9/29/2025

All homework problem's for math 220a will be coming from John Conway's book *functions of one complex variable* (See *item 13* in the bibliography).

Exercise I.6.4: Let Λ be a circle lying in S^2. Then (by the definition of a circle) there is a unique plane P in $\mathbb{R}^3$ such that $P \cap S^2 = \Lambda$. So, take $P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1\beta_1 + x_2\beta_2 + x_3\beta_3 = \ell\}$ where $(\beta_1, \beta_2, \beta_3)$ is a unit vector orthogonal to P and $\ell \in \mathbb{R}$. Then show that if Λ contains $(0,0,1)$, it's projection to $\mathbb{C}_\infty$ is a straight line, and meanwhile if Λ doesn't contain $(0,0,1)$ then it's projection to $\mathbb{C}_\infty$ is another circle.

To start off, any point $z = x + iy \in \mathbb{C}$ corresponds to a point in Λ iff:

$$\frac{2x}{|z|^2+1}\beta_1 + \frac{2y}{|z|^2+1}\beta_2 + \frac{|z|^2-1}{|z|^2+1}\beta^3 = \ell$$

After some rearranging this becomes $2\beta_1 x + 2\beta_2 y + (x^2+y^2-1)\beta_3 = \ell(x^2+y^2+1)$. Or in other words $(\beta_3 - \ell)x^2 + 2\beta_1 x + (\beta_3 - \ell)y^2 + 2\beta_2 y = \ell + \beta_3$. And now we break off into two cases.

1. Suppose $\beta_3 = \ell$. Then we know that $0\beta_1 + 0\beta_2 + 1\beta_3 = \ell$. So $(0,0,1) \in \Lambda$. At the same time, all the square terms in our condition cancel and we are left with the requirement $2\beta_1 x + 2\beta_2 y = \ell + \beta_3$. This is the equation of a line. Hence, we've shown that the projection of Λ onto $\mathbb{C}_\infty$ is a straight line unioned with the point at infinity.

2. Suppose $\beta_3 \neq \ell$. Then we know that $0\beta_1 + 0\beta_2 + 1\beta_3 \neq \ell$ and hence $(0,0,1) \notin \Lambda$. At the same time, we can now divide our equation from before by $\beta - \ell$ in order to get that $x^2 + 2\frac{\beta_1}{\beta_3-\ell}x + y^2 + 2\frac{\beta_2}{\beta_3-\ell}y = \frac{\ell+\beta_3}{\beta_3-\ell}$. And by completing the square we have:

$$(x + \frac{\beta_1}{\beta_3-\ell})^2 + (y + \frac{\beta_2}{\beta_3-\ell})^2 = \frac{\beta_3+\ell}{\beta_3-\ell} + \frac{\beta_1^2+\beta_2^2}{(\beta_3-\ell)^2} = \frac{1-\ell^2}{(\beta_3-\ell)^2}.$$

   So, the projection of Λ onto $\mathbb{C}_\infty$ is a circle of radius $\frac{\sqrt{1-\ell^2}}{\beta_3-\ell}$ centered at $\frac{-\beta_1}{\beta_3-\ell} + i\frac{-\beta_2}{\beta_3-\ell}$.

   As a side note, let $\mathbf{b} = (b_1, b_2, b_3)$ and consider any $\mathbf{y} = (y_1, y_2, y_3) \in \Lambda$. Then note by the Cauchy Schwarts inequality that $\ell^2 = (\mathbf{b}\cdot\mathbf{y})^2 \leq \|\mathbf{b}\|^2\|\mathbf{y}\|^2 = 1 \cdot 1 = 1$. Hence, $\sqrt{1-\ell^2}$ is a well defined real number.

I'll continue with this class on *page* ___

Math 200a (lecture 2):

Given a group G and $a \in G$, we denote the order of a as $o(a) := o(\langle a \rangle)$ Now let C_n be a cyclic group of order n. In other words, $C_n = \langle a \rangle$ where $o(a) = n$. What is $\mathrm{Aut}(C_n)$?

Set 1 problem 5:

(a) Suppose $\theta \in \mathrm{Aut}(C_n)$ and pick any $m \in \mathbb{Z}$ with $\theta(a) = a^m$. Then $\gcd(m,n) = 1$.

Since θ is a bijection, we know that $\theta(a^k)$ for each $k \in \{0, \dots, n-1\}$ is distinct. But also since θ is a group homomorphism, we know that $\theta(a^0) = a^0 = 1$ and $\theta(a^k) = (\theta(a))^k = (a^m)^k$ for $k > 0$. Hence, we must have that $o(a^m) = n$. But now recall also that $o(a^m) = \frac{o(a)}{\gcd(o(a),m)} = \frac{n}{\gcd(n,m)}$. So, we must have that $\gcd(n,m) = 1$.

(b) Suppose $m \in \mathbb{Z}$ satisfies that $\gcd(m, n) = 1$ and define $\theta_m(a) : C_n \to C_n$ by $\theta_m(a^k) = (a^k)^m$. Then $\theta \in \mathrm{Aut}(C_n)$.

To see that θ_m is a group homomorphism, just note that:

$$\begin{aligned}\theta_m(a^{k_1}a^{k_2}) &= \theta_m(a^{k_1+k_2}) \\ &= (a^{k_1+k_2})^m = a^{m(k_1+k_2)} \\ &= a^{mk_1+mk_2} = (a^{k_1})^m(a^{k_2})^m = \theta_m(a^{k_1})\theta_m(a^{k_2})\end{aligned}$$

Meanwhile note that $o(a^m) = \frac{o(a)}{\gcd(o(a),m)} = \frac{n}{1} = n$.

Therefore, because $\theta_m(a^k) = (\theta_m(a))^k = (a^m)^k$ since θ_m is a group homomorphism, we know by pigeonhole principle that θ_m is both injective and surjective.

(c) The mapping $(\mathbb{Z}/n\mathbb{Z})^\times \to \mathrm{Aut}(C_n)$ is an isomorphism.