Math 100A Notes (Professor: Aaron Pollack)

Isabelle Mills

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Lecture 1 Notes: 9/27/2024

Motivation for this class:
Let $\mathcal{F}$ be any figure in $\mathbb{R}^2$. We want some way of talking about the symmetries of $\mathcal{F}$.

Letting d be the standard metric for $\mathbb{R}^2$, we say $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is distance preserving if $d(P,Q) = d(f(P), f(Q))$ for all $P, Q \in \mathbb{R}^2$. If f is distance-preserving and $f(\mathcal{F}) = \mathcal{F}$, then we call f a symmetry of $\mathcal{F}$.

We define $\mathrm{Sym}(\mathcal{F})$ to be the set of symmetries of $\mathcal{F}$.

Lemma 2: The set $\mathrm{Sym}(\mathcal{F})$ has the following properties:

1. The identity map Id is in $\mathrm{Sym}(\mathcal{F})$
2. If $f \in \mathrm{Sym}(\mathcal{F})$, then $f^{-1} \in \mathrm{Sym}(\mathcal{F})$.

   I realize we haven't yet shown that every $f \in \mathrm{Sym}(\mathcal{F})$ is a bijection. Given such an f, it's easy to see that f must be injective. After all, the distance preserving property of f means that $f(P) = f(Q) \implies P = Q$. Showing that f is surjective is harder. By assumption, we know that f is surjective when restricted to $\mathcal{F}$. More complicatedly, we can show that f must have a certain form which happens to be surjective. Perhaps I'll prove that later.

   Once, you've accepted that f^{-1} exists, then it's clearly true that f^{-1} is also distance preserving with $f^{-1}(\mathcal{F}) = \mathcal{F}$.
3. If $f_1, f_2 \in \mathrm{Sym}(\mathcal{F})$, then $f_1 \circ f_2 \in \mathrm{Sym}(\mathcal{F})$ and $f_2 \circ f_1 \in \mathrm{Sym}(\mathcal{F})$.

   This is pretty trivial to show.

Now while it's all good that we have a concrete way of describing the symmetries of a figure, our current terminology is not the most useful. After all, suppose $\mathcal{S}$ and $\mathcal{S}'$ are two squares such that $\mathcal{S}$ is centered at the origin and $\mathcal{S}'$ is centered at the point $(5,5)$. Then even though we know both $\mathcal{S}$ and $\mathcal{S}'$ have symmetries in the form of rotating and reflecting, the particular functions in $\mathrm{Sym}(\mathcal{S})$ and $\mathrm{Sym}(\mathcal{S})$ will be different (except for Id). So, how do we compare the symmetries of those two squares?

Aside start…

Proof that all symmetries are surjective (taken from our textbook):

Note:

- Our textbook calls a distance-preserving function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ an isometry.
- Rather than writing $f_1 \circ f_2$ to represent function composition, our textbook just writes $f_1 f_2$.

Some Facts:

(a) Orthogonal linear operators are isometries.

Let φ be n orthogonal linear map. φ being linear means that $\varphi(u)-\varphi(v)=\varphi(u-v)$. Meanwhile, φ being orthogonal means that $|\varphi(u-v)|=\sqrt{\varphi(u-v)\cdot\varphi(u-v)}=\sqrt{(u-v)\cdot(u-v)}=|u-v|$. So, for any $u,v\in\mathbb{R}^n$, we have that $|\varphi(u)-\varphi(v)|=|u-v|$.

(b) The translation t_a by a vector a defined by $t_a(x)=x+a$ is an isometry.

For any $u,v\in\mathbb{R}^n$, we have $|t_a(u)-t_a(v)|=|u+a-v-a|=|u-v|$.

(c) The composition of isometries is an isometry.

If f_1,f_2 are isometries, then for all $u,v\in\mathbb{R}^n$, we have that $|f_1(f_2(u))-f_1(f_2(v))|=|f_2(u)-f_2(v)|=|u-v|$.

Theorem 6.2.3: The following conditions on a map $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ are equivalent:

(a) φ is an isometry such that $\varphi(0)=0$.

(b) φ preserves dot products: $\varphi(u)\cdot\varphi(w)=u\cdot w$ for all $u,w\in\mathbb{R}^n$.

(c) φ is an orthogonal linear operator.

Proof:
(c) $\Longrightarrow$ (a)
This comes both from the first fact on this page plus the fact that all linear operators map 0 to 0.

(b) $\Longrightarrow$ (c)
Our challenge here is to show that such a φ has to be linear operator.

Lemma: For $x,y\in\mathbb{R}^n$, if $(x\cdot x)=(x\cdot y)=(y\cdot y)$, then $x=y$.

Proof: $|x-y|^2=(x-y)\cdot(x-y)=(x\cdot x)-2(x\cdot y)+(y\cdot y)$.

Consider any $u,v\in\mathbb{R}^n$ and set $w=u+v$. Then set $u'=\varphi(u)$, $v'=\varphi(v)$, and $w'=\varphi(w)$. To show that $w'=v'+u'$, we shall show that $(w'\cdot w')=(w'\cdot(u'+v'))=((u'+v')\cdot(u'+v'))$.

Firstly, simplify our equation to:

$$(w'\cdot w')=(w'\cdot u')+(w'\cdot v')=(u'\cdot u')+2(u'\cdot v')+(v'\cdot v')$$

Next, since φ is assumed to preserve dot products, we can thus simplify our equation to:

$$(w\cdot w)=(w\cdot u)+(w\cdot v)=(u\cdot u)+2(u\cdot v)+(v\cdot v)$$

And since $w=u+b$, all of those equalities are true. Hence, we know by our lemma above that $w'=u'+v'$.

Meanwhile, let $v \in \mathbb{R}^n$ and set $u = cv$ where c is a constant. Then define u' and v' as before. Then we can do a few trivial simplications to show that $(u' \cdot u')$, $(u' \cdot cv')$ and $(cv' \cdot cv')$ are all equal to $c^2(v \cdot v)$. So, $u' = cv'$.

(a) $\Longrightarrow$ (b)
Since φ is distance preserving, we know that $\forall u, v \in \mathbb{R}^n$,

$$(\varphi(u) - \varphi(v)) \cdot (\varphi(u) - \varphi(v)) = (u - v) \cdot (u - v)|.$$

By plugging in $v = 0$, this simplifies to $(\varphi(u) \cdot \varphi(u)) = (u \cdot u)$. Similarly, by plugging in $u = 0$, we can get that $(\varphi(v) \cdot \varphi(v)) = (v \cdot v)$. So, by expanding and canceling out parts of our above expression, we get that:

$$-2(\varphi(u) \cdot \varphi(v)) = -2(u \cdot v).$$

Corollary 6.2.7: Every isometry f of $\mathbb{R}^n$ is the composition of an orthogonal linear operator and a translation. Specifically, if $f(0) = a$, then $f = t_a\varphi$ where t_a is a translation and φ is an orthogonal linear operator.

Proof:
Let f be an isometry, let $a = f(0)$, and define $\varphi = t_{-a}f$. Then clearly $t_a\varphi = f$. So, we just need to show that φ is an orthogonal linear operator. To prove this, first note that φ is the composition of two isometries, and is thus an isometry itself. Also, $\varphi(0) = -a + f(0) = -a + a = 0$. So applying theorem 6.2.3, we know that φ is an orthogonal linear operator.

Now we've proven in other classes that both translations and linear orthogonal operators on $\mathbb{R}^n$ are surjective. So, all isometries are the composition of surjections, meaning they are surjective themselves. And since we also previously proved that all isometries are injective, we know they are bijective and have inverses.

Aside over...

Our textbook is *Algebra, Second Edition* by Michael Artin.