Math 240A Notes (Professor: Luca Spolaor)

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Lecture 1 Notes: 9/26/2024

Given an indexed family of sets $\{X_\alpha\}_{\alpha\in A}$, we define its Cartesian Product to be:

$$\prod_{\alpha\in A} X_\alpha = \{f : A \longrightarrow \bigcup_{\alpha\in A} X_\alpha \mid f(\alpha \in X_\alpha)\}$$

A projection is a function $\pi_\alpha : \prod_{\alpha\in A} X_\alpha \longrightarrow X_\alpha$ satisfying that $f \mapsto f(\alpha)$.

If X, Y are sets, we define:

- $\operatorname{card}(X) \leq \operatorname{card}(Y)$ if there exists an injection $f : X \longrightarrow Y$.
- $\operatorname{card}(X) \geq \operatorname{card}(Y)$ if there exists a surjection $f : X \longrightarrow Y$.
- $\operatorname{card}(X) = \operatorname{card}(Y)$ if there exists a bijection $f : X \longrightarrow Y$.

Note that $\operatorname{card}(X) \leq \operatorname{card}(Y) \iff \operatorname{card}(Y) \geq \operatorname{card}(X)$. After all, given an injection in one direction, we can easily make a surjection in the other direction. Or given a surjection in one direction, we can (using A.O.C (axiom of choice)) easily make an injection in the other direction.

Also, if $\operatorname{card}(X) \leq \operatorname{card}(Y)$ and $\operatorname{card}(Y) \leq \operatorname{card}(X)$, then we know that $\operatorname{card}(Y) = \operatorname{card}(X)$.

Proof:
We know there exists $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ which are both injective. Hence, $g \circ f$ is an injection from X to $g(Y) \subseteq X$. By an exercise done in my math journal on page 8, we thus there exists a bijection h from X to $g(Y)$. And letting g^{-1} be any left-inverse of g, we then have that $g^{-1} \circ h$ is a bijection from X to Y.

We say X has the cardinality of the continuum if $\operatorname{card}(X) = \operatorname{card}(\mathbb{R})$.

Proposition: $\operatorname{card}(\mathcal{P}(\mathbb{N})) = \operatorname{card}(\mathbb{R})$.

Our textbook goes about proving this by constructing two functions: an injection and a surjection, from $\mathcal{P}(\mathbb{N})$ to $\mathbb{R}$ based on the binary expansion of any real number. That way, we know that $\operatorname{card}(\mathcal{P}(\mathbb{N})) \leq \operatorname{card}(\mathbb{R})$ and $\operatorname{card}(\mathcal{P}(\mathbb{N})) \geq \operatorname{card}(\mathbb{R})$.

Given a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ we know there exists: $\limsup x_n = \inf_{k\geq 1}(\sup_{n\geq k} x_n)$ and $\liminf x_n = \sup_{k\geq 1}(\inf_{n\geq k} x_n)$.

Also, given a function $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, we can define:

$$\limsup_{x\to a} f(x) = \inf_{\delta>0}\left(\sup_{0<|x-a|<\delta} f(x)\right).$$

If X is an arbitrary set and $f : X \longrightarrow [0, \infty]$, we define:

$$\sum_{x\in X} f(x) = \sup\left\{\sum_{x\in F} f(x) \mid F \subseteq X \ s.t.\ F \text{ is finite}\right\}.$$

Cool Proposition from textbook (not covered in lecture):

Let $A = \{x \in X \mid f(x) > 0\}$. If A is uncountable, then $\sum\limits_{x\in X} f(x) = \infty$.

If A is countably infinite and $g : \mathbb{N} \longrightarrow A$ is a bijection, then $\sum\limits_{x\in X} f(x) = \sum\limits_{n=1}^{\infty} f(g(n))$.

Proof of first statement:

$A = \bigcup\limits_{n\in\mathbb{N}} A_n$ where $A_n = \{x \in X \mid f(x) > \frac{1}{n}\}$.

If A is uncountable, we must have that some A_n is uncountable. But then for any finite set $F \subseteq X$, we have that $\sum\limits_{x\in F} f(x) > \frac{\mathrm{card}(F)}{n}$. So $\sum\limits_{x\in X} f(x)$ is unbounded.

A metric space (X, ρ) is a set X equipped with a distance function $\rho : X \times X \longrightarrow [0, \infty)$. We denote the open ball of radius r about x to be $B(r, x) = \{y \in X \mid \rho(x, y) < r\}$. And you remember our definitions from 140A... right?

Proposition 0.21: Every open set in $\mathbb{R}$ is a countable union of disjoint open intervals.

We proved this as part of a homework exercise in Math 140A.

Given a metric space (X, ρ), an element $x \in X$, and sets $F, E \subseteq X$, we can define:

- $\rho(x, E) = \rho_E(x) = \inf\{\rho(x, y) \mid y \in E\}$.
- $\rho(F, E) = \inf\{\rho_E(y) \mid y \in F\}$.

Exercise: $g(x, E) = 0 \iff x \in \overline{E}$.

Proof:

If $\inf\{\rho(x, y) \mid y \in E\} = 0$, then there exists a sequence $\{y_n\}$ in E such that $\rho(x, y_n) \to 0$. This implies $x \in \overline{E}$. Similarly, if $x \in \overline{E}$, we can construct a sequence $\{y_n\}$ such that $\rho(x, y_n) < \frac{1}{n}$ for all n. Then:

$$0 \le \inf\{\rho(x, y) \mid y \in E\} \le \inf\{\rho(x, y_n) \mid n \in \mathbb{N}\} = 0.$$

Given a subset E of a metric space (X, ρ), we define:

$$\mathrm{diam}(E) = \sup\{\rho(x, y) \mid x, y \in E\}.$$

If $\mathrm{diam}(E) < \infty$, we say E is <u>bounded</u>. If $\forall\varepsilon > 0$, E can be covered by finitely many balls of radius ε, then we say E is <u>totally bounded</u>.

Exercise: E being totally bounded implies E is bounded.

Pick $\varepsilon > 0$ and let $\{z_1, \ldots, z_n\}$ be the set of points such that $E \subseteq \bigcup_{k=1}^{n} B(\varepsilon, z_n)$.

Then given any $x, y \in E$, we can assume that $x \in B(\varepsilon, z_i)$ and $y \in B(\varepsilon, z_j)$. So, $\rho(x, y) \leq \rho(x, z_i) + \rho(z_i, z_j) + \rho(z_j, y) < 2\varepsilon + \max\{\rho(z_i, z_j) \mid 1 \leq i, j \leq n\}$.

The converse is not generally true. For instance, if you use the discrete metric, then any set with more than one element will have a diameter of 1. But if $0 < \varepsilon < 1$, then it will be impossible to cover an infinite set with finitely many balls.

Lecture 2 Notes: 10/1/2024

Proposition: Suppose E is a subset of a metric space (X, ρ). Then the following are equivalent.

1. E is complete and totally bounded
2. All sequences $(x_n) \subseteq E$, have a convergent subsequence.
3. For all open covers $\{V_\alpha\}_{\alpha \in A}$ of E, there exists $V_{\alpha_1}, \ldots, V_{\alpha_n}$ such that $E \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$.

Proof:
(1) $\Longrightarrow$ (2):

Lemma:
If E is totally bounded and $F \subseteq E$, then F is totally bounded.

Given any $\varepsilon > 0$, let $\{x_1, \ldots, x_n\}$ be a subset of E such that $E \subseteq \bigcup_{i=1}^{n} B(\varepsilon/2, x_i)$. Then consider the collection of sets:

$$\{F \cap B(\varepsilon/2, x_i)\} - \{\emptyset\}.$$

We know the diameter of each $F \cap B(\varepsilon/2, x_i)$ is at most ε. So in each set, pick $y_i \in F \cap B(\varepsilon/2, x_i)$. Then for some $m \leq n$:

$$F \subseteq \bigcup_{i=1}^{m} B(\varepsilon, y_i)$$

Let $A_1 = E$. Then for $k \geq 2$ we recursively define A_k as follows:

Assuming $A_{k-1} \cap (x_n)_{n\in\mathbb{N}}$ is infinite and A_{k-1} is totally bounded, choose $\{y_1, \ldots, y_m\}$ in A_k such that $A_k \subseteq \bigcup_{i=1}^{m} B(2^{-n}, y_i)$. Importantly, since $(x_n)_{n\in\mathbb{N}} \cap A_{k-1}$ is infinite, we know one of those open balls contains infinitely many points in our sequence. So set A_k equal to that ball intersected with E. Note that by our lemma, A_k is totally bounded.

Now pick any x_{n_1} and then for all $k \geq 2$ pick $x_{n_k} \in A_k$ such that $n_k > n_{k-1}$. That way, $(x_{n_k})_{k\in\mathbb{Z}_+}$ is a subsequence of $(x_n)_{n\in\mathbb{Z}_+}$. Also, we know that $(x_{n_k})_{k\in\mathbb{Z}_+}$ is Cauchy. Hence, since E is complete, we know that it converges to some x in E.

(2) $\Longrightarrow$ (1):

Firstly, suppose E is not complete. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ that is Cauchy but does not converge in E. Importantly, because $(x_n)_{n\in\mathbb{N}}$ is Cauchy, if there was a convergent subsequence, we know the limit of that subsequence would have to be the limit of the whole sequence. But that doesn't exist. So, we know (2) can't be true.

Secondly, suppose E is not totally bounded. Then there exists $\varepsilon > 0$ such that it is impossible to cover E in balls of radius ε. So, we can recursively define a sequence $(x_n)_{n\in\mathbb{N}}$ in E satisfying that:

$$x_n \in E - \bigcup_{i=1}^{n-1} B(\varepsilon, x_i).$$

Importantly, for all natural numbers $n \neq m$, we have that $\rho(x_n, x_m) \geq \varepsilon$. So, it is impossible to find a convergent subsequence of (x_n), meaning (2) is false.

(1) and (2) $\Longrightarrow$ (3):

Let $\{V_\alpha\}_{\alpha\in A}$ be an open cover of E.

Suppose for the sake of contradiction that for all $n \in \mathbb{N}$, there is a ball B_n of radius 2^{-n} centered in E such that $B_n \cap E \neq \emptyset$ but $B_n \not\subseteq V_\alpha$ for all $\alpha \in A$. Then we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $x_n \in B_n \cap E$ for all $n \in \mathbb{N}$. By (2), we know there is a subsequence that converges to some $x \in E$. Importantly, we know $x \in V_\alpha$ for some $\alpha \in A$, and because V_α is open, there is $\varepsilon > 0$ such that $B(\varepsilon, x) \subseteq V_\alpha$. But now we get a contradiction because by picking n such that $2^{-n} < \varepsilon/3$ and $\rho(x, x_n) < \varepsilon/3$, we have for all $y \in B_n$ that:

$$\rho(x, y) \leq \rho(x, x_n) + \rho(x_n, y) < 2^{-n} + 2^{-n+1} < \varepsilon$$

So $B_n \subseteq B(\varepsilon, x) \subseteq V_\alpha$.

We've thus shown that for some $n \in N$, all balls of radius 2^{-n} centered in E are contained by some V_α. And assuming (1), we can cover E with finitely many balls of radius 2^{-n} It follows that by picking a V_α containing a ball for each ball covering E, we've found a finite covering E using the sets in $\{V_\alpha\}_{\alpha\in A}$.

(3) $\Longrightarrow$ (2):

Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in E with no convergent subsequence. Then for each $x \in E$, there must exist $\varepsilon_x > 0$ such that $B(\varepsilon_x, x) \cap (x_n)_{n\in\mathbb{N}}$ is finite. (If ε_x didn't exist, we could construct a Cauchy subsequence converging to x).

But now $\{B(\varepsilon_x, x)\}_{x\in E}$ is an open cover of E with no finite subcover of E because it will take an infinite cover to cover all of $(x_n)_{n\in\mathbb{N}}$.

If E satisfies all three of the above properties, we say E is compact.

Corollary: $K \subseteq \mathbb{R}^n$ is compact iff it's closed and bounded.

Roughly speaking, we want a measure to be a function $\mu : \mathcal{P}(\mathbb{R}^n) \longrightarrow [0, \infty)$ such that $E \mapsto \mu(E) =$ "the area of E". Also, we would like it if:

(i) $\mu([0,1)^n) = 1$

(ii) $\mu($rotation, translation, or reflection of $A) = \mu(A)$

(iii) $\mu(\bigcup\limits_{i=1}^{\infty} A_i) = \sum\limits_{i=1}^{\infty} \mu(A_i)$ if $A_i \cap A_j \neq \emptyset \Longrightarrow i = j$.

Unfortunately, the properties as written above are inconsistent.

Vitali Sets:
Defining $x \sim y$ iff $x - y \in \mathbb{Q}$, let $N \subseteq [0,1)$ be a set such that $N \cap [x]$ has precisely one element for all $x \in \mathbb{R}$. Next let $R = [0,1) \cap \mathbb{Q}$, and for all $r \in R$ define:

$$N_r = \{x + r \mid x \in N \cap [0, 1-r)\} \cup \{x + r - 1 \mid x \in N \cap [1-r, 1)\}.$$

Importantly, note that $N_r \subseteq [0,1)$. Plus, the two sets being unioned over to make N_r are both disjoint and can be translated around so that they are still disjoint but their union forms N. Hence assuming $\mu : \mathcal{P}(\mathbb{R}^n) \longrightarrow [0,\infty)$ satisfying (ii) and (iii), we know $\mu(N_r) = \mu(N)$.

Also, for all $y \in [0,1)$, if $x \in N \cap [y]$, we know that $y \in N_r$ where $r = x - y$ if $x \geq y$, or $r = x - y + 1$ if $x < y$. Hence, $[0,1) = \bigcup\limits_{r\in R} N_r$.

Also, given any N_r and N_s, if $x \in N_r \cap N_s$, then we'd be able to show that both $x - r$ or $x - r + 1$ and $x - s$ or $x - s + 1$ are distinct elements of N in the same equivalence class, which contradicts how we defined N.

You work through the scratch work of the different cases on your own! :P

So supposing μ satisfies (i) and (iii) and because R is countable, we have that:

$$1 = \sum_{r\in R} \mu(N_r) = \sum_{r\in R} (N) = 0 \text{ or } \infty.$$

This is a contradiction.

Furthermore, the problem is not the countable union property as is demonstrated by the Banach-Tarsky paradox:

Theorem: Let U and V be arbitrary bounded sets in $\mathbb{R}^n$ where $n \geq 3$. Then there exists $E_1, \ldots, E_N, F_1, \ldots, F_N$ in $\mathbb{R}^n$ such that:

- $E_i \cap E_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{N} E_i = U$
- $F_i \cap F_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{N} F_i = V$
- E_i and F_i are congruent for all $i \in \{1, \ldots, N\}$.

Supposing that $\mu(E_j)$ and $\mu(F_j)$ exists for all j and that μ satisfies (i), (ii), and (iii) except only for finite unions, then that would suggest all sets have the same "area", which we know doesn't make sense.

What we will do to fix this issue is only define μ on a subset of $\mathcal{P}(\mathbb{R}^n)$.

Let $X \neq \emptyset$. An algebra of sets in X is a nonempty collection $\mathcal{A} \subseteq \mathcal{P}(X)$ which is closed under finite unions and complements. If is $\mathcal{A}$ is also closed under countable unions, we say $\mathcal{A}$ is a σ-algebra.

Observations:

1. Algebras of sets are closed under finite intersections and σ-algebras are closed under countable intersection. (This also means algebras of sets are closed under set differences.)

   This is because $\bigcap_{n\in\mathbb{N}} A_n = \left(\bigcup_{n\in\mathbb{N}} A_n^{\mathsf{C}}\right)^{\mathsf{C}}$.

2. If $\mathcal{A}$ is closed under disjoint countable union, then it's closed under arbitrary countable unions.

   This is because $\bigcup_{n\in\mathbb{N}} A_n = A_1 \cup \bigcup_{n\geq 2}\left(A_n \cap \left(\bigcup_{i=1}^{n-1} A_i\right)^{\mathsf{C}}\right)$

3. If $\{\mathcal{E}_\alpha\}_{\alpha\in A}$ is a collection of σ-algebras, then $\bigcap_{\alpha\in A} \mathcal{E}_\alpha$ is a σ-algebra.

   This is pretty trivial to prove. It should remind you of topologies.

Exercise 1.1: A family of sets $\mathcal{R} \subseteq \mathcal{P}(X)$ is called a ring if it is closed under finite unions and difference. If $\mathcal{R}$ is also closed under countable unions, it is called a σ-ring.

(a) Rings are closed under finite intersections and σ-rings are closed under countable intersections.

If $\mathcal{R}$ is a ring and $A_1, \ldots, A_n \in \mathcal{R}$, then:

$$\bigcap_{i=1}^{n} A_n = A_1 - \bigcup_{i=2}^{n}(A_1 - A_i) \in \mathcal{R}$$

This is because each $A_1 - A_i \in \mathcal{R}$, meaning $\bigcup_{i=2}^{n}(A_1 - A_i) \in \mathcal{R}$, and so finally $A_1 - \bigcup_{i=2}^{n}(A_1 - A_i) \in \mathcal{R}$.

If $\mathcal{R}$ is a σ-algebra, we can replace the finite intersection and union used in the prior reasoning with a countable intersection and union.

(b) If $\mathcal{R}$ is a ring (or σ-ring), then $\mathcal{R}$ is an algebra (or σ-algebra) iff $X \in \mathbb{R}$.

($\Longrightarrow$) Suppose $\mathcal{R}$ is an algebra. Then note that $\emptyset \in \mathcal{R}$ because for any $A \in \mathcal{R}$, $A - A \in \mathcal{R}$. So taking complements, we get that $X \in \mathcal{R}$.

($\Longleftarrow$) Suppose $X \in \mathcal{R}$. Then for any $A \in \mathcal{R}$, we know that $A^{\mathsf{C}} = X - A \in \mathcal{R}$. So $\mathcal{R}$ is an algebra (or σ-algebra).

(c) If $\mathcal{R}$ is a σ-ring, then $\mathcal{A} = \{E \subseteq X \mid E \in \mathcal{R}$ or $E^{\mathsf{C}} \in \mathcal{R}\}$ is a σ-algebra.

To start, we know that $\mathcal{A}$ is closed under complements because for any $A \in \mathcal{A}$,

$$A \in \mathcal{R} \Longrightarrow (A^{\mathsf{C}})^{\mathsf{C}} \in \mathcal{R} \Longrightarrow A^{\mathsf{C}} \in \mathcal{A}$$
$$A \notin \mathcal{R} \Longrightarrow A^{\mathsf{C}} \in \mathcal{R} \Longrightarrow A^{\mathsf{C}} \in \mathcal{A}$$

Also, let $(E_n)_{n\in\mathbb{N}}$ be a countable collection of sets in $\mathcal{A}$. Then define $A = \{n \in \mathbb{N} \mid E_n^{\mathsf{C}} \notin \mathcal{R}\}$ and $B = \{n \in \mathbb{N} \mid E_n^{\mathsf{C}} \in \mathcal{R}\}$. Clearly, we have that:

$$\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in A} E_n \cup \bigcup_{n\in B} E_n = \bigcup_{n\in A} E_n \cup \bigcup_{n\in B} (E_n^{\mathsf{C}})^{\mathsf{C}}$$

Also $\bigcup_{n\in B} (E_n^{\mathsf{C}})^{\mathsf{C}} = \left(\bigcap_{n\in B} E_n^{\mathsf{C}}\right)^{\mathsf{C}}$, and by part (a), we know that $E_B \coloneqq \bigcap_{n\in B} E_n^{\mathsf{C}} \in \mathcal{R}$.

Similarly, we know $E_A \coloneqq \bigcup_{n\in A} E_n \in \mathcal{R}$. So, we've shown that $\bigcup_{n\in\mathbb{N}} E_n = E_A \cup E_B^{\mathsf{C}}$ where $E_A, E_B \in \mathcal{R}$.

Finally, note that $E_A \cup E_B^{\mathsf{C}} = (E_B - E_A)^{\mathsf{C}}$. Since $E_B - E_A \in \mathcal{R}$, we know that $(E_B - E_A)^{\mathsf{C}} \in \mathcal{A}$.

(d) If $\mathcal{R}$ is a σ-ring, then $\mathcal{A} = \{E \subseteq X \mid E \cap F \in \mathcal{R}$ for all $F \in \mathcal{R}\}$ is a σ-algebra.

To start if $E \in \mathcal{A}$, then $E^{\mathsf{C}} \in \mathcal{A}$ because for all $F \in \mathcal{R}$ we have that:

$$E^{\mathsf{C}} \cap F = F - E = F - (E \cap F) \in \mathcal{R}.$$

Also, let $(E_n)_{n\in\mathbb{N}}$ be a countable collection of sets in $\mathcal{A}$. Then for all $F \in \mathcal{R}$, we have that $\left(\bigcup_{n\in\mathbb{N}} E_n\right) \cap F = \bigcup_{n\in\mathbb{N}} (E_n \cap F) \in \mathcal{R}$. So $\mathcal{A}$ is closed under countable union.

Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of sets. Since the intersection of σ-algebras is still a σ-algebra, we define $\mathcal{M}(\mathcal{E})$ to be the smallest σ-algebra that contains $\mathcal{E}$. In other words, $\mathcal{M}(\mathcal{E})$ is the intersection of all σ-algebras that contain $\mathcal{E}$.

We call $\mathcal{M}(\mathcal{E})$ the σ-algebra generated by $\mathcal{E}$.

Lemma: if $\mathcal{E} \in \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$.

Let (X, ρ) be a metric space. We define the Borel σ-algebra on X: $\mathcal{B}_X$, to be the σ-algebra generated by the collection of all open sets, or equivalently the collection of all closed sets.

- A set is G_δ if it is a countable intersection of open sets.
- A set is F_σ if it is a countable union of closed sets.
- A set is $G_{\delta\sigma}$ if it is a countable union of G_δ sets.
- A set is $F_{\sigma\delta}$ if it is a countable intersection of F_σ sets.

You can hopefully see the pattern. Also the professor isn't sure how much we'll use this δ and σ notation in class.

Exercise 1.2: $\mathcal{B}_\mathbb{R}$ is generated by each of the following:

(a) the set of open intervals: $\mathcal{E}_1 = \{(a, b) \mid a < b\}$

(b) the set of closed intervals: $\mathcal{E}_2 = \{[a, b] \mid a < b\}$

(c) the set of half-open intervals:

  (i) $\mathcal{E}_3 = \{(a, b] \mid a < b\}$
  (ii) $\mathcal{E}_4 = \{[a, b) \mid a < b\}$

(c) the set of open rays:

  (i) $\mathcal{E}_5 = \{(a, \infty) \mid a \in \mathbb{R}\}$
  (ii) $\mathcal{E}_6 = \{(-\infty, a) \mid a \in \mathbb{R}\}$

(d) the set of closed rays:

  (i) $\mathcal{E}_7 = \{[a, \infty) \mid a \in \mathbb{R}\}$
  (ii) $\mathcal{E}_8 = \{(-\infty, a] \mid a \in \mathbb{R}\}$

Proof:
We trivially have that $\mathcal{M}(\mathcal{E}_1), \mathcal{M}(\mathcal{E}_2), \mathcal{M}(\mathcal{E}_5), \mathcal{M}(\mathcal{E}_6), \mathcal{M}(\mathcal{E}_7), \mathcal{M}(\mathcal{E}_8) \subseteq \mathcal{B}_\mathbb{R}$ since each of them contain either only open sets or only closed sets. As for the other inclusions, we must do more work.

(a) Note that $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$. Hence, a countable base of $\mathbb{R}$ is the set: $\mathcal{F} = \{(p-q, p+q) \subset \mathbb{R} \mid p, q \in \mathbb{Q} \text{ and } q > 0\}$. In other words, given any open set $E \subseteq \mathbb{R}$, there is a countable subcollection of $\mathcal{F}$ whose union is E.

To see why, let $x \in E$. Since E is open, there exists $r > 0$ with $B(r, x) \subseteq E$. Next, pick $p \in (x, x + \frac{r}{2}) \cap \mathbb{Q}$, followed by $q \in (p - x, r - p) \cap \mathbb{Q}$. Then $x \in (p-q, p+q) \in \mathcal{F}$ and $(p-q, p+q) \subseteq (x-r, x+r)$.

With that, we've now shown that for all $x \in E$, there exists $F \in \mathcal{F}$ such that $x \in F \subseteq E$. If we choose such an F_x for all $x \in E$, we then get that $E = \bigcup\limits_{x \in E} F_x$. So E is the union of a subcollection of $\mathcal{F}$. But since $\mathcal{F}$ is countable, the set $\{F_x \in \mathcal{F} \mid x \in E\}$ is also countable.

Importantly, $\mathcal{F} \subset \mathcal{E}_1$. So $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E}_1)$. However as shown above, we must have that $\mathcal{M}(\mathcal{F})$ includes all open sets. So by our lemma on the previous page, $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E}_1)$.

(b) Given any $E = (a, b) \in \mathcal{E}_1$, we can write that $E = \bigcup\limits_{n \in \mathbb{Z}_+} [a + \frac{1}{n}, b - \frac{1}{n}]$. Thus, $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_2)$, meaning $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_2)$.

(c) Remember that for these two, we still need to show that $\mathcal{M}(\mathcal{E}_1), \mathcal{M}(\mathcal{E}_2) \in \mathcal{B}_{\mathbb{R}}$.

(i) Firstly note that if $F = (a, b] \in \mathcal{E}_3$, then $= \bigcap\limits_{n \in \mathbb{Z}_+} (a, b + \frac{1}{n})$. So $\mathcal{E}_3 \subseteq \mathcal{M}(\mathcal{E}_1)$.

On the other hand, if $E = (a, b) \in \mathcal{E}_1$, we have that $E = \bigcup\limits_{n \in \mathbb{Z}_+} (a, b - \frac{1}{n}]$. So $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_3)$.

By our lemma on the previous page, we thus have that:

$$\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_1) = \mathcal{B}_{\mathbb{R}}.$$

(ii) Mostly identical reasoning as with $\mathcal{E}_3$ shows that:

$$\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_1) = \mathcal{B}_{\mathbb{R}}$$

(d)

(i) If $E = (a, b) \in \mathcal{E}_1$, then we know that:

$$E = (a, \infty) \cap \big(\bigcap_{n \in \mathbb{Z}_+} (b - \tfrac{1}{n}, \infty)\big)^{\mathsf{C}} \in \mathcal{M}(\mathcal{E}_5).$$

So $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_5)$, meaning $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_5)$.

(ii) Analogous reasoning to that with $\mathcal{E}_5$ shows that $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_6)$.

(e)

(i) If $E = (a, \infty) \in \mathcal{E}_6$, then we have that $E = \bigcup\limits_{n \in \mathbb{Z}_+} [a + \frac{1}{n}, \infty)$. So $\mathcal{E}_5 \subseteq \mathcal{M}(\mathcal{E}_7)$, meaning that $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{M}(\mathcal{E}_7)$.

(ii) Analogous reasoning as with $\mathcal{E}_7$ shows that $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_6) \subseteq \mathcal{M}(\mathcal{E}_8)$.

Exercise 1.3: Let $\mathcal{A}$ be an infinite σ-algebra on X.

(a) $\mathcal{A}$ contains an infinite sequence of disjoint sets.

By the Hausdorff maximum principle, we know there is a subcollection $\mathcal{S}$ of $\mathcal{A}$ which is simply ordered by proper subset and is not contained in any other collection of $\mathcal{A}$ which is simply ordered by proper subset.

We claim $\mathcal{S}$ can't be finite. For suppose $\mathcal{S} = \{A_1, \ldots, A_n\}$ is a sequence of sets in $\mathcal{A}$ simply ordered by proper subset which are indexed such that:

- $A_1 = \emptyset$
- $A_i \subset A_{i+1}$ for all $i \in \{1, \ldots, n-1\}$
- $A_n = X$.

(If $\mathcal{S}$ is maximal, we know $\emptyset, X \in \mathcal{S}$)

Then choose $B \notin \mathcal{A} - \mathcal{M}(\{A_1, \ldots, A_n\})$. We know we can do this because $\mathcal{M}(\{A_1, \ldots, A_n\})$ is finite while $\mathcal{A}$ is infinite.

Next let $k = \min\{i \in \{1, \ldots, n\} \mid B \subset A_i\}$. In other words, let k be such that $B \subset A_k$ but $B \not\subset A_{k-1}$. If $A_{k-1} \cap B \neq \emptyset$, then because $B \not\subseteq A_{k-1}$, we know that $A_{k-1} \subset A_{k-1} \cup B \subset A_k$. Meanwhile, if $A_{k-1} \cap B = \emptyset$, then note that: $A_{k-1} \cup B = A_k \Longrightarrow B = A_k - A_{k-1} \in \mathcal{M}(\{A_1, \ldots, A_n\})$. So we know that $A_{k-1} \cup B \neq A_k$. At the same time, since $B \neq \emptyset$, we know that $A_{k-1} \subset A_{k-1} \cup B \subset A_k$.

By transitivity, we know that $A_{k-1} \cup B$ is comparable via proper subset with A_i for all $i \in \{1, \ldots, n\}$. Hence, we've shown that $\mathcal{S} \cup \{A_{k-1} \cup B\}$ is a sequence of sets in $\mathcal{A}$ simply ordered by proper subset. But this contradicts that $\mathcal{S}$ is maximal.

Now that we know $\mathcal{S}$ is infinite, let $(E_n)_{n \in \mathbb{Z}_+}$ be a sequence of sets in $\mathcal{S}$ satisfying that $E_n \subset E_{n+1}$. Then we have that $(E_{n+1} - E_n)_{n \in \mathbb{Z}_+}$ is an infinite sequence of disjoint sets in $\mathcal{A}$.

(b) Show that $\mathrm{card}(\mathcal{A}) \geq \mathfrak{c}$.

Let $(E_n)_{n \in \mathbb{N}}$ be a sequence of disjoint sets in $\mathcal{A}$. Then if we define the map $f : [0,1]^{\mathbb{N}} \longrightarrow \mathcal{A}$ such that $(a_0, a_1, a_2, \ldots)$ is mapped to the union of all E_n such that $a_n = 1$, we have that f is an injection.

Hence, $\mathrm{card}(\mathcal{A}) \geq \mathrm{card}([0,1]^{\mathbb{N}})$. And since there is a trivial bijection from $[0,1]^{\mathbb{N}}$ and $\mathcal{P}(\mathbb{N})$, plus the fact that we proved early on in the class that $\mathrm{card}(\mathcal{P}(\mathbb{N})) = \mathrm{card}(\mathbb{R})$, we thus know that $\mathrm{card}(\mathcal{A}) \geq \mathfrak{c}$.

Exercise 1.4: An algebra $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is closed under countable increasing unions (meaning $E_1 \subseteq E_2 \subseteq \ldots$).

The rightward implication is true since $\mathcal{A}$ being a σ-algebra means that $\mathcal{A}$ is closed under all countable unions. As for showing the leftward implication, suppose $\{A_n\}_{n\in\mathbb{Z}_+}$ is a countable collection of sets in $\mathcal{A}$. Then for all $n \in \mathbb{Z}_+$, define $E_n = A_1 \cup \ldots \cup A_n$.

Since each E_n are finite unions of sets in $\mathcal{A}$, we know that each E_n is in $\mathcal{A}$. Also, we clearly have that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ In order to make the sets strictly increasing, let $S = \{1\} \cup \{k \in \mathbb{Z} \mid k > 1 \text{ and } E_k - E_{k-1} \neq \emptyset\}$. Then for any $n, m \in S$, we know that $n < m \Longrightarrow E_n \subset E_m$.

Finally, $\bigcup\limits_{n\in\mathbb{Z}_+} A_n = \bigcup\limits_{n\in\mathbb{Z}_+} E_n = \bigcup\limits_{n\in S} E_n$.

Importantly, S is either finite or countably infinite, and S consists of strictly increasing sets. So by the right hypothesis, we know $\bigcup\limits_{n\in S} E_n \in \mathcal{A}$. Hence, the union over $\{A_n\}_{n\in\mathbb{Z}_+}$ is in $\mathcal{A}$.

Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of nonempty sets, and define $X = \prod\limits_{\alpha\in A} X_\alpha$.

If $\mathcal{M}_\alpha$ is a σ-algebra in X_α for all $\alpha \in A$, then we define the product σ-algebra on X to be: $\bigotimes\limits_{\alpha\in A} \mathcal{M}_\alpha = \mathcal{M}(\{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{M}_\alpha \text{ and } \alpha \in A\})$.

To get a better geometric intuition for this definition, consider if $A = \{1, 2\}$. Then:

$$\begin{aligned}\bigotimes_{\alpha\in A} \mathcal{M}_\alpha &= \{\pi_1^{-1}(E_1) \mid E_1 \in \mathcal{M}_1\} \cup \{\pi_2^{-1}(E_2) \mid E_2 \in \mathcal{M}_2\} \\ &= \{E_1 \times X_2 \mid E_1 \in \mathcal{M}_1\} \cup \{X_1 \times E_2 \mid E_2 \in \mathcal{M}_2\}\end{aligned}$$

Also, the motivation for this definition is that $\bigotimes\limits_{\alpha\in A} \mathcal{M}_\alpha$ is the smallest σ-algebra where π_α is "measurable" for all α. We'll learn what that means shortly...

Proposition:

(i) A is countable implies $\bigotimes\limits_{\alpha\in A} \mathcal{M}_\alpha = \mathcal{M}(\{\prod\limits_{\alpha\in A} E_\alpha \mid \forall\alpha \in A,\ \ E_\alpha \in \mathcal{M}_\alpha\})$

Proof:
If $E_\alpha \in \mathcal{M}_\alpha$, then $\pi_\alpha^{-1}(E_\alpha) = \prod\limits_{\beta\in A} E_\beta$ where $E_\beta = X_\beta$ if $\beta \neq \alpha$ (and $E_\beta = E_\alpha$ if $\beta = \alpha$).

So $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{M}(\{\prod\limits_{\alpha\in A} E_\alpha \mid \forall\alpha \in A,\ \ E_\alpha \in \mathcal{M}_\alpha\})$

On the other hand, $\prod\limits_{\alpha\in A} E_\alpha = \bigcap\limits_{\alpha\in A} \pi_\alpha^{-1}(E_\alpha)$.

Since A is countable, we thus know that if $E_\alpha \in \mathcal{M}_\alpha$ for all $\alpha \in A$, then $\prod\limits_{\alpha\in A} E_\alpha \in \bigotimes\limits_{\alpha\in A} \mathcal{M}_\alpha$.

(ii) Suppose $\mathcal{M}_\alpha = \mathcal{M}(\mathcal{E}_\alpha)$ for all $\alpha \in A$. Then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by $\mathcal{F} = \{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{E}_\alpha \text{ and } \alpha \in A\}$.

Proof:

Since $\mathcal{F} \subseteq \{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{M}_\alpha \text{ and } \alpha \in A\}$, we trivially have that $\mathcal{M}(\mathcal{F}) \subseteq \bigotimes_{\alpha \in A} \mathcal{M}_\alpha$.

As for showing the other inclusion, define for each $\alpha \in A$:

$$\mathcal{F}_\alpha = \{E \subseteq X_\alpha \mid \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})\}.$$

Note that $\mathcal{F}_\alpha$ is a σ-algebra on X_α that contains $\mathcal{E}_\alpha$.

This is because for any $F \in \mathcal{F}_\alpha$ and $(E_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}_\alpha$, we know that:

- $(\pi_\alpha^{-1}(F))^{\mathsf{C}} = \pi_\alpha^{-1}(F^{\mathsf{C}})$
- $\bigcup_{n \in \mathbb{N}} \pi_\alpha^{-1}(E_n) = \left(\pi_\alpha^{-1}(\bigcup_{n \in \mathbb{N}} E_n)\right)$

Also, for any $E \subseteq X_\alpha$, $E \in \mathcal{E}_\alpha \Longrightarrow \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})$.

By definition, we thus know that $\mathcal{M}_\alpha \subseteq \mathcal{F}_\alpha$. So for all $\alpha \in A$ and $E_\alpha \in \mathcal{M}_\alpha$, we know that $E_\alpha \in \mathcal{F}_\alpha$, which means that $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{M}(\mathcal{F})$. So $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha \subseteq \mathcal{M}(\mathcal{F})$.

(iii) We can also combine the first two parts of this proposition. If A is countable and $\mathcal{M}_\alpha = \mathcal{M}(\mathcal{E}_\alpha)$ for all $\alpha \in A$, then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by:

$$\{\prod_{\alpha \in A} E_\alpha \mid \forall \alpha \in A, \;\; E_\alpha \in \mathcal{E}_\alpha\}$$

Lecture 3 Notes: 10/3/2024

Proposition: Let $X_1, \ldots, X_n$ be metric spaces, and define $X = \prod_{i=1}^{n} X_i$ to be the metric space equipped with the product metric.

The product metric defines the distance between any $\boldsymbol{x}, \boldsymbol{y} \in \prod_{i=1}^{n}$ to be the max distance between a coordinate of $\boldsymbol{x}$ and the corresponding coordinate in $\boldsymbol{y}$.

- $\bigotimes_{i=1}^{n} \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$.

  Proof:

  By the previous proposition: $\bigotimes_{i=1}^{n} \mathcal{B}_{X_i}$ is generated by the collection:

  $$\{\pi_i^{-1}(U_i) \mid i \in \{1, \ldots, n\} \text{ and } U_i \subseteq X_i \text{ is open}\}.$$

  Also, by the definition of a product topology, we know that each $\pi_i^{-1}(U_i)$ is open in X. So by the lemma on page 9, we know that $\bigotimes_{i=1}^{n} \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$.

- If each X_i is separable, then $\bigotimes_{i=1}^{n} \mathcal{B}_{X_i} = \mathcal{B}_X$.

  Proof:
  Let $C_i \subseteq X_i$ be countable with $\overline{C_i} = X_i$ for all $i \in \{1, \ldots, n\}$. Then define $\mathcal{E}_i = \{B(p, x) \mid x \in C_i \text{ and } p \in \mathbb{Q}_+\}$ for each i. Since $\mathcal{E}_i$ is countable and all open sets in X_i are the union of a subcollection of $\mathcal{E}_i$, we know that any open set in X_i is also in $\mathcal{M}(\mathcal{E}_i)$. So, $\mathcal{B}_{X_i} \subseteq \mathcal{M}(\mathcal{E}_i)$. And since $\mathcal{E}_i$ contains only open sets of X_i, the reverse inclusion holds too.

  Also, $C = \prod_{i=1}^{n} C_i$ is a countable dense subset of X.

  Defining $\mathcal{E} = \{B(p, \boldsymbol{x}) \mid \boldsymbol{x} \in C \text{ and } p \in \mathbb{Q}_+\}$, we have that $\mathcal{E}$ is countable and any open set in X is also in $\mathcal{M}(\mathcal{E})$. So, $\mathcal{B}_X \subseteq \mathcal{M}(\mathcal{E})$. And like before since $\mathcal{E}$ contains only open sets of X, the reverse inclusion holds too.

  But now note that given, $B(p, (x_1, \ldots, x_n)) \in \mathcal{E}$, we know that $B(p, (x_1, \ldots, x_n)) = \prod_{i=1}^{n} B(p, x_i)$ where $(p, x_i) \in \mathcal{E}_i$ for all i.

  So applying part 3 of the previous proposition and the lemma on page 9:
  $$\mathcal{B}_X = \mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}\left(\prod_{i=1}^{n} E_i \mid E_i \in \mathcal{E}_i \text{ for all } i\right) = \bigotimes_{i=1}^{n} \mathcal{M}(\mathcal{E}_i) = \bigotimes_{i=1}^{n} \mathcal{B}_{X_i}$$

Corollary: $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{i=1}^{n} \mathcal{B}_{\mathbb{R}}$.

This is because the product metric ρ_1 of $\prod_{i=1}^{n} \mathbb{R}$ is equivalent to the standard metric ρ_2 of $\mathbb{R}^n$, meaning that:
$$\exists C, C' > 0 \text{ such that } C\rho_1 \leq \rho_2 \leq C'\rho_1.$$

In the specific case of this corollary, set $C = \sqrt{1/n}$ and $C' = 1$.

The fact relevant here is that given the metrics ρ_1, ρ_2 on a set X, if ρ_1 is equivalent to ρ_2, then (X, ρ_1) and (X, ρ_2) have the same open sets (this is really trivial to prove).

An elementary family is a collection $\mathcal{E}$ of subsets of a set X such that:

- $\emptyset \in \mathcal{E}$
- If $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$.
- If $E \in \mathcal{E}$, then E^{C} is a finite disjoint union of members of $\mathcal{E}$.

If $\mathcal{E}$ is an elementary collection, then $\mathcal{A}$ equal to the collection of finite disjoint unions of $\mathcal{E}$ is an algebra.

Proof:
Firstly, given any $A, B \in \mathcal{E}$, we have that $A \cup B = (A \cap B^{\mathsf{C}}) \cup B$. Also, $(B - A) = (A \cap \bigcup_{i=1}^{k} C_i)$ where each C_i is disjoint. This shows that $A - B \in \mathcal{E}$. So $A \cup B$ is a finite union of disjoint sets in $\mathcal{E}$.

By induction, we get that for any $A_1, \ldots, A_n \in \mathcal{E}$, $A_1 \cup \ldots \cup A_n$ is a finite union of disjoint sets in $\mathcal{E}$. So $\mathcal{A}$ actually equals the set of all finite unions of $\mathcal{E}$. It follows that $\mathcal{A}$ is closed under finite unions.

I really don't want to write down the proof that $\mathcal{A}$ is closed under complements. It's what you would expect but just heavy on notation.

Exercise 1.5: If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

For the sake of convenience, I will write the union of σ-algebras generated by countable subsets of $\mathcal{E}$ as: $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

To start, since each $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E})$, we trivially know $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E}) = \mathcal{M}$. On the other hand, $\mathcal{E} \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ since each countable $\mathcal{F} \subseteq \mathcal{E}$ is contained in $\mathcal{M}(\mathcal{F}) \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$. So, if we can show that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is a σ-algebra, then we will know that: $\mathcal{M} = \mathcal{M}(\mathcal{E}) \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

Fortunately, it's trivial to show that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is closed under complements. Given any $E \in \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$, we know there exists $\mathcal{M}(\mathcal{F})$ with $E \in \mathcal{M}(\mathcal{F})$. Then $E^{\mathsf{C}} \in \mathcal{M}(\mathcal{F}) \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

Meanwhile, the proof that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is closed under countable unions is more involved:

Suppose $\{E_n\}_{n \in \mathbb{N}}$ is a countable collection of sets in $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$. Then for each $n \in \mathbb{N}$, there exists $\mathcal{F}_n$ such that $E_n \in \mathcal{M}(\mathcal{F}_n)$. Importantly, $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ is still countable. So, setting $\mathcal{F}' = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, we have that:

$$\mathcal{M}(\mathcal{F}') \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$$

Since $\mathcal{F}_n \subseteq \mathcal{F}'$ for all n, we know that $\mathcal{M}(\mathcal{F}_n) \subseteq \mathcal{M}(\mathcal{F}')$ for all n. So, $\{E_n\}_{n \in \mathbb{N}}$ is contained in $\mathcal{M}(\mathcal{F}')$. It follows that $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{M}(\mathcal{F}') \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

Let $X \neq \emptyset$ and $\mathcal{M}$ be a σ-algebra on X. A measure $\mu : \mathcal{M} \longrightarrow [0, \infty]$ is a function satisfying that:

- $\mu(\emptyset) = 0$
- $\mu(\bigcup\limits_{j=1}^{\infty} E_j) = \sum\limits_{j=1}^{\infty} \mu(E_j)$ if $E_j \in \mathcal{M}$ for all j and $E_j \cap E_i = \emptyset$ for all $i \neq j$

$(X, \mathcal{M})$ is called a measurable space and $(X, \mathcal{M}, \mu)$ is called a measure space.

Let $(X, \mathcal{M}, \mu)$ be a measure space.

- μ is called finite if $\mu(X) < \infty$.

  It follows if μ is finite that $\mu(E) < \infty$ for all $E \in \mathcal{M}$ since $E \subseteq X$.
  In probability theory, most measure spaces are finite.

- μ is called σ-finite if $X = \bigcup\limits_{j=1}^{\infty} E_j$, such that $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j.

- μ is called semifinite if $\forall E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $F \subset E$ such that $F \in \mathcal{M}$, and $0 < \mu(F) < \infty$.

Example: Let $X \neq \emptyset$ and $\mathcal{M} = \mathcal{P}(X)$. Then given a function $\rho : X \longrightarrow [0, \infty]$, $\mu(E) = \sum\limits_{x \in E} \rho(x)$ is a measure.

- μ is semifinite if and only if $f(x) < \infty$ for all $x \in X$.
- μ is σ-finite if and only if it is semifinite and $\{x \in X \mid f(x) > 0\}$ is countable.

If $\rho(x) = 1$ for all x, then μ is called the counting measure.

If $\rho(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0, \end{cases}$ then μ is called the Dirac measure at x_0: δ_{x_0}.

Theorem: Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:

1. If $E, F \in \mathcal{M}$ with $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
2. If $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}$, then $\mu(\bigcup\limits_{j=1}^{\infty} E_j) \leq \sum\limits_{j=1}^{\infty} \mu(E_j)$.
3. If $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}$ with $E_j \subseteq E_{j+1}$ for all $j \in \mathbb{N}$, then $\mu(\bigcup\limits_{j=1}^{\infty} E_j) = \lim\limits_{j \to \infty} \mu(E_j)$.
4. If $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}$ and $\mu(E_1) < \infty$ and $E_{j+1} \subseteq E_j$ for all $j \in \mathbb{N}$, then $\mu(\bigcap\limits_{j=1}^{\infty} E_j) = \lim\limits_{j \to \infty} \mu(E_j)$.

Proofs:
(1) Suppose $E, F \in \mathcal{M}$ with $E \subseteq F$. Then $F = (F - E) \cup E$ is a disjoint union of sets in $\mathcal{M}$, meaning $\mu(F) = \mu(F - E) + \mu(E) \geq \mu(E)$.

(2) Set $F_1 = E_1$ and $F_m = E_m - \bigcup_{i=1}^{m-1} E_i$ for all $m > 1$. Then $(F_i)_{i\in\mathbb{N}}$ is pairwise disjoint and $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$. So $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(F_i)$. On the other hand, $F_i \subseteq E_i$ for all i. So $\sum_{i=1}^{\infty} \mu(F_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$

(3) Setting $E_0 = \emptyset$, we have that $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i - E_{i-1})$. Also, $\mu(E_n) = \sum_{i=1}^{n} \mu(E_i - E_{i-1})$. So:

$$\lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i - E_{i-1}) = \sum_{i=1}^{\infty} \mu(E_i - E_{i-1}) = \mu(\bigcup_{i=1}^{\infty} E_i).$$

(4) Let $F_j = E_1 - E_j$ for all $j \in \mathbb{N}$. Then for all $j \in \mathbb{N}$, $F_j \subseteq F_{j+1}$, $\mu(E_1) = \mu(F_j) + \mu(E_j)$, and $\bigcup_{j=1}^{\infty} F_j = E_1 - \bigcap_{j=1}^{\infty} E_j$. We can thus conclude that:

$$\begin{aligned}\mu(E_1) &= \mu(\bigcap_{j=1}^{\infty} E_j) + \mu(\bigcup_{j=1}^{\infty} F_j) \\ &= \mu(\bigcap_{j=1}^{\infty} E_j) + \lim_{j\to\infty}(F_j) = \mu(\bigcap_{j=1}^{\infty} E_j) + \lim_{j\to\infty}(\mu(E_1) - \mu(E_j))\end{aligned}$$

Since $\mu(E_1) < \infty$, we can subtract it out of the expression to get: $\mu(\bigcap_{j=1}^{\infty} E_j) - \lim_{j\to\infty}(\mu(E_j)) = 0$. Also, we know $\mu(\bigcap_{j=1}^{\infty} E_j) < \infty$ since it's a subset of E_j. So, we can rearrange to get: $\mu(\bigcap_{j=1}^{\infty} E_j) = \lim_{j\to\infty}(\mu(E_j))$.

Exercise 1.9: If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then we have that $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

We know $\mu(E) = \mu(E - f) + \mu(E \cap F)$ and $\mu(F) = \mu(F - E) + \mu(F \cap E)$. Adding those equations together we get that:

$$\begin{aligned}\mu(E) + \mu(F) &= (\mu(E - F) + \mu(E \cap F) + \mu(F - E)) + \mu(E \cap F) \\ &= \mu(E \cup F) + \mu(E \cap F).\end{aligned}$$

Exercise 1.14: If μ is a semifinite measure and $\mu(E) = \infty$, then for any $C > 0$ there exists $F \subset E$ in $\mathcal{M}$ with $C < \mu(F) < \infty$.

Let S be the set of $C > 0$ for which there exists $F \subset E$ in $\mathcal{M}$ with $C < \mu(F) < \infty$. By the definition of semifiniteness, we know S isn't empty. Meanwhile, if for some C we had that there didn't exist a set $F \subset E$ in $\mathcal{M}$ with $C < \mu(F) < \infty$, then we'd know that S is bounded above. Hence, we'd know there exists $\alpha = \sup(S)$.

Now firstly, for all $n \in \mathbb{N}$, choose $G_n \subset E$ in $\mathcal{M}$ such that $\alpha - \frac{1}{n} < \mu(G_n) < \infty$. After that, define $F_n = \bigcup\limits_{i=1}^{n} G_i$ for all $n \in \mathbb{N}$. Since $\mathcal{M}$ is closed under finite unions, we know each F_n is in $\mathcal{M}$. So then observe:

1. $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$
2. For each $n \in \mathbb{N}$, $\alpha - \frac{1}{n} < \mu(F_n) \leq \alpha$

This is because for each $n \in \mathbb{N}$, $\mu(F_n) < \sum\limits_{i=1}^{n} \mu(G_i)$ which is a finite sum of finite quantities. So $\mu(F_n) < \infty$. At the same time, $F_n \subset E$ since each G_i is a subset of E (we know it is a proper subset because it has a different measure than E). So, if $\mu(F_n) > \alpha$, then $\frac{1}{2}(\mu(F_n) + \alpha)$ would be an element of S greater than α, thus contradicting that $\alpha = \sup(S)$. As for the other inequality, note that $G_n \subseteq F_n$. Thus $\mu(F_n) \geq \mu(G_n) > \alpha - \frac{1}{n}$.

Now $\bigcup\limits_{n=1}^{\infty} F_n \in \mathcal{M}$ due to $\mathcal{M}$ being closed under countable sums. Also, by the two observations above, we know $\mu(\bigcup\limits_{n=1}^{\infty} F_n) = \lim\limits_{n\to\infty} \mu(F_n) = \alpha$. And finally, note that $\bigcup\limits_{n=1}^{\infty} F_n$ is a proper subset of E (we know this because each $F_n \subset E$ and $\bigcup\limits_{n=1}^{\infty} F_n$ can't equal E since their measures are different).

So, we have now proven the existence of a set $F \in \mathcal{M}$ such that $F \subset E$ and $\mu(F) = \alpha$. But now note that $\mu(E - F)$ must be infinite since:

$$\mu(E - F) + \alpha = \mu(E - F) + \mu(F) = \mu(E) = \infty.$$

Because μ is semifinite, there exists $F' \subset F - E$ in $\mathcal{M}$ with $0 < \mu(F') < \infty$. But then because F and F' are disjoint subsets of E in $\mathcal{M}$, we know $F \cup F' \in \mathcal{M}$ and $\mu(F \cup F') = \mu(F) + \mu(F') > \alpha$. Plus $F \cup F'$ is a proper subset of E. (It can't equal E because it's measure isn't equal to E. But, both F and F' individually are subsets of E.)

Hence, we have that $\frac{1}{2}(\alpha, \mu(F) + \mu(F)')$ is an element of S greater than α, thus contradicting that α was the supremum of S. We conclude therefore that α does not exist, meaning S is unbounded.