Math 188 Notes (Professor: Steven Sam)

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Lecture 1 Notes: 9/27/2024

Linear Recurrence Relations:

A sequence $(a_n)_{n\geq 0}$ satisfies a linear recurrence relation of order d if there exists $c_1, \ldots, c_d$ with $c_d \neq 0$ such that for all $n \geq d$:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_d a_{n-d}$$

(For $0 \leq n < d$, we usually explicitely specify a_n.)

To start this course, we're gonna discuss finding explicit (non-recursive) solutions.

Firstly, if $d = 1$, then this problem is easy. We can just plug in previous elements repeatedly to get that:

$$a_n = c_1 a_{n-1} = c_1^2 a_{n-2} = \ldots = c_1^n a_0$$

If $d = 2$, then plugging in previous elements doesn't help us really anymore. So how do we solve this problem now?

Theorem: Consider the characteristic polynomial $t^2 - c_1 t - c_2$ and let r_1, r_2 be the roots of that polynomial. If $r_1 \neq r_2$, then there exists α_1, α_2 such that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for all $n \geq 0$.

To solve for α_1 and α_2, plug in different values of n into our equation. Since $r_1 \neq r_2$, we know the below linear system has a unique solution:

$$a_0 = \alpha_1 + \alpha_2$$
$$a_1 = \alpha_1 r_1 + \alpha_2 r_2$$

Now backing up, why does the above method work?

Approach 1: (Vector Spaces)

The set of sequences $(a_n)_{n\geq 0}$ form a vector space. Furthermore given any constants c_1 and c_2, we know that the set of sequences satisfying $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ for all $n \geq 2$ is a subspace.

Proof:

Suppose (a_n) and (b_n) both satisfy that $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and $b_n = c_1 b_{n-1} + c_2 b_{n-2}$. Then given any constants γ and δ, we have that:

$$(\gamma a_n + \delta b_n) = c_1(\gamma a_{n-1} + \delta b_{n-1}) + c_2(\gamma a_{n-2} + \delta b_{n-2})$$

Hence, all linear combinations of any two sequences satisfying our linear recurrence relation also satisfies our linear recurrence relation.

Now what our above theorem is stating is that the sequences (r_1^n) and (r_2^n) span the subspace of solutions to our linear recurrence relation.

To see this, first note that (r_1^n) and (r_2^n) satisfy our recurrence relation.

If $n \geq 2$, then $r_i^n - c_1 r_i^{n-1} - c_2 r_i^{n-2} = r_i^{n-2}(r_i^2 - c_1 r_i - c_2) = r_i^{n-2}(0)$.

Hence, we know that $r_i^n = c_1 r_i^{n-1} + c_2 r_i^{n-2}$ for all $n \geq 2$.

Also, since we assumed $r_1 \neq r_2$, we know that (r_1^n) is linearly independent to (r_2^n). And finally, as mentioned before, we can solve a linear system of equations to find coffecients for a linear combination of (r_1^n) and (r_2^n) equal to any other sequence satisfying our recurrence relation.

Approach 2: (Formal Power Series)
Define the power series $A(x) = \sum_{n \geq 0} a_n x^n$. We call $A(x)$ a generating function of the sequence (a_n).

(We'll treat the formal power series more rigorously later...)

Now note that:

$$\begin{aligned} A(x) &= a_0 + a_1 x + \sum_{n \geq 2} a_n x^n \\ &= a_0 + a_1 x + \sum_{n \geq 2} (c_1 a_{n-1} + c_2 a_{n-2}) x^n \\ &= a_0 + a_1 x + c_1 \sum_{n \geq 2} a_{n-1} x^n + c_2 \sum_{n \geq 2} a_{n-2} x^n \\ &= a_0 + a_1 x + c_1 (A(x) - a_0) x + c_2 (A(x)) x^2 \end{aligned}$$

Isolating $A(x)$, we get the equation: $A(x) = \dfrac{a_0 + a_1 x - a_0 c_1 x}{1 - c_1 x - c_2 x^2}$.

Next, let's do fraction decomposition on our equation for $A(x)$.

Issue: We defined r_1 and r_2 as the roots of $t^2 - c_1 t - c_2 = (t - r_1)(t - r_2)$.

Trick: Plug in $t = \frac{1}{x}$. That way, we have that:

$$x^{-2} - c_1 x^{-1} - c_2 = (x^{-1} - r_1)(x^{-1} - r_2).$$

After that, multiply both sides of our equation by x^2 to get that:

$$1 - c_1 x - c_2 x^2 = (1 - r_1 x)(1 - r_2 x)$$

Since we're assuming $r_1 \neq r_2$, we know that for some constants α_1 and α_2, we have that:

$$A(x) = \frac{\alpha_1}{1 - r_1 x} + \frac{\alpha_2}{1 - r_2 x}$$

(If $r_1 = r_2$, then this step is where things will go differently.)

Now finally, we can rewrite $\frac{\alpha_1}{1 - r_1 x}$ as the geometric series $\alpha_1 \sum_{n \geq 0} (r_1 x)^n$. Doing likewise with $\frac{\alpha_2}{1 - r_2 x}$, we get that:

$$A(x) = \sum_{n \geq 0} a_n x^n = \alpha_1 \sum_{n \geq 0} (r_1 x)^n + \alpha_2 \sum_{n \geq 0} (r_2 x)^n = \sum_{n \geq 0} (\alpha_1 r_1^n + \alpha_2 r_2^n) x^n$$

Hence, we have for each n that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$.

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Approach 3: (Matrices)

If $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then we can say that: $\begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$

Letting $\boldsymbol{C} = \begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix}$, we thus know that: $\boldsymbol{C}^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$

Notably, the characteristic polynomial of $\boldsymbol{C}$ is $t^2 - c_1 t - c_2$. So the eigenvalues of $\boldsymbol{C}$ are r_1 and r_2. Because we assumed r_1 and r_2 are distinct, we know $\boldsymbol{C}$ is diagonalizable. Hence there exists an invertible matrix $\boldsymbol{B}$ such that:

$$\boldsymbol{B} \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \boldsymbol{B}^{-1} = \boldsymbol{C}$$

Now set $\begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{B}^{-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$. Then we can see that:

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \boldsymbol{C}^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \boldsymbol{B}\boldsymbol{D}^n \begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{B} \begin{bmatrix} r_1^n x \\ r_2^n y \end{bmatrix} = \begin{bmatrix} b_{1,1} r_1^n x + b_{1,2} r_2^n y \\ b_{2,1} r_1^n x + b_{2,2} r_2^n y \end{bmatrix}$$

Setting $\alpha_1 = b_{2,1} x$ and $\alpha_2 = b_{2,2} y$, we have thus found constants α_1 and α_2 such that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$.

Now some further questions to ask about recurrence relations are:

1. What if $r_1 = r_2$?
2. What if $d \geq 3$?
3. What if the recurrence relation is non-homogeneous or non-linear?

To start, let's answer question 1.

Theorem: Suppose r_1 and r_2 are the roots of $t^2 - c_1 t - c_2$ with $r_1 = r_2$. Then there exists α_1, α_2 such that $a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n$ for all $n \geq 0$.

As was true when $r_1 \neq r_2$, you can solve for α_1 and α_2 by plugging in different values of n into the equation in order to get a linear system of equations.

To explain why this is, let's revisit two of our previous approaches.

The Formal Power Series Approach Revisited:

Before, we were able to show that $A(x) = \dfrac{a_0 + (a_1 - a_0 c_1)x}{(1 - r_1 x)(1 - r_2 x)}$ without assuming anything about r_1 and r_2.

But when we assume $r_1 = r_2$, we then get a different partial fraction decomposition for $A(x)$. Specifically, we have that there exists constants β_1, β_2 such that:

$$A(x) = \frac{\beta_1}{1 - r_1 x} + \frac{\beta_2}{(1 - r_1 x)^2}$$

Now we'll go into more rigor later. But for now, accept that:

$$\tfrac{1}{(1-y)^2} = \tfrac{d}{dy}\left(\tfrac{1}{1-y}\right) = \tfrac{d}{dy}\left(\sum_{n\geq 0} y^n\right) = \sum_{n\geq 1} n y^{n-1} = \sum_{n\geq 0} (n+1) y^n$$

From the perspective of real analysis, this should make sense because derivatives of power series behave nicely when the power series converges.

Hence, we can write $A(x) = \sum\limits_{n\geq 0} a_n x^n = (\beta_1 + \beta_2) \sum\limits_{n\geq 0} r_1^n x^n + \beta_2 \sum\limits_{n\geq 0} n r_1^n x^n$.

Or in other words, setting $\alpha_1 = \beta_1 + \beta_2$ and $\alpha_2 = \beta_2$, we have that:

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n$$

The Matrix Approach Revisited:

If $r_1 = r_2$, then we must hav ethat the matrix $\boldsymbol{C}$ is not diagonalizable. For suppose it was, meaning there exists an invertible matrix $\boldsymbol{B}$ such that:

$$\boldsymbol{C} = \boldsymbol{B} \begin{bmatrix} r_1 & 0 \\ 0 & r_1 \end{bmatrix} \boldsymbol{B}^{-1}$$

Then we'd have to have that $\boldsymbol{C} = r_1 \boldsymbol{B}\boldsymbol{B}^{-1} = \begin{bmatrix} r_1 & 0 \\ 0 & r_1 \end{bmatrix}$. But we know $\boldsymbol{C}$ isn't that.

Since we know $\boldsymbol{C}$ Is not diagonalizable, we will instead use the *Jordan-normal form* of $\boldsymbol{C}$. Specifically, we know there exists an invertible matrix $\boldsymbol{B}$ such that:

$$\boldsymbol{C} = \boldsymbol{B} \begin{bmatrix} r_1 & 1 \\ 0 & r_1 \end{bmatrix} \boldsymbol{B}^{-1}$$

Don't worry for the time being about how to prove the Jordan-normal form of a matrix always exists.

This tells us that $\boldsymbol{C}^n = \boldsymbol{B} \begin{bmatrix} r_1 & 1 \\ 0 & r_1 \end{bmatrix}^n \boldsymbol{B}^{-1}$.

Also, you can show by induction that $\begin{bmatrix} r_1 & 1 \\ 0 & r_1 \end{bmatrix}^n = \begin{bmatrix} r_1^n & n r_1^{n-1} \\ 0 & r_1^n \end{bmatrix}$.

So finally, defining $\begin{bmatrix} x \\ y \end{bmatrix}$ as before and expanding out the expression, you can get an explicit equation for a_n.

As for answering question 2, if $d \geq 3$, then our characteristic polynomial becomes $t^d - c_1 t^{d-1} - \ldots - c_d$. We'll assume this polynomial has distinct roots $r_1, \ldots, r_m$ with multiplicities $s_1, \ldots, s_m$ respectively.

Theorem: There exists constants $\alpha_1, \ldots, \alpha_d$ such that:

$$a_n = \sum_{i=1}^{s_1} \alpha_i n^{i-1} r_1^n + \ldots + \sum_{i=s_1+\ldots+s_{m-1}+1}^{s_1+\ldots+s_m} \alpha_i n^{i-1} r_m^n$$

As before, to solve for α_1 through α_d, you can plug in values of n and solve a linear system of equations.

The approaches to prove this are the same as when $d = 2$. However, there are just more terms floating around that need to be dealt with.

Special case: suppose the characteristic polynomial is $(t-1)^d$.

In that case, because the root of the polynomial r is 1, there exists $\alpha_1, \ldots, \alpha_d$ such that

$$a_n = \alpha_1 + n\alpha_2 + n^2\alpha_3 + \ldots + n^{d-1}\alpha_d.$$

In other words, the formula for a_n is a polynomial in n.

Another perspective on the characteristic polynomial:

Let V be the vector space of sequences $(a_n)_{n\geq 0}$, and define the translation operator $T : V \longrightarrow V$ such that $(a_n)_{n\geq 0} \mapsto (a_{n+1})_{n\geq 0}$. Now, given $\boldsymbol{a} \in V$ and the recurrence relation $a_n = c_1 a_{n-1} + \ldots + c_d a_{n-d}$ for all $n \geq d$, we have that $\boldsymbol{a}$ satisfies our recurrence relation if and only if:

$$T^d \boldsymbol{a} = c_1 T^{d-1} \boldsymbol{a} + c_2 T^{d-2} \boldsymbol{a} + \ldots + c_d \boldsymbol{A}$$

In other words, we must have that $\boldsymbol{a} \in \ker(T^d - c_1 T^{d-1} - \ldots - c_d)$.

Homework 1:

(1) Find a closed formula for the following recurrence relation:

$$a_0 = 1, \;\; a_1 = 0, \;\; a_2 = 2,$$
$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} \;\; \text{for } n \geq 3$$