My Notes on Paolo Aluffi's Algebra Chapter 0

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A multiset is a collection of elements which like a set is unordered but unlike a set can contain duplicate elements.

One way to define a multiset is as a function $f : A \to \mathbb{N}$ such that each $\alpha \in A$ is mapped to the number of times that α appears in the multiset. Then, given the multisets $f_1 : A \to \mathbb{N}$ and $f_2 : B \to \mathbb{N}$, we can define the following operations:

- $\alpha \in f_1 \iff \alpha \in A$
- $f_1 \subseteq f_2 \iff \forall \alpha \in f_1,\ \alpha \in f_2$ and $f_1(\alpha) \leq f_2(\alpha)$
- $f_1 \cup f_2 : (A \cup B) \longrightarrow \mathbb{N}$ such that for $\alpha \in A \cup B$, if $\alpha \in A \cap B$, then $(f_1 \cup f_2)(\alpha) = f_1(\alpha) + f_2(\alpha)$. As for if $\alpha \notin A \cap B$, then $(f_1 \cup f_2)(\alpha)$ equals whatever α was mapped to in the multiset it originally came from.
- $f_1 \cap f_2 : (A \cap B) \longrightarrow \mathbb{N}$ such that for $\alpha \in A \cap B$, we have that $(f_1 \cap f_2)(\alpha) = \min(f_1(\alpha), f_2(\alpha))$
- $f_1 \setminus f_2 : ((A \setminus B) \cup \{\alpha \in A \cap B \mid f_1(\alpha) > f_2(\alpha)\}) \longrightarrow \mathbb{N}$ such that for each $\alpha \in f_1 \setminus f_2$, if $\alpha \in f_2$, then $(f_1 \setminus f_2)(\alpha) = f_1(\alpha) - f_2(\alpha)$. As for if $\alpha \notin f_2$, then $(f_1 \setminus f_2)(\alpha) = f_1(\alpha)$

A practical example of a multiset is the prime factorization of any positive integer.

We say that two sets A and B are isomorphic if and only if there exists a bijection between A and B. We denote this by writing $A \cong B$. Additionally, we can refer to any bijection f between A and B as an isomorphism between the two sets.

A function $f : A \to B$ is a monomorphism (a.k.a a monic) if for all sets Z and all functions a' and $a'' : Z \to A$, we have that $f \circ a' = f \circ a'' \implies a' = a''$.

Proposition 1: A function is injective if and only if it is a monomorphism.

Proof: Let's say we have a function $f : A \to B$.

First, let us assume f is injective.

Then let us assume we have two functions a' and a'' from some set Z to A such that $f \circ a' = f \circ a''$. Because f is injective, we know it has a left-hand inverse $g : B \to A$ such that $g \circ f = \mathrm{Id}_A$. Composing g with the previous equation, we get that:

$$a' = \mathrm{Id}_A \circ a' = g \circ (f \circ a') = g \circ (f \circ a'') = \mathrm{Id}_A \circ a'' = a''$$

Thus, we've shown that f is a monomorphism.

Next, we shall assume f is a monomorphism.

Based on this, we can say that for any two functions a' and a'' mapping a set Z to A, we have that $f \circ a' = f \circ a'' \implies a' = a''$. However, now note that if we make Z a singleton, meaning it only contains one element, then a' and a'' can each only take on one value. So, we can effectively rewrite $f \circ a' = f \circ a'' \Rightarrow a' = a''$ as:

$$f(a') = f(a'') \Rightarrow a' = a''$$

This is the definition of an injective function. ■

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A function $f : A \to B$ is an epimorphism (a.k.a an epi) if for all sets Z and all functions a' and $a'' : B \to Z$, we have that $a' \circ f = a'' \circ f \Rightarrow a' = a''$.

Proposition 2: A function is a surjection if and only if it is an epimorphism.

Proof: Let's say we have a function $f : A \to B$.

First, let us assume f is surjective.

Then let's assume we have two functions a' and a'' from B to some set Z such that $a' \circ f = a'' \circ f$. Because f is surjective, we know it has a right-hand inverse $h : B \to A$ such that $f \circ h = \mathrm{Id}_B$. Composing h with the previous equation, we get that:

$$a' = a' \circ \mathrm{Id}_B = (a' \circ f) \circ h = (a'' \circ f) \circ h = a'' \circ \mathrm{Id}_B = a''$$

So f is an epimorphism.

Next, assume f is not surjective.

Then there exists $\beta \in B$ such that for all $\alpha \in A$, we have that $f(\alpha) \neq \beta$. Importantly, as $f(\alpha) \in B$, we know $|B| \neq 1$. So set a' equal to Id_B and define a'' as a function mapping each element of $B \setminus \{\beta\}$ to itself and β to any of the other elements in B. Now, $a' \circ f = f = a'' \circ f$ but $a' \neq a''$. So f is not epimorphic. ■

Sometimes, to indicate that a function $f : A \to B$ is a monomorphism, epimorphism, or isomorphism, we use the following notation:

- Monomorphism: $f : A \hookrightarrow B$
- Epimorphism: $f : A \twoheadrightarrow B$
- Isomorphism: $f : A \xrightarrow{\sim} B$

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A relation on a set S is a subset R of the cartesian product $S \times S$. Specifically, we use the notation $x \, R \, y$ to mean that $(x, y) \in R$. Certain types of relations are especially important and thus are represented with their own symbol.

- An equivalence relation, typically denoted $\sim$, on a set S has the properties:
  ∘ $\forall a \in S, \;\; a \sim a$ ∘ $a \sim b \Longrightarrow b \sim a$ ∘ $a \sim b$ and $b \sim c \Longrightarrow a \sim c$
- An order relation, typically denoted $<$, on a set S has the properties:
  ∘ $\forall a, b \in S$, exactly one of the following is true: $a < b$, $b < a$, or $a = b$.
  ∘ $a < b$ and $b < c$ implies that $a < c$.

Given a set S, an equivalence relation $\sim$, and an element $a \in S$, we define the equivalence class of a with respect to $\sim$ to be the set $[a]_\sim = \{b \in S \mid a \sim b\}$. Also, we define the quotient of S with respect to the equivalence relation $\sim$ as the set of equivalence classes with respect to $\sim$.

$$S/{\sim} = \{[a]_\sim \mid a \in S\}$$

Given any function $f : A \longrightarrow B$, define $a \sim b \Longleftrightarrow f(a) = f(b)$.

Proposition 3: Every function f can be decomposed as follows:

$$\begin{tikzcd} A \arrow[r, two heads, "g"'] \arrow[rrr, bend left, "f"] & (A/{\sim}) \arrow[r, "\sim", "\widetilde{f}"'] & \operatorname{im} f \arrow[r, hook, "h"'] & B \end{tikzcd}$$

(in other words, $f = h \circ \widetilde{f} \circ g$)

...where g is the surjection mapping a to $[a]_\sim$ for all $a \in A$, h is the inclusion function (which is injective) mapping the image of f to B, and $\widetilde{f}$ is the function mapping $[a]_\sim$ to $f(a)$ where $a \in [a]_\sim$.