Math 200a (lecture 6)

Sylow's 2nd. Theorem: Suppose $P_0 \in \mathrm{Syl}_p(G)$ and Q is a p-subgroup of G. Then there exists $x \in G$ such that $Q \subseteq xP_0x^{-1}$.

Proof:
$Q \curvearrowright G/P_0$ by left-translation. Then by the theorem in the middle of *page 271*, we have that $|G/P_0| \equiv |(G/P_0)^Q| \pmod p$. But since $P_0 \in \mathrm{Syl}_p(G)$, we have that $|G/P_0| \not\equiv 0 \pmod p$. Hence, there must exist some $gP_0 \in (G/P_0)^Q$.

In turn, $xgP = gP$ for all $x \in Q$. Or in other words, $g \in gPg^{-1}$ for all $x \in Q$. Hence, $Q \subseteq gPg^{-1}$. ■

Corollary: If $P_1, P_2 \in \mathrm{Syl}_p(G)$ then there exists $g \in G$ such that $gP_1g^{-1} = P_2$.

Proof:
By Sylow's 2nd theorem we know there exists $g \in G$ such that $P_2 \subseteq gP_1g^{-1}$. And since $|P_2| = |P_1|$ we deduce $P_2 = gP_1g^{-1}$. ■

Note the following observations:

- If $\theta \in \mathrm{Aut}(G)$ and $P \in \mathrm{Syl}_p(G)$ then $\theta(P) \in \mathrm{Syl}_p(G)$.
- $G \curvearrowright \mathrm{Syl}_p(G)$ by conjugation and this actions is transitive (by the last corollary).
- A subgroup $H < G$ is called a characteristic subgroup if $\forall\theta \in \mathrm{Aut}(G)$ we have that $\theta(H) = H$. By the last two observations, if $\mathrm{Syl}_p(G) = \{P\}$, then P is a characteristic subgroup of G (which automatically means P is normal since conjugation is an automorphism of G).

Corollary: If $P \lhd G$ and $P \in \mathrm{Syl}_p(G)$, then P is a characteristic subgroup of G.

Proof:
Since $P \lhd G$, $P \in \mathrm{Syl}_p(G)$, and $G \curvearrowright \mathrm{Syl}_p(G)$ transitively via conjugation, we must have that $\mathrm{Syl}_p(G) = \{P\}$. Hence P is a characteristic subgroup of G. ■

Lemma: If $P \in \mathrm{Syl}_p(G)$, then $\mathrm{Syl}_p(N_G(P)) = \{P\}$.

Proof:
We know $|P| = p^{\nu_p(|G|)}$. Also, $P < N_G(P) < G$ means that $|P|$ divides $|N_G(P)|$ and $|N_G(P)|$ divides $|G|$. Thus $\nu_p(|G|) = \nu_p(|N_G(P)|)$ and so $P \in \mathrm{Syl}_p(N_G(P))$. Finally, since $P \lhd N_G(P)$, we know from the last corollary that $\mathrm{Syl}_p(N_G(P)) = \{P\}$. ■

Lemma: If $P_0 \in \mathrm{Syl}_p(G)$ and we consider $P_0 \curvearrowright \mathrm{Syl}_p(G)$ by conjugation, then $(\mathrm{Syl}_p(G))^{P_0} = \{P_0\}$.

Proof:
$P \in (\mathrm{Syl}_p(G))^{P_0}$ if and only if for all $x \in P_0$, $xPx^{-1} = P$. That's to say, iff $P_0 \subseteq N_G(P)$. But that would mean $P_0 \in \mathrm{Syl}_p(N_G(P)) = \{P\}$. So $(\mathrm{Syl}_p(G))^{P_0} = \{P_0\}$. ■

Sylow's 3rd. Theorem: If G is a finite group, $|\text{Syl}_p(G)| \equiv 1 \pmod{p}$.

Proof:

Suppose $P_0 \in \text{Syl}_p(G)$. Then $|\text{Syl}_p(G) \equiv |(\text{Syl}_p(G))^{P_0} \pmod{p}$. But from the prior lemma we know $|(\text{Syl}_p(G))^{P_0}| = 1$. ■

So as a recap, suppose G is a finite group and p is a prime number dividing $|G|$. Then:

- Sylow's first theorem guarentees that $\text{Syl}_p(G) \neq \emptyset$.
- Sylow's third theorem guarentees that $|\text{Syl}_p(G)| \equiv 1 \pmod{p}$.
- Sylow's second theorem guarentees that $|\text{Syl}_p(G)|$ equals the number of conjugates of P_0 where $P_0 \in \text{Syl}_p(G)$. Thus (see *page 271*), we have for any $P_0 \in \text{Syl}_p(G)$ that $|\text{Syl}_p(G)| = [G : N_G(P_0)]$. And in particular, since $P_0 < N_G(P_0) < G$, we have that $|\text{Syl}_p(G)| = \frac{|G|}{|N_G(P_0)|} = \frac{[G:P_0]|P_0|}{[N_G(P_0):P_0]|P_0|} = \frac{[G:P_0]}{[N_G(P_0):P_0]}$. So $|\text{Syl}_p(G)|$ divides $[G : P_0]$.

Proposition: $P \in \text{Syl}_p(G) \implies N_G(N_G(P)) = N_G(P)$.

Proof:

The $\supseteq$ inclusion is obvious. Meanwhile, $x \in N_G(N_G(P))$ implies that $xN_G(P)x^{-1} = N_G(P)$. But note that if $\theta \in \text{Aut}(G)$ and $H < G$, then $\theta(N_G(H)) = N_G(\theta(H))$.

If $x \in N_G(H)$ then we know that $xHx^{-1} = H$. So:

$$\phi(x)\phi(H)\phi(x)^{-1} = \phi(xHx^{-1}) = \phi(H).$$

This shows that $\phi(x) \in N_G(\phi(H))$ and hence $\phi(N_G(H)) \subseteq N_G(\phi(H))$ whenever $\phi \in \text{Aut}(G)$. Using this fact, now note that for any $\phi \in \text{Aut}(G)$, we have that:

$$N_G(H) = \phi^{-1}(\phi(N_G(H))) \subseteq \phi^{-1}(N_G(\phi(H))) \subseteq N_G(\phi^{-1}(\phi(H))) = N_G(H)$$

So, $N_G(H) = \phi^{-1}(N_G(\phi(H)))$. And by composing ϕ we get that:

$$\phi(N_G(H)) = N_G(\phi(H)).$$

It follows that $N_G(xPx^{-1}) = xN_G(P)x^{-1} = N_G(P)$ whenever $x \in N_G(N_G(P))$. But in that case we have that $\text{Syl}_p(N_G(xPx^{-1})) = \text{Syl}_p(N_G(P))$. And as P and xPx^{-1} are both Sylow p-groups, we conclude $xPx^{-1} = P$. So $x \in N_G(P)$

I probably should have been taught this in math 100a but never was. So, I guess I'll just refresh myself now. The book I'm following along with is *Abstract Algebra* by Dummit and Foote.

Suppose G is a group and H, K are subgroups of G. Then we define:

$$HK := \{hk \in G : h \in H \text{ and } k \in K\}.$$

Proposition 3.2.13: If H and K are finite subgroups of a group, then $|HK| = \frac{|H||K|}{|H \cap K|}$.

Proof:

Note that $HK = \bigcup_{h \in H} hK$. Thus $|HK|$ equals $|K|$ times the number of distinct left cosets hK where $h \in H$. But note that for any $h_1, h_2 \in H$:

$$h_1K = h_2K \iff h_2^{-1}h_1 \in H \cap K \iff h_1(H \cap K) = h_2(H \cap K).$$

Hence $|HK| = |K| \cdot [H : H \cap K] = |K|\frac{|H|}{|H \cap K|}$ by Lagrange's theorem. ■

Proposition 3.2.14: If H and K are subgroups of G, then $HK < G$ iff $HK = KH$.

$(\Longrightarrow)$
Suppose $HK < G$. Since $K < HK$ and $H < HK$, we thus know that $KH \subseteq HK$. Meanwhile, suppose $h \in H$ and $k \in K$. Since HK is a group, we know $(hk)^{-1} \in HK$. So there exists $h' \in H$ and $k' \in K$ such that $(hk)^{-1} = h'k'$. But then $hk = (k')^{-1}(h')^{-1}$ which is in KH. So $HK \subseteq KH$.

$(\Longleftarrow)$
Assume $HK = KH$ and let $a, b \in HK$. Then there exists $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that $a = h_1k_1$ and $b = h_2k_2$. Now it's clear that $1_G \in HK$. So, if we can show that $ab^{-1} \in HK$, then we will know that HK is a group.

Fortunately, $ab^{-1} = h_1k_1k_2^{-1}h_2^{-1}$. And since $KH = HK$, we know there is $h_3 \in H$ and $k_3 \in K$ such that $(k_1k_2^{-1})h_2^{-1} = h_3k_3$. Thus, $ab^{-1} = (h_1h_3)(k_3) \in HK$. ■

Corollary 3.2.15: If H and K are subgroups of G and $H < N_G(K)$, then HK is a subgroup of G. In particular, if $K \lhd G$ then $HK < G$ for any $H < G$.

Proof:
Let $h \in H$ and $k \in K$. Then $hkh^{-1} \in K$. So $hk = (hkh^{-1})h \in KH$ and we've proven that $HK = KH$. ■

Second Isomorphism Theorem: Let G be a group, let A and B be subgroups of G, and assume $A < N_G(B)$. Then $AB < G$, $B \lhd AB$, $A \cap B \lhd A$, and $AB/B \cong A/(A \cap B)$.

Proof:
By the last corollary we know that $AB < G$. Also, since $A < N_G(B)$ and $B < N_G(B)$, it follows $AB < N_G(B)$. Hence $B \lhd AB$.

Now we know there is a well-defined group homomorphism $\phi : A \to AB/B$ given by $\phi(a) = aB$. Clearly ϕ is surjective. Meanwhile, it's easy to see that $\ker(\phi) = A \cap B$. So by the first isomorphism theorem, we have that $A \cap B \lhd A$ and that:
$$AB/B \cong A/(A \cap B).\ ■$$

Here is one more miscellaneous result before getting back to the lecture:

Lemma: If $N_1, N_2 \lhd G$, then $\forall x \in N_1$ and $\forall y \in N_2$ we have that $xyx^{-1}y^{-1} \in N_1 \cap N_2$.

Proof:
$(xyx^{-1}) \in N_2$ and $(yx^{-1}y^{-1}) \in N_1$ since both N_1 and N_2 are normal. Hence:
$$(xyx^{-1})y^{-1} = x(yx^{-1}y^{-1}) \in N_1 \cap N_2.\ ■$$

Corollary: If $N_1, N_2 \lhd G$ and $N_1 \cap N_2 = \{1\}$, then $xy = yx$ for all $x \in N_1$ and $y \in N_2$.

So here are some uses of Sylow's theorems:

- Suppose $p < q$ are distinct primes with $p \nmid q - 1$. If $|G| = pq$ then $G \cong C_{pq}$.

  Let s_q and s_p be shorthand for $|\mathrm{Syl}_q(G)|$ and $|\mathrm{Syl}_p(G)|$. Now we know by Sylow's third theorem that $s_q \equiv 1 \pmod q$.

Also, we know that $s_q \mid p$ by Sylow's second theorem. And since $p < q$, we must have that $s_q = 1$. Hence $\mathrm{Syl}_q(G) = \{Q\}$ for some $Q \lhd G$ such that $|Q| = q$ and Q is cylic of order q.

Next, note once again by Sylow's second theorem that $s_p \mid q$. Hence, we must have that either $s_p = 1$ or $s_p = q$. That said, we know $q - 1 \not\equiv 0 \pmod{p}$ by assumption and that $s_p \equiv 1 \pmod{p}$ by Sylow's third theorem. So, we must have that $s_p = 1$ and it follows that $\mathrm{Syl}_p(G) = \{P\}$ for some $P \lhd G$ such that $|P| = p$ and P is cyclic of order p.

Now $|P \cap Q| \mid \gcd(|P|, |Q|) = 1$. So $P \cap Q = \{1\}$. And by our prior corollary, this means that $xy = yx$ for all $x \in P$ and $y \in Q$.

Now consider the map $f : P \times Q \to G$ given by $(x, y) \mapsto xy$. We claim this is a group isomorphism.

- Note that:
$$\begin{aligned} f(x_1, y_1)f(x_2, y_2) = x_1y_1x_2y_2 &= x_1x_2y_1y_2 \\ &= f(x_1x_2, y_1y_2) = f((x_1, y_1)(x_2, y_2)). \end{aligned}$$

  Thus f is a group homomorphism.

- Suppose $f(x, y) = 1$. Then $xy = 1$ which means that $x = y^{-1}$. But now $x, y^{-1} \in P \cap Q = \{1\}$. So $(x, y) = (1, 1)$ and we've shown that f is injective.

- $|\mathrm{im}(f)| = |PQ| = \frac{|P||Q|}{|P \cap Q|} = \frac{pq}{1} = |G|$. So f is surjective.

It follows that $G \cong P \times Q \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}/pq\mathbb{Z}$. (The last equivalence is from Chinese remainder theorem...)

I am not fully caught up with this class yet. But, I'll stop here for now so that I can go back to taking functional analysis notes. For more math 200a notes go to *page* ___.

10/14/2025

Math 241a (lectures 3-5 continued):

If $\mathcal{X}$ is a topological vector space, then here is one more topology on $\mathcal{X}^*$ to be aware of.

> Let $\mathcal{A}$ be the collection of all (Von-Neumann) bounded sets in $\mathcal{X}$ and then for each $A \in \mathcal{A}$ define the seminorm $p_A(\lambda) = \sup_{x \in A} |\lambda(x)|$ on $\mathcal{X}^*$. Since every singleton is bounded, we know this defines a sufficient family. Also, the topology generated by that family is finer than the weak* topology. So, we call it the strong topology on $\mathcal{X}^*$.

(Definition 1.2.19:) If $\mathcal{X}$, $\mathcal{Y}$ are topological (K-)vector spaces and $T \in B(\mathcal{X}, \mathcal{Y})$, then T's adjoint is defined as the map $T^* : \mathcal{Y}^* \to \mathcal{X}^*$ given by $T^*(\lambda) = \lambda \circ T$.

Note that:

- T^* is a well-defined linear operator.

To show that T^* is well defined, suppose $c_1, c_2 \in K$ and $x_1, x_2 \in \mathcal{X}$. Then:

$$T^*(\lambda)(c_1x_1 + c_2x_2) = \lambda(T(c_1x_1 + c_2x_2)) \\ = c_1\lambda(T(x_1)) + c_2\lambda(T(x_2)) = c_1T^*(\lambda)(x_1) + c_2T^*(\lambda)(x_2).$$

Next, to show that T^* is linear, suppose $c_1, c_2 \in K$ and $\lambda_1, \lambda_2 \in \mathcal{Y}^*$. Then for any $x \in \mathcal{X}$ we have that:

$$T^*(c_1\lambda_1 + c_2\lambda_2)(x) = (c_1\lambda_1 + c_2\lambda_2)(T(x)) \\ = c_1\lambda_1(T(x)) + c_2\lambda_2(T(x)) = c_1T^*(\lambda_1)(x) + c_2T^*(\lambda_2)(x) \\ = (c_1T^*(\lambda_1) + c_2T^*(\lambda_2))(x)$$

- T^* is continuous if $\mathcal{X}^*$ and $\mathcal{Y}^*$ are equipped with the weak* topologies.

This is because if $\lambda_\beta(y) \to \lambda(y)$ for all $y \in \mathcal{Y}$ then $\lambda_\beta(Tx) \to \lambda(\beta)(Tx)$ for all $x \in \mathcal{X}$. Hence, we have for any weak*-ly convergent net $\langle \lambda_\beta \rangle$ that $\langle T^*(\lambda_\beta) \rangle$ is also weak*-ly convergent.

- T^* is also continuous if $\mathcal{X}^*$ and $\mathcal{Y}^*$ are equipped with the strong topologies.

This is because if T is continuous, then T maps bounded sets to bounded sets.

Proof:
Suppose $A \subseteq \mathcal{X}$ is bounded and let N be any neighborhood of $0 \in \mathcal{Y}$. Because T is continuous, we know that $T^{-1}(N)$ is a neighborhood of $0 \in \mathcal{X}$. And since A is bounded, there is some $r > 0$ such that $A \subseteq sT^{-1}(N)$ for all $s \in K$ with $|s| \geq r$. In turn, $T(A) \subseteq T(sT^{-1}(N)) = sT(T6{-}1(N)) = sN$ whenever $|s| \geq r$. And this proves that $T(A) \subseteq \mathcal{Y}$ is bounded.

Thus by similar logic to the last bullet point, if $\langle \lambda_\beta \rangle$ is a strongly convergent net then $\langle T^*(\lambda_\beta) \rangle$ is also a strongly convergent net.

As a side note, technically only the third bullet point actually required the continuity of T.

<u>Lemma 1.2.21:</u> If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in B(\mathcal{X}, \mathcal{Y})$, then $\|T^*\|_{\text{op}} = \|T\|_{\text{op}}$.

Proof:
For all $x \in \mathcal{X}$ and $\lambda \in \mathcal{Y}^*$, we have that $|T^*(\lambda)(x)| = |\lambda(Tx)| \leq \|\lambda\|\|T\|\|x\|$. So $\|T^*(\lambda)\| \leq \|\lambda\|\|T\|$ for all $\lambda \in \mathcal{Y}^*$. And this shows that $\|T^*\| \leq \|T\|$.

On the other hand, for all $\varepsilon > 0$ there exists $x \in E$ such that $\|x\| = 1$ and $\|Tx\| \geq \|T\| - \varepsilon$. Also, as a consequence of the Hahn Banach theorem (see Folland theorem 5.8 in my math 240b notes), there exists $\lambda \in \mathcal{Y}^*$ such that $\lambda(Tx) = \|Tx\|$ and $\|\lambda\| = 1$. So:

$$\|T^*\| \geq \|\lambda\|^{-1}\|T^*(\lambda)\| \\ = 1 \cdot \|T^*(\lambda)\| \geq \|x\|^{-1}|T^*(\lambda)(x)| \\ = 1 \cdot |T^*(\lambda)(x)| = |\lambda(Tx)| = \|Tx\| \geq \|T\| - \varepsilon. \blacksquare$$

Let $\mathcal{H}$ be a real of complex Hilbert space. Then recall that there is an isometric bijection $i : \mathcal{H} \to \mathcal{H}^*$ where we identify every $x \in \mathcal{H}$ with the linear functional $i(x) := \langle \cdot, x \rangle$. Therefore, when working on Hilbert spaces it's often convenient to just identify $\mathcal{H} \cong \mathcal{H}^*$.

As an example of this, consider any $T \in B(\mathcal{H})$ and define $T' = i^{-1} \circ T^* \circ i$. Then $T' \in B(\mathcal{H})$ as well. Also, since $i \circ T' = T^* \circ i$, we have that:

$$\langle Tx, y \rangle = (i(y))(Tx) = (T^*(i(y)))(x) = (i(T'(y)))(x) = \langle x, T'y \rangle$$

Now by a typical abuse of notation, we just say $T' \cong T^*$.

Note: if $\{e_i\}_{i \in I}$ is an orthonormal basis for $\mathcal{H}$, then:

$$T^*_{i,j} := \langle T^* e_j, e_i \rangle = \overline{\langle e_i, T^* e_j \rangle} = \overline{\langle T e_j, e_i \rangle} =: \overline{T_{j,i}}$$

So, if we "expressed T^* and T as matrices", then T^* would be the conjugate transpose of T.

We say $T \in B(\mathcal{H})$ is self-adjoint if $T^* = T$.

We say $U \in B(\mathcal{H})$ is unitary if U is an isometric isomorphism. Also, we often denote $\mathrm{Iso}(\mathcal{H})$ as $U(\mathcal{H})$ when working on Hilbert spaces.

Proposition: $U \in B(\mathcal{H})$ is unitary if and only if U is surjective and $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all x, y.

($\Longleftarrow$)
We know that U is an isometry since $\|Ux\| = \langle Ux, Ux \rangle = \langle x, x \rangle = \|x\|$ for all $x \in \mathcal{H}$. This also proves that U is injective and continuous. And when we then consider that U is also surjective, we know by the open map theorem that U^{-1} is continuous. Hence $U \in U(\mathcal{H})$.

($\Longrightarrow$)
Since U is an isomorphism, we automatically know U is surjective. Meanwhile, to see that U preserves inner products, note that:

$$\langle x, y \rangle = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$$

To prove this, note that:

- $$\frac{\|x+y\|^2 - \|x-y\|^2}{4} = \frac{\|x\|^2 + \langle x,y \rangle + \langle y,x \rangle + \|y\|^2 - \|x\|^2 + \langle x,y \rangle + \langle y,x \rangle - \|y\|^2}{4}$$
  $$= \frac{2\langle x,y \rangle + 2\langle y,x \rangle}{4} = \frac{\langle x,y \rangle + \langle y,x \rangle}{2} = \mathrm{Re}(\langle x,y \rangle)$$
- $$\frac{\|x+iy\|^2 - \|x-iy\|^2}{4} = \frac{\|x\|^2 + \langle x,iy \rangle + \langle iy,x \rangle + \|iy\|^2 - \|x\|^2 + \langle x,iy \rangle + \langle iy,x \rangle - \|iy\|^2}{4}$$
  $$= \frac{-2i\langle x,y \rangle + 2i\langle y,x \rangle}{4} = -i\frac{\langle x,y \rangle - \langle y,x \rangle}{2} = \mathrm{Im}(\langle x,y \rangle)$$

And since $\langle x, y \rangle = \mathrm{Re}(\langle x, y \rangle) + \mathrm{Im}(\langle x, y \rangle)i$, our claimed identity falls out. And now it is clear that by preserving norms U also preserves inner products. ■

Proposition: $U \in B(\mathcal{H})$ is unitary iff $U^{-1} = U^*$.

$(\Longrightarrow)$

Suppose U is unitary. Then for all $x, y \in \mathcal{H}$ we have that:

- $(U^*Ux)(y) = \langle y, U^*Ux\rangle = \langle Uy, Ux\rangle = \langle y, x\rangle = x(y)$,
- $(UU^*x)(y) = \langle y, UU^*x\rangle = \langle U^{-1}y, U^*x\rangle = \langle UU^{-1}y, x\rangle = \langle y, x\rangle = x(y)$

Thus $U^*U(x) = x = UU^*(x)$ for all $x \in \mathcal{H}$. And this proves that $U^{-1} = U^*$.

$(\Longleftarrow)$

Since U has an inverse, we automatically know that U is surjective. Also note that for any $x, y \in \mathcal{H}$, since $U^*Uy = y$, we have that $\langle x, y\rangle = \langle x, U^*Uy\rangle = \langle Ux, Uy\rangle$. ■

Suppose X is a measure space and let $\mathcal{H} = L^2(X)$. Then recalling *example 1.2.1* on page 284, let $\varphi \in L^\infty(X)$ and consider the linear operator $M_\varphi \in B(L^2(X))$. Then note for all $f, g \in \mathcal{H}$ that:

$$\langle g, M_\varphi^* f\rangle = \langle M_\varphi g, f\rangle = \int M_\varphi g\overline{f} = \int \varphi g\overline{f} = \int g\overline{\overline{\varphi} f} = \langle g, \overline{\varphi} f\rangle = \langle g, M_{\overline{\varphi}} f\rangle$$

This implies that $M_\varphi^* = M_{\overline{\varphi}}$. So, we are able to say that M_φ is self-adjoint iff φ is real a.e. and M_φ is unitary iff $|\varphi| = 1$ a.e.

(Definition 1.3.1:) Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces. Then we define the following topologies on $B(\mathcal{X}, \mathcal{Y})$.

(a) The norm topology on $B(\mathcal{X}, \mathcal{Y})$ is the topology defined by the operator norm on $B(E, F)$.

(b) For all $x \in \mathcal{X}$ define the seminorm $p_x(T) := \|Tx\|$ for all $T \in B(\mathcal{X}, \mathcal{Y})$. This is a well-defined seminorm and the family of these seminorms is sufficient on $B(\mathcal{X}, \mathcal{Y})$. The topology generated by that family is the strong operator topology.

(Note that strong operator convergence on $B(\mathcal{X}, \mathcal{Y})$ is equivalent to pointwise convergence on $\mathcal{X}$…)

(c) For all $x \in \mathcal{X}$ and $\lambda \in \mathcal{Y}^*$ let $p_{x,\lambda}(T) := |\lambda(Tx)|$. By the Hahn-Banach theorem, this defines a sufficient family of seminorms. And the topology generated by that family is the weak operator topology.

(Note that weak operator convergence on $B(\mathcal{X}, \mathcal{Y})$ is equivalent to weak pointwise convergence on $\mathcal{X}$…)

Note: if $\mathcal{Y} = K$, then both the strong operator topology and the weak operator topology are just the weak* topology on $\mathcal{X}^*$.

(Example 1.3.2:) Suppose $\mathcal{H}$ is a Hilbert space with an orthonormal basis $\{e_i\}_{i\in I}$, and let $T, T_n \in B(\mathcal{H})$ for all $n \in \mathbb{N}$ with $\|T_n\|, \|T\| \leq 1$.

- If $T_n \to T$ in operator norm, then $T_n e_i \to Te_i$ uniformly over the $i \in I$.

  This is just because for all $i \in I$ we have that $\|T_n e_i - Te_i\| \leq \|T_n - T\|$ for all n.

- $T_n \to T$ operator strongly if and only if $T_n e_i \to Te_i$ for all $i \in I$.

  The $(\Longrightarrow)$ direction is obvious. To prove the converse, we need to show that if $T_n e_i \to Te_i$ for all $i \in I$, then $T_n x \to Tx$ for all $x \in \mathcal{H}$. Fortunately, note that there is some countable collection $\{i_k\}_{k\in\mathbb{N}}$ such that $x = \sum_{k\in\mathbb{N}}\langle x, e_{i_k}\rangle e_{i_k}$ and the latter sum converges absolutely.

  Since T and each T_n are continuous, we have that:
  $$T(x) = T(\textstyle\sum_{k\in\mathbb{N}}\langle x, e_{i_k}\rangle e_{i_k}) = \sum_{k\in\mathbb{N}}\langle x, e_{i_k}\rangle T(e_{i_k})$$

  And similarly we have $T_n(x) = \sum_{k\in\mathbb{N}}\langle x, e_{i_k}\rangle T_n(e_{i_k})$.

  Now, you can use somewhat standard analysis arguments to show $T_n x \to Tx$. I'm gonna skip doing that...

- $T_n \to T$ operator weakly if and only if $\langle T_n e_i, e_j\rangle \to \langle Te_i, e_j\rangle$ for all $i, j \in I$.

  Once again the $(\Longrightarrow)$ direction is obvious. As for the other direction, we need to show that for any $x, y \in \mathcal{H}$ we have that $\langle T_n x, y\rangle \to \langle Tx, y\rangle$. If I'm inspired, I'll prove this later on *page* ___. But I'm tired. Goodnight.

10/15/2025

I need to work on math 200a again so I will be taking a break from the math 241a notes. See *page* ___ to skip ahead to more functional analysis notes.

Math 200a (lectures 7-8):

Examples of uses of Sylow's theorems continued:

- Suppose p is prime and $|G| = p(p-1)$. Then there exists $P \lhd G$ such that $|P| = p$.

  By Sylow's theorems, we know that $s_p \mid p-1$ and $s_p \equiv 1 \pmod{p}$. Together, that tells us that $s_p = 1$. Hence, G has a unique Sylow p-subgroup which we'll call P. Also $P \lhd G$ and $|P| = p$.

- Suppose p is prime and $|G| = p(p+1)$. Then G has a normal subgroup of order either p or $p+1$.

  We may assume that $s_p \neq 1$ since otherwise we'd know that G has a unique subgroup of order p which is automatically normal.

  Now by Sylow's theorems, we have that $s_p \mid p+1$ (which means that $s_p \leq p+1$) and that $s_p \equiv 1 \pmod{p}$ (which means that $s_p \in \{1, p+1, 2p+1, \ldots\}$). Since we already assumed $s_p \neq 1$, this means that $s_p = p+1$. Hence, we may say that $\mathrm{Syl}_p(G) = \{P_1, \ldots, P_{p+1}\}$.

Now note that each $P_i \cong C_p$ (i.e. each P_i is cyclic with order p). As a consequence, we have that $P_i \cap P_j = \{1\}$ if $i \neq j$. So, let $X := G - (\bigcup_{i=1}^{p+1} P_i - \{1\})$. Also note that $|X| = p(p+1) - (p+1)(p-1) = p^2 + p - p^2 + 1 = p + 1$.

Note: For every finite group G, $\bigcup_{P \in \mathrm{Syl}_p(G)} P = \{x \in G : o(x) \text{ is a power of } p\}$.

To see why, first note that if $x \in P \in \mathrm{Syl}_p(G)$, then $o(x) \mid |P| = p^k$ for some $k \in \mathbb{N}$. Hence, the $\subseteq$ inclusion is obvious. Meanwhile, the other inclusion is just a direct application of Sylow's second theorem.

Hence, $X = \{x \in G : o(x) \neq p\}$. And from that we also know $\mathrm{Cl}(x) \subseteq X - \{1\}$ for all $x \in X - \{1\}$.

(As a reminder, $\mathrm{Cl}(x) := \{gxg^{-1} : g \in G\}$...)

Now by Sylow's second theorem, $p+1 = s_p = [G : N_G(P_i)]$ for all P_i. But also note that $P_i \subseteq N_G(P_i)$ and $[G : P_i] = p+1$. Hence, $N_G(P_i) = P_i$ for all $P_i \in \mathrm{Syl}_p(G)$. But note that since P_i has prime order, if $y \in P_i - \{1\}$ then $\langle y \rangle = P_i$. Also, note that for any $y \in G$ and positive integer n we have that $C_G(y) \subseteq C_G(y^n)$.

This is because if $gy = yg$, then $gy^2 = ygy = y(yg) = y^2 g$. And continuing by induction, if $gy^n = y^n g$, then $gy^{n+1} = y^n gy = y^n(yg) = y^{n+1}g$.

It follows for any $y \in P_i - \{1\}$ that the elements of $C_G(y)$ must commute with all the elements of P_i. So, $C_G(y) \subseteq N_G(P_i) = P_i$. But also since P_i is abelian (since it's cyclic), we have that $P_i \subseteq C_G(y)$. So, $C_G(y) = P_i$ for all $y \in P_i - \{1\}$.

Now from that we also know $C_G(x) \subseteq X$ for all $x \in X - \{1\}$. After all, if $x, y \in G$ then we have that $x \in C_g(y) \iff y \in C_g(x)$.

This is because $x \in C_G(y) \implies xy = yx \iff x \in C_G(y)$.

But we know that any $x \in X - \{1\}$ isn't in $C_G(y)$ for any $y \in \bigcup_{i=1}^{p+1} P_i$. Hence, $C_G(x) \subseteq X = G - \bigcup_{i=1}^{p+1} P_i$.

Now since for $\mathrm{Cl}(x) \subseteq X - \{1\}$ and $C_G(x) \subseteq X$ for all $x \in X - \{1\}$, we in turn know that $|\mathrm{Cl}(x)| \leq p$ and $|C_G(x)| \leq p+1$ for all $x \in X - \{1\}$.

Now by the orbit stabilizer theorem (when considering the *action $G \curvearrowright G$ by conjugation*), we know $|\mathrm{Cl}(x)| = [G : C_G(x)]$. Also, by Lagrange we have that $|C_G(x)|[G : C_G(x)] = |G| = p(p+1)$. So, $|C_G(x)| \cdot |\mathrm{Cl}(x)| = p(p+1)$. And this implies that $|C_G(x)| = p+1$ and $|\mathrm{Cl}(x)| = p$ for all $x \in X - \{1\}$. Hence, $X = C_G(x)$ and $X - \{1\} = \mathrm{Cl}(x)$ for all $x \in X - \{1\}$.

This proves that X is an abelian normal subgroup of G with order $p+1$. ■

As a side note, the case where $s_p = p+1$ is actually going to be really rare. To see this, note that if p is odd, then $2 \mid p+1$ and hence there exists $x_0 \in X$ such that $o(x_0) = 2$ (by *Cauchy's theorem*). But then since $\mathrm{Cl}(x_0) = X - \{1\}$ and conjugation preserves the order of elements, we must have that $o(x) = 2$ for all $x \in X - \{1\}$. And so, by another application of Cauchy's theorem we know that $|X|$ must have no prime factor other than 2. Or in other words, $|X| = 2^n$ for some $n \in \mathbb{N}$.

This shows that in the prior example, we can only have that $s_p = p+1$ if p is a Mersenne prime (i.e. a prime number such that $p = 2^n - 1$ for some $n \in \mathbb{N}$).

An exact sequence is a commutative diagram:

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \dots \xrightarrow{f_k} G_{k+1}$$

where the nodes of the diagram are groups, the arrows are group homomorphisms, and $\mathrm{im}(f_i) = \ker(f_{i+1})$ for all $i \in \{1, \dots, k-1\}$.

If the first and last groups in an exact sequence are trivial, then we call that exact sequence a short exact sequence (or S.E.S.).

If G is a group and $N \lhd G$, then the standard S.E.S. is:

$$\{1\} \longrightarrow N \overset{i}{\hookrightarrow} G \overset{\pi}{\twoheadrightarrow} G/N \longrightarrow \{1\}$$

where i is the inclusion map and π is the projection map $x \mapsto xN$.

Note: $\hookrightarrow$ denotes an injective (i.e. monomorphic) homormorphism and $\twoheadrightarrow$ denotes a surjective (i.e. epimorphic) homomorphism.

Given two S.E.Ss (which for now I'll just take to have length 5), we say a homomorphism between those S.E.Ss is a ordered collection $(\theta_1, \theta_2, \theta_3)$ of group homomorphisms $\theta_i : G_i \to G_i'$ such that the diagram below commutes:

$$\begin{tikzcd}
\{1\} \arrow[r] & G_1 \arrow[r, "f_1"] \arrow[d, "\theta_1"'] & G_2 \arrow[r, "f_2"] \arrow[d, "\theta_2"'] & G_3 \arrow[r] \arrow[d, "\theta_3"'] & \{1\} \\
\{1\} \arrow[r] & G_1' \arrow[r, "f_1'"'] & G_2' \arrow[r, "f_2'"'] & G_3' \arrow[r] & \{1\}
\end{tikzcd}$$

Short Five Lemma: Suppose $(\theta_1, \theta_2, \theta_3)$ is an S.E.S. homomorphism from one S.E.S to another as shown in the above commutative diagram.

(a) If θ_1, θ_3 are injective, then so is θ_2.

Proof:
It suffices to show that $\ker(\theta_2)$ is trivial. So suppose $x_2 \in \ker(\theta_2)$. Then $\theta_3(f_2(x_2)) = f_2'(\theta_2(x_2)) = 1$. And since θ_3 is injective, this implies that $f_2(x_2) = 1$. Hence, $x_2 \in \ker(f_2) = \mathrm{im}(f_1)$.

Now pick $x_1 \in G_1$ such that $x_2 = f_1(x_1)$. Notably, $f_1'(\theta_1(x)) = \theta_2(f_1(x_1)) = 1$. So, $\theta_1(x) \in \ker(f_1')$. And since $\ker(f_1') = \operatorname{im}(\{1\} \to G_1')$ is trivial, this means that $\theta_1(x_1) = 1$. In turn, since θ_1 is injective, $x_1 = 1$. So, $x_2 = f_1(x_1) = f_1(1) = 1$.

(b) If θ_1, θ_3 are surjective, then so is θ_2.

Proof:
Let $x_2' \in G_2'$. Then since θ_3 is surjective, there exists $x_3 \in G_3$ such that $\theta_3(x_3) = f_2'(x_2')$. Also, since $\operatorname{im}(f_2) = \ker(G_3 \to \{1\}) = G_3$, we know there exists $x_2 \in G_2$ such that $f(x_2) = x_3$. And now:

$$f_2'(\theta_2(x_2)) = \theta_3(f_2(x_2)) = \theta_3(x_3) = f_2'(x_2')$$

We thus know that $\theta_2(x_2^{-1})x_2' \in \ker(f_2') = \operatorname{im}(f_1')$. Hence, there exists $x_1' \in G_1'$ such that $f_1'(x_1') = \theta_2(x_2^{-1})x_2'$. Also, since θ_1 is surjective, we know there exists $x_1 \in G_1$ such that $\theta_1(x_1) = x_1'$. And now:

$$\theta_2(f_1(x_1)) = f_1'(\theta_1(x_1)) = f_1'(x_1') = \theta_2(x_2^{-1})x_2'.$$

So $x_2' = \theta_2(x_2)\theta_2(f_1(x_1)) = \theta_2(x_2 f_1(x_1))$. ■

Note that every length five S.E.S. is isomorphic to a standard S.E.S.

Note that $\ker(f_1) = \operatorname{im}(\{1\} \to G_1) = \{1\}$ and so f_1 is injective. It follows that $G_1 \cong \operatorname{im}(f_1) = \ker(f_2)$ by the map $x \mapsto f_1(x)$. So, just define $\bar{f}_1$ to be f_1 with it's codomain restricted.

Meanwhile, note that $\operatorname{im}(f_2) = \ker(G_3 \to \{1\}) = G_3$. So, by the first isomorphism theorem we have that $G_2/\ker(f_2) \cong G_3$ via the map $x\ker(f_2) \mapsto f_2(x)$. We'll call this map $\bar{f}_2$.

Now our claim is that the following diagrams commute:

$$\begin{tikzcd}
\{1\} \arrow[r] & G_1 \arrow[r, "f_1"] & G_2 \arrow[r, "f_2"] & G_3 \arrow[r] & \{1\} \\
\{1\} \arrow[r] & \ker(f_2) \arrow[u, "\bar{f}_1^{-1}"] \arrow[r, hook, "i"'] & G_2 \arrow[u, "\mathrm{Id}"'] \arrow[r, two heads, "\pi"'] & G_2/\ker(f_2) \arrow[u, "\bar{f}_2"] \arrow[r] & \{1\}
\end{tikzcd}$$

$$\begin{tikzcd}
\{1\} \arrow[r] & G_1 \arrow[r, "f_1"] \arrow[d, "\bar{f}_1"] & G_2 \arrow[r, "f_2"] \arrow[d, "\mathrm{Id}"] & G_3 \arrow[r] \arrow[d, "\bar{f}_2^{-1}"] & \{1\} \\
\{1\} \arrow[r] & \ker(f_2) \arrow[r, hook, "i"'] & G_2 \arrow[r, two heads, "\pi"'] & G_2/\ker(f_2) \arrow[r] & \{1\}
\end{tikzcd}$$

To prove this, it suffices to show that each square commutes (I'll prove this by induction after I'm done with this). Fortunately though, it is easy to see at a glance that each square commutes.

Consider the following diagram:

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & \cdots \arrow[r, "f_{n-1}"] & A_n \arrow[r, "f_n"] \arrow[d, "h_n"] & A_{n+1} \arrow[d, "h_{n+1}"] \\
B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & \cdots \arrow[r, "g_{n-1}"'] & B_n \arrow[r, "g_n"'] & B_{n+1}
\end{tikzcd}$$

To express composing two arrows i and j together (where j starts at where i ends), we write ij. Also, we can identify certain compositions of arrows with each other. For example, we always say $(ij)k = i(jk)$, thus making arrow composition associative. And hence, it is well defined to just write of a composition ijk.

Now we can also identify other arrow compositions with each other. For example, we say a diagram <u>commutes</u> if given any two compositions of arrows $i_1 \cdots i_r$ and $j_1 \cdots j_s$ starting and ending at the same node of our diagram we have that $i_1 \cdots i_r = j_1 \cdots j_s$.

We claim that specifically for the diagram above, the arrows in this diagram commute iff $f_i h_{i+1} = h_i g_i$ for all $i \in \{1, \ldots, n\}$.

Proof:
The $(\Longrightarrow)$ implication is trivial. Meanwhile, to show the other implication we proceed by induction. For our base case, we have that the claim is trivial if $n = 1$. Meanwhile, suppose we've already proven our desired claim for a diagram of the form (where $k \leq n$):

$$\begin{tikzcd}
A'_1 \arrow[r, "f'_1"] \arrow[d, "h_1"] & A'_2 \arrow[r, "f'_2"] \arrow[d, "h_2"] & \cdots \arrow[r, "f'_{k-2}"] & A'_{k-1} \arrow[r, "f'_{k-1}"] \arrow[d, "h_{k-1}"] & A'_k \arrow[d, "h_k"] \\
B'_1 \arrow[r, "g'_1"'] & B'_2 \arrow[r, "g'_2"'] & \cdots \arrow[r, "g'_{k-2}"'] & B'_{k-1} \arrow[r, "g'_{k-1}"'] & B'_k
\end{tikzcd}$$

Then in the $n + 1$ case, by overlaying that smaller diagram we can show that every path from A_{k_1} or B_{k_1} to A_{k_2} or B_{k_2} commutes so long as $k_2 - k_1 < n$. Hence, we just need to show that any two walks in the diagram from A_1 to A_{n+1} or from A_1 to B_{n+1} or from B_1 to B_{n+1} are considered equivalent. But since there is only one walk from A_1 to A_{n+1} and B_1 to B_{n+1}, the only actual nontrivial thing to prove is that all walks from A_1 to B_{n+1} are considered equivalent.

So consider any walk in our diagram going from A_1 to B_{n+1}. Then we know there exists $r \in \{1, \ldots, n+1\}$ such that the walk is equal to $f_1 \cdots f_{r-1} h_r g_r \ldots g_n$. And then if $r \leq n$, we can say that $f_1 \cdots f_{r-1}(h_r g_r) \ldots g_n = (f_1 \cdots f_{r-1} f_r g_{r+1} g_{r+1} \ldots g_n)$.

By another induction argument, you can thus show that every walk from A_1 to B_{n+1} is considered equivalent to $f_1 \cdots f_n h_{n+1}$. ■

We say the following S.E.S. splits if there exists a group homomorphism $f : G_3 \to G_2$ such that $f_2 \circ f = \mathrm{Id}_{G_3}$:

$$1 \longrightarrow G_1 \xrightarrow{f_1} G_2 \underset{f}{\overset{f_2}{\rightleftarrows}} G_3 \longrightarrow 1$$

Note that we don't necessarily have that $f \circ f_2 = \mathrm{Id}_{G_2}$. After all, f_2 is not necessarily injective so it may not have a left inverse. f_2 is always surjective though so the question of whether f exists can be summed up as: does f_2 have a right inverse that's also a group homomorphism.

For more 200a notes, go to *page* ___.

Math 220a (lecture 8):

Using power series we can define more interesting holomorphic functions. For example (and I'm only doing this because I didn't take notes on this in math 140b) let:

- $\exp(z) := \sum_{n=0}^{\infty} \frac{1}{n!} z^n$,
- $\cos(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$,
- $\sin(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$.

These power series have infinite radii of convergence. To see that, note by a basic induction argument that $(n+k)! \geq k^n$ for all positive integers n, k. Therefore, we can say for all $k \in \mathbb{N}$ that:

$$\limsup_{n\to\infty} \sqrt[n]{\tfrac{1}{n!}} \leq \limsup_{n\to\infty} \sqrt[n]{\tfrac{1}{k^{n-k}}} = \limsup_{n\to\infty} \frac{\sqrt[n]{k^k}}{k} = \frac{1}{k}.$$

And by taking $k \to \infty$ we get that all three power series have radius of convergence $1/0 = \infty$.

Lemma: If $G \subseteq \mathbb{C}$ is a region, $f : G \to \mathbb{C}$ is differentiable, and $f' = 0$ on G, then f equals a constant c.

Proof:

By corollary 9.20 in my math 140c notes, we know that this is true when G is convex. As for the general case, consider that every point in G has a convex neighborhood on which f is constant. So if $A \subseteq G$ is the set of points where $f(z) = f(w)$ for some arbitrary $w \in G$, then we can easily show that A is open. At the same time though, if $f(z') \neq f(w)$, then there must be a neighborhood around z' where $f \neq f(w)$. Hence, A^{C} is open too.

Since G is connected, the only way this is possible is if $A^{\mathsf{C}} = \emptyset$. ■

Proposition: $\exp(z, w) = \exp(z)\exp(w)$ for all $z, w \in \mathbb{C}$.

Proof:

Fix any $\alpha \in \mathbb{C}$ and note by product and chain rule that:

$$\frac{\mathrm{d}}{\mathrm{d}z}(\exp(z)\exp(a-z)) = \exp(z)\exp(a-z) - \exp(z)\exp(a-z) = 0$$

Thus $\exp(z)\exp(a-z)$ equals some constant. And by plugging in $z = 0$ we can then calculate that this constant is $\exp(a)$. Hence, $\exp(z)\exp(a-z) = \exp(a)$ for all $z, a \in \mathbb{C}$.

To complete the proof, set $a = z + w$. ■

Since $\exp(1) = e$ and $\exp(0) = 1$, we thus typically just use the abuse of notation that $\exp(z) = e^z$.

Also note that $e^{iz} = \cos(z) + i\sin(z)$ for all $z \in \mathbb{C}$.

I'd love to take more notes on rigorously defining $\exp$, $\cos$, and $\sin$ since I never got around to taking notes on that back when I took undergrad real analysis. But unfortunately I don't have time right now. Maybe one day in the future I'll work through it finally.

Math 200a Homework:

Set 3 Problem 2: Suppose G is a finite group, $N \lhd G$, and $P \in \mathrm{Syl}_p(N)$. Then $G = N_G(P)N$.

Proof:

Note that for any $g \in G$, we have that $gPg^{-1} \subseteq N$ since $P \subseteq N$ and N is normal. That said, we also know for any $P' \in \mathrm{Syl}_p(G)$ that there exists $g \in G$ such that $gPg^{-1} = P'$. Hence, we must have that $\mathrm{Syl}_p(G) = \mathrm{Syl}_p(N)$, and from now on in this proof I'll refer to their common value s_p.

Now we know $\frac{|N|}{|N_N(P)|} = [N : N_N(P)] = s_p = [G : N_G(P)] = \frac{|G|}{|N_G(P)|}$. It thus follows that $|N_G(P)| = |N_N(P)| \cdot [G : N]$. But also note that $N_N(P) = N_G(P) \cap N$. Therefore:

$$|N_G(P)N| = \frac{|N_G(P)||N|}{|N_G(P) \cap N|} = \frac{(|N_G(P) \cap N|[G:N]) \cdot |N|}{|N_G(P) \cap N|} = [G : N] \cdot |N| = |G|.$$

It follows that $N_G(P)N = G$. ■

Set 3 Problem 6: Suppose p and q are prime numbers and G is a group with $|G| = p^2 q$. Prove that G is not simple.

Proof:

Let s_p and s_q denote $|\mathrm{Syl}_p(G)|$ and $|\mathrm{Syl}_q(G)|$ respectively. Now by Sylow's theorems, we have that $s_p \equiv 1 \pmod{p}$ and that $s_p \in \{1, q\}$. But if $s_p = 1$, then we are already done showing that G is not simple. Hence, we can without loss of generality assume that $s_p = q$ and therefore $p \mid q - 1$.

Next, $s_q \equiv 1 \pmod{q}$ and $s_1 \in \{1, p, p^2\}$ by Sylow's theorems. But also like before, if $s_q = 1$ then we're already done showing that G is not simple. Hence, we shall assume $s_q \neq 1$. In turn, this means that either $q \mid p - 1$ (if $s_q = p$) or $q \mid p^2 - 1 = (p-1)(p+1)$ (if $s_q = p^2$). Or equivalently, this means that $q \mid p - 1$ or $q \mid p + 1$.

But note that if $q \mid p-1$, then $q+1 \leq p$. Yet this contradicts that $p \leq q-1$ (which we know since $p \mid q-1$). Hence, we must instead have that $q \mid p+1$. Firstly, this guarentees that $s_q = p^2$. Secondly, by also considering the fact that $p \mid q-1$, we know that $p+1 = q$. And since p and q are both prime numbers, this must mean that $p = 2$ and $q = 3$.

Finally though, we now have that $|G| = 12$ and that there are $s_q(q-1) = 4(3-1) = 8$ elements of G with order 3. This is a contradiction since there aren't enough elements leftover for s_p to be greater than 1 and we already assumed $s_p = q = 3$. ■

Set 3 Problem 1: Suppose $p < q < \ell$ are three primes, G is a group, and $|G| = pq\ell$. Then G has a normal Sylow ℓ-subgroup.

Proof:

By Sylow's second theorem, we know that $|\mathrm{Syl}_\ell(G)| =: s_\ell \in \{1, p, q, pq\}$. But we also know by Sylow's third theorem that $s_\ell \equiv 1 \pmod{\ell}$. Since $1 < p, q < \ell$, this means that the only actual options that s_ℓ could be are 1 and pq. In the former case that $s_\ell = 1$, we'd already be done since the unique $L \in \mathrm{Syl}_\ell(G)$ would automatically be normal. Hence, we'll instead assume for the sake of contradiction that $s_\ell = pq$.

Next note that for any two distinct $L, L' \in \mathrm{Syl}_\ell(G)$, since L and L' are cyclic with prime order, we must have that $L \cap L' = \{1\}$. It follows that if $X = G - \bigcup_{L \in \mathrm{Syl}_\ell(G)}(L - \{1\})$ then we have that $|X| = pq\ell - pq(\ell - 1) = pq$. But also since X contains precisely the elements of G with order not equal to ℓ, we know that any Sylow q-groups must be entirely contained in X.

We now consider $|\mathrm{Syl}_q(G)| =: s_q$. By Sylow's theorems, we have that $s_1 \equiv 1 \pmod{q}$ and that $s_q \in \{1, p, \ell, p\ell\}$. But since $1 < p < q$, we automatically can rule out that $s_q = p$. By a slightly more involved argument, we can also rule out that $s_q = \ell$ or $p\ell$.

To see why, note that for any distinct $Q, Q' \in \mathrm{Syl}_q(G)$, since Q and Q' are cyclic with prime order, we must have that $Q \cap Q' = \{1\}$. Hence, if $Y - \bigcup_{Q \in \mathrm{Syl}_q(G)} Q$, then we must have that $|Y| = s_q(q-1) + 1$.

But also note that $Y \subseteq X$ and therefore $|Y| \leq |X| = pq$. Hence, we must have that $pq \geq s_1(q-1) + 1 \geq s_1 p + 1$ (where the last inequality follows since $q > p$). And thus s_1 equaling ℓ or ℓp (which are both greater than q) would be a contradiction.

It follows that $s_q = 1$ and hence there is a unique group $Q \in \mathrm{Syl}_q(G)$ which is automatically normal. And to finish off our proof, we now consider the subgroups QL_1 and QL_2 of G where L_1 and L_2 are distinct groups in $\mathrm{Syl}_\ell(G)$. Note that QL_i is a group for both i since $Q \lhd G$. Also, once again since Q and L_i are distinct cyclic groups of prime order, we know that $Q \cap L_i = \{1\}$ for both i. Hence $|QL_1| = |QL_2| = ql$.

Since $(QL_1) \cap (QL_2)$ is a subgroup of QL_1, we know $|(QL_1) \cap (QL_2)| \in \{1, q, \ell, q\ell\}$. However, we also know that $(QL_1)(QL_2) \subseteq G$ and hence:

$$|(QL_1)(QL_2)| = \frac{q^2\ell^2}{|(QL_1) \cap (QL_2)|} \leq |G| = pql.$$

Since $q^2\ell^2$, $q\ell^2$, and $q^2\ell$ are all greater than $pq\ell$, it must be that $|(QL_1) \cap (QL_2)| = q\ell$. But that implies that $QL_1 = QL_2$, which in turn gives us a different contradiction. After all, since $QL_1 = QL_2$ is a group and $L_1, L_2 \subseteq QL_1 = QL_2$, we have that $L_1L_2 \subseteq QL_1$. However, we already went over that $L_1 \cap L_2 = \{1\}$. Hence $|L_1L_2| = \ell^2$ and we've shown that $\ell^2 = |L_1L_2| \leq |QL_1| = q\ell$. But that contradicts that $q < \ell$. ■

Set 3 Problem 7:

(a) Suppose $N \lhd G$ and K is a characteristic subgroup of N. Then $K \lhd G$.

Since $N \lhd G$, we know that conjugation by x is an automorphism of N for all $x \in G$. And since K is a characteristic subgroup of N, this means that $xKx^{-1} = K$ for all $x \in G$. Hence, $K \lhd G$.

(b) We say a group is characteristically simple if the only characteristic subgroups of H are 1 and H. Suppose N is a *minimal* normal subgroup of G, meaning that if $M < N$ and $M \lhd G$ then $M = \{1\}$ or N. Then N is characteristically simple.

Let M be a characteristic subgroup of N. Then by part (a) we know that $M \lhd G$. And since N is minimally normal, then means that either $M = \{1\}$ or $M = N$. ■

Math 220a Homework:

Exercise II.5.7: Let G be an open subset of $\mathbb{C}$ and P be a polygon (recall the definition on *page 247* of my journal) in G going from a to b. Then show that there is a polygon $Q \subseteq G$ from a to b which is composed of line segments which are parallel to either the real or imaginary axes.

For now, we'll just focus on the case that P is a line segment $[a, b]$. Then note that $[a, b]$ is precisely the image of the map $f(t) = tb + (1 - t)a$ from $[0, 1] \subseteq \mathbb{R}$.

Since f is continuous and $[0, 1]$ is compact, it follows that $[a, b]$ is compact as well and that f is actually uniformly continuous. So firstly, for every $z \in [a, b]$ consider picking $r_z > 0$ such that the open ball $B_{r_z}(z) \subseteq G$. Then let $\mathcal{U}$ be an open cover of $[a, b]$ consisting of smaller balls: $\{B_{\frac{r_z}{3}}(z) : z \in [a, b]\}$.

By the Lebesgue number lemma, we know there is some $\varepsilon > 0$ such that whenever $w_1, w_2 \in [a, b]$ satisfy that $|w_1 - w_2| < \varepsilon$, then w_1, w_2 are contained in a single ball $B_{\frac{r_z}{3}}(z)$. And importantly in that case, if $w_1 = x_1 + iy_1$ and $w_2 = x_2 + iy_2$, then the polygon $[x_1 + iy_1, x_2 + iy_1, x_2 + iy_2]$ going from w_1 to w_2 is contained in G and clearly consists of line segments parallel to the real and imaginary axes.

Why? Since $B_{r_z}(z)$ is convex, it suffices to show that $x_2 + iy_1 \in B_{r_z}(z)$. But luckily, note that:

$$\begin{aligned}
|x_2 + iy_1 - z| &= |x_2 - x_1 + x_1 + iy_1 - z| \\
&\leq |\mathrm{Re}(w_2 - w_1)| + |w_1 - z| \\
&\leq |w_2 - w_1| + |w_1 - z| \leq |w_2 - z| + 2|w_1 - z| < 3\tfrac{r_z}{3} = r_z
\end{aligned}$$

Next, using the uniform continuity of f, pick $\delta > 0$ such that $|f(t_2) - f(t_1)| < \varepsilon$ when $|t_2 - t_1| \leq \delta$. In particular, this means that if $n \in \mathbb{N}$ satisfies that $n\delta \leq 1$ but $(n+1)\delta > 1$, then we apply the above observation to the points $f(0) = a, f(\delta), f(2\delta), \ldots, f(n\delta)$, $f(1) = b$ to construct a polygon from a to b contained in G which consists of $2(n + 1)$ line segments parallel to either the real or imaginary axes.

To generalize this to when the polygon P isn't a single line segment, just apply the prior reasoning to each line segment making up P. ■

Exercise II.6.1: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of uniformly continuous functions from the metric space (X, d) to the metric space (Ω, p), and suppose $f_n \to f$ uniformly. Then f is uniformly continuous.

Proof:
For any $\varepsilon > 0$ pick $n \in \mathbb{N}$ such that $p(f(x), f_n(x)) < \varepsilon/3$ for all $x \in X$. Then since f_n is uniformly continuous, pick $\delta > 0$ such that $p(f_n(x), f_n(y)) < \varepsilon/3$ whenever $d(x, y) < \delta$. Then, we can see that $p(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$. Hence f is uniformly continuous.

Furthermore, if each f_n is a Lipschitz function with constant M_n and $\sup M_n < \infty$, then f is a Lipschitz function.

Proof:
Pick $M \geq \sup M_n$ and then note for all $n \in \mathbb{N}$ that:

$$\begin{aligned} p(f(x), f(y)) &\leq p(f(x), f_n(x)) + p(f_n(x), f_n(y)) + p(f(y), f_n(y)) \\ &\leq p(f(x), f_n(x)) + Md(x, y) + p(f(y), f_n(y)) \end{aligned}$$

And by taking $n \to \infty$ we get that $p(f(x), f_n(x)) \to 0$ and $p(f(y), f_n(y)) \to 0$. Hence $p(f(x), f(y)) \leq Md(x, y)$.

Finally, if $\sup M_n = \infty$ then f can fail to be Lipschitz.

Proof:
Let $X = [0, \infty)$, $\Omega = \mathbb{R}$, and define $f_n(x) := \sqrt{x + \frac{1}{n}}$ for all $x \in X$ and $n \in \mathbb{N}$. Our first claim is that $f_n \to f$ uniformly where $f(x) = \sqrt{x}$.

To see this, note that for all $x \in X$ and $n \in \mathbb{N}$ that:

$$\left(\sqrt{x + \tfrac{1}{n}} - \sqrt{x}\right)^2 = 2x + \tfrac{1}{n} - 2\sqrt{x^2 + \tfrac{x}{n}} \leq \tfrac{1}{n}.$$

Hence $|f_n(x) - f(x)| < n^{-1/2}$ for all $n \in \mathbb{N}$ and $x \in X$.

Next, we claim that each f_n is Lipschitz on X with the the constant $\frac{\sqrt{n}}{2}$.

To see this, note that $f_n'(x) = \frac{1}{2\sqrt{x + \frac{1}{n}}}$ for all $x \in X$.

It follows that $f_n'(x)$ attains a maximum of $\frac{\sqrt{n}}{2}$ at $x = 0$. And by the mean value theorem it follows that $\frac{\sqrt{n}}{2}$ is a Lipschitz constant for f on X.

That said, $\frac{\sqrt{n}}{2} \to \infty$ as $n \to \infty$. Also note that f is not Lipschitz on X.

To see this, note that f is differentiable when $x \neq 0$ and that $f'(x) = \frac{1}{2}x^{-1/2}$. But now $f'(x) \to 0$ as $x \to 0$. Hence for any $M > 0$ there is some interval $[a, b] \subseteq X$ such that $f'(x) > M$ for all $x \in [a, b]$. And in turn, by the mean value theorem we have that $|f(b) - f(a)| > M|b - a|$. So, M cannot be a Lipschitz constant for f and this proves f isn't Lipschitz.

Exercise III.1.5: If $(a_n)_{n \in \mathbb{N}}$ is a convergent sequence in $\mathbb{R}$ and $a = \lim a_n$, show that $a = \liminf a_n = \limsup a_n$.

To start off, how the hell is this a grad level problem? Like god I know the professor said that he reviews everything cause "A lOt Of PeOpLe ArE rUsTy" or something. But it's not his job to unrust us! Literally, I would argue that since math 140c is a prerequisite for this class, the professor should be obligated to assume we all have a working proficiency at undergrad real-analysis. Otherwise, why not just make the class have zero prerequisites?

Anyways the definition of $\liminf$ and $\limsup$ which Conway gives is that:

$$\liminf a_n = \lim_{n\to\infty} \inf_{k \geq n} a_k \text{ and } \limsup a_n = \lim_{n\to\infty} \sup_{k \geq n} a_k.$$

Now let $\varepsilon > 0$ and note that because $a_n \to a$ as $n \to \infty$, we know there exists $N \in \mathbb{N}$ such that $a - \varepsilon < a_n < a + \varepsilon$ for all $n \geq N$. Hence $\inf\{a_n, a_{n+1}, \ldots\} \geq a - \varepsilon$ and $\sup\{a_n, a_{n+1}, \ldots\} \leq a + \varepsilon$ for all $n \geq N$.

This in turn means that $\liminf a_n \geq a - \varepsilon$ and $\limsup a_n \leq a + \varepsilon$ for any $\varepsilon > 0$. Taking $\varepsilon \to 0$ and noting that $\liminf a_n \leq \limsup a_n$ just by comparison test, we have that:

$$a \leq \liminf a_n \leq \limsup a_n \leq a. \blacksquare$$

Exercise III.1.7: Show that the radius of convergence of the power series $f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$ is 1 and discuss convergence for $z = 1, -1,$ and i.

Firstly, consider the power series $g(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n$ since it's simpler than f. Importantly, $|\frac{(-1)^n}{n}|^{1/n} = \frac{1}{\sqrt[n]{n}} \to 1$ as $n \to \infty$. Hence, the radius of convergence of g is $1^{-1} = 1$.

This in turn also tells us that the radius of convergence of f is 1. After all, if $|z| < 1$ then $|z^{n(n+1)}| \leq |z|^n$ and so we know by comparison test with $g(|z|)$ that $f(z)$ converges. So, the radius of convergence of f is at least 1. Meanwhile, if $|z| > 1$ then $|z^{n(n+1)}| \geq |z|^n$ and so by comparison test with $g(|z|)$ we know that $f(z)$ doesn't absolutely converge. Hence, the radius of convergence is at most $|z|$ for any $z \in \mathbb{C}$ with $|z| > 1$.

Next we examine the convergence of $f(1)$, $f(-1)$, and $f(i)$.

- $f(1)$ is the alternating harmonic series. So it converges but not absolutely to $ln(2)$.
- $f(-1) = \sum_{n=1}^{\infty} \frac{1}{n}(-1)^n(-1)^{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n(n+2)}$. But no matter if n is even or odd, $n(n+2)$ is even. So $(-1)^{n(n+2)} = 1$ and thus $f(-1)$ is the harmonic series which diverges.
- $f(i) = \sum_{n=1}^{\infty} \frac{1}{n} i^{2n} i^{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} i^{n(n+3)} = \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{\psi(n)}$ where $\psi(n) = 0$ if $n \equiv 0 \text{ or } 1 \pmod 4$ and $\psi(n) = 1$ otherwise. In other words:
$$f(i) = 1 - \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} - \cdots$$

  Now note that the partial sums $\sum_{k=1}^{n}(-1)^{\psi(k)}$ are bounded between -1 and 1. Also, $(\frac{1}{n})_{n\to\infty}$ is a decreasing sequence of nonnegative numbers converging to 0. Therefore, by the result below from my math 140a notes we know that $f(i)$ converges (although again not absolutely).

Proposition 57: If the partial sums of Σa_n are bounded and we have a sequence $b_0 \geq b_1 \geq b_2 \geq \cdots$ such that $b_n \to 0$, then $\sum a_n b_n$ will converge.

Proof:

Set $A_n = \sum\limits_{k=0}^{n} a_k$. Then pick $M > 0$ such that $\forall n, \ \ |A_n| < M$.

Given $\varepsilon > 0$, pick N with $b_N < \frac{\varepsilon}{2M}$. Then when $q \geq p \geq N$, we have:

$$\begin{aligned}\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\leq \sum_{n=p}^{q-1} |A_n|\,(b_n - b_{n+1}) + |A_q| b_q + |A_{p-1}| b_p \\ &\leq M(b_p - b_q + b_q + b_p) = 2Mb_p \leq 2Mb_N < \varepsilon\end{aligned}$$

Exercise III.2.1: Show that $f(z) = |z|^2$ is complex differentiable only at the origin.

Identify $\mathbb{C}$ with $\mathbb{R}$ and consider f as the function $f(x, y) = (x^2 + y^2, 0)$ going from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then $f \in C^\infty(\mathbb{R}^2)$ with a derivative matrix $\left(\begin{smallmatrix} 2x & 2y \\ 0 & 0 \end{smallmatrix}\right)$. Now for f to satisfy the Cauchy-Riemann equations (see the theorem on *page 296*) at a point (x, y), we must have that $2x = 0$ and $-2y = 0$. Hence the only point where f is complex differentiable is at $(0, 0) = 0 + i0$.

Exercise III.2.3: Show that $\lim_{n\to\infty} n^{1/n} = 1$.

If you want to prove this theorem without using logarithms (because you hadn't defined logarithms yet when you first relied on this fact), then here is the proof from math 140a:

(C) $\sqrt[n]{n} \to 1$

Proof:
Let $x_n = \sqrt[n]{n} - 1$. Then $x_n \geq 0$ and by the binomial theorem:

$$\frac{n(n-1)}{2}(x_n)^2 \leq \sum_{k=0}^{n} \binom{n}{k}(x_n)^k = (x_n + 1)^n = n$$

Then we have that $0 \leq x_n \leq \sqrt{\frac{2n}{n(n-1)}} = \sqrt{\frac{2}{n-1}}$ when $n \geq 2$.

Now, $\sqrt{\frac{2}{n-1}} \to 0$.

Proof: Let $\varepsilon > 0$. Then $\sqrt{\frac{2}{n-1}} < \varepsilon$ whenever $n > 1 + \frac{2}{\varepsilon^2}$.

Therefore, by proposition 43, we know that $x_n \to 0$. So finally, we conclude that:

$$\sqrt[n]{n} \to \lim_{n\to\infty}(x_n) + 1 = 0 + 1$$

If you are willing to rely on logarithms and calculus though, then here is a slicker proof:

Note that $\log(n^{1/n}) = \frac{1}{n}\log(n)$ for all n. Then by L'Hôpital's rule we have that $\lim_{x\to\infty} x^{-1}\log(x) = \lim_{x\to\infty}(1)^{-1}\frac{1}{x} = 0$. And hence $\log(n^{1/n}) \to 0$ as $n \to \infty$.

Finally, since $\exp$ is continuous, we have that $n^{1/n} = \exp(\log(n^{1/n})) \to \exp(0) = 1$ as $n \to \infty$.

Exercise III.2.19: Let G be a region and define $G^* = \{z : \overline{z} \in G\}$. If $f : G \to \mathbb{C}$ is holomorphic prove that $f^* : G^* \to \mathbb{C}$ defined by $f^*(z) = \overline{f(\overline{z})}$ is also holomorphic.

Once again identify $\mathbb{C}$ with $\mathbb{R}^2$ and write f as $f(x, y) = (u(x, y), v(x, y))$. Then we have that $f^*(x, y) = (u(x, -y), -v(x, -y))$. And since f is C^1, we can calculate that the derivative matrix of f^* is $\mathrm{D}(f^*) = \begin{pmatrix} u_x(x,-y) & -u_y(x,-y) \\ -v_x(x,-y) & v_y(x,y) \end{pmatrix}$.

Firstly, this shows that f^* is also C^1 since all the partial derivatives of f^* are continuous. Also, this shows that if f satisfies the Cauchy-Riemann ewquations, then so does f^*. Hence f being holomorphic on G implies f^* is as well.