Math 140A Lecture Notes (Professor: Brandon Seward)

Isabelle Mills

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Lecture 1: 1/8/2024

An order on a set S, typically denoted as $<$, is a binary relation satisfying:

1. $\forall x, y \in S$, exactly one of the following is true:
   - $x < y$
   - $x = y$
   - $y < x$
2. given $x, y, z \in S$, we have that $x < y < z \Rightarrow x < z$

As a shorthand, we will specify that

- $x > y \Leftrightarrow y < x$
- $x \leq y \Leftrightarrow x < y$ or $x = y$
- $x \geq y \Leftrightarrow x > y$ or $x = y$

An ordered set is a set with a specified ordering. Let S be an ordered set and E be a nonempty subset of S.

- If $b \in S$ has the property that $\forall x \in E,\ x \leq b$, then we call b an upperbound to E and say that E is bounded above by b.
- if $b \in S$ has the property that $\forall x \in E,\ x \geq b$, then we call b an lower bound to E and say that E is bounded below by b.
- We call $\beta \in S$ the least upperbound to E if β is an upper bound to E and β is the least of all upperbounds to E. In this case, we also commonly call β the supremum of E and denote it as $\sup E$.
- We call $\beta \in S$ the greatest lower bound to E if β is an lower bound to E and β is the greatest of all lower bounds to E. In this case, we also commonly call β the infimum of E and denote it as $\inf E$.
- We call $e \in E$ the maximum of E if $\forall x \in E,\ x \leq e$
- We call $e \in E$ the minimum of E if $\forall x \in E,\ x \geq e$

Fact: For an ordered set S and nonempty $E \subseteq S$, either:

- neither $\max E$ nor $\sup E$ exists
- $\sup E$ exists but $\max E$ does not exist
- $\max E$ exists and $\sup E = \max E$

Using $\mathbb{Q}$ as our ordered set...

- For $E = \{q \in \mathbb{Q} \mid 0 < q < 1\}$, $\max E$ does not exist but $\sup E$ exists and equals 1.

  To understand why, note that the set of all upper bounds of E is equal to $\{q \in \mathbb{Q} \mid q \geq 1\}$ and 1 is obviously the smallest element of that set. Thus, 1 is the supremum of E. However, $1 \notin E$. Thus, if $\max E$ did exist, it would have to not equal 1. But that would contradict 1 being the least greatest bound.

- For $E = \{q \in \mathbb{Q} \mid 0 < q \leq 1\}$, $\max E$ and $\sup E$ exist and they both are equal to 1

  The reasoning for this is similar to that for the previous set.

- For $E = \{q \in \mathbb{Q} \mid q^2 < 2\}$, neither $\max E$ and $\sup E$ exist.

  To prove this, we can show there exists a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $\forall q \in \mathbb{Q}^+$, $q^2 < 2 \Rightarrow q^2 < (f(q))^2 < 2$ and $2 < q^2 \Rightarrow 2 < (f(q))^2 < q^2$. That way we can give a counter example to any possible claimed supremum or maximum of E.

  Now instead of being like Rudin and simply providing the desired function, I want to present how one may come up with a function that works for this proof themselves.

  Firstly, note that for the following reasons, we know our desired function must be a rational function:

  - ◇ $\forall q \in \mathbb{Q}$, $f(q) \in \mathbb{Q}$. Based on this, we can't use any radicals, trig functions, logarithms, or exponentials in our desired function.

  - ◇ $q^2 > 2 \Rightarrow f(q) < q$. In other words, f needs to grow slower than a linear function. Thus, we can rule out the possibility of f being a polynomial.

  - ◇ If we wanted f to be a linear function, it would have to have the form $f(q) = \alpha(q - \sqrt{2}) + \sqrt{2}$ where α is some constant. This is because when $q^2 = 2$, $f(q) = q$. However, there is no value one can set α to which both eliminates the presence of irrational numbers in that function while simultaneously making $f(q) \neq q$ when $q^2 \neq 2$. So no linear function can possibly work for this proof.

  Having narrowed our search, let's now pick some convenient properties we would wish our proof function to have. Specifically, let's force f to be constantly increasing, have a y-intercept of 1, and approach a horizontal asymptote of $y = 2$. Doing this, we can now say that an acceptable function will have the following form where α is an unknown constant:

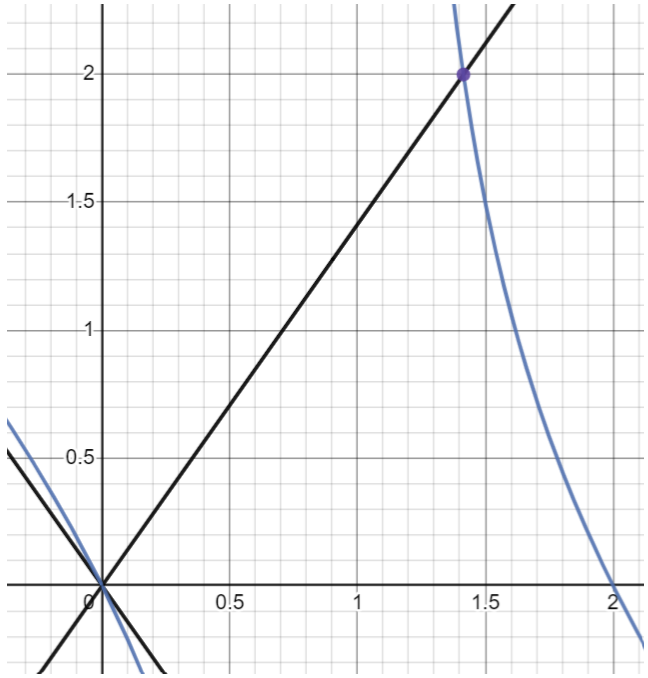
  $$f(q) = 1 + \frac{q}{q + \alpha}$$

  And finally, we can solve for α using the following system of equations:

  $$(1 + \tfrac{q}{q+\alpha})^2 = 2$$

  $$1 + \tfrac{q}{q+\alpha} = q$$

Now here's where a graphing calculator like Desmos can be very useful. Instead of painstakely having to solve for α, we can use a graphing calculator to approximate the value of α that satisfies our system of equations.



Based on the graph above, it looks like $f(q) = 1 + \frac{q}{q+2}$ will work for our proof. And sure enough it does. Furthermore, we can verify that the function we came up with is equivalent to that which Rudin presents.

We say an ordered set S has the least upperbound property if and only if when $E \subseteq S$ is nonempty and bounded above, then the supremum of E exists in S. Additionally, we say an ordered set S has the greatest lower bound property if and only if when $E \subseteq S$ is nonempty and bounded below, then the infimum of E exists in S.

When we define the set of real numbers, this will be one of the fundamental properties of that set.

Lecture 2: 1/10/2024

Proposition 1: S has the least upperbound property if and only if S has the greatest lower bound property.

Proof: Let's say we have an ordered set S

Assume S has the least upperbound property. Then, let $B \subseteq S$ be a nonempty subset which is bounded below. Additionally, let $A \subseteq S$ be the set of all lower bounds of B.

We know that $A \neq \emptyset$ because we assumed that B is bounded below. Thus, at least one lower bound to B exists and belongs to A. Additionally, because we assumed B is nonempty, we can say that each $b \in B$ is an upper bound to A. Thus, A is bounded above. Because of these two facts, we can apply the greatest lower bound property to say that the supremum of A exists.

Let's define $\alpha := \sup A$. With that, our goal is now to show that $\alpha = \inf B$. To do this, we need to show firstly that α is a lower bound to B and secondly that it is greater than all other lower bounds of B.

1. For each $b \in B$, we have that b is an upperbound to A. And since $\alpha = \sup A$ is the least upperbound to A, we must have that $\alpha \leq b$. Thus α is a lower bound to B.

2. If $x \in S$ is a lower bound to B, then $x \in A$. And since $\alpha = \sup A$, $x \leq \alpha$. This shows that α is greater than or equal to all other lower bounds.

Hence, α is the infimum of B. And since we did this for a general $B \subseteq S$, we can thus say that S has the greatest lower bound property.

Now we skipped doing the reverse direction proof because it is almost identical to the foward direction proof. However, just know that the above proposition is an if and only if statement. ■

A field is a set F equipped with 2 binary operations, denoted $+$ and $\cdot$, and containing two elements $0 \neq 1 \in F$ satisfying the following conditions for all $x, y, z \in F$:

- Associativity: $$(x+y)+z = x+(y+z)$$ $$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

- Commutativity: $$x + y = y + x$$ $$x \cdot y = y \cdot x$$

- Identity: $$0 + x = x$$ $$1 \cdot x = x$$

- Inverses: $$\forall x \in F,\ \ \exists {-x} \in F\ \ s.t.\ \ x + -x = 0$$ $$\forall x \neq 0 \in F,\ \ \exists \frac{1}{x} \in F\ \ s.t.\ \ x \cdot \frac{1}{x} = 1$$

- Distributivity: $$x \cdot (y+z) = (x \cdot y) + (x \cdot z)$$

We shall assign the following notation:

We write ______	to mean ______
$x - y$	$x + -y$
$\frac{x}{y}$	$x \cdot \frac{1}{y}$
2	$1 + 1$
$2x$	$x + x$
x^2	$x \cdot x$
xy	$x \cdot y$

Now what follows is a number of propositions concerning the arithmetic properties of a field...

For a field F and elements $x, y, z \in F$, we have the following propositions:

Proposition 2.1: $x + y = x + z \Rightarrow y = z$

Proof: Assume $x + y = x + z$. Then...

$$\begin{aligned} y &= 0 + y && \text{(addition identity property)} \\ &= (-x + x) + y && \text{(addition inverse property)} \\ &= -x + (x + y) && \text{(addition associative property)} \\ &= -x + (x + z) && \text{(by our assumption)} \\ &= (-x + x) + z && \text{(addition associative property)} \\ &= 0 + z && \text{(addition inverse property)} \\ &= z && \text{(addition identity property)} \end{aligned}$$

Proposition 2.2: $x + y = x \Rightarrow y = 0$

Proof: Plug in $z = 0$ into proposition 2.1. in order to get that $y = z = 0$.

Proposition 2.3: $x + y = 0 \Rightarrow y = -x$

Proof: Plug in $z = -x$ into proposition 2.1. in order to get that $y = z = -x$.

Proposition 2.4: $-(-x) = x$

Proof: Observe that $x + -x = -x + x = 0$ by the inverse and commutative properties of addition. Then, by proposition 2.3, we know that $-x + x = 0 \Rightarrow x = -(-x)$.

Proposition 2.5: $x \cdot y = x \cdot z$ and $x \neq 0 \Rightarrow y = z$

Proof: Assume $x \cdot y = x \cdot z$ and $x \neq 0$. Then...

$$\begin{aligned} y &= 1 \cdot y && \text{(multiplication identity property)} \\ &= (\tfrac{1}{x} \cdot x) \cdot y && \text{(multiplication inverse property)} \\ &= \tfrac{1}{x} \cdot (x \cdot y) && \text{(multiplication associative property)} \\ &= \tfrac{1}{x} \cdot (x \cdot z) && \text{(by our assumption)} \\ &= (\tfrac{1}{x} \cdot x) \cdot z && \text{(multiplication associative property)} \\ &= 1 \cdot z && \text{(multiplication inverse property)} \\ &= z && \text{(multiplication identity property)} \end{aligned}$$

Note that to use the multiplication inverse property, we have to assume $x \neq 0$!!

Proposition 2.6: $x \cdot y = x \Rightarrow y = 1$

Proof: Plug in $z = 1$ into proposition 2.5. in order to get that $y = z = 1$.

Proposition 2.7: $x \cdot y = 1 \Rightarrow y = \dfrac{1}{x}$

Proof: Plug in $z = \frac{1}{x}$ into proposition 2.5. in order to get that $y = z = \frac{1}{x}$.

Proposition 2.8: $\dfrac{1}{\frac{1}{x}} = x$

Proof: Observe that $x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$ by the inverse and commutative properties of multiplication. Then, by proposition 2.7, we know that $\frac{1}{x} \cdot x = 1 \Rightarrow x = \dfrac{1}{\frac{1}{x}}$.

Proposition 2.9: $0 \cdot x = 0$

Proof: $(0 \cdot x) + (0 \cdot x) = (0 + 0) \cdot x = 0 \cdot x$. Thus we have an expression of the form $a + b = a$ which we can use proposition 2.2 on. Hence, we can conclude $0 \cdot x = 0$.

Proposition 2.10: $x \neq 0$ and $y \neq 0 \Rightarrow x \cdot y \neq 0$

Proof: since $x, y \neq 0$, we can say that $x \cdot y \cdot \frac{1}{x} \cdot \frac{1}{y} = 1 \neq 0$. Now by proposition 2.9, $x \cdot y = 0 \Rightarrow (x \cdot y) \cdot \left(\frac{1}{x} \cdot \frac{1}{y}\right) = 0$. However, we know that is not the case. So $x \cdot y$ can't equal zero.

Lecture 3: 1/12/2024

Proposition 2.11: $(-x)y = -(xy) = x(-y)$

Proof: $xy + (-x)y = (x + -x)y = 0y = 0$. Thus by proposition 2.3, $(-x)y = -(xy)$. We can make a similar argument to also say that $x(-y) = -(xy)$.

Proposition 2.12: $(-x)(-y) = xy$

Proof: Using proposition 2.11, we can say that $(-x)(-y) = -(x(-y)) = -(-(xy))$. Then by proposition 2.4, we can conclude $-(-(xy)) = xy$.

An ordered field is a field F equipped with an ordering $<$ satisfying $\forall x, y, z \in F$:

OF1. $y < z \Rightarrow y + x < z + x$

OF2. $(x > 0 \text{ and } y > 0) \Rightarrow xy > 0$

For x in an ordered field, we call x positive if and only if $x > 0$. Similarly, we call x negative if and only if $x < 0$.

Proposition 3: For an ordered field F and $x, y, z \in F$, we have:

1. $x < y \Leftrightarrow -y < -x$

   Proof: By property OF1 of an ordered field, we can say that $x < y \Rightarrow x + (-x + -y) < y + (-x + -y) \Rightarrow -y < -x$.

2. $(x > 0 \text{ and } y < z) \Rightarrow xy < xz$

   Proof: By property OF1 of an ordered field, $y < z \Rightarrow y - y < z - y$. Or in other words, $0 < z - y$. Therefore, since x is also positive by assumption, property OF2 of an ordered field tells us that $x(z-y) > 0$. Finally, adding xy to both sides by property OF1 and then distributing gives us: $xz - xy + xy = xz < xy$.

3. $(x < 0 \text{ and } y < z) \Rightarrow xy > xz$

   Proof: Since $x < 0$, we have $-x > 0$ by proposition 3.1. Then by applying proposition 3.2, we know that $(-x > 0 \text{ and } y < z) \Rightarrow -xy < -xz$. Finally, by reapplying proposition 3.1, this becomes $xy > xz$.

4. $x \neq 0 \Rightarrow x^2 > 0$

   Proof: If $x > 0$, then $x^2 = xx > 0x = 0$ by property OF2 of an ordered field. Meanwhile, if $x < 0$, then $-x > 0$ by proposition 3.1. So $(-x)(-x) > 0$ by property OF2. But $(-x)(-x) = x^2$ by proposition 2.12. So $x^2 > 0$.

5. $0 < x < y \Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$

Proof: Since $y > 0$ and $y \cdot \frac{1}{y} = 1 > 0 = 0 \cdot \frac{1}{y}$, we must have $\frac{1}{y} > 0$ by propositions 3.2 and 3.3. Note that $\frac{1}{y} \neq 0$ because if it did, $y \cdot \frac{1}{y} = 0$. Similarly, we can show $\frac{1}{x} > 0$. Now multiply both sides of $x < y$ by the positive element $\frac{1}{x} \cdot \frac{1}{y}$ and apply proposition 3.2 to get that $\frac{1}{y} < \frac{1}{x}$.

Theorem: There is (up to isomorphism) precisely one ordered field that contains $\mathbb{Q}$ and has the least upper bound property. We denote this field $\mathbb{R}$ and we call its elements real numbers.

In other words, this theorem is stating that $\mathbb{R}$ exists and is unique. Unfortunately, the proof for this is very long and so won't be covered in lecture.

Proposition 4.1: If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$. This is called the archimedean property.

Proof: We proceed by looking for a contradiction. Let $A = \{nx \mid n \in \mathbb{Z}^+\}$ and assume $\nexists n \in \mathbb{Z}^+$ such that $nx > y$. In that case we know y is an upper bound of A. Additionally, since A is bounded above, we know by the least upper bound property of the real numbers that $\sup A$ exists. So, let $\alpha = \sup A$.

Now because $\mathbb{R}$ is an ordered field, we know that:
$x > 0 \Rightarrow -x < 0 \Rightarrow \alpha - x < \alpha$. Therefore, because α is the least upper bound, we know that $\alpha - x$ is not an upper bound for A. Or in other words, there exists $n \in \mathbb{Z}^+$ such that $nx > \alpha - x$. But this contradicts that α is the least upper bound of A because $nx > \alpha - x \Rightarrow (n+1)x > \alpha$ and $(n+1)x \in A$. So we conclude that the supremum of A can't exist, which by the contrapositive of the least upper bound property, means that A is not bounded above.

Proposition 4.2: If $x, y \in \mathbb{R}$ and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$. In other words, we say that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof: Since $x < y$, we have that $0 < y - x$. Then because $y - x$ is positive, we can use the archimedean property to say that there exists an integer n such that $n(y - x) > 1$. Note for later that this means $ny > 1 + nx$.

Now note that since $1 > 0$ and nx is a real number, we can use the archimedean property twice to get positive integers m_1 and m_2 such that $m_1 \cdot 1 > -nx$ and $m_2 \cdot 1 > +nx$. Thus, we get the expression $-m_1 < nx < m_2$. So now consider the set $B = \{m \in \mathbb{Z} \mid -m_1 \geq nx \geq m_2 \text{ and } m > nx\}$. We know that B has finitely many elements and that B contains at least one element: m_2. So B must have a minimum element. We'll refer to that minimum element as m. Notably, as m is the minimum element of B, we know that $m - 1 \notin B$, meaning that $m - 1 \leq nx < m$

We now combine inequalities as follows: $m - 1 \leq nx \Rightarrow m \leq nx + 1$. So we have that $nx < m \leq nx + 1$. But now remember from the previous page that $ny > 1 + nx$. So we can say that $nx < m \leq nx + 1 < ny$. Finally, because $n > 0$, we can multiply the inequality by $\frac{1}{n}$ to get that $x < \frac{m}{n} < y$. ■

Lecture 4: 1/17/2024

Theorem: If $x \in \mathbb{R}$, $x > 0$, $n \in \mathbb{Z}$, and $n > 0$, then there is a unique $y \in \mathbb{R}$ with $y > 0$ and $y^n = x$. This number y is denoted $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

To prove this, first note the following lemma about positive integers n and $a, b \in \mathbb{R}$:

$$b^n - a^n = (b - a)(b^{n-1} + ab^{n-2} + \ldots + a^{n-2}b + a^{n-1})$$

To prove this, one can either use induction or just calculate it out by hand to verify that the equality holds.

Additionally, also consider that if n is a positive integer and $0 \leq a \leq b$ where $a, b \in \mathbb{R}$, then we have that $a^n \leq b^n$. Combining this fact with the lemma above, we can say that $0 \leq a \leq b$ implies that $b^n - a^n \leq (b - a)nb^{n-1}$. Or in other words: $a^n \leq b^n \leq a^n + (b - a)nb^{n-1}$.

This comes from replasing every a in the expression $(b^{n-1} + ab^{n-2} + \ldots + a^{n-2}b + a^{n-1})$ with b in order to get that $b^n - a^n \leq (b - a)(b^{n-1} + b^{n-1} + \ldots + b^{n-1})$

Now set $E = \{t \in \mathbb{R} \mid t > 0, t^n \leq x\}$.

We can show that E is nonempty...

- If $x \geq 1$, then $t = 1 \in E$ since $1^n = 1 \leq x$.
- If $x < 1$, then $x \in E$ since $x < 1 \Rightarrow x^{n-1} < 1^{n-1} = 1$. But then $x^n < x$.

Thus, we know $E \neq \emptyset$.

We can also show that E is bounded above. Consider $t = 1 + x$. In that case, $t > 1$, which implies that $t^{n-1} > 1^{n-1} = 1$. Therefore, $t^n > t$, meaning that $t^n > x$. So $t = x + 1$ is an upper bound for E.

Thus by the least upper bound property of the real numbers, we know $y = \sup E$ exists.

Claim 1: $y^n \geq x$.

To prove this, we shall procede towards a contradiction. Assume $y^n < x$

Then pick some h such that $0 < h < \gamma$ and γ is some mystery constant for us to find. Then, we can say that $y < y + h$, meaning by the lemma on the previous page that $y^n \leq (y+h)^n \leq y^n + (y+h-y)n(y+h)^n - 1$. Or in other words, $(y+h)^n \leq y^n + hn(y+h)^n - 1$.

Now we shall make our first assumption about γ: let $\gamma \leq 1$. That way, we know that $(y+h)^n \leq y^n + hn(y+h)^n - 1 < y^n + hn(y+1)^n$. And since, we are assuming that $y^n < x$, we know there must exist some value of h such that $y^n + hn(y+1)^{n-1} < x$. Putting this limitation on h, we get that $h < \frac{x-y^n}{n(y+1)^{n-1}}$ (Remember that $x - y^n$, y, and n are all positive). So finally, we say that $\gamma = \min\left(1, \frac{x-y^n}{n(y+1)^{n-1}}\right)$. This is so that for $0 < h < \gamma$, we have that $(y+h)^n < x$.

Thus, we have a contradiction as we assumed that y is the supremum of E and yet we just proved that $y + h \in E$. So, y^n cannot be less than x, meaning that that $y^n \geq x$.

Claim 2: $y^n \leq x$.

To prove this, we shall again proceed towards a contradiction. Assume $y^n > x$.

Then for some h such that $0 < h < \gamma$ where γ is a new mystery constant, consider $y - h$.

I now realize that I need to prove this lemma: for a positive integer n and real numbers a and b such that $a \geq b$, we have that $(a-b)^n \geq a^n - bna^{n-1}$. We can prove this through induction.

Firstly for $n = 1$: we have that $(a-b)^1 = a^1 - b(1)a^0$.

Now assume that for $k \geq 1$, $(a-b)^k \geq a^k - bka^{k-1}$.
Then $(a-b)^{k+1} = (a-b)(a-b)^k$. And since $(a-b) > 1$, we know that $(a-b)^{k+1} = (a-b)(a-b)^k \geq (a-b)(a^k - bka^{k-1})$.

Now let's expand out our lesser term to get that:
$(a-b)^{k+1} \geq a^{k+1} - bka^k - ba^k + b^2ka^{k-1}$. Thus, we know that $(a-b)^{k+1} \geq a^{k+1} - b(k+1)a^k + b^2ka^{k-1} > a^{k+1} - b(k+1)a^k$. Hence, we have shown that $(a-b)^{k+1} \geq a^{k+1} - b(k+1)a^k$.

Based on the lemma covered right before this, we have that $(y-h)^n \geq y^n - hny^{n-1}$. But now let's require that $y^n - hny^{n-1} > x$. Thus, we can say that $h < \frac{y^n - x}{ny^{n-1}}$.

So setting $\gamma = \frac{y^n - x}{ny^{n-1}}$, we have that for $0 < h < \gamma,\ \ (y-h)^n > x$. But this now leads to a contradiction as $y - h$ must be an upper bound to E.

(If some number z is greater than $y-h$, than $z^n > (y-h)^n > x$. So $z \notin E$.)

However, $y - h$ can't be an upper bound to E as we specified that y is the least upper bound of E. So we conclude that y^n cannot be greater than x, thus meaning $y^n \leq x$.

So since $y^n \leq x$ and $y^n \geq x$, we conclude that $y^n = x$.

Finally, we now shall mention that y is obviously the unique number such that $y^n = x$. After all, for $0 < a < y < b$, we have that $a^n < y^n < b^n$. So, there can only be one number y such that $y^n = x$.

Lecture 5: 1/19/2024

Decimal representations of real numbers:

- Each $x \in \mathbb{R}$ such that $x > 0$ can be written $x = n_0.n_1n_2n_3\ldots$ where $n_0 \in \mathbb{Z}$ and $\forall i \geq 1$, $n_i \in \{0,1,2,3,4,5,6,7,8,9\}$.

  Specifically, let n_0 be the largest integer with $n \leq x$. Then inductively, pick n_k to be the max element in $\{0,1,2,3,4,5,6,7,8,9\}$ such that:

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_{k-1}}{10^{k-1}} + \frac{n_k}{10^k} \leq x$$

- Conversely, suppose $n_0 \in \mathbb{Z}$ and $\forall i \geq 1, n_i \in \{0,1,2,3,4,5,6,7,8,9\}$. Then, defining $E = \{n_0, n_0 + \frac{n_1}{10}, \ldots, n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k}, \ldots\}$, we have that $n_0.n_1n_2n_3\ldots = x \in \mathbb{R}$ where $x = \sup E$.

We will rarely ever use decimal representations though.

The extended real number system is the set $\mathbb{R} \cup \{-\infty, +\infty\}$ where for all $x \in \mathbb{R}$:

- $-\infty < x < +\infty$
- $x + \infty = +\infty$
- $x - \infty = -\infty$
- $\frac{x}{+\infty} = 0 = \frac{x}{-\infty}$
- $x > 0 \Rightarrow x(+\infty) = +\infty$
- $x > 0 \Rightarrow x(-\infty) = -\infty$
- $+\infty + \infty = +\infty$
- $+\infty(+\infty) = +\infty$
- $x < 0 \Rightarrow x(+\infty) = -\infty$
- $x < 0 \Rightarrow x(-\infty) = +\infty$
- $-\infty - \infty = -\infty$
- $+\infty(-\infty) = -\infty$

All other operation involving $+\infty$ and $-\infty$ are left undefined.

◇ Sometimes, we denote the extended real number system $\overline{\mathbb{R}}$.
◇ The extended real number system is not a field.
◇ To distinguish $x \in \mathbb{R}$ from ∞ or $-\infty$, we call $x \in \mathbb{R}$ finite.

The set of complex numbers, denoted $\mathbb{C}$, is the set of all things of the form $a + bi$ where $a, b \in \mathbb{R}$ and i is a symbol satisfying $i^2 = -1$.

To be more rigorous about this definition, what we would do is define the set of complex numbers to be the set of pairs of real numbers equipped with the following operations:

For $z, u \in \mathbb{C}$ such that $z = (a, b)$ and $u = (c, d)$:

- $z + u = (a + c, b + d)$
- $z \cdot u = (ac - bd, ad + bc)$

Having done that, we would then:

1. Define $0 = (0, 0)$ and $1 = (1, 0)$
2. Prove that $\mathbb{C}$ satisfies our field axioms
3. Say that $i = (0, 1)$ and then show that $i^2 = (-1, 0)$
4. And finally show that for $a, b \in \mathbb{R}$, $a(1) + b(i) = (a, b)$

(Thus it makes sense to denote $z \in \mathbb{C}$ as $z = a + bi$)

However, we're behind and so not going to spend time doing that in class.

For $z = a + bi$, we denote $\mathrm{Re}(z) = a$ the real part of z. On the other hand, we denote $\mathrm{Im}(z) = b$ the imaginary part of z.

The complex conjugate of $z = a + bi$ is $\overline{z} = a - bi$.

Proposition 5: If $z, w \in \mathbb{C}$, then:

1. $\overline{z+w} = \overline{z} + \overline{w}$
2. $\overline{zw} = \overline{z} \cdot \overline{w}$
3. $z + \overline{z} = 2\mathrm{Re}(z)$
4. $z - \overline{z} = 2\mathrm{Im}(z)i$
5. $z\overline{z} \in \mathbb{R}$ and $z\overline{z} > 0$ when $z \neq 0$.

Proof:
Points 1-4 can be verified by direct computation.
As for point 5, note that if $z = a+bi$, then $z\overline{z} = (a+bi)(a-bi) = a^2 + b^2$.
Now as $a, b \in \mathbb{R}$, we know that $a^2+b^2 \in \mathbb{R}$. But $a^2+b^2 > 0$ if $b \neq a \neq 0$. Meanwhile, $a^2 + b^2 = 0$ if $a = b = 0$. So $z\overline{z} > 0$ if $z \neq 0$

The absolute value of $z = a + bi$ is $|z| = \sqrt{z\,\overline{z}}$

Propostion 6: For $z, w \in \mathbb{C}$, we have that:

1. $|0| = 0$ and $|z| > 0$ when $z \neq 0$.
2. $|z| = |\overline{z}|$
3. $|zw| = |z||w|$
4. $|\mathrm{Re}(z)| \leq |z|$
5. $|\mathrm{Im}(z)| \leq |z|$
6. $|z + w| \leq |z| + |w|$

This last bullet is the triangle inequality.

Proof:
Claims 1, 2, and 3 can be verified through direct computation.

To prove claim 4, note that $a^2 \leq a^2 + b^2$. So, $|a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$. We can repeat this but with b^2 to prove claim 5.

Lastly, to prove claim 6, note that $|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\overline{z}+\overline{w})$. Now, we can distribute to get that $|z+w|^2 = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w}$. So, we know that $|z+w|^2 = |z|^2 + z\overline{w} + w\overline{z} + |w|^2$.

But now observe that $w\overline{z} = \overline{z\overline{w}}$. So $z\overline{w} + w\overline{z} = 2\mathrm{Re}(z\overline{w})$. But by claim 4, we know that $\mathrm{Re}(z\overline{w}) \leq |z\overline{w}|$. Additionally, by claims 2 and 3, we have that $|z\overline{w}| = |z||\overline{w}| = |z||w|$. So, we know that $|z+w|^2 \leq |z|^2 + 2|z||w| + |w|^2$. This simplifies to $|z+w|^2 \leq (|z|+|w|)^2$. Hence, $|z+w| \leq |z| + |w|$.

Lecture 6: 1/22/2024

Theorem: (the Cauchy-Schwarz Inequality)
If $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}$, then:

$$\left|\sum_{j=1}^{n} a_j\overline{b_j}\right|^2 \leq \sum_{j=1}^{n}|a_j|^2 \sum_{j=1}^{n}|b_j|^2$$

Proof:
Define $A = \sum_{j=1}^{n}|a_j|^2$, $B = \sum_{j=1}^{n}|b_j|^2$, and $C = \sum_{j=1}^{n} a_j\overline{b_j}$.
Note that $A, B \in \mathbb{R}$ such that $A, B \geq 0$. Meanwhile, $C \in \mathbb{C}$.

If $B = 0$, then $b_1 = \ldots = b_n = 0$. Thus $C = 0$ as well and so the inequality is trivially true.

So now consider if $B > 0$. Then we can make a series of manipulations starting with: $0 \leq \sum_{j=1}^{n}|Ba_j - Cb_j|^2$.

(The professor said not to worry about how Rudin thought of using this formula.)

$$\begin{aligned}
0 &\leq \sum_{j=1}^{n}|Ba_j - Cb_j|^2 \\
&= \sum_{j=1}^{n}(Ba_j - Cb_j)(B\overline{a_j} - \overline{C}\overline{b_j}) \\
&= B^2\sum_{j=1}^{n}|a_j|^2 - BC\sum_{j=1}^{n}\overline{a_j}b_j - B\overline{C}\sum_{j=1}^{n}a_j\overline{b_j} + |C|^2\sum_{j=1}^{n}|b_j|^2 \\
&= B^2A - BC\overline{C} - B\overline{C}C + |C|^2B \\
&= B^2A - B|C|^2 \\
&= B(AB - |C|^2)
\end{aligned}$$

Thus, since we're assuming $B > 0$, we know that $AB - |C|^2 \geq 0$.
So, $AB \geq |C|^2$. ■

We call elements $\vec{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ vectors or points. The x_i are the coordinates of $\vec{x}$.

The inner product or dot product of $\vec{x}$, $\vec{y} \in \mathbb{R}^k$ is: $\vec{x} \cdot \vec{y} = \sum_{i=1}^{k} x_i y_i$

The norm of $x \in \mathbb{R}^k$ is $\|\vec{x}\| = (\vec{x} \cdot \vec{x})^{\frac{1}{2}}$

Proposition 7: If $\vec{x}$, $\vec{y}$, $\vec{z} \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$, then:

1. $\|\vec{0}\| = 0$ and $\|\vec{x}\| > 0$ when $\vec{x} \neq \vec{0}$.
2. $\|\alpha\vec{x}\| = \alpha\|\vec{x}\|$
3. $\|\vec{x} \cdot \vec{y}\| \leq \|\vec{x}\|\|\vec{y}\|$
4. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$
5. $\|\vec{x} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\|$

The proofs for 1-4. are nearly identical to those for complex numbers so we won't cover them here.

As for 5, note that $\vec{x} + (-\vec{z}) = \vec{x} - \vec{y} + \vec{y} - \vec{z}$.

For sets X, Y and a function $f : X \to Y$, we shall write:

- for $A \subseteq X,\ \ f(A) = \{f(a) \mid a \in A\}$ (This is the image of A.)
- for $B \subseteq Y,\ \ f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ (This is the preimage of A.)
- for $y \in Y$, we write $f^{-1}(y)$ for $f^{-1}(\{y\})$

We say two sets A and B have equal cardinality, denoted $|A| = |B|$ if there is a bijection f from A onto B.

- A is finite if it has equal cardinality with $\{1, \ldots, n\}$ for some $n \in \mathbb{Z}^+$ or if $A = \emptyset$.
- A is countable if either A is finite or A has equal cardinality with $\mathbb{Z}^+$.
- A is uncountable if its not countable.

A sequence is a function f having domain $\mathbb{Z}^+$. If $f(n) = x_n \in A$ for each integer n, it is typical to denote f by $(x_n)_{n \in \mathbb{Z}^+}$ or more simply by (x_n).

Proposition 8: If A is countable and $E \subseteq A$, then E is countable.

Proof:

If E is finite, then E is countable and we're done. So assume E is infinite. Then as $E \subseteq A$, we know A is infinite as well.

Since A is countable, we can enumerate A as $x_1, x_2, x_3, \ldots$.
Set $n_1 = \min\{m \in \mathbb{Z}^+ \mid x_m \in E\}$. Then, inductively set $n_{k+1} = \min\{m \in \mathbb{Z}^+ \mid m > n_k$ and $x_m \in E\}$. Finally, define $f : \mathbb{Z}^+ \to E$ by the rule $f(k) = x_{n_k}$. That way f is a bijection.

Proposition 9: $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Proof:

Define $g : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $g(i,j) = \left(\sum_{k=1}^{i+j-2} k\right) + j$.

This is a bijection.

Here's what the bijection looks like:

$$\begin{array}{cccc}
(1,1) & (1,2) & (1,3) & \cdots \\
(2,1) & (2,2) & (2,3) & \cdots \\
(3,1) & (3,2) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

Lecture 7: 1/24/2024

Proposition 10: If A is a countable set, B any set, and $g : A \to B$ is a surjection, then B is countable.

Proof:

If A is finite, then $B = g(A)$ is finite as well. So the proposition is trivially true. Now assume A is infinite. Then since A is countable, there is a bijection $f : \mathbb{Z}^+ \to A$. Now define $\phi = g \circ f : \mathbb{Z}^+ \to B$. We know that ϕ is a surjection as it is the composition of two surjections.

Let $E \subseteq \mathbb{Z}^+$ be any set that contains precisely one element of $\phi^{-1}(b)$ for each $b \in B$. For instance, we can define E as the set:

$$\{n \in \mathbb{Z}^+ \mid \forall m \in \mathbb{Z}^+,\ m < n \Rightarrow \phi(m) \neq \phi(n)\}$$

Now by proposition 8, we know that E is countable as E is a subset of a countable set. But additionally we have that ϕ acts as a bijection from E to B. Therefore, $|E| = |B|$, meaning B is countable.

Proposition 11: A set A is countable if and only if there exists a surjection from $\mathbb{Z}^+$ onto A.

Proof:
($\Longleftarrow$) Since $\mathbb{Z}^+$ is the definition of a countable set, if there is a surjection from $\mathbb{Z}^+$ to A, then we have by proposition 10 that A is also countable.

($\Longrightarrow$) Assume A is countable. If A is finite, then we can number the elements of A as $\{a_1, a_2, \ldots, a_n\}$. So, we may define the surjection $f : \mathbb{Z}^+ \to A$ with the correspondance rule:

$$f(k) = \begin{cases} a_k & \text{if } k \leq n \\ a_n & \text{if } k > n \end{cases}$$

Meanwhile if A is infinite, then by definition there exists a bijection from $\mathbb{Z}^+$ to A. So, no matter if A is infinite or finite, if A is countable, then there exists a bijection from $\mathbb{Z}^+$ to A.

Proposition 12: If E_n is a countable set for each $n \in \mathbb{Z}^+$, then $\displaystyle\bigcup_{n \in \mathbb{Z}^+} E_n$ is countable.

Proof:
For each $n \in \mathbb{Z}^+$, there is a surjection $f_n : \mathbb{Z}^+ \to E_n$.
Define $g : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \displaystyle\bigcup_{n \in \mathbb{Z}^+} E_n$ by $g(n, k) = f_n(k)$.

Then as g is a surjection and $\mathbb{Z} \times \mathbb{Z}$ is countable by proposition 9, we know by proposition 10 that $\displaystyle\bigcup_{n \in \mathbb{Z}^+} E_n$ is countable.

In other words, the union of countably many countable sets is countable.

Proposition 13: If A is countable, then for every $n \in \mathbb{Z}^+$, the set $A^n = A \times A \times \ldots \times A$ is countable.

Proof: (we can proceed by induction)
When $n = 1$, then $A^n = A^1 = A$ is obviously countable.
Now assume the proposition is true for $n - 1$, meaning A^{n-1} is countable.
Then: $A^n = \displaystyle\bigcup_{a \in A} \{a\} \times A^{n-1}$ is countable by proposition 12.

Corollary: $\mathbb{Q}$ is countable.

Proof:

Define $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}^+$ by setting $f(p) = (n, m)$ where n, m are the unique coprime integers with $m > 0$, $\frac{n}{m} = p$. Also define $f(0) = (1, 0)$ Then $f(\mathbb{Q}) \subset \mathbb{Z} \times \mathbb{Z}^+$ and the latter set is countable. So $f(\mathbb{Q})$ is countable. Since f is injective, f is a bijection between $\mathbb{Q}$ and a countable set. Thus $\mathbb{Q}$ is countable.

Given sets A and B, we write A^B to denote the set of all functions from B to A.

Proposition 14: $\{0,1\}^{\mathbb{Z}^+}$ is uncountable.

Proof: Let $\{f_1, f_2, \ldots\}$ be any countable subset of $\{0,1\}^{\mathbb{Z}^+}$. Then define $g \in \{0,1\}^{\mathbb{Z}^+}$ by the rule $g(n) = 1 - f_n(n)$. Since $g(n) \neq f_n(n)$, we have that $g \neq f_n$. Since this holds for all $n \in \mathbb{Z}^+$, we can thus conclude that $g \notin \{f_1, f_2, \ldots\}$. We thus conclude that any countable subset of $\{0,1\}^{\mathbb{Z}^+}$ is a proper subset. So $\{0,1\}^{\mathbb{Z}^+}$ must be uncountable.

Lecture 8: 1/26/2024

A metric space is a set X equipped with a function $d : X \times X \longrightarrow [0, \infty)$ satisfying:

1. $\forall p, q \in X \quad p \neq q \Rightarrow d(p,q) > 0$ whereas $p = q \Rightarrow d(p,q) = 0$
2. $\forall p, q \in X \quad d(p,q) = d(q,p)$
3. $\forall p, q, s \in X \quad d(p,q) \leq d(p,s) + d(s,q)$

The function d is called a distance function or metric.

Examples:

- $\mathbb{R}^k$ is a metric space (we have several metrics to choose from):
  - $d_p(\vec{x}, \vec{y}) = (\sum |x_i - y_i|^p)^{\frac{1}{p}}$

    $d_2(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$
  - $d_\infty(\vec{x}, \vec{y}) = \max\limits_{1 \leq i \leq k} |x_i - y_i|$
- Any set X is a metric space when equipped with the discrete metric:
$$d(p,q) = \begin{cases} 0 \text{ if } p = q \\ 1 \text{ if } p \neq q \end{cases}$$
- The set of all functions from $[0,1] \to [0,1]$ can be equipped with the metric:
$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

Let X be a metric space. Then for $p \in X$ and $r > 0$, the (open) ball of radius r around p is $B_r(p) = \{q \in X \mid d(p,q) < r\}$.

$p \in X$ is a limit point of $E \subseteq X$ if there are points in $E \setminus \{p\}$ that are arbitrarily close to p. Or in other words, if p is a limit point, then

$$\forall r > 0, \quad ((B_r(p) \setminus \{p\}) \cap E) \neq \emptyset.$$

The set of limit points of $E \subseteq X$ is denoted E'.

- E is closed if $E' \subseteq E$.
- E is perfect if $E = E'$.
- We say E is dense in X if $E \cup E' = X$

$p \in X$ is an interior point of $E \subseteq X$ if $\exists r > 0 \ \ s.t. \ \ B_r(p) \subseteq E$.

The set of interior points of E is denoted E°.

- E is open if $E^\circ = E$.
- E is a neighborhood of p if $p \in E^\circ$.

The complement of E is $E^{\mathsf{C}} = X \setminus E$.
E is bounded if there is a point $p \in X$ and $R > 0$ with $E \subseteq B_R(p)$.
If $p \in E$, we say p is an isolated point of E if $\exists r > 0 \ \ s.t. \ \ B_r(p) \cap E = \{p\}$.

Proposition 15: If X is a metric space, $p \in X$, and $r > 0$, then $B_r(p)$ is open.

Proof:
Consider a point $q \in B_r(p)$. We claim that $B_{(r-d(p,q))}(q) \subseteq B_r(p)$.
To prove this consider that for $z \in B_{(r-d(p,q))}(q)$, we have that $d(p,z) \leq d(p,q) + d(q,z) < d(p,q) + (r - d(p,q)) = r$. Thus, $z \in B_r(p)$.
And, since we can do this for any $z \in B_{(r-d(p,q))}(q)$, we know that $B_{(r-d(p,q))}(q) \subseteq B_r(p)$. Therefore q is an interior point of $B_r(p)$. And, since we can say this for any $q \in B_r(p)$, we thus conclude that $B_r(p)$ consists of interior points. So $B_r(p)$ is open.

Lecture 9: 1/29/2024

Let X be a metric with metric d and let $E \subseteq X$...

Proposition 16: If $p \in E'$, then $(B_r(p) \setminus \{p\}) \cap E$ is infinite for every $r > 0$.

Proof (by contrapositive):
Let $p \in X$ and suppose $\exists r > 0$ with $(B_r(p) \setminus \{p\}) \cap E$ finite.
Then set $t = \min\{d(p,q) \mid q \in (B_r(p) \setminus \{p\}) \cap E\}$. That way, we must have that $t > 0$. But at the same time, $B_t(p) \setminus \{p\} \cap E$ is empty. Therefore $p \notin E'$.

Corollary: If E is finite, then $E' = \emptyset$. This means that finite sets are always closed.

Propostion 17: E is open if and only if E^{C} is closed.

Proof:

$$
\begin{aligned}
E^{\mathsf{C}} \text{ is closed} &\iff \left(E^{\mathsf{C}}\right)' \subseteq E^{\mathsf{C}} \\
&\iff \left(E^{C}\right)' \cap E = \emptyset \\
&\iff \forall p \in E,\ \ p \notin \left(E^{\mathsf{C}}\right)' \\
&\iff \forall p \in E,\ \ \exists r > 0\ \ s.t.\ \ (B_r(p) \setminus \{p\}) \cap E^{\mathsf{C}} = \emptyset \\
&\iff \forall p \in E,\ \ \exists r > 0\ \ s.t.\ \ B_r(p) \setminus \{p\} \subseteq E \\
&\iff \forall p \in E,\ \ \exists r > 0\ \ s.t.\ \ B_r(p) \subseteq E \\
&\iff \forall p \in E,\ \ p \in E^{\circ} \\
&\iff E \text{ is open}
\end{aligned}
$$

Corollary: E is closed if and only if E^{C} is open.

Proposition 18: Let A be any set.

1. If $u_\alpha \subseteq X$ is an open set for each $\alpha \in A$, then $\bigcup_{\alpha \in A} u_\alpha$ is open.

   Proof:
   Let $p \in \bigcup_{\alpha \in A} u_\alpha$. Pick $\beta \in A$ with $p \in u_\beta$.
   Since u_β is open, we know that $\exists r > 0\ \ s.t.\ \ B_r(p) \subseteq u_\beta \subseteq \bigcup_{\alpha \in A} u_\alpha$.
   So p is an interior point. Hence, we conclude that $\bigcup_{\alpha \in A} u_\alpha$ is open.

2. If $F_\alpha \subseteq X$ is a closed set for each $\alpha \in A$, then $\bigcap_{\alpha \in A} F_\alpha$ is closed.

   Proof:
   $\left(\bigcap_{\alpha \in A} F_\alpha\right)^{\mathsf{C}} = \bigcup_{\alpha \in A} (F_\alpha)^{\mathsf{C}}$ by De Morgan's laws.

Since each F_α is closed, we know each $(F_\alpha)^C$ is open.
So by proposition 18.1, we know that $\bigcup_{\alpha \in A} (F_\alpha)^C$ is open.

Then, by proposition 17, we know that its complement, $\bigcap_{\alpha \in A} F_\alpha$ is closed.

3. If $u_1, u_2, \ldots, u_n \subseteq X$ are open, then $\bigcap_{i=1}^{n} u_i$ is open.

   Proof:
   Let $p \in \bigcap_{i=1}^{n} u_i$. Then $p \in u_i$ for every i.
   Since u_i is open, $\exists r_i > 0 \;\; s.t. \;\; B_{r_i}(p) \subseteq u_i$. Therefore, set $r = \min\{r_i \mid 1 \le i \le n\}$ so that for all i, $B_r(p) \subseteq B_{r_i}(p) \subseteq u_i$.
   Hence, $B_r(p) \subseteq \bigcap_{i=1}^{n} u_i$. We thus conclude that $\bigcap_{i=1}^{n} u_i$ is open.

4. If $F_1, F_2, \ldots, F_n \subseteq X$ are closed, then $\bigcup_{i=1}^{n} F_i$ is closed.

   The proof of this follows from proposition 18.3 in the same way that proposition 18.2 follows from proposition 18.1.

Lecture 10: 2/2/2024

Given a metric space X, the closure of $E \subseteq X$ is $\overline{E} := E \cup E'$.

Proposition 19.1: $\overline{E}$ is closed.

Proof:
Let $p \in (\overline{E})^C$. Thus, $p \notin E'$, meaning that we can fix $r > 0$ so that $(B_r(p) \backslash \{p\}) \cap E = \emptyset$. Additionally, since $p \notin E$, we have that $B_r(p) \cap E = \emptyset$.

Now consider any $q \in B_r(p)$. Setting $t = r - d(p,q)$, we have that $B_t(q) \subseteq B_r(p)$. Therefore, since $B_r(p) \cap E = \emptyset$, we know $B_t(q) \cap E = \emptyset$. This tells us that $q \notin E'$. Hence, $B_r(p) \cap E' = \emptyset$.

We've now shown that $B_r(p) \cap E = \emptyset$ and that $B_r(p) \cap E' = \emptyset$.
Therefore, $B_r(p) \cap (E \cup E') = B_r(p) \cap \overline{E} = \emptyset$, meaning that $B_r(p) \subseteq (\overline{E})^C$.
So $(\overline{E})^C$ is open, meaning that $\overline{E}$ is closed.

Proposition 19.2: $E = \overline{E}$ if and only if E is closed.

Proof:

($\Longrightarrow$) If $\overline{E}$ is closed by proposition 19.1. So $E = \overline{E}$ implies E is closed.

($\Longleftarrow$) If E is closed, then $E' \subseteq E$. Hence, $\overline{E} = E \cup E' = E$

Proposition 19.3: If F is closed and $F \supseteq E$, then $F \supseteq \overline{E}$.

Proof:

Observe that if F is any set and $E \subseteq F$, then $E' \subseteq F'$. Thus, if F is also closed, we have that $E' \subseteq F' \subseteq F$. Therefore, $F = F \cup F' \supseteq E \cup E' = \overline{E}$.

Note that in this class, unless it is mentioned otherwise, you should assume that we are equipping $\mathbb{R}$ or $\mathbb{R}^k$ with the Euclidean metric: d_2.

Proposition 20: If $E \subseteq \mathbb{R}$ is nonempty and bounded above, then $\sup E \in \overline{E}$.

Proof:

Set $y = \sup E$. If $y \in E$, then we are done. So assume $y \notin E$.
Consider any $r > 0$. Since $y - r < y = \sup E$, we know $y - r$ is not an upperbound to E. Hence, there is $e \in E$ with $y - r < e < y$. Therefore, $(B_r(y) \setminus \{y\}) \cap E \neq \emptyset$. Hence, we conclude that $y \in E' \subseteq \overline{E}$.

Note that if X is a metric space with metric d and $Y \subseteq X$, then Y is also a metric space with d when d is restricted to Y.

$E \subseteq Y \subseteq X$ is open/closed/etc. relative to Y if E is open/closed/etc. in the metric space Y.

If $Y \subseteq X$ and $B_r(p)$ denotes the ball of radius r around $p \in Y$ in the metric space X, then the ball of radius r around p in the metric space Y is $B_r(p) \cap Y$.

Proposition 21: Let $E \subseteq Y \subseteq X$. Then E is open relative to Y if and only if there is an open set $U \subseteq X$ with $E = U \cap Y$.

Proof:

($\Longrightarrow$) For each $p \in E$, pick $r(p) > 0$ so that $B_{r(p)}(p) \cap Y \subseteq E$. Then, setting $U = \bigcup_{p \in E} B_{r(p)}(p)$, we have that U is open and that

$$E = \bigcup_{p \in E} \{p\} \subseteq \bigcup_{p \in E} B_{r(p)}(p) \cap Y = U \cap Y \subseteq E$$

So $U \cap Y = E$.

($\Longleftarrow$) Now say that $E = U \cap Y$ where $U \subseteq X$ is open. Also let $p \in E$. We know $p \in U$. Additionally, since U is open, there is $r > 0$ with $B_r(p) \subseteq U$. Consequently, $B_r(p) \cap Y \subseteq U \cap Y = E$. So, p is an interior point of E relative to Y. We conclude that E is open relative to Y.

Let X be a metric space. An open cover of $E \subseteq X$ is a collection $\{u_\alpha \mid \alpha \in A\}$ of open sets u_α satisfying:

$$E \subseteq \bigcup_{\alpha \in A} u_\alpha$$

$K \subseteq X$ is compact if every open cover of K containsa finite subcover of K.

More precisely: K is compact if and only if for ever open cover $\{u_\alpha \mid \alpha \in A\}$ of K, there is $n \in \mathbb{Z}^+$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ such that:

$$K \subseteq \bigcup_{i=1}^{n} u_{\alpha_i}$$

As an aside, compactness often acts as a generalization of finiteness in topology.

Lecture 11: 2/5/2024

Finite sets are compact.

Proposition 22: compactness is an intrinisic property, meaning if $K \subseteq Y \subseteq X$, then K is compact relative to X if and only if K is compact relative to Y.

Proof:

($\Longrightarrow$) Consider any collection of sets $v_\alpha \subseteq Y$ that are open relative to Y and satisfy that $K \subseteq \bigcup\limits_{\alpha \in A} v_\alpha$.

By a previous theorem, we know there are sets w_α open relative to X such that $v_\alpha = w_\alpha \cap Y$. So we have that $K \subseteq \bigcup\limits_{\alpha \in A} v_\alpha \subseteq \bigcup\limits_{\alpha \in A} w_\alpha$.

If K is compact relative to X, then there exists $n \in \mathbb{Z}^+$ and $\alpha_1, \ldots, \alpha_n \in A$ such that $K \subseteq \bigcup\limits_{i=1}^{n} w_{\alpha_i}$. And since $K \subseteq Y$, we have that:

$$K = K \cap Y \subseteq \left(\bigcup_{i=1}^{n} w_{\alpha_i}\right) \cap Y = \left(\bigcup_{i=1}^{n} v_{\alpha_i}\right)$$

Hence, K is compact relative to Y.

($\Longleftarrow$) Now consider any set K which is compact relative to Y and open cover $\{w_\alpha \mid \alpha \in A\}$ such that $w_\alpha \subseteq X$ and $K \subseteq \bigcup_{\alpha \in A} w_\alpha$.

By proposition 21, we know that $v_\alpha = w_\alpha \cap Y$ is open relative to Y. So as $K \subseteq Y$, we have that $K = K \cap Y \subseteq \bigcup_{\alpha \in A} w_\alpha \cap Y = \bigcup_{\alpha \in A} v_\alpha$.

But that means that $\{v_\alpha \mid \alpha \in A\}$ forms an open cover of K relative to Y. So, there exists $n \in \mathbb{Z}^+$ and $\alpha_1, \ldots, \alpha_n \in A$ such that $\{v_{\alpha_1}, \ldots, v_{\alpha_n}\}$ is a finite cover of K. Then note that:

$$K \subseteq \bigcup_{i=1}^{n} v_{\alpha_i} \subseteq \bigcup_{i=1}^{n} w_{\alpha_i}$$

So, $\{w_{\alpha_1}, \ldots, w_{\alpha_n}\}$ forms a finite subcover of K using sets in our original arbitrary open cover. Therefore, we conclude that K is compact relative to X.

Proposition 23: Compact sets are closed.

Proof:
Let $K \subseteq X$ be compact. It then suffices to show that K^{C} is open. So, consider any $p \in K^{\mathsf{C}}$. We know that $\{B_{\frac{1}{3}d(p,q)}(q) \mid q \in K\}$ forms an open cover of K. Additionally, because K is compact, there exists $n \in \mathbb{Z}^+$ and $q_1, \ldots, q_n \in K$ such that:

$$K \subseteq \bigcup_{i=1}^{n} B_{\frac{1}{3}d(p,q_i)}(q_i)$$

Thus, let $r = \min\{d(p, q_i) \mid 1 \leq i \leq n\}$. That way, $\frac{1}{3}r > 0$ and

$$\left(\bigcup_{i=1}^{n} B_{\frac{1}{3}d(p,q_i)}(q_i)\right) \cap B_{\frac{1}{3}r}(p) = \emptyset.$$

This then means that $K \cap B_{\frac{1}{3}r}(p) = \emptyset$, meaning that $B_{\frac{1}{3}r}(p) \subseteq K^{\mathsf{C}}$. So p is an interior point of K^{C}. We thus conclude that K^{C} is open.

Proposition 24: K is compact and $F \subseteq K$ is closed implies that F is compact.

Proof:
Consider any open cover $\{v_\alpha \mid \alpha \in A\}$ of F. Since F is closed, F^{C} is open. So, we can say that $\{F^{\mathsf{C}}\} \cup \{v_\alpha \mid \alpha \in A\}$ is an open cover of K as:

$$\left(\bigcup_{\alpha \in A} v_\alpha\right) \cup F^{\mathsf{C}} \supseteq F \cup F^{\mathsf{C}} \supseteq K$$

Since K is compact, there exists $n \in \mathbb{Z}^+$ and $\alpha_1, \dots, \alpha_n \in A$ such that:

$$K \subseteq \left(\bigcup_{i=1}^{n} v_{\alpha_i} \right) \cup F^{\mathsf{C}}$$

F^{C} may or may not be needed to cover K. However, its inclusion doesn't effect the finiteness of the cover.

Therefore $F \subseteq \bigcup_{i=1}^{n} v_{\alpha_i}$. So, F is compact.

Corollary: K is compact and F is closed implies that $K \cap F$ is compact.

Proof: K being compact means that K is closed. Thus $K \cap F$ is closed. And as $K \cap F$ is a subset of K, by the above theorem we have that $K \cap F$ is compact.

Theorem (the Finite Intersection Property): If $\{K_\alpha \mid \alpha \in A\}$ is any collection of compact sets in X having the property that the intersection of any finitely many of the K_α's is nonempty, then:

$$\bigcap_{\alpha \in A} K_\alpha \neq \emptyset$$

Proof: (we shall proceed by proving the contrapositive...)
Assume that $\bigcap_{\alpha \in A} K_\alpha = \emptyset$. Thus, taking complements gives: $\bigcup_{\alpha \in A} (X \setminus K_\alpha) = X$.

Pick any $\alpha_0 \in A$. Then $\{X \setminus K_\alpha \mid \alpha \in A\}$ is an open cover of K_{α_0} because $K_{\alpha_0} \subseteq X$ and because each $X \setminus K_\alpha$ must be open due to K_α being closed.

As K_{α_0} is compact, there exists $n \in \mathbb{Z}^+$ and $\alpha_1, \dots, \alpha_n \in A$ such that:

$$K_{\alpha_0} \subseteq \bigcup_{i=1}^{n} (X \setminus K_{\alpha_i})$$

Taking complements again, we get that: $X \setminus K_{\alpha_0} \supseteq \bigcap_{i=1}^{n} (K_{\alpha_i})$. So:

$$\left(\bigcap_{i=1}^{n} (K_{\alpha_i}) \right) \cap K_{\alpha_0} = \bigcap_{i=0}^{n} (K_{\alpha_i}) = \emptyset$$

Proposition 25: If K is compact and $E \subseteq K$ is infinite, then $E' \neq \emptyset$.

Proof: (we shall proceed by proving the contrapositive...)
Let $E \subseteq K$ and suppose $E' = \emptyset$. Then for each $q \in K$, since $q \notin E'$, we can pick $r(q) > 0$ such that $(B_{r(q)}(q) \setminus \{q\}) \cap E = \emptyset$. In particular, $(B_{r(q)}(q)) \cap E \subseteq \{q\}$.

Now note that $\bigcup_{q \in K} B_{r(q)}(q)$ is an open cover of K.

Since K is compact, we can pick $q_1, \ldots, q_n \in K$ so that $K \subseteq \bigcup_{i=1}^{n} B_{r(q_i)}(q_i)$.

Then, $E = E \cap K \subseteq E \cap \left(\bigcup_{i=1}^{n} B_{r(q_i)}(q_i) \right) = \bigcup_{i=1}^{n} \left(B_{r(q_i)}(q_i) \cap E \right) \subseteq \bigcup_{i=1}^{n} q_i$.

Hence E is finite.

Lecture 12: 2/7/2024

In $\mathbb{R}$, we define the interval $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$.

Proposition 26: If $I_n = [a_n, b_n] \neq \emptyset$ and $I_{n+1} \subseteq I_n$ for all n, then $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

Proof: For all $n, m \in \mathbb{N}$, we have $a_n \leq a_{n+m} \leq b_{n+m} \leq b_m$. Thus, for all m we have that b_m is an upperbound to $\{a_n \mid n \in \mathbb{N}\}$. This means that by the least upper bound property of $\mathbb{R}$, we know that $\alpha = \sup \{a_n \mid n \in \mathbb{N}\}$ exists and that $a_m \leq \alpha \leq b_m$ for all m. Hence, $\alpha \in \bigcap_{n \in \mathbb{N}} I_n$, which means $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

Corollary: If $C_n = [a_{n,1}, b_{n,1}] \times [a_{n,2}, b_{n,2}] \times \cdots \times [a_{n,k}, b_{n,k}] \subseteq \mathbb{R}^k$ and $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$.

Proof: $$\bigcap_{n \in \mathbb{N}} C_n = \left(\bigcap_{n \in \mathbb{N}} [a_{n,1}, b_{n,1}] \right) \times \cdots \times \left(\bigcap_{n \in \mathbb{N}} [a_{n,1}, b_{n,1}] \right) \neq \emptyset$$

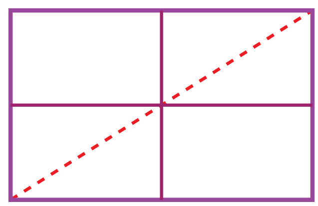
Proposition 27: $C = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k] \subseteq \mathbb{R}^k$ is compact.

Proof: (we'll proceed by finding a contradiction) Suppose $\{u_\alpha \mid \alpha \in A\}$ is an open cover of C containing no finite subcover of C.

Then set $\delta = \sqrt{\sum_{i=1}^{k} b_i - a_i}$

(this is the length of the largest diagonal of C.)

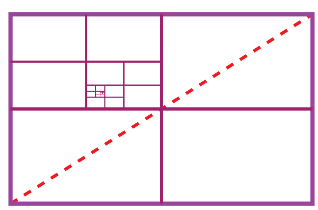
Since C forms a k-dimensional rectangle in $\mathbb{R}^k$, we can divide C into 2^k many pieces by cutting each side of C at its midpoints. Then each smaller piece will have a longest diagonal of length $\frac{1}{2}\delta$.



Since C can't be covered by finitely many u_α, there must be a piece, call it C_1, which cannot be covered by finitely many u_α. Also $C_1 \subseteq C$.

Now proceed inductively to build a sequence C_n such that:

1. $C_{n+1} \subset C_n$ for all n
2. The largest diagonal of C_n is $2^{-n}\delta$
3. C_n cannot be covered by finitely many u_α



By the above corollary, $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$. Thus consider $z \in \bigcap_{n \in \mathbb{N}} C_n \subseteq C$.

Pick $\alpha \in A$ with $z \in u_\alpha$. Since u_α is open, there exists $r > 0$ with $B_r(z) \subseteq u_\alpha$. Now pick n with $2^{-n}\delta < r$. Then as $z \in C_n$, we know $C_n \subseteq B_r(z) \subseteq u_\alpha$. This contradicts the point that C_n cannot be covered by finitely many u_α.

So we conclude C is compact.

Proposition 28: For $E \in \mathbb{R}^k$, the following are equivalent:

a. E is closed and bounded.

b. E is compact

c. Every infinite subset of E has a limit point in E.

Proof:

(a. $\implies$ b.) E is bounded means $E \subseteq C$ for some bounded rectangle C. Since E is closed and C is compact, E is compact by proposition 24.

(b. $\implies$ c.) This is just proposition 25.

(c. $\implies$ a.) Firstly, let us show that E is bounded.

Suppose E is not bounded. Then for all $n \in \mathbb{N}$, pick $\vec{x}_n \in E$ with $\|\vec{x}_n\| \geq n$. For any $\vec{y} \in \mathbb{R}^k$, we have that: $\vec{x}_n \in B_1(\vec{y}) \implies \|\vec{x}_n\| \leq \|\vec{y}\| + 1$, which in turn implies that $n \leq \|\vec{y}\| + 1$. So $(B_1(\vec{y}) \setminus \{\vec{y}\}) \cap \{\vec{x}_n \mid n \in \mathbb{N}\}$ is finite. As a result, we know $\vec{y} \notin \{\vec{x}_n \mid n \in \mathbb{N}\}'$. Thus, $\{\vec{x}_n \mid n \in \mathbb{N}\}$ has no limit points. But this is a contradiction because $\{\vec{x}_n \mid n \in \mathbb{N}\}$ is an infinite subset of E.

Now, let us show that E is closed.

Let $y \in E'$. Then for each $n \in \mathbb{Z}^+$, we can pick $\vec{x}_n \in B_{\frac{1}{n}}(\vec{y}) \cap E$. Now if $\vec{z} \in \mathbb{R}^k$ and $\vec{z} \neq \vec{y}$, then:

$$\begin{aligned}
\vec{x}_n \in B_{\frac{1}{2}\|\vec{y}-\vec{z}\|}(\vec{z}) \quad &\Rightarrow \quad \|\vec{y} - \vec{z}\| \leq \|\vec{y} - \vec{x}_n\| + \|\vec{x}_n - \vec{z}\| \\
&\qquad\qquad\qquad < \|\vec{y} - \vec{x}_n\| + \tfrac{1}{2}\|\vec{y} - \vec{z}\| \\
&\Rightarrow \quad \tfrac{1}{2}\|\vec{y} - \vec{z}\| < \|\vec{y} - \vec{x}_n\| < \tfrac{1}{n} \\
&\Rightarrow \quad n < \tfrac{2}{\|\vec{y} - \vec{x}_n\|}
\end{aligned}$$

Therefore: $\left(B_{\frac{1}{2}\|\vec{y}-\vec{z}\|}(\vec{z}) \setminus \{\vec{z}\}\right) \cap \{\vec{x}_n \mid n \in \mathbb{Z}^+\}$ is finite.

So $\vec{z} \notin \{\vec{x}_n \mid n \in \mathbb{N}\}'$, which means that $\vec{y}$ is the unique limit point of $\{\vec{x}_n \mid n \in \mathbb{N}\}$. Finally, since we assumed that any infinite subset of E has at least one limit point inside E, we know that $\vec{y} \in E$ because it is the only possible limit point that can fulfill this requirement.

Proposition 29: (<u>Bolzano-Weierstrauss Theorem</u>): Every bounded infinite subset of $\mathbb{R}^k$ has a limit point.

Proof: Let E be a bounded infinite subset of $\mathbb{R}^k$. Then $\overline{E}$ is closed and bounded, meaning that every infinite subset of $\overline{E}$ has a limit point in $\overline{E}$, meaning $(\overline{E})' \neq \emptyset$. Finally, we know from a homework question last week that $(\overline{E})' = E'$. So, $E' \neq \emptyset$.

The Cantor Set is very important as a counter example in topology. It is constructed as follows:

Let $E_0 = \{[0,1]\}$. Then for $n > 0$, inductively define E_n as a set containing closed intervals of the first and last thirds of each interval in E_{n-1}.

Additionally, for $0 \leq i$, define $C_i = \bigcup_{I \in E_i} I$.

Here are the first few iterations:

$C_0 = [0,1]$
$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$
$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$
$C_3 = [0, \frac{1}{27}] \cup [\frac{2}{27}, \frac{1}{9}] \cup [\frac{2}{9}, \frac{7}{27}] \cup [\frac{8}{27}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{19}{27}] \cup [\frac{20}{27}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{25}{27}] \cup [\frac{26}{27}, 1]$

Then we define the Cantor set as $C = \bigcap_{n \in \mathbb{Z}^+} C_n$.

Lecture 13: 2/9/2024

The Cantor set C is closed.

Each C_n is closed and the intercept of countably many closed sets is closed.

C is compact.

$C \subseteq [0,1]$. Therefore C is a bounded closed set in $\mathbb{R}$

$C \neq \emptyset$.

We know this by the finite intersection property. Each C_n is compact and the intersect of any finitely many C_n is nonempty as it will contain 0 or 1.

If x is an endpoint of an interval from E_n, then $x \in C$.

C contains no intervals.

Any interval in C must be contained in each E_n. Hence it must have length less than 3^{-n} for all n.

C is perfect.

Because C is closed, we know $C' \subseteq C$. Now, consider any $x \in C$ and let $r > 0$. Picking n with $3^{-n} < r$, we can specify I to be the interval of E_n containing x. The two end points of I are within distance r of x and belong to C. At least one is not x. Thus, $(B_r(x) \setminus \{x\}) \cap C \neq \emptyset$. So, $x \in C'$, meaning that $C \subseteq C'$.

C is uncountable.

We know this because...

Proposition 30: If $P \subseteq \mathbb{R}^k$ is perfect and nonempty, then P is uncountable.

To prove this, we first need two lemmas...

Lemma A: If $\vec{p}_n \in \mathbb{R}^k$ and $r_n > 0$ satisfy that $\overline{B_{r_{n+1}}(\vec{p}_{n+1})} \subseteq B_{r_n}(\vec{p}_n)$, and that $B_{r_n}(\vec{p}_n) \cap P \neq \emptyset$, then:

$$P \cap \left(\bigcap_{n \in \mathbb{N}} \overline{B_{r_n}(p_n)} \right) \neq \emptyset$$

Proof:

P is closed since P is perfect. So for all n, $P \cap \overline{B_{r_n}(\vec{p}_n)}$ is compact. Also by the assumption of the lemma, $B_{r_{n+1}}(\vec{p}_{n+1}) \neq \emptyset$

Meanwhile, $P \cap \overline{B_{r_{n+1}}(\vec{p}_{n+1})} \subseteq P \cap B_{r_n}(\vec{p}_n) \subseteq P \cap \overline{B_{r_n}(\vec{p}_n)}$. Thus, we can use the finite intersection property to say that:

$$P \cap \left(\bigcap_{n \in \mathbb{N}} \overline{B_{r_n}(\vec{p}_n)} \right) = \bigcap_{n \in \mathbb{N}} \left(P \cap \overline{B_{r_n}(\vec{p}_n)} \right) \neq \emptyset$$

Lemma B: Say $\vec{x} \neq \vec{p} \in \mathbb{R}^k$ and $r > 0$.

If $\vec{q} \in B_r(p) \setminus \{\vec{x}\}$, then there is $s > 0$ with $\overline{B_s(\vec{q})} \subseteq B_r(\vec{p}) \setminus \{\vec{x}\}$

Proof:

Set $s = \frac{1}{2} \min \{r - d(\vec{p}, \vec{q}), \quad d(\vec{x}, \vec{q})\}$.

Now consider any countable set of points in P: $\vec{x}_1, \vec{x}_2, \ldots$ We will inductively choose $\vec{p}_n \in P$ and $r_n > 0$ satisfying:

- $\vec{x}_n \notin \overline{B_{r_{n+1}}(\vec{p}_{n+1})}$
- $\overline{B_{r_{n+1}}(\vec{p}_{n+1})} \subseteq B_{r_n}(\vec{p}_n)$

To do this, first pick any $\vec{p}_1 \in P$ and $r_1 > 0$. Then for any $n \geq 1$, since P is perfect, we know that $\vec{p}_n \in P \Rightarrow \vec{p}_n \in P'$. So there are infinitely many points in $B_{r_n}(\vec{p}_n) \cap P$. Pick $\vec{p}_{n+1} \in B_{r_n}(\vec{p}_n) \cap P$ such that $\vec{p}_{n+1} \neq \vec{x}_n$. Then, using lemma B, we can define $B_{r_{n+1}}(\vec{p}_{n+1})$ satisfying our two requirements above.

By lemma A, we know that the intercept of all $B_{r_n}(\vec{p}_n)$ is nonempty. However, we also know that each $\vec{x}_n$ is not in $B_{r_{n+1}}(\vec{p}_{n+1})$. So, the point in the intercept of all $B_{r_n}(\vec{p}_n)$ is an element of P not included in our countable subset of P.

Hence, all countable subsets of P are proper. So, we conclude P is uncountable.

Note: A real number x is in the Cantor set if and only if it is between $0 \leq x \leq 1$, and if in base 3, all of the digits of x are either 0 or 2.

Hopefully it is clear from this how an irrational number could be found in the Cantor set.

Let X be a metric space. $A, B \subseteq X$ are separated if $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.

$E \subseteq X$ is connected if whenever $E = A \cup B$, either A and B are not separated or else one of A, B is empty.

For example, $(0, 1)$ and $(1, 2)$ are separated.
Meanwhile, $(0, 1]$ and $(1, 2)$ are disjoint but not separated.

Proposition 31: $E \subseteq \mathbb{R}$ is connected if and only if $\forall x, y \in E$ with $x < y$, $[x, y] \subseteq E$

Proof: (for both directions we will prove the contrapositive)

($\Longrightarrow$) Suppose $x, y \in E$, $x < y$, and $[x, y] \nsubseteq E$. Pick $z \in [x, y] \setminus E$. Since $(-\infty, z)$ and $(z, +\infty)$ are separated, so are $A = E \cap (-\infty, z)$ and $B = E \cap (z, +\infty)$. Additionally, since $z \notin E$, we have that $E = A \cup B$. However, as $A \neq \emptyset \neq B$ since $x \in A$ and $y \in B$, we conclude that E is not connected.

($\Longleftarrow$) Now suppose E is not connected. Say $A \neq \emptyset \neq B$ are separated and $A \cup B = E$. Pick $x \in A$ and $y \in B$. Without loss of generality, we can assume $x < y$. Define $z = \sup (A \cap [x, y])$. By proposition 20, we have that $z \in \overline{A}$. So as A and B are separated, we have $z \notin B$.

If $z \notin A$, then we know that $z \notin E$. So as $x \leq z < y$, we know that $[x, y] \nsubseteq E$. Meanwhile, if $z \in A$, then $z \notin \overline{B}$. So, there exists z_1 such that $z < z_1 < y$ and $z_1 \notin B$. Then, $z_1 \notin A \cup B$. So, $z_1 \notin E$, meaning $[x, y] \nsubseteq E$.

Lecture 14: 2/12/2024

A sequence $(p_n)_{n \in \mathbb{N}}$ in a metric space X converges if there is $p \in X$ such that:

$$\forall \varepsilon > 0, \ \ \exists N \in \mathbb{N} \ \ s.t. \ \forall n \geq N, \ \ p_n \in B_\varepsilon(p)$$

When this occurs, we say that (p_n) converges to p or that p is a limit of (p_n), and we write this as: $p_n \to p$ or $\lim\limits_{n \to \infty} p_n = p$.

If (p_n) does not converge, then we say it diverges.

The range of $(p_n)_{n\in\mathbb{N}}$ is defined as the set $\{p_n \mid n \in \mathbb{N}\}$.

Basically, the range just ignores the order part of a sequence.

$(p_n)_{n\in\mathbb{N}}$ is called bounded if $\{p_n \mid n \in \mathbb{N}\}$ is bounded.

Proposition 32:

(A): (p_n) converges to p if and only if every ball around p contains all but finitely many p_n.

Proof:
This is just the definition worded of a sequence converging but worded slightly differently.

(B): If (p_n) converges to p and p', then $p = p'$. (In other words, p is unique.)

Proof:
Let $\varepsilon > 0$. Pick $N, N' \in \mathbb{N}$ with:

$$\forall n \geq N \ \ d(p, p_n) < \varepsilon/2$$
$$\forall n \geq N' \ \ d(p', p_n) < \varepsilon/2$$

Setting $n = \max(N, N')$, we have that:

$$d(p, p') \leq d(p, p_n) + d(p_n, p') < \varepsilon 2 + \varepsilon 2 = \varepsilon.$$

And, as ε was arbitrary, we have that $0 \leq d(p, p') \leq \inf \{\varepsilon \mid \varepsilon > 0\} = 0$.
So, $d(p, p') = 0$ and thus $p = p'$.

(C): (p_n) converges $\Longrightarrow$ (p_n) is bounded.

Proof:
Say $p_n \to p$. Pick $N \in \mathbb{N}$ with $\forall n \geq N, \ \ d(p_n, p) < 1$.
Then set $r = \max \{d(p, p_1), d(p, p_2), \ldots, d(p, p_{N-1}), 1\}$.
Therefore, we have that $\forall n \in \mathbb{N} \ \ d(p_n, p) \leq r$.

(D): If $E \subseteq X$ and $p \in \overline{E}$, then there exists a sequence (p_n) in E with $p_n \to p$.

Proof:
Suppose $p \in E$. Then for all n, define $p_n = p$.

Now suppose $p \in E'$. Then, for each $n \in \mathbb{N}$, we must have that $(B_{\frac{1}{n+1}}(p) \setminus \{p\}) \cap E \neq \emptyset$. So, we can pick $p_n \in (B_{\frac{1}{n+1}}(p) \setminus \{p\}) \cap E$.
Then, $p_n \to p$.

(E): If (p_n) is a sequence in $E \subseteq X$ and $p_n \to p$, then $p \in \overline{E}$.

Proof:
Say $p_n \in E$ and $p_n \to p$. If $p \in E$, then we are done. So suppose $p \notin E$.
For every $r > 0$, there is n with $p_n \in B_r(p) \cap E = (B_r(p) \setminus \{p\}) \cap E$. So $p \in E'$.

Proposition 33: Suppose (s_n) and (t_n) are sequences in $\mathbb{C}$ with $s_n \to s$ and $t_n \to t$. Then:

1. $s_n + t_n \to s + t$

   Proof:
   Let $\varepsilon > 0$. Pick $N_1, N_2 \in \mathbb{N}$ such that:
   $\forall n \geq N_1 \;\; d(s, s_n) < \varepsilon/2$
   $\forall n \geq N_2 \;\; d(t, t_n) < \varepsilon/2$

   Then for all $n \geq \max(N_1, N_2)$, we have that:
   $|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \varepsilon 2 + \varepsilon 2 = \varepsilon$.

2. $s_n t_n \to st$

   Proof:
   Let $\varepsilon > 0$. Since $t_n \to t$, (t_n) is bounded. So, there exists $M > 0$ such that $|s| \leq M$ and $\forall n, \;\; |t_n| < M$. Pick N_1, N_2 with:
   $\forall n \geq N_1 \;\; d(s, s_n) < \varepsilon/2M$
   $\forall n \geq N_2 \;\; d(t, t_n) < \varepsilon/2M$

   Then for $n \geq \max(N_1, N_2)$, we have that:
   $$\begin{aligned} |s_n t_n - st| &= |s_n t_n - s t_n + s t_n - st| \\ &\leq |s_n - s||t_n| + |s||t_n - t| \\ &< \varepsilon/2M \cdot M + \varepsilon/2M \cdot M = \varepsilon \end{aligned}$$

3. $cs_n \to cs$ for all $c \in \mathbb{C}$.

   Proof:
   This follows from 33.2.

4. If $s \neq 0$, then $\frac{1}{s_n} \to \frac{1}{s}$.

   Proof:
   Let $\varepsilon > 0$. Pick N_1 so that $\forall n \geq N, \;\; |s_n - s| < \frac{1}{2}|s|$. Then $\forall n \geq N$, we have that $|s_n| \geq |s| - |s_n - s| > \frac{1}{2}|s|$. Next, pick N_2 so that $\forall n \geq N$, $|s - s_n| < \frac{1}{2}\varepsilon|s|^2$. Then $\forall n \geq \max(N_1, N_2)$ we have that:

   $$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|s_n s|} < \frac{|s - s_n|}{\frac{1}{2}|s|^2} < \frac{\frac{1}{2}\varepsilon|s|^2}{\frac{1}{2}|s|^2} = \varepsilon$$

5. If $\forall n \in \mathbb{N}, \;\; s_n, t_n \in \mathbb{R}$ and $s_n \leq t_n$, then $s \leq t$.

   Proof:
   $t_n - s_n \in [0, \infty)$. So by propositions 32.E and 33.1, we have that:
   $t - s = \lim\limits_{n\to\infty}(t_n - s_n) \in \overline{[0, \infty)} = [0, \infty)$. Hence, $t \geq s$.

Proposition 34:

(A) If $\vec{x}_n = (\alpha_{1,n}, \alpha_{2,n}, \ldots, \alpha_{k,n}) \in \mathbb{R}^k$, then $\vec{x}_n \to \vec{x}$ if and only if for all $1 \leq i \leq k$, $\alpha_{i,n} \to \alpha_i$. In other words, covergence is coordinate-wise.

Proof:
For all $1 \leq i \leq k$, we have that $|\alpha_{i,n} - \alpha_i| \leq \|\vec{x}_n - \vec{x}\|$. So $(\vec{x}_n)$ converging implies that each $(\alpha_{i,n})$ converges.

Meanwhile, $\|\vec{x}_n - \vec{x}\| = \left(\sum_{i=1}^{k} |\alpha_{i,n} - \alpha_i|^2\right)^{\frac{1}{2}} \leq \sqrt{k} \cdot \max_{1 \leq i \leq k} |\alpha_{i,n} - \alpha_i|$.
Thus if each $(\alpha_{i,n})$ converges, then $(\vec{x}_n)$ converges.

(B) If $\vec{x}_n, \vec{y}_n \in \mathbb{R}^k$, $\vec{x}_n \to \vec{x}$, and $\vec{y}_n \to \vec{y}$, then:

- $\vec{x}_n + \vec{y}_n \to \vec{x} + \vec{y}$
- $\vec{x}_n \cdot \vec{y}_n \to \vec{x} \cdot \vec{y}$
- $\beta_n \in \mathbb{R}$ and $\beta_n \to \beta$ implies that $\beta_n \vec{x}_n \to \beta \vec{x}$.

Proof:
This follows from propositions 33 and 34.A.

If $n_1 < n_2 < \ldots$ are positive integers, then $(p_{n_i})_{i \in \mathbb{Z}^+}$ is called a subsequence of $(p_n)_{n \in \mathbb{Z}^+}$. If (p_{n_i}) converges to p, we call p a subsequential limit of (p_n).

For example, if $x_n = (-1)^n$, then (x_n) does not converge. However, -1 and 1 are subsequential limits of (x_n).

Also, observe that $(p_n) \to p$ if and only if every subsequence of (p_n) converges to p.

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Propostion 35: q is a subsequential limit of (p_n) if and only if for all $r > 0$, $\{n \in \mathbb{N} \mid p_n \in B_r(q)\}$ is infinite.

Proof:
($\Longrightarrow$) Say $p_{n_i} \to q$. Then for all $r > 0$, $B_r(q)$ contains p_{n_i} for all but finitely many i. So, $\{n \in \mathbb{N} \mid p_n \in B_r(q)\}$ is infinite.

($\Longleftarrow$) Pick n_1 with $p_{n_1} \in B_1(q)$. Then for $i > 1$, pick $n_i > n_{i-1}$ with $p_{n_i} \in B_{1/i}(q)$. Thus, (p_{n_i}) is a subsequence converging to q.

Corollary: $q \in \{p_n \mid n \in \mathbb{N}\}'$ implies that q is a subsequential limit of (p_n).

Proposition 36:

(A) If (p_n) is a sequence in a compact space X, then (p_n) has a subsequential limit.

Proof:
Set $E = \{p_n \mid n \in \mathbb{N}\}$.

If E is finite, there are $n_1 < n_2 < \ldots$ such that $\forall i, j, \;\; p_{n_i} = p_{n_j}$.
Therefore, $p_{n_i} \to p$ for some $p \in E$.

Meanwhile, if E is infinte, then $E' \neq \emptyset$ by proposition 25. Thus, by the corollary to proposition 35, we have that $p \in E'$ is a subsequential limit of (p_n)

(B) Every bounded sequence in $\mathbb{R}^k$ has a subsequential limit.

Proof:
Define E as before. Then because $\overline{E} \subseteq \mathbb{R}^k$ is bounded and closed, we know that $\overline{E}$ is compact. So, we can apply proposition 36.A to $E \subseteq \overline{E}$.

Proposition 37: For any metric space X, the set of all subsequential limits of (p_n) is closed.

Proof:
Let $x \in X$ be a limit point of a set of subsequential limits of (p_n). Also, fix $r > 0$. There must be a subsequential limit q of (p_n) with $q \in B_r(x)$. Setting $s = r - d(x, q)$ we have that $B_s(q) \subseteq B_r(x)$.

Since q is a subsequential limit, by proposition 35, we know that the set: $\{n \in \mathbb{N} \mid p_n \in B_s(q)\}$, is infinite. Thus, $\{n \in \mathbb{N} \mid p_n \in B_r(x)\}$ is infinite. So proposition 35, x is a subsequential limit of (p_n).

A sequence (p_n) is Cauchy if $\forall \varepsilon > 0, \;\; \exists N \in \mathbb{N} \;\; s.t. \;\; \forall n, m > N, \;\; d(p_n, p_m) < \varepsilon$.

The diameter of $\emptyset \neq E \subseteq X$ is $\operatorname{diam}(E) = \sup \{d(p, q) \mid p, q \in E\}$. Note that $\operatorname{diam}(E) \in [0, \infty]$.

Observe: (p_n) is Cauchy if and only if $\lim\limits_{n \to \infty} (\operatorname{diam}(\{p_m \mid m \geq n\})) = 0$.

Proposition 38:

(A) For $\emptyset \neq E \subseteq X$, $\operatorname{diam}(\overline{E}) = \operatorname{diam}(E)$.

Proof:
Let $p, q \in \overline{E}$ and $\varepsilon > 0$. Pick $p', q' \in E$ with $d(p, p'), d(q, q') < \varepsilon$. Then $d(p, q) \leq d(p, p') + d(p', q') + d(q, q') < \varepsilon + \operatorname{diam}(E) + \varepsilon = 2\varepsilon + \operatorname{diam}(E)$. Since $\varepsilon > 0$ was arbitrary, we find that $d(p, q) \leq \operatorname{diam}(E)$.

Hence $\operatorname{diam}(\overline{E}) \leq \operatorname{diam}(E)$.

Meanwhile, its obvious that $\operatorname{diam}(E) \leq \operatorname{diam}(\overline{E})$. So $\operatorname{diam}(E) = \operatorname{diam}(\overline{E})$.

(B) If for all $n \in \mathbb{N}$, we have that K_n is compact and nonempty, $K_{n+1} \subseteq K_n$, and $\operatorname{diam}(K_n) \to 0$, then $\bigcap\limits_{n \in \mathbb{N}} K_n$ is a singleton.

Proof:
$\bigcap\limits_{n \in \mathbb{N}} K_n \neq \emptyset$ by the finite intersection property.

Also, $\operatorname{diam}(\bigcap\limits_{n \in \mathbb{N}} K_n) \leq \operatorname{diam}(K_n)$ for all n. So, $\operatorname{diam}(\bigcap\limits_{n \in \mathbb{N}} K_n) = 0$.

Thus $\bigcap\limits_{n \in \mathbb{N}} K_n$ contains a single point.

Proposition 39: Let X be a metric space.

1. If (p_n) converges, then (p_n) is Cauchy.

   Proof:
   Assume that $p_n \to p$. Let $\varepsilon > 0$. Pick N with $\forall n \geq N$, $d(p_n, p) < \varepsilon/2$. Then for all $n, m \geq N$, $d(p_n, p_m) \leq d(p_n, p) + d(q_n, q) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

2. If X is compact, then (p_n) being Cauchy implies that (p_n) converges.

   Proof:
   Set $E_n = \{p_n, p_{n+1}, \ldots\}$. Since (p_n) is Cauchy, we know that:
   $\operatorname{diam}(\overline{E_n}) = \operatorname{diam}(E_n) \to 0$. Also, since X is compact and $\overline{E_n} \subseteq X$, we know by proposition 24 that $\overline{E_n}$ is compact. Additionally, $\overline{E_{n+1}} \subseteq \overline{E_n}$.

   Therefore, by proposition 38.B, $\bigcap\limits_{n \in \mathbb{N}} \overline{E_n} = \{p\}$ for some $p \in X$.

   Now let $\varepsilon > 0$. Pick N with $\operatorname{diam}(\overline{E_n}) < \varepsilon$. Then for all $n \geq N$, we have that $p_n, p \in \overline{E_n}$. So $d(p_n, p) < \operatorname{diam}(\overline{E_n}) < \varepsilon$. Hence, $p_n \to p$.

3. If $X = \mathbb{R}^k$, then (p_n) being Cauchy implies that (p_n) converges.

Proof:
Since (p_n) is Cauchy, pick N with $\operatorname{diam}(\{p_N, p_{N+1}, \dots\}) < 1$. Setting $r = \max\{1, d(p_1, p_N), \dots, d(p_{N-1}, p_N)\}$, we have that for all n, $d(p_n, p_N) \leq r$. So (p_n) is bounded. This means that the closure of the range of (p_n) is compact. So (p_n) is contained in a compact metric space, meaning that (p_n) converges by proposition 39.2.

A metric space X is complete if every Cauchy sequence in X converges.

Proposition 39 says that compact metric spaces and Euclidean spaces are complete.

Fact 1: $\mathbb{R}$ is the smallest complete metric space containing $\mathbb{Q}$.
Fact 2: $\mathbb{C}$ is also complete.

A sequence (s_n) in $\mathbb{R}$ is called:

- monotone increasing if $\forall n, \;\; s_n \leq s_{n+1}$.
- monotone decreasing if $\forall n, \;\; s_n \geq s_{n+1}$.
- monotone if either of the above.

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Proposition 40: Suppose $(s_n) \subseteq \mathbb{R}$ is monotone. Then (s_n) converges if and only if (s_n) is bounded.

Proof:
($\Longrightarrow$) This is just proposition 32.C.

($\Longleftarrow$) We'll assume (s_n) is monotone increasing because the other case is basically identical but with flipped inequalities.

Set $s = \sup\{s_n \mid n \in \mathbb{N}\}$. We know this exists because (s_n) is bounded and $\mathbb{R}$ has the least upper bound property. Next let $\varepsilon > 0$. Since $s - \varepsilon$ is not an upper bound to $\{s_n \mid n \in \mathbb{N}\}$, we know there is N with $s - \varepsilon < s_N$. Hence, $\forall n \geq N, \;\; s - \varepsilon < s_N \leq s_n \leq s$. Thus, $s_n \to s$.

For a sequence (s_n) in $\mathbb{R}$, we write:

- $s_n \to \infty$ or $\lim\limits_{n\to\infty} (s_n) = \infty$ if $\;\forall M \in \mathbb{R}, \;\; \exists N \in \mathbb{N} \;\; s.t. \;\; \forall n \geq N, \;\; s_n > M$.
- $s_n \to -\infty$ or $\lim\limits_{n\to\infty} (s_n) = -\infty$ if $\;\forall M \in \mathbb{R}, \;\; \exists N \in \mathbb{N} \;\; s.t. \;\; \forall n \geq N, \;\; s_n < M$.

In both of the above cases, we still say that s_n diverges.

Let (s_n) be a sequence in $\mathbb{R}$. Let E be the set of all subsequential limits of (s_n) in $\mathbb{R} \cup \{-\infty, \infty\}$.

The upper limit / limit supremum of (s_n) is: $\limsup s_n = \sup E$. Meanwhile, the lower limit / limit infimum of (s_n) is $\liminf s_n = \inf E$.

Importantly, the limit supremum and limit infimum always exist. This is because if (s_n) is not bounded above, then $+\infty \in E$. Meanwhile if (s_n) is not bounded below, then $-\infty \in E$. Finally, if (s_n) is bounded, then by the Bolzano-Weierstrauss Theorem (proposition 29), (s_n) has a limit point. Thus, by proposition 35, that limit point is a subsequential limit.

All in all, this means that E is never empty. And as the extended real numbers have the least upper bound property and all nonempty sets in the extended real numbers are bounded, we thus know that both the limit supremum and limit infimum always exist.

Proposition 41: Let (s_n) and E be defined as above…

(A) $\limsup s_n \in E$.

Proof:

If (s_n) is not bounded above, then $+\infty \in E$. So $\limsup s_n = +\infty$ which is in E. So let's assume (s_n) is bounded above.

If $\limsup s_n = -\infty$, then $E = \{-\infty\}$. So $\limsup s_n \in E$.

Finally, consider if $\limsup s_n \in \mathbb{R}$. Then $E \subseteq [-\infty, \limsup s_n]$. So, $\limsup s_n = \sup E = \sup (E \cap \mathbb{R}) \in \overline{E \cap \mathbb{R}}$. Then, as $\mathbb{R}$ is closed and E is closed by proposition 37, we have that $E \cap \mathbb{R} = \overline{E \cap \mathbb{R}}$. Therefore, $\limsup s_n \in E \cap \mathbb{R} \subseteq E$.

(B) If $x > \limsup s_n$, then there exists an integer N such that $\forall n \geq N, \;\; s_n < x$.

Proof:

Let $x > \limsup s_n$. Then $E \cap [x, +\infty] = \emptyset$. Now towards a contradiction, suppose $s_n \geq x$ for infinitely many n. Then, (s_n) has a subsequence in $[x, +\infty)$. Therefore, (s_n) has a subsequential limit y in $[x, +\infty]$. But this is a contradiction because $y \in E$ and $y > \sup E$.

(C) $\limsup s_n$ is the unique element of E satisfying propositions 41.B.

Proof:

Suppose towards a contradiction that both p and q satisfy proposition 41.B and are in E. Without loss of generality, let $p < q$. Then consider x with $p < x < q$. Applying proposition 41.B to p and x, we find that all but finitely many s_n are less than x. Hence, every subsequential limit is at most x. This contradicts that $q \in E$.

Also, one can obviously make analogous propositions for $\liminf s_n$.

Proposition 42: If $s_n \geq t_n$ for all $n \geq N$, then $\limsup s_n \geq \limsup t_n$ and $\liminf s_n \geq \liminf t_n$.

Proof:

Use proposition 33.5...

Consider the sequence: $s_n = \dfrac{(-1)^n}{1-\frac{1}{n}}$...

For every s_n we have that $|s_n| > |1|$. Yet observe that:

- $\limsup s_n = +1$
- $\liminf s_n = -1$

This demonstrates that the limite supremum or infinimum of a sequence is not the same as supremum or infimum of the range of a sequence.

Binomial Theorem: For $z, w \in \mathbb{C}$ and $n \in \mathbb{Z}^+$, $(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$.

Proof:

Use induction...

Proposition 43: If there exists $N > 0$ such that for all $n > N$, $0 \leq x_n \leq s_n$ and $s_n \to 0$, then $x_n \to 0$.

Proof:

Pick $\varepsilon > 0$. Then, we know there exists M such that for all $n > M$ $0 \leq s_n \leq \varepsilon$. However, $0 \leq x_n \leq s_n$. So for all $n > M$, we have that $0 \leq x_n \leq s_n < \varepsilon$. Hence, x_n converges to 0.

Proposition 44:

(A) If $p > 0$, then $\frac{1}{n^p} \to 0$.

Proof:

Let $\varepsilon > 0$. Then $0 < \frac{1}{n^p} < \varepsilon$ whenever $(\frac{1}{\varepsilon})^{\frac{1}{p}} < n$. Hence $\frac{1}{n^p} \to 0$.

(B) If $p > 0$, then $\sqrt[n]{p} \to 1$.

Proof:

If $p > 1$, then $x_n = \sqrt[n]{p} - 1 > 0$. Also, $p = (x_n + 1)^n$. Therefore by the binomial theorem: $1 + nx_n \leq (x_n + 1)^n = 1 + nx_n + \ldots = p$. This means that $0 < x_n \leq (p-1)\frac{1}{n}$.

Using proposition 33.3 and the limit found above, we have that $\frac{p-1}{n} \to 0$. And as each $0 < x_n \leq \frac{p-1}{n}$, we know by proposition 43 that $x_n \to 0$. Therefore, $\sqrt[n]{p} = x_n + 1 \to 0 + 1 = 1$

As for if $0 < p < 1$, then we know from above that $\frac{1}{\sqrt[n]{p}} = \sqrt[n]{\frac{1}{p}} \to 1$. Therefore, $\sqrt[n]{p} = \frac{1}{1} = 1$ by proposition 33.4.

Finally, if $p = 1$, then the limit is 1 trivially.

(C) $\sqrt[n]{n} \to 1$

Proof:
Let $x_n = \sqrt[n]{n} - 1$. Then $x_n \geq 0$ and by the binomial theorem:

$$\frac{n(n-1)}{2}(x_n)^2 \leq \sum_{k=0}^{n} \binom{n}{k}(x_n)^k = (x_n + 1)^n = n$$

Then we have that $0 \leq x_n \leq \sqrt{\frac{2n}{n(n-1)}} = \sqrt{\frac{2}{n-1}}$ when $n \geq 2$.

Now, $\sqrt{\frac{2}{n-1}} \to 0$.

Proof: Let $\varepsilon > 0$. Then $\sqrt{\frac{2}{n-1}} < \varepsilon$ whenever $n > 1 + \frac{2}{\varepsilon^2}$.

Therefore, by proposition 43, we know that $x_n \to 0$. So finally, we conclude that:

$$\sqrt[n]{n} \to \lim_{n\to\infty}(x_n) + 1 = 0 + 1$$

(D) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\frac{n^\alpha}{(1+p)^n} \to 0$.

Proof:
Fix an integer $k > \max(\alpha, 0)$. When $n > 2k$, we have $n - k + 1 > \frac{n}{2}$.

You should just be able to intuit the above inequality but here's proof:
$n > 2k \implies \frac{n}{2} > k \implies n - \frac{n}{2} = \frac{n}{2} < n - k < n - k + 1$.

By the binomial theorem, we have that:

$$(1+p)^n > \binom{n}{k}p^k = \frac{n(n-1)\cdots(n-k+1)}{k!}p^k$$

Applying the above inequality, we can then say that:

$$\frac{n(n-1)\cdots(n-k+1)}{k!}p^k > \left(\frac{n}{2}\right) \cdot \frac{1}{k!}p^k = \frac{p^k}{2^k k!}n^k$$

So, $\frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k}n^\alpha n^{-k}$. Now note that as $k > \alpha$, we have that $\alpha - k < 0$. Therefore, by proposition 44.A, we know that $n^\alpha n^{-k} = n^{\alpha-k} \to 0$.

Multiplying this by the constant $\frac{2^k k!}{p^k}$ and applying proposition 33.3, we then get that $\frac{2^k k!}{p^k} n^\alpha n^{-k} \to 0$. Finally, note that $\frac{n^\alpha}{(1+p)^n} > 0$ because $(1+p) > 0 \implies (1+p)^n > 0$ and $n > 0 \implies n^\alpha > 0$. Hence, we can apply proposition 43 to get that $\frac{n^\alpha}{(1+p)^n} \to 0$.

(E) If $z \in \mathbb{C}$ and $|z| < 1$, then $z^n \to 0$.

$|z| < 1 \implies \frac{1}{|z|} - 1 > 0$. So, using $p = \frac{1}{|z|} - 1$ and $\alpha = 0$, we know from proposition 44.D that $\frac{n^0}{(1+\frac{1}{|z|}-1)^n} = |z|^n \to 0$. Now note that $0 \leq d(0, z^n) = |z^n| = |z|^n$. Therefore, $d(0, z^n) \to 0$, meaning that z^n converges to 0.

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Given a sequence (a_n) in $\mathbb{C}$, we write $\sum_{n=p}^{q} a_n$ to denote $a_p + a_{p+1} + \ldots + a_q$ where p and q are integers such that $p \leq q$.

We associate a sequence (a_n) in $\mathbb{C}$ with a sequence (s_n) such that $s_n = \sum_{k=1}^{n} a_k$. We call each s_n a partial sum.

$a_1 + a_2 + \ldots$ and $\sum_{n=1}^{\infty} a_n$ are called series and they denote the value $\lim_{n \to \infty} s_n$ when that limit exists.

We say that $\sum_{n=1}^{\infty} a_n$ converges / diverges if (s_n) converges / diverges.

Series are also notated with other starting indexes. For example: $\sum_{n=0}^{\infty} a_n$.
Additionally, when we don't want to worry about the first index in the series, we typically refer to (s_n) as Σa_n.

Proposition 45: Σa_n converges if and only if:

$$\forall \varepsilon > 0,\ \ \exists N \in \mathbb{N}\ \ s.t.\ \ \forall n \geq m \geq N,\ \ \left|\sum_{k=m}^{n} a_k\right| < \varepsilon$$

This is just proposition 39 (also called the Cauchy Criterion) applied to the sequence (s_n). After all, (s_n) is in a complete metric space.

Proposition 46: If Σa_n converges, then $a_n \to 0$.

Note, the converse isn't true.
For example: $\frac{1}{n} \to 0$ but $\Sigma \frac{1}{n}$ diverges.

Proof:
Consider proposition 45 with $n = m$. Then we have that for any $\varepsilon > 0$, there exists N such that $|a_k| < \varepsilon$ for all $k > N$.

Proposition 47: If $\forall n, \;\; a_n \geq 0$, then Σa_n converges if and only if its partial sums are bounded.

Proof:
Σa_n is monotone increasing. Thus Σa_n converges if it is bounded.

Proposition 48: (Comparison Test)

1. If $|a_n| \leq c_n$ for all $n \geq N$ and if Σc_n converges, then Σa_n converges.

   Proof:
   Let $\varepsilon > 0$. Pick $M \geq N$ with $\forall n \geq m \geq M, \;\; \left| \sum\limits_{k=m}^{n} c_k \right| < \varepsilon$.
   Then for $n \geq m \geq M$, we have:
   $$\left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} c_k = \left| \sum_{k=m}^{n} c_k \right| < \varepsilon.$$
   So, Σa_n converges by proposition 45.

2. If $a_n \geq d_n \geq 0$ for all $n \geq N$ and Σd_n diverges, then Σa_n diverges.

   Proof:
   This is the contrapositive of proposition 48.1.

Proposition 49: For $z \in \mathbb{C}$, the series $\sum\limits_{n=0}^{\infty} z^n$ is called a geometric series.

- If $|z| < 1$, then the geometric series converges.

  Proof:
  Assume $|z| < 1$. Then $s_n = \sum\limits_{k=0}^{n} z^k = 1 + z + z^2 + \ldots + z^n$.
  Multiplying and dividing the partial sum by $1 - z$, we can cancel a lot of stuff out in order to get that: $s_n = \frac{1-z^{n+1}}{1-z}$. Then:
  $$\lim_{n\to\infty} (s_n) = \frac{1 - z \cdot \lim\limits_{n\to\infty} (z^n)}{1-z} = \frac{1-0z}{1-z} = \frac{1}{1-z}$$

- If $|z| \geq 1$, then the geometric series diverges.

  Proof:
  When $|z| \geq 1$, then $|z|^n \not\to 0$. To see this, note that $|z|^n \geq |z| \geq 1$. Thus, every element of the sequence $(|z|^n)$ is at least 1, which means that if the sequence did converge, it would have to converge in $[1, \infty)$. So, $d(z^n, 0)$ doesn't converge to 0, which in turn means that z^n does not stay close to 0. Hence, z^n doesn't converge to 0, which means that by proposition 46, Σz^n diverges.

Proposition 50: Suppose $a_1 \geq a_2 \geq \ldots \geq 0$.
Then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Proof:
By proposition 47, each series converges if and only if its partial sums are bounded. Set $s_n = a_1 + \ldots + a_n$, and $t_k = a_1 + 2a_2 + \ldots + 2^k a_{2^k}$.

When $n < 2^k$,

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + (a_{2^k} + \ldots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + 4a_4 + \ldots + 2^k a_{2^k} = t_k \end{aligned}$$

Thus $s_n \leq t_k$ when $n < 2^k$.

When $n > 2^k$:

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + \ldots + (a_{2^{k-1}+1} + \ldots + a_{2^k}) \\ &\geq \tfrac{1}{2}a_1 + a_2 + 2a_4 + \ldots + 2^{k-1} a_{2^k} = \tfrac{1}{2}t_k \end{aligned}$$

Thus $2s_n \geq t_k$ when $n > 2^k$.

Therefore, (s_n) is bounded above if and only if t_k is bounded above.

Proposition 51: $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof:
By proposition 50, $\sum \frac{1}{n^p}$ converges if and only if $\sum 2^k \frac{1}{2^{kp}}$ converges. But $\sum 2^k \frac{1}{2^{kp}} = \sum 2^{k-kp} = \sum \left(2^{1-p}\right)^k$ is a geometric series. So $\sum \left(2^{1-p}\right)^k$ converges if and only if $2^{1-p} < 1$, and this inequality is only true when $p > 1$.

(We haven't officially covered logarithms yet but...)

Proposition 52: $\sum_{n=1}^{\infty} \frac{1}{n(\log(n))^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof:
By proposition 50, $\sum \frac{1}{n(\log(n))^p}$ converges if and only if $\sum 2^k \frac{1}{2^k(\log(2^k))^p}$ converges. But $2^k \frac{1}{2^k(\log(2^k))^p} = \frac{1}{(k\log(2))^p} = \frac{1}{(\log(2))^p}\frac{1}{k^p}$. Therefore, $\sum 2^k \frac{1}{2^k(\log(2^k))^p}$ is a geometric sequence and only converges if $p > 1$.

We define $e = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.71828.$

We know that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges...

$$s_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^{n-1}} = 1 + \frac{1}{1-\frac{1}{2}} = 3.$$

Here are three facts we won't spend more time on because we're behind.

- $(1 + \frac{1}{n})^n \to e$.
- e is irrational.
- e is not algebraic.

(meaning e is not the root of a polynomial with rational coefficients.)

Lecture 18: 2/23/2024

Proposition 53: (Root Test)
Consider a series Σa_n and let $\alpha = \limsup \sqrt[n]{|a_n|}$.

(A) If $\alpha < 1$, then Σa_n converges.

(B) If $\alpha > 1$, then Σa_n diverges.

(C) If $\alpha = 1$, this gives no information.

Proof:

(A) Since $\alpha < 1$, we can pick $\alpha < \beta < 1$. Then by proposition 41.B, there exists N with $\forall n \geq N$, $\sqrt[n]{|a_n|} < \beta$. Or in other words, $\forall n \geq N$, $|a_n| < \beta^n$. $\Sigma \beta^n$ is a geometric sequence that converges. So by the comparison test (proposition 48), Σa_n converges.

(B) If $\limsup \sqrt[n]{|a_n|} > 1$, then there is a subsequence of $\sqrt[n]{|a_n|}$ that converges to a value greater than 1. This in turn means there is a subsequence of $|a_n|$ that converges to a value greater than 1.
So, $\limsup |a_n| > 1$. Hence, we know that $a_n \not\to 0$, which means Σa_n diverges.

(C) $\alpha = 1$ for both $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. However, $\sum \frac{1}{n}$ diverges while $\sum \frac{1}{n^2}$ converges.

Proposition 54: (Ratio Test)
Suppose $\forall n, \;\; a_n \neq 0$.

1. The series Σa_n converges if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. The series Σa_n diverges if $\exists N \;\; s.t. \;\; \forall n \geq N, \;\; \left|\frac{a_{n+1}}{a_n}\right| \geq 1$.

Proof:

1. Pick β with $\limsup \left|\frac{a_{n+1}}{a_n}\right| < \beta < 1$. By proposition 41.B, there is N with $\forall n \geq N, \;\; \left|\frac{a_{n+1}}{a_n}\right| < \beta$. So, consider any $n \geq N$ and set $k = n - N$. That way, we have that $n = N + k$.

   Now notice that for $n \geq N$:
   $$|a_{N+k}| < \beta |a_{N+k-1}| < \beta^2 |a_{N+k-2}| < \cdots < \beta^k |a_N|$$

   Therefore, $\forall n \geq N, \;\; |a_n| < \beta^{n-N} |a_n|$. However, $\beta^{n-N}|a_n|$ is a geometric sequence whose series converges. Hence, by the comparison test, we know that a_n converges.

2. If $\exists N \;\; s.t. \;\; \forall n \geq N, \;\; \left|\frac{a_{n+1}}{a_n}\right| \geq 1$, then we must have that $a_n \not\to 0$. So Σa_n diverges.

The ratio test can often be easier to apply. However, the root test is almost always more accurate. Specifically, the root test will always give an answer if the ratio test gives an answer. However, the reverse is not true.

This is because for any sequence (c_n) of positive numbers:

- $\liminf \frac{c_{n+1}}{c_n} \leq \liminf \sqrt[n]{c_n}$
- $\limsup \sqrt[n]{c_n} \leq \limsup \frac{c_{n+1}}{c_n}$

(Unfortunately, we're behind and so won't be proving this...)

For a sequence $c_n \in \mathbb{C}$ and any $z \in \mathbb{C}$, the series $\sum\limits_{n=0}^{\infty} c_n z^n$ is called a power series.

Note that convergence/divergence of a power series depends on the value of z.

Proposition 55: For a power series $\sum\limits_{n=0}^{\infty} c_n z^n$, set $\alpha = \limsup \sqrt[n]{|c_n|}$ and $R = \frac{1}{\alpha}$. (If $\alpha = +\infty$, then define $R = 0$, whereas if $\alpha = 0$, then define $R = +\infty$.) Then $\sum c_n z^n$ converges when $|z| < R$ and diverges when $|z| > R$.

R is called the radius of convergence. Also, convergence/divergence is more complicated when $|z| = R$.

Proof: (apply the root test)
$\limsup \sqrt[n]{|c_n z^n|} = |z| \cdot \limsup \sqrt[n]{|c_n|} = \frac{|z|}{R}$. Thus, $\limsup \sqrt[n]{|c_n z^n|} < 1$ if and only if $|z| < R$.

Examples:

- For $\sum \frac{z^n}{n!}$, we have by the ratio test that $R = +\infty$.
- For $\sum n^n z^n$, we have that $R = 0$.
- For $\sum z^n$, we have that $R = 1$. Also, it diverges when $|z| = 1$.
- For $\sum \frac{z^n}{n^2}$, we have that $R = 1$. Also, it converges when $|z| = 1$.
- For $\sum \frac{z^n}{n}$, we have that $R = 1$. Also, if $z = 1$, the series diverges. But if $z \neq 1$ and $|z| = 1$, then the series converges.

Proposition 56: Given the sequences (a_n) and (b_n), set $A_{-1} = 0$ and then for $n \geq 0$, let $A_n = \sum\limits_{k=0}^{n} a_k$. Then for $0 \leq p \leq q$, we have:

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Notice how similar this is to integration by parts (where $f = F'$):

$$\int_a^b f g dx = -\int_a^b F g' dx + F(b)g(b) - F(a)g(a)$$

Proof:

$$\begin{aligned}\sum_{n=p}^{q} a_n b_n &= \sum_{n=p}^{q} (A_n - A_{n-1}) b_n \\ &= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\ &= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \quad \blacksquare\end{aligned}$$

Proposition 57: If the partial sums of Σa_n are bounded and we have a sequence $b_0 \geq b_1 \geq b_2 \geq \cdots$ such that $b_n \to 0$, then $\sum a_n b_n$ will converge.

Proof:

Set $A_n = \sum_{k=0}^{n} a_k$. Then pick $M > 0$ such that $\forall n, \ |A_n| < M$.

Given $\varepsilon > 0$, pick N with $b_N < \frac{\varepsilon}{2M}$. Then when $q \geq p \geq N$, we have:

$$\begin{aligned}\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\leq \sum_{n=p}^{q-1} |A_n| (b_n - b_{n+1}) + |A_q| b_q + |A_{p-1}| b_p \\ &\leq M(b_p - b_q + b_q + b_p) = 2M b_p \leq 2M b_N < \varepsilon\end{aligned}$$

Lecture 19: 2/26/2024

For the sake of brevity, from now on I will write $\lim(c_n)$ to denote $\lim_{n\to\infty}(c_n)$ when c_n is a sequence and it is clear from context what index variable is getting arbitrarily large.

Proposition 58: (Alternating Series Test)
Suppose $|c_1| \geq |c_2| \geq \ldots, \ \lim(c_n) = 0$, and $c_{2n+1} \geq 0$ and $c_{2n} \leq 0$ for all n. Then Σc_n converges.

Proof:

Let $a_n = (-1)^{n+1}$ and $b_n = |c_n|$. Since Σa_n is bounded and $b_n \to 0$, we can apply proposition 57 to get that $\Sigma c_n = \sum a_n b_n$ converges.

Proposition 59: Suppose $\sum_{n=0}^{\infty} c_n z^n$ has a radius of convergence: 1, and that $c_0 \geq c_1 \geq c_2 \geq \dots$ and $c_n \to 0$. Then $\sum c_n z^n$ converges for all $z \in \mathbb{C}$ with $|z| = 1$ except possibly $z = 1$.

Proof:
Let $a_n = z^n$ and $b_n = c_n$. Obviously, $c_n \to 0$. Meanwhile, when $|z| = 1$ and $z \neq 1$, we have:

$$\left|\sum_{k=0}^{n} a_k\right| = \left|\sum_{k=0}^{n} z^k\right| = \left|\frac{1-z^{n+1}}{1-z}\right| \leq \frac{2}{|1-z|}$$

Hence the partials sums of Σa_n are bounded. Therefore, applying proposition 57, we get that $\Sigma c_n z^n = \Sigma a_n b_n$ converges.

We say that $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Proposition 60: If Σa_n converges absolutely, then Σa_n converges.

Proof:
Let $b_n = |a_n|$. Then as Σb_n converges and $|a_n| \leq b_n$, we can use the comparison test to say that Σa_n converges.

Note that the comparison, root, and ratio tests all test for absolute convergence. Also, the radius of convergence of a power series is also the "radius of absolute convergence".

Proposition 61:

- If $\Sigma a_n = A$ and $\Sigma b_n = B$, then $\sum (a_n + b_n) = A + B$.

  Proof:
  Set $A_n = \sum_{k=0}^{n} a_k$ and $B_n = \sum_{k=0}^{n} b_k$. Then $A_n \to A$ and $B_n \to B$.

  Therefore:

$$\sum (a_n + b_n) = \lim_{n\to\infty} \left(\sum_{k=0}^{n} (a_k + b_k)\right) = \lim_{n\to\infty} (A_n + B_n) = A + B.$$

- Let $\Sigma a_n = A$ and $c \in \mathbb{C}$. Then $\sum c a_n = cA$.

  Set $A_n = \sum_{k=0}^{n} a_k$. Then $A_n \to A$. So:

$$\sum c a_n = \lim_{n\to\infty} \left(\sum_{k=0}^{n} c a_k\right) = \lim_{n\to\infty} c A_n = cA$$

The Cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is $\sum_{n=0}^{\infty} c_n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

The motivation for this definition is that if $\sum_{n=0}^{\infty} c_n$ is the Cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, then assuming we can regroup terms:

$$\begin{aligned}&\left(a_0 + a_1 z + a_2 z^2 + \dots\right)\left(b_0 + b_1 z + b_2 z^2 + \dots\right)\\ &\quad = a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots\\ &\quad = c_0 + c_1 z + c_2 z^2 + \dots\end{aligned}$$

That said, sometimes $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) \neq \left(\sum_{n=0}^{\infty} c_n\right)$.

We'll prove a theorem related to this in 140B.

For example: Suppose $a_n = b_n = \left(\frac{(-1)^n}{\sqrt{n+1}}\right)$, and Σc_n is the Cauchy product of Σa_n and Σb_n.

Σa_n and Σb_n converge by the alternating series test (although they don't converge absolutely). However:

$$|c_n| = \left|\sum_{k=0}^{n} a_k b_{n-k}\right| = \left|\sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}}\right|$$

And since $(k+1)(n-k+1) = (\frac{n}{2}+1)^2 - (\frac{n}{2}-k)^2 \leq (\frac{n}{2}+1)^2$, we thus can say that:

$$\left|\sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}}\right| \geq \sum_{k=0}^{n} \frac{1}{\frac{n}{2}+1} = \frac{2(n+1)}{n+2}.$$

Thus as $\frac{2(n+1)}{n+2} \not\to 0$ and $c_n \geq \frac{2(n+1)}{n+2}$, we know that $c_n \not\to 0$. So, $\sum c_n$ diverges.

Proposition 62: (Merten's Theorem)
Suppose $\sum a_n = A$ and $\sum b_n = B$ with $\sum a_n$ converging absolutely. Let $\sum c_n$ be the Cauchy product of $\sum a_n$ and $\sum b_n$. Then $\sum c_n = AB$.

Proof:

Set $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, and $C_n = \sum_{k=0}^{n} c_k$. Also set $\beta_n = B_n - B$.

Then:

$$\begin{aligned} C_n &= c_0 + c_1 + \ldots + c_n \\ &= a_0b_0 + (a_0b_1 + a_1b_0) + \ldots + (a_0b_n + \ldots a_nb_0) \\ &= a_0B_n + a_1B_{n-1} + \ldots + a_nB_0 \\ &= a_0(\beta_n + B) + a_1(\beta_{n-1} + B) + \ldots + a_n(\beta_0 + B) \\ &= A_nB + a_0\beta_n + a_1\beta_{n-1} + \ldots + a_n\beta_0 \end{aligned}$$

Since $A_nB \to AB$, it suffices to show that $\gamma_n \to 0$ where

$$\gamma_n = a_0\beta_n + a_1\beta_{n-1} + \ldots + a_n\beta_0.$$

Let $\varepsilon > 0$ and set $\alpha = \sum_{n=0}^{\infty} |a_n|$.

Since $\beta_n \to 0$, we can pick N such that $\forall n \geq N, \quad |\beta_n| < \varepsilon$. Then, for $n \geq N$, we have that:

$$\begin{aligned} |\gamma_n| &= |a_0\beta_n + a_1\beta_{n-1} + \ldots + a_{n-N}\beta_N + a_{n-N+1}\beta_{N-1} + \ldots + a_n\beta_0| \\ &\leq |a_0||\beta_n| + |a_1||\beta_{n-1}| + \ldots + |a_{n-N}||\beta_N| + |a_{n-N+1}\beta_{N-1} + \ldots + a_n\beta_0| \\ &< \varepsilon \sum_{k=0}^{n-N} (|a_k|) + |a_{n-N+1}\beta_{N-1} + \ldots + a_n\beta_0| \\ &\leq \varepsilon\alpha + |a_{n-N+1}\beta_{N-1} + \ldots + a_n\beta_0| \end{aligned}$$

Now imporantly, $|a_{n-N+1}\beta_{N-1} + \ldots + a_n\beta_0|$ has exactly $N-1$ terms which all approach a limit of 0 because $a_k \to 0$. Thus, whatever it is, we know that $\limsup |\gamma_n| \leq \varepsilon\alpha$.

However, ε was arbitrary. Thus, $\limsup |\gamma_n| = 0$. And as $\liminf |\gamma_n| \geq 0$ trivially, we can conclude that $|\gamma_n| \to 0$.

Lecture 20: 3/1/2024

If (k_n) is a sequence in $\mathbb{N}$ using each natural number precisely once and if we set $a'_n = a_{k_n}$, then the series $\sum a'_n$ is called a rearrangement of $\sum a_n$.

Proposition 63: (Riemann Theorem)
Suppose $\forall n, \quad a_n \in \mathbb{R}$, and that $\sum a_n$ converges but not absolutely. Let $-\infty \leq \alpha \leq \beta \leq \infty$. Then there exists a rearrangement $\sum a'_n$ having partial sums s'_n satisfying:

- $\liminf(s'_n) = \alpha$
- $\limsup(s'_n) = \beta$

Proof:

Set $p_n = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{otherwise} \end{cases}$ and $q_n = \begin{cases} -a_n & \text{if } a_n \leq 0 \\ 0 & \text{otherwise} \end{cases}$

Note that for all n, we have that $a_n = p_n - q_n$ and $p_n, q_n \geq 0$.

If $\sum p_n$ converges, then $\sum q_n = \sum(p_n - a_n)$ must converge (see proposition 61). And since $p_n = |p_n|$ and $q_n = |q_n|$, we can use proposition 61 again to get that $\sum(|p_n| + |q_n|)$ converges. However, this is a contradiction because $|a_n| \leq |p_n| + |q_n|$. So by comparison test, $|a_n|$ must converge absolutely which we assumed was not the case.

So, $\sum p_n$ must diverge. By similar reasoning, $\sum q_n$ must diverge.

Note that $\sum p_n$ and $\sum q_n$ specifically approach $+\infty$ because their partial sums are monotonically increasing.

Let $P_1, P_2, \ldots$ be the list (in order) of the non-negative terms from $a_1, a_2, \ldots$ Also let $Q_1, Q_2, \ldots$ be the list (in order) of the absolute values of the strictly negative terms from $a_1, a_2, \ldots$ Then $\sum P_n$ and $\sum Q_n$ diverge since they only differ from $\sum p_n$ and $\sum q_n$ by 0 terms.

Choose sequences $(\alpha_n), (\beta_n)$ in $\mathbb{R}$ with $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\beta_1 > 0$, $\alpha_{n-1} < \beta_n$, and $\alpha_n < \beta_n$.

Next, pick $m_1, k_1 \in \mathbb{Z}^+$ to be the least integers such that:
$P_1 + P_2 + \ldots + P_{m_1} > \beta_1$ and $P_1 + P_2 + \ldots + P_{m_1} - Q_1 - Q_2 - \ldots - Q_{k_1} < \alpha_1$.

Continue inductively, picking the least $m_n, k_n \in \mathbb{Z}^+$ such that:

(1) $P_1 + P_{m_1} - Q_1 - \ldots - Q_k + \ldots - Q_{k_{n-1}} + P_{m_{n-1}+1} + P_{m_n} > \beta_n$

(2) $P_1 + P_{m_1} - Q_1 - \ldots - Q_k + \ldots - Q_{k_{n-1}} + P_{m_{n-1}+1} + P_{m_n} - Q_{k_{n-1}+1} - \ldots - Q_{k_n} < \alpha_n$

Let x_n be the left side of equation (1) written just above and y_n be the left side of equation (2) written just above. We claim that the rearrangement $P_1 + \ldots + P_{m_1} - Q_1 - \ldots - Q_{k_1} + P_{m_1+1} + \cdots\cdots$ is what we want.

Clearly x_n and y_n are partial sums. Also since m_n was chosen to be least, we have that $x_n - P_{m_n} \leq \beta_n$. So, $\beta_n < x_n \leq \beta_n + P_{m_n}$. Similarly, $\alpha_n - Q_{k_n} \leq y_n < \alpha_n$. Since $\sum a_n$ converges, we have that $a_n \to 0$. Hence, $P_n \to 0$ and $Q_n \to 0$. It follows that $x_n \to \lim \beta_n = \beta_n$ and $y_n \to \lim \alpha_n = \alpha$.

So β and α are subsequential limits of the partial sums s'_n.

If $\beta = +\infty$, then we must have that $\limsup s'_n = +\infty = \beta$. Now suppose $\beta \neq +\infty$. Since $\limsup x_n = \lim x_n = \beta$, by proposition 41, for any real number $r > \beta$, there exists N with $\forall n \geq N,\ \ x_n < r$. Let $r > \beta$ and let N be as above.

Pick I with $s'_I = x_N$ Now consider any $i \geq I$. Then there is $n \geq N$ with:

$$s'_i = \begin{cases} y_{n-1} + P_{m_{n-1}+1} + \ldots + P_j & \text{for some } m_{n-1}+1 \leq j \leq m_n \\ y_{n-1} + P_{m_{n-1}+1} + \ldots + P_j - Q_{k_{n-1}+1} - \ldots - Q_j & \text{for some } k_{n-1}+1 \leq j \leq k_n \end{cases}$$

In both cases, $s'_i \leq y_{n-1} + P_{m_{n-1}+1} + \ldots + P_m = x_n < r$. Therefore, $\limsup s'_i = \beta$ by proposition 41.

We can do similar reasoning to show that $\liminf s'_i = \alpha$. ■

Proposition 64: If $\forall n,\ \ a_n \in \mathbb{C}$ and if $\sum a_n$ converges absolutely, then for any rearrangement $\sum a'_n$, we have that $\sum a'_n$ converges to $\sum a_n$.

Proof:
Let (k_n) be a sequence in $\mathbb{Z}^+$ using each element of $\mathbb{Z}^+$ precisely once.

Set $A = \sum a_n$, let $\varepsilon > 0$, and pick N_1 with $\forall n \geq N_1,\ \ \left|\sum\limits_{k=1}^{n} (a_n) - A\right| < \varepsilon/2$.

By the Cauchy criterion, we can pick N_2 with $\forall m \geq N_2,\ \sum\limits_{k=N_2}^{m} |a_k| < \varepsilon/2$.

Thus for all finite $I \subset \{N_2, N_2+1, \ldots\}$, we have that $\sum\limits_{k \in I} |a_k| < \varepsilon/2$.

Set $N = \max(N_1, N_2)$ and pick $p \in \mathbb{Z}^+$ large enough that $\{k_1, \ldots, k_p\} \supseteq \{1, 2, \ldots, N\}$. Then for all $q \geq p$, we have that:

$$\begin{aligned} \left|\sum_{n=1}^{q} (a_{k_n}) - A\right| &\leq \left|\sum_{n=1}^{q} (a_{k_n}) - \sum_{n=1}^{N} (a_n)\right| + \left|\sum_{n=1}^{N} (a_n) - A\right| \\ &< \left(\sum_{n \in \{k_1,\ldots,k_p\} \setminus \{1,2,\ldots,N\}} |a_n|\right) + \varepsilon/2 \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

We thus conclude that $\sum a_{k_n}$ converges to A.

Lecture 21: 3/4/2024

Suppose X and Y are metric spaces, $E \subseteq X$, and $f : E \to Y$. Let $p \in E'$ and $q \in Y$. We say the limit of f at p is q, written as:

$$\lim_{x \to p} f(x) = q \text{ or alternatively } f(x) \to q \text{ as } x \to p,$$

if $\forall \varepsilon > 0, \ \exists \delta > 0 \ s.t. \ \forall x \in E, \ 0 < d_X(x, p) < \delta \implies d_Y(f(x), q) < \varepsilon$.

Proposition 65: $\lim_{x \to p} f(x) = q$ if and only if for all sequences (p_n) in E satisfying $\forall n, \ p_n \neq p$ and $p_n \to p$, we have that $\lim_{n \to \infty} f(p_n) = q$.

Proof:
($\Longrightarrow$) Consider a sequence (p_n) in E with $\forall n, \ p_n \neq p$ and $p_n \to p$.
Let $\varepsilon > 0$. Since $\lim_{x \to p} f(x) = q$, there is $\delta > 0$ such that
$\forall x \in E, \ 0 < d_X(x, p) < \delta \implies d_Y(f(x), q) < \varepsilon$.

Since $p_n \to p$, pick N with $\forall n \geq N$, $d_X(p_n, p) < \delta$. Then for $n \geq N$, we have $0 < d_X(x, p) < \delta$, and therefore $d_Y(f(p_n), q) < \varepsilon$. Thus $\lim_{n \to \infty} f(p_n) = q$.

($\Longleftarrow$) Assume the statement is false. Then:
$\exists \varepsilon > 0 \ s.t. \ \forall \delta > 0, \ \exists x \in E \ s.t. \ 0 < d_X(x, p) < \delta$ and $d_Y(f(x), q) \geq \varepsilon$.

For each $n \in \mathbb{Z}^+$, using $\delta = \frac{1}{n}$, we obtain a point $p_n \in E$ with $0 < d_X(p_n, p) < \frac{1}{n}$ and $d_Y(f(p_n), q) \geq \varepsilon$. Then (p_n) is a sequence in E with $\forall n, \ p_n \neq p$ and $\lim(p_n) = p$ but $(f(p_n))$ does not converge to q. So, we've shown the contrapositive.

Corollary: If $\lim_{x \to p} f(x) = q_1$ and $\lim_{x \to p} f(x) = q_2$, then $q_1 = q_2$.

Proof:
Limits of sequences are unique. So, we can apply the previous theorem to easily show this corollary.

For functions $f, g : E \to \mathbb{C}$, we can build new functions:

- $(f + g)(x) = f(x) + g(x)$
- $(f - g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (defined on $\{x \in E \mid g(x) \neq 0\}$)

Proposition 66: Let X be a metric space, $E \subseteq X$, and $f, g : E \to \mathbb{C}$. Let $p \in E'$. If $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$, then:

- $\lim_{x \to p} (f + g)(x) = A + B$
- $\lim_{x \to p} (f - g)(x) = A - B$
- $\lim_{x \to p} (fg)(x) = AB$
- $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ when $B \neq 0$.

Proof:
Apply the corresponding parts of proposition 33, a proposition about sequences, as well as proposition 65.

Similarly, if $\vec{f}, \vec{g} : E \to \mathbb{R}^k$, then:

- $\lim_{x \to p} (\vec{f} + \vec{g})(x) = A + B$
- $\lim_{x \to p} (\vec{f} \cdot \vec{g})(x) = AB$

Proof:
Apply proposition 34 and proposition 65.

Lecture 22: 3/6/2024

Let X, Y be metric spaces, $E \subseteq X$, and $f : E \to Y$. For $p \in E$, we say f is continuous at p if:

$$\forall \varepsilon > 0, \;\; \exists \delta > 0 \;\; s.t. \;\; \forall x \in E, \;\; 0 \leq d_X(x, p) < \delta \Longrightarrow d_Y(f(x), q) < \varepsilon.$$

If f is continuous at every point in its domain E, then we call f continuous.

Proposition 67: If $p \in E \setminus E'$, then every function $f : E \to Y$ will be continuous at p. If $p \in E \cap E'$, then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

Proof:
First suppose $p \in E \setminus E'$. Then there exists $\delta > 0$ with $(B_\delta(p) \setminus \{p\}) \cap E = \emptyset$. Hence, $B_\delta(p) \cap E = \{p\}$. Then for any $x \in E$ and function f,

$$d_X(x, p) < \delta \Rightarrow x \in B_\delta(p) \cap E \Rightarrow x = p \Rightarrow f(x) = f(p) \Rightarrow d_y(f(x), f(p)) = 0.$$

Thus f is continuous at p.

Next, notice that for $p \in E \cap E'$ and any function $f : E \to Y$, we have that:

$$\lim_{x \to p} f(x) = f(p)$$
$$\Updownarrow$$
$$\forall \varepsilon > 0, \;\; \exists \delta > 0 \;\; s.t. \;\; \forall x \in E, \;\; 0 < d_X(x,p) < \delta \Longrightarrow d_Y(f(x), f(p)) < \varepsilon$$
$$\Updownarrow$$
$$\forall \varepsilon > 0, \;\; \exists \delta > 0 \;\; s.t. \;\; \forall x \in E, \;\; 0 \leq d_X(x,p) < \delta \Longrightarrow d_Y(f(x), f(p)) < \varepsilon$$
$$\Updownarrow$$

f is continuous at p

Proposition 68: Let X, Y, Z be metric spaces, let $E_x \subseteq X$, $E_y \subseteq Y$, $f : E_x \to E_y$, and $g : E_y \to Z$. Define $h : E_x \to Z$ by $h(x) = (g \circ f)(x)$. If f is continuous at p and g is continuous at $f(p)$, then h is continuous at p.

Proof:

Let $\varepsilon > 0$. Since g is continuous at $f(p)$, there is $\kappa > 0$ with:

$$\forall y \in E_y, d_Y(y, f(p)) < \kappa \Rightarrow d_Z(g(y), g(f(p))) < \varepsilon$$

Then since f is continuous at p, there is $\delta > 0$ with:

$$\forall x \in E_x, d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \kappa \Rightarrow d_Z(g(f(x)), g(f(p))) < \varepsilon$$

So $\forall \varepsilon > 0, \;\; \exists \delta > 0 \;\; s.t. \;\; d_X(x,p) < \delta \Longrightarrow d_Z(h(x), h(p)) < \varepsilon$.

Proposition 69: $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is an open subset of X whenever V is an open subset of Y.

Proof:

($\Longrightarrow$) Let $V \subseteq Y$ be open. Consider any $p \in f^{-1}(V)$. Then $f(p) \in V$. And since V is open, there is some $\varepsilon > 0$ with $B_\varepsilon(f(p)) \subseteq V$.

Since f is continuous, there is $\delta > 0$ such that for all $x \in X$, $d_X(x,p) < \delta \Longrightarrow d_Y(f(x), f(p)) < \varepsilon$. Thus, $f(B_\delta(p)) \subseteq B_\varepsilon(f(p)) \subseteq V$, meaning that $B_\delta(p) \subseteq f^{-1}(V)$. Thus p is an interior point of $f^{-1}(V)$. So we conclude that $f^{-1}(V)$ is open.

($\Longleftarrow$) Let $p \in X$ and let $\varepsilon > 0$. Set $V = B_\varepsilon(f(p))$. Then V is open since V is a ball, which in turn means that f^{-1} is open. Clearly, $p \in f^{-1}(V)$, and since this set is open, there exists $\delta > 0$ with $B_\delta(p) \subseteq f^{-1}(V)$.

Then $\forall x \in X$,

$$\begin{aligned} d_X(x,p) < \delta &\implies x \in B_\delta(p) \subseteq f^{-1}(V) \\ &\implies x \in f^{-1}(V) \\ &\implies f(x) \in V = B_\varepsilon(f(p)) \\ d_Y(f(x), f(p)) < \varepsilon. \end{aligned}$$

Thus f is continuous at p. So f is a continuous function.

Corollary: $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed whenever $C \subseteq Y$ is closed.

Proof:
For any set $D \subseteq Y$, we have that $f^{-1}(Y \setminus D) = X \setminus f^{-1}(D)$. Therefore, this corollary follows from the previous proposition in addition to the duality of open / closed sets under complements.

Proposition 70: If $f, g : X \to \mathbb{C}$ are continuous, then so are $f+g$, $f-g$, fg, and $\frac{f}{g}$.

Proof:
At isolated points, there is nothing to prove. At limit points, this follows from proposition 66.

Proposition 71:

(a) Let $f_1, f_2, \ldots, f_k : X \to \mathbb{R}$ and define $\vec{f} : X \to \mathbb{R}^k$ by $\vec{f}(x) = (f_1(x), f_2(x), \ldots, f_k(x))$. Then $\vec{f}$ is continuous if and only if $\forall 1 \leq i \leq k, \quad f_i$ is continuous.

Proof: Use proposition 67, then 65, and then 34.

(b) If $\vec{f}$, $\vec{g} : X \to \mathbb{R}^k$ are continuous, then so are $\vec{f} + \vec{g}$, $\vec{f} - \vec{g}$, and $\vec{f} \cdot \vec{g}$.

Proof: Use proposition 67 and then 66.

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The map sending $\vec{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ to x_i is continuous (this is trivial to check). Therefore, for $n_1, n_2, \ldots, n_k \in \mathbb{Z}^+$, the map:

$$\vec{x} = (x_1, x_2, \ldots, x_k) \mapsto x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

is continuous by proposition 70.

In turn, this means that given a polynomial:

$$p(\vec{x}) = \sum_{(n_1,\ldots,n_k)\in\mathbb{Z^+}^k} c_{(n_1,\ldots,n_k)} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

where all but finitely many $(n_1, \ldots, n_k) \in \mathbb{Z^+}^k$ are zero, we have that $p(\vec{x})$ is continuous by proposition 70 and the previous observation.

Next, this means that given a rational function $\frac{p(\vec{x})}{q(\vec{x})}$ where p and q are polynomials, we have that that rational function is continuous on its domain (once again because of proposition 70).

Another continuous function is: $\vec{x} \in \mathbb{R}^k \mapsto \|\vec{x}\|$ (this is trivial to check).

A function $f : E \to Y$ (where Y is a metric space) is bounded if its image $f(E)$ is a bounded set in Y.

Proposition 72: Let X, Y be metric spaces. If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.

Proof:
Let $\{V_\alpha \mid \alpha \in A\}$ be an open cover of $f(X)$. Then by proposition 69, each $f^{-1}(V_\alpha)$ is open. So $\{f^{-1}(V_\alpha) \mid \alpha \in A\}$ is an open cover of X.

Since X is compact, there are $\alpha_1, \ldots, \alpha_n \in A$ with $X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$.

Therefore, $f(X) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$. So, we conclude that $f(X)$ is compact.

Proposition 73: If X is a compact metric space and $f : X \to \mathbb{R}^k$ is continuous, then $f(X)$ is closed and bounded.

Proof:
This is just proposition 72 combined with proposition 28 (basically applying the previous proposition to $\mathbb{R}^k$).

Proposition 74: (Extreme Value Theorem)
If X is a compact metric space and $f : X \to \mathbb{R}$ is continuous, then f achieves a minimum and maximum value, meaning there are $p, q \in X$ with $f(p) = \sup f(X)$ and $f(q) = \inf f(X)$.

Proof:
By proposition 73, we know that $f(X)$ is closed and bounded. Thus as $\inf f(X)$ and $\sup f(X)$ are in $\overline{f(X)}$, we have that $\sup f(X) \in f(X)$ and $\inf f(X) \in f(X)$.

Proposition 75: Let X and Y be metric spaces. If $f : X \to Y$ is a continuous bijection and X is compact, then $f^{-1} : Y \to X$ is continuous.

Proof:
By proposition 69, it suffices to show that $\left(f^{-1}\right)^{-1}(U) = f(U)$ is open whenever $U \subseteq X$ is open.

Let $U \subseteq X$ be open. Then $X \setminus U$ is closed. Since X is compact, $X \setminus U$ is compact. Therefore, $f(X \setminus U)$ is compact by proposition 72, which means that $f(X \setminus U)$ is closed.

In turn, $Y \setminus f(X \setminus U)$ is open. And, since f is bijective, $f(X \setminus U) = Y \setminus f(U)$. So, $Y \setminus f(X \setminus U) = Y \setminus (Y \setminus f(U)) = f(U)$. Thus, $F(U)$ is open and we can conclude that f^{-1} is continuous.

Generally, it is not always true that the inverse of a continuous function is continuous. For example, consider $f : (-1, 0] \cup [1, 2] \to [0, 2]$ with the rule: $f(x) = |x|$. f is continuous on its domain but f^{-1} is not continuous at $x = 1$.

Let X and Y be metric spaces and $f : X \to Y$. We say f is uniformly continuous if $\forall \varepsilon > 0,\ \exists \delta > 0\ \ s.t.\ \forall x_1, x_2 \in X,\ \ d_X(x_1, x_2) < \delta \Longrightarrow d_y(f(x_1), f(x_2)) < \varepsilon.$

Proposition 76: Let X and Y be metric spaces. If X is compact and $f : X \to Y$ is continuous, then f is uniformly continuous.

Proof:
Consider the following lemma: If X and Y are metric spaces and $f : X \to Y$ is continuous, then for every $p \in X$ and $\varepsilon > 0$, there is $\delta > 0$ such that:

$$\forall x_1 \in B_\delta(p),\ \ \forall x_2 \in X,\ \ d_X(x_1, x_2) < \delta \Longrightarrow d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Proof:
Since f is continuous at p, we can find $\delta > 0$ with:
$\forall x \in X,\ \ d_X(x, p) < 2\delta \Longrightarrow d_Y(f(x), f(p)) < \frac{\varepsilon}{2}.$

Now consider $x_1 \in B_\delta(p)$ and $x_2 \in X$ with $d_X(x_1, x_2) < \delta$.
Then $d_X(x_1, p) < \delta$ and $d_X(x_2, p) \leq d_X(x_2, x_1) + d_X(x_1, p) < 2\delta$. So $d_Y(f(x_1), f(p)) < \frac{\varepsilon}{2}$ and $d_Y(f(x_2), f(p)) < \frac{\varepsilon}{2}$. Finally, by the triangle inequality: $d_Y(f(x_1), f(x_2)) \leq d_Y(f(x_1), f(p)) + d_Y(f(x_2), f(p)) < \varepsilon$.

Now let $\varepsilon > 0$. Then for each $x \in X$, pick $\delta_x > 0$ as in the above the lemma. Note that $\{B_{\delta_x} \mid x \in X\}$ is an open cover of X. So, there are $x_1, \ldots, x_n$ in X with $X = \bigcap_{i=1}^{n} B_{\delta_{x_i}}(x_i)$.

Set $\delta = \min_{1 \leq i \leq n} \delta_{x_i}$. Then consider any $x_1 \in X$ and pick an i such that $x_1 \in B_{\delta_{x_i}}(x_i)$. For any $x_2 \in X$, with $d(x_1, x_2) < \delta$, we know by the above lemma that $d_Y(f(x_1), f(x_2)) < \varepsilon$.

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Proposition 77: Let $E \subseteq \mathbb{R}$ be non-compact. Then:

(A) $\exists f : E \to \mathbb{R}$ such that f is continuous but not bounded.

(B) $\exists f : E \to \mathbb{R}$ such that f is continuous and bounded but has no maximum.

(C) If we additionally assume E is bounded, then $\exists f : E \to \mathbb{R}$ such that f is continuous but not uniformly continuous.

Proof:
First assume E is bounded.

If E is also closed, then by the Heine-Borel theorem (proposition 28), we would have that E is compact since $E \subseteq \mathbb{R}$. So, by our assumption that E is not compact, we can conclude that $E' \setminus E \neq \emptyset$. Thus, pick $x_0 \in E' \setminus E$.

(A) Define $f(x) = \frac{1}{x - x_0}$ for $x \in E$. Because $x_0 \notin E$, $f(x)$ is defined over all E. Additionally, we can show that $\frac{1}{x-x_0}$ is continuous over its domain. Finally, because x_0 is a limit point of E, we can find $x \in E$ arbitrarily close to x_0, meaning that $\frac{1}{x-x_0}$ is unbounded. Hence we have defined a continuous unbounded $f : E \to \mathbb{R}$.

(B) Define $f(x) = \frac{1}{1+|x-x_0|}$ for $x \in E$. f is defined over all E because $1 + |x - x_0| > 0$. Also, f can be shown to be continuous. Meanwhile $\forall x \in E$, we have that $0 < f(x) \leq 1$. So f is bounded. Because $x_0 \notin E$, we have that $f(x) \neq 1$ for any x. But because $x_0 \in E'$, we can say that $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \frac{1}{1+|x-x_0|} = 1$. So f doesn't acheive a maximum.

(C) Use the same f as in (A). Let $\varepsilon > 0$. Consider any $\delta > 0$. Since f is not bounded on $B_{\frac{\delta}{2}}(x_0) \cap E$, we can pick $p, q \in B_{\frac{\delta}{2}}(x_0) \cap E$ with $|f(p) - f(q)| \geq \varepsilon$. This shows f is not uniform continuous.

Now assume E is not bounded.

(A) $f(x) = x$ for $x \in E$ is continuous but not bounded.

(B) Define $f(x) = \frac{|x|}{1+|x|}$ for $x \in E$. f is defined over all E because $1+|x| > 0$. Also f is continuous. Meanwhile, $\forall x \in E$, we have that $0 \leq f(x) < 1$. So f is bounded. Now note that we can rewrite f as $f(x) = 1 - \frac{1}{1+|x|}$. And since x is unbounded, we thus have that $\sup\{f(x) \mid x \in E\} = 1$. Thus f does not achieve a maximum.

Observation: $\mathbb{Z} \subseteq \mathbb{R}$ is not compact and not bounded but every function $f : \mathbb{Z} \to \mathbb{R}$ is uniformly continuous.

Proof: Take $\delta < 1$ for all ε.

As a result, this shows that proposition 77.C is not necessarily true if E is unbounded.

Proposition 78: Let X and Y be metric spaces and $f : X \to Y$ be continuous. If $E \subseteq X$ is connected, then $f(E)$ is connected.

Proof:

We'll prove the contrapositive. Suppose $f(E)$ is not connected. Then there are separated sets A, and B that are nonempty and satisfy $A \cap B = f(E)$. Set $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$. Since f is continuous, $f^{-1}(\overline{A})$ is closed. That along with the fact that $G \subseteq f^{-1}(\overline{A})$ means that $\overline{G} \subseteq f^{-1}(\overline{A})$. Now as $\overline{A} \cap B = \emptyset$, we have:

$$\overline{G} \cap H = f^{-1}(\overline{A}) \cap f^{-1}(B) = f^{-1}(\overline{A} \cap B) = f^{-1}(\emptyset) = \emptyset.$$

We can use similar reasoning to show $G \cap \overline{H} = \emptyset$. Therefore, G, H are nonempty and separated. Finally, note that:

$$E = E \cap f^{-1}(f(E)) = E \cap f^{-1}(A \cup B) = G \cup H.$$

Therefore E is not connected.

Proposition 79: (Intermediate Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then for $c \in \mathbb{R}$ such that c is between $f(a)$ and $f(b)$, there is $x \in [a, b]$ with $f(x) = c$.

Proof:
$[a,b]$ is connected because it satisfies proposition 31. Thus applying proposition 78, we get that $f([a,b]) \subseteq \mathbb{R}$ is connected. Finally, by proposition 31 again, we have that $[f(a), f(b)] \subseteq f([a,b])$. So there exists x such that $f(x) = c$.

Observation: The converse of proposition 79 is false.

Define $f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$

For any $-1 \leq c \leq 1$, we can find $x \in [-1,1]$. But f is not continuous at 0.

If x is in the domain of f and f is not continuous at x, then we call x a discontinuity of f and say f is discontinuous at x.

Let $f : (a,b) \to \mathbb{R}$. If $a \leq x < b$, we define the right-hand limit of f at x and denote this $f(x+)$ or $\lim\limits_{t \to x^+} f(t)$ to be $\lim\limits_{t \to x} g(t)$ where g is the restriction of f to (x,b).

Similarly, if $a < x \leq b$, we define the left-hand limit of f at x and denote this $f(x-)$ or $\lim\limits_{t \to x^-} f(t)$ to be $\lim\limits_{t \to x} h(t)$ where h is the restriction of f to (a,x).

Observation: $\lim\limits_{t \to x}$ exists if and only if both $f(x-)$ and $f(x+)$ exist and are equal to each other. In this case, all three are equal to each other.

If f is discontinuous at x and both $f(x-)$ and $f(x+)$ are defined, then we say that x is a discontinuity of the first kind or a simple discontinuity. Otherwise, we say x is a discontinuity of the second kind.

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A function $f : (a,b) \to \mathbb{R}$ is monotone increasing if $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$. Similarly, f is monotone decreasing if $f(x_1) \geq f(x_2)$ whenever $x_1 \leq x_2$.

Proposition 80: Let $f : (a,b) \to \mathbb{R}$ be monotone increasing. Then:

1. For every $x \in (a,b)$, both $f(x+)$ and $f(x-)$ exists and
$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t).$$

Proof:
Fix $x \in (a,b)$. Set $A = \sup\limits_{a<t<x} f(t)$ and let $\varepsilon > 0$. Since $A - \varepsilon$ is not an upperbound to $\{f(t) \mid a < t < x\}$, there is $\delta > 0$ so that $a < x - \delta$ and $f(x-\delta) > A - \varepsilon$. Since f is monotone increasing, we have that $\forall t \in (x-\delta, x),\ \ A - \varepsilon < f(x-\delta) \leq f(t) \leq A$. Hence $t \in (x-\delta, x) \implies |f(t) - A| < \varepsilon$. So $\lim\limits_{t \to x^-} f(t) = A$.

We can similarly show that $f(x+) = \inf\limits_{x<t<b} f(t)$.

Now because f is monotone increasing, we know that $f(x) \geq f(t)$ for all $t \in (a,x)$. By the definition of a supremum, we thus have that $f(x) \geq \sup\limits_{a<t<x} f(t)$. By similar reasoning, we have that $f(x) \leq f(x+)$.

2. If $x < y$, then $f(x+) \leq f(y-)$.

   Proof:
   Pick any $x < c < y$. Since f is monotone increasing:
   $$f(x+) = \inf_{x<t<b} f(t) \leq f(c) \leq \sup_{a<t<y} f(t) = f(y-)$$

Obviously there is an analogous proposition for a monotone decreasing function.

Proposition 81: If $f : (a,b) \to \mathbb{R}$ is monotone, then it has at most countably many discontinuities (and they are all simple by the previous proposition).

Proof:
Say f is monotone increasing (the proof is mostly the same if f is monotone decreasing but with some flipped inequalities). Let E be the set of discontinuities of f.

For each $x \in E$, we have $f(x-) < f(x+)$. So we can pick $r(x) \in \mathbb{Q}$ with $f(x-) < r(x) < f(x+)$. If $x, y \in E$ and $x < y$, then $r(x) < f(x+) \leq f(y-) < r(y)$. Therefore $r : E \to \mathbb{Q}$ is injective, and hence E is countable.

Observation: Given any countable $E \subseteq (a,b)$, there is a monotone increasing function $f : (a,b) \to \mathbb{R}$ such that E is the set of discontinuities of f.

Pick any sequence (c_n) of positive real numbers such that $\sum c_n < \infty$. Also let $x_1, x_2, \ldots$ be an enumeration of E. Then define:
$$f(x) = \sum_{\{n \in \mathbb{N} \,|\, x_n < x\}} c_n$$

f is monotone increasing. Also, f is continuous at every point in $(a,b) \setminus E$ and for each $x_n \in E$, we have $f(x_n+) - f(x_n-) = c_n$.

Lecture 26: 3/15/2024

For $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Let $x \in E'$ or $x = +\infty$ if E is not bounded above or $x = -\infty$ if E is not bounded below. Then for $y \in \mathbb{R} \cup \{-\infty, \infty\}$, we say the limit of f at x is y and write $\lim_{t \to x} f(t) = y$ if for every sequence (t_n) with $\lim t_n = x$ and with $\forall n,\ \ t_n \neq x$ and $t_n \in E$, we have that $\lim f(t_n) = y$.

Proposition 82: Let E and x be as above. Let $f, g : E \to \mathbb{R}$ and assume $\lim_{t \to x} f(t) = A$ and $\lim_{t \to x} g(t) = B$ where $A, B \in \mathbb{R} \cap \{-\infty, \infty\}$. Then:

- $\lim_{t \to x} (f + g)(t) = A + B$
- $\lim_{t \to x} (fg)(t) = AB$
- $\lim_{t \to x} (\frac{f}{g})(t) = \frac{A}{B}$

...provided that the right hand sides of the above expressions are defined.

Also $\lim_{t \to x} f(t) = A' \implies A' = A$.

The proof for this has been left as an exercise by the professor and I don't have time to do it. :)

An Incomplete List of How The Proposition Numbering in my Notes Lines up With Our Textbook:

Proposition Number	Label in Textbook	Proposition Number	Label in Textbook
1		2	
3		4	1.20
5		6	
7		8	
9		10	
11		12	2.12
13	2.13	14	
15		16	2.20
17	2.23	18	
19	2.27	20	2.28
21	2.30	22	
23		24	
25	2.37	26	2.38
27	2.40	28	2.41
29	2.42	30	2.43
31	2.47	32	3.2
33	3.3	34	3.4
35	n.a.	36	3.6
37	3.7	38	3.10
39	3.11	40	3.14
41	3.17	42	3.19
43	n.a	44	3.20
45	3.22	46	3.23
47	3.24	48	3.25
49	3.26	50	3.27
51	3.28	52	3.29
53	3.33	54	3.34
55	3.39	56	3.41
57	3.42	58	3.43

Proposition Number	Label in Textbook	Proposition Number	Label in Textbook
59	3.44	60	3.45
61	3.47	62	3.50
63	3.54	64	3.55
65	4.2	66	4.4
67	4.6	68	4.7
69	4.8	70	4.9
71	4.10	72	4.14
73	4.15	74	4.16
75	4.17	76	4.19
77	4.20	78	4.22
79	4.23	80	4.29
81	4.30	82	4.34

Our textbook is *Principles of Mathematical Analysis* by Walter Rudin.