Math 140B Lecture Notes (Professor: Brandon Seward)

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Lecture 1: 4/1/2024

Let $f : E \longrightarrow \mathbb{R}$ where $E \subseteq \mathbb{R}$.

Since E is the domain of f, we shall also refer to it as $\mathrm{dom}(f)$.

Fix a point $x \in E \cap E'$. Then consider the function $\frac{f(t)-f(x)}{t-x}$ for $t \in \mathrm{dom}(f) \setminus \{x\}$ and define the derivative of f at x to be $f'(x) = \lim\limits_{t \to x} \left(\frac{f(t)-f(x)}{t-x} \right)$ provided that this limit exists. When the above limit exists, we say f is differentiable at x.

We say f is differentiable on $D \subseteq E$ if f is differentiable at every point in D, and if f is differentiable on its entire domain, then we call f differentiable.

The function $f'(x) = \lim\limits_{t \to x} \left(\frac{f(t)-f(x)}{t-x} \right)$ is called the derivative of f.

Proposition 83: If f is differentiable at x, then f is continuous at x.

Proof:
Note that $\lim\limits_{t \to x} (f(t)) = \lim\limits_{t \to x} \left((t-x)\frac{f(t)-f(x)}{t-x} + f(x) \right)$.

Now $\lim\limits_{t \to x}(t-x) = 0$ and we know $\lim\limits_{t \to x} \frac{f(t)-f(x)}{t-x} = f'(x)$ exists because f is differentiable at x. Also, obviously $\lim\limits_{t \to x} f(x) = f(x)$.

Thus by proposition 66 (check 140A notes), we know that:

$$\begin{aligned} \lim_{t \to x} \left((t-x)\frac{f(t)-f(x)}{t-x} + f(x) \right) &= \lim_{t \to x}(t-x) \lim_{t \to x} \left(\frac{f(t)-f(x)}{t-x} \right) + \lim_{t \to x} f(x) \\ &= 0 \cdot f'(x) + f(x) \\ &= f(x) \end{aligned}$$

Thus, f is continuous at x.

Notes:

1. The above proposition says that differentiability is stronger than continuity.
2. The converse of this proposition is false. For example, the function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

Proposition 84: Suppose f and g are real valued functions with $\mathrm{dom}(f), \mathrm{dom}(g) \subseteq \mathbb{R}$. Also suppose f and g are differentiable at x. Then $f+g$, fg, and (when $g(x) \neq 0$) $\frac{f}{g}$ are differentiable at x with:

(A) $(f+g)'(x) = f'(x) + g'(x)$ (sum rule)

(B) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ (product rule)

(C) $\left(\dfrac{f}{g}\right)'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ (quotient rule)

Proof:

(A) Since both f and g are differentiable, we know that both $f'(x) = \lim\limits_{t\to x} \frac{f(t)-f(x)}{t-x}$ and $g'(x) = \lim\limits_{t\to x} \frac{g(t)-g(x)}{t-x}$ exist. So by proposition 66:

$$(f+g)'(x) = \lim_{t\to x} \frac{f(t)+g(t)-f(x)-g(x)}{t-x} = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} + \lim_{t\to x} \frac{g(t)-g(x)}{t-x}$$

This means $(f+g)'(x) = f'(x) + g'(x)$.

(B) Note that:

$$\begin{aligned}(fg)'(x) &= \lim_{t\to x} \frac{f(t)g(t)-f(x)g(x)}{t-x} \\ &= \lim_{t\to x} \frac{f(t)g(t)-f(x)g(t)+f(x)g(t)-f(x)g(x)}{t-x} \\ &= \lim_{t\to x} \left(g(t)\frac{f(t)-f(x)}{t-x} + f(x)\frac{g(t)-g(x)}{t-x} \right)\end{aligned}$$

By proposition 83, $g(t) \to g(x)$ as $t \to x$. Also, since both f and g are differentiable, we know $f'(x) = \lim\limits_{t\to x} \frac{f(t)-f(x)}{t-x}$ and $g'(x) = \lim\limits_{t\to x} \frac{g(t)-g(x)}{t-x}$ exist. So by proposition 66:

$$\lim_{t\to x} \left(g(t)\frac{f(t)-f(x)}{t-x} + f(x)\frac{g(t)-g(x)}{t-x} \right) = f'(x)g(x) + f(x)g'(x).$$

(C) Note that:

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \lim_{t\to x} \frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{t-x} \\ &= \lim_{t\to x} \left(\frac{1}{g(x)g(t)} \frac{f(t)g(x)-f(x)g(t)}{t-x} \right) \\ &= \lim_{t\to x} \left(\frac{1}{g(x)g(t)} \frac{f(t)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(t)}{t-x} \right) \\ &= \lim_{t\to x} \left(\frac{1}{g(x)g(t)} \left(g(x)\frac{f(t)-f(x)}{t-x} - f(x)\frac{g(t)-g(x)}{t-x} \right) \right)\end{aligned}$$

Now, for the same reasons as before, we can use propositions 83 and 66 to separate the parts of the above limit to get that the above limit equals:

$$\frac{1}{(g(x))^2} \left(g(x)f'(x) - f(x)g'(x) \right)$$

If $f(x) = \alpha$ where $\alpha \in \mathbb{R}$ is constant, then trivially $f'(x) = 0$ for all x.
Meanwhile, if $f(x) = x$, then we can trivially find that $f'(x) = 1$.

Claim 1: For all $n \in \mathbb{Z}^+$, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Proof: (we proceed by induction)

Base Case:

If $n = 1$, then for $f(x) = x^1$, we have that $f'(x) = 1 \cdot x^0$.

Induction:

Now assume $n > 1$, and for $f(x) = x^{n-1}$, we have that $f'(x) = (n-1)x^{n-2}$.
For the rest of this proof, I'll abreviate the derivative of x^n as $(x^n)'$ and the derivative of x^{n-1} as $(x^{n-1})'$. Then using product rule, we know that:

$$(x^n)' = x(x^{n-1})' + 1 \cdot x^{n-1} = x \cdot (n-1)x^{n-2} + x^{n-1} = ((n-1)+1)x^{n-1} = nx^{n-1}$$

Claim 2: If f is differentiable at x and $\alpha \in \mathbb{R}$, then $(\alpha f)'(x) = \alpha f'(x)$.

Proof:

By the product rule: $(\alpha f)'(x) = \alpha f' + (\alpha)' f = \alpha f' + 0 \cdot f = \alpha f'$.

These combined with proposition 84 tells us that both polynomials and rational functions are differentiable over their domains.

Proposition 85: (chain rule)
Let f and g be real-valued functions with $\mathrm{dom}(f), \mathrm{dom}(g) \subseteq \mathbb{R}$. Let $x \in \mathbb{R}$. Suppose that f is differentiable at x and that g is differentiable at $f(x)$. Then $g \circ f$ is differentiable at x and $(g \circ f)'(x) = g'(f(x))f'(x)$.

Intuition:

$$\lim_{t \to x} \left(\frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \cdot \frac{f(t) - f(x)}{t - x} \right) = g'(f(t)) \cdot f'(t).$$

That said, the issue with this intuition is that we need to address the possibility that $f(t) - f(x) = 0$.

Proof:

Set $y = f(x)$ and define $v(s) = \begin{cases} \frac{g(s)-g(y)}{s-y} - g'(y) & \text{if } s \neq y \\ 0 & \text{if } s = y \end{cases}$

Note that v is continuous at y. This is because g being differentiable at $f(x) = y$ means that:

$$\lim_{s \to y} v(s) = \lim_{s \to y} \left(\frac{g(s)-g(y)}{s-y} - g'(y) \right) = g'(y) - g'(y) = 0 = v(y).$$

Also, since f is differentiable at x, we know that f is continuous at x. Therefore, $v \circ f$ is continuous at x by proposition 68. Additionally, setting $s = f(t)$, we know that $s \to y$ as $t \to x$ because f is continuous at x. Thus:

$$\lim_{t \to x} v(f(t)) = \lim_{s \to y} v(s) = 0$$

Finally, note that $g(s) - g(y) = (s-y)(g'(y) + v(s))$ for all s. Thus by substituting that into our limit:

$$\begin{aligned}(g \circ f)'(x) &= \lim_{t \to x} \tfrac{g(f(t)) - g(f(x))}{t-x} \\ &= \lim_{t \to x} \tfrac{f(t)-f(x)}{t-x}(g'(f(x)) + v(f(t))) \\ &= f'(x)\,(g'(f(x)) + 0) \qquad \text{(by proposition 66)}\end{aligned}$$

Lecture 2: 4/3/2024

To start off lecture, here is some intuition about the behavior of derivatives. We'll formally define sine and cosine later (on page ___) but for this section please take for granted that $(\sin(x))' = \cos(x)$. Additionally, please take for granted that the power rule holds for non-positive integer exponents.

1. Define $f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

   When $x \neq 0$, we have by chain rule that $f'(x) = \sin(\frac{1}{x}) - \frac{1}{x}\cos(\frac{1}{x})$.
   Meanwhile if $x = 0$, then $\frac{f(t)-f(0)}{t-0} = \frac{t\sin(\frac{1}{t})}{t} = \sin(\frac{1}{t})$ when $t \neq 0$.
   So $\lim\limits_{t \to 0}\left(\frac{f(t)-f(0)}{t-0}\right)$ does not exist, meaning f is not differentiable at x.

   This shows that $\mathrm{dom}(f')$ can be a proper subset of $\mathrm{dom}(f)$.

2. Define $g(x) = \begin{cases} x^2\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

   When $x \neq 0$, we have by chain rule that $g'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$.
   Meanwhile when $t \neq 0$:

   $$\left|\tfrac{g(t)-g(0)}{t-0}\right| = \left|\tfrac{t^2\sin(\frac{1}{t})}{t}\right| = \left|t\sin(\tfrac{1}{t})\right| \leq |t|.$$

   Thus $0 = \lim\limits_{t \to 0}(-t) \leq \lim\limits_{t \to 0}\left(\frac{g(t)-g(0)}{t-0}\right) \leq \lim\limits_{t \to 0}(t) = 0$, meaning $g'(0) = 0$.
   So $\mathrm{dom}(g') = \mathrm{dom}(g)$. That said, note that g' has a discontinuity of the second kind at 0. Therefore, because g is continuous, this shows that the derivative of a continuous function does not have to be continuous.

Let X be a metric space. A function $f : X \longrightarrow \mathbb{R}$ has a local maximum at $p \in X$ if $\exists \delta > 0 \;\; s.t. \;\; \forall x \in B_\delta(p), \;\; f(x) \leq f(p)$. Similarly, f has a local minimum if $\exists \delta > 0 \;\; s.t. \;\; \forall x \in B_\delta(p), \;\; f(x) \geq f(p)$.

Proposition 86: Let $f : (a, b) \longrightarrow \mathbb{R}$. If f has a local maximum at x and f is differentiable at x, then $f'(x) = 0$.

Proof:
Let $\delta > 0$ so that $\forall t \in B_\delta(x), \;\; f(t) \leq f(x)$. Then for all $t \in (x - \delta, x)$, $\frac{f(t)-f(x)}{t-x} \geq 0$. So $f'(x) \geq 0$. Similarly for all $t \in (x, x + \delta)$, we have $\frac{f(t)-f(x)}{t-x} \leq 0$. Thus $f'(x) \leq 0$.

Hence $f'(x) = 0$.

Note that analogous reasoning can show that if f has a local minimum at x and f is differentiable at x, then $f'(x) = 0$.

Proposition 87: If $f, g : [a, b] \longrightarrow \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then there exists $x \in (a, b)$ with:

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x).$$

Proof:
Define $h : [a, b] \longrightarrow \mathbb{R}$ by $h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$.
Then $h(a) = f(b)g(a) - g(b)f(a) = h(b)$.

Notice that h is continuous on $[a, b]$ and differentiable on (a, b) because of propositions 70 and 84. Since $h'(x) = (f(b) - f(a))g'(x) - (g(b) - g(a))f'(x)$, for all $x \in (a, b)$ it now suffices to show that there exists $x \in (a, b)$ with $h'(x) = 0$.

Since h is continuous on a compact set $[a, b]$, we know that h attains a maximum value and a minimum value over the interval $[a, b]$.

Case 1: If h is constant on $[a, b]$, then $h'(x) = 0$ for all $x \in (a, b)$.

Case 2: If there is $t \in (a, b)$ with $h(t) > h(a) = h(b)$, then $h(a)$ and $h(b)$ can't be the max. value that h attains on $[a, b]$. So h has a maximum at some point $x \in (a, b)$. Then by the last theorem, $h'(x) = 0$.

Case 3: If there is $t \in (a, b)$ with $h(t) < h(a) = h(b)$, then $h(a)$ and $h(b)$ can't be the min. value that h attains on $[a, b]$. So h has a minimum at some point $x \in (a, b)$. Then by the last theorem, $h'(x) = 0$.

Proposition 88: (Mean Value Theorem)
If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there is $x \in (a, b)$ with $f(b) - f(a) = (b - a)f'(x)$.

To prove this, apply the previous proposition with $g(x) = x$.

Proposition 89: Suppose $f(a, b) \longrightarrow \mathbb{R}$ is differentiable. Then:

- If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotone increasing.
- If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotone decreasing.
- If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

Proof:
For all $a < x_1 < x_2 < b$, we know by the mean value theorem that there exists $t \in (x_1, x_2)$ with $f(x_2) - f(x_1) = (x_2 - x_1)f'(t)$. Then since $x_2 - x_1 > 0$, the sign of $f(x_2) - f(x_1)$ depends entirely on $f'(t)$.

Exercise 5.2 Let $f : (a, b) \longrightarrow \mathbb{R}$ be differentiable with $f'(x) > 0$. Then f is strictly increasing.

For all $a < x_1 < x_2 < b$, we know by the mean value theorem that there exists $t \in (x_1, x_2)$ with $f(x_2) - f(x_1) = (x_2 - x_1)f'(t)$. Since $(x_2 - x_1)$ and $f'(t)$ are positive, we thus have that $f(x_2) - f(x_1) > 0$.

As a consequence of f being strictly increasing, we know f is injective. Thus if we restrict the codomain of f to $f((a, b))$, then f is bijective, meaning there exists a function $g = f^{-1}$ such that $(g \circ f)(x) = x = (f \circ g)(x)$. Now we show that g is continuous.

Let $y \in \mathrm{dom}(g)$. Then there exists $x \in \mathrm{dom}(f)$ such that $f(x) = y$. But, since $\mathrm{dom}(f)$ is an open interval, we know that there exists δ such that $[x - \delta, x + \delta] \subseteq \mathrm{dom}(f)$. If we let $\hat{f}$ be the restriction of f whose domain is $[x - \delta, x + \delta]$ and whose codomain is the image of $[x - \delta, x + \delta]$

A List of How The Proposition Numbering in my Notes Lines up With Our Textbook:

Proposition Number	Label in Textbook	Proposition Number	Label in Textbook
83	5.2	84	5.3
85	5.5	86	5.8
87	5.9	88	5.10
89	5.11	90	
91		92	

Our textbook is *Principles of Mathematical Analysis* by Walter Rudin.