My Notes on Paolo Aluffi's Algebra Chapter 0

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A multiset is a collection of elements which like a set is unordered but unlike a set can contain duplicate elements.

One way to define a multiset is as a function $f : A \to \mathbb{N}$ such that each $\alpha \in A$ is mapped to the number of times that α appears in the multiset. Then, given the multisets $f_1 : A \to \mathbb{N}$ and $f_2 : B \to \mathbb{N}$, we can define the following operations:

- $\alpha \in f_1 \iff \alpha \in A$
- $f_1 \subseteq f_2 \iff \forall \alpha \in f_1,\ \alpha \in f_2$ and $f_1(\alpha) \leq f_2(\alpha)$
- $f_1 \cup f_2 : (A \cup B) \longrightarrow \mathbb{N}$ such that for $\alpha \in A \cup B$, if $\alpha \in A \cap B$, then $(f_1 \cup f_2)(\alpha) = f_1(\alpha) + f_2(\alpha)$. As for if $\alpha \notin A \cap B$, then $(f_1 \cup f_2)(\alpha)$ equals whatever α was mapped to in the multiset it originally came from.
- $f_1 \cap f_2 : (A \cap B) \longrightarrow \mathbb{N}$ such that for $\alpha \in A \cap B$, we have that $(f_1 \cap f_2)(\alpha) = \min(f_1(\alpha), f_2(\alpha))$
- $f_1 \setminus f_2 : ((A \setminus B) \cup \{\alpha \in A \cap B \mid f_1(\alpha) > f_2(\alpha)\}) \longrightarrow \mathbb{N}$ such that for each $\alpha \in f_1 \setminus f_2$, if $\alpha \in f_2$, then $(f_1 \setminus f_2)(\alpha) = f_1(\alpha) - f_2(\alpha)$. As for if $\alpha \notin f_2$, then $(f_1 \setminus f_2)(\alpha) = f_1(\alpha)$

A practical example of a multiset is the prime factorization of any positive integer.

We say that two sets A and B are isomorphic if and only if there exists a bijection between A and B. We denote this by writing $A \cong B$. Additionally, we can refer to any bijection f between A and B as an isomorphism between the two sets.

A function $f : A \to B$ is a monomorphism (a.k.a a monic) if for all sets Z and all functions a' and $a'' : Z \to A$, we have that $f \circ a' = f \circ a'' \implies a' = a''$.

Proposition 1: A function is injective if and only if it is a monomorphism.

Proof: Let's say we have a function $f : A \to B$.

First, let us assume f is injective.

Then let us assume we have two functions a' and a'' from some set Z to A such that $f \circ a' = f \circ a''$. Because f is injective, we know it has a left-hand inverse $g : B \to A$ such that $g \circ f = \mathrm{Id}_A$. Composing g with the previous equation, we get that:

$$a' = \mathrm{Id}_A \circ a' = g \circ (f \circ a') = g \circ (f \circ a'') = \mathrm{Id}_A \circ a'' = a''$$

Thus, we've shown that f is a monomorphism.

Next, we shall assume f is a monomorphism.

Based on this, we can say that for any two functions a' and a'' mapping a set Z to A, we have that $f \circ a' = f \circ a'' \Longrightarrow a' = a''$. However, now note that if we make Z a singleton, meaning it only contains one element, then a' and a'' can each only take on one value. So, we can effectively rewrite $f \circ a' = f \circ a'' \Rightarrow a' = a''$ as:

$$f(a') = f(a'') \Rightarrow a' = a''$$

This is the definition of an injective function. ■

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A function $f : A \to B$ is an epimorphism (a.k.a an epi) if for all sets Z and all functions b' and $b'' : B \to Z$, we have that $b' \circ f = b'' \circ f \Rightarrow b' = b''$.

Proposition 2: A function is a surjection if and only if it is an epimorphism.

Proof: Let's say we have a function $f : A \to B$.

First, let us assume f is surjective.

Then let's assume we have two functions b' and b'' from B to some set Z such that $b' \circ f = b'' \circ f$. Because f is surjective, we know it has a right-hand inverse $h : B \to A$ such that $f \circ h = \mathrm{Id}_B$. Composing h with the previous equation, we get that:

$$b' = b' \circ \mathrm{Id}_B = (b' \circ f) \circ h = (b'' \circ f) \circ h = b'' \circ \mathrm{Id}_B = b''$$

So f is an epimorphism.

Next, assume f is not surjective.

Then there exists $\beta \in B$ such that for all $\alpha \in A$, we have that $f(\alpha) \neq \beta$. Importantly, as $f(\alpha) \in B$, we know $|B| \neq 1$. So set b' equal to Id_B and define b'' as a function mapping each element of $B \setminus \{\beta\}$ to itself and β to any of the other elements in B. Now, $b' \circ f = f = b'' \circ f$ but $b' \neq b''$. So f is not an epimorphism. ■

Sometimes, to indicate that a function $f : A \to B$ is a monomorphism, epimorphism, or isomorphism, we use the following notation:

- Monomorphism: $f : A \hookrightarrow B$
- Epimorphism: $f : A \twoheadrightarrow B$
- Isomorphism: $f : A \xrightarrow{\sim} B$

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A relation on a set S is a subset R of the cartesian product $S \times S$. Specifically, we use the notation $x \, R \, y$ to mean that $(x, y) \in R$. Certain types of relations are especially important and thus are represented with their own symbol.

- An equivalence relation, typically denoted $\sim$, on a set S has the properties:
  ∘ $\forall a \in S, \;\; a \sim a$ ∘ $a \sim b \Longrightarrow b \sim a$ ∘ $a \sim b$ and $b \sim c \Longrightarrow a \sim c$
- An order relation, typically denoted $<$, on a set S has the properties:
  ∘ $\forall a, b \in S$, exactly one of the following is true: $a < b$, $b < a$, or $a = b$.
  ∘ $a < b$ and $b < c$ implies that $a < c$.

Given a set S, an equivalence relation $\sim$, and an element $a \in S$, we define the equivalence class of a with respect to $\sim$ to be the set $[a]_\sim = \{b \in S \mid a \sim b\}$. Also, we define the quotient of S with respect to the equivalence relation $\sim$ as the set of equivalence classes with respect to $\sim$.

$$S/\!\sim \; = \{[a]_\sim \mid a \in S\}$$

Given any function $f : A \longrightarrow B$, define $a \sim b \Longleftrightarrow f(a) = f(b)$.

Proposition 3: Every function f can be decomposed as follows:

$$\begin{tikzcd} A \arrow[r, two heads, "g"'] \arrow[rrr, bend left, "f"] & (A/\sim) \arrow[r, "\sim", "\widetilde{f}"'] & \operatorname{im} f \arrow[r, hook, "h"'] & B \end{tikzcd}$$

(in other words, $f = h \circ \widetilde{f} \circ g$)

...where g is the surjection mapping a to $[a]_\sim$ for all $a \in A$, h is the inclusion function (which is injective) from the image of f to B, and $\widetilde{f}$ is a bijective function defined as the mapping $[a]_\sim$ to $f(a)$ where $a \in [a]_\sim$.

Proof:
$(A/\!\sim)$ is defined as the range of g. So g is automatically surjective. Also, inclusion functions like h are always injective.

Now we show $\widetilde{f}$ is well defined and bijective.

1. Assume $a', a'' \in A$ such that $[a'] = [a'']$. Then by how we defined $\sim$, $f(a') = f(a'')$. So $[a'] = [a''] \Longrightarrow \widetilde{f}([a']) = \widetilde{f}([a''])$, meaning $\widetilde{f}$ is well defined.

2. Assume $\widetilde{f}([a']) = \widetilde{f}([a''])$. Then $f(a') = f(a'')$, meaning $a' \sim a''$. Hence $[a'] = [a'']$, meaning $\widetilde{f}$ is injective.

3. Given any $b \in \operatorname{im} f$, there exists $a \in A$ such that $f(a) = b$. Then $\widetilde{f}([a]_\sim) = f(a) = b$. So $\widetilde{f}$ is surjective.

Finally, it's clear that $f = h \circ \widetilde{f} \circ g$. So we're done.

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A category C consists of a class $\mathrm{Obj}(\mathsf{C})$ of objects of the category, and for every two objects A, B of C, a set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ of morphisms with the following properties:

- For every object A of C, there exists a morphism $1_A \in \mathrm{Hom}_{\mathsf{C}}(A, A)$ called the identity on A.
- Morphisms can be composed, meaning $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ means that $gf \in \mathrm{Hom}_{\mathsf{C}}(A, C)$
- Composition is associative, meaning if $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathsf{C}}(B, C)$, and $h \in \mathrm{Hom}_{\mathsf{C}}(C, D)$, then $(hg)f = h(gf)$.
- The identity morphisms are identities with respect to composition, meaning for all $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, $f1_A = f$ and $1_B f = f$.
- $\mathrm{Hom}_{\mathsf{C}}(A, B)$ and $\mathrm{Hom}_{\mathsf{C}}(C, D)$ are disjoint unless $A = C$ and $B = D$.

We use the word "class" because by Russell's paradox, there are many sets which aren't well defined. For example, there is no set of sets. So we instead make a class of all sets.

Also, we write category names in sans-serif font to better distinguish them.

A morphism of an object A of a category C to itself is called an endomorphism. Thus we denote $\mathrm{Hom}_{\mathsf{C}}(A, A)$ as $\mathrm{End}_{\mathsf{C}}(A)$.

Note that by the composition rules of a category, if $f, g \in \mathrm{End}_{\mathsf{C}}(A)$, then $fg, gf \in \mathrm{End}_{\mathsf{C}}(A)$.

We can denote a morphism $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ as $f : A \to B$.

Examples of Categories:

- We define the category of sets: Set, such that $\mathrm{Obj}(\mathsf{Set})$ is the class of all sets and for A and B in $\mathrm{Obj}(\mathsf{Set})$, $\mathrm{Hom}_{\mathsf{Set}}(A, B)$ is the set of all functions from A to B (abbreviated as B^A).

- If S is a set and $\sim$ is an equivalence relation on S, then we can define a category whose objects are the elements of S, and for $a, b \in S$, $\mathrm{Hom}(a, b)$ equals $\{(a, b)\}$ when $a \sim b$ and $\emptyset$ otherwise.

  Note that for this category, we need to define what it means to compose morphisms. So let's say that if $f = \{(a, b)\}$ and $g = \{(b, c)\}$, then $gf = \{(a, c)\}$.

- Let C be a category and let A be an object of C. Then we can define a category C_A as follows:
  - $\mathrm{Obj}(\mathsf{C}_A) =$ all morphisms from any object of C to A
  - If $f_1 : Z_1 \longrightarrow A$ and $f_2 : Z_2 \longrightarrow A$ are objects of C_A, then $\mathrm{Hom}_{\mathsf{C}_A}(f, g)$ is the set of morphisms $\sigma : Z_1 \to Z_2$ such that $f_1 = f_2\sigma$.

  Thus the morphisms of C_A are <u>commutative diagrams</u> with the objects Z_1, Z_2, and A.

  To prove that this is a category, first consider that each object $f : Z \longrightarrow A$ has an identity morphism:

$$\begin{tikzcd}
Z \arrow[rr, "1_Z"] \arrow[dr, "f"'] & & Z \arrow[dl, "f"] \\
& A &
\end{tikzcd}$$

  Also, the morphisms of C_A compose. If the diagram with σ is in $\mathrm{Hom}_{\mathsf{C}_A}(f_1, f_2)$ and the diagram with τ is in $\mathrm{Hom}_{\mathsf{C}_A}(f_2, f_3)$, then we define their composition in $\mathrm{Hom}_{\mathsf{C}_A}(f_2, f_3)$ as the diagram with $\tau\sigma$.

$$\begin{tikzcd}
Z_1 \arrow[r, "\sigma"] \arrow[dr, "f_1"'] & Z_2 \arrow[r, "\tau"] \arrow[d, "f_2"] & Z_3 \arrow[dl, "f_3"] \\
& A &
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
Z_1 \arrow[rr, "\tau\sigma"] \arrow[dr, "f_1"'] & & Z_3 \arrow[dl, "f_3"] \\
& A &
\end{tikzcd}$$

  As is hopefully apparent, the identity morphisms compose as is required for C_A to be a category.

  Finally, composing morphisms of C_A is associative.

$$\begin{tikzcd}
Z_1 \arrow[r, "\sigma"] \arrow[rr, bend left, "\tau\sigma"] \arrow[rrr, bend left=50, "\upsilon\tau\sigma"] \arrow[drr, "f_1"'] & Z_2 \arrow[r, "\tau"] \arrow[rr, bend left, "\upsilon\tau"] \arrow[dr, "f_2"] & Z_3 \arrow[r, "\upsilon"] \arrow[d, "f_3"'] & Z_4 \arrow[dl, "f_4"] \\
& & A &
\end{tikzcd}$$

- Categories like the one in the previous example are called slice categories. We can similarly define coslice categories as follows:

  Let C be a category and let A be an object of C. Then we can define a category C^A such that:
  - $\mathrm{Obj}(\mathsf{C}^A) =$ all morphisms from A to any object of C
  - If $f_1 : A \longrightarrow Z_1$ and $f_2 : A \longrightarrow Z_2$ are objects of C^A, then $\mathrm{Hom}_{\mathsf{C}^A}(f, g)$ is the set of morphisms $\sigma : Z_1 \to Z_2$ such that $\sigma f_1 = f_2$.

    In other words, we're now considering commutative diagrams of the form:

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ Z_1 \arrow[rr, "\sigma"'] & & Z_2 \end{tikzcd}$$

Problem 3.8: A subcategory C' of a category C consists of a collection of objects of C with morphisms $\mathrm{Hom}_{\mathsf{C}'}(A, B) \subseteq \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all objects A, B in $\mathrm{Obj}(\mathsf{C}')$ such that C' has all the necessary identities and compositions to be a category. A subcategory C' is full if $\mathrm{Hom}_{\mathsf{C}'}(A, B) = \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all A, B in $\mathrm{Obj}(\mathsf{C}')$.

Let Set' be the category of infinite sets.

- $\mathrm{Obj}(\mathsf{Set}')$ is the class of all infinite sets.
- For all A, B in $\mathrm{Obj}(\mathsf{Set}')$, $\mathrm{Hom}_{\mathsf{Set}'}(A, B)$ is the set of all functions from A to B.

Now given the infinite sets A and B, any morphism $f \in \mathrm{Hom}_{\mathsf{Set}}(A, B)$ is also a morphism of $\mathrm{Hom}_{\mathsf{Set}'}(A, B)$. So Set' is a full subcategory of Set.

Problem 3.1: Let C be a category. Then consider C^{op} with

- $\mathrm{Obj}(\mathsf{C}^{op}) = \mathrm{Obj}(\mathsf{C})$
- for A, B in $\mathrm{Obj}(\mathsf{C}^{op})$, $\mathrm{Hom}_{\mathsf{C}^{op}}(A, B) = \mathrm{Hom}_{\mathsf{C}}(B, A)$.

Let A, B, and C be objects of C^{op}. Given $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$ and $h \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, C)$, define the composition $hg \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, C)$ to be the morphism $gh \in \mathrm{Hom}_{\mathsf{C}}(C, A)$.

To see why this is well defined note that if $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$, then $g \in \mathrm{Hom}_{\mathsf{C}}(B, A)$. Similarly, if $h \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, C)$, then $h \in \mathrm{Hom}_{\mathsf{C}}(C, B)$. As C is a category, there must exist a morphism $gh \in \mathrm{Hom}_{\mathsf{C}}(C, A)$, which in turn means that the morphism we defined as the composition $hg \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, C)$ exists.

$$\begin{tikzcd} & B \arrow[r, "g"] & A \\ C \arrow[ur, "h"] \arrow[urr, bend right, "gh"'] & & \end{tikzcd}$$

So by how we defined composition of morphisms in C^{op}, we know C^{op} satisfies the composition property of a category. Now what's left to show is that C^{op} has the other properties of a category.

For any object A in $\mathrm{Obj}(\mathsf{C}^{op})$, $\mathrm{End}_{\mathsf{C}^{op}}(A) = \mathrm{End}_{\mathsf{C}}(A)$. So, A inherits a morphism 1_A from C.

Consider $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$. Then $g1_A$ in $\mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$ is equal to $1_A g = g$ in $\mathrm{Hom}_{\mathsf{C}}(B, A)$. So in C^{op}, we have that $g1_A = g$.

Similarly, consider $h \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, A)$. Then $1_A h \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, A)$ is equal to $h1_A = h$ in $\mathrm{Hom}_{\mathsf{C}}(A, B)$. So in C^{op}, we have that $1_A h = h$.

Finally, observe that given the morphisms $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$, $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, C)$, and $h \in \mathrm{Hom}_{\mathsf{C}^{op}}(C, D)$, we know that in C:

$$
\begin{tikzcd}
& C \arrow[rr, bend left, "fg"] \arrow[dr, "g"] & & A \\
D \arrow[ur, "h"] \arrow[rr, bend right, "gh"'] & & B \arrow[ur, "f"] &
\end{tikzcd}
$$

$(gf) \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, C)$ refers to the morphism $fg \in \mathrm{Hom}_{\mathsf{C}}(C, A)$. So, $h(gf) \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, D)$ refers to the morphism $(fg)h \in \mathrm{Hom}_{\mathsf{C}}(D, A)$. At the same time, $(hg) \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, D)$ refers to the morphism $gh \in \mathrm{Hom}_{\mathsf{C}}(D, B)$. So, $(hg)f \in \mathrm{Hom}_{\mathsf{C}}(D, A)$ refers to the morphism $f(gh) \in \mathrm{Hom}_{\mathsf{C}}(D, A)$. Thus as $(fg)h = f(gh)$ in C, we have that $h(gf) = (hg)f$ in C^{op}.

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A morphism $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ is an isomorphism if it has a two sided inverse under composition (i.e. $\exists g \in \mathrm{Hom}_{\mathsf{C}}(B, A)$ such that $gf = 1_A$ and $fg = 1_B$).

Proposition 4: The inverse of an isomorphism is unique.

Proof:

Suppose $g_1, g_2 : B \longrightarrow A$ both act as inverses of $f : A \longrightarrow B$. Then:

$$g_1 = g_1 1_B = g_1(fg_2) = (g_1 f)g_2 = 1_A g_2 = g_2$$

Corollary: If f has a left-hand inverse g_1 and a righthand inverse g_2, then f must be an isomorphism and $g_1 = g_2$ must be the unique inverse of f.

(Our proof from before also shows this.)

Since the inverse of f is unique, we denote it f^{-1}.

Proposition 5:

(A) Each identity 1_A is an isomorphism with itself being its own inverse.

(B) If f is an isomorphism, then f^{-1} is an isomorphism and $(f^{-1})^{-1} = f$.

(C) If $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ are isomorphisms, then the composition gf is an isomorphism and $(gf)^{-1} = f^{-1}g^{-1}$.

To prove any of these, just show that the proposed inverses are in fact an inverse. For example:

- $1_A 1_A = 1_A$
- $(gf)(f^{-1}g^{-1}) = g(ff^{-1})g^{-1} = g1_B g^{-1} = gg^{-1} = 1_C$

Two objects A and B of a category are isomorphic if there is an isomorphism $f : A \longrightarrow B$. We denote this by writing $A \cong B$.

An automorphism of an object A of a category C is an isomorphism from A to itself. The set of automorphisms of A is denoted $\mathrm{Aut}_{\mathsf{C}}(A)$.

Note:

- $\mathrm{Aut}_{\mathsf{C}}(A) \subseteq \mathrm{End}_{\mathsf{C}}(A)$
- If $f, g \in \mathrm{Aut}_{\mathsf{C}}(A)$, then fg and gf are in $\mathrm{Aut}_{\mathsf{C}}(A)$.
- $1_A \in \mathrm{Aut}_{\mathsf{C}}(A)$
- For each $f \in \mathrm{Aut}_{\mathsf{C}}(A)$, there exists $f^{-1} \in \mathrm{Aut}_{\mathsf{C}}(A)$.

Spoiler: The last three points mean that $\mathrm{Aut}_{\mathsf{C}}(A)$ forms a group.

The definitions of surjections and injections don't translate into category theory because the objects of a category don't necessarily have elements. However, the definitions of monomorphisms and epimorphisms do hold in category theory.

Let C be a category and $f : A \to B$ a morphism.

- f is a monomorphism if for any object Z of C and morphisms $\alpha', \alpha'' \in \mathrm{Hom}_{\mathsf{C}}(Z, A)$, we have that $f\alpha' = f\alpha'' \Longrightarrow \alpha' = \alpha''$.

- f is a epimorphism if for any object Z of C and morphisms $\beta', \beta'' \in \mathrm{Hom}_{\mathsf{C}}(B, Z)$, we have that $\beta' f = \beta'' f \Longrightarrow \beta' = \beta''$.

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f being both a monomorphism and epimorphism does not necessarily imply that f is isomorphism.

For example, consider a category whose objects are all the elements of $\mathbb{Z}$, and where for $a, b \in \mathbb{Z}$, $\mathrm{Hom}(a, b)$ equals $\{(a, b)\}$ if $a \leq b$ and $\emptyset$ otherwise.

Also we define the composition of $\{(a, b)\}$ and $\{(b, c)\}$ to be $\{(a, c)\}$.