Math 240A Notes (Professor: Luca Spolaor)

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Lecture 1 Notes: 9/26/2024

Given an indexed family of sets $\{X_\alpha\}_{\alpha \in A}$, we define its Cartesian Product to be:

$$\prod_{\alpha \in A} X_\alpha = \{f : A \longrightarrow \bigcup_{\alpha \in A} X_\alpha \mid f(\alpha \in X_\alpha)\}$$

A projection is a function $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \longrightarrow X_\alpha$ satisfying that $f \mapsto f(\alpha)$.

If X, Y are sets, we define:

- $\mathrm{card}(X) \leq \mathrm{card}(Y)$ if there exists an injection $f : X \longrightarrow Y$.
- $\mathrm{card}(X) \geq \mathrm{card}(Y)$ if there exists a surjection $f : X \longrightarrow Y$.
- $\mathrm{card}(X) = \mathrm{card}(Y)$ if there exists a bijection $f : X \longrightarrow Y$.

Note that $\mathrm{card}(X) \leq \mathrm{card}(Y) \iff \mathrm{card}(Y) \geq \mathrm{card}(X)$. After all, given an injection in one direction, we can easily make a surjection in the other direction. Or given a surjection in one direction, we can (using A.O.C (axiom of choice)) easily make an injection in the other direction.

Also, if $\mathrm{card}(X) \leq \mathrm{card}(Y)$ and $\mathrm{card}(Y) \leq \mathrm{card}(X)$, then we know that $\mathrm{card}(Y) = \mathrm{card}(X)$.

Proof:
We know there exists $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ which are both injective. Hence, $g \circ f$ is an injection from X to $g(Y) \subseteq X$. By an exercise done in my math journal on page 8, we thus there exists a bijection h from X to $g(Y)$. And letting g^{-1} be any left-inverse of g, we then have that $g^{-1} \circ h$ is a bijection from X to Y.

We say X has the cardinality of the continuum if $\mathrm{card}(X) = \mathrm{card}(\mathbb{R})$.

Proposition: $\mathrm{card}(\mathcal{P}(\mathbb{N})) = \mathrm{card}(\mathbb{R})$.

Our textbook goes about proving this by constructing two functions: an injection and a surjection, from $\mathcal{P}(\mathbb{N})$ to $\mathbb{R}$ based on the binary expansion of any real number. That way, we know that $\mathrm{card}(\mathcal{P}(\mathbb{N})) \leq \mathrm{card}(\mathbb{R})$ and $\mathrm{card}(\mathcal{P}(\mathbb{N})) \geq \mathrm{card}(\mathbb{R})$.

Given a sequence $\{x_n\}$ in $\mathbb{R}$ we know there exists: $\limsup x_n = \inf_{k \geq 1}(\sup_{n \geq k} x_n)$ and $\liminf x_n = \sup_{k \geq 1}(\inf_{n \geq k} x_n)$.

Also, given a function $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, we can define:

$$\limsup_{x \to a} f(x) = \inf_{\delta > 0} \left(\sup_{0 < |x-a| < \delta} f(x) \right).$$

If X is an arbitrary set and $f : X \longrightarrow [0, \infty]$, we define:
$$\sum_{x \in X} f(x) = \sup \left\{ \sum_{x \in F} f(x) \mid F \subseteq X \ s.t. \ F \text{ is finite} \right\}.$$

Cool Proposition from textbook (not covered in lecture):
Let $A = \{x \in X \mid f(x) > 0\}$. If A is uncountable, then $\sum_{x \in X} f(x) = \infty$.

If A is countably infinite and $g : \mathbb{N} \longrightarrow A$ is a bijection, then $\sum_{x \in X} f(x) = \sum_{n=1}^{\infty} f(g(n))$.

Proof of first statement:
$A = \bigcup_{n \in \mathbb{N}} A_n$ where $A_n = \{x \in X \mid f(x) > \frac{1}{n}\}$.

If A is uncountable, we must have that some A_n is uncountable. But then for any finite set $F \subseteq X$, we have that $\sum_{x \in F} f(x) > \frac{\operatorname{card}(F)}{n}$. So $\sum_{x \in X} f(x)$ is unbounded.

A metric space (X, ρ) is a set X equipped with a distance function $\rho : X \times X \longrightarrow [0, \infty)$. We denote the open ball of radius r about x to be $B(r, x) = \{y \in X \mid \rho(x, y) < r\}$. And you remember our definitions from 140A... right?

Proposition 0.21: Every open set in $\mathbb{R}$ is a countable union of disjoint open intervals.
We proved this as part of a homework exercise in Math 140A.

Given a metric space (X, ρ), an element $x \in X$, and sets $F, E \subseteq X$, we can define:

- $\rho(x, E) = \rho_E(x) = \inf\{\rho(x, y) \mid y \in E\}$.
- $\rho(F, E) = \inf\{\rho_E(y) \mid y \in F\}$.

Exercise: $g(x, E) = 0 \iff x \in \overline{E}$.

Proof:
If $\inf\{\rho(x, y) \mid y \in E\} = 0$, then there exists a sequence $\{y_n\}$ in E such that $\rho(x, y_n) \to 0$. This implies $x \in \overline{E}$. Similarly, if $x \in \overline{E}$, we can construct a sequence $\{y_n\}$ such that $\rho(x, y_n) < \frac{1}{n}$ for all n. Then:
$$0 \leq \inf\{\rho(x, y) \mid y \in E\} \leq \inf\{\rho(x, y_n) \mid n \in \mathbb{N}\} = 0.$$

Given a subset E of a metric space (X, ρ), we define:
$$\operatorname{diam}(E) = \sup\{\rho(x, y) \mid x, y \in E\}.$$

If $\operatorname{diam}(E) < \infty$, we say E is bounded. If $\forall \varepsilon > 0$, E can be covered by finitely many balls of radius ε, then we say E is totally bounded.

Exercise: E being totally bounded implies E is bounded.

Pick $\varepsilon > 0$ and let $\{z_1, \dots, z_n\}$ be the set of points such that $E \subseteq \bigcup_{k=1}^{n} B(\varepsilon, z_n)$.

Then given any $x, y \in E$, we can assume that $x \in B(\varepsilon, z_i)$ and $y \in B(\varepsilon, z_j)$. So, $\rho(x, y) \leq \rho(x, z_i) + \rho(z_i, z_j) + \rho(z_j, y) < 2\varepsilon + \max\{\rho(z_i, z_j) \mid 1 \leq i, j \leq n\}$.

The converse is not generally true. For instance, if you use the discrete metric, then any set with more than one element will have a diameter of 1. But if $0 < \varepsilon < 1$, then it will be impossible to cover an infinite set with finitely many balls.

Lecture 2 Notes: 10/1/2024