Math 158 Lecture Notes (Professor: Jacques Verstraete)

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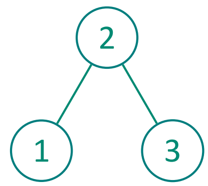
Lecture 1: 1/9/2024

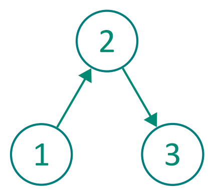
A graph is a pair (V, E) where V is a set of vertices and E is a set of unordered pairs of elements of V called edges. For $u, v \in V$, we say u and v are adjacent if $\{u, v\} \in E$.

For example: $G = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}\})$



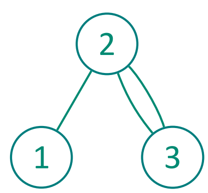
A directed graph (a.k.a a digraph) is a pair (V, E) where V is a set of vertices and E is a set of ordered pairs of elements of V.

For example: $G = (\{1, 2, 3\}, \{(1, 2), (2, 3)\})$



A multigraph is a pair (V, E) where V is a set of vertices and E is a multiset of unordered pairs of elements of V.

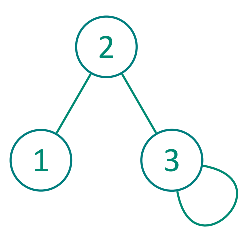
For example: $G = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}, \{2, 3\}\})$



A pseudograph is like a graph and multigraph except that the pairs in E are multisets.

Essentially, an element $\{a, a\}$ can belong to E in a pseudograph. This type of edge is called a loop.

For example: $G = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}, \{3, 3\}\})$



If $G = (V, E)$ and $v \in V$, the neighborhood of v is $N_G(v) = \{w \in V \mid \{v, w\} \in E\}$.

The degree of v is $d_G(v) = |N_G(v)|$. Or in other words, v's degree is equal to the number of edges connecting to v.

The Handshaking lemma states that for any graph (V, E):

$$\sum_{v \in V} d_G(v) = 2|E|$$

The reason for this is that each edge increments the degrees of exactly two vertices. So the above sum counts every edge twice.

Lemma: Every graph has an even number of vertices with odd degrees.

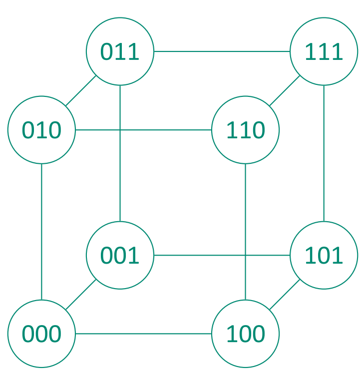
Proof: We can split the vertices of any graph into two categories: those with odd degrees, and those with even degrees.

Now recall that an even number plus an even number always equals an even number, as does an odd number plus an odd number. However, an odd number plus an even numbers equals an odd number. Based on this fact, we can guarentee that the sum of even degrees in any graph is even. And since the sum of even degrees plus the sum of odd degrees must be even as it equals $2|E|$ by the Handshaking lemma, we thus know that the sum of odd degrees must be even. Hence, it must be the case that there are an even number of vertices with odd degree because otherwise the sum of their degrees won't be even.

A graph is called r-regular if all of its vertices have degree r.

Note that the number of edges in any n-vertex r-regular graph is $\dfrac{rn}{2}$.

An r-dimensional cube graph, denoted as Q_r, is a graph such that $V(\mathrm{Q}_r)$, the set of vertices in Q_r, is equal to the set of binary strings of length r; and $E(\mathrm{Q}_r)$, the set of edges in Q_r, is equal to the set of pairs of binary strings which differ in only one position.



$$|V(\mathrm{Q}_r)| = 2^r$$

$$|E(\mathrm{Q}_r)| = \tfrac{2^r r}{2} = 2^{r-1} r$$

Note that Q_r is r-regular.

If $G = (V, E)$, then $H = (W, F)$ is a subgraph of G if $W \subseteq V$ and $F \subseteq E$.

If $W = V$, then H is a spanning subgraph of G (meaning that H has the same vertices as G but is lacking some of G's edges)

We define subtracting a set of vertices from a graph as follows:

For $G = (V, E)$ and $X \subset V$, we define...

$$G - X = (V \setminus X, \{\{u, v\} \in E \mid \{u, v\} \cap X = \emptyset\})$$

We define subtracting a set of edges from a graph as follows:

For $G = (V, E)$ and $L \subset E$, we define...

$$G - L = (V, E \setminus L)$$

Lecture 2: 1/11/2024

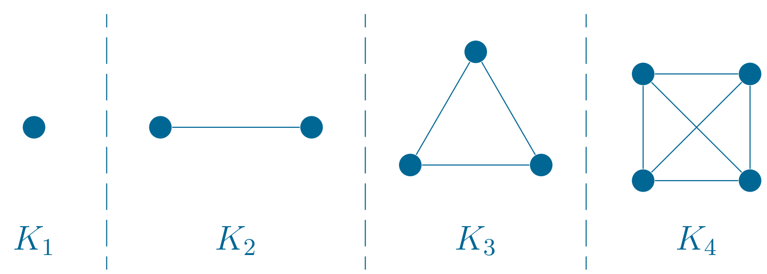
We shall notate that H is a subgraph of G by writing $H \subseteq G$.

An induced subgraph of $G = (V, E)$ is a subgraph $G[X] = G - (V \setminus X)$ where $X \subseteq V$. Alternatively, this is called the subgraph induced by X.

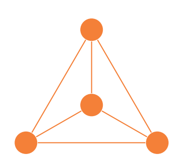
Given $G = (V, E)$ and $F \subseteq E$, the subgraph spanned by F is the subgraph whose edge set is F and whose vertex set is $\bigcup_{e \in F} e$.

Here are some basic classes of graphs:

- Complete graphs / cliques, denoted K_n, are graphs where every possible edge is present between n vertices.



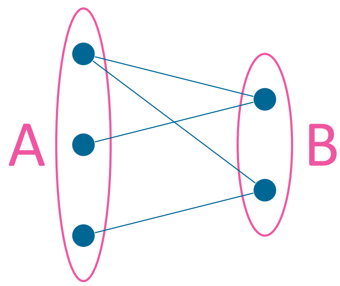
Note we can also draw K_4 such that there are no edge interceptions as follows:



$$|V(K_n)| = n$$

$$|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$$

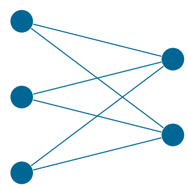
- A graph $G = (V, E)$ is bipartite if there exists a partition (A, B) of V such that every edge in E has one end in A and one end in B.



The partition (A, B) is called the bipartition of G. Then A and B are called the parts of G.

- A Complete bipartite graphs $K_{s,t}$, is the bipartite graph with parts A and B where $|A| = s$, $|B| = t$, and all possible edges between A and B exist.

For example, $K_{3,2}$:

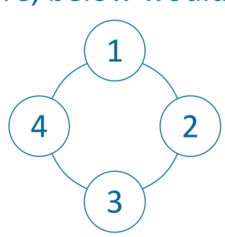


- A path P_k of length k has a vertex set $V = \{v_1, v_2, \ldots, v_k, v_{k+1}\}$ and an edge set $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_{k+1}\}\}$.

Note that $|V(P_k)| = k + 1$ and $|E(P_k)| = k$.
Therefore, below would be P_3...

- A cycle C_k of length k has a vertex set $V = \{v_1, v_2, \ldots, v_k\}$ and an edge set $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_1\}\}$.

Note that $|V(C_k)| = k$ and $|E(C_k)| = k$.
Therefore, below would be C_4...



Here is some terminology before the next lemma. For the graph $G = (V, E)$...

- $\delta(G) = \min\{d_G(v) \mid v \in V\}$ is the minimum degree of G.
- $\Delta(G) = \max\{d_G(v) \mid v \in V\}$ is the maximum degree of G.
- The degree sequence of G is the sequence of degrees of vertices G in non-increasing order.

Lemma (part 1): If $G = (V, E)$ is a graph of minimum degree $k \geq 2$, then G contains a cycle of length at least $k + 1$.

Proof: Let P be a longest possible path in G, say:

$$V(P) = \{v_1, v_2, \ldots, v_r\}$$

Then $N(v_r) \subseteq V(P)$. After all, if this were not the case, we'd be able to extend the path to the vertex in $N(v_r)$ but not in $V(P)$, thus contradicting the fact that P is a longest path.

Let v_i be the first neighbor of v_r along the path from v_1 to v_r. Then $\{v_i, v_{i+1}, \ldots, v_r\}$ are the vertices of a cycle C.

Now note that because $N(v_r) \subseteq P$ and v_i was the first element in the path P to belong to $N(v_r)$, we know that C contains all the elements of P that $N(v_r)$ also has. So, $N(v_r) \subseteq C$.

But now note that $|N(v_r)| \geq \delta(G) = k$. Plus, v_r itself is not in $N(v_r)$. Combining these facts together, we can say that the cycle C has at least $k + 1$ vertices.

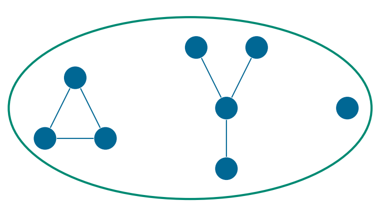
Lemma (part 2): The cycle length $k + 1$ is the longest we can guarentee based on the minimum degree of the graph being k.

Proof: Take the graph K_{k+1} which has a minimum degree k. Obviously, the longest cycle in K_{k+1} is the cycle containing all $k + 1$ elements of K_{k+1}. Thus, we have shown that there are graphs with minimum degree k which don't have cycles of length greater than $k + 1$.

A connected graph is a graph in which any two vertices are the ends of a path.

The components of a graph are the maximal connected subgraphs. For example:

Let us define G as:



As can be seen, G has three components.

A tree is a connected graph with no cycles (a.k.a it is acyclic).

Some examples of small trees include: K_1, K_2, $K_{1,2}$, P_3, and $K_{1,3}$.

Lemma: Every tree with n vertices has exactly $n-1$ edges.

Proof: We shall proceed by induction.

If $n=1$, the tree is K_1, meaning that it has $0=n-1$ edges.

Now assume the lemma is true for all trees with n vertices, and let T be a tree with $n+1$ vertices. Then, we shall remove a vertex v of T with degree 1. (Note that we know such a vertex must exist since otherwise the minimum degree of T would be at least 2 and that would guarentee a cycle exists of at least length 3. This of course contradicts the fact that T is acyclic.)

Then $T-\{v\}$ is a tree with n vertices as it must be acyclic and connected. So by induction it has $n-1$ edges. And because v has degree 1, we know that
$|E(T)| = 1 + |E(T-\{v\})| = 1 + (n-1) = n.$

Lemma: Any connected graph with finite vertices has a spanning tree.

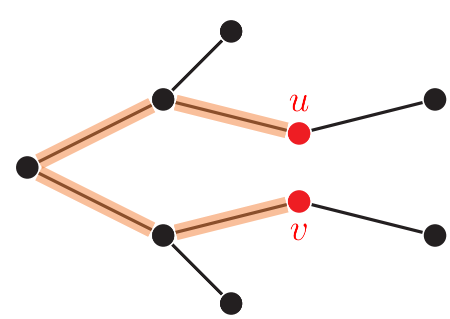
Proof:

Firstly consider the case that the graph G has no cycle. Then, it is a tree by definition.

Now, consider if G has a cycle C. Then for any edge $e \in E(C)$, we have that $G-\{e\}$ is still connected. So, we can now go back to the top of the proof and ask: does $G-\{e\}$ have any cycles? We can repeatedly do this until the graph has no cycles since taking away edges does not remove any vertices.

This actually acts as an algorithm for finding a spanning tree of any connected graph.

If u and v are two vertices in a connected graph, the distance from u to v is the length of a shortest path with ends at u and v.



Let $d_G(u,v)$ be the distance between u and v.

Distance is a metric, meaning:

1. $d_G(u,v) = 0 \iff u = v$
2. $d_G(u,v) = d_G(v,u)$
3. $\forall w \in V, \;\; d_G(u,v) \leq d_G(u,w) + d_G(w,v)$

The diameter of a connected graph G is the maximum distance between any two vertices of G. Or in other words, $\max\{d_G(u,v) \mid u,v \in V(G)\}$.

The radius of G is equal to $\min\{\max\{d_g(u,v) \mid u \in V(G)\} \mid v \in V(G)\}$. What that means is that the radius of G measures the smallest distance path one could limit themselves to drawing while still being able to have that path have one end at some fixed vertex and its other end at any arbitrary vertex in the graph.

Examples:

1. The radius of K_n is 1. The diameter of K_n is n.

2. The diameter of P_k is k. The radius, can be computed as follows:

   The middle vertex of a path will have the fastest access to either end of the path. So, we shall measure the radius from the vertex: $v_{\lceil \frac{k+1}{2} \rceil}$. Then, we can see that v_{k+1} is going to be a farthest element from $v_{\lceil \frac{k+1}{2} \rceil}$. So the radius of P_k equals $k + \lceil \frac{k+1}{2} \rceil$.

   Now you can consider what happens when k is even and odd. But what's important is that it works out that the radius is $\lceil \frac{k}{2} \rceil$.

We can use a search tree to more generally find the radii and diameters of graphs.

Breadth-First-Search

Here's how to find a spanning tree in a connected graph with a root vertex v such that the tree "preserves" all distances from v. (This tree is called a BFS tree).

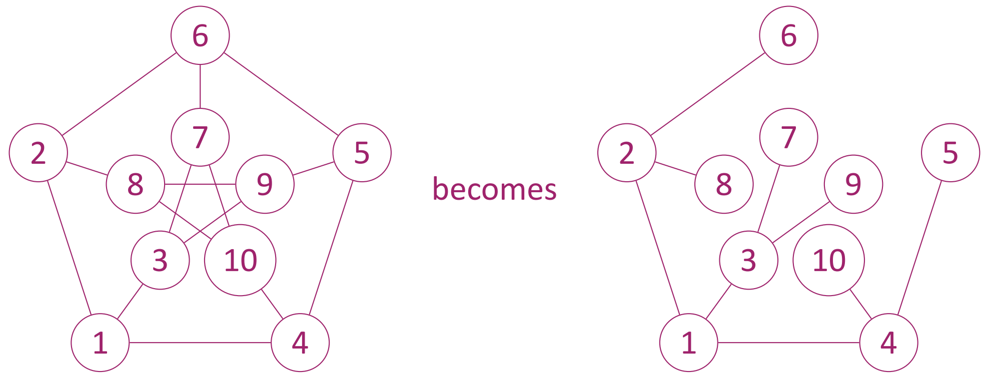
Let G be a connected graph and let $(v_1, v_2, v_3, \ldots, v_n)$ be any ordering of the vertices of G.

Pick a vertex $v = v_1$ to be the root of the BFS tree.

Now, at any stage in constructing this tree, we will have a vertex set $V(T) = \{v_1, v_2 \ldots, v_k\}$ (when we first start, $V(T)$ will only contain v_0. So don't worry about that). Now if $V(T) = V(G)$ we can stop. Otherwise though, we can say that there is a smallest integer i such that for $v_i \in V(T), \;\; N(v_i) \setminus V(T) \neq \emptyset$. Choose v_{k+1} to be the smallest neighbor (by the ordering of $V(G)$) of v_i not in T and add the edge $\{v_i, v_{k+1}\}$ to T. Then we repeat this paragraph.

Beware the ordering we are creating in our tree will often be different from the order of the graph you started with.

Worked Demonstration:



Properties of BFS:

- If the root is v, then $d_T(v, w) = d_G(v, w)$. In otherwords, a BFS tree preserves distances from its root.

- The Tree with root v has layers $N_i(v) = \{w \in V(G) \mid d_G(v, w) = i\}$. Furthermore all edges in the original graph either stay inside a single layer $N_i(v)$ or go between adjacent layers (i.e. from $N_i(v)$ to $N_{i+1}(v)$).
If an edge did "jump over" a layer, that violate the fact that distance is a metric.

- The diameter of G equals the maximum number of layers of all BFS trees (not including the 0-layer).

- The radius of G equals the minimum number of layers of all BFS trees (also not including the 0-layer).

Lecture 3: 1/16/2024

Note that a tree is "minimally connecting" as subtracting any edge from a tree will produce a disconnected graph.

We know this is the case because if we could remove an edge and still have the graph be connected, then that would imply the existence of a path between two neighboring vertices that doesn't go through their shared edge. But then, we'd be able to make a cycle subgraph by adding their shared edge to that path.

Depth-First-Search

Here is alternate algorithm for generating a spanning tree of a connected graph. A resulting tree of this algorithm is called a DFS tree.

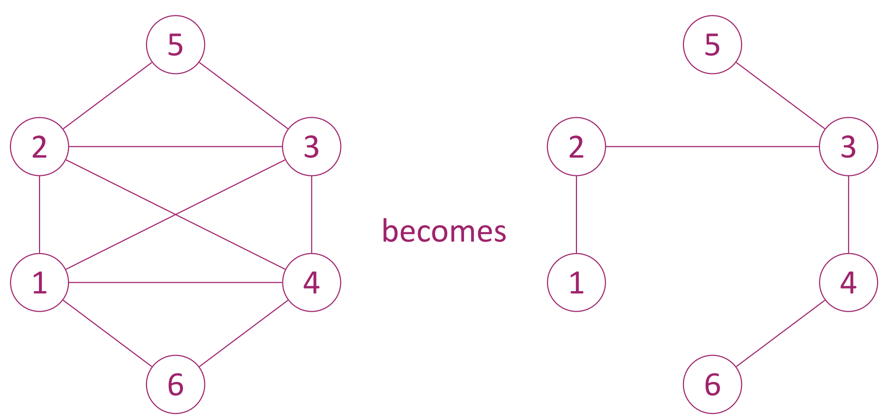
Let G be a connected graph and let $(v_1, v_2, \ldots, v_n)$ be any ordering of the vertices of G.

Pick a vertex $v = v_1$ to be the root of the DFS-tree.

Now, at any stage in constructing this tree, we will have a vertex set $V(T) = \{v_1, v_2, \ldots, v_k\}$. If $V(T) = V(G)$, we can stop. Otherwise though, we select i to be the largest integer such that for $v_i \in V(T)$, $N(v_i) \setminus V(T) \neq \emptyset$. Then, choose v_{k+1} to be the smallest neighbor (by the ordering of $V(G)$) of v_i not in $V(T)$ and add the edge $\{v_i, v_{k+1}\}$ to T. Then we repeat this paragraph.

Once again beware the ordering we are creating in our tree will typically be different from the order of the graph you started with.

Worked Demonstration:



Theorem: A graph is bipartite if and only if it contains no odd cycles.

Proof:

($\Longrightarrow$) First note that an odd cycle isn't bipartite. Thus, any graph containing an odd cycle is not bipartite.

($\Longleftarrow$) Now supposed we are given some graph G with no odd cycles. Then, assuming G is connected (if G isn't connected, we can break G up into its component subgraphs and do this process for each component), we can construct a BFS-tree in G rooted at some $v \in V(G)$. Let us name this tree T.

Now as noted before, T will have layers L_i where each $L_i = \{u \in V(G) \mid d_G(v,u) = i\}$. Using those layers, we can partition T into two subsets A and B where A is the union of all L_i where i is even and B is the union of all L_j where j is odd. So, T is clearly bipartite.

Now, let's reinsert the removed edges from G back into T. Note that for each re-inserted edge e, it must be the case that either e is a subset of some L_i or that e goes between some L_i and L_{i+1}. Importantly, edges of the latter case do not violate our partition. So, if all the edges in $E(G) \setminus E(T)$ go between layers, then we can conclude that G is definitely bipartite just like T.

With that, we now intend to show that an edge G having an edge belonging to a single layer L_i guarentees that G contains an odd cycle.

Assume the graph G has an edge $\{u,w\} \subseteq L_i$ where L_i is the ith layer of a BFS tree rooted at v. Then, we know that there exists a path P_1 contained in that BFS tree going from v to u and a path P_2 contained in that BFS tree going from v to w. In order to draw a cycle from this information, let x be the vertex of some L_j such that $x \in V(P_1)$, $x \in V(P_2)$, and j is as large as possible. That way, by defining the subpaths ${P_1}'$ going from x to u and ${P_2}'$ going from x to w, we can get the following cyclic subgraph of G :

$$C = (V({P_1}') \cup V({P_2}'), E({P_1}') \cup E({P_2}') \cup \{u,w\})$$

However, now note that $|E({P_1}')| = |E({P_2}')| = i - j$. Hence, $|E(C)| = 2(i-j)+1$, which in turn means that C has an odd number of edges. So, we have shown that if a graph G contains an edge within a single layer L_i, then we can give an example of an odd cycle within G.

So in conclusion, if we assume G has no odd cycles, then G can't have any edges which are subsets of a single layer L_i. But that means that every edge in G respects the partition we made to show that T is bipartite. So, G must also be bipartite with the same partition as T. ■

A Hamiltonian cycle is a spanning cycle of a graph. We say a graph is Hamiltonian if it contains such a cycle.

A Hamiltonian path is a spanning path of a graph. We say a graph is traceable if it has a hamiltonian path.

A walk is a sequence of vertices and edges: i.e. $(v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, \ldots)$

Note that a walk can go over the same edge or vertex multiple times.

A trail is a walk with no repeated edge.

Interestingly, all paths are trails and all trails are walks. So a trail is kind of a middle concept between being a walk or a path.

A tour is a trail with the same first and last vertex.

So, all cycles are tours and all tours are walks.

An Eulerian tour of a graph is a tour which contains all the edges of the graph.

For context, the name Eulerian is in honor of Leonhard Euler because he was the first mathematician to ask when a graph would have an Eulerian tour (look up the Seven Bridges of Königsburg problem).

We call a graph Eulerian if for every vertex v in the graph: $d(v)$ is even.

Lecture 4: 1/18/2024

Before finding conditions for the existence of an eulerian tour of a graph, let's establish some terminology for digraphs so that we can study Eulerian tours in digraphs as well.

Firstly, a walk, trail, and tour are defined almost identically in a digraph as in a graph. The one difference is that given some edge (u, v) of a digraph, a walk, trail, and path are only allowed to traverse that edge going from u to v.

Given a digraph (V, E), the "out" and "in" neighborhoods of a vertex $v \in V$ are:

out: $N^+(v) = \{w \in V \mid (v, w) \in E\}$

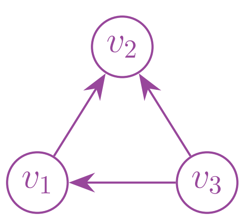
in: $N^-(v) = \{w \in V \mid (w, v) \in E\}$

Similarly, the out-degree and in-degree of $v \in V$ are:

out: $d^+(v) = |N^+(v)|$

in: $d^-(v) = |N^-(v)|$

A digraph is called Eulerian if for each vertex v , $d^+(v) = d^-(v)$.



$d^+(v_1) = 1$ $\quad d^+(v_2) = 0$ $\quad d^+(v_3) = 2$

$d^-(v_1) = 1$ $\quad d^-(v_2) = 2$ $\quad d^-(v_3) = 0$

This digraph is not Eulerian.

An orientation of a graph G is a digraph with the same vertices as G but where each $\{u, v\} \in E(G)$ is replaced with either (u, v) or (v, u).

The underlying graph (or multigraph) of a digraph G is the graph (or multigraph) such that $\{u, v\}$ is an edge whenever $(u, v) \in G$.

Theorem:

1. A graph has an Eulerian tour if and only if it is connected and Eulerian.
2. A digraph has an Eulerian tour if and only if it has a connected underlying graph and if it is Eulerian.

Proof of statement 1:

($\Longrightarrow$) If G has an Eulerian tour $v_0e_0v_1e_1v_2e_2 \ldots v_ke_kv_0$, then for any v_i, the tour has to use an edge into v_i and an edge out of v_i each time it visits v_i. So, $d(v_i)$ is even for all i.

($\Longleftarrow$) Now suppose G is connected and all vertices have even degree. Then let $T = v_0e_0v_1e_1v_2e_2 \ldots v_le_lv_{l+1}$ be a longest trail in G.

If $v_{l+1} \neq v_0$, then we know that v_{l+1} has an odd degree in T as the trail goes into v_{l+1} and doesn't leave. However, because we assumed that all vertices in G have an even degree, we know there must be an even number of edges coming out of v_{l+1}. So, we can add another edge to our trail to get a longer trail. But this contradicts our assumption that T is a longest trail in G. Hence, we conclude that $v_{l+1} = v_0$, meaning T is a tour.

Now consider if T is not an Eulerian tour. In that case, there is an edge of G not in T. Additionally, because G is connected, we know that that edge will have the form $e = \{v_i, w\}$. So now consider a new trail: T' defined as $v_ie_iv_{i+1}e_{i+1} \ldots v_0e_0v_1e_1 \ldots v_iew$. Importantly, T' is a longer trail than T. So we have a contradiction as T is not a longest trail.

Therefore, the longest trail T in G must be an Eulerian tour.

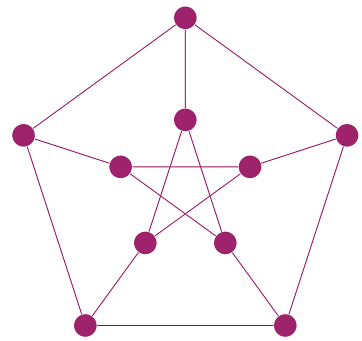
The proof of statement 2 is nearly identical.

Note that this proof can be interpretted as giving an algorithm for finding an Eulerian tour.

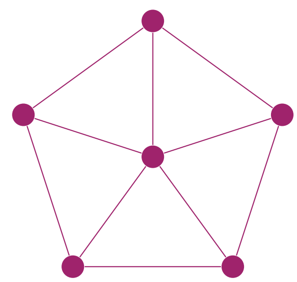
1. Make a trail T.
2. Add edges to T until you get stuck at a vertex. Then you know that your trail forms a tour.
3. If T is not an Eulerian tour, then going by the steps in the proof above, define T'. Then do step 2. on T^prime.
4. If T is an Eulerian tour, you're done.

A harder problem is whether a graph has a Hamiltonian (spanning) cycle or not.

This graph does not have a Hamiltonian cycle.



This graph does have a Hamiltonian cycle.



Dirac's Theorem: Let $n \geq 3$ and let G be an n-vertex graph of minimum degree at least $\frac{n}{2}$. Then G is hamiltonian.

Proof:
Suppose the theorem is false. Let G be a counter-example with as many edges as possible (a maximal counter-example). Then, we know there exists an edge $\{u, v\} \notin E(G)$ as G cannot equal K_n since K_n is hamiltonian. Furthermore, we know that $G + \{u, v\}$ is hamiltonian since G was maximal. So, there is a hamiltonian cycle $C \subseteq G + \{u, v\}$ using the edge $\{u, v\}$. This in turn means that $C - \{u, v\}$ is a hamiltonian path belonging to G.

Let $P = v_1 e_1 v_2 e_2 \ldots e_n v_n$ be the hamiltonian path in G from u to v and let $N^+(w)$ denote the set of vertices immediately following a neighbor of w on the path P. In other words: $N^+(w) = \{v_{i+1} \mid v_i \in N_G(w)\}$.

By the theorem's assumption about the minimum degree of the graph, we know that $|N_G(u)| \geq \frac{n}{2}$. Meanwhile on the other end of P, since $v = v_n \notin N_G(v)$, we know that every neighbor of v has an element following it on the path. So $|N^+(v)| \geq \frac{n}{2}$. Thus $|N_G(u)| + |N^+(v)| \geq n$. But now note that u does not belong to either of the above sets. So $|N_G(u) \cup N^+(v)| \leq n - 1$. As a consequence of this, $|N_G(u) \cap N^+(v)| > 0$.

Let v_i be a vertex belonging to $(N_G(v) \cap N^+(u))$. Then we can draw a cycle in G visiting all the vertices of G in the following order:

$$v_1, v_2, \ldots, v_{i-1}, v_n, v_{n-1}, \ldots, v_i, v_1$$

This contradicts our assumption that G would be a counter example and thus not Hamiltonian. So, we assume no such counter example exists.

We can also show that Dirac's Theorem gives the best possible minimum degree for a graph to be guarenteably Hamiltonian. Consider a graph G containing two copies of K_m sharing exactly one vertex. In that case, $n = |V(G)| = 2m - 1$ and $\delta(G) = m - 1 = \frac{n-1}{2}$. However, this graph does not have a spanning subcycle as any spanning cycle would have to cross that shared vertex twice.

Let P be a longest path in G going from a vertex u to a vertex v. Additionally, for any $w \in V(P)$, let w^+ be the vertex following w as one travels from u to v along P. Importantly, since P is a longest path, we know that $(N_G(u) \cup N_G(v)) \subseteq V(P)$. Furthermore, for each $w \in N_G(v)$, we can define $Q = P - \{w, w^+\} + \{v, w\}$ where Q is a longest path of G going from u to w^+ instead of u to v. We call Q a rotation of P at v.

Pósa's Rotation Lemma: Suppose G is a graph and for every $S \subseteq V(G)$ with $|S| \leq t$, $|N(S)| > 2|S|$. Then G contains a path of length $3t + 1$.

$N(S)$ is referring to the union of the neighborhoods of each $v \in S$ minus any vertices in S.

Proof:
Let P be a longest path ending at a vertex v. Also, let S be the set of end vertices of all possible longest paths that could be obtained through any number of rotations starting with P. Finally, let S^+ and S^- denoted the vertices of P immediately after and immediately before vertices in S respectively.

Obviously, $|S^+| \leq |S|$ and $|S^-| \leq |S|$ as all vertices except the first and last vertex of P have exactly one vertex before and after them in P. Also note that $N(S) \subseteq S^+ \cup S^-$. This is because if there did exist $w \in N(S)$ such that $w \notin S^+ \cup S^-$, then we would know that no rotation of P made it so that w was not proceeded by w^- and followed by w^+. So, doing a rotation with the vertex w, we would show that either w^+ or w^- belonged to S, thus reaching a contradiction.

Overall, this means that $|N(S)| \leq |S^+ \cup S^-| \leq |S^+| + |S^-| \leq 2|S|$. However, note that by the theorem's assumption about G, we know that $|S| \geq t$ because otherwise we'd have that $2\,|S| < |N(S)|$. So, let T be a subset of S such that $|T| = t$. Then, because T and $N(T)$ are disjoint subsets of $(S \cup N(S))$ which itself is a subset of $V(P)$, we know that $|V(P)| \geq |T| + |N(T)|$. And since $|N(T)| > 2\,|T| = 2t$ by the theorem's assumption about G, we thus can say that $|V(P)| > 3t$. ■

Lecture 5: 1/23/2024

Theorem: If for every set S of vertices in a graph G, we have that $|N(S)| \geq \min\{2\,|S| + 1,\ |V(G) \setminus S|\}$, then G has a hamiltonian path.

Proof:
Once again let us define P as a longest path of G, as well as S as the set of end vertices of all possible rotations of P. Then by the same reasoning as before, we know that $|N(S)| \leq 2\,|S|$. Therefore, since $|N(S)|$ isn't greater than or equal to $2\,|S| + 1$, we know by the assumption of the theorem that $N(S)$ is greater than or equal to $|V(G) \setminus S|$.

Now $S \cup (V(G) \setminus S) = V(G)$ and $(S \cup N(S)) \subseteq V(P)$. Additionally, S and $(V(G) \setminus S)$ are disjoint to each other, as is S and $N(S)$. Thus, we can say that

$$|V(G)| = |S| + |(V(G) \setminus S)| \geq |S| + |N(S)| \leq |V(P)|$$

Therefore the longest path P must cover every vertex of G, meaning that it is a Hamiltonian path. ■

This is what is used to find Hamiltonian paths in random graphs.

A graph is uniquely Hamiltonian if it has exactly one Hamiltonian cycle.

Theorem: If all vertices in a graph G have odd degree, then every edge is in an even number of hamiltonian cycles.

In other words such a graph is not uniquely Hamiltonian.

Proof:
If the graph G in the theorem has no hamiltonian cycles, then we're done. Every edge is in 0 hamiltonian cycles.

Now pick an edge and suppose that there is a Hamiltonian cycle C containing it. We'll call that edge $\{u, v\}$ and let w be the vertex coming before u on C.

Then, let us define a new graph H whose vertices are Hamiltonian paths in G which start with the edge $\{u, v\}$. For example, $(C - \{u, w\}) \in V(H)$. Additionally, let $\{P, Q\}$ be an edge of H if P and Q are rotations of each other.

If $P \in V(H)$ is a hamiltonian path in G ending in a vertex $x \in V(G)$, then:

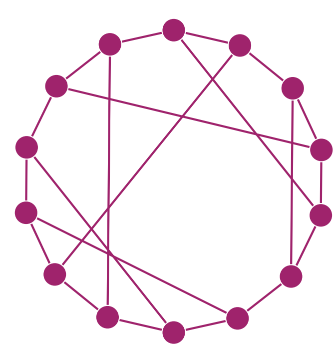
$$d_H(P) = \begin{cases} d_G(x) - 1 & x \notin N_G(u) \\ d_G(x) - 2 & x \in N_G(u) \end{cases}$$

Essentially, for every edge connecting to x except for the one already used by P, there is a rotation of P. We can then say that that rotation is a vertex of H if it includes the edge $\{u, v\}$ (In other words, a rotation including the edge $\{u, x\}$ would not be included in H).

If x is adjacent to u, then $P + \{\{u, x\}\}$ is a hamiltonian cycle containing $\{u, v\}$.

Now here is the clever part: since $d_G(x)$ is assumed to be odd for all $v \in V(G)$, we have that $d_H(P)$ is even if x is not adjacent to u and odd if x is adjacent to u. But now note that every graph has an even number of vertices of odd degree because of the handshaking lemma. So, there must be an even number of paths including the edge $\{u, v\}$ and ending in a vertex x such that x is adjacent to u. Or in other words, G has an even number of hamiltonian cyles containing the edge $\{u, v\}$. ■

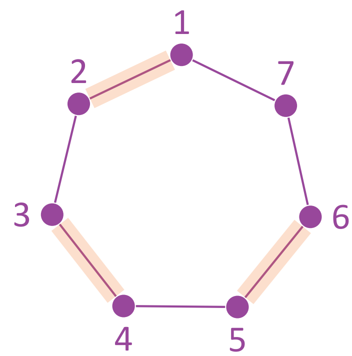
For example, by the above theorem we know that this graph is not uniquely Hamiltonian.



One note about the above theorem is that it can be interpretted as giving an algorithm for finding a second hamiltonian cycle given one hamiltonian cycle in a graph where all degrees are odd.

A matching in a graph is a set of vertex disjoint edges in the graph.

A vertex is saturated by a matching if one of the edges of the matching contains the vertex. Otherwise, we say the vertex is exposed by the matching.



The matching shown to the left is:
$M = \{\{1,2\},\{3,4\},\{5,6\}\}$

In the matching to the left, 7 is exposed and all other vertices are saturated.

A maximum matching in a graph is a matching with the maximum number of edges.

A perfect matching (or 1-factor) is a matching which saturates all the vertices of a graph.

Proposition: for a graph to have a perfect matching, it must have an even number of vertices.

Lecture 6: 1/30/2024

Hall's Theorem: Let G be a bipartite graph with parts A and B. Then G has a matching saturating A if and only if for every set $X \subseteq A$,

$$|N_G(X)| \geq |X|.$$

Note: we refer to the below statement as Hall's condition:
$\forall X \subseteq A,\ |N_G(X)| \geq |X|.$

Proof:
($\Longrightarrow$) If G has a matching M containing A, then for any $X \subseteq A$, we trivially have that $|N_G(X)| \geq |N_m(X)| = |X|$.

($\Longleftarrow$) To prove the other way, we proceed by induction on A while assuming Hall's condition is true.

Base case: assume that $|A| = 1$. In that case, because Hall's condition is assumed to be true, we know that $|N_G(A)| \geq |A| = 1$. Thus, for some $a \in A$ and $b \in B$, we know there exists an edge $\{a, b\} \in E(G)$. The matching of just that edge saturates A.

Induction step: assume that Hall's theorem works if $1 \leq |A| < n$. Then assume that $|A| = n$. Since we are assuming Hall's condition to be true, we know that for all proper subsets $X \subset A$, either $|N_G(X)| > |X|$ or $|N_G(X)| = |X|$ is true.

- Case 1: For all proper subsets X of A, $|N_G(X)| > |X|$...
  Pick an edge $\{a, b\} \in E(G)$ and consider $H = G - \{a\} - \{b\}$. We know that H will be a bipartite graph with two sets of vertices: $A' = A \setminus \{a\}$, and $B' = B \setminus \{b\}$. Additionally, given any $X \subseteq A'$, we know that $|N_H(X)| \geq |N_G(X)| - 1 \geq |X|$. So, by our inductive hypothesis, H has a matching covering A'. Now add the edge $\{a, b\}$ to this matching to get a matching in G covering A.

- Case 2: There exists a proper subset X of A such that $|N_G(X)| = |X|$...
  In this second case our reasoning for case 1 breaks down because given $X \subseteq A'$, we can no longer guarentee that $|N_G(X)| - 1 \geq |X|$. So, using that same set X, consider G_1 the induced graph of $N_G(X) \cup X$ and G_2 equal to $G - V(G_1)$.

  For any $Y \subseteq X$, we have that $|N_{G_1}(Y)| = |N_G(Y)| \geq |Y|$ due to Hall's condition. Thus, by our inductive hypothesis there exists a matching M_1 saturating X in G_1.

  Additionally, for any $Y \subseteq A \setminus X$, consider $N_G(Y \cup X)$. By assuming Hall's condition, we know that $|N_G(Y \cup X)| \geq |Y \cup X|$. And, because X and Y are disjoint, $|Y \cup X| = |Y| + |X|$. On the other hand, we also know that $N_G(Y \cup X) = N_{G_2}(Y) + N_G(X)$. So, $|N_{G_2}(Y)| + |N_G(X)| \geq |Y| + |X|$. Finally, because $|N_G(X)| = |X|$, we can cancel terms to get that: $|N_{G_2}(Y)| \geq |Y|$. Hence by our inductive hypothesis, we know there exists a matching M_2 saturating $A \setminus X$ in G_2.

  We now can combine M_1 and M_2 in order to get a matching in G covering A.

Side proposition: if in additional to Hall's condition holding, $|A| = |B|$, then the matching for G generated by the above proof is perfect. Meanwhile, if $|A| \neq |B|$, then it is impossible to make a perfect matching. To prove this, assume without loss of generality that $|A| < |B|$. Then, because $|N_G(B)|$ is a most $|A|$, we know that $|N_G(B)| < |B|$. So no matching can exist covering B.