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Math 241a Notes:

Suppose V is a finite dimensional vector space and $\pi : G \to \mathrm{GL}(V)$ is a representation. Then π is called irreducible if the only $\pi(G)$-invariant subspaces are $\{0\}$ and V. π is called completely reducible if $V = \bigoplus V_i$ where V_i is a $\pi(G)$-invariant irreducible subspace.

Also if $V = \mathbb{C}^n$ or $\mathbb{R}^n$, then I shall denote $\mathrm{GL}(V)$ as $\mathrm{GL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{R})$ respectively. Similarly, I shall denote $U(V)$ as $U(n)$.

Proposition 2.2.11: If G is a group and $\pi : G \to U(n)$ is a unitary representation, then:

(i) every $\pi(G)$-invariant subspace has a $\pi(G)$-invariant orthogonal complement.

Proof:
Suppose V is invariant and $w \in V^\perp$. Then as $\pi(g)$ is unitary (which means $\pi(g)^* = \pi(g)^{-1}$) for each $g \in G$, we know:

$$\langle \pi(g)w, v\rangle = \langle w, \pi(g)^* v\rangle = \langle w, \pi(g^{-1})v\rangle = 0.$$

It follows that $V^\perp$ is G-invariant.

(ii) π is completely reducible.

Proof:
We can prove this by induction. If $\mathbb{C}^n$ isn't irreducible then we can write $\mathbb{C}^n = V \oplus V^\perp \cong \mathbb{C}^k \oplus \mathbb{C}^{n-k}$ where both V and $V^\perp$ are G-invariant. Then we just repeat this reasoning on the smaller subspaces. ■

Proposition 2.2.12: If G is a compact group, V is a finite dimensional real or complex Hausdorff topological vector space, and $\pi : G \to \mathrm{GL}(V)$ is a (strong operator) continuous representation, then π is completely reducible.

Proof:
Using *corollary 2.2.8 on page 485*, let $\langle\ ,\ \rangle$ be a G-invariant inner product on V. Then π is a unitary representation with respect to this inner product. So, we can apply the prior proposition. ■

Let $\mathcal{X}$ be a real or complex vector space and let $A \subseteq \mathcal{X}$ be convex.

- Given any $x, y \in \mathcal{X}$ we let $[x, y] \coloneqq \{ty + (1-t)x : 0 \leq t \leq 1\}$. Also, we let $(x, y) \coloneqq \{ty + (1-t)x : 0 < t < 1\}$.
- We say $x \in A$ is an extreme point if for any $y, z \in A$ we have that $x \in [y, z]$ iff $x = y$ or $x = z$. We denote the set of such points as $\mathrm{ex}(A)$.
- We say $\emptyset \neq B \subseteq A$ is an extreme set if for any $y, z \in A$ we have that:
$$(y, z) \cap B \neq \emptyset \Longrightarrow [y, z] \cap B \neq \emptyset.$$

Given a set $E \subseteq \mathcal{X}$ where $\mathcal{X}$ is a topological real or complex vector space, we define $\overline{\mathrm{conv}}(A)$ to be the smallest closed convex set containing A. This is well-defined because arbitrary intersections of closed convex sets are closed and convex.