Math 140A Lecture Notes (Professor: Brandon Seward)

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Lecture 1: 1/8/2024

An order on a set S, typically denoted as $<$, is a binary relation satisfying:

1. $\forall x, y \in S$, exactly one of the following is true:
   - $x < y$
   - $x = y$
   - $y < x$
2. given $x, y, z \in S$, we have that $x < y < z \Rightarrow x < z$

As a shorthand, we will specify that

- $x > y \Leftrightarrow y < x$
- $x \leq y \Leftrightarrow x < y$ or $x = y$
- $x \geq y \Leftrightarrow x > y$ or $x = y$

An ordered set is a set with a specified ordering. Let S be an ordered set and E be a nonempty subset of S.

- If $b \in S$ has the property that $\forall x \in E,\ x \leq b$, then we call b an upperbound to E and say that E is bounded above by b.
- if $b \in S$ has the property that $\forall x \in E,\ x \geq b$, then we call b an lower bound to E and say that E is bounded below by b.
- We call $\beta \in S$ the least upperbound to E if β is an upper bound to E and β is the least of all upperbounds to E. In this case, we also commonly call β the supremum of E and denote it as $\sup E$.
- We call $\beta \in S$ the greatest lower bound to E if β is an lower bound to E and β is the greatest of all lower bounds to E. In this case, we also commonly call β the infimum of E and denote it as $\inf E$.
- We call $e \in E$ the maximum of E if $\forall x \in E,\ x \leq e$
- We call $e \in E$ the minimum of E if $\forall x \in E,\ x \geq e$

Fact: For an ordered set S and nonempty $E \subseteq S$, either:

- neither $\max E$ nor $\sup E$ exists
- $\sup E$ exists but $\max E$ does not exist
- $\max E$ exists and $\sup E = \max E$

Using $\mathbb{Q}$ as our ordered set...

- For $E = \{q \in \mathbb{Q} \mid 0 < q < 1\}$, $\max E$ does not exist but $\sup E$ exists and equals 1.

  To understand why, note that the set of all upper bounds of E is equal to $\{q \in \mathbb{Q} \mid q \geq 1\}$ and 1 is obviously the smallest element of that set. Thus, 1 is the supremum of E. However, $1 \notin E$. Thus, if $\max E$ did exist, it would have to not equal 1. But that would contradict 1 being the least greatest bound.

- For $E = \{q \in \mathbb{Q} \mid 0 < q \leq 1\}$, $\max E$ and $\sup E$ exist and they both are equal to 1

  The reasoning for this is similar to that for the previous set.

- For $E = \{q \in \mathbb{Q} \mid q^2 < 2\}$, neither $\max E$ and $\sup E$ exist.

  To prove this, we can show there exists a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $\forall q \in \mathbb{Q}^+$, $q^2 < 2 \Rightarrow q^2 < (f(q))^2 < 2$ and $2 < q^2 \Rightarrow 2 < (f(q))^2 < q^2$. That way we can give a counter example to any possible claimed supremum or maximum of E.

  Now instead of being like Rudin and simply providing the desired function, I want to present how one may come up with a function that works for this proof themselves.

  Firstly, note that for the following reasons, we know our desired function must be a rational function:

  - $\forall q \in \mathbb{Q}$, $f(q) \in \mathbb{Q}$. Based on this, we can't use any radicals, trig functions, logarithms, or exponentials in our desired function.

  - $q^2 > 2 \Rightarrow f(q) < q$. In other words, f needs to grow slower than a linear function. Thus, we can rule out the possibility of f being a polynomial.

  - If we wanted f to be a linear function, it would have to have the form $f(q) = \alpha(q - \sqrt{2}) + \sqrt{2}$ where α is some constant. This is because when $q^2 = 2$, $f(q) = q$. However, there is no value one can set α to which both eliminates the presence of irrational numbers in that function while simultaneously making $f(q) \neq q$ when $q^2 \neq 2$. So no linear function can possibly work for this proof.

  Having narrowed our search, let's now pick some convenient properties we would wish our proof function to have. Specifically, let's force f to be constantly increasing, have a y-intercept of 1, and approach a horizontal asymptote of $y = 2$. Doing this, we can now say that an acceptable function will have the following form where α is an unknown constant:

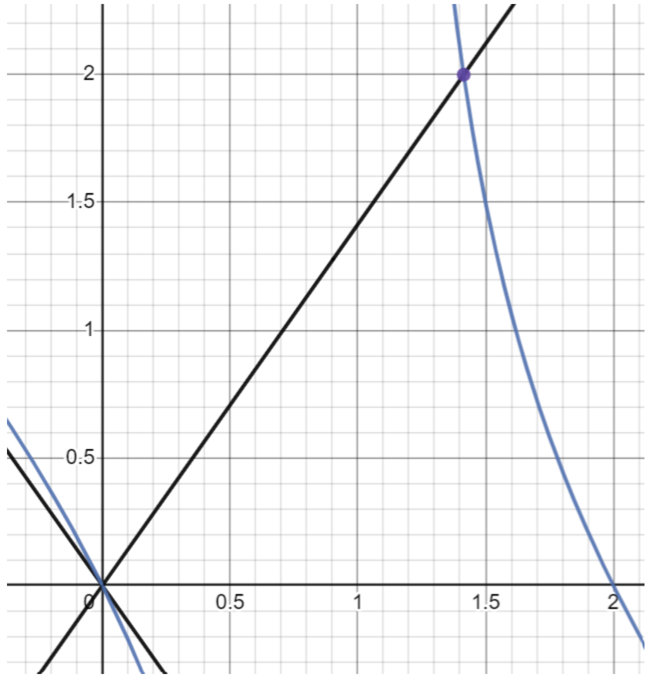
  $$f(q) = 1 + \frac{q}{q + \alpha}$$

  And finally, we can solve for α using the following system of equations:

  $$(1 + \tfrac{q}{q+\alpha})^2 = 2$$

  $$1 + \tfrac{q}{q+\alpha} = q$$

Now here's where a graphing calculator like Desmos can be very useful. Instead of painstakely having to solve for α, we can use a graphing calculator to approximate the value of α that satisfies our system of equations.



Based on the graph above, it looks like $f(q) = 1 + \frac{q}{q+2}$ will work for our proof. And sure enough it does. Furthermore, we can verify that the function we came up with is equivalent to that which Rudin presents.

We say an ordered set S has the least upperbound property if and only if when $E \subseteq S$ is nonempty and bounded above, then the supremum of E exists in S. Additionally, we say an ordered set S has the greatest lower bound property if and only if when $E \subseteq S$ is nonempty and bounded below, then the infimum of E exists in S.

When we define the set of real numbers, this will be one of the fundamental properties of that set.

Lecture 2: 1/10/2024

Proposition 1: S has the least upperbound property if and only if S has the greatest lower bound property.

Proof: Let's say we have an ordered set S

Assume S has the least upperbound property. Then, let $B \subseteq S$ be a nonempty subset which is bounded below. Additionally, let $A \subseteq S$ be the set of all lower bounds of B.

We know that $A \neq \emptyset$ because we assumed that B is bounded below. Thus, at least one lower bound to B exists and belongs to A. Additionally, because we assumed B is nonempty, we can say that each $b \in B$ is an upper bound to A. Thus, A is bounded above. Because of these two facts, we can apply the greatest lower bound property to say that the supremum of A exists.

Let's define $\alpha := \sup A$. With that, our goal is now to show that $\alpha = \inf B$. To do this, we need to show firstly that α is a lower bound to B and secondly that it is greater than all other lower bounds of B.

1. For each $b \in B$, we have that b is an upperbound to A. And since $\alpha = \sup A$ is the least upperbound to A, we must have that $\alpha \leq b$. Thus α is a lower bound to B.

2. If $x \in S$ is a lower bound to B, then $x \in A$. And since $\alpha = \sup A$, $x \leq \alpha$. This shows that α is greater than or equal to all other lower bounds.

Hence, α is the infimum of B. And since we did this for a general $B \subseteq S$, we can thus say that S has the greatest lower bound property.

Now we skipped doing the reverse direction proof because it is almost completely identical to the foward direction proof. However, just know that the above proposition is an if and only if statement. ■

A field is a set F equipped with 2 binary operations, denoted $+$ and $\cdot$, and containing two elements $0 \neq 1 \in F$ satisfying the following conditions for all $x, y, z \in F$:

- Associativity: $$(x+y)+z = x+(y+z)$$ $$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

- Commutativity: $$x+y = y+x$$ $$x \cdot y = y \cdot x$$

- Identity: $$0+x = x$$ $$1 \cdot x = x$$

- Inverses: $$\forall x \in F, \ \ \exists -x \in F \ \ s.t. \ \ x + -x = 0$$ $$\forall x \neq 0 \in F, \ \ \exists \frac{1}{x} \in F \ \ s.t. \ \ x \cdot \frac{1}{x} = 1$$

- Distributivity: $$x \cdot (y+z) = (x \cdot y) + (x \cdot z)$$

We shall assign the following notation:

We write ______	to mean ______
$x - y$	$x + -y$
$\dfrac{x}{y}$	$x \cdot \dfrac{1}{y}$
2	$1 + 1$
$2x$	$x + x$
x^2	$x \cdot x$
xy	$x \cdot y$

Now what follows is a number of propositions concerning the arithmetic properties of a field...

For a field F and elements $x, y, z \in F$, we have the following propositions:

Proposition 2.1: $x + y = x + z \Rightarrow y = z$

Proof: Assume $x + y = x + z$. Then...

$$\begin{aligned}
y &= 0 + y && \text{(addition identity property)} \\
&= (-x + x) + y && \text{(addition inverse property)} \\
&= -x + (x + y) && \text{(addition associative property)} \\
&= -x + (x + z) && \text{(by our assumption)} \\
&= (-x + x) + z && \text{(addition associative property)} \\
&= 0 + z && \text{(addition inverse property)} \\
&= z && \text{(addition identity property)}
\end{aligned}$$

Proposition 2.2: $x + y = x \Rightarrow y = 0$

Proof: Plug in $z = 0$ into proposition 2.1. in order to get that $y = z = 0$.

Proposition 2.3: $x + y = 0 \Rightarrow y = -x$

Proof: Plug in $z = -x$ into proposition 2.1. in order to get that $y = z = -x$.

Proposition 2.4: $-(-x) = x$

Proof: Observe that $x + -x = -x + x = 0$ by the inverse and commutative properties of addition. Then, by proposition 2.3, we know that $-x + x = 0 \Rightarrow x = -(-x)$.

Proposition 2.5: $x \cdot y = x \cdot z$ and $x \neq 0 \Rightarrow y = z$

Proof: Assume $x \cdot y = x \cdot z$ and $x \neq 0$. Then...

$$\begin{aligned} y &= 1 \cdot y && \text{(multiplication identity property)} \\ &= (\tfrac{1}{x} \cdot x) \cdot y && \text{(multiplication inverse property)} \\ &= \tfrac{1}{x} \cdot (x \cdot y) && \text{(multiplication associative property)} \\ &= \tfrac{1}{x} \cdot (x \cdot z) && \text{(by our assumption)} \\ &= (\tfrac{1}{x} \cdot x) \cdot z && \text{(multiplication associative property)} \\ &= 1 \cdot z && \text{(multiplication inverse property)} \\ &= z && \text{(multiplication identity property)} \end{aligned}$$

Note that to use the multiplication inverse property, we have to assume $x \neq 0$!!

Proposition 2.6: $x \cdot y = x \Rightarrow y = 1$

Proof: Plug in $z = 1$ into proposition 2.5. in order to get that $y = z = 1$.

Proposition 2.7: $x \cdot y = 1 \Rightarrow y = \dfrac{1}{x}$

Proof: Plug in $z = \frac{1}{x}$ into proposition 2.5. in order to get that $y = z = \frac{1}{x}$.

Proposition 2.8: $\dfrac{1}{\frac{1}{x}} = x$

Proof: Observe that $x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$ by the inverse and commutative properties of multiplication. Then, by proposition 2.7, we know that $\frac{1}{x} \cdot x = 1 \Rightarrow x = \dfrac{1}{\frac{1}{x}}$.

Proposition 2.9: $0 \cdot x = 0$

Proof: $(0 \cdot x) + (0 \cdot x) = (0 + 0) \cdot x = 0 \cdot x$. Thus we have an expression of the form $a + b = a$ which we can use proposition 2.2 on. Hence, we can conclude $0 \cdot x = 0$.

Proposition 2.10: $x \neq 0$ and $y \neq 0 \Rightarrow x \cdot y \neq 0$

Proof: since $x, y \neq 0$, we can say that $x \cdot y \cdot \frac{1}{x} \cdot \frac{1}{y} = 1 \neq 0$. Now by proposition 2.9, $x \cdot y = 0 \Rightarrow (x \cdot y) \cdot \left(\frac{1}{x} \cdot \frac{1}{y}\right) = 0$. However, we know that is not the case. So $x \cdot y$ can't equal zero.

Lecture 3: 1/12/2024

Proposition 2.11: $(-x)y = -(xy) = x(-y)$

Proof: $xy + (-x)y = (x + -x)y = 0y = 0$. Thus by proposition 2.3, $(-x)y = -(xy)$. We can make a similar argument to also say that $x(-y) = -(xy)$.

Proposition 2.12: $(-x)(-y) = xy$

Proof: Using proposition 2.11, we can say that $(-x)(-y) = -(x(-y)) = -(-(xy))$. Then by proposition 2.4, we can conclude $-(-(xy)) = xy$.

An ordered field is a field F equipped with an ordering $<$ satisfying $\forall x, y, z \in F$:

OF1. $y < z \Rightarrow y + x < z + x$

OF2. $(x > 0 \text{ and } y > 0) \Rightarrow xy > 0$

For x in an ordered field, we call x positive if and only if $x > 0$. Similarly, we call x negative if and only if $x < 0$.

Proposition 3: For an ordered field F and $x, y, z \in F$, we have:

1. $x < y \Leftrightarrow -y < -x$

   Proof: By property OF1 of an ordered field, we can say that $x < y \Rightarrow x + (-x + -y) < y + (-x + -y) \Rightarrow -y < -x$.

2. $(x > 0 \text{ and } y < z) \Rightarrow xy < xz$

   Proof: By property OF1 of an ordered field, $y < z \Rightarrow y - y < z - y$. Or in other words, $0 < z - y$. Therefore, since x is also positive by assumption, property OF2 of an ordered field tells us that $x(z-y) > 0$. Finally, adding xy to both sides by property OF1 and then distributing gives us: $xz - xy + xy = xz < xy$.

3. $(x < 0 \text{ and } y < z) \Rightarrow xy > xz$

   Proof: Since $x < 0$, we have $-x > 0$ by proposition 3.1. Then by applying proposition 3.2, we know that $(-x > 0 \text{ and } y < z) \Rightarrow -xy < -xz$. Finally, by reapplying proposition 3.1, this becomes $xy > xz$.

4. $x \neq 0 \Rightarrow x^2 > 0$

   Proof: If $x > 0$, then $x^2 = xx > 0x = 0$ by property OF2 of an ordered field. Meanwhile, if $x < 0$, then $-x > 0$ by proposition 3.1. So $(-x)(-x) > 0$ by property OF2. But $(-x)(-x) = x^2$ by proposition 2.12. So $x^2 > 0$.

5. $0 < x < y \Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$

Proof: Since $y > 0$ and $y \cdot \frac{1}{y} = 1 > 0 = 0 \cdot \frac{1}{y}$, we must have $\frac{1}{y} > 0$ by propositions 3.2 and 3.3. Note that $\frac{1}{y} \neq 0$ because if it did, $y \cdot \frac{1}{y} = 0$. Similarly, we can show $\frac{1}{x} > 0$. Now multiply both sides of $x < y$ by the positive element $\frac{1}{x} \cdot \frac{1}{y}$ and apply proposition 3.2 to get that $\frac{1}{y} < \frac{1}{x}$.

Theorem: There is (up to isomorphism) precisely one ordered field that contains $\mathbb{Q}$ and has the least upper bound property. We denote this field $\mathbb{R}$ and we call its elements real numbers.

In other words, this theorem is stating that $\mathbb{R}$ exists and is unique. Unfortunately, the proof for this is very long and so won't be covered in lecture. However, the professor has left some resources to cover it. So, I will have the proof of this theorem later in these notes.
See page: <research how to cite a page>

Proposition 4.1: If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$. This is called the archimedean property.

Proof: We proceed by looking for a contradiction. Let $A = \{nx \mid n \in \mathbb{Z}^+\}$ and assume $\nexists n \in \mathbb{Z}^+$ such that $nx > y$. In that case we know y is an upper bound of A. Additionally, since A is bounded above, we know by the least upper bound property of the real numbers that $\sup A$ exists. So, let $\alpha = \sup A$.

Now because $\mathbb{R}$ is an ordered field, we know that:
$x > 0 \Rightarrow -x < 0 \Rightarrow \alpha - x < \alpha$. Therefore, because α is the least upper bound, we know that $\alpha - x$ is not an upper bound for A. Or in other words, there exists $n \in \mathbb{Z}^+$ such that $nx > \alpha - x$. But this contradicts that α is the least upper bound of A because $nx > \alpha - x \Rightarrow (n+1)x > \alpha$ and $(n+1)x \in A$. So we conclude that the supremum of A can't exist, which by the contrapositive of the least upper bound property, means that A is not bounded above.

Proposition 4.2: If $x, y \in \mathbb{R}$ and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$. In other words, we say that Q is dense in R.

Proof: Since $x < y$, we have that $0 < y - x$. Then because $y - x$ is positive, we can use the archimedean property to say that there exists an integer n such that $n(y - x) > 1$. Note for later that this means $ny > 1 + nx$.

Now note that since $1 > 0$ and nx is a real number, we can use the archimedean property twice to get positive integers m_1 and m_2 such that $m_1 \cdot 1 > -nx$ and $m_2 \cdot 1 > +nx$. Thus, we get the expression $-m_1 < nx < m_2$ which can be rewritten as $0 < nx + m_1 < m_1 + m_2$. Next, because the set of positive integers is well ordered, meaning every nonempty subset of them has a least element, and $m_1 + m_2$ belongs to the set of positive integers greater than $nx + m_1$, we know there exists some least positive integer greater than $nx+m_1$. Or in other words, there exists m' such that $m'-1 \leq nx+m_1 < m'$. This then leads us to conclude that there is an integer $m = m'-m_1$ such that $m - 1 \leq nx < m$.

We now combine inequalities as follows: $m-1 \leq nx \Rightarrow m \leq nx+1$. So we have that $nx < m \leq nx+1$. But now remember from the previous page that $ny > 1 + nx$. So we can say that $nx < m \leq nx + 1 < ny$. Finally, because $n > 0$, we can multiply the inequality by $\frac{1}{n}$ to get that $x < \frac{m}{n} < y$. ■