Math 140B Lecture Notes (Professor: Brandon Seward)

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April 2, 2024

Lecture 1: 4/1/2024

Let $f : E \longrightarrow \mathbb{R}$ where $E \subseteq \mathbb{R}$.

Since E is the domain of f, we shall also refer to it as $\mathrm{dom}(f)$.

Fix a point $x \in E \cap E'$. Then consider the function $\frac{f(t)-f(x)}{t-x}$ for $t \in \mathrm{dom}(f) \setminus \{x\}$ and define the derivative of f at x to be $f'(x) = \lim\limits_{t \to x} \left(\frac{f(t)-f(x)}{t-x}\right)$ provided that this limit exists. When the above limit exists, we say f is differentiable at x.

We say f is differentiable on $D \subseteq E$ if f is differentiable at every point in D, and if f is differentiable on its entire domain, then we call f differentiable.

The function $f'(x) = \lim\limits_{t \to x} \left(\frac{f(t)-f(x)}{t-x}\right)$ is called the derivative of f.

Proposition 83: If f is differentiable at x, then f is continuous at x.

Proof:
Note that $\lim\limits_{t \to x} (f(t)) = \lim\limits_{t \to x} \left((t-x)\frac{f(t)-f(x)}{t-x} + f(x)\right)$.

Now $\lim\limits_{t \to x}(t-x) = 0$ and we know $\lim\limits_{t \to x} \frac{f(t)-f(x)}{t-x} = f'(x)$ exists because f is differentiable at x. Also, obviously $\lim\limits_{t \to x} f(x) = f(x)$.

Thus by proposition 66 (check 140A notes), we know that:

$$\begin{aligned}
\lim_{t \to x} \left((t-x)\frac{f(t)-f(x)}{t-x} + f(x)\right) &= \lim_{t \to x}(t-x) \lim_{t \to x} \left(\frac{f(t)-f(x)}{t-x}\right) + \lim_{t \to x} f(x) \\
&= 0 \cdot f'(x) + f(x) \\
&= f(x)
\end{aligned}$$

Thus, f is continuous at x.

Notes:

1. The above proposition says that differentiability is stronger than continuity.
2. The converse of this proposition is false. For example, the function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

Proposition 84: Suppose f and g are real valued functions with $\mathrm{dom}(f), \mathrm{dom}(g) \subseteq \mathbb{R}$. Also suppose f and g are differentiable at x. Then $f+g$, fg, and (when $g(x) \neq 0$) $\frac{f}{g}$ are differentiable at x with:

(A) $(f+g)'(x) = f'(x) + g'(x)$ (sum rule)

(B) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ (product rule)

(C) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ (quotient rule)

Proof:

(A) Since both f and g are differentiable, we know that both $f'(x) = \lim\limits_{t\to x} \frac{f(t)-f(x)}{t-x}$ and $g'(x) = \lim\limits_{t\to x} \frac{g(t)-g(x)}{t-x}$ exist. So by proposition 66:

$$(f+g)'(x) = \lim_{t\to x} \frac{f(t)+g(t)-f(x)-g(x)}{t-x} = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} + \lim_{t\to x} \frac{g(t)-g(x)}{t-x}$$

This means $(f+g)'(x) = f'(x) + g'(x)$.

(B) Note that:

$$\begin{aligned}(fg)'(x) &= \lim_{t\to x} \frac{f(t)g(t)-f(x)g(x)}{t-x} \\ &= \lim_{t\to x} \frac{f(t)g(t)-f(x)g(t)+f(x)g(t)-f(x)g(x)}{t-x} \\ &= \lim_{t\to x} \left(g(t)\frac{f(t)-f(x)}{t-x} + f(x)\frac{g(t)-g(x)}{t-x}\right)\end{aligned}$$

By proposition 83, $g(t) \to g(x)$ as $t \to x$. Also, since both f and g are differentiable, we know $f'(x) = \lim\limits_{t\to x} \frac{f(t)-f(x)}{t-x}$ and $g'(x) = \lim\limits_{t\to x} \frac{g(t)-g(x)}{t-x}$ exist. So by proposition 66:

$$\lim_{t\to x} \left(g(t)\frac{f(t)-f(x)}{t-x} + f(x)\frac{g(t)-g(x)}{t-x}\right) = f'(x)g(x) + f(x)g'(x).$$

(C) Note that:

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \lim_{t\to x} \frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{t-x} \\ &= \lim_{t\to x} \left(\frac{1}{g(x)g(t)} \frac{f(t)g(x)-f(x)g(t)}{t-x}\right) \\ &= \lim_{t\to x} \left(\frac{1}{g(x)g(t)} \frac{f(t)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(t)}{t-x}\right) \\ &= \lim_{t\to x} \left(\frac{1}{g(x)g(t)} \left(g(x)\frac{f(t)-f(x)}{t-x} - f(x)\frac{g(t)-g(x)}{t-x}\right)\right)\end{aligned}$$

Now, for the same reasons as before, we can use propositions 83 and 66 to separate the parts of the above limit to get that the above limit equals:

$$\frac{1}{(g(x))^2}\left(g(x)f'(x) - f(x)g'(x)\right)$$

If $f(x) = \alpha$ where $\alpha \in \mathbb{R}$ is constant, then trivially $f'(x) = 0$ for all x.
Meanwhile, if $f(x) = x$, then we can trivially find that $f'(x) = 1$.

Claim: for all $n \in \mathbb{Z}^+$, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Proof: (we proceed by induction)

Base Case:

If $n = 1$, then for $f(x) = x^1$, we have that $f'(x) = 1 \cdot x^0$.

Induction:

Now assume $n > 1$, and for $f(x) = x^{n-1}$, we have that $f'(x) = (n-1)x^{n-2}$.
For the rest of this proof, I'll abreviate the derivative of x^n as $(x^n)'$ and the derivative of x^{n-1} as $(x^{n-1})'$. Then using product rule, we know that:

$$(x^n)' = x(x^{n-1})' + 1 \cdot x^{n-1} = x \cdot (n-1)x^{n-2} + x^{n-1} = ((n-1)+1)x^{n-1} = nx^{n-1}$$

This combined with proposition 84 tells us that both polynomials and rational functions are differentiable over their domains.

Proposition 85: (chain rule)
Let f and g be real-valued functions with $\mathrm{dom}(f), \mathrm{dom}(g) \subseteq \mathbb{R}$. Let $x \in \mathbb{R}$.
Suppose that f is differentiable at x and that g is differentiable at $f(x)$. Then $g \circ f$ is differentiable at x and $(g \circ f)'(x) = g'(f(x))f'(x)$.

A List of How The Proposition Numbering in my Notes Lines up With Our Textbook:

Proposition Number	Label in Textbook	Proposition Number	Label in Textbook
83	5.2	84	5.3
85	5.5	86	
87		88	
89		90	
91		92	

Our textbook is *Principles of Mathematical Analysis* by Walter Rudin.