Math 140A Lecture Notes (Professor: Brandon Seward)

Isabelle Mills

February 21, 2024

Lecture 1: 1/8/2024

An order on a set S, typically denoted as $<$, is a binary relation satisfying:

1. $\forall x, y \in S$, exactly one of the following is true:
   - $x < y$
   - $x = y$
   - $y < x$
2. given $x, y, z \in S$, we have that $x < y < z \Rightarrow x < z$

As a shorthand, we will specify that

- $x > y \Leftrightarrow y < x$
- $x \leq y \Leftrightarrow x < y$ or $x = y$
- $x \geq y \Leftrightarrow x > y$ or $x = y$

An ordered set is a set with a specified ordering. Let S be an ordered set and E be a nonempty subset of S.

- If $b \in S$ has the property that $\forall x \in E,\ x \leq b$, then we call b an upperbound to E and say that E is bounded above by b.
- if $b \in S$ has the property that $\forall x \in E,\ x \geq b$, then we call b an lower bound to E and say that E is bounded below by b.
- We call $\beta \in S$ the least upperbound to E if β is an upper bound to E and β is the least of all upperbounds to E. In this case, we also commonly call β the supremum of E and denote it as $\sup E$.
- We call $\beta \in S$ the greatest lower bound to E if β is an lower bound to E and β is the greatest of all lower bounds to E. In this case, we also commonly call β the infimum of E and denote it as $\inf E$.
- We call $e \in E$ the maximum of E if $\forall x \in E,\ x \leq e$
- We call $e \in E$ the minimum of E if $\forall x \in E,\ x \geq e$

Fact: For an ordered set S and nonempty $E \subseteq S$, either:

- neither $\max E$ nor $\sup E$ exists
- $\sup E$ exists but $\max E$ does not exist
- $\max E$ exists and $\sup E = \max E$

Using $\mathbb{Q}$ as our ordered set...

- For $E = \{q \in \mathbb{Q} \mid 0 < q < 1\}$, $\max E$ does not exist but $\sup E$ exists and equals 1.

  To understand why, note that the set of all upper bounds of E is equal to $\{q \in \mathbb{Q} \mid q \geq 1\}$ and 1 is obviously the smallest element of that set. Thus, 1 is the supremum of E. However, $1 \notin E$. Thus, if $\max E$ did exist, it would have to not equal 1. But that would contradict 1 being the least greatest bound.

- For $E = \{q \in \mathbb{Q} \mid 0 < q \leq 1\}$, $\max E$ and $\sup E$ exist and they both are equal to 1

  The reasoning for this is similar to that for the previous set.

- For $E = \{q \in \mathbb{Q} \mid q^2 < 2\}$, neither $\max E$ and $\sup E$ exist.

  To prove this, we can show there exists a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $\forall q \in \mathbb{Q}^+$, $q^2 < 2 \Rightarrow q^2 < (f(q))^2 < 2$ and $2 < q^2 \Rightarrow 2 < (f(q))^2 < q^2$. That way we can give a counter example to any possible claimed supremum or maximum of E.

  Now instead of being like Rudin and simply providing the desired function, I want to present how one may come up with a function that works for this proof themselves.

  Firstly, note that for the following reasons, we know our desired function must be a rational function:

  - ◇ $\forall q \in \mathbb{Q}$, $f(q) \in \mathbb{Q}$. Based on this, we can't use any radicals, trig functions, logarithms, or exponentials in our desired function.
  - ◇ $q^2 > 2 \Rightarrow f(q) < q$. In other words, f needs to grow slower than a linear function. Thus, we can rule out the possibility of f being a polynomial.
  - ◇ If we wanted f to be a linear function, it would have to have the form $f(q) = \alpha(q - \sqrt{2}) + \sqrt{2}$ where α is some constant. This is because when $q^2 = 2$, $f(q) = q$. However, there is no value one can set α to which both eliminates the presence of irrational numbers in that function while simultaneously making $f(q) \neq q$ when $q^2 \neq 2$. So no linear function can possibly work for this proof.

  Having narrowed our search, let's now pick some convenient properties we would wish our proof function to have. Specifically, let's force f to be constantly increasing, have a y-intercept of 1, and approach a horizontal asymptote of $y = 2$. Doing this, we can now say that an acceptable function will have the following form where α is an unknown constant:

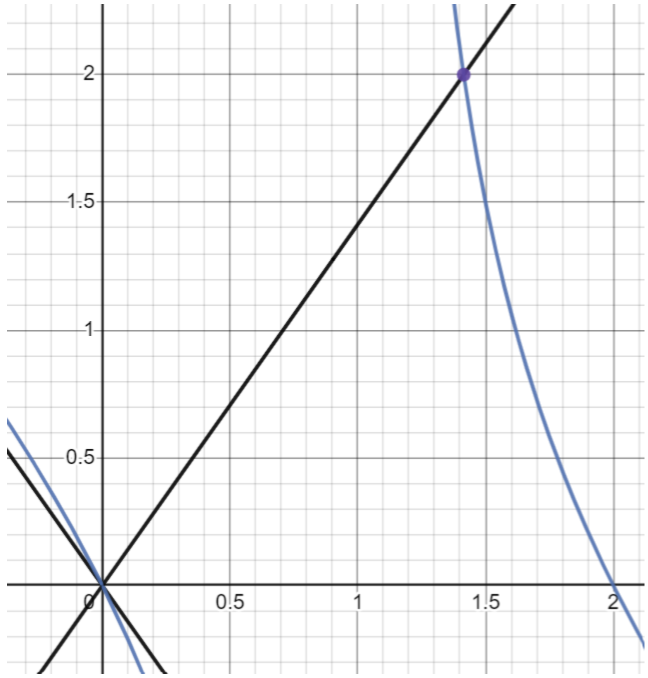
  $$f(q) = 1 + \frac{q}{q + \alpha}$$

  And finally, we can solve for α using the following system of equations:

  $$(1 + \tfrac{q}{q+\alpha})^2 = 2$$

  $$1 + \tfrac{q}{q+\alpha} = q$$

Now here's where a graphing calculator like Desmos can be very useful. Instead of painstakely having to solve for α, we can use a graphing calculator to approximate the value of α that satisfies our system of equations.



Based on the graph above, it looks like $f(q) = 1 + \frac{q}{q+2}$ will work for our proof. And sure enough it does. Furthermore, we can verify that the function we came up with is equivalent to that which Rudin presents.

We say an ordered set S has the least upperbound property if and only if when $E \subseteq S$ is nonempty and bounded above, then the supremum of E exists in S. Additionally, we say an ordered set S has the greatest lower bound property if and only if when $E \subseteq S$ is nonempty and bounded below, then the infimum of E exists in S.

When we define the set of real numbers, this will be one of the fundamental properties of that set.

Lecture 2: 1/10/2024

Proposition 1: S has the least upperbound property if and only if S has the greatest lower bound property.

Proof: Let's say we have an ordered set S

Assume S has the least upperbound property. Then, let $B \subseteq S$ be a nonempty subset which is bounded below. Additionally, let $A \subseteq S$ be the set of all lower bounds of B.

We know that $A \neq \emptyset$ because we assumed that B is bounded below. Thus, at least one lower bound to B exists and belongs to A. Additionally, because we assumed B is nonempty, we can say that each $b \in B$ is an upper bound to A. Thus, A is bounded above. Because of these two facts, we can apply the greatest lower bound property to say that the supremum of A exists.

Let's define $\alpha := \sup A$. With that, our goal is now to show that $\alpha = \inf B$. To do this, we need to show firstly that α is a lower bound to B and secondly that it is greater than all other lower bounds of B.

1. For each $b \in B$, we have that b is an upperbound to A. And since $\alpha = \sup A$ is the least upperbound to A, we must have that $\alpha \leq b$. Thus α is a lower bound to B.

2. If $x \in S$ is a lower bound to B, then $x \in A$. And since $\alpha = \sup A$, $x \leq \alpha$. This shows that α is greater than or equal to all other lower bounds.

Hence, α is the infimum of B. And since we did this for a general $B \subseteq S$, we can thus say that S has the greatest lower bound property.

Now we skipped doing the reverse direction proof because it is almost identical to the foward direction proof. However, just know that the above proposition is an if and only if statement. ■

A field is a set F equipped with 2 binary operations, denoted $+$ and $\cdot$, and containing two elements $0 \neq 1 \in F$ satisfying the following conditions for all $x, y, z \in F$:

- Associativity: $$(x+y)+z = x+(y+z)$$ $$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$
- Commutativity: $$x + y = y + x$$ $$x \cdot y = y \cdot x$$
- Identity: $$0 + x = x$$ $$1 \cdot x = x$$
- Inverses: $$\forall x \in F, \ \exists {-x} \in F \ \ s.t. \ \ x + -x = 0$$ $$\forall x \neq 0 \in F, \ \exists \frac{1}{x} \in F \ \ s.t. \ \ x \cdot \frac{1}{x} = 1$$
- Distributivity: $$x \cdot (y+z) = (x \cdot y) + (x \cdot z)$$

We shall assign the following notation:

We write ______	to mean ______
$x - y$	$x + -y$
$\dfrac{x}{y}$	$x \cdot \dfrac{1}{y}$
2	$1 + 1$
$2x$	$x + x$
x^2	$x \cdot x$
xy	$x \cdot y$

Now what follows is a number of propositions concerning the arithmetic properties of a field...

For a field F and elements $x, y, z \in F$, we have the following propositions:

Proposition 2.1: $x + y = x + z \Rightarrow y = z$

Proof: Assume $x + y = x + z$. Then...

$$\begin{aligned}
y &= 0 + y && \text{(addition identity property)} \\
&= (-x + x) + y && \text{(addition inverse property)} \\
&= -x + (x + y) && \text{(addition associative property)} \\
&= -x + (x + z) && \text{(by our assumption)} \\
&= (-x + x) + z && \text{(addition associative property)} \\
&= 0 + z && \text{(addition inverse property)} \\
&= z && \text{(addition identity property)}
\end{aligned}$$

Proposition 2.2: $x + y = x \Rightarrow y = 0$

Proof: Plug in $z = 0$ into proposition 2.1. in order to get that $y = z = 0$.

Proposition 2.3: $x + y = 0 \Rightarrow y = -x$

Proof: Plug in $z = -x$ into proposition 2.1. in order to get that $y = z = -x$.

Proposition 2.4: $-(-x) = x$

Proof: Observe that $x + -x = -x + x = 0$ by the inverse and commutative properties of addition. Then, by proposition 2.3, we know that $-x + x = 0 \Rightarrow x = -(-x)$.

Proposition 2.5: $x \cdot y = x \cdot z$ and $x \neq 0 \Rightarrow y = z$

Proof: Assume $x \cdot y = x \cdot z$ and $x \neq 0$. Then...

$$\begin{aligned} y &= 1 \cdot y && \text{(multiplication identity property)} \\ &= (\tfrac{1}{x} \cdot x) \cdot y && \text{(multiplication inverse property)} \\ &= \tfrac{1}{x} \cdot (x \cdot y) && \text{(multiplication associative property)} \\ &= \tfrac{1}{x} \cdot (x \cdot z) && \text{(by our assumption)} \\ &= (\tfrac{1}{x} \cdot x) \cdot z && \text{(multiplication associative property)} \\ &= 1 \cdot z && \text{(multiplication inverse property)} \\ &= z && \text{(multiplication identity property)} \end{aligned}$$

Note that to use the multiplication inverse property, we have to assume $x \neq 0$!!

Proposition 2.6: $x \cdot y = x \Rightarrow y = 1$

Proof: Plug in $z = 1$ into proposition 2.5. in order to get that $y = z = 1$.

Proposition 2.7: $x \cdot y = 1 \Rightarrow y = \dfrac{1}{x}$

Proof: Plug in $z = \frac{1}{x}$ into proposition 2.5. in order to get that $y = z = \frac{1}{x}$.

Proposition 2.8: $\dfrac{1}{\frac{1}{x}} = x$

Proof: Observe that $x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$ by the inverse and commutative properties of multiplication. Then, by proposition 2.7, we know that $\frac{1}{x} \cdot x = 1 \Rightarrow x = \dfrac{1}{\frac{1}{x}}$.

Proposition 2.9: $0 \cdot x = 0$

Proof: $(0 \cdot x) + (0 \cdot x) = (0 + 0) \cdot x = 0 \cdot x$. Thus we have an expression of the form $a + b = a$ which we can use proposition 2.2 on. Hence, we can conclude $0 \cdot x = 0$.

Proposition 2.10: $x \neq 0$ and $y \neq 0 \Rightarrow x \cdot y \neq 0$

Proof: since $x, y \neq 0$, we can say that $x \cdot y \cdot \frac{1}{x} \cdot \frac{1}{y} = 1 \neq 0$. Now by proposition 2.9, $x \cdot y = 0 \Rightarrow (x \cdot y) \cdot \left(\frac{1}{x} \cdot \frac{1}{y}\right) = 0$. However, we know that is not the case. So $x \cdot y$ can't equal zero.

Lecture 3: 1/12/2024

Proposition 2.11: $(-x)y = -(xy) = x(-y)$

Proof: $xy + (-x)y = (x + -x)y = 0y = 0$. Thus by proposition 2.3, $(-x)y = -(xy)$. We can make a similar argument to also say that $x(-y) = -(xy)$.

Proposition 2.12: $(-x)(-y) = xy$

Proof: Using proposition 2.11, we can say that $(-x)(-y) = -(x(-y)) = -(-(xy))$. Then by proposition 2.4, we can conclude $-(-(xy)) = xy$.

An ordered field is a field F equipped with an ordering $<$ satisfying $\forall x, y, z \in F$:

OF1. $y < z \Rightarrow y + x < z + x$

OF2. $(x > 0$ and $y > 0) \Rightarrow xy > 0$

For x in an ordered field, we call x positive if and only if $x > 0$. Similarly, we call x negative if and only if $x < 0$.

Proposition 3: For an ordered field F and $x, y, z \in F$, we have:

1. $x < y \Leftrightarrow -y < -x$

   Proof: By property OF1 of an ordered field, we can say that $x < y \Rightarrow x + (-x + -y) < y + (-x + -y) \Rightarrow -y < -x$.

2. $(x > 0$ and $y < z) \Rightarrow xy < xz$

   Proof: By property OF1 of an ordered field, $y < z \Rightarrow y - y < z - y$. Or in other words, $0 < z - y$. Therefore, since x is also positive by assumption, property OF2 of an ordered field tells us that $x(z - y) > 0$. Finally, adding xy to both sides by property OF1 and then distributing gives us: $xz - xy + xy = xz < xy$.

3. $(x < 0$ and $y < z) \Rightarrow xy > xz$

   Proof: Since $x < 0$, we have $-x > 0$ by proposition 3.1. Then by applying proposition 3.2, we know that $(-x > 0$ and $y < z) \Rightarrow -xy < -xz$. Finally, by reapplying proposition 3.1, this becomes $xy > xz$.

4. $x \neq 0 \Rightarrow x^2 > 0$

   Proof: If $x > 0$, then $x^2 = xx > 0x = 0$ by property OF2 of an ordered field. Meanwhile, if $x < 0$, then $-x > 0$ by proposition 3.1. So $(-x)(-x) > 0$ by property OF2. But $(-x)(-x) = x^2$ by proposition 2.12. So $x^2 > 0$.

5. $0 < x < y \Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$

Proof: Since $y > 0$ and $y \cdot \frac{1}{y} = 1 > 0 = 0 \cdot \frac{1}{y}$, we must have $\frac{1}{y} > 0$ by propositions 3.2 and 3.3. Note that $\frac{1}{y} \neq 0$ because if it did, $y \cdot \frac{1}{y} = 0$. Similarly, we can show $\frac{1}{x} > 0$. Now multiply both sides of $x < y$ by the positive element $\frac{1}{x} \cdot \frac{1}{y}$ and apply proposition 3.2 to get that $\frac{1}{y} < \frac{1}{x}$.

Theorem: There is (up to isomorphism) precisely one ordered field that contains $\mathbb{Q}$ and has the least upper bound property. We denote this field $\mathbb{R}$ and we call its elements real numbers.

In other words, this theorem is stating that $\mathbb{R}$ exists and is unique. Unfortunately, the proof for this is very long and so won't be covered in lecture. However, the professor has left some resources to cover it. So, I will have the proof of this theorem later in these notes. See page: <research how to cite a page>

Proposition 4.1: If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$. This is called the archimedean property.

Proof: We proceed by looking for a contradiction. Let $A = \{nx \mid n \in \mathbb{Z}^+\}$ and assume $\nexists n \in \mathbb{Z}^+$ such that $nx > y$. In that case we know y is an upper bound of A. Additionally, since A is bounded above, we know by the least upper bound property of the real numbers that $\sup A$ exists. So, let $\alpha = \sup A$.

Now because $\mathbb{R}$ is an ordered field, we know that:
$x > 0 \Rightarrow -x < 0 \Rightarrow \alpha - x < \alpha$. Therefore, because α is the least upper bound, we know that $\alpha - x$ is not an upper bound for A. Or in other words, there exists $n \in \mathbb{Z}^+$ such that $nx > \alpha - x$. But this contradicts that α is the least upper bound of A because $nx > \alpha - x \Rightarrow (n+1)x > \alpha$ and $(n+1)x \in A$. So we conclude that the supremum of A can't exist, which by the contrapositive of the least upper bound property, means that A is not bounded above.

Proposition 4.2: If $x, y \in \mathbb{R}$ and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$. In other words, we say that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof: Since $x < y$, we have that $0 < y - x$. Then because $y - x$ is positive, we can use the archimedean property to say that there exists an integer n such that $n(y - x) > 1$. Note for later that this means $ny > 1 + nx$.

Now note that since $1 > 0$ and nx is a real number, we can use the archimedean property twice to get positive integers m_1 and m_2 such that $m_1 \cdot 1 > -nx$ and $m_2 \cdot 1 > +nx$. Thus, we get the expression $-m_1 < nx < m_2$. So now consider the set $B = \{m \in \mathbb{Z} \mid -m_1 \geq nx \geq m_2 \text{ and } m > nx\}$. We know that B has finitely many elements and that B contains at least one element: m_2. So B must have a minimum element. We'll refer to that minimum element as m. Notably, as m is the minimum element of B, we know that $m - 1 \notin B$, meaning that $m - 1 \leq nx < m$

We now combine inequalities as follows: $m - 1 \leq nx \Rightarrow m \leq nx + 1$. So we have that $nx < m \leq nx + 1$. But now remember from the previous page that $ny > 1 + nx$. So we can say that $nx < m \leq nx + 1 < ny$. Finally, because $n > 0$, we can multiply the inequality by $\frac{1}{n}$ to get that $x < \frac{m}{n} < y$. ■

Lecture 4: 1/17/2024

Theorem: If $x \in \mathbb{R}$, $x > 0$, $n \in \mathbb{Z}$, and $n > 0$, then there is a unique $y \in \mathbb{R}$ with $y > 0$ and $y^n = x$. This number y is denoted $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

To prove this, first note the following lemma about positive integers n and $a, b \in \mathbb{R}$:

$$b^n - a^n = (b - a)(b^{n-1} + ab^{n-2} + \ldots + a^{n-2}b + a^{n-1})$$

To prove this, one can either use induction or just calculate it out by hand to verify that the equality holds.

Additionally, also consider that if n is a positive integer and $0 \leq a \leq b$ where $a, b \in \mathbb{R}$, then we have that $a^n \leq b^n$. Combining this fact with the lemma above, we can say that $0 \leq a \leq b$ implies that $b^n - a^n \leq (b - a)nb^{n-1}$. Or in other words: $a^n \leq b^n \leq a^n + (b - a)nb^{n-1}$.

This comes from replasing every a in the expression $(b^{n-1} + ab^{n-2} + \ldots + a^{n-2}b + a^{n-1})$ with b in order to get that $b^n - a^n \leq (b - a)(b^{n-1} + b^{n-1} + \ldots + b^{n-1})$

Now set $E = \{t \in \mathbb{R} \mid t > 0, t^n \leq x\}$.

We can show that E is nonempty...

- If $x \geq 1$, then $t = 1 \in E$ since $1^n = 1 \leq x$.
- If $x < 1$, then $x \in E$ since $x < 1 \Rightarrow x^{n-1} < 1^{n-1} = 1$. But then $x^n < x$.

Thus, we know $E \neq \emptyset$.

We can also show that E is bounded above. Consider $t = 1 + x$. In that case, $t > 1$, which implies that $t^{n-1} > 1^{n-1} = 1$. Therefore, $t^n > t$, meaning that $t^n > x$. So $t = x + 1$ is an upper bound for E.

Thus by the least upper bound property of the real numbers, we know $y = \sup E$ exists.

Claim 1: $y^n \geq x$.

To prove this, we shall procede towards a contradiction. Assume $y^n < x$

Then pick some h such that $0 < h < \gamma$ and γ is some mystery constant for us to find. Then, we can say that $y < y + h$, meaning by the lemma on the previous page that $y^n \leq (y+h)^n \leq y^n + (y+h-y)n(y+h)^n - 1$. Or in other words, $(y+h)^n \leq y^n + hn(y+h)^n - 1$.

Now we shall make our first assumption about γ: let $\gamma \leq 1$. That way, we know that $(y+h)^n \leq y^n + hn(y+h)^n - 1 < y^n + hn(y+1)^n$. And since, we are assuming that $y^n < x$, we know there must exist some value of h such that $y^n + hn(y+1)^{n-1} < x$. Putting this limitation on h, we get that $h < \frac{x-y^n}{n(y+1)^{n-1}}$ (Remember that $x - y^n$, y, and n are all positive). So finally, we say that $\gamma = \min\left(1, \frac{x-y^n}{n(y+1)^{n-1}}\right)$. This is so that for $0 < h < \gamma$, we have that $(y+h)^n < x$.

Thus, we have a contradiction as we assumed that y is the supremum of E and yet we just proved that $y + h \in E$. So, y^n cannot be less than x, meaning that that $y^n \geq x$.

Claim 2: $y^n \leq x$.

To prove this, we shall again proceed towards a contradiction. Assume $y^n > x$.

Then for some h such that $0 < h < \gamma$ where γ is a new mystery constant, consider $y - h$.

I now realize that I need to prove this lemma: for a positive integer n and real numbers a and b such that $a \geq b$, we have that $(a-b)^n \geq a^n - bna^{n-1}$. We can prove this through induction.

Firstly for $n = 1$: we have that $(a-b)^1 = a^1 - b(1)a^0$.

Now assume that for $k \geq 1$, $(a-b)^k \geq a^k - bka^{k-1}$.
Then $(a-b)^{k+1} = (a-b)(a-b)^k$. And since $(a-b) > 1$, we know that $(a-b)^{k+1} = (a-b)(a-b)^k \geq (a-b)(a^k - bka^{k-1})$.

Now let's expand out our lesser term to get that:
$(a-b)^{k+1} \geq a^{k+1} - bka^k - ba^k + b^2ka^{k-1}$. Thus, we know that $(a-b)^{k+1} \geq a^{k+1} - b(k+1)a^k + b^2ka^{k-1} > a^{k+1} - b(k+1)a^k$. Hence, we have shown that $(a-b)^{k+1} \geq a^{k+1} - b(k+1)a^k$.

Based on the lemma covered right before this, we have that $(y-h)^n \geq y^n - hny^{n-1}$. But now let's require that $y^n - hny^{n-1} > x$. Thus, we can say that $h < \frac{y^n - x}{ny^{n-1}}$.

So setting $\gamma = \frac{y^n - x}{ny^{n-1}}$, we have that for $0 < h < \gamma$, $\;(y-h)^n > x$. But this now leads to a contradiction as $y - h$ must be an upper bound to E.

(If some number z is greater than $y-h$, than $z^n > (y-h)^n > x$. So $z \notin E$.)

However, $y - h$ can't be an upper bound to E as we specified that y is the least upper bound of E. So we conclude that y^n cannot be greater than x, thus meaning $y^n \leq x$.

So since $y^n \leq x$ and $y^n \geq x$, we conclude that $y^n = x$.

Finally, we now shall mention that y is obviously the unique number such that $y^n = x$. After all, for $0 < a < y < b$, we have that $a^n < y^n < b^n$. So, there can only be one number y such that $y^n = x$.

Lecture 5: 1/19/2024

Decimal representations of real numbers:

- Each $x \in \mathbb{R}$ such that $x > 0$ can be written $x = n_0.n_1n_2n_3\ldots$ where $n_0 \in \mathbb{Z}$ and $\forall i \geq 1, n_i \in \{0,1,2,3,4,5,6,7,8,9\}$.

  Specifically, let n_0 be the largest integer with $n \leq x$. Then inductively, pick n_k to be the max element in $\{0,1,2,3,4,5,6,7,8,9\}$ such that:

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_{k-1}}{10^{k-1}} + \frac{n_k}{10^k} \leq x$$

- Conversely, suppose $n_0 \in \mathbb{Z}$ and $\forall i \geq 1, n_i \in \{0,1,2,3,4,5,6,7,8,9\}$. Then, defining $E = \{n_0, n_0 + \frac{n_1}{10}, \ldots, n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k}, \ldots\}$, we have that $n_0.n_1n_2n_3\ldots = x \in \mathbb{R}$ where $x = \sup E$.

We will rarely ever use decimal representations though.

The extended real number system is the set $\mathbb{R} \cup \{-\infty, +\infty\}$ where for all $x \in \mathbb{R}$:

- $-\infty < x < +\infty$
- $x + \infty = +\infty$
- $x - \infty = -\infty$
- $\frac{x}{+\infty} = 0 = \frac{x}{-\infty}$
- $x > 0 \Rightarrow x(+\infty) = +\infty$
- $x > 0 \Rightarrow x(-\infty) = -\infty$
- $+\infty + \infty = +\infty$
- $+\infty(+\infty) = +\infty$
- $x < 0 \Rightarrow x(+\infty) = -\infty$
- $x < 0 \Rightarrow x(-\infty) = +\infty$
- $-\infty - \infty = -\infty$
- $+\infty(-\infty) = -\infty$

All other operation involving $+\infty$ and $-\infty$ are left undefined.

◇ Sometimes, we denote the extended real number system $\overline{\mathbb{R}}$.
◇ The extended real number system is not a field.
◇ To distinguish $x \in \mathbb{R}$ from ∞ or $-\infty$, we call $x \in \mathbb{R}$ finite.

The set of complex numbers, denoted $\mathbb{C}$, is the set of all things of the form $a + bi$ where $a, b \in \mathbb{R}$ and i is a symbol satisfying $i^2 = -1$.

To be more rigorous about this definition, what we would do is define the set of complex numbers to be the set of pairs of real numbers equipped with the following operations:

For $z, u \in \mathbb{C}$ such that $z = (a, b)$ and $u = (c, d)$:
- $z + u = (a + c, b + d)$
- $z \cdot u = (ac - bd, ad + bc)$

Having done that, we would then:
1. Define $0 = (0, 0)$ and $1 = (1, 0)$
2. Prove that $\mathbb{C}$ satisfies our field axioms
3. Say that $i = (0, 1)$ and then show that $i^2 = (-1, 0)$
4. And finally show that for $a, b \in \mathbb{R}$, $a(1) + b(i) = (a, b)$

(Thus it makes sense to denote $z \in \mathbb{C}$ as $z = a + bi$)

However, we're behind and so not going to spend time doing that in class.

For $z = a + bi$, we denote $\mathrm{Re}(z) = a$ the real part of z. On the other hand, we denote $\mathrm{Im}(z) = b$ the imaginary part of z.

The complex conjugate of $z = a + bi$ is $\overline{z} = a - bi$.

Proposition 5: If $z, w \in \mathbb{C}$, then:

1. $\overline{z+w} = \overline{z} + \overline{w}$
2. $\overline{zw} = \overline{z} \cdot \overline{w}$
3. $z + \overline{z} = 2\mathrm{Re}(z)$
4. $z - \overline{z} = 2\mathrm{Im}(z)i$
5. $z\overline{z} \in \mathbb{R}$ and $z\overline{z} > 0$ when $z \neq 0$.

Proof:
Points 1-4 can be verified by direct computation.
As for point 5, note that if $z = a+bi$, then $z\overline{z} = (a+bi)(a-bi) = a^2 + b^2$.
Now as $a, b \in \mathbb{R}$, we know that $a^2+b^2 \in \mathbb{R}$. But $a^2+b^2 > 0$ if $b \neq a \neq 0$.
Meanwhile, $a^2 + b^2 = 0$ if $a = b = 0$. So $z\overline{z} > 0$ if $z \neq 0$

The absolute value of $z = a + bi$ is $|z| = \sqrt{z\,\overline{z}}$

Propostion 6: For $z, w \in \mathbb{C}$, we have that:

1. $|0| = 0$ and $|z| > 0$ when $z \neq 0$.
2. $|z| = |\overline{z}|$
3. $|zw| = |z||w|$
4. $|\mathrm{Re}(z)| \leq |z|$
5. $|\mathrm{Im}(z)| \leq |z|$
6. $|z + w| \leq |z| + |w|$

This last bullet is the triangle inequality.

Proof:
Claims 1, 2, and 3 can be verified through direct computation.

To prove claim 4, note that $a^2 \leq a^2 + b^2$. So, $|a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$.
We can repeat this but with b^2 to prove claim 5.

Lastly, to prove claim 6, note that $|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\overline{z}+\overline{w})$.
Now, we can distribute to get that $|z+w|^2 = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w}$. So, we know that $|z+w|^2 = |z|^2 + z\overline{w} + w\overline{z} + |w|^2$.

But now observe that $w\overline{z} = \overline{z\overline{w}}$. So $z\overline{w} + w\overline{z} = 2\mathrm{Re}(z\overline{w})$. But by claim 4, we know that $\mathrm{Re}(z\overline{w}) \leq |z\overline{w}|$. Additionally, by claims 2 and 3, we have that $|z\overline{w}| = |z||\overline{w}| = |z||w|$. So, we know that $|z+w|^2 \leq |z|^2 + 2|z||w| + |w|^2$.
This simplifies to $|z+w|^2 \leq (|z|+|w|)^2$. Hence, $|z+w| \leq |z| + |w|$.

Lecture 6: 1/22/2024

Theorem: (the Cauchy-Schwarz Inequality)

If $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{C}$, then:

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

Proof:

Define $A = \sum_{j=1}^{n} |a_j|^2$, $B = \sum_{j=1}^{n} |b_j|^2$, and $C = \sum_{j=1}^{n} a_j \overline{b_j}$.

Note that $A, B \in \mathbb{R}$ such that $A, B \geq 0$. Meanwhile, $C \in \mathbb{C}$.

If $B = 0$, then $b_1 = \ldots = b_n = 0$. Thus $C = 0$ as well and so the inequality is trivially true.

So now consider if $B > 0$. Then we can make a series of manipulations starting with: $0 \leq \sum_{j=1}^{n} |Ba_j - Cb_j|^2$.

(The professor said not to worry about how Rudin thought of using this formula.)

$$\begin{aligned}
0 &\leq \sum_{j=1}^{n} |Ba_j - Cb_j|^2 \\
&= \sum_{j=1}^{n} (Ba_j - Cb_j)(B\overline{a_j} - \overline{C}\overline{b_j}) \\
&= B^2 \sum_{j=1}^{n} |a_j|^2 - BC \sum_{j=1}^{n} \overline{a_j} b_j - B\overline{C} \sum_{j=1}^{n} a_j \overline{b_j} + |C|^2 \sum_{j=1}^{n} |b_j|^2 \\
&= B^2 A - BC\overline{C} - B\overline{C}C + |C|^2 B \\
&= B^2 A - B|C|^2 \\
&= B(AB - |C|^2)
\end{aligned}$$

Thus, since we're assuming $B > 0$, we know that $AB - |C|^2 \geq 0$.

So, $AB \geq |C|^2$. ■

We call elements $\vec{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ vectors or points. The x_i are the coordinates of $\vec{x}$.

The inner product or dot product of $\vec{x}$, $\vec{y} \in \mathbb{R}^k$ is: $\vec{x} \cdot \vec{y} = \sum_{i=1}^{k} x_i y_i$

The norm of $x \in \mathbb{R}^k$ is $\|\vec{x}\| = (\vec{x} \cdot \vec{x})^{\frac{1}{2}}$

Proposition 7: If $\vec{x}$, $\vec{y}$, $\vec{z} \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$, then:

1. $\|\vec{0}\| = 0$ and $\|\vec{x}\| > 0$ when $\vec{x} \neq \vec{0}$.
2. $\|\alpha \vec{x}\| = \alpha \|\vec{x}\|$
3. $\|\vec{x} \cdot \vec{y}\| \leq \|\vec{x}\| \|\vec{y}\|$
4. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$
5. $\|\vec{x} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\|$

The proofs for 1-4. are nearly identical to those for complex numbers so we won't cover them here.

As for 5, note that $\vec{x} + (-\vec{z}) = \vec{x} - \vec{y} + \vec{y} - \vec{z}$.

For sets X, Y and a function $f : X \to Y$, we shall write:

- for $A \subseteq X$, $f(A) = \{f(a) \mid a \in A\}$ (This is the image of A.)
- for $B \subseteq Y$, $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ (This is the preimage of A.)
- for $y \in Y$, we write $f^{-1}(y)$ for $f^{-1}(\{y\})$

We say two sets A and B have equal cardinality, denoted $|A| = |B|$ if there is a bijection f from A onto B.

- A is finite if it has equal cardinality with $\{1, \ldots, n\}$ for some $n \in \mathbb{Z}^+$ or if $A = \emptyset$.
- A is countable if either A is finite or A has equal cardinality with $\mathbb{Z}^+$.
- A is uncountable if its not countable.

A sequence is a function f having domain $\mathbb{Z}^+$. If $f(n) = x_n \in A$ for each integer n, it is typical to denote f by $(x_n)_{n \in \mathbb{Z}^+}$ or more simply by (x_n).

Proposition 8: If A is countable and $E \subseteq A$, then E is countable.

Proof:

If E is finite, then E is countable and we're done. So assume E is infinite. Then as $E \subseteq A$, we know A is infinite as well.

Since A is countable, we can enumerate A as $x_1, x_2, x_3, \ldots$. Set $n_1 = \min\{m \in \mathbb{Z}^+ \mid x_m \in E\}$. Then, inductively set $n_{k+1} = \min\{m \in \mathbb{Z}^+ \mid m > n_k$ and $x_m \in E\}$. Finally, define $f : \mathbb{Z}^+ \to E$ by the rule $f(k) = x_{n_k}$. That way f is a bijection.

Proposition 9: $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Proof:

Define $g : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $g(i,j) = \left(\sum_{k=1}^{i+j-2} k\right) + j$.

This is a bijection.

Here's what the bijection looks like:

$$\begin{array}{cccc}
(1,1) & (1,2) & (1,3) & \cdots \\
(2,1) & (2,2) & (2,3) & \cdots \\
(3,1) & (3,2) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

Lecture 7: 1/24/2024

Proposition 10: If A is a countable set, B any set, and $g : A \to B$ is a surjection, then B is countable.

Proof:

If A is finite, then $B = g(A)$ is finite as well. So the proposition is trivially true. Now assume A is infinite. Then since A is countable, there is a bijection $f : \mathbb{Z}^+ \to A$. Now define $\phi = g \circ f : \mathbb{Z}^+ \to B$. We know that ϕ is a surjection as it is the composition of two surjections.

Let $E \subseteq \mathbb{Z}^+$ be any set that contains precisely one element of $\phi^{-1}(b)$ for each $b \in B$. For instance, we can define E as the set:

$$\{n \in \mathbb{Z}^+ \mid \forall m \in \mathbb{Z}^+,\ m < n \Rightarrow \phi(m) \neq \phi(n)\}$$

Now by proposition 8, we know that E is countable as E is a subset of a countable set. But additionally we have that ϕ acts as a bijection from E to B. Therefore, $|E| = |B|$, meaning B is countable.

Proposition 11: A set A is countable if and only if there exists a surjection from $\mathbb{Z}^+$ onto A.

Proof:
($\Longleftarrow$) Since $\mathbb{Z}^+$ is the definition of a countable set, if there is a surjection from $\mathbb{Z}^+$ to A, then we have by proposition 10 that A is also countable.

($\Longrightarrow$) Assume A is countable. If A is finite, then we can number the elements of A as $\{a_1, a_2, \ldots, a_n\}$. So, we may define the surjection $f : \mathbb{Z}^+ \to A$ with the correspondance rule:

$$f(k) = \begin{cases} a_k & \text{if } k \leq n \\ a_n & \text{if } k > n \end{cases}$$

Meanwhile if A is infinite, then by definition there exists a bijection from $\mathbb{Z}^+$ to A. So, no matter if A is infinite or finite, if A is countable, then there exists a bijection from $\mathbb{Z}^+$ to A.

Proposition 12: If E_n is a countable set for each $n \in \mathbb{Z}^+$, then $\displaystyle\bigcup_{n\in\mathbb{Z}^+} E_n$ is countable.

Proof:
For each $n \in \mathbb{Z}^+$, there is a surjection $f_n : \mathbb{Z}^+ \to E_n$.
Define $g : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \displaystyle\bigcup_{n\in\mathbb{Z}^+} E_n$ by $g(n, k) = f_n(k)$.

Then as g is a surjection and $\mathbb{Z} \times \mathbb{Z}$ is countable by proposition 9, we know by proposition 10 that $\displaystyle\bigcup_{n\in\mathbb{Z}^+} E_n$ is countable.

In other words, the union of countably many countable sets is countable.

Proposition 13: If A is countable, then for every $n \in \mathbb{Z}^+$, the set $A^n = A \times A \times \ldots \times A$ is countable.

Proof: (we can proceed by induction)
When $n = 1$, then $A^n = A^1 = A$ is obviously countable.
Now assume the proposition is true for $n - 1$, meaning A^{n-1} is countable.
Then: $A^n = \displaystyle\bigcup_{a\in A} \{a\} \times A^{n-1}$ is countable by proposition 12.

Corollary: $\mathbb{Q}$ is countable.

Proof:

Define $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}^+$ by setting $f(p) = (n, m)$ where n, m are the unique coprime integers with $m > 0$, $\frac{n}{m} = p$. Also define $f(0) = (1, 0)$ Then $f(\mathbb{Q}) \subset \mathbb{Z}\times\mathbb{Z}^+$ and the latter set is countable. So $f(\mathbb{Q})$ is countable. Since f is injective, f is a bijection between $\mathbb{Q}$ and a countable set. Thus $\mathbb{Q}$ is countable.

Given sets A and B, we write A^B to denote the set of all functions from B to A.

Proposition 14: $\{0,1\}^{\mathbb{Z}^+}$ is uncountable.

Proof: Let $\{f_1, f_2, \ldots\}$ be any countable subset of $\{0,1\}^{\mathbb{Z}^+}$. Then define $g \in \{0,1\}^{\mathbb{Z}^+}$ by the rule $g(n) = 1 - f_n(n)$. Since $g(n) \neq f_n(n)$, we have that $g \neq f_n$. Since this holds for all $n \in \mathbb{Z}^+$, we can thus conclude that $g \notin \{f_1, f_2, \ldots\}$. We thus conclude that any countable subset of $\{0,1\}^{\mathbb{Z}^+}$ is a proper subset. So $\{0,1\}^{\mathbb{Z}^+}$ must be uncountable.

Lecture 8: 1/26/2024

A metric space is a set X equipped with a function $d : X \times X \longrightarrow [0, \infty)$ satisfying:

1. $\forall p, q \in X \quad p \neq q \Rightarrow d(p,q) > 0$ whereas $p = q \Rightarrow d(p,q) = 0$
2. $\forall p, q \in X \quad d(p,q) = d(q,p)$
3. $\forall p, q, s \in X \quad d(p,q) \leq d(p,s) + d(s,q)$

The function d is called a distance function or metric.

Examples:

- $\mathbb{R}^k$ is a metric space (we have several metrics to choose from):
  - ◇ $d_p(\vec{x}, \vec{y}) = (\sum |x_i - y_i|^p)^{\frac{1}{p}}$

    $d_2(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$
  - ◇ $d_\infty(\vec{x}, \vec{y}) = \max\limits_{1 \leq i \leq k} |x_i - y_i|$
- Any set X is a metric space when equipped with the discrete metric:

$$d(p,q) = \begin{cases} 0 \text{ if } p = q \\ 1 \text{ if } p \neq q \end{cases}$$

- The set of all functions from $[0,1] \to [0,1]$ can be equipped with the metric:

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

Let X be a metric space. Then for $p \in X$ and $r > 0$, the (open) ball of radius r around p is $B_r(p) = \{q \in X \mid d(p,q) < r\}$.

$p \in X$ is a limit point of $E \subseteq X$ if there are points in $E \setminus \{p\}$ that are arbitrarily close to p. Or in other words, if p is a limit point, then

$$\forall r > 0, \quad ((B_r(p) \setminus \{p\}) \cap E) \neq \emptyset.$$

The set of limit points of $E \subseteq X$ is denoted E'.

- E is closed if $E' \subseteq E$.
- E is perfect if $E = E'$.
- We say E is dense in X if $E \cup E' = X$

$p \in X$ is an interior point of $E \subseteq X$ if $\exists r > 0 \ \ s.t. \ \ B_r(p) \subseteq E$.

The set of interior points of E is denoted E°.

- E is open if $E^\circ = E$.
- E is a neighborhood of p if $p \in E^\circ$.

The complement of E is $E^{\mathsf{C}} = X \setminus E$.
E is bounded if there is a point $p \in X$ and $R > 0$ with $E \subseteq B_R(p)$.
If $p \in E$, we say p is an isolated point of E if $\exists r > 0 \ \ s.t. \ \ B_r(p) \cap E = \{p\}$.

Proposition 15: If X is a metric space, $p \in X$, and $r > 0$, then $B_r(p)$ is open.

Proof:
Consider a point $q \in B_r(p)$. We claim that $B_{(r-d(p,q))}(q) \subseteq B_r(p)$.
To prove this consider that for $z \in B_{(r-d(p,q))}(q)$, we have that $d(p,z) \leq d(p,q) + d(q,z) < d(p,q) + (r - d(p,q)) = r$. Thus, $z \in B_r(p)$.
And, since we can do this for any $z \in B_{(r-d(p,q))}(q)$, we know that $B_{(r-d(p,q))}(q) \subseteq B_r(p)$. Therefore q is an interior point of $B_r(p)$. And, since we can say this for any $q \in B_r(p)$, we thus conclude that $B_r(p)$ consists of interior points. So $B_r(p)$ is open.

Lecture 9: 1/29/2024

Let X be a metric with metric d and let $E \subseteq X$...

Proposition 16: If $p \in E'$, then $(B_r(p) \setminus \{p\}) \cap E$ is infinite for every $r > 0$.

Proof (by contrapositive):
Let $p \in X$ and suppose $\exists r > 0$ with $(B_r(p) \setminus \{p\}) \cap E$ finite.
Then set $t = \min\{d(p,q) \mid q \in (B_r(p) \setminus \{p\}) \cap E\}$. That way, we must have that $t > 0$. But at the same time, $B_t(p) \setminus \{p\} \cap E$ is empty. Therefore $p \notin E'$.

Corollary: If E is finite, then $E' = \emptyset$. This means that finite sets are always closed.

Propostion 17: E is open if and only if E^{C} is closed.

Proof:

$$
\begin{aligned}
E^{\mathsf{C}} \text{ is closed} &\iff \left(E^{\mathsf{C}}\right)' \subseteq E^{\mathsf{C}} \\
&\iff \left(E^{C}\right)' \cap E = \emptyset \\
&\iff \forall p \in E,\ \ p \notin \left(E^{\mathsf{C}}\right)' \\
&\iff \forall p \in E,\ \ \exists r > 0\ \ s.t.\ \ (B_r(p) \setminus \{p\}) \cap E^{\mathsf{C}} = \emptyset \\
&\iff \forall p \in E,\ \ \exists r > 0\ \ s.t.\ \ B_r(p) \setminus \{p\} \subseteq E \\
&\iff \forall p \in E,\ \ \exists r > 0\ \ s.t.\ \ B_r(p) \subseteq E \\
&\iff \forall p \in E,\ \ p \in E^{\circ} \\
&\iff E \text{ is open}
\end{aligned}
$$

Corollary: E is closed if and only if E^{C} is open.

Proposition 18: Let A be any set.

1. If $u_\alpha \subseteq X$ is an open set for each $\alpha \in A$, then $\bigcup_{\alpha \in A} u_\alpha$ is open.

   Proof:
   Let $p \in \bigcup_{\alpha \in A} u_\alpha$. Pick $\beta \in A$ with $p \in u_\beta$.
   Since u_β is open, we know that $\exists r > 0\ \ s.t.\ \ B_r(p) \subseteq u_\beta \subseteq \bigcup_{\alpha \in A} u_\alpha$.
   So p is an interior point. Hence, we conclude that $\bigcup_{\alpha \in A} u_\alpha$ is open.

2. If $F_\alpha \subseteq X$ is a closed set for each $\alpha \in A$, then $\bigcap_{\alpha \in A} F_\alpha$ is closed.

   Proof:
   $\left(\bigcap_{\alpha \in A} F_\alpha\right)^{\mathsf{C}} = \bigcup_{\alpha \in A} (F_\alpha)^{\mathsf{C}}$ by De Morgan's laws.

Since each F_α is closed, we know each $(F_\alpha)^\mathsf{C}$ is open.
So by proposition 18.1, we know that $\bigcup_{\alpha \in A} (F_\alpha)^\mathsf{C}$ is open.

Then, by proposition 17, we know that its complement, $\bigcap_{\alpha \in A} F_\alpha$ is closed.

3. If $u_1, u_2, \ldots, u_n \subseteq X$ are open, then $\bigcap_{i=1}^{n} u_i$ is open.

   Proof:
   Let $p \in \bigcap_{i=1}^{n} u_i$. Then $p \in u_i$ for every i.
   Since u_i is open, $\exists r_i > 0 \;\; s.t. \;\; B_{r_i}(p) \subseteq u_i$. Therefore, set $r = \min\{r_i \mid 1 \leq i \leq n\}$ so that for all i, $B_r(p) \subseteq B_{r_i}(p) \subseteq u_i$.
   Hence, $B_r(p) \subseteq \bigcap_{i=1}^{n} u_i$. We thus conclude that $\bigcap_{i=1}^{n} u_i$ is open.

4. If $F_1, F_2, \ldots, F_n \subseteq X$ are closed, then $\bigcup_{i=1}^{n} F_i$ is closed.

   The proof of this follows from proposition 18.3 in the same way that proposition 18.2 follows from proposition 18.1.

Lecture 10: 2/2/2024

Given a metric space X, the closure of $E \subseteq X$ is $\overline{E} := E \cup E'$.

Proposition 19.1: $\overline{E}$ is closed.

Proof:
Let $p \in (\overline{E})^\mathsf{C}$. Thus, $p \notin E'$, meaning that we can fix $r > 0$ so that $(B_r(p) \backslash \{p\}) \cap E = \emptyset$. Additionally, since $p \notin E$, we have that $B_r(p) \cap E = \emptyset$.

Now consider any $q \in B_r(p)$. Setting $t = r - d(p, q)$, we have that $B_t(q) \subseteq B_r(p)$. Therefore, since $B_r(p) \cap E = \emptyset$, we know $B_t(q) \cap E = \emptyset$. This tells us that $q \notin E'$. Hence, $B_r(p) \cap E' = \emptyset$.

We've now shown that $B_r(p) \cap E = \emptyset$ and that $B_r(p) \cap E' = \emptyset$.
Therefore, $B_r(p) \cap (E \cup E') = B_r(p) \cap \overline{E} = \emptyset$, meaning that $B_r(p) \subseteq (\overline{E})^\mathsf{C}$.
So $(\overline{E})^\mathsf{C}$ is open, meaning that $\overline{E}$ is closed.

Proposition 19.2: $E = \overline{E}$ if and only if E is closed.

Proof:

($\Longrightarrow$) If $\overline{E}$ is closed by proposition 19.1. So $E = \overline{E}$ implies E is closed.

($\Longleftarrow$) If E is closed, then $E' \subseteq E$. Hence, $\overline{E} = E \cup E' = E$

Proposition 19.3: If F is closed and $F \supseteq E$, then $F \supseteq \overline{E}$.

Proof:

Observe that if F is any set and $E \subseteq F$, then $E' \subseteq F'$. Thus, if F is also closed, we have that $E' \subseteq F' \subseteq F$. Therefore, $F = F \cup F' \supseteq E \cup E' = \overline{E}$.

Note that in this class, unless it is mentioned otherwise, you should assume that we are equipping $\mathbb{R}$ or $\mathbb{R}^k$ with the Euclidean metric: d_2.

Proposition 20: If $E \subseteq \mathbb{R}$ is nonempty and bounded above, then $\sup E \in \overline{E}$.

Proof:

Set $y = \sup E$. If $y \in E$, then we are done. So assume $y \notin E$.
Consider any $r > 0$. Since $y - r < y = \sup E$, we know $y - r$ is not an upperbound to E. Hence, there is $e \in E$ with $y - r < e < y$. Therefore, $(B_r(y) \setminus \{y\}) \cap E \neq \emptyset$. Hence, we conclude that $y \in E' \subseteq \overline{E}$.

Note that if X is a metric space with metric d and $Y \subseteq X$, then Y is also a metric space with d when d is restricted to Y.

$E \subseteq Y \subseteq X$ is open/closed/etc. relative to Y if E is open/closed/etc. in the metric space Y.

If $Y \subseteq X$ and $B_r(p)$ denotes the ball of radius r around $p \in Y$ in the metric space X, then the ball of radius r around p in the metric space Y is $B_r(p) \cap Y$.

Proposition 21: Let $E \subseteq Y \subseteq X$. Then E is open relative to Y if and only if there is an open set $U \subseteq X$ with $E = U \cap Y$.

Proof:

($\Longrightarrow$) For each $p \in E$, pick $r(p) > 0$ so that $B_{r(p)}(p) \cap Y \subseteq E$. Then, setting $U = \bigcup_{p \in E} B_{r(p)}(p)$, we have that U is open and that

$$E = \bigcup_{p \in E} \{p\} \subseteq \bigcup_{p \in E} B_{r(p)}(p) \cap Y = U \cap Y \subseteq E$$

So $U \cap Y = E$.

($\Longleftarrow$) Now say that $E = U \cap Y$ where $U \subseteq X$ is open. Also let $p \in E$. We know $p \in U$. Additionally, since U is open, there is $r > 0$ with $B_r(p) \subseteq U$. Consequently, $B_r(p) \cap Y \subseteq U \cap Y = E$. So, p is an interior point of E relative to Y. We conclude that E is open relative to Y.

Let X be a metric space. An open cover of $E \subseteq X$ is a collection $\{u_\alpha \mid \alpha \in A\}$ of open sets u_α satisfying:

$$E \subseteq \bigcup_{\alpha \in A} u_\alpha$$

$K \subseteq X$ is compact if every open cover of K containsa finite subcover of K.

More precisely: K is compact if and only if for ever open cover $\{u_\alpha \mid \alpha \in A\}$ of K, there is $n \in \mathbb{Z}^+$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ such that:

$$K \subseteq \bigcup_{i=1}^{n} u_{\alpha_i}$$

As an aside, compactness often acts as a generalization of finiteness in topology.

Lecture 11: 2/5/2024

Finite sets are compact.

Proposition 22: compactness is an intrinisic property, meaning if $K \subseteq Y \subseteq X$, then K is compact relative to X if and only if K is compact relative to Y.

Proof:

($\Longrightarrow$) Consider any collection of sets $v_\alpha \subseteq Y$ that are open relative to Y and satisfy that $K \subseteq \bigcup_{\alpha \in A} v_\alpha$.

By a previous theorem, we know there are sets w_α open relative to X such that $v_\alpha = w_\alpha \cap Y$. So we have that $K \subseteq \bigcup_{\alpha \in A} v_\alpha \subseteq \bigcup_{\alpha \in A} w_\alpha$.

If K is compact relative to X, then there exists $n \in \mathbb{Z}^+$ and $\alpha_1, \ldots, \alpha_n \in A$ such that $K \subseteq \bigcup_{i=1}^{n} w_{\alpha_i}$. And since $K \subseteq Y$, we have that:

$$K = K \cap Y \subseteq \left(\bigcup_{i=1}^{n} w_{\alpha_i} \right) \cap Y = \left(\bigcup_{i=1}^{n} v_{\alpha_i} \right)$$

Hence, K is compact relative to Y.

($\Longleftarrow$) Now consider any set K which is compact relative to Y and open cover $\{w_\alpha \mid \alpha \in A\}$ such that $w_\alpha \subseteq X$ and $K \subseteq \bigcup\limits_{\alpha \in A} w_\alpha$.

By proposition 21, we know that $v_\alpha = w_\alpha \cap Y$ is open relative to Y. So as $K \subseteq Y$, we have that $K = K \cap Y \subseteq \bigcup\limits_{\alpha \in A} w_\alpha \cap Y = \bigcup\limits_{\alpha \in A} v_\alpha$.

But that means that $\{v_\alpha \mid \alpha \in A\}$ forms an open cover of K relative to Y. So, there exists $n \in \mathbb{Z}^+$ and $\alpha_1, \ldots, \alpha_n \in A$ such that $\{v_{\alpha_1}, \ldots, v_{\alpha_n}\}$ is a finite cover of K. Then note that:

$$K \subseteq \bigcup_{i=1}^{n} v_{\alpha_i} \subseteq \bigcup_{i=1}^{n} w_{\alpha_i}$$

So, $\{w_{\alpha_1}, \ldots, w_{\alpha_n}\}$ forms a finite subcover of K using sets in our original arbitrary open cover. Therefore, we conclude that K is compact relative to X.

Proposition 23: Compact sets are closed.

Proof:
Let $K \subseteq X$ be compact. It then suffices to show that K^{C} is open. So, consider any $p \in K^{\mathsf{C}}$. We know that $\{B_{\frac{1}{3}d(p,q)}(q) \mid q \in K\}$ forms an open cover of K. Additionally, because K is compact, there exists $n \in \mathbb{Z}^+$ and $q_1, \ldots, q_n \in K$ such that:

$$K \subseteq \bigcup_{i=1}^{n} B_{\frac{1}{3}d(p,q_i)}(q_i)$$

Thus, let $r = \min\{d(p, q_i) \mid 1 \leq i \leq n\}$. That way, $\frac{1}{3}r > 0$ and

$$\left(\bigcup_{i=1}^{n} B_{\frac{1}{3}d(p,q_i)}(q_i)\right) \cap B_{\frac{1}{3}r}(p) = \emptyset.$$

This then means that $K \cap B_{\frac{1}{3}r}(p) = \emptyset$, meaning that $B_{\frac{1}{3}r}(p) \subseteq K^{\mathsf{C}}$. So p is an interior point of K^{C}. We thus conclude that K^{C} is open.

Proposition 24: K is compact and $F \subseteq K$ is closed implies that F is compact.

Proof:
Consider any open cover $\{v_\alpha \mid \alpha \in A\}$ of F. Since F is closed, F^{C} is open. So, we can say that $\{F^{\mathsf{C}}\} \cup \{v_\alpha \mid \alpha \in A\}$ is an open cover of K as:

$$\left(\bigcup_{\alpha \in A} v_\alpha\right) \cup F^{\mathsf{C}} \supseteq F \cup F^{\mathsf{C}} \supseteq K$$

Since K is compact, there exists $n \in \mathbb{Z}^+$ and $\alpha_1, \ldots, \alpha_n \in A$ such that:

$$K \subseteq \left(\bigcup_{i=1}^{n} v_{\alpha_i} \right) \cup F^{\mathsf{C}}$$

F^{C} may or may not be needed to cover K. However, its inclusion doesn't effect the finiteness of the cover.

Therefore $F \subseteq \bigcup_{i=1}^{n} v_{\alpha_i}$. So, F is compact.

Corollary: K is compact and F is closed implies that $K \cap F$ is compact.

Proof: K being compact means that K is closed. Thus $K \cap F$ is closed. And as $K \cap F$ is a subset of K, by the above theorem we have that $K \cap F$ is compact.

Theorem (the Finite Intersection Property): If $\{K_\alpha \mid \alpha \in A\}$ is any collection of compact sets in X having the property that the intersection of any finitely many of the K_α's is nonempty, then:

$$\bigcap_{\alpha \in A} K_\alpha \neq \emptyset$$

Proof: (we shall proceed by proving the contrapositive...)

Assume that $\bigcap_{\alpha \in A} K_\alpha = \emptyset$. Thus, taking complements gives: $\bigcup_{\alpha \in A} (X \setminus K_\alpha) = X$.

Pick any $\alpha_0 \in A$. Then $\{X \setminus K_\alpha \mid \alpha \in A\}$ is an open cover of K_{α_0} because $K_{\alpha_0} \subseteq X$ and because each $X \setminus K_\alpha$ must be open due to K_α being closed.

As K_{α_0} is compact, there exists $n \in \mathbb{Z}^+$ and $\alpha_1, \ldots, \alpha_n \in A$ such that:

$$K_{\alpha_0} \subseteq \bigcup_{i=1}^{n} (X \setminus K_{\alpha_i})$$

Taking complements again, we get that: $X \setminus K_{\alpha_0} \supseteq \bigcap_{i=1}^{n} (K_{\alpha_i})$. So:

$$\left(\bigcap_{i=1}^{n} (K_{\alpha_i}) \right) \cap K_{\alpha_0} = \bigcap_{i=0}^{n} (K_{\alpha_i}) = \emptyset$$

Proposition 25: If K is compact and $E \subseteq K$ is infinite, then $E' \neq \emptyset$.

Proof: (we shall proceed by proving the contrapositive...)
Let $E \subseteq K$ and suppose $E' = \emptyset$. Then for each $q \in K$, since $q \notin E'$, we can pick $r(q) > 0$ such that $(B_{r(q)}(q) \setminus \{q\}) \cap E = \emptyset$. In particular, $(B_{r(q)}(q)) \cap E \subseteq \{q\}$.

Now note that $\bigcup_{q \in K} B_{r(q)}(q)$ is an open cover of K.

Since K is compact, we can pick $q_1, \ldots, q_n \in K$ so that $K \subseteq \bigcup_{i=1}^{n} B_{r(q_i)}(q_i)$.

Then, $E = E \cap K \subseteq E \cap \left(\bigcup_{i=1}^{n} B_{r(q_i)}(q_i) \right) = \bigcup_{i=1}^{n} \left(B_{r(q_i)}(q_i) \cap E \right) \subseteq \bigcup_{i=1}^{n} q_i$.

Hence E is finite.

Lecture 12: 2/7/2024

In $\mathbb{R}$, we define the interval $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$.

Proposition 26: If $I_n = [a_n, b_n] \neq \emptyset$ and $I_{n+1} \subseteq I_n$ for all n, then $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

Proof: For all $n, m \in \mathbb{N}$, we have $a_n \leq a_{n+m} \leq b_{n+m} \leq b_m$. Thus, for all m we have that b_m is an upperbound to $\{a_n \mid n \in \mathbb{N}\}$. This means that by the least upper bound property of $\mathbb{R}$, we know that $\alpha = \sup \{a_n \mid n \in \mathbb{N}\}$ exists and that $a_m \leq \alpha \leq b_m$ for all m. Hence, $\alpha \in \bigcap_{n \in \mathbb{N}} I_n$, which means $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

Corollary: If $C_n = [a_{n,1}, b_{n,1}] \times [a_{n,2}, b_{n,2}] \times \cdots \times [a_{n,k}, b_{n,k}] \subseteq \mathbb{R}^k$ and $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$.

Proof: $\bigcap_{n \in \mathbb{N}} C_n = \left(\bigcap_{n \in \mathbb{N}} [a_{n,1}, b_{n,1}] \right) \times \cdots \times \left(\bigcap_{n \in \mathbb{N}} [a_{n,1}, b_{n,1}] \right) \neq \emptyset$

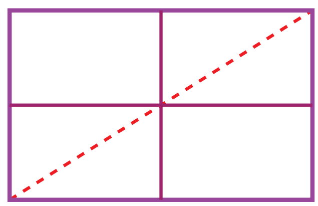
Proposition 27: $C = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k] \subseteq \mathbb{R}^k$ is compact.

Proof: (we'll proceed by finding a contradiction) Suppose $\{u_\alpha \mid \alpha \in A\}$ is an open cover of C containing no finite subcover of C.

Then set $\delta = \sqrt{\sum_{i=1}^{k} b_i - a_i}$

(this is the length of the largest diagonal of C.)

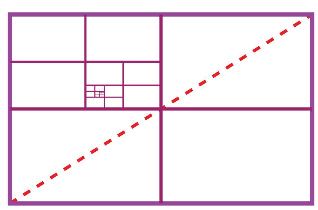
Since C forms a k-dimensional rectangle in $\mathbb{R}^k$, we can divide C into 2^k many pieces by cutting each side of C at its midpoints. Then each smaller piece will have a longest diagonal of length $\frac{1}{2}\delta$.



Since C can't be covered by finitely many u_α, there must be a piece, call it C_1, which cannot be covered by finitely many u_α. Also $C_1 \subseteq C$.

Now proceed inductively to build a sequence C_n such that:

1. $C_{n+1} \subset C_n$ for all n
2. The largest diagonal of C_n is $2^{-n}\delta$
3. C_n cannot be covered by finitely many u_α



By the above corollary, $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$. Thus consider $z \in \bigcap_{n\in\mathbb{N}} C_n \subseteq C$.

Pick $\alpha \in A$ with $z \in u_\alpha$. Since u_α is open, there exists $r > 0$ with $B_r(z) \subseteq u_\alpha$. Now pick n with $2^{-n}\delta < r$. Then as $z \in C_n$, we know $C_n \subseteq B_r(z) \subseteq u_\alpha$. This contradicts the point that C_n cannot be covered by finitely many u_α.

So we conclude C is compact.

Proposition 28: For $E \in \mathbb{R}^k$, the following are equivalent:

a. E is closed and bounded.

b. E is compact

c. Every infinite subset of E has a limit point in E.

Proof:

(a. $\Longrightarrow$ b.) E is bounded means $E \subseteq C$ for some bounded rectangle C. Since E is closed and C is compact, E is compact by proposition 24.

(b. $\Longrightarrow$ c.) This is just proposition 25.

(c. $\Longrightarrow$ a.) Firstly, let us show that E is bounded.

Suppose E is not bounded. Then for all $n \in \mathbb{N}$, pick $\vec{x}_n \in E$ with $\|\vec{x}_n\| \geq n$. For any $\vec{y} \in \mathbb{R}^k$, we have that: $\vec{x}_n \in B_1(\vec{y}) \Longrightarrow \|\vec{x}_n\| \leq \|\vec{y}\| + 1$, which in turn implies that $n \leq \|\vec{y}\| + 1$. So $(B_1(\vec{y}) \setminus \{\vec{y}\}) \cap \{\vec{x}_n \mid n \in \mathbb{N}\}$ is finite. As a result, we know $\vec{y} \notin \{\vec{x}_n \mid n \in \mathbb{N}\}'$. Thus, $\{\vec{x}_n \mid n \in \mathbb{N}\}$ has no limit points. But this is a contradiction because $\{\vec{x}_n \mid n \in \mathbb{N}\}$ is an infinite subset of E.

Now, let us show that E is closed.

Let $y \in E'$. Then for each $n \in \mathbb{Z}^+$, we can pick $\vec{x}_n \in B_{\frac{1}{n}}(\vec{y}) \cap E$. Now if $\vec{z} \in \mathbb{R}^k$ and $\vec{z} \neq \vec{y}$, then:

$$\begin{aligned} \vec{x}_n \in B_{\frac{1}{2}\|\vec{y}-\vec{z}\|}(\vec{z}) \quad &\Rightarrow \quad \|\vec{y}-\vec{z}\| \leq \|\vec{y}-\vec{x}_n\| + \|\vec{x}_n - \vec{z}\| \\ &\qquad\qquad < \|\vec{y}-\vec{x}_n\| + \tfrac{1}{2}\|\vec{y}-\vec{z}\| \\ &\Rightarrow \quad \tfrac{1}{2}\|\vec{y}-\vec{z}\| < \|\vec{y}-\vec{x}_n\| < \tfrac{1}{n} \\ &\Rightarrow \quad n < \tfrac{2}{\|\vec{y}-\vec{x}_n\|} \end{aligned}$$

Therefore: $\left(B_{\frac{1}{2}\|\vec{y}-\vec{z}\|}(\vec{z}) \setminus \{\vec{z}\}\right) \cap \{\vec{x}_n \mid n \in \mathbb{Z}^+\}$ is finite.

So $\vec{z} \notin \{\vec{x}_n \mid n \in \mathbb{N}\}'$, which means that $\vec{y}$ is the unique limit point of $\{\vec{x}_n \mid n \in \mathbb{N}\}$. Finally, since we assumed that any infinite subset of E has at least one limit point inside E, we know that $\vec{y} \in E$ because it is the only possible limit point that can fulfill this requirement.

Proposition 29: (<u>Bolzano-Weierstrauss Theorem</u>): Every bounded infinite subset of $\mathbb{R}^k$ has a limit point.

Proof: Let E be a bounded infinite subset of $\mathbb{R}^k$. Then $\overline{E}$ is closed and bounded, meaning that every infinite subset of $\overline{E}$ has a limit point in $\overline{E}$, meaning $(\overline{E})' \neq \emptyset$. Finally, we know from a homework question last week that $(\overline{E})' = E'$. So, $E' \neq \emptyset$.

The Cantor Set is very important as a counter example in topology. It is constructed as follows:

Let $E_0 = \{[0,1]\}$. Then for $n > 0$, inductively define E_n as a set containing closed intervals of the first and last thirds of each interval in E_{n-1}.

Additionally, for $0 \leq i$, define $C_i = \bigcup_{I \in E_i} I$.

Here are the first few iterations:

$C_0 = [0,1]$

$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

$C_3 = [0, \frac{1}{27}] \cup [\frac{2}{27}, \frac{1}{9}] \cup [\frac{2}{9}, \frac{7}{27}] \cup [\frac{8}{27}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{19}{27}] \cup [\frac{20}{27}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{25}{27}] \cup [\frac{26}{27}, 1]$

Then we define the Cantor set as $C = \bigcap_{i=0}^{n} C_i$.

Lecture 13: 2/9/2024

The Cantor set C is closed.

Each C_n is closed and the intercept of countably many closed sets is closed.

C is compact.

$C \subseteq [0,1]$. Therefore C is a bounded closed set in $\mathbb{R}$

$C \neq \emptyset$.

We know this by the finite intersection property. Each C_n is compact and the intersect of any finitely many C_n is nonempty as it will contain 0 or 1.

If x is an endpoint of an interval from E_n, then $x \in C$.

C contains no intervals.

Any interval in C must be contained in each E_n. Hence it must have length less than 3^{-n} for all n.

C is perfect.

Because C is closed, we know $C' \subseteq C$. Now, consider any $x \in C$ and let $r > 0$. Picking n with $3^{-n} < r$, we can specify I to be the interval of E_n containing x. The two end points of I are within distance r of x and belong to C. At least one is not x. Thus, $(B_r(x) \setminus \{x\}) \cap C \neq \emptyset$. So, $x \in C'$, meaning that $C \subseteq C'$.

C is uncountable.

We know this because...

Proposition 30: If $P \subseteq \mathbb{R}^k$ is perfect and nonempty, then P is uncountable.

To prove this, we first need two lemmas...

Lemma A: If $\vec{p}_n \in \mathbb{R}^k$ and $r_n > 0$ satisfy that $\overline{B_{r_{n+1}}(\vec{p}_{n+1})} \subseteq B_{r_n}(\vec{p}_n)$, and that $B_{r_n}(\vec{p}_n) \cap P \neq \emptyset$, then:

$$P \cap \left(\bigcap_{n \in \mathbb{N}} \overline{B_{r_n}(p_n)} \right) \neq \emptyset$$

Proof:

P is closed since P is perfect. So for all n, $P \cap \overline{B_{r_n}(\vec{p}_n)}$ is compact. Also by the assumption of the lemma, $B_{r_{n+1}}(\vec{p}_{n+1}) \neq \emptyset$

Meanwhile, $P \cap \overline{B_{r_{n+1}}(\vec{p}_{n+1})} \subseteq P \cap B_{r_n}(\vec{p}_n) \subseteq P \cap \overline{B_{r_n}(\vec{p}_n)}$. Thus, we can use the finite intersection property to say that:

$$P \cap \left(\bigcap_{n \in \mathbb{N}} \overline{B_{r_n}(\vec{p}_n)} \right) = \bigcap_{n \in \mathbb{N}} \left(P \cap \overline{B_{r_n}(\vec{p}_n)} \right) \neq \emptyset$$

Lemma B: Say $\vec{x} \neq \vec{p} \in \mathbb{R}^k$ and $r > 0$.

If $\vec{q} \in B_r(p) \setminus \{\vec{x}\}$, then there is $s > 0$ with $\overline{B_s(\vec{q})} \subseteq B_r(\vec{p}) \setminus \{\vec{x}\}$

Proof:

Set $s = \frac{1}{2} \min \{ r - d(\vec{p}, \vec{q}), \quad d(\vec{x}, \vec{q}) \}$.

Now consider any countable set of points in P: $\vec{x}_1, \vec{x}_2, \ldots$ We will inductively choose $\vec{p}_n \in P$ and $r_n > 0$ satisfying:

- $\vec{x}_n \notin \overline{B_{r_{n+1}}(\vec{p}_{n+1})}$
- $\overline{B_{r_{n+1}}(\vec{p}_{n+1})} \subseteq B_{r_n}(\vec{p}_n)$

To do this, first pick any $\vec{p}_1 \in P$ and $r_1 > 0$. Then for any $n \geq 1$, since P is perfect, we know that $\vec{p}_n \in P \Rightarrow \vec{p}_n \in P'$. So there are infinitely many points in $B_{r_n}(\vec{p}_n) \cap P$. Pick $\vec{p}_{n+1} \in B_{r_n}(\vec{p}_n) \cap P$ such that $\vec{p}_{n+1} \neq \vec{x}_n$. Then, using lemma B, we can define $B_{r_{n+1}}(\vec{p}_{n+1})$ satisfying our two requirements above.

By lemma A, we know that the intercept of all $B_{r_n}(\vec{p}_n)$ is nonempty. However, we also know that each $\vec{x}_n$ is not in $B_{r_{n+1}}(\vec{p}_{n+1})$. So, the point in the intercept of all $B_{r_n}(\vec{p}_n)$ is an element of P not included in our countable subset of P.

Hence, all countable subsets of P are proper. So, we conclude P is uncountable.

Note: A real number x is in the Cantor set if and only if it is between $0 \leq x \leq 1$, and if in base 3, all of the digits of x are either 0 or 2.

Hopefully it is clear from this how an irrational number could be found in the Cantor set.

Let X be a metric space. $A, B \subseteq X$ are separated if $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.

$E \subseteq X$ is connected if whenever $E = A \cup B$, either A and B are not separated or else one of A, B is empty.

For example, $(0, 1)$ and $(1, 2)$ are separated.
Meanwhile, $(0, 1]$ and $(1, 2)$ are disjoint but not separated.

Proposition 31: $E \subseteq \mathbb{R}$ is connected if and only if $\forall x, y \in E$ with $x < y$, $[x, y] \subseteq E$

Proof: (for both directions we will prove the contrapositive)

($\Longrightarrow$) Suppose $x, y \in E$, $x < y$, and $[x, y] \nsubseteq E$. Pick $z \in [x, y] \setminus E$. Since $(-\infty, z)$ and $(z, +\infty)$ are separated, so are $A = E \cap (-\infty, z)$ and $B = E \cap (z, +\infty)$. Additionally, since $z \notin E$, we have that $E = A \cup B$. However, as $A \neq \emptyset \neq B$ since $x \in A$ and $y \in B$, we conclude that E is not connected.

($\Longleftarrow$) Now suppose E is not connected. Say $A \neq \emptyset \neq B$ are separated and $A \cup B = E$. Pick $x \in A$ and $y \in B$. Without loss of generality, we can assume $x < y$. Define $z = \sup{(A \cap [x, y])}$. By proposition 20, we have that $z \in \overline{A}$. So as A and B are separated, we have $z \notin B$.

If $z \notin A$, then we know that $z \notin E$. So as $x \leq z < y$, we know that $[x, y] \nsubseteq E$. Meanwhile, if $z \in A$, then $z \notin \overline{B}$. So, there exists z_1 such that $z < z_1 < y$ and $z_1 \notin B$. Then, $z_1 \notin A \cup B$. So, $z_1 \notin E$, meaning $[x, y] \nsubseteq E$.

Lecture 14: 2/12/2024

A sequence $(p_n)_{n \in \mathbb{N}}$ in a metric space X converges if there is $p \in X$ such that:

$$\forall \varepsilon > 0, \;\; \exists N \in \mathbb{N} \;\; s.t. \;\; \forall n \geq N, \;\; p_n \in B_\varepsilon(p)$$

When this occurs, we say that (p_n) converges to p or that p is a limit of (p_n), and we write this as: $p_n \to p$ or $\lim\limits_{n \to \infty} p_n = p$.

If (p_n) does not converge, then we say it diverges.

The range of $(p_n)_{n\in\mathbb{N}}$ is defined as the set $\{p_n \mid n \in \mathbb{N}\}$.

Basically, the range just ignores the order part of a sequence.

$(p_n)_{n\in\mathbb{N}}$ is called bounded if $\{p_n \mid n \in \mathbb{N}\}$ is bounded.

Proposition 32:

(A): (p_n) converges to p if and only if every ball around p contains all but finitely many p_n.

Proof:
This is just the definition worded of a sequence converging but worded slightly differently.

(B): If (p_n) converges to p and p', then $p = p'$. (In other words, p is unique.)

Proof:
Let $\varepsilon > 0$. Pick $N, N' \in \mathbb{N}$ with:

$$\forall n \geq N \;\; d(p, p_n) < \varepsilon/2$$
$$\forall n \geq N' \;\; d(p', p_n) < \varepsilon/2$$

Setting $n = \max(N, N')$, we have that:

$$d(p, p') \leq d(p, p_n) + d(p_n, p') < \varepsilon 2 + \varepsilon 2 = \varepsilon.$$

And, as ε was arbitrary, we have that $0 \leq d(p, p') \leq \inf\{\varepsilon \mid \varepsilon > 0\} = 0$.
So, $d(p, p') = 0$ and thus $p = p'$.

(C): (p_n) converges $\Longrightarrow$ (p_n) is bounded.

Proof:
Say $p_n \to p$. Pick $N \in \mathbb{N}$ with $\forall n \geq N, \;\; d(p_n, p) < 1$.
Then set $r = \max\{d(p, p_1), d(p, p_2), \ldots, d(p, p_{N-1}), 1\}$.
Therefore, we have that $\forall n \in \mathbb{N} \;\; d(p_n, p) \leq r$.

(D): If $E \subseteq X$ and $p \in \overline{E}$, then there exists a sequence (p_n) in E with $p_n \to p$.

Proof:
Suppose $p \in E$. Then for all n, define $p_n = p$.

Now suppose $p \in E'$. Then, for each $n \in \mathbb{N}$, we must have that $(B_{\frac{1}{n+1}}(p) \setminus \{p\}) \cap E \neq \emptyset$. So, we can pick $p_n \in (B_{\frac{1}{n+1}}(p) \setminus \{p\}) \cap E$.
Then, $p_n \to p$.

(E): If (p_n) is a sequence in $E \subseteq X$ and $p_n \to p$, then $p \in \overline{E}$.

Proof:
Say $p_n \in E$ and $p_n \to p$. If $p \in E$, then we are done. So suppose $p \notin E$.
For every $r > 0$, there is n with $p_n \in B_r(p) \cap E = (B_r(p) \setminus \{p\}) \cap E$. So $p \in E'$.

Proposition 33: Suppose (s_n) and (t_n) are sequences in $\mathbb{C}$ with $s_n \to s$ and $t_n \to t$. Then:

1. $s_n + t_n \to s + t$

   Proof:
   Let $\varepsilon > 0$. Pick $N_1, N_2 \in \mathbb{N}$ such that:
   $\forall n \geq N_1 \;\; d(s, s_n) < \varepsilon/2$
   $\forall n \geq N_2 \;\; d(t, t_n) < \varepsilon/2$

   Then for all $n \geq \max(N_1, N_2)$, we have that:
   $|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \varepsilon 2 + \varepsilon 2 = \varepsilon$.

2. $s_n t_n \to st$

   Proof:
   Let $\varepsilon > 0$. Since $t_n \to t$, (t_n) is bounded. So, there exists $M > 0$ such that $|s| \leq M$ and $\forall n, \;\; |t_n| < M$. Pick N_1, N_2 with:
   $\forall n \geq N_1 \;\; d(s, s_n) < \varepsilon/2M$
   $\forall n \geq N_2 \;\; d(t, t_n) < \varepsilon/2M$

   Then for $n \geq \max(N_1, N_2)$, we have that:
   $$\begin{aligned} |s_n t_n - st| &= |s_n t_n - s t_n + s t_n - st| \\ &\leq |s_n - s||t_n| + |s||t_n - t| \\ &< \varepsilon/2M \cdot M + \varepsilon/2M \cdot M = \varepsilon \end{aligned}$$

3. $cs_n \to cs$ for all $c \in \mathbb{C}$.

   Proof:
   This follows from 33.2.

4. If $s \neq 0$, then $\frac{1}{s_n} \to \frac{1}{s}$.

   Proof:
   Let $\varepsilon > 0$. Pick N_1 so that $\forall n \geq N, \;\; |s_n - s| < \frac{1}{2}|s|$. Then $\forall n \geq N$, we have that $|s_n| \geq |s| - |s_n - s| > \frac{1}{2}|s|$. Next, pick N_2 so that $\forall n \geq N$, $|s - s_n| < \frac{1}{2}\varepsilon|s|^2$. Then $\forall n \geq \max(N_1, N_2)$ we have that:
   $$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|s_n s|} < \frac{|s - s_n|}{\frac{1}{2}|s|^2} < \frac{\frac{1}{2}\varepsilon|s|^2}{\frac{1}{2}|s|^2} = \varepsilon$$

5. If $\forall n \in \mathbb{N}, \;\; s_n, t_n \in \mathbb{R}$ and $s_n \leq t_n$, then $s \leq t$.

   Proof:
   $t_n - s_n \in [0, \infty)$. So by propositions 32.E and 33.1, we have that:
   $t - s = \lim_{n\to\infty} (t_n - s_n) \in \overline{[0, \infty)} = [0, \infty)$. Hence, $t \geq s$.

Proposition 34:

(A) If $\vec{x}_n = (\alpha_{1,n}, \alpha_{2,n}, \ldots, \alpha_{k,n}) \in \mathbb{R}^k$, then $\vec{x}_n \to \vec{x}$ if and only if for all $1 \leq i \leq k$, $\alpha_{i,n} \to \alpha_i$. In other words, covergence is coordinate-wise.

Proof:
For all $1 \leq i \leq k$, we have that $|\alpha_{i,n} - \alpha_i| \leq |\vec{x}_n - \vec{x}|$. So $(\vec{x}_n)$ converging implies that each $(\alpha_{i,n})$ converges.

Meanwhile, $|\vec{x}_n - \vec{x}| = \left(\sum_{i=1}^{k} |\alpha_{i,n} - \alpha_i|^2 \right)^{\frac{1}{2}} \leq \sqrt{k} \cdot \max_{1 \leq i \leq k} |\alpha_{i,n} - \alpha_i|$.
Thus, for $(\vec{x}_n)$ to converge, each $(\alpha_{i,n})$ must converge.

(B) If $\vec{x}_n, \vec{y}_n \in \mathbb{R}^k$, $\vec{x}_n \to \vec{x}$, and $\vec{y}_n \to \vec{y}$, then:

- $\vec{x}_n + \vec{y}_n \to \vec{x} + \vec{y}$
- $\vec{x}_n \cdot \vec{y}_n \to \vec{x} \cdot \vec{y}$
- $\beta_n \in \mathbb{R}$ and $\beta_n \to \beta$ implies that $\beta_n \vec{x}_n \to \beta \vec{x}$.

Proof:
This follows from propositions 33 and 34.A.

If $n_1 < n_2 < \ldots$ are positive integers, then $(p_{n_i})_{i \in \mathbb{Z}^+}$ is called a subsequence of $(p_n)_{n \in \mathbb{Z}^+}$. If (p_{n_i}) converges to p, we call p a subsequential limit of (p_n).

For example, if $x_n = (-1)^n$, then (x_n) does not converge. However, -1 and 1 are subsequential limits of (x_n).

Also, observe that $(p_n) \to p$ if and only if every subsequence of (p_n) converges to p.

Lecture 15: 2/14/2024

Propostion 35: q is a subsequential limit of (p_n) if and only if for all $r > 0$, $\{n \in \mathbb{N} \mid p_n \in B_r(q)\}$ is infinite.

Proof:
($\Longrightarrow$) Say $p_{n_i} \to q$. Then for all $r > 0$, $B_r(q)$ contains p_{n_i} for all but finitely many i. So, $\{n \in \mathbb{N} \mid p_n \in B_r(q)\}$ is infinite.

($\Longleftarrow$) Pick n_1 with $p_{n_1} \in B_1(q)$. Then for $i > 1$, pick $n_i > n_{i-1}$ with $p_{n_i} \in B_{1/i}(q)$. Thus, (p_{n_i}) is a subsequence converging to q.

Corollary: $q \in \{p_n \mid n \in \mathbb{N}\}'$ implies that q is a subsequential limit of (p_n).

Proposition 36:

(A) If (p_n) is a sequence in a compact space X, then (p_n) has a subsequential limit.

Proof:
Set $E = \{p_n \mid n \in \mathbb{N}\}$.

If E is finite, there are $n_1 < n_2 < \dots$ such that $\forall i, j, \;\; p_{n_i} = p_{n_j}$.
Therefore, $p_{n_i} \to p$ for some $p \in E$.

Meanwhile, if E is infinte, then $E' \neq \emptyset$ by proposition 25. Thus, by the corollary to proposition 35, we have that $p \in E'$ is a subsequential limit of (p_n)

(B) Every bounded sequence in $\mathbb{R}^k$ has a subsequential limit.

Proof:
Define E as before. Then because $\overline{E} \subseteq \mathbb{R}^k$ is bounded and closed, we know that $\overline{E}$ is compact. So, we can apply proposition 36.A to $E \subseteq \overline{E}$.

Proposition 37: For any metric space X, the set of all subsequential limits of (p_n) is closed.

Proof:
Let $x \in X$ be a limit point of a set of subsequential limits of (p_n). Also, fix $r > 0$. There must be a subsequential limit q of (p_n) with $q \in B_r(x)$. Setting $s = r - d(x, q)$ we have that $B_s(q) \subseteq B_r(x)$.

Since q is a subsequential limit, by proposition 35, we know that the set: $\{n \in \mathbb{N} \mid p_n \in B_s(q)\}$, is infinite. Thus, $\{n \in \mathbb{N} \mid p_n \in B_r(x)\}$ is infinite. So proposition 35, x is a subsequential limit of (p_n).

A sequence (p_n) is Cauchy if $\forall \varepsilon > 0, \;\; \exists N \in \mathbb{N} \;\; s.t. \;\; \forall n, m > N, \;\; d(p_n, p_m) < \varepsilon$.

The diameter of $\emptyset \neq E \subseteq X$ is $\operatorname{diam}(E) = \sup\{d(p,q) \mid p, q \in E\}$. Note that $\operatorname{diam}(E) \in [0, \infty]$.

Observe: (p_n) is Cauchy if and only if $\lim\limits_{n \to \infty} (\operatorname{diam}(\{p_m \mid m \geq n\})) = 0$.

Proposition 38:

(A) For $\emptyset \neq E \subseteq X$, $\mathrm{diam}(\overline{E}) = \mathrm{diam}(E)$.

Proof:

Let $p, q \in \overline{E}$ and $\varepsilon > 0$. Pick $p', q' \in E$ with $d(p, p'), d(q, q') < \varepsilon$. Then $d(p, q) \leq d(p, p') + d(p', q') + d(q, q') < \varepsilon + \mathrm{diam}(E) + \varepsilon = 2\varepsilon + \mathrm{diam}(E)$. Since $\varepsilon > 0$ was arbitrary, we find that $d(p, q) \leq \mathrm{diam}(E)$.

Hence $\mathrm{diam}(\overline{E}) \leq \mathrm{diam}(E)$.

Meanwhile, its obvious that $\mathrm{diam}(E) \leq \mathrm{diam}(\overline{E})$. So $\mathrm{diam}(E) = \mathrm{diam}(\overline{E})$.

(B) If for all $n \in \mathbb{N}$, we have that K_n is compact and nonempty, $K_{n+1} \subseteq K_n$, and $\mathrm{diam}(K_n) \to 0$, then $\bigcap_{n \in \mathbb{N}} K_n$ is a singleton.

Proof:

$\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$ by the finite intersection property.

Also, $\mathrm{diam}(\bigcap_{n \in \mathbb{N}} K_n) \leq \mathrm{diam}(K_n)$ for all n. So, $\mathrm{diam}(\bigcap_{n \in \mathbb{N}} K_n) = 0$.

Thus $\bigcap_{n \in \mathbb{N}} K_n$ contains a single point.

Proposition 39: Let X be a metric space.

1. If (p_n) converges, then (p_n) is Cauchy.

Proof:

Assume that $p_n \to p$. Let $\varepsilon > 0$. Pick N with $\forall n \geq N$, $d(p_n, p) < \varepsilon/2$. Then for all $n, m \geq N$, $d(p_n, p_m) \leq d(p_n, p) + d(q_n, q) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

2. If X is compact, then (p_n) being Cauchy implies that (p_n) converges.

Proof:

Set $E_n = \{p_n, p_{n+1}, \ldots\}$. Since (p_n) is Cauchy, we know that: $\mathrm{diam}(\overline{E_n}) = \mathrm{diam}(E_n) \to 0$. Also, since X is compact and $\overline{E_n} \subseteq X$, we know by proposition 24 that $\overline{E_n}$ is compact. Additionally, $\overline{E_{n+1}} \subseteq \overline{E_n}$.

Therefore, by proposition 38.B, $\bigcap_{n \in \mathbb{N}} \overline{E_n} = \{p\}$ for some $p \in X$.

Now let $\varepsilon > 0$. Pick N with $\mathrm{diam}(\overline{E_n}) < \varepsilon$. Then for all $n \geq N$, we have that $p_n, p \in \overline{E_n}$. So $d(p_n, p) < \mathrm{diam}(\overline{E_n}) < \varepsilon$. Hence, $p_n \to p$.

3. If $X = \mathbb{R}^k$, then (p_n) being Cauchy implies that (p_n) converges.

   Proof:
   Since (p_n) is Cauchy, pick N with $\mathrm{diam}(\{p_N, p_{N+1}, \ldots\}) < 1$. Setting $r = \max\{1, d(p_1, p_N), \ldots, d(p_{N-1}, p_N)\}$, we have that for all n, $d(p_n, p_N) \leq r$. So (p_n) is bounded.

A metric space X is complete if every Cauchy sequence in X converges.

Fact: $\mathbb{R}$ is the smallest complete metric space containing $\mathbb{Q}$.

A sequence (s_n) in $\mathbb{R}$ is called:

- monotone increasing if $\forall n, \quad s_n \leq s_{n+1}$.
- monotone decreasing if $\forall n, \quad s_n \geq s_{n+1}$.
- monotone if either of the above.

Lecture 16: 2/16/2024

Proposition 40: Suppose $(s_n) \subseteq \mathbb{R}$ is monotone. Then (s_n) converges if and only if (s_n) is bounded.

Proof:
($\Longrightarrow$) This is just proposition 32.C.

($\Longleftarrow$) We'll assume (s_n) is monotone increasing because the other case is basically identical but with flipped inequalities.

Set $s = \sup\{s_n \mid n \in \mathbb{N}\}$. We know this exists because (s_n) is bounded and $\mathbb{R}$ has the least upper bound property. Next let $\varepsilon > 0$. Since $s - \varepsilon$ is not an upper bound to $\{s_n \mid n \in \mathbb{N}\}$, we know there is N with $s - \varepsilon < s_N$. Hence, $\forall n \geq N, \quad s - \varepsilon < s_N \leq s_n \leq s$. Thus, $s_n \to s$.

For a sequence (s_n) in $\mathbb{R}$, we write:

- $s_n \to \infty$ or $\lim\limits_{n\to\infty} (s_n) = \infty$ if $\forall M \in \mathbb{R}, \quad \exists N \in \mathbb{N} \;\; s.t. \;\; \forall n \geq N, \quad s_n > M$.
- $s_n \to -\infty$ or $\lim\limits_{n\to\infty} (s_n) = -\infty$ if $\forall M \in \mathbb{R}, \quad \exists N \in \mathbb{N} \;\; s.t. \;\; \forall n \geq N, \quad s_n < M$.

In both of the above cases, we still say that s_n diverges.

Let (s_n) be a sequence in $\mathbb{R}$. Let E be the set of all subsequential limits of (s_n) in $\mathbb{R} \cup \{-\infty, \infty\}$.

The upper limit / limit supremum of (s_n) is: $\limsup s_n = \sup E$. Meanwhile, the lower limit / limit infimum of (s_n) is $\liminf s_n = \inf E$.

Importantly, the limit supremum and limit infimum always exist. This is because if (s_n) is not bounded above, then $+\infty \in E$. Meanwhile if (s_n) is not bounded below, then $-\infty \in E$. Finally, if (s_n) is bounded, then by the Bolzano-Weierstrauss Theorem (proposition 29), (s_n) has a limit point. Thus, by proposition 35, that limit point is a subsequential limit.

All in all, this means that E is never empty. And as the extended real numbers have the least upper bound property and all nonempty sets in the extended real numbers are bounded, we thus know that both the limit supremum and limit infimum always exist.

Proposition 41: Let (s_n) and E be defined as above...

(A) $\limsup s_n \in E$.

Proof:

If (s_n) is not bounded above, then $+\infty \in E$. So $\limsup s_n = +\infty$ which is in E. So let's assume (s_n) is bounded above.

If $\limsup s_n = -\infty$, then $E = \{-\infty\}$. So $\limsup s_n \in E$.

Finally, consider if $\limsup s_n \in \mathbb{R}$. Then $E \subseteq [-\infty, \limsup s_n]$. So, $\limsup s_n = \sup E = \sup (E \cap \mathbb{R}) \in \overline{E \cap \mathbb{R}}$. Then, as $\mathbb{R}$ is closed and E is closed by proposition 37, we have that $E \cap \mathbb{R} = \overline{E \cap \mathbb{R}}$. Therefore, $\limsup s_n \in E \cap \mathbb{R} \subseteq E$.

(B) If $x > \limsup s_n$, then there exists an integer N such that $\forall n \geq N, \;\; s_n < x$.

Proof:

Let $x > \limsup s_n$. Then $E \cap [x, +\infty] = \emptyset$. Now towards a contradiction, suppose $s_n \geq x$ for infinitely many n. Then, (s_n) has a subsequence in $[x, +\infty)$. Therefore, (s_n) has a subsequential limit y in $[x, +\infty]$. But this is a contradiction because $y \in E$ and $y > \sup E$.

(C) $\limsup s_n$ is the unique element of E satisfying propositions 41.B.

Proof:

Suppose towards a contradiction that both p and q satisfy proposition 41.B and are in E. Without loss of generality, let $p < q$. Then consider x with $p < x < q$. Applying proposition 41.B to p and x, we find that all but finitely many s_n are less than x. Hence, every subsequential limit is at most x. This contradicts that $q \in E$.

Also, one can obviously make analogous propositions for $\liminf s_n$.

Consider the sequence: $s_n = \dfrac{(-1)^n}{1 - \frac{1}{n}}$...

For every s_n we have that $|s_n| > |1|$. Yet observe that:

- $\limsup s_n = +1$
- $\liminf s_n = -1$

This demonstrates that the limite supremum or infinimum of a sequence is not the same as supremum or infimum of the range of a sequence.

An Incomplete List of How The Proposition Numbering in my Notes Lines up With Our Textbook:

Proposition Number	Label in Textbook	Proposition Number	Label in Textbook
1		2	
3		4	1.20
5		6	
7		8	
9		10	
11		12	2.12
13	2.13	14	
15		16	2.20
17	2.23	18	
19	2.27	20	2.28
21	2.30	22	
23		24	
25	2.37	26	2.38
27	2.40	28	2.41
29	2.42	30	2.43
31	2.47	32	3.2
33	3.3	34	3.4
35	n.a.	36	3.6
37	3.7	38	3.10
39	3.11	40	3.14
41	3.17	42	
43		44	