1/26/2025

Math 220b Notes:

Given a collection of points $a_1, \ldots, a_N \in \mathbb{C}$ and positive integers $m_1, \ldots, m_N$, we can always find an entire function $f : \mathbb{C} \to \mathbb{C}$ which only has zeros at the a_n and with multiplicities given by m_n. After all, we can just consider the polynomial:

$$f(z) = \prod_{n=1}^{n} (z - a_n)^{m_n}.$$

A natural folowup question to ask is if we can find an entire function with infinitely many prescribed zeros of varying multiplicities. As it turns out the answer is yes.

A natural first idea one might have for proving this is to try setting:

$$f(z) = \prod_{a=1}^{\infty} (1 - \tfrac{z}{a_n})^{m_n}.$$

The issue with this approach though is that that product may diverge. As an example, consider setting $a_n = -n$ and $m_n = 1$ for all $n \in \mathbb{N}$. Then the product $f(z) \coloneqq \prod_{n=1}^{\infty} (1 - \frac{z}{a_n})^{m_n} = \prod_{n=1}^{\infty} (1 + \frac{z}{n})$ diverges to ∞ at $z = 1$. After all, we get that $f(1) = 2(\frac{3}{2})(\frac{4}{3})(\frac{5}{4}) \cdots$.

By noting that $\operatorname{ord}(\prod_{k=1}^{\infty} g_k, a) = \sum_{k=1}^{\infty} \operatorname{ord}(g_k, a)$, a natural way of modifying our first idea is to try and multiply each $(1 - \frac{z}{a_n})^{m_n}$ term by some other function with no zeros. That way, we don't add any unwanted zeros to the function we are constructing and we can maybe coerce the product into converging.

It turns out this modified approach will work. Although surprisingly, by the homework problem on *pages 549-550*, we know these other function we're multiplying onto $(1 - \frac{z}{a_n})^{m_n}$ would have to have the form e^{g_n} where $g_n \in O(\mathbb{C})$ for all n.

Given any $p \in \mathbb{Z}_{\geq 0}$, we define the pth. Weierstrass primary/elementary factor to be:

$$E_p(z) = \begin{cases} (1 - z) & \text{if } p = 0 \\ (1 - z)\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{z}) & \text{if } p \neq 0 \end{cases}.$$

Note that if $a \in \mathbb{C} - \{0\}$, then $E_p(\frac{z}{a})$ is entire and it's only zero is a a simple one at $z = a$.

For motivation on why we are defining $E_p(z)$, note that the Taylor series for $-\operatorname{Log}(1 - z)$ about 0 is $\sum_{n=1}^{\infty} \frac{z^n}{n}$. Therefore, we'll ideally have that $E_p(z)$ is approximately equal to $1 = \frac{1-z}{1-z}$ when p is large and $|z|$ is small. The next lemma will make this idea more concrete.

Lemma A: $|E_p(z) - 1| \leq |z|^{p+1}$ if $|z| \leq 1$.

Proof:

If $p = 0$, then $|E_p(z) - 1| = |-z| = |z|$. So our proposed inequality trivialy holds.

Suppose $p > 0$ and set $u(z) = z + \frac{z^2}{2} + \cdots + \frac{z^p}{z}$. Then $E_p(z) = (1 - z)e^{u(z)}$ and we want to express $E_p(z)$ as a power series $\sum_{k=0}^{\infty} a_k z^k$ (which will have infinite radius of convergence since $E_p(z)$ is entire).

Claim 1: $a_0 = 1$.

Note that $a_0 = E_p(0) = (1-0)e^{u(0)} = 1$.

Claim 2: $a_1 = a_2 = \cdots = a_p = 0$.

By differentiating $u(z)$, we get that $u'(z) = 1 + z + \ldots + z^{p-1}$. In turn:
$$(1-z)u'(z) = 1 - z^p.$$

But that implies that:
$$E_p'(z) = -e^{u(z)} + (1-z)u'(z)e^{u(z)} = -e^{u(z)} + (1-z^p)e^{u(z)} = -z^p e^{u(z)}$$

Therefore, if we look at the taylor expansion $\sum_{k=1}^{\infty} k a_k z^{k-1}$ for $E_p'(z)$ about 0 and note that $e^{u(0)} = 1 \neq 0$, we must have that the lowest degree term in that power series is the z^p term. In other words, $a_1 = a_2 = \cdots = a_p = 0$.

Claim 3: $a_k \leq 0$ in $\mathbb{R}$ for all $k \geq p+1$.

We showed in claim 2 that:
$$\textstyle\sum_{k=p+1}^{\infty} k a_k z^{k-1} = -z^p e^{u(z)}.$$

In turn, $\sum_{k=p+1}^{\infty} k a_k z^{k-(p+1)} = -e^{u(z)} = -\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p})$. Or in other words, we have that a_k equals $\frac{-1}{k}$ times the $(k-(p+1))$th. coefficient in the Taylor expansion of $\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p})$ for all $k \geq p+1$.

If we can show that all coefficients in the Taylor expansion of $\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p})$ are positive, we will be done. Fortunately, note that:
$$\exp(z + \tfrac{z^2}{2} + \cdots + \tfrac{z^p}{p}) = e^z e^{\frac{z^2}{2}} \cdots e^{\frac{z^p}{p}} = \prod_{n=1}^{p} \left(\sum_{\ell=0}^{\infty} \tfrac{1}{\ell! n^\ell} z^{n\ell} \right).$$

By taking successive Cauchy products of those series (i.e. foiling), we'll get that the coefficients of the power series for $\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p})$ are all sums of products of the $\frac{1}{\ell! n^\ell}$ (which are all positive). Hence, those coefficients are positive.

Unnecessary tangent: Here's how we can actually calculate the power series expansion $\sum_{k=0}^{\infty} c_k z^k$ for $e^{u(z)}$.

To start off, the derivative of $e^{u(z)}$ is $(1 + z + \ldots + z^{p-1})e^{u(z)}$. Therefore, we must have that $\sum_{k=0}^{\infty}(k+1)c_{k+1}z^k = (1 + z + \ldots + z^{p-1}) \sum_{k=0}^{\infty} c_k z^k$. Also, when you consider that $e^{u(0)} = 1$, we thus get the following recurrence relation:

- $c_0 = 1$
- $c_{k+1} = \frac{1}{k+1} \sum\limits_{\ell=0}^{\min(k,p-1)} c_{k-\ell}$

Next, we can easily calculate from that relation that $c_1 = \cdots = c_p = 1$. Furthermore, note that for any $k \geq p-1$ that:
$$c_{k+2} - \tfrac{k+1}{k+2} c_{k+1} = \tfrac{1}{k+2} c_{k+1} - \tfrac{1}{k+2} c_{k-p+1}.$$

In other words, $c_{k+2} = c_{k+1} - \frac{1}{k+2} c_{k-p+1}$. So finally (and this is how much simplified I'm capable of getting it before my attention span runs out), we have that the coefficients c_k are given by the following recurrence relation:

- $c_0 = c_1 = \cdots = c_p = 1$;
- $c_{k+1} = c_k - \frac{1}{k+1}c_{k-p}$ if $k \geq p$.

Claim 4: $\sum_{k=p+1}^{\infty} |a_k| = 1$

Note that $0 = E_p(1) = 1 + \sum_{k=p+1}^{\infty} a_k$. Therefore:

$$\sum_{k=p+1}^{\infty} |a_k| = -\sum_{k=p+1}^{\infty} a_k = 1$$

Finally, note that when $|z| \leq 1$, we have that:

$$|E_p(z) - 1| = |z^{p+1} \sum_{k=p+1}^{\infty} a_k z^{k-p-1}|$$

$$\leq |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k||z^{k-p-1}| \leq |z|^{p+1} \sum_{k=p+1}^{\infty} 1|a_k| = |z|^{p+1}. \blacksquare$$

Lemma B: Given any sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C} - \{0\}$ such that $|a_n| \to \infty$, we have for all $r > 0$ that $\sum_{n=1}^{\infty} (\frac{r}{|a_n|})^n < \infty$.

Proof:

Since $|a_n| \to \infty$, we know there exists N such that $|a_n| \geq 2r$ for all $n \geq N$. In turn, for all $n \geq N$ we have that $(\frac{r}{|a_n|})^n \leq \frac{1}{2^n}$. And since $\sum_{n=0}^{\infty} \frac{1}{2^n} < \infty$, we can conclude by the comparison test that $\sum_{n=1}^{\infty} (\frac{r}{|a_n|})^n < \infty$. $\blacksquare$

With that we are ready to prove our desired theorem:

Weierstraß Factorization Problem: Let $\{a_n\}_{n\in\mathbb{N}}$ be any sequence of distinct elements in $\mathbb{C}$ with $|a_n| \to \infty$, and also let $\{m_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}_{>0}$.

Note that the condition that $|a_n| \to \infty$ is equivalent to guarenteeing that the set $\{a_n : n \in \mathbb{N}\}$ has no limit points in $\mathbb{C}$.

We claim there exists an entire function $f : \mathbb{C} \to \mathbb{C}$ which only has zeros at the a_n and with multiplicity given by m_n.

Proof:

To start off, it will be convenient (for the sake of notation) to replace $\{a_n\}_{n\in\mathbb{N}}$ with the sequence:

$$\underbrace{a_1, \ldots, a_1}_{m_1 \text{ times}}, \underbrace{a_2, \ldots, a_2}_{m_2 \text{ times}}, \underbrace{a_3, \ldots, a_3}_{m_3 \text{ times}} \ldots$$

Importantly, doing this relabeling won't change the fact that $|a_n| \to \infty$ as $n \to \infty$. One other note is that we can without loss of generality assume $a_n \neq 0$ for any n. After all, if not we can just remove those terms from our sequence, solve the factorization problem to get an entire function $h(z)$, and then finally set $f(z) = z^m h(z)$ where m is the number of zero terms we removed from the sequence.

Now our goal is to pick a sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}_{\geq 0}$ such that $f(z) = \prod_{n=1}^{\infty} E_{p_n}(\frac{z}{a_n})$ converges absolutely locally uniformly. Equivalently, we need $\sum_{n=1}^{\infty} |E_{p_n}(\frac{z}{a_n}) - 1|$ to converge locally uniformly.

Set $p_n = n-1$ for all n. This will guarentee that $\sum_{n=1}^{\infty} |E_{p_n}(\frac{z}{a_n}) - 1|$ converges normally (and hence locally uniformly).

Indeed let $K \subseteq \mathbb{C}$ be compact. Then there exists $r > 0$ such that $K \subseteq \overline{\Delta}(0, r)$. Since $|a_n| \to \infty$, there exists N such that $|a_n| \geq r$ if $n \geq N$. And now by lemma A, we have that:

$$|E_{p_n}(\tfrac{z}{a_n}) - 1| \leq |\tfrac{z}{a_n}|^{p_n+1} \leq (\tfrac{r}{|a_n|})^n \text{ for all } z \in K$$

Finally normal converge follows from lemma B.

It follows that $f(z) := \prod_{n=1}^{\infty} E_{p_n}(\frac{z}{a_n})$ is an entire function with a zero at each a_n. ■

Note that by no means did we show that the p_n used in the above proof are unique. On the contrary, it is easy to justify that we can always modify at least finitely many of the p_n.

By combining the above proof with the homework problem on *pages 549-550*, we can rewrite our prior result in the following more versatile form:

Theorem: If f is entire with countably infinite zeros, then $f(z) = z^m e^{h(z)} \prod_{n\in\mathbb{N}} E_{p_n}(\frac{z}{a_n})$ for some $h \in O(\mathbb{C})$, $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C} - \{0\}$, $m \in \mathbb{Z}_{\geq 0}$ and $\{p_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}_{\geq 0}$.

Taking a break from the lecture and looking at (Conway) Theorem VII.5.15, we can also solve the Weierstraß factorization problem on an arbitrary region $G \subseteq \mathbb{C}$.

Theorem VII.5.15: Let G be a region and let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of distinct points in G with no limit point in G. Also let $\{m_n\}_{n\in\mathbb{N}}$ be a sequence of positive integers. Then there is a function $f \in O(G)$ whose only zeros are at the a_n and with multiplicity given by m_n

Proof:

Part 1: Suppose there exists $R > 0$ such that $\{z \in \mathbb{C} : |z| > R\} \subseteq G$ and $|a_n| \leq R$ for all n.

Note that this guarentees that G^{C} is a nonempty compact set. After all, G^{C} is clearly closed and bounded. Also, G^{C} can't be empty as G^{C} must contain at least one limit point of the a_n in the comapct set $\overline{\Delta}(0, R)$.

Then we claim there is a function $f \in H(G)$ solving the Weierstraß problem and which has the further property that $\lim_{z\to\infty} f(z) = 1$.

Like in the proof of the Weierstraß problem for entire functions, we replace $\{a_n\}_{n\in\mathbb{N}}$ with the sequence:

$$\underbrace{a_1, \ldots, a_1}_{m_1 \text{ times}}, \underbrace{a_2, \ldots, a_2}_{m_2 \text{ times}}, \underbrace{a_3, \ldots, a_3}_{m_3 \text{ times}} \ldots$$

Doing this will not change the fact that $\{a_n\}_{n\in\mathbb{N}}$ has no limit points. Next as G^{C} is compact, we know for each $n \in \mathbb{N}$ that there exists a point $b_n \in G^{\mathsf{C}}$ with:

$$|b_n - a_n| = \min_{w\in G^{\mathsf{C}}} |w - a_n|.$$

Also $\min_{w\in G^{\mathsf{C}}} |w - a_n| \to 0$ as $n \to \infty$ or else we'd know the a_n have a limit point in G. Therefore, $|a_n - b_n| \to 0$ as $n \to \infty$.

Now set $p_n = n$ and define $f(z) := \prod_{n=1}^{\infty} E_{p_n}(\frac{a_n - b_n}{z - b_n})$

Note that:
$E_p(\frac{a_n-b_n}{z-b_n}) = (1-\frac{a_n-b_n}{z-b_n})\exp(\frac{a_n-b_n}{z-b_n}+\frac{1}{2}\left(\frac{a_n-b_n}{z-b_n}\right)^2+\cdots+\frac{1}{p}\left(\frac{a_n-b_n}{z-b_n}\right)^p)$.

The exponential factor contributes no zeros. Meanwhile $(1-\frac{a_n-b_n}{z-b_n})$ only has a zero at $z=a_n$. And since $(1-\frac{a_n-b_n}{z-b_n})' = \frac{-(a_n-b_n)}{(z-b_n)^2}$, we also know that zero is simple.

Based on the above reasoning, we know that if f converges absolutely locally uniformly on G, then the only zeros of f will be at the a_n and with the multiplicity we want. In other words, it suffices to show that if $K\subseteq G$ is compact, then:
$\sum_{n=1}^{\infty}|E_{p_n}(\frac{a_n-b_n}{z-b_n})-1|$ converges uniformly on K.

Let $\varepsilon = \inf\{|z-w| : z\in K, w\in G^{\mathsf{C}}\}$ and note that $\varepsilon>0$. Then:
$|\frac{a_n-b_n}{z-b_n}| \leq |a_n-b_n|\varepsilon^{-1}$.

Next as $|a_n-b_n|\to 0$, we know there exists N such that $|a_n-b_n|<\varepsilon/2$ for all $n\geq N$. So, by *lemma A*, we have for $n\geq N$
$|E_{p_n}(\frac{a_n-b_n}{z-b_n})-1| \leq (\frac{\varepsilon}{2}\cdot\varepsilon^{-1})^{p_n+1} = \frac{1}{2^{n+1}}$.

By the Weierstrass M-test, we can conclude that $\sum_{n=1}^{\infty}|E_{p_n}(\frac{a_n-b_n}{z-b_n})-1|$ converges uniformly on K.

To finish this part, we need to show $f(z)=\prod_{n=1}^{\infty}E_{p_n}(\frac{a_n-b_n}{z-b_n})\to 1$ as $|z|\to\infty$. Therefore, consider any $\varepsilon>0$ and let $\kappa_1>0$ satisfy that $|e^w-1|<\varepsilon$ when $|w|<\kappa_1$.

Like on page 546 (see $(*)$), we can choose some $\delta\in(0,1)$ such that:
$\frac{1}{2}|z| \leq |\operatorname{Log}(1+z)| \leq \frac{3}{2}|z|$ when $|z|<\delta$.

Next note that if we fix $0<\kappa_2<\delta$, then we can set $R_1>0$ large enough so that $2R<\kappa_2(R_1-R)$. And since $a_n, b_n\in\overline{\Delta}(0,R)$ for all n we know that:
$|\frac{a_n-b_n}{z-b_n}| \leq \frac{2R}{R_1-R} < \kappa_2$ for all $n\in\mathbb{N}$ and $|z|\geq R_1$.

In particular, this has the advantage that since $|E_{p_n}(\frac{a_n-b_n}{z-b_n})-1| < \kappa_2^{p_n+1} < 1$, we know $\operatorname{Log}(E_{p_n}(\frac{a_n-b_n}{z-b_n}))$ is well-defined for all $n\in\mathbb{N}$ and $|z|\geq R_1$. So, we can write $f(z)=\exp(\sum_{n=1}^{\infty}\operatorname{Log}(E_{p_n}(\frac{a_n-b_n}{z-b_n})))$ when $|z|\geq R_1$ and then note for all $|z|\geq R_1$ that:

$$\begin{aligned}
|\sum_{n=1}^{\infty}\operatorname{Log}(E_{p_n}(\tfrac{a_n-b_n}{z-b_n}))| &\leq \sum_{n=1}^{\infty}|\operatorname{Log}(E_{p_n}(\tfrac{a_n-b_n}{z-b_n}))| \\
&\leq \tfrac{3}{2}\sum_{n=1}^{\infty}|E_{p_n}(\tfrac{a_n-b_n}{z-b_n})-1| \leq \tfrac{3}{2}\sum_{n=1}^{\infty}|\tfrac{a_n-b_n}{z-b_n}|^{p_n+1} \leq \tfrac{3}{2}\sum_{n=1}^{\infty}\kappa_2^{n+1} \\
&= \tfrac{3}{2}\cdot\tfrac{\kappa_2^2}{1-\kappa_2}
\end{aligned}$$

By choosing κ_2 small enough, we can guarentee that $\frac{3}{2} \cdot \frac{\kappa_2^2}{1-\kappa_2} < \kappa_1$. Therefore, $|f(z) - 1| < \varepsilon$ when $|z| \geq R_1$.

Part 2:
Now suppose G is an arbitrary region in $\mathbb{C}$ with $\{a_n\}_{n\in\mathbb{N}}$ a sequence of distinct points in G with no limit point, and let $\{m_n\}_{n\in\mathbb{N}}$ be a sequence of positive integers. Then there is some disk $\overline{\Delta}(a,r) \subseteq G$ with $\Delta(a,r)$ containing none of the a_n. So, consider the Möbius transformation $Tz = (z-a)^{-1}$ and put $G_1 = T(G)$ and $b_n = Ta_n = (a_n - a)^{-1}$ for all n.

Part 1 clearly applies to the region G_1 and the sequences $\{b_n\}_{n\in\mathbb{N}}$ and $\{m_n\}_{n\in\mathbb{N}}$.
We know $\{b_n\}_{n\in\mathbb{N}}$ has no limit points in G_1 because T has a continuous inverse Möbius transformation. Suppose $b_{n_k} \to \beta$ where there exists $\alpha \in G$ with $T\alpha = \beta$. Then $a_{n_k} = T^{-1}(b_{n_k}) \to T^{-1}\beta = \alpha$. But that's a contradiction.

So, we can pick a function $g \in O(G_1)$ whose only zeros are at the b_n and with multiplicity m_n. Next, we set $f(z) = g(Tz)$. Then $f \in O(G - \{a\})$ and has only zeros at the a_n and with multiplicity m_n.

Finally, $f(z) = g(\frac{1}{z-a}) \to 1$ as $z \to a$, we know f has a removable singularity at a. So, we can extend f to be a function in $O(G)$ that has exactly the zeros we want. ■

One example of the significance of the Weierstraß factorization problem is as follows:
Suppose that $G \subseteq \mathbb{C}$ is a region and then note that $O(G)$ is an integral domain.
To see why, note that G cannot be countable as any open ball contained in G is not meager (recall the Baire category theorem). That said, any set S without a limit point in G must be countable. This is because G is σ-compact. As a consequence, if $f, g \in O(G)$ are not the constant zero functions, then fg can only have countably many zeros and is thus also not the constant zero function.

Claim: The field of fractions of $O(G)$ is precisely the set of all meromorphic functions on G.

Proof:
It's easy to see that if $g, h \in O(G)$ with $h \not\equiv 0$, then g/h is meromorphic in G. So, the field of fractions of $O(G)$ can be embedded in the set of meromorphic functions on G.

Conversely, suppose f is meromorphic in G and let S be the set of poles of f. Then there exists a function $h \in O(G)$ with zeros only at the points in S and which satisfies that the multiplicity of any zero in S of h in equal to the order of the corresponding pole of f.

Next set $g = fh$ on $G - S$. Then as all the singularities of g are removable, we can holomorphically extend g to being a function in $O(G)$. At last, we have found two holomorphic functions $g, h \in O(G)$ with $f = \frac{g}{h}$. ■

Another interpretation goes as follows:

A divisor on G is a formal sum $D = \sum_{p\in G} n_p[p]$ where $n_p \in \mathbb{Z}$ for all $p \in G$ and the set $\{p \in G : n_p \neq 0\}$ does not have a limit point.

We say a divisor $D = \sum_{p\in G} n_p[p]$ is effective (denoted $D \geq 0$) if $n_p \geq 0$ for all p.

Note that divisors can be added formally.

In particular, if S_1 and S_2 are sets in G without a limit point, then $S_1 \cup S_2$ also can't have a limit point. After all, if p was a limit point of $S_1 \cup S_2$, then any open set U containing p would have to contain infinitely many points in $S_1 \cup S_2$. But by making U small enough, we can guarentee that U contains only finitely many points in S_1 or S_2.

It follows that the divisors form an abelian group which we'll denote $(\mathrm{Div}, +)$.

A divisor D is principal if there exists a meromorphic function f on G with:

$$D = \sum_{p\in G} \mathrm{ord}(f,p)[p].$$

Note: We define $\mathrm{ord}(f,p)$ to be a negative integer if f has a pole at p and $\mathrm{ord}(f,p)$ is a positive integer if f has a zero at p.

We shall denote $\sum_{p\in G} \mathrm{ord}(f,p)[p] = \mathrm{div}(f)$.

Importantly, note that $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$. Therefore, the collection of principal divisors forms a subgroup $(\mathrm{Prin}, +)$ of $(\mathrm{Div}, +)$.

Finally, we define the divisor class subgroup to be quotient group $(\mathrm{Cl}, +) := (\mathrm{Div}/\mathrm{Prin}, +)$.

Corollary of Weierstraß: $\mathrm{Cl} = \{0\}$.

Proof:

We show $\mathrm{Div} = \mathrm{Prin}$. To start off, if D is a divisor, then we can write $D = D_+ - D_-$ where both D_+ and D_- are effective. Next, by Weierstraß there exists $g, h \in O(G)$ with $D_+ = \mathrm{div}(g)$ and $D_- = \mathrm{div}(h)$. Finally, $\mathrm{div}(g/h) = D$. ■

You may wonder why mathematicians went to the effort of defining Cl. Essentially, it's because we can more generally define holomorphic and meromorphic functions on objects called Riemann surfaces.

Math 220b Homework: