Math 158 Lecture Notes (Professor: Jacques Verstraete)

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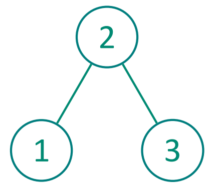
Lecture 1: 1/9/2024

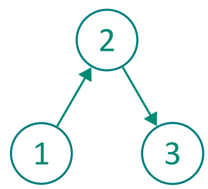
A graph is a pair (V, E) where V is a set of vertices and E is a set of unordered pairs of elements of V called edges. For $u, v \in V$, we say u and v are adjacent if $\{u, v\} \in E$.

For example: $G = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}\})$



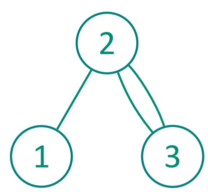
A directed graph (a.k.a a digraph) is a pair (V, E) where V is a set of vertices and E is a set of ordered pairs of elements of V.

For example: $G = (\{1, 2, 3\}, \{(1, 2), (2, 3)\})$



A multigraph is a pair (V, E) where V is a set of vertices and E is a multiset of unordered pairs of elements of V.

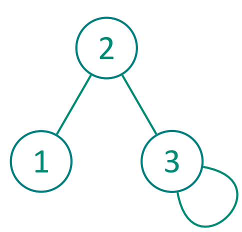
For example: $G = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}, \{2, 3\}\})$



A pseudograph is like a graph and multigraph except that the pairs in E are multisets.

Essentially, an element $\{a, a\}$ can belong to E in a pseudograph. This type of edge is called a loop.

For example: $G = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}, \{3, 3\}\})$



If $G = (V, E)$ and $v \in V$, the neighborhood of v is $N_G(v) = \{w \in V \mid \{v, w\} \in E\}$.

The degree of v is $d_G(v) = |N_G(v)|$. Or in other words, v's degree is equal to the number of edges connecting to v.

The Handshaking lemma states that for any graph (V, E):

$$\sum_{v \in V} d_G(v) = 2|E|$$

The reason for this is that each edge increments the degrees of exactly two vertices. So the above sum counts every edge twice.

Lemma: Every graph has an even number of vertices with odd degrees.

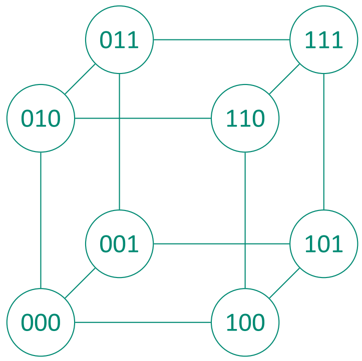
Proof: We can split the vertices of any graph into two categories: those with odd degrees, and those with even degrees.

Now recall that an even number plus an even number always equals an even number, as does an odd number plus an odd number. However, an odd number plus an even numbers equals an odd number. Based on this fact, we can guarentee that the sum of even degrees in any graph is even. And since the sum of even degrees plus the sum of odd degrees must be even as it equals $2|E|$ by the Handshaking lemma, we thus know that the sum of odd degrees must be even. Hence, it must be the case that there are an even number of vertices with odd degree because otherwise the sum of their degrees won't be even.

A graph is called r-regular if all of its vertices have degree r.

Note that the number of edges in any n-vertex r-regular graph is $\frac{rn}{2}$.

An r-dimensional cube graph, denoted as Q_r, is a graph such that $V(Q_r)$, the set of vertices in Q_r, is equal to the set of binary strings of length r; and $E(Q_r)$, the set of edges in Q_r, is equal to the set of pairs of binary strings which differ in only one position.



$$|V(Q_r)| = 2^r$$

$$|E(Q_r)| = \frac{2^r r}{2} = 2^{r-1} r$$

Note that Q_r is r-regular.

If $G = (V, E)$, then $H = (W, F)$ is a subgraph of G if $W \subseteq V$ and $F \subseteq E$.

If $W = V$, then H is a spanning subgraph of G (meaning that H has the same vertices as G but is lacking some of G's edges)

We define subtracting a set of vertices from a graph as follows:
For $G = (V, E)$ and $X \subset V$, we define...

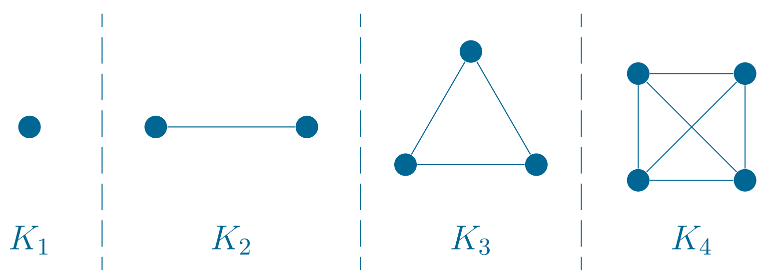
$$G - X = (V \setminus X, \{\{u, v\} \in E \mid \{u, v\} \cap X = \emptyset\})$$

We define subtracting a set of edges from a graph as follows:
For $G = (V, E)$ and $L \subset E$, we define...

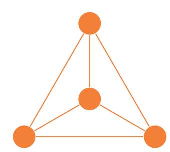
$$G - L = (V, E \setminus L)$$

Here are some basic classes of graphs:

- Complete graphs / cliques are graphs where every possible edge is present.



Note we can also draw K_4 such that there are no edge interceptions as follows:



- Complete bipartite graphs, denoted $K_{s,t}$, contain all edges with one end in a set of vertices of size s and the other end in a set of vertices of size t.

For example, $K_{3,2}$: