12/14/2025

Math 241a Notes:

Suppose V is a finite dimensional vector space and $\pi : G \to \mathrm{GL}(V)$ is a representation. Then π is called irreducible if the only $\pi(G)$-invariant subspaces are $\{0\}$ and V. π is called completely reducible if $V = \bigoplus V_i$ where V_i is a $\pi(G)$-invariant irreducible subspace.

Also if $V = \mathbb{C}^n$ or $\mathbb{R}^n$, then I shall denote $\mathrm{GL}(V)$ as $\mathrm{GL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{R})$ respectively. Similarly, I shall denote $U(V)$ as $U(n)$.

Proposition 2.2.11: If G is a group and $\pi : G \to U(n)$ is a unitary representation, then:

(i) every $\pi(G)$-invariant subspace has a $\pi(G)$-invariant orthogonal complement.

Proof:
Suppose V is invariant and $w \in V^\perp$. Then as $\pi(g)$ is unitary (which means $\pi(g)^* = \pi(g)^{-1}$) for each $g \in G$, we know:

$$\langle \pi(g)w, v\rangle = \langle w, \pi(g)^* v\rangle = \langle w, \pi(g^{-1})v\rangle = 0.$$

It follows that $V^\perp$ is G-invariant.

(ii) π is completely reducible.

Proof:
We can prove this by induction. If $\mathbb{C}^n$ isn't irreducible then we can write $\mathbb{C}^n = V \oplus V^\perp \cong \mathbb{C}^k \oplus \mathbb{C}^{n-k}$ where both V and $V^\perp$ are G-invariant. Then we just repeat this reasoning on the smaller subspaces. ■

Proposition 2.2.12: If G is a compact group, V is a finite dimensional real or complex Hausdorff topological vector space, and $\pi : G \to \mathrm{GL}(V)$ is a (strong operator) continuous representation, then π is completely reducible.

Proof:
Using *corollary 2.2.8 on page 485*, let $\langle\ ,\ \rangle$ be a G-invariant inner product on V. Then π is a unitary representation with respect to this inner product. So, we can apply the prior proposition. ■

Let $\mathcal{X}$ be a real or complex vector space and let $A \subseteq \mathcal{X}$ be convex.

- Given any $x, y \in \mathcal{X}$ we let $[x,y] \coloneqq \{ty + (1-t)x : 0 \leq t \leq 1\}$. Also, we let $(x,y) \coloneqq \{ty + (1-t)x : 0 < t < 1\}$.
- We say $x \in A$ is an extreme point if for any $y, z \in A$ we have that $x \in [y,z]$ iff $x = y$ or $x = z$. We denote the set of such points as $\mathrm{ex}(A)$.
- We say $\emptyset \neq B \subseteq A$ is an extreme set if for any $y, z \in A$ we have that:

$$(y,z) \cap B \neq \emptyset \Longrightarrow [y,z] \subseteq B.$$

Given a set $E \subseteq \mathcal{X}$ where $\mathcal{X}$ is a topological real or complex vector space, we define $\overline{\mathrm{conv}}(A)$ to be the smallest closed convex set containing A. This is well-defined because arbitrary intersections of closed convex sets are closed and convex.

Clearly, if one has a convex polyhedron in $\mathbb{R}^3$, then the faces of the polyhedron are extreme sets and the extreme points are precisely the vertices.

Exercise 2.2.10: Let X be a compact Hausdorff space and let $M(X)$ denote the set of Radon probability measures on X. Then $\mathrm{ex}(M(X)) = \{\delta_x : x \in X\}$ where δ_x is the Dirac delta measure at x.

Proof:
Let μ_0 and μ_1 be probability measures on X, and suppose that $\delta_x \in [\mu_0, \mu_1]$. Hence, there exists $t \in [0,1]$ such that $t\mu_1 + (1-t)\mu_0 = \delta_x$. If $t = 0$ or $t = 1$, there is nothing to show. So suppose $t \in (0,1)$. As $\delta_x(\{x\}^{\mathsf{C}}) = 0$, we know that $t\mu_1(\{x\}^{\mathsf{C}}) = -(1-t)\mu_0(\{x\}^{\mathsf{C}})$. That said, we also must have that $\mu_1(\{x\}^{\mathsf{C}}) \geq 0$ and $\mu_0(\{x\}^{\mathsf{C}}) \geq 0$. In turn, the left side of our equation must be nonnegative and the right side must be nonpositive. The only way this works out is if $t\mu_1(\{x\}^{\mathsf{C}}) = 0 = -(1-t)\mu_0(\{x\}^{\mathsf{C}})$. And since $t \neq 0$ and $-(1-t) \neq 0$, we can conclude that $\mu_1(\{x\}^{\mathsf{C}}) = 0 = \mu_0(\{x\}^{\mathsf{C}})$. And now it is clear that $\mu_0 = \delta_x = \mu_1$ since all three have total measure 1. This proves that $\delta_x \in \mathrm{ex}(M(X))$ for any Dirac delta measure δ_x.

To show the converse, we first introduce a lemma. Suppose ν is a Borel Radon probability measure on X. Then $\nu(E) \in \{0,1\}$ for all sets $E \in \mathcal{B}_X$ if and only if ν is a Dirac delta measure.

Proof:
The $(\Longleftarrow)$ claim is obvious. To show the other claim, you could just use the reasoning on *pages 444-445*. However, I wrote a different proof before realizing that.

Let $\mathcal{F}$ be the set of all compact subsets of X with measure 1. This collection is partially ordered by inclusion, and by Zorn's lemma we can conclude that there is a minimal set F in $\mathcal{F}$.

Suppose $\mathcal{F}_0$ is a chain in $\mathcal{F}$ and let $K' = \bigcap_{K \in \mathcal{F}_0}$. I claim that $\mu(K') = 1$. This will be a compact subset of X since it is a closed subset of X. We also claim $\mu(K') = 1$. After all, if not then by the outer regularity of μ plus the fact that $\mu(E) \in \{0,1\}$ for all sets $E \in \mathcal{B}_X$ we know there exists an open set $U \supseteq K'$ with $\mu(U) = 0$. Next, by the compactness of X we know there are finitely many sets $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n$ in $\mathcal{F}_0$ such that $X = U \cup \bigcup_{j=1}^{n} K_j^{\mathsf{C}}$. Finally, we know that $K' \neq K_n$ since $\mu(K') \neq \mu(K_n)$. So, there must exist $K \in \mathcal{F}_0$ with $K' \subseteq K \subsetneq K_n$. But in turn we must have that $K \subseteq U$. This implies that $\mu(K) = 0$, which is a contradiction as $K \in \mathcal{F}_0$ means that $\mu(K) = 1$.

We conclude that $K' \in \mathcal{F}$. And clearly K' is a bound to $\mathcal{F}_0$.

Finally, suppose there exists distinct $x, y \in F$. Then we know that $\{x\}$ is a proper compact subset of F. Hence, $\mu(\{x\}) = 0$. Also, by similar arguments to before we know that there is an open set $V \supseteq \{x\}$ such that $\mu(V) = 0$. And, by intersecting V with a neighborhood of x not containing y (which we know exists since X is T_1), we can assume $y \notin V$. But now $F - V$ is a compact subset of X properly contained in F such that $\mu(F - V) = 1$. This contradicts the minimality of F. Hence, we conclude that there does not exist two distinct elements in F.

That said, F isn't empty since otherwise we'd have that $\mu(F) = 0$. So, we conclude that F is a singleton $\{x\}$ and $\mu = \delta_x$.

Now suppose for the sake of contradiction that ν is any measure in $M(X)$ that isn't a Dirac delta measure. Then by our prior lemma we know that there exists a set $E \subseteq X$ such that $0 < \nu(E) < 1$. In turn, we now know there exists well-defined probability measures $\mu_0(A) := (\nu(E))^{-1}\nu(A \cap E)$ and $\mu_1(A) := (\nu(X - E))^{-1}\nu(A - E)$ which are distinct from ν. Finally, by setting $t = \nu(X - E)$ we have that $0 < t < 1$ and $\nu(E) = 1 - t$. And then for all $A \in \mathcal{B}_X$ we have that:

$$\begin{aligned}\nu(A) = \nu(A \cap E) + \nu(A - E) &= \nu(E)\mu_0(A) + \nu(X - E)\mu_1(A) \\ &= (1 - t)\mu_0(A) + t\mu_1(A)\end{aligned}$$

This shows that $\nu \in [\mu_0, \mu_1]$ but $\nu \neq \mu_0$ and $\nu \neq \mu_1$. So, ν is not an extreme point of $M(X)$ if ν is not a Dirac delta measure. ■

Lemma 2.3.5: Suppose $\mathcal{X}$ is a locally convex topological real vector space, $A \subseteq \mathcal{X}$ is closed and convex, and $x \in \mathcal{X} - A$. Then there exists $f \in \mathcal{X}^*$ and $c \in \mathbb{R}$ with $f(y) < c < f(x)$ for all $y \in A$.

Proof:

Let U be an open neighborhood of $0 \in \mathcal{X}$ such that $(x + U) \cap A = \emptyset$. By local convexity we can restrict U to a convex open subset containing 0. And by the reasoning on *page 230-232*, we can further restrict U to also ensure that U is balanced.

Next note that because U is balanced, we can equivalently say that $x \notin U + A$. But we want slightly more wiggle room so we'll instead consider the set $\frac{1}{2}U + A$. A fact we will use later is that because because U is convex, we have that $\frac{1}{2}U + \frac{1}{2}U \subseteq U$.

Note that $\frac{1}{2}U + A$ is open since $\frac{1}{2}U + A = \bigcup_{a \in A}(a + \frac{1}{2}U)$. It's also convex because $t(a_1 + \frac{1}{2}u_1) + (1-t)(a_0 + \frac{1}{2}u_0) = (ta_1 + (1-t)a_0) + \frac{1}{2}(tu_1 + (1-t)u_0) \in A + \frac{1}{2}U$ for all $t \in [0, 1]$, $a \in A$, and $u \in U$ since both U and A are convex. Going a step further, we can assume without loss of generality that $0 \in \frac{1}{2}U + A$.

To see why, note that if $0 \notin \frac{1}{2}U + A$ then we can translate our entire vector space by some fixed $\frac{1}{2}u + a \in U + A$. Then after doing the later reasoning, we will have a linear functional f and $c \in \mathbb{R}$ such that $f(y - (a + \frac{1}{2}u)) < c < f(x - (a + \frac{1}{2}u))$ for all $y \in A$. In turn $f(y - x) < c - f(x - (a + \frac{1}{2}u)) < 0$ and thus:

$$f(y) < c - f(x - (a + \tfrac{1}{2}u)) + f(x) < f(x) \text{ for all } y \in A$$

where $c - f(x - (a + \frac{1}{2}u)) + f(x)$ is another fixed constant in $\mathbb{R}$.

But now if p is the Minkowski functional associated to $\frac{1}{2}U + A$, we can follow the reasoning on *page 233* to see that p satisfies the triangle inequality and is continuous. And while

$p(cy) \neq |c|p(y)$ if c is negative since $U + A$ isn't balanced, we do at least have that $p(cy) = cp(y)$ if $c \geq 0$. Hence, we know that p is a well-defined sublinear functional on $\mathcal{X}$.

Now it's obvious that $p(x) \geq 1$ and that $p(y) \leq 1$ for all $y \in A$. What's less obvious is that these inequalities are strict.

- To see that $p(x) > 1$, suppose to the contrary that $x \in c(\frac{1}{2}U + A)$ for all $c > 1$. Equivalently, this means that $cx \in \frac{1}{2}U + A$ for all $c < 1$. But now as $cx - x \to 0$ as $c \to 1$ and $\frac{1}{2}U$ is a neighborhood of 0 in $\mathcal{X}$, we know that eventually $cx - x \in \frac{1}{2}U$. So, we can pick c close enough to 1 such that $cx - x = \frac{1}{2}u'$ for some $u' \in U$. At the same time, as $cx \in \frac{1}{2}U + A$ we know there exists $u \in U$ and $a \in A$ such that $cx = \frac{1}{2}u + a$. Hence, we get a contradiction as:
$$x = \tfrac{1}{2}u - \tfrac{1}{2}u' + a \in \tfrac{1}{2}U + \tfrac{1}{2}U + A \subseteq U + A.$$

- To see that $p(y) < 1$ for any fixed $y \in A$, note again that because $cy - y \to 0$ as $c \to 1$ and $\frac{1}{2}U$ is a neighborhood of 0, we know that there is some $\varepsilon_y > 0$ such that $cy - y \in \frac{1}{2}U$ when $c < 1 + \varepsilon_y$. In turn, $cy \in \frac{1}{2}U + y \subseteq \frac{1}{2}U + A$ when $c < 1 + \varepsilon_y$. And finally, we have that $y \in c(\frac{1}{2}U + A)$ if $c > (1 + \varepsilon_y)^{-1}$ where the latter is strictly less than 1.

Finally, we actually create our linear functional. Let $\mathcal{M} = \{cx : c \in \mathbb{R}\}$ and then define $g : \mathcal{M} \to \mathbb{R}$ by $g(cx) = cp(x)$. Then g is a linear functional on the subspace $\mathcal{M}$. Also since $p(cx) \geq 0 > g(cx)$ when $c < 0$ and we know from the sublinearity of p that $g(cx) = p(cx)$ when $c \geq 0$, we can conclude that $g \leq p$ on $\mathcal{M}$. So, by the real Hahn-Banach theorem we know there exists a linear functional $f : \mathcal{X} \to \mathbb{R}$ with $f(y) \leq p(y)$ for all $y \in \mathcal{X}$ and $f(cx) = g(cx)$ for all $c \in \mathbb{R}$.

Note that $|f(y)| = \max(-f(y), f(y)) = \max(f(-y), f(y)) \leq \max(p(-y), p(y))$ and that p is continuous, meaning that $p(-y) \to 0$ and $p(y) \to 0$ as $y \to 0$. Hence, we can conclude that f is continuous. Also, $f(x) = p(x) > 1 > p(y) \geq f(y)$ for all $y \in A$. ■

Krein-Millman Theorem: Let $\mathcal{X}$ be a topological vector space whose topology is defined by a sufficient family of seminorms. If $A \subseteq \mathcal{X}$ is compact and convex, then $\overline{\text{conv}}(\text{ex}(A)) = A$.

Proof:
Without loss of generality, we may assume $\mathcal{X}$ is a real vector space.

Claim: If B is a closed convex extreme subset of A, then $B \cap \text{ex}(A) \neq \emptyset$.

To prove this we use Zorn's lemma. Let $\mathcal{F}$ be the collection of all closed convex extreme subsets of A. Also partially order $\mathcal{F}$ by inclusion. Then we claim $\mathcal{F}$ has a minimal element.

Let $\mathcal{F}_0$ be a chain in $\mathcal{F}$ and set $C = \bigcap_{B \in \mathcal{F}_0} B$. Then C is not empty by the finite intersection property of A (since A is compact). Also C is closed and convex since it is the intersection of closed convex sets. Finally, suppose $y, z \in A$ satisfy that $(y, z) \cap C \neq \emptyset$. Then for any $B \in \mathcal{F}_0$ we know $(y, z) \cap B \neq \emptyset$. In turn, $[y, z] \subseteq B$ for all $B \in \mathcal{F}_0$. And this proves that $[y, z] \subseteq C$. All of this shows that $C \in \mathcal{F}$.

Now let D be a minimal set in $\mathcal{F}$. If D is a singleton $\{x\}$, then we will be done as $x \in B \cap \mathrm{ex}(A)$.

Suppose for the sake of contradiction that x, y are distinct elements of D. Then by lemma 2.3.5, there exists $f \in \mathcal{X}^*$ such that $f(x) < f(y)$. Since D is compact, we know that $M = \max\{f(z) : z \in D\}$ exists. So, let $E = \{z \in D : f(z) = M\}$. Then E is a proper subset of D as $x \notin E$. We also claim that E is an extreme set, thus contradicting that minimality of D.

E is compact since it is a closed subset of D. Also note that E is convex because if $z_0, z_1 \in D$ and $t \in [0,1]$ then:

$$f(tz_1 + (1-t)z_0) = tf(z_1) + (1-t)z_0 = tM + (1-t)M = M.$$

Finally, suppose $z_0, z_1 \in A$ and $tz_1 + (1-t)z_0 \in E$ for some $t \in (0,1)$. As $D \supseteq E$ is an extreme set we must have that $z_0, z_1 \in D$. And now as $M = f(tz_1 + (1-t)z_0) = tf(z_1) + (1-t)f(z_0)$ and both $f(z_0) \leq M$ and $f(z_1) \leq M$, we must have that $f(z_0) = M = f(z_1)$. So $[z_0, z_1] \subseteq E$.

Now it's clear that $\overline{\mathrm{conv}}(\mathrm{ex}(A)) \subseteq A$ (since A is a closed convex set containing $\mathrm{ex}(A)$). But suppose for the sake of contradiction that there exists $x \in A$ with $x \notin \overline{\mathrm{conv}}(\mathrm{ex}(A))$. By lemma 2.3.5. again we can find a linear functional $f \in \mathcal{X}^*$ such that $f(y) < \alpha < f(x)$ for all $y \in \overline{\mathrm{conv}}(\mathrm{ex}(A))$ (where $\alpha \in \mathbb{R}$). And since A is compact we know like before that $M = \max\{f(x) : x \in A\}$ exists.

By identical reasoning to before we know that $B = \{x \in A : f(x) = M\}$ is an extreme set. So by our claim, we have that $B \cap \mathrm{ex}(A) \neq \emptyset$. Yet this is a contradiction because $\mathrm{ex}(A) \subseteq \overline{\mathrm{conv}}(\mathrm{ex}(A))$ is disjoint from B. ■

Obvious Corollary: If A is a compact convex subset of a topological vector space $\mathcal{X}$ whose topology is generated by a sufficient family of seminorms, then $\mathrm{ex}(A) \neq \emptyset$.

A small lemma worth noting is that if $\mathcal{X}$ is a topological vector space and $A \subseteq \mathcal{X}$ is convex, then so is $\overline{A}$.

To see this, suppose $x, y \in \overline{A}$. Then we know that there are nets $\langle x_i \rangle_{i \in I}$ and $\langle y_J \rangle_{j \in J}$ contained in A and converging to x and y respectively. In turn, by considering the product net $\langle x_i, y_j \rangle_{I \times J}$ we have for any $t \in [0,1]$ that $ty_j + (1-t)x_i \to ty + (1-t)x$. And since $ty_j + (1-t)x_i \in A$ for all $(i,j) \in I \times J$ we have shown that $ty + (1-t)x \in \overline{A}$. So, $\overline{A}$ is convex.

Consequently, we always have that $\overline{\mathrm{conv}(E)} \supseteq \overline{\mathrm{conv}}(E)$ for any set $E \subseteq \mathcal{X}$. And this lets us rephrase the Krein Millman theorem in a slightly more useful way. If $\mathcal{X}$ is as stated in the theorem and $A \subseteq \mathcal{X}$ is compact and convex, then $\overline{\mathrm{conv}(\mathrm{ex}(A))} = A$.

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For this section assume that all vector spaces are Banach spaces.

Suppose $\mathcal{X}, \mathcal{Y}$ are Banach spaces. Then a bounded linear operator $T : \mathcal{X} \to \mathcal{Y}$ is called compact if $\overline{T(\mathcal{X}_1)}$ is compact in $\mathcal{Y}$. (See *page 469* for a reminder of what $\mathcal{X}_1$ means...)

Note: if T has finite rank (meaning $T(\mathcal{X})$ has finitely many dimensions), then T is compact.

Why?

Since $T(\mathcal{X})$ is a finite dimensional subspace, we know by (Rudin) Theorem 1.21 on *page 442* that $T(\mathcal{X})$ is a closed set. Hence, $C := \overline{T(\mathcal{X}_1)}$ is a closed subset of $T(\mathcal{X})$. Furthermore, $C \subseteq \{y \in T(\mathcal{X}) : \|y\| \leq \|T\|_{\text{op}}\}$. So, if we consider any bijective linear isometric map between $\mathbb{C}^n$ (or $\mathbb{R}^n$) and $T(\mathcal{X})$, then we will get that C is homeomorphic to a closed and bounded subset of $\mathbb{C}^n$ (or $\mathbb{R}^n$). By Heine-Borel we thus have that C is compact. ■

As a side note, you can use similar reasoning to show that any closed and bounded set in a finite dimensioned normed vector space is compact.

Lemma 3.1.3: Suppose $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of compact maps in $B(\mathcal{X}, \mathcal{Y})$ and $\|T_n - T\|_{\text{op}} \to 0$ as $n \to \infty$. Then T is also compact.

Proof:

It suffices to prove that $T(\mathcal{X}_1)$ is totally bounded since that will imply $\overline{T(\mathcal{X}_1)}$ is totally bounded (and we already have completeness just from the fact it's a closed set in a complete metric space $\mathcal{Y}$). Note that for all $x, y \in \mathcal{X}_1$, we have that:

$$\begin{aligned}
\|Tx - Ty\| &\leq \|Tx - T_nx\| + \|T_nx - T_ny\| + \|T_ny - Ty\| \\
&\leq \|T - T_n\|_{\text{op}}\|x\| + \|T_nx - T_ny\| + \|T_n - T\|_{\text{op}}\|y\| \\
&\leq 2\|T - T_n\|_{\text{op}} + \|T_nx - T_ny\|.
\end{aligned}$$

Given any $\varepsilon > 0$, fix n large enough so that $\|T_n - T\|_{\text{op}} < \varepsilon/4$. Then using the fact that T_n is a compact operator, pick $x_1, \ldots, x_m \in \mathcal{X}_1$ such that any $y \in T_n(\mathcal{X}_1)$ is within $\varepsilon/2$ from some T_nx_i. It then follows that any $y \in T(\mathcal{X}_1)$ is within ε from some Tx_i. ■

As an application of the above points, suppose $\mathcal{H}$ is a Hilbert space with orthonormal basis $\{e_i\}_{i\in I}$ and $T \in B(\mathcal{H})$ is given by a diagonal matrix $[\,\lambda_i\delta_{i,j}\,]$ (in other words $Te_i = \lambda e_i$ for all $i \in I$). Then T is compact iff $\{i \in I : |\lambda_i| > \varepsilon\}$ is finite for all $\varepsilon > 0$.

Lemma: If S is a linear operator on $\mathcal{H}$ given by a diagonal matrix $[\,\mu_i\delta_{i,j}\,]$ where the μ_i are bounded, then $\|S\|_{\text{op}} = \sup_{i\in I}|\mu_i|$.

Proof:

We can use *example 1.2.1 on page 284*. Specifically, recall that $\mathcal{H}$ is unitarily isomorphic to $\ell^2(I)$ by a natural map U. Furthermore, S is unitarily equivalent to multiplication by the element $\mu \in \ell^\infty(I)$ where $\mu = \{\mu_i\}_{i\in I}$.

In other words, $S = U^{-1}M_\mu U$.

Therefore, we have that $\|S\|_{\text{op}} = \|M_\mu\|_{\text{op}} = \|\mu\|_{\text{op}} = \sup_{i\in I}|\mu_i|$. ■

$(\Longleftarrow)$

If the latter is true then the set of i for which $\lambda_i \neq 0$ must be countable. Hence we can enumerate those i as $\{i_n\}_{n\in\mathbb{N}} \subseteq I$. Next, for each n we define T_n by letting $T_ne_{i_k} = \lambda_{i_k}$ for all $k \leq n$ and $T_ne_i = 0$ for all other $i \in I$. Then each T_n is bounded with finite rank, and is thus compact. Also, $\|T - T_n\|_{\text{op}} = \sup_{k>n}|\lambda_{i_k}| \to 0$ as $n \to \infty$. So, T is compact.

$(\Longrightarrow)$
Suppose that there is some $\varepsilon > 0$ such that $S = \{i \in I : \lambda_i \geq \varepsilon\}$ is an infinite set. Then for all $i, j \in S$ we have that $\|Te_i - Te_j\|^2 = |\lambda_i|^2 + |\lambda_j|^2 \geq 2\varepsilon^2$. So if we pick a nonrepeating sequence $\{e_{i_n}\}_{n \in \mathbb{N}}$ where each $i_n \in S$, then this sequence has no subsequential limits. This proves that $\overline{T(\mathfrak{X}_1)}$ is not compact. ■

Recall from my math 240b notes that if $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, $p \in [1, \infty]$, K is a $(\mathcal{M} \otimes \mathcal{N})$-measurable function on $X \times Y$, and there exists $C > 0$ such that $\int |K(x,y)| \mathrm{d}\mu(x) \leq C$ for a.e. y and $\int |K(x,y)| \mathrm{d}\nu(y)$ for a.e. x, then we have that $(T_K f)(x) \coloneqq \int_Y K(x,y) f(y) \mathrm{d}\nu(y)$ is a linear operator in $B(L^p(\nu), L^p(\mu))$ such that $\|T_K\|_{\mathrm{op}} \leq C$.
(This was Folland theorem 6.18).

The following theorem goes into more depth on this linear operator.

Theorem 3.1.5: Let X and Y be LCH spaces with σ-finite Borel measure μ and ν. Also assume μ and ν are finite on compact sets. If $K \in C_c(X \times Y)$ and $p \in [1, \infty]$ then the integral operator $T_K : L^p(\nu) \to L^p(\mu)$ is compact.

Proof:
Firstly, recall (Folland) proposition 7.22 on *pages 183-184* to see that K is $(\mathcal{M} \otimes \mathcal{N})$-measurable. Furthermore, let $U \subseteq X$ and $V \subseteq Y$ be precompact open sets such that $\mathrm{supp}(K) \subseteq U \times V$. Then given any $\widetilde{K} \in C_c(X, Y)$ with $\mathrm{supp}(\widetilde{K}) \subseteq U \times V$, we have that:

$$\int |\widetilde{K}(x,y)| \mathrm{d}\nu(y) \leq \|\widetilde{K}\|_u \nu(\overline{V}) \text{ and } \int |\widetilde{K}(x,y)| \mathrm{d}\mu(x) \leq \|\widetilde{K}\|_u \mu(\overline{U})$$

Therefore, for all $\widetilde{K} \in C_c(X, Y)$ with $\mathrm{supp}(\widetilde{K}) \subseteq U \times V$, we have that:

$$\|T_{\widetilde{K}}\|_{\mathrm{op}} \leq \|\widetilde{K}\|_u \cdot \max(\mu(\overline{U}), \nu(\overline{V})).$$

But also note by (Folland) proposition 7.21 (also on *page 183*) that there is a sequence of functions $\{K_n\}_{n \in \mathbb{N}}$ in $C_c(X)$ converging uniformly to K and satisfying for all $n \in \mathbb{N}$ that:

- $\mathrm{supp}(K_n) \subseteq U \times V$
- there exists $m \in \mathbb{N}$ such that $K_n(x,y) = \sum_{i=1}^m \phi(x)\psi(y)$ where $\phi_i \in C_c(X)$ and $\psi_i \in C_c(Y)$ for all $i \in \{1, \ldots, n\}$.

Importantly, note that $(T_{K_n} f)(x) = \sum_{i=1}^n (\int_X \psi_i f \mathrm{d}\mu) \phi_i(x)$. It thus follows that each T_{K_n} is a bounded linear operator with finite rank. Additionally,

$$\|T_K - T_{K_n}\|_{\mathrm{op}} = \|T_{(K-K_n)}\|_{\mathrm{op}} \leq \|K - K_n\|_u \cdot \max(\mu(\overline{U}), \nu(\overline{V})) \to 0 \text{ as } n \to \infty.$$

By lemma 3.1.3 we thus know that T_K is compact. ■

There are some other theorems for determining when an integral operator $T_K f(x) = \int K(x,y) f(y) \mathrm{d}\nu(y)$ is a well-defined bounded map.

Example 1.2.14: Let $p \in [1,\infty)$. Then suppose (X,μ) and (Y,ν) are σ-finite measure spaces and $K \in L^p(X \times Y, \mu \times \nu)$. If q is the conjugate exponent of p, then we have that $T_K : L^q(Y) \to L^p(X)$ defined by $(T_K f)(x) = \int_Y K(x,y)f(y)\mathrm{d}\nu(y)$ is a bounded linear map with $\|T_K\|_{\mathrm{op}} \leq \|K\|_{L^p(X\times Y)}$.

This is because for all $f \in L^q(Y)$:

$$\begin{aligned}
\|T_K f\|_p^p &= \int |\int K(x,y)f(y)\mathrm{d}\nu(y)|^p \mathrm{d}\mu(x) \\
&\leq \int \left(\int |K(x,y)f(y)|\mathrm{d}\nu(y)\right)^p \mathrm{d}\mu(x) \\
&\leq \int \left(\int |K(x,y)|^p \mathrm{d}\nu(y)\right)^{p/p} \cdot \|f\|_q^p \mathrm{d}\mu(x) = \|f\|_q^p \iint |K(x,y)|^p \mathrm{d}\nu(y)\mathrm{d}\mu(x) \\
&= \|f\|_q^p \|K\|_{L^p(X\times Y)}^p
\end{aligned}$$
