

# My Notes on Paolo Aluffi's Algebra Chapter 0

Isabelle Mills

March 26, 2024

# 1/7/2024

A multiset is a collection of elements which like a set is unordered but unlike a set can contain duplicate elements.

One way to define a multiset is as a function  $f : A \rightarrow \mathbb{N}$  such that each  $\alpha \in A$  is mapped to the number of times that  $\alpha$  appears in the multiset. Then, given the multisets  $f_1 : A \rightarrow \mathbb{N}$  and  $f_2 : B \rightarrow \mathbb{N}$ , we can define the following operations:

- $\alpha \in f_1 \iff \alpha \in A$
- $f_1 \subseteq f_2 \iff \forall \alpha \in f_1, \alpha \in f_2 \text{ and } f_1(\alpha) \leq f_2(\alpha)$
- $f_1 \cup f_2 : (A \cup B) \rightarrow \mathbb{N}$  such that for  $\alpha \in A \cup B$ , if  $\alpha \in A \cap B$ , then  $(f_1 \cup f_2)(\alpha) = f_1(\alpha) + f_2(\alpha)$ . As for if  $\alpha \notin A \cap B$ , then  $(f_1 \cup f_2)(\alpha)$  equals whatever  $\alpha$  was mapped to in the multiset it originally came from.
- $f_1 \cap f_2 : (A \cap B) \rightarrow \mathbb{N}$  such that for  $\alpha \in A \cap B$ , we have that  $(f_1 \cap f_2)(\alpha) = \min(f_1(\alpha), f_2(\alpha))$
- $f_1 \setminus f_2 : ((A \setminus B) \cup \{\alpha \in A \cap B \mid f_1(\alpha) > f_2(\alpha)\}) \rightarrow \mathbb{N}$  such that for each  $\alpha \in f_1 \setminus f_2$ , if  $\alpha \in f_2$ , then  $(f_1 \setminus f_2)(\alpha) = f_1(\alpha) - f_2(\alpha)$ . As for if  $\alpha \notin f_2$ , then  $(f_1 \setminus f_2)(\alpha) = f_1(\alpha)$

A practical example of a multiset is the prime factorization of any positive integer.

We say that two sets  $A$  and  $B$  are isomorphic if and only if there exists a bijection between  $A$  and  $B$ . We denote this by writing  $A \cong B$ . Additionally, we can refer to any bijection  $f$  between  $A$  and  $B$  as an isomorphism between the two sets.

A function  $f : A \rightarrow B$  is a monomorphism (a.k.a a monic) if for all sets  $Z$  and all functions  $a'$  and  $a'' : Z \rightarrow A$ , we have that  $f \circ a' = f \circ a'' \implies a' = a''$ .

**Proposition 1:** A function is injective if and only if it is a monomorphism.

**Proof:** Let's say we have a function  $f : A \rightarrow B$ .

First, let us assume  $f$  is injective.

Then let us assume we have two functions  $a'$  and  $a''$  from some set  $Z$  to  $A$  such that  $f \circ a' = f \circ a''$ . Because  $f$  is injective, we know it has a left-hand inverse  $g : B \rightarrow A$  such that  $g \circ f = \text{Id}_A$ . Composing  $g$  with the previous equation, we get that:

$$a' = \text{Id}_A \circ a' = g \circ (f \circ a') = g \circ (f \circ a'') = \text{Id}_A \circ a'' = a''$$

Thus, we've shown that  $f$  is a monomorphism.

Next, we shall assume  $f$  is a monomorphism.

Based on this, we can say that for any two functions  $a'$  and  $a''$  mapping a set  $Z$  to  $A$ , we have that  $f \circ a' = f \circ a'' \implies a' = a''$ . However, now note that if we make  $Z$  a singleton, meaning it only contains one element, then  $a'$  and  $a''$  can each only take on one value. So, we can effectively rewrite  $f \circ a' = f \circ a'' \implies a' = a''$  as:

$$f(a') = f(a'') \implies a' = a''$$

This is the definition of an injective function. ■

## 1/8/2024

A function  $f : A \rightarrow B$  is an epimorphism (a.k.a an epi) if for all sets  $Z$  and all functions  $a'$  and  $a'' : B \rightarrow Z$ , we have that  $a' \circ f = a'' \circ f \implies a' = a''$ .

**Proposition 2:** A function is a surjection if and only if it is an epimorphism.

**Proof:** Let's say we have a function  $f : A \rightarrow B$ .

First, let us assume  $f$  is surjective.

Then let's assume we have two functions  $a'$  and  $a''$  from  $B$  to some set  $Z$  such that  $a' \circ f = a'' \circ f$ . Because  $f$  is surjective, we know it has a right-hand inverse  $h : B \rightarrow A$  such that  $f \circ h = \text{Id}_B$ . Composing  $h$  with the previous equation, we get that:

$$a' = a' \circ \text{Id}_B = (a' \circ f) \circ h = (a'' \circ f) \circ h = a'' \circ \text{Id}_B = a''$$

So  $f$  is an epimorphism.

Next, assume  $f$  is not surjective.

Then there exists  $\beta \in B$  such that for all  $\alpha \in A$ , we have that  $f(\alpha) \neq \beta$ . Importantly, as  $f(\alpha) \in B$ , we know  $|B| \neq 1$ . So set  $a'$  equal to  $\text{Id}_B$  and define  $a''$  as a function mapping each element of  $B \setminus \{\beta\}$  to itself and  $\beta$  to any of the other elements in  $B$ . Now,  $a' \circ f = f = a'' \circ f$  but  $a' \neq a''$ . So  $f$  is not epimorphic. ■

Sometimes, to indicate that a function  $f : A \rightarrow B$  is a monomorphism, epimorphism, or isomorphism, we use the following notation:

- Monomorphism:  $f : A \hookrightarrow B$
- Epimorphism:  $f : A \twoheadrightarrow B$
- Isomorphism:  $f : A \xrightarrow{\sim} B$

# 3/24/2024

A relation on a set  $S$  is a subset  $R$  of the cartesian product  $S \times S$ . Specifically, we use the notation  $x R y$  to mean that  $(x, y) \in R$ . Certain types of relations are especially important and thus are represented with their own symbol.

- An equivalence relation, typically denoted  $\sim$ , on a set  $S$  has the properties:
  - $\forall a \in S, a \sim a$       ◦  $a \sim b \implies b \sim a$       ◦  $a \sim b$  and  $b \sim c \implies a \sim c$
- An order relation, typically denoted  $<$ , on a set  $S$  has the properties:
  - $\forall a, b \in S$ , exactly one of the following is true:  $a < b$ ,  $b < a$ , or  $a = b$ .
  - $a < b$  and  $b < c$  implies that  $a < c$ .

Given a set  $S$ , an equivalence relation  $\sim$ , and an element  $a \in S$ , we define the equivalence class of  $a$  with respect to  $\sim$  to be the set  $[a]_{\sim} = \{b \in S \mid a \sim b\}$ . Also, we define the quotient of  $S$  with respect to the equivalence relation  $\sim$  as the set of equivalence classes with respect to  $\sim$ .

$$S/\sim = \{[a]_{\sim} \mid a \in S\}$$

Given any function  $f : A \longrightarrow B$ , define  $a \sim b \iff f(a) = f(b)$ .

**Proposition 3:** Every function  $f$  can be decomposed as follows:

$$A \xrightarrow{g} (A/\sim) \xrightarrow[\tilde{f}]{\sim} \text{im} f \xhookrightarrow{h} B$$

(in other words,  $f = h \circ \tilde{f} \circ g$ )

...where  $g$  is the surjection mapping  $a$  to  $[a]_{\sim}$  for all  $a \in A$ ,  $h$  is the inclusion function (which is injective) from the image of  $f$  to  $B$ , and  $\tilde{f}$  is a bijective function defined as the mapping  $[a]_{\sim}$  to  $f(a)$  where  $a \in [a]_{\sim}$ .

**Proof:**

$(A/\sim)$  is defined as the range of  $g$ . So  $g$  is automatically surjective. Also, inclusion functions like  $h$  are always injective.

Now we show  $\tilde{f}$  is well defined and bijective.

1. Assume  $a', a'' \in A$  such that  $[a'] = [a'']$ . Then by how we defined  $\sim$ ,  $f(a') = f(a'')$ . So  $[a'] = [a''] \implies \tilde{f}([a']) = \tilde{f}([a''])$ , meaning  $\tilde{f}$  is well defined.

2. Assume  $\tilde{f}([a']) = \tilde{f}([a''])$ . Then  $f(a') = f(a'')$ , meaning  $a' \sim a''$ .  
Hence  $[a'] = [a'']$ , meaning  $\tilde{f}$  is injective.

3. Given any  $b \in \text{im } f$ , there exists  $a \in A$  such that  $f(a) = b$ . Then  $\tilde{f}([a]_{\sim}) = f(a) = b$ . So  $\tilde{f}$  is surjective.

Finally, it's clear that  $f = h \circ \tilde{f} \circ g$ . So we're done.

## 3/25/2024

A category  $\mathcal{C}$  consists of a class  $\text{Obj}(\mathcal{C})$  of objects of the category, and for every two objects  $A, B$  of  $\mathcal{C}$ , a set  $\text{Hom}_{\mathcal{C}}(A, B)$  of morphisms with the following properties:

- For every object  $A$  of  $\mathcal{C}$ , there exists a morphism  $1_A \in \text{Hom}_{\mathcal{C}}(A, A)$  called the identity on  $A$ .
- Morphisms can be composed, meaning  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $g \in \text{Hom}_{\mathcal{C}}(B, C)$  means that  $gf \in \text{Hom}_{\mathcal{C}}(A, C)$
- Composition is associative, meaning if  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ ,  $g \in \text{Hom}_{\mathcal{C}}(B, C)$ , and  $h \in \text{Hom}_{\mathcal{C}}(C, D)$ , then  $(hg)f = h(gf)$ .
- The identity morphisms are identities with respect to composition, meaning for all  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ ,  $f1_A = f$  and  $1_B f = f$ .
- $\text{Hom}_{\mathcal{C}}(A, B)$  and  $\text{Hom}_{\mathcal{C}}(C, D)$  are disjoint unless  $A = C$  and  $B = D$ .

We use the word "class" because by Russell's paradox, there are many sets which aren't well defined. For example, there is no set of sets. So we instead make a class of all sets.

Also, we write category names in sans-serif font to better distinguish them.

A morphism of an object  $A$  of a category  $\mathcal{C}$  to itself is called an endomorphism. Thus we denote  $\text{Hom}_{\mathcal{C}}(A, A)$  as  $\text{End}_{\mathcal{C}}(A)$ .

Note that by the composition rules of a category, if  $f, g \in \text{End}_{\mathcal{C}}(A)$ , then  $fg, gf \in \text{End}_{\mathcal{C}}(A)$ .

We can denote a morphism  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  as  $f : A \rightarrow B$ .

Examples of Categories:

- We define the category of sets:  $\text{Set}$ , such that  $\text{Obj}(\text{Set})$  is the class of all sets and for  $A$  and  $B$  in  $\text{Obj}(\text{Set})$ ,  $\text{Hom}_{\text{Set}}(A, B)$  is the set of all functions from  $A$  to  $B$  (abbreviated as  $B^A$ ).

- If  $S$  is a set and  $\sim$  is an equivalence relation on  $S$ , then we can define a category whose objects are the elements of  $S$ , and for  $a, b \in S$ ,  $\text{Hom}(a, b)$  equals  $\{a, b\}$  when  $a \sim b$  and  $\emptyset$  otherwise.

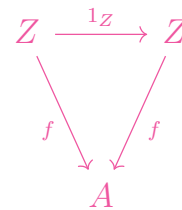
Note that for this category, we need to define what it means to compose morphisms. So let's say that if  $f = \{a, b\}$  and  $g = \{b, c\}$ , then  $gf = \{a, c\}$ .

- Let  $C$  be a category and let  $A$  be an object of  $C$ . Then we can define a category  $C_A$  as follows:

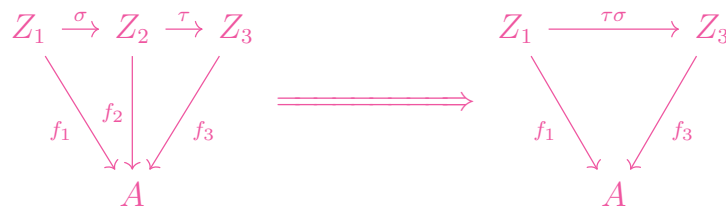
- $\text{Obj}(C_A) = \text{all morphisms from any object of } C \text{ to } A$
- If  $f_1 : Z_1 \rightarrow A$  and  $f_2 : Z_2 \rightarrow A$  are objects of  $C_A$ , then  $\text{Hom}_{C_A}(f_1, f_2)$  is the set of morphisms  $\sigma : Z_1 \rightarrow Z_2$  such that  $f_1 = f_2 \sigma$ .

Thus the morphisms of  $C_A$  are commutative diagrams with the objects  $Z_1, Z_2$ , and  $A$ .

To prove that this is a category, first consider that each object  $f : Z \rightarrow A$  has an identity morphism:

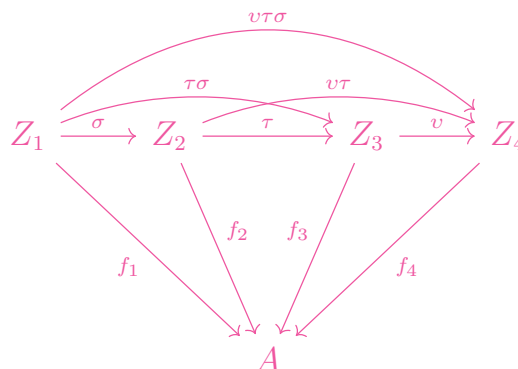


Also, the morphisms of  $C_A$  compose. If the diagram with  $\sigma$  is in  $\text{Hom}_{C_A}(f_1, f_2)$  and the diagram with  $\tau$  is in  $\text{Hom}_{C_A}(f_2, f_3)$ , then we define their composition in  $\text{Hom}_{C_A}(f_1, f_3)$  as the diagram with  $\tau\sigma$ .



As is hopefully apparent, the identity morphisms compose as is required for  $C_A$  to be a category.

Finally, composing morphisms of  $C_A$  is associative.

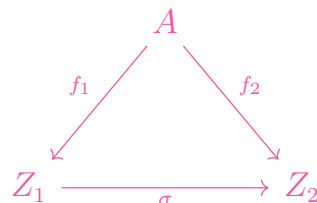


- Categories like the one in the previous example are called slice categories. We can similarly define coslice categories as follows:

Let  $C$  be a category and let  $A$  be an object of  $C$ . Then we can define a category  $C^A$  such that:

- $\text{Obj}(C^A)$  = all morphisms from  $A$  to any object of  $C$
- If  $f_1 : A \rightarrow Z_1$  and  $f_2 : A \rightarrow Z_2$  are objects of  $C^A$ , then  $\text{Hom}_{C^A}(f, g)$  is the set of morphisms  $\sigma : Z_1 \rightarrow Z_2$  such that  $\sigma f_1 = f_2$ .

In other words, we're now considering commutative diagrams of the form:



**Problem 3.8:** A subcategory  $C'$  of a category  $C$  consists of a collection of objects of  $C$  with morphisms  $\text{Hom}_{C'}(A, B) \subseteq \text{Hom}_C(A, B)$  for all objects  $A, B$  in  $\text{Obj}(C')$  such that  $C'$  has all the necessary identities and compositions to be a category. A subcategory  $C'$  is full if  $\text{Hom}_{C'}(A, B) = \text{Hom}_C(A, B)$  for all  $A, B$  in  $\text{Obj}(C')$ .

Let  $\text{Set}'$  be the category of infinite sets.

- $\text{Obj}(\text{Set}')$  is the class of all infinite sets.
- For all  $A, B$  in  $\text{Obj}(\text{Set}')$ ,  $\text{Hom}_{\text{Set}'}(A, B)$  is the set of all functions from  $A$  to  $B$ .

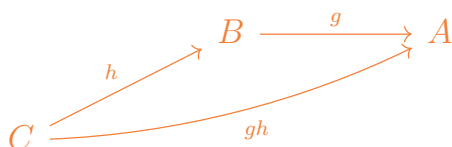
Now given the infinite sets  $A$  and  $B$ , any morphism  $f \in \text{Hom}_{\text{Set}}(A, B)$  is also a morphism of  $\text{Hom}_{\text{Set}'}(A, B)$ . So  $\text{Set}'$  is a full subcategory of  $\text{Set}$ .

**Problem 3.1:** Let  $C$  be a category. Then consider  $C^{op}$  with

- $\text{Obj}(C^{op}) = \text{Obj}(C)$
- for  $A, B$  in  $\text{Obj}(C^{op})$ ,  $\text{Hom}_{C^{op}}(A, B) = \text{Hom}_C(B, A)$ .

Let  $A, B$ , and  $C$  be objects of  $C^{op}$ . Given  $g \in \text{Hom}_{C^{op}}(A, B)$  and  $h \in \text{Hom}_{C^{op}}(B, C)$ , define the composition  $hg \in \text{Hom}_{C^{op}}(A, C)$  to be the morphism  $gh \in \text{Hom}_C(C, A)$ .

To see why this is well defined note that if  $g \in \text{Hom}_{C^{op}}(A, B)$ , then  $g \in \text{Hom}_C(B, A)$ . Similarly, if  $h \in \text{Hom}_{C^{op}}(B, C)$ , then  $h \in \text{Hom}_C(C, B)$ . As  $C$  is a category, there must exist a morphism  $gh \in \text{Hom}_C(C, A)$ , which in turn means that the morphism we defined as the composition  $hg \in \text{Hom}_{C^{op}}(A, C)$  exists.



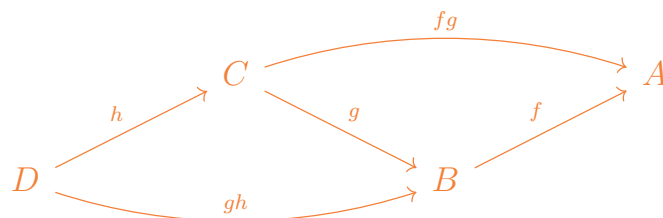
So by how we defined composition of morphisms in  $C^{op}$ , we know  $C^{op}$  satisfies the composition property of a category. Now what's left to show is that  $C^{op}$  has the other properties of a category.

For any object  $A$  in  $\text{Obj}(C^{op})$ ,  $\text{End}_{C^{op}}(A) = \text{End}_C(A)$ . So,  $A$  inherits a morphism  $1_A$  from  $C$ .

Consider  $g \in \text{Hom}_{C^{op}}(A, B)$ . Then  $g1_A$  in  $\text{Hom}_{C^{op}}(A, B)$  is equal to  $1_A g = g$  in  $\text{Hom}_C(B, A)$ . So in  $C^{op}$ , we have that  $g1_A = g$ .

Similarly, consider  $h \in \text{Hom}_{C^{op}}(B, A)$ . Then  $1_A h \in \text{Hom}_{C^{op}}(B, A)$  is equal to  $h1_A = h$  in  $\text{Hom}_C(A, B)$ . So in  $C^{op}$ , we have that  $1_A h = h$ .

Finally, observe that given the morphisms  $f \in \text{Hom}_{C^{op}}(A, B)$ ,  $g \in \text{Hom}_{C^{op}}(B, C)$ , and  $h \in \text{Hom}_{C^{op}}(C, D)$ , we know that in  $C$ :



$(gf) \in \text{Hom}_{C^{op}}(A, C)$  refers to the morphism  $fg \in \text{Hom}_C(C, A)$ . So,  $h(gf) \in \text{Hom}_{C^{op}}(A, D)$  refers to the morphism  $(fg)h \in \text{Hom}_C(D, A)$ . At the same time,  $(hg) \in \text{Hom}_{C^{op}}(B, D)$  refers to the morphism  $gh \in \text{Hom}_C(D, B)$ . So,  $(hg)f \in \text{Hom}_C(D, A)$  refers to the morphism  $f(gh) \in \text{Hom}_C(D, A)$ . Thus as  $(fg)h = f(gh)$  in  $C$ , we have that  $(hg)f = h(gf)$  in  $C^{op}$ .

---

**3/26/2024**