<u>(Conway) Cauchy's Theorem (Third Version)</u> Let f be analytic in a region G and let $\gamma_0, \gamma_1 : [0,1] \to G$ be closed piecewise C^1 paths such that $\gamma_0 \sim_G \gamma_1$. Then $\int_{\gamma_0} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z$.

Proof:

Let $\Gamma : [0,1]^2 \to G$ be the homotopy.

Note that a difficulty with proving this theorem is that $\gamma_t(s) := \Gamma(s,t)$ is not guarenteed to be piecewise C^1 for any $t \neq 0, 1$.

Since Γ is continuous and $[0,1]^2$ is compact, we know that $\Gamma([0,1]^2)$ is compact in G. It follows that $\varepsilon = \inf\{|x-y| : x \in \Gamma([0,1]^2), y \in \mathbb{C} - G\} > 0$. It also follows that Γ is uniformly continuous. And by uniform continuity we can find n such that given any square $I_{j,k} := [\frac{j}{n}, \frac{j+1}{n}] \times [\frac{k}{n}, \frac{k+1}{n}]$ we have that $\Gamma(I_{j,k}) \subseteq B_\varepsilon(z_{j,k}) \subseteq G$ for all $j, k \in \{0, \ldots, n-1\}$ where $z_{j,k} := \Gamma(\frac{j}{n}, \frac{k}{n})$.

Now we approximate $\gamma_t(s) := \Gamma(s,t)$ where $t = \frac{k}{n}$ by taking the closed polygonal path $P_k = [z_{0,k}, z_{1,k}] + \ldots + [z_{n-1,k}, z_{n,k}]$. Note that $[z_{j,k}, z_{j,k+1}] \subseteq B_\varepsilon(z_{j,k})$ for all j, k. Hence, $\{P_k\} \subseteq G$ for each k (meaning we can integrate f along these paths). Our claim is that:

$$\int_{\gamma_0} f\mathrm{d}z = \int_{P_0} f\mathrm{d}z = \int_{P_1} f\mathrm{d}z = \cdots = \int_{P_n} f\mathrm{d}z = \int_{\gamma_1} f\mathrm{d}z$$

<u>Part 1:</u> $\int_{\gamma_0} f\mathrm{d}z = \int_{P_0} f\mathrm{d}z$ and $\int_{\gamma_1} f\mathrm{d}z = \int_{P_n} f\mathrm{d}z$.

The proof of both equalities is the same so I'll focus on the first equation. Let $\gamma_0^{(j)}$ be the restriction of γ to $[\frac{j}{n}, \frac{j+1}{n}]$. Then after some rearranging we get that:

$$\int_{\gamma_0} f\mathrm{d}z - \int_{P_0} f\mathrm{d}z = \sum_{j=0}^{n-1} \left(\int_{\gamma_0^{(j)}} f\mathrm{d}z + \int_{[z_{j+1,0}, z_{j,0}]} f\mathrm{d}z\right)$$

But note that $\gamma_0^{(j)}$ starts and ends at $z_{j,0}$ and $z_{j+1,0}$ respectively. Thus $\gamma_0^{(j)} + [z_{j+1,0}, z_{j,0}]$ is a closed C^1 path. And as $\{\gamma_0^{(j)}\} \subseteq \Gamma(I_{j,k}) \subseteq B_\varepsilon(z_{j,0})$, we know that the trace of $\gamma_0^{(j)} + [z_{j+1,0}, z_{j,0}]$ is contained in a convex disc contained in G. So by Cauchy's theorem, we have that $\left(\int_{\gamma_0^{(j)}} f\mathrm{d}z + \int_{[z_{j+1,0}, z_{j,0}]} f\mathrm{d}z\right) = \int_{\gamma_0^{(j)} + [z_{j+1,0}, z_{j,0}]} f\mathrm{d}z = 0$ for all j.

<u>Part 2:</u> $\int_{P_k} f\mathrm{d}z = \int_{P_{k+1}} f\mathrm{d}z$ for all k.

Note that the polygon $Q_{j,k} := [z_{j,k}, z_{j+1,k}, z_{j+1,k+1}, z_{j,k+1}, z_{j,k}] \subseteq B_\varepsilon(z_{j,k}) \subseteq G$ for all j, k. And as $B_\varepsilon(z_{j,k})$ is convex, we thus know by Cauchy's theorem that:

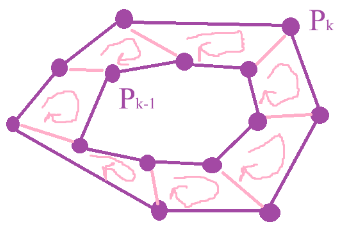
$$\int_{Q_{j,k}} f\mathrm{d}z = 0 \text{ for all } j, k.$$

But now note that after some rearranging we have that:

$$\begin{aligned}\int_{P_k} f\mathrm{d}z - \int_{P_{k+1}} f\mathrm{d}z &= \int_{[z_{n,k}, z_{n,k+1}]} f\mathrm{d}z - \int_{[z_{0,k}, z_{0,k+1}]} f\mathrm{d}z + \sum_{j=1}^{n-1} \int_{Q_{j,k}} f\mathrm{d}z \\ &= \int_{[z_{n,k}, z_{n,k+1}]} f\mathrm{d}z - \int_{[z_{0,k}, z_{0,k+1}]} f\mathrm{d}z + 0\end{aligned}$$

But as $\Gamma(1,t) = \Gamma(0,t)$ for all $t \in [0,1]$ we know that $z_{0,k} = z_{n,k}$ and $z_{0,k+1} = z_{n,k+1}$. Therefore, $\int_{[z_{n,k}, z_{n,k+1}]} f\mathrm{d}z - \int_{[z_{0,k}, z_{0,k+1}]} f\mathrm{d}z = 0$ as well.

Here is a picture to hopefully help describe this part:



Corollary: If $\gamma : [0,1] \to G$ is a closed piecewise C^1 curve and $\gamma \sim_G 0$, then $n(\gamma; a) = 0$ for all $a \in \mathbb{C} - G$.

Proof:

Just apply the previous theorem to the function $f(z) = \frac{1}{2\pi i(z-a)}$. Then as any path integral along a constant curve always evaluates to zero, we are done.

(Conway) Cauchy's Theorem (Second Version) If $f : G \to \mathbb{C}$ is an analytic function and γ is closed C^1 curve in G with $\gamma \sim_G 0$, then $\int_\gamma f = 0$.

Proof:

Apply Cauchy's integral theorem plus the last corollary.

Corollary: If $G \subseteq \mathbb{C}$ is open and simply connected, then $\int_\gamma f \mathrm{d}z = 0$ for any closed piecewise C^1 curve γ in G and analytic function f on G.

Munkres definition of being path homotopic (see *page 117*) is equivalent to Conway's definition of being Fixed-End-Point (F.E.P.) homotopic. Note that if γ_1 and γ_2 are closed curves rooted at the same point, then γ_1, γ_2 being F.E.P. homotopic implies $\gamma_1 \sim_G \gamma_2$. Also note that if γ_1 and γ_2 are F.E.P. homotopic then $\gamma_1 + (-\gamma_2)$ is F.E.P. homotopic to a constant curve. In turn, we get the following theorem:

Independence of Path Theorem: If γ_0, γ_1 are two piecewise C^1 curves in an open set $G \subseteq \mathbb{C}$ from a to b and $\gamma_0 \sim_G \gamma_1$, then $\int_{\gamma_0} f = \int_{\gamma_1} f$ for any analytic function f on G.

Proof:

$\int_{\gamma_0} f \mathrm{d}z - \int_{\gamma_1} f \mathrm{d}z = \int_{\gamma_0 + (-\gamma_1)} f \mathrm{d}z = 0$ by the last corollary.

When G is simply connected (so that all curves in G from a point a to a point b are path homotopic), we thus have that $\int_\gamma f$ depends only on the endpoints of γ and not on the particular path taken. This has the following consequences:

Theorem: If G is simply connected then every analytic function f has a primitive F.

Proof:

Fix $a \in G$ and then for every $z \in G$ define $F(z) = \int_{\gamma_z} f \mathrm{d}w$ where γ_z is any piecewise C^1 curve from a to z.

Recall from *theorem II.2.3* on page 247 that if G is connected then we can always find a polygonal path in G going between any two points of G. Thus, we don't need to worry about if a piecewise C^1 curve from a to z exists.

We claim F is a primitive of f. After all, given any fixed z_0, let $r > 0$ be such that $B_r(z_0) \subseteq G$. Now by the corollary following Cauchy's theorem (second version), since $\gamma_z + [z, z_0] + (-\gamma_{z_0})$ is a closed piecewise C^1 curve in G for any arbitrary piecewise C^1 curves γ_z and γ_{z_0} in G going from a to z and z_0 respectively, we know that:

$$F(z) + \int_{[z,z_0]} f \mathrm{d}w - F(z_0) = 0 \text{ for all } z \in B_r(z_0).$$

In other words, $\frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{z-z_0} \int_{[z_0,z]} f(w)\mathrm{d}w$. Then after subtracting $f(z_0)$ from both sides we get that:

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0} \int_{[z_0,z]} f(w) - f(z_0)\mathrm{d}w$$

Finally, since f is continuous at z_0, we know for any $\varepsilon > 0$ that there exists $0 < \delta < r$ such that when $|w - z_0| < \delta$ then $|f(w) - f(z_0)| < \varepsilon$. In turn, for all $z \in B_\delta(z_0)$ we have that:

$$|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)| \leq \frac{1}{|z-z_0|} \int_{[z_0,z]} |f(w) - f(z_0)|\mathrm{d}w < \frac{1}{|z-z_0|} \cdot |z_0 - z|\varepsilon = \varepsilon.$$

This proves that F is differentiable at z_0 with $F'(z_0) = f(z_0)$. ■

Theorem: If $G \subseteq \mathbb{C}$ is simply connected and f is an analytic nowhere vanishing function in G, then there exists a branch of $\log(f)$ on G (i.e. an analytic function g on G such that $e^{g(z)} = f(z)$).

Proof:

Since $f \neq 0$ in G, we know $\frac{f'}{f}$ is analytic on G. Hence by the prior theorem there exists $g : G \to \mathbb{C}$ such that $g_1' = \frac{f'}{f}$.

Next, pick $z_0 \in G$ and $w_0 \in \mathbb{C}$ such that $f(z_0) = e^{w_0}$. Since g will still be a primitive even after adding a constant, we can without loss of generality assume $g(z_0) = w_0$. That way, $f(z_0) = e^{g(z_0)}$.

Finally, consider $h(z) = e^{g(z)}$. Then:

$$(\tfrac{h}{f})' = \frac{h'f - hf'}{f^2} = \frac{g'e^g f - hf'}{f^2} = \frac{g'h}{f} - \frac{h}{f}\frac{f'}{f} = \frac{h}{f}(g' - \tfrac{f'}{f}) = \frac{h}{f}(0) = 0$$

Since G is connected, this shows that $\frac{h}{f}$ is constant on G. And since $\frac{h(z_0)}{f(z_0)} = 1$, we've proven that $h = f$ everywhere on G. ■

Math 200a Notes:

Given any integer $k > 0$, we let F_k denote the free group generated by k elements.

Theorem: $\langle \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right] \rangle \cong F_2$.

Proof:

Let $G = \langle \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right] \rangle$. Then note that $G \curvearrowright \mathbb{R}^2$ by linear transformations. In particular, if ℓ is a line passing through 0, then each element of G sends ℓ to another line. So, we can actually say that $G \curvearrowright X := \mathbb{RP}$.

Next, let $G_1 = \langle \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right] \rangle$ and $G_2 = \langle \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right] \rangle$. Then note that $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]^n = \left[\begin{smallmatrix} 1 & 2n \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]^n = \left[\begin{smallmatrix} 1 & 0 \\ 2n & 1 \end{smallmatrix}\right]$ (you can easily show this via induction).

Thus, $G_1 = \{\left[\begin{smallmatrix} 1 & 2n \\ 0 & 1 \end{smallmatrix}\right] : n \in \mathbb{Z}\}$ and $G_2 = \{\left[\begin{smallmatrix} 1 & 0 \\ 2n & 1 \end{smallmatrix}\right] : n \in \mathbb{Z}\}$. In particular, this means $G_1 \cong \mathbb{Z}$, $G_2 \cong \mathbb{Z}$.

Recall from *page 336* that any line in $\mathbb{RP}$ passing through (x, y) can be uniquely represented by the homogeneous coordinates $[x : y] = x/y$. Then as $\left[\begin{smallmatrix} 1 & 2n \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x+2ny \\ y \end{smallmatrix}\right]$, we have that $\left[\begin{smallmatrix} 1 & 2n \\ 0 & 1 \end{smallmatrix}\right][1 : 0] = [1 : 0]$ and $\left[\begin{smallmatrix} 1 & 2n \\ 0 & 1 \end{smallmatrix}\right][k : 1] = [k + 2n : 1]$.

Similarly, we have that $\left[\begin{smallmatrix} 1 & 0 \\ 2n & 1 \end{smallmatrix}\right][0 : 1] = [0 : 1]$ and $\left[\begin{smallmatrix} 1 & 2n \\ 0 & 1 \end{smallmatrix}\right][1 : k] = [1 : k + 2n]$. So finally, let $X_1 = \{[1 : 0]\} \cup \{[k : 1] : |k| \geq 1\}$ and $X_2 = \{[0 : 1]\} \cup \{[1 : k] : |k| \geq 1\}$.

If $g \in G_1 - \{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\}$, then $g \cdot X_2 \subseteq X_1$. (After all, $[1 : k] = [1/k : 1]$ and $|x + 2n| \geq 1$ for all $n \in \mathbb{Z}$ if $|x| \leq 1$). Similarly, $(G_2 - \{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\}) \cdot X_1 \subseteq X_2$.

By the ping pong lemma we conclude:

$$G = \langle \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right] \rangle = \langle G_1, G_2 \rangle \cong G_1 * G_2 \cong \mathbb{Z} * \mathbb{Z} = F_2. \blacksquare$$

$\mathrm{SL}_n(\mathbb{Z})$ refers to the collection of nxn matrices with determinant 1 and integer coefficients. At least for $\mathrm{SL}_2(\mathbb{Z})$ I already know how to show that $\mathrm{SL}_2(\mathbb{Z})$ is a group with respect to matrix multiplication.

In slightly more generallity, given any commutative ring R, the formula for matrix multiplication and the determinant of a matrix can still be carried out in R and the formula for the determinant of a matrix in R still makes sense. It follows that we can define $\mathrm{SL}_n(R)$ to be the collection of nxn matrices with determinant $1 \in R$ and coefficients in R.

Again, I don't know enough linear algebra to prove $\mathrm{SL}_n(R)$ is a group for arbitrary n. That said, if $n = 2$ then it is easy to see that $\mathrm{SL}_2(R)$ is a group.

- $$\begin{aligned} \det(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left[\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right]) &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= aecf + adeh + bgcf + bgdh - afce - afdg - bhce - bhdg \\ &= adeh + bgcf - afdg - bhce \\ &= ad(eh - fg) - bc(eh - fg) = \det(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right])\det(\left[\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right]) = 1. \end{aligned}$$
- $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right] = \left[\begin{smallmatrix} ad-bc & -ba+ab \\ cd-dc & ad-bc \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

Next we define $\mathrm{PSL}_2(\mathbb{Z}) := SL_2(\mathbb{Z})/\{\pm I\}$. Note that $\{\pm I\} = Z(SL_2(\mathbb{Z}))$ and is thus a normal subgroup. Hence, $\mathrm{PSL}_2(\mathbb{Z})$ is well-defined. Also we denote $\overline{A} = A\{\pm I\} \in \mathrm{PSL}_2(\mathbb{Z})$. Note that $\overline{A} = \{A, -A\}$.

<u>Theorem:</u> $\langle \overline{\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]}, \overline{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]} \rangle \cong \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

Proof:
Let $G_1 = \langle \overline{\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]} \rangle$ and $G_2 = \langle \overline{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]} \rangle$. We already know from the last proof that:

$$G_1 = \left\{ \overline{\left[\begin{smallmatrix} 1 & 2n \\ 0 & 1 \end{smallmatrix}\right]} : n \in \mathbb{Z} \right\} \cong \mathbb{Z}.$$

Meanwhile, $(\overline{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]})^2 = \overline{\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]} = \overline{I}$. Thus $G_2 \cong \mathbb{Z}/2\mathbb{Z}$.

Next, note that $\mathrm{PSL}_2(\mathbb{R}) \curvearrowright H := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ by Möbius transformations (recall *problem 3 on the second set*).

In particular, $\overline{\left[\begin{smallmatrix} 1 & 2n \\ 0 & 1 \end{smallmatrix}\right]} \cdot z = z + 2n$ and $\overline{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]} \cdot z = \frac{-1}{z}$. Thus $(G_1 - \{\overline{I}\}) \cdot X_2 \subseteq X_1$ and $(G_2 - \{\overline{I}\}) \cdot X_1 \subseteq X_2$ where $X_1 = \{z \in H : |z| > 1\}$ and $X_2 = \{z \in H : |z| < 1\}$.

By the ping pong lemma we are done. ■

We say a group Γ is residually $\mathcal{C}$ if for all $x \in \Gamma - \{1\}$ there exists a finite group G which satisfies $\mathcal{C}$ and a group homomorphism $\phi : \Gamma \to G$ such that $\phi(x) \neq 1$.

We say Γ is residually finite if $\forall x \in G - \{1\}$ there exists a finite group G and a group homomorphism $\phi : \Gamma \to G$ such that $\phi(x) \neq 1$.

(By first isomorphism theorem, this is equivalent ot saying that for all $x \in \Gamma - \{1\}$ there exists a group $N \lhd \Gamma$ of finite index such that $x \notin N$.)

Theorem: F_2 is residually finite.

Proof:

Recall $F_2 \cong \langle \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right] \rangle \subseteq \mathrm{SL}_2(\mathbb{Z})$. Thus, we can define a group homomorphism $\phi_n : F_2 \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ such that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{smallmatrix}\right]$ where $\overline{x} = x + n\mathbb{Z}$.

Since the mod map preserves addition and multiplication, it's clear that $\phi_n(AB) = \phi_n(A)\phi_n(B)$ and that:

$$\det(A) = 1 \in \mathbb{Z} \Longrightarrow \det(\phi_n(A)) = 1 \in \mathbb{Z}/n\mathbb{Z}.$$

Hence ϕ_n is a well-defined group homomorphism into $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.

But now $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ has less than n^4 elements. Also, if $x \in F_2 - \{I\}$ then we can choose n large enough so that $\phi_n(x) \neq \left[\begin{smallmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{smallmatrix}\right]$. ■

A group Γ is virtually $\mathcal{C}$ if there exists $\Lambda < \Gamma$ with finite index such that Λ satisfies $\mathcal{C}$.

Γ is virtually solvable if there exists $\Lambda \lhd \Gamma$ such that $[\Gamma : \Lambda] < \infty$ and Λ is solvable.

Note that it is not a restriction to assume Λ is a normal subgroup. After all, suppose $\Lambda < \Gamma$ and Λ is solvable. Then if we consider the group action $\Gamma \curvearrowright \Gamma/\Lambda$ by left translation, we get a group homomorphism $\phi : \Gamma \to S_{\Gamma/\Lambda}$. In turn, $\mathrm{core}_\Gamma(\Lambda) = \ker(\phi)$ is a normal subgroup of Γ whose index is finite as $|\mathrm{im}(\phi)|$ divides $[\Gamma : \Lambda]! < \infty$. And as $\mathrm{core}_\Gamma(\Lambda) < \Lambda$ we know that $\mathrm{core}_\Gamma(\Lambda)$ is solvable.

One other observation: If Γ is virtually solvable then so is any quotient of Γ.

Why?

Consider any subgroup $N \lhd \Gamma$. Then $\Lambda N/N \cong \Lambda/N \cap \Lambda$, and the latter is solvable. Hence $\Lambda N/N$ is solvable (see *problem 3 on the sixth set*). At the same time, $(\Lambda N)/N \lhd \Gamma/N$ as Λ and N are both normal subgroups of Γ. So, $(\Gamma/N)/(\Lambda N)/N \cong \Gamma/(\Lambda N)$ and the latter clearly has less elements than the finitely many in Γ/Λ. So, $(\Lambda N)/N$ satisfies the requirements for Γ/N to be virtually solvable.

Lemma: F_2 is not virtually solvable.

Proof:

Recall that $N \lhd S_n$ with N solvable implies that $N = \{1\}$ when $n \geq 5$. That said, we also have that $S_n = \langle (1\ \ 2), (1\ \ 2\ \ \cdots\ \ n) \rangle$. (For a proof of this see page 53 of my math 100c notes.) So, by the universal property of free groups we know there exists a surjective group homomorphism $\Phi : F_{\{a,b\}} \to S_n$ such that $\phi(a) = (1\ \ 2)$ and $\phi(b) = (1\ \ 2\ \ \cdots\ \ n)$.

($F_{\{a,b\}}$ is just notation for F_2 that makes it explicit what the generators of F_2 are...)

Suppose there exists $\Lambda \lhd F_{\{a,b\}}$ such that $[F_{\{a,b\}} : \Lambda] = m$ and Λ is solvable. Then we'd have that $\Phi(\Lambda)$ is solvable and $\Phi(\Lambda) \lhd S_n$. And when $n \geq 5$, this means that $\Phi(\Lambda) = \{\mathrm{Id}\}$. So, $\Lambda < \ker(\Phi)$ and in turn there is a surjective mapping:

$$F_{\{a,b\}}/\Lambda \twoheadrightarrow F_{\{a,b\}}/\ker(\Phi) \cong \mathrm{im}(\Phi) = S_n.$$

Consequently, we must have that $m \geq n!$ for any $n \geq 5$. This is a contradiction. ■

If G is a group and $R \subseteq G$, then we say $\langle\!\langle R \rangle\!\rangle$ is the smallest normal subgroup of G containing R. Next, given the sets S and $R \subseteq \mathcal{F}(S)$ (where $\mathcal{F}(S)$ is the free group of S), we define $\langle S | R \rangle \coloneqq \mathcal{F}(S)/\langle\!\langle R \rangle\!\rangle$. Also, we call $\langle S | R \rangle$ a presentation.

In other words, $\langle\!\langle R \rangle\!\rangle$ is the set of all words in $\mathcal{F}(S)$ identified with 1. Also, note that a common abuse of notation is to list an element of R as "word 1" = "word 2" as opposed to ("word 1")("word 2")$^{-1}$. Given this abuse of notation, it shouldn't be surprising that we call R the set of defining relations of $\langle S | R \rangle$.

When trying to prove what group a presentation is isomorphic to, there is a general procedure that works.

1. Already have an idea that $\langle S | R \rangle \cong G$. (Unfortunately, this procedure can only verify hunches one already has).
2. Let $S' \subseteq G$ be a generating set for G such that there exists a bijection $f : S \to S'$. Then using the universal property of free groups, let $\Phi : \mathcal{F}(S) \to G$ be a group homomorphism such that $\Phi(x) = f(x)$ for all $x \in S$. This group homomorphism is a surjection.
3. Check the relations to make sure that $R < \ker(\Phi)$. That way, we know that $\langle\!\langle R \rangle\!\rangle < \ker(\Phi)$. And in turn, there is a well-defined surjective group homomorphism $\overline{\Phi} : \langle S | R \rangle \to G$ such that $\overline{\Phi}(x) = \Phi(x) = f(x)$ for all $x \in S$.
4. Finally, find a trick to show that $\overline{\Phi}$ is injective.

Example 1: $\langle x | x^n = 1 \rangle \cong C_n$.

Let a be a generator for C_n. Then there is surjective homomorphism $\Phi : \mathcal{F}(\{x\}) \to C_n$ given by $\Phi(x) = a$. Also, it is clear that $\Phi(x^n) = a^n = 1$. So $\langle\!\langle x^n \rangle\!\rangle \subseteq \ker(\Phi)$ and we can define a surjective group homomorphism $\overline{\Phi} : \langle x | x^n = 1 \rangle \to C_n$ such that $\overline{\Phi}(x) = a$. Finally, note that $|\langle x | x^n = 1 \rangle| = n = |C_n|$. So by pigeonhole we know $\overline{\Phi}$ is a bijection.

Example 2: $\langle x, y \mid x^n = 1, y^2 = 1, yxy = x^{-1}\rangle \cong D_{2n}$.

Show this yourself. The proof is mostly identical to the prior example. :p

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Set 8 Problem 1: Prove that $\langle a, b \mid [a, b]\rangle \cong \mathbb{Z} \times \mathbb{Z}$.

By the universal property of free groups, we know there is a group homomorphism $f : F_{\{a,b\}} \to \mathbb{Z} \times \mathbb{Z}$ such that $f(a) = (1, 0)$ and $f(b) = (0, 1)$. Furthermore, we then have that $f([a, b]) = f(a)f(b)f(a^{-1})f(b^{-1}) = (1, 0) + (0, 1) - (1, 0) - (0, 1) = 0$. Hence, by quotienting out $\ll[a, b]\gg$ we can get a well-defined group homomorphism:

$\widetilde{f} : \langle a, b \mid [a, b]\rangle \to \mathbb{Z} \times \mathbb{Z}$ such that $f(a) = (1, 0)$ and $f(b) = (0, 1)$.

Also note that as $\langle(1, 0), (0, 1)\rangle = \mathbb{Z} \times \mathbb{Z}$, we know that f and in turn $\widetilde{f}$ are surjective.

What's left to show is that $\widetilde{f}$ is a bijection. So first we note that the following relevant commutators are in $\ll[a, b]\gg$.

$[b, a] = ([a, b])^{-1}$, $a^{-1}[a, b]a = [b, a^{-1}]$, $b^{-1}[a, b]b = [b^{-1}, a]$, and $(a^{-1}b^{-1})[b, a]ba = [b^{-1}, a^{-1}]$.

This shows that $a^{e_1}b^{e_2} = b^{e_2}a^{e_1}$ where $e_1, e_2 \in \{\pm 1\}$. Then by induction on k we can conclude that for all $k \in \mathbb{N}$:

- $b^k a = b^{k-1}ba(b^{-1}a^{-1}ab) = b^{k-1}ab = ab^{k-1}b = ab^k$,
- $b^{-k}a = b^{-k+1}b^{-1}a(ba^{-1}ab^{-1}) = b^{-k+1}ab^{-1} = ab^{-k+1}b^{-1} = ab^{-k}$,
- $b^k a^{-1} = b^{k-1}ba^{-1}(b^{-1}aa^{-1}b) = b^{k-1}a^{-1}b = a^{-1}b^{k-1}b = a^{-1}b^k$,
- $b^{-k}a^{-1} = b^{-k+1}b^{-1}a^{-1}(baa^{-1}b^{-1}) = b^{-k+1}a^{-1}b^{-1} = a^{-1}b^{-k+1}b^{-1} = a^{-1}b^{-k}$.

Another round of induction then shows that $a^m b^n = b^n a^m$ for all $m, n \in \mathbb{Z}$. And finally, this lets us show (again through induction) that every element of $\langle a, b \mid [a, b]\rangle$ can be represented by a word of the form $a^m b^n$ where $m, n \in \mathbb{Z}$.

We also claim that $a^m b^n = 1$ iff $m = 0 = n$. To see this, note that we can define the "a-power" and "b-power" of any word in $F_{\{a,b\}}$ by adding up the powers of all the a terms and b terms respectively.

Technically I'm overlooking the fact that the elements of $F_{\{a,b\}}$ are equivalence classes of words. That said, the two manipulations that let you go between any two words in the same equivalence class preserve "a-power" and "b-power". So, this technicality doesn't really matter.

But now if we let $N \subseteq F_{\{a,b\}}$ be the collection of all words with an a-power and b-power of 0, then we have that N is closed under word concatenation, inversing, and conjugation. Also $[a, b] \in N$. So $\ll[a, b]\gg < N \lhd F_{\{a,b\}}$. And in turn, we know that if $a^m b^n = 1$ in $\langle a, b \mid [a, b]\rangle$ then we must have that $a^m b^n \in N$ when considered as an element of $F_{\{a,b\}}$. But that implies that $m = 0 = n$.

Consequently, we know that if $a^{m_1}b^{n_1} = a^{m_2}b^{n_2}$ then $m_1 = m_2$ and $n_1 = n_2$. Hence by all the prior reasoning, if we define $g : \mathbb{Z} \times \mathbb{Z} \to \langle a, b \mid [a, b]\rangle$ by $g(m, n) = a^m b^n$ then we know g is an injective and surjective function satisfying that $\widetilde{f} \circ g = \mathrm{Id}_{\mathbb{Z}\times\mathbb{Z}}$. In turn, $\widetilde{f} = g^{-1}$ and this proves that $\widetilde{f}$ is a bijection. ■

Set 8 Problem 2: Suppose X_1 and X_2 are two disjoint sets. Prove that:
$$\langle X_1 \mid R_1\rangle * \langle X_2 \mid R_2\rangle \cong \langle X_1 \cup X_2 \mid R_1 \cup R_2\rangle$$

I shall start by proving something I think my professor meant for me to take as obvious. Consider the natural inclusion maps $j_i : \mathcal{F}(X_i) \hookrightarrow \mathcal{F}(X_1 \cup X_2)$ for each i.

To see that each j_i is an injection, note that adding symbols to an alphabet X does not change the reduced form of words already in $\mathcal{F}(X)$. And since each word equivalence class in $\mathcal{F}(X_i)$ has a unique reduced form (see *page* ___ for more on this), we know that j_i is an embedding. That said, the fact that j_i is an injection isn't important to the proof.

Then we know that $j_i^{-1}(\ll j_i(R_i) \gg)$ is a normal subgroup of $\mathcal{F}(X_i)$ containing R_i. Hence, there is a well-defined map $\overline{j_i} : \langle X_i \mid R_i\rangle \to \langle X_1 \cup X_2 \mid j_i(R_i)\rangle$ such that:
$$\overline{j_i}(\omega \ll R_i \gg) = j_i(\omega) \ll j_i(R_i) \gg \text{ for all words } \omega.$$

Furthermore, since $\ll j_i(R_i) \gg \subseteq \ll j_1(R_1) \cup j_2(R_2) \gg$ for both i, we know that there are well defined maps $k_i : \langle X_1 \cup X_2 \mid j_i(R_i)\rangle \to \langle X_1 \cup X_2 \mid j_1(R_1) \cup j_2(R_2)\rangle$ with $k_i(\omega \ll j_i(R_i) \gg) = \omega \ll j_1(R_1) \cup j_2(R_2) \gg$ for all words ω.

Now by setting $\theta_i = k_i \circ \overline{j_i}$ for both i, we now have shown that the obvious inclusion function $\langle X_i \mid R_i\rangle \to \langle X_1 \cup X_2 \mid j_1(R_1) \cup j_2(R_2)\rangle$ given by $\theta_i(\omega) = j_i(\omega) \ll j_1(R_1) \cup j_2(R_2) \gg$ is a well-defined group homomorphism.

With that out of the way I'm now going to identify $j_i(\omega)$ with ω for all $\omega \in \mathcal{F}(X_i)$. Also, I'll just write $\theta_i(\omega)$ as ω.

By the universal property of free products there exists a group homomorphism $\theta : \langle X_1 \mid R_1\rangle * \langle X_2 \mid R_2\rangle \to \langle X_1 \cup X_2 \mid R_1 \cup R_2\rangle$ such that $\theta(x_1) = x_1$ and $\theta(x_2) = x_2$ in $\langle X_1 \cup X_2 \mid R_1 \cup R_2\rangle$ for all $x_1 \in X_1$ and $x_2 \in X_2$.

Meanwhile, by the universal property of free groups there exists a group homomorphism $\phi : \mathcal{F}(X_1 \cup X_2) \to \langle X_1 \mid R_1\rangle * \langle X_2 \mid R_2\rangle$ such that $\phi(x_1) = x_1$ and $\phi(x_2) = x_2$ in $\langle X_1 \mid R_1\rangle * \langle X_2 \mid R_2\rangle$ for all $x_1 \in X_1$ and $x_2 \in X_2$. Also, note that if $\omega \in R_1 \cup R_2$ then $\phi(\omega) = 1$. Hence, by quotienting out $\ll R_1 \cup R_2 \gg$ we get a well-defined map
$$\widetilde{\phi} : \langle X_1 \cup X_2 \mid R_1 \cup R_2\rangle \to \langle X_1 \mid R_1\rangle * \langle X_2 \mid R_2\rangle$$

...with $\widetilde{\phi}(x_1) = x_1$ and $\widetilde{\phi}(x_2) = x_2$ in $\langle X_1 \mid R_1\rangle * \langle X_2 \mid R_2\rangle$ for all $x_1 \in X_1$ and $x_2 \in X_2$.

Finally, $\widetilde{\phi} \circ \theta(x) = x$ and $\theta \circ \widetilde{\phi}(x) = x$ for all $x \in X_1 \cup X_2$. And as $X_1 \cup X_2$ is a generating subset of both $\langle X_1 \mid R_1\rangle * \langle X_2 \mid R_2\rangle$ and $\langle X_1 \cup X_2 \mid R_1 \cup R_2\rangle$, we can extrapolate that $\widetilde{\phi} \circ \theta = \mathrm{Id}$ and $\theta \circ \widetilde{\phi} = \mathrm{Id}$. So, θ and $\widetilde{\phi}$ are isomorphisms. ■

Set 8 Problem 3: Prove that the subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ which is generated by $\overline{\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]}$ and $\overline{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]}$ has the presentation $\langle a, b \mid b^2\rangle$.

Recall from *page 417* that $\langle \overline{\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]}$ and $\overline{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]} \rangle = \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Then as $\mathbb{Z} \cong \langle a \mid \emptyset\rangle$ and $\mathbb{Z}/2\mathbb{Z} \cong \langle b \mid b^2\rangle$, we have by the prior problem that $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong \langle a, b \mid b^2\rangle$. ■

Before moving on to the next problem, I want to show that $\mathrm{PSL}_2(\mathbb{Z})$ is generated by the matrices $\sigma := \overline{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]}$ and $\tau := \overline{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]}$.

Why?

Note that $\sigma^n = \overline{\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right]}$. In turn, given any matrix $\overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} \in \mathrm{PSL}_2(\mathbb{Z})$ we have that:
$$\sigma^n \overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} = \overline{\left[\begin{smallmatrix} a+nc & b+nd \\ c & d \end{smallmatrix}\right]} \text{ and } \tau \overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} = \overline{\left[\begin{smallmatrix} c & d \\ -a & -b \end{smallmatrix}\right]}.$$

This suggests the following construction. Suppose $\overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]}$ is any matrix in $\mathrm{PSL}_2(\mathbb{Z})$ such that $|a| \geq |c| > 0$. Then we know there exists $n \in \mathbb{Z}$ such that $\sigma^n \overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} = \overline{\left[\begin{smallmatrix} a+nc & b+nd \\ c & d \end{smallmatrix}\right]} = \overline{\left[\begin{smallmatrix} a' & b' \\ c & d \end{smallmatrix}\right]}$ where $|a'| < |c'|$. (This is a consequence of the division algorithm). In turn:

$$\tau\sigma^n \overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} = \overline{\left[\begin{smallmatrix} c & d \\ -a' & -b' \end{smallmatrix}\right]}, \text{ where } |-a'| < c \leq |a|.$$

As for the case that $|a| < |c|$ initially, then we can just apply the prior reasoning to $\tau \overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]}$. Either way, if $G := \langle \sigma, \tau \rangle \subseteq \mathrm{PSL}_2(\mathbb{Z})$ and $\overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} \in \mathrm{PSL}_2(\mathbb{Z})$, then we've proven that there is a matrix $g \in G$ such that $g \overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} = \overline{\left[\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right]}$ satisfies that $|c'| < c$.

By induction on $|c|$, we can thus conclude for any $\overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} \in \mathrm{PSL}_2(\mathbb{Z})$ that there exists $g_1, \ldots, g_n \in G$ such that $g_n \cdots g_1 \overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} = \overline{\left[\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right]}$.

But as $a'd' - 0b' = a'd' = 1$ and both a' and d' are integers, we may assume $a' = d' = 1$. Hence, we actually have that:

$$g_n \cdots g_1 \overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} = \overline{\left[\begin{smallmatrix} 1 & b' \\ 0 & 1 \end{smallmatrix}\right]} = \sigma^{b'}$$

And finally $\overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} = g_1^{-1} \cdots g_n^{-1} \sigma^{b'} \in G$. This proves that $\mathrm{PSL}_2(\mathbb{Z}) = \langle \overline{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]}, \overline{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]} \rangle$.

Set 8 Problem 4: Prove that $\mathrm{PSL}_2(\mathbb{Z}) = \langle a, b \mid a^2, b^3 \rangle$.

Let $\sigma := \overline{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]}$ and $\tau := \overline{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]}$ like before. Then set $\omega = \sigma\tau$ and define $G_1 := \langle \tau \rangle$ and $G_2 := \langle \omega \rangle$.

Claim 1: $\langle G_1, G_2 \rangle = \langle \tau, \omega \rangle = \mathrm{PSL}_2(\mathbb{Z})$.

Why? We already know $\mathrm{PSL}_2(\mathbb{Z}) = \langle \tau, \sigma \rangle$. Also, $\tau = \tau^{-1}$. Therefore, $\sigma = \omega\tau$ is in $\langle \tau, \omega \rangle$. And this proves that:

$$\mathrm{PSL}_2(\mathbb{Z}) = \langle \tau, \sigma \rangle \subseteq \langle \tau, \omega \rangle \subseteq \mathrm{PSL}_2(\mathbb{Z})$$

Claim 2: $G_1 \cong C_2$ and $G_2 \cong C_3$.

Why? We already know from class that $\tau^2 = \overline{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]}$. Meanwhile $\omega = \sigma\tau = \overline{\left[\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right]}$. In turn, $\omega^3 = \overline{\left[\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right]}\,\overline{\left[\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right]}\,\overline{\left[\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right]} = \overline{\left[\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right]}\,\overline{\left[\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right]} = \overline{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]}$. And as $o(\omega)$ divides 3 and doesn't equal 1, we know $o(\omega) = 3$.

Now consider the action $\mathrm{PSL}_2(\mathbb{Z}) \curvearrowright \mathbb{R} \cup \{\infty\}$ by Möbius transformations.

In other words, $\overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} \cdot x = \frac{ax+b}{cx+d}$ for all $x \in \mathbb{R}$ and $\overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} \cdot \infty = \frac{a}{c}$ (and if any of the right-hand expressions are undefined, then $\overline{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]}$ sends the element of $\mathbb{R} \cup \{\infty\}$ to ∞.)

Recall from *page 335* that if $T(x) = \frac{a_1x+b_1}{c_1x+d_1}$ and $S(x) = \frac{a_2x+b_2}{c_2x+d_2}$, then:

$$(T \circ S)(x) = \frac{(a_1a_2+b_1c_2)x+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)x+(c_1b_2+d_1d_2)}.$$

So, we do have that $\overline{\left[\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right]} \cdot (\overline{\left[\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right]} \cdot x) = (\overline{\left[\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right]}\,\overline{\left[\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right]}) \cdot x$ and $\overline{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]} \cdot x = x$.

Let $X_1 = (-\infty, 0]$ and $X_2 = (0, \infty) \cup \{\infty\}$. Then $\tau \cdot x = \frac{1}{-x}$ and so $(G_1 - \{\overline{I}\}) \cdot X_2 \subseteq X_1$. Meanwhile, $\omega \cdot x = \frac{x-1}{x}$ and $\omega^2 \cdot x = \frac{-1}{x-1}$ and so $(G_2 - \{\overline{I}\}) \cdot X_1 \subseteq X_2$. Thus by ping pong lemma, we have that:

$$C_2 * C_3 \cong G_1 * G_2 = \langle G_1, G_2 \rangle = \mathrm{PSL}_2(\mathbb{Z}).$$

Finally, by problem 2 we know that $C_2 * C_3 \cong \langle a \mid a^2 \rangle * \langle b \mid b^3 \rangle \cong \langle a, b \mid a^2, b^3 \rangle$. ■

Interestingly, this and the last problem shows that $\langle a, b \mid a^2 \rangle$ is isomorphic to a subgroup of $\langle a, b \mid a^2, b^3 \rangle$. So that's cool.

Set 8 Problem 5: Prove that the group of Euclidean symmetries of the integers is isomorphic to $\langle a, b \mid a^2, b^2 \rangle$.

To start off, a Euclidean symmetry of the integers is an isometry $\theta : \mathbb{R} \to \mathbb{R}$ satisfying that $\theta(\mathbb{Z}) = \mathbb{Z}$. Note that all isometries are equal to an orthogonal linear function followed by translation by constant. So, we must have that $\theta(x) = ax + b$ where $a \in \{\pm 1\}$ and $b \in \mathbb{R}$. Also, as $\theta(0) \in \mathbb{Z}$ we must have that $b \in \mathbb{Z}$. Hence, the group the problem is asking us about is D_∞ from *problem 5 of the sixth problem set*.

Now, we already know from that prior homework set that D_∞ is generated by the maps $r(x) = -x$ and $s(x) = -x + 1$ where both r and s have order 2. So using the universal property of free groups, let $\Phi : F_{\{a,b\}} \to D_\infty$ be a group homomorphism such that $\Phi(a) = r$ and $\Phi(b) = s$. This map is surjective because r and s generate D_∞. Also, since $\Phi(a^2) = \mathrm{Id} = \Phi(b^2)$ we know there is a well-defined surjective group homomorphism $\overline{\Phi} : \langle a, b \mid a^2, b^2 \rangle \to D_\infty$ with $\overline{\Phi}(a) = r$ and $\overline{\Phi}(b) = s$.

But now because $a^2 = b^2 = 1$, all words in $\langle a, b \mid a^2, b^2 \rangle$ can be reduced to the form $(ab)^k a$, $(ab)^k$, $(ba)^k b$ or $(ba)^k$ where k is a nonnegative integer. Also, note that:

$$(ab)^{-k} = ((ab)^k)^{-1} = (ba)^k \text{ and } (ab)^{-k}a = (ba)^k a = (ba)^{k-1} b$$

Therefore, we can actually write that:

$$\langle a, b \mid a^2, b^2 \rangle = \{(ab)^n : n \in \mathbb{Z}\} \cup \{(ab)^n : n \in \mathbb{Z}\}a$$

And finally, $\overline{\Phi}$ sends each of the elements in the above two cosets to different isometries in D_∞. Specifically, $(\overline{\Phi}((ab)^n))(x) = x - n$ while $(\overline{\Phi}((ab)^n a))(x) = -x - n$. (This was shown in my write up for that prior homework set).

So, $\overline{\Phi}$ is an injection and we are done.

Set 8 Problem 6: Let $G_n \coloneqq \langle s_1, \ldots, s_{n-1} \mid s_i^2, (s_i s_j)^2 \text{ if } |i - j| > 1; (s_i s_{i+1})^3 \rangle$ (where $n \geq 2$). Prove that $G_n \cong S_n$.

Let $\tau_i \coloneqq (i \;\; i+1)$ for all $1 \leq i < n$. Then using the universal property of free groups, let $\Phi : F_{\{s_1, \ldots, s_{n-1}\}} \to S_n$ be the unique group homomorphism such that $\Phi(s_i) = \tau_i$ for each i. Note that Φ is surjective since the τ_i generate all of S_n.

If you really doubt that, recall that $\tau_1 = (1 \;\; 2)$ and $\tau_1 \tau_2 \cdots \tau_n = (1 \;\; 2 \;\; \cdots \;\; n)$ generate all of S_n.

Also note that $\tau_i^2 = \mathrm{Id}$. And if $|i - j| > 1$ then $\tau_i \tau_j$ has cycle type $(2 \geq 2 \geq 1 \geq \cdots \geq 1)$. So, $(\tau_i \tau_j)^2 = \mathrm{Id}$. And finally, $\tau_i \tau_{i+1}$ is a three cycle so $(\tau_i \tau_{i+1})^3 = \mathrm{Id}$ for each i. All in all, this shows that all the defining relations of the proposed presentation are in the kernel of Φ. Hence, after quotienting out the normal subgroup generated by them we get a well defined surjective group homomorphism $\overline{\Phi} : G_n \to S_n$ such that $\overline{\Phi}(s_i) = \tau_i$ for each $1 \leq i < n$.

Now to prove that $\overline{\Phi}$ is injective, we proceed by induction on n to show that $|G_n| \leq n!$. That way the only way for $\overline{\Phi}$ to also be surjective is if $|G_n| = n!$ and $\overline{\Phi}$ is one-to-one. For our base case, note that $G_2 = \langle s_1 \mid s_1^2 \rangle \cong C_2$ and $|C_2| = 2 = 2!$

Meanwhile for the inductive step, let H_{n-1} be the subgroup of G_n generated by $s_1, \ldots, s_{n-2}$. Then using the universal property of free groups, let $\Psi : F_{\{s_1, \ldots, s_{n-2}\}} \to H_{n-1}$ be the unique group homomorphism such that $\Psi(s_i) = s_i$ for each i. Again, Ψ is surjective.

It's clear that all the relations defining G_{n-1} are in the kernel of Ψ. Thus, after quotienting them out we get a well-defined surjective group homomorphism $\overline{\Psi} : G_{n-1} \to H_{n-1}$ such that $\overline{\Psi}(s_i) = s_i$ for all i. And by induction, this proves that $|H_{n-1}| \leq (n-1)!$.

Next let $H_{n-1}^{(n-j)}$ be the coset $s_{n-j} \cdots s_{n-1} H_{n-1}$ and also denote $H_{n-1}^{(n)} = H_{n-1}$. Then set $X_n \coloneqq \{H_{n-1}^{(1)}, \ldots, H_{n-1}^{(n)}\} \subseteq G_n / H_{n-1}$. We can easily see that $s_i H_{n-1}^{(i+1)} = H_{n-1}^{(i)}$. And as $s_i^2 = 1$ we can also see that $s_i H_{n-1}^{(i)} = H_{n-1}^{(i+1)}$.

To show the other cases, note that if $j \leq i-2$, then $s_j s_i = s_i s_j$. Thus since $s_j \in H_{n-1}$ for all $j \leq n-2$, we know that:

$$\begin{aligned} s_j H_{n-1}^{(i)} = s_j s_i s_{i+1} \cdots s_{n-1} H_{n-1} &= s_i s_{i+1} \cdots s_{n-1} s_j H_{n-1} \\ &= s_i s_{i+1} \cdots s_{n-1} H_{n-1} = H_{n-1}^{(i)} \text{ when } j \leq i-2. \end{aligned}$$

As for if $j > i$, then we can write $s_j s_i s_{i+1} \cdots s_{n-1} = s_i \cdots s_{j-2} s_j s_{j-1} s_j \cdots s_{n-1}$ using the identity from the previous paragraph. After that, as $(s_{j-1} s_j)^3 = 1$, we know that $s_j s_{j-1} s_j = s_{j-1} s_j s_{j-1}$. Hence:

$$\begin{aligned} s_i \cdots s_{j-2} s_j s_{j-1} s_j s_{j+1} \cdots s_{n-1} &= s_i \cdots s_{j-2} s_{j-1} s_j s_{j-1} s_{j+1} \cdots s_{n-1} \\ &= s_i \cdots s_{j-2} s_{j-1} s_j s_{j+1} \cdots s_{n-1} s_{j-1} \end{aligned}$$

And as $s_{j-1} \in H_{n-1}$, this shows that $s_j H_{n-1}^{(i)} = H_{n-1}^{(i)}$ when $j > i$.

All in all, this proves that $s_j X_n = X_n$ for all $1 \leq j < n$. And since the s_j generate all of G_n, we in turn know that $\omega X_n = X_n$ for all words $\omega \in G_n$. In particular, this means $\omega H_{n-1} \in X_n$ for all $\omega \in G_n$. So, $[G_n : H_{n-1}] \leq |X_n| \leq n$.

Thus $|G_n| = |H_{n-1}|[G_n : H_{n-1}] \leq (n-1)! \cdot n = n!$. ■

Math 200a notes:

In this class, we define a ring to be a set A equipped with operations $+, \cdot$ such that $(A, +)$ is an abelian group and $(A, \cdot)$ is a semigroup (i.e. a set with an associative operation) such that $0 \cdot a = 0 = a \cdot 0$, $c \cdot (a+b) = ca + cb$, and $(a+b) \cdot c = ac + bc$.

Note, that we shall make a distinction between unital rings and non-unital rings (also called rng's). Specifically, a unital ring has a multiplicative identity element 1 whereas a non-unital ring doesn't. (So in other words we won't take it by definition that a ring has an element 1.)

Usually, we shall assume we are working with commutative unital rings. That said, there are cases where we sometimes want to drop those assumptions.

- Given any ring A, we can define a ring $M_n(A)$ of $n \times n$ matrices of A using standard matrix addition and multiplication. In other words, $[a_{i,j}] + [b_{i,j}] = [(a_{i,j} + b_{i,j})]$ and $[a_{i,j}] \cdot [b_{i,j}] = [(\sum_{k=1}^{n} a_{i,k} b_{k,j})]$. Note that $M_n(A)$ is usually not a commutative even if A is.

  I'm not gonna show these operations satisfy the ring axioms.

- A common counter example is the rng where multiplication sends all pairs of elements to 0.

If G is a group or M is a monoid, then given a ring A we call $A[M]$ or $A[G]$ the monoid ring or group ring where $A[M]$ (resp. $A[G]$) is the collection of formal sums $\sum_{m\in M} a_m m$ (resp. $\sum_{g\in G} a_g g$) where each $a_m \in A$ and $a_m = 0$ for all but finitely many $m \in M$. To turn $A[M]$ (resp. $A[G]$) into a ring, we define:

- $\sum\limits_{m\in M} a_m m + \sum\limits_{m\in M} a'_m m := \sum\limits_{m\in M} (a_m + a'_m)m$,
- $(\sum\limits_{m\in M} a_m m)(\sum\limits_{m\in M} a'_m m) = \sum\limits_{m\in M} \left(\sum\limits_{m_1\cdot m_2 = m} a_{m_1} a'_{m_2}\right) m$. (This is called a convolution...)

  I'm not gonna show these operations satisfy the ring axioms.

  (Also note that if A is a commutative ring and M (or G) is abelian, then $A[M]$ (resp. $A[G]$) is a commutative ring.)

If $M = (\mathbb{Z}_{\geq 0})^k \cong \{x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} : (i_1, \ldots, i_k) \in (\mathbb{Z}_{\geq 0})^k\}$, then $A[(\mathbb{Z}_{\geq 0})^k] \cong A[x_1, \ldots, x_k]$ is the polynomial ring.

Given two rings A_1, A_2, we say $\phi : A_1 \to A_2$ is a ring homomorphism if $\phi(x+y) = \phi(x) + \phi(y)$ and $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A_1$. Also, we say ϕ is a unital ring homomorphism if A_1, A_2 are unital rings and $\phi(1_{A_1}) = 1_{A_2}$.

In other words, unlike in math 100b we are not assuming by default that ring homomorphisms are unital.

For example: if B is a commutative ring and $A \subseteq B$ is a subring, then for all $b \in B$ we have that the map $e_b : A[x] \to B$ given by $e_b(f) := f(b)$ is a ring homomorphism. And if B and A share a multiplicative identity, then e_b is also unital.

You can see my 100b notes on why this is a homomorphism.

Also, something that's not at all clear is how the professor defines a subring since we've loosened our definition of a ring and ring homomorphism. In this class we say $A \subseteq B$ is a subring if A is closed under multiplication and a subgroup of B with respect to addition.

We say $\mathfrak{a}$ is an ideal of A (also written as $\mathfrak{a} \lhd A$) if $(\mathfrak{a}, +)$ is a subgroup of $(A, +)$ and $ax, xa \in \mathfrak{a}$ for all $x \in \mathfrak{a}$ and $a \in A$.

Note that if A is a unital ring then it suffices to show A is closed under addition and has the mentioned multiplication property. After all, we then have that $-x = (-1)\cdot x \in \mathfrak{a}$ for all $x \in \mathfrak{a}$.

Lemma: If $\phi : A_1 \to A_2$ is a ring homomorphism then $\mathrm{im}(\phi)$ is a subring of A_2 and $\ker(\phi)$ is an ideal of A.

Proof:
Since $\phi : (A_1, +) \to (A_2, +)$ is a group homomorphism, we know that $\mathrm{im}(\phi)$ and $\ker(\phi)$ are subgroups of A_1 and A_2 respectively with respect to $+$. To show that $\ker(\phi)$ is an ideal, note that $\phi(ax) = \phi(a)\phi(x) = 0_{A_2} = \phi(x)\phi(a) = \phi(xa)$ for all $x \in A_1$ and $a \in \ker(\phi)$. Meanwhile, to show that $\mathrm{im}(\phi)$ is a subring, note that if $\phi(x) = a$ and $\phi(y) = b$ then $\phi(xy) = ab$. ■

Switching our perspective, note that if $\mathfrak{a}$ is an ideal of a ring A, then we can define a quotient ring $A/\mathfrak{a}$ by defining $(x+\mathfrak{a})\cdot(y+\mathfrak{a}) = xy+\mathfrak{a}$ on the abelian quotient group $(A/\mathfrak{a}, +)$.

See my math 100b notes for why this is well-defined.

Then the natural projection map $j : A \twoheadrightarrow A/\mathfrak{a}$ satisfies that $\ker(j) = \mathfrak{a}$. Hence, all ideals are kernels of some ring homomorphism.

Returning to the evaluation map $e_b : A[x] \to B$ where B is a commutative ring and $A \subseteq B$ is a subring, one can fairly easy see that $\operatorname{im}(e_b)$ is the smallest subring of B containing A and b. We denote $\operatorname{im}(e_b)$ as $A[b]$.

11/24/2025

Math 220a Notes:

If G is an open set, then we say γ is homologous to zero (denoted $\gamma \approx_G 0$) iff $n(\gamma; w) = 0$ for all $w \in \mathbb{C} - G$.

Note that by the first corollary on *page 415*, we have that $\gamma \sim_G 0 \implies \gamma \approx_G 0$.

Suppose $G \subseteq \mathbb{C}$ is a region and $f : G \to \mathbb{C}$ is analytic on G with the zeros $a_1, \ldots, a_n$ (where the a_k are allowed to be repeated). As noted on *page 382* of my journal as well as my spring notes, we can then find an analytic function $g : G \to \mathbb{C}$ with no zeros such that $f(z) = (z-a_1)\cdots(z-a_n)g(z)$. Then by product rule, we get that:

$$f'(z) = \sum_{k=1}^{n}\Big(\prod_{i\neq k}(z-a_i)\Big)g(z) + g'(z)\prod_{k=1}^{n}(z-a_k)$$

And dividing both sides by $f(z)$ we get that:

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \sum_{k=1}^{n}(z-a_k)^{-1} \text{ when } z \neq a_1, \ldots, a_n$$

(Conway) Theorem IV.7.2: Let G be a region and let f be an analytic function on G with zeros $a_1, \ldots, a_n$ (repeated according to multiplicity) like above. If γ is a closed piecewise C^1 curve in G which does not pass through any point a_k and if $\gamma \approx_G 0$ then:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \sum_{k=1}^{m} n(\gamma; a_k).$$

Proof:

Letting g be as above, we know that:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\mathrm{d}z + \sum_{k=1}^{n}\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a_k}\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\mathrm{d}z + \sum_{k=1}^{m} n(\gamma; a_k)$$

Then since $g(z) \neq 0$ for any $z \in G$, we know that $\frac{g'}{g}$ is analytic on G. Hence, we have that $\int_\gamma \frac{g'(z)}{g(z)}\mathrm{d}z = 0$. ■

<u>(Conway) Corollary IV.7.3:</u> Let f, G, γ be as in the last theorem but let $a_1, \ldots, a_n$ (repeated according to multiplicity) be all the points where f equals α. In other words, $a_1, \ldots, a_n$ are the zeros of $f(z) - \alpha$. Then:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}\mathrm{d}z = \sum_{k=1}^{m} n(\gamma; a_k).$$

Note that if $f : G \to \mathbb{C}$ is an analytic non-constant function on G, it is possible for f to have infinitely many zeros in G. That said, because the set of zeros can't have a limit point in G, we know that if $K \subseteq G$ is compact then f can only have finitely many limit points in K. Consequently, if $\gamma \approx_G 0$ we can show that $f(z) = \alpha$ must have only finitely many solutions in G such that $n(\gamma; z) \neq 0$.

Exercise IV.7.2: Let $G \subseteq \mathbb{C}$ be open and suppose γ is a closed piecewise C^1 curve in G such that $\gamma \approx_G 0$. Set $H \coloneqq \{z \in \mathbb{C} : n(\gamma; z) = 0\}$.

(a) Suppose G is a proper subset of $\mathbb{C}$ and define $r \coloneqq \inf(\{|z - w| : z \in \{\gamma\}, w \in \partial G\})$. Note that r exists and is positive because $\partial G, \{\gamma\}$ are closed disjoint nonempty sets with $\{\gamma\}$ compact. Now show that $\{z \in \overline{G} : \inf\{|z - w| : w \in \partial G\} < \frac{r}{2}\} \subseteq H$

It suffices to show that if $\inf\{|z-w| : w \in \partial G\} < \frac{r}{2}$ then z is in the same component of $G - \{\gamma\}$ as some $w \in \partial G$. After all, as $w \in G^{\mathsf{C}}$ and $f \approx_G 0$ we know that $n(\gamma; w) = 0$. Also, as $n(\gamma; z)$ is constant on each component of $\mathbb{C} - G$, we would thus have that $n(\gamma; z) = n(\gamma; w)$. Fortunately, we can just pick $w \in \partial G$ such that $|z - w| < \frac{1}{2}r$. Next, we note that the line segment $[z, w]$ can't intercept $\{\gamma\}$ as that would contradict how we defined r. So, z, w must be in the same component of $\mathbb{C} - \{\gamma\}$.

(b) Use part (a) to show that if $f : G \to \mathbb{C}$ is analytic and non-constant then $f(z) = \alpha$ has at most a finite number of solutions z such that $n(\gamma; z) \neq 0$.

Since γ is bounded, we can find an open ball $B \subseteq \mathbb{C}$ of finite radius with $\{\gamma\} \subseteq B$. Then as B is convex, we know that $\gamma \sim_B 0$. Hence $G - B \subseteq H$.

Meanwhile, let $r \coloneqq \inf(\{|z - w|\} : z \in \{\gamma\}, w \in \partial(B \cap G))$. Then by part (a) we know that $V \coloneqq \{z \in \overline{B \cap G} : \inf\{|z-w| \in \overline{B \cap G} : w \in \partial(B \cap G)\} < \frac{r}{2}\} \subseteq H$. Hence $K \coloneqq \overline{B \cap G} - V$ must contain H^{C}. But also note that V is an open subset of $\overline{B \cap G}$. Hence, K is a closed subset relative to the compact set $\overline{B \cap G}$. In turn, K is compact. Also as $\partial(B \cap G) \subseteq V$ we know that $K \subseteq B \cap G$.

With that, we've proven there is a compact set $K \subseteq G$ with $n(\gamma; z) = 0$ outside of K. And as noted before, $f(z) = \alpha$ can only have finitely many solution on K as any infinite subset of a compact set has a limit point. ■

A simple root of $f(z) = \xi$ is a zero of $f(z) - \xi$ with multiplicity 1.

(Conway) Theorem IV.7.4: Suppose $f : G \to \mathbb{C}$ is analytic and $z_0 \in G$ is such that $f(z) - w_0$ has a zero of multiplicity m at z_0. Then there exists $\varepsilon, \delta > 0$ such that for all $w \in B_\varepsilon(w_0)$ the equation $f(z) - w$ has exactly m zeros in $B_\delta(z_0)$ which furthermore are all simple if $w \neq w_0$.

Proof:

To start off, we may pick $\delta > 0$ such that $f(z) - w_0 \neq 0$ for all $z \in \overline{B_\delta(z_0)} - \{z_0\} \subseteq G$. Then let $\gamma(s) = z_0 + \delta e^{is}$ and note that $\sigma := f \circ \gamma$ is a closed piecewise C^1 curve not passing through w_0. Hence, $\varepsilon := \inf\{|w - w_0| : w \in \{\sigma\}\} > 0$ and in turn $g(w) := \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)-w} \mathrm{d}z$ is continuous (by Leibniz's rule) as w ranges over $B_\varepsilon(w_0)$. Yet also recall from our prior theorems that $g(w)$ is integer valued. Hence, we know g is constant on $B_\varepsilon(w_0)$.

But now note that $n(\gamma; z) = 1$ for all $z \in B_\delta(z_0)$ and $n(\gamma; z) = 0$ for all other $z \in \mathbb{C} - \{\gamma\}$. Therefore, we can calculate that $g(w_0) = \sum_{k=1}^m n(\gamma; z_0) = m \cdot 1$. And this proves that $g(w) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)-w} \mathrm{d}z = m$ for all $w \in B_\varepsilon(w_0)$.

Next note that if $a_1, \ldots, a_n$ are the zeros in $B_\delta(z_0)$ (repeated according to multiplicity) of $f(z) - w$, then since $n(\gamma; a_k) = 1$ for each k, we have for all $w \in B_\varepsilon(w_0)$ that:

$$m = g(w) = \sum_{k=1}^n n(\gamma; a_k) = n$$

So, there are exactly m solutions in $B_\delta(z_0)$ to the equation $f(z) = w$ for all $w \in B_\varepsilon(w_0)$.

Finally, if $m = 1$ then there is nothing to prove. Meanwhile, if $m > 1$ then we can easily show that $f'(z_0) = 0$ (see the exercise below). In turn, as f' is analytic we can say that if we had initially started with a small enough δ then we'd have that $f'(z) \neq 0$ for all $z \in \overline{B}_\delta(z_0) - \{z_0\}$. In turn, each root of $f(z) - w$ must be simple when $w \neq w_0$. ■

Exercise IV.7.3: Let f be analytic in $B_R(a)$ and suppose that $f(a) = 0$. Show that a is a zero of multiplicity m iff $f^{(m-1)}(a) = \cdots = f^{(1)}(a) = f(a) = 0$ and $f^{(m)}(a) \neq 0$.

$(\Longleftarrow)$

Write f as a power series $\sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!}(z-a)^n$. If $f^{(m-1)}(a) = \cdots = f^{(1)}(a) = f(a) = 0$ and $f^{(m)}(a) \neq 0$ then we can factor out $(z-a)^m$ and get a power series which is nonzero at a.

$(\Longrightarrow)$

Write $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$ and both f and g are analytic. Then we can express g as a power series $\sum_{n=0}^\infty \frac{g^{(n)}(a)}{n!}(z-a)^n$. In turn:

$$f(z) = \sum_{n=0}^{m-1} \frac{0}{n!}(z-a)^n + \sum_{n=0}^\infty \frac{g^{(n)}(a)}{(n)!}(z-a)^{n+m}.$$

And by looking at each coefficient in the power series, we see that $f^{(n)}(a) = 0$ for all $n < m$ and $f^{(m)}(a) = m! g(a) \neq 0$. ■

Open Mapping Theorem: Let G be a region and suppose that $f : G \to \mathbb{C}$ is a non-constant analytic function on G. Then for any open set $U \subseteq G$ we have that $f(U)$ is open in $\mathbb{C}$.

Proof:

By the last theorem, for all $z \in G$ we can find $\varepsilon, \delta > 0$ such that:

$$B_\varepsilon(f(z)) \subseteq f(B_\delta(z)) \subseteq f(G).$$

One more comment before I start with the 220 homework. Conway finally proves that being complex differentiable a single time on an open set makes a function holomoprhic on that open set. The proof he uses doesn't have any new ideas from my notes from last Spring though.

Math 220a Homework:

Exercise IV.6.1: Let G be a region and let $\sigma_1, \sigma_2 : [0,1] \to G$ be the constant curves at a and b in G. Show that if γ is a closed piecewise C^1 curve and $\gamma \sim_G \omega_1$ then $\gamma \sim_G \omega_2$.

Proof:

Since G is a connected open subset of $\mathbb{C}$, we know G is path connected. Then letting $\omega : [0,1] \to G$ be any path going from a to b, we have that $\Gamma(s,t) = \omega(t)$ is a homotopy from σ_1 to σ_2. Hence, $\sigma_1 \sim_G \sigma_2$.

Then as $\sim_G$ is an equivalence relation and $\gamma \sim_G \sigma_1 \sim_G \sigma_2$, we are done. ■

Exercise IV.6.4: Let $G = \mathbb{C} - \{0\}$ and show that every closed curve in G is homotopic to a closed curve whose trace is contained in $\{z : |z| = 1\}$.

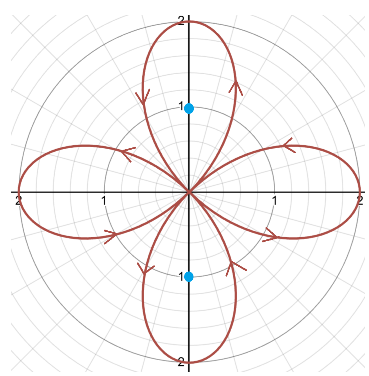
Define $\Gamma(s,t) := (1-t)\gamma(s) + t\frac{\gamma(s)}{|\gamma(s)|}$. Since $\gamma(s) \neq 0$ ever, we know that Γ is continuous. Also, $\Gamma(s,0) = \gamma(s)$ and $|\Gamma(s,1)| = |0 + 1\frac{\gamma(s)}{|\gamma(s)|}| = 1$. So, the curve $\gamma_1(s) = \Gamma(s,1)$ is a continuous curve whose trace is contained in $\{z : |z| = 1\}$. Finally, as $\gamma(0) = \gamma(1)$ we know that $\Gamma(0,t) = \Gamma(1,t)$ for all t. Hence, Γ is a homotopy.

Exercise IV.6.5: Evaluate the integral $\int_\gamma \frac{\mathrm{d}z}{z^2+1}$ where $\gamma(\theta) = 2|\cos(2\theta)|e^{i\theta}$ for $0 \leq \theta \leq 2\pi$.

Note that $\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$. Then if $a(z+i) + b(z-i) = 1$, we must have that $a + b = 0$ and $i(a-b) = 1$. In particular $a = -b$ and so $i(2a) = 1$. In turn $a = \frac{1}{2i}$ and $b = \frac{-1}{2i}$. And this lets us conclude that:

$$\int_\gamma \frac{\mathrm{d}z}{z^2+1} = \frac{1}{2i}\int_\gamma \frac{\mathrm{d}z}{z-i} - \frac{1}{2i}\int_\gamma \frac{\mathrm{d}z}{z+i} = \pi n(\gamma; i) - \pi n(\gamma; -i)$$

But now note that $\gamma(\theta) = 2|\cos(2\theta)|e^{i\theta}$ traces out the same curve as the polar graph $r(\theta) = 2|cos(2\theta)|$. Specifically, that curve has 4 equally spaced petals flower drawn counter-clockwise about the origin as shown to the right.



From here it is clear that $n(\gamma; i) = 1$ and $n(\gamma; -i) = 1$. So:

$$\pi n(\gamma; i) - \pi n(\gamma; -i) = \pi - \pi = 0.$$

Exercise IV.7.4: Suppose that $f : G \to \mathbb{C}$ is analytic and injective. Then $f'(z) \neq 0$ for any $z \in G$.

Proof:

Suppose to the contrary that $f'(z_0) = 0$ for some $z_0 \in G$ and let $w_0 = f(z_0)$. Then we'd know that $f(z) - w_0$ has a zero of multiplicity $m > 1$ at $z = z_0$. So in turn, there exists $\varepsilon, \delta > 0$ such that $f(z) - w$ has two simple roots for in $B_\delta(z_0)$ for all $w \in B_\varepsilon(w_0)$. But that also means that f isn't injective on $B_\delta(z_0) \subseteq G$.

This proves that $f'(z) \neq 0$ anywhere on G is a necessary condition for an analytic function $f : G \to \mathbb{C}$ to be injective.

Exercise IV.7.5: Let X and Ω be metric spaces and suppose that $f : X \to \Omega$ is a bijection. Then f is an open map iff f is a closed map.

To start off, for any set $E \subseteq X$ we have that $f(E^{\mathsf{C}}) = f(X) - f(E)$ since f is injective. Then as f is surjective we have that $f(X) - f(E) = (f(E))^{\mathsf{C}}$. Hence, we've shown that set complements commute in and out of the function.

$(\Longrightarrow)$
Suppose $f(U)$ is open for all open U. Then given any closed set C, we know that $f(C^{\mathsf{C}})$ is open. But we also know that $f(C^{\mathsf{C}}) = (f(C))^{\mathsf{C}}$. So, $f(C)$ is closed.

$(\Longleftarrow)$
Literally do the same reasoning but swap the words open and closed.

I want to finish taking notes on Haar measures now. See *page 364* for where I'm starting from. As a reminder, I'm following Folland's real analysis book. Also, if G is a topological group then e is the identity element of G.

Theorem 11.9: If μ and ν are left Haar measures on a locally compact group G then there exists $c > 0$ such that $\mu = c\nu$.

(Proof for when μ is both left- and right-invariant [which will for example happen if G is abelian]):

Pick $h \in C_c^+(G)$ such that $h(x) = h(x^{-1})$. (One way of doing this would be to just define $h(x) = g(x) + g(x^{-1})$ where $g \in C_c^+(G)$). Then for any $f \in C_c(G)$, we have that:

$$\begin{aligned}\int h\mathrm{d}\nu \cdot \int f\mathrm{d}\mu &= \iint h(y)f(x)\mathrm{d}\mu(x)\mathrm{d}\nu(y)\\ &= \iint h(y)f(xy)\mathrm{d}\mu(x)\mathrm{d}\nu(y) \text{ (by right-invariance of } \mu)\\ &= \iint h(y)f(xy)\mathrm{d}\nu(y)\mathrm{d}\mu(x) \text{ (by Fubini's theorem)}\\ &= \iint h(x^{-1}y)f(x^{-1}xy)\mathrm{d}\nu(y)\mathrm{d}\mu(x) \text{ (by left invariance of } \nu)\\ &= \iint h(y^{-1}x)f(y)\mathrm{d}\nu(y)\mathrm{d}\mu(x) \text{ (by how we chose } h)\\ &= \iint h(y^{-1}x)f(y)\mathrm{d}\mu(x)\mathrm{d}\nu(y) \text{ (by Fubini's theorem)}\\ &= \iint h(yy^{-1}x)f(y)\mathrm{d}\mu(x)\mathrm{d}\nu(y) \text{ (by left-invariance of } \mu)\\ &= \iint h(x)f(y)\mathrm{d}\mu(x)\mathrm{d}\nu(y) = \int h\mathrm{d}\mu \cdot \int f\mathrm{d}\nu\end{aligned}$$

Hence $\int f\mathrm{d}\mu = c\int f\mathrm{d}\nu$ for all $f \in C_c^+(G)$ where $c = (\int h\mathrm{d}\mu)/(\int h\mathrm{d}\nu)$. (Recall that $\int h\mathrm{d}\nu > 0$ by *proposition 11.4(c) on page 353*). In turn, this implies that $\mu = c\nu$ (since μ and ν are Radon measures).

(General Proof:)

Note that $\mu = c\nu$ iff the ratio $r_f := (\int f\mathrm{d}\mu)/(\int f\mathrm{d}\nu)$ is independent of $f \in C_c^+(G)$.

The $(\Longleftarrow)$ implication is obvious. Meanwhile, to see the other direction, note that for any nonempty open set we can find a sequence of functions such that $(\int f_n\mathrm{d}\mu)/(\int f_n\mathrm{d}\nu) \to \mu(U)/\nu(U)$ as $n \to \infty$ (again, U has nonzero ν measure by *proposition 11.4(c)*). So the right side statement would imply $\mu(U) = r_f\nu(U)$ for all open sets U. Then by the outer regularity of μ and ν we'd have that $\mu = r_f\nu$.

So, suppose $f, g \in C_c^+(G)$. Then fix a compact symmetric neighborhood V_0 of e and set $A := (\operatorname{supp}(f))V_0 \cup V_0(\operatorname{supp}(f))$ and $B := (\operatorname{supp}(g))V_0 \cup V_0(\operatorname{supp}(g))$.

Note by the continuity of $x \mapsto x^{-1}$ that if N is a compact neighborhood of e then so is N^{-1}. So, we have no issue defining $V_0 = N \cap N^{-1}$ like in *proposition 11.1(b)*. Similarly, A and B are compact by proposition 11.1(f).

Now for any $y \in V_0$ the functions $x \mapsto f(xy) - f(yx)$ and $x \mapsto g(xy) - g(yx)$ are supported in A and B. Also by *proposition 11.2*, given any $\varepsilon > 0$ we can get a symmetric compact neighborhood $V \subseteq V_0$ of e such that:

$\sup_{x\in G} |f(xy) - f(yx)| < \varepsilon$ and $\sup_{x\in G} |g(xy) - g(yx)| < \varepsilon$ for all $y \in V$.

To get V, first just take the intersection of V_0 with four different neighborhoods gotten by proposition 11.2. Then use LCH space properties to get compact neighborhood of e contained in that intersection. And finally, use proposition 11.1(b) to get a compact symmetric neighborhood.

Pick $h \in C_C^+(G)$ with $\operatorname{supp}(h) \subseteq V$ and $h(x) = h(x^{-1})$. Similarly to the last page, you can do this by defining $h(x) = g(x) + g(x^{-1})$ where $g \in C_c^+(G)$ satisfies that $\operatorname{supp}(g) \subseteq V$. Then:

$$\begin{aligned}\int h\mathrm{d}\nu \int f\mathrm{d}\mu &= \iint h(y)f(x)\mathrm{d}\mu(x)\mathrm{d}\nu(y)\\ &= \iint h(y)f(yx)\mathrm{d}\mu(x)\mathrm{d}\nu(y) \text{ (by the left-invariance of } \mu)\end{aligned}$$

But also note that:

$$\begin{aligned}\int h\mathrm{d}\mu \int f\mathrm{d}\nu &= \iint h(x)f(y)\mathrm{d}\mu(x)\mathrm{d}\nu(y)\\ &= \iint h(y^{-1}x)f(y)\mathrm{d}\mu(x)\mathrm{d}\nu(y) \text{ (by the left-invariance of } \mu)\\ &= \iint h(y^{-1}x)f(y)\mathrm{d}\nu(y)\mathrm{d}\mu(x) \text{ (by Fubini's theorem)}\\ &= \iint h(x^{-1}y)f(y)\mathrm{d}\nu(y)\mathrm{d}\mu(x) \text{ (by how we chose } h)\\ &= \iint h(xx^{-1}y)f(xy)\mathrm{d}\nu(y)\mathrm{d}\mu(x) \text{ (by left-invariance of } \nu)\\ &= \iint h(y)f(xy)\mathrm{d}\mu(x)\mathrm{d}\nu(y) \text{ (by Fubini's theorem)}\end{aligned}$$

Therefore, we have that:

$$\begin{aligned}\left|\int h\mathrm{d}\mu \int f\mathrm{d}\nu - \int h\mathrm{d}\nu \int f\mathrm{d}\mu\right| &= \left|\iint h(y)\cdot(f(xy) - f(yx))\mathrm{d}\mu(x)\mathrm{d}\nu(y)\right|\\ &\leq \varepsilon\mu(A)\int h\mathrm{d}\nu\end{aligned}$$

By identical reasoning we can also conclude that:

$$\left|\int h\mathrm{d}\mu \int g\mathrm{d}\nu - \int h\mathrm{d}\nu \int g\mathrm{d}\mu\right| \leq \varepsilon\mu(B)\int h\mathrm{d}\nu.$$

So, divide these inequalities by $(\int h\mathrm{d}\nu)(\int f\mathrm{d}\nu)$ and $(\int h\mathrm{d}\nu)(\int g\mathrm{d}\nu)$ respectively to get that:

$$\left|\frac{\int h\mathrm{d}\mu}{\int h\mathrm{d}\nu} - \frac{\int f\mathrm{d}\mu}{\int f\mathrm{d}\nu}\right| \leq \frac{\varepsilon\mu(A)}{\int f\mathrm{d}\nu} \text{ and } \left|\frac{\int h\mathrm{d}\mu}{\int h\mathrm{d}\nu} - \frac{\int g\mathrm{d}\mu}{\int g\mathrm{d}\nu}\right| \leq \frac{\varepsilon\mu(B)}{\int g\mathrm{d}\nu}$$

In turn, by triangle inequality we know that $\left|\frac{\int f\mathrm{d}\mu}{\int f\mathrm{d}\nu} - \frac{\int g\mathrm{d}\mu}{\int g\mathrm{d}\nu}\right| \leq \varepsilon(\frac{\mu(A)}{\int f\mathrm{d}\nu} + \frac{\mu(B)}{\int g\mathrm{d}\nu})$. And to finish the proof we take $\varepsilon \to 0$ (which we can do because A and B were chosen before we considered ε). ■

If μ is a left Haar measure on G and $x \in G$, then the measure $\mu_x(E) = \mu(Ex)$ is another left Haar measure. Hence by the prior theorem there exists a number $\Delta(x)$ such that $\mu_x = \Delta(x)\mu$. Also by the prior theorem, $\Delta(x)$ is independent of our choice of left Haar measure μ.

We call $\Delta : G \to (0, \infty)$ the modular function of G.

Proposition 11.10: Δ is a continuous homomorphism from G to the multiplicative group of positive real numbers. Moreover, if μ is a left Haar measure on G, for any $f \in L^1(\mu)$ and $y \in G$ we have that $\int(R_y f)\mathrm{d}\mu = \Delta(y^{-1})\int f\mathrm{d}\mu$.

Proof:

For any $x, y \in G$ and $E \in \mathcal{B}_G$ we have that:

$$\Delta(xy)\mu(E) = \mu(Exy) = \Delta(y)\mu(Ex) = \Delta(y)\Delta(x)\mu(E) = \Delta(x)\Delta(y)\mu(E).$$

Hence, Δ is a group homomorphism from G to $(0, \infty)$.

Next note that $\mu_{y^{-1}}$ is just the image (or pushfoward) measure of the function $x \mapsto xy$. Hence by *proposition 10.1 on page 193*:

$$\int(R_y f)\mathrm{d}\mu = \int f\mathrm{d}\mu_{y^{-1}} = \Delta(y^{-1})\int f\mathrm{d}\mu$$

Finally, the below exercise plus the above formula shows that the map $y \mapsto \Delta(y^{-1})\int f\mathrm{d}\mu$ is continuous for any $f \in L^1(\mu)$. After fixing f so that $\int f\mathrm{d}\mu = 1$ and composing this map from the inside with the continuous inversion map, we get that Δ is continuous.

Exercise 11.2: If μ is a Radon measure on the locally compact group G and $f \in C_c(G)$ then the functions $x \mapsto \int(L_x f)\mathrm{d}\mu$ and $x \mapsto \int(R_x f)\mathrm{d}\mu$ are continuous.

The proof is analogous for the left translation and right translation cases. So I'll just focus on the map $x \mapsto \int(R_x f)\mathrm{d}\mu$.

Given $f \in C_c(G)$, consider any fixed $x_0 \in G$ and $\varepsilon > 0$. By *proposition 11.2* we can find a neighborhood V of e such that for all $y \in V$:

$$\|R_y(R_{x_0}f) - (R_{x_0}f)\|_u = \|R_{yx_0}f - (R_{x_0}f)\|_u < \frac{\varepsilon}{\mu(\mathrm{supp}(R_{x_0}f))}.$$

In particular, this means for any x in the neighborhood Vx_0 of x_0 that

$$|\int(R_x f)\mathrm{d}\mu - \int(R_{x_0}f)\mathrm{d}\mu| \leq \frac{\varepsilon}{\mu(\mathrm{supp}(R_{x_0}f))} \cdot \mu(\mathrm{supp}(R_{x_0}f)) = \varepsilon.$$

And this proves that $x \mapsto \int(R_x f)\mathrm{d}\mu$ is continuous at $x = x_0$. ■

Any left Haar measure of a locally compact group G is also a right Haar measure iff $\mathrm{im}(\Delta) = 1$, in which case G is called unimodular. Now it's obvious that all abelian locally compact groups are unimodular. But interestingly enough, we can also show that if a group becomes not abelian enough, then it's also guarenteed to be unimodular.

Proposition 11.12: Let G be a locally compact group. If $G/[G,G]$ is finite then G is unimodular.

Since Δ is a homomorphism from G to an abelian group, we must have that the commutator subgroup $[G,G]$ is contained in the kernel of Δ. Hence, by quotienting out $[G,G]$ we get a well-defined homomorphism $\widetilde{\Delta} : G/[G,G] \to (0,\infty)$. But now as $G/[G,G]$ is finite, we must have that $\mathrm{im}(\widetilde{\Delta}) = \mathrm{im}(\Delta)$ is a finite subgroup of $(0,\infty)$. Yet, the only finite subgroup of the multiplicative group of positive real numbers is $\{1\}$. So, $\Delta(g) = 1$ for all $g \in G$. ■

Another useful case is as follows:

Proposition 11.13: If G is a compact group then G is unimodular.

Proof:
Let μ be a left Haar measure. Then for any $x \in G$ we have that $\mu(G) = \mu(Gx^{-1}) = \Delta(x)\mu(G)$. And since $0 < \mu(G) < \infty$, this means that $\Delta(x) = 1$ for all $x \in G$. ■

This is where I'm going to stop covering Folland again and instead switch over to the math 241 class (which I'm still in by the way).

11/26/2025

Math 241a Notes:

In this class we'll assume topological groups are always Hausdorff. Recall *page 351* for why this isn't must of a restriction.

(Example 1.3.4:) Here are some relevant examples of topological groups.

- Note that $\mathrm{GL}_n(\mathbb{R})$ is a group with an obvious embedding into $\mathbb{R}^{n^2}$. Furthermore, matrix multiplication and inversion can be written such that each component of the resulting matrix is a rational function of the components of the input matrices. Hence, giving $\mathrm{GL}_n(\mathbb{R})$ the Euclidean topology induced by $\mathbb{R}^{n^2}$ turns $\mathrm{GL}_n(\mathbb{R})$ into a topological group.
- If G is a topological group and $H < G$, then H equipped with the subspace topology will be a topological group. In particular, this means any subgroup of $\mathrm{GL}_n(\mathbb{R})$ is a topological group.

  Side note, on *page 92* I showed that the set of all orthogonal nxn matrices $O_n(\mathbb{R})$ is a smooth compact manifold in $\mathbb{R}^{n^2}$. And since the group operations on $O_n(\mathbb{R})$ are smooth, we say $O_n(\mathbb{R})$ is a lie group.

- If $\mathcal{X}$ is a normed vector space, then $\mathrm{Iso}(\mathcal{X})$ is a topological group when equipped with the strong operator topology.

  Proof:
  Let $\langle (T_i, S_i) \rangle_{i \in I}$ be a net in $\mathrm{Iso}(\mathcal{X}) \times \mathrm{Iso}(\mathcal{X})$ converging to (T, S) operator strongly. Then we claim that $T_i S_i \to TS$ operator strongly. After all, fix any $x \in \mathcal{X}$ and $\varepsilon > 0$. Then as T_i is an isometry for each i, we have that:
  $$\begin{aligned} \|T_i(S_i(x)) - T(S(x))\| &\leq \|T_i(S_i(x)) - T_i(S(x))\| + \|T_i(S(x)) - T(S(x)) \\ &= \|S_i(x) - S(x)\| + \|T_i(S(x)) - T(S(x))\| \end{aligned}$$

  Then because $S_i \to S$ and $T_i \to T$ operator strongly, we know that $\|S_i(x) - S(x)\| \to 0$ and $\|T_i(S(x)) - T(S(x))\| \to 0$.

  Next, let $\langle T_i \rangle_{i \in I}$ be a net in $\mathrm{Iso}(\mathcal{X})$ converging to T operator strongly. Then since T_i is an isometry, for any fixed $x \in \mathcal{X}$ we have that:
  $$\|T_i^{-1}(x) - T^{-1}(x)\| = \|x - T_i(T^{-1}(x))\| = \|T(T^{-1}(x)) - T_i(T^{-1}(x))\|$$

  And since $T_i \to T$ operator strongly, $\|T(T^{-1}(x)) - T_i(T^{-1}(x))\| \to 0$. Hence $T_i^{-1} \to T^{-1}$ operator strongly. ■

(Zimmer) Exercise 1.21: Let $\mathcal{H}$ be a Hilbert space and let $U(\mathcal{H})$ be the group of unitary linear operators on $\mathcal{H}$. Then the strong and weak operator topologies are the same on $U(\mathcal{H})$.

Proof:
We already know the strong operator topology is finer than the weak operator topology. Meanwhile, to show the other direction it suffices to show by the corollary on *page 229* that weak operator convergence in $U(\mathcal{H})$ implies strong operator convergence in $U(\mathcal{H})$.

Let $\{e_i\}_{i \in I}$ be an orthonormal basis for $\mathcal{H}$. Then consider any net $\langle T_\alpha \rangle_{\alpha \in A}$ in $U(\mathcal{H})$ converging to T. Since $\|T_\alpha\|, \|T\| = 1$ for all $\alpha \in A$, we know by example 1.3.2 on *pages 303-304* that if $\|T_\alpha e_i - Te_i\| \to 0$ for all i then $T_\alpha \to T$ operator strongly.

As a side note, none of the reasoning I wrote on pages 303 and 304 breaks down if you use nets instead of sequences. I just used sequences because the professor prefers them.

Fortunately, note that:
$$\begin{aligned} \|T_\alpha e_i - Te_i\|^2 &= \langle T_\alpha e_i - Te_i, T_\alpha e_i - Te_i \rangle \\ &= \langle T_\alpha e_i, T_\alpha e_i - Te_i \rangle - \langle Te_i, T_\alpha e_i - Te_i \rangle \\ &= \langle T_\alpha e_i, T_\alpha e_i \rangle - \langle T_\alpha e_i, Te_i \rangle - \langle Te_i, T_\alpha e_i \rangle + \langle Te_i, Te_i \rangle \\ &= \langle e_i, e_i \rangle - \langle T_\alpha e_i, Te_i \rangle - \langle Te_i, T_\alpha e_i \rangle + \langle e_i, e_i \rangle \\ &= 2 - (\langle T_\alpha e_i, Te_i \rangle - \langle Te_i, T_\alpha e_i \rangle) \\ &= 2 - (\langle T_\alpha e_i, Te_i \rangle - \overline{\langle T_\alpha e_i, Te_i \rangle}) \end{aligned}$$

Since $T_\alpha \to T$ operator weakly, we know that $\langle T_\alpha x, y \rangle \to \langle Tx, y \rangle$ for all $x, y \in \mathcal{H}$. In particular, setting $x = e_i$ and $y = Te_i$ we have that $\langle T_\alpha e_i, Te_i \rangle \to \langle Te_i, Te_i \rangle$. And as T is unitary, the latter is equal to $\langle e_i, e_i \rangle = 1$. This shows that:
$$2 - (\langle T_\alpha e_i, Te_i \rangle - \overline{\langle T_\alpha e_i, Te_i \rangle}) \to 2 - (1 + 1) = 0. \blacksquare$$

Given a topological group G and a topological space X, we say an action $G \curvearrowright X$ is continuous if it's corresponding induced map $G \times X \to X$ is continuous. Note in that case that the map $\varphi_g(x) := g \cdot x$ is a homeomorphism on X with inverse $\varphi_{g^{-1}}$.

If G is a group and V is a vector space, a representation of G on V is a homomorphism $G \to \mathrm{GL}(V)$ where $\mathrm{GL}(V)$ is the group of invertible linear maps on V. If $\mathcal{X}$ is a topological vector space (which is always assumed to be over $\mathbb{R}$ or $\mathbb{C}$ in this class), a representation of G on $\mathcal{X}$ is a homomorphism $\pi : G \to \mathrm{Aut}(\mathcal{X})$. When G is also a topological group, we can talk about π as being continuous with respect to an operator topology on $\mathrm{Aut}(\mathcal{X})$.

If $\mathcal{X}$ is a normed vector space, a representation $\pi : G \to \mathcal{X}$ is called an isometric representation if $\pi(G) \subseteq \mathrm{Iso}(\mathcal{X})$. We similarly define unitary representations into Hilbert spaces.

(Zimmer) Exercise 1.12: If G is a topological group and $\mathcal{X}$ is a normed space, show that a representation $\pi : G \to \mathrm{Aut}(\mathcal{X})$ is continuous iff it is continuous at the identity e of G.

The ($\Longleftarrow$) direction with respect to each of the three settings below is trivial. Meanwhile, given any net $\langle g_i \rangle_{i \in I}$ in G converging to some element g, note that:

- $\|\pi(g_i) - \pi(g)\|_{\mathrm{op}} = \|\pi(g_i g^{-1})\pi(g) - \pi(g)\|_{\mathrm{op}} \leq \|\pi(g_i g^{-1}) - \mathrm{Id}\|_{\mathrm{op}} \cdot \|\pi(g)\|_{\mathrm{op}}$,
- $\|(\pi(g_i))(x) - (\pi(g))(x)\| = \|(\pi(g_i g^{-1}))((\pi(g))(x)) - (\pi(g))(x)\|$
  $= \|(\pi(g_i g^{-1}) - \mathrm{Id})((\pi(g))(x))\|$ for all $x \in \mathcal{X}$
- $|f((\pi(g_i))(x) - (\pi(g))(x))| = |f((\pi(g_i g^{-1}))((\pi(g))(x)) - (\pi(g))(x))|$
  $= |f\left((\pi(g_i g^{-1}) - \mathrm{Id})((\pi(g))(x))\right)|$ for all $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$

Thus, $\pi(g_i) \to \pi(g)$ in norm, operator strongly, or operator weakly if $\pi(g_i g^{-1}) \to \pi(e)$ in norm, operator strongly, or operator weakly respectively. Fortunately, the latter happens if π is continuous at e. ■

Proposition 1.3.9: Let G be a topological group acting continuously on an LCH space X. Then let $\pi : G \to \mathrm{Iso}(C_c(X))$ be given by $(\pi(g))(f) := f(g^{-1} \cdot x)$. Now π is a continuous representation when $\mathrm{Iso}(C_c(X))$ has the strong operator topology.

Proof:

To start off, recall *example 1.2.4* on page 284 for why $\pi(g) \in \mathrm{Iso}(C_c(X))$ for each g.

Technically, on page 284 I showed that $\pi(g)$ would be an isometric isomorphism on $BC(X)$. That said, as $x \mapsto g \cdot x$ and $x \mapsto g^{-1} \cdot x$ are continuous maps, we know that $\mathrm{supp}(f)$ is compact iff $g \cdot \mathrm{supp}(f)$ is compact. Hence, $\pi(g)$ maps $C_c(X)$ bijectively into $C_c(X)$.

Meanwhile, it's easy to see π is a group homomorphism. So, all that's left to show is that π is continuous, and to do that it suffices by the prior exercise to show π is continuous at $e \in G$. Thus, we want to show that if $f \in C_c(X)$ and $\varepsilon > 0$ then there is a neighborhood V of e with $\|(\pi(g))(f) - f\|_u < \varepsilon$ for all $g \in V$.

Fortunately, since $\mathrm{supp}(f)$ is compact and X is locally compact, we can find a precompact open set $U \subseteq X$ containing $\mathrm{supp}(f)$. Then for each $x \in \mathrm{supp}(f)$, continuity of the group action implies there is an open neighborhood U_x of x in X and an open neighborhood W_x of e in G such that $W_x \cdot U_x \subseteq U$.

$W_x \times U_x$ is an open neighborhood of (e, x) which is in the preimage of U with respect to the group action.

Next, by the compactness of $\mathrm{supp}(f)$ there exists $x_1, \ldots, x_n \in \mathrm{supp}(f)$ such that $\mathrm{supp}(f) \subseteq \bigcup_{i=1}^n U_{x_i}$. In turn, $W := \bigcap_{i=1}^n W_{x_i}$ is an open neighborhood of e such that $W \cdot \mathrm{supp}(f) \subseteq U$. And in particular, after making W symmetric (remember *proposition 11.1(b)* from Folland), we can say that $\mathrm{supp}((\pi(g))(f)) \subseteq \overline{U}$ for all $g \in W$.

Now we just need to find an open neighborhood $V \subseteq W$ of e such that:

$$|f(g^{-1} \cdot x) - f(x)| < \varepsilon \text{ for all } x \in \overline{U}.$$

To do that, note by the continuity of f that for each $x \in \overline{U}$ we can choose an open neighborhood U'_x of x such that $|f(y) - f(x)| < \varepsilon/2$ for all $y \in U'_x$. Then by the continuity of the group action, we can find open neighborhoods Z_x of e in G and Y_x of x in X such that $Z_x \cdot Y_x \subseteq U'_x$.

Using the compactness of $\overline{U}$, choose a new finite set $x_1, \ldots, x_m \in \overline{U}$ such that $\overline{U} \subseteq \bigcup_{i=1}^m Y_{x_i}$. Then set $W' = W \cap \bigcap_{i=1}^m Z_{x_i}$ and define $V := W' \cap (W')^{-1}$. Now V is an open neighborhood of e in G. Also if $g \in V$ and $y \in \overline{U}$, then because $y \in Y_{x_i} \subseteq U_{x_i}$ for some i (which also means $g^{-1} \cdot y \in Y_{X_i} \subseteq U_{x_i}$), we know that:

$$|f(g^{-1} \cdot y) - f(y)| \leq |f(g^{-1}y) - f(x_i)| + |f(x_i) - f(y)| < \varepsilon/2 + \varepsilon/2. \blacksquare$$

A basic corollary of the above proposition is that every $f \in C_c(\mathbb{R}^n)$ is uniformly continuous. After all, we can apply the above proposition to the action $\mathbb{R}^n \curvearrowright \mathbb{R}^n$ by translation. That said, I already proved this corollary in my notes from Spring 2025.

In a similar vein, the next two results will prove a generalization of Folland proposition 8.5 from my math 240c notes from last spring.

(Zimmer) exercise 1.13: Suppose $\mathcal{X}$ is a normed topology and $\langle T_i \rangle_{i \in I}$ is a net in $B(\mathcal{X})$. Also suppose that $T \in B(\mathcal{X})$ and there exists $C > 0$ with $\|T_i\|, \|T\| < C$ for all $i \in I$. Then $T_i \to T$ operator strongly if and only if there is a dense set $\mathcal{X}_0 \subseteq \mathcal{X}$ such that $T_i x \to Tx$ for all $x \in \mathcal{X}$.

Proof:

The $(\Longrightarrow)$ direction is trivial. As for the other direction, consider any $x \in \mathcal{X}$ and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{X}_0$ converging to x. Then, we know that $\|x_n - x\| < \varepsilon/2C$ for some $n \in \mathbb{N}$. In turn:

$$\begin{aligned}\|T_i x - Tx\| &\leq \|T_i x - T_i x_n\| + \|T_i x_n - Tx_n\| + \|Tx_n - Tx\| \\ &\leq \|T_i\|\|x - x_n\| + \|T_i x_n - Tx_n\| + \|T\|\|x_n - x\| \\ &< C\tfrac{\varepsilon}{2C} + \|T_i x_n - Tx_n\| + C\tfrac{\varepsilon}{2C} = \|T_i x_n - Tx_n\| + \varepsilon.\end{aligned}$$

And since $T_i x_n \to Tx_n$, we thus know that $\|T_i x - Tx\| < 2\varepsilon$ eventually for all $\varepsilon > 0$. $\blacksquare$

Proposition 1.3.10: Let G be a topological group acting continuously on an LCH space X. Suppose μ is a measure on X which is G-invariant (i.e φ_g is measure preserving for each $g \in G$ [recall *page 263*]). Suppose further that $\mu(K) < \infty$ for every compact $K \subseteq X$ (which is true if μ is Radon). Then for $1 \leq p < \infty$, the representation $\pi : G \to \mathrm{Iso}(L^p(X))$ given by $(\pi(g)(f))(x) \coloneqq f(g^{-1} \cdot x)$ is continuous for the strong operator topology.

Proof:

To see that π really does map G into $\mathrm{Iso}(L^p(X))$ just apply lemma 2.6 on *page 264* to $|f(x)|^p$ and $|f(g^{-1} \cdot x)|^p$. Also, π is seen to be a group homomorphism identically as in the last proposition. So, we just need to show that π is continuous for the strong operator topology. This is equivalent to saying that $g \mapsto (\pi(g))(f)$ is a continuous map from G to $L^p(X)$ for all $f \in L^p(X)$.

Fortunately, like in the last proposition it suffices to show that $\|(\pi(g_i))(f) - f\|_p \to 0$ for any net $\langle g_i \rangle_{i \in I}$ converging to e in G. Also, by the prior exercise plus the fact that $C_c(X)$ is dense in $L^p(X)$ for $1 \leq p < \infty$ (see my math 240c notes), it suffices to assume that $f \in C_c(X)$.

But now we already know from the proof of the last proposition that $(\pi(g_i))(f) \to f$ uniformly and that we can find a compact set $\overline{U}$ such that $\mathrm{supp}(f) \subseteq \overline{U}$ and $\mathrm{supp}((\pi(g_i))(f)) \subseteq \overline{U}$ eventually. This implies L^p convergence. ■

Note that every representation $\pi : G \to \mathrm{Aut}(\mathcal{X})$ is a group action $G \times \mathcal{X} \to \mathcal{X}$ by linear automorphisms (i.e. $g \cdot x = (\pi(g))x$) and vice versa. Thus, when we talk about fixed points of representations we are really talking about the fixed points of their induced group action.

Kakutani-Markov Fixed Point Theorem: Let $\mathcal{X}$ be a topological vector space whose topology is defined by a sufficient family of seminorms. Suppose G is an abelian group and $\pi : G \to \mathrm{Aut}(\mathcal{X})$ is a representation. Let $A \subseteq \mathcal{X}$ be a compact convex set that is G-invariant (i.e. $(\pi(g))(A) \subseteq A$ for all $g \in G$). Then there is a G-fixed point in A.

Proof:

For each $g \in G$ and $n \geq 0$, define $M_{n,g} \in B(\mathcal{X})$ by $M_{n,g} = \frac{1}{n}\sum_{i=0}^{n-1} \pi(g^i)$. Since A is convex and G-invariant, we have that $M_{n,g}(A) \subseteq A$ for all n, g. Next, let G^* be the semigroup of operators generated by $\{M_{n,g} : n \geq 0, g \in G\}$ (i.e. the collection of all finite compositions of such operators).

(Technically G^* would be a monoid as $M_{1,e} = \mathrm{Id}$. That said, we don't necessarily know if the operators in G^* have inverses, or if they do have inverses which aren't in G^*).

Since G is abelian we know that G^* is commutative. After all:

$$\begin{aligned} M_{n_1,g_1} M_{n_2,g_2} &= \left(\frac{1}{n_1}\sum_{i=0}^{n_1-1} \pi(g_1^i)\right)\left(\frac{1}{n_2}\sum_{j=0}^{n_2-1} \pi(g_2^j)\right) \\ &= \frac{1}{n_1 n_2}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} \pi(g_1^i)\pi(g_2^j) = \frac{1}{n_2 n_1}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} \pi(g_2^j)\pi(g_1^i) \end{aligned}$$

$$= \left(\frac{1}{n_2} \sum_{j=0}^{n_2-1} \pi(g_2^j) \right) \left(\frac{1}{n_1} \sum_{i=0}^{n_1-1} \pi(g_1^i) \right)$$

$$= M_{n_2,g_2} M_{n_1,g_1}$$

Now we claim that $F := \bigcap_{T \in G^*} T(A) \neq \emptyset$ and that every element of F is a G-fixed point. To see this, note that as A is compact and each $T \in G^*$ is continuous, we know that $T(A)$ is compact for each $T \in G^*$. Then since $\mathcal{X}$ is Hausdorff, we know that $T(A)$ is closed for each $T \in G^*$. And since $T(A) \subseteq A$ for each $T \in G^*$ and A is compact, we will be able to use the finite intersection property to prove F is nonempty if for any finite set $T_1, \ldots, T_n \in G^*$ we have that $\bigcap_{i=1}^n T_i(A) \neq \emptyset$.

Fortunately, if we let $S = T_1 \circ \cdots \circ T_n \in G^*$ then we know that:

$$S(A) = T_1(T_2 \circ \cdots \circ T_n(A)) \subseteq T_1(A).$$

Also note that as G^* is commutative, we can rewrite $S = T_i \circ T_1 \circ \cdots \circ T_{i-1} \circ T_{i+1} \circ \cdots \circ T_n$ and apply the prior line's reasoning to get that $S(A) \subseteq T_i(A)$ for any i. Therefore:

$$\emptyset \neq S(A) \subseteq \bigcap_{i=1}^n T_i(A).$$

With that we know F is nonempty. So, we now consider any $y \in F$. By the definition of F, for each $n \geq 0$ and $g \in G$ there is some $x \in A$ such that $y = \frac{1}{n}(x + \cdots + \pi(g^{n-1})x)$. In turn:

$$\pi(g)y - y = \frac{\pi(g)x + \cdots + \pi(g^n)x}{n} - \frac{x + \cdots + \pi(g^{n-1})x}{n} = \frac{\pi(g^n)x - x}{n}$$

Now if p is any of the seminorms defining the topology, then for each n we have that $p(\pi(g)y - y) \leq 2B_p/n$ where $B_p = \sup\{p(a) : a \in A\}$. (Note that B_p is well-defined (and finite) since A is compact and p is continuous.) As n was arbitrary, we can take $n \to \infty$ to get that $p(\pi(g)y - y) = 0$. And since p was an arbitrary seminorm in the sufficient family defining the topology on $\mathcal{X}$, we have that $\pi(g)y = y$. Finally, as g was arbitrary we know y is a G-fixed point. ■

As a side note, I proved a similar fact on the second problem set for math 200a (see *page 287*). To briefly compare the two results, this result doesn't assume G is a finite group like the homework problem did, or that we are working in $\mathbb{R}^n$. That said, the homework problem didn't require A to be compact and G to be abelian. Also, the homework problem assumed G acted by affine transformations as opposed to linear transformations.

Although, it is worth noting that none of the reasoning in this proof breaks down if we assume $\pi(g)$ is a continuous affine transformation on $\mathcal{X}$ instead of a linear transformation.

As shown on page 286, every affine transformation is a linear transformation minus a constant and vice versa. So, we know that summing, scaling, and composing continuous affine transformations always yields a continuous affine transformation. Furthermore, our evaluation of $\pi(g)y - y$ doesn't change and we still have that each $M_{n,g}$ maps A into A.

One particular compact convex subset of a topological vector space we care about is the set of probability measures $M(X) \subseteq C(X)^*$ (where X is a compact metric space). (It is easy to see that $M(X)$ is convex and we know from *corollary 1.1.29 (see page 283)* that $M(X)$ is compact in the weak* topology.)

Before continuing on, I want to actually prove that all probability measures on X are regular (and thus Radon) since I forgot to show that back on *page 283*.

Claim: If X is a compact metric space, then X is second countable.

Proof:
Since X is totally bounded, we know that there is a finite collection $\mathcal{U}_n$of open balls of radius $1/n$ covering X for all $n \in \mathbb{N}$. Then we claim that $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ is a countable base for X. After all, we already know each $x \in X$ is contained in some $B \in \mathcal{U}$. On the other hand, suppose $x \in X$ and $V \subseteq X$ is an open set containing x. Then we know $B_r(x) \subseteq V$ for some $r > 0$. In turn, if we fix $k \in \mathbb{N}$ such that $1/k < r/2$ then we know there is some ball B of radius $1/k$ in $\mathcal{U}$ satisfying that $x \in B \subseteq B_r(x) \subseteq V$.

Consequently, we know by Folland theorem 7.8 and the comment right after it in my math 240c notes that any Borel measure on a compact metric space X that is finite on compact sets is regular. In particular, this means all finite Borel measures on compact metric spacess are regular.

(Definition 1.3.12:) If $\pi : G \to \mathrm{GL}(V)$ is a representation, we define the adjoint of π by $\pi^* : G \to \mathrm{GL}(V^*)$ where $\pi^*(g) := \pi(g^{-1})^*$. ($\pi(g^{-1})^*$ denotes the transpose of $\pi(g^{-1})$ [see *page 102* and *page 300*.])

Note that π^* is a representation. After all, recalling *proposition 1.3.18 on page 103*, we know that:

$$\pi^*(gh) = \pi(h^{-1}g^{-1})^* = (\pi(h^{-1})\pi(g^{-1}))^* = \pi(g^{-1})^*\pi(h^{-1})^* = \pi^*(g)\pi^*(h)$$

When $V = \mathcal{X}$ is a topological vector space (so that our definition of a representation requires π to map G into $\mathrm{Aut}(\mathcal{X})$), then π^* still defines a representation if either:

- $\mathcal{X}^*$ has the weak* topology;
- $\mathcal{X}$ is normed and $\mathcal{X}^*$ has the operator norm topology.

To see this just recall *page 301* and also note that $\pi^*(g)$ has the continuous inverse $\pi^*(g^{-1})$ for all $g \in G$.

Now let's return to considering a compact metric space X and suppose G is a topological group acting continuously on X. By using *proposition 1.3.9* we get a representation $\pi : G \to \mathrm{Iso}(C(X))$ given by $(\pi(g))(f) := f(g^{-1} \cdot x)$. Then, by taking the adjoint of π we get a representation $\pi^* : G \to \mathrm{Aut}(C(X)^*)$ given by:

$$(\pi^*(g))(\lambda) = (\pi(g^{-1}))^*(\lambda) = \lambda \circ \pi(g^{-1}).$$ (FYI this construction continues on the next page...)

As a bit of notation, if μ is a measure on X and $\varphi : X \to Y$ is a measurable function, then I'll follow Zimmer and denote $\varphi_*\mu(E) \coloneqq \mu(\varphi^{-1}(E))$ to be the pushforward / image measure of the function φ. Also, as G is acting continuously on X I'll let $g_*\mu$ denote the pushforward of $\varphi_g(x) \coloneqq g \cdot x$ (in other words $g_*\mu(E) = \mu(g^{-1} \cdot E)$).

Now recall by the Riesz-representation theorem that every linear functional in $C(X)^*$ corresponds to integration by a complex Radon measure and vice versa. (Also as mentioned on the prior page, all complex Borel measures on X are Radon.) Therefore, we now ask what does the action of π^* look like on a given measure μ?

Note for any $f \in C(X)$ that:

$$((\pi^*(g))(\textstyle\int \cdot \, \mathrm{d}\mu))(f) = \int (\pi(g^{-1}))(f)\mathrm{d}\mu = \int f(g \cdot x)\mathrm{d}\mu = \int f \mathrm{d}g_*\mu$$

Thus $(\pi^*(g))(\mu) = g_*\mu$ for all $\mu \in C(X)^*$.

Importantly, recall that pushforwards preserve the positivity of a measure and the total measure of the space. Thus $M(X)$ is invariant under π^* and we arrive at the following result:

Corollary 2.1.6: Let G be an abelian group acting continuously on a compact metric space X. Then there is a G-invariant probability measure on X.

Proof:
Apply the Kakutani-Markov fixed point theorem to $M(X) \subseteq C(X)^*$ (equipped with the weak* topology) and the representation $\pi^* : G \to \mathrm{Aut}(C(X)^*)$ described above. Then the resulting probability measure satisfies that $g_*\mu(E) \coloneqq \mu(g^{-1} \cdot E) = \mu(E)$ for all $g \in G$. ■

11/27/2025

It's Thanksgiving so let's do some more representation theory before I have to go back to taking notes for the classes with actual due dates. (In other words I'm still taking math 241a notes…)

To start off, I already proved in math math 240b homework that there is only one norm topology for any finite dimensioned real or complex vector space. Now, I would like to prove the more general statement that there is only one Hausdorff topology on a finite dimensioned real or complex vector space $\mathcal{X}$ such that $\mathcal{X}$ is a topological vector space.

First, I'd like to go over some quick facts about topological vector spaces now that I've learned some topological group theory.

Lemma 1: If G, G' are topological groups and $\phi : G \to G'$ is a group homomorphism, then ϕ is continuous iff ϕ is continuous at $e \in G$.

Proof:
One direction is trivial. Meanwhile, to show the other direction suppose $\langle g_i \rangle_{i \in I}$ is any net in G converging to g. Then $\phi(g_i) \to \phi(g)$ if and only if $\phi(g_i)\phi(g)^{-1} = \phi(g_i g^{-1}) \to e' \in G'$. But note that $g_i \to g$ implies that $g_i g^{-1} \to e \in G$. So, if ϕ is continuous at $e \in G$ then we know that $\phi(g_i g^{-1}) \to \phi(e) = e'$. ■

A common application of this fact is that if $T : \mathcal{X} \to \mathcal{Y}$ is a linear map between topological vector spaces, then T is continuous iff T is continuous at 0. After all, topological vector spaces are just abelian topological groups which also satisfy that scalar multiplication by numbers other than ± 1 are continuous.

Lemma 2: A topological vector space $\mathcal{X}$ is Hausdorff iff $\{0\}$ is closed.

Proof:
By (Folland) proposition 11.3 on *page 351*, we know that $\mathcal{X}$ is Hausdorff iff $\mathcal{X}$ is T_1. And the latter is true iff every singleton is closed. Finally, note that as translation on $\mathcal{X}$ is a homeomorphism, we have that all singletons in $\mathcal{X}$ are closed iff $\{0\}$ is closed. ■

Lemma 3: Hausdorff topological groups (and therefore Hausdorff topological vector spaces) are regular (i.e. T_3). Specifically, given any closed subset F of a topological group G and $x \notin F$, we claim there is an open neighborhood V of e such that xV and FV are disjoint open sets containing x and F respectively.

Proof:
$U = F^{\mathsf{C}}$ is an open neighborhood of x. In turn, $x^{-1}U$ is an open neighborhood of e. So, by applying (Folland) proposition 11.1(b) and (c) on *page 333*, we can find a neighborhood V of $e \in G$ such that $VV \subseteq x^{-1}U$ and V is symmetric.

Now xV is a neighborhood of x in G. Also we claim that FV is disjoint from xV. After all, suppose $y \in xV \cap FV$. Then we can write $y = xv_1 = gv_2$ where $g \in F$ and $v_1, v_2 \in V$. But now $xv_1v_2^{-1} = g$. This is a contradiction as V is symmetric and $VV \subseteq x^{-1}U$ implies $xVV \subseteq U = F^{\mathsf{C}}$.

Finally, $FV = \bigcup_{g \in F} gV$ is an open set containing F. This finishes showing that G is T_3. ■

Next, I'm going to follow grandpa Rudin for a little bit. (see *citation 19* in the bibliography).

Lemma (unnumbered): Suppose V and W are K-vector spaces (where $K = \mathbb{R}$ or $\mathbb{C}$). Then suppose $T : V \to W$ is a linear map, $A \subseteq V$, and $B \subseteq W$. Then:

- If A is a subspace, convex set, or balanced set then so is $T(A)$.
- If B is a subspace, convex set, or balanced set then so is $T^{-1}(B)$.

Proof:
I think the only nontrivial claim to see here is that the preimage of a balanced set with respect to T is still balanced. (By the way see *page 230* for a reminder of what it means for a set to be balanced.)

Suppose $c \in K$ With $|c| \leq 1$. Then if $x \in cT^{-1}(B)$ we know that $T(c^{-1}x) \in B$. In turn, $T(x) \in cB \subseteq B$ and hence $x \in T^{-1}(B)$. Thus, $cT^{-1}(B) \subseteq T^{-1}(B)$. ■

<u>Theorem 1.18</u> Let λ be a linear functional on a Hausdorff topological vector space $\mathcal{X}$ such that $\lambda(x) \neq 0$ for some $x \in \mathcal{X}$. Then the following are equivalent:

(a) λ is continuous,

(b) $\ker(\lambda)$ is closed,

(c) $\ker(\lambda)$ is not dense in $\mathcal{X}$,

(d) λ is bounded on some neighborhood V of 0 in $\mathcal{X}$ (meaning $\sup_{x \in V} \lambda(x) < \infty$).

$(a \implies b)$
Since λ is continuous and $\{0\}$ is closed, we know that $\lambda^{-1}(\{0\}) = \ker(\lambda)$ is closed.

$(b \implies c)$
Suppose $\ker(\lambda)$ is closed. Then since there exists $x \in \mathcal{X}$ such that $x \notin \ker(\lambda)$, we know that $\ker(\lambda)$ is not dense in $\mathcal{X}$.

$(c \implies d)$
Suppose $x \notin \overline{\ker(\lambda)}$. Then by lemma 3 on the prior page plus the reasoning on *pages 230-232*, we can find a balanced neighborhood V of 0 such that $x + V$ and $\overline{\ker(\lambda)} + V$ are disjoint open sets containing x and $\overline{\ker(\lambda)}$ respectively. But also note that as V is balanced, we know that $\lambda(V)$ is balanced. The only way this is possible is if $\lambda(V)$ is an open or closed ball in $\mathbb{R}$ or $\mathbb{C}$ about 0 or if $\lambda(V)$ is all of $\mathbb{R}$ or $\mathbb{C}$.

Suppose the latter is true (that $\lambda(V) = \mathbb{R}$ or $\mathbb{C}$). Then there exists $y \in V$ such that $\lambda(y) = -\lambda(x)$. But then $x + y \in \overline{\ker(\lambda)}$ and that contradicts that $x + V$ is disjoint from $\overline{\ker(\lambda)} + V$. Hence, we conclude that λ is bounded on V.

$(d \implies a)$
Let $M > 0$ be such that $|\lambda(x)| \leq M$ for all $x \in V$. Then for any $r > 0$ we know that $W := (r/M)V$ is a neighborhood of 0 satisfying that $|\lambda(x)| < r$ for all $x \in W$. Hence, λ is continuous at 0. ■

Set 6 Problem 6: Suppose G is a group. For all $x, y \in G$, let $[x, y] \coloneqq xyx^{-1}y^{-1}$ and ${}^{x}y \coloneqq xyx^{-1}$. Then Hall's equation asserts that:

$$[[x, y], {}^{y}z][[y, z], {}^{z}x][[z, x], {}^{x}y] = 1.$$

To prove this, first note that:

$$\begin{aligned}[[a, b], {}^{b}c] &= (aba^{-1}b^{-1})(bcb^{-1})(bab^{-1}a^{-1})(bc^{-1}b^{-1}) \\ &= (aba^{-1})c(ab^{-1}a^{-1})(bc^{-1}b^{-1}) = {}^{a}b \cdot c \cdot {}^{a}(b^{-1}) \cdot {}^{b}(c^{-1})\end{aligned}$$

Also note that ${}^{b}(a^{-1}) \cdot {}^{b}a = bab^{-1} \cdot ba^{-1}b^{-1} = 1$. Therefore:

$$\begin{aligned}&[[x, y], {}^{y}z][[y, z], {}^{z}x][[z, x], {}^{x}y] \\ &= ({}^{x}y \cdot z \cdot {}^{x}(y^{-1}) \cdot {}^{y}(z^{-1}))({}^{y}z \cdot x \cdot {}^{y}(z^{-1}) \cdot {}^{z}(x^{-1}))({}^{z}x \cdot y \cdot {}^{z}(x^{-1}) \cdot {}^{x}(y^{-1})) \\ &= ({}^{x}y \cdot z \cdot {}^{x}(y^{-1}))(x \cdot {}^{y}(z^{-1}))(y \cdot {}^{z}(x^{-1}) \cdot {}^{x}(y^{-1})) \\ &= (xyx^{-1}zxy^{-1}x^{-1})(xyz^{-1}y^{-1})(yzx^{-1}z^{-1}xy^{-1}x^{-1}) \\ &= (xyx^{-1}zxy^{-1})(yz^{-1})(zx^{-1}z^{-1}xy^{-1}x^{-1}) \\ &= (xyx^{-1}zx)(x^{-1}z^{-1}xy^{-1}x^{-1}) = 1\end{aligned}$$

Next consider the lower central series $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$ for all i.

Note that $[H_1, H_2] = [H_2, H_1]$ for any subgroups $H_1, H_2 < G$ since $([h_1, h_2])^{-1} = [h_2, h_1]$. So this definition is equivalent to the one in class.