# Math 140B Lecture Notes (Professor: Brandon Seward)

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# Lecture 1: 4/1/2024

Let $f : E \longrightarrow \mathbb{R}$ where $E \subseteq \mathbb{R}$.

Since $E$ is the domain of $f$, we shall also refer to it as $\mathrm{dom}(f)$.

Fix a point $x \in E \cap E'$. Then consider the function $\frac{f(t)-f(x)}{t-x}$ for $t \in \mathrm{dom}(f) \setminus \{x\}$ and define the derivative of $f$ at $x$ to be $f'(x) = \lim\limits_{t \to x} \left(\frac{f(t)-f(x)}{t-x}\right)$ provided that this limit exists. When the above limit exists, we say $f$ is differentiable at $x$.

We say $f$ is differentiable on $D \subseteq E$ if $f$ is differentiable at every point in $D$, and if $f$ is differentiable on its entire domain, then we call $f$ differentiable.

The function $f'(x) = \lim\limits_{t \to x} \left(\frac{f(t)-f(x)}{t-x}\right)$ is called the derivative of $f$.

Proposition 83: If $f$ is differentiable at $x$, then $f$ is continuous at $x$.

Proof:
Note that $\lim\limits_{t \to x} (f(t)) = \lim\limits_{t \to x} \left((t-x)\frac{f(t)-f(x)}{t-x} + f(x)\right)$.

Now $\lim\limits_{t \to x}(t-x) = 0$ and we know $\lim\limits_{t \to x} \frac{f(t)-f(x)}{t-x} = f'(x)$ exists because $f$ is differentiable at $x$. Also, obviously $\lim\limits_{t \to x} f(x) = f(x)$.

Thus by proposition 66 (check 140A notes), we know that:

$$\begin{aligned}\lim_{t \to x}\left((t-x)\frac{f(t)-f(x)}{t-x} + f(x)\right) &= \lim_{t \to x}(t-x)\lim_{t \to x}\left(\frac{f(t)-f(x)}{t-x}\right) + \lim_{t \to x} f(x) \\ &= 0 \cdot f'(x) + f(x) \\ &= f(x)\end{aligned}$$

Thus, $f$ is continuous at $x$.

Notes:

1. The above proposition says that differentiability is stronger than continuity.
2. The converse of this proposition is false. For example, the function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

Proposition 84: Suppose $f$ and $g$ are real valued functions with $\mathrm{dom}(f), \mathrm{dom}(g) \subseteq \mathbb{R}$. Also suppose $f$ and $g$ are differentiable at $x$. Then $f+g$, $fg$, and (when $g(x) \neq 0$) $\frac{f}{g}$ are differentiable at $x$ with:

(A) $(f+g)'(x) = f'(x) + g'(x)$ (sum rule)

(B) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ (product rule)

(C) $\left(\frac{f}{g}\right)'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ (quotient rule)

Proof:

(A) Since both $f$ and $g$ are differentiable, we know that both $f'(x) = \lim\limits_{t \to x} \frac{f(t)-f(x)}{t-x}$ and $g'(x) = \lim\limits_{t \to x} \frac{g(t)-g(x)}{t-x}$ exist. So by proposition 66:

$$(f+g)'(x) = \lim_{t \to x} \frac{f(t)+g(t)-f(x)-g(x)}{t-x} = \lim_{t \to x} \frac{f(t)-f(x)}{t-x} + \lim_{t \to x} \frac{g(t)-g(x)}{t-x}$$

This means $(f+g)'(x) = f'(x) + g'(x)$.

(B) Note that:

$$\begin{aligned}(fg)'(x) &= \lim_{t \to x} \frac{f(t)g(t)-f(x)g(x)}{t-x} \\ &= \lim_{t \to x} \frac{f(t)g(t)-f(x)g(t)+f(x)g(t)-f(x)g(x)}{t-x} \\ &= \lim_{t \to x} \left( g(t)\frac{f(t)-f(x)}{t-x} + f(x)\frac{g(t)-g(x)}{t-x} \right)\end{aligned}$$

By proposition 83, $g(t) \to g(x)$ as $t \to x$. Also, since both $f$ and $g$ are differentiable, we know $f'(x) = \lim\limits_{t \to x} \frac{f(t)-f(x)}{t-x}$ and $g'(x) = \lim\limits_{t \to x} \frac{g(t)-g(x)}{t-x}$ exist. So by proposition 66:

$$\lim_{t \to x} \left( g(t)\frac{f(t)-f(x)}{t-x} + f(x)\frac{g(t)-g(x)}{t-x} \right) = f'(x)g(x) + f(x)g'(x).$$

(C) Note that:

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \lim_{t \to x} \frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{t-x} \\ &= \lim_{t \to x} \left( \frac{1}{g(x)g(t)} \frac{f(t)g(x)-f(x)g(t)}{t-x} \right)\end{aligned}$$

# A List of How The Proposition Numbering in my Notes Lines up With Our Textbook:

| Proposition Number | Label in Textbook | Proposition Number | Label in Textbook |
|---|---|---|---|
| 83 | 5.2 | 84 | 5.3 |

Our textbook is *Principles of Mathematical Analysis* by Walter Rudin.