

(Audit) Math 140C Lecture Notes (Professor: Luca Spolaor)

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Lecture 1: 4/2/2024

A set $X \subseteq \mathbb{R}^n$ where $X \neq \emptyset$ is a vector space if:

- $\vec{x}, \vec{y} \in X \implies \vec{x} + \vec{y} \in X$
- $\vec{x} \in X$ and $c \in \mathbb{R} \implies c\vec{x} \in X$.

If $\phi = \{\vec{x}_1, \dots, \vec{x}_k\} \subset \mathbb{R}^n$, then we define:

$$\text{span } \phi = \text{span}\{\vec{x}_1, \dots, \vec{x}_k\} = \{c_1\vec{x}_1 + \dots + c_k\vec{x}_k \mid c_1, \dots, c_k \in \mathbb{R}\}.$$

If $E \subseteq \mathbb{R}^n$ and $E = \text{span } \phi$, then we say ϕ generates E .

Note that $\text{span}\{\vec{x}_1, \dots, \vec{x}_2\}$ forms a vector space (this is trivial to check).

$\{\vec{x}_1, \dots, \vec{x}_k\} \subseteq \mathbb{R}^n$ is called linearly independent if:

$$\sum_{i=1}^k c_i \vec{x}_i = 0 \implies \forall i \in \{1, \dots, k\}, c_i = 0.$$

If the above implication does not hold, then we call the set linearly dependent.

If $X \subseteq \mathbb{R}^n$ is a vector space, then we define the dimension of X as:

$$\dim(X) = \sup\{k \in \mathbb{N} \cup \{0\} \mid \exists \{\vec{x}_1, \dots, \vec{x}_k\} \subset X \text{ which is linearly independent}\}.$$

Also, we define any set containing $\vec{0}$ to be automatically linearly dependent.

This includes the singleton: $\{\vec{0}\}$.

$Q = \{\vec{x}_1, \dots, \vec{x}_k\}$ is a basis for X if:

- Q is linearly independent.
- $\text{span } Q = X$

As an example of a basis, for \mathbb{R}^n we define the standard basis as the set $\{e_1, e_2, \dots, e_n\}$ where e_i is the vector whose i th element is 1 and whose other elements are 0. It is pretty trivial to check that this set is in fact a basis of \mathbb{R}^n .

Proposition: If $B = \{\vec{x}_1, \dots, \vec{x}_k\}$ is a basis of a vector space X , then:

$$1. \forall \vec{v} \in X, c_1, \dots, c_k \in \mathbb{R} \text{ s.t. } \vec{v} = \sum_{i=1}^k c_i \vec{x}_i$$

This is true because $X = \text{span } B$. So by definition of a span, \vec{v} can be expressed as a linear combination of the vectors of B .

2. The c_i such that $\vec{v} = \sum_{i=1}^k c_i \vec{x}_i$ are unique.

Suppose that $\vec{v} = \sum c_i \vec{x}_i = \sum \alpha_i \vec{x}_i$. Then $\vec{0} = \sum (c_i - \alpha_i) \vec{x}_i$.
Then since $\{\vec{x}_1, \dots, \vec{x}_k\}$ are linearly independent, we know for all i that $c_i - \alpha_i = 0$. Hence, $c_i = \alpha_i$ for each i .

Theorem 9.2: Let $k \in \mathbb{N} \cup \{0\}$. If $X = \text{span}\{\vec{x}_1, \dots, \vec{x}_k\}$, then $\dim(X) \leq k$.

Proof:

Suppose for the sake of contradiction that there exists a linearly independent set $Q = \{\vec{y}_1, \dots, \vec{y}_{k+1}\} \subset X$ which spans X . Then, define $S_0 = \{\vec{x}_1, \dots, \vec{x}_k\}$ and note that S_0 spans X .

Now by induction, assume for $i \in \{0, 1, \dots, k-1\}$, that S_i contains the first i vectors of Q in addition to $k-i$ vectors of S_0 , and that $\text{span } S_i = X$. Then since S_i spans X , we know that $\vec{y}_{i+1} \in X$ is in the span of S_i . So, letting $\vec{x}_{n_1}, \dots, \vec{x}_{n_{k-i}}$ be the elements from S_0 in S_i , we know that there exists scalars $a_1, \dots, a_{i+1}, b_1, \dots, b_{k-i} \in \mathbb{R}$ where $a_{i+1} = 1$ such that:

$$\sum_{j=1}^{i+1} a_j \vec{y}_j + \sum_{j=1}^{k-i} b_j \vec{x}_{n_j} = \vec{0}$$

If all $b_j = 0$, then we have a contradiction. This is because $\{\vec{y}_1, \dots, \vec{y}_{k+1}\}$ is assumed to be linearly independent. So, having all $b_j = 0$ implies that:

$$\sum_{j=1}^{i+1} a_j \vec{y}_j = \sum_{j=1}^{i+1} a_j \vec{y}_j + \sum_{j=i+2}^{k+1} 0 \cdot \vec{y}_j = \vec{0}$$

In turn this means that all $a_j = 0$, which contradicts that $a_{i+1} = 1$.

So, not all $b_j = 0$. This means that for some j we must have that \vec{x}_{n_j} is in the span of $(S_i \setminus \{\vec{x}_{n_j}\}) \cup \{\vec{y}_{i+1}\}$. Call this set S_{i+1} . Clearly, S_{i+1} contains the first $i+1$ vectors of Q . Also:

$$\text{span } S_{i+1} = \text{span}(S_i \cup \{\vec{y}_{i+1}\}) = \text{span } S_i = X.$$

So S_{i+1} satisfies the same conditions S_i did.

Now we get to the contradiction. Using the above reasoning, we will eventually construct $S_k = \{\vec{y}_1, \dots, \vec{y}_k\}$ which still spans X . However, since $\vec{y}_{k+1} \in X$, that means that \vec{y}_{k+1} equals some linear combination of the other \vec{y} in Q . This contradicts that Q is linearly independent. ■

Corollary: If $B = \{\vec{x}_1, \dots, \vec{x}_k\}$ is a basis for X , then $\dim(X) = k$.

Proof:

Since B is linearly independent, by definition $\dim(X) \geq k$. Meanwhile, since B spans X , we know by the above theorem that $\dim(X) \leq k$. So $\dim(X) = k$.

Theorem 9.3: Suppose X is a vector space and $\dim(X) = n$. Then:

- (A) For $E = \{\vec{x}_1, \dots, \vec{x}_n\} \subset X$, we have that $X = \text{span } E$ if and only if E is linearly independent.

Proof:

First, assume E is linearly independent. Then, note that for any $\vec{y} \in X$, we must have that $E \cup \{\vec{y}\}$ is linearly dependent because $|E \cup \{\vec{y}\}| > \dim(X)$. So, there exists $c_1, \dots, c_n, c_{n+1} \in \mathbb{R}$ such that at least one c_i is nonzero and:

$$\sum_{i=1}^n c_i \vec{x}_i + c_{n+1} \vec{y} = \vec{0}$$

Now if $c_{n+1} = 0$, we have a contradiction because E is linearly independent. So, we conclude that $c_{n+1} \neq 0$. Thus, by rearranging terms we can express y as a linear combination of the vectors of E . Therefore, $\text{span } E = X$ since y can be any vector in X .

Secondly, assume E is not linearly independent. Then for some $\vec{x}_i \in E$, we have that $\text{span } E = \text{span}(E \setminus \{\vec{x}_i\})$. However, $|E \setminus \{\vec{x}_i\}| = n - 1$. So if $X = \text{span } E$, then $\dim(X) \leq |E \setminus \{\vec{x}_i\}| = n - 1$, which contradicts our assumption that $\dim(X) = n$. Hence, $X \neq \text{span } E$.

- (B) X has a basis and every basis of X consists of n vectors.

Proof:

By the definition of $\dim(X)$, we know that there exists a linearly independent set of n vectors. By the previous part of this theorem, we also know that that set spans X . So, it is a basis of X . Meanwhile, by the corollary to theorem 9.2, we know that the number of vectors in a basis of X equals the dimension of X . Hence, all bases of X must have n vectors.

- (C) If $1 \leq m \leq n$ and $\{\vec{y}_1, \dots, \vec{y}_m\} \subset X$ is linearly independent, then X has a basis that contains $\vec{y}_1, \dots, \vec{y}_m$.

Proof:

Let $S_0 = \{\vec{x}_1, \dots, \vec{x}_n\}$ be a basis of X and $Q = \{\vec{y}_1, \dots, \vec{y}_m\}$. Then by the same induction which we used to prove theorem 9.2, we can construct a basis: S_m , of X which contains $\vec{y}_1, \dots, \vec{y}_m$.

Let X and Y be vector spaces. A map $\mathbf{A} : X \longrightarrow Y$ is linear if $\mathbf{A}(c_1 \vec{x}_1 + c_2 \vec{x}_2) = c_1 \mathbf{A}(\vec{x}_1) + c_2 \mathbf{A}(\vec{x}_2)$ for all $\vec{x}_1, \vec{x}_2 \in X$ and $c_1, c_2 \in \mathbb{R}$.

Observations:

1. A linear map sends $\vec{0}$ to $\vec{0}$. This is because:

$$\mathbf{A}(\vec{0}) = \mathbf{A}(\vec{v} - \vec{v}) = \mathbf{A}(\vec{v}) - \mathbf{A}(\vec{v}) = \vec{0}.$$

2. If $\mathbf{A} : X \rightarrow Y$ is a linear map and $B = \{\vec{x}_1, \dots, \vec{x}_k\}$ is a basis of X ,

$$\text{then } \mathbf{A} \left(\sum_{i=1}^k (c_i \vec{x}_i) \right) = \sum_{i=1}^k c_i \mathbf{A}(\vec{x}_i) \text{ for all } c_1, \dots, c_k \in \mathbb{R}.$$

Given two vector spaces X and Y , we define $L(X, Y)$ to be the set of all linear transformations from X into Y . Also, we shall abbreviate $L(X, X)$ as $L(X)$.

$$\mathcal{N}(\mathbf{A}) = \text{"null space / kernel of } \mathbf{A}\text{"} = \{\vec{x} \in X \mid \mathbf{A}(\vec{x}) = \vec{0}\}.$$

$$\mathcal{R}(\mathbf{A}) = \text{"range of } \mathbf{A}\text{"} = \{\vec{y} \in Y \mid \exists \vec{x} \in X \text{ s.t. } \mathbf{A}\vec{x} = \vec{y}\}.$$

Proposition: For any linear map $\mathbf{A} : X \rightarrow Y$, $\mathcal{N}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A})$ are vector spaces.

Proof:

- Assume $\vec{x}_1, \vec{x}_2 \in \mathcal{N}(\mathbf{A}) \subset X$ and $c \in \mathbb{R}$. Then:
 - $\mathbf{A}(\vec{x}_1 + \vec{x}_2) = \mathbf{A}(\vec{x}_1) + \mathbf{A}(\vec{x}_2) = \vec{0} + \vec{0} = \vec{0}$, which means that $\vec{x}_1 + \vec{x}_2 \in \mathcal{N}(\mathbf{A})$.
 - $\mathbf{A}(c\vec{x}_1) = c\mathbf{A}(\vec{x}_1) = c\vec{0} = \vec{0}$. So $c\vec{x}_1 \in \mathcal{N}(\mathbf{A})$.
 This shows that $\mathcal{N}(\mathbf{A})$ is a vector space.
- Assume $\vec{y}_1, \vec{y}_2 \in \mathcal{R}(\mathbf{A}) \subset Y$ and $c \in \mathbb{R}$. Then:
 - We know there exists $\vec{x}_1, \vec{x}_2 \in X$ such that $\mathbf{A}(\vec{x}_1) = \vec{y}_1$ and $\mathbf{A}(\vec{x}_2) = \vec{y}_2$. In turn, $\mathbf{A}(\vec{x}_1 + \vec{x}_2) = \mathbf{A}(\vec{x}_1) + \mathbf{A}(\vec{x}_2) = \vec{y}_1 + \vec{y}_2$. So $\vec{y}_1 + \vec{y}_2 \in \mathcal{R}(\mathbf{A})$.
 - Now continue letting $\vec{x}_1 \in X$ be a vector such that $\mathbf{A}(\vec{x}_1) = \vec{y}_1$. Then $\mathbf{A}(c\vec{x}_1) = c\mathbf{A}(\vec{x}_1) = c\vec{y}_1$. So $c\vec{y}_1 \in \mathcal{R}(\mathbf{A})$.
 This shows that $\mathcal{R}(\mathbf{A})$ is a vector space.

$$\text{rk}(\mathbf{A}) = \text{"rank of } \mathbf{A}\text{"} = \dim(\mathcal{R}(\mathbf{A})).$$

$$\text{null}(\mathbf{A}) = \text{"nullity of } \mathbf{A}\text{"} = \dim(\mathcal{N}(\mathbf{A})).$$

Rank-Nullity Theorem: Given any $\mathbf{A} \in L(X, Y)$, we have that

$$\dim(X) = \text{rk}(\mathbf{A}) + \text{null}(\mathbf{A}).$$

Proof:

Let $\dim(X) = n$.

$\mathcal{N}(\mathbf{A}) \subseteq X$ is a vector space. So pick a basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ for $\mathcal{N}(\mathbf{A})$ where $k = \text{null}(\mathbf{A}) \leq \dim(X)$. Then by theorem 9.3, choose $\vec{w}_1, \dots, \vec{w}_{m-k}$ such that $\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_1, \dots, \vec{w}_{m-k}\}$ is a basis of X . Note that $\dim(X) = n$.

Claim: $B = \{\mathbf{A}(\vec{w}_1), \dots, \mathbf{A}(\vec{w}_{n-k})\}$ is a basis of $\mathcal{R}(\mathbf{A})$.

- $\mathbf{A}(\vec{v}_i) = \vec{0}$ for all $i \in \{1, \dots, k\}$. So:

$$\begin{aligned}\mathcal{R}(\mathbf{A}) &= \text{span}\{\mathbf{A}(\vec{v}_1), \dots, \mathbf{A}(\vec{v}_k), \mathbf{A}(\vec{w}_1), \dots, \mathbf{A}(\vec{w}_{n-k})\} \\ &= \text{span}\{\mathbf{A}(\vec{w}_1), \dots, \mathbf{A}(\vec{w}_{n-k})\} = \text{span } B\end{aligned}$$

- B is linearly independent.

To see this, note that: $\sum_{i=1}^{n-k} (c_i \mathbf{A}(\vec{w}_i)) = \vec{0} \implies \mathbf{A}\left(\sum_{i=1}^{n-k} c_i \vec{w}_i\right) = \vec{0}$

Since we picked each $\vec{w}_1, \dots, \vec{w}_{n-k} \in B$ so that they were not in $\mathcal{N}(\mathbf{A})$, we know that any vector in the span of B is not mapped to 0 by \mathbf{A} unless it is the zero vector. So

$$\sum_{i=1}^{n-k} c_i \vec{w}_i = \vec{0}$$

And since all the \vec{w}_i are linearly independent, all constants c_i equal 0 .

So $\text{rk}(\mathbf{A}) = n - k = \dim(X) - \text{null}(\mathbf{A})$.

Lecture 2: 4/4/2024

Proposition: Given $\mathbf{A} \in L(X, Y)$, then:

- \mathbf{A} is injective if and only if $\text{null}(\mathbf{A}) = 0$.

Proof:

(\implies) If \mathbf{A} is injective, then since $\mathbf{A}(\vec{0}) = \vec{0}$, we have that any vector $\vec{v} \neq \vec{0}$ is not in $\mathcal{N}(\mathbf{A})$. So $\mathcal{N}(\mathbf{A}) = \{\vec{0}\}$, meaning $\text{null}(\mathbf{A}) = 0$.

(\impliedby) If $\text{null}(\mathbf{A}) = 0$, then $\mathbf{A}(\vec{v}) = \vec{0} \implies \vec{v} = \vec{0}$. So now assume $\mathbf{A}(\vec{v}) = \mathbf{A}(\vec{u})$. Then $\mathbf{A}(\vec{v} - \vec{u}) = \vec{0}$, meaning $\vec{v} = \vec{u}$. Hence \mathbf{A} is injective.

- \mathbf{A} is surjective if and only if $\text{rk}(\mathbf{A}) = \dim Y$.

Proof:

(\implies) If \mathbf{A} is surjective then $\mathcal{R}(\mathbf{A}) = Y$. So we automatically have that $\text{rk}(\mathbf{A}) = \dim(Y)$

(\impliedby) If $\text{rk}(\mathbf{A}) = \dim(Y)$, then there exists a linearly independent set of vectors $B \subset \mathcal{R}(\mathbf{A})$ containing $\dim(Y)$ many vectors and spanning $\mathcal{R}(\mathbf{A})$. Then by theorem 9.3, since $B \subset \mathcal{R}(\mathbf{A}) \subseteq Y$, we know $\text{span } B = Y$. So, $\mathcal{R}(\mathbf{A}) = Y$, meaning \mathbf{A} is surjective.

Corollary: Let $\mathbf{A} \in L(X)$. Then \mathbf{A} is bijective if and only if $\text{null}(\mathbf{A}) = 0$.

Proof: (let $\mathbf{A} : X \longrightarrow X$ be a linear map)

(\implies) If \mathbf{A} is bijective, then automatically \mathbf{A} is injective. So $\text{null}(\mathbf{A}) = 0$ by the previous proposition.

(\impliedby) If $\text{null}(\mathbf{A}) = 0$, then by the rank-nullity theorem, we know that $\text{rk}(\mathbf{A}) = \dim(X)$. Thus \mathbf{A} is both injective and surjective, meaning \mathbf{A} is bijective.

For $\mathbf{A} \in L(X)$, when $\text{null}(\mathbf{A}) = 0$, we call \mathbf{A} invertible and define $\mathbf{A}^{-1} : X \longrightarrow X$ by $\mathbf{A}^{-1}(\mathbf{A}(\vec{x})) = \vec{x}$ for all $\vec{x} \in X$.

Because \mathbf{A} must be a bijective set function, we know that \mathbf{A}^{-1} must also be a right-inverse of \mathbf{A} , meaning $\mathbf{A}(\mathbf{A}^{-1}(\vec{x})) = \vec{x}$.

Additionally, consider any $\vec{x}_1, \vec{x}_2 \in X$. Then let $\vec{x}'_1 = \mathbf{A}^{-1}(\vec{x}_1)$ and $\vec{x}'_2 = \mathbf{A}^{-1}(\vec{x}_2)$. Then since \mathbf{A} is a linear mapping, we know that for any $c_1, c_2 \in \mathbb{R}$:

$$\mathbf{A}(c_1 \vec{x}'_1 + c_2 \vec{x}'_2) = c_1 \mathbf{A}(\mathbf{A}^{-1}(\vec{x}_1)) + c_2 \mathbf{A}(\mathbf{A}^{-1}(\vec{x}_2)) = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

So: $\mathbf{A}^{-1}(c_1 \vec{x}_1 + c_2 \vec{x}_2) = c_1 \vec{x}'_1 + c_2 \vec{x}'_2 = c_1 \mathbf{A}^{-1}(\vec{x}_1) + c_2 \mathbf{A}^{-1}(\vec{x}_2)$. Hence, we've shown that \mathbf{A}^{-1} is a linear mapping, meaning that $\mathbf{A}^{-1} \in L(X)$.

Let $\mathbf{A} \in L(X, Y)$ and $\mathbf{B} \in L(Y, Z)$. Then we define $\mathbf{BA} : X \longrightarrow Z$ by the rule that $\vec{x} \mapsto \mathbf{B}(\mathbf{A}(\vec{x}))$.

We can trivially show that \mathbf{BA} is a linear mapping. Consider any $\vec{x}_1, \vec{x}_2 \in X$ and $c_1, c_2 \in \mathbb{R}$. Then:

$$\begin{aligned} \mathbf{BA}(c_1 \vec{x}_1 + c_2 \vec{x}_2) &= \mathbf{B}(c_1 \mathbf{A}(\vec{x}_1) + c_2 \mathbf{A}(\vec{x}_2)) \\ &= c_1 \mathbf{B}(\mathbf{A}(\vec{x}_1)) + c_2 \mathbf{B}(\mathbf{A}(\vec{x}_2)) \\ &= c_1 \mathbf{BA}(\vec{x}_1) + c_2 \mathbf{BA}(\vec{x}_2) \end{aligned}$$

This means that $\mathbf{BA} \in L(X, Z)$.

Let $\mathbf{A}, \mathbf{B} \in L(X, Y)$ and $c_1, c_2 \in \mathbb{R}$. Then we define $(c_1 \mathbf{A} + c_2 \mathbf{B}) : X \longrightarrow Y$ by the rule: $\vec{x} \mapsto c_1 \mathbf{A}(\vec{x}) + c_2 \mathbf{B}(\vec{x})$.

It is even more trivial to show that $(c_1 \mathbf{A} + c_2 \mathbf{B})$ is a linear map.

Let $\mathbf{A} \in L(\mathbb{R}^n, \mathbb{R}^m)$. We define the norm of \mathbf{A} as:

$$\|\mathbf{A}\| = \sup \{ \|\mathbf{A}(\vec{x})\| \mid \vec{x} \in \mathbb{R}^n \text{ and } \|\vec{x}\| \leq 1 \}.$$

Throughout this section, we shall prove that $\|\cdot\| : L(\mathbb{R}^n, \mathbb{R}^m) \longrightarrow \mathbb{R}$ is well-defined and fulfills the properties of a general norm function.

Proposition: If $\mathbf{A} \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\|\mathbf{A}\|$ exists and is finite.

Proof:

Let $\{e_1, \dots, e_n\}$ be the standard basis in \mathbb{R}^n . Then for any $\vec{x} \in \mathbb{R}^n$, there are unique $c_1, \dots, c_n \in \mathbb{R}$ such that $\vec{x} = c_1 e_1 + \dots + c_n e_n$.

Since we are working with the standard basis, we know: $\|\vec{x}\| = \sqrt{\sum_{i=1}^n c_i^2}$.

Thus, for $\|\vec{x}\| \leq 1$, we must have that $|c_i| \leq 1$ for each c_i . This means:

$$\|\mathbf{A}(\vec{x})\| = \left\| \sum_{i=1}^n c_i \mathbf{A}(e_i) \right\| \leq \sum_{i=1}^n \|c_i \mathbf{A}(e_i)\| = \sum_{i=1}^n |c_i| \|\mathbf{A}(e_i)\| \leq \sum_{i=1}^n \|\mathbf{A}(e_i)\|$$

Importantly, we must have that $\sum_{i=1}^n \|\mathbf{A}(e_i)\|$ is finite. Additionally, it is an upper bound to the set: $\{\|\mathbf{A}(\vec{x})\| \mid \vec{x} \in \mathbb{R}^n \text{ and } \|\vec{x}\| \leq 1\} \subseteq \mathbb{R}$.

So, we showed that the above set is bounded above. Also, the above set is nonempty because it must contain $\|\vec{0}\| = 0$. Thus by the least upper bound property of \mathbb{R} , we know that the supremum of this set exists in \mathbb{R} .

Hence, $\|\mathbf{A}\|$ exists and is finite.

Side note, the above proof also shows that $\|\mathbf{A}\| \geq 0$.

Lemma: For $\mathbf{A} \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $\vec{x} \in \mathbb{R}^n$, we have that $\|\mathbf{A}(\vec{x})\| \leq \|\mathbf{A}\| \|\vec{x}\|$.

Proof:

Case 1: $\vec{x} \neq \vec{0}$.

Then since $\|\vec{x}\| \neq 0$, we can say that:

$$\|\mathbf{A}(\vec{x})\| = \left\| \mathbf{A} \left(\|\vec{x}\| \frac{\vec{x}}{\|\vec{x}\|} \right) \right\| = \left\| \|\vec{x}\| \mathbf{A} \left(\frac{\vec{x}}{\|\vec{x}\|} \right) \right\| = \left\| \mathbf{A} \left(\frac{\vec{x}}{\|\vec{x}\|} \right) \right\| \|\vec{x}\|$$

Now $\frac{\vec{x}}{\|\vec{x}\|} \in \mathbb{R}^n$ and $\left\| \frac{\vec{x}}{\|\vec{x}\|} \right\| = 1$. So, $\left\| \mathbf{A} \left(\frac{\vec{x}}{\|\vec{x}\|} \right) \right\| \|\vec{x}\| \leq \|\mathbf{A}\| \|\vec{x}\|$

Case 2: $\vec{x} = \vec{0}$.

Then trivially $\|\mathbf{A}(\vec{x})\| = \|\mathbf{A}(\vec{0})\| = 0 = \|\mathbf{A}\| \|\vec{0}\| = \|\mathbf{A}\| \|\vec{x}\|$

Proposition: If $\mathbf{A} \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $0 \leq \|\mathbf{A}\|$. Also $\|\mathbf{A}\| = 0$ if and only if \mathbf{A} is the unique function mapping all of \mathbb{R}^n to $\vec{0}$.

Proof:

We already showed previously that $\|\mathbf{A}\| \geq 0$. So, it now suffices to show that $\|\mathbf{A}\| = 0 \iff \mathcal{N}(\mathbf{A}) = \mathbb{R}^n$.

(\implies) Assume that $\mathcal{N}(\mathbf{A}) \neq \mathbb{R}^n$. Then there exists $\vec{x} \in \mathbb{R}^n$ such that $\mathbf{A}(\vec{x}) \neq \vec{0}$. Since \vec{x} can't be $\vec{0}$, consider the vector $\hat{x} = \frac{\vec{x}}{\|\vec{x}\|}$. By the linearity of \mathbf{A} , we know $\mathbf{A}(\hat{x}) = \frac{1}{\|\vec{x}\|} \mathbf{A}(\vec{x}) \neq \vec{0}$. So, $\|\mathbf{A}(\hat{x})\| > 0$. But $\|\mathbf{A}(\hat{x})\|$ is in the set that $\|\mathbf{A}\|$ is a supremum of, which means that $\|\mathbf{A}\| \geq \|\mathbf{A}(\hat{x})\| > 0$. Or in other words, $\|\mathbf{A}\| \neq 0$.

(\impliedby) Assume that $\mathcal{N}(\mathbf{A}) = \mathbb{R}^n$. Then,

$$\sup \{ \|\mathbf{A}(\vec{x})\| \mid \vec{x} \in \mathbb{R}^n \text{ and } \|\vec{x}\| \leq 1 \} = \sup \{ 0 \} = 0$$

Corollary: Given $\mathbf{A} \in L(\mathbb{R}^n, \mathbb{R}^m)$, we have that \mathbf{A} is uniformly continuous.

Proof:

Case 1: $\|\mathbf{A}\| \neq 0$, meaning we can divide by $\|\mathbf{A}\|$.

By the previous proposition, $\|\mathbf{A}(\vec{x}) - \mathbf{A}(\vec{y})\| \leq \|\mathbf{A}\| \|\vec{x} - \vec{y}\|$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. Hence, for any $\varepsilon > 0$, if we make $\|\vec{x} - \vec{y}\| < \frac{\varepsilon}{\|\mathbf{A}\|}$, then $\|\mathbf{A}(\vec{x}) - \mathbf{A}(\vec{y})\| < \varepsilon$.

Case 2: $\|\mathbf{A}\| = 0$.

Then \mathbf{A} is a constant function, making it automatically uniformly continuous.

Subcorollary: Given $\mathbf{A} \in L(\mathbb{R}^n, \mathbb{R}^m)$, there exists $\vec{x} \in \mathbb{R}^n$ with $\|\vec{x}\| \leq 1$ such that $\|\mathbf{A}(\vec{x})\| = \|\mathbf{A}\|$.

Proof:

Let $S = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| \leq 1 \}$ and consider the restriction $\mathbf{A}|_S$.

Since S is a closed and bounded subset of \mathbb{R}^n , we know that S is compact by the Heine-Borel theorem (see proposition 28 in Math 140A notes).

This combined with the fact that $\mathbf{A}|_S$ is still continuous means that by the extreme value theorem, there is $\vec{x} \in S$ with:

$$\mathbf{A}(\vec{x}) = \mathbf{A}|_S(\vec{x}) = \sup \{ \|\mathbf{A}(\vec{x})\| \mid \vec{x} \in \mathbb{R}^n \text{ and } \|\vec{x}\| \leq 1 \}.$$

Proposition: If $\mathbf{A}, \mathbf{B} \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$.

Proof:

Let $\vec{x} \in \mathbb{R}^n$ be a vector such that $\|\vec{x}\| \leq 1$ and $\|\mathbf{A}(\vec{x})\| = \|\mathbf{A}\|$. Then:

$$\begin{aligned} \|\mathbf{A} + \mathbf{B}\| &= \|(\mathbf{A} + \mathbf{B})(\vec{x})\| = \|\mathbf{A}(\vec{x}) + \mathbf{B}(\vec{x})\| \\ &\leq \|\mathbf{A}(\vec{x})\| + \|\mathbf{B}(\vec{x})\| \leq \|\mathbf{A}\| + \|\mathbf{B}\| \end{aligned}$$

Proposition: If $\mathbf{A} \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $c \in \mathbb{R}$, then $\|c\mathbf{A}\| = |c|\|\mathbf{A}\|$.

Proof:

Pick $\vec{x} \in \mathbb{R}^n$ satisfying $\|\vec{x}\| \leq 1$ and $\|\mathbf{A}(\vec{x})\| = \|\mathbf{A}\|$. Then:

$$|c|\|\mathbf{A}\| = |c|\|\mathbf{A}(\vec{x})\| = \|c\mathbf{A}(\vec{x})\| = \|(c\mathbf{A})(\vec{x})\| \leq \|c\mathbf{A}\|.$$

Next, pick $\vec{y} \in \mathbb{R}^n$ satisfying $\|\vec{y}\| \leq 1$ and $\|(c\mathbf{A})(\vec{y})\| = \|c\mathbf{A}\|$. Then:

$$\|c\mathbf{A}\| = \|(c\mathbf{A})(\vec{y})\| = \|c\mathbf{A}(\vec{y})\| = |c|\|\mathbf{A}(\vec{y})\| \leq |c|\|\mathbf{A}\|.$$

Specifically because of the four propositions above, we have shown that $\|\cdot\| : L(\mathbb{R}^n, \mathbb{R}^m) \rightarrow \mathbb{R}$ is well-defined and a valid norm. Consequently, by defining $d(\mathbf{A}, \mathbf{B}) = \|\mathbf{A} - \mathbf{B}\|$ for all $\mathbf{A}, \mathbf{B} \in L(\mathbb{R}^n, \mathbb{R}^m)$, we can say that $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space.

Before moving on, here is another corollary of the above statements.

Corollary: If $\mathbf{A} \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $\mathbf{B} \in L(\mathbb{R}^m, \mathbb{R}^k)$, then $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|$.

Proof:

Pick $\vec{x} \in \mathbb{R}^n$ satisfying $\|\vec{x}\| \leq 1$ and $\|(\mathbf{BA})(\vec{x})\| = \|\mathbf{BA}\|$. Then:

$$\|\mathbf{BA}\| = \|(\mathbf{BA})(\vec{x})\| = \|\mathbf{B}(\mathbf{A}(\vec{x}))\| \leq \|\mathbf{B}\|\|\mathbf{A}(\vec{x})\| \leq \|\mathbf{B}\|\|\mathbf{A}\|.$$

Theorem 9.8: Let $\Omega \subset L(\mathbb{R}^n)$ be the set of all invertible linear mappings on \mathbb{R}^n .

(A) If $\mathbf{A} \in \Omega$, $\mathbf{B} \in L(\mathbb{R}^n)$, and $\|\mathbf{B} - \mathbf{A}\|\|\mathbf{A}^{-1}\| < 1$, then $\mathbf{B} \in \Omega$.

(B) Ω is an open subset of $L(\mathbb{R}^n)$, and the mapping over Ω with the rule:
 $\mathbf{A} \mapsto \mathbf{A}^{-1}$, is continuous.