Math 140A Lecture Notes (Professor: Brandon Seward)

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Lecture 1: 1/8/2024

An order on a set S, typically denoted as $<$, is a binary relation satisfying:

1. $\forall x, y \in S$, exactly one of the following is true:
   - $x < y$
   - $x = y$
   - $y < x$
2. given $x, y, z \in S$, we have that $x < y < z \Rightarrow x < z$

As a shorthand, we will specify that

- $x > y \Leftrightarrow y < x$
- $x \leq y \Leftrightarrow x < y$ or $x = y$
- $x \geq y \Leftrightarrow x > y$ or $x = y$

An ordered set is a set with a specified ordering. Let S be an ordered set and E be a nonempty subset of S.

- If $b \in S$ has the property that $\forall x \in E,\ x \leq b$, then we call b an upperbound to E and say that E is bounded above by b.

- if $b \in S$ has the property that $\forall x \in E,\ x \geq b$, then we call b an lower bound to E and say that E is bounded below by b.

- We call $\beta \in S$ the least upperbound to E if β is an upper bound to E and β is the least of all upperbounds to E. In this case, we also commonly call β the supremum of E and denote it as $\sup E$.

- We call $\beta \in S$ the greatest lower bound to E if β is an lower bound to E and β is the greatest of all lower bounds to E. In this case, we also commonly call β the infimum of E and denote it as $\inf E$.

- We call $e \in E$ the maximum of E if $\forall x \in E,\ x \leq e$

- We call $e \in E$ the minimum of E if $\forall x \in E,\ x \geq e$

Fact: For an ordered set S and nonempty $E \subseteq S$, either:

- neither $\max E$ nor $\sup E$ exists
- $\sup E$ exists but $\max E$ does not exist
- $\max E$ exists and $\sup E = \max E$

Using $\mathbb{Q}$ as our ordered set...

- For $E = \{q \in \mathbb{Q} \mid 0 < q < 1\}$, $\max E$ does not exist but $\sup E$ exists and equals 1.

  To understand why, note that the set of all upper bounds of E is equal to $\{q \in \mathbb{Q} \mid q \geq 1\}$ and 1 is obviously the smallest element of that set. Thus, 1 is the supremum of E. However, $1 \notin E$. Thus, if $\max E$ did exist, it would have to not equal 1. But that would contradict 1 being the least greatest bound.

- For $E = \{q \in \mathbb{Q} \mid 0 < q \leq 1\}$, $\max E$ and $\sup E$ exist and they both are equal to 1

  The reasoning for this is similar to that for the previous set.

- For $E = \{q \in \mathbb{Q} \mid q^2 < 2\}$, neither $\max E$ and $\sup E$ exist.

  To prove this, we can show that there exists a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $\forall q \in \mathbb{Q}^+$, we have that $q^2 < 2 \Rightarrow q^2 < (f(q))^2 < 2$ and $2 < q^2 \Rightarrow 2 < (f(q))^2 < q^2$. Thus, we can show that the set of upper bounds to E has no minimum element (meaning $\sup E$ is undefined) and E itself has no maximum element.

  Now instead of being like Rudin and simply providing the desired function, I want to present how one may come up with a function that works for this proof themselves.

  Firstly, note that for the following reasons, we know our desired function must be a rational function:

  ◇ $\forall q \in \mathbb{Q}$, $f(q) \in \mathbb{Q}$. Based on this, we can't use any radicals, trig functions, logarithms, or exponentials in our desired function.

  ◇ $q^2 > 2 \Rightarrow f(q) < q$. In other words, f needs to grow slower than a linear function. Thus, we can rule out the possibility of f being a polynomial.

  ◇ If we wanted f to be a linear function, it would have to have the form $f(q) = \alpha(q - \sqrt{2}) + \sqrt{2}$ where α is some constant. This is because when $q^2 = 2$, $f(q) = q$. However, there is no value one can set α to which both eliminates the presence of irrational numbers in that function while simultaneously making $f(q) \neq q$ when $q^2 \neq 2$. So no linear function can possibly work for this proof.

  Having narrowed our search, let's now pick some convenient properties we would wish our proof function to have. Specifically, let's force f to be constantly increasing, have a y-intercept of 1, and approach a horizontal asymptote of $y = 2$. Doing this, we can now say that an acceptable function will have the following form where α is an unknown constant:

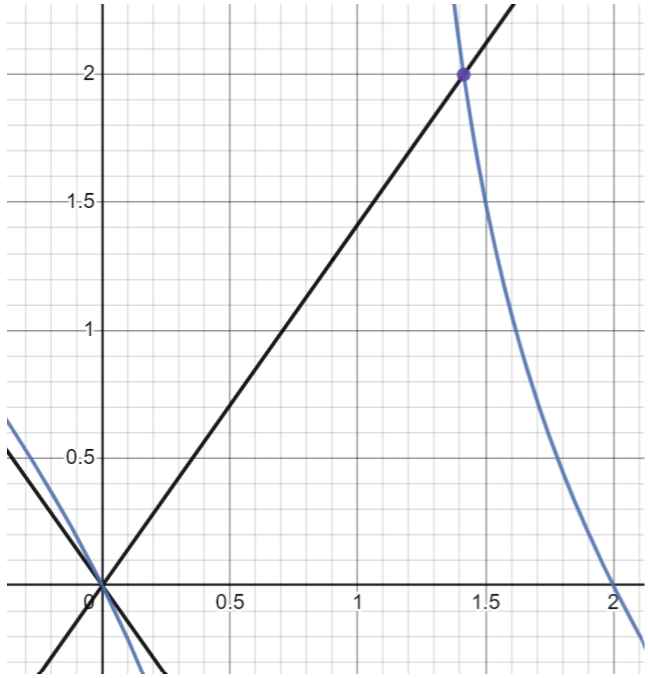
  $$f(q) = 1 + \frac{q}{q + \alpha}$$

  And finally, we can solve for α using the following system of equations:

  $$(1 + \tfrac{q}{q+\alpha})^2 = 2$$

  $$1 + \tfrac{q}{q+\alpha} = q$$

Now here's where a graphing calculator like Desmos can be very useful. Instead of painstakely having to solve for α, we can use a graphing calculator to approximate the value of α that satisfies our system of equations.



Based on the graph above, it looks like $f(q) = 1 + \frac{q}{q+2}$ will work for our proof. And sure enough it does. Furthermore, we can verify that the function we came up with is equivalent to that which Rudin presents.

We say an ordered set S has the least upperbound property if and only if when $E \subseteq S$ is nonempty and bounded above, then the supremum of E exists in S. Additionally, we say an ordered set S has the greatest lower bound property if and only if when $E \subseteq S$ is nonempty and bounded below, then the infimum of E exists in S.

When we define the set of real numbers, this will be one of the fundamental properties of that set.

Lecture 2: 1/10/2024

Proposition 1: S has the least upperbound property if and only if S has the greatest lower bound property.

Proof: Let's say we have an ordered set S

Assume S has the least upperbound property. Then, let $B \subseteq S$ be a nonempty subset which is bounded below. Additionally, let $A \subseteq S$ be the set of all lower bounds of B.

We know that $A \neq \emptyset$ because we assumed that B is bounded below. Thus, at least one lower bound to B exists and belongs to A. Additionally, because we assumed B is nonempty, we can say that each $b \in B$ is an upper bound to A. Thus, A is bounded above. Because of these two facts, we can apply the greatest lower bound property to say that the supremum of A exists.

Let's define $\alpha \coloneqq \sup A$. With that, our goal is now to show that $\alpha = \inf B$. To do this, we need to show firstly that α is a lower bound to B and secondly that it is greater than all other lower bounds of B.

1. For each $b \in B$, we have that b is an upperbound to A. And since $\alpha = \sup A$ is the least upperbound to A, we must have that $\alpha \leq b$. Thus α is a lower bound to B.

2. If $x \in S$ is a lower bound to B, then $x \in A$. And since $\alpha = \sup A$, $x \leq \alpha$. This shows that α is greater than or equal to all other lower bounds.

Hense, α is the infimum of B. And since we did this for a general $B \subseteq S$, we can thus say that S has the greatest lower bound property.

Now we skipped doing the reverse direction proof because it is almost completely identical to the foward direction proof. However, just know that the above proposition is an if and only if statement. ■

A field is a set F equipped with 2 binary operations, denoted $+$ and $\cdot$, and containing two elements $0 \neq 1 \in F$ satisfying the following conditions for all $x, y, z \in F$:

- Associativity: $$(x+y)+z = x+(y+z)$$ $$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$
- Commutativity: $$x+y = y+x$$ $$x \cdot y = y \cdot x$$
- Identity: $$0+x = x$$ $$1 \cdot x = x$$
- Inverses: $$\forall x \in F, \ \ \exists {-x} \in F \ \ s.t. \ \ x + -x = 0$$ $$\forall x \neq 0 \in F, \ \ \exists \frac{1}{x} \in F \ \ s.t. \ \ x \cdot \frac{1}{x} = 1$$
- Distributivity: $$x \cdot (y+z) = (x \cdot y) + (x \cdot z)$$

We shall assign the following notation:

We write ______	to mean ______
$x - y$	$x + -y$
$\frac{x}{y}$	$x \cdot \frac{1}{y}$
2	$1 + 1$
$2x$	$x + x$
x^2	$x \cdot x$

Now what follows is a number of propositions concerning the arithmetic properties of a field...

For a field F and elements $x, y, z \in F$, we have the following propositions:

Proposition 2.1: $x + y = x + z \Rightarrow y = z$

Proof: Assume $x + y = x + z$. Then...

$$\begin{aligned} y &= 0 + y && \text{(addition identity property)} \\ &= (-x + x) + y && \text{(addition inverse property)} \\ &= -x + (x + y) && \text{(addition associative property)} \\ &= -x + (x + z) && \text{(by our assumption)} \\ &= (-x + x) + z && \text{(addition associative property)} \\ &= 0 + z && \text{(addition inverse property)} \\ &= z && \text{(addition identity property)} \end{aligned}$$

Proposition 2.2: $x + y = x \Rightarrow y = 0$

Proof: Plug in $z = 0$ into proposition 2.1. in order to get that $y = z = 0$.

Proposition 2.3: $x + y = 0 \Rightarrow y = -x$

Proof: Plug in $z = -x$ into proposition 2.1. in order to get that $y = z = -x$.

Proposition 2.4: $-(-x) = x$

Proof: Observe that $x + -x = -x + x = 0$ by the inverse and commutative properties of addition. Then, by proposition 2.3, we know that $-x + x = 0 \Rightarrow x = -(-x)$.

Proposition 2.5: $x \cdot y = x \cdot z$ and $x \neq 0 \Rightarrow y = z$

Proof: Assume $x \cdot y = x \cdot z$ and $x \neq 0$. Then...

$$\begin{aligned} y &= 1 \cdot y && \text{(multiplication identity property)} \\ &= (\tfrac{1}{x} \cdot x) \cdot y && \text{(multiplication inverse property)} \\ &= \tfrac{1}{x} \cdot (x \cdot y) && \text{(multiplication associative property)} \\ &= \tfrac{1}{x} \cdot (x \cdot z) && \text{(by our assumption)} \\ &= (\tfrac{1}{x} \cdot x) \cdot z && \text{(multiplication associative property)} \\ &= 1 \cdot z && \text{(multiplication inverse property)} \\ &= z && \text{(multiplication identity property)} \end{aligned}$$

Note that to use the multiplication inverse property, we have to assume $x \neq 0$!!

Proposition 2.6: $x \cdot y = x \Rightarrow y = 1$

Proof: Plug in $z = 1$ into proposition 2.5. in order to get that $y = z = 1$.

Proposition 2.7: $x \cdot y = 1 \Rightarrow y = \dfrac{1}{x}$

Proof: Plug in $z = \dfrac{1}{x}$ into proposition 2.5. in order to get that $y = z = \dfrac{1}{x}$.

Proposition 2.8: $\dfrac{1}{\frac{1}{x}} = x$

Proof: Observe that $x \cdot \dfrac{1}{x} = \dfrac{1}{x} \cdot x = 0$ by the inverse and commutative properties of multiplication. Then, by proposition 2.7, we know that $\dfrac{1}{x} \cdot x = 0 \Rightarrow x = \dfrac{1}{\frac{1}{x}}$.

Proposition 2.9: $0 \cdot x = 0$

Proof: $(0 \cdot x) + (0 \cdot x) = (0 + 0) \cdot x = 0 \cdot x$. Thus we have an expression of the form $a + b = a$ which we can use proposition 2.2 on. Hence, we can conclude $0 \cdot x = 0$.

Proposition 2.10: $x \neq 0$ and $y \neq 0 \Rightarrow x \cdot y \neq 0$

Proof: since $x, y \neq 0$, we can say that $x \cdot y \cdot \dfrac{1}{x} \cdot \dfrac{1}{y} = 1 \neq 0$. Now by proposition 2.9, $x \cdot y = 0 \Rightarrow (x \cdot y) \cdot (\dfrac{1}{x} \cdot \dfrac{1}{y}) = 0$. However, we know that is not the case. So $x \cdot y$ can't equal zero.

Lecture 3: 1/12/2024