) + \varepsilon \ge \sum_{j=1}^{\infty} \mu_0(B_j) = \sum_{j=1}^{\infty} \mu_0(B_j \cap A) + \sum_{j=1}^{\infty} \mu_0(B_j - A)$$
   $$\ge \mu^*(E \cap A) + \mu^*(E - A)$$

   Taking $\varepsilon \to 0$, we get that $\mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E - A)$.

Theorem: Suppose $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra and $\mu_0 : \mathcal{A} \longrightarrow [0, \infty]$ is a premeasure. Then there exists $\mu : \mathcal{M}(\mathcal{A}) \longrightarrow [0, \infty]$ such that:

- $\mu|_{\mathcal{A}} = \mu_0$
- if $\nu : \mathcal{M}(\mathcal{A}) \longrightarrow [0, \infty]$ is a measure with $\nu|_{\mathcal{A}} = \mu|_{\mathcal{A}}$, then $\nu \le \mu$ (with equality if $\mu(E) < \infty$).
- If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}(\mathcal{A})$.

   Proof:

   1. The first claim is true by Carathéodory's Theorem and the last proposition. Specifically, define $\mu = \mu^*|_{\mathcal{M}(\mathcal{A})}$ where μ^* is the outer measure induced by μ_0. Since $\mathcal{A}$ is a subset of the σ-algebra $\mathcal{M}$ of μ^* measurable sets, we know that $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}$. So μ is a measure over $\mathcal{M}(\mathcal{A})$. Also, we know that $\mu(A) = \mu^*(A) = \mu_0(A)$ for all $A \in \mathcal{A}$ by the last proposition.

   2. To show the second claim, let $E \in \mathcal{M}(\mathcal{A})$ and $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ be a covering of E such that $\sum_{j=1}^{\infty} \mu_0(A_j) \le \mu(E) + \varepsilon$ for a given $\varepsilon > 0$. Then:

   $$\nu(E) \le \sum_{j=1}^{\infty} \nu(A_j) = \sum_{j=1}^{\infty} \mu_0(A_j) \le \mu(E) + \varepsilon.$$

   Taking $\varepsilon \to 0$, we get that $\nu(E) \le \mu(E)$.

As for the other inequality, consider that for any $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$, we know by part 3 of the theorem on page 16 and the fact that $\nu(A_j) = \mu(A_j)$ for all j that if $A = \bigcup\limits_{j\in\mathbb{N}} A_j$, then:

$$\nu(A) = \lim_{m\to\infty} (\nu(\bigcup_{j=1}^{m} A_j)) = \lim_{m\to\infty} (\mu(\bigcup_{j=1}^{m} A_j)) = \mu(A).$$

Also, if $\mu(E)$ is finite, then we can choose the covering $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ of E so that all A_j are disjoint and $A = \bigcup\limits_{j\in\mathbb{N}} A_j$ satisfies for a given $\varepsilon > 0$ that:

$$\mu(E) \leq \mu(A) = \sum_{j=1}^{\infty} \mu(A_j) \leq \mu(E) + \varepsilon$$

It follows that $\mu(A - E) < \varepsilon$. So:

$$\mu(E) \leq \mu(A) = \nu(A) = \nu(E) + \nu(A - E) \leq \nu(E) + \mu(A - E) \leq \nu(E) + \varepsilon.$$

Taking $\varepsilon \to 0$, we get that $\mu(E) \leq \nu(E)$.

3. For the third claim, suppose $X = \bigcup\limits_{j\in\mathbb{N}} A_j$ with $\mu_0(A_j) < \infty$ and all A_j being disjoint. Then for any $E \in \mathcal{M}(\mathcal{A})$:

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E \cap A_j) = \sum_{j=1}^{\infty} \nu(E \cap A_j) = \nu(E)$$

Exercise 1.16: Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subseteq X$ is called locally measurable if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets. Trivially, we know $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$. If $\mathcal{M} = \widetilde{\mathcal{M}}$, then μ is called saturated.

(a) If μ is σ-finite, then μ is saturated.

Let $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}$ satisfy that $\mu(A_j) < \infty$ for all j, and that $X = \bigcup\limits_{j\in\mathbb{N}} A_j$. Then if E is locally measurable, we know: $E = \bigcup\limits_{j\in\mathbb{N}} (E \cap A_j) \in \mathcal{M}$.

(b) $\widetilde{\mathcal{M}}$ is a σ-algebra.

- If $E \in \widetilde{\mathcal{M}}$, then given any $A \in \mathcal{M}$ with $\mu(A) < \infty$, we know $E \cap A \in \mathcal{M}$. It follows that $E^{\mathsf{C}} \cap A = A - E = A - (E \cap A) \in \mathcal{M}$. So $E^{\mathsf{C}} \in \widetilde{\mathcal{M}}$.

- Suppose $(E_j)_{j\in\mathbb{N}}$ is a sequence of sets in $\widetilde{\mathcal{M}}$ and $A \in \mathcal{M}$ satisfies that $\mu(A) < \infty$. Then: $(\bigcup\limits_{j\in\mathbb{N}} E_j) \cap A = \bigcup\limits_{j\in\mathbb{N}} (E_j \cap A) \in \mathcal{M}$.

So, $\bigcup\limits_{j\in\mathbb{N}} E_j \in \widetilde{\mathcal{M}}$

(c) Define $\widetilde{\mu}$ on $\widetilde{\mathcal{M}}$ by $\widetilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\widetilde{\mu}(E) = \infty$ otherwise. Then $\widetilde{\mu}$ is a saturated measure on $\mathcal{M}$ called the saturation of μ.

Since $\emptyset \in \mathcal{M}$, we know $\widetilde{\mu}(\emptyset) = \mu(\emptyset) = 0$.

Note that if $A, B \in \widetilde{\mathcal{M}}$ with $A \subseteq B$ and $A \notin \mathcal{M}$ but $B \in \mathcal{M}$, then we immediately get a contradiction since that would suggest $A = A \cap B \in \mathcal{M}$. As a result, supposing $(E_j)_{j \in \mathbb{N}}$ is a sequence of disjoint sets in $\widetilde{\mathcal{M}}$, we have that if any $E_j \notin \mathcal{M}$, then:

$$\widetilde{\mu}(\bigcup_{j \in \mathbb{N}} E_j) = \infty = \sum_{j=1}^{\infty} \widetilde{\mu}(E_j).$$

Meanwhile, if all sets of $(E_j)_{j \in \mathbb{N}}$ are in $\mathcal{M}$, then:

$$\widetilde{\mu}(\bigcup_{j \in \mathbb{N}} E_j) = \mu(\bigcup_{j \in \mathbb{N}} E_j) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \widetilde{\mu}(E_j).$$

(d) If μ is complete, then so is $\widetilde{\mu}$.

This fact is obvious because by the way we defined $\widetilde{\mu}$, we know a set is $\widetilde{\mu}$-null if and only if it is μ-null.

(e) Suppose that μ is semifinite. For $E \in \widetilde{\mathcal{M}}$, define:

$$\underline{\mu}(E) = \sup\{\mu(A) \mid A \in \mathcal{M} \text{ and } A \subseteq E\}.$$

This is well defined because $\mu(\emptyset)$ is always in the above set and $\mu(X)$ is an upper-bound. Then $\underline{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ that extends μ.

Firstly, we show $\underline{\mu}$ is a measure. To start, it's trivial to see that $\underline{\mu}(\emptyset) = \mu(\emptyset) = 0$.

Lemma: If $E \in \widetilde{\mathcal{M}}$ and $\underline{\mu}(E) = \infty$, then there exists a set $A \in \mathcal{M}$ such that $A \subseteq E$ and $\mu(A) = \infty$.

To show this, construct a sequence of "increasing" sets $(A_j)_{j \in \mathbb{N}}$ satisfying that $A_j \subseteq E$ and $\mu(A_j) \geq j$. Then the union A of that sequence will satisfy that $A \in \mathcal{M}$, that $A \subseteq E$, and that $\mu(A) = \infty$.

Because of that lemma, we don't need to deal with the edge case that a least upper bound equaling infinity doesn't mean a set contains infinity. So, let $(E_j)_{j \in \mathbb{N}}$ be a sequence of disjoint sets in $\widetilde{\mathcal{M}}$ with $E = \bigcup_{j \in \mathbb{N}} E_j$. Then let $\varepsilon > 0$.

To show one inequality, pick a sequence $(A_j)_{j \in \mathbb{N}}$ of sets in $\mathcal{M}$ satisfying that $A_j \subseteq E_j$ and $\underline{\mu}(E_j) - \varepsilon/2^j \leq \mu(A_j)$. Since $A = \bigcup_{j \in \mathbb{N}} A_j \subseteq E$, and each A_j is disjoint, we thus have:

$$-\varepsilon + \sum_{j=1}^{\infty} \underline{\mu}(E_j) \leq \sum_{j=1}^{\infty} \mu(A_j) = \mu(A) \leq \underline{\mu}(E)$$

To show the other inequality, pick $B \in \mathcal{M}$ satisfying that $B \subseteq E$ and $\underline{\mu}(E) - \varepsilon < \mu(B)$. Because $E, E_j \in \widetilde{\mathcal{M}}$ for each j, we know that $B \cap E$ and $B \cap E_j$ are in $\mathcal{M}$ for each j. So:

$$\underline{\mu}(E) - \varepsilon < \mu(B) = \mu(B \cap E) = \sum_{j=1}^{\infty} \mu(B \cap E_j) \leq \sum_{j=1}^{\infty} \underline{\mu}(E_j)$$

Taking $\varepsilon \to 0$, we thus get that $\underline{\mu}(E) = \sum_{j=1}^{\infty} \underline{\mu}(E_j)$.

Proving that $\mu(E) = \underline{\mu}(E)$ when $E \in \mathcal{M}$ is trivial. Obviously, $\mu(E) \leq \underline{\mu}(E)$. Meanwhile for any $F \in \mathcal{M}$ satisfying that $F \subseteq E$, we know that $\mu(F) \leq \mu(E)$. So, there does not exists a subset of E in $\mathcal{M}$ with greater measure than $\mu(E)$.

Note that $\widetilde{\mu}$ and $\underline{\mu}$ are not necessarily equal. Part (f) of this problem gives a relatively simple counterexample.

Exercise 1.17: If μ^* is an outer measure on X and $(A_j)_{j \in \mathbb{N}}$ is a sequence of disjoint μ^*-measurable sets, then:

$$\mu^*(E \cap \bigcup_{j \in \mathbb{N}} A_j) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \text{ for any } E \subseteq X.$$

Note that by induction, we can show that for any $n \in \mathbb{N}$:

$$\mu^*(E \cap \bigcup_{j=1}^{\infty} A_j) = \mu^*(E \cap \bigcup_{j=1}^{\infty} A_j \cap A_1) + \mu^*(E \cap \bigcup_{j=1}^{\infty} A_j - A_1)$$

$$= \mu^*(E \cap A_1) + \mu^*(E \cap \bigcup_{j=2}^{\infty} A_j)$$

$$= \sum_{j=1}^{2} \mu^*(E \cap A_j) + \mu^*(E \cap \bigcup_{j=3}^{\infty} A_j)$$

$$= \cdots = \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap \bigcup_{j=n+1}^{\infty} A_j)$$

Thus, we clearly have for all n that $\sum_{j=1}^{n} \mu^*(E \cap A_j) \leq \mu^*(E \cap \bigcup_{j=1}^{\infty} A_j)$.

Taking the limit as $n \to \infty$, we thus know $\sum_{j=1}^{\infty} \mu^*(E \cap A_j) \leq \mu^*(E \cap \bigcup_{j \in \mathbb{N}} A_j)$.

The other inequality is obvious from the subadditivity property of outer measures.

Exercise 1.18: Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ be be the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ be the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

(a) For any $E \subseteq X$ and $\varepsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ with $E \subseteq A$ and $\mu^*(A) \leq \mu^*(E) + \varepsilon$.

Let $(A_j)_{j \in \mathbb{N}}$ be a sequence of sets in $\mathcal{A}$ which cover E and satisfy that $\sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \varepsilon$. In turn, by the subadditivity of outer measures, and the fact that $\mu^*(A) = \mu_0(A)$ for all $A \in \mathcal{A}$, we know that:

$$E \subseteq \bigcup_{j \in \mathbb{N}} A_j \in \mathcal{A}_\sigma \text{ and } \mu^*(\bigcup_{j \in \mathbb{N}} A_j) \leq \sum_{j=1}^{\infty} \mu^*(A_j) = \sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \varepsilon$$

(b) If $\mu^*(E) < \infty$, then E is μ^*-measurable if and only if there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B - E) = 0$.

Suppose E is μ^*-measurable. Then for all $j \in \mathbb{N}$, pick $A_j \in \mathcal{A}_\sigma$ satisfying that $E \subseteq A_j$ and $\mu^*(E) \leq \mu^*(A_j) \leq \mu^*(E) + 1/j$. Since E is μ^*-measurable, we know that for all $j \in \mathbb{N}$:

$$\mu^*(A_j) = \mu^*(A_j \cap E) + \mu^*(A_j - E) = \mu^*(E) + \mu^*(A_j - E)$$

In turn, because $\mu^*(E) < \infty$, this tell us that $\mu^*(A_j - E) \leq 1/j$. Also, because $\bigcap_{j \in \mathbb{N}} A_j \subseteq A_n$ for all $n \in \mathbb{N}$, we know that $\mu^*(\bigcap_{j \in \mathbb{N}} A_j - E) \leq 1/n$ for all $n \in \mathbb{N}$.

As a result, we know that $\bigcap_{j \in \mathbb{N}} A_j \in \mathcal{A}_{\sigma\delta}$, $E \subseteq \bigcap_{j \in \mathbb{N}} A_j$, and $\mu^*(\bigcap_{j \in \mathbb{N}} A_j - E) = 0$.

To prove the reverse implication, suppose there exists a μ^*-separable set B satisfying that $E \subseteq B$ and $\mu^*(B - E) = 0$ (any set in $\mathcal{A}_{\sigma\delta}$ will be μ^*-separable because $\mathcal{A}_{\sigma\delta} \subseteq \mathcal{M}(\mathcal{A})$). Then given any set F, we have that:

$$\mu^*(F - E) = \mu^*(F \cap E^\mathsf{C} \cap B) + \mu^*(F \cap E^\mathsf{C} \cap B^\mathsf{C})$$
$$= \mu^*(F \cap (B - E)) + \mu^*(F - B) = \mu^*(F - B)$$

Also, since $F \cap E \subseteq F \cap B$, we know that $\mu^*(F \cap E) \leq \mu^*(F \cap B)$.

So, $\mu^*(F \cap E) + \mu^*(F - E) \leq \mu^*(F \cap B) + \mu^*(F - B) = \mu^*(F)$. Hence, E is μ^*-measurable.

(c) If μ_0 is σ-finite, we can remove the requirement in part (b) that $\mu^*(E) < \infty$.

Because the backwards implication proof never required $\mu^*(E)$ to be finite, it suffices to show that E being μ^*-measurable implies there exists $B \in A_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B - E) = 0$.

To start, let $(C_i)_{i \in \mathbb{N}}$ satisfy that $\mu_0(C_i) < \infty$ and $\bigcup_{i=1}^{\infty} C_i = X$.

Next for all $j, i \in \mathbb{N}$, pick $A_j^{(i)} \in \mathcal{A}_\sigma$ satisfying that $E \cap C_i \subseteq A_j^{(i)}$ and $\mu^*(E) \leq \mu^*(A_j^{(i)}) \leq \mu^*(E) + 1/j2^i$. Since $\mu^*(E \cap C_i)$ is finite, we can use the same reasoning as in part (b) to say that $\mu^*(A_j^{(i)} - (E \cap C_i)) \leq 1/j2^i$.

Importantly, $A_j^{(i)} - E \subseteq A_j^{(i)} - (E \cap C_i)$. for all i. Therefore:

$$\mu^*((\bigcup_{i \in \mathbb{N}} A_j^{(i)}) - E) = \mu^*(\bigcup_{i \in \mathbb{N}} (A_j^{(i)} - E))$$
$$\leq \mu^*(\bigcup_{i \in \mathbb{N}} (A_j^{(i)} - (E \cap C_i)))$$
$$\leq \sum_{i \in \mathbb{N}} \mu^*(A_j^{(i)} - (E \cap C_i)) \leq \sum_{i \in \mathbb{N}} \frac{1}{j2^i} = \frac{1}{j}$$

Since $E \subseteq \bigcup_{i \in \mathbb{N}} A_j^{(i)} \in \mathcal{A}_\sigma$, we've thus shown for all $j \in \mathbb{N}$ that there exists a set $A_j \in \mathcal{A}_\sigma$ satisfying that $\mu^*(A_j - E) \leq 1/j$ and $E \subseteq A_j$.

Finally, intersecting all those A_j like in part (b), we get our set satisfying the right-side of the implication.

Exercise 1.19: Let μ^* be an outer measure on X induced from a finite premeasure μ_0 defined on an algebra $\mathcal{A}$. If $E \subseteq X$, define the <u>inner measure</u> of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^\mathsf{C})$. Then E is μ^*-measurable if and only if $\mu^*(E) = \mu_*(E)$.

($\Longrightarrow$)
If E is μ^*-measurable, then we have $\mu^*(X \cap E) + \mu^*(X - E) = \mu^*(X)$. Because $\mu^*(X) = \mu_0(X)$, we thus have that $\mu_*(E) = \mu_0(X) - \mu^*(X - E) = \mu_*(E)$.

($\Longleftarrow$)
By part (a) of the previous exercise, we know there exists $A_j \in \mathcal{A}_\sigma$ satisfying that $E \subseteq A_j$ and $\mu^*(A_j) \leq \mu^*(E) + 1/j$.

Note that A_j is μ^*-measurable because $\mathcal{A}_\sigma \subseteq \mathcal{M}(\mathcal{A})$. This means that:
- $\mu_0(X) = \mu^*(X) = \mu^*(X \cap A_j) + \mu^*(X - A_j) = \mu^*(A_j) + \mu^*(A_j^\mathsf{C})$.
- $\mu^*(E^\mathsf{C}) = \mu^*(E^\mathsf{C} \cap A_j) + \mu^*(A_j^\mathsf{C} \cap E^\mathsf{C}) = \mu^*(A_j - E) + \mu^*(A_j^\mathsf{C})$.

Supposing that $\mu^*(E) = \mu_*(E) = \mu_0(X) - \mu^*(X - E)$ and plugging in the first bullet-pointed identity, we get that:

$$\mu^*(E^\mathsf{C}) = \mu^*(A_j) + \mu^*(A_j^\mathsf{C}) - \mu^*(E).$$

Substituting that into the second bullet-pointed identity, we have:

$$\mu^*(A_j) - \mu^*(E) = \mu^*(A_j - E).$$

And finally, using the inequality: $\mu^*(A_j) \leq \mu^*(E) + 1/j$, we get $\mu^*(A_j - E) \leq 1/j$.

Hence, we've shown that there exists a set $A_j \in \mathcal{A}_\sigma$ such that $E \subseteq A_j$ and $\mu^*(A_j - E) < 1/j$ for all $j \in \mathbb{N}$. From there, we can proceed exactly like in part (b) of exercise 1.18. Pick such an A_j for all $j \in \mathbb{N}$ and then intersect them together. The result will be a set $B \in \mathcal{A}_{\sigma\delta}$ satisfying that $E \subseteq B$ and $\mu^*(B - E) = 0$. Since such a set exists, we know by the conclusion of part (b) of exercise 1.18 that E is μ^*-measurable.

Exercise 1.21: Let μ^* be an outer measure induced from a premeasure defined on an algebra $\mathcal{A}$ and $\overline{\mu}$ be the restriction of μ^* to the collection $\mathcal{M}$ of μ^*-measurable sets. Then $\overline{\mu}$ is saturated.

Let E be a locally $\overline{\mu}$-measurable set and choose any $F \subseteq X$. Given any $\varepsilon > 0$, by part (a) of exercise 1.18, we know that there exists a μ^*-measurable set $A \in \mathcal{A}_\sigma \subseteq \mathcal{M}$ such that $F \subseteq A$ and $\mu^*(A) \leq \mu^*(F) + \varepsilon$.

Assuming without loss of generality that $\mu^*(F)$ is finite, we thus know that $\mu^*(A) < \infty$. So, since E is locally $\overline{\mu}$-measurable, we have that $E \cap A \in \mathcal{M}$.

Now, we first note that because $F \cap E \subseteq A \cap E$ and $F - E \subseteq A - E$, we have that:
$$\mu^*(F \cap E) + \mu^*(F - E) \leq \mu^*(A \cap E) + \mu^*(A - E)$$

Next we note that:

- $A \cap (A \cap E) = A \cap E$
- $A \cap (A \cap E)^{\mathsf{C}} = (A \cap A^{\mathsf{C}}) \cup (A \cap E^{\mathsf{C}}) = A - E$

So: $\mu^*(A \cap E) + \mu^*(A - E) = \mu^*(A \cap (A \cap E)) + \mu^*(A \cap (A \cap E)^{\mathsf{C}}) = \mu^*(A)$.

And finally, since $\mu^*(A) \leq \mu^*(F) + \varepsilon$, we can thus conclude that:
$$\mu^*(F \cap E) + \mu^*(F - E) \leq \mu^*(F) + \varepsilon.$$

Taking $\varepsilon \to 0$, we get that $\mu^*(F \cap E) + \mu^*(F - E) \leq \mu^*(F)$ So, E is μ^*-measurable, meaning $E \in \mathcal{M}$.

Consider the collection $H = \{\emptyset, (a, b], (a, \infty) \mid -\infty \leq a < b < \infty\}$ of "half-open-intervals" of $\mathbb{R}$.

This forms an elementary family.

- We specified in the definition that $\emptyset \in H$.
- If $x \in (a, b] \cap (c, d] \neq \emptyset$, then we know $a < x < d$ and $c < x < b$. So $(a, b] \cap (c, d] = (\max(a, c), \min(b, d)] \in H$.

- Given $(a, b] \in H$, we have that $(a, b]^C = (-\infty, a] \cup (b, \infty)$.

  For the sake of time, I'm ignoring edge cases of a right bound of infinity since they are still trivial.

By exercise 1.2, we know that $\mathcal{M}(H) = \mathcal{B}_{\mathbb{R}}$. And, by a previous proposition, we know that $\mathcal{A}$ equal to the collection of finite disjoint unions of H is an algebra.

Proposition: Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a monotonically increasing and right continuous function (meaning $\lim_{t \to x^+} F(t) = F(x)$ for all $x \in \mathbb{R}$). Also, define:
$$F(-\infty) = \lim_{t \to -\infty} F(t) \text{ and } F(\infty) = \lim_{t \to \infty} F(t).$$

If $(a_j, b_j]$ for $j = 1, \ldots, n$ are disjoint intervals in H, define:
$$\mu_0\left(\bigcup_{j=1}^{n} (a_j, b_j]\right) = \sum_{j=1}^{n} F(b_j) - F(a_j)$$

Also let $\mu_0(\emptyset) = 0$. And if $F(\infty) = \infty$ and $F(-\infty) = -\infty$, define $\mu_0(\mathbb{R}) = \infty$. Then this is a premeasure.

Proof:

1. μ_0 is well defined.

   Suppose $(a_j, b_j]$ for $j = 1, \ldots, n$ are disjoint intervals in H satisfying that $(a, b] = \bigcup_{j=1}^{n} (a_j b_j] = (a, b]$. Then after indexing those half intervals in a certain way, we must have that:
   $$a = a_1 < b_1 = a_2 < \ldots < b_{n-1} = a_n < b_{n-1} = b$$

   It follows that:
   $$F(b) - F(a) = \mu_0\left(\bigcup_{j=1}^{n} (a_j, b_j]\right) = \sum_{j=1}^{n} \mu_0((a_j, b_j]) = \sum_{j=1}^{n} F(b_j) - F(a_j)$$

   In other words, μ_0 is well defined for individual intervals.

   Now suppose I_i and J_j for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ are disjoint intervals in H satisfying that $\bigcup_{i=1}^{m} I_i = \bigcup_{j=1}^{n} J_j$.

   For each I_i, we can repeat the same reasoning as above with the collection of sets $I_i \cap J_j$ for $j = 1, \ldots, n$ in order to get that:
   $$\mu_0(I_i) = \sum_{j=1}^{n} \mu_0(I_i \cap J_j)$$

   Similarly, we can show for each J_j that:
   $$\mu_0(J_j) = \sum_{i=1}^{m} \mu_0(I_i \cap J_j)$$

   Thus: $\sum_{i=1}^{m} \mu_0(I_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_0(I_i \cap J_j) = \sum_{j=1}^{n} \mu(J_j)$.

2. If $(I_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ is a disjoint sequence satisfying that $\bigcup_{j\in\mathbb{N}} I_j \in \mathcal{A}$, then:

$$\mu_0(\bigcup_{j\in\mathbb{N}} I_j) = \sum_{j=1}^{\infty} \mu_0(I_j).$$

Since $\bigcup_{j\in\mathbb{N}} I_j \in \mathcal{A}$, we know it is equal to a finite union of disjoint intervals of H. By considering those intervals separately, we can thus assume without loss of generality that $\bigcup_{j\in\mathbb{N}} I_j = (a, b]$ where not both $a = -\infty$ and $b = \infty$.

Also, without loss of generality we can assume each I_j is one interval.

Now it's obvious from the construction of μ_0 that μ_0 is additive. And since $(a, b] - \bigcup_{j=1}^{m} I_j \in \mathcal{A}$ for all m, we thus know that:

$$\mu_0((a, b]) = \mu_0(\bigcup_{j=1}^{m} I_j) + \mu_0((a, b] - \bigcup_{j=1}^{m} I_j) \geq \mu_0(\bigcup_{j=1}^{m} I_j) = \sum_{j=1}^{m} \mu_0(I_j)$$

Taking the limit as $m \to \infty$, we get that: $\sum_{j=1}^{\infty} \mu_0(I_j) \leq \mu_0(\bigcup_{j\in\mathbb{N}} I_j)$.

Lecture 6 Notes: 10/15/2024

To show the reverse inequality, suppose $a, b \in \mathbb{R}$ (a.k.a. finite), and let $\varepsilon > 0$. Since F is right-continuous, there exists $\delta > 0$ such that $F(a + \delta) - F(a) < \varepsilon$. Similarly, given that $I_j = (a_j, b_j]$, there exists $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < \frac{\varepsilon}{2^j}$ for all $j \in \mathbb{N}$.

Next, note that the collection $\{(a_j, b_j + \delta_j)\}_{j\in\mathbb{N}}$ of open intervals covers the set $[a + \delta, b]$. Thus by compactness, there is a finite subcover. In other words,

$$(a_1, b_1 + \delta_1), \ldots, (a_N, b_N + \delta_N) \text{ cover } [a + \delta, b]$$

Furthermore, by removing intervals in that finite subcover which are subsets of other intervals and by reindexing, we can assume that:

$$b_j + \delta_j \in (a_{j+1}, b_{j+1} + \delta_{j+1}) \text{ for all } j = 1, \ldots, N - 1.$$

Then:
$$\begin{aligned}
\mu_0((a, b]) &= F(b) - F(a) \\
&< F(b) - F(a + \delta) + \varepsilon \\
&\leq F(b_N + \delta_N) - F(a_1) + \varepsilon \qquad \text{(since } F \text{ is monotone increasing)} \\
&= F(b_N + \delta_N) - f(a_N) + \sum_{j=1}^{N-1} (F(a_{j+1}) - F(a_j)) + \varepsilon \\
&\leq F(b_N + \delta_N) - f(a_N) + \sum_{j=1}^{N-1} (F(b_j + \delta_j) - F(a_j)) + \varepsilon \\
&\qquad\qquad\qquad\qquad \text{(again since } F \text{ is monotone increasing)} \\
&= \sum_{j=1}^{N} (F(b_j + \delta_j) - F(a_j)) + \varepsilon \\
&< \sum_{j=1}^{N} (F(b_j) + \tfrac{\varepsilon}{2^j} - F(a_j)) + \varepsilon < \sum_{j=1}^{\infty} \mu(I_j) + 2\varepsilon
\end{aligned}$$

Since ε is arbitrary, we've now shown the reverse inequality when a and b are finite. To extend this result to when $a = -\infty$ or $b = \infty$, note that the invervals $(a_j, b_j + \delta_j)$ cover $[-M + \delta, b]$ or $[a + \delta, M]$ for all M in either $(-\infty, b]$ or (a, ∞).

So, doing the same manipulations as before, since $\sum\limits_{j=1}^{\infty} \mu(I_j) + 2\varepsilon$ is an upper bound of $\mu_0((-M, b])$ or $\mu_0((a, M])$, we know that the limit of $\mu_0((-M, b])$ or $\mu_0((a, M])$ as $M \to \infty$ will not exceed that upper bound. Then taking $\varepsilon \to 0$, we get the same result as before.

Plus, based on how we defined $F(\infty)$ and $F(-\infty)$, we know that $\lim\limits_{M \to \infty} \mu_0((-M, b]) = \mu_0((-\infty, b])$ and $\lim\limits_{M \to \infty} \mu_0((a, M]) = \mu_0((a, \infty))$.

Theorem:

1. If F is a monotone increasing and right-continuous like above, there is a unique Borel measure μ_F on $\mathcal{B}_{\mathbb{R}}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all $a < b$.

2. If G is another such monotone increasing and right-continuous, then $\mu_G = \mu_F$ if and only if $F - G$ is a constant.

3. If μ is a $\mathcal{B}_{\mathbb{R}}$ measure that is finite on all bounded sets, then we can define:
$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu((x, 0]) & \text{if } x < 0 \end{cases}$$

Then F is a monotone increasing and right-continuous like above with $\mu = \mu_F$.

Proof:

1. By the previous proposition, we know F induces a premeasure μ_F. Also, μ_F is a σ-finite premeasure. Thus, by the theorem on page 24, we know μ_F induces a unique measure on $\mathcal{M}(H) = \mathcal{B}_{\mathbb{R}}$.

2. Clearly, if $G(b) - G(a) = F(b) - F(a)$ for all $-\infty \le a < b \le \infty$, then we must have that $G - F$ is constant.

3. Finally, by the theorem at the bottom of page 16, we can fairly easily show that F is right-continuous and monotone increasing.

Given a monotone increasing and right-continuous function F, we call μ_F the <u>Lesbesgue-Stieltjes measure</u>. If $F(x) = x$, we just write m and call it the <u>Lesbesgue measure</u>.

Also, we'll almost always use μ_F to refer to the completion of $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_F)$. So, this measure is actually defined on more than just $\mathcal{B}_{\mathbb{R}}$.

Let μ be a Lesbesgue-Stieltjes measure associated with a monotone increasing, right-continuous function F, and let $\mathcal{M}_\mu$ be the set of μ-measureable sets. Here are some nice properties of μ:

Lemma: Suppose $\nu(E) = \inf(\{\sum_{j=1}^{\infty} \mu((a_j, b_j)) \mid E \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j)\})$. Then $\nu(E) = \mu(E)$ for all $E \in \mathcal{M}_\mu$.

Proof:

Fix $\varepsilon > 0$. Then, there exists $((a_j, b_j])_{j \in \mathbb{N}}$ such that $E \subseteq \bigcup_{j \in \mathbb{N}}(a_j, b_j]$ and $\sum_{j=1}^{\infty} \mu((a_j, b_j]) \leq \mu(E) + \varepsilon$.

Then for all $j \in \mathbb{N}$, pick $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) \leq \varepsilon/2^j$. Thus:

$$\nu(E) \leq \sum_{j=1}^{\infty} \mu((a_j, b_j + \delta_j)) \leq \sum_{j=1}^{\infty} \mu((a_j, b_j + \delta_j])$$
$$\leq \sum_{j=1}^{\infty} \mu((a_j, b_j]) + \varepsilon \leq \mu(E) + 2\varepsilon$$

Taking $\varepsilon \to 0$, we get that $\nu(E) \leq \mu(E)$.

To get the reverse inequality, note that we can write any open interval as the countable union of a sequence of disjoint half-open-intervals. It follows that $\mu(E) \leq \nu(E)$.

Theorem: If $E \subseteq \mathcal{M}_\mu$, then:

- $\mu(E) = \inf(\{\mu(U) \mid E \subseteq U$ and U is open$\})$
- $\mu(E) = \sup(\{\mu(K) \mid K \subseteq E$ and K is compact$\})$

Proof:

The first bullet point follows almost immediately from the previous lemma because unions of open sets are open.

To show the second bullet point, first suppose E is bounded. If E is closed, then the equality is trivial. So suppose $\overline{E} - E \neq \emptyset$. Then given any $\varepsilon > 0$, let U be an open set such that $\overline{E} - E \subseteq U$ and $\mu(U) \leq \mu(\overline{E} - E) + \varepsilon$. (Note that $\overline{E} \in \mathcal{B}_{\mathbb{R}}$ and so $\overline{E} - E \in \mathcal{M}_\mu$.)

We have that $K = \overline{E} - U$ is a closed bounded set satisfying that $K \subseteq E$ and:

$$\mu(K) = \mu(E) - \mu(E \cap U) = \mu(E) - (\mu(U) - \mu(U - E))$$
$$\geq \mu(E) - \mu(U) + \mu(\overline{E} - E) \geq \mu(E) - \varepsilon$$

If E is unbounded, then consider $E_j \cap (j, j+1]$ for all $j \in \mathbb{Z}$. By the previous reasoning, given any $\varepsilon > 0$ there exists a compact set K_j such that $K_j \subseteq E_j$ and $\mu(K_j) \geq \mu(E) - \varepsilon/2^{|j|}$.

Now define $H_n = \bigcup\limits_{j=-n}^{n} K_j$. That way H_n is a compact subset of E and:

$$\mu(H_n) \geq \mu(\bigcup\limits_{j=-n}^{n} E_j) - 3\varepsilon.$$

Since $\mu(E) = \lim\limits_{n\to\infty} (\mu(\bigcup\limits_{j=-n}^{n} E_j)) \leq \lim\limits_{n\to\infty} (\mu(H_n)) + 3\varepsilon$, the bulllet pointed claim follows.

Theorem: If $E \subseteq \mathbb{R}$, then the following are equivalent:

(a) $E \in \mathcal{M}_\mu$.

(b) $E = V - N_1$ where V is a G_δ set and $\mu(N_1) = 0$.

(c) $E = H \cup N_2$ where H is an F_σ set and $\mu(N_2) = 0$.

Proof:
We trivially have that (b) $\Longrightarrow$ (a) and (c) $\Longrightarrow$ (a) because $V, H \in \mathcal{B}_\mathbb{R}$ and $N_1, N_2 \in \mathcal{M}_\mu$.

To show the other direction, suppose $E \subseteq \mathcal{M}_\mu$. Firstly, we'll suppose $\mu(E) < \infty$. Then for all $j \in \mathbb{N}$, we can choose an open set $U_j \supseteq E$ and a compact set $K_j \subseteq E$ such that:

$$\mu(U_j) - {}^1\!/_{2^j} \leq \mu(E) \leq \mu(K_j) + {}^1\!/_{2^j}$$

Define $V = \bigcap\limits_{j \in \mathbb{N}} U_j$ and $H = \bigcup\limits_{j \in \mathbb{N}} K_j$.

Then $H \subseteq E \subseteq V$ and $\mu(H) = \mu(E) = \mu(V)$. It follows since $\mu(E) < \infty$ that:
- Letting $N_1 = V - E$, we have that $E = V - N_1$ and
$$\mu(V) = \mu(N_1) + \mu(E) \Longrightarrow 0 = \mu(V) - \mu(E) = \mu(N_1).$$
- Letting $N_2 = E - H$, we have that $E = H \cup N_2$ and
$$\mu(E) = \mu(H) + \mu(N_2) \Longrightarrow 0 = \mu(E) - \mu(H) = \mu(N_2).$$

The rest is **Exercise 1.25**.
Now suppose $\mu(E) = \infty$.

For all $j \in \mathbb{Z}$, define $E_j = E \cap (j, j+1]$. That way E is the disjoint union of all E_j. Importantly, each $\mu(E_j) \leq \mu((j, j+1]) < \infty$. So, using the previous logic, for all j we can find an F_σ set H_j satisfying that $H_j \subseteq E_j$ and:
- Letting $N_2^{(j)} = E_j - H_j$, we have that $E_j = H_j \cup N_2^{(j)}$ and $\mu(N_2^{(j)}) = 0$.

Setting $H = \bigcup\limits_{j \in \mathbb{Z}} H_j$, we have that H is a countable union of countable unions of closed sets. Thus, H is an F_σ set.

Also $N_2 = \bigcup\limits_{j \in \mathbb{Z}} N_2^{(j)}$ is still a null set since the set of null sets is closed under countable union. And finally, $H \cup N_2 = E$ becuase each $E_j = H_j \cup N_2^{(j)}$. Thus, we've shown (c).

Meanwhile, to show (b), consider that if $E \in \mathcal{M}_\mu$, then $E^C \in \mathcal{M}_\mu$. So, we can find an F_σ set H and null set N_1 such that $E^C = H \cup N_1$. Taking complements of both sides, we get that $E = H^C - N_1$. And importantly, the complement of a union of sets is equivalent to the intersection of the complements of those sets. So, since the complement of a closed set is open, H^C is a G_δ set.

Note that we call measures with the property described in the above two theorems regular.

Proposition: If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for all $\varepsilon > 0$, there is a set A that is a finite union of open intervals such that $\mu(E \,\Delta\, A) < \varepsilon$.

Note that $E\,(\Delta)A = (E - A) \cup (A - E) = (E \cup A) - (E \cap A)$.

The proof is **Exercise 1.26**

Let U be an open set satisfying that $E \subseteq U$ and $\mu(U) \leq \mu(E) + \varepsilon/2$. Then, a fact about the standard metric space of $\mathbb{R}$ is that any open set is a countable union of disjoint open intervals.

(writing this so I have it in my notes)
To see this, given the open set $U \subseteq \mathbb{R}$, and any $x \in U$, define $I_x = (a, b)$ where $a = \inf(\alpha \in \overline{\mathbb{R}} \mid (\alpha, x) \subset U)$ and $b = \sup(\beta \in \overline{\mathbb{R}} \mid (x, \beta) \subset U)$.

Then consider the collection $\{I_x\}_{x \in U}$. Given $I_x = (a_x, b_x)$ and $I_y = (a_y, b_y)$, supposing $t \in I_x \cap I_y$, we must have that:
$$(\min(a_x, a_y), t] \cup [t, \max(b_x, b_y)) = (\min(a_x, a_y), \max(b_x, b_y)) \subseteq U.$$

Because of how we defined a_x, b_x, a_y, b_y, and the fact that obviously $I_x, I_y \subseteq (\min(a_x, a_y), \max(b_x, b_y))$, we must have that $a_x = \min(a_x, a_y) = a_y$ and $b_x = \min(b_x, b_y) = b_y$. So, $I_x = I_y$.

Hence, $\{I_x\}_{x \in U}$ is a disjoint collection of intervals. Also, because each interval can be injectively mapped to a rational number inside that interval, we know this collection is countable.

So, let $(I_j)_{j \in \mathbb{N}}$ be a sequence of disjoint open intervals whose union is U. Then observe that $\lim_{n \to \infty} (\mu(\bigcup_{j=1}^{n})I_j) = \mu(U)$. So, we can pick n large enough that for $A = \bigcup_{j=1}^{n} I_j$, we have that A is a finite union of open intervals and:
$$\mu(U) - \varepsilon/2 \leq \mu(A) \leq \mu(U) + \varepsilon/2$$

Now since $A \subseteq U$ and $E \subseteq U$, we know that $E - A \subseteq U - A$ and $A - E \subseteq U - A$. So:

- $\mu(E) + \mu(A - E) \leq \mu(E) + \mu(U - E) = \mu(U) \leq \mu(E) + \varepsilon/2$.
- $\mu(E - A) \leq \mu(U - A) = \mu(U) - \mu(A) \leq \mu(U) - \mu(U) + \varepsilon/2 = \varepsilon/2$
  (because $\mu(U) \leq \mu(E) + \varepsilon/2$ is finite)

Since $\mu(E)$ is finite, we can subtract it out of the first bulleted inequality and add it to the second to get:
$$\mu(A \,\Delta\, E) = \mu(A - E) + \mu(E - A) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Exercise 1.28: Let F be increasing and right continuous, and let μ_F be the associated measure. Then:

(a) $\mu_F(a) = F(a) - F(a-)$ (Note that this is a special case of (c))

(b) $\mu_F([a, b)) = F(b-) - F(a-)$

(c) $\mu_F([a, b]) = F(b) - F(a-)$

(d) $\mu_F((a, b)) = F(b-) - F(a)$

To start let's show (c).
Note that $[a, b] = \bigcap_{n \in \mathbb{N}} (a - \frac{1}{n}, b]$ and $((a - \frac{1}{n}, b])_{n \in \mathbb{N}}$ is a decreasing sequence of sets. So:
$$\mu_F([a, b]) = \lim_{n \to \infty} (\mu_F((a - \tfrac{1}{n}, b])) = \lim_{n \to \infty} (F(b) - F(a - \tfrac{1}{n})) = F(b) - F(a-)$$

Next, let's show (b).
Note that $[a, b) = \bigcup_{n \in \mathbb{N}} [a, b - \frac{1}{n}]$ and $([a, b - \frac{1}{n}])_{n \in \mathbb{N}}$ is an increasing sequence of sets. So:
$$\mu_F([a, b)) = \lim_{n \to \infty} (\mu_F([a, b - \tfrac{1}{n}])) = \lim_{n \to \infty} (F(b - \tfrac{1}{n}) - F(a-)) = F(b-) - F(a-)$$

Finally, let's show (d).
Note that $(a, b) = \bigcup_{n \in \mathbb{N}} (a, b - \frac{1}{n}]$ and $((a, b - \frac{1}{n}])_{n \in \mathbb{N}}$ is an increasing sequence of sets. So:
$$\mu_F((a, b)) = \lim_{n \to \infty} (\mu_F((a, b - \tfrac{1}{n}])) = \lim_{n \to \infty} (F(b - \tfrac{1}{n}) - F(a)) = F(b-) - F(a)$$

If $\mu = m$ is the Lesbesgue measure (a.k.a when $F(x) = x$), we denote the collection of Lesbesgue measurable sets $\mathcal{L}$ to be the completion of $\mathcal{B}_{\mathbb{R}}$.

Theorem: If $E \in \mathcal{L}$ and $s, r \in \mathbb{R}$, we define:
$$E + s = \{x + s \mid x \in E\}, \qquad rE = \{rx \mid x \in E\}.$$

Then $E + s, rE \in \mathcal{L}$, $m(E + s) = m(E)$ and $m(rE) = |r| m(E)$.

Proof: (the professor skipped doing this proof and the book doesn't seem to prove this well. So I'm going on my own here.)

Supposing $E + s \in \mathcal{L}$, define $m_s(E) = m(E + s)$. Similarly define $m^r(E) = m(rE)$ if $rE \in \mathcal{L}$.

Let $\mathcal{A}$ be the collection of disjoint unions of half-open-intervals, and define m^* as the outer measure induced by $m|_{\mathcal{A}}$. Then you can fairly trivially (albeit with a lot of notation) show that $m^*(E + s) = m^*(E)$ and $m^*(rE) = |r| m^*(E)$. Also, we know that $m^*(E)$ must equal $m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}}$ since any measure on $\mathcal{B}_{\mathbb{R}}$ is uniquely determined by it's restriction to $\mathcal{A}$ and m^* is a measure on $\mathcal{B}_{\mathbb{R}}$ (see the theorem on the bottom of page 24).

Next, note that $E \in \mathcal{B}_\mathbb{R} \implies E + s, rE \in \mathcal{B}_\mathbb{R}$. The reason why is that as seen in a bonus proposition below, we have that a set $E \in \mathcal{B}_\mathbb{R}$ if and only if it can be gained by taking a countable amount of countable unions, countable intersections, and complements of sets of $\mathcal{A}$. But note that given any $A \in \mathbb{R}$:

- $(A + s)^\mathsf{C} = A^\mathsf{C} + s$ and $(rA)^\mathsf{C} = rA^\mathsf{C}$
- $\displaystyle\bigcup_{j \in \mathbb{N}} (A_j + s) = (\bigcup_{j \in \mathbb{N}} A_j) + s$ and $\displaystyle\bigcup_{j \in \mathbb{N}} (rA_j) = r(\bigcup_{j \in \mathbb{N}} A_j)$
- $\displaystyle\bigcap_{j \in \mathbb{N}} (A_j + s) = (\bigcap_{j \in \mathbb{N}} A_j) + s$ and $\displaystyle\bigcap_{j \in \mathbb{N}} (rA_j) = r(\bigcap_{j \in \mathbb{N}} A_j)$

So, by replacing all the sets from $\mathcal{A}$ in our expression for E with their translated or scaled sets which are also in $\mathcal{A}$, we can show that $E + s$ or rE is also in $\mathcal{B}_\mathbb{R}$.

Combining that with the reasoning on the previous page, we know that $m(E) = m(E + s)$ and $m(rE) = |r|m(E)$ for all $E \in \mathcal{B}_\mathbb{R}$. What's left to show now is that for all sets in the completion $\mathcal{L}$ of $\mathcal{B}_\mathbb{R}$, we have that $E \in \mathcal{L} \implies E + s, rE \in \mathcal{L}$, $m(E + s) = m(E)$, and $m(rE) = |r|m(E)$.

This is not hard to show and I'm bored of this. So do it yourself.

Some fairly trivial observations:

1. $m(\{x\}) = 0$ for all $x \in \mathbb{R}$.

2. All countable sets have a Lesbesgue measure of zero (this includes $\mathbb{Q}$).

3. There exists uncountable sets with a Lesbesgue measure of zero (example: the Cantor set).

Given a collection of sets: $\mathcal{A}$, here is how to actually construct $\mathcal{M}(\mathcal{A})$.

Bonus Proposition: Let Ω be a minimal uncountable well-ordered set. Denoting 0 as the least element of Ω, define $\mathcal{E}_0 = \mathcal{A}$. Then for all $\alpha \in \Omega - \{0\}$, if α has an immediate predecessor β, define $\mathcal{E}_\alpha$ as the collection of all countable unions of sets and complements of sets in $\mathcal{E}_\beta$. Otherwise, define $\mathcal{E}_\alpha = \displaystyle\bigcup_{\beta \in S_\alpha} \mathcal{E}_\beta$ (where S_α is the set $\{\beta \in \Omega \mid \beta < \alpha\}$).

Then $\mathcal{M} = \mathcal{M}(\mathcal{A}) = \displaystyle\bigcup_{\alpha \in \Omega} \mathcal{E}_\alpha$.

Proof:

Suppose for all $\beta \in S_\alpha$, we have that $\mathcal{E}_\beta \subseteq \mathcal{M}$.

- Suppose α has no immediate predecessor. If $S_\alpha = \emptyset$, then we know that $\mathcal{E} = \mathcal{A} \subseteq \mathcal{M}$. Otherwise, we trivially have that:
$$\mathcal{E}_\alpha = \bigcup_{\beta \in S_\alpha} \mathcal{E}_\beta \subseteq \mathcal{M}.$$

- If α has a predecessor β', then because $\mathcal{M}$ is closed under countable unions and complements and $\mathcal{E}_{\beta'} \subseteq \mathcal{M}$, we know that $\mathcal{E}_\alpha \subseteq \mathcal{M}$.

It follows by transfinite induction that $\mathcal{E}_\alpha \subseteq \mathcal{M}$ for all $\alpha \in \Omega$. So, we've shown that $\displaystyle\bigcup_{\alpha \in \Omega} \mathcal{E}_\alpha \subseteq \mathcal{M}$.

To show the other inclusion, it suffices to show that $\bigcup_{\alpha\in\Omega}\mathcal{E}_\alpha$ is a σ-algebra (we know it contains $\mathcal{A}$ since $\mathcal{E}_0 = \mathcal{A}$).

Firstly, note that if $A \in \mathcal{E}_\alpha$ for any $\alpha \in \Omega$, then denoting β be the immediate successor of α, we have that $A^{\mathsf{C}} \in \mathcal{E}_\beta$.

Next, suppose $(A_j)_{j\in\mathbb{N}}$ is a sequence of sets in $\bigcup_{\alpha\in\Omega}\mathcal{E}_\alpha$ satisfying that $A_j \in \mathcal{E}_{\alpha_j}$ and $\alpha_j \in \Omega$ for all $j \in \mathbb{N}$.

Now every countable sequence in Ω has a supremum. So let α' be the supremum of the sequence $(\alpha_j)_{j\in\mathbb{N}}$. Next, let β' be the least element of Ω which is both greater than α' and has no immediate successor. That way $\mathcal{E}_{\alpha_j} \subseteq \mathcal{E}_{\beta'}$ for all $j \in \mathbb{N}$. Finally, let γ' be the successor of β'. Then:

$$\bigcup_{j\in\mathbb{N}} A_j \in \mathcal{E}_{\gamma'} \subseteq \bigcup_{\alpha\in\Omega}\mathcal{E}_\alpha$$

So, we've shown that $\bigcup_{\alpha\in\Omega}\mathcal{E}_\alpha$ is a σ-algebra containing $\mathcal{A}$.

It follows from this proposition that if a set is in $\mathcal{M}(\mathcal{A})$, then it can be gained through a procedure of countably many steps of taking countable unions, intersections, and complements of sets in $\mathcal{A}$.

Measurable Functions:

$f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ is called $(\mathcal{M}, \mathcal{N})$ measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

Proposition: If $\mathcal{N}$ is generated by $\mathcal{E}$, then $f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{M})$ is $(\mathcal{M}, \mathcal{N})$ measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Proof:
The rightward implication is trivial. Meanwhile suppose the right statement holds. Then consider the collection $\{E \subseteq Y \mid f^{-1}(E) \in \mathcal{M}\}$. By assumption we know it contains $\mathcal{E}$.

Firstly, this collection is closed under countable unions because:

$$f^{-1}(\bigcup_{j\in\mathbb{N}} A_j) = \bigcup_{j\in\mathbb{N}} f^{-1}(A_j).$$

Secondly, this collection is closed under complements. To prove this, first note that since $\emptyset \in \mathcal{E}$, we have that $Y - \emptyset = Y$ is a finite union of sets in $\mathcal{E}$. It follows then that $F^{-1}(Y) \in \mathcal{M}$. Next consider any $E \subseteq Y$ satisfying that $f^{-1}(E) \in \mathcal{M}$. Then $f^{-1}(E^{\mathsf{C}}) = f^{-1}(Y) - f^{-1}(E) \in \mathcal{M}$.

So, we've shown that the collection $\{E \subseteq Y \mid f^{-1}(E) \in \mathcal{M}\}$ is a σ-algebra containing $\mathcal{E}$. It follows that it contains $\mathcal{N} = \mathcal{M}(\mathcal{E})$.

Corollary: If X, Y are topological spaces, then every continuous function $f : X \longrightarrow Y$ is $(\mathcal{B}_X, \mathcal{B}_y)$ measurable.

We say a function $f : (X, \mathcal{M}) \longrightarrow \mathbb{R}$ (or $\mathbb{C}$ or $\mathbb{R}^n$) is $\mathcal{M}$-measurable if it is $(\mathcal{M}, \mathcal{B}_\mathbb{R}$ (or $\mathcal{B}_\mathbb{C}$ or $\mathcal{B}_{\mathbb{R}^n}$)) measurable. If it's obvious what $\mathcal{M}$ is, we can omit saying it.

In other words, the measurable sets of the range of a real or complex function is assumed to be the collection of Borel sets.

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Lesbesgue measurable if f is $(\mathcal{L}, \mathcal{B}_\mathbb{R})$ measurable.

Meanwhile, $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Borel measurable if f is $(\mathcal{B}_\mathbb{R}, \mathcal{B}_\mathbb{R})$ measurable.

Lecture 7 Notes: 10/17/2024

Exercise: Suppose $f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ is $(\mathcal{M}, \mathcal{N})$ measurable and $g : (Y, \mathcal{N}) \longrightarrow (Z, \mathcal{O})$ is $(\mathcal{N}, \mathcal{O})$ measurable. Then $g \circ f$ is $(\mathcal{M}, \mathcal{O})$ measurable.

Proof:
Given any $E \in \mathcal{O}$, because g is measurable, we know that $g^{-1}(E) \in \mathcal{N}$. Therefore, because f is measurable, we know $f^{-1}(g^{-1}(E)) = (g \circ f)^{-1}(E) \in \mathcal{M}$.

Proposition: Suppose $(X, \mathcal{M})$ is a measurable space. Given $f : X \longrightarrow \mathbb{R}$, the following are equivalent:

- f is measurable.
- $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
- $f^{-1}([a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
- $f^{-1}((-\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
- $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

Proof:
It's trivial that the first bullet point implies the others. Meanwhile, to go from any other bullet point back to the first, recall exercise 1.2 and the proposition on the bottom of page 39.

Suppose $(Y_\alpha, \mathcal{N}_\alpha)_{\alpha \in A}$ is a collection of measurable spaces. Then given a set X and functions $f_\alpha : X \longrightarrow Y_\alpha$ for all $\alpha \in A$, there exists a smallest σ-algebra $\mathcal{M}$ on X such that each f_α is measurable. Specifically:
$$\mathcal{M} = \mathcal{M}(\{f_\alpha^{-1}(E_\alpha) \mid \alpha \in A \text{ and } E_\alpha \in \mathcal{N}_\alpha\}).$$

Note that if $X = \prod_{\alpha \in A} Y_\alpha$ and $f_\alpha = \pi_\alpha$ for all $\alpha \in A$, then the $\mathcal{M}$ defined above is equal to $\bigotimes_{\alpha \in A} \mathcal{N}_\alpha$.

Proposition: Suppose $(Y_\alpha, \mathcal{N}_\alpha)_{\alpha \in A}$ is a collection of measurable spaces. Then define $Y = \prod_{\alpha \in A} Y_\alpha$ and $\mathcal{N} = \bigotimes_{\alpha \in A} \mathcal{N}_\alpha$.

We claim $f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ is $(\mathcal{M}, \mathcal{N})$ measurable if and only if $f_\alpha := \pi_\alpha \circ f$ is $(\mathcal{M}, \mathcal{N}_\alpha)$ measurable for all $\alpha \in A$.

Proof:
We already showed at the beginning of lecture that the composition of measurable functions is measurable. So, the rightward implication is obvious. Conversely, if $f_\alpha := \pi_\alpha \circ f$ is $(\mathcal{M}, \mathcal{N}_\alpha)$ measurable for all $\alpha \in A$, then given any $E_\alpha \in \mathcal{N}_\alpha$ for any $\alpha \in A$, we know that $f_\alpha^{-1}(E_\alpha) = f^{-1}(\pi_\alpha^{-1}(E_\alpha)) \in \mathcal{M}$. Now $\mathcal{N}$ is generated by the collection of all such $\pi_\alpha^{-1}(E_\alpha)$. So by the proposition on the bottom of page 39, we know that f is $(\mathcal{M}, \mathcal{N})$ measurable.

Corollary: $f : X \longrightarrow \mathbb{C}$ is measurable if and only if $\mathrm{Re}(f), \mathrm{Im}(f)$ are measurable.

In the extended real numbers $\overline{\mathbb{R}}$, we define $\mathcal{B}_{\overline{\mathbb{R}}} = \{E \in \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$. In other words, $\mathcal{B}_{\overline{\mathbb{R}}} = \{A, A \cup \{\infty\}, A \cup \{-\infty\}, A \cup \{\infty, -\infty\} \mid A \in \mathcal{B}_{\mathbb{R}}\}$.

We define $f : (X, \mathcal{M}) \longrightarrow \overline{\mathbb{R}}$ to be $\mathcal{M}$-measurable if f is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$ measurable.

Given a measurable space $(X, \mathcal{M})$ and set $E \in \mathcal{M}$, we say $f : X \longrightarrow \mathbb{R}$ (or $\mathbb{C}$ or $\mathbb{R}^n$) is <u>measurable on E</u> if $f^{-1}(B) \cap E \in \mathcal{M}$ for each Borel set B.

Exercise 2.1: Suppose $(X, \mathcal{M})$ is a measurable space. Let $f : X \longrightarrow \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then f is measurable if and only if $f^{-1}(\{-\infty\}) \in \mathcal{M}$, $f^{-1}(\{\infty\}) \in M$, and f is measurable on Y.

The forward implication is trivial. After all, $\{\infty\}$ and $\{-\infty\}$ are in $\mathcal{B}_{\overline{\mathbb{R}}}$. So, f being measurable implies $f^{-1}(\{-\infty\}) \in \mathcal{M}$ and $f^{-1}(\{\infty\}) \in \mathcal{M}$. Also, we know $\mathbb{R} \in \mathcal{B}_{\overline{\mathbb{R}}}$. So if $f^{-1}(B) \in \mathcal{M}$ for all $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, then:
$$f^{-1}(B) \cap Y = f^{-1}(B) \cap f^{-1}(\mathbb{R}) \in \mathcal{M} \text{ for all } B \in \mathcal{B}_{\overline{\mathbb{R}}}.$$

Now suppose $f^{-1}(\{-\infty\}), f^{-1}(\{\infty\}) \in \mathcal{M}$ and f is measurable on Y. Then given any set $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, we have that:
$$f^{-1}(B) = f^{-1}(B \cap \{-\infty\}) \cup f^{-1}(B \cap \{\infty\}) \cup (f^{-1}(B \cap \mathbb{R}))$$

Note that: $f^{-1}(B \cap \mathbb{R}) = f^{-1}(B) \cap Y$. So, since we know that $f^{-1}(B \cap \{-\infty\}), f^{-1}(B \cap \{\infty\})$, and $(f^{-1}(B \cap \mathbb{R}))$ are in $\mathcal{M}$ for all $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, we thus know $f^{-1}(B) \in \mathcal{M}$ for all $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, meaning f is measurable.

Now it's a bit frustrating but I don't think I have enough knowledge of topology right now to show this all fully rigorously (I'm running into the problem that $\overline{\mathbb{R}}$ is not a metric space). So right now I don't have a working definition of what it means for a subset of $\overline{\mathbb{R}}$ to be open in this context. I'll just leave these as notes and maybe in the future I'll come back and prove them.

Also neither the professor nor Folland seem to care…

Lemma 1: The collections $\{(a, \infty] \mid a \in \mathbb{R}\}$ and $\{[\infty, a) \mid a \in \mathbb{R}\}$ are both bases of the σ-algebra $\mathcal{B}_{\overline{\mathbb{R}}}$.

Lemma 2: Suppose $(X, \mathcal{M})$ is a measurable space. Given $f : X \longrightarrow \overline{\mathbb{R}}$, the following are equivalent:

- f is measurable.
- $f^{-1}((a, \infty]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
- $f^{-1}([-\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

Proposition: Suppose $f, g : X \longrightarrow \mathbb{C}$ are measurable. Then $f + g$ and fg are measurable.

Proof:
Define $F(x) = (f(x), g(x))$. Then F is $(X, \mathcal{B}_{\mathbb{C}^2} = \bigotimes_{i=1}^{2} \mathcal{B}_{\mathbb{C}})$ measurable by the above proposition.

Also, defining $G(z, w) = z + w$ and $H(z, w) = zw$ for all $z, w \in \mathbb{C}$, we have that G and H are continuous functions and thus $(\mathcal{B}_{\mathbb{C}^2}, \mathcal{B}_{\mathbb{C}})$ measurable. Since the composition of measurable functions is also measurable, this proposition follows.

Proposition: Suppose $(f_j)_{j \in \mathbb{N}}$ is a sequence of measurable functions from $(X, \mathcal{M})$ to $\overline{\mathbb{R}}$. Then the following functions are measurable.
$$g_1(x) = \sup_{j \in \mathbb{N}}(f_j(x)), \qquad g_2(x) = \inf_{j \in \mathbb{N}}(f_j(x)),$$
$$g_3(x) = \limsup_{j \to \infty}(f_j(x)), \quad g_4(x) = \liminf_{j \to \infty}(f_j(x)).$$

Proof:
Note that $g_1^{-1}((a, \infty]) = \bigcup_{j=1}^{\infty} f_j^{-1}((a, \infty])$ and $g_2^{-2}([-\infty, a)) = \bigcup_{j=1}^{\infty} f_j^{-1}([-\infty, a))$.

I'll show where the first of those equalities come from. Showing the other would be basically the same.

Suppose $x \in \bigcup\limits_{j=1}^{\infty} f_j^{-1}((a, \infty])$. Then there exists f_j with $f_j(x) > a$. It follows that $g_1(x) \geq f_j(x) > a$ and thus $x \in g_1^{-1}((a, \infty])$. So:

$$\bigcup_{j=1}^{\infty} f_j^{-1}((a, \infty]) \subseteq g_1^{-1}((a, \infty]).$$

On the other hand, suppose $x \in g_1^{-1}((a, \infty])$. We'll first address if $x \neq \infty$. Then letting $0 < \varepsilon < x - a$, we know there exists f_j in the sequence such that $f_j(x) \geq g(x) - \varepsilon$. It follows that $x \in \bigcup\limits_{j=1}^{\infty} f_j^{-1}((a, \infty])$. On the other hand, if $x = \infty$, then we know there exists f_j satisfying $f_j(x) > M$ for any $M \in \mathbb{R}$. Setting $M = a$, we get that $x \in \bigcup\limits_{j=1}^{\infty} f_j^{-1}((a, \infty])$. In both cases, we've thus shown that:

$$g_1^{-1}((a, \infty]) \subseteq \bigcup_{j=1}^{\infty} f_j^{-1}((a, \infty]).$$

Since all f_j are measurable, it follows that $g_1^{-1}((a, \infty])$ and $g_2^{-1}([\infty, a))$ are in $\mathcal{M}$ for all $a \in \mathbb{R}$. By our lemma on the previous page, we thus have that g_1 and g_2 are measurable.

To show that g_3 is measurable, define $h_k(x) = \sup\limits_{n \geq k}(f_n(x))$ for all $k \in \mathbb{N}$. Then note that $g_3(x) = \inf\limits_{k \in \mathbb{N}}(h_k(x))$.

By our previous reasoning about g_1, we know all h_k are measurable functions. Hence, it follows that g_3 is measurable by the same reasoning that g_2 is measurable. An analogous argument shows that g_4 is measurable.

Also if $f(x) = \lim\limits_{j \to \infty}(f_j(x))$ exists for all $x \in X$, then $f(x)$ is measurable. This is because $f(x)$ existing for all $x \in X$ means that $f(x) = g_3(x) = g_4(x)$.

Corollary 1: If $f, g : X \longrightarrow \overline{\mathbb{R}}$ are measurable, then $\max(f, g)$ and $\min(f, g)$ are measurable.

Corollary 2: If $(f_j)_{j \in \mathbb{N}}$ is a sequence of measurable functions from X to $\mathbb{C}$, and $f(x) = \lim\limits_{j \to \infty}(f_j(x))$ for all $x \in X$, then f is measurable.

To show this, apply the corollary on page 41 to consider the components of f and of each f_j. Then use the previous proposition.

Here are two important decompositions of functions:

1. Given a function $f : X \longrightarrow \overline{\mathbb{R}}$, define $f^+(x) := \max(f(x), 0)$ and $f^-(x) := \max(-f(x), 0)$. Then $f = f^+(x) - f^-(x)$ and $|f| = f^+(x) + f^-(x)$.

2. Given $f : X \longrightarrow \mathbb{C}$, we define it's <u>polar decomposition</u>:
$$f(z) = \operatorname{sgn}(f(z))|f(z)| \text{ where } \operatorname{sgn}(z) = \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

A function $f : X \longrightarrow \mathbb{C}$ is called <u>simple</u> if it is measurable and has a finite image. A standard representation of f is given by:
$$\sum_{j=1}^{n} z_j \chi_{E_j}$$

...where $E_j = f^{-1}(\{z_j\})$ and $\chi_E = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$

Observation: If $f, g : X \longrightarrow \mathbb{C}$ are simple functions, then so are $f + g$ and fg.

Theorem: Let $(X, \mathcal{M})$ be a measurable space.

1. If $f : X \longrightarrow [0, \infty]$ be a measurable function, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}}$ of simple functions such that $0 \leq \phi_1 \leq \ldots \leq f$ and $\phi_n \to f$ pointwise as $n \to \infty$. If f is bounded, then $\phi_n \to f$ uniformly.

2. If $f : X \longrightarrow \mathbb{C}$ is measurable, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}}$ of simple functions such that $0 \leq |\phi_1| \leq \ldots \leq |f|$ and $\phi_n \to f$ pointwise. If f is bounded, then $\phi_n \to f$ uniformly.

    Proof:

1. For all $n \in \mathbb{N}$, define ϕ_n as follows:
   Define $E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-n}])$ for all $k \in \{0, \ldots, 2^{2n} - 1\}$. Also define $F_n = f^{-1}((2^n, \infty])$. Then finally, set:
   $$\phi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n} \chi_{E_n^k} + 2^m \chi_{F_m}$$

   To check that $\phi_n \leq \phi_{n+1}$, first consider the case that $f(x) \leq 2^n$. Note that we can rewrite the expression $\phi_n(x) = k2^{-n} < f(x) \leq (k+1)2^{-n}$ as:
   $(2k)2^{-(n+1)} < f(x) \leq (2k+2)2^{-(n+1)}$.

   But then note that $\phi_{n+1}(x)$ is either $(2k)2^{-(n+1)}$ or $(2k+1)2^{-(n+1)}$. So, $\phi_{n+1}(x) \geq \phi_n(x)$. As for if $f(x) > 2^n$, then note that $\phi_{n+1}(x) \geq 2^n = \phi_n(x)$ because $E_{n+1}^{(2^{2n+1})} = f^{-1}(((2^{2n+1})2^{-(n+1)}, (2^{2n+1} + 1)2^{-(n+1)}])$

   Additionally, note that $\phi_n(x) \leq f(x)$ for all $x \in X$. And, when $f(x) < 2^n$, we clearly have that $|f(x) - \phi_n(x)| < 2^{-n}$. So it's clear that $(\phi_n)_{n \in \mathbb{N}}$ meets our convergence requirements.

2. Given $f(x) = g(x) + ih(x)$, apply part 1 of this theorem to g^+, g^-, h^+, and h^- to get the sequences (ψ_n^+), (ψ_n^-), (ζ_n^+), and (ζ_n^-) respectively of simple functions. Then letting $\phi_n = (\psi_n^+ - \psi_n^-) + i(\zeta_n^+ - \zeta_n^-)$ for all n satisfies our theorem.

We've mostly been working without regard to a particular measure. Here are some theorems which do require that we are working with a complete measure space $(X, \mathcal{M}, \mu)$:

Exercise 2.10: Prove the following are true if and only if $(X, \mathcal{M}, \mu)$ is complete:

(a) Suppose f is measurable and $f = g$ μ-a.e., then g is measurable.

> Note that if we say f is "measurable", it's implied that f is a function to $\mathbb{R}$, $\overline{\mathbb{R}}$, $\mathbb{C}$, or $\mathbb{R}^n$. We'll work here in $\mathbb{R}$ (and $\overline{\mathbb{R}}$) since this can all be extended to $\mathbb{C}$ and $\mathbb{R}^n$ by working component wise.

Proof:

($\Longrightarrow$)

Suppose $(X, \mathcal{M}, \mu)$ is complete. Then given such an f and g above, we know $\{x \mid f(x) \neq g(x)\}$ is a subset of a null set, meaning that all it's subsets are measurable. Next note that given any $a \in \mathbb{R}$, we have that:

$$\{x \mid g(x) > a\} = \{x \mid g(x) > a, f(x) = g(x)\} \cup \{x \mid g(x) > a, f(x) \neq g(x)\}.$$

Importantly, $\{x \mid g(x) > a, f(x) \neq g(x)\}$ is measurable since it's a subset of $\{x \mid f(x) \neq g(x)\}$. On the other hand, note that:

$$\{x \mid g(x) > a, f(x) = g(x)\} = \{x \mid f(x) > a\} - \{x \mid f(x) > a, f(x) \neq g(x)\}$$

Thus $\{x \mid g(x) > a, f(x) = g(x)\}$ is measurable because f is a measurable function and $\{x \mid f(x) > a, f(x) \neq g(x)\}$ is a subset of $\{x \mid f(x) \neq g(x)\}$. Hence, we've shown that $\{x \mid g(x) > a\}$ is measurable for all $a \in \mathbb{R}$, meaning g is measurable.

($\Longleftarrow$)

Let N be a null set of $(X, \mathcal{M}, \mu)$ and $F \subseteq N$. Then, let $f = \chi_N$ and $g = \chi_F$. Clearly, $f(x) = g(x)$ for all $x \in X - N$. And looking back at the definition of something being true μ-a.e., that means that $f(x) = g(x)$ μ-a.e. So by assumption, we know g is measurable, meaning that $g^{-1}(1) = F \in \mathcal{M}$.

(b) If f_n is measurable for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} f_n(x) = f(x)$ μ-a.e., then f is measurable.

($\Longrightarrow$)

Given $a \in \mathbb{R}$, note that:

$$\{x \mid f(x) > a\} = \{x \mid f(x) > a, f_n(x) \to f(x)\} \cup \{x \mid f(x) > a, f_n(x) \not\to f(x)\}.$$

The first of those two sets is measurable by the proposition on pages 42 and 43. The second is a subset of a null set. So since $(X, \mathcal{M}, \mu)$ is complete, it follows that $\{x \mid f(x) > a\} \in \mathcal{M}$.

$(\Longleftarrow)$
Let N be a null set of $(X, \mathcal{M}, \mu)$ and $F \subseteq N$. Then define $f_n = 0$ for all $n \in \mathbb{N}$ and $f = \chi_F$. Then $f_n(x) \to f(x)$ for all $x \in N^{\mathsf{C}}$, meaning $f_n(x) \to f(x)$ μ-a.e. So by assumption, f is measurable, meaning that $f^{-1}(1) = F \in \mathcal{M}$.

Proposition: If $(X, \mathcal{M}, \mu)$ is a measure space and $(X, \overline{\mathcal{M}}, \overline{\mu})$ is its completion. If f is $\overline{\mathcal{M}}$-measurable, then there exists g that is $\mathcal{M}$-measurable such that $f = g$ $\overline{\mu}$-a.e.

If $f = \chi_E$ for $E \in \mathcal{M}$, then this is obvious.

$E = E' \cup F'$ where $E' \in \mathcal{M}$ and $F' \subseteq N' \in \mathcal{M}$ with $\mu(N') = 0$. So define $g = \chi_{E'}$. Then g is $\mathcal{M}$-measurable and $f(x) = g(x)$ for all $x \in (N')^{\mathsf{C}}$. So $f(x) = g(x)$ μ-a.e. and thus also $\overline{\mu}$-a.e.

Now let f be any $\overline{\mathcal{M}}$-measurable function and let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence of $\overline{\mathcal{M}}$-measurable simple functions converging pointwise to f. Next, for each n, let ψ_n be an $\mathcal{M}$-simple function satisfying that $\psi_n(x) = \phi_n(x)$ except on a null set E_n of $\overline{\mathcal{M}}$.
Remember above definition for why we can do that.

Choose a μ-null set $N \in \mathcal{M}$ such that $\bigcup_{n \in \mathbb{N}} E_n \subseteq N$. Then define $g = \lim_{n \to \infty} \chi_{N^{\mathsf{C}}} \psi_n(x)$.

By the second corollary on page 43, we thus know that g is $\mathcal{M}$-measurable with $g(x) = f(x)$ for all $x \in N^{\mathsf{C}}$.

Lecture 8 Notes: 10/22/2024

Fixing our measure space as $(X, \mathcal{M}, \mu)$, define $L^+ := \{f : X \to [0, \infty] : f \text{ is measurable}\}$. We say the function ϕ is <u>simple</u> if $\phi = \sum_{j=1}^m a_j \chi_{E_j}$ where all $a_j \in \mathbb{C}$ and all $E_j \in \mathcal{M}$.

If $\phi = \sum_{j=1}^m a_j \chi_{E_j}$ where all $a_j \in [0, \infty]$ and all $E_j \in \mathcal{M}$ are disjoint, we define:
$$\int \phi = \int \phi \mathrm{d}\mu = \int \phi(x) \mathrm{d}\mu(x) := \sum_{j=1}^m a_j \mu(E_j)$$

If $A \in \mathcal{M}$, then we define $\int_A \phi := \int \chi_A \phi$. Note that if ϕ, ψ are simple functions in L^+, then you can show:

1. $c \geq 0 \implies \int c\phi = c \int \phi$

2. $\int (\phi + \psi) = \int \phi + \int \psi$

3. $\phi \leq \psi \implies \int \phi \leq \int \psi$.

Given any $f \in L^+$, we now define:
$$\int f \mathrm{d}\mu := \sup \left\{ \int \phi \mathrm{d}\mu : 0 \leq \phi \leq f \text{ and } \phi \text{ is simple} \right\}$$

Proposition: If $f \in L^+$ and $\int f < \infty$, then $\{x \mid f(x) = \infty\}$ is a null set and $\{x \mid f(x) > 0\}$ is σ-finite.

The proof is **Exercise 2.12**:

Firstly, suppose $\mu(\{x \mid f(x) = \infty\}) = \alpha > 0$. Then for all $n \in \mathbb{N}$, we can define: $\phi_n = n\chi_{\{x \mid f(x) = \infty\}}$. Thus, (ϕ_n) is an increasing sequence of simple functions less than f satisfying that $\int f \geq \int \phi_n = n\alpha$ for all n. It follows that $\int f \geq \lim_{n \to \infty}(n\alpha) = \infty$, meaning that $\int f = \infty$.

Hence, we've shown that $\int f < \infty \implies \mu(\{x \mid f(x) = \infty\}) = 0$.

Next, suppose that for some $\beta > 0$, we have that $\mu(\{x \mid f(x) > \beta\}) = \infty$. Then $\phi = \beta\chi_{\{x \mid f(x) > \beta\}}$ is a simple function less than f satisfying that:

$$\infty = \beta\infty = \int \phi \leq \int f.$$

It follows that $\int f < \infty \implies \mu(\{x \mid f(x) > \beta\}) < \infty$ for all $\beta > 0$.

Finally, note that $\{x \mid f(x) > 0\} = \bigcup_{n \in \mathbb{N}} \{x \mid f(x) > \frac{1}{n}\}$ where each $\mu(\{x \mid f(x) > \frac{1}{n}\}) < \infty$.

So, $\{x \mid f(x) > 0\}$ is σ-finite.

Exercise 2.13: Suppose $(f_n)_{n \in \mathbb{N}} \subset L^+$, $f_n \to f$ pointwise, and $\int f = \lim_{n \to \infty} \int f_n < \infty$. Then $\int_E f = \lim_{n \to \infty} \int_E f_n$ for all $E \in \mathcal{M}$.

Firstly let $E \in \mathcal{M}$. Then $\int f\chi_E + \int f\chi_{E^c} = \int f$ and $\int f_n\chi_E + \int f_n\chi_{E^c} = \int f_n$ for all $n \in \mathbb{N}$. Based on that, we can show:

$$\limsup_{n \to \infty} \int f_n\chi_E + \liminf_{n \to \infty} \int f_n\chi_{E^c} = \lim_{n \to \infty} \int f_n = \int f$$

To see this, consider any subsequence $(\int f_{n_k}\chi_E)_{k \in \mathbb{N}}$ converging to $\limsup_{n \to \infty} \int f_n\chi_E$.

Because $\int f$ is finite, we know there is $N_1 \in \mathbb{N}$ such that $\int f_n < \infty$ for all $n > N_1$. In turn, we know that $\int f_n\chi_E < \infty$ for all $n > N_1$ and thus $\limsup_{n \to \infty} \int f_n\chi_E < \infty$.

Now since $(\int f_{n_k}\chi_E)_{k \in \mathbb{N}}$ and $(\int f_n)_{n \in \mathbb{N}}$ have finite limits, given any $\varepsilon > 0$, we can pick $N_2 \in \mathbb{N}$ greater than N_1 such that for all $k \geq N_2$, we have that:

$$\left|\int f_{n_k}\chi_E - \limsup_{n \to \infty} \int f_n\chi_E\right| < \varepsilon/2 \text{ and } \left|\int f_{n_k} - \int f\right| < \varepsilon/2$$

It follows that because $\int f_{n_k} - \int f_{n_k}\chi_E = \int f_{n_k}\chi_{E^c}$, we have that:

$$\left(\int f - \limsup_{n \to \infty} \int f_n\chi_E\right) - \varepsilon < \int f_{n_k}\chi_{E^c} < \left(\int f - \limsup_{n \to \infty} \int f_n\chi_E\right) + \varepsilon$$

So $\lim_{k \to \infty} \int f_{n_k}\chi_{E^c}$ exists and:

$$\liminf_{n \to \infty} \int f_n\chi_{E^c} \leq \lim_{k \to \infty} \int f_{n_k}\chi_{E^c} = \int f - \limsup_{n \to \infty} \int f_n\chi_E.$$

Meanwhile, consider any subsequence $(\int f_{n_j} \chi_{E^c})_{j \in \mathbb{N}}$ converging to $\liminf_{n \to \infty} \int f_n \chi_{E^c}$.

By analogous reasoning to before, we know that $\liminf_{n \to \infty} \int f_n \chi_{E^c}$ is finte.

Also similarly to before, we can show that $\lim_{j \to \infty} \int f_{n_j} \chi_E$ exists and:

$$\int f - \liminf_{n \to \infty} \int f_n \chi_{E^c} = \lim_{j \to \infty} \int f_{n_j} \chi_E \leq \limsup_{n \to \infty} \int f_n \chi_E$$

Finally, by rearranging terms, we get that:

$$\int f \leq \limsup_{n \to \infty} \int f_n \chi_E + \liminf_{n \to \infty} \int f_n \chi_{E^c} \leq \int f.$$

Next, note that $f_n \chi_E \to f\chi_E$ and $f_n \chi_{E^c} \to f\chi_{E^c}$ pointwise as $n \to \infty$. Thus by Fatou's lemma, we immediately have that:

$$\int f\chi_E = \int \lim_{n \to \infty} (f_n \chi_E) \leq \liminf_{n \to \infty} \int f_n \chi_E$$

Also, more round-aboutly we get that:

$$\int f\chi_E = \int f - \int f\chi_{E^c} = \int f - \int \lim_{n \to \infty} f_n \chi_{E^c}$$
$$\geq \int f - \liminf_{n \to \infty} \int f_n \chi_{E^c} = \limsup_{n \to \infty} \int f_n \chi_E$$

Thus, $\int f\chi_E \leq \liminf_{n \to \infty} f_n \chi_E \leq \limsup_{n \to \infty} f_n \chi_E \leq \int f\chi_E$.

And so, we have shown that $\lim_{n \to \infty} \int f_n \chi_E$ exists and that $\int f\chi_E = \lim_{n \to \infty} \int f_n \chi_E$

However, this need not be true if $\int f = \lim_{n \to \infty} \int f_n = \infty$.

Suppose $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{L}, m)$ and then define $f_n = \chi_{(-\infty,0)} + \chi_{(n,n+1]}$ for all $n \in \mathbb{N}$ and $f = \chi_{(-\infty,0)}$. Then note that $f_n \to f$ pointwise as $n \to \infty$. Also, for all $n \in \mathbb{N}$ we have that $\int f = \infty = \int f_n$. However, $E = [0, \infty) \in \mathcal{L}$ and $\int_E f = 0$ while $\int_E f_n = 1$ for all n. So, $\int_E f \neq \lim_{n \to \infty} \int_E f_n$.

Exercise 2.14: If $f \in L^+$, let $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$ and for any $g \in L^+$, $\int g d\lambda = \int f g d\mu$.

To start, we show λ is measure. Clearly, $\lambda(\emptyset) = \int_\emptyset f d\mu = \int f\chi_\emptyset d\mu = \int 0 d\mu = 0$. Meanwhile, suppose $(E_n)_{n \in \mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{M}$ whose union is E.

Note that $\chi_E = \sum_{n=0}^{\infty} \chi_{E_n}$. So:

$$\lambda(E) = \int_E f d\mu = \int f\chi_E d\mu = \int \left(\sum_{n=0}^{\infty} f\chi_{E_n} \right) d\mu = \sum_{n=0}^{\infty} \int f\chi_{E_n} d\mu = \sum_{n=0}^{\infty} \lambda(E_n)$$

Hence, we've shown that λ is a measure on $\mathcal{M}$.

Now, in order to show that $\int g d\lambda = \int f g d\mu$ for all $g \in L^+$, let's first consider a simple function $\phi \in L^+$.

Suppose $\phi = \sum_{k=1}^{N} x_k \chi_{E_k}$ where each E_k is disjoint. Then:

$$\int \sum_{k=1}^{N} \chi_{E_k} d\lambda = \sum_{k=1}^{N} x_k \lambda(E_k)$$
$$= \sum_{k=1}^{N} x_k \int f \chi_{E_k} d\mu = \sum_{k=1}^{N} \int x_k f \chi_{E_k} d\mu$$
$$= \int \sum_{k=1}^{N} x_k f \chi_{E_k} d\mu = \int f \sum_{k=1}^{N} x_k \chi_{E_k} d\mu$$

Hence, $\int \phi d\lambda = \int f \phi d\mu$.

To extend this to any function $g \in L^+$, let $(\phi_n)_{n \in \mathbb{N}}$ be an increasing sequence of simple functions such that $\phi_n \to g$ pointwise as $n \to \infty$. Importantly, we also have that $f\phi_n \to fg$ pointwise as $n \to \infty$ with $f\phi_n \le f\phi_{n+1}$ for all $n \in \mathbb{N}$. So, applying the monotone convergence theorem, we have that:

$$\int g d\lambda = \lim_{n \to \infty} \left(\int \phi_n d\lambda \right) = \lim_{n \to \infty} \left(\int \phi_n f d\mu \right) = \int f g d\mu$$

Exercise 2.25: Let $f(x) = x^{-1/2}$ if $0 < x < 1$ and $f(x) = 0$ otherwise. Let $(r_n)_{n \in \mathbb{N}}$ be an enumeration of the rational numbers, and set $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$.

(a) $g \in L^1(m)$ and in particular $g < \infty$ a.e.

Note that $\int |2^{-n} f(x - r_n)| dm = 2^{-n} \int |f(x - r_n)| dm$ since $2^{-n} > 0$. Next, note that by the monotone convergence theorem, if $(a_k)_{k \in \mathbb{N}}$ is any decreasing sequence of numbers converging to r_n, then:

$$\int |f(x - r_n)| dm = \lim_{k \to \infty} \int |f(x - r_n)| \chi_{\{x | x > a_k\}} dm$$
$$= \lim_{k \to \infty} \int_{a_k}^{r_n + 1} |f(x - r_n)| dx$$
$$= \lim_{a \to r_n +} \int_a^{r_n + 1} f(x - r_n) dx$$
$$= \lim_{a \to 0+} \int_a^1 f(x) dx = \lim_{a \to 0+} \int_a^1 x^{-1/2} dx = \lim_{a \to 0+} (2x|_a^1) = 2$$

It follows that:

$$\int \sum_{n=1}^{\infty} |2^{-n} f(x - r_n)| dm = \sum_{n=1}^{\infty} \int |2^{-n} f(x - r_n)| dm = \sum_{n=1}^{\infty} 2^{-n} \cdot 2 = 1.$$

So by theorem 2.25, we know that $\sum_{n=1}^{\infty} 2^{-n} f(x - r_n) \in L^1(m)$.

By an earlier exercise, since $\sum_{n=1}^{\infty} 2^{-n} f(x - r_n) \geq 0$ for all x, we know that it has a finite integral if and only if it equals infinity on a null set. Hence, it follows that:

$$g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n) < \infty \text{ almost everywhere.}$$

(b) g is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lesbesgue null set.

To start, let's fix $x \in \mathbb{R}$ and $\delta \in (0, 1)$. Then consider any $N > 0$.

Note that there exists $r_n \in (r_n)_{n \in \mathbb{N}}$ such that $x < r_n < x + \delta$. Then, note that if $r_n < x < \min(r_n + \frac{1}{N^2} 2^{-2n}, r_n + 1)$, we have that $2^{-n} f(x - r_n) > N$. Since $x + \delta < r_n + 1$, we thus know that $g(s) > N$ for all s in

$$(r_n, \min(x + \delta, r_n + \tfrac{1}{N^2} 2^{-2n})) \subseteq (x, x + \delta).$$

Also note that we can pick s in that interval such that $g(s)$ is finite. Furthermore, if we change g on a null set N, then we can also select s to be not in N.

To see this, first note that $m(r_n, \min(r_n + N^2 2^{2n}, x + \delta))$ is an open interval and thus has positive measure. Also, by the completeness of m, we know that $\{x \mid g(x) = \infty\}$ is a subset of a null set and thus itself a null set.

Finally, suppose we have another null set N. Then taking the difference of the interval with the two null sets sets must give a set with the positive measure of the original interval. And thus the difference can't be the empty set which has a measure of zero.

It immediately follows that even after modifying g on a null set, we still have that g is unbounded on all non-singleton intervals since we can fix x to be any interior point of the interval and δ such that $(x, x+\delta)$ is a subset of that interval.

Also, if $g(x)$ is finite and $\varepsilon > 0$. Then we've shown that even if we modify g, we still have that for all $\delta > 0$, there exists $s \in (x - \delta, x + \delta)$ with $g(x) + \varepsilon < g(s) < \infty$. So g is not continuous at x.

We haven't defined continuity yet for when g can equal ∞ on an interval. But here's my current intuition on what being continuous at x when $g(x) = \infty$ should mean.

Finally if $g(x) = \infty$, then by the previous reasoning, even if we modify g on a null set, we can still find s in any neighborhood of x such that $g(s) < \infty$. Thus, g is not continuous at x.

(c) $g^2 < \infty$ a.e. but g^2 is not integrable on any interval.

We know $g^2 < \infty$ a.e. because $g < \infty$ a.e.

Now note that since $2^{-n} f(x - r_n) \geq 0$ for all $n \in \mathbb{N}$, we can say that for all $N \in \mathbb{N}$:
$$\sum_{n=1}^{N} (2^{-n} f(x - r_n))^2 \leq \left(\sum_{n=1}^{N} 2^{-n} f(x - r_n) \right)^2$$

Taking $N \to \infty$, we thus get that $\sum_{n=1}^{\infty} (2^{-n} f(x - r_n))^2 \leq (g(x))^2$.

Also, given any interval I, we can multiply our above inequality by χ_I and it won't flip because $\chi_I \geq 0$. Then as $(2^{-n} f(x - r_n))^2 \chi_I \geq 0$ for all $n \in \mathbb{N}$, we can thus conclude that:
$$\sum_{n=1}^{\infty} \int_I (2^{-n} f(x - r_n))^2 dm = \int_I \sum_{n=1}^{\infty} (2^{-n} f(x - r_n))^2 dm \leq \int_I (g(x))^2 dm$$

But now fix n such that there exists $\varepsilon \in (0, 1)$ with $[r_n - \varepsilon, r_n + \varepsilon] \subseteq I$. Then note that:
$$\begin{aligned}
\int_I (2^{-n} f(x - r_n))^2 dm &\geq 2^{-2n} \cdot \lim_{a \to r_n+} \int_a^{r_n + \varepsilon} (f(x - r_n))^2 dx \\
&= 2^{-2n} \cdot \lim_{a \to r_n+} \int_a^{r_n + \varepsilon} \frac{1}{x - r_n} dx \\
&= 2^{-2n} \cdot \lim_{a \to r_n^+} (\log(\varepsilon) - \log(a - r_n)) = \infty
\end{aligned}$$

So $\int_I (2^{-n} f(x - r_n))^2 dm = \infty$. Hence, so does $\int_I (g(x))^2 dm = \infty$. And thus, $(g(x))^2$ is not integrable on the interval I.

Exercise 2.26: If $f \in L^1(m)$ and $F(x) = \int_{-\infty}^x f(t)dt$, then F is continuous on $\mathbb{R}$.

Let $(x_k)_{k \in \mathbb{N}}$ be any sequence of points converging to a given point x. Importantly, $f \in L^1(m)$ implies that $\int |f| < \infty$. Also, we have that $|f \chi_{(-\infty, x_k)}| \leq |f|$ for all $k \in \mathbb{N}$. Plus, $f \chi_{(-\infty, x_k)} \to f \chi_{(-\infty, x)}$ as $k \to \infty$.

So, by applying the dominated convergence theorem, we have that:
$$\lim_{k \to \infty} (F(x_k)) = \lim_{k \to \infty} (\int f \chi_{(-\infty, x_k)}) = \int f \chi_{(-\infty, x)} = F(x)$$

Hence, $\lim_{t \to x}(F(t)) = F(x)$ and thus F is continuous at x.

Exercise 2.20: Generalized Dominated Convergence Theorem If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \le g_n$, and $\int g_n \to \int g$. Then $\int f_n \to \int f$.

The most murky part of this theorem was saying that we could assume f is measurable. Luckily, the problem statement tells me out right that all the functions are measurable so I don't need to struggle to justify that.

Now by seperating the functions into their real and imaginary parts, we can without loss of generality assume f_n and f are real-valued. Also, the problem statement doesn't make sense if the g_n aren't real-valued.

Since $|f_n| < g_n$ for all n, we know that $g_n + f_n > 0$ and $g_n - f_n > 0$ for all n. Hence, applying Fatou's lemma we know that:

- $\int g + \int f = \int \lim_{n\to\infty} (g_n + f_n) = \int \liminf_{n\to\infty}(g_n + f_n) \le \liminf_{n\to\infty} \int (g_n + f_n)$
- $\int g - \int f = \int \lim_{n\to\infty} (g_n - f_n) = \int \liminf_{n\to\infty}(g_n - f_n) \le \liminf_{n\to\infty} \int (g_n - f_n)$

Next, here's a lemma. If $(a_n), (b_n)$ are sequences of real numbers satisfying that $\lim_{n\to\infty} a_n \to a \in \mathbb{R}$ and $\liminf_{n\to\infty} b_n = \beta \in \overline{\mathbb{R}}$, then $\liminf_{n\to\infty}(a_n + b_n) = a + \beta$.

Proof:
Let $(a_{n_k} + b_{n_k})$ be a subsequence of $(a_n + b_n)$ converging to $\liminf_{n\to\infty}(a_n + b_n)$. Then because $a_{n_k} \to a$ as $n \to \infty$, we must have that $b_{n_k} \to \liminf_{n\to\infty}(a_n + b_n) - a \ge \beta$. In other words, $\liminf_{n\to\infty}(a_n + b_n) \ge a + \beta$

On the other hand, let (b_{n_j}) be a subsequence of (b_n) converging to β. Then $a_{n_j} + b_{n_j} \to a + \beta \ge \liminf_{n\to\infty}(a_n + b_n)$ as $n \to \infty$.

So:

- $\liminf_{n\to\infty} \int (g_n + f_n) = \liminf_{n\to\infty}(\int g_n + \int f_n) = \int g + \liminf_{n\to\infty} \int f_n$
- $\liminf_{n\to\infty} \int (g_n - f_n) = \liminf_{n\to\infty}(\int g_n - \int f_n) = \int g + \liminf_{n\to\infty}(-\int f_n)$

Finally, $\liminf_{n\to\infty}(-\int f_n) = -\limsup_{n\to\infty} \int f_n$.

Therefore subtracting out $\int g$ which we can do because $g \in L^1$ and thus $\int g$ is finite:

- $\int g + \int f \le \int g + \liminf_{n\to\infty} \int f_n \implies \int f \le \liminf_{n\to\infty} \int f_n$
- $\int g - \int f \le \int g + \liminf_{n\to\infty}(-\int f_n) \implies \limsup_{n\to\infty} \int f_n \le \int f$

Hence $\lim_{n\to\infty} \int f_n = \int f$.

Exercise 2.21: Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.

(⟹)

Note that $|f_n| \leq |f_n - f| + |f|$ for all n, $|f_n - f| + |f| \to 0 + |f| = |f|$ a.e., and $\int (|f_n - f| + |f|) \to \int 0 + \int |f| = \int |f|$ by assumption. Then by exercise 2.20, we know that $\lim_{n\to\infty} \int |f_n| = \int \lim_{n\to\infty} |f_n| = \int |f|$.

(⟸)

Note that $|f_n - f| \leq |f_n| + |f|$ for all n, $|f_n| + |f| \to 2|f|$ a.e., and by assumption $\int (|f_n| + |f|) \to \int 2|f| = \int \lim_{n\to\infty} (|f_n| + |f|)$. So by exercise 2.20, we know that $\lim_{n\to\infty} \int |f_n - f| = \int \lim_{n\to\infty} |f_n - f| = \int 0 = 0$.

Exercise 2.28: Compute the following limits:

(a) $\lim_{n\to\infty} \int_0^\infty (1 + x/n)^{-n} \sin(x/n) dx$

Note that $|(1 + x/n)^{-n} \sin(x/n)| \leq (1 + x/n)^{-n}$ for all $n \geq 1$ and $x \geq 0$ because $|\sin(x/n)| \leq 1$ and $(1 + x/n)^{-n} \geq 0$ when n and x satisfy the above inequalities.

Next, note that $(1 + x/n)^n$ is a strictly increasing sequence for all $x > 0$.

To see this, note that because $\frac{x}{n+1} < \frac{x}{n}$, we have that:

$(1 + \frac{x}{n+1})^{n+1} - (1 + \frac{x}{n})^n \geq (1 + \frac{x}{n+1})^{n+1} - (1 + \frac{x}{n+1})^n = (1 + \frac{x}{n+1})^n \cdot \frac{x}{n+1} > 0$

As a result, $(1 + x/n)^{-n}$ is a strictly decreasing sequence for all $x > 0$.

Now, while $\int_0^\infty (1 + x)^{-1} dx = \infty$ and so we can't use that as our upperbound, note that $(1 + x/n)^{-n} \leq (1 + x/2)^{-2}$ for all $n \geq 2$ by our above reasoning and:

$\int_0^\infty (1 + x/2)^{-2} dx = 2 \int_1^\infty u^{-2} du = 2 \lim_{N\to\infty} (\frac{-1}{N} + 1) = 2$

Hence, $g(x) = (1 + \frac{x}{2})^{-2} \in L^1$ with $|f_n| \leq g$ for all $n \geq 2$.

Clearly the subsequence leaving out when $n = 1$ will still have identical limiting behavior as the full sequence including $n = 1$. So by applying D.C.T (technically to the subsequence without $n = 1$), we get that:

$\lim_{n\to\infty} \int_0^\infty (1 + x/n)^{-n} \sin(x/n) dx = \int_0^\infty (\lim_{n\to\infty} (1 + x/n)^{-n} \sin(x/n)) dx$

Now $\sin$ is continuous and $x/n \to 0$ pointwise. Thus $\sin(x/n) \to \sin(0) = 0$ pointwise as $n \to \infty$. At the same time, note that $(1 + x/n)^n \to e^x$ as $n \to \infty$.

Small proof:
By L'Hôpital's rule: $\lim_{y\to 0} \frac{\log(1+ay)}{y} = \lim_{y\to 0} \frac{a}{1+ay} = a$.
So, since $\exp$ is continuous, we have that:

$\lim_{y\to 0} (1 + ay)^{1/y} = \lim_{y\to 0} \exp(\frac{\log(1+ay)}{y}) = \exp(\lim_{y\to 0} \frac{\log(1+ay)}{y}) = \exp(a)$

Thus $(1 + x/n)^{-n} \sin(x/n) \to \frac{0}{e^{-x}} = 0$. And so:
$$\int_0^\infty (\lim_{n\to\infty} (1 + x/n)^{-n} \sin(x/n))dx = \int_0^\infty 0 dx = 0.$$

(b) $\lim_{n\to\infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n}dx$

Note that $(1 + nx^2)(1 + x^2)^{-n}$ is a decreasing sequence of functions.

Proof: $\frac{1+(n+1)x^2}{(1+x^2)^{n+1}} - \frac{1+nx^2}{(1+x^2)^n} = \frac{1+nx^2+x^2-1-x^2-nx^2-nx^4}{(1+x^2)^{n+1}} = \frac{-nx^4}{(1+x^2)^{n+1}} \leq 0$

Also, when $n = 1$, we have that $(1 + nx^2)(1 + x^2)^{-n} = 1$ and $\int_0^1 1 dx = 1$. So, by applying D.C.T, we get that:
$$\lim_{n\to\infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n}dx = \int_0^1 \lim_{n\to\infty} ((1 + nx^2)(1 + x^2)^{-n})dx$$

If $x = 0$, then $(1 + nx^2)(1 + x^2)^{-n} = 1$ for all n. Meanwhile if $x \neq 0$, then $1 + x > 0$ and thus:
$$\frac{1+nx^2}{(1+x^2)^n} = \frac{1}{(1+x^2)^n} + x^2\frac{n}{(1+x^2)^n} \to 0 + x^2 \cdot 0 = 0$$

It follows that $\int_0^1 \lim_{n\to\infty} ((1+nx^2)(1+x^2)^{-n})dx = \int 0 dx = 0$ (we can change the inside function at the point $x = 0$ without changing the value of the integral).

Exercise 2.29: Show that $\int_0^\infty x^n e^{-x}dx = n!$ by differentiating the equation $\int_0^\infty e^{-tx}dx = 1/t$. Similarly show that $\int_{-\infty}^\infty x^{2n}e^{-x^2}dx = \frac{(2n)!\sqrt{\pi}}{4^n n!}$ by differentiating the equation $\int_{-\infty}^\infty e^{-tx^2}dx = \sqrt{\pi/t}$.

Note that for all $x \geq 0$ and $t \geq 1$, $x^k e^{-tx} \leq x^k e^{-x} \leq \frac{x^k}{\frac{x^k}{k!} + \frac{x^{k+2}}{(k+2)!}} \leq \frac{(k+2)!}{(k+2)(k+1)+x^2}$.

Similarly, for all $x \geq 0$ and $t \geq 1$, we have:
$$x^{2k}e^{-tx^2} \leq x^{2k}e^{-x^2} \leq \frac{x^{2k}}{\frac{x^{2k}}{k!} + \frac{x^{2k+2}}{(k+1)!}} = \frac{(k+1)!}{1+k+x^2}.$$

Note that these upper bounds are gotten by considering the taylor series of e^x.

Now importantly $\int_0^\infty \frac{1}{a^2+x^2}dx = \frac{1}{a}(\arctan(x/a))|_0^\infty = \frac{\pi}{2a}$ for all real nonzero a.
Similarly, $\int_{-\infty}^\infty \frac{1}{a^2+x^2}dx = \frac{1}{a}(\arctan(x/a))|_{-\infty}^\infty = \frac{\pi}{a}$

Thus, $\frac{(k+2)!}{(k+2)(k+1)+x^2}dx$ is integrable on $[0, \infty)$ and $\int_{-\infty}^\infty \frac{(k+1)!}{1+k+x^2}dx$ is integrable on $(-\infty, \infty)$ for all k.

So, we've shown that for all $t \geq 1$:
- $\frac{d}{dt}\int_0^\infty x^{k-1}e^{-tx}dx = \int_0^\infty \frac{\partial}{\partial t}x^{k-1}e^{-tx}dx = \int_0^\infty -x^k e^{-tx}dx$
- $\frac{d}{dt}\int_0^\infty x^{2k-2}e^{-tx^2}dx = \int_0^\infty \frac{\partial}{\partial t}x^{2k-2}e^{-tx^2}dx = \int_0^\infty -x^{2k}e^{-tx^2}dx$

And now the rest of the problem is just repeatedly differentiating the equations the problem tells you to.

Firstly, note that:

$$\int_0^\infty e^{-tx}dx = t^{-1} \xoverset{\partial/\partial t}{\Longrightarrow} \int_0^\infty -xe^{-tx}dx = -t^{-2} \Longrightarrow \int_0^\infty -1(x)e^{-tx}dx = 1t^{-2}$$

$$\xoverset{\partial/\partial t}{\Longrightarrow} \int_0^\infty -x^2 e^{-tx}dx = -2!t^{-3} \Longrightarrow \int_0^\infty x^2 e^{-tx}dx = 2!t^{-2}$$

$$\cdots \xoverset{\partial/\partial t}{\Longrightarrow} \int_0^\infty -x^n e^{-tx}dx = -n!t^{-n-1} \Longrightarrow \int_0^\infty x^n e^{-tx}dx = n!t^{-n-1}$$

Plugging in $t = 1$, we then get $\int_0^\infty x^n e^{-x}dx = n!$.

Secondly, note that:

$$\int_{-\infty}^\infty e^{-tx^2}dx = \sqrt{\pi/t} \xoverset{\partial/\partial t}{\Longrightarrow} \int_{-\infty}^\infty -x^2 e^{-tx^2}dx = -\sqrt{\pi}(\tfrac{1}{2})t^{-3/2}$$

$$\Longrightarrow \int_{-\infty}^\infty x^2 e^{-tx^2}dx = \sqrt{\pi}(\tfrac{1}{2})t^{-3/2}$$

$$\xoverset{\partial/\partial t}{\Longrightarrow} \int_{-\infty}^\infty -x^{2(2)} e^{-tx^2}dx = -\sqrt{\pi}(\tfrac{1}{2})(\tfrac{3}{2})t^{-5/2}$$

$$\Longrightarrow \int_{-\infty}^\infty x^{2(2)} e^{-tx^2}dx = \sqrt{\pi}(\tfrac{1}{2})(\tfrac{3}{2})t^{-5/2}$$

$$\cdots \xoverset{\partial/\partial t}{\Longrightarrow} \int_{-\infty}^\infty -x^{2(n)} e^{-tx^2}dx = -\sqrt{\pi}(\tfrac{1}{2})(\tfrac{3}{2})\cdots(\tfrac{2n-1}{2})t^{-(2n+1)/2}$$

$$\Longrightarrow \int_{-\infty}^\infty x^{2(n)} e^{-tx^2}dx = \sqrt{\pi}(\tfrac{1}{2})(\tfrac{3}{2})\cdots(\tfrac{2n-1}{2})t^{-(2n+1)/2}$$

Plugging in $t = 1$, we get:

$$\int_{-\infty}^\infty x^{2n}e^{-x^2}dx = \sqrt{\pi}(\tfrac{1}{2})(\tfrac{3}{2})\cdots(\tfrac{2n-1}{2})$$

$$= \tfrac{\sqrt{\pi}}{2^n}(2n-1)(2n-3)\cdots(3)(1) = \tfrac{\sqrt{\pi}}{2^n}\cdot\tfrac{(2n)!}{2^n n!} = \tfrac{\sqrt{\pi}(2n)!}{4^n n!}$$

Exercise 2.31.a: Derive the following formula by expanding part of the integrand into an infinite series and justifying the term-by term integration:

$$\text{For } a > 0, \int_{-\infty}^\infty e^{-x^2}\cos(ax)dx = \sqrt{\pi}e^{-a^2/4}$$

Note that:

$$\int_{-\infty}^\infty e^{-x^2}\cos(ax)dx = \int_{-\infty}^\infty e^{-x^2}\sum_{i=0}^\infty \tfrac{(-1)^i a^{2i}x^{2i}}{(2i)!}dx = \int_{-\infty}^\infty \sum_{i=0}^\infty \tfrac{a^{2i}(-1)^i}{(2i)!}e^{-x^2}x^{2i}dx$$

Next, consider the sequence of functions: $\left(\tfrac{a^{2i}(-1)^i}{(2i)!}e^{-x^2}x^{2i}\right)_i$:

By exercise 29, we know:

$$\sum_{i=0}^\infty \int_{-\infty}^\infty \left|\tfrac{a^{2i}(-1)^i}{(2i)!}e^{-x^2}x^{2i}\right|dx = \sum_{i=0}^\infty \tfrac{a^{2i}}{(2i)!}\cdot\tfrac{\sqrt{\pi}(2i)!}{4^i i!} = \sqrt{\pi}\sum_{i=0}^\infty \tfrac{a^{2i}}{4^i i!} = \sqrt{\pi}e^{a^2/4} < \infty$$

Therefore:

$$\int_{-\infty}^\infty \sum_{i=0}^\infty \tfrac{a^{2i}(-1)^i}{(2i)!}e^{-x^2}x^{2i}dx = \sum_{i=0}^\infty \tfrac{a^{2i}(-1)^i}{(2i)!}\int_{-\infty}^\infty e^{-x^2}x^{2i}dx$$

$$= \sum_{i=0}^\infty \tfrac{a^{2i}(-1)^i}{4^i i!} = \sum_{i=0}^\infty \tfrac{1}{i!}\left(\tfrac{-a^2}{4}\right)^i = e^{-a^2/4}$$

Exercise 2.32: Suppose $\mu(X) < \infty$. If f and g are complex-valued measurable functions on X, define

$$\rho(f, g) = \int \frac{|f-g|}{1+|f-g|} d\mu$$

Then ρ is a metric on the space of measurable functions if we identify functions are equal a.e., and $f_n \to f$ with respect to this metric iff $f_n \to f$ in measure.

First we show ρ is a metric. To start, a reoccuring relevant fact is that $\frac{|f-g|}{1+|f-g|} \geq 0$. So:

- $\rho(f, g) \geq 0$
- $\rho(f, g) = 0 \iff \frac{|f-g|}{1+|f-g|} = 0$ a.e. $\iff |f - g| = 0$ a.e. $\iff f = g$ a.e.

It's also trivial that $\rho(f, g) = \rho(g, f)$ since $|f - g| = |g - f|$.

Finally, we show the triangle inequality:

Consider any $x, y, z \in \mathbb{C}$.
- If $|x - y| \geq |x - z|$, then we have that $\frac{|x-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-y|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}$.
- If $|y - z| \geq |x - z|$, then we have that $\frac{|x-z|}{1+|x-z|} \leq \frac{|y-z|}{1+|y-z|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}$.
- If both $|x - y| < |x - z|$ and $|y - z| < |x - z|$, then by triangle inequality:
$$\frac{|x-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-z|} + \frac{|y-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}$$

Thus in all cases, we have that $\frac{|x-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}$.

It follows that $\frac{|f-h|}{1+|f-h|} \leq \frac{|f-g|}{1+|f-g|} + \frac{|g-h|}{1+|g-h|}$. And since all these terms are nonnegative:
$$\rho(f, h) = \int \frac{|f-h|}{1+|f-h|} d\mu \leq \int \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} d\mu$$
$$= \int \frac{|x-y|}{1+|x-y|} d\mu + \int \frac{|y-z|}{1+|y-z|} d\mu = \rho(f, g) + \rho(g, h)$$

Having shown ρ is a metric, we now prove the two implications.

($\Longleftarrow$)
Fix $\varepsilon > 0$ and define for some $\gamma > 0$ the sets $E_n = \{x \mid |f_n(x) - f(x)| > \gamma\}$ for all n. Then note that for all n:
$$\int \frac{|f_n-f|}{1+|f_n-f|} d\mu = \int_{E_n} \frac{|f_n-f|}{1+|f_n-f|} d\mu + \int_{E_n^{\complement}} \frac{|f_n-f|}{1+|f_n-f|} d\mu$$
$$\leq \int_{E_n} \frac{|f_n-f|}{1+|f_n-f|} d\mu + \int_{E_n^{\complement}} \frac{\gamma}{1+\gamma} d\mu$$
$$\leq \int_{E_n} 1 d\mu + \int_{E_n^{\complement}} \frac{\gamma}{1+\gamma} d\mu = \mu(E_n) + \mu(E_n^{\complement})\frac{\gamma}{1+\gamma}$$
$$\leq \mu(E_n) + \mu(X)\frac{\gamma}{1+\gamma}$$

Note that if $\gamma < \frac{\varepsilon}{1-\varepsilon}$ (assuming that $\varepsilon < 1$), then $\frac{\gamma}{1+\gamma} < \varepsilon$. So by fixing γ sufficiently small (and because $\mu(X)$ is finite), we can force $\mu(X)\frac{\gamma}{1+\gamma} < \mu(X)\varepsilon$. Then, because $f_n \to f$ in measure, we know that $\mu(E_n) \to 0$ as $n \to \infty$. So, we can pick N such that $\forall n \geq N$, $\mu(E_n) < \varepsilon$.

Therefore, for all $n \geq N$, we have that:
$$\int \frac{|f_n - f|}{1 + |f_n - f|} d\mu \leq \varepsilon(1 + \mu(X))$$

$1 + \mu(X)$ is a finite constant and ε was arbitrary. So, we've shown that $\int \frac{|f_n - f|}{1 + |f_n - f|} d\mu \to 0$ as $n \to \infty$.

($\Longrightarrow$)
Fix $\varepsilon > 0$ and then for all n define $E_n = \{x \mid |f_n(x) - f(x)| > \varepsilon\}$. Then note that:
$$\int \frac{|f_n - f|}{1 + |f_n - f|} d\mu = \int_{E_n} \frac{|f_n - f|}{1 + |f_n - f|} d\mu + \int_{E_n^{\complement}} \frac{|f_n - f|}{1 + |f_n - f|} d\mu$$
$$\geq \int_{E_n} \frac{|f_n - f|}{1 + |f_n - f|} d\mu \geq \int_{E_n} \frac{\varepsilon}{1 + \varepsilon} = \mu(E_n) \frac{\varepsilon}{1 + \varepsilon}$$

Hence, we have that $0 \leq \mu(E_n) \leq \frac{1 + \varepsilon}{\varepsilon} \int \frac{|f_n - f|}{1 + |f_n - f|} d\mu$.

And now we are done because by our hypothesis, $\frac{1 + \varepsilon}{\varepsilon} \int \frac{|f_n - f|}{1 + |f_n - f|} d\mu \to 0$ as $n \to \infty$. Therefore, so must $\mu(E_n)$.

Side note: only the backward direction required the fact that $\mu(X)$ is finite.

Exercise 2.34: Suppoose $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure.

(a) $\int f = \lim_{n \to \infty} \int f_n$.

A hopefully clear fact is that $f_n \to f$ in measure only if (f_n) is Cauchy in measure.

To prove this, fix any $\varepsilon > 0$ and note that:
$$\{x \mid |f_n - f_m| \geq \varepsilon\} \subseteq \{x \mid |f_n - f| \geq \varepsilon/2\} \cup \{x \mid |f_m - f| \geq \varepsilon/2\}$$

Since $f_n \to f$ in measure, for all $\delta > 0$ there exists N such that when $n, m \geq N$, then:
$$\mu(\{x \mid |f_n - f| \geq \varepsilon/2\} \cup \{x \mid |f_m - f| \geq \varepsilon/2\})$$
$$\leq \mu(\{x \mid |f_n - f| \geq \varepsilon/2\}) + \mu(\{x \mid |f_m - f| \geq \varepsilon/2\}) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

Another relevant fact is that if $f_n \to f$ in measure, then we must also have that that $f_{n_j} \to f$ in measure for any subsequence (f_{n_j}) of (f_n). As a result, we know that given any subsequence (f_{n_j}) of f_n, it must be Cauchy and thus it must have subsequence $(f_{n_{j_k}})$ such that there exists a function g such that $f_{n_{j_k}} \to h$ pointwise and $h = f$ a.e.. Importantly, by D.C.T we then know that $f \in L^1$ and $\int f_{n_{j_k}} \to \int h = \int f$.

Now consider the sequence $(\int f_n)_{n \in \mathbb{N}}$. Then consider two subsequences $(\int f_{n_j})_{j \in \mathbb{N}}$ and $(\int f_{n_J})_{J \in \mathbb{N}}$ such that $\int f_{n_j} \to \liminf \int f_n$ and $\int f_{n_J} \to \limsup \int f_n$.

By prior reasoning, we know there exists a subsequence of $(f_{n_{j_k}})$ such that $\int f_{n_{j_k}} \to \int f$. But at the same time, since $\int f_{n_{j_k}}$ is a subsequence of an already convergent sequence, we know that $\int f_{n_{j_k}} \to \lim_{j \to \infty} \int f_{n_j} = \liminf \int f_n$. So, we must have that $\int f = \liminf \int f_n$.

By analogous reasoning to $(\int f_{n_J})$, we can show that $\int f = \limsup \int f_n$. So, $\liminf \int f_n = \int f = \limsup \int f_n \implies \lim_{n \to \infty} \int f_n = \int f$.

(b) $f_n \to f$ in L^1.

Note that $|f_n - f| \le |f_n| + |f| \le g + |f|$. In the previous part, we showed that $f \in L^1$. Also g is assumed to be nonnegative and in L^1. So

$$\int (g + |f|) = \int g + \int |f| < \infty.$$

Now recall that given any subsequence (f_{n_j}) of (f_n) we can find a sub-subsequence $(f_{n_{j_k}})$ such that $f_{n_{j_k}}$ converges to a function h pointwise a.e and $h = f$ a.e. In simpler terms we can just say that $f_{n_{j_k}} \to f$ pointwise a.e.. But then we have that $|f_{n_{j_k}} - f| \to 0$ pointwise a.e., and by applying D.C.T (which we can do because of the previous paragraph), we know that $\int |f_{n_{j_k}} - f| \to \int 0 = 0$.

Finally, consider the sequence $(\int |f_n - f|)_n$ and let $(\int |f_{n_j} - f|)_j$ and $(\int |f_{n_J} - f|)_J$ be subsequences converging to $\liminf \int |f_n - f|$ and $\limsup \int |f_n - f|$ respectively.

From before, we know there exists sub-subsequences $(\int |f_{n_{j_k}} - f|)_k$ and $(\int |f_{n_{J_K}} - f|)_K$ which both converge to 0. But those subsequences must also converge to $\liminf \int |f_n - f|$ and $\limsup \int |f_n - f|$. So:

$$\liminf \int |f_n - f| = 0 = \limsup \int |f_n - f| \implies \lim_{n \to \infty} \int |f_n - f| = 0$$

Exercise 2.38: Suppose $f_n \to f$ in measure and $g_n \to g$ in measure.

(a) $f_n + g_n \to f + g$ in measure.

Fix $\varepsilon > 0$ and note that:
$$\{x \mid |f_n + g_n - f - g| \ge \varepsilon\} \subseteq \{x \mid |f_n - f| + |g_n - g| \ge \varepsilon\}$$
$$\subseteq \{x \mid |f_n - f| \ge \varepsilon/2\} \cup \{x \mid |g_n - g| \ge \varepsilon/2\}$$

Therefore, we have that:
$$\mu(\{x \mid |f_n + g_n - f - g| \ge \varepsilon\}) \le \mu(\{x \mid |f_n - f| \ge \varepsilon/2\} \cup \{x \mid |g_n - g| \ge \varepsilon/2\})$$
$$\le \mu(\{x \mid |f_n - f| \ge \varepsilon/2\}) + \mu(\{x \mid |g_n - g| \ge \varepsilon/2\})$$

Since $f_n \to f$ in measure and $g_n \to g$ in measure, the two terms on the right go to 0 as $n \to \infty$. Thus, we also have that $\mu(\{x \mid |f_n + g_n - f - g| > \varepsilon\}) \to 0$ as $n \to \infty$.

(b) $f_n g_n \to fg$ in measure if $\mu(X) < \infty$ but not necessarily if $\mu(X) = \infty$.

Note that:
$$\begin{aligned}
|f_n g_n - fg| &= |f_n g_n + f_n g + fg_n + fg - f_n g - fg_n - fg - fg| \\
&\leq |f_n g_n + fg - f_n g - fg_n| + |f_n g - fg| + |fg_n - fg| \\
&\leq |f_n(g_n - g) - f(g_n - g)| + |g(f_n - f)| + |f(g_n - g)| \\
&\leq |f - f_n||g - g_n| + |g||f_n - f| + |f||g_n - g|
\end{aligned}$$

Thus, fixing $\varepsilon > 0$ we have that:
$$\begin{aligned}
\mu(\{x : |f_n g_n - fg| > \varepsilon\}) &\leq \mu(\{x : |f - f_n||g - g_n| \geq \varepsilon/3\}) \\
&\quad + \mu(\{x : |g||f_n - f| \geq \varepsilon/3\}) + \mu(\{x : |f||g_n - g| \geq \varepsilon/3\})
\end{aligned}$$

Now, define $E_n := \{|f| > n\}$ and note that E_n is a decreasing sequence of sets. Also $\mu(E_1) < \mu(X) < \infty$ and $\bigcap_{n \geq 1} E_n = \emptyset$ It follows that $\lim_{n \to \infty} \mu(E_n) = \mu(\bigcap_{n \geq 1} E_n) = 0$.

Doing the same reasoning with $F_n := \{|g| > n\}$, we get that $\mu(F_n) \to 0$ as $n \to \infty$. So given any $\delta > 0$, there exists integers M_1 and M_2 such that for all $n \geq M_1$ and $m \geq M_2$, $\mu(E_n) < \delta$ and $\mu(F_n) < \delta$. Set $N_1 = \max(M_1, M_2)$. Thus $\mu(E_{N_1}) < \delta$, $\mu(F_{N_2}) < \delta$, and outside E_{N_1} and F_{N_1} we have that $|f| \leq N_1$ and $|g| \leq N_1$ respectively.

In rapid succession, use the fact that $f_n \to f$ and $g_n \to g$ in measure in order to find integers M_3, M_4, M_5, M_6 such that:

- $n \geq M_3 \implies \mu(\{x : |f_n - f| \geq \frac{\varepsilon}{3N_1}\}) < \delta$
- $n \geq M_4 \implies \mu(\{x : |g_n - g| \geq \frac{\varepsilon}{3N_1}\}) < \delta$
- $n \geq M_5 \implies \mu(\{x : |f_n - f| \geq \sqrt{\varepsilon/3}\}) < \delta$
- $n \geq M_6 \implies \mu(\{x : |g_n - g| \geq \sqrt{\varepsilon/3}\}) < \delta$

Take $N_2 = \max(M_3, M_4, M_5, M_6)$. Then finally, note that:

- $\begin{aligned}[t]
\mu(\{x : |f - f_n||g - g_n| \geq \varepsilon/3\}) &\leq \mu(\{x : |f - f_n| \geq \sqrt{\varepsilon/3}\} \cup \{x : |g - g_n| \geq \sqrt{\varepsilon/3}\}) \\
&\leq \mu(\{x : |f - f_n| \geq \sqrt{\varepsilon/3}\}) + \mu(\{x : |g - g_n| \geq \sqrt{\varepsilon/3}\}) \\
&< \delta + \delta = 2\delta \text{ when } n \geq N_2.
\end{aligned}$

- $\begin{aligned}[t]
\mu(\{x : |g||f - f_n| \geq \varepsilon/3\}) &\leq \mu(\{x : |g| \geq N_1\} \cup \{x : |f - f_n| \geq \frac{\varepsilon}{3N_1}\}) \\
&\leq \mu(\{x : |g| \geq N_1\}) + \mu(\{x : |f - f_n| \geq \frac{\varepsilon}{3N_1}\}) \\
&< \delta + \delta = 2\delta \text{ when } n \geq N_2.
\end{aligned}$

- $\mu(\{x : |f||g - g_n| \geq \varepsilon/3\}) \leq \mu(\{x : |f| \geq N_1\} \cup \{x : |g - g_n| \geq \frac{\varepsilon}{3N_1}\})$
$$\leq \mu(\{x : |f| \geq N_1\}) + \mu(\{x : |g - g_n| \geq \frac{\varepsilon}{3N_1}\})$$
$$< \delta + \delta = 2\delta \text{ when } n \geq N_2.$$

So for $n \geq N_2$, we have that $\mu(\{x : |f_n g_n - fg| > \varepsilon\}) < 6\delta$. And since δ is arbitrary, we thus know that $\mu(\{x : |f_n g_n - fg| > \varepsilon\}) \to 0$ as $n \to \infty$.

Now in our work above we assumed $\mu(X) < \infty$. To see that this isn't necessarily true if $\mu(X) = \infty$, let us consider the real Lesbesgue measure space restricted to $(0, \infty)$.

Define $f_n(x) = x + \frac{1}{n}\chi_{(n,n+1)} = g_n(X)$. Then clearly $f_n \to x$ and $g_n \to x$ in measure. However, $f_n g_n = x^2 + \frac{2x}{n}\chi_{(n,n+1)} + \frac{1}{n^2}\chi_{n,n+1}$.

For all $x \in (n, n + 1)$, we have that
$$x^2 + \tfrac{2x}{n}\chi_{(n,n+1)} + \tfrac{1}{n^2}\chi_{n,n+1} - x^2 \geq \tfrac{2x}{n}\chi_{(n,n+1)} > 2.$$

So $\mu(|f_n g_n - x^2| \geq 2) > \mu((n, n + 1)) = 1$ for all $n \in \mathbb{N}$.

Exercise 2.51: Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be arbitrary measure spaces (not necessarily σ-finite).

(a) If $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable, $g : Y \to \mathbb{C}$ is $\mathcal{N}$-measurable, and $h(x, y) = f(x)g(y)$, then h is $\mathcal{M} \otimes \mathcal{N}$-measurable.

Define $F(x, y) = f(x)$ and $G(x, y) = g(y)$. Then given any set $B \in \mathcal{B}_{\mathbb{C}}$, we have that $F^{-1}(B) = f^{-1}(B) \times Y$ and $G^{-1}(B) = X \times g^{-1}(B)$. And since both f and g are measurable, we know that $f^{-1}(B) \in \mathcal{M}$ and $g^{-1}(B) \in \mathcal{N}$. So, $F^{-1}(B)$ and $G^{-1}(B)$ are both rectangles and thus in $\mathcal{M} \otimes \mathcal{N}$.

Next, note that $h(x, y) = F(x, y)G(x, y)$. Thus, h is the product of two $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}_{\mathbb{C}})$-measurable functions and thus itself also a $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}_{\mathbb{C}})$-measurable function.

(b) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and $\int h \, d(\mu \times \nu) = [\int f \, d\mu][\int g \, d\nu]$.

To start, if f and g are nonnegative simple functions, then this is true. To see this, write $\phi_n = \sum_{i=1}^{n} a_i \chi_{E_i}$ and $\varphi_n = \sum_{j=1}^{m} b_j \chi_{F_j}$ where all a_i, b_j are complex numbers, all $E_i \subseteq X$, and all $F_i \subseteq Y$. Then:
$$fg = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \chi_{E_i}(x)\chi_{F_j}(y) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \chi_{E_i \times F_j}((x, y))$$

So, $\int fg d(\mu \times \nu) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \mu \times \nu(E_i \times F_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \mu(E_i) \nu(F_j)$.

But at the same time,

$$(\int f d\mu)(\int g d\nu) = (\sum_{i=1}^{n} a_i \mu(E_i))(\sum_{j=1}^{m} b_i \nu(F_j)) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \mu(E_i) \nu(F_j).$$

To extend this to nonnegative real-valued functions, let (ϕ_n) and (φ_n) be increasing sequences of nonnegative simple functions converging pointwise to f and g respectively. Then by M.C.T., we know that $\int \phi_n d\mu \to \int f d\mu$ and $\int \varphi_n d\nu \to \int g d\nu$ as $n \to \infty$.

Next, note that $\phi_n \varphi_n \to fg$ pointwise as $n \to \infty$. Also, each $\phi_n \varphi_n$ is a simple function as is demontrated on the previous line, and $\phi_n(x) \varphi_n(y)$ monotonically increases as $n \to \infty$. So by applying M.C.T., we know that:

$$\int \phi_n \varphi_n d(\mu \times \nu) \to \int fg d(\mu \times \nu) = \int h d(\mu \times \nu)$$

Thus, we can say that:

$$\int f d\mu \int g d\nu = \lim_{n \to \infty} (\int \phi_n d\mu \int \varphi_n d\nu) = \lim_{n \to \infty} (\int \phi_n \varphi_n d(\mu \times \nu)) = \int h d(\mu \times \nu).$$

Now consider if f, g are real-valued functions. Then $|h(x,y)| = |f(x)||g(y)|$ and so from before: $\int |h| d(\mu \times \nu) = (\int |f| d\mu)(\int |g| d\nu)$. If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then the right side of that expression is finite. So $\int |h| < \infty$ and thus $h \in L^1(\mu \times \nu)$.

Assuming f, g are integrable, then note that $h^+ = f^+ g^+ + f^- g^-$ and $h^- = f^+ g^- + f^- g^+$. So applying our prior reasoning to those parts:

$$\int h d(\mu \times \nu) = \int h^+ d(\mu \times \nu) - \int h^- d(\mu \times \nu)$$
$$= \int f^+ d\mu \int g^+ d\nu + \int f^- d\mu \int g^- d\nu - \int f^+ d\mu \int g^- d\nu - \int f^- d\mu \int g^+ d\nu$$
$$= \int f^+ d\mu(\int g^+ d\nu - \int g^- d\nu) - \int f^- d\mu(\int g^+ d\nu - \int g^- d\nu)$$
$$= (\int f^+ d\mu - \int f^- d\mu)(\int g^+ d\nu - \int g^- d\nu) = (\int f d\mu)(\int g d\nu)$$

Finally, we consider when f, g are complex valued functions. Like before, we have that $|h(x,y)| = |f(x)||g(y)|$ and so by the same reasoning as when f, g were real valued, if $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $\int |h| d(\mu \times \nu) < \infty$ and thus $h \in L^1(\mu \times \nu)$.

Finally, supposing that f, g are integrable one last time, note that:

$$\mathrm{Re}(h) = \mathrm{Re}(f) \mathrm{Re}(g) - \mathrm{Im}(f) \mathrm{Im}(g) \text{ and } \mathrm{Im}(h) = \mathrm{Re}(f) \mathrm{Im}(g) + \mathrm{Im}(f) \mathrm{Re}(g).$$

By doing a bunch of similar manipulations to the real-valued case, you can then show that:

$$\int \mathrm{Re}(h)d(\mu \times \nu) + i \int \mathrm{Im}(h)d(\mu \times \nu) =$$
$$(\int \mathrm{Re}(f)d\mu + i \int \mathrm{Im}(f)d\mu)(\int \mathrm{Re}(g)d\mu + i \int \mathrm{Im}(g)d\mu).$$

But I'm out of time so do them yourself :P.

Exercise 2.55: Let $E = [0,1] \times [0,1]$. Investigate the existence and equality of $\int_E f dm^2$, $\int_0^1 \int_0^1 f(x,y)dxdy$, and $\int_0^1 \int_0^1 f(x,y)dydx$ for the following f.

(a) $f(x,y) = (x^2 - y^2)(x^2 + y^2)^{-2}$

Note that:

$$\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}dx = \int_0^{\tan^{-1}(1/y)} \frac{y^3(\tan^2(\theta)-1)\sec^2(\theta)}{(y^2(\tan^2(\theta)+1))^2}d\theta \qquad \text{(substituting } x = y\tan(\theta))$$

$$= \frac{1}{y}\int_0^{\tan^{-1}(1/y)} \frac{(\tan^2(\theta)-1)\sec^2(\theta)}{\sec^4(\theta)}d\theta$$

$$= \frac{1}{y}\int_0^{\tan^{-1}(1/y)} \frac{\tan^2(\theta)-1}{\sec^2(\theta)}d\theta$$

$$= \frac{1}{y}\int_0^{\tan^{-1}(1/y)} \sin^2(\theta) - \cos^2(\theta)d\theta$$

$$= \frac{1}{y}\int_0^{\tan^{-1}(1/y)} -\cos(2\theta)d\theta$$

$$= \frac{-1}{2y}(\sin(2\tan^{-1}(1/y)) - 0)$$

$$= \frac{-1}{y}\sin(\tan^{-1}(1/y))\cos(\tan^{-1}(1/y)) \qquad \text{(getting rid of the 2)}$$

$$= \frac{-1}{y}\tan(\tan^{-1}(1/y))\cos^2(\tan^{-1}(1/y))$$

$$= \frac{-1}{y^2}\frac{1}{1+\tan^2(\tan^{-1}(1/y))} \qquad (\cos^2 = \frac{1}{\sec^2} = \frac{1}{1+\tan^2})$$

$$= \frac{-1}{y^2} \cdot \frac{1}{1+y^{-2}} = \frac{-1}{y^2+1}$$

Luckily we don't need to do that again because:

$$\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dy = -\int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}dy = \frac{1}{x^2+1}.$$

Next note that $\int_0^1 \frac{-1}{y^2+1}dy = -\tan^{-1}(1) + \tan^{-1}(0) = -\pi/4$. In turn this means that $\int_0^1 \frac{1}{x^2+1}dx = \pi/4$.

So, we've shown that $\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dydx \neq \int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dxdy$

The reason why is that $\frac{x^2-y^2}{(x^2+y^2)^2} \notin L^1(m^2)$.

To see why, note that:

$$\int_0^1 \left| \frac{x^2-y^2}{(x^2+y^2)^2} \right| dx = -\int_0^y \frac{x^2-y^2}{(x^2+y^2)^2} dx + \int_y^1 \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= -\frac{-1}{2y} \sin(2\tan^{-1}(1)) + \frac{-1}{2y} \sin(2\tan^{-1}(1/y)) - \frac{-1}{2y} \sin(2\tan^{-1}(1))$$

$$= \frac{1}{y} - \frac{1}{1+y^2}$$

But then $\int_0^1 \frac{1}{y} - \frac{1}{1+y^2} dy = \ln(1) - \ln(0) - \tan^{-1}(1) + \tan^{-1}(0) = \infty$.

Assuming $\left| \frac{x^2-y^2}{(x^2+y^2)^2} \right| \in L^+$, we know by Tonelli-theorem that:

$$\int_E \left| \frac{x^2-y^2}{(x^2+y^2)^2} \right| dm^2 = \int_0^1 \int_0^1 \left| \frac{x^2-y^2}{(x^2+y^2)^2} \right| dxdy.$$

So, our function is not integrable with respect to m^2.

(b) $f(x,y) = (1-xy)^{-a}$ with $a > 0$.

$(1-xy)^{-a}$ is nonnegative for all $x,y \in [0,1]$ and continuous almost every-where, we know $(1-xy)^{-a} \in L^+(E)$. By Tonelli's theorem, we thus know that the following three integrals exist and:

$$\int (1-xy)^{-a} dm^2 = \int_0^1 \int_0^1 (1-xy)^{-a} dxdy = \int_0^1 \int_0^1 (1-xy)^{-a} dydx$$

That said, it could be that they are all infinite.

(c) $f(x,y) = (x-\frac{1}{2})^{-3}$ if $0 < y < |x-\frac{1}{2}|$ and $f(x,y) = 0$ otherwise.

Note that when $y \leq \frac{1}{2}$, then:

$$\int_0^1 f(x,y)dx = \int_0^{\frac{1}{2}-y} (x-\tfrac{1}{2})^{-3} dx + \int_{\frac{1}{2}+y}^1 (x-\tfrac{1}{2})^{-3} dx$$

$$= \frac{1}{-2}(x-\tfrac{1}{2})^{-2} \big|_0^{1/2-y} + \frac{1}{-2}(x-\tfrac{1}{2})^{-2} \big|_{y+1/2}^1 = \frac{-1}{2y^2} + 2 - 2 - \frac{-1}{2y^2} = 0$$

Also, when $y \geq \frac{1}{2}$, then we trivially have that $\int_0^1 f(x,y)dx = \int_0^1 0dx = 0$. So $\int_0^1 \int_0^1 f(x,y)dxdy = 0$.

On the other hand, note that $\int_0^1 f(x,y)dy = \int_0^{|x-1/2|} (x-\tfrac{1}{2})^{-3}dy = \frac{|x-\frac{1}{2}|}{(x-\frac{1}{2})^3}$.

I didn't have time to do the calculations but plugging it into an online calculator:
$\int_0^1 \frac{|x-\frac{1}{2}|}{(x-\frac{1}{2})^3}dx = \infty$
So $\int_0^1 \int_0^1 f(x,y)dxdy \neq \int_0^1 \int_0^1 f(x,y)dydx$ and $f(x,y) \notin L^1(m^2)$

Exercise 2.59: Let $f(x) = x^{-1}\sin(x)$.

(a) Show that $\int_0^\infty |f(x)|dx = \infty$

(b) Show that $\lim_{b \to \infty} \int_0^b f(x)dx = \frac{1}{2}\pi$ by integrating $e^{-xy}\sin(x)$ with respect to x and y.

Homework 5

Exercise 3.8: Given a signed measure ν and positive measure μ on the measurable space $(X, \mathcal{M})$, we have that: $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Let ν^+ and ν^- be the positive and negative variations of ν. Also let P and N be a Hahn decomposition for ν. Finally, let $E \in \mathcal{M}$ be such that $\mu(E) = 0$.

Suppose $\nu \ll \mu$. Thus $\nu(E) = 0$. Then consider if $\nu^+(E) \neq 0$. Since $\nu(E) = \nu^+(E) - \nu^-(E)$ and either ν^+ or ν^- is finite, we have that $\nu^+(E) = M = \nu^-(E)$ where $0 < M < \infty$. But $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = \nu(E \cap N)$. So $\nu(E \cap P) = m \neq 0$ and $\nu(E \cap N) = m \neq 0$. But at the same time, $\mu(E \cap P) = \mu(E \cap N) = 0$ since $E \cap P \subseteq E$ and $E \cap N \subseteq N$. This contradicts that $\nu \ll \mu$. So, $\nu \ll \mu \implies \nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Next, suppose $\mu(E) = 0 \implies |\nu|(E) = 0$. Then since $\nu^+(E) \geq 0$, $\nu^-(E) \geq 0$, and $|\nu|(E) = \nu^+(E) + \nu^-(E)$, we must have that $\nu^+(E) = 0\nu^-(E)$. So $|\nu|(E) \ll \mu \implies \nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Finally, it's trivial that $\nu^+ \ll \mu$ and $\nu^- \ll \mu \implies |\nu| \ll \mu$ because:
$$\mu(E) \implies \nu^+(E) = 0 = \nu^-(E) \implies |\nu|(E) = \nu^+(E) + \nu^-(E) = 0 + 0.$$

Similarly, $\nu^+ \ll \mu$ and $\nu^- \ll \mu \implies \nu \ll \mu$ because:
$$\mu(E) \implies \nu^+(E) = 0 = \nu^-(E) \implies \nu(E) = \nu^+(E) - \nu^-(E) = 0 - 0.$$

Exercise 3.9: Let $(X, \mathcal{M})$ be a measurable space and suppose μ is a signed measure and $(\nu_j)_{j \in \mathbb{N}}$ is a sequence of of positive measures.

If $\nu_j \perp \mu$ for all j, then $\sum_{j=1}^{\infty} \nu_j \perp \mu$, and if $\nu_j \ll \mu$ for all j, then $\sum_{j=1}^{\infty} v_j \ll \mu$.

First let's consider if $\nu_j \perp \mu$ for all j.

For each $j \in \mathbb{N}$, let E_j and F_j be such that $E_j \cup F_j = X$, $E_j \cap F_j = \emptyset$, $\nu_j(F_j) = 0$, and E_j is a μ-null set.

Define $E := \bigcup_{j \in \mathbb{N}} E_j$. Then E is a μ-null set.

Also, $F := X - E = X - \bigcup_{j \in \mathbb{N}} E_j = \bigcap_{j \in \mathbb{N}} X - E_j = \bigcap_{j \in \mathbb{N}} F_j$.

Importantly, $E \cap F = \emptyset$, $E \cup F = X$ and $F \subseteq F_j$ for all $j \in \mathbb{N}$. Thus $\nu_j(F) = 0$ for all $j \in \mathbb{N}$, and so $\sum_{j=1}^{\infty} \nu_j(F) = 0$. We've thus shown that $\sum_{j=1}^{\infty} \nu_j \perp \mu$.

Next, let's consider if $\nu_j \ll \mu$ for all j.

Then $\mu(E) = 0 \implies \nu_j(E) = 0$ for all $j \in \mathbb{N} \implies \sum_{j=1}^{\infty} \nu_j(E) = 0$. So $\sum_{j=1}^{\infty} \nu_j \ll \mu$.

Exercise 3.3: Let ν be a signed measure on $(X, \mathcal{M})$. Show the following:

(a) $L^1(\nu) = L^1(|\nu|)$

Let ν^+ and ν^- be the positive and negative variations of ν. Recall that $f \in L^1(\nu)$ iff $\int f d\nu = \int f d\nu^+ - \int f d\nu^-$ exists and is finite. The latter statement is true if and only if $f \in L^1(\nu^+) \cap L^1(\nu^-)$.

Meanwhile $|\nu| = \nu^+ + \nu^-$. As a result, when $f = \chi_E$, then:
$$\int f d|\nu| = |\nu|(E) = \nu^+(E) + \nu^-(E) = \int f d\nu^+ + \int f d\nu^-.$$

By linearity, we thus conclude that $\int f d|\nu| = \int f d\nu^+ + \int f d\nu^-$ when f is a simple function. And by considering increasing sequences of simple functions and using the M.C.T, we can conclude that $\int f d|\nu| = \int f d\nu^+ + \int f d\nu^-$ holds for all nonnegative f.

Now suppose f is a real-valued function. If all the terms below are finite, then we have that:
$$\int f d|\nu| = \int f^+ d|\nu| - \int f^- d|\nu| = \left(\int f^+ d\nu^+ + \int f^+ d\nu^-\right) - \left(\int f^- d\nu^+ + \int f^- d\nu^-\right)$$
$$= \left(\int f^+ d\nu^+ - \int f^- d\nu^+\right) + \left(\int f^+ d\nu^- - \int f^- d\nu^-\right)$$
$$= \int f d\nu^+ + \int f d\nu^-$$

Importantly, $\int f^+ d|\nu| < \infty$ if and only if $\int f^+ d\nu^+ < \infty$ and $\int f^+ d\nu^- < \infty$. Similarly, $\int f^- d|\nu| < \infty$ if and only if $\int f^- d\nu^+ < \infty$ and $\int f^- d\nu^- < \infty$. Thus:
$$f \in L^1(|\nu|) \iff \int f^+ d|\nu| < \infty \text{ and } \int f^- d|\nu| < \infty$$
$$\iff \int f^+ d\nu^+ < \infty, \int f^+ d\nu^- < \infty, \int f^- d\nu^+ < \infty, \text{ and } \int f^- d\nu^- < \infty$$
$$\iff f \in L^1(\nu^+) \text{ and } f \in L^1(\nu^-) \iff f \in L^1(\nu^+) \cap L^1(\nu^-).$$

Finally, note that if f is complex-valued, then assuming none of the integrals below are infinite:
$$\int f d|\nu| = \int \mathrm{Re}(f) d|\nu| + i \int \mathrm{Im}(f) d|\nu|$$
$$= \left(\int \mathrm{Re}(f) d\nu^+ + \int \mathrm{Re}(f) d\nu^-\right) + i\left(\int \mathrm{Im}(f) d\nu^+ + \int \mathrm{Im}(f) d\nu^-\right)$$
$$= \left(\int \mathrm{Re}(f) d\nu^+ + i \int \mathrm{Im}(f) d\nu^+\right) + \left(\int \mathrm{Re}(f) d\nu^- + i \int \mathrm{Im}(f) d\nu^-\right)$$
$$= \int f d\nu^+ + \int f d\nu^-$$

Importantly:
$$f \in L^1(|\nu|) \iff \int \mathrm{Re}(f) d|\nu| < \infty \text{ and } \int \mathrm{Im}(f) d|\nu| < \infty$$
$$\iff \int \mathrm{Re}(f) d\nu^+ < \infty, \int \mathrm{Re}(f) d\nu^- < \infty, \int \mathrm{Im}(f) d\nu^+ < \infty,$$
$$\text{and } \int \mathrm{Im}(f) d\nu^- < \infty$$
$$\iff f \in L^1(\nu^+) \text{ and } f \in L^1(\nu^-) \iff f \in L^1(\nu^+) \cap L^1(\nu^-).$$

So $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-) = L^1(\nu)$ and if $f \in L^1(|\nu|)$, then $\int f d|\nu| = \int f d\nu^+ + \int f d\nu^-$.

(b) If $f \in L^1(\nu)$, then $|\int f d\nu| \leq \int |f| d|\nu|$.

If $f \in L^1(\nu)$, then from before we know $f \in L^1(|\nu|)$. So:
$$|\textstyle\int f d\nu| = |\int f d\nu^+ - \int f d\nu^-| \leq |\int f d\nu^+ - 0| + |\int f d\nu^- - 0|$$
$$= \textstyle\int f d\nu^+ + \int f d\nu^- = \int f d|\nu|$$

Next, from our theorems on integrals using positive measure, we know that $|f| \in L^1(|\nu|)$ and $\left|\int f d|\nu|\right| \leq \int |f| d|\nu|$. So:
$$|\textstyle\int f d\nu| \leq \int f d|\nu| \leq \left|\int f d|\nu|\right| \leq \int |f| d|\nu|$$

(c) If $E \in \mathcal{M}$, $|\nu|(E) = \sup\{\int_E f d\nu : |f| \leq 1\}$.

Let P and N give a Hahn decomposition for ν on X with P being positive and N being negative. Then define $f = \chi_P - \chi_N$. Importantly, $|f| \leq 1$ and $\int_E f d\nu = \nu(E \cap P) - \nu(E \cap N) = \nu^+(E) + \nu^-(E) = |\nu|(E)$.

Hence, $|\nu|(E) \leq \sup\{\int_E f d\nu : |f| \leq 1\}$

On the other hand, if $|f| \leq 0$, then by part (b) we know:
$$\textstyle\int_E f d\nu \leq |\int_E f d\nu| \leq \int_E |f| d|\nu| \leq \int_E 1 \cdot d|\nu| = |\nu|(E)$$

So $\sup\{\int_E f d\nu : |f| \leq 1\} \leq |\nu|(E)$.

Exercise 3.6: Suppose $\nu(E) = \int_E f d\mu$ where μ is a positive measure on $(X, \mathcal{M})$ and f is an extended μ-integrable function. Describe the Hahn decompositions of ν and the positive, negative, and total variations of ν in terms of f and ν.

Recall that $\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$. Thus, defining $\nu^+(E) = \int_E f^+ d\mu$ and $\nu^-(E) = \int_E f^- d\mu$, we have that ν^+ and ν^- are positive measures such that $\nu = \nu^+ - \nu^-$.

Additionally, let $P = f^{-1}([0, \infty])$ and $N = f^{-1}([-\infty, 0))$. Since f is measurable, we know both P and N are in $\mathcal{M}$. Also, $P \cap N = \emptyset$ and $P \cup N = \text{domain}(f) = X$. And thirdly, $f^+ = 0$ on N and $f^- = 0$ on P. Thus, if $E \subseteq N$, then $\nu^+(E) = \int_E f^+ d\mu = \int 0 d\mu = 0$. Similarly, if $E \subseteq P$, then $\nu^-(E) = \int_E f^- d\mu = \int 0 d\mu = 0$.

Hence, we've shown that $\nu^+(E) = \int_E f^+ d\mu$ and $\nu^-(E) = \int_E f^- d\mu$ are positive measures such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. By the Jordan decomposition theorem, we thus know that ν^+ and ν^- give the unique Jordan decomposition of ν. In other words, the positive variation of ν is given by $\int_E f^+ d\mu$ and the negative variation of ν is given by $\int_E f^- d\mu$.

In turn, the total variation of ν is $\int_E f^+ d\mu + \int_E f^- d\mu = \int_E |f| d\mu$

Now we turn to describing the Hahn decompositions for ν.

Claim: E is positive iff $\mu(E \cap f^{-1}([-\infty, 0))) = 0$.

Proof:

Suppose $\mu(E \cap f^{-1}([-\infty, 0))) = 0$. Then given any $F \in \mathcal{M}$ with $F \subseteq E$, we have that $\int_F f d\mu = \int_F f^+ d\mu - \int_F f^- d\mu \geq 0 - \int_E f^- d\mu = 0$ since $f^- = 0$ a.e. on E. So $\nu(F) \geq 0$. Since F was arbitrary, we've shown E is positive.

Conversely, suppose $\mu(E \cap f^{-1}([-\infty, 0))) > 0$. Then because f is measurable, we can fix $F = E \cap f^{-1}([-\infty, 0))$ and have that $F \in \mathcal{M}$ and $F \subseteq E$. But now note that $\nu(F) = \int_F f d\mu = 0 - \int_F f^{-1} d\mu$. And since $f^- > 0$ on all of F (with F not being null), we do not have that $f^{-1} = 0$ a.e. on F. That combined with the fact that $f^{-1} \in L^+$ means that $\int_F f^{-1} d\mu > 0$ and thus $\nu(F) < 0$. So, E isn't positive.

Analogously, we can show that E is negative iff $\mu(E \cap f^{-1}((0, \infty])) = 0$.

So two subsets P and N of X will be a Hahn decomposition for ν if and only if $P = A_1 \cup f^{-1}((0, \infty]) \cup B_2 - B_1$ and $N = A_2 \cup f^{-1}([-\infty, 0)) \cup B_1 - B_2$ where B_1 and B_2 are null-subsets of $f^{-1}((0, \infty])$ and $f^{-1}([-\infty, 0))$ respectively and A_1 and A_2 are disjoint sets in $\mathcal{M}$ partitioning $f^{-1}(\{0\})$.

Recall that earlier in this problem, we defined $P = f^{-1}([0, \infty])$ and $N = f^{-1}([-\infty, 0))$. We know from before that $P \cap N = \emptyset$ and $P \cup N = X$. Also, we know that P is positive and N is negative due to our claims right above. So, $f^{-1}([0, \infty])$ and $f^{-1}([-\infty, 0))$ give a Hahn decomposition for ν.

Exercise 2.60: Show that $\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1} dt$ for $x, y > 0$.

As a consequence of exercise 2.51, we can rewrite:
$$\Gamma(x)\Gamma(y) = \int_0^\infty s^{x-1} e^{-s} ds \int_0^\infty s^{y-1} e^{-s} ds = \int_0^\infty \int_0^\infty t^{x-1} e^{-t} s^{y-1} e^{-s} ds dt$$
$$= \int_0^\infty \int_0^\infty t^{x-1} s^{y-1} e^{-(s+t)} ds dt$$

Now we want to do a change of variables. The clues to focus on is that we want to consolidate the exponential by making $(t + s) = u$, and we want to have the term $(1 - w)^{y-1}$ in our final integral (meaning s is a multiple of $1 - w$).

Define $G(u, w) = (uw, u(1 - w))$. Then $D_{(u,w)}G = \begin{bmatrix} w & u \\ 1-w & -u \end{bmatrix}$. Thus, G is $\mathcal{C}^1$ continuous and $|\det(D_{(u,w)}G)| = |-uw - u + uw| = |u|$.

Also, G is injective when $u \neq 0$.

Proof:

Suppose $uw = u'w'$ and $u(1 - w) = u'(1 - w')$.

If $w = 0$, then $u'w' = 0 = uw$. So $u(1 - w) = u'(1 - w')$ becomes $u = u'$ and in turn $u'w' = uw \implies w' = w$.

Meanwhile, if $w \neq 0$, then $u = \frac{u'w'}{w}$. So:
$$u' - u'w' = u'(1 - w') = u(1 - w) = \frac{u'w'}{w}(1 - w) = \frac{u'w'}{w} - u'w'.$$

This says that $u'w = u'w'$. By assumption, $u' \neq 0$. So, canceling u' we get that $w = w'$. Then since $w \neq 0$, $uw = u'w' \implies u = u'$.

Finally, G maps $(0, \infty) \times (0, 1)$ to $(0, \infty) \times (0, \infty)$.

Proof:

Given $(t, s) \in (0, \infty) \times (0, \infty)$, let $u = t + s$ and $w = \frac{t}{t+s}$. Then $u \in (0, \infty)$, $v \in (0, 1)$, and $g(u, v) = ((t + s)\frac{t}{t+s}, (t + s)(1 - \frac{t}{t+s})) = (t, t + s - t) = (t, s)$.

So:
$$\int_0^\infty \int_0^\infty t^{x-1} s^{y-1} e^{-(s+t)} ds dt = \int_0^1 \int_0^\infty u^{x-1} w^{x-1} u^{y-1} (1 - w)^{y-1} e^{-u} |u| du dw$$
$$= \int_0^1 w^{x-1}(1 - w)^{y-1} dw \int_0^\infty u^x u^{y-1} e^{-u} du$$
$$= \int_0^1 w^{x-1}(1 - w)^{y-1} dw \cdot \Gamma(x + y)$$

Exercise 2.63: The technique used to prove proposition 2.54 can also be used to integrate any polynomial over S^{n-1}. Suppose $f(x) = \prod_{j=1}^n x_j^{\alpha_j}$. Then $\int f d\sigma = 0$ if any α_j is odd, and if all α_j's are even, then:
$$\int_{S_n} f d\sigma = \frac{2\Gamma(\beta_1) \cdots \Gamma(\beta_n)}{\Gamma(\beta_1 + \ldots + \beta_n)} \text{ where } \beta_j = \frac{\alpha_j + 1}{2}.$$

Note that $\int_{\mathbb{R}^n} f(x) e^{-|x|^2} dx = \int_{\mathbb{R}^n} x_1^{\alpha_1} \cdots x_n^{\alpha_n} e^{-x_1^2 - \cdots - x_n^2} dx$.

If we use the Fubini-Tonelli theorem, we can simplify the above expression a lot. But first we need to prove that $f(x) e^{-|x|^2}$ is integrable on $\mathbb{R}^n$.

One way to do this is show that $|t|^\alpha e^{-t^2} \in L^1$ for all integers $\alpha \in \mathbb{N} \cup \{0\}$.

If $\alpha = 2\beta - 1$, meaning that $\beta = \frac{\alpha+1}{2}$, then:
$$\int_{-\infty}^\infty |t|^\alpha e^{-t^2} dt = \int_{-\infty}^\infty |t|^{2\beta-1} e^{-t^2} dt = 2\int_0^\infty t^{2\beta-1} e^{-t^2} dt.$$

Now subsitute in $s = t^2$. Then our expression becomes $\frac{2}{2}\int_0^\infty t^{2\beta-2} e^{-s} ds = \int_0^\infty s^{\beta-1} e^{-s} ds$. When $\alpha \geq 0$, then $\beta \geq \frac{1}{2} > 0$. So $\int_0^\infty s^{\beta-1} e^{-s} ds = \Gamma(\beta) < \infty$.

Then by exercise 2.51, since $x_1^{\alpha_1} e^{-x_1^2} \in L^1(m)$ and $x_2^{\alpha_2} e^{-x_2^2} \in L^1(m)$, we have that $\left(x_1^{\alpha_1} e^{-x_1^2}\right)\left(x_2^{\alpha_2} e^{-x_2^2}\right) = x_1^{\alpha_1} x_2^{\alpha_2} e^{-x_1^2 - x_2^2} \in L^1(m \times m)$. Additionally, since $x_3^{\alpha_3} e^{-x_3^2} \in L^1(m)$, we thus have that:
$$\left(x_1^{\alpha_1} x_2^{\alpha_2} e^{-x_1^2 - x_2^2}\right)\left(x_3^{\alpha_3} e^{-x_3^2}\right) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} e^{-x_1^2 - x_2^2 - x_3^2} \in L^1((m \times m) \times m).$$

Because m is σ-finite, if we equate $(A \times B) \times C$ and $A \times B \times C$, we have that $(m \times m) \times m = m \times m \times m$. Thus $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} e^{-x_1^2 - x_2^2 - x_3^2} \in L^1(m \times m \times m)$.

Continuing likewise, we get that $f(x)e^{-|x|^2} \in L^1(m \times \ldots \times m) \subseteq L^1(m^n)$.

With that, we can now apply Fubini's theorem to get that:
$$\int_{\mathbb{R}^n} x_1^{\alpha_1} \cdots x_n^{\alpha_n} e^{-x_1^2 - \cdots - x_n^2} dx = \prod_{j=1}^n \int_{-\infty}^\infty x_j^{\alpha_j} e^{-x_j^2} dx_j$$

If some α_j is odd, then for that j we have $\int_{-\infty}^\infty x_j^{\alpha_j} e^{-x_j^2} dx_j = 0$. So the entire product cancels.

Meanwhile, if α_j is even, then $x_j^{\alpha_j} = |x_j|^{\alpha_j}$ and so from before we know that $\int_{-\infty}^\infty x_j^{\alpha_j} e^{-x_j^2} dx_j = \Gamma(\beta_j)$ where $\beta_j = \frac{\alpha_j + 1}{2}$. Hence, if all α_j are even, then our original integral evaluates to $\Gamma(\beta_1) \cdots \Gamma(\beta_n)$.

Next, we integrate $f(x)e^{-|x|^2}$ with respect to polar coordinates. To start, note that if $x' = (x_1', \ldots, x_n')$, then:
$$f(rx') = r^{\alpha_1}(x_1')^{\alpha_1} \cdots r^{\alpha_n}(x_n')^{\alpha_n} = r^{\alpha_1 + \cdots + \alpha_n}(x_1')^{\alpha_1} \cdots (x_n')^{\alpha_n}$$

Therefore:
$$\begin{aligned}
\Gamma(\beta_1) \cdots \Gamma(\beta_n) &= \int_{S^{n-1}} \int_0^\infty r^{\alpha_1 + \cdots + \alpha_n}(x_1')^{\alpha_1} \cdots (x_n')^{\alpha_n} r^{n-1} e^{-r^2} dr d\sigma(x') \\
&= \int_{S^{n-1}} (x_1')^{\alpha_1} \cdots (x_n')^{\alpha_n} d\sigma(x') \cdot \int_0^\infty r^{\alpha_1 + \cdots + \alpha_n} r^{n-1} e^{-r^2} dr \\
&= \int_{S^{n-1}} f d\sigma \cdot \int_0^\infty r^{(\alpha_1 + 1) + \cdots + (\alpha_n + 1) - 1} e^{-r^2} dr \\
&= \int_{S^{n-1}} f d\sigma \cdot \int_0^\infty r^{2(\beta_1 + \cdots + \beta_n) - 1} e^{-r^2} dr \\
&= \tfrac{1}{2} \int_{S^{n-1}} f d\sigma \cdot \int_0^\infty r^{2(\beta_1 + \cdots + \beta_n) - 2} e^{-s} ds \\
&= \tfrac{1}{2} \int_{S^{n-1}} f d\sigma \cdot \int_0^\infty s^{(\beta_1 + \cdots + \beta_n) - 1} e^{-s} ds \\
&= \tfrac{1}{2} \int_{S^{n-1}} f d\sigma \cdot \Gamma(\beta_1 + \ldots + \beta_n)
\end{aligned}$$

Exercise 2.64: For which real values of a and b is $|x|^a |\log|x||^b$ integrable over $\{x \in \mathbb{R}^n : |x| < \frac{1}{2}\}$ or over $\{x \in \mathbb{R}^n : |x| > 2\}$.

Let $B_1 = \{x \in \mathbb{R}^n : |x| < \frac{1}{2}\}$ and $B_2 = \{x \in \mathbb{R}^n : |x| > 2\}$. Note that $|\log|x|| > \log(2)$ for all $x \in B_1 \cup B_2$. As a consequence, when $b > 0$, we have that $|\log|x||^b > (\log(2))^b > 0$. Meanwhile, when $b \leq 0$, we have that $0 \leq |\log|x||^b < (\log(2))^b$

As a result, assuming $a > -n$ and $b > 0$, then:
$$|x|^a |\log|x||^b \geq |x|^a (\log(2))^b \geq |x|^{-n} (\log(2))^b \text{ for all } x \in B_1.$$

So by corollary 2.52, $|x|^a |\log|x||^b$ wouldn't be integrable on B_1.

Similarly, assuming $a > -n$ and $b \leq 0$, we have:
$$|x|^a |\log|x||^b \leq (\log(2))^b |x|^a \text{ for all } x \in B^1.$$

So by the same corollary, $|x|^a|\log|x||^b$ would be integrable on B_1.

Next suppose $a \leq -n$. If $b > 0$. Then $|x|^a|\log|x||^b \geq |x|^a(\log(2))^b$. But
$\int_0^{1/2} r^a(\log(2))^b r^{n-1}dr = (\log(2))^b \int_0^{1/2} r^{a+n-1}dr \leq (\log(2))^b \int_0^{1/2} r^{-1}dr = +\infty$.
So $|x|^a|\log|x||^b$ wouldn't be integrable on B_1.

Fourthly, if $a \leq -n$ and $b \leq 0$ then:
$$\int_{\mathbb{R}^n} |x|^a|\log|x||^b dx = \sigma(S^{n-1})\int_0^{1/2} r^a|\log(r)|^b r^{n-1}dr$$
$$= \sigma(S^{n-1})\int_0^{1/2} r^{a+n-1}(\log(r^{-1}))^b dr$$
$$= \sigma(S^{n-1})\int_2^\infty y^{-a-n+1}(\log(y))^b \frac{1}{y^2}dy \text{ (substituting } r = \frac{1}{y^2}\text{)}$$
$$= \sigma(S^{n-1})\int_2^\infty y^{-a-n-1}(\log(y))^b dy$$
$$\leq (\log(2))^b \sigma(S^{n-1})\int_2^\infty y^{-a-n-1}dy \text{ (since } b \leq 0\text{)}$$

But $-a \geq n$ and so $y^{-a-n-1} \geq y^{-1}$ for all $y > 2$. Hence, $\int_2^\infty y^{-a-n-1}dy \geq \int_2^\infty y^{-1}dy = \infty$. Thus, if $a \leq -n$ and $b \leq 0$, then $|x|^a|\log|x||^b$ is not integrable on B_1.

We conclude that $|x|^a|\log|x||^b$ is integrable on B_1 if and only if $a > -n$ and $b \leq 0$.

Finally, note that:
$$\int_{B_2} |x|^a|\log|x||^b dx = \sigma(S^{n-1})\int_2^\infty r^a(\log(r))^b r^{n-1}dr$$
$$= \sigma(S^{n-1})\int_0^{1/2} y^{-a}(\log(y^{-1}))^b y^{-n+1}\frac{1}{y^2}dr$$
$$= \sigma(S^{n-1})\int_0^{1/2} y^{-a}(\log(y^{-1}))^b y^{-n-1}dr$$
$$= \sigma(S^{n-1})\int_0^{1/2} y^{-a-2n}(\log(y^{-1}))^b y^{-n+2n-1}dr$$
$$= \sigma(S^{n-1})\int_0^{1/2} y^{-a-2n}(\log(y^{-1}))^b y^{n-1}dr$$
$$= \int_{B_1} |x|^{-a-2n}|\log|x||^b dx$$

So $|x|^a|\log|x||^b$ on B_2 if and only if $|x|^{-a-2n}|\log|x||^b$ on B_1, and that happens when $b \leq 0$ and $-a - 2n > -n \implies -a > n \implies a < -n$.

Thus, $|x|^a|\log|x||^b$ is integrable on B_1 if and only if $a > -n$ and $b \leq 0$. Also, $|x|^a|\log|x||^b$ is integrable on B_2 if and only if $a < -n$ and $b \leq 0$.

Homework 6

Exercise 3.13: Let $X = [0,1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, $m =$ Lebesgue measure and $\mu = $ counting measure on $\mathcal{M}$.

(a) $m \ll \mu$ but $\mathrm{d}m \neq f\mathrm{d}\mu$ for any f.

The only null set of μ is the emptyset and the empty set has measure zero in all measures. So all measures are absolutely continuous with respect to μ.

Next, assume such an f as in the prompt exists. Then for all $x_0 \in X$, we have:
$$0 = m(\{x_0\}) = \int_{\{x_0\}} f\mathrm{d}\mu = \int f\chi_{\{x_0\}}\mathrm{d}\mu$$
$$= \int f(x_0)\chi_{\{x_0\}}\mathrm{d}\mu = f(x_0)\mu(\{x_0\}) = f(x_0)$$

The second line of those manipulations follows because $f(x_0)\chi_{\{x_0\}}$ is a simple function.

So we've shown that we must have $f = 0$. In turn, for all $E \in \mathcal{M}$ we have that:
$$\int_E f\mathrm{d}\mu = \sup\left\{\sum_{x \in F} f(x) : F \subseteq E \text{ with } F \text{ finite}\right\} = 0.$$

But if $E = X$, then $m(E) = 1 \neq 0$ (a contradiction).

(b) μ has no Lebesgue decomposition with respect to m.

Suppose λ and ρ are signed measures with $\mu = \lambda + \rho$, $\lambda \perp m$, and $\rho \ll m$.

Since $m(E) = 0$ for all at most finite sets F, we must have that $\rho(F) = 0$ as well. In turn for any finite set F we know:
$$\mu(F) = \lambda(F) + \rho(F) \implies \mu(F) = \lambda(F) + 0 \implies \mu(F) = \lambda(F)$$

In turn, we've shown that the only set λ is null on is the emptyset, because given any nonempty set $E \in \mathcal{M}$, if we consider a finite nonempty subset of E, then λ will be positive on that subset (and defined since all finite subsets are closed and thus in $\mathcal{B}_{[0,1]} = \mathcal{M}$). So, $\lambda \perp m$ implies m is null on X. But this contradicts that $m(X) = m([0,1]) = 1$.

Exercise 3.16: Suppose that μ, ν are measures on $(X, \mathcal{M})$ with $\nu \ll \mu$, and let $\lambda = \mu + \nu$. If $f = \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}$, then $0 \leq f < 1$ μ-a.e. and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \frac{f}{(1-f)}$.

To start, here's a quick sanity test:

If $g = \chi_E$ where $E \in \mathcal{M}$, then:
$$\int g\mathrm{d}(\mu + \nu) = \mu + \nu(E) = \mu(E) + \nu(E) = \int g\mathrm{d}\mu + \int g\mathrm{d}\nu.$$

By linearity this extends to all simple functions. Then by M.C.T. this extends to all measurable nonnegative extended-real-valued functions.

Next note that if g is $\mathbb{C}$-valued or $\overline{\mathbb{R}}$-valued, then by the previous reasoning: $\int |g| \mathrm{d}(\mu + \nu) = \int |g| \mathrm{d}\mu + \int |g| \mathrm{d}\nu$. This tells us that $L^1(\mu + \nu) = L^1(\mu) \cap L^1(\nu)$. So assuming $g \in L^1(\mu)$ and $g \in L^1(\nu)$, by splitting into positive / negative and real / imaginary components, we can show that $\int g \mathrm{d}(\mu + \nu) = \int g \mathrm{d}\mu + \int g \mathrm{d}\nu$.

Now note $\nu(E) = \int_E f d\lambda$. This tells us that $\int_E f d\lambda$ is nonnegative for all $E \in \mathcal{M}$ since $\nu(E)$ is nonnegative for all $E \in \mathcal{M}$. In turn, this means that $f \geq 0$ λ-a.e.

Suppose $f < 0$ on a set E with $\lambda(E) > 0$. Then define $A_1 = \{x \in E : f^-(x) \geq 1\}$ and $A_n = \{x \in E : \frac{1}{n-1} > f^-(x) \geq \frac{1}{n}\}$ for all $n > 1$.

Then $\lambda(E) = \sum\limits_{n=1}^{\infty} \lambda(A_n) > 0$. So there exists A_n with $\lambda(A_n) > 0$.

But then $\int_E f \mathrm{d}\lambda = - \int_E f^- \mathrm{d}\lambda \leq - \int_{A_n} f \mathrm{d}\lambda \leq \frac{-1}{n}\lambda(A_n) < 0$, thus contradicting that $\int_E f \mathrm{d}\lambda$ is nonnegative for all $E \in \mathcal{M}$.

Also observe that if $\mathcal{N}(\lambda), \mathcal{N}(\mu), \mathcal{N}(\nu)$ are the collections of null sets of λ, μ, and ν respectively, then $\mathcal{N}(\lambda) = \mathcal{N}(\mu) \cap \mathcal{N}(\nu)$. Thus, we've shown that something being true λ-a.e. implies it is true μ-a.e. and ν-a.e.

So $f \geq 0$ μ-a.e. and this means that for any $E \in \mathcal{M}$:
$$\int_E d\nu = \nu(E) = \int_E f \mathrm{d}\lambda = \int_E f \mathrm{d}\mu + \int_E f \mathrm{d}\nu \implies \int_E (1-f) \mathrm{d}\nu = \int_E f \mathrm{d}\mu \geq 0$$

In turn, we have $1 - f \geq 0$ ν-a.e. But from before we know $f \geq 0$ ν-a.e. Hence, $0 \leq f \leq 1$ ν-a.e.

Then, let $A = \{x \in X : f(x) = 1\}$, $B = \{x \in X : f(x) > 1\}$. We can show:
- $0 = 0 \cdot \nu(A) = \int_A (1-f) \mathrm{d}\nu = \int_A f \mathrm{d}\mu = \int_A \mathrm{d}\mu = \mu(E)$
- $0 = \int_B (1-f) \mathrm{d}\nu = \int_B f \mathrm{d}\mu \geq \int_B 1 \mathrm{d}\mu = \mu(B) \geq 0$

Hence $0 \leq f < 1$ μ-a.e.

Now by similar logic to how we showed $f \geq 0$ λ-a.e., we can show $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \geq 0$ μ-a.e. Also, we must have $\int (1-f)\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\mathrm{d}\mu = \int f \mathrm{d}\mu$. And since the integrands are all nonnegative (after redefining on a μ-null set), we thus know: $(1-f)\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = f$ μ-a.e.

Therefore, $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is given by the equivalence class of $\frac{f}{1-f}$ with respect to μ.

Exercise 3.17: Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $\mathcal{N}$ a sub-σ-algebra of $\mathcal{M}$, and $\nu = \mu|_{\mathcal{N}}$. If $f \in L^1(\mu)$, there exists $g \in L^1(\nu)$ (thus g is $\mathcal{N}$-measurable) such that $\int_E f \mathrm{d}\mu = \int_E g \mathrm{d}\nu$ for all $E \in \mathcal{N}$; if g' is another such function then $g = g'$ ν-a.e. (In probability theory, g is called the conditional expectation of f on $\mathcal{N}$.)

If $f \in L^1(\mu)$, then we know that $\rho(E) = \int_E f \mathrm{d}\mu$ is a finite signed measure on $\mathcal{M}$. Also set $\lambda(E) = \rho(E)|_{\mathcal{N}}$. Importantly, $\lambda(X) = \rho(X) < \infty$. So λ is finite.

Also if we know ν is σ-finite, then we know there exists a Radon-Nikodym derivative of λ with respect to ν. So we define $g = \frac{d\lambda}{d\nu}$. Then $g \in L^1(\nu)$ and for all $E \in \mathcal{N}$ we have $\int_E f d\mu = \lambda(E) = \int_E g d\nu$. Additionally, the Lebesgue-Radon-Nikodym guarentees that given another function g' satisfying this, $g' = g$ ν-a.e.

But you see, here Folland is a dumb idiot who doesn't proof-read his own writing. And hell there's nothing in the book's errata either to address this.

Firstly, we can't prove that ν is σ-finite based on the information given.
Let $\mathcal{N} = \{X, \emptyset\}$ be our sub-σ-algebra of $\mathcal{M}$. If μ is finite, then we still have that ν is finite. But if $\mu(X) = \infty$, then $\nu(X) = \infty$ and there is no sequence of sets of finite measure in $\mathcal{N}$ whose union is X. Hence ν is not σ-finite.

Secondly, if don't restrict ν in some way (such as demanding it's σ-finite), then the prompt is false.
Keeping $\mathcal{N} = \{X, \emptyset\}$, note that the only $\mathcal{N}$-measurable functions are constant functions. After all, if x and y are two distinct points in X and $h(x) = a$ and $h(y) = b \neq a$, then $\emptyset \subset \{y\} \subseteq h^{-1}(\{b\}) \subseteq X - \{x\} \subset X$. Hence, h is not $\mathcal{N}$-measurable since $h^{-1}(\{b\}) \notin \mathcal{N}$.

As a consequence, the only function $g \in L^1(\nu)$ is $g = 0$. Thus we must have $\int g d\nu = 0$. However, this gives a contradiction since it's possible $\int f d\mu \neq 0$.

Finally, I think Folland overlooked this issue because he was blinded by the probability application. After all in probability ν will be finite because μ is finite and then we don't have any issues.

Exercise 3.22 If $f \in L^1(\mathbb{R}^n)$, $f \neq 0$, there exists $C, R > 0$ such that $Hf(x) \geq C|x|^{-n}$ for $|x| > R$. Hence $m(\{x : Hf(x) > \alpha\}) \geq \frac{C'}{\alpha}$ when α is small, so the estimate in the maximal theorem is essentially sharp.

I'm going to asssume Folland means $f \neq 0$ a.e. cause otherwise the problem doesn't make sense. Plus, I really should be getting use to thinking about equivalence classes of functions.

Since $f \neq 0$ a.e., there exists $R > 1$ such that $\int_{B(R,0)} |f(y)| dy > c > 0$ for some c in $\mathbb{R}^n$. Then when $|x| > R$, because $B(R, 0) \subseteq B(2|x|, x)$, we have:

$$Hf(x) \geq \frac{1}{m(B(2|x|,x))} \int_{B(2|x|,x)} |f(y)| dy$$

$$\geq \frac{1}{m(B(2|x|,x))} \int_{B(R,0)} |f(y)| dy$$

$$= \frac{1}{2^n |x|^n m(B(1,0))} \int_{B(R,0)} |f(y)| dy > \frac{c}{2^n m(B(1,0))} |x|^{-n}$$

Setting $C = \frac{c}{2^n m(B(1,0))}$, we've thus shown the first claim.

Next, consider when $\alpha < \frac{C}{2R^n} < \frac{C}{R^n}$. Then since $R > 1$, we know that $\sqrt[n]{C/\alpha} > R$. So for $R < |x| < \sqrt[n]{C/\alpha}$, we have:

$$Hf(x) > C|x|^{-n} > C\left(\sqrt[n]{C/\alpha}\right)^{-n} = C\frac{\alpha}{C} = \alpha$$

In turn:
$$m(\{x : Hf(x) > \alpha\}) \geq m(R < |x| < \sqrt[n]{C/\alpha})$$
$$= m(\sqrt[n]{C/\alpha}, 0) - m(R, 0) = (\tfrac{C}{\alpha} - R^n)m(B(1,0))$$

Now $\alpha < \frac{C}{2R^n} \implies R^n < \frac{C}{2\alpha}$. So finally we have that:
$$(\tfrac{C}{\alpha} - R^n)m(B(1,0)) \geq (\tfrac{C}{\alpha} - \tfrac{C}{2\alpha})m(B(1,0)) = \tfrac{1}{\alpha}\tfrac{Cm(B(1,0))}{2}$$

Set $C' = \frac{Cm(B(1,0))}{2}$ and we are done.

Exercise 3.23: A useful variant of the Hardy-Littlewood maximal function is:
$$H^*f(x) = \sup\left\{\tfrac{1}{m(B)}\int_B |f(y)|dy : B \text{ is a ball and } x \in B\right\}$$

Show that $Hf \leq H^*f \leq 2^n Hf$.

It's trivial that $Hf(x) \leq H^*f(x)$ because all the balls $Hf(x)$ is taking a sup over are included in the set of balls that $H^*f(x)$ is taking a sup over.

Meanwhile, given any ball $B(R, z)$ containing x, we know $B(R, z) \subseteq B(2R, x)$, and thus:
$$\tfrac{1}{m(B(R,z))}\int_{B(R,z)} |f(y)|dy = \tfrac{1}{m(B(R,x))}\int_{B(R,z)} |f(y)|dy$$
$$\leq \tfrac{1}{m(B(R,x))}\int_{B(2R,x)} |f(y)|dy$$
$$= \tfrac{2^n}{m(B(2R,x))}\int_{B(2R,x)} |f(y)|dy \leq 2^n Hf(x)$$

So $H^*f(x) \leq 2^n Hf(x)$.

Exercise 3.25: If E is a Borel set in $\mathbb{R}^n$, the <u>density</u> $D_E(x)$ of E at x is defined (whenever the limit exists) as:
$$D_E(x) = \lim_{r \to 0} \tfrac{m(E \cap B(r,x))}{m(B(r,x))}$$

(a) Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x)$ for a.e. $x \in E^\complement$.

Define the measure $\nu(F) = m(E \cap F)$. Importantly, ν is still σ-finite and also $m(F) = 0 \implies m(E \cap F) = \nu(F) = 0$, meaning $\nu \ll m$. Thus by the Radon-Nikodym Theorem (and the fact that ν is positive), there exists $f \in L^+$ such that $d\nu = f dm$.

We also have $f \in L^1_{\text{loc}}$. After all, if K is a bounded measurable set, then $0 \leq \int_K f dm = \nu(K) < m(K) < \infty$.

Thus, by a result in the book ν is regular, and in turn by theorem 3.22, we know that:

$$\lim_{r \to 0} \frac{\nu(B(r,x))}{m(B(r,x))} = \lim_{r \to 0} \frac{m(E \cap B(r,x))}{m(B(r,x))} = f(x) \text{ for } m\text{-a.e. } x.$$

As for what $f(x)$ is, note:

- $\int_{E^C} f \, \mathrm{d}m = \nu(E^C) = 0$. Thus $f = 0$ m-a.e. on E^C.
- Let $\{V_n\}_n$ be a sequence of subsets of E with finite measure such that $E = \bigcup_{n \in \mathbb{N}} V_n$. Then for any measurable $F \subseteq V_n$, we can say that:

$$\int_F |f - 1| \, \mathrm{d}m = \int_F f \, \mathrm{d}m - \int_F \mathrm{d}m = \nu(V_n) - m(V_n) = 0.$$

 This tells us that $f - 1 = 0$ m-a.e. on V_n. So $f = 1$ m-a.e. on V_n. Then repeating this for all V_n tells us that $f = 1$ m-a.e. on E.

(b) Find examples of E and x such that $D_E(x)$ is a given number $\alpha \in (0, 1)$, or such that $D_E(x)$ does not exist.

Exercise 3.26: If λ and μ are positive, mutually singular Borel measures on $\mathbb{R}^n$ and $\lambda + \mu$ is regular, then so are λ and μ.

Let K be a compact subset of $E \in \mathcal{B}_{\mathbb{R}^n}$. Thus we know that $\lambda(K) + \mu(K) < \infty$. And because $0 \leq \lambda(K)$ and $0 \leq \mu(K)$, we thus must have that:

$$\lambda(K), \mu(K) \leq \lambda(K) + \mu(K) < \infty.$$

Thus the requirement of regularity is satisfied. Technically this is enough because the first requirement implies the second according to the textbook. However, we haven't proven that yet. So I'm going to try and prove the second requirement.

Let $E, F \in \mathcal{B}_{\mathbb{R}^n}$ be such that $E \cap F = \emptyset$, $E \cup F = \mathbb{R}^n$, $\lambda(E) = 0$ and $\mu(E) = 0$.

...actually nevermind I don't have time to prove this before my extended deadline runs out...