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Math 220b Notes:

Given a collection of points $a_1, \ldots, a_N \in \mathbb{C}$ and positive integers $m_1, \ldots, m_N$, we can always find an entire function $f : \mathbb{C} \to \mathbb{C}$ which only has zeros at the a_n and with multiplicities given by m_n. After all, we can just consider the polynomial:

$$f(z) = \prod_{n=1}^{n}(z - a_n)^{m_n}.$$

A natural folowup question to ask is if we can find an entire function with infinitely many prescribed zeros of varying multiplicities. As it turns out the answer is yes.

A natural first idea one might have for proving this is to try setting:

$$f(z) = \prod_{a=1}^{\infty}(1 - \tfrac{z}{a_n})^{m_n}.$$

The issue with this approach though is that that product may diverge. As an example, consider setting $a_n = -n$ and $m_n = 1$ for all $n \in \mathbb{N}$. Then the product $f(z) \coloneqq \prod_{n=1}^{\infty}(1 - \frac{z}{a_n})^{m_n} = \prod_{n=1}^{\infty}(1 + \frac{z}{n})$ diverges to ∞ at $z = 1$. After all, we get that $f(1) = 2(\frac{3}{2})(\frac{4}{3})(\frac{5}{4})\cdots$.

By noting that $\operatorname{ord}(\prod_{k=1}^{\infty} g_k, a) = \sum_{k=1}^{\infty} \operatorname{ord}(g_k, a)$, a natural way of modifying our first idea is to try and multiply each $(1 - \frac{z}{a_n})^{m_n}$ term by some other function with no zeros. That way, we don't add any unwanted zeros to the function we are constructing and we can maybe coerce the product into converging.

It turns out this modified approach will work. Although surprisingly, by the homework problem on *pages 549-550*, we know these other function we're multiplying onto $(1 - \frac{z}{a_n})^{m_n}$ would have to have the form e^{g_n} where $g_n \in O(\mathbb{C})$ for all n.

Given any $p \in \mathbb{Z}_{\geq 0}$, we define the pth. Weierstrass primary/elementary factor to be:

$$E_p(z) = \begin{cases} (1 - z) & \text{if } p = 0 \\ (1 - z)\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{z}) & \text{if } p \neq 0 \end{cases}.$$

Note that if $a \in \mathbb{C} - \{0\}$, then $E_p(\frac{z}{a})$ is entire and it's only zero is a a simple one at $z = a$.

For motivation on why we are defining $E_p(z)$, note that the Taylor series for $-\operatorname{Log}(1 - z)$ about 0 is $\sum_{n=1}^{\infty} \frac{z^n}{n}$. Therefore, we'll ideally have that $E_p(z)$ is approximately equal to $1 = \frac{1-z}{1-z}$ when p is large and $|z|$ is small. The next lemma will make this idea more concrete.

Lemma A: $|E_p(z) - 1| \leq |z|^{p+1}$ if $|z| \leq 1$.

Proof:

If $p = 0$, then $|E_p(z) - 1| = |-z| = |z|$. So our proposed inequality trivialy holds.

Suppose $p > 0$ and set $u(z) = z + \frac{z^2}{2} + \cdots + \frac{z^p}{z}$. Then $E_p(z) = (1 - z)e^{u(z)}$ and we want to express $E_p(z)$ as a power series $\sum_{k=0}^{\infty} a_k z^k$ (which will have infinite radius of convergence since $E_p(z)$ is entire).

Claim 1: $a_0 = 1$.

Note that $a_0 = E_p(0) = (1-0)e^{u(0)} = 1$.

Claim 2: $a_1 = a_2 = \cdots = a_p = 0$.

By differentiating $u(z)$, we get that $u'(z) = 1 + z + \ldots + z^{p-1}$. In turn:
$$(1-z)u'(z) = 1 - z^p.$$

But that implies that:
$$E_p'(z) = -e^{u(z)} + (1-z)u'(z)e^{u(z)} = -e^{u(z)} + (1-z^p)e^{u(z)} = -z^p e^{u(z)}$$

Therefore, if we look at the taylor expansion $\sum_{k=1}^{\infty} k a_k z^{k-1}$ for $E_p'(z)$ about 0 and note that $e^{u(0)} = 1 \neq 0$, we must have that the lowest degree term in that power series is the z^p term. In other words, $a_1 = a_2 = \cdots = a_p = 0$.

Claim 3: $a_k \leq 0$ in $\mathbb{R}$ for all $k \geq p+1$.

We showed in claim 2 that:
$$\sum_{k=p+1}^{\infty} k a_k z^{k-1} = -z^p e^{u(z)}.$$

In turn, $\sum_{k=p+1}^{\infty} k a_k z^{k-(p+1)} = -e^{u(z)} = -\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p})$. Or in other words, we have that a_k equals $\frac{-1}{k}$ times the $(k-(p+1))$th. coefficient in the Taylor expansion of $\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p})$ for all $k \geq p+1$.

If we can show that all coefficients in the Taylor expansion of $\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p})$ are positive, we will be done. Fortunately, note that:
$$\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}) = e^z e^{\frac{z^2}{2}} \cdots e^{\frac{z^p}{p}} = \prod_{n=1}^{p} \left(\sum_{\ell=0}^{\infty} \frac{1}{\ell! n^\ell} z^{n\ell} \right).$$

By taking successive Cauchy products of those series (i.e. foiling), we'll get that the coefficients of the power series for $\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p})$ are all sums of products of the $\frac{1}{\ell! n^\ell}$ (which are all positive). Hence, those coefficients are positive.

Unnecessary tangent: Here's how we can actually calculate the power series expansion $\sum_{k=0}^{\infty} c_k z^k$ for $e^{u(z)}$.

To start off, the derivative of $e^{u(z)}$ is $(1 + z + \ldots + z^{p-1})e^{u(z)}$. Therefore, we must have that $\sum_{k=0}^{\infty}(k+1)c_{k+1}z^k = (1 + z + \ldots + z^{p-1}) \sum_{k=0}^{\infty} c_k z^k$. Also, when you consider that $e^{u(0)} = 1$, we thus get the following recurrence relation:

- $c_0 = 1$
- $c_{k+1} = \frac{1}{k+1} \sum_{\ell=0}^{\min(k,p-1)} c_{k-\ell}$

Next, we can easily calculate from that relation that $c_1 = \cdots = c_p = 1$. Furthermore, note that for any $k \geq p-1$ that:
$$c_{k+2} - \frac{k+1}{k+2} c_{k+1} = \frac{1}{k+2} c_{k+1} - \frac{1}{k+2} c_{k-p+1}.$$

In other words, $c_{k+2} = c_{k+1} - \frac{1}{k+2} c_{k-p+1}$. So finally (and this is how much simplified I'm capable of getting it before my attention span runs out), we have that the coefficients c_k are given by the following recurrence relation:

- $c_0 = c_1 = \cdots = c_p = 1$;
- $c_{k+1} = c_k - \frac{1}{k+1}c_{k-p}$ if $k \geq p$.

Claim 4: $\sum_{k=p+1}^{\infty} |a_k| = 1$

Note that $0 = E_p(1) = 1 + \sum_{k=p+1}^{\infty} a_k$. Therefore:

$$\sum_{k=p+1}^{\infty} |a_k| = -\sum_{k=p+1}^{\infty} a_k = 1$$

Finally, note that when $|z| \leq 1$, we have that:

$$|E_p(z) - 1| = |z^{p+1} \sum_{k=p+1}^{\infty} a_k z^{k-p-1}|$$

$$\leq |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k||z^{k-p-1}| \leq |z|^{p+1} \sum_{k=p+1}^{\infty} 1|a_k| = |z|^{p+1}. \blacksquare$$

Lemma B: Given any sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C} - \{0\}$ such that $|a_n| \to \infty$, we have for all $r > 0$ that $\sum_{n=1}^{\infty} (\frac{r}{|a_n|})^n < \infty$.

Proof:

Since $|a_n| \to \infty$, we know there exists N such that $|a_n| \geq 2r$ for all $n \geq N$. In turn, for all $n \geq N$ we have that $(\frac{r}{|a_n|})^n \leq \frac{1}{2^n}$. And since $\sum_{n=0}^{\infty} \frac{1}{2^n} < \infty$, we can conclude by the comparison test that $\sum_{n=1}^{\infty} (\frac{r}{|a_n|})^n < \infty$. $\blacksquare$

With that we are ready to prove our desired theorem:

Weierstraß Factorization Problem: Let $\{a_n\}_{n\in\mathbb{N}}$ be any sequence of distinct elements in $\mathbb{C}$ with $|a_n| \to \infty$, and also let $\{m_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}_{>0}$.

Note that the condition that $|a_n| \to \infty$ is equivalent to guarenteeing that the set $\{a_n : n \in \mathbb{N}\}$ has no limit points in $\mathbb{C}$.

We claim there exists an entire function $f : \mathbb{C} \to \mathbb{C}$ which only has zeros at the a_n and with multiplicities given by m_n.

Proof:

To start off, it will be convenient (for the sake of notation) to replace $\{a_k\}_{k\in\mathbb{N}}$ with the sequence:

$$\underbrace{a_1, \ldots, a_1}_{m_1 \text{ times}}, \underbrace{a_2, \ldots, a_2}_{m_2 \text{ times}}, \underbrace{a_3, \ldots, a_3}_{m_3 \text{ times}} \ldots$$

Importantly, doing this relabeling won't change the fact that $|a_n| \to \infty$ as $n \to \infty$. One other note is that we can without loss of generality assume $a_n \neq 0$ for any n. After all, if not we can just remove those terms from our sequence, solve the factorization problem to get an entire function $h(z)$, and then finally set $f(z) = z^m h(z)$ where m is the number of zero terms we removed from the sequence.

Now our goal is to pick a sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}_{\geq 0}$ such that $f(z) = \prod_{n=1}^{\infty} E_{p_n}(\frac{z}{a_n})$ converges absolutely locally uniformly. Equivalently, we need $\sum_{n=1}^{\infty} |E_{p_n}(\frac{z}{a_n}) - 1|$ to converge locally uniformly.

Set $p_n = n-1$ for all n. This will guarentee that $\sum_{n=1}^{\infty} |E_{p_n}(\frac{z}{a_n}) - 1|$ converges normally (and hence locally uniformly).

Indeed let $K \subseteq \mathbb{C}$ be compact. Then there exists $r > 0$ such that $K \subseteq \overline{\Delta}(0, r)$. Since $|a_n| \to \infty$, there exists N such that $|a_n| \geq r$ if $n \geq N$. And now by lemma A, we have that:

$$|E_{p_n}(\tfrac{z}{a_n}) - 1| \leq |\tfrac{z}{a_n}|^{p_n+1} \leq (\tfrac{r}{|a_n|})^n \text{ for all } z \in K$$

Finally normal converge follows from lemma B.

It follows that $f(z) := \prod_{n=1}^{\infty} E_{p_n}(\frac{z}{a_n})$ is an entire function with a zero at each a_n. ■

Note that by no means did we show that the p_n used in the above proof are unique. On the contrary, it is easy to justify that we can always modify at least finitely many of the p_n.

By combining the above proof with the homework problem on *pages 549-550*, we can rewrite our prior result in the following more versatile form:

Theorem: If f is entire with countably infinite zeros, then $f(z) = z^m e^{h(z)} \prod_{n \in \mathbb{N}} E_{p_n}(\frac{z}{a_n})$ for some $h \in O(\mathbb{C})$, $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C} - \{0\}$, $m \in \mathbb{Z}_{\geq 0}$ and $\{p_n\}_{n \in \mathbb{N}} \subseteq \mathbb{Z}_{\geq 0}$.