(Audit) Math 140C Lecture Notes (Professor: Luca Spolaor)

Isabelle Mills

April 3, 2024

Lecture 1: 4/2/2024

A set $X \subseteq \mathbb{R}^n$ where $X \neq \emptyset$ is a vector space if:

- $\vec{x}, \vec{y} \in X \implies \vec{x} + \vec{y} \in X$
- $\vec{x} \in X$ and $c \in \mathbb{R} \implies c\vec{x} \in X$.

If $\phi = \{\vec{x}_1, \ldots, \vec{x}_k\} \subset \mathbb{R}^n$, then we define:

$$\text{span}\, \phi = \text{span}\{\vec{x}_1, \ldots, \vec{x}_k\} = \{c_1\vec{x}_1 + \ldots + c_k\vec{x}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}.$$

If $E \subseteq \mathbb{R}^n$ and $E = \text{span}\, \phi$, then we say ϕ generates E.

Note that $\text{span}\{\vec{x}_1, \ldots, \vec{x}_2\}$ forms a vector space (this is trivial to check).

$\{\vec{x}_1, \ldots, \vec{x}_k\} \subseteq \mathbb{R}^n$ is called linearly independent if:

$$\sum_{i=1}^{k} c_i \vec{x}_i = 0 \implies \forall i \in \{1, \ldots, k\}, \;\; c_i = 0.$$

If the above implication does not hold, then we call the set linearly dependent.

If $X \subseteq \mathbb{R}^n$ is a vector space, then we define the dimension of X as:
$\dim(X) = \sup\{k \in \mathbb{N} \cup \{0\} \mid \exists\{\vec{x}_1, \ldots, \vec{x}_k\} \subset X$ which is linearly independent$\}$.

Also, we define any set containing $\vec{0}$ to be automatically linearly dependent.
This includes the singleton: $\{\vec{0}\}$.

$Q = \{\vec{x}_1, \ldots, \vec{x}_k\}$ is a basis for X if:

- Q is linearly independent.
- $\text{span}\, Q = X$

As an example of a basis, for $\mathbb{R}^n$ we define the standard basis as the set $\{e_1, e_2, \ldots, e_n\}$ where e_i is the vector whose ith element is 1 and whose other elements are 0. It is pretty trivial to check that this set is in fact a basis of $\mathbb{R}^n$.

Proposition: If $B = \{\vec{x}_1, \ldots, \vec{x}_k\}$ is a basis of a vector space X, then:

1. $\forall \vec{v} \in X, \;\; c_1, \ldots, c_k \in \mathbb{R} \;\; s.t. \;\; \vec{v} = \sum_{i=1}^{k} c_i \vec{x}_i$

   This is true because $X = \text{span}\, B$. So by definition of a span, $\vec{v}$ can be expressed as a linear combination of the vectors of B.

2. The c_i such that $\vec{v} = \sum_{i=1}^{k} c_i \vec{x}_i$ are unique.

   Suppose that $\vec{v} = \sum c_i \vec{x}_i = \sum \alpha_i \vec{x}_i$. Then $\vec{0} = \sum (c_i - \alpha_i)\vec{x}_i$. Then since $\{\vec{x}_1, \ldots, \vec{x}_k\}$ are linearly independent, we know for all i that $c_i - \alpha_i = 0$. Hence, $c_i = \alpha_i$ for each i.

Theorem 9.2: Let $k \in \mathbb{N} \cup \{0\}$. If $X = \operatorname{span}\{\vec{x}_1, \ldots, \vec{x}_k\}$, then $\dim(X) \leq k$.

Proof:
Suppose for the sake of contradiction that there exists a linearly independent set $Q = \{\vec{y}_1, \ldots, \vec{y}_{k+1}\} \subset X$ which spans X. Then, define $S_0 = \{\vec{x}_1, \ldots, \vec{x}_k\}$ and note that S_0 spans X.

Now by induction, assume for $i \in \{0, 1, \ldots, k-1\}$, that S_i contains the first i vectors of Q in addition to $k - i$ vectors of S_0, and that $\operatorname{span} S_i = X$. Then since S_i spans X, we know that $y_{i+1} \in X$ is in the span of S_i. So, letting $\vec{x}_{n_1}, \ldots, \vec{x}_{n_{k-i}}$ be the elements from S_0 in S_i, we know that there exists scalars $a_1, \ldots, a_{i+1}, b_1, \ldots, b_{k-i} \in \mathbb{R}$ where $a_{i+1} = 1$ such that:

$$\sum_{j=1}^{i+1} a_j \vec{y}_j + \sum_{j=1}^{k-i} b_j \vec{x}_{n_j} = \vec{0}$$

If all $b_j = 0$, then we have a contradiction. This is because $\{\vec{y}_1, \ldots, \vec{y}_{k+1}\}$ is assumed to be linearly independent. So, having all $b_j = 0$ implies that:

$$\sum_{j=1}^{i+1} a_j \vec{y}_j = \sum_{j=1}^{i+1} a_j \vec{y}_j + \sum_{j=i+2}^{k+1} 0 \cdot \vec{y}_j = \vec{0}$$

In turn this means that all $a_j = 0$, which contradicts that $a_{i+1} = 1$.

So, not all $b_j = 0$. This means that for some j we must have that $\vec{x}_{n_j}$ is in the span of $(S_i \setminus \{\vec{x}_{n_j}\}) \cup \{\vec{y}_{i+1}\}$. Call this set S_{i+1}. Clearly, S_{i+1} contains the first $i+1$ vectors of Q. Also:

$$\operatorname{span} S_{i+1} = \operatorname{span}(S_i \cup \{\vec{y}_{i+1}\}) = \operatorname{span} S_i = X.$$

So S_{i+1} satisfies the same conditions S_i did.

Now we get to the contradiction. Using the above reasoning, we will eventually construct $S_k = \{\vec{y}_1, \ldots, \vec{y}_k\}$ which still spans X. However, since $\vec{y}_{k+1} \in X$, that means that $\vec{y}_{k+1}$ equals some linear combination of the other $\vec{y}$ in Q. This contradicts that Q is linearly independent. ■

Corollary: If $B = \{\vec{x}_1, \ldots, \vec{x}_k\}$ is a basis for X, then $\dim(X) = k$.

Proof:
Since B is linearly independent, by definition $\dim(X) \geq k$. Meanwhile, since B spans X, we know by the above theorem that $\dim(X) \leq k$.
So $\dim(X) = k$.

Theorem 9.3: Suppose X is a vector space and $\dim(X) = n$. Then:

(A) For $E = \{\vec{x}_1, \dots, \vec{x}_n\} \subset X$, we have that $X = \operatorname{span} E$ implies that E is linearly independent.

Proof:
First, assume E is linearly independent. Then, note that for any $\vec{y} \in X$, we must have that $E \cup \{\vec{y}\}$ is linearly dependent because $|E \cup \{\vec{y}\}| > \dim(X)$. So, there exists $c_1, \dots, c_n, c_{n+1} \in \mathbb{R}$ such that at least one c_i is nonzero and:

$$\sum_{i=1}^{n} c_i \vec{x}_i + c_{n+1} \vec{y} = \vec{0}$$

Now if $c_{n+1} = 0$, we have a contradiction because E is linearly independent. So, we conclude that $c_{n+1} \neq 0$. Thus, by rearranging terms we can express y as a linear combination of the vectors of E. Therefore, $\operatorname{span} E = X$ since y can be any vector in X.

Secondly, assume E is not linearly independent. Then for some $\vec{x}_i \in E$, we have that $\operatorname{span} E = \operatorname{span}(E \setminus \{\vec{x}_i\})$. However, $|E \setminus \{\vec{x}_i\}| = n - 1$. So if $X = \operatorname{span} E$, then $\dim(X) \leq |E \setminus \{\vec{x}_i\}| = n - 1$, which contradicts our assumption that $\dim(X) = n$. Hence, $X \neq \operatorname{span} E$.

(B) X has a basis and every basis of X consists of n vectors.

Proof:
By the definition of $\dim(X)$, we know that there exists a linearly independent set of n vectors. By the previous part of this theorem, we also know that that set spans X. So, it is a basis of X. Meanwhile, by the corollary to theorem 9.2, we know that the number of vectors in a basis of X equals the dimension of X. Hence, all bases of X must have n vectors.

(C) If $1 \leq m \leq n$ and $\{\vec{y}_1, \dots, \vec{y}_m\} \subset X$ is linearly independent, then X has a basis that contains $\vec{y}_1, \dots, \vec{y}_m$.

Proof:
Let $S_0 = \{\vec{x}_1, \dots, \vec{x}_n\}$ be a basis of X and $Q = \{\vec{y}_1, \dots, \vec{y}_m\}$. Then by the same induction which we used to prove theorem 9.2, we can construct a basis: S_m, of X which contains $\vec{y}_1, \dots, \vec{y}_m$.

Let X and Y be vector spaces. A map $\mathbf{A} : X \longrightarrow Y$ is linear if $\mathbf{A}(c_1 \vec{x}_1 + c_2 \vec{x}_2) = c_1 \mathbf{A}(\vec{x}_1) + c_2 \mathbf{A}(\vec{x}_2)$ for all $\vec{x}_1, \vec{x}_2 \in X$ and $c_1, c_2 \in \mathbb{R}$.

Observations:

1. A linear map sends $\vec{0}$ to $\vec{0}$. This is because:
$$\mathbf{A}(\vec{0}) = \mathbf{A}(\vec{v} - \vec{v}) = \mathbf{A}(\vec{v}) - \mathbf{A}(\vec{v}) = \vec{0}.$$

2. If $\mathbf{A} : X \longrightarrow Y$ is a linear map and $B = \{\vec{x}_1, \ldots, \vec{x}_k\}$ is a basis of X, then $\mathbf{A}\left(\sum_{i=1}^{k}(c_i\vec{x}_i)\right) = \sum_{i=1}^{k} c_i\mathbf{A}(\vec{x}_i)$ for all $c_1, \ldots, c_k \in \mathbb{R}$.

Given two vector spaces X and Y, we define $L(X, Y)$ to be the set of all linear transformations from X into Y. Also, we shall abbreviate $L(X, X)$ as $L(X)$.

$\mathscr{N}(\mathbf{A})$ = "null space of $\mathbf{A}$" $= \{\vec{x} \in X \mid \mathbf{A}(\vec{x}) = \vec{0}\}$.

$\mathscr{R}(\mathbf{A})$ = "range of $\mathbf{A}$" $= \{\vec{y} \in Y \mid \exists \vec{x} \in X \;\; s.t. \;\; \mathbf{A}\vec{x} = \vec{y}\}$.

Proposition: For any linear map $\mathbf{A} : X \longrightarrow Y$, $\mathscr{N}(\mathbf{A})$ and $\mathscr{R}(\mathbf{A})$ are vector spaces.

Proof:

- Assume $\vec{x}_1, \vec{x}_2 \in \mathscr{N}(\mathbf{A}) \subset X$ and $c \in \mathbb{R}$. Then:
  - $\mathbf{A}(\vec{x}_1 + \vec{x}_2) = \mathbf{A}(\vec{x}_1) + \mathbf{A}(\vec{x}_2) = \vec{0} + \vec{0} = \vec{0}$, which means that $\vec{x}_1 + \vec{x}_2 \in \mathscr{N}(\mathbf{A})$.
  - $\mathbf{A}(c\vec{x}_1) = c\mathbf{A}(\vec{x}_1) = c\vec{0} = \vec{0}$. So $c\vec{x}_1 \in \mathscr{N}(\mathbf{A})$.

  This shows that $\mathscr{N}(\mathbf{A})$ is a vector space.

- Assume $\vec{y}_1, \vec{y}_2 \in \mathscr{R}(\mathbf{A}) \subset Y$ and $c \in \mathbb{R}$. Then:
  - We know there exists $\vec{x}_1, \vec{x}_2 \in X$ such that $\mathbf{A}(\vec{x}_1) = \vec{y}_1$ and $\mathbf{A}(\vec{x}_2) = \vec{y}_2$. In turn, $\mathbf{A}(\vec{x}_1 + \vec{x}_2) = \mathbf{A}(\vec{x}_1) + \mathbf{A}(\vec{x}_2) = \vec{y}_1 + \vec{y}_2$. So $\vec{y}_1 + \vec{y}_2 \in \mathscr{R}(\mathbf{A})$.
  - Now continue letting $\vec{x}_1 \in X$ be a vector such that $\mathbf{A}(\vec{x}_1) = \vec{y}_1$. Then $\mathbf{A}(c\vec{x}_1) = c\mathbf{A}(\vec{x}_1) = c\vec{y}_1$. So $c\vec{y}_1 \in \mathscr{R}(\mathbf{A})$.

  This shows that $\mathscr{R}(\mathbf{A})$ is a vector space.

$\mathrm{rk}(\mathbf{A})$ = "rank of $\mathbf{A}$" $= \dim(\mathscr{R}(\mathbf{A}))$.

$\mathrm{null}(\mathbf{A})$ = "nullity of $\mathbf{A}$" $= \dim(\mathscr{N}(\mathbf{A}))$.

Rank-Nullity Theorem: Given any $\mathbf{A} \in L(X, Y)$, we have that
$$\dim(X) = \mathrm{rk}(\mathbf{A}) + \mathrm{null}(\mathbf{A}).$$

Proof: