Math 240A Notes (Professor: Luca Spolaor)

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Lecture 1 Notes: 9/26/2024

Given an indexed family of sets $\{X_\alpha\}_{\alpha \in A}$, we define its Cartesian Product to be:

$$\prod_{\alpha \in A} X_\alpha = \{f : A \longrightarrow \bigcup_{\alpha \in A} X_\alpha \mid f(\alpha \in X_\alpha)\}$$

A projection is a function $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \longrightarrow X_\alpha$ satisfying that $f \mapsto f(\alpha)$.

If X, Y are sets, we define:

- $\mathrm{card}(X) \leq \mathrm{card}(Y)$ if there exists an injection $f : X \longrightarrow Y$.
- $\mathrm{card}(X) \geq \mathrm{card}(Y)$ if there exists a surjection $f : X \longrightarrow Y$.
- $\mathrm{card}(X) = \mathrm{card}(Y)$ if there exists a bijection $f : X \longrightarrow Y$.

Note that $\mathrm{card}(X) \leq \mathrm{card}(Y) \iff \mathrm{card}(Y) \geq \mathrm{card}(X)$. After all, given an injection in one direction, we can easily make a surjection in the other direction. Or given a surjection in one direction, we can (using A.O.C (axiom of choice)) easily make an injection in the other direction.

Also, if $\mathrm{card}(X) \leq \mathrm{card}(Y)$ and $\mathrm{card}(Y) \leq \mathrm{card}(X)$, then we know that $\mathrm{card}(Y) = \mathrm{card}(X)$.

Proof:
We know there exists $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ which are both injective. Hence, $g \circ f$ is an injection from X to $g(Y) \subseteq X$. By an exercise done in my math journal on page 8, we thus there exists a bijection h from X to $g(Y)$. And letting g^{-1} be any left-inverse of g, we then have that $g^{-1} \circ h$ is a bijection from X to Y.

We say X has the cardinality of the continuum if $\mathrm{card}(X) = \mathrm{card}(\mathbb{R})$.

Proposition: $\mathrm{card}(\mathcal{P}(\mathbb{N})) = \mathrm{card}(\mathbb{R})$.

Our textbook goes about proving this by constructing two functions: an injection and a surjection, from $\mathcal{P}(\mathbb{N})$ to $\mathbb{R}$ based on the binary expansion of any real number. That way, we know that $\mathrm{card}(\mathcal{P}(\mathbb{N})) \leq \mathrm{card}(\mathbb{R})$ and $\mathrm{card}(\mathcal{P}(\mathbb{N})) \geq \mathrm{card}(\mathbb{R})$.

Given a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ we know there exists: $\limsup x_n = \inf_{k \geq 1}(\sup_{n \geq k} x_n)$ and $\liminf x_n = \sup_{k \geq 1}(\inf_{n \geq k} x_n)$.

Also, given a function $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, we can define:

$$\limsup_{x \to a} f(x) = \inf_{\delta > 0}\left(\sup_{0 < |x-a| < \delta} f(x)\right).$$

If X is an arbitrary set and $f : X \longrightarrow [0, \infty]$, we define:

$$\sum_{x\in X} f(x) = \sup\left\{\sum_{x\in F} f(x) \mid F \subseteq X \; s.t. \; F \text{ is finite}\right\}.$$

Cool Proposition from textbook (not covered in lecture):

Let $A = \{x \in X \mid f(x) > 0\}$. If A is uncountable, then $\sum_{x\in X} f(x) = \infty$.

If A is countably infinite and $g : \mathbb{N} \longrightarrow A$ is a bijection, then $\sum_{x\in X} f(x) = \sum_{n=1}^{\infty} f(g(n))$.

Proof of first statement:
$A = \bigcup_{n\in\mathbb{N}} A_n$ where $A_n = \{x \in X \mid f(x) > \frac{1}{n}\}$.

If A is uncountable, we must have that some A_n is uncountable. But then for any finite set $F \subseteq X$, we have that $\sum_{x\in F} f(x) > \frac{\text{card}(F)}{n}$. So $\sum_{x\in X} f(x)$ is unbounded.

A metric space (X, ρ) is a set X equipped with a distance function $\rho : X \times X \longrightarrow [0, \infty)$. We denote the open ball of radius r about x to be $B(r, x) = \{y \in X \mid \rho(x, y) < r\}$. And you remember our definitions from 140A... right?

Proposition 0.21: Every open set in $\mathbb{R}$ is a countable union of disjoint open intervals.

We proved this as part of a homework exercise in Math 140A.

Given a metric space (X, ρ), an element $x \in X$, and sets $F, E \subseteq X$, we can define:

- $\rho(x, E) = \rho_E(x) = \inf\{\rho(x, y) \mid y \in E\}$.
- $\rho(F, E) = \inf\{\rho_E(y) \mid y \in F\}$.

Exercise: $g(x, E) = 0 \iff x \in \overline{E}$.

Proof:
If $\inf\{\rho(x, y) \mid y \in E\} = 0$, then there exists a sequence $\{y_n\}$ in E such that $\rho(x, y_n) \to 0$. This implies $x \in \overline{E}$. Similarly, if $x \in \overline{E}$, we can construct a sequence $\{y_n\}$ such that $\rho(x, y_n) < \frac{1}{n}$ for all n. Then:

$$0 \leq \inf\{\rho(x, y) \mid y \in E\} \leq \inf\{\rho(x, y_n) \mid n \in \mathbb{N}\} = 0.$$

Given a subset E of a metric space (X, ρ), we define:

$$\text{diam}(E) = \sup\{\rho(x, y) \mid x, y \in E\}.$$

If $\text{diam}(E) < \infty$, we say E is bounded. If $\forall \varepsilon > 0$, E can be covered by finitely many balls of radius ε, then we say E is totally bounded.

Exercise: E being totally bounded implies E is bounded.

Pick $\varepsilon > 0$ and let $\{z_1, \ldots, z_n\}$ be the set of points such that $E \subseteq \bigcup_{k=1}^{n} B(\varepsilon, z_n)$.

Then given any $x, y \in E$, we can assume that $x \in B(\varepsilon, z_i)$ and $y \in B(\varepsilon, z_j)$. So, $\rho(x, y) \leq \rho(x, z_i) + \rho(z_i, z_j) + \rho(z_j, y) < 2\varepsilon + \max\{\rho(z_i, z_j) \mid 1 \leq i, j \leq n\}$.

The converse is not generally true. For instance, if you use the discrete metric, then any set with more than one element will have a diameter of 1. But if $0 < \varepsilon < 1$, then it will be impossible to cover an infinite set with finitely many balls.

Lecture 2 Notes: 10/1/2024

Proposition: Suppose E is a subset of a metric space (X, ρ). Then the following are equivalent.

1. E is complete and totally bounded
2. All sequences $(x_n) \subseteq E$, have a convergent subsequence.
3. For all open covers $\{V_\alpha\}_{\alpha \in A}$ of E, there exists $V_{\alpha_1}, \ldots, V_{\alpha_n}$ such that $E \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$.

Proof:
(1) $\implies$ (2):

Lemma:
If E is totally bounded and $F \subseteq E$, then F is totally bounded.

Given any $\varepsilon > 0$, let $\{x_1, \ldots, x_n\}$ be a subset of E such that $E \subseteq \bigcup_{i=1}^{n} B(\varepsilon/2, x_i)$. Then consider the collection of sets:

$$\{F \cap B(\varepsilon/2, x_i)\} - \{\emptyset\}.$$

We know the diameter of each $F \cap B(\varepsilon/2, x_i)$ is at most ε. So in each set, pick $y_i \in F \cap B(\varepsilon/2, x_i)$. Then for some $m \leq n$:

$$F \subseteq \bigcup_{i=1}^{m} B(\varepsilon, y_i)$$

Let $A_1 = E$. Then for $k \geq 2$ we recursively define A_k as follows:

Assuming $A_{k-1} \cap (x_n)_{n\in\mathbb{N}}$ is infinite and A_{k-1} is totally bounded, choose $\{y_1, \ldots, y_m\}$ in A_k such that $A_k \subseteq \bigcup_{i=1}^{m} B(2^{-n}, y_i)$. Importantly, since $(x_n)_{n\in\mathbb{N}} \cap A_{k-1}$ is infinite, we know one of those open balls contains infinitely many points in our sequence. So set A_k equal to that ball intersected with E. Note that by our lemma, A_k is totally bounded.

Now pick any x_{n_1} and then for all $k \geq 2$ pick $x_{n_k} \in A_k$ such that $n_k > n_{k-1}$. That way, $(x_{n_k})_{k\in\mathbb{Z}_+}$ is a subsequence of $(x_n)_{n\in\mathbb{Z}_+}$. Also, we know that $(x_{n_k})_{k\in\mathbb{Z}_+}$ is Cauchy. Hence, since E is complete, we know that it converges to some x in E.

(2) $\Longrightarrow$ (1):

Firstly, suppose E is not complete. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ that is Cauchy but does not converge in E. Importantly, because $(x_n)_{n\in\mathbb{N}}$ is Cauchy, if there was a convergent subsequence, we know the limit of that subsequence would have to be the limit of the whole sequence. But that doesn't exist. So, we know (2) can't be true.

Secondly, suppose E is not totally bounded. Then there exists $\varepsilon > 0$ such that it is impossible to cover E in balls of radius ε. So, we can recursively define a sequence $(x_n)_{n\in\mathbb{N}}$ in E satisfying that:

$$x_n \in E - \bigcup_{i=1}^{n-1} B(\varepsilon, x_i).$$

Importantly, for all natural numbers $n \neq m$, we have that $\rho(x_n, x_m) \geq \varepsilon$. So, it is impossible to find a convergent subsequence of (x_n), meaning (2) is false.

(1) and (2) $\Longrightarrow$ (3):

Let $\{V_\alpha\}_{\alpha\in A}$ be an open cover of E.

Suppose for the sake of contradiction that for all $n \in \mathbb{N}$, there is a ball B_n of radius 2^{-n} centered in E such that $B_n \cap E \neq \emptyset$ but $B_n \not\subseteq V_\alpha$ for all $\alpha \in A$. Then we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $x_n \in B_n \cap E$ for all $n \in \mathbb{N}$. By (2), we know there is a subsequence that converges to some $x \in E$. Importantly, we know $x \in V_\alpha$ for some $\alpha \in A$, and because V_α is open, there is $\varepsilon > 0$ such that $B(\varepsilon, x) \subseteq V_\alpha$. But now we get a contradiction because by picking n such that $2^{-n} < \varepsilon/3$ and $\rho(x, x_n) < \varepsilon/3$, we have for all $y \in B_n$ that:

$$\rho(x, y) \leq \rho(x, x_n) + \rho(x_n, y) < 2^{-n} + 2^{-n+1} < \varepsilon$$

So $B_n \subseteq B(\varepsilon, x) \subseteq V_\alpha$.

We've thus shown that for some $n \in N$, all balls of radius 2^{-n} centered in E are contained by some V_α. And assuming (1), we can cover E with finitely many balls of radius 2^{-n} It follows that by picking a V_α containing a ball for each ball covering E, we've found a finite covering E using the sets in $\{V_\alpha\}_{\alpha\in A}$.

(3) $\Longrightarrow$ (2):

Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in E with no convergent subsequence. Then for each $x \in E$, there must exist $\varepsilon_x > 0$ such that $B(\varepsilon_x, x) \cap (x_n)_{n\in\mathbb{N}}$ is finite. (If ε_x didn't exist, we could construct a Cauchy subsequence converging to x).

But now $\{B(\varepsilon_x, x)\}_{x\in E}$ is an open cover of E with no finite subcover of E because it will take an infinite cover to cover all of $(x_n)_{n\in\mathbb{N}}$.

If E satisfies all three of the above properties, we say E is compact.

Corollary: $K \subseteq \mathbb{R}^n$ is compact iff it's closed and bounded.

Roughly speaking, we want a measure to be a function $\mu : \mathcal{P}(\mathbb{R}^n) \longrightarrow [0, \infty)$ such that $E \mapsto \mu(E) =$ "the area of E". Also, we would like it if:

(i) $\mu([0,1)^n) = 1$

(ii) $\mu($rotation, translation, or reflection of $A) = \mu(A)$

(iii) $\mu(\bigcup\limits_{i=1}^{\infty} A_i) = \sum\limits_{i=1}^{\infty} \mu(A_i)$ if $A_i \cap A_j \neq \emptyset \Longrightarrow i = j$.

Unfortunately, the properties as written above are inconsistent.

Vitali Sets:
Defining $x \sim y$ iff $x - y \in \mathbb{Q}$, let $N \subseteq [0,1)$ be a set such that $N \cap [x]$ has precisely one element for all $x \in \mathbb{R}$. Next let $R = [0,1) \cap \mathbb{Q}$, and for all $r \in R$ define:

$$N_r = \{x + r \mid x \in N \cap [0, 1-r)\} \cup \{x + r - 1 \mid x \in N \cap [1-r, 1)\}.$$

Importantly, note that $N_r \subseteq [0,1)$. Plus, the two sets being unioned over to make N_r are both disjoint and can be translated around so that they are still disjoint but their union forms N. Hence assuming $\mu : \mathcal{P}(\mathbb{R}^n) \longrightarrow [0,\infty)$ satisfying (ii) and (iii), we know $\mu(N_r) = \mu(N)$.

Also, for all $y \in [0,1)$, if $x \in N \cap [y]$, we know that $y \in N_r$ where $r = x - y$ if $x \geq y$, or $r = x - y + 1$ if $x < y$. Hence, $[0,1) = \bigcup\limits_{r\in R} N_r$.

Also, given any N_r and N_s, if $x \in N_r \cap N_s$, then we'd be able to show that both $x - r$ or $x - r + 1$ and $x - s$ or $x - s + 1$ are distinct elements of N in the same equivalence class, which contradicts how we defined N.

You work through the scratch work of the different cases on your own! :P

So supposing μ satisfies (i) and (iii) and because R is countable, we have that:

$$1 = \sum_{r\in R} \mu(N_r) = \sum_{r\in R} (N) = 0 \text{ or } \infty.$$

This is a contradiction.

Furthermore, the problem is not the countable union property as is demonstrated by the Banach-Tarsky paradox:

Theorem: Let U and V be arbitrary bounded sets in $\mathbb{R}^n$ where $n \geq 3$. Then there exists $E_1, \ldots, E_N, F_1, \ldots, F_N$ in $\mathbb{R}^n$ such that:

- $E_i \cap E_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{N} E_i = U$
- $F_i \cap F_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{N} F_i = V$
- E_i and F_i are congruent for all $i \in \{1, \ldots, N\}$.

Supposing that $\mu(E_j)$ and $\mu(F_j)$ exists for all j and that μ satisfies (i), (ii), and (iii) except only for finite unions, then that would suggest all sets have the same "area", which we know doesn't make sense.

What we will do to fix this issue is only define μ on a subset of $\mathcal{P}(\mathbb{R}^n)$.

Let $X \neq \emptyset$. An algebra of sets in X is a nonempty collection $\mathcal{A} \subseteq \mathcal{P}(X)$ which is closed under finite unions and complements. If is $\mathcal{A}$ is also closed under countable unions, we say $\mathcal{A}$ is a σ-algebra.

Observations:

1. Algebras of sets are closed under finite intersections and σ-algebras are closed under countable intersection. (This also means algebras of sets are closed under set differences.)

   This is because $\bigcap_{n\in\mathbb{N}} A_n = \left(\bigcup_{n\in\mathbb{N}} A_n^{\mathsf{C}}\right)^{\mathsf{C}}$.

2. If $\mathcal{A}$ is closed under disjoint countable union, then it's closed under arbitrary countable unions.

   This is because $\bigcup_{n\in\mathbb{N}} A_n = A_1 \cup \bigcup_{n\geq 2} \left(A_n \cap \left(\bigcup_{i=1}^{n-1} A_i\right)^{\mathsf{C}}\right)$

3. If $\{\mathcal{E}_\alpha\}_{\alpha\in A}$ is a collection of σ-algebras, then $\bigcap_{\alpha\in A} \mathcal{E}_\alpha$ is a σ-algebra.

   This is pretty trivial to prove. It should remind you of topologies.

Exercise 1.1: A family of sets $\mathcal{R} \subseteq \mathcal{P}(X)$ is called a ring if it is closed under finite unions and difference. If $\mathcal{R}$ is also closed under countable unions, it is called a σ-ring.

(a) Rings are closed under finite intersections and σ-rings are closed under countable intersections.

If $\mathcal{R}$ is a ring and $A_1, \ldots, A_n \in \mathcal{R}$, then:

$$\bigcap_{i=1}^{n} A_n = A_1 - \bigcup_{i=2}^{n} (A_1 - A_i) \in \mathcal{R}$$

This is because each $A_1 - A_i \in \mathcal{R}$, meaning $\bigcup\limits_{i=2}^{n}(A_1 - A_i) \in \mathcal{R}$, and so finally $A_1 - \bigcup\limits_{i=2}^{n}(A_1 - A_i) \in \mathcal{R}$.

If $\mathcal{R}$ is a σ-algebra, we can replace the finite intersection and union used in the prior reasoning with a countable intersection and union.

(b) If $\mathcal{R}$ is a ring (or σ-ring), then $\mathcal{R}$ is an algebra (or σ-algebra) iff $X \in \mathbb{R}$.

($\Longrightarrow$) Suppose $\mathcal{R}$ is an algebra. Then note that $\emptyset \in \mathcal{R}$ because for any $A \in \mathcal{R}$, $A - A \in \mathcal{R}$. So taking complements, we get that $X \in \mathcal{R}$.

($\Longleftarrow$) Suppose $X \in \mathcal{R}$. Then for any $A \in \mathcal{R}$, we know that $A^{\mathsf{C}} = X - A \in \mathcal{R}$. So $\mathcal{R}$ is an algebra (or σ-algebra).

(c) If $\mathcal{R}$ is a σ-ring, then $\mathcal{A} = \{E \subseteq X \mid E \in \mathcal{R} \text{ or } E^{\mathsf{C}} \in \mathcal{R}\}$ is a σ-algebra.

To start, we know that $\mathcal{A}$ is closed under complements because for any $A \in \mathcal{A}$,

$$A \in \mathcal{R} \Longrightarrow (A^{\mathsf{C}})^{\mathsf{C}} \in \mathcal{R} \Longrightarrow A^{\mathsf{C}} \in \mathcal{A}$$
$$A \notin \mathcal{R} \Longrightarrow A^{\mathsf{C}} \in \mathcal{R} \Longrightarrow A^{\mathsf{C}} \in \mathcal{A}$$

Also, let $(E_n)_{n\in\mathbb{N}}$ be a countable collection of sets in $\mathcal{A}$. Then define $A = \{n \in \mathbb{N} \mid E_n^{\mathsf{C}} \notin \mathcal{R}\}$ and $B = \{n \in \mathbb{N} \mid E_n^{\mathsf{C}} \in \mathcal{R}\}$. Clearly, we have that:

$$\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in A} E_n \cup \bigcup_{n\in B} E_n = \bigcup_{n\in A} E_n \cup \bigcup_{n\in B} (E_n^{\mathsf{C}})^{\mathsf{C}}$$

Also $\bigcup\limits_{n\in B}(E_n^{\mathsf{C}})^{\mathsf{C}} = \left(\bigcap\limits_{n\in B} E_n^{\mathsf{C}}\right)^{\mathsf{C}}$, and by part (a), we know that $E_B := \bigcap\limits_{n\in B} E_n^{\mathsf{C}} \in \mathcal{R}$.

Similarly, we know $E_A := \bigcup\limits_{n\in A} E_n \in \mathcal{R}$. So, we've shown that $\bigcup\limits_{n\in\mathbb{N}} E_n = E_A \cup E_B^{\mathsf{C}}$ where $E_A, E_B \in \mathcal{R}$.

Finally, note that $E_A \cup E_B^{\mathsf{C}} = (E_B - E_A)^{\mathsf{C}}$. Since $E_B - E_A \in \mathcal{R}$, we know that $(E_B - E_A)^{\mathsf{C}} \in \mathcal{A}$.

(d) If $\mathcal{R}$ is a σ-ring, then $\mathcal{A} = \{E \subseteq X \mid E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

To start if $E \in \mathcal{A}$, then $E^{\mathsf{C}} \in \mathcal{A}$ because for all $F \in \mathcal{R}$ we have that:

$$E^{\mathsf{C}} \cap F = F - E = F - (E \cap F) \in \mathcal{R}.$$

Also, let $(E_n)_{n\in\mathbb{N}}$ be a countable collection of sets in $\mathcal{A}$. Then for all $F \in \mathcal{R}$, we have that $\left(\bigcup\limits_{n\in\mathbb{N}} E_n\right) \cap F = \bigcup\limits_{n\in\mathbb{N}} (E_n \cap F) \in \mathcal{R}$. So $\mathcal{A}$ is closed under countable union.

Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of sets. Since the intersection of σ-algebras is still a σ-algebra, we define $\mathcal{M}(\mathcal{E})$ to be the smallest σ-algebra that contains $\mathcal{E}$. In other words, $\mathcal{M}(\mathcal{E})$ is the intersection of all σ-algebras that contain $\mathcal{E}$.

We call $\mathcal{M}(\mathcal{E})$ the σ-algebra generated by $\mathcal{E}$.

Lemma: if $\mathcal{E} \in \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$.

Let (X, ρ) be a metric space. We define the Borel σ-algebra on X: $\mathcal{B}_X$, to be the σ-algebra generated by the collection of all open sets, or equivalently the collection of all closed sets.

- A set is G_δ if it is a countable intersection of open sets.
- A set is F_σ if it is a countable union of closed sets.
- A set is $G_{\delta\sigma}$ if it is a countable union of G_δ sets.
- A set is $F_{\sigma\delta}$ if it is a countable intersection of F_σ sets.

You can hopefully see the pattern. Also the professor isn't sure how much we'll use this δ and σ notation in class.

Exercise 1.2: $\mathcal{B}_\mathbb{R}$ is generated by each of the following:

(a) the set of open intervals: $\mathcal{E}_1 = \{(a, b) \mid a < b\}$

(b) the set of closed intervals: $\mathcal{E}_2 = \{[a, b] \mid a < b\}$

(c) the set of half-open intervals:

 (i) $\mathcal{E}_3 = \{(a, b] \mid a < b\}$

 (ii) $\mathcal{E}_4 = \{[a, b) \mid a < b\}$

(c) the set of open rays:

 (i) $\mathcal{E}_5 = \{(a, \infty) \mid a \in \mathbb{R}\}$

 (ii) $\mathcal{E}_6 = \{(-\infty, a) \mid a \in \mathbb{R}\}$

(d) the set of closed rays:

 (i) $\mathcal{E}_7 = \{[a, \infty) \mid a \in \mathbb{R}\}$

 (ii) $\mathcal{E}_8 = \{(-\infty, a] \mid a \in \mathbb{R}\}$

Proof:
We trivially have that $\mathcal{M}(\mathcal{E}_1), \mathcal{M}(\mathcal{E}_2), \mathcal{M}(\mathcal{E}_5), \mathcal{M}(\mathcal{E}_6), \mathcal{M}(\mathcal{E}_7), \mathcal{M}(\mathcal{E}_8) \subseteq \mathcal{B}_\mathbb{R}$ since each of them contain either only open sets or only closed sets. As for the other inclusions, we must do more work.

(a) Note that $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$. Hence, a countable base of $\mathbb{R}$ is the set: $\mathcal{F} = \{(p-q, p+q) \subset \mathbb{R} \mid p, q \in \mathbb{Q}$ and $q > 0\}$. In other words, given any open set $E \subseteq \mathbb{R}$, there is a countable subcollection of $\mathcal{F}$ whose union is E.

To see why, let $x \in E$. Since E is open, there exists $r > 0$ with $B(r, x) \subseteq E$. Next, pick $p \in (x, x + \frac{r}{2}) \cap \mathbb{Q}$, followed by $q \in (p - x, r - p) \cap \mathbb{Q}$. Then $x \in (p-q, p+q) \in \mathcal{F}$ and $(p-q, p+q) \subseteq (x - r, x + r)$.

With that, we've now shown that for all $x \in E$, there exists $F \in \mathcal{F}$ such that $x \in F \subseteq E$. If we choose such an F_x for all $x \in E$, we then get that $E = \bigcup\limits_{x \in E} F_x$. So E is the union of a subcollection of $\mathcal{F}$. But since $\mathcal{F}$ is countable, the set $\{F_x \in \mathcal{F} \mid x \in E\}$ is also countable.

Importantly, $\mathcal{F} \subset \mathcal{E}_1$. So $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E}_1)$. However as shown above, we must have that $\mathcal{M}(\mathcal{F})$ includes all open sets. So by our lemma on the previous page, $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E}_1)$.

(b) Given any $E = (a, b) \in \mathcal{E}_1$, we can write that $E = \bigcup\limits_{n \in \mathbb{Z}_+} [a + \frac{1}{n}, b - \frac{1}{n}]$. Thus, $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_2)$, meaning $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_2)$.

(c) Remember that for these two, we still need to show that $\mathcal{M}(\mathcal{E}_1), \mathcal{M}(\mathcal{E}_2) \in \mathcal{B}_{\mathbb{R}}$.

(i) Firstly note that if $F = (a, b] \in \mathcal{E}_3$, then $= \bigcap\limits_{n \in \mathbb{Z}_+} (a, b + \frac{1}{n})$. So $\mathcal{E}_3 \subseteq \mathcal{M}(\mathcal{E}_1)$.

On the other hand, if $E = (a, b) \in \mathcal{E}_1$, we have that $E = \bigcup\limits_{n \in \mathbb{Z}_+} (a, b - \frac{1}{n}]$. So $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_3)$.

By our lemma on the previous page, we thus have that:

$$\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_1) = \mathcal{B}_{\mathbb{R}}.$$

(ii) Mostly identical reasoning as with $\mathcal{E}_3$ shows that:

$$\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_1) = \mathcal{B}_{\mathbb{R}}$$

(d)

(i) If $E = (a, b) \in \mathcal{E}_1$, then we know that:

$$E = (a, \infty) \cap (\bigcap_{n \in \mathbb{Z}_+} (b - \tfrac{1}{n}, \infty))^{\mathsf{C}} \in \mathcal{M}(\mathcal{E}_5).$$

So $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_5)$, meaning $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_5)$.

(ii) Analogous reasoning to that with $\mathcal{E}_5$ shows that $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_6)$.

(e)

(i) If $E = (a, \infty) \in \mathcal{E}_6$, then we have that $E = \bigcup\limits_{n \in \mathbb{Z}_+} [a + \frac{1}{n}, \infty)$. So $\mathcal{E}_5 \subseteq \mathcal{M}(\mathcal{E}_7)$, meaning that $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{M}(\mathcal{E}_7)$.

(ii) Analogous reasoning as with $\mathcal{E}_7$ shows that $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_6) \subseteq \mathcal{M}(\mathcal{E}_8)$.

Exercise 1.3: Let $\mathcal{M}$ be an infinite σ-algebra on X.

(a) $\mathcal{M}$ contains an infinite sequence of disjoint sets.

By the Hausdorff maximum principle, we know there is a subcollection $\mathcal{S}$ of $\mathcal{M}$ which is simply ordered by proper subset and is not contained in any other collection of $\mathcal{M}$ which is simply ordered by proper subset.

We claim $\mathcal{S}$ can't be finite. For suppose $\mathcal{S} = \{E_1, \ldots, E_n\}$ is a sequence of sets in $\mathcal{M}$ simply ordered by proper subset which are indexed such that $E_i \subset E_{i+1}$ for all $i \in \{1, \ldots, n-1\}$.

(Note: if $\mathcal{S}$ is maximal, then we must have $E_1 = \emptyset$ and $E_n = X$.)

We can partition $\mathcal{M}$ into collections $\mathcal{M}_1, \ldots \mathcal{M}_n$ such that $A \in \mathcal{M}_i$ iff i is the least integer for which $A \subseteq E_i$. Importantly, all sets in $\mathcal{M}$ will fall into a partition because all sets from $\mathcal{M}$ are contained in E_n. Also note that while there are infinitely many $A \in \mathcal{M}$, there are only n many partitions. So, there must be a least integer k such that $\mathcal{M}_k$ contains infinitely many $A \in \mathcal{M}$.

And since $\mathcal{M}_1 = \{\emptyset\}$, we know $k \geq 2$.

The fact that $\mathcal{M}_i$ is finite for all $i < k$ means that there are only finitely many sets from $\mathcal{M}$ contained in E_{k-1}. Thus, we can pick a set $B \in \mathcal{A}_k$ such that $B \neq (E_k - E_{k-1}) \cup A$ for any $A \in \mathcal{M}$ that is a subset of E_{k-1}.

Note that since $E_{k-1} \cap B$ is a set in $\mathcal{M}$, we must have that $(E_k - E_{k-1}) \not\subseteq B$ or else B would be the union of $(E_k - E_{k-1})$ and a set from $\mathcal{M}$. Thus, we know E_k contains points that neither B nor E_{k-1} have. At the same time, we know B has points that E_{k-1} doesn't have. It follows that: $E_{k-1} \subset E_{k-1} \cup B \subset E_k$.

Via transitivity, $E_{k-1} \cup B$ is comparable via proper subset with E_i for all $i \in \{1, \ldots, n\}$. Hence, we've shown that $\mathcal{S} \cup \{E_{k-1} \cup B\}$ is a sequence of sets in $\mathcal{M}$ simply ordered by proper subset. But this contradicts that $\mathcal{S}$ is maximal.

Now that we know $\mathcal{S}$ is infinite, let $(E_n)_{n \in \mathbb{Z}_+}$ be a sequence of sets in $\mathcal{S}$ satisfying that $E_n \subset E_{n+1}$. Then we have that $(E_{n+1} - E_n)_{n \in \mathbb{Z}_+}$ is an infinite sequence of nonempty disjoint sets in $\mathcal{M}$.

(b) Show that $\mathrm{card}(\mathcal{M}) \geq \mathfrak{c}$.

Let $(E_n)_{n \in \mathbb{N}}$ be a sequence of nonempty disjoint sets in $\mathcal{M}$. Then if we define the map $f : [0,1]^{\mathbb{N}} \longrightarrow \mathcal{M}$ such that $(a_0, a_1, a_2, \ldots)$ is mapped to the union of all E_n such that $a_n = 1$, we have that f is an injection.

Hence, $\mathrm{card}(\mathcal{M}) \geq \mathrm{card}([0,1]^{\mathbb{N}})$. And since there is a trivial bijection from $[0,1]^{\mathbb{N}}$ and $\mathcal{P}(\mathbb{N})$, plus the fact that we proved early on in the class that $\mathrm{card}(\mathcal{P}(\mathbb{N})) = \mathrm{card}(\mathbb{R})$, we thus know that $\mathrm{card}(\mathcal{M}) \geq \mathfrak{c}$.

Exercise 1.4: An algebra $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is closed under countable increasing unions (meaning $E_1 \subseteq E_2 \subseteq \ldots$).

The rightward implication is true since $\mathcal{A}$ being a σ-algebra means that $\mathcal{A}$ is closed under all countable unions. As for showing the leftward implication, suppose $\{A_n\}_{n\in\mathbb{Z}_+}$ is a countable collection of sets in $\mathcal{A}$. Then for all $n \in \mathbb{Z}_+$, define $E_n = A_1 \cup \ldots \cup A_n$.

Since each E_n are finite unions of sets in $\mathcal{A}$, we know that each E_n is in $\mathcal{A}$. Also, we clearly have that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ In order to make the sets strictly increasing, let $S = \{1\} \cup \{k \in \mathbb{Z} \mid k > 1 \text{ and } E_k - E_{k-1} \neq \emptyset\}$. Then for any $n, m \in S$, we know that $n < m \Longrightarrow E_n \subset E_m$.

Finally, $\bigcup\limits_{n\in\mathbb{Z}_+} A_n = \bigcup\limits_{n\in\mathbb{Z}_+} E_n = \bigcup\limits_{n\in S} E_n$.

Importantly, S is either finite or countably infinite, and S consists of strictly increasing sets. So by the right hypothesis, we know $\bigcup\limits_{n\in S} E_n \in \mathcal{A}$. Hence, the union over $\{A_n\}_{n\in\mathbb{Z}_+}$ is in $\mathcal{A}$.

Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of nonempty sets, and define $X = \prod\limits_{\alpha\in A} X_\alpha$.

If $\mathcal{M}_\alpha$ is a σ-algebra in X_α for all $\alpha \in A$, then we define the product σ-algebra on X to be: $\bigotimes\limits_{\alpha\in A} \mathcal{M}_\alpha = \mathcal{M}(\{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{M}_\alpha \text{ and } \alpha \in A\})$.

To get a better geometric intuition for this definition, consider if $A = \{1, 2\}$. Then:

$$\begin{aligned}\bigotimes_{\alpha\in A} \mathcal{M}_\alpha &= \{\pi_1^{-1}(E_1) \mid E_1 \in \mathcal{M}_1\} \cup \{\pi_2^{-1}(E_2) \mid E_2 \in \mathcal{M}_2\} \\ &= \{E_1 \times X_2 \mid E_1 \in \mathcal{M}_1\} \cup \{X_1 \times E_2 \mid E_2 \in \mathcal{M}_2\}\end{aligned}$$

Also, the motivation for this definition is that $\bigotimes\limits_{\alpha\in A} \mathcal{M}_\alpha$ is the smallest σ-algebra where π_α is "measurable" for all α. We'll learn what that means shortly...

Proposition:

(i) A is countable implies $\bigotimes\limits_{\alpha\in A} \mathcal{M}_\alpha = \mathcal{M}(\{\prod\limits_{\alpha\in A} E_\alpha \mid \forall\alpha \in A,\ \ E_\alpha \in \mathcal{M}_\alpha\})$

Proof:
If $E_\alpha \in \mathcal{M}_\alpha$, then $\pi_\alpha^{-1}(E_\alpha) = \prod\limits_{\beta\in A} E_\beta$ where $E_\beta = X_\beta$ if $\beta \neq \alpha$ (and $E_\beta = E_\alpha$ if $\beta = \alpha$).

So $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{M}(\{\prod\limits_{\alpha\in A} E_\alpha \mid \forall\alpha \in A,\ \ E_\alpha \in \mathcal{M}_\alpha\})$
On the other hand, $\prod\limits_{\alpha\in A} E_\alpha = \bigcap\limits_{\alpha\in A} \pi_\alpha^{-1}(E_\alpha)$.

Since A is countable, we thus know that if $E_\alpha \in \mathcal{M}_\alpha$ for all $\alpha \in A$, then $\prod\limits_{\alpha\in A} E_\alpha \in \bigotimes\limits_{\alpha\in A} \mathcal{M}_\alpha$.

(ii) Suppose $\mathcal{M}_\alpha = \mathcal{M}(\mathcal{E}_\alpha)$ for all $\alpha \in A$. Then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by $\mathcal{F} = \{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{E}_\alpha$ and $\alpha \in A\}$.

Proof:

Since $\mathcal{F} \subseteq \{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{M}_\alpha$ and $\alpha \in A\}$, we trivially have that $\mathcal{M}(\mathcal{F}) \subseteq \bigotimes_{\alpha \in A} \mathcal{M}_\alpha$.

As for showing the other inclusion, define for each $\alpha \in A$:

$$\mathcal{F}_\alpha = \{E \subseteq X_\alpha \mid \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})\}.$$

Note that $\mathcal{F}_\alpha$ is a σ-algebra on X_α that contains $\mathcal{E}_\alpha$.

This is because for any $F \in \mathcal{F}_\alpha$ and $(E_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}_\alpha$, we know that:

- $(\pi_\alpha^{-1}(F))^{\mathsf{C}} = \pi_\alpha^{-1}(F^{\mathsf{C}})$
- $\bigcup_{n \in \mathbb{N}} \pi_\alpha^{-1}(E_n) = \left(\pi_\alpha^{-1}(\bigcup_{n \in \mathbb{N}} E_n)\right)$

Also, for any $E \subseteq X_\alpha$, $E \in \mathcal{E}_\alpha \Longrightarrow \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})$.

By definition, we thus know that $\mathcal{M}_\alpha \subseteq \mathcal{F}_\alpha$. So for all $\alpha \in A$ and $E_\alpha \in \mathcal{M}_\alpha$, we know that $E_\alpha \in \mathcal{F}_\alpha$, which means that $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{M}(\mathcal{F})$. So $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha \subseteq \mathcal{M}(\mathcal{F})$.

(iii) We can also combine the first two parts of this proposition. If A is countable and $\mathcal{M}_\alpha = \mathcal{M}(\mathcal{E}_\alpha)$ for all $\alpha \in A$, then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by:

$$\{\prod_{\alpha \in A} E_\alpha \mid \forall \alpha \in A, \;\; E_\alpha \in \mathcal{E}_\alpha\}$$

Lecture 3 Notes: 10/3/2024

Proposition: Let $X_1, \ldots, X_n$ be metric spaces, and define $X = \prod_{i=1}^n X_i$ to be the metric space equipped with the product metric.

The product metric defines the distance between any $\boldsymbol{x}, \boldsymbol{y} \in \prod_{i=1}^n$ to be the max distance between a coordinate of $\boldsymbol{x}$ and the corresponding coordinate in $\boldsymbol{y}$.

- $\bigotimes_{i=1}^n \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$.

Proof:

By the previous proposition: $\bigotimes_{i=1}^n \mathcal{B}_{X_i}$ is generated by the collection:

$$\{\pi_i^{-1}(U_i) \mid i \in \{1, \ldots, n\} \text{ and } U_i \subseteq X_i \text{ is open}\}.$$

Also, by the definition of a product topology, we know that each $\pi_i^{-1}(U_i)$ is open in X. So by the lemma on page 9, we know that $\bigotimes_{i=1}^n \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$.

- If each X_i is separable, then $\bigotimes_{i=1}^{n} \mathcal{B}_{X_i} = \mathcal{B}_X$.

  Proof:
  Let $C_i \subseteq X_i$ be countable with $\overline{C_i} = X_i$ for all $i \in \{1, \dots, n\}$. Then define $\mathcal{E}_i = \{B(p, x) \mid x \in C_i \text{ and } p \in \mathbb{Q}_+\}$ for each i. Since $\mathcal{E}_i$ is countable and all open sets in X_i are the union of a subcollection of $\mathcal{E}_i$, we know that any open set in X_i is also in $\mathcal{M}(\mathcal{E}_i)$. So, $\mathcal{B}_{X_i} \subseteq \mathcal{M}(\mathcal{E}_i)$. And since $\mathcal{E}_i$ contains only open sets of X_i, the reverse inclusion holds too.

  Also, $C = \prod_{i=1}^{n} C_i$ is a countable dense subset of X.

  Defining $\mathcal{E} = \{B(p, \boldsymbol{x}) \mid \boldsymbol{x} \in C \text{ and } p \in \mathbb{Q}_+\}$, we have that $\mathcal{E}$ is countable and any open set in X is also in $\mathcal{M}(\mathcal{E})$. So, $\mathcal{B}_X \subseteq \mathcal{M}(\mathcal{E})$. And like before since $\mathcal{E}$ contains only open sets of X, the reverse inclusion holds too.

  But now note that given, $B(p, (x_1, \dots, x_n)) \in \mathcal{E}$, we know that $B(p, (x_1, \dots, x_n)) = \prod_{i=1}^{n} B(p, x_i)$ where $(p, x_i) \in \mathcal{E}_i$ for all i.

  So applying part 3 of the previous proposition and the lemma on page 9:
  $$\mathcal{B}_X = \mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}\left(\prod_{i=1}^{n} E_i \mid E_i \in \mathcal{E}_i \text{ for all } i\right) = \bigotimes_{i=1}^{n} \mathcal{M}(\mathcal{E}_i) = \bigotimes_{i=1}^{n} \mathcal{B}_{X_i}$$

Corollary: $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{i=1}^{n} \mathcal{B}_{\mathbb{R}}$.

This is because the product metric ρ_1 of $\prod_{i=1}^{n} \mathbb{R}$ is equivalent to the standard metric ρ_2 of $\mathbb{R}^n$, meaning that:
$$\exists C, C' > 0 \text{ such that } C\rho_1 \leq \rho_2 \leq C'\rho_1.$$

In the specific case of this corollary, set $C = \sqrt{1/n}$ and $C' = 1$.

The fact relevant here is that given the metrics ρ_1, ρ_2 on a set X, if ρ_1 is equivalent to ρ_2, then (X, ρ_1) and (X, ρ_2) have the same open sets (this is really trivial to prove).

An elementary family is a collection $\mathcal{E}$ of subsets of a set X such that:

1. $\emptyset \in \mathcal{E}$
2. If $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$.
3. If $E \in \mathcal{E}$, then E^{C} is a finite disjoint union of members of $\mathcal{E}$.

If $\mathcal{E}$ is an elementary collection, then $\mathcal{A}$ equal to the collection of finite disjoint unions of $\mathcal{E}$ is an algebra.

Proof:
Firstly, given any $A, B \in \mathcal{E}$, we have that $A \cup B = (A - B) \cup B$. Also, by property 3 of elementary families, $(A - B) = (A \cap B^{\mathsf{C}}) = (A \cap \bigcup_{i=1}^{k} C_i)$ where each $C_i \in \mathcal{E}$ and disjoint. By property 2 of elementary families, we thus know $A \cap C_i \in \mathcal{E}$ for all i. So $(A - B)$ is a finite union of disjoint sets in $\mathcal{E}$. In turn, so is $(A - B) \cup B$. Hence, $A \cup B \in \mathcal{A}$.

By induction, we get that for any $A_1, \ldots, A_n \in \mathcal{E}$, $A_1 \cup \ldots \cup A_n$ is a finite union of disjoint sets in $\mathcal{E}$. So $\mathcal{A}$ actually equals the set of all finite unions of $\mathcal{E}$. It follows that $\mathcal{A}$ is closed under finite unions.

I really don't want to write down the proof that $\mathcal{A}$ is closed under complements. It's what you would expect but just heavy on notation.

Exercise 1.5: If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

For the sake of convenience, I will write the union of σ-algebras generated by countable subsets of $\mathcal{E}$ as: $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

To start, since each $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E})$, we trivially know $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E}) = \mathcal{M}$. On the other hand, $\mathcal{E} \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ since each countable $\mathcal{F} \subseteq \mathcal{E}$ is contained in $\mathcal{M}(\mathcal{F}) \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$. So, if we can show that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is a σ-algebra, then we will know that: $\mathcal{M} = \mathcal{M}(\mathcal{E}) \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

Fortunately, it's trivial to show that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is closed under complements. Given any $E \in \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$, we know there exists $\mathcal{M}(\mathcal{F})$ with $E \in \mathcal{M}(\mathcal{F})$. Then $E^{\mathsf{C}} \in \mathcal{M}(\mathcal{F}) \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

Meanwhile, the proof that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is closed under countable unions is more involved:

Suppose $\{E_n\}_{n \in \mathbb{N}}$ is a countable collection of sets in $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$. Then for each $n \in \mathbb{N}$, there exists $\mathcal{F}_n$ such that $E_n \in \mathcal{M}(\mathcal{F}_n)$. Importantly, $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ is still countable. So, setting $\mathcal{F}' = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, we have that:

$$\mathcal{M}(\mathcal{F}') \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$$

Since $\mathcal{F}_n \subseteq \mathcal{F}'$ for all n, we know that $\mathcal{M}(\mathcal{F}_n) \subseteq \mathcal{M}(\mathcal{F}')$ for all n. So, $\{E_n\}_{n \in \mathbb{N}}$ is contained in $\mathcal{M}(\mathcal{F}')$. It follows that $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{M}(\mathcal{F}') \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

Let $X \neq \emptyset$ and $\mathcal{M}$ be a σ-algebra on X. A measure $\mu : \mathcal{M} \longrightarrow [0, \infty]$ is a function satisfying that:

- $\mu(\emptyset) = 0$
- $\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)$ if $E_j \in \mathcal{M}$ for all j and $E_j \cap E_i = \emptyset$ for all $i \neq j$

$(X, \mathcal{M})$ is called a measurable space and $(X, \mathcal{M}, \mu)$ is called a measure space.

Let $(X, \mathcal{M}, \mu)$ be a measure space.

- μ is called finite if $\mu(X) < \infty$.

  It follows if μ is finite that $\mu(E) < \infty$ for all $E \in \mathcal{M}$ since $E \subseteq X$.
  In probability theory, most measure spaces are finite.

- μ is called σ-finite if $X = \bigcup_{j=1}^{\infty} E_j$, such that $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j.

- μ is called semifinite if $\forall E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $F \subset E$ such that $F \in \mathcal{M}$, and $0 < \mu(F) < \infty$.

Example: Let $X \neq \emptyset$ and $\mathcal{M} = \mathcal{P}(X)$. Then given a function $\rho : X \longrightarrow [0, \infty]$, $\mu(E) = \sum_{x \in E} \rho(x)$ is a measure.

- μ is semifinite if and only if $\rho(x) < \infty$ for all $x \in X$.
- μ is σ-finite if and only if it is semifinite and $\{x \in X \mid \rho(x) > 0\}$ is countable.

If $\rho(x) = 1$ for all x, then μ is called the counting measure.

If $\rho(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0, \end{cases}$ then μ is called the Dirac measure at x_0: δ_{x_0}.

Theorem: Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:

1. If $E, F \in \mathcal{M}$ with $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
2. If $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}$, then $\mu(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} \mu(E_j)$.
3. If $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}$ with $E_j \subseteq E_{j+1}$ for all $j \in \mathbb{N}$, then $\mu(\bigcup_{j=1}^{\infty} E_j) = \lim_{j \to \infty} \mu(E_j)$.
4. If $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}$ and $\mu(E_1) < \infty$ and $E_{j+1} \subseteq E_j$ for all $j \in \mathbb{N}$, then $\mu(\bigcap_{j=1}^{\infty} E_j) = \lim_{j \to \infty} \mu(E_j)$.

Proofs:
(1) Suppose $E, F \in \mathcal{M}$ with $E \subseteq F$. Then $F = (F - E) \cup E$ is a disjoint union of sets in $\mathcal{M}$, meaning $\mu(F) = \mu(F - E) + \mu(E) \geq \mu(E)$.

(2) Set $F_1 = E_1$ and $F_m = E_m - \bigcup_{i=1}^{m-1} E_i$ for all $m > 1$. Then $(F_i)_{i \in \mathbb{N}}$ is pairwise disjoint and $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$. So $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(F_i)$. On the other hand, $F_i \subseteq E_i$ for all i. So $\sum_{i=1}^{\infty} \mu(F_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$

(3) Setting $E_0 = \emptyset$, we have that $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i - E_{i-1})$. Also, $\mu(E_n) = \sum_{i=1}^{n} \mu(E_i - E_{i-1})$. So:

$$\lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i - E_{i-1}) = \sum_{i=1}^{\infty} \mu(E_i - E_{i-1}) = \mu(\bigcup_{i=1}^{\infty} E_i).$$

(4) Let $F_j = E_1 - E_j$ for all $j \in \mathbb{N}$. Then for all $j \in \mathbb{N}$, $F_j \subseteq F_{j+1}$, $\mu(E_1) = \mu(F_j) + \mu(E_j)$, and $\bigcup_{j=1}^{\infty} F_j = E_1 - \bigcap_{j=1}^{\infty} E_j$. We can thus conclude that:

$$\begin{aligned}\mu(E_1) &= \mu(\bigcap_{j=1}^{\infty} E_j) + \mu(\bigcup_{j=1}^{\infty} F_j)\\ &= \mu(\bigcap_{j=1}^{\infty} E_j) + \lim_{j\to\infty}(\mu(F_j)) = \mu(\bigcap_{j=1}^{\infty} E_j) + \lim_{j\to\infty}(\mu(E_1) - \mu(E_j))\end{aligned}$$

Since $\mu(E_1) < \infty$, we can subtract it out of the expression to get: $\mu(\bigcap_{j=1}^{\infty} E_j) - \lim_{j\to\infty}(\mu(E_j)) = 0$. Also, we know $\mu(\bigcap_{j=1}^{\infty} E_j) < \infty$ since it's a subset of E_j. So, we can rearrange to get: $\mu(\bigcap_{j=1}^{\infty} E_j) = \lim_{j\to\infty}(\mu(E_j))$.

Exercise 1.9: If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then we have that $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

We know $\mu(E) = \mu(E - f) + \mu(E \cap F)$ and $\mu(F) = \mu(F - E) + \mu(F \cap E)$. Adding those equations together we get that:

$$\begin{aligned}\mu(E) + \mu(F) &= (\mu(E - F) + \mu(E \cap F) + \mu(F - E)) + \mu(E \cap F)\\ &= \mu(E \cup F) + \mu(E \cap F).\end{aligned}$$

Exercise 1.14: If μ is a semifinite measure and $\mu(E) = \infty$, then for any $C > 0$ there exists $F \subset E$ in $\mathcal{M}$ with $C < \mu(F) < \infty$.

Let S be the set of $C > 0$ for which there exists $F \subset E$ in $\mathcal{M}$ with $C < \mu(F) < \infty$. By the definition of semifiniteness, we know S isn't empty. Meanwhile, if for some C we had that there didn't exist a set $F \subset E$ in $\mathcal{M}$ with $C < \mu(F) < \infty$, then we'd know that S is bounded above. Hence, we'd know there exists $\alpha = \sup(S)$.

Now firstly, for all $n \in \mathbb{N}$, choose $G_n \subset E$ in $\mathcal{M}$ such that $\alpha - \frac{1}{n} < \mu(G_n) < \infty$. After that, define $F_n = \bigcup_{i=1}^{n} G_i$ for all $n \in \mathbb{N}$. Since $\mathcal{M}$ is closed under finite unions, we know each F_n is in $\mathcal{M}$. So then observe:

1. $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$
2. For each $n \in \mathbb{N}$, $\alpha - \frac{1}{n} < \mu(F_n) \leq \alpha$

This is because for each $n \in \mathbb{N}$, $\mu(F_n) < \sum_{i=1}^{n} \mu(G_i)$ which is a finite sum of finite quantities. So $\mu(F_n) < \infty$. At the same time, $F_n \subset E$ since each G_i is a subset of E (we know it is a proper subset because it has a different measure than E). So, if $\mu(F_n) > \alpha$, then $\frac{1}{2}(\mu(F_n) + \alpha)$ would be an element of S greater than α, thus contradicting that $\alpha = \sup(S)$. As for the other inequality, note that $G_n \subseteq F_n$. Thus $\mu(F_n) \geq \mu(G_n) > \alpha - \frac{1}{n}$.

Now $\bigcup_{n=1}^{\infty} F_n \in \mathcal{M}$ due to $\mathcal{M}$ being closed under countable sums. Also, by the two observations above, we know $\mu(\bigcup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \mu(F_n) = \alpha$. And finally, note that $\bigcup_{n=1}^{\infty} F_n$ is a proper subset of E (we know this because each $F_n \subset E$ and $\bigcup_{n=1}^{\infty} F_n$ can't equal E since their measures are different).

So, we have now proven the existence of a set $F \in \mathcal{M}$ such that $F \subset E$ and $\mu(F) = \alpha$. But now note that $\mu(E - F)$ must be infinite since:

$$\mu(E - F) + \alpha = \mu(E - F) + \mu(F) = \mu(E) = \infty.$$

Because μ is semifinite, there exists $F' \subset E - F$ in $\mathcal{M}$ with $0 < \mu(F') < \infty$. But because F and F' are disjoint subsets of E in $\mathcal{M}$, we know $F \cup F' \in \mathcal{M}$ and $\mu(F \cup F') = \mu(F) + \mu(F') > \alpha$. Plus $F \cup F'$ is a proper subset of E. (It can't equal E because it's measure isn't equal to E. But, both F and F' individually are subsets of E.)

Hence, we have that $\frac{1}{2}(\alpha, \mu(F) + \mu(F)')$ is an element of S greater than α, thus contradicting that α was the supremum of S. We conclude therefore that α does not exist, meaning S is unbounded.

Given a measure space $(X, \mathcal{M}, \mu)$, a set $E \in \mathcal{M}$ satisfying that $\mu(E) = 0$ is called a null set (or μ-null set if we want more precision).

By subadditivity (a.k.a. the fact that for all $(E_j)_{j\in\mathbb{N}} \subset \mathcal{M}$, $\mu(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} \mu(E_j)$), we know countable unions of null sets are also null sets.

Given a proposition $P(x)$, if there exists a null set $E \in \mathcal{M}$ satisfying that $P(x)$ is true for all $x \in X - E$, then we say P is true almost everywhere (abbreviated as μ-a.e. or just a.e. if the measure being used is clear).

A measure space is complete if given any $E \subseteq X$, we have that $N \in \mathcal{M}$ with $\mu(N) = 0$ and $E \subsetneq N$ implies that $E \in \mathcal{M}$.

Proposition: Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Let:

- $\mathcal{N} = \{N \in \mathcal{M} \mid \mu(N) = 0\}$
- $\overline{\mathcal{M}} = \{E \cup F \mid E \in \mathcal{M}$ and $F \subseteq N$ where $N \in \mathcal{N}\}$.

Then $\overline{\mathcal{M}}$ is a σ-algebra and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

Proof:

Claim 1: $\overline{\mathcal{M}}$ is a σ-algebra.

To see that $\overline{\mathcal{M}}$ is closed under countable union, let $(E_i \cup F_i)_{i \in \mathbb{N}}$ be a sequence of sets in $\overline{\mathcal{M}}$ with each $E_i \in \mathcal{M}$ and $F_i \subseteq N_i$ for some $N_i \in \mathcal{N}$. Then $\bigcup\limits_{i \in \mathbb{N}} (E_i \cup F_i) = \bigcup\limits_{i \in \mathbb{N}} E_i \cup \bigcup\limits_{i \in \mathbb{N}} F_i$.

Importantly, since $\mathcal{M}$ and $\mathcal{N}$ are closed under countable union, we know that $\bigcup\limits_{i \in \mathbb{N}} E_i \in \mathcal{M}$ and $\bigcup\limits_{i \in \mathbb{N}} F_i \subseteq \bigcup\limits_{i \in \mathbb{N}} N_i \in \mathcal{N}$. So, $\bigcup\limits_{i \in \mathbb{N}} (E_i \cup F_i) \in \overline{\mathcal{M}}$.

To show that $\overline{\mathcal{M}}$ is closed under complements, let $E \cup F \in \overline{\mathcal{M}}$ with $E \in \mathcal{M}$ and $F \subseteq N$ for some $N \in \mathcal{N}$. Also note that we can assume $E \cap N = \emptyset$. After all, if $E \cap N \neq \emptyset$, then define $N' = N - E$ and $F' = F - E$. Since $N' \subseteq N$ and $N' \in \mathcal{M}$, we know that $\mu(N') = 0$. Also, $E \cup F = E \cup F'$ with $F' \subseteq N'$. So, E, F', and N' fulfil the same properties we picked E, F, and N for having. But also $E \cap N' = \emptyset$.

Now, $(E \cup F)^{\mathsf{C}} = (E \cup N)^{\mathsf{C}} \cup (N - F)$ where $(E \cup N)^{\mathsf{C}} \in \mathcal{M}$ and $(N - F) \subset N$. So $(E \cup F)^{\mathsf{C}} \in \overline{\mathcal{M}}$.

Now given any $E \cup F \in \overline{\mathcal{M}}$ with $E \in \mathcal{M}$ and $F \subset N$ for some $N \in \mathcal{N}$, define $\overline{\mu}(E \cup F) = \mu(E)$.

Claim 2: $\overline{\mu}$ is well-defined.

Suppose $E_1 \cup F_1 = E_2 \cup F_2$ where for $j \in \{1, 2\}$ we have $E_j \in \mathcal{M}$ and $F_j \subset N_j$ for some $N_j \in \mathcal{N}$. Then $E_1 \subseteq E_2 \cup N_2$, meaning that $\mu(E_1) \leq \mu(E_2) + \mu(N_2) = \mu(E_2)$. By similar reasoning, we can say that $\mu(E_2) \leq \mu(E_1)$. So $\overline{\mu}(E_1 \cup F_1) = \overline{\mu}(E_2 \cup F_2)$.

(The rest is exercise 1.6:)

Claim 3: $\overline{\mu}$ is a complete measure on $\overline{\mathcal{M}}$.

It's easy to show that $\overline{\mu}$ is a measure. After all, $\emptyset \in \mathcal{M} \cap \mathcal{N}$. So, $\overline{\mu}(\emptyset) = \overline{\mu}(\emptyset \cup \emptyset) = \mu(\emptyset) = 0$. Also, suppose $(E_i \cup F_i)_{i \in \mathbb{N}}$ is a sequence of disjoint sets in $\overline{\mathcal{M}}$ with $E_i \in \mathcal{M}$ and $F_i \subset N_i$ for some $N_i \in \mathcal{N}$.

Then $\bigcup_{i\in\mathbb{N}} E_i \in \mathcal{M}$ where each E_i is disjoint and $\bigcup_{i\in\mathbb{N}} F_i \subseteq \bigcup_{i\in\mathbb{N}} N_i \in \mathcal{N}$. So:

$$\begin{aligned}\overline{\mu}(\bigcup_{i\in\mathbb{N}}(E_i \cup F_i)) &= \overline{\mu}(\bigcup_{i\in\mathbb{N}} E_i \cup \bigcup_{i\in\mathbb{N}} F_i) \\ &= \mu(\bigcup_{i\in\mathbb{N}} E_i) = \sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{\infty} \overline{\mu}(E_i \cup F_i)\end{aligned}$$

Finally, to show that $(X, \overline{\mathcal{M}}, \overline{\mu})$ is complete, suppose $A \subseteq X$ and $N_1 \in \overline{\mathcal{M}}$ with $\overline{\mu}(N_1) = 0$ and $A \subseteq N_1$. By definition, we know $N_1 = E \cup F$ where $E \in \mathcal{M}$ and $F \subseteq N_2$ for some $N_2 \in \mathcal{N}$. However, we can also assume $E = \emptyset$. For if $E \neq \emptyset$, then because $\mu(E \cup N_2) \leq \mu(E) + \mu(N_2) \leq 0$, we can define $N_2' = E \cup N_2$ and $F' = E \cup F$. Then $N_1 = \emptyset \cup F'$ and $F' \subseteq N_2'$ where $N_2' \in \mathcal{N}$.

So $A \subset F \subset N_2$ where $N_2 \in \mathcal{N}$. It follows that $A = \emptyset \cup A \in \overline{\mathcal{M}}$.

Claim 4: $\overline{\mu}$ is the unique measure on $\overline{\mathcal{M}}$ that extends μ.

Suppose $\overline{\overline{\mu}}$ is another measure on $\overline{\mathcal{M}}$ such that $\overline{\overline{\mu}}|_{\mathcal{M}} = \mu$. Then consider any $E \cup F \in \overline{M}$ such that $E \in \mathcal{M}$ and $F \subseteq N$ for some $N \in \mathcal{N}$. As shown before, we can assume without loss of generality that $E \cap N = \emptyset$ and thus also $E \cap F = \emptyset$. So, we have that:

$$\overline{\overline{\mu}}(E \cup F) = \overline{\overline{\mu}}(E) + \overline{\overline{\mu}}(F)$$

Next, note that:

$$\mu(E) = \overline{\overline{\mu}}(E) \leq \overline{\overline{\mu}}(E) + \overline{\overline{\mu}}(F) \leq \overline{\overline{\mu}}(E) + \overline{\overline{\mu}}(N) = \mu(E) + \mu(N) = \mu(E)$$

Hence, we know that $\overline{\overline{\mu}}(E \cup F) = \mu(E)$. But also $\overline{\mu}(E \cup F) = \mu(E)$. So $\overline{\overline{\mu}}(E \cup F) = \overline{\mu}(E \cup F)$ for all $E \cup F \in \overline{\mathcal{M}}$.

Note: We call $\overline{\mu}$ the completion of μ and $\overline{\mathcal{M}}$ the completion of $\mathcal{M}$ with respect to μ.

Lecture 4 Notes: 10/8/2024

An outer measure on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \longrightarrow [0, \infty]$ satisfying that:

1. $\mu^*(\emptyset) = 0$.
2. $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$.
3. $\mu^*(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$. (this property is called subadditivity)

Proposition: Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of elementary sets and $\mu : \mathcal{E} \longrightarrow [0, \infty]$ be a function satisfying that $\mu(\emptyset) = 0$.

The textbook only assumes that $\emptyset$ and X are in $\mathcal{E}$.

Then define $\mu^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \mu(E_j) \,\middle|\, E_j \in \mathcal{E} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$. This is an outer measure.

Proof:

Given any set $A \subseteq \mathcal{P}(X)$, define $A^* = \left\{ \sum_{j=1}^{\infty} \mu(E_j) \,\middle|\, E_j \in \mathcal{E} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$. That way $\mu^*(A) = \inf(A^*)$.

1. $\mu^*(A)$ is well defined because $\mu(X) \in A^*$ and $\mu(\emptyset) = 0$ is a lower bound of A^*.
2. Since $0 \in \emptyset^*$, we know that $\mu^*(\emptyset) = \inf(\emptyset^*) = 0$.
3. Suppose $A \subseteq B$. Then given any $(E_j)_{j \in \mathbb{N}}$ of sets in $\mathcal{E}$ covering B, we know that they will also cover A. So, $A^* \subseteq B^*$, meaning $\mu^*(A) \leq \mu^*(B)$.
4. Suppose $(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{P}(X)$. Then fix $\varepsilon > 0$. For all $j \in \mathbb{N}$, let $(E_j^{(k)})_{k \in \mathbb{N}}$ be a sequence of sets in $\mathcal{E}$ such that $A_j \subseteq \bigcup_{k \in \mathbb{N}} E_j^{(k)}$ and:
$$\mu^*(A_j) \leq \sum_{k=1}^{\infty} \mu(E_j^{(k)}) \leq \mu^*(A_j) + \varepsilon/2^j.$$
   Note that $\bigcup_{j \in \mathbb{N}} A_j \subseteq \bigcup_{j \in \mathbb{N}} (\bigcup_{k \in \mathbb{N}} E_j^{(k)})$.

   So, $(\bigcup_{j \in \mathbb{N}} A_j)^* \ni \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mu(E_j^{(k)}) \leq \sum_{j=1}^{\infty} (\mu^*(A_j) + \varepsilon/2^j) = \varepsilon + \sum_{j=1}^{\infty} \mu^*(A_j)$.

   Since ε was arbitrary, we thus know that:
$$\mu^*(\bigcup_{j \in \mathbb{N}} A_j) = \inf(\bigcup_{j \in \mathbb{N}} A_j)^* \leq \sum_{j=1}^{\infty} \mu^*(A_j).$$

Let μ^* be an outer measure on a nonempty set X. Then $A \subseteq X$ is called μ^*-measurable if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E - A)$ for all $E \subseteq X$.

Note that we trivially have $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E - A)$ for all $E \in \mathcal{P}(X)$. Also, $\mu^*(E \cap A) + \mu^*(E - A) \leq \mu^*(E)$ holds trivially if $\mu^*(E)$. So μ^*-measurability just means that $\mu^*(E \cap A) + \mu^*(E - A) \leq \mu^*(E)$ holds even if $\mu^*(E) < \infty$.

Carathéodory's Theorem: If μ^* is an outer measure on X, then the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

Proof:

Part 1: $\mathcal{M}$ is an algebra and μ^* is additive on $\mathcal{M}$ (meaning $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$ implies that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$).

We know $\mathcal{M}$ is an algebra because:

- $\emptyset \in \mathcal{M}$ because $\mu^*(E) = \mu^*(E \cap \emptyset) + \mu^*(E - \emptyset) = 0 + \mu^*(\emptyset)$ for all $E \subseteq X$.

- Both $A^{\mathsf{C}} \in \mathcal{M}$ $A \in \mathcal{M}$ are equivalent to us having for all $E \subseteq X$ that $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^{\mathsf{C}})$.

- Suppose A and B are sets in $\mathcal{M}$. Then given $E \subseteq X$, we have:
$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E - A)\\ &= \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) - B)\\ &\qquad + \mu^*((E - A) \cap B) + \mu^*((E - A) - B)\end{aligned}$$

  Now $(E - A) - B = E \cap A^{\mathsf{C}} \cap B^{\mathsf{C}} = E \cap (A \cup B)^{\mathsf{C}}$. Meanwhile, $(E \cap A) - B = E \cap (A - B)$ and $(E - A) \cap B = E \cap (B - A)$. So, by subadditivity, we have that:
$$\begin{aligned}&\mu^*(E \cap (A \cup B)) + \mu(E - (A \cup B))\\ &\qquad \leq \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) - B)\\ &\qquad\quad + \mu^*((E - A) \cap B) + \mu^*((E - A) - B)\end{aligned}$$

  Hence, $\mu^*(E \cap (A \cup B)) + \mu^*(E - (A \cup B)) \leq \mu^*(E)$. So, $A \cup B \in \mathcal{M}$.

Next, to show that μ is additive on $\mathcal{M}$, consider any $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$. Then:
$$\mu^*(A \cup B) = \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) - A) = \mu^*(A) + \mu^*(B)$$

Part 2: $\mathcal{M}$ is a σ-algebra and μ^* is σ-additive (think countably additive) on X.

To show that $\mathcal{M}$ is a σ-algebra, it suffices to show that $\mathcal{M}$ is closed under countable disjoint unions. So let $(A_j)_{j \in \mathbb{N}}$ be a sequence of disjoint sets in $\mathcal{M}$. If E is any set in X and $m > 1$, then:
$$\mu^*(E) = \mu^*(E \cap (\bigcup_{j=1}^{m} A_j)) + \mu^*(E - (\bigcup_{j=1}^{m} A_j))$$

But then note that because $\mathcal{M}$ is an algebra, we know $\bigcup_{j=1}^{m} A_j \in \mathcal{M}$. So:
$$\begin{aligned}\mu^*(E \cap (\bigcup_{j=1}^{m} A_j)) &= \mu^*(E \cap (\bigcup_{j=1}^{m} A_j) \cap A_m) + \mu^*(E \cap (\bigcup_{j=1}^{m} A_j) \cap A_m^{\mathsf{C}})\\ &= \mu^*(E \cap A_m) + \mu^*(E \cap (\bigcup_{j=1}^{m-1} A_j))\end{aligned}$$

By induction, we thus have that $\mu^*(E \cap \bigcup_{j=1}^{m} A_j) = \sum_{j=1}^{m} \mu^*(E \cap A_j)$. Also, since $E - (\bigcup_{j=1}^{m} A_j) \supset E - (\bigcup_{j \in \mathbb{N}} A_j)$, we thus know that:
$$\mu^*(E) \geq \sum_{j=1}^{m} \mu^*(E \cap A_j) + \mu^*(E - \bigcup_{j \in \mathbb{N}} A_j)$$

Taking the limit as $m \to \infty$, we thus get that:
$$\begin{aligned}\mu^*(E) &\geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(E - \bigcup_{j \in \mathbb{N}} A_j)\\ &\geq \mu^*(E \cap (\bigcup_{j \in \mathbb{N}} A_j)) + \mu^*(E - \bigcup_{j \in \mathbb{N}} A_j)\end{aligned}$$

So, $\bigcup_{j\in\mathbb{N}} A_j$ is μ^*-measurable. Hence, $\mathcal{M}$ is a σ-algebra.

Also, in order to show that $\mu^*(\bigcup_{j\in\mathbb{N}} A_j) = \sum_{j=1}^{\infty} \mu^*(A_j)$, just substitute $E = \bigcup_{j\in\mathbb{N}} A_j$ into the expression at the bottom of the last page.

Part 3: $(X, \mathcal{M}, \mu^*)$ is a complete measure space.

Suppose $\mu^*(A) = 0$. Then given $E \subseteq X$, we have that:

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E - A) \leq \mu^*(A) + \mu^*(E) \leq 0 + \mu^*(E)$$

It follows that $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E - A)$ for all E. So $A \in \mathcal{M}$.

Now if $\mu^*(A) = 0$, then $\mu^*(E) = 0$ for all $E \subseteq A$. It follows that all subsets of μ^*-null sets are in $\mathcal{M}$.

The moral of the story is that we'll just call μ^* a measure because it is when restricted to the right σ-algebra.

A premeasure $\mu_0 : \mathcal{A} \longrightarrow [0, \infty]$ is a function on an algebra satisfying that:

- $\mu_0(\emptyset) = 0$
- if $(A_j)_{j\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{A}$ with $\bigcup_{j\in\mathbb{N}} A_j \in \mathcal{A}$, then $\mu_0(\bigcup_{j\in\mathbb{N}} A_j) = \sum_{j=1}^{\infty} \mu_0(A_j)$.

By setting all but finitely many A_j to the emptyset, we can show that μ_0 must be finitely additive. In turn, this is enough to show that $\mu_0(A) \leq \mu_0(B)$ if $A \subseteq B$ for any $A, B \in \mathcal{A}$.

We say μ^* is induced by μ_0 if $\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(E_j) \;\middle|\; E_j \in \mathcal{A} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$.

Note that μ^* is an outer measure by a previous proposition.

Proposition: In this situation:

1. $\mu^*|_{\mathcal{A}} = \mu_0$

Proof:

Suppose $E \in \mathcal{A}$ and let $(A_j)_{j\in\mathbb{N}}$ be a sequence of sets of in $\mathcal{A}$ covering E. It's trivial that $\mu^*(E) \leq \mu_0(E)$ because we could just let $A_1 = E$ and $A_n = \emptyset$ for all $n \geq 2$.

On the other hand, letting $B_m = E \cap A_1$ and $B_m = E \cap A_m - \bigcup_{j=1}^{m-1} A_j$, we have that $(B_j)_{j\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{A}$ whose union is E. It follows from the second property of a premeasure and the fact that $B_j \subseteq A_j$ for all j that:

$$\mu_0(E) = \sum_{j=1}^{\infty} \mu_0(B_j) \le \sum_{j=1}^{\infty} \mu_0(A_j)$$

Since $(A_j)_{j\in\mathbb{N}}$ was not specified, it follows that $\mu_0(E) \le \mu^*(E)$.

2. Every set in $\mathcal{A}$ is μ^*-measurable.

   Proof:
   Suppose $A \in \mathcal{A}, E \subseteq X$, and $\varepsilon > 0$. Then there is a sequence $(B_j)_{j\in\mathbb{N}}$ of sets in $\mathcal{A}$ with $E \subseteq \bigcup_{j=1}^{\infty} B_j$ and $\sum_{j=1}^{\infty} \mu_0(B_j) \le \mu^*(E) + \varepsilon$. Since μ_0 is additive on $\mathcal{A}$, $(B_j \cap A)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ covers $E \cap A$, and $(B_j - A)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ covers $E - A$, we have that:

$$\begin{aligned}\mu^*(E) + \varepsilon \ge \sum_{j=1}^{\infty} \mu_0(B_j) &= \sum_{j=1}^{\infty} \mu_0(B_j \cap A) + \sum_{j=1}^{\infty} \mu_0(B_j - A) \\ &\ge \mu^*(E \cap A) + \mu^*(E - A)\end{aligned}$$

   Taking $\varepsilon \to 0$, we get that $\mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E - A)$.

Theorem: Suppose $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra and $\mu_0 : \mathcal{A} \longrightarrow [0, \infty]$ is a premeasure. Then there exists $\mu : \mathcal{M}(\mathcal{A}) \longrightarrow [0, \infty]$ such that:

- $\mu|_{\mathcal{A}} = \mu_0$
- if $\nu : \mathcal{M}(\mathcal{A}) \longrightarrow [0, \infty]$ is a measure with $\nu|_{\mathcal{A}} = \mu|_{\mathcal{A}}$, then $\nu \le \mu$ (with equality if $\mu(E) < \infty$).
- If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$.

  Proof:

1. The first claim is true by Carathéodory's Theorem and the last proposition. Specifically, define $\mu = \mu^*|_{\mathcal{M}(\mathcal{A})}$ where μ^* is the outer measure induced by μ_0. Since $\mathcal{A}$ is a subset of the σ-algebra $\mathcal{M}$ of μ^* measurable sets, we know that $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}$. So μ is a measure over $\mathcal{M}(\mathcal{A})$. Also, we know that $\mu(A) = \mu^*(A) = \mu_0(A)$ for all $A \in \mathcal{A}$ by the last proposition.

2. To show the second claim, let $E \in \mathcal{M}(\mathcal{A})$ and $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ be a covering of E such that $\sum_{j=1}^{\infty} \mu_0(A_j) \le \mu(E) + \varepsilon$ for a given $\varepsilon > 0$. Then:

$$\nu(E) \le \sum_{j=1}^{\infty} \nu(A_j) = \sum_{j=1}^{\infty} \mu_0(A_j) \le \mu(E) + \varepsilon.$$

   Taking $\varepsilon \to 0$, we get that $\nu(E) \le \mu(E)$.

As for the other inequality, consider that for any $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$, we know by part 3 of the theorem on page 16 and the fact that $\nu(A_j) = \mu(A_j)$ for all j that if $A = \bigcup_{j\in\mathbb{N}} A_j$, then:

$$\nu(A) = \lim_{m\to\infty}(\nu(\bigcup_{j=1}^{m} A_j)) = \lim_{m\to\infty}(\mu(\bigcup_{j=1}^{m} A_j)) = \mu(A).$$

Also, if $\mu(E)$ is finite, then we can choose the covering $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ of E so that all A_j are disjoint and $A = \bigcup_{j\in\mathbb{N}} A_j$ satisfies for a given $\varepsilon > 0$ that:

$$\mu(E) \leq \mu(A) = \sum_{j=1}^{\infty} \mu(A_j) \leq \mu(E) + \varepsilon$$

It follows that $\mu(A - E) < \varepsilon$. So:

$$\mu(E) \leq \mu(A) = \nu(A) = \nu(E) + \nu(A - E) \leq \nu(E) + \mu(A - E) \leq \nu(E) + \varepsilon.$$

Taking $\varepsilon \to 0$, we get that $\mu(E) \leq \nu(E)$.

3. For the third claim, suppose $X = \bigcup_{j\in\mathbb{N}} A_j$ with $\mu_0(A_j) < \infty$ and all A_j being disjoint. Then for any $E \in \mathcal{M}(\mathcal{A})$:

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E \cap A_j) = \sum_{j=1}^{\infty} \nu(E \cap A_j) = \nu(E)$$

Exercise 1.16: Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subseteq X$ is called locally measurable if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets. Trivially, we know $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$. If $\mathcal{M} = \widetilde{\mathcal{M}}$, then μ is called saturated.

(a) If μ is σ-finite, then μ is saturated.

Let $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}$ satisfy that $\mu(A_j) < \infty$ for all j, and that $X = \bigcup_{j\in\mathbb{N}} A_j$. Then if E is locally measurable, we know: $E = \bigcup_{j\in\mathbb{N}} (E \cap A_j) \in \mathcal{M}$.

(b) $\widetilde{\mathcal{M}}$ is a σ-algebra.

- If $E \in \widetilde{\mathcal{M}}$, then given any $A \in \mathcal{M}$ with $\mu(A) < \infty$, we know $E \cap A \in \mathcal{M}$. It follows that $E^{\mathsf{C}} \cap A = A - E = A - (E \cap A) \in \mathcal{M}$. So $E^{\mathsf{C}} \in \widetilde{\mathcal{M}}$.
- Suppose $(E_j)_{j\in\mathbb{N}}$ is a sequence of sets in $\widetilde{\mathcal{M}}$ and $A \in \mathcal{M}$ satisfies that $\mu(A) < \infty$. Then: $(\bigcup_{j\in\mathbb{N}} E_j) \cap A = \bigcup_{j\in\mathbb{N}} (E_j \cap A) \in \mathcal{M}$.

  So, $\bigcup_{j\in\mathbb{N}} E_j \in \widetilde{\mathcal{M}}$

(c) Define $\widetilde{\mu}$ on $\widetilde{\mathcal{M}}$ by $\widetilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\widetilde{\mu}(E) = \infty$ otherwise. Then $\widetilde{\mu}$ is a saturated measure on $\mathcal{M}$ called the saturation of μ.

Since $\emptyset \in \mathcal{M}$, we know $\widetilde{\mu}(\emptyset) = \mu(\emptyset) = 0$.

Note that if $A, B \in \widetilde{\mathcal{M}}$ with $A \subseteq B$ and $A \notin \mathcal{M}$ but $B \in \mathcal{M}$, then we immediately get a contradiction since that would suggest $A = A \cap B \in \mathcal{M}$. As a result, supposing $(E_j)_{j\in\mathbb{N}}$ is a sequence of disjoint sets in $\widetilde{\mathcal{M}}$, we have that if any $E_j \notin \mathcal{M}$, then:

$$\widetilde{\mu}(\bigcup_{j\in\mathbb{N}} E_j) = \infty = \sum_{j=1}^{\infty} \widetilde{\mu}(E_j).$$

Meanwhile, if all sets of $(E_j)_{j\in\mathbb{N}}$ are in $\mathcal{M}$, then:

$$\widetilde{\mu}(\bigcup_{j\in\mathbb{N}} E_j) = \mu(\bigcup_{j\in\mathbb{N}} E_j) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \widetilde{\mu}(E_j).$$

(d) If μ is complete, then so is $\widetilde{\mu}$.

This fact is obvious because by the way we defined $\widetilde{\mu}$, we know a set is $\widetilde{\mu}$-null if and only if it is μ-null.

(e) Suppose that μ is semifinite. For $E \in \widetilde{\mathcal{M}}$, define:

$$\underline{\mu}(E) = \sup\{\mu(A) \mid A \in \mathcal{M} \text{ and } A \subseteq E\}.$$

This is well defined because $\mu(\emptyset)$ is always in the above set and $\mu(X)$ is an upper-bound. Then $\underline{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ that extends μ.

Firstly, we show $\underline{\mu}$ is a measure. To start, it's trivial to see that $\underline{\mu}(\emptyset) = \mu(\emptyset) = 0$.

Lemma: If $E \in \widetilde{\mathcal{M}}$ and $\underline{\mu}(E) = \infty$, then there exists a set $A \in \mathcal{M}$ such that $A \subseteq E$ and $\mu(A) = \infty$.

To show this, construct a sequence of "increasing" sets $(A_j)_{j\in\mathbb{N}}$ satisfying that $A_j \subseteq E$ and $\mu(A_j) \geq j$. Then the union A of that sequence will satisfy that $A \in \mathcal{M}$, that $A \subseteq E$, and that $\mu(A) = \infty$.

Because of that lemma, we don't need to deal with the edge case that a least upper bound equaling infinity doesn't mean a set contains infinity. So, let $(E_j)_{j\in\mathbb{N}}$ be a sequence of disjoint sets in $\widetilde{\mathcal{M}}$ with $E = \bigcup_{j\in\mathbb{N}} E_j$. Then let $\varepsilon > 0$.

To show one inequality, pick a sequence $(A_j)_{j\in\mathbb{N}}$ of sets in $\mathcal{M}$ satisfying that $A_j \subseteq E_j$ and $\underline{\mu}(E_j) - \varepsilon/2^j \leq \mu(A_j)$. Since $A = \bigcup_{j\in\mathbb{N}} A_j \subseteq E$, and each A_j is disjoint, we thus have:

$$-\varepsilon + \sum_{j=1}^{\infty} \underline{\mu}(E_j) \leq \sum_{j=1}^{\infty} \mu(A_j) = \mu(A) \leq \underline{\mu}(E)$$

To show the other inequality, pick $B \in \mathcal{M}$ satisfying that $B \subseteq E$ and $\underline{\mu}(E) - \varepsilon < \mu(B)$. Because $E, E_j \in \widetilde{\mathcal{M}}$ for each j, we know that $B \cap E$ and $B \cap E_j$ are in $\mathcal{M}$ for each j. So:

$$\underline{\mu}(E) - \varepsilon < \mu(B) = \mu(B \cap E) = \sum_{j=1}^{\infty} \mu(B \cap E_j) \leq \sum_{j=1}^{\infty} \underline{\mu}(E_j)$$

Taking $\varepsilon \to 0$, we thus get that $\underline{\mu}(E) = \sum\limits_{j=1}^{\infty} \underline{\mu}(E_j)$.

Proving that $\mu(E) = \underline{\mu}(E)$ when $E \in \mathcal{M}$ is trivial. Obviously, $\mu(E) \leq \underline{\mu}(E)$. Meanwhile for any $F \in \mathcal{M}$ satisfying that $F \subseteq E$, we know that $\mu(F) \leq \mu(E)$. So, there does not exists a subset of E in $\mathcal{M}$ with greater measure than $\mu(E)$.

Note that $\widetilde{\mu}$ and $\underline{\mu}$ are not necessarily equal. Part (f) of this problem gives a relatively simple counterexample.

Exercise 1.17: If μ^* is an outer measure on X and $(A_j)_{j \in \mathbb{N}}$ is a sequence of disjoint μ^*-measurable sets, then:

$$\mu^*(E \cap \bigcup_{j \in \mathbb{N}} A_j) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \text{ for any } E \subseteq X.$$

Note that by induction, we can show that for any $n \in \mathbb{N}$:

$$\begin{aligned} \mu^*(E \cap \bigcup_{j=1}^{\infty} A_j) &= \mu^*(E \cap \bigcup_{j=1}^{\infty} A_j \cap A_1) + \mu^*(E \cap \bigcup_{j=1}^{\infty} A_j - A_1) \\ &= \mu^*(E \cap A_1) + \mu^*(E \cap \bigcup_{j=2}^{\infty} A_j) \\ &= \sum_{j=1}^{2} \mu^*(E \cap A_j) + \mu^*(E \cap \bigcup_{j=3}^{\infty} A_j) \\ &= \cdots = \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap \bigcup_{j=n+1}^{\infty} A_j) \end{aligned}$$

Thus, we clearly have for all n that $\sum\limits_{j=1}^{n} \mu^*(E \cap A_j) \leq \mu^*(E \cap \bigcup\limits_{j=1}^{\infty} A_j)$.

Taking the limit as $n \to \infty$, we thus know $\sum\limits_{j=1}^{\infty} \mu^*(E \cap A_j) \leq \mu^*(E \cap \bigcup\limits_{j \in \mathbb{N}} A_j)$.

The other inequality is obvious from the subadditivity property of outer measures.

Exercise 1.18: Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ be be the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ be the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

(a) For any $E \subseteq X$ and $\varepsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ with $E \subseteq A$ and $\mu^*(A) \leq \mu^*(E) + \varepsilon$.

Let $(A_j)_{j\in\mathbb{N}}$ be a sequence of sets in $\mathcal{A}$ which cover E and satisfy that $\sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \varepsilon$. In turn, by the subadditivity of outer measures, and the fact that $\mu^*(A) = \mu_0(A)$ for all $A \in \mathcal{A}$, we know that:

$$E \subseteq \bigcup_{j\in\mathbb{N}} A_j \in \mathcal{A}_\sigma \text{ and } \mu^*(\bigcup_{j\in\mathbb{N}} A_j) \leq \sum_{j=1}^{\infty} \mu^*(A_j) = \sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \varepsilon$$

(b) If $\mu^*(E) < \infty$, then E is μ^*-measurable if and only if there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B - E) = 0$.

Suppose E is μ^*-measurable. Then for all $j \in \mathbb{N}$, pick $A_j \in \mathcal{A}_\sigma$ satisfying that $E \subseteq A_j$ and $\mu^*(E) \leq \mu^*(A_j) \leq \mu^*(E) + {}^1\!/\!_j$. Since E is μ^*-measurable, we know that for all $j \in \mathbb{N}$:

$$\mu^*(A_j) = \mu^*(A_j \cap E) + \mu^*(A_j - E) = \mu^*(E) + \mu^*(A_j - E)$$

In turn, because $\mu^*(E) < \infty$, this tell us that $\mu^*(A_j - E) \leq {}^1\!/\!_j$. Also, because $\bigcap_{j\in\mathbb{N}} A_j \subseteq A_n$ for all $n \in \mathbb{N}$, we know that $\mu^*(\bigcap_{j\in\mathbb{N}} A_j - E) \leq {}^1\!/\!_n$ for all $n \in \mathbb{N}$.

As a result, we know that $\bigcap_{j\in\mathbb{N}} A_j \in \mathcal{A}_{\sigma\delta}$, $E \subseteq \bigcap_{j\in\mathbb{N}} A_j$, and $\mu^*(\bigcap_{j\in\mathbb{N}} A_j - E) = 0$.

To prove the reverse implication, suppose there exists a μ^*-separable set B satisfying that $E \subseteq B$ and $\mu^*(B - E) = 0$ (any set in $\mathcal{A}_{\sigma\delta}$ will be μ^*-separable because $\mathcal{A}_{\sigma\delta} \subseteq \mathcal{M}(\mathcal{A})$). Then given any set F, we have that:

$$\begin{aligned}\mu^*(F - E) &= \mu^*(F \cap E^{\mathsf{C}} \cap B) + \mu^*(F \cap E^{\mathsf{C}} \cap B^{\mathsf{C}}) \\ &= \mu^*(F \cap (B - E)) + \mu^*(F - B) = \mu^*(F - B)\end{aligned}$$

Also, since $F \cap E \subseteq F \cap B$, we know that $\mu^*(F \cap E) \leq \mu^*(F \cap B)$.

So, $\mu^*(F \cap E) + \mu^*(F - E) \leq \mu^*(F \cap B) + \mu^*(F - B) = \mu^*(F)$. Hence, E is μ^*-measurable.

(c) If μ_0 is σ-finite, we can remove the requirement in part (b) that $\mu^*(E) < \infty$.

Because the backwards implication proof never required $\mu^*(E)$ to be finite, it suffices to show that E being μ^*-measurable implies there exists $B \in A_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B - E) = 0$.

To start, let $(C_i)_{i\in\mathbb{N}}$ satisfy that $\mu_0(C_i) < \infty$ and $\bigcup_{i=1}^{\infty} C_i = X$.

Next for all $j, i \in \mathbb{N}$, pick $A_j^{(i)} \in \mathcal{A}_\sigma$ satisfying that $E \cap C_i \subseteq A_j^{(i)}$ and $\mu^*(E) \leq \mu^*(A_j^{(i)}) \leq \mu^*(E) + 1/j2^i$. Since $\mu^*(E \cap C_i)$ is finite, we can use the same reasoning as in part (b) to say that $\mu^*(A_j^{(i)} - (E \cap C_i)) \leq 1/j2^i$.

Importantly, $A_j^{(i)} - E \subseteq A_j^{(i)} - (E \cap C_i)$. for all i. Therefore:

$$\begin{aligned}\mu^*((\bigcup_{i \in \mathbb{N}} A_j^{(i)}) - E) &= \mu^*(\bigcup_{i \in \mathbb{N}} (A_j^{(i)} - E)) \\ &\leq \mu^*(\bigcup_{i \in \mathbb{N}} (A_j^{(i)} - (E \cap C_i))) \\ &\leq \sum_{i \in \mathbb{N}} \mu^*(A_j^{(i)} - (E \cap C_i)) \leq \sum_{i \in \mathbb{N}} \tfrac{1}{j2^i} = \tfrac{1}{j}\end{aligned}$$

Since $E \subseteq \bigcup_{i \in \mathbb{N}} A_j^{(i)} \in \mathcal{A}_\sigma$, we've thus shown for all $j \in \mathbb{N}$ that there exists a set $A_j \in \mathcal{A}_\sigma$ satisfying that $\mu^*(A_j - E) \leq 1/j$ and $E \subseteq A_j$.

Finally, intersecting all those A_j like in part (b), we get our set satisfying the right-side of the implication.

Exercise 1.19: Let μ^* be an outer measure on X induced from a finite premeasure μ_0 defined on an algebra $\mathcal{A}$. If $E \subseteq X$, define the inner measure of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^\complement)$. Then E is μ^*-measurable if and only if $\mu^*(E) = \mu_*(E)$.

($\Longrightarrow$)
If E is μ^*-measurable, then we have $\mu^*(X \cap E) + \mu^*(X - E) = \mu^*(X)$. Because $\mu^*(X) = \mu_0(X)$, we thus have that $\mu_*(E) = \mu_0(X) - \mu^*(X - E) = \mu_*(E)$.

($\Longleftarrow$)
By part (a) of the previous exercise, we know there exists $A_j \in \mathcal{A}_\sigma$ satisfying that $E \subseteq A_j$ and $\mu^*(A_j) \leq \mu^*(E) + 1/j$.

Note that A_j is μ^*-measurable because $\mathcal{A}_\sigma \subseteq \mathcal{M}(\mathcal{A})$. This means that:

- $\mu_0(X) = \mu^*(X) = \mu^*(X \cap A_j) + \mu^*(X - A_j) = \mu^*(A_j) + \mu^*(A_j^\complement)$.
- $\mu^*(E^\complement) = \mu^*(E^\complement \cap A_j) + \mu^*(A_j^\complement \cap E^\complement) = \mu^*(A_j - E) + \mu^*(A_j^\complement)$.

Supposing that $\mu^*(E) = \mu_*(E) = \mu_0(X) - \mu^*(X - E)$ and plugging in the first bullet-pointed identity, we get that:

$$\mu^*(E^\complement) = \mu^*(A_j) + \mu^*(A_j^\complement) - \mu^*(E).$$

Substituting that into the second bullet-pointed identity, we have:

$$\mu^*(A_j) - \mu^*(E) = \mu^*(A_j - E).$$

And finally, using the inequality: $\mu^*(A_j) \leq \mu^*(E) + {}^1\!/\!_j$, we get $\mu^*(A_j - E) \leq {}^1\!/\!_j$.

Hence, we've shown that there exists a set $A_j \in \mathcal{A}_\sigma$ such that $E \subseteq A_j$ and $\mu^*(A_j - E) < {}^1\!/\!_j$ for all $j \in \mathbb{N}$. From there, we can proceed exactly like in part (b) of exercise 1.18. Pick such an A_j for all $j \in \mathbb{N}$ and then intersect them together. The result will be a set $B \in \mathcal{A}_{\sigma\delta}$ satisfying that $E \subseteq B$ and $\mu^*(B - E) = 0$. Since such a set exists, we know by the conclusion of part (b) of exercise 1.18 that E is μ^*-measurable.

Exercise 1.21: Let μ^* be an outer measure induced from a premeasure defined on an algebra $\mathcal{A}$ and $\overline{\mu}$ be the restriction of μ^* to the collection $\mathcal{M}$ of μ^*-measurable sets. Then $\overline{\mu}$ is saturated.

Let E be a locally $\overline{\mu}$-measurable set and choose any $F \subseteq X$. Given any $\varepsilon > 0$, by part (a) of exercise 1.18, we know that there exists a μ^*-measurable set $A \in \mathcal{A}_\sigma \subseteq \mathcal{M}$ such that $F \subseteq A$ and $\mu^*(A) \leq \mu^*(F) + \varepsilon$.

Assuming without loss of generality that $\mu^*(F)$ is finite, we thus know that $\mu^*(A) < \infty$. So, since E is locally $\overline{\mu}$-measurable, we have that $E \cap A \in \mathcal{M}$.

Now, we first note that because $F \cap E \subseteq A \cap E$ and $F - E \subseteq A - E$, we have that:

$$\mu^*(F \cap E) + \mu^*(F - E) \leq \mu^*(A \cap E) + \mu^*(A - E)$$

Next we note that:

- $A \cap (A \cap E) = A \cap E$
- $A \cap (A \cap E)^{\mathsf{C}} = (A \cap A^{\mathsf{C}}) \cup (A \cap E^{\mathsf{C}}) = A - E$

So: $\mu^*(A \cap E) + \mu^*(A - E) = \mu^*(A \cap (A \cap E)) + \mu^*(A \cap (A \cap E)^{\mathsf{C}}) = \mu^*(A)$.

And finally, since $\mu^*(A) \leq \mu^*(F) + \varepsilon$, we can thus conclude that:

$$\mu^*(F \cap E) + \mu^*(F - E) \leq \mu^*(F) + \varepsilon.$$

Taking $\varepsilon \to 0$, we get that $\mu^*(F \cap E) + \mu^*(F - E) \leq \mu^*(F)$ So, E is μ^*-measurable, meaning $E \in \mathcal{M}$.

Consider the collection $H = \{\emptyset, (a, b], (a, \infty) \mid -\infty \leq a < b < \infty\}$ of "half-open-intervals" of $\mathbb{R}$.

This forms an elementary family.

- We specified in the definition that $\emptyset \in H$.
- If $x \in (a, b] \cap (c, d] \neq \emptyset$, then we know $a < x < d$ and $c < x < b$. So $(a, b] \cap (c, d] = (\max(a, c), \min(b, d)] \in H$.

- Given $(a, b] \in H$, we have that $(a, b]^{\mathsf{C}} = (-\infty, a] \cup (b, \infty)$.

  For the sake of time, I'm ignoring edge cases of a right bound of infinity since they are still trivial.

By exercise 1.2, we know that $\mathcal{M}(H) = \mathcal{B}_{\mathbb{R}}$. And, by a previous proposition, we know that $\mathcal{A}$ equal to the collection of finite disjoint unions of H is an algebra.

Proposition: Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a monotonically increasing and right continuous function (meaning $\lim\limits_{t \to x^+} F(t) = F(x)$ for all $x \in \mathbb{R}$). Also, define $F(-\infty)$ and $F(\infty)$ respecting the monotonicity of F.

If $(a_j, b_j]$ for $j = 1, \ldots, n$ are disjoint intervals in H, define:

$$\mu_0(\bigcup_{j=1}^{n} (a_j, b_j]) = \sum_{j=1}^{n} F(b_j) - F(a_j)$$

Also let $\mu_0(\emptyset) = 0$. Then this is a premeasure.

Proof:

1. μ_0 is well defined.

   Suppose $(a_j, b_j]$ for $j = 1, \ldots, n$ are disjoint intervals in H satisfying that $(a, b] = \bigcup\limits_{j=1}^{n} (a_j b_j] = (a, b]$. Then after indexing those half intervals in a certain way, we must have that:

   $$a = a_1 < b_1 = a_2 < \ldots < b_{n-1} = a_n < b_{n-1} = b$$

   It follows that:

   $$F(b) - F(a) = \mu_0(\bigcup_{j=1}^{n} (a_j, b_j]) = \sum_{j=1}^{n} \mu_0((a_j, b_j]) = \sum_{j=1}^{n} F(b_j) - F(a_j)$$

   In other words, μ_0 is well defined for individual intervals.

   Now suppose I_i and J_j for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ are disjoint intervals in H satisfying that $\bigcup\limits_{i=1}^{m} I_i = \bigcup\limits_{j=1}^{n} J_j$.

   For each I_i, we can repeat the same reasoning as above with the collection of sets $I_i \cap J_j$ for $j = 1, \ldots, n$ in order to get that:

   $$\mu_0(I_i) = \sum_{j=1}^{n} \mu_0(I_i \cap J_j)$$

   Similarly, we can show for each J_j that:

   $$\mu_0(J_j) = \sum_{i=1}^{m} \mu_0(I_i \cap J_j)$$

   Thus: $\sum\limits_{i=1}^{m} \mu_0(I_i) = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \mu_0(I_i \cap J_j) = \sum\limits_{j=1}^{n} \mu(J_j)$.

2. If $(I_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ is a disjoint sequence satisfying that $\bigcup\limits_{j\in\mathbb{N}} I_j \in \mathcal{A}$, then:

$$\mu_0(\bigcup_{j\in\mathbb{N}} I_j) = \sum_{j=1}^{\infty} \mu_0(I_j).$$

Since $\bigcup\limits_{j\in\mathbb{N}} I_j \in \mathcal{A}$, we know it is equal to a finite union of disjoint intervals of H. By considering those intervals separately, we can thus assume without loss of generality that $\bigcup\limits_{j\in\mathbb{N}} I_j = (a,b]$.

Now it's obvious from the construction of μ_0 that μ_0 is additive. And since $(a,b] - \bigcup\limits_{j=1}^{m} I_j \in \mathcal{A}$ for all m, we thus know that:

$$\mu_0((a,b]) = \mu_0(\bigcup_{j=1}^{m} I_j) + \mu_0((a,b] - \bigcup_{j=1}^{m} I_j) \geq \mu_0(\bigcup_{j=1}^{m} I_j) = \sum_{j=1}^{m} \mu_0(I_j)$$

Taking the limit as $m \to \infty$, we get that: $\sum\limits_{j=1}^{\infty} \mu_0(I_j) \leq \mu_0(\bigcup\limits_{j\in\mathbb{N}} I_j)$.

Lecture 6 Notes: 10/15/2024

To show the reverse inequality, suppose $a, b \in \mathbb{R}$ (a.k.a. finite), and let $\varepsilon > 0$. Since F is right-continuous, there exists $\delta > 0$ such that $F(a+\delta) - F(a) < \varepsilon$. Similarly, given that $I_j = (a_j, b_j]$, there exists $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < \frac{\varepsilon}{2^j}$ for all $j \in \mathbb{N}$.

Next, note that the collection $\{(a_j, b_j + \delta_j)\}_{j\in\mathbb{N}}$ of open intervals covers the set $[a+\delta, b]$. Thus by compactness, there is a finite subcover. In other words,

$(a_1, b_1 + \delta_1), \dots, (a_N, b_N + \delta_N)$ cover $[a+\delta, b]$

Furthermore, by removing intervals in that finite subcover which are subsets of other intervals and by reindexing, we can assume that:

$b_j + \delta_j \in (a_{j+1}, b_{j+1} + \delta_{j+1})$ for all $j = 1, \dots, N-1$.

Then:

$$\begin{aligned}
\mu_0((a,b]) &= F(b) - F(a) \\
&< F(b) - F(a+\delta) + \varepsilon \\
&\leq F(b_N + \delta_N) - F(a_1) + \varepsilon \qquad \text{(since } F \text{ is monotone increasing)} \\
&= F(b_N + \delta_N) - f(a_N) + \sum_{j=1}^{N-1} (F(a_{j+1}) - F(a_j)) + \varepsilon \\
&\leq F(b_N + \delta_N) - f(a_N) + \sum_{j=1}^{N-1} (F(b_j + \delta_j) - F(a_j)) + \varepsilon \\
&\qquad \text{(again since } F \text{ is monotone increasing)} \\
&= \sum_{j=1}^{N} (F(b_j + \delta_j) - F(a_j)) + \varepsilon \\
&< \sum_{j=1}^{N} (F(b_j) + \tfrac{\varepsilon}{2^j} - F(a_j)) + \varepsilon < \sum_{j=1}^{\infty} \mu(I_j) + 2\varepsilon
\end{aligned}$$

Since ε is arbitrary, we've now shown the reverse inequality when a and b are finite. To extend this result to when $a = -\infty$ or $b = \infty$, note that the invervals $(a_j, b_j + \delta_j)$ cover $[-M + \delta, b]$ or $[a + \delta, M]$.

So, doing the same manipulations as before, since $\sum_{j=1}^{\infty} \mu(I_j) + 2\varepsilon$ is an upper bound of $\mu_0((-M, b])$ or $\mu_0((a, M])$, we know that the limit of $\mu_0((-M, b])$ or $\mu_0((a, M])$ as $M \to \infty$ will not exceed that upper bound. Then taking $\varepsilon \to 0$, we get the same result as before.

Theorem:

1. If F is a monotone increasing and right-continuous, there is a unique Borel measure μ_F on $\mathcal{B}_{\mathbb{R}}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all $a < b$.
2. If G is another monotone increasing and right-continuous, then $\mu_G = \mu_F$ if and only if $F - G$ is a constant.
3. If μ is a $\mathcal{B}_{\mathbb{R}}$ measure that is finite on all bounded sets, then we can define:
$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu((x, 0]) & \text{if } x < 0 \end{cases}$$

   Then F is monotone increasing and right-continuous with $\mu = \mu_F$.