Math 140A Lecture Notes (Professor: Brandon Seward)

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Lecture 1: 1/8/2024

An order on a set S, typically denoted as $<$, is a binary relation satisfying:

1. $\forall x, y \in S$, exactly one of the following is true:
   - $x < y$
   - $x = y$
   - $y < x$
2. given $x, y, z \in S$, we have that $x < y < z \Rightarrow x < z$

As a shorthand, we will specify that

- $x > y \Leftrightarrow y < x$
- $x \leq y \Leftrightarrow x < y$ or $x = y$
- $x \geq y \Leftrightarrow x > y$ or $x = y$

An ordered set is a set with a specified ordering. Let S be an ordered set and E be a nonempty subset of S.

- If $b \in S$ has the property that $\forall x \in E,\ x \leq b$, then we call b an upperbound to E and say that E is bounded above by b.
- if $b \in S$ has the property that $\forall x \in E,\ x \geq b$, then we call b an lower bound to E and say that E is bounded below by b.
- We call $\beta \in S$ the least upperbound to E if β is an upper bound to E and β is the least of all upperbounds to E. In this case, we also commonly call β the supremum of E and denote it as $\sup E$.
- We call $\beta \in S$ the greatest lower bound to E if β is an lower bound to E and β is the greatest of all lower bounds to E. In this case, we also commonly call β the infimum of E and denote it as $\inf E$.
- We call $e \in E$ the maximum of E if $\forall x \in E,\ x \leq e$
- We call $e \in E$ the minimum of E if $\forall x \in E,\ x \geq e$

Fact: For an ordered set S and nonempty $E \subseteq S$, either:

- neither $\max E$ nor $\sup E$ exists
- $\sup E$ exists but $\max E$ does not exist
- $\max E$ exists and $\sup E = \max E$

Using $\mathbb{Q}$ as our ordered set...

- For $E = \{q \in \mathbb{Q} \mid 0 < q < 1\}$, $\max E$ does not exist but $\sup E$ exists and equals 1.

  To understand why, note that the set of all upper bounds of E is equal to $\{q \in \mathbb{Q} \mid q \geq 1\}$ and 1 is obviously the smallest element of that set. Thus, 1 is the supremum of E. However, $1 \notin E$. Thus, if $\max E$ did exist, it would have to not equal 1. But that would contradict 1 being the least greatest bound.

- For $E = \{q \in \mathbb{Q} \mid 0 < q \leq 1\}$, $\max E$ and $\sup E$ exist and they both are equal to 1

  The reasoning for this is similar to that for the previous set.

- For $E = \{q \in \mathbb{Q} \mid q^2 < 2\}$, neither $\max E$ and $\sup E$ exist.

  To prove this, we can show there exists a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $\forall q \in \mathbb{Q}^+$, $q^2 < 2 \Rightarrow q^2 < (f(q))^2 < 2$ and $2 < q^2 \Rightarrow 2 < (f(q))^2 < q^2$. That way we can give a counter example to any possible claimed supremum or maximum of E.

  Now instead of being like Rudin and simply providing the desired function, I want to present how one may come up with a function that works for this proof themselves.

  Firstly, note that for the following reasons, we know our desired function must be a rational function:

  - ◇ $\forall q \in \mathbb{Q}$, $f(q) \in \mathbb{Q}$. Based on this, we can't use any radicals, trig functions, logarithms, or exponentials in our desired function.

  - ◇ $q^2 > 2 \Rightarrow f(q) < q$. In other words, f needs to grow slower than a linear function. Thus, we can rule out the possibility of f being a polynomial.

  - ◇ If we wanted f to be a linear function, it would have to have the form $f(q) = \alpha(q - \sqrt{2}) + \sqrt{2}$ where α is some constant. This is because when $q^2 = 2$, $f(q) = q$. However, there is no value one can set α to which both eliminates the presence of irrational numbers in that function while simultaneously making $f(q) \neq q$ when $q^2 \neq 2$. So no linear function can possibly work for this proof.

  Having narrowed our search, let's now pick some convenient properties we would wish our proof function to have. Specifically, let's force f to be constantly increasing, have a y-intercept of 1, and approach a horizontal asymptote of $y = 2$. Doing this, we can now say that an acceptable function will have the following form where α is an unknown constant:

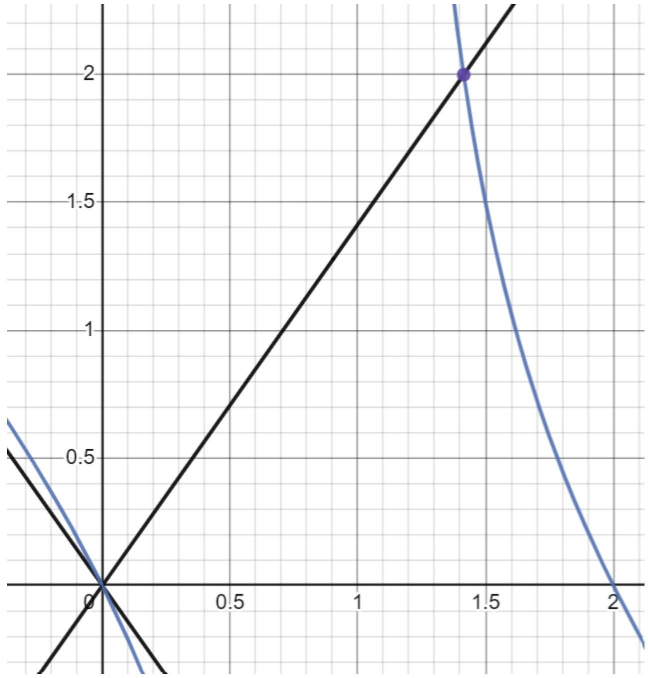
  $$f(q) = 1 + \frac{q}{q + \alpha}$$

  And finally, we can solve for α using the following system of equations:

  $$(1 + \tfrac{q}{q+\alpha})^2 = 2$$

  $$1 + \tfrac{q}{q+\alpha} = q$$

Now here's where a graphing calculator like Desmos can be very useful. Instead of painstakely having to solve for α, we can use a graphing calculator to approximate the value of α that satisfies our system of equations.



Based on the graph above, it looks like $f(q) = 1 + \frac{q}{q+2}$ will work for our proof. And sure enough it does. Furthermore, we can verify that the function we came up with is equivalent to that which Rudin presents.

We say an ordered set S has the least upperbound property if and only if when $E \subseteq S$ is nonempty and bounded above, then the supremum of E exists in S. Additionally, we say an ordered set S has the greatest lower bound property if and only if when $E \subseteq S$ is nonempty and bounded below, then the infimum of E exists in S.

When we define the set of real numbers, this will be one of the fundamental properties of that set.

Lecture 2: 1/10/2024

Proposition 1: S has the least upperbound property if and only if S has the greatest lower bound property.

Proof: Let's say we have an ordered set S

Assume S has the least upperbound property. Then, let $B \subseteq S$ be a nonempty subset which is bounded below. Additionally, let $A \subseteq S$ be the set of all lower bounds of B.

We know that $A \neq \emptyset$ because we assumed that B is bounded below. Thus, at least one lower bound to B exists and belongs to A. Additionally, because we assumed B is nonempty, we can say that each $b \in B$ is an upper bound to A. Thus, A is bounded above. Because of these two facts, we can apply the greatest lower bound property to say that the supremum of A exists.

Let's define $\alpha \coloneqq \sup A$. With that, our goal is now to show that $\alpha = \inf B$. To do this, we need to show firstly that α is a lower bound to B and secondly that it is greater than all other lower bounds of B.

1. For each $b \in B$, we have that b is an upperbound to A. And since $\alpha = \sup A$ is the least upperbound to A, we must have that $\alpha \leq b$. Thus α is a lower bound to B.

2. If $x \in S$ is a lower bound to B, then $x \in A$. And since $\alpha = \sup A$, $x \leq \alpha$. This shows that α is greater than or equal to all other lower bounds.

Hence, α is the infimum of B. And since we did this for a general $B \subseteq S$, we can thus say that S has the greatest lower bound property.

Now we skipped doing the reverse direction proof because it is almost identical to the foward direction proof. However, just know that the above proposition is an if and only if statement. ■

A field is a set F equipped with 2 binary operations, denoted $+$ and $\cdot$, and containing two elements $0 \neq 1 \in F$ satisfying the following conditions for all $x, y, z \in F$:

- Associativity: $$(x+y)+z = x+(y+z)$$ $$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$
- Commutativity: $$x+y = y+x$$ $$x \cdot y = y \cdot x$$
- Identity: $$0+x = x$$ $$1 \cdot x = x$$
- Inverses: $$\forall x \in F,\ \ \exists {-x} \in F \ \ s.t.\ \ x + -x = 0$$ $$\forall x \neq 0 \in F,\ \ \exists \frac{1}{x} \in F \ \ s.t.\ \ x \cdot \frac{1}{x} = 1$$
- Distributivity: $$x \cdot (y+z) = (x \cdot y) + (x \cdot z)$$

We shall assign the following notation:

We write ______	to mean ______
$x - y$	$x + -y$
$\frac{x}{y}$	$x \cdot \frac{1}{y}$
2	$1 + 1$
$2x$	$x + x$
x^2	$x \cdot x$
xy	$x \cdot y$

Now what follows is a number of propositions concerning the arithmetic properties of a field...

For a field F and elements $x, y, z \in F$, we have the following propositions:

Proposition 2.1: $x + y = x + z \Rightarrow y = z$

Proof: Assume $x + y = x + z$. Then...

$$\begin{aligned} y &= 0 + y && \text{(addition identity property)} \\ &= (-x + x) + y && \text{(addition inverse property)} \\ &= -x + (x + y) && \text{(addition associative property)} \\ &= -x + (x + z) && \text{(by our assumption)} \\ &= (-x + x) + z && \text{(addition associative property)} \\ &= 0 + z && \text{(addition inverse property)} \\ &= z && \text{(addition identity property)} \end{aligned}$$

Proposition 2.2: $x + y = x \Rightarrow y = 0$

Proof: Plug in $z = 0$ into proposition 2.1. in order to get that $y = z = 0$.

Proposition 2.3: $x + y = 0 \Rightarrow y = -x$

Proof: Plug in $z = -x$ into proposition 2.1. in order to get that $y = z = -x$.

Proposition 2.4: $-(-x) = x$

Proof: Observe that $x + -x = -x + x = 0$ by the inverse and commutative properties of addition. Then, by proposition 2.3, we know that $-x + x = 0 \Rightarrow x = -(-x)$.

Proposition 2.5: $x \cdot y = x \cdot z$ and $x \neq 0 \Rightarrow y = z$

Proof: Assume $x \cdot y = x \cdot z$ and $x \neq 0$. Then...

$$\begin{aligned}
y &= 1 \cdot y && \text{(multiplication identity property)} \\
&= (\tfrac{1}{x} \cdot x) \cdot y && \text{(multiplication inverse property)} \\
&= \tfrac{1}{x} \cdot (x \cdot y) && \text{(multiplication associative property)} \\
&= \tfrac{1}{x} \cdot (x \cdot z) && \text{(by our assumption)} \\
&= (\tfrac{1}{x} \cdot x) \cdot z && \text{(multiplication associative property)} \\
&= 1 \cdot z && \text{(multiplication inverse property)} \\
&= z && \text{(multiplication identity property)}
\end{aligned}$$

Note that to use the multiplication inverse property, we have to assume $x \neq 0$!!

Proposition 2.6: $x \cdot y = x \Rightarrow y = 1$

Proof: Plug in $z = 1$ into proposition 2.5. in order to get that $y = z = 1$.

Proposition 2.7: $x \cdot y = 1 \Rightarrow y = \dfrac{1}{x}$

Proof: Plug in $z = \frac{1}{x}$ into proposition 2.5. in order to get that $y = z = \frac{1}{x}$.

Proposition 2.8: $\dfrac{1}{\frac{1}{x}} = x$

Proof: Observe that $x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$ by the inverse and commutative properties of multiplication. Then, by proposition 2.7, we know that $\frac{1}{x} \cdot x = 1 \Rightarrow x = \dfrac{1}{\frac{1}{x}}$.

Proposition 2.9: $0 \cdot x = 0$

Proof: $(0 \cdot x) + (0 \cdot x) = (0 + 0) \cdot x = 0 \cdot x$. Thus we have an expression of the form $a + b = a$ which we can use proposition 2.2 on. Hence, we can conclude $0 \cdot x = 0$.

Proposition 2.10: $x \neq 0$ and $y \neq 0 \Rightarrow x \cdot y \neq 0$

Proof: since $x, y \neq 0$, we can say that $x \cdot y \cdot \frac{1}{x} \cdot \frac{1}{y} = 1 \neq 0$. Now by proposition 2.9, $x \cdot y = 0 \Rightarrow (x \cdot y) \cdot \left(\frac{1}{x} \cdot \frac{1}{y}\right) = 0$. However, we know that is not the case. So $x \cdot y$ can't equal zero.

Lecture 3: 1/12/2024

Proposition 2.11: $(-x)y = -(xy) = x(-y)$

Proof: $xy + (-x)y = (x + -x)y = 0y = 0$. Thus by proposition 2.3, $(-x)y = -(xy)$. We can make a similar argument to also say that $x(-y) = -(xy)$.

Proposition 2.12: $(-x)(-y) = xy$

Proof: Using proposition 2.11, we can say that $(-x)(-y) = -(x(-y)) = -(-(xy))$. Then by proposition 2.4, we can conclude $-(-(xy)) = xy$.

An ordered field is a field F equipped with an ordering $<$ satisfying $\forall x, y, z \in F$:

OF1. $y < z \Rightarrow y + x < z + x$

OF2. $(x > 0 \text{ and } y > 0) \Rightarrow xy > 0$

For x in an ordered field, we call x positive if and only if $x > 0$. Similarly, we call x negative if and only if $x < 0$.

Proposition 3: For an ordered field F and $x, y, z \in F$, we have:

1. $x < y \Leftrightarrow -y < -x$

   Proof: By property OF1 of an ordered field, we can say that $x < y \Rightarrow x + (-x + -y) < y + (-x + -y) \Rightarrow -y < -x$.

2. $(x > 0 \text{ and } y < z) \Rightarrow xy < xz$

   Proof: By property OF1 of an ordered field, $y < z \Rightarrow y - y < z - y$. Or in other words, $0 < z - y$. Therefore, since x is also positive by assumption, property OF2 of an ordered field tells us that $x(z - y) > 0$. Finally, adding xy to both sides by property OF1 and then distributing gives us: $xz - xy + xy = xz < xy$.

3. $(x < 0 \text{ and } y < z) \Rightarrow xy > xz$

   Proof: Since $x < 0$, we have $-x > 0$ by proposition 3.1. Then by applying proposition 3.2, we know that $(-x > 0 \text{ and } y < z) \Rightarrow -xy < -xz$. Finally, by reapplying proposition 3.1, this becomes $xy > xz$.

4. $x \neq 0 \Rightarrow x^2 > 0$

   Proof: If $x > 0$, then $x^2 = xx > 0x = 0$ by property OF2 of an ordered field. Meanwhile, if $x < 0$, then $-x > 0$ by proposition 3.1. So $(-x)(-x) > 0$ by property OF2. But $(-x)(-x) = x^2$ by proposition 2.12. So $x^2 > 0$.

5. $0 < x < y \Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$

Proof: Since $y > 0$ and $y \cdot \frac{1}{y} = 1 > 0 = 0 \cdot \frac{1}{y}$, we must have $\frac{1}{y} > 0$ by propositions 3.2 and 3.3. Note that $\frac{1}{y} \neq 0$ because if it did, $y \cdot \frac{1}{y} = 0$. Similarly, we can show $\frac{1}{x} > 0$. Now multiply both sides of $x < y$ by the positive element $\frac{1}{x} \cdot \frac{1}{y}$ and apply proposition 3.2 to get that $\frac{1}{y} < \frac{1}{x}$.

Theorem: There is (up to isomorphism) precisely one ordered field that contains $\mathbb{Q}$ and has the least upper bound property. We denote this field $\mathbb{R}$ and we call its elements real numbers.

In other words, this theorem is stating that $\mathbb{R}$ exists and is unique. Unfortunately, the proof for this is very long and so won't be covered in lecture. However, the professor has left some resources to cover it. So, I will have the proof of this theorem later in these notes. See page: <research how to cite a page>

Proposition 4.1: If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$. This is called the archimedean property.

Proof: We proceed by looking for a contradiction. Let $A = \{nx \mid n \in \mathbb{Z}^+\}$ and assume $\nexists n \in \mathbb{Z}^+$ such that $nx > y$. In that case we know y is an upper bound of A. Additionally, since A is bounded above, we know by the least upper bound property of the real numbers that $\sup A$ exists. So, let $\alpha = \sup A$.

Now because $\mathbb{R}$ is an ordered field, we know that:
$x > 0 \Rightarrow -x < 0 \Rightarrow \alpha - x < \alpha$. Therefore, because α is the least upper bound, we know that $\alpha - x$ is not an upper bound for A. Or in other words, there exists $n \in \mathbb{Z}^+$ such that $nx > \alpha - x$. But this contradicts that α is the least upper bound of A because $nx > \alpha - x \Rightarrow (n+1)x > \alpha$ and $(n+1)x \in A$. So we conclude that the supremum of A can't exist, which by the contrapositive of the least upper bound property, means that A is not bounded above.

Proposition 4.2: If $x, y \in \mathbb{R}$ and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$. In other words, we say that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof: Since $x < y$, we have that $0 < y - x$. Then because $y - x$ is positive, we can use the archimedean property to say that there exists an integer n such that $n(y - x) > 1$. Note for later that this means $ny > 1 + nx$.

Now note that since $1 > 0$ and nx is a real number, we can use the archimedean property twice to get positive integers m_1 and m_2 such that $m_1 \cdot 1 > -nx$ and $m_2 \cdot 1 > +nx$. Thus, we get the expression $-m_1 < nx < m_2$. So now consider the set $B = \{m \in \mathbb{Z} \mid -m_1 \geq nx \geq m_2 \text{ and } m > nx\}$. We know that B has finitely many elements and that B contains at least one element: m_2. So B must have a minimum element. We'll refer to that minimum element as m. Notably, as m is the minimum element of B, we know that $m - 1 \notin B$, meaning that $m - 1 \leq nx < m$

We now combine inequalities as follows: $m - 1 \leq nx \Rightarrow m \leq nx + 1$. So we have that $nx < m \leq nx + 1$. But now remember from the previous page that $ny > 1 + nx$. So we can say that $nx < m \leq nx + 1 < ny$. Finally, because $n > 0$, we can multiply the inequality by $\frac{1}{n}$ to get that $x < \frac{m}{n} < y$. ■

Lecture 4: 1/17/2024

Theorem: If $x \in \mathbb{R}$, $x > 0$, $n \in \mathbb{Z}$, and $n > 0$, then there is a unique $y \in \mathbb{R}$ with $y > 0$ and $y^n = x$. This number y is denoted $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

To prove this, first note the following lemma about positive integers n and $a, b \in \mathbb{R}$:

$$b^n - a^n = (b - a)(b^{n-1} + ab^{n-2} + \ldots + a^{n-2}b + a^{n-1})$$

To prove this, one can either use induction or just calculate it out by hand to verify that the equality holds.

Additionally, also consider that if n is a positive integer and $0 \leq a \leq b$ where $a, b \in \mathbb{R}$, then we have that $a^n \leq b^n$. Combining this fact with the lemma above, we can say that $0 \leq a \leq b$ implies that $b^n - a^n \leq (b - a)nb^{n-1}$. Or in other words: $a^n \leq b^n \leq a^n + (b - a)nb^{n-1}$.

This comes from replasing every a in the expression $(b^{n-1} + ab^{n-2} + \ldots + a^{n-2}b + a^{n-1})$ with b in order to get that $b^n - a^n \leq (b - a)(b^{n-1} + b^{n-1} + \ldots + b^{n-1})$

Now set $E = \{t \in \mathbb{R} \mid t > 0, t^n \leq x\}$.

We can show that E is nonempty...

- If $x \geq 1$, then $t = 1 \in E$ since $1^n = 1 \leq x$.
- If $x < 1$, then $x \in E$ since $x < 1 \Rightarrow x^{n-1} < 1^{n-1} = 1$. But then $x^n < x$.

Thus, we know $E \neq \emptyset$.

We can also show that E is bounded above. Consider $t = 1 + x$. In that case, $t > 1$, which implies that $t^{n-1} > 1^{n-1} = 1$. Therefore, $t^n > t$, meaning that $t^n > x$. So $t = x + 1$ is an upper bound for E.

Thus by the least upper bound property of the real numbers, we know $y = \sup E$ exists.

Claim 1: $y^n \geq x$.

To prove this, we shall procede towards a contradiction. Assume $y^n < x$

Then pick some h such that $0 < h < \gamma$ and γ is some mystery constant for us to find. Then, we can say that $y < y + h$, meaning by the lemma on the previous page that $y^n \leq (y+h)^n \leq y^n + (y+h-y)n(y+h)^n - 1$. Or in other words, $(y+h)^n \leq y^n + hn(y+h)^n - 1$.

Now we shall make our first assumption about γ: let $\gamma \leq 1$. That way, we know that $(y+h)^n \leq y^n + hn(y+h)^n - 1 < y^n + hn(y+1)^n$. And since, we are assuming that $y^n < x$, we know there must exist some value of h such that $y^n + hn(y+1)^{n-1} < x$. Putting this limitation on h, we get that $h < \frac{x-y^n}{n(y+1)^{n-1}}$ (Remember that $x - y^n$, y, and n are all positive). So finally, we say that $\gamma = \min\left(1, \frac{x-y^n}{n(y+1)^{n-1}}\right)$. This is so that for $0 < h < \gamma$, we have that $(y+h)^n < x$.

Thus, we have a contradiction as we assumed that y is the supremum of E and yet we just proved that $y + h \in E$. So, y^n cannot be less than x, meaning that that $y^n \geq x$.

Claim 2: $y^n \leq x$.

To prove this, we shall again proceed towards a contradiction. Assume $y^n > x$.

Then for some h such that $0 < h < \gamma$ where γ is a new mystery constant, consider $y - h$.

I now realize that I need to prove this lemma: for a positive integer n and real numbers a and b such that $a \geq b$, we have that $(a-b)^n \geq a^n - bna^{n-1}$. We can prove this through induction.

Firstly for $n = 1$: we have that $(a-b)^1 = a^1 - b(1)a^0$.

Now assume that for $k \geq 1$, $(a-b)^k \geq a^k - bka^{k-1}$.
Then $(a-b)^{k+1} = (a-b)(a-b)^k$. And since $(a-b) > 1$, we know that $(a-b)^{k+1} = (a-b)(a-b)^k \geq (a-b)(a^k - bka^{k-1})$.

Now let's expand out our lesser term to get that:
$(a-b)^{k+1} \geq a^{k+1} - bka^k - ba^k + b^2ka^{k-1}$. Thus, we know that $(a-b)^{k+1} \geq a^{k+1} - b(k+1)a^k + b^2ka^{k-1} > a^{k+1} - b(k+1)a^k$. Hence, we have shown that $(a-b)^{k+1} \geq a^{k+1} - b(k+1)a^k$.

Based on the lemma covered right before this, we have that $(y-h)^n \geq y^n - hny^{n-1}$. But now let's require that $y^n - hny^{n-1} > x$. Thus, we can say that $h < \frac{y^n - x}{ny^{n-1}}$.

So setting $\gamma = \frac{y^n - x}{ny^{n-1}}$, we have that for $0 < h < \gamma, \quad (y-h)^n > x$. But this now leads to a contradiction as $y - h$ must be an upper bound to E.

(If some number z is greater than $y-h$, than $z^n > (y-h)^n > x$. So $z \notin E$.)

However, $y - h$ can't be an upper bound to E as we specified that y is the least upper bound of E. So we conclude that y^n cannot be greater than x, thus meaning $y^n \leq x$.

So since $y^n \leq x$ and $y^n \geq x$, we conclude that $y^n = x$.

Finally, we now shall mention that y is obviously the unique number such that $y^n = x$. After all, for $0 < a < y < b$, we have that $a^n < y^n < b^n$. So, there can only be one number y such that $y^n = x$.

Lecture 5: 1/19/2024

Decimal representations of real numbers:

- Each $x \in \mathbb{R}$ such that $x > 0$ can be written $x = n_0.n_1n_2n_3\ldots$ where $n_0 \in \mathbb{Z}$ and $\forall i \geq 1$, $n_i \in \{0,1,2,3,4,5,6,7,8,9\}$.

  Specifically, let n_0 be the largest integer with $n \leq x$. Then inductively, pick n_k to be the max element in $\{0,1,2,3,4,5,6,7,8,9\}$ such that:

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_{k-1}}{10^{k-1}} + \frac{n_k}{10^k} \leq x$$

- Conversely, suppose $n_0 \in \mathbb{Z}$ and $\forall i \geq 1, n_i \in \{0,1,2,3,4,5,6,7,8,9\}$. Then, defining $E = \{n_0, n_0 + \frac{n_1}{10}, \ldots, n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k}, \ldots\}$, we have that $n_0.n_1n_2n_3\ldots = x \in \mathbb{R}$ where $x = \sup E$.

We will rarely ever use decimal representations though.

The extended real number system is the set $\mathbb{R} \cup \{-\infty, +\infty\}$ where for all $x \in \mathbb{R}$:

- $-\infty < x < +\infty$
- $x + \infty = +\infty$
- $x - \infty = -\infty$
- $\frac{x}{+\infty} = 0 = \frac{x}{-\infty}$
- $x > 0 \Rightarrow x(+\infty) = +\infty$
- $x > 0 \Rightarrow x(-\infty) = -\infty$
- $+\infty + \infty = +\infty$
- $+\infty(+\infty) = +\infty$
- $x < 0 \Rightarrow x(+\infty) = -\infty$
- $x < 0 \Rightarrow x(-\infty) = +\infty$
- $-\infty - \infty = -\infty$
- $+\infty(-\infty) = -\infty$

All other operation involving $+\infty$ and $-\infty$ are left undefined.

◇ Sometimes, we denote the extended real number system $\overline{\mathbb{R}}$.
◇ The extended real number system is not a field.
◇ To distinguish $x \in \mathbb{R}$ from ∞ or $-\infty$, we call $x \in \mathbb{R}$ finite.

The set of complex numbers, denoted $\mathbb{C}$, is the set of all things of the form $a + bi$ where $a, b \in \mathbb{R}$ and i is a symbol satisfying $i^2 = -1$.

To be more rigorous about this definition, what we would do is define the set of complex numbers to be the set of pairs of real numbers equipped with the following operations:

For $z, u \in \mathbb{C}$ such that $z = (a, b)$ and $u = (c, d)$:

- $z + u = (a + c, b + d)$
- $z \cdot u = (ac - bd, ad + bc)$

Having done that, we would then:

1. Define $0 = (0, 0)$ and $1 = (1, 0)$
2. Prove that $\mathbb{C}$ satisfies our field axioms
3. Say that $i = (0, 1)$ and then show that $i^2 = (-1, 0)$
4. And finally show that for $a, b \in \mathbb{R}$, $a(1) + b(i) = (a, b)$

(Thus it makes sense to denote $z \in \mathbb{C}$ as $z = a + bi$)

However, we're behind and so not going to spend time doing that in class.

For $z = a + bi$, we denote $\mathrm{Re}(z) = a$ the real part of z. On the other hand, we denote $\mathrm{Im}(z) = b$ the imaginary part of z.

The complex conjugate of $z = a + bi$ is $\overline{z} = a - bi$.

Proposition 5: If $z, w \in \mathbb{C}$, then:

1. $\overline{z+w} = \overline{z} + \overline{w}$
2. $\overline{zw} = \overline{z} \cdot \overline{w}$
3. $z + \overline{z} = 2\mathrm{Re}(z)$
4. $z - \overline{z} = 2\mathrm{Im}(z)i$
5. $z\overline{z} \in \mathbb{R}$ and $z\overline{z} > 0$ when $z \neq 0$.

Proof:
Points 1-4 can be verified by direct computation.
As for point 5, note that if $z = a+bi$, then $z\overline{z} = (a+bi)(a-bi) = a^2 + b^2$.
Now as $a, b \in \mathbb{R}$, we know that $a^2+b^2 \in \mathbb{R}$. But $a^2+b^2 > 0$ if $b \neq a \neq 0$.
Meanwhile, $a^2 + b^2 = 0$ if $a = b = 0$. So $z\overline{z} > 0$ if $z \neq 0$

The absolute value of $z = a + bi$ is $|z| = \sqrt{z\overline{z}}$

Propostion 6: For $z, w \in \mathbb{C}$, we have that:

1. $|0| = 0$ and $|z| > 0$ when $z \neq 0$.
2. $|z| = |\overline{z}|$
3. $|zw| = |z||w|$
4. $|\mathrm{Re}(z)| \leq |z|$
5. $|\mathrm{Im}(z)| \leq |z|$
6. $|z + w| \leq |z| + |w|$

This last bullet is the triangle inequality.

Proof:
Claims 1, 2, and 3 can be verified through direct computation.

To prove claim 4, note that $a^2 \leq a^2 + b^2$. So, $|a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$.
We can repeat this but with b^2 to prove claim 5.

Lastly, to prove claim 6, note that $|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\overline{z}+\overline{w})$.
Now, we can distribute to get that $|z + w|^2 = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w}$. So, we know that $|z + w|^2 = |z|^2 + z\overline{w} + w\overline{z} + |w|^2$.

But now observe that $w\overline{z} = \overline{z\overline{w}}$. So $z\overline{w} + w\overline{z} = 2\mathrm{Re}(z\overline{w})$. But by claim 4, we know that $\mathrm{Re}(z\overline{w}) \leq |z\overline{w}|$. Additionally, by claims 2 and 3, we have that $|z\overline{w}| = |z||\overline{w}| = |z||w|$. So, we know that $|z + w|^2 \leq |z|^2 + 2|z||w| + |w|^2$.
This simplifies to $|z + w|^2 \leq (|z| + |w|)^2$. Hence, $|z + w| \leq |z| + |w|$.

Lecture 6: 1/22/2024

Theorem: (the Cauchy-Schwarz Inequality)

If $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}$, then:

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

Proof:

Define $A = \sum_{j=1}^{n} |a_j|^2$, $B = \sum_{j=1}^{n} |b_j|^2$, and $C = \sum_{j=1}^{n} a_j \overline{b_j}$.

Note that $A, B \in \mathbb{R}$ such that $A, B \geq 0$. Meanwhile, $C \in \mathbb{C}$.

If $B = 0$, then $b_1 = \ldots = b_n = 0$. Thus $C = 0$ as well and so the inequality is trivially true.

So now consider if $B > 0$. Then we can make a series of manipulations starting with: $0 \leq \sum_{j=1}^{n} |Ba_j - Cb_j|^2$.

(The professor said not to worry about how Rudin thought of using this formula.)

$$\begin{aligned}
0 &\leq \sum_{j=1}^{n} |Ba_j - Cb_j|^2 \\
&= \sum_{j=1}^{n} (Ba_j - Cb_j)(B\overline{a_j} - \overline{C}\overline{b_j}) \\
&= B^2 \sum_{j=1}^{n} |a_j|^2 - BC \sum_{j=1}^{n} \overline{a_j} b_j - B\overline{C} \sum_{j=1}^{n} a_j \overline{b_j} + |C|^2 \sum_{j=1}^{n} |b_j|^2 \\
&= B^2 A - BC\overline{C} - B\overline{C}C + |C|^2 B \\
&= B^2 A - B|C|^2 \\
&= B(AB - |C|^2)
\end{aligned}$$

Thus, since we're assuming $B > 0$, we know that $AB - |C|^2 \geq 0$.

So, $AB \geq |C|^2$. ■

We call elements $\vec{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ vectors or points. The x_i are the coordinates of $\vec{x}$.

The inner product or dot product of $\vec{x}$, $\vec{y} \in \mathbb{R}^k$ is: $\vec{x} \cdot \vec{y} = \sum_{i=1}^{k} x_i y_i$

The norm of $x \in \mathbb{R}^k$ is $\|\vec{x}\| = (\vec{x} \cdot \vec{x})^{\frac{1}{2}}$

Proposition 7: If $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$, then:

1. $\|\vec{0}\| = 0$ and $\|\vec{x}\| > 0$ when $\vec{x} \neq \vec{0}$.
2. $\|\alpha\vec{x}\| = \alpha\|\vec{x}\|$
3. $\|\vec{x} \cdot \vec{y}\| \leq \|\vec{x}\|\|\vec{y}\|$
4. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$
5. $\|\vec{x} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\|$

The proofs for 1-4. are nearly identical to those for complex numbers so we won't cover them here.

As for 5, note that $\vec{x} + (-\vec{z}) = \vec{x} - \vec{y} + \vec{y} - \vec{z}$.

For sets X, Y and a function $f : X \to Y$, we shall write:

- for $A \subseteq X, \;\; f(A) = \{f(a) \mid a \in A\}$ (This is the image of A.)
- for $B \subseteq Y, \;\; f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ (This is the preimage of A.)
- for $y \in Y$, we write $f^{-1}(y)$ for $f^{-1}(\{y\})$

We say two sets A and B have equal cardinality, denoted $|A| = |B|$ if there is a bijection f from A onto B.

- A is finite if it has equal cardinality with $\{1, \ldots, n\}$ for some $n \in \mathbb{Z}^+$ or if $A = \emptyset$.
- A is countable if either A is finite or A has equal cardinality with $\mathbb{Z}^+$.
- A is uncountable if its not countable.

A sequence is a function f having domain $\mathbb{Z}^+$. If $f(n) = x_n \in A$ for each n, it is typical to denote f by $(x_n)_{n \in \mathbb{Z}^+}$

Proposition 8: If A is countable and $E \subseteq A$, then E is countable.

Proof:

If E is finite, then E is countable and we're done. So assume E is infinite. Then as $E \subseteq A$, we know A is infinite as well.

Since A is countable, we can enumerate A as $x_1, x_2, x_3, \ldots$.
Set $n_1 = \min\{m \in \mathbb{Z}^+ \mid x_m \in E\}$. Then, inductively set $n_{k+1} = \min\{m \in \mathbb{Z}^+ \mid m > n_k$ and $x_m \in E\}$. Finally, define $f : \mathbb{Z}^+ \to E$ by the rule $f(k) = x_{n_k}$. That way f is a bijection.

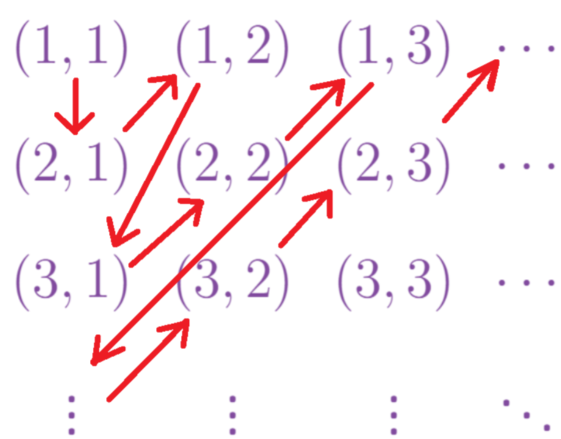
Proposition 9: $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Proof:

Define $g : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $g(i,j) = \left(\sum_{k=1}^{i+j-2} k\right) + j$.

This is a bijection.

Here's what the bijection looks like:



Lecture 7: 1/24/2024

Proposition 10: If A is a countable set, B any set, and $g : A \to B$ is a surjection, then B is countable.

Proof:

If A is finite, then $B = g(A)$ is finite as well. So the proposition is trivially true. Now assume A is infinite. Then since A is countable, there is a bijection $f : \mathbb{Z}^+ \to A$. Now define $\phi = g \circ f : \mathbb{Z}^+ \to B$. We know that ϕ is a surjection as it is the composition of two surjections.

Let $E \subseteq \mathbb{Z}^+$ be any set that contains precisely one element of $\phi^{-1}(b)$ for each $b \in B$. For instance, we can define E as the set:

$$\{n \in \mathbb{Z}^+ \mid \forall m \in \mathbb{Z}^+,\ m < n \Rightarrow \phi(m) \neq \phi(n)\}$$

Now by proposition 8, we know that E is countable as E is a subset of a countable set. But additionally we have that ϕ acts as a bijection from E to B. Therefore, $|E| = |B|$, meaning B is countable.

Proposition 11: A set A is countable if and only if there exists a surjection from $\mathbb{Z}^+$ onto A.

Proof:

($\Longleftarrow$) Since $\mathbb{Z}^+$ is the definition of a countable set, if there is a surjection from $\mathbb{Z}^+$ to A, then we have by proposition 10 that A is also countable.

($\Longrightarrow$) Assume A is countable. If A is finite, then we can number the elements of A as $\{a_1, a_2, \ldots, a_n\}$. So, we may define the surjection $f : \mathbb{Z}^+ \to A$ with the correspondance rule:

$$f(k) = \begin{cases} a_k & \text{if } k \leq n \\ a_n & \text{if } k > n \end{cases}$$

Meanwhile if A is infinite, then by definition there exists a bijection from $\mathbb{Z}^+$ to A. So, no matter if A is infinite or finite, if A is countable, then there exists a bijection from $\mathbb{Z}^+$ to A.

Proposition 12: If E_n is a countable set for each $n \in \mathbb{Z}^+$, then $\displaystyle\bigcup_{n \in \mathbb{Z}^+} E_n$ is countable.

Proof:

For each $n \in \mathbb{Z}^+$, there is a surjection $f_n : \mathbb{Z}^+ \to E_n$.

Define $g : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \displaystyle\bigcup_{n \in \mathbb{Z}^+} E_n$ by $g(n, k) = f_n(k)$.

Then as g is a surjection and $\mathbb{Z} \times \mathbb{Z}$ is countable by proposition 9, we know by proposition 10 that $\displaystyle\bigcup_{n \in \mathbb{Z}^+} E_n$ is countable.

In other words, the union of countably many countable sets is countable.

Proposition 13: If A is countable, then for every $n \in \mathbb{Z}^+$, the set $A^n = A \times A \times \ldots \times A$ is countable.

Proof: (we can proceed by induction)

When $n = 1$, then $A^n = A^1 = A$ is obviously countable.

Now assume the proposition is true for $n - 1$, meaning A^{n-1} is countable.

Then: $A^n = \displaystyle\bigcup_{a \in A} \{a\} \times A^{n-1}$ is countable by proposition 12.

Corollary: $\mathbb{Q}$ is countable.

Proof:

Define $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}^+$ by setting $f(p) = (n, m)$ where n, m are the unique coprime integers with $m > 0$, $\frac{n}{m} = p$. Also define $f(0) = (1, 0)$ Then $f(\mathbb{Q}) \subset \mathbb{Z} \times \mathbb{Z}^+$ and the latter set is countable. So $f(\mathbb{Q})$ is countable. Since f is injective, f is a bijection between $\mathbb{Q}$ and a countable set. Thus $\mathbb{Q}$ is countable.

Given sets A and B, we write A^B to denote the set of all functions from B to A.

Proposition 14: $\{0, 1\}^{\mathbb{Z}^+}$ is uncountable.

Proof: Let $\{f_1, f_2, \ldots\}$ be any countable subset of $\{0, 1\}^{\mathbb{Z}^+}$. Then define $g \in \{0, 1\}^{\mathbb{Z}^+}$ by the rule $g(n) = 1 - f_n(n)$. Since $g(n) \neq f_n(n)$, we have that $g \neq f_n$. Since this holds for all $n \in \mathbb{Z}^+$, we can thus conclude that $g \notin \{f_1, f_2, \ldots\}$. We thus conclude that any countable subset of $\{0, 1\}^{\mathbb{Z}^+}$ is a proper subset. So $\{0, 1\}^{\mathbb{Z}^+}$ must be uncountable.

Lecture 8: 1/26/2024

A metric space is a set X equipped with a function $d : X \times X \longrightarrow [0, \infty)$ satisfying:

1. $\forall p, q \in X \quad p \neq q \Rightarrow d(p, q) > 0$ whereas $p = q \Rightarrow d(p, q) = 0$
2. $\forall p, q \in X \quad d(p, q) = d(q, p)$
3. $\forall p, q, s \in X \quad d(p, q) \leq d(p, s) + d(s, q)$

The function d is called a distance function or metric.

Examples:

- $\mathbb{R}^k$ is a metric space (we have several metrics to choose from):
  - ◇ $d_p(\vec{x}, \vec{y}) = (\sum |x_i - y_i|^p)^{\frac{1}{p}}$

    $d_2(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$
  - ◇ $d_\infty(\vec{x}, \vec{y}) = \max_{1 \leq i \leq k} |x_i - y_i|$
- Any set X is a metric space when equipped with the discrete metric:

$$d(p, q) = \begin{cases} 0 \text{ if } p = q \\ 1 \text{ if } p \neq q \end{cases}$$

- The set of all functions from $[0, 1] \to [0, 1]$ can be equipped with the metric:

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

Let X be a metric space. Then for $p \in X$ and $r > 0$, the (open) ball of radius r around p is $B_r(p) = \{q \in X \mid d(p,q) < r\}$.

$p \in X$ is a limit point of $E \subseteq X$ if there are points in $E \setminus \{p\}$ that are arbitrarily close to p. Or in other words, if p is a limit point, then

$$\forall r > 0, \quad ((B_r(p) \setminus \{p\}) \cap E) \neq \emptyset.$$

The set of limit points of $E \subseteq X$ is denoted E'.

- E is closed if $E' \subseteq E$.
- E is perfect if $E = E'$.
- We say E is dense in X if $E \cup E' = X$

$p \in X$ is an interior point of $E \subseteq X$ if $\exists r > 0 \ \ s.t. \ \ B_r(p) \subseteq E$.

The set of interior points of E is denoted E°.

- E is open if $E^\circ = E$.
- E is a neighborhood of p if $p \in E^\circ$.

The complement of E is $E^{\mathsf{C}} = X \setminus E$.
E is bounded if there is a point $p \in X$ and $R > 0$ with $E \subseteq B_R(p)$.
If $p \in E$, we say p is an isolated point of E if $\exists r > 0 \ \ s.t. \ \ B_r(p) \cap E = \{p\}$.

Proposition 15: If X is a metric space, $p \in X$, and $r > 0$, then $B_r(p)$ is open.

Proof:
Consider a point $q \in B_r(p)$. We claim that $B_{(r-d(p,q))}(q) \subseteq B_r(p)$.
To prove this consider that for $z \in B_{(r-d(p,q))}(q)$, we have that $d(p,z) \leq d(p,q) + d(q,z) < d(p,q) + (r - d(p,q)) = r$. Thus, $z \in B_r(p)$.
And, since we can do this for any $z \in B_{(r-d(p,q))}(q)$, we know that $B_{(r-d(p,q))}(q) \subseteq B_r(p)$. Therefore q is an interior point of $B_r(p)$. And, since we can say this for any $q \in B_r(p)$, we thus conclude that $B_r(p)$ consists of interior points. So $B_r(p)$ is open.

Lecture 9: 1/29/2024

Let X be a metric with metric d and let $E \subseteq X$...

Proposition 16: If $p \in E'$, then $(B_r(p) \setminus \{p\}) \cap E$ is infinite for every $r > 0$.

Proof (by contrapositive):
Let $p \in X$ and suppose $\exists r > 0$ with $(B_r(p) \setminus \{p\}) \cap E$ finite.
Then set $t = \min\{d(p,q) \mid q \in (B_r(p) \setminus \{p\}) \cap E\}$. That way, we must have that $t > 0$. But at the same time, $B_t(p) \setminus \{p\} \cap E$ is empty. Therefore $p \notin E'$.

Corollary: If E is finite, then $E' = \emptyset$. This means that finite sets are always closed.

Propostion 17: E is open if and only if E^{C} is closed.

Proof:

$$
\begin{aligned}
E^{\mathsf{C}} \text{ is closed} &\iff \left(E^{\mathsf{C}}\right)' \subseteq E^{\mathsf{C}} \\
&\iff \left(E^{C}\right)' \cap E = \emptyset \\
&\iff \forall p \in E,\ \ p \notin \left(E^{\mathsf{C}}\right)' \\
&\iff \forall p \in E,\ \ \exists r > 0 \ \ s.t.\ \ (B_r(p) \setminus \{p\}) \cap E^{\mathsf{C}} = \emptyset \\
&\iff \forall p \in E,\ \ \exists r > 0 \ \ s.t.\ \ B_r(p) \setminus \{p\} \subseteq E \\
&\iff \forall p \in E,\ \ \exists r > 0 \ \ s.t.\ \ B_r(p) \subseteq E \\
&\iff \forall p \in E,\ \ p \in E^{\circ} \\
&\iff E \text{ is open}
\end{aligned}
$$

Corollary: E is closed if and only if E^{C} is open.

Proposition 18: Let A be any set.

1. If $u_\alpha \subseteq X$ is an open set for each $\alpha \in A$, then $\bigcup_{\alpha \in A} u_\alpha$ is open.

   Proof:
   Let $p \in \bigcup_{\alpha \in A} u_\alpha$. Pick $\beta \in A$ with $p \in u_\beta$.
   Since u_β is open, we know that $\exists r > 0 \ \ s.t.\ \ B_r(p) \subseteq u_\beta \subseteq \bigcup_{\alpha \in A} u_\alpha$.
   So p is an interior point. Hence, we conclude that $\bigcup_{\alpha \in A} u_\alpha$ is open.

2. If $F_\alpha \subseteq X$ is a closed set for each $\alpha \in A$, then $\bigcap_{\alpha \in A} F_\alpha$ is closed.

   Proof:
   $\left(\bigcap_{\alpha \in A} F_\alpha\right)^{\mathsf{C}} = \bigcup_{\alpha \in A} (F_\alpha)^{\mathsf{C}}$ by De Morgan's laws.

Since each F_α is closed, we know each $(F_\alpha)^\mathsf{C}$ is open.
So by proposition 18.1, we know that $\bigcup_{\alpha\in A} (F_\alpha)^\mathsf{C}$ is open.
Then, by proposition 17, we know that its complement, $\bigcap_{\alpha\in A} F_\alpha$ is closed.

3. If $u_1, u_2, \ldots, u_n \subseteq X$ are open, then $\bigcap_{i=1}^{n} u_i$ is open.

   Proof:
   Let $p \in \bigcap_{i=1}^{n} u_i$. Then $p \in u_i$ for every i.
   Since u_i is open, $\exists r_i > 0 \;\; s.t. \;\; B_{r_i}(p) \subseteq u_i$. Therefore, set $r = \min\{r_i \mid 1 \leq i \leq n\}$ so that for all i, $B_r(p) \subseteq B_{r_i}(p) \subseteq u_i$.
   Hence, $B_r(p) \subseteq \bigcap_{i=1}^{n} u_i$. We thus conclude that $\bigcap_{i=1}^{n} u_i$ is open.

4. If $F_1, F_2, \ldots, F_n \subseteq X$ are closed, then $\bigcup_{i=1}^{n} F_i$ is closed.

   The proof of this follows from proposition 18.3 in the same way that proposition 18.2 follows from proposition 18.1.

Lecture 10: 2/2/2024

Given a metric space X, the closure of $E \subseteq X$ is $\overline{E} := E \cup E'$.

Proposition 19.1: $\overline{E}$ is closed.

Proof:
Let $p \in (\overline{E})^\mathsf{C}$. Thus, $p \notin E'$, meaning that we can fix $r > 0$ so that $(B_r(p)\backslash\{p\}) \cap E = \emptyset$. Additionally, since $p \notin E$, we have that $B_r(p) \cap E = \emptyset$.

Now consider any $q \in B_r(p)$. Setting $t = r - d(p,q)$, we have that $B_t(q) \subseteq B_r(p)$. Therefore, since $B_r(p) \cap E = \emptyset$, we know $B_t(q) \cap E = \emptyset$. This tells us that $q \notin E'$. Hence, $B_r(p) \cap E' = \emptyset$.

We've now shown that $B_r(p) \cap E = \emptyset$ and that $B_r(p) \cap E' = \emptyset$.
Therefore, $B_r(p) \cap (E \cup E') = B_r(p) \cap \overline{E} = \emptyset$, meaning that $B_r(p) \subseteq (\overline{E})^\mathsf{C}$.
So $(\overline{E})^\mathsf{C}$ is open, meaning that $\overline{E}$ is closed.

Proposition 19.2: $E = \overline{E}$ if and only if E is closed.

Proof:

($\Longrightarrow$) If $\overline{E}$ is closed by proposition 19.1. So $E = \overline{E}$ implies E is closed.

($\Longleftarrow$) If E is closed, then $E' \subseteq E$. Hence, $\overline{E} = E \cup E' = E$

Proposition 19.3: If F is closed and $F \supseteq E$, then $F \supseteq \overline{E}$.

Proof:

Observe that if F is any set and $E \subseteq F$, then $E' \subseteq F'$. Thus, if F is also closed, we have that $E' \subseteq F' \subseteq F$. Therefore, $F = F \cup F' \supseteq E \cup E' = \overline{E}$.

Note that in this class, unless it is mentioned otherwise, you should assume that we are equipping $\mathbb{R}$ or $\mathbb{R}^k$ with the Euclidean metric: d_2.

Proposition 20: If $E \subseteq \mathbb{R}$ is nonempty and bounded above, then $\sup E \in \overline{E}$.

Proof:

Set $y = \sup E$. If $y \in E$, then we are done. So assume $y \notin E$.

Consider any $r > 0$. Since $y - r < y = \sup E$, we know $y - r$ is not an upperbound to E. Hence, there is $e \in E$ with $y - r < e < y$. Therefore, $(B_r(y) \setminus \{y\}) \cap E \neq \emptyset$. We therefore conclude $y \in E' \subseteq \overline{E}$.

Note that if X is a metric space with metric d and $Y \subseteq X$, then Y is also a metric space with d when d is restricted to Y.

$E \subseteq Y \subseteq X$ is open/closed/etc. relative to Y if E is open/closed/etc. in the metric space Y.

If $Y \subseteq X$ and $B_r(p)$ denotes the ball of radius r around $p \in Y$ in the metric space X, then the ball of radius r around p in the metric space Y is $B_r(p) \cap Y$.

Proposition 21: Let $E \subseteq Y \subseteq X$. Then E is open relative to Y if and only if there is an open set $U \subseteq X$ with $E = U \cap Y$.

Proof:

($\Longrightarrow$) For each $p \in E$, pick $r(p) > 0$ so that $B_{r(p)}(p) \cap Y \subseteq E$. Then, setting $U = \bigcup_{p \in E} B_{r(p)}(p)$, we have that U is open and that

$$E = \bigcup_{p \in E} \{p\} \subseteq \bigcup_{p \in E} B_{r(p)}(p) \cap Y = U \cap Y \subseteq E$$

So $U \cap Y = E$.

($\Longleftarrow$) Now say that $E = U \cap Y$ where $U \subseteq X$ is open. Also let $p \in E$. We know $p \in U$. Additionally, since U is open, there is $r > 0$ with $B_r(p) \subseteq U$. Consequently, $B_r(p) \cap Y \subseteq U \cap Y = E$. So, p is an interior point of E relative to Y. We conclude that E is open relative to Y.

Let X be a metric space. An open cover of $E \subseteq X$ is a collection $\{u_\alpha \mid \alpha \in A\}$ of open sets u_α satisfying:

$$E \subseteq \bigcup_{\alpha \in A} u_\alpha$$

$K \subseteq X$ is compact if every open cover of K containsa finite subcover of K.

More precisely: K is compact if and only if for ever open cover $\{u_\alpha \mid \alpha \in A\}$ of K, there is $n \in \mathbb{Z}^+$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ such that:

$$K \subseteq \bigcup_{i=1}^{n} u_{\alpha_i}$$

As an aside, compactness often acts as a generalization of finiteness in topology.

Lecture 11: 2/5/2024

Finite sets are compact.

Proposition 22: compactness is an intrinisic property, meaning if $K \subseteq Y \subseteq X$, then K is compact relative to X if and only if K is compact relative to Y.

Proof:

($\Longrightarrow$) Consider any collection of sets $v_\alpha \subseteq Y$ that are open relative to Y and satisfy that $K \subseteq \bigcup_{\alpha \in A} v_\alpha$.

By a previous theorem, we know there are sets w_α open relative to X such that $v_\alpha = w_\alpha \cap Y$. So we have that $K \subseteq \bigcup_{\alpha \in A} v_\alpha \subseteq \bigcup_{\alpha \in A} w_\alpha$.

If K is compact relative to X, then there exists $n \in \mathbb{Z}^+$ and $\alpha_1, \ldots, \alpha_n \in A$ such that $K \subseteq \bigcup_{i=1}^{n} w_{\alpha_i}$. And since $K \subseteq Y$, we have that:

$$K = K \cap Y \subseteq \left(\bigcup_{i=1}^{n} w_{\alpha_i} \right) \cap Y = \left(\bigcup_{i=1}^{n} v_{\alpha_i} \right)$$

Hence, K is compact relative to Y.

($\Longleftarrow$) Now consider any set K which is compact relative to Y and open cover $\{w_\alpha \mid \alpha \in A\}$ such that $w_\alpha \subseteq X$ and $K \subseteq \bigcup\limits_{\alpha \in A} w_\alpha$.

By proposition 21, we know that $v_\alpha = w_\alpha \cap Y$ is open relative to Y. So as $K \subseteq Y$, we have that $K = K \cap Y \subseteq \bigcup\limits_{\alpha \in A} w_\alpha \cap Y = \bigcup\limits_{\alpha \in A} v_\alpha$.

But that means that $\{v_\alpha \mid \alpha \in A\}$ forms an open cover of K relative to Y. So, there exists $n \in \mathbb{Z}^+$ and $\alpha_1, \ldots, \alpha_n \in A$ such that $\{v_{\alpha_1}, \ldots, v_{\alpha_n}\}$ is a finite cover of K. Then note that:

$$K \subseteq \bigcup_{i=1}^{n} v_{\alpha_i} \subseteq \bigcup_{i=1}^{n} w_{\alpha_i}$$

So, $\{w_{\alpha_1}, \ldots, w_{\alpha_n}\}$ forms a finite subcover of K using sets in our original arbitrary open cover. Therefore, we conclude that K is compact relative to X.

Proposition 23: Compact sets are closed.

Proof:
Let $K \subseteq X$ be compact. It then suffices to show that K^{C} is open. So, consider any $p \in K^{\mathsf{C}}$. We know that $\{B_{\frac{1}{3}d(p,q)}(q) \mid q \in K\}$ forms an open cover of K. Additionally, because K is compact, there exists $n \in \mathbb{Z}^+$ and $q_1, \ldots, q_n \in K$ such that:

$$K \subseteq \bigcup_{i=1}^{n} B_{\frac{1}{3}d(p,q_i)}(q_i)$$

Thus, let $r = \min\{d(p, q_i) \mid 1 \leq i \leq n\}$. That way, $\frac{1}{3}r > 0$ and

$$\left(\bigcup_{i=1}^{n} B_{\frac{1}{3}d(p,q_i)}(q_i)\right) \cap B_{\frac{1}{3}r}(p) = \emptyset.$$

This then means that $K \cap B_{\frac{1}{3}r}(p) = \emptyset$, meaning that $B_{\frac{1}{3}r}(p) \subseteq K^{\mathsf{C}}$. So p is an interior point of K^{C}. We thus conclude that K^{C} is open.

Proposition 24: K is compact and $F \subseteq K$ is closed implies that F is compact.

Proof:
Consider any open cover $\{v_\alpha \mid \alpha \in A\}$ of F. Since F is closed, F^{C} is open. So, we can say that $\{F^{\mathsf{C}}\} \cup \{v_\alpha \mid \alpha \in A\}$ is an open cover of K as:

$$\left(\bigcup_{\alpha \in A} v_\alpha\right) \cup F^{\mathsf{C}} \supseteq F \cup F^{\mathsf{C}} \supseteq K$$

Since K is compact, there exists $n \in \mathbb{Z}^+$ and $\alpha_1, \ldots, \alpha_n \in A$ such that:

$$K \subseteq \left(\bigcup_{i=1}^{n} v_{\alpha_i} \right) \cup F^{\mathsf{C}}$$

F^{C} may or may not be needed to cover K. However, its inclusion doesn't effect the finiteness of the cover.

Therefore $F \subseteq \bigcup_{i=1}^{n} v_{\alpha_i}$. So, F is compact.

Corollary: K is compact and F is closed implies that $K \cap F$ is compact.

Proof: K being compact means that K is closed. Thus $K \cap F$ is closed. And as $K \cap F$ is a subset of K, by the above theorem we have that $K \cap F$ is compact.

<u>Theorem (the Finite Intersection Property)</u>: If $\{K_\alpha \mid \alpha \in A\}$ is any collection of compact sets in X having the property that the intersection of any finitely many of the K_α's is nonempty, then:

$$\bigcap_{\alpha \in A} K_\alpha \neq \emptyset$$

Proof: (we shall proceed by proving the contrapositive...)

Assume that $\bigcap_{\alpha \in A} K_\alpha = \emptyset$. Thus, taking complements gives: $\bigcup_{\alpha \in A} (X \setminus K_\alpha) = X$.

Pick any $\alpha_0 \in A$. Then $\{X \setminus K_\alpha \mid \alpha \in A\}$ is an open cover of K_{α_0} because $K_{\alpha_0} \subseteq X$ and because each $X \setminus K_\alpha$ must be open due to K_α being closed.

As K_{α_0} is compact, there exists $n \in \mathbb{Z}^+$ and $\alpha_1, \ldots, \alpha_n \in A$ such that:

$$K_{\alpha_0} \subseteq \bigcup_{i=1}^{n} (X \setminus K_{\alpha_i})$$

Taking complements again, we get that: $X \setminus K_{\alpha_0} \supseteq \bigcap_{i=1}^{n} (K_{\alpha_i})$. So:

$$\left(\bigcap_{i=1}^{n} (K_{\alpha_i}) \right) \cap K_{\alpha_0} = \bigcap_{i=0}^{n} (K_{\alpha_i}) = \emptyset$$