Math 188 Notes (Professor: Steven Sam)

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Lecture 1 Notes: 9/27/2024

Linear Recurrence Relations:

A sequence $(a_n)_{n\geq 0}$ satisfies a linear recurrence relation of order d if there exists $c_1, \ldots, c_d$ with $c_d \neq 0$ so that $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_d a_{n-d}$ for all $n \geq d$.

(For $0 \leq n < d$, we usually explicitely specify a_n. Also, it seems like we are assuming all a_n and c_n are complex numbers right now)

To start this course, we're gonna discuss finding explicit (non-recursive) solutions.

Firstly, if $d = 1$, then this problem is easy. We can just plug in previous elements repeatedly to get that:

$$a_n = c_1 a_{n-1} = c_1^2 a_{n-2} = \ldots = c_1^n a_0$$

If $d = 2$, then plugging in previous elements doesn't help us really anymore. So how do we solve this problem now?

Theorem: Consider the characteristic polynomial $t^2 - c_1 t - c_2$ and let r_1, r_2 be the roots of that polynomial. If $r_1 \neq r_2$, then there exists α_1, α_2 such that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for all $n \geq 0$.

To solve for α_1 and α_2, plug in different values of n into our equation. Since $r_1 \neq r_2$, we know the below linear system has a unique solution:

$$a_0 = \alpha_1 + \alpha_2$$
$$a_1 = \alpha_1 r_1 + \alpha_2 r_2$$

Now backing up, why does the above method work?

Approach 1: (Vector Spaces)

The set of sequences $(a_n)_{n\geq 0}$ form a vector space. Furthermore given any constants c_1 and c_2, we know that the set of sequences satisfying $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ for all $n \geq 2$ is a subspace.

Proof:

Suppose (a_n) and (b_n) both satisfy that $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and $b_n = c_1 b_{n-1} + c_2 b_{n-2}$. Then given any constants γ and δ, we have that:

$$(\gamma a_n + \delta b_n) = c_1(\gamma a_{n-1} + \delta b_{n-1}) + c_2(\gamma a_{n-2} + \delta b_{n-2})$$

Hence, all linear combinations of any two sequences satisfying our linear recurrence relation also satisfies our linear recurrence relation.

Now what our above theorem is stating is that the sequences (r_1^n) and (r_2^n) span the subspace of solutions to our linear recurrence relation.

To see this, first note that (r_1^n) and (r_2^n) satisfy our recurrence relation.

If $n \geq 2$, then $r_i^n - c_1 r_i^{n-1} - c_2 r_i^{n-2} = r_i^{n-2}(r_i^2 - c_1 r_i - c_2) = r_i^{n-2}(0)$.

Hence, we know that $r_i^n = c_1 r_i^{n-1} + c_2 r_i^{n-2}$ for all $n \geq 2$.

Also, since we assumed $r_1 \neq r_2$, we know that (r_1^n) is linearly independent to (r_2^n). And finally, as mentioned before, we can solve a linear system of equations to find coffecients for a linear combination of (r_1^n) and (r_2^n) equal to any other sequence satisfying our recurrence relation.

Approach 2: (Formal Power Series)
Define the power series $A(x) = \sum_{n\geq 0} a_n x^n$. We call $A(x)$ a generating function of the sequence (a_n).

(We'll treat the formal power series more rigorously later...)

Now note that:

$$\begin{aligned} A(x) &= a_0 + a_1 x + \sum_{n\geq 2} a_n x^n \\ &= a_0 + a_1 x + \sum_{n\geq 2} (c_1 a_{n-1} + c_2 a_{n-2}) x^n \\ &= a_0 + a_1 x + c_1 \sum_{n\geq 2} a_{n-1} x^n + c_2 \sum_{n\geq 2} a_{n-2} x^n \\ &= a_0 + a_1 x + c_1 (A(x) - a_0) x + c_2 (A(x)) x^2 \end{aligned}$$

Isolating $A(x)$, we get the equation: $A(x) = \dfrac{a_0 + a_1 x - a_0 c_1 x}{1 - c_1 x - c_2 x^2}$.

Next, let's do fraction decomposition on our equation for $A(x)$.

Issue: We defined r_1 and r_2 as the roots of $t^2 - c_1 t - c_2 = (t - r_1)(t - r_2)$.

Trick: Plug in $t = \frac{1}{x}$. That way, we have that:

$$x^{-2} - c_1 x^{-1} - c_2 = (x^{-1} - r_1)(x^{-1} - r_2).$$

After that, multiply both sides of our equation by x^2 to get that:

$$1 - c_1 x - c_2 x^2 = (1 - r_1 x)(1 - r_2 x)$$

Since we're assuming $r_1 \neq r_2$, we know that for some constants α_1 and α_2, we have that:

$$A(x) = \frac{\alpha_1}{1 - r_1 x} + \frac{\alpha_2}{1 - r_2 x}$$

(If $r_1 = r_2$, then this step is where things will go differently.)

Now finally, we can rewrite $\frac{\alpha_1}{1 - r_1 x}$ as the geometric series $\alpha_1 \sum_{n\geq 0} (r_1 x)^n$. Doing likewise with $\frac{\alpha_2}{1 - r_2 x}$, we get that:

$$A(x) = \sum_{n\geq 0} a_n x^n = \alpha_1 \sum_{n\geq 0} (r_1 x)^n + \alpha_2 \sum_{n\geq 0} (r_2 x)^n = \sum_{n\geq 0} (\alpha_1 r_1^n + \alpha_2 r_2^n) x^n$$

Hence, we have for each n that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$.

Lecture 2: 9/30/2024

Approach 3: (Matrices)

If $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then we can say that: $\begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$

Letting $\boldsymbol{C} = \begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix}$, we thus know that: $\boldsymbol{C}^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$

Notably, the characteristic polynomial of $\boldsymbol{C}$ is $t^2 - c_1 t - c_2$. So the eigenvalues of $\boldsymbol{C}$ are r_1 and r_2. Because we assumed r_1 and r_2 are distinct, we know $\boldsymbol{C}$ is diagonalizable. Hence there exists an invertible matrix $\boldsymbol{B}$ such that:

$$\boldsymbol{B} \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \boldsymbol{B}^{-1} = \boldsymbol{C}$$

Now set $\begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{B}^{-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$. Then we can see that:

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \boldsymbol{C}^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \boldsymbol{B}\boldsymbol{D}^n \begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{B} \begin{bmatrix} r_1^n x \\ r_2^n y \end{bmatrix} = \begin{bmatrix} b_{1,1} r_1^n x + b_{1,2} r_2^n y \\ b_{2,1} r_1^n x + b_{2,2} r_2^n y \end{bmatrix}$$

Setting $\alpha_1 = b_{2,1} x$ and $\alpha_2 = b_{2,2} y$, we have thus found constants α_1 and α_2 such that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$.

Now some further questions to ask about recurrence relations are:

1. What if $r_1 = r_2$?
2. What if $d \geq 3$?
3. What if the recurrence relation is non-homogeneous or non-linear?

To start, let's answer question 1.

Theorem: Suppose r_1 and r_2 are the roots of $t^2 - c_1 t - c_2$ with $r_1 = r_2$. Then there exists α_1, α_2 such that $a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n$ for all $n \geq 0$.

As was true when $r_1 \neq r_2$, you can solve for α_1 and α_2 by plugging in different values of n into the equation in order to get a linear system of equations.

To explain why this is, let's revisit two of our previous approaches.

The Formal Power Series Approach Revisited:

Before, we were able to show that $A(x) = \dfrac{a_0 + (a_1 - a_0 c_1)x}{(1 - r_1 x)(1 - r_2 x)}$ without assuming anything about r_1 and r_2.

But when we assume $r_1 = r_2$, we then get a different partial fraction decomposition for $A(x)$. Specifically, we have that there exists constants β_1, β_2 such that:

$$A(x) = \frac{\beta_1}{1 - r_1 x} + \frac{\beta_2}{(1 - r_1 x)^2}$$

Now we'll go into more rigor later. But for now, note that:

$$\tfrac{1}{(1-y)^2} = \left(\tfrac{1}{1-y}\right)' = \left(\sum_{n\geq 0} y^n\right)' = \sum_{n\geq 1} n y^{n-1} = \sum_{n\geq 0}(n+1)y^n$$

Comment from the future: as we'll cover two lectures from now, the definition of a derivative of a formal power series is different from the analysis definition we're familiar with.

Hence, we can write $A(x) = \sum_{n\geq 0} a_n x^n = (\beta_1 + \beta_2)\sum_{n\geq 0} r_1^n x^n + \beta_2 \sum_{n\geq 0} n r_1^n x^n$.

Or in other words, setting $\alpha_1 = \beta_1 + \beta_2$ and $\alpha_2 = \beta_2$, we have that:

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n$$

The Matrix Approach Revisited:

If $r_1 = r_2$, then we must hav ethat the matrix $\boldsymbol{C}$ is not diagonalizable. For suppose it was, meaning there exists an invertible matrix $\boldsymbol{B}$ such that:

$$\boldsymbol{C} = \boldsymbol{B}\begin{bmatrix} r_1 & 0 \\ 0 & r_1 \end{bmatrix}\boldsymbol{B}^{-1}$$

Then we'd have to have that $\boldsymbol{C} = r_1 \boldsymbol{B}\boldsymbol{B}^{-1} = \begin{bmatrix} r_1 & 0 \\ 0 & r_1 \end{bmatrix}$. But we know $\boldsymbol{C}$ isn't that.

Since we know $\boldsymbol{C}$ Is not diagonalizable, we will instead use the *Jordan-normal form* of $\boldsymbol{C}$. Specifically, we know there exists an invertible matrix $\boldsymbol{B}$ such that:

$$\boldsymbol{C} = \boldsymbol{B}\begin{bmatrix} r_1 & 1 \\ 0 & r_1 \end{bmatrix}\boldsymbol{B}^{-1}$$

Don't worry for the time being about how to prove the Jordan-normal form of a matrix always exists.

This tells us that $\boldsymbol{C}^n = \boldsymbol{B}\begin{bmatrix} r_1 & 1 \\ 0 & r_1 \end{bmatrix}^n \boldsymbol{B}^{-1}$.

Also, you can show by induction that $\begin{bmatrix} r_1 & 1 \\ 0 & r_1 \end{bmatrix}^n = \begin{bmatrix} r_1^n & n r_1^{n-1} \\ 0 & r_1^n \end{bmatrix}$.

So finally, defining $\begin{bmatrix} x \\ y \end{bmatrix}$ as before and expanding out the expression, you can get an explicit equation for a_n.

As for answering question 2, if $d \geq 3$, then our characteristic polynomial becomes $t^d - c_1 t^{d-1} - \ldots - c_d$. We'll assume this polynomial has distinct roots $r_1, \ldots, r_m$ with multiplicities $s_1, \ldots, s_m$ respectively.

Theorem: There exists constants $\alpha_1, \ldots, \alpha_d$ such that:

$$a_n = \sum_{i=1}^{s_1} \alpha_i n^{i-1} r_1^n + \ldots + \sum_{i=s_1+\ldots+s_{m-1}+1}^{s_1+\ldots+s_m} \alpha_i n^{i-1} r_m^n$$

As before, to solve for α_1 through α_d, you can plug in values of n and solve a linear system of equations.

The approaches to prove this are the same as when $d = 2$. However, there are just more terms floating around that need to be dealt with.

Special case: suppose the characteristic polynomial is $(t-1)^d$.

In that case, because the root of the polynomial r is 1, there exists $\alpha_1, \ldots, \alpha_d$ such that

$$a_n = \alpha_1 + n\alpha_2 + n^2\alpha_3 + \ldots + n^{d-1}\alpha_d.$$

In other words, the formula for a_n is a polynomial in n.

Another perspective on the characteristic polynomial:

Let V be the vector space of sequences $(a_n)_{n\geq 0}$, and define the translation operator $T : V \longrightarrow V$ such that $(a_n)_{n\geq 0} \mapsto (a_{n+1})_{n\geq 0}$. Now, given $\boldsymbol{a} \in V$ and the recurrence relation $a_n = c_1 a_{n-1} + \ldots + c_d a_{n-d}$ for all $n \geq d$, we have that $\boldsymbol{a}$ satisfies our recurrence relation if and only if:

$$T^d \boldsymbol{a} = c_1 T^{d-1} \boldsymbol{a} + c_2 T^{d-2} \boldsymbol{a} + \ldots + c_d \boldsymbol{A}$$

In other words, we must have that $\boldsymbol{a} \in \ker(T^d - c_1 T^{d-1} - \ldots - c_d)$.

If $r_1, \ldots, r_d$ are the roots of the characteristic polynomial $t^d - c_1 T^{d-1} - \ldots - c_d$, then we can rewrite this as:

$$(T - r_1) \cdots (T - r_d)\boldsymbol{a} = \boldsymbol{0}$$

Proposition: Given a sequence $\boldsymbol{a} = (a_n)_{n\geq 0}$, there exists a polynomial $p(n)$ of degree at most $d-1$ such that $a_n = p(n)$ if and only if $(T-1)^d \boldsymbol{a} = \boldsymbol{0}$.

We already saw in the special case above one direction of this statement. As for the other direction, suppose $p(n) = \alpha_d n^{d-1} + \alpha_{d-1} n^{d-2} + \ldots + \alpha_1$. Then $(T-1)$ applied to the sequence $(p(n))_{n\geq 0}$ is the sequence $(p(n+1) - p(n))_{n\geq 0}$ Importantly, $p(n+1)$ is also a polynomial of degree $d-1$ with α_d as the coefficient in front of n^{d-1}. So the difference is a polynomial of degree at most $d-2$.

Proceeding by induction, we know that $(T-1)^d(p(n))_{n\geq 0} = \mathbf{0}$.

Note that the operator $(T-1)$ can be thought of as the taking the "derivative" of a sequence $\boldsymbol{a}$. Going by that analogy, the previous proposition is saying that a sequence $\boldsymbol{a}$ is given by a polynomial if and only if a derivative of some order of the sequence is zero. Interestingly, the same is true of differential equations.

Lecture 3: 10/2/2024

To quickly address question 3, in general there is no unified approach to dealing with nonlinear recurrence relations. However, we can often solve non-homogeneous linear recurrence relations.

This will be addressed by the homework (see HW 1: Exercise (2)).

Formal Power Series:

A formal power series in the variable x is an expression of the form $A(x) = \sum_{n\geq 0} a_n x^n$ where a_n is a sequence of elements of a field.

Technically, we can go more general to a commutative ring (but we won't).

We call $A(x)$ the generating function of $(a_n)_{n\geq 0}$.

If $A(x)$ and $B(x)$ are the generating functions of $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ respectively, then:

- $A(x) = B(x)$ iff $a_n = b_n$ for all n.
- $A(x) + B(x) := \sum_{n\geq 0} (a_n + b_n)x^n$.
- $A(x)B(x) := \sum_{n\geq 0} c_n x^n$ where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$.

(in other words this is the Cauchy Product of $A(x)$ and $B(x)$)

Note that polynomials and constants are special cases of formal power series with the sequence generating that function being eventually zero.

Also, sums and products of formal power series satisfy the commutative, associative, and distributive properties of a field.

The only one of those properties that's non-trivial to show is the associativity of products. One way that you can prove this property is to show that :

$$\sum_{i=0}^{n}\sum_{j=0}^{i} a_j b_{i-j} c_{n-i} = \sum_{(p,q,r)\in I} a_p b_q c_r = \sum_{i=0}^{n}\sum_{j=0}^{i} a_{n-i} b_j c_{i-j},$$

where $I = \{(p,q,r) \in \mathbb{Z}^3 \mid p+q+r = n \text{ and } p,q,r \geq 0\}$.

Plus, letting $0+0x+\dots$ be the additive identity, then given any formal power series $A(x) = \sum_{n\geq 0} a_n x^n$, we have that:
$-A(x) = (-1+0x+\dots)\sum_{n\geq 0} a_n x^n = \sum_{n\geq 0}(-a_n)x^n$ is the additive inverse of $A(x)$.

Finally, we clearly have that $1+0x+0x^2+\dots$ is a multiplicative identity of the set of formal power series.

Therefore, to make this perfectly clear, the set of formal power series is not actually a set of functions. Rather, given a field (or commutative ring) F, the set of formal power series with coefficients in that ring is a commutative ring on the set F^ω of infinite sequences of elements of F using the $+$ and $\cdot$ operations defined above.

In other words, it doesn't make sense to plug values of x into formal power series. Rather, the power series notation is just to make it clearer where the definitions of different operations are coming from.

Also, since the set of formal power series on a commutative ring is itself a commutative ring, you can define the set of formal power series on the set of formal power series on that commutative ring. Spoiler: this is a way to define multivariate formal power series.

A formal power series $A(x)$ is invertible if there exists a formal power series $B(x)$ such that $A(x)B(x) = 1$. We write $B(x) = A(x)^{-1} = \frac{1}{A(x)}$, and call $B(x)$ the inverse of $A(x)$.

Example: If $A(x) = \sum_{n\geq 0} x^n$, then $A(x)$ is invertible with inverse $B(x) = 1-x$.

Proof:

$$\begin{aligned} A(x)B(x) &= (1+x+x^2+\dots)(1-x) \\ &= 1+x-x+x^2-x^2+x^3-x^3+\dots = 1 \end{aligned}$$

So $\sum_{n\geq 0} x^n = \frac{1}{1-x}$.

Theorem: $A(x) = \sum_{n\geq 0} a_n$ is invertible if and only if a_0 has a multiplicative inverse.

Proof:
If there exists $B(x)$ such that $A(x)B(x) = 1$, then we must have that:

$$\begin{aligned} &a_0 b_0 = 1 \\ &a_0 b_1 + a_1 b_0 = 0 \\ &a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \\ &\qquad\qquad\vdots \end{aligned}$$

If $\frac{1}{a_0}$ exists then we can inductively solve for each b_n. Specifically, $b_0 = \frac{1}{a_0}$ and $b_n = -\frac{1}{a_0}\sum_{i=1}^{n} a_i b_{n-i}$. Then $B(x) = \sum_{n\geq 0} b_n x^n$ satisfies that $A(x)B(x) = 1$.

If $\frac{1}{a_0}$ doesn't exist, then there is no choice of b_0 such that $A(x)(\sum_{n\geq 0} b_n x^n) = 1$. So $A(x)$ has no inverse.

Lecture 4: 10/4/2024

If $A(x)$ is a power series and $n \geq 0$, then $[x^n]A(x)$ refers to the coefficient a_n in front of x^n.

Let $A_0(x), A_1(x), \ldots$ be a sequence of formal power series. We say the sequence formally converges to $A(x)$ if:

$$\forall n \geq 0, \exists N \geq 0 \; s.t. \; i \geq N \implies [x^n]A_i(x) = [x^n]A(x)$$

We also write this as $\lim_{i\to\infty} A_i(x) = A(x)$

Note that this definition is different from the familiar definition of convergence in 140. For instance, the sequence $A_i(x) = \frac{1}{i+1}$ doesn't formally converge.

Lemma: Suppose $\lim_{i\to\infty} A_i(x) = A(x)$ and $\lim_{i\to\infty} B_i(x) = B(x)$. Then:

- $\lim_{i\to\infty}(A_i(x) + B_i(x)) = A(x) + B(x)$
- $\lim_{i\to\infty}(A_i(x)B_i(x)) = A(x)B(x)$

The proof for this is rather trivial. So do it yourself. :P

Continuing to let $A_0(x), A_1(x), \ldots$, be a sequence of formal power series, we define:

$$\sum_{i\geq 0} A_i(x) \coloneqq \lim_{i\to\infty}\left(\sum_{j=0}^{i} A_j(x)\right)$$

$$\prod_{i\geq 0} A_i(x) \coloneqq \lim_{i\to\infty}\left(\prod_{j=0}^{i} A_j(x)\right)$$

Lemma: (This is just reapplying the previous lemma for sequences and using the commutative property...)

- If $\sum_{i\geq 0} A_i(x)$ and $\sum_{i\geq 0} B_i(x)$ exist, then:
$$\sum_{i\geq 0}(A_i(x) + B_i(x)) = \sum_{i\geq 0} A_i(x) + \sum_{i\geq 0} B_i(x).$$
- If $\prod_{i\geq 0} A_i(x)$ and $\prod_{i\geq 0} B_i(x)$ exist, then:
$$\prod_{i\geq 0}(A_i(x)B_i(x)) = \left(\prod_{i\geq 0} A_i(x)\right)\left(\prod_{i\geq 0} B_i(x)\right).$$

Given a formal power series $A(x)$, we define:

$$\mathrm{mdeg}\ A(x) := \inf(\{n \in \mathbb{Z}_+ \cup \{0\} \mid [x^n]A(x) \neq 0\} \cup \{\infty\}).$$

Proposition: Suppose $A_0(x), A_1(x), \ldots$ is a sequence of formal power series.

- $\sum\limits_{i \geq 0} A_i(x)$ exists if and only if $\lim\limits_{i\to\infty} \mathrm{mdeg}\ A_i(x) = \infty$.

  Proof: (The professor skipped this because he thinks it's boring.)

  ($\Longleftarrow$)

  Suppose $\lim\limits_{j\to\infty} \mathrm{mdeg}\ A_j(x) = \infty$. Then for all $n \geq 0$, there exists $N \geq 0$ such that $\mathrm{mdeg}\ A_j(x) > n$ for all $j > N$. So:

  $$[x^n]\left(\sum_{j=0}^{i} A_j(x)\right) = [x^n]\left(\sum_{j=0}^{N} A_j(x)\right) \text{ for all } i > N.$$

  ($\Longrightarrow$)

  Suppose that $\lim\limits_{j\to\infty} \mathrm{mdeg}\ A_j(x)$ either doesn't exist or doesn't equal infinity if it does exist. Then we know there must exist N such that $\mathrm{mdeg}\ A_j(x) < N$ for infinitely many $j \geq 0$. In turn, for some $n \in \{0, 1, \ldots, N-1\}$, there must be infinitely many $j \geq 0$ such that $\mathrm{mdeg}\ A_j(x) = n$. Thus, there does not exist $M \geq 0$ such that:

  $$[x^n]\left(\sum_{j=0}^{i} A_j(x)\right) \text{ is the same for all } i \geq M.$$

- Assume each A_i has no constant term. Then $\prod\limits_{i\geq 0}(1 + A_i(x))$ exists if and only if $\lim\limits_{i\to\infty} \mathrm{mdeg}\ A_i(x) = \infty$.

  Proof: (btw I'm having to figure this all out without any outside help)

  Lemma: Suppose $B(x)$ and $C(x)$ are formal power series such that $[x^0]B(x) = 1$ and $\mathrm{mdeg}\ C(x) = n$. Then $\mathrm{mdeg}\ B(x)C(x) = n$ with $[x^n](B(x)C(x)) = [x^n](C(x))$.

  Corollary 1: Given $B(x)$ and $C(x)$ defined as before, for all $0 \leq i < n$:

  $$[x^i](B(x)(1 + C(x))) = [x^i](B(x) + B(x)C(x)) = [x^i](B(x)).$$

  Corollary 2: $[x^0]\left(\prod\limits_{j=0}^{i}(1 + A_j)\right) = 1$ for all $i \geq 0$.

  ($\Longleftarrow$)

  Suppose $\lim\limits_{j\to\infty} \mathrm{mdeg}\ A_j(x) = \infty$. Then for any $n \geq 0$, there exists $N \geq 0$ such that $\mathrm{mdeg}\ A_j(x) > n$ for all $j > N$. So given any $i > N$, we can inductively show using the above lemma and corollaries that:

$$[x^n]\left(\prod_{j=0}^{i}(1+A_j(x))\right) = [x^n]\left(\prod_{j=0}^{i-1}(1+A_j(x))\right)$$
$$= \cdots = [x^n]\left(\prod_{j=0}^{N}(1+A_j(x))\right)$$

($\Longrightarrow$)
As before, we can show there must be infinitely many $i \geq 0$ such that $\text{mdeg } A_i(x) = n$ for some n. And for any such i, we have by the above lemma that:

$$[x^n]\left(\prod_{j=0}^{i}(1+A_j(x))\right)$$
$$= [x^n]\left(\prod_{j=0}^{i-1}(1+A_j(x)) + \left(\prod_{j=0}^{i-1}(1+A_j(x))\right)A_i(x)\right)$$
$$= [x^n]\left(\prod_{j=0}^{i-1}(1+A_j(x))\right) + [x^n]A_i(x)$$
$$\neq [x^n]\left(\prod_{j=0}^{i-1}(1+A_j(x))\right)$$

So there is no $N \geq 0$ such that:

$$[x^n]\left(\prod_{j=0}^{i}(1+A_j(x))\right) \text{ is the same for all } i \geq N.$$

Suppose $A(x)$ and $B(x)$ are formal power series such that $A(x)$ has no constant term and $B(x) = \sum_{n\geq 0} b_n x^n$. Then we define their composition:

$$(B \circ A)(x) = B(A(x)) := \sum_{n\geq 0} b_n A(x)^n$$

This is well defined because $\text{mdeg } A(x) \geq 1 \Longrightarrow \text{mdeg } A(x)^n \geq n$. Therefore, $\lim_{n\to\infty} b_n A(x)^n = \infty$, meaning we can apply the previous proposition.

Special Case: If $A(x) = 0$, then $(B \circ A)(x) = b_0$.

Proposition: If $A(x)$, $B(x)$, and $C(x)$ are power series generated by $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, and $(c_n)_{n\geq 0}$ respectively such that $(B \circ A)(x)$ and $(C \circ A)(x)$ are defined, then:

- $((B+C)\circ A)(x) = (B\circ A)(x) + (C \circ A)(x)$

  Proof:
  By the second lemma on page 9, we know that:
  $$\sum_{n\geq 0}(b_n+c_n)A(x)^n = \sum_{n\geq 0}(b_nA(x)^n + c_nA(x)^n) = \sum_{n\geq 0} b_nA(x)^n + \sum_{n\geq 0} c_nA(x)^n$$

- $((BC)\circ A)(x) = (B\circ A)(x)(C\circ A)(x)$

Proof:
By the first lemma on page 9, we know that:

$$\begin{aligned}(B\circ A)(x)(C\circ A(x)) &= \left(\lim_{n\to\infty}\left(\sum_{i=0}^{n} b_iA(x)^i\right)\right)\left(\lim_{n\to\infty}\left(\sum_{i=0}^{n} c_iA(x)^i\right)\right)\\ &= \lim_{n\to\infty}\left(\left(\sum_{i=0}^{n} b_iA(x)^i\right)\left(\sum_{i=0}^{n} c_iA(x)^i\right)\right)\end{aligned}$$

Next note that given any $n \geq 0$, there exists a formal power series $R_n(x)$ with $\operatorname{mdeg} R_n(x) > n$ such that:

$$\left(\sum_{i=0}^{n} b_iA(x)^i\right)\left(\sum_{i=0}^{n} c_iA(x)^i\right) = \sum_{i=0}^{n}\left(\sum_{j=0}^{i} b_jc_{i-j}\right)A(x)^i + R_n(x)$$

Since $\lim\limits_{n\to\infty}\left(\sum_{i=0}^{n}\left(\sum_{j=0}^{i} b_jc_{i-j}\right)A(x)^i\right) = ((BC)\circ A)(x)$ and $\lim\limits_{n\to\infty}(R_n(x)) = 0$, we can thus apply the first lemma on page 9 again to get that:

$$\begin{aligned}&\lim_{n\to\infty}\left(\left(\sum_{i=0}^{n} b_iA(x)^i\right)\left(\sum_{i=0}^{n} c_iA(x)^i\right)\right)\\ &\qquad = \lim_{n\to\infty}\left(\sum_{i=0}^{n}\left(\sum_{j=0}^{i} b_jc_{i-j}\right)A(x)^i\right) + \lim_{n\to\infty}(R_n(x))\\ &\qquad = ((BC)\circ A)(x) + 0\end{aligned}$$

Suppose $A(x)$ is a formal power series. We define its derivative:

$$(DA)(x) = A'(x) := \sum_{n\geq 1} na_nx^{n-1} = \sum_{n\geq 0}(n+1)a_{n+1}x^n$$

Note that for $n \in \mathbb{Z}^+$ and $a_n \in F$, we define na_n via repeated addition.

Proposition: The following rules hold for any two formal power series $A(x)$ and $B(x)$:

- Sum Rule: $(A+B)'(x) = A'(x) + B'(x)$

  This identity is hopefully obvious.

- Product Rule: $(AB)'(x) = A'(x)B(x) + A(x)B'(x)$

  The proof for this identity requires rearranging sums strategically.

- Power Rule: $(A^n)'(x) = nA^{n-1}(x)A'(x)$ if $n > 0$ and $(A^n)'(x) = 0$ if $n = 0$.

  To prove this, do induction on n using the product rule.

Also if $[x^0]A(x) = 0$, then:

- Chain Rule: $(B \circ A)'(x) = A'(x)B'(A(x))$

  Proof: (seriously I'm doing this proof on my own...I order you to give me pity.)

  Lemma: If $A_0(x), A_1(x), \ldots$ are a sequence of formal power series that converges to $A(x)$, then $\lim\limits_{n\to\infty} A_n'(x) = A'(x)$.

  The proof for this is rather trivial. However, this is notably different from the convergence of derivatives of sequences of functions in math 140.

  Now suppose $B(x) = \sum\limits_{n\geq 0} b_n x^n$, and for each n, set $B_n(x) = \sum\limits_{i=0}^{n} b_i x^i$.

  By definition, we know that:
  $$\lim_{n\to\infty}((B_n \circ A)(x)) = \lim_{n\to\infty}\left(\sum_{i=0}^{n} b_i A(x)^i\right) = (B \circ A)(x).$$

  Hence, applying the previous lemma, we know that:
  $$\lim_{n\to\infty}((B_n \circ A)'(x)) = (B \circ A)'(x).$$

  But now note that by the sum and power rules:
  $$\begin{aligned}(B_n \circ A)'(x) &= \sum_{i=0}^{n} b_i (A^i)'(x)\\ &= \sum_{i=1}^{n} i b_i A(x)^{i-1} A'(x) = A'(x)\sum_{i=0}^{n-1}(i+1)b_{i+1}A(x)^i\\ &= A'(x)(B_n' \circ A)(x)\end{aligned}$$

  Finally, by definition we know that:
  $$\lim_{n\to\infty}((B_n' \circ A)(x)) = \lim_{n\to\infty}\left(\sum_{i=0}^{n-1}(i+1)b_{i+1}A(x)^i\right) = (B' \circ A)(x).$$

  So by applying the first lemma on page 9, we know that:
  $$\lim_{n\to\infty}((B_n \circ A)'(x)) = \lim_{n\to\infty}(A'(x)(B_n' \circ A)(x)) = A'(x)(B' \circ A)(x)$$

Meanwhile, if $A(x)$ is invertible, then:

- Multiplicative inverse rule: $(\frac{1}{A(x)})' = \frac{-A'(x)}{A(x)^2}$

  To prove this, just apply product rule to the expression $A(x)(\frac{1}{A(x)}) = 1$.

Examples of proving identities:

1. Since $\frac{1}{1-x} = \sum\limits_{n\geq 0} x^n$, we know $-\frac{-1}{(1-x)^2} = (\frac{1}{1-x})' = \left(\sum\limits_{n\geq 0} x^n\right)' = \sum\limits_{n\geq 0}(n+1)x^n$.
2. $\sum\limits_{n\geq 0} nx^n = \sum\limits_{n\geq 0}(n+1)x^n - \sum\limits_{n\geq 0} x^n = \frac{1}{(1-x)^2} - \frac{1}{1-x}\left(\frac{1-x}{1-x}\right) = \frac{x}{(1+x)^2}$

Let F be the field or commutative ring that our formal power series are defined over.

Note that given $n, m \in \mathbb{Z}_+$, we define $n, m \in F$ by repeatedly adding $1 \in F$ to itself n and m times respectively. Then, we can define $nm = mn \in F$ by doing repeated addition of n or m with itself. With that in mind, defining $n! \in F$ in a similar fashion and assuming that there exists $\frac{1}{n!} \in F$, we have that:

$$[x^n]A(x) = \frac{(D^n A)(0)}{n!} = \frac{(D^n A)(0+0x+0x^2+\ldots)}{n!}.$$

This is a random thought I had outside of lecture and wanted to write down:

Note: For the sake of clarity, I looked this up on wikipedia. If R is a commutative ring, then the set of formal power series in the variable x over R is written: $R[[x]]$.

Now given $A(x), B(x) \in R[[x]]$, we can define a metric $\rho(A(x), B(x)) = 2^{-n}$ where the nth coefficients of $A(x)$ and $B(x)$ are the first to differ, or if no such n exists, then we define $\rho(A(x), B(x)) = 0$. This somewhat trivially satisfies that:

- $\rho(A(x), B(x)) = 0 \iff A(x) = B(x)$.
- $\rho(A(x), B(x)) = \rho(B(x), A(x))$ for all $A(x), B(x) \in R[[x]]$.
- $\rho(A(x), B(x)) \leq \rho(A(x), C(x)) + \rho(C(x), B(x))$.

Also, we clearly have from of our definition of convergence that a sequence $(A_n(x))_{n\geq 0}$ in $R[[x]]$ converges if it is Cauchy. So this metric space is complete.

If I think of anything more to do with this, I'll add it to my notes.

10/7/2024

Homework 1:

(1) Find a closed formula for the following recurrence relation:

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = 2,$$
$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} \quad \text{for } n \geq 3$$

The characteristic polynomial of this relation is $t^3 - 5t^2 + 8t - 4$.

Because I wanted to trust that the professor wouldn't give us a messy polynomial, I used the rational root theorem to get a list of candidate roots to test. Those candidates are ± 1, ± 2, and ± 4.

After testing, I found that $(t-1)(t-2) = t^2 - 3t + 2$ is a factor of the characteristic polynomial. Doing polynomial long division, I then got that the other factor is $(t-2)$. So, our characteristic polynomial equals $(t-1)(t-2)^2$.

With that, we now know that $a_n = \beta_1 + \beta_2 2^n + \beta_3 n 2^n$. Plugging in $n = 0$, 1, and 2 respectively, we get the following system of equations:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 1 & 4 & 8 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

To solve this, I wrote the following code:

```
C:\Users\jmill>py
Python 3.12.1 (tags/v3.12.1:2305ca5, Dec  7 2023, 22:03:25) [MSC v.1937 64 bit (AMD64)] on win32
Type "help", "copyright", "credits" or "license" for more information.
>>>
>>> import numpy as np
>>>
>>> M = np.array([[1, 1, 0], [1, 2, 2], [1, 4, 8]])
>>> y = np.array([[1], [0], [2]])
>>>
>>> print((np.linalg.inv(M))@y)
[[ 6.]
 [-5.]
 [ 2.]]
>>>
```

So, $a_n = 6 - (5 + 2n)2^n$.

(2) Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be sequences. Assume that $(b_n)_{n\geq 0}$ satisfies a linear recurrence relation of order e. Then let $c_1, \ldots, c_d$ be scalars with $c_d \neq 0$ and assume that $(a_n)_{n\geq 0}$ satisfies:

$$a_n = c_1 a_{n-1} + \ldots + c_d a_{n-d} + b_n \text{ for all } n \geq d$$

Prove that $(a_n)_{n\geq 0}$ satisfies a linear recurrence relation of order $d + e$.

To start, let's show some smaller facts:

Observation 1: Suppose $a_n = c_1 a_{n-1} + \ldots + c_d a_{n-d} + p_m(n) r^n + F(n)$ where F is some arbitrary function, p_m is a polynomial of degree m and r is a nonzero constant. Then a_n satisfies a recurrence relation of degree $e = d + m + 1$ where $F'(n)$ is an arbitrary function $F(n)$ determined by m, r, and $F(n)$, and:

$$a_n = c_1' a_{n-1} + \ldots + c_e' a_{n-e} + F'(n) \text{ for all } n \geq e.$$

Proof by induction:

Lemma: If $p_m(n)$ is a polynomial of degree $m > 0$, then $p_m(n) - p_m(n-1)$ is a polynomial of degree $m - 1$.

This is because for all $N > 0$ and coefficients b, we have that: $bn^N - b(n-1)^N = bn^N - bn^N + q(n)$ where q is a polynomial of degree $N - 1$. Meanwhile, the case where $N = 0$ is trivial.

Base Case: If $m = 0$, meaning $p_m(n) = b$ where b is a constant, then by taking the difference of a_n and ra_{n-1} for all $n \geq d+1$, we get that:

$$a_n = (c_1 + r)a_{n-1} + (c_2 - rc_1)a_{n-2} + \ldots + (c_d - rc_{d-1})a_{n-d} - rc_d a_{n-d-1} + F(n) - rF(n-1).$$

Setting $c_1' = c_1 + 1, c_2' = c_2 - rc_1, \ \ldots, \ c_d' = c_d - rc_{d-1}, c_{d+1}' = -rc_d$, and $F'(n) = F(n) - rF(n-1)$, we thus get that:

$$a_n = c_1' a_{n-1} + \ldots + c_{d+1}' a_{n-d-1} + F'(n) \text{ for all } n \geq d+1.$$

Also $c_{d+1}' \neq 0$ because neither r nor c_d equal 0.

Induction on m: If $m > 0$, then by taking the difference of a_n and ra_{n-1} for all $n \geq d+1$, since $r^n(p(n)) - rr^n(p(n-1)) = r^n(p(n) - p(n-1))$, we get by our lemma above that:

$$a_n = (c_1 + r)a_{n-1} + (c_2 - rc_1)a_{n-2} + \ldots + (c_d - rc_{d-1})a_{n-d} - rc_d a_{n-d-1} + q(n)r^n + F'(n)$$

where q is a polynomial of degree $m - 1$ and $F'(n) = F(n) - rF(n-1)$.

And same as before, $-rc_d \neq 0$ because $r \neq 0$ and $c_d \neq 0$.

But now we can conclude by induction that a_n satisfies a (possibly inhomogeneous) recurrence relation of order $e = (d+1) + ((m-1)+1) = d+m+1$ such that $F''(n)$ is some function determined by m, r and $F(n)$, and:

$$a_n = c_1'' a_{n-1} + \ldots + c_e'' a_{n-e} + F''(n) \text{ for all } n \geq e.$$

One important observation from above is that if $F(n) = 0$, then $F'(n) = 0$. In some other situations, $F'(n)$ also behaves nicely.

Observation 2: Let $p_1(n), \ldots, p_k(n)$ be polynomials of degree $m_1, \ldots, m_k$ respectively. Also let $r_1, \ldots r_k$ be distinct nonzero constants. If:

$$a_n = c_1 a_{n-1} + \ldots + c_d a_{n-d} + p_1(n) r_1^n + F(n)$$

where $F(n) = \sum_{i=2}^{k} p_i(n) r_i^n$, then part 1 will make $F'(n) = \sum_{i=2}^{k} q_i(n) r_i^n$ where $q_2(n), \ldots, q_k(n)$ are also polynomials of degree $m_2, \ldots, m_k$ respectively.

To see why, note that:

$$\begin{aligned} F(n) - r_1 F(n-1) &= \sum_{i=2}^{k} p_i(n) r_i^n - \sum_{i=2}^{k} p_i(n-1) r_1 r_i^{n-1} \\ &= \sum_{i=2}^{k} (p_i(n) - \tfrac{r_1}{r_i} p_i(n-1)) r_i^n \end{aligned}$$

Because $r_i \neq r_1$, we know $\frac{r_1}{r_i} \neq 1$, meaning that the degree m term of $p_i(n) - \frac{r_1}{r_i}p_i(n-1)$ doesn't cancel. So $q_i(n) := p_i(n) - \frac{r_1}{r_i}p_i(n-1)$ is still a degree m polynomial. If the process in part 1 takes more steps, then we can just repeat this reasoning.

Combining observations 1 and 2 together, we can inductively show that if

$$a_n = c_1 a_{n-1} + \ldots + c_d a_{n-d} + \sum_{i=1}^{k} p_i(n) r_i^n$$

where $r_1, \ldots, r_k$ are distinct nonzero constants and $p_1(n), \ldots, p_k(n)$ are polynomials of degree $m_1, \ldots, m_k$ respectively, then letting $e = \sum_{i=1}^{k}(m_i + 1)$, there exists constants $c'_1, \ldots c'_{d+e}$ such that:

$$a_n = c'_1 a_{n-1} + \ldots + c'_{e+d} a_{n-d-e}$$

But now note that if $(b_n)_{n\geq 0}$ satisfies a linear recurrence relation of order e, we can write $b_n = \sum_{i=1}^{k} p_i(n) r_i^n$ for some polynomials $p_1(n), \ldots p_k(n)$ with degrees $m_1, \ldots, m_k$, as well as some distinct nonzero constants $r_1, \ldots, r_k$.

We know that each r_i is nonzero because the constant term in the characteristic polynomial for (b_n)'s recurrence relation must be nonzero.

Also, as we showed in class, $\sum_{i=1}^{k}(m_i + 1) = e =$ the order of the recurrence relation of $(b_n)_{n\geq 0}$.

So, we've shown that $a_n = c_1 a_{n-1} + \ldots + c_d a_{n-d} + b_n$ can be rewritten as a homogenous linear recurrence relation of order $(d + e)$.

(3) Let $(f_n)_{n\geq 0}$ be the Fibonacci numbers, and define $a_n = \sum_{i=0}^{n} f_i$.

(a) Find a linear recurrence relation of order 3 that $(a_n)_{n\geq 0}$ satisfies.

Note that $(a_n)_{n\geq 0}$ satisfies the relation $a_n = a_{n-1} + f_n$ for all a_n. Also, we showed in the first lecture that:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

So we know that: $a_n = a_{n-1} + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

Firstly, taking the difference of a_n and $\frac{1+\sqrt{5}}{2}a_{n-1}$, after a lot of simplifying we have that:

$$a_n = (1 + \tfrac{1+\sqrt{5}}{2})a_{n-1} - \tfrac{1+\sqrt{5}}{2}a_{n-2} + \left(-\tfrac{1}{\sqrt{5}} + \tfrac{1+\sqrt{5}}{2\sqrt{5}} \cdot \tfrac{2}{1-\sqrt{5}}\right)\left(\tfrac{1-\sqrt{5}}{2}\right)^n$$

$$= \tfrac{3+\sqrt{5}}{2}a_{n-1} - \tfrac{1+\sqrt{5}}{2}a_{n-2} + \left(\tfrac{1-\sqrt{5}}{2}\right)^{n-1}$$

Secondly, we take the difference of a_n and $\frac{1-\sqrt{5}}{2}a_{n-1}$, and after a lot more simplifying get:

$$\begin{aligned} a_n &= \left(\tfrac{3+\sqrt{5}}{2} + \tfrac{1-\sqrt{5}}{2}\right) a_{n-1} - \left(\tfrac{1+\sqrt{5}}{2} + \tfrac{1-\sqrt{5}}{2} \cdot \tfrac{3+\sqrt{5}}{2}\right) a_{n-2} + \left(\tfrac{1-\sqrt{5}}{2} \cdot \tfrac{1+\sqrt{5}}{2}\right) a_{n-3} \\ &= 2a_{n-1} + 0a_{n-2} - a_{n-3} \end{aligned}$$

(b) Find a closed formula for a_n.

Method 1:

The characteristic polynomial of $a_n = 2a_{n-1} - a_{n-3}$ is $t^3 - 2t^2 + 1$. Just by looking at it, I can already see that $(t-1)$ is a factor of that polynomial. So after doing polynomial long division, we have that $(t-1)(t^2-t-1) = t^3 - 2t^2 + 1$.

By quadratic formula, the remaining roots are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ (a.k.a the same roots as with the Fibonacci recurrence relation).

Finally, we get a system of linear equations:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & \left(\frac{1+\sqrt{5}}{2}\right)^2 & \left(\frac{1-\sqrt{5}}{2}\right)^2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

To solve this, I finally started learning sympy:

```
C:\Users\jmill\Desktop\py test> virt_math_test\Scripts\activate.bat

(virt_math_test) C:\Users\jmill\Desktop\py test>py
Python 3.12.1 (tags/v3.12.1:2305ca5, Dec  7 2023, 22:03:25) [MSC v.1937 64 bit (AMD64)] on win32
Type "help", "copyright", "credits" or "license" for more information.
>>>
>>> from sympy.matrices import Matrix
>>> from sympy import sqrt
>>> from sympy import pprint
>>> from sympy import simplify
>>>
>>> M = Matrix([[1, 1, 1], [1, ((1 + sqrt(5)) / 2), ((1 - sqrt(5)) / 2)], [1, ((1 + sqrt(5)) / 2
)**2, ((1 - sqrt(5)) / 2)**2] ])
>>> y = Matrix([[0], [1], [2]])
>>>
>>> x = simplify(simplify(M ** -1) * y)
>>>
>>> pprint(x)
[    -1    ]
[          ]
[1   3⋅√5  ]
[─ + ────  ]
[2    10   ]
[          ]
[1   3⋅√5  ]
[─ - ────  ]
[2    10   ]
>>>
>>> simplify(M * x) == y
True
>>>
>>> exit()

(virt_math_test) C:\Users\jmill\Desktop\py test> virt_math_test\Scripts\deactivate.bat
C:\Users\jmill\Desktop\py test>
```

So, assuming I've not made a silly error somewhere, we should have that:

$$a_n = -1 + \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Method 2: (The hinted route)

Note that $\frac{1-r^{n+1}}{1-r} = \sum_{i=0}^{n} r^i$. Using this fact, we can see that:

$$\begin{aligned} a_n = \sum_{i=0}^{n} f_i &= \frac{1}{\sqrt{5}} \sum_{i=0}^{n} \left(\frac{1+\sqrt{5}}{2}\right)^i - \frac{1}{\sqrt{5}} \sum_{i=0}^{n} \left(\frac{1-\sqrt{5}}{2}\right)^i \\ &= \frac{1}{\sqrt{5}} \cdot \frac{1}{1-\frac{1+\sqrt{5}}{2}} \left(1 - \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\right) - \frac{1}{\sqrt{5}} \cdot \frac{1}{1-\frac{1-\sqrt{5}}{2}} \left(1 - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) \end{aligned}$$

Now technically we are done since that is a closed formula for a_n. However, it looks ugly. So I'm going to learn more sympy so it can symplify this:

```
>>>
>>> from sympy import Symbol
>>> from sympy import sqrt
>>> from sympy import simplify
>>> from sympy import expand
>>> from sympy import pprint
>>>
>>> n = Symbol('n')
>>> p1 = (1 / sqrt(5)) * (1 / (1 - ((1+sqrt(5))/2) )) * (1 - (((1 + sqrt(5))/2) ** (n + 1)))
>>> p2 = (1 / sqrt(5)) * (1 / (1 - ((1-sqrt(5))/2) )) * (1 - (((1 - sqrt(5))/2) ** (n + 1)))
>>>
>>> pprint(simplify(expand(p1 - p2)))
 -n ⎛        n              n            n            n              n⎞
2  ⋅⎝- 10⋅2  - 3⋅√5⋅(1 - √5)  + 5⋅(1 - √5)  + 5⋅(1 + √5)  + 3⋅√5⋅(1 + √5) ⎠
───────────────────────────────────────────────────────────────────────────
                                     10
>>>
```

Hence we get the same answer as before:

$$a_n = -1 + \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$