My Notes on Paolo Aluffi's Algebra Chapter 0

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April 1, 2024

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A multiset is a collection of elements which like a set is unordered but unlike a set can contain duplicate elements.

One way to define a multiset is as a function $f : A \to \mathbb{N}$ such that each $\alpha \in A$ is mapped to the number of times that α appears in the multiset. Then, given the multisets $f_1 : A \to \mathbb{N}$ and $f_2 : B \to \mathbb{N}$, we can define the following operations:

- $\alpha \in f_1 \iff \alpha \in A$
- $f_1 \subseteq f_2 \iff \forall \alpha \in f_1,\ \alpha \in f_2$ and $f_1(\alpha) \leq f_2(\alpha)$
- $f_1 \cup f_2 : (A \cup B) \longrightarrow \mathbb{N}$ such that for $\alpha \in A \cup B$, if $\alpha \in A \cap B$, then $(f_1 \cup f_2)(\alpha) = f_1(\alpha) + f_2(\alpha)$. As for if $\alpha \notin A \cap B$, then $(f_1 \cup f_2)(\alpha)$ equals whatever α was mapped to in the multiset it originally came from.
- $f_1 \cap f_2 : (A \cap B) \longrightarrow \mathbb{N}$ such that for $\alpha \in A \cap B$, we have that $(f_1 \cap f_2)(\alpha) = \min(f_1(\alpha), f_2(\alpha))$
- $f_1 \setminus f_2 : ((A \setminus B) \cup \{\alpha \in A \cap B \mid f_1(\alpha) > f_2(\alpha)\}) \longrightarrow \mathbb{N}$ such that for each $\alpha \in f_1 \setminus f_2$, if $\alpha \in f_2$, then $(f_1 \setminus f_2)(\alpha) = f_1(\alpha) - f_2(\alpha)$. As for if $\alpha \notin f_2$, then $(f_1 \setminus f_2)(\alpha) = f_1(\alpha)$

A practical example of a multiset is the prime factorization of any positive integer.

We say that two sets A and B are isomorphic if and only if there exists a bijection between A and B. We denote this by writing $A \cong B$. Additionally, we can refer to any bijection f between A and B as an isomorphism between the two sets.

A function $f : A \to B$ is a monomorphism (a.k.a a monic) if for all sets Z and all functions a' and $a'' : Z \to A$, we have that $f \circ a' = f \circ a'' \implies a' = a''$.

Proposition 1: A function is injective if and only if it is a monomorphism.

Proof: Let's say we have a function $f : A \to B$.

First, let us assume f is injective.

Then let us assume we have two functions a' and a'' from some set Z to A such that $f \circ a' = f \circ a''$. Because f is injective, we know it has a left-hand inverse $g : B \to A$ such that $g \circ f = \mathrm{Id}_A$. Composing g with the previous equation, we get that:

$$a' = \mathrm{Id}_A \circ a' = g \circ (f \circ a') = g \circ (f \circ a'') = \mathrm{Id}_A \circ a'' = a''$$

Thus, we've shown that f is a monomorphism.

Next, we shall assume f is a monomorphism.

Based on this, we can say that for any two functions a' and a'' mapping a set Z to A, we have that $f \circ a' = f \circ a'' \implies a' = a''$. However, now note that if we make Z a singleton, meaning it only contains one element, then a' and a'' can each only take on one value. So, we can effectively rewrite $f \circ a' = f \circ a'' \Rightarrow a' = a''$ as:

$$f(a') = f(a'') \Rightarrow a' = a''$$

This is the definition of an injective function. ■

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A function $f : A \to B$ is an epimorphism (a.k.a an epi) if for all sets Z and all functions b' and $b'' : B \to Z$, we have that $b' \circ f = b'' \circ f \Rightarrow b' = b''$.

Proposition 2: A function is a surjection if and only if it is an epimorphism.

Proof: Let's say we have a function $f : A \to B$.

First, let us assume f is surjective.

Then let's assume we have two functions b' and b'' from B to some set Z such that $b' \circ f = b'' \circ f$. Because f is surjective, we know it has a right-hand inverse $h : B \to A$ such that $f \circ h = \mathrm{Id}_B$. Composing h with the previous equation, we get that:

$$b' = b' \circ \mathrm{Id}_B = (b' \circ f) \circ h = (b'' \circ f) \circ h = b'' \circ \mathrm{Id}_B = b''$$

So f is an epimorphism.

Next, assume f is not surjective.

Then there exists $\beta \in B$ such that for all $\alpha \in A$, we have that $f(\alpha) \neq \beta$. Notably, this mean $|B| \geq 1$.

If $A = \emptyset$, then define b' to be the function from B to $\{0\}$ and b'' to be the function from B to $\{1\}$. Then, $b' \circ f = f = b'' \circ f$ but $b' \neq b''$.

Meanwhile if $A \neq \emptyset$, then there exists $f(\alpha) \in B \setminus \{\beta\}$. So, $|B| \geq 2$, meaning we can set b' equal to Id_B and define b'' as a function mapping each element of $B \setminus \{\beta\}$ to itself and β to any of the other elements in B. Now, $b' \circ f = f = b'' \circ f$ but $b' \neq b''$.

Hence, we have shown that f is not an epimorphism.

Sometimes, to indicate that a function $f : A \to B$ is a monomorphism, epimorphism, or isomorphism, we use the following notation:

- Monomorphism: $f : A \hookrightarrow B$
- Epimorphism: $f : A \twoheadrightarrow B$
- Isomorphism: $f : A \xrightarrow{\sim} B$

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A relation on a set S is a subset R of the cartesian product $S \times S$. Specifically, we use the notation $x \, R \, y$ to mean that $(x, y) \in R$. Certain types of relations are especially important and thus are represented with their own symbol.

- An equivalence relation, typically denoted $\sim$, on a set S has the properties:
  ◦ $\forall a \in S, \;\; a \sim a$ ◦ $a \sim b \Longrightarrow b \sim a$ ◦ $a \sim b$ and $b \sim c \Longrightarrow a \sim c$
- An order relation, typically denoted $<$, on a set S has the properties:
  ◦ $\forall a, b \in S$, exactly one of the following is true: $a < b$, $b < a$, or $a = b$.
  ◦ $a < b$ and $b < c$ implies that $a < c$.

Given a set S, an equivalence relation $\sim$, and an element $a \in S$, we define the equivalence class of a with respect to $\sim$ to be the set $[a]_\sim = \{b \in S \mid a \sim b\}$. Also, we define the quotient of S with respect to the equivalence relation $\sim$ as the set of equivalence classes with respect to $\sim$.

$$S/{\sim} = \{[a]_\sim \mid a \in S\}$$

Given any function $f : A \longrightarrow B$, define $a \sim b \Longleftrightarrow f(a) = f(b)$.

Proposition 3: Every function f can be decomposed as follows:

$$\begin{tikzcd} A \arrow[r, two heads, "g"'] \arrow[rrr, bend left, "f"] & (A/\sim) \arrow[r, "\sim", "\widetilde{f}"'] & \operatorname{im} f \arrow[r, hook, "h"'] & B \end{tikzcd}$$

(in other words, $f = h \circ \widetilde{f} \circ g$)

...where g is the surjection mapping a to $[a]_\sim$ for all $a \in A$, h is the inclusion function (which is injective) from the image of f to B, and $\widetilde{f}$ is a bijective function defined as the mapping $[a]_\sim$ to $f(a)$ where $a \in [a]_\sim$.

Proof:
$(A/{\sim})$ is defined as the range of g. So g is automatically surjective. Also, inclusion functions like h are always injective.

Now we show $\widetilde{f}$ is well defined and bijective.

1. Assume $a', a'' \in A$ such that $[a'] = [a'']$. Then by how we defined $\sim$, $f(a') = f(a'')$. So $[a'] = [a''] \Longrightarrow \widetilde{f}([a']) = \widetilde{f}([a''])$, meaning $\widetilde{f}$ is well defined.

2. Assume $\widetilde{f}([a']) = \widetilde{f}([a''])$. Then $f(a') = f(a'')$, meaning $a' \sim a''$. Hence $[a'] = [a'']$, meaning $\widetilde{f}$ is injective.

3. Given any $b \in \text{im} f$, there exists $a \in A$ such that $f(a) = b$. Then $\widetilde{f}([a]_\sim) = f(a) = b$. So $\widetilde{f}$ is surjective.

Finally, it's clear that $f = h \circ \widetilde{f} \circ g$. So we're done.

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A category **C** consists of a class $\text{Obj}(\mathsf{C})$ of objects of the category, and for every two objects A, B of **C**, a set $\text{Hom}_{\mathsf{C}}(A, B)$ of morphisms with the following properties:

- For every object A of **C**, there exists a morphism $1_A \in \text{Hom}_{\mathsf{C}}(A, A)$ called the identity on A.
- Morphisms can be composed, meaning $f \in \text{Hom}_{\mathsf{C}}(A, B)$ and $g \in \text{Hom}_{\mathsf{C}}(B, C)$ means that $gf \in \text{Hom}_{\mathsf{C}}(A, C)$
- Composition is associative, meaning if $f \in \text{Hom}_{\mathsf{C}}(A, B)$, $g \in \text{Hom}_{\mathsf{C}}(B, C)$, and $h \in \text{Hom}_{\mathsf{C}}(C, D)$, then $(hg)f = h(gf)$.
- The identity morphisms are identities with respect to composition, meaning for all $f \in \text{Hom}_{\mathsf{C}}(A, B)$, $f1_A = f$ and $1_B f = f$.
- $\text{Hom}_{\mathsf{C}}(A, B)$ and $\text{Hom}_{\mathsf{C}}(C, D)$ are disjoint unless $A = C$ and $B = D$.

We use the word "class" because by Russell's paradox, there are many sets which aren't well defined. For example, there can be no set of all sets. So we instead define a class of all sets.

Also, we write category names in sans-serif font to better distinguish them.

A morphism of an object A of a category **C** to itself is called an endomorphism. Thus we denote $\text{Hom}_{\mathsf{C}}(A, A)$ as $\text{End}_{\mathsf{C}}(A)$.

Note that by the composition rules of a category, if $f, g \in \text{End}_{\mathsf{C}}(A)$, then $fg, gf \in \text{End}_{\mathsf{C}}(A)$.

We can denote a morphism $f \in \text{Hom}_{\mathsf{C}}(A, B)$ as $f : A \to B$.

Examples of Categories:

- We define the category of sets: **Set**, such that $\text{Obj}(\mathsf{Set})$ is the class of all sets and for A and B in $\text{Obj}(\mathsf{Set})$, $\text{Hom}_{\mathsf{Set}}(A, B)$ is the set of all functions from A to B (abbreviated as B^A).

- If S is a set and $\sim$ is an equivalence relation on S, then we can define a category whose objects are the elements of S, and for $a, b \in S$, $\mathrm{Hom}(a, b)$ equals $\{(a, b)\}$ when $a \sim b$ and $\emptyset$ otherwise.

  Note that for this category, we need to define what it means to compose morphisms. So let's say that if $f = \{(a, b)\}$ and $g = \{(b, c)\}$, then $gf = \{(a, c)\}$.

- Let **C** be a category and let A be an object of **C**. Then we can define a category $\mathbf{C}_A$ as follows:
  - $\mathrm{Obj}(\mathbf{C}_A) =$ all morphisms from any object of **C** to A
  - If $f_1 : Z_1 \longrightarrow A$ and $f_2 : Z_2 \longrightarrow A$ are objects of $\mathbf{C}_A$, then $\mathrm{Hom}_{\mathbf{C}_A}(f, g)$ is the set of morphisms $\sigma : Z_1 \to Z_2$ such that $f_1 = f_2\sigma$.

  Thus the morphisms of $\mathbf{C}_A$ are <u>commutative diagrams</u> with the objects Z_1, Z_2, and A.

  To prove that this is a category, first consider that each object $f : Z \longrightarrow A$ has an identity morphism:

$$\begin{tikzcd} Z \arrow[rr, "1_Z"] \arrow[dr, "f"'] & & Z \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

  Also, the morphisms of $\mathbf{C}_A$ compose. If the diagram with σ is in $\mathrm{Hom}_{\mathbf{C}_A}(f_1, f_2)$ and the diagram with τ is in $\mathrm{Hom}_{\mathbf{C}_A}(f_2, f_3)$, then we define their composition in $\mathrm{Hom}_{\mathbf{C_A}}(f_2, f_3)$ as the diagram with the composed morphism $\tau\sigma$ in **C**.

$$\begin{tikzcd} Z_1 \arrow[r, "\sigma"] \arrow[dr, "f_1"'] & Z_2 \arrow[r, "\tau"] \arrow[d, "f_2"] & Z_3 \arrow[dl, "f_3"] \\ & A & \end{tikzcd} \Longrightarrow \begin{tikzcd} Z_1 \arrow[rr, "\tau\sigma"] \arrow[dr, "f_1"'] & & Z_3 \arrow[dl, "f_3"] \\ & A & \end{tikzcd}$$

  As is hopefully apparent, the identity morphisms compose as is required for $\mathbf{C}_A$ to be a category.

  Finally, composing morphisms of $\mathbf{C}_A$ is associative because $(\upsilon\tau)\sigma = \upsilon(\tau\sigma)$ in the category **C**.

$$\begin{tikzcd} Z_1 \arrow[r, "\sigma"] \arrow[rr, bend left, "\tau\sigma"] \arrow[rrr, bend left=50, "\upsilon\tau\sigma"] \arrow[ddr, "f_1"'] & Z_2 \arrow[r, "\tau"] \arrow[rr, bend left, "\upsilon\tau"] \arrow[dd, "f_2"] & Z_3 \arrow[r, "\upsilon"] \arrow[ddl, "f_3"] & Z_4 \arrow[ddll, "f_4"] \\ & & & \\ & A & & \end{tikzcd}$$

- Categories like the one in the previous example are called <u>slice categories</u>. We can similarly define <u>coslice categories</u> as follows:

  Let **C** be a category and let A be an object of **C**. Then we can define a category $\mathbf{C}^A$ such that:

  - $\mathrm{Obj}(\mathbf{C}^A) =$ all morphisms from A to any object of **C**
  - If $f_1 : A \longrightarrow Z_1$ and $f_2 : A \longrightarrow Z_2$ are objects of $\mathbf{C}^A$, then $\mathrm{Hom}_{\mathbf{C}^A}(f, g)$ is the set of morphisms $\sigma : Z_1 \to Z_2$ such that $\sigma f_1 = f_2$.

  In other words, we're now considering commutative diagrams of the form:

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ Z_1 \arrow[rr, "\sigma"'] & & Z_2 \end{tikzcd}$$

Problem 3.8: A <u>subcategory</u> $\mathbf{C}'$ of a category **C** consists of a collection of objects of **C** with morphisms $\mathrm{Hom}_{\mathbf{C}'}(A, B) \subseteq \mathrm{Hom}_{\mathbf{C}}(A, B)$ for all objects A, B in $\mathrm{Obj}(\mathbf{C}')$ such that $\mathbf{C}'$ has all the necessary identities and compositions to be a category. A subcategory $\mathbf{C}'$ is <u>full</u> if $\mathrm{Hom}_{\mathbf{C}'}(A, B) = \mathrm{Hom}_{\mathbf{C}}(A, B)$ for all A, B in $\mathrm{Obj}(\mathbf{C}')$.

Let $\mathbf{Set}'$ be the category of infinite sets.

- $\mathrm{Obj}(\mathbf{Set}')$ is the class of all infinite sets.
- For all A, B in $\mathrm{Obj}(\mathbf{Set}')$, $\mathrm{Hom}_{\mathbf{Set}'}(A, B)$ is the set of all functions from A to B.

Now given the infinite sets A and B, any morphism $f \in \mathrm{Hom}_{\mathbf{Set}}(A, B)$ is also a morphism of $\mathrm{Hom}_{\mathbf{Set}'}(A, B)$. So $\mathbf{Set}'$ is a full subcategory of **Set**.

Problem 3.1: Let **C** be a category. Then consider $\mathbf{C}^{op}$ with

- $\mathrm{Obj}(\mathbf{C}^{op}) = \mathrm{Obj}(\mathbf{C})$
- for A, B in $\mathrm{Obj}(\mathbf{C}^{op})$, $\mathrm{Hom}_{\mathbf{C}^{op}}(A, B) = \mathrm{Hom}_{\mathbf{C}}(B, A)$.

Let A, B, and C be objects of $\mathbf{C}^{op}$. Given $g \in \mathrm{Hom}_{\mathbf{C}^{op}}(A, B)$ and $h \in \mathrm{Hom}_{\mathbf{C}^{op}}(B, C)$, define the composition $hg \in \mathrm{Hom}_{\mathbf{C}^{op}}(A, C)$ to be the morphism $gh \in \mathrm{Hom}_{\mathbf{C}}(C, A)$.

To see why this is well defined note that if $g \in \mathrm{Hom}_{\mathbf{C}^{op}}(A, B)$, then $g \in \mathrm{Hom}_{\mathbf{C}}(B, A)$. Similarly, if $h \in \mathrm{Hom}_{\mathbf{C}^{op}}(B, C)$, then $h \in \mathrm{Hom}_{\mathbf{C}}(C, B)$. As **C** is a category, there must exist a morphism $gh \in \mathrm{Hom}_{\mathbf{C}}(C, A)$, which in turn means that the morphism we defined as the composition $hg \in \mathrm{Hom}_{\mathbf{C}^{op}}(A, C)$ exists.

$$\begin{tikzcd} & B \arrow[r, "g"] & A \\ C \arrow[ur, "h"] \arrow[urr, bend right, "gh"'] & & \end{tikzcd}$$

So by how we defined composition of morphisms in $\mathbf{C}^{op}$, we know $\mathbf{C}^{op}$ satisfies the composition property of a category. Now what's left to show is that $\mathbf{C}^{op}$ has the other properties of a category.

For any object A in $\mathrm{Obj}(\mathbf{C}^{op})$, $\mathrm{End}_{\mathbf{C}^{op}}(A) = \mathrm{End}_{\mathbf{C}}(A)$. So, A inherits a morphism 1_A from $\mathbf{C}$.

Consider $g \in \mathrm{Hom}_{\mathbf{C}^{op}}(A, B)$. Then $g1_A$ in $\mathrm{Hom}_{\mathbf{C}^{op}}(A, B)$ is equal to $1_A g = g$ in $\mathrm{Hom}_{\mathbf{C}}(B, A)$. So in $\mathbf{C}^{op}$, we have that $g1_A = g$.

Similarly, consider $h \in \mathrm{Hom}_{\mathbf{C}^{op}}(B, A)$. Then $1_A h \in \mathrm{Hom}_{\mathbf{C}^{op}}(B, A)$ is equal to $h1_A = h$ in $\mathrm{Hom}_{\mathbf{C}}(A, B)$. So in $\mathbf{C}^{op}$, we have that $1_A h = h$.

Finally, observe that given the morphisms $f \in \mathrm{Hom}_{\mathbf{C}^{op}}(A, B)$, $g \in \mathrm{Hom}_{\mathbf{C}^{op}}(B, C)$, and $h \in \mathrm{Hom}_{\mathbf{C}^{op}}(C, D)$, we know that in $\mathbf{C}$:

$$\begin{tikzcd}
& C \arrow[rr, bend left, "fg"] \arrow[dr, "g"] & & A \\
D \arrow[ur, "h"] \arrow[rr, bend right, "gh"'] & & B \arrow[ur, "f"] &
\end{tikzcd}$$

$(gf) \in \mathrm{Hom}_{\mathbf{C}^{op}}(A, C)$ refers to the morphism $fg \in \mathrm{Hom}_{\mathbf{C}}(C, A)$. So, $h(gf) \in \mathrm{Hom}_{\mathbf{C}^{op}}(A, D)$ refers to the morphism $(fg)h \in \mathrm{Hom}_{\mathbf{C}}(D, A)$. At the same time, $(hg) \in \mathrm{Hom}_{\mathbf{C}^{op}}(B, D)$ refers to the morphism $gh \in \mathrm{Hom}_{\mathbf{C}}(D, B)$. So, $(hg)f \in \mathrm{Hom}_{\mathbf{C}}(D, A)$ refers to the morphism $f(gh) \in \mathrm{Hom}_{\mathbf{C}}(D, A)$. Thus as $(fg)h = f(gh)$ in $\mathbf{C}$, we have that $h(gf) = (hg)f$ in $\mathbf{C}^{op}$.

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A morphism $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ is an isomorphism if it has a two sided inverse under composition (i.e. $\exists g \in \mathrm{Hom}_{\mathbf{C}}(B, A)$ such that $gf = 1_A$ and $fg = 1_B$).

Proposition 4: The inverse of an isomorphism is unique.

Proof:

Suppose $g_1, g_2 : B \longrightarrow A$ both act as inverses of $f : A \longrightarrow B$. Then:

$$g_1 = g_1 1_B = g_1(fg_2) = (g_1 f)g_2 = 1_A g_2 = g_2$$

> Corollary: If f has a left-hand inverse g_1 and a righthand inverse g_2, then f must be an isomorphism and $g_1 = g_2$ must be the unique inverse of f.
>
> (Our proof from before also shows this.)

Since the inverse of f is unique, we denote it f^{-1}.

Proposition 5:

(A) Each identity 1_A is an isomorphism with itself being its own inverse.

(B) If f is an isomorphism, then f^{-1} is an isomorphism and $(f^{-1})^{-1} = f$.

(C) If $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathbf{C}}(B, C)$ are isomorphisms, then the composition gf is an isomorphism and $(gf)^{-1} = f^{-1}g^{-1}$.

To prove any of these, just show that the proposed inverses are in fact an inverse. For example:

- $1_A 1_A = 1_A$
- $(gf)(f^{-1}g^{-1}) = g(ff^{-1})g^{-1} = g1_B g^{-1} = gg^{-1} = 1_C$

Two objects A and B of a category are isomorphic if there is an isomorphism $f : A \longrightarrow B$. We denote this by writing $A \cong B$.

An automorphism of an object A of a category **C** is an isomorphism from A to itself. The set of automorphisms of A is denoted $\mathrm{Aut}_{\mathbf{C}}(A)$.

Note:

- $\mathrm{Aut}_{\mathbf{C}}(A) \subseteq \mathrm{End}_{\mathbf{C}}(A)$
- If $f, g \in \mathrm{Aut}_{\mathbf{C}}(A)$, then fg and gf are in $\mathrm{Aut}_{\mathbf{C}}(A)$.
- $1_A \in \mathrm{Aut}_{\mathbf{C}}(A)$
- For each $f \in \mathrm{Aut}_{\mathbf{C}}(A)$, there exists $f^{-1} \in \mathrm{Aut}_{\mathbf{C}}(A)$.

Spoiler: The last three points mean that $\mathrm{Aut}_{\mathbf{C}}(A)$ forms a group.

The definitions of surjections and injections don't translate into category theory because the objects of a category don't necessarily have elements. However, the definitions of monomorphisms and epimorphisms do hold in category theory.

Let **C** be a category and $f : A \to B$ a morphism.

- f is a monomorphism if for any object Z of **C** and morphisms $\alpha', \alpha'' \in \mathrm{Hom}_{\mathbf{C}}(Z, A)$, we have that $f\alpha' = f\alpha'' \Longrightarrow \alpha' = \alpha''$.

- f is a epimorphism if for any object Z of **C** and morphisms $\beta', \beta'' \in \mathrm{Hom}_{\mathbf{C}}(B, Z)$, we have that $\beta' f = \beta'' f \Longrightarrow \beta' = \beta''$.

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f being both a monomorphism and epimorphism does not necessarily imply that f is isomorphism.

For example, consider a category whose objects are all the elements of $\mathbb{Z}$, and where for $a, b \in \mathbb{Z}$, $\mathrm{Hom}(a, b)$ equals $\{(a, b)\}$ if $a \leq b$ and $\emptyset$ otherwise.

Also we define the composition of (a, b) and (b, c) to be (a, c).

Let $f : a \longrightarrow b$ be a morphism and consider any object z of the category. Since there is only at most one morphism possible in $\mathrm{Hom}(z, a)$, f is automatically a monomorphism. Similarly, f is automatically an epimorphism because there is only at most one morphism possible in $\mathrm{Hom}(b, z)$. That said, the only isomorphisms are the morphisms: $(a, a) \in \mathrm{End}(a)$ for each $a \in \mathbb{Z}$.

Another thing the above category demonstrates is that monomorphisms don't necessarily have left-hand inverses and epimorphisms don't necessarily have right-hand inverses.

Problem 4.3: Let **C** be a category and $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ be a morphism. Prove that if f has a right-inverse, then f is an epimorphism.

Assume f has a right-inverse $g : B \longrightarrow A$. Then consider two morphisms $\beta', \beta'' : B \longrightarrow Z$ such that $\beta' f = \beta'' f$. Thus:

$$\beta' = \beta' 1_B = (\beta' f)g = (\beta'' f)g = \beta'' 1_B = \beta''$$

By similar reasoning, we can show that f having a left-inverse implies that f is a monomorphism.

Problem 4.4:

- Prove that the composition of two monomorphisms is a monomorphism.

  Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be monomorphisms. Then consider two morphisms $\alpha', \alpha'' : Z \longrightarrow A$ such that $(gf)\alpha' = (gf)\alpha''$.

  Since g is a monomorphism, $g(f\alpha') = g(f\alpha'') \Longrightarrow f\alpha' = f\alpha''$.
  Then as f is a monomorphism, $f\alpha' = f\alpha'' \Longrightarrow \alpha' = \alpha''$.
  So gf is a monomorphism.

  By similar reasoning, we can show that the composition of two epimorphisms is an epimorphism.

- Deduce that we can define a subcategory $\mathbf{C}_{\mathrm{mono}}$ of a category **C** such that:
  - $\mathrm{Obj}(\mathbf{C}_{\mathbf{mono}}) = \mathrm{Obj}(\mathbf{C})$
  - For each A, B in $\mathrm{Obj}(\mathbf{C}_{\mathbf{mono}})$, $\mathrm{Hom}_{\mathbf{C}_{\mathbf{mono}}}(A, B)$ is the subset of $\mathrm{Hom}_{\mathbf{C}}(A, B)$ consisting of only monomorphisms.

  Having followed the recipe above for making $\mathbf{C}_{\mathrm{mono}}$, we need to show that $\mathbf{C}_{\mathrm{mono}}$ satisfies the properties of a category.

  By the previous part of the problem, we know that all morphisms in C_{mono} compose with each other to give other morphisms in C_{mono}. Also, because morphism composition in **C** is associative, we also have that morphism composition in $\mathbf{C}_{\mathrm{mono}}$ is associative. So, what we have left to show is that each object in $\mathbf{C}_{\mathrm{mono}}$ has an identity morphism.

By problem 4.3, we know that isomorphisms are automatically both monomorphisms and epimorphisms because they have both a right-inverse and a left-inverse. This means that since the identity morphisms of **C** are isomorphisms, we know that they are also morphisms in $\mathbf{C}_{\text{mono}}$. So each object A in $\text{Obj}(\mathbf{C}_{\text{mono}})$ has an identity morphism 1_A. Additionally, for every morphism $f : A \longrightarrow B$ in $\mathbf{C}_{\text{mono}}$, we have that $1_B f = f$ and $f 1_A = f$ because that's how those morphisms would compose in **C**.

So $\mathbf{C}_{\text{mono}}$ satisfies the properties of a category. Hence we conclude that we can define it as a subcategory of **C**.

Let **C** be a category. We say that an object I of **C** is initial in **C** if for every object A of **C**, there exists exactly one morphism $I \longrightarrow A$ in **C**. Meanwhile, we say that an object F of **C** is final in **C** if for every object A of **C**, there exists exactly one morphism $A \longrightarrow F$ in **C**.

One can use the word terminal to describe either I or F.

Examples:

In the category **Set**, $\emptyset$ is initial because there is a single morphism: the empty function $\emptyset$, going from $\emptyset$ to every other set. Also, every other object of **Set** is not initial since they all have at least two morphisms towards any set of size 2.

Meanwhile, every singleton $\{a\}$ in the category of **Set** is final since for every other set S, there is exactly one morphism from S to $\{a\}$. Specifically, that morphism is the function assigning all elements of S to a.

Proposition 6: Let **C** be a category.

- If I_1 and I_2 are both initial objects in **C**, then $I_1 \cong I_2$.
- If F_1 and F_2 are both final objects in **C**, then $F_1 \cong F_2$.

Furthermore, these isomorphisms are uniquely determined.

Proof:

By the definition of a category, all objects have an identity morphism. So if F is final, then the unique morphism $F \longrightarrow F$ must be the identity morphism 1_F.

Now assume F_1 and F_2 are both final in **C**. Then there is a unique morphism $f : F_1 \longrightarrow F_2$ and a unique morphism $g : F_2 \longrightarrow F_1$. Now, gf is a morphism from F_1 to F_1. So, gf must equal 1_{F_1}. By similar reasoning, $fg = 1_{F_2}$. This tells us that $g = f^{-1}$ and f is an isomorphism. So $F_1 \cong F_2$.

The proof for initial objects is entirely analogous.

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A big reason we care about category theory is that it allows us to characterize concepts at "satisfying a universal property" or "being the solution to a universal problem" without worrying about the specifics of how we construct that concept.

We say a construction satisfies a universal property if it may be viewed as a terminal object of a category. A general outline for what might be said is:

> "Object X is univeral with respect to the following property: for any Y such that..., there exists a unique morphism $Y \to X$ such that...."

Notably, the above statement doesn't say what the category is that X and Y are in. This is normal because category theorists are fucking lazy. In fact, category theorists actually often write even less because they think it's other people's responsibility to try and decode whatever cryptic, terse statements they write.

Example:
Let $\sim$ be an equivalence relation defined on a set A. Then "the quotient $A/\sim$ is universal with respect to the property of mapping A to a set in such a way that equivalent elements have the same image."

> In this statement, the other objects in the category are functions $\varphi : A \longrightarrow S$ satisfying that $a' \sim a'' \implies \varphi(a') = \varphi(a'')$. Thus, we can conclude that the object satisfying the universal property is not literally the quotient $A/\sim$ but instead some function from A to $A/\sim$.
>
> Now, the objects of the category in above statement are morphisms of the category **Set** which are coming out of a specific object A in **Set** and satisfying an additional property. Thus, (according to Aluffi) the only category that our above statement could reasonably be alluding to is a subcategory of $\mathbf{C}^A$ (see page 7). Letting **C** be the category of the statement above, we have that:
>
> - $\mathrm{Obj}(\mathbf{C})$ = all functions / morphisms φ from A to any object of **Set** satisfying that $a' \sim a'' \implies \varphi(a') = \varphi(a'')$.
> - For $f_1 : A \to S_1$ and $f_2 : A \to S_2$ in $\mathrm{Obj}(\mathbf{C})$, $\mathrm{Hom}_{\mathbf{C}}(f_1, f_2)$ = the set of morphisms $\sigma : S_1 \to S_2$ such that $\sigma \circ f_1 = f_2$.
>
> Finally, what function from A to $A/\sim$ could the above statement possibly care about other than the canonical projection π mapping a to $[a]_\sim$? So, to prove this statement we now prove that π is a terminal object of **C**.

Firstly, unless $\sim \, = A \times A$, we do not have that π is a final object of **C**. This is because all the final objects of **C** are functions mapping A to a singleton. So, we instead want to show that π is an initial object of **C**.

Let $\varphi : A \longrightarrow S$ be an object of **C**. If there is a function $\overline{\varphi}$ such that $\overline{\varphi} \circ \pi = \varphi$, then $\overline{\varphi}([a]_\sim) = \varphi(a)$ for all $[a]_\sim \in A/\sim$. Thus $\overline{\varphi}$ must be unique if it exists. Meanwhile, $[a_1]_\sim = [a_2]_\sim \implies a_1 \sim a_2 \implies \varphi(a_1) = \varphi(a_2)$. Hence $\overline{\varphi}$ is well-defined.

So, π is an initial object of **C**.

As for how the previous statement can be useful, consider a function $f : A \to B$ and equivalence relation $\sim$ defined such that $a_1 \sim a_2 \iff f(a_1) = f(a_2)$. We can show that the function from A to $\mathrm{im} f$ mapping a to $f(a)$ also satisfies the same universal property as the canonical projection from A to $A/\sim$. So the two functions must be isomorphic in the category of the statement.

In turn, this gives us another proof for proposition 3 on pages 4 and 5.

$$\begin{tikzcd} A/\sim \arrow[r, bend left, "\tilde{f}"] & \mathrm{im} f \arrow[l, bend left, "\tilde{f}^{-1}"] \arrow[r] & B \\ & A \arrow[ul] \arrow[u] & \end{tikzcd}$$

Products and Coproducts of Categories:

Let **C** be a category and A and B be objects in **C**. Similar to how we defined $\mathbf{C}_A$ and $\mathbf{C}^A$ earlier, we can define $\mathbf{C}_{A,B}$ and $\mathbf{C}^{A,B}$ as follows:

$\mathbf{C}_{A,B}$

$\mathrm{Obj}(\mathbf{C}_{A,B}) =$ all diagrams of the form to the right where Z is an object in **C**, $f \in \mathrm{Hom}_{\mathbf{C}}(Z, A)$ and $g \in \mathrm{Hom}_{\mathbf{C}}(Z, B)$.

$$\begin{tikzcd} & A \\ Z \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & B \end{tikzcd}$$

A morphism from
$$\begin{tikzcd} & A \\ Z_1 \arrow[ur, "f_1"] \arrow[dr, "g_1"'] & \\ & B \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} & A \\ Z_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"'] & \\ & B \end{tikzcd}$$

is a morphism $\sigma \in \mathrm{Hom}_{\mathbf{C}}(Z_1, Z_2)$ such that $f_2\sigma = f_1$ and $g_2\sigma = g_1$.

In other words, a morphism of $\mathbf{C}_{A,B}$ is a commutative diagram:

$$\begin{tikzcd}
 & & A \\
Z_1 \arrow[urr, bend left, "f_1"] \arrow[r, "\sigma"] \arrow[drr, bend right, "g_1"'] & Z_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"'] & \\
 & & B
\end{tikzcd}$$

$\mathbf{C}^{A,B}$

$\mathrm{Obj}(\mathbf{C}^{A,B}) =$ all diagrams of the form to the right where Z is an object in $\mathbf{C}$, $f \in \mathrm{Hom}_{\mathbf{C}}(A, Z)$ and $g \in \mathrm{Hom}_{\mathbf{C}}(B, Z)$.

$$\begin{tikzcd}
A \arrow[dr, "f"] & \\
 & Z \\
B \arrow[ur, "g"'] &
\end{tikzcd}$$

A morphism from
$$\begin{tikzcd}
A \arrow[dr, "f_1"] & \\
 & Z_1 \\
B \arrow[ur, "g_1"'] &
\end{tikzcd}
\quad \text{to} \quad
\begin{tikzcd}
A \arrow[dr, "f_2"] & \\
 & Z_2 \\
B \arrow[ur, "g_2"'] &
\end{tikzcd}$$

is a morphism $\sigma \in \mathrm{Hom}_{\mathbf{C}}(Z_1, Z_2)$ such that $f_2 = \sigma f_1$ and $g_2 = \sigma g_1$.

In other words, a morphism of $\mathbf{C}^{A,B}$ is a commutative diagram:

$$\begin{tikzcd}
A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] & & \\
 & Z_1 \arrow[r, "\sigma"] & Z_2 \\
B \arrow[ur, "g_1"] \arrow[urr, bend right, "g_2"'] & &
\end{tikzcd}$$

The proof that these are valid categories is similar to the proof that C_A and C^A are valid categories.

We say a category $\mathbf{C}$ has (finite) products if for all objects A and B in $\mathbf{C}$, the category $\mathbf{C}_{A,B}$ has final objects. Such a final object consists of an object of $\mathbf{C}$ which is called a product of A and B and usually denoted $A \times B$, and two morphisms: $A \times B \longrightarrow A$ and $A \times B \longrightarrow B$.

Examples of products in categories:

1. We define the product of two sets A and B as:
$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

   Having done that, we can then define the natural projection functions:

   - $\pi_A : A \times B \longrightarrow A$ with the correspondence rule: $(a,b) \mapsto a$
   - $\pi_B : A \times B \longrightarrow B$ with the correspondence rule: $(a,b) \mapsto b$

   Claim: In the category **Set**, the diagram below is a final object of $\textbf{Set}_{A,B}$:

$$\begin{tikzcd} & A \\ A \times B \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"'] & \\ & B \end{tikzcd}$$

   Proof:
   For any set Z and morphisms $f_A : Z \longrightarrow A$ and $f_B : Z \longrightarrow B$, define $\sigma(z) = (f_A(z), f_B(z))$ for all $z \in Z$. Then the diagram below commutes:

$$\begin{tikzcd} & & A \\ Z \arrow[urr, "f_A"] \arrow[drr, "f_B"'] \arrow[r, "\sigma"] & A \times B \arrow[ur, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ & & B \end{tikzcd}$$

   Furthermore, σ is the only morphism that works to make the above diagram commute because if any element of Z is mapped to $A \times B$ differently, then the above diagram won't commute.

2. Define a category as follows:

   Let $\mathrm{Obj}(\mathbf{C}) = \mathbb{Z}$ and for all $a, b \in \mathbb{Z}$, let $\mathrm{Hom}_{\mathbf{C}}(a,b) = \{(a,b)\}$ if $a \leq b$ and $\emptyset$ otherwise. Also define the composition of (a,b) and (b,c) to be (a,c).

   (Recall pages 9 and 10)

   For any two integers a and b, the unique final object of $\mathbf{C}_{a,b}$ is the integer $\min(a,b)$ along with the morphisms: $(\min(a,b), a)$ and $(\min(a,b), b)$.

The prefix "co-" in category theory usually indicates that we are reversing arrows. Hence, given a category **C** and objects A and B in **C**, we define a coproduct of A and B: denoted $A \amalg B$, as an object of **C** endowed with two morphisms: $i_A : A \longrightarrow A \amalg B$ and $i_B : B \longrightarrow A \amalg B$, such that the below diagram is an initial object of $\mathbf{C}^{A,B}$:

$$\begin{tikzcd}
A \arrow[dr, "i_A"] & \\
& A \amalg B \\
B \arrow[ur, "i_B"'] &
\end{tikzcd}$$

We say a category **C** has coproducts if for all objects A and B of **C**, $\mathbf{C}^{A,B}$ has initial objects.

Examples of coproducts in categories:

1. We define the disjoint union of two sets A and B as a set $A \amalg B$ obtained by first producing copies A' and B' of the sets A and B such that $A' \cap B' = \emptyset$ and then taking the union of A' and B'.

   One way we can implement this is by defining $A' = \{0\} \times A$ and $B' = \{1\} \times B$.

   Having done that, we can then define the natural injection functions:

   - $i_A : A \longrightarrow A \amalg B$ which maps each $a \in A$ to its corresponding element of $A \amalg B$
   - $i_B : B \longrightarrow A \amalg B$ which maps each $b \in B$ to its corresponding element of $A \amalg B$.

   If $A' = \{0\} \times A$ and $B' = \{1\} \times B$, then $i_A(a) = (0, a)$ and $i_B(b) = (1, b)$ for each $a \in A$ and $b \in B$.

   Claim: The disjoint union is a coproduct of **Set**.

   Proof:
   Consider two arbitrary morphisms $f_A : A \longrightarrow Z$ and $f_B : B \longrightarrow Z$ going to a common target. Then if we define $\sigma : A \amalg B \Longrightarrow Z$ such that $\forall c \in A \amalg B$:

$$\sigma(c) = \begin{cases} f_A(c) & \text{if } c \text{ came from } A \\ f_B(c) & \text{if } c \text{ came from } B \end{cases}$$

   then the diagram at the top of the next page commutes:

$$\begin{tikzcd}
A \arrow[dr, "i_A"'] \arrow[drr, bend left, "f_A"] & & \\
& A \amalg B \arrow[r, "\sigma"'] & Z \\
B \arrow[ur, "i_B"] \arrow[urr, bend right, "f_B"'] & &
\end{tikzcd}$$

Furthermore, σ is the only morphism that works to make the above diagram commute because if any element of $A \amalg B$ is mapped to Z differently, then the above diagram won't commute.

2. Define a category as follows:

   Let $\mathrm{Obj}(\mathbf{C}) = \mathbb{Z}$ and for all $a, b \in \mathbb{Z}$, let $\mathrm{Hom}_{\mathbf{C}}(a, b) = \{(a, b)\}$ if $a \leq b$ and $\emptyset$ otherwise. Also define the composition of (a, b) and (b, c) to be (a, c).

   (Recall pages 9-10, 15)

   For any two integers a and b, the unique initial object of $\mathbf{C}^{a,b}$ is the integer $\max(a, b)$ along with the morphisms: $(a, \max(a, b))$ and $(b, \max(a, b))$.

3/31/2024