# Math 240A Notes (Professor: Luca Spolaor)

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# Lecture 1 Notes: 9/26/2024

Given an indexed family of sets $\{X_\alpha\}_{\alpha \in A}$, we define its Cartesian Product to be:

$$\prod_{\alpha \in A} X_\alpha = \{f : A \longrightarrow \bigcup_{\alpha \in A} X_\alpha \mid f(\alpha \in X_\alpha)\}$$

A projection is a function $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \longrightarrow X_\alpha$ satisfying that $f \mapsto f(\alpha)$.

If $X, Y$ are sets, we define:

- $\operatorname{card}(X) \leq \operatorname{card}(Y)$ if there exists an injection $f : X \longrightarrow Y$.
- $\operatorname{card}(X) \geq \operatorname{card}(Y)$ if there exists a surjection $f : X \longrightarrow Y$.
- $\operatorname{card}(X) = \operatorname{card}(Y)$ if there exists a bijection $f : X \longrightarrow Y$.

Note that $\operatorname{card}(X) \leq \operatorname{card}(Y) \iff \operatorname{card}(Y) \geq \operatorname{card}(X)$. After all, given an injection in one direction, we can easily make a surjection in the other direction. Or given a surjection in one direction, we can (using A.O.C (axiom of choice)) easily make an injection in the other direction.

Also, if $\operatorname{card}(X) \leq \operatorname{card}(Y)$ and $\operatorname{card}(Y) \leq \operatorname{card}(X)$, then we know that $\operatorname{card}(Y) = \operatorname{card}(X)$.

Proof:
We know there exists $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ which are both injective. Hence, $g \circ f$ is an injection from $X$ to $g(Y) \subseteq X$. By an exercise done in my math journal on page 8, we thus there exists a bijection $h$ from $X$ to $g(Y)$. And letting $g^{-1}$ be any left-inverse of $g$, we then have that $g^{-1} \circ h$ is a bijection from $X$ to $Y$.

We say $X$ has the cardinality of the continuum if $\operatorname{card}(X) = \operatorname{card}(\mathbb{R})$.

Proposition: $\operatorname{card}(\mathcal{P}(\mathbb{N})) = \operatorname{card}(\mathbb{R})$.

Our textbook goes about proving this by constructing two functions: an injection and a surjection, from $\mathcal{P}(\mathbb{N})$ to $\mathbb{R}$ based on the binary expansion of any real number. That way, we know that $\operatorname{card}(\mathcal{P}(\mathbb{N})) \leq \operatorname{card}(\mathbb{R})$ and $\operatorname{card}(\mathcal{P}(\mathbb{N})) \geq \operatorname{card}(\mathbb{R})$.

Given a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ we know there exists: $\limsup x_n = \inf_{k \geq 1}(\sup_{n \geq k} x_n)$ and $\liminf x_n = \sup_{k \geq 1}(\inf_{n \geq k} x_n)$.

Also, given a function $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, we can define:

$$\limsup_{x \to a} f(x) = \inf_{\delta > 0} \left( \sup_{0 < |x - a| < \delta} f(x) \right).$$

If $X$ is an arbitrary set and $f : X \longrightarrow [0, \infty]$, we define:

$$\sum_{x \in X} f(x) = \sup \left\{ \sum_{x \in F} f(x) \mid F \subseteq X \ s.t. \ F \text{ is finite} \right\}.$$

Cool Proposition from textbook (not covered in lecture):

Let $A = \{x \in X \mid f(x) > 0\}$. If $A$ is uncountable, then $\sum_{x \in X} f(x) = \infty$.

If $A$ is countably infinite and $g : \mathbb{N} \longrightarrow A$ is a bijection, then $\sum_{x \in X} f(x) = \sum_{n=1}^{\infty} f(g(n))$.

Proof of first statement:
$A = \bigcup_{n \in \mathbb{N}} A_n$ where $A_n = \{x \in X \mid f(x) > \frac{1}{n}\}$.

If $A$ is uncountable, we must have that some $A_n$ is uncountable. But then for any finite set $F \subseteq X$, we have that $\sum_{x \in F} f(x) > \frac{\mathrm{card}(F)}{n}$. So $\sum_{x \in X} f(x)$ is unbounded.

A metric space $(X, \rho)$ is a set $X$ equipped with a distance function $\rho : X \times X \longrightarrow [0, \infty)$. We denote the open ball of radius $r$ about $x$ to be $B(r, x) = \{y \in X \mid \rho(x, y) < r\}$. And you remember our definitions from 140A... right?

**Proposition 0.21:** Every open set in $\mathbb{R}$ is a countable union of disjoint open intervals.

We proved this as part of a homework exercise in Math 140A.

Given a metric space $(X, \rho)$, an element $x \in X$, and sets $F, E \subseteq X$, we can define:

- $\rho(x, E) = \rho_E(x) = \inf\{\rho(x, y) \mid y \in E\}$.
- $\rho(F, E) = \inf\{\rho_E(y) \mid y \in F\}$.

**Exercise:** $g(x, E) = 0 \iff x \in \overline{E}$.

Proof:
If $\inf\{\rho(x, y) \mid y \in E\} = 0$, then there exists a sequence $\{y_n\}$ in $E$ such that $\rho(x, y_n) \to 0$. This implies $x \in \overline{E}$. Similarly, if $x \in \overline{E}$, we can construct a sequence $\{y_n\}$ such that $\rho(x, y_n) < \frac{1}{n}$ for all $n$. Then:

$$0 \leq \inf\{\rho(x, y) \mid y \in E\} \leq \inf\{\rho(x, y_n) \mid n \in \mathbb{N}\} = 0.$$

Given a subset $E$ of a metric space $(X, \rho)$, we define:

$$\mathrm{diam}(E) = \sup\{\rho(x, y) \mid x, y \in E\}.$$

If $\mathrm{diam}(E) < \infty$, we say $E$ is bounded. If $\forall \varepsilon > 0$, $E$ can be covered by finitely many balls of radius $\varepsilon$, then we say $E$ is totally bounded.

**Exercise:** $E$ being totally bounded implies $E$ is bounded.

Pick $\varepsilon > 0$ and let $\{z_1, \ldots, z_n\}$ be the set of points such that $E \subseteq \bigcup_{k=1}^{n} B(\varepsilon, z_n)$.

Then given any $x, y \in E$, we can assume that $x \in B(\varepsilon, z_i)$ and $y \in B(\varepsilon, z_j)$. So, $\rho(x, y) \leq \rho(x, z_i) + \rho(z_i, z_j) + \rho(z_j, y) < 2\varepsilon + \max\{\rho(z_i, z_j) \mid 1 \leq i, j \leq n\}$.

The converse is not generally true. For instance, if you use the discrete metric, then any set with more than one element will have a diameter of $1$. But if $0 < \varepsilon < 1$, then it will be impossible to cover an infinite set with finitely many balls.

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# Lecture 2 Notes: 10/1/2024

**Proposition:** Suppose $E$ is a subset of a metric space $(X, \rho)$. Then the following are equivalent.

1. $E$ is complete and totally bounded
2. All sequences $(x_n) \subseteq E$, have a convergent subsequence.
3. For all open covers $\{V_\alpha\}_{\alpha \in A}$ of $E$, there exists $V_{\alpha_1}, \ldots, V_{\alpha_n}$ such that $E \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$.

Proof:

(1) $\Longrightarrow$ (2):

Lemma:

If $E$ is totally bounded and $F \subseteq E$, then $F$ is totally bounded.

Given any $\varepsilon > 0$, let $\{x_1, \ldots, x_n\}$ be a subset of $E$ such that $E \subseteq \bigcup_{i=1}^{n} B(\varepsilon/2, x_i)$. Then consider the collection of sets:

$$\{F \cap B(\varepsilon/2, x_i)\} - \{\emptyset\}.$$

We know the diameter of each $F \cap B(\varepsilon/2, x_i)$ is at most $\varepsilon$. So in each set, pick $y_i \in F \cap B(\varepsilon/2, x_i)$. Then for some $m \leq n$:

$$F \subseteq \bigcup_{i=1}^{m} B(\varepsilon, y_i)$$

Let $A_1 = E$. Then for $k \geq 2$ we recursively define $A_k$ as follows:

Assuming $A_{k-1} \cap (x_n)_{n \in \mathbb{N}}$ is infinite and $A_{k-1}$ is totally bounded, choose $\{y_1, \ldots, y_m\}$ in $A_k$ such that $A_k \subseteq \bigcup_{i=1}^{m} B(2^{-n}, y_i)$. Importantly, since $(x_n)_{n \in \mathbb{N}} \cap A_{k-1}$ is infinite, we know one of those open balls contains infinitely many points in our sequence. So set $A_k$ equal to that ball intersected with $E$. Note that by our lemma, $A_k$ is totally bounded.

Now pick any $x_{n_1}$ and then for all $k \geq 2$ pick $x_{n_k} \in A_k$ such that $n_k > n_{k-1}$. That way, $(x_{n_k})_{k\in\mathbb{Z}_+}$ is a subsequence of $(x_n)_{n\in\mathbb{Z}_+}$. Also, we know that $(x_{n_k})_{k\in\mathbb{Z}_+}$ is Cauchy. Hence, since $E$ is complete, we know that it converges to some $x$ in $E$.

(2) $\Longrightarrow$ (1):

Firstly, suppose $E$ is not complete. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ that is Cauchy but does not converge in $E$. Importantly, because $(x_n)_{n\in\mathbb{N}}$ is Cauchy, if there was a convergent subsequence, we know the limit of that subsequence would have to be the limit of the whole sequence. But that doesn't exist. So, we know (2) can't be true.

Secondly, suppose $E$ is not totally bounded. Then there exists $\varepsilon > 0$ such that it is impossible to cover $E$ in balls of radius $\varepsilon$. So, we can recursively define a sequence $(x_n)_{n\in\mathbb{N}}$ in $E$ satisfying that:

$$x_n \in E - \bigcup_{i=1}^{n-1} B(\varepsilon, x_i).$$

Importantly, for all natural numbers $n \neq m$, we have that $\rho(x_n, x_m) \geq \varepsilon$. So, it is impossible to find a convergent subsequence of $(x_n)$, meaning (2) is false.

(1) and (2) $\Longrightarrow$ (3):

Let $\{V_\alpha\}_{\alpha\in A}$ be an open cover of $E$.

Suppose for the sake of contradiction that for all $n \in \mathbb{N}$, there is a ball $B_n$ of radius $2^{-n}$ centered in $E$ such that $B_n \cap E \neq \emptyset$ but $B_n \not\subseteq V_\alpha$ for all $\alpha \in A$. Then we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ in $E$ such that $x_n \in B_n \cap E$ for all $n \in \mathbb{N}$. By (2), we know there is a subsequence that converges to some $x \in E$. Importantly, we know $x \in V_\alpha$ for some $\alpha \in A$, and because $V_\alpha$ is open, there is $\varepsilon > 0$ such that $B(\varepsilon, x) \subseteq V_\alpha$. But now we get a contradiction because by picking $n$ such that $2^{-n} < \varepsilon/3$ and $\rho(x, x_n) < \varepsilon/3$, we have for all $y \in B_n$ that:

$$\rho(x, y) \leq \rho(x, x_n) + \rho(x_n, y) < 2^{-n} + 2^{-n+1} < \varepsilon$$

So $B_n \subseteq B(\varepsilon, x) \subseteq V_\alpha$.

We've thus shown that for some $n \in N$, all balls of radius $2^{-n}$ centered in $E$ are contained by some $V_\alpha$. And assuming (1), we can cover $E$ with finitely many balls of radius $2^{-n}$ It follows that by picking a $V_\alpha$ containing a ball for each ball covering $E$, we've found a finite covering $E$ using the sets in $\{V_\alpha\}_{\alpha\in A}$.

(3) $\Longrightarrow$ (2):

Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in $E$ with no convergent subsequence. Then for each $x \in E$, there must exist $\varepsilon_x > 0$ such that $B(\varepsilon_x, x) \cap (x_n)_{n\in\mathbb{N}}$ is finite. (If $\varepsilon_x$ didn't exist, we could construct a Cauchy subsequence converging to $x$).

But now $\{B(\varepsilon_x, x)\}_{x\in E}$ is an open cover of $E$ with no finite subcover of $E$ because it will take an infinite cover to cover all of $(x_n)_{n\in\mathbb{N}}$.

If $E$ satisfies all three of the above properties, we say $E$ is compact.

**Corollary**: $K \subseteq \mathbb{R}^n$ is compact iff it's closed and bounded.

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Roughly speaking, we want a measure to be a function $\mu : \mathcal{P}(\mathbb{R}^n) \longrightarrow [0, \infty)$ such that $E \mapsto \mu(E) =$ "the area of $E$". Also, we would like it if:

(i) $\mu([0,1)^n) = 1$

(ii) $\mu($rotation, translation, or reflection of $A) = \mu(A)$

(iii) $\mu(\bigcup\limits_{i=1}^{\infty} A_i) = \sum\limits_{i=1}^{\infty} \mu(A_i)$ if $A_i \cap A_j \neq \emptyset \Longrightarrow i = j$.

Unfortunately, the properties as written above are inconsistent.

**Vitali Sets:**
Defining $x \sim y$ iff $x - y \in \mathbb{Q}$, let $N \subseteq [0,1)$ be a set such that $N \cap [x]$ has precisely one element for all $x \in \mathbb{R}$. Next let $R = [0,1) \cap \mathbb{Q}$, and for all $r \in R$ define:

$$N_r = \{x + r \mid x \in N \cap [0, 1-r)\} \cup \{x + r - 1 \mid x \in N \cap [1-r, 1)\}.$$

Importantly, note that $N_r \subseteq [0,1)$. Plus, the two sets being unioned over to make $N_r$ are both disjoint and can be translated around so that they are still disjoint but their union forms $N$. Hence assuming $\mu : \mathcal{P}(\mathbb{R}^n) \longrightarrow [0,\infty)$ satisfying (ii) and (iii), we know $\mu(N_r) = \mu(N)$.

Also, for all $y \in [0,1)$, if $x \in N \cap [y]$, we know that $y \in N_r$ where $r = x - y$ if $x \geq y$, or $r = x - y + 1$ if $x < y$. Hence, $[0,1) = \bigcup\limits_{r\in R} N_r$.

Also, given any $N_r$ and $N_s$, if $x \in N_r \cap N_s$, then we'd be able to show that both $x - r$ or $x - r + 1$ and $x - s$ or $x - s + 1$ are distinct elements of $N$ in the same equivalence class, which contradicts how we defined $N$.

You work through the scratch work of the different cases on your own! :P

So supposing $\mu$ satisfies (i) and (iii) and because $R$ is countable, we have that:

$$1 = \sum_{r\in R} \mu(N_r) = \sum_{r\in R} (N) = 0 \text{ or } \infty.$$

This is a contradiction.

Furthermore, the problem is not the countable union property as is demonstrated by the Banach-Tarsky paradox:

**Theorem:** Let $U$ and $V$ be arbitrary bounded sets in $\mathbb{R}^n$ where $n \geq 3$. Then there exists $E_1, \ldots, E_N, F_1, \ldots, F_N$ in $\mathbb{R}^n$ such that:

- $E_i \cap E_j = \emptyset$ for all $i \neq j$ and $\bigcup\limits_{i=1}^{N} E_i = U$
- $F_i \cap F_j = \emptyset$ for all $i \neq j$ and $\bigcup\limits_{i=1}^{N} F_i = V$
- $E_i$ and $F_i$ are congruent for all $i \in \{1, \ldots, N\}$.

Supposing that $\mu(E_j)$ and $\mu(F_j)$ exists for all $j$ and that $\mu$ satisfies (i), (ii), and (iii) except only for finite unions, then that would suggest all sets have the same "area", which we know doesn't make sense.

What we will do to fix this issue is only define $\mu$ on a subset of $\mathcal{P}(\mathbb{R}^n)$.

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Let $X \neq \emptyset$. An algebra of sets in $X$ is a nonempty collection $\mathcal{A} \subseteq \mathcal{P}(X)$ which is closed under finite unions and complements. If is $\mathcal{A}$ is also closed under countable unions, we say $\mathcal{A}$ is a $\sigma$-algebra.

Observations:

1. Algebras of sets are closed under finite intersections and $\sigma$-algebras are closed under countable intersection. (This also means algebras of sets are closed under set differences.)

   This is because $\bigcap\limits_{n\in\mathbb{N}} A_n = \left(\bigcup\limits_{n\in\mathbb{N}} A_n^{\mathsf{C}}\right)^{\mathsf{C}}$.

2. If $\mathcal{A}$ is closed under disjoint countable union, then it's closed under arbitrary countable unions.

   This is because $\bigcup\limits_{n\in\mathbb{N}} A_n = A_1 \cup \bigcup\limits_{n\geq 2} \left(A_n \cap \left(\bigcup\limits_{i=1}^{n-1} A_i\right)^{\mathsf{C}}\right)$

3. If $\{\mathcal{E}_\alpha\}_{\alpha\in A}$ is a collection of $\sigma$-algebras, then $\bigcap\limits_{\alpha\in A} \mathcal{E}_\alpha$ is a $\sigma$-algebra.

   This is pretty trivial to prove. It should remind you of topologies.

**Exercise 1.1:** A family of sets $\mathcal{R} \subseteq \mathcal{P}(X)$ is called a ring if it is closed under finite unions and difference. If $\mathcal{R}$ is also closed under countable unions, it is called a $\sigma$-ring.

(a) Rings are closed under finite intersections and $\sigma$-rings are closed under countable intersections.

If $\mathcal{R}$ is a ring and $A_1, \ldots, A_n \in \mathcal{R}$, then:

$$\bigcap_{i=1}^{n} A_n = A_1 - \bigcup_{i=2}^{n} (A_1 - A_i) \in \mathcal{R}$$

This is because each $A_1 - A_i \in \mathcal{R}$, meaning $\bigcup_{i=2}^{n}(A_1 - A_i) \in \mathcal{R}$, and so finally $A_1 - \bigcup_{i=2}^{n}(A_1 - A_i) \in \mathcal{R}$.

If $\mathcal{R}$ is a $\sigma$-algebra, we can replace the finite intersection and union used in the prior reasoning with a countable intersection and union.

(b) If $\mathcal{R}$ is a ring (or $\sigma$-ring), then $\mathcal{R}$ is an algebra (or $\sigma$-algebra) iff $X \in \mathbb{R}$.

($\Longrightarrow$) Suppose $\mathcal{R}$ is an algebra. Then note that $\emptyset \in \mathcal{R}$ because for any $A \in \mathcal{R}$, $A - A \in \mathcal{R}$. So taking complements, we get that $X \in \mathcal{R}$.

($\Longleftarrow$) Suppose $X \in \mathcal{R}$. Then for any $A \in \mathcal{R}$, we know that $A^{\mathsf{C}} = X - A \in \mathcal{R}$. So $\mathcal{R}$ is an algebra (or $\sigma$-algebra).

(c) If $\mathcal{R}$ is a $\sigma$-ring, then $\mathcal{A} = \{E \subseteq X \mid E \in \mathcal{R} \text{ or } E^{\mathsf{C}} \in \mathcal{R}\}$ is a $\sigma$-algebra.

To start, we know that $\mathcal{A}$ is closed under complements because for any $A \in \mathcal{A}$,

$$A \in \mathcal{R} \Longrightarrow (A^{\mathsf{C}})^{\mathsf{C}} \in \mathcal{R} \Longrightarrow A^{\mathsf{C}} \in \mathcal{A}$$
$$A \notin \mathcal{R} \Longrightarrow A^{\mathsf{C}} \in \mathcal{R} \Longrightarrow A^{\mathsf{C}} \in \mathcal{A}$$

Also, let $(E_n)_{n\in\mathbb{N}}$ be a countable collection of sets in $\mathcal{A}$. Then define $A = \{n \in \mathbb{N} \mid E_n^{\mathsf{C}} \notin \mathcal{R}\}$ and $B = \{n \in \mathbb{N} \mid E_n^{\mathsf{C}} \in \mathcal{R}\}$. Clearly, we have that:

$$\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in A} E_n \cup \bigcup_{n\in B} E_n = \bigcup_{n\in A} E_n \cup \bigcup_{n\in B} (E_n^{\mathsf{C}})^{\mathsf{C}}$$

Also $\bigcup_{n\in B} (E_n^{\mathsf{C}})^{\mathsf{C}} = \left(\bigcap_{n\in B} E_n^{\mathsf{C}}\right)^{\mathsf{C}}$, and by part (a), we know that $E_B \coloneqq \bigcap_{n\in B} E_n^{\mathsf{C}} \in \mathcal{R}$.

Similarly, we know $E_A \coloneqq \bigcup_{n\in A} E_n \in \mathcal{R}$. So, we've shown that $\bigcup_{n\in\mathbb{N}} E_n = E_A \cup E_B^{\mathsf{C}}$ where $E_A, E_B \in \mathcal{R}$.

Finally, note that $E_A \cup E_B^{\mathsf{C}} = (E_B - E_A)^{\mathsf{C}}$. Since $E_B - E_A \in \mathcal{R}$, we know that $(E_B - E_A)^{\mathsf{C}} \in \mathcal{A}$.

(d) If $\mathcal{R}$ is a $\sigma$-ring, then $\mathcal{A} = \{E \subseteq X \mid E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a $\sigma$-algebra.

To start if $E \in \mathcal{A}$, then $E^{\mathsf{C}} \in \mathcal{A}$ because for all $F \in \mathcal{R}$ we have that:

$$E^{\mathsf{C}} \cap F = F - E = F - (E \cap F) \in \mathcal{R}.$$

Also, let $(E_n)_{n\in\mathbb{N}}$ be a countable collection of sets in $\mathcal{A}$. Then for all $F \in \mathcal{R}$, we have that $\left(\bigcup_{n\in\mathbb{N}} E_n\right) \cap F = \bigcup_{n\in\mathbb{N}} (E_n \cap F) \in \mathcal{R}$. So $\mathcal{A}$ is closed under countable union.

Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of sets. Since the intersection of $\sigma$-algebras is still a $\sigma$-algebra, we define $\mathcal{M}(\mathcal{E})$ to be the smallest $\sigma$-algebra that contains $\mathcal{E}$. In other words, $\mathcal{M}(\mathcal{E})$ is the intersection of all $\sigma$-algebras that contain $\mathcal{E}$.

We call $\mathcal{M}(\mathcal{E})$ the $\sigma$-algebra generated by $\mathcal{E}$.

**Lemma:** if $\mathcal{E} \in \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$.

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Let $(X, \rho)$ be a metric space. We define the Borel $\sigma$-algebra on $X$: $\mathcal{B}_X$, to be the $\sigma$-algebra generated by the collection of all open sets, or equivalently the collection of all closed sets.

- A set is $G_\delta$ if it is a countable intersection of open sets.
- A set is $F_\sigma$ if it is a countable union of closed sets.
- A set is $G_{\delta\sigma}$ if it is a countable union of $G_\delta$ sets.
- A set is $F_{\sigma\delta}$ if it is a countable intersection of $F_\sigma$ sets.

You can hopefully see the pattern. Also the professor isn't sure how much we'll use this $\delta$ and $\sigma$ notation in class.

**Exercise 1.2:** $\mathcal{B}_\mathbb{R}$ is generated by each of the following:

(a) the set of open intervals: $\mathcal{E}_1 = \{(a,b) \mid a < b\}$

(b) the set of closed intervals: $\mathcal{E}_2 = \{[a,b] \mid a < b\}$

(c) the set of half-open intervals:

  (i) $\mathcal{E}_3 = \{(a,b] \mid a < b\}$
  (ii) $\mathcal{E}_4 = \{[a,b) \mid a < b\}$

(c) the set of open rays:

  (i) $\mathcal{E}_5 = \{(a,\infty) \mid a \in \mathbb{R}\}$
  (ii) $\mathcal{E}_6 = \{(-\infty,a) \mid a \in \mathbb{R}\}$

(d) the set of closed rays:

  (i) $\mathcal{E}_7 = \{[a,\infty) \mid a \in \mathbb{R}\}$
  (ii) $\mathcal{E}_8 = \{(-\infty,a] \mid a \in \mathbb{R}\}$

Proof:
We trivially have that $\mathcal{M}(\mathcal{E}_1), \mathcal{M}(\mathcal{E}_2), \mathcal{M}(\mathcal{E}_5), \mathcal{M}(\mathcal{E}_6), \mathcal{M}(\mathcal{E}_7), \mathcal{M}(\mathcal{E}_8) \subseteq \mathcal{B}_\mathbb{R}$ since each of them contain either only open sets or only closed sets. As for the other inclusions, we must do more work.

(a) Note that $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$. Hence, a countable base of $\mathbb{R}$ is the set: $\mathcal{F} = \{(p-q, p+q) \subset \mathbb{R} \mid p, q \in \mathbb{Q}$ and $q > 0\}$. In other words, given any open set $E \subseteq \mathbb{R}$, there is a countable subcollection of $\mathcal{F}$ whose union is $E$.

To see why, let $x \in E$. Since $E$ is open, there exists $r > 0$ with $B(r, x) \subseteq E$. Next, pick $p \in (x, x + \frac{r}{2}) \cap \mathbb{Q}$, followed by $q \in (p - x, r - p) \cap \mathbb{Q}$. Then $x \in (p-q, p+q) \in \mathcal{F}$ and $(p-q, p+q) \subseteq (x-r, x+r)$.

With that, we've now shown that for all $x \in E$, there exists $F \in \mathcal{F}$ such that $x \in F \subseteq E$. If we choose such an $F_x$ for all $x \in E$, we then get that $E = \bigcup_{x \in E} F_x$. So $E$ is the union of a subcollection of $\mathcal{F}$. But since $\mathcal{F}$ is countable, the set $\{F_x \in \mathcal{F} \mid x \in E\}$ is also countable.

Importantly, $\mathcal{F} \subset \mathcal{E}_1$. So $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E}_1)$. However as shown above, we must have that $\mathcal{M}(\mathcal{F})$ includes all open sets. So by our lemma on the previous page, $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E}_1)$.

(b) Given any $E = (a, b) \in \mathcal{E}_1$, we can write that $E = \bigcup_{n \in \mathbb{Z}_+} [a + \frac{1}{n}, b - \frac{1}{n}]$. Thus, $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_2)$, meaning $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_2)$.

(c) Remember that for these two, we still need to show that $\mathcal{M}(\mathcal{E}_1), \mathcal{M}(\mathcal{E}_2) \in \mathcal{B}_{\mathbb{R}}$.

(i) Firstly note that if $F = (a, b] \in \mathcal{E}_3$, then $F = \bigcap_{n \in \mathbb{Z}_+} (a, b + \frac{1}{n})$. So $\mathcal{E}_3 \subseteq \mathcal{M}(\mathcal{E}_1)$.

On the other hand, if $E = (a, b) \in \mathcal{E}_1$, we have that $E = \bigcup_{n \in \mathbb{Z}_+} (a, b - \frac{1}{n}]$. So $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_3)$.

By our lemma on the previous page, we thus have that:
$$\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_1) = \mathcal{B}_{\mathbb{R}}.$$

(ii) Mostly identical reasoning as with $\mathcal{E}_3$ shows that:
$$\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_1) = \mathcal{B}_{\mathbb{R}}$$

(d)

(i) If $E = (a, b) \in \mathcal{E}_1$, then we know that:
$$E = (a, \infty) \cap (\bigcap_{n \in \mathbb{Z}_+} (b - \tfrac{1}{n}, \infty))^{\mathsf{C}} \in \mathcal{M}(\mathcal{E}_5).$$
So $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_5)$, meaning $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_5)$.

(ii) Analogous reasoning to that with $\mathcal{E}_5$ shows that $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_6)$.

(e)

(i) If $E = (a, \infty) \in \mathcal{E}_6$, then we have that $E = \bigcup_{n \in \mathbb{Z}_+} [a + \frac{1}{n}, \infty)$. So $\mathcal{E}_5 \subseteq \mathcal{M}(\mathcal{E}_7)$, meaning that $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{M}(\mathcal{E}_7)$.

(ii) Analogous reasoning as with $\mathcal{E}_7$ shows that $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_6) \subseteq \mathcal{M}(\mathcal{E}_8)$.

**Exercise 1.3:** Let $\mathcal{M}$ be an infinite $\sigma$-algebra on $X$.

(a) $\mathcal{M}$ contains an infinite sequence of disjoint sets.

By the Hausdorff maximum principle, we know there is a subcollection $\mathcal{S}$ of $\mathcal{M}$ which is simply ordered by proper subset and is not contained in any other collection of $\mathcal{M}$ which is simply ordered by proper subset.

We claim $\mathcal{S}$ can't be finite. For suppose $\mathcal{S} = \{E_1, \ldots, E_n\}$ is a sequence of sets in $\mathcal{M}$ simply ordered by proper subset which are indexed such that $E_i \subset E_{i+1}$ for all $i \in \{1, \ldots, n-1\}$.

(Note: if $\mathcal{S}$ is maximal, then we must have $E_1 = \emptyset$ and $E_n = X$.)

We can partition $\mathcal{M}$ into collections $\mathcal{M}_1, \ldots \mathcal{M}_n$ such that $A \in \mathcal{M}_i$ iff $i$ is the least integer for which $A \subseteq E_i$. Importantly, all sets in $\mathcal{M}$ will fall into a partition because all sets from $\mathcal{M}$ are contained in $E_n$. Also note that while there are infinitely many $A \in \mathcal{M}$, there are only $n$ many partitions. So, there must be a least integer $k$ such that $\mathcal{M}_k$ contains infinitely many $A \in \mathcal{M}$.

And since $\mathcal{M}_1 = \{\emptyset\}$, we know $k \geq 2$.

The fact that $\mathcal{M}_i$ is finite for all $i < k$ means that there are only finitely many sets from $\mathcal{M}$ contained in $E_{k-1}$. Thus, we can pick a set $B \in \mathcal{A}_k$ such that $B \neq (E_k - E_{k-1}) \cup A$ for any $A \in \mathcal{M}$ that is a subset of $E_{k-1}$.

Note that since $E_{k-1} \cap B$ is a set in $\mathcal{M}$, we must have that $(E_k - E_{k-1}) \not\subseteq B$ or else $B$ would be the union of $(E_k - E_{k-1})$ and a set from $\mathcal{M}$. Thus, we know $E_k$ contains points that neither $B$ nor $E_{k-1}$ have. At the same time, we know $B$ has points that $E_{k-1}$ doesn't have. It follows that: $E_{k-1} \subset E_{k-1} \cup B \subset E_k$.

Via transitivity, $E_{k-1} \cup B$ is comparable via proper subset with $E_i$ for all $i \in \{1, \ldots, n\}$. Hence, we've shown that $\mathcal{S} \cup \{E_{k-1} \cup B\}$ is a sequence of sets in $\mathcal{M}$ simply ordered by proper subset. But this contradicts that $\mathcal{S}$ is maximal.

Now that we know $\mathcal{S}$ is infinite, let $(E_n)_{n \in \mathbb{Z}_+}$ be a sequence of sets in $\mathcal{S}$ satisfying that $E_n \subset E_{n+1}$. Then we have that $(E_{n+1} - E_n)_{n \in \mathbb{Z}_+}$ is an infinite sequence of nonempty disjoint sets in $\mathcal{M}$.

(b) Show that $\mathrm{card}(\mathcal{M}) \geq \mathfrak{c}$.

Let $(E_n)_{n \in \mathbb{N}}$ be a sequence of nonempty disjoint sets in $\mathcal{M}$. Then if we define the map $f : [0,1]^{\mathbb{N}} \longrightarrow \mathcal{M}$ such that $(a_0, a_1, a_2, \ldots)$ is mapped to the union of all $E_n$ such that $a_n = 1$, we have that $f$ is an injection.

Hence, $\mathrm{card}(\mathcal{M}) \geq \mathrm{card}([0,1]^{\mathbb{N}})$. And since there is a trivial bijection from $[0,1]^{\mathbb{N}}$ and $\mathcal{P}(\mathbb{N})$, plus the fact that we proved early on in the class that $\mathrm{card}(\mathcal{P}(\mathbb{N})) = \mathrm{card}(\mathbb{R})$, we thus know that $\mathrm{card}(\mathcal{M}) \geq \mathfrak{c}$.

**Exercise 1.4**: An algebra $\mathcal{A}$ is a $\sigma$-algebra if and only if $\mathcal{A}$ is closed under countable increasing unions (meaning $E_1 \subseteq E_2 \subseteq \ldots$).

The rightward implication is true since $\mathcal{A}$ being a $\sigma$-algebra means that $\mathcal{A}$ is closed under all countable unions. As for showing the leftward implication, suppose $\{A_n\}_{n \in \mathbb{Z}_+}$ is a countable collection of sets in $\mathcal{A}$. Then for all $n \in \mathbb{Z}_+$, define $E_n = A_1 \cup \ldots \cup A_n$.

Since each $E_n$ are finite unions of sets in $\mathcal{A}$, we know that each $E_n$ is in $\mathcal{A}$. Also, we clearly have that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ In order to make the sets strictly increasing, let $S = \{1\} \cup \{k \in \mathbb{Z} \mid k > 1$ and $E_k - E_{k-1} \neq \emptyset\}$. Then for any $n, m \in S$, we know that $n < m \Longrightarrow E_n \subset E_m$.

Finally, $\bigcup\limits_{n \in \mathbb{Z}_+} A_n = \bigcup\limits_{n \in \mathbb{Z}_+} E_n = \bigcup\limits_{n \in S} E_n$.

Importantly, $S$ is either finite or countably infinite, and $S$ consists of strictly increasing sets. So by the right hypothesis, we know $\bigcup\limits_{n \in S} E_n \in \mathcal{A}$. Hence, the union over $\{A_n\}_{n \in \mathbb{Z}_+}$ is in $\mathcal{A}$.

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Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of nonempty sets, and define $X = \prod\limits_{\alpha \in A} X_\alpha$.

If $\mathcal{M}_\alpha$ is a $\sigma$-algebra in $X_\alpha$ for all $\alpha \in A$, then we define the product $\sigma$-algebra on $X$ to be: $\bigotimes\limits_{\alpha \in A} \mathcal{M}_\alpha = \mathcal{M}(\{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{M}_\alpha$ and $\alpha \in A\})$.

To get a better geometric intuition for this definition, consider if $A = \{1, 2\}$. Then:

$$\begin{aligned}\bigotimes_{\alpha \in A} \mathcal{M}_\alpha &= \{\pi_1^{-1}(E_1) \mid E_1 \in \mathcal{M}_1\} \cup \{\pi_2^{-1}(E_2) \mid E_2 \in \mathcal{M}_2\} \\ &= \{E_1 \times X_2 \mid E_1 \in \mathcal{M}_1\} \cup \{X_1 \times E_2 \mid E_2 \in \mathcal{M}_2\}\end{aligned}$$

Also, the motivation for this definition is that $\bigotimes\limits_{\alpha \in A} \mathcal{M}_\alpha$ is the smallest $\sigma$-algebra where $\pi_\alpha$ is "measurable" for all $\alpha$. We'll learn what that means shortly...

**Proposition:**

(i) $A$ is countable implies $\bigotimes\limits_{\alpha \in A} \mathcal{M}_\alpha = \mathcal{M}(\{\prod\limits_{\alpha \in A} E_\alpha \mid \forall \alpha \in A,\ \ E_\alpha \in \mathcal{M}_\alpha\})$

Proof:

If $E_\alpha \in \mathcal{M}_\alpha$, then $\pi_\alpha^{-1}(E_\alpha) = \prod\limits_{\beta \in A} E_\beta$ where $E_\beta = X_\beta$ if $\beta \neq \alpha$ (and $E_\beta = E_\alpha$ if $\beta = \alpha$).

So $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{M}(\{\prod\limits_{\alpha \in A} E_\alpha \mid \forall \alpha \in A,\ \ E_\alpha \in \mathcal{M}_\alpha\})$

On the other hand, $\prod\limits_{\alpha \in A} E_\alpha = \bigcap\limits_{\alpha \in A} \pi_\alpha^{-1}(E_\alpha)$.

Since $A$ is countable, we thus know that if $E_\alpha \in \mathcal{M}_\alpha$ for all $\alpha \in A$, then $\prod\limits_{\alpha \in A} E_\alpha \in \bigotimes\limits_{\alpha \in A} \mathcal{M}_\alpha$.

(ii) Suppose $\mathcal{M}_\alpha = \mathcal{M}(\mathcal{E}_\alpha)$ for all $\alpha \in A$. Then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by $\mathcal{F} = \{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{E}_\alpha \text{ and } \alpha \in A\}$.

Proof:

Since $\mathcal{F} \subseteq \{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{M}_\alpha \text{ and } \alpha \in A\}$, we trivially have that $\mathcal{M}(\mathcal{F}) \subseteq \bigotimes_{\alpha \in A} \mathcal{M}_\alpha$.

As for showing the other inclusion, define for each $\alpha \in A$:

$$\mathcal{F}_\alpha = \{E \subseteq X_\alpha \mid \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})\}.$$

Note that $\mathcal{F}_\alpha$ is a $\sigma$-algebra on $X_\alpha$ that contains $\mathcal{E}_\alpha$.

This is because for any $F \in \mathcal{F}_\alpha$ and $(E_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}_\alpha$, we know that:

- $(\pi_\alpha^{-1}(F))^\mathsf{C} = \pi_\alpha^{-1}(F^\mathsf{C})$
- $\bigcup_{n \in \mathbb{N}} \pi_\alpha^{-1}(E_n) = \left(\pi_\alpha^{-1}(\bigcup_{n \in \mathbb{N}} E_n)\right)$

Also, for any $E \subseteq X_\alpha$, $E \in \mathcal{E}_\alpha \Longrightarrow \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})$.

By definition, we thus know that $\mathcal{M}_\alpha \subseteq \mathcal{F}_\alpha$. So for all $\alpha \in A$ and $E_\alpha \in \mathcal{M}_\alpha$, we know that $E_\alpha \in \mathcal{F}_\alpha$, which means that $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{M}(\mathcal{F})$. So $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha \subseteq \mathcal{M}(\mathcal{F})$.

(iii) We can also combine the first two parts of this proposition. If $A$ is countable and $\mathcal{M}_\alpha = \mathcal{M}(\mathcal{E}_\alpha)$ for all $\alpha \in A$, then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by:

$$\{\prod_{\alpha \in A} E_\alpha \mid \forall \alpha \in A, \quad E_\alpha \in \mathcal{E}_\alpha\}$$

# Lecture 3 Notes: 10/3/2024

**Proposition**: Let $X_1, \ldots, X_n$ be metric spaces, and define $X = \prod_{i=1}^n X_i$ to be the metric space equipped with the product metric.

The product metric defines the distance between any $\boldsymbol{x}, \boldsymbol{y} \in \prod_{i=1}^n$ to be the max distance between a coordinate of $\boldsymbol{x}$ and the corresponding coordinate in $\boldsymbol{y}$.

- $\bigotimes_{i=1}^n \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$.

Proof:

By the previous proposition: $\bigotimes_{i=1}^n \mathcal{B}_{X_i}$ is generated by the collection:

$$\{\pi_i^{-1}(U_i) \mid i \in \{1, \ldots, n\} \text{ and } U_i \subseteq X_i \text{ is open}\}.$$

Also, by the definition of a product topology, we know that each $\pi_i^{-1}(U_i)$ is open in $X$. So by the lemma on page 9, we know that $\bigotimes_{i=1}^n \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$.

- If each $X_i$ is separable, then $\bigotimes_{i=1}^{n} \mathcal{B}_{X_i} = \mathcal{B}_X$.

  Proof:
  Let $C_i \subseteq X_i$ be countable with $\overline{C_i} = X_i$ for all $i \in \{1, \ldots, n\}$. Then define $\mathcal{E}_i = \{B(p, x) \mid x \in C_i \text{ and } p \in \mathbb{Q}_+\}$ for each $i$. Since $\mathcal{E}_i$ is countable and all open sets in $X_i$ are the union of a subcollection of $\mathcal{E}_i$, we know that any open set in $X_i$ is also in $\mathcal{M}(\mathcal{E}_i)$. So, $\mathcal{B}_{X_i} \subseteq \mathcal{M}(\mathcal{E}_i)$. And since $\mathcal{E}_i$ contains only open sets of $X_i$, the reverse inclusion holds too.

  Also, $C = \prod_{i=1}^{n} C_i$ is a countable dense subset of $X$.

  Defining $\mathcal{E} = \{B(p, \boldsymbol{x}) \mid \boldsymbol{x} \in C \text{ and } p \in \mathbb{Q}_+\}$, we have that $\mathcal{E}$ is countable and any open set in $X$ is also in $\mathcal{M}(\mathcal{E})$. So, $\mathcal{B}_X \subseteq \mathcal{M}(\mathcal{E})$. And like before since $\mathcal{E}$ contains only open sets of $X$, the reverse inclusion holds too.

  But now note that given, $B(p, (x_1, \ldots, x_n)) \in \mathcal{E}$, we know that
  $B(p, (x_1, \ldots, x_n)) = \prod_{i=1}^{n} B(p, x_i)$ where $(p, x_i) \in \mathcal{E}_i$ for all $i$.

  So applying part 3 of the previous proposition and the lemma on page 9:
  $$\mathcal{B}_X = \mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}\left(\prod_{i=1}^{n} E_i \mid E_i \in \mathcal{E}_i \text{ for all } i\right) = \bigotimes_{i=1}^{n} \mathcal{M}(\mathcal{E}_i) = \bigotimes_{i=1}^{n} \mathcal{B}_{X_i}$$

**Corollary:** $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{i=1}^{n} \mathcal{B}_{\mathbb{R}}$.

This is because the product metric $\rho_1$ of $\prod_{i=1}^{n} \mathbb{R}$ is equivalent to the standard metric $\rho_2$ of $\mathbb{R}^n$, meaning that:
$$\exists C, C' > 0 \text{ such that } C\rho_1 \leq \rho_2 \leq C'\rho_1.$$

In the specific case of this corollary, set $C = \sqrt{1/n}$ and $C' = 1$.

The fact relevant here is that given the metrics $\rho_1, \rho_2$ on a set $X$, if $\rho_1$ is equivalent to $\rho_2$, then $(X, \rho_1)$ and $(X, \rho_2)$ have the same open sets (this is really trivial to prove).

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An elementary family is a collection $\mathcal{E}$ of subsets of a set $X$ such that:

1. $\emptyset \in \mathcal{E}$
2. If $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$.
3. If $E \in \mathcal{E}$, then $E^{\mathsf{C}}$ is a finite disjoint union of members of $\mathcal{E}$.

If $\mathcal{E}$ is an elementary collection, then $\mathcal{A}$ equal to the collection of finite disjoint unions of $\mathcal{E}$ is an algebra.

Proof:
Firstly, given any $A, B \in \mathcal{E}$, we have that $A \cup B = (A - B) \cup B$. Also, by property 3 of elementary families, $(A - B) = (A \cap B^{\mathsf{C}}) = (A \cap \bigcup_{i=1}^{k} C_i)$ where each $C_i \in \mathcal{E}$ and disjoint. By property 2 of elementary families, we thus know $A \cap C_i \in \mathcal{E}$ for all $i$. So $(A - B)$ is a finite union of disjoint sets in $\mathcal{E}$. In turn, so is $(A - B) \cup B$. Hence, $A \cup B \in \mathcal{A}$.

By induction, we get that for any $A_1, \ldots, A_n \in \mathcal{E}$, $A_1 \cup \ldots \cup A_n$ is a finite union of disjoint sets in $\mathcal{E}$. So $\mathcal{A}$ actually equals the set of all finite unions of $\mathcal{E}$. It follows that $\mathcal{A}$ is closed under finite unions.

I really don't want to write down the proof that $\mathcal{A}$ is closed under complements. It's what you would expect but just heavy on notation.

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**Exercise 1.5:** If $\mathcal{M}$ is the $\sigma$-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the $\sigma$-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

For the sake of convenience, I will write the union of $\sigma$-algebras generated by countable subsets of $\mathcal{E}$ as: $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

To start, since each $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E})$, we trivially know $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E}) = \mathcal{M}$. On the other hand, $\mathcal{E} \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ since each countable $\mathcal{F} \subseteq \mathcal{E}$ is contained in $\mathcal{M}(\mathcal{F}) \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$. So, if we can show that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is a $\sigma$-algebra, then we will know that: $\mathcal{M} = \mathcal{M}(\mathcal{E}) \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

Fortunately, it's trivial to show that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is closed under complements. Given any $E \in \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$, we know there exists $\mathcal{M}(\mathcal{F})$ with $E \in \mathcal{M}(\mathcal{F})$. Then $E^{\mathsf{C}} \in \mathcal{M}(\mathcal{F}) \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

Meanwhile, the proof that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is closed under countable unions is more involved:

Suppose $\{E_n\}_{n \in \mathbb{N}}$ is a countable collection of sets in $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$. Then for each $n \in \mathbb{N}$, there exists $\mathcal{F}_n$ such that $E_n \in \mathcal{M}(\mathcal{F}_n)$. Importantly, $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ is still countable. So, setting $\mathcal{F}' = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, we have that:

$$\mathcal{M}(\mathcal{F}') \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$$

Since $\mathcal{F}_n \subseteq \mathcal{F}'$ for all $n$, we know that $\mathcal{M}(\mathcal{F}_n) \subseteq \mathcal{M}(\mathcal{F}')$ for all $n$. So, $\{E_n\}_{n \in \mathbb{N}}$ is contained in $\mathcal{M}(\mathcal{F}')$. It follows that $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{M}(\mathcal{F}') \subseteq \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$.

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Let $X \neq \emptyset$ and $\mathcal{M}$ be a $\sigma$-algebra on $X$. A measure $\mu : \mathcal{M} \longrightarrow [0, \infty]$ is a function satisfying that:

- $\mu(\emptyset) = 0$
- $\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)$ if $E_j \in \mathcal{M}$ for all $j$ and $E_j \cap E_i = \emptyset$ for all $i \neq j$

$(X, \mathcal{M})$ is called a measurable space and $(X, \mathcal{M}, \mu)$ is called a measure space.

Let $(X, \mathcal{M}, \mu)$ be a measure space.

- $\mu$ is called finite if $\mu(X) < \infty$.

  It follows if $\mu$ is finite that $\mu(E) < \infty$ for all $E \in \mathcal{M}$ since $E \subseteq X$.
  In probability theory, most measure spaces are finite.

- $\mu$ is called $\sigma$-finite if $X = \bigcup_{j=1}^{\infty} E_j$, such that $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all $j$.

- $\mu$ is called semifinite if $\forall E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $F \subset E$ such that $F \in \mathcal{M}$, and $0 < \mu(F) < \infty$.

Example: Let $X \neq \emptyset$ and $\mathcal{M} = \mathcal{P}(X)$. Then given a function $\rho : X \longrightarrow [0, \infty]$, $\mu(E) = \sum_{x \in E} \rho(x)$ is a measure.

- $\mu$ is semifinite if and only if $\rho(x) < \infty$ for all $x \in X$.
- $\mu$ is $\sigma$-finite if and only if it is semifinite and $\{x \in X \mid \rho(x) > 0\}$ is countable.

If $\rho(x) = 1$ for all $x$, then $\mu$ is called the counting measure.

If $\rho(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0, \end{cases}$ then $\mu$ is called the Dirac measure at $x_0$: $\delta_{x_0}$.

**Theorem:** Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:

1. If $E, F \in \mathcal{M}$ with $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
2. If $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}$, then $\mu(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} \mu(E_j)$.
3. If $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}$ with $E_j \subseteq E_{j+1}$ for all $j \in \mathbb{N}$, then $\mu(\bigcup_{j=1}^{\infty} E_j) = \lim_{j \to \infty} \mu(E_j)$.
4. If $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}$ and $\mu(E_1) < \infty$ and $E_{j+1} \subseteq E_j$ for all $j \in \mathbb{N}$, then $\mu(\bigcap_{j=1}^{\infty} E_j) = \lim_{j \to \infty} \mu(E_j)$.

Proofs:
(1) Suppose $E, F \in \mathcal{M}$ with $E \subseteq F$. Then $F = (F - E) \cup E$ is a disjoint union of sets in $\mathcal{M}$, meaning $\mu(F) = \mu(F - E) + \mu(E) \geq \mu(E)$.

(2) Set $F_1 = E_1$ and $F_m = E_m - \bigcup_{i=1}^{m-1} E_i$ for all $m > 1$. Then $(F_i)_{i\in\mathbb{N}}$ is pairwise disjoint and $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$. So $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(F_i)$. On the other hand, $F_i \subseteq E_i$ for all $i$. So $\sum_{i=1}^{\infty} \mu(F_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$

(3) Setting $E_0 = \emptyset$, we have that $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i - E_{i-1})$. Also, $\mu(E_n) = \sum_{i=1}^{n} \mu(E_i - E_{i-1})$. So:

$$\lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i - E_{i-1}) = \sum_{i=1}^{\infty} \mu(E_i - E_{i-1}) = \mu(\bigcup_{i=1}^{\infty} E_i).$$

(4) Let $F_j = E_1 - E_j$ for all $j \in \mathbb{N}$. Then for all $j \in \mathbb{N}$, $F_j \subseteq F_{j+1}$, $\mu(E_1) = \mu(F_j) + \mu(E_j)$, and $\bigcup_{j=1}^{\infty} F_j = E_1 - \bigcap_{j=1}^{\infty} E_j$. We can thus conclude that:

$$\begin{aligned}\mu(E_1) &= \mu(\bigcap_{j=1}^{\infty} E_j) + \mu(\bigcup_{j=1}^{\infty} F_j) \\ &= \mu(\bigcap_{j=1}^{\infty} E_j) + \lim_{j\to\infty}(\mu(F_j)) = \mu(\bigcap_{j=1}^{\infty} E_j) + \lim_{j\to\infty}(\mu(E_1) - \mu(E_j))\end{aligned}$$

Since $\mu(E_1) < \infty$, we can subtract it out of the expression to get:

$\mu(\bigcap_{j=1}^{\infty} E_j) - \lim_{j\to\infty}(\mu(E_j)) = 0$. Also, we know $\mu(\bigcap_{j=1}^{\infty} E_j) < \infty$ since it's a subset of $E_j$. So, we can rearrange to get: $\mu(\bigcap_{j=1}^{\infty} E_j) = \lim_{j\to\infty}(\mu(E_j))$.

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**Exercise 1.9:** If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then we have that $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

We know $\mu(E) = \mu(E - f) + \mu(E \cap F)$ and $\mu(F) = \mu(F - E) + \mu(F \cap E)$. Adding those equations together we get that:

$$\begin{aligned}\mu(E) + \mu(F) &= (\mu(E - F) + \mu(E \cap F) + \mu(F - E)) + \mu(E \cap F) \\ &= \mu(E \cup F) + \mu(E \cap F).\end{aligned}$$

**Exercise 1.14:** If $\mu$ is a semifinite measure and $\mu(E) = \infty$, then for any $C > 0$ there exists $F \subset E$ in $\mathcal{M}$ with $C < \mu(F) < \infty$.

Let $S$ be the set of $C > 0$ for which there exists $F \subset E$ in $\mathcal{M}$ with $C < \mu(F) < \infty$. By the definition of semifiniteness, we know $S$ isn't empty. Meanwhile, if for some $C$ we had that there didn't exist a set $F \subset E$ in $\mathcal{M}$ with $C < \mu(F) < \infty$, then we'd know that $S$ is bounded above. Hence, we'd know there exists $\alpha = \sup(S)$.

Now firstly, for all $n \in \mathbb{N}$, choose $G_n \subset E$ in $\mathcal{M}$ such that $\alpha - \frac{1}{n} < \mu(G_n) < \infty$. After that, define $F_n = \bigcup\limits_{i=1}^{n} G_i$ for all $n \in \mathbb{N}$. Since $\mathcal{M}$ is closed under finite unions, we know each $F_n$ is in $\mathcal{M}$. So then observe:

1. $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$
2. For each $n \in \mathbb{N}$, $\alpha - \frac{1}{n} < \mu(F_n) \leq \alpha$

This is because for each $n \in \mathbb{N}$, $\mu(F_n) < \sum\limits_{i=1}^{n} \mu(G_i)$ which is a finite sum of finite quantities. So $\mu(F_n) < \infty$. At the same time, $F_n \subset E$ since each $G_i$ is a subset of $E$ (we know it is a proper subset because it has a different measure than $E$). So, if $\mu(F_n) > \alpha$, then $\frac{1}{2}(\mu(F_n) + \alpha)$ would be an element of $S$ greater than $\alpha$, thus contradicting that $\alpha = \sup(S)$. As for the other inequality, note that $G_n \subseteq F_n$. Thus $\mu(F_n) \geq \mu(G_n) > \alpha - \frac{1}{n}$.

Now $\bigcup\limits_{n=1}^{\infty} F_n \in \mathcal{M}$ due to $\mathcal{M}$ being closed under countable sums. Also, by the two observations above, we know $\mu(\bigcup\limits_{n=1}^{\infty} F_n) = \lim\limits_{n\to\infty} \mu(F_n) = \alpha$. And finally, note that $\bigcup\limits_{n=1}^{\infty} F_n$ is a proper subset of $E$ (we know this because each $F_n \subset E$ and $\bigcup\limits_{n=1}^{\infty} F_n$ can't equal $E$ since their measures are different).

So, we have now proven the existence of a set $F \in \mathcal{M}$ such that $F \subset E$ and $\mu(F) = \alpha$. But now note that $\mu(E - F)$ must be infinite since:

$$\mu(E - F) + \alpha = \mu(E - F) + \mu(F) = \mu(E) = \infty.$$

Because $\mu$ is semifinite, there exists $F' \subset E - F$ in $\mathcal{M}$ with $0 < \mu(F') < \infty$. But because $F$ and $F'$ are disjoint subsets of $E$ in $\mathcal{M}$, we know $F \cup F' \in \mathcal{M}$ and $\mu(F \cup F') = \mu(F) + \mu(F') > \alpha$. Plus $F \cup F'$ is a proper subset of $E$. (It can't equal $E$ because it's measure isn't equal to $E$. But, both $F$ and $F'$ individually are subsets of $E$.)

Hence, we have that $\frac{1}{2}(\alpha, \mu(F) + \mu(F)')$ is an element of $S$ greater than $\alpha$, thus contradicting that $\alpha$ was the supremum of $S$. We conclude therefore that $\alpha$ does not exist, meaning $S$ is unbounded.

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Given a measure space $(X, \mathcal{M}, \mu)$, a set $E \in \mathcal{M}$ satisfying that $\mu(E) = 0$ is called a null set (or $\mu$-null set if we want more precision).

By subadditivity (a.k.a. the fact that for all $(E_j)_{j\in\mathbb{N}} \subset \mathcal{M}$, $\mu(\bigcup\limits_{j=1}^{\infty} E_j) \leq \sum\limits_{j=1}^{\infty} \mu(E_j)$), we know countable unions of null sets are also null sets.

Given a proposition $P(x)$, if there exists a null set $E \in \mathcal{M}$ satisfying that $P(x)$ is true for all $x \in X - E$, then we say $P$ is true almost everywhere (abbreviated as $\mu$-a.e. or just a.e. if the measure being used is clear).

A measure space is complete if given any $E \subseteq X$, we have that $N \in \mathcal{M}$ with $\mu(N) = 0$ and $E \subseteq N$ implies that $E \in \mathcal{M}$.

**Proposition:** Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Let:

- $\mathcal{N} = \{N \in \mathcal{M} \mid \mu(N) = 0\}$
- $\overline{\mathcal{M}} = \{E \cup F \mid E \in \mathcal{M}$ and $F \subseteq N$ where $N \in \mathcal{N}\}$.

Then $\overline{\mathcal{M}}$ is a $\sigma$-algebra and there is a unique extension $\overline{\mu}$ of $\mu$ to a complete measure on $\overline{\mathcal{M}}$.

Proof:
Claim 1: $\overline{\mathcal{M}}$ is a $\sigma$-algebra.

To see that $\overline{\mathcal{M}}$ is closed under countable union, let $(E_i \cup F_i)_{i\in\mathbb{N}}$ be a sequence of sets in $\overline{\mathcal{M}}$ with each $E_i \in \mathcal{M}$ and $F_i \subseteq N_i$ for some $N_i \in \mathcal{N}$. Then $\bigcup\limits_{i\in\mathbb{N}}(E_i \cup F_i) = \bigcup\limits_{i\in\mathbb{N}} E_i \cup \bigcup\limits_{i\in\mathbb{N}} F_i$.

Importantly, since $\mathcal{M}$ and $\mathcal{N}$ are closed under countable union, we know that $\bigcup\limits_{i\in\mathbb{N}} E_i \in \mathcal{M}$ and $\bigcup\limits_{i\in\mathbb{N}} F_i \subseteq \bigcup\limits_{i\in\mathbb{N}} N_i \in \mathcal{N}$. So, $\bigcup\limits_{i\in\mathbb{N}}(E_i \cup F_i) \in \overline{\mathcal{M}}$.

To show that $\overline{\mathcal{M}}$ is closed under complements, let $E \cup F \in \overline{\mathcal{M}}$ with $E \in \mathcal{M}$ and $F \subseteq N$ for some $N \in \mathcal{N}$. Also note that we can assume $E \cap N = \emptyset$. After all, if $E \cap N \neq \emptyset$, then define $N' = N - E$ and $F' = F - E$. Since $N' \subseteq N$ and $N' \in \mathcal{M}$, we know that $\mu(N') = 0$. Also, $E \cup F = E \cup F'$ with $F' \subseteq N'$. So, $E$, $F'$, and $N'$ fulfil the same properties we picked $E$, $F$, and $N$ for having. But also $E \cap N' = \emptyset$.

Now, $(E \cup F)^{\mathsf{C}} = (E \cup N)^{\mathsf{C}} \cup (N - F)$ where $(E \cup N)^{\mathsf{C}} \in \mathcal{M}$ and $(N - F) \subset N$. So $(E \cup F)^{\mathsf{C}} \in \overline{\mathcal{M}}$.

Now given any $E \cup F \in \overline{\mathcal{M}}$ with $E \in \mathcal{M}$ and $F \subset N$ for some $N \in \mathcal{N}$, define $\overline{\mu}(E \cup F) = \mu(E)$.

Claim 2: $\overline{\mu}$ is well-defined.

Suppose $E_1 \cup F_1 = E_2 \cup F_2$ where for $j \in \{1, 2\}$ we have $E_j \in \mathcal{M}$ and $F_j \subset N_j$ for some $N_j \in \mathcal{N}$. Then $E_1 \subseteq E_2 \cup N_2$, meaning that $\mu(E_1) \leq \mu(E_2) + \mu(N_2) = \mu(E_2)$. By similar reasoning, we can say that $\mu(E_2) \leq \mu(E_1)$. So $\overline{\mu}(E_1 \cup F_1) = \overline{\mu}(E_2 \cup F_2)$.

(The rest is exercise 1.6:)
Claim 3: $\overline{\mu}$ is a complete measure on $\overline{\mathcal{M}}$.

It's easy to show that $\overline{\mu}$ is a measure. After all, $\emptyset \in \mathcal{M} \cap \mathcal{N}$. So, $\overline{\mu}(\emptyset) = \overline{\mu}(\emptyset \cup \emptyset) = \mu(\emptyset) = 0$. Also, suppose $(E_i \cup F_i)_{i\in\mathbb{N}}$ is a sequence of disjoint sets in $\overline{\mathcal{M}}$ with $E_i \in \mathcal{M}$ and $F_i \subset N_i$ for some $N_i \in \mathcal{N}$.

Then $\bigcup_{i\in\mathbb{N}} E_i \in \mathcal{M}$ where each $E_i$ is disjoint and $\bigcup_{i\in\mathbb{N}} F_i \subseteq \bigcup_{i\in\mathbb{N}} N_i \in \mathcal{N}$. So:

$$\overline{\mu}(\bigcup_{i\in\mathbb{N}}(E_i \cup F_i)) = \overline{\mu}(\bigcup_{i\in\mathbb{N}} E_i \cup \bigcup_{i\in\mathbb{N}} F_i)$$
$$= \mu(\bigcup_{i\in\mathbb{N}} E_i) = \sum_{i=1}^{\infty}\mu(E_i) = \sum_{i=1}^{\infty}\overline{\mu}(E_i \cup F_i)$$

Finally, to show that $(X, \overline{\mathcal{M}}, \overline{\mu})$ is complete, suppose $A \subseteq X$ and $N_1 \in \overline{\mathcal{M}}$ with $\overline{\mu}(N_1) = 0$ and $A \subseteq N_1$. By definition, we know $N_1 = E \cup F$ where $E \in \mathcal{M}$ and $F \subseteq N_2$ for some $N_2 \in \mathcal{N}$. However, we can also assume $E = \emptyset$. For if $E \neq \emptyset$, then because $\mu(E \cup N_2) \leq \mu(E) + \mu(N_2) \leq 0$, we can define $N_2' = E \cup N_2$ and $F' = E \cup F$. Then $N_1 = \emptyset \cup F'$ and $F' \subseteq N_2'$ where $N_2' \in \mathcal{N}$.

So $A \subset F \subset N_2$ where $N_2 \in \mathcal{N}$. It follows that $A = \emptyset \cup A \in \overline{\mathcal{M}}$.

Claim 4: $\overline{\mu}$ is the unique measure on $\overline{\mathcal{M}}$ that extends $\mu$.

Suppose $\overline{\overline{\mu}}$ is another measure on $\overline{\mathcal{M}}$ such that $\overline{\overline{\mu}}|_{\mathcal{M}} = \mu$. Then consider any $E \cup F \in \overline{M}$ such that $E \in \mathcal{M}$ and $F \subseteq N$ for some $N \in \mathcal{N}$. As shown before, we can assume without loss of generality that $E \cap N = \emptyset$ and thus also $E \cap F = \emptyset$. So, we have that:

$$\overline{\overline{\mu}}(E \cup F) = \overline{\overline{\mu}}(E) + \overline{\overline{\mu}}(F)$$

Next, note that:

$$\mu(E) = \overline{\overline{\mu}}(E) \leq \overline{\overline{\mu}}(E) + \overline{\overline{\mu}}(F) \leq \overline{\overline{\mu}}(E) + \overline{\overline{\mu}}(N) = \mu(E) + \mu(N) = \mu(E)$$

Hence, we know that $\overline{\overline{\mu}}(E \cup F) = \mu(E)$. But also $\overline{\mu}(E \cup F) = \mu(E)$. So $\overline{\overline{\mu}}(E \cup F) = \overline{\mu}(E \cup F)$ for all $E \cup F \in \overline{\mathcal{M}}$.

Note: We call $\overline{\mu}$ the completion of $\mu$ and $\overline{\mathcal{M}}$ the completion of $\mathcal{M}$ with respect to $\mu$.

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# Lecture 4 Notes: 10/8/2024

An outer measure on a nonempty set $X$ is a function $\mu^* : \mathcal{P}(X) \longrightarrow [0, \infty]$ satisfying that:

1. $\mu^*(\emptyset) = 0$.
2. $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$.
3. $\mu^*(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty}\mu^*(A_j)$. (this property is called subadditivity)

**Proposition:** Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of elementary sets and $\mu : \mathcal{E} \longrightarrow [0, \infty]$ be a function satisfying that $\mu(\emptyset) = 0$.

The textbook only assumes that $\emptyset$ and $X$ are in $\mathcal{E}$.

Then define $\mu^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \mu(E_j) \,\middle|\, E_j \in \mathcal{E} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$. This is an outer measure.

Proof:

Given any set $A \subseteq \mathcal{P}(X)$, define $A^* = \left\{ \sum_{j=1}^{\infty} \mu(E_j) \,\middle|\, E_j \in \mathcal{E} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$. That way $\mu^*(A) = \inf(A^*)$.

1. $\mu^*(A)$ is well defined because $\mu(X) \in A^*$ and $\mu(\emptyset) = 0$ is a lower bound of $A^*$.
2. Since $0 \in \emptyset^*$, we know that $\mu^*(\emptyset) = \inf(\emptyset^*) = 0$.
3. Suppose $A \subseteq B$. Then given any $(E_j)_{j\in\mathbb{N}}$ of sets in $\mathcal{E}$ covering $B$, we know that they will also cover $A$. So, $A^* \subseteq B^*$, meaning $\mu^*(A) \leq \mu^*(B)$.
4. Suppose $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{P}(X)$. Then fix $\varepsilon > 0$. For all $j \in \mathbb{N}$, let $(E_j^{(k)})_{k\in\mathbb{N}}$ be a sequence of sets in $\mathcal{E}$ such that $A_j \subseteq \bigcup_{k\in\mathbb{N}} E_j^{(k)}$ and:

$$\mu^*(A_j) \leq \sum_{k=1}^{\infty} \mu(E_j^{(k)}) \leq \mu^*(A_j) + \varepsilon/2^j.$$

Note that $\bigcup_{j\in\mathbb{N}} A_j \subseteq \bigcup_{j\in\mathbb{N}} (\bigcup_{k\in\mathbb{N}} E_j^{(k)})$.

So, $(\bigcup_{j\in\mathbb{N}} A_j)^* \ni \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mu(E_j^{(k)}) \leq \sum_{j=1}^{\infty} (\mu^*(A_j) + \varepsilon/2^j) = \varepsilon + \sum_{j=1}^{\infty} \mu^*(A_j)$.

Since $\varepsilon$ was arbitrary, we thus know that:

$$\mu^*(\bigcup_{j\in\mathbb{N}} A_j) = \inf(\bigcup_{j\in\mathbb{N}} A_j)^* \leq \sum_{j=1}^{\infty} \mu^*(A_j).$$

Let $\mu^*$ be an outer measure on a nonempty set $X$. Then $A \subseteq X$ is called $\mu^*$-measurable if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E - A)$ for all $E \subseteq X$.

Note that we trivially have $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E - A)$ for all $E \in \mathcal{P}(X)$. Also, $\mu^*(E \cap A) + \mu^*(E - A) \leq \mu^*(E)$ holds trivially if $\mu^*(E)$. So $\mu^*$-measurability just means that $\mu^*(E \cap A) + \mu^*(E - A) \leq \mu^*(E)$ holds even if $\mu^*(E) < \infty$.

**Carathéodory's Theorem:** If $\mu^*$ is an outer measure on $X$, then the collection $\mathcal{M}$ of $\mu^*$-measurable sets is a $\sigma$-algebra and the restriction of $\mu^*$ to $\mathcal{M}$ is a complete measure.

Proof:

Part 1: $\mathcal{M}$ is an algebra and $\mu^*$ is additive on $\mathcal{M}$ (meaning $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$ implies that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$).

We know $\mathcal{M}$ is an algebra because:

- $\emptyset \in \mathcal{M}$ because $\mu^*(E) = \mu^*(E \cap \emptyset) + \mu^*(E - \emptyset) = 0 + \mu^*(\emptyset)$ for all $E \subseteq X$.

- Both $A^{\mathsf{C}} \in \mathcal{M}$ $A \in \mathcal{M}$ are equivalent to us having for all $E \subseteq X$ that $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^{\mathsf{C}})$.

- Suppose $A$ and $B$ are sets in $\mathcal{M}$. Then given $E \subseteq X$, we have:
$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E - A) \\ &= \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) - B) \\ &\qquad + \mu^*((E - A) \cap B) + \mu^*((E - A) - B)\end{aligned}$$

  Now $(E - A) - B = E \cap A^{\mathsf{C}} \cap B^{\mathsf{C}} = E \cap (A \cup B)^{\mathsf{C}}$. Meanwhile, $(E \cap A) - B = E \cap (A - B)$ and $(E - A) \cap B = E \cap (B - A)$. So, by subadditivity, we have that:
$$\begin{aligned}&\mu^*(E \cap (A \cup B)) + \mu(E - (A \cup B)) \\ &\qquad \leq \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) - B) \\ &\qquad\quad + \mu^*((E - A) \cap B) + \mu^*((E - A) - B)\end{aligned}$$

  Hence, $\mu^*(E \cap (A \cup B)) + \mu^*(E - (A \cup B)) \leq \mu^*(E)$. So, $A \cup B \in \mathcal{M}$.

Next, to show that $\mu$ is additive on $\mathcal{M}$, consider any $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$. Then:
$$\mu^*(A \cup B) = \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) - A) = \mu^*(A) + \mu^*(B)$$

Part 2: $\mathcal{M}$ is a $\sigma$-algebra and $\mu^*$ is $\sigma$-additive (think countably additive) on $X$.

To show that $\mathcal{M}$ is a $\sigma$-algebra, it suffices to show that $\mathcal{M}$ is closed under countable disjoint unions. So let $(A_j)_{j \in \mathbb{N}}$ be a sequence of disjoint sets in $\mathcal{M}$. If $E$ is any set in $X$ and $m > 1$, then:
$$\mu^*(E) = \mu^*(E \cap (\bigcup_{j=1}^{m} A_j)) + \mu^*(E - (\bigcup_{j=1}^{m} A_j))$$

But then note that because $\mathcal{M}$ is an algebra, we know $\bigcup_{j=1}^{m} A_j \in \mathcal{M}$. So:
$$\begin{aligned}\mu^*(E \cap (\bigcup_{j=1}^{m} A_j)) &= \mu^*(E \cap (\bigcup_{j=1}^{m} A_j) \cap A_m) + \mu^*(E \cap (\bigcup_{j=1}^{m} A_j) \cap A_m^{\mathsf{C}}) \\ &= \mu^*(E \cap A_m) + \mu^*(E \cap (\bigcup_{j=1}^{m-1} A_j))\end{aligned}$$

By induction, we thus have that $\mu^*(E \cap \bigcup_{j=1}^{m} A_j) = \sum_{j=1}^{m} \mu^*(E \cap A_j)$. Also, since $E - (\bigcup_{j=1}^{m} A_j) \supset E - (\bigcup_{j \in \mathbb{N}} A_j)$, we thus know that:
$$\mu^*(E) \geq \sum_{j=1}^{m} \mu^*(E \cap A_j) + \mu^*(E - \bigcup_{j \in \mathbb{N}} A_j)$$

Taking the limit as $m \to \infty$, we thus get that:
$$\begin{aligned}\mu^*(E) &\geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(E - \bigcup_{j \in \mathbb{N}} A_j) \\ &\geq \mu^*(E \cap (\bigcup_{j \in \mathbb{N}} A_j)) + \mu^*(E - \bigcup_{j \in \mathbb{N}} A_j)\end{aligned}$$

So, $\bigcup_{j\in\mathbb{N}} A_j$ is $\mu^*$-measurable. Hence, $\mathcal{M}$ is a $\sigma$-algebra.

Also, in order to show that $\mu^*(\bigcup_{j\in\mathbb{N}} A_j) = \sum_{j=1}^{\infty} \mu^*(A_j)$, just substitute $E = \bigcup_{j\in\mathbb{N}} A_j$ into the expression at the bottom of the last page.

Part 3: $(X, \mathcal{M}, \mu^*)$ is a complete measure space.

Suppose $\mu^*(A) = 0$. Then given $E \subseteq X$, we have that:

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E - A) \leq \mu^*(A) + \mu^*(E) \leq 0 + \mu^*(E)$$

It follows that $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E - A)$ for all $E$. So $A \in \mathcal{M}$.

Now if $\mu^*(A) = 0$, then $\mu^*(E) = 0$ for all $E \subseteq A$. It follows that all subsets of $\mu^*$-null sets are in $\mathcal{M}$.

The moral of the story is that we'll just call $\mu^*$ a measure because it is when restricted to the right $\sigma$-algebra.

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A premeasure $\mu_0 : \mathcal{A} \longrightarrow [0, \infty]$ is a function on an algebra satisfying that:

- $\mu_0(\emptyset) = 0$
- if $(A_j)_{j\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{A}$ with $\bigcup_{j\in\mathbb{N}} A_j \in \mathcal{A}$, then $\mu_0(\bigcup_{j\in\mathbb{N}} A_j) = \sum_{j=1}^{\infty} \mu_0(A_j)$.

By setting all but finitely many $A_j$ to the emptyset, we can show that $\mu_0$ must be finitely additive. In turn, this is enough to show that $\mu_0(A) \leq \mu_0(B)$ if $A \subseteq B$ for any $A, B \in \mathcal{A}$.

We say $\mu^*$ is induced by $\mu_0$ if $\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(E_j) \;\middle|\; E_j \in \mathcal{A} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$.

Note that $\mu^*$ is an outer measure by a previous proposition.

**Proposition:** In this situation:

1. $\mu^*|_{\mathcal{A}} = \mu_0$

   Proof:

   Suppose $E \in \mathcal{A}$ and let $(A_j)_{j\in\mathbb{N}}$ be a sequence of sets of in $\mathcal{A}$ covering $E$. It's trivial that $\mu^*(E) \leq \mu_0(E)$ because we could just let $A_1 = E$ and $A_n = \emptyset$ for all $n \geq 2$.

On the other hand, letting $B_m = E \cap A_1$ and $B_m = E \cap A_m - \bigcup_{j=1}^{m-1} A_j$, we have that $(B_j)_{j\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{A}$ whose union is $E$. It follows from the second property of a premeasure and the fact that $B_j \subseteq A_j$ for all $j$ that:

$$\mu_0(E) = \sum_{j=1}^{\infty} \mu_0(B_j) \leq \sum_{j=1}^{\infty} \mu_0(A_j)$$

Since $(A_j)_{j\in\mathbb{N}}$ was not specified, it follows that $\mu_0(E) \leq \mu^*(E)$.

2. Every set in $\mathcal{A}$ is $\mu^*$-measurable.

Proof:
Suppose $A \in \mathcal{A}, E \subseteq X$, and $\varepsilon > 0$. Then there is a sequence $(B_j)_{j\in\mathbb{N}}$ of sets in $\mathcal{A}$ with $E \subseteq \bigcup_{j=1}^{\infty} B_j$ and $\sum_{j=1}^{\infty} \mu_0(B_j) \leq \mu^*(E) + \varepsilon$. Since $\mu_0$ is additive on $\mathcal{A}$, $(B_j \cap A)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ covers $E \cap A$, and $(B_j - A)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ covers $E - A$, we have that:

$$\begin{aligned}\mu^*(E) + \varepsilon \geq \sum_{j=1}^{\infty} \mu_0(B_j) &= \sum_{j=1}^{\infty} \mu_0(B_j \cap A) + \sum_{j=1}^{\infty} \mu_0(B_j - A) \\ &\geq \mu^*(E \cap A) + \mu^*(E - A)\end{aligned}$$

Taking $\varepsilon \to 0$, we get that $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E - A)$.

**Theorem:** Suppose $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra and $\mu_0 : \mathcal{A} \longrightarrow [0, \infty]$ is a premeasure. Then there exists $\mu : \mathcal{M}(\mathcal{A}) \longrightarrow [0, \infty]$ such that:

- $\mu|_{\mathcal{A}} = \mu_0$
- if $\nu : \mathcal{M}(\mathcal{A}) \longrightarrow [0, \infty]$ is a measure with $\nu|_{\mathcal{A}} = \mu|_{\mathcal{A}}$, then $\nu \leq \mu$ (with equality if $\mu(E) < \infty$).
- If $\mu_0$ is $\sigma$-finite, then $\mu$ is the unique extension of $\mu_0$ to a measure on $\mathcal{M}(\mathcal{A})$.

Proof:

1. The first claim is true by Carathéodory's Theorem and the last proposition. Specifically, define $\mu = \mu^*|_{\mathcal{M}(\mathcal{A})}$ where $\mu^*$ is the outer measure induced by $\mu_0$. Since $\mathcal{A}$ is a subset of the $\sigma$-algebra $\mathcal{M}$ of $\mu^*$ measurable sets, we know that $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}$. So $\mu$ is a measure over $\mathcal{M}(\mathcal{A})$. Also, we know that $\mu(A) = \mu^*(A) = \mu_0(A)$ for all $A \in \mathcal{A}$ by the last proposition.

2. To show the second claim, let $E \in \mathcal{M}(\mathcal{A})$ and $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ be a covering of $E$ such that $\sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu(E) + \varepsilon$ for a given $\varepsilon > 0$. Then:

$$\nu(E) \leq \sum_{j=1}^{\infty} \nu(A_j) = \sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu(E) + \varepsilon.$$

Taking $\varepsilon \to 0$, we get that $\nu(E) \leq \mu(E)$.

As for the other inequality, consider that for any $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$, we know by part 3 of the theorem on page 16 and the fact that $\nu(A_j) = \mu(A_j)$ for all $j$ that if $A = \bigcup_{j\in\mathbb{N}} A_j$, then:

$$\nu(A) = \lim_{m\to\infty}(\nu(\bigcup_{j=1}^{m} A_j)) = \lim_{m\to\infty}(\mu(\bigcup_{j=1}^{m} A_j)) = \mu(A).$$

Also, if $\mu(E)$ is finite, then we can choose the covering $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ of $E$ so that all $A_j$ are disjoint and $A = \bigcup_{j\in\mathbb{N}} A_j$ satisfies for a given $\varepsilon > 0$ that:

$$\mu(E) \leq \mu(A) = \sum_{j=1}^{\infty} \mu(A_j) \leq \mu(E) + \varepsilon$$

It follows that $\mu(A - E) < \varepsilon$. So:

$$\mu(E) \leq \mu(A) = \nu(A) = \nu(E) + \nu(A - E) \leq \nu(E) + \mu(A - E) \leq \nu(E) + \varepsilon.$$

Taking $\varepsilon \to 0$, we get that $\mu(E) \leq \nu(E)$.

3. For the third claim, suppose $X = \bigcup_{j\in\mathbb{N}} A_j$ with $\mu_0(A_j) < \infty$ and all $A_j$ being disjoint. Then for any $E \in \mathcal{M}(\mathcal{A})$:

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E \cap A_j) = \sum_{j=1}^{\infty} \nu(E \cap A_j) = \nu(E)$$

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**Exercise 1.16:** Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subseteq X$ is called locally measurable if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets. Trivially, we know $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$. If $\mathcal{M} = \widetilde{\mathcal{M}}$, then $\mu$ is called saturated.

(a) If $\mu$ is $\sigma$-finite, then $\mu$ is saturated.

Let $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}$ satisfy that $\mu(A_j) < \infty$ for all $j$, and that $X = \bigcup_{j\in\mathbb{N}} A_j$. Then if $E$ is locally measurable, we know: $E = \bigcup_{j\in\mathbb{N}} (E \cap A_j) \in \mathcal{M}$.

(b) $\widetilde{\mathcal{M}}$ is a $\sigma$-algebra.

- If $E \in \widetilde{\mathcal{M}}$, then given any $A \in \mathcal{M}$ with $\mu(A) < \infty$, we know $E \cap A \in \mathcal{M}$. It follows that $E^{\mathsf{C}} \cap A = A - E = A - (E \cap A) \in \mathcal{M}$. So $E^{\mathsf{C}} \in \widetilde{\mathcal{M}}$.
- Suppose $(E_j)_{j\in\mathbb{N}}$ is a sequence of sets in $\widetilde{\mathcal{M}}$ and $A \in \mathcal{M}$ satisfies that $\mu(A) < \infty$. Then: $(\bigcup_{j\in\mathbb{N}} E_j) \cap A = \bigcup_{j\in\mathbb{N}} (E_j \cap A) \in \mathcal{M}$.

  So, $\bigcup_{j\in\mathbb{N}} E_j \in \widetilde{\mathcal{M}}$

(c) Define $\widetilde{\mu}$ on $\widetilde{\mathcal{M}}$ by $\widetilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\widetilde{\mu}(E) = \infty$ otherwise. Then $\widetilde{\mu}$ is a saturated measure on $\mathcal{M}$ called the saturation of $\mu$.

Since $\emptyset \in \mathcal{M}$, we know $\widetilde{\mu}(\emptyset) = \mu(\emptyset) = 0$.

Note that if $A, B \in \widetilde{\mathcal{M}}$ with $A \subseteq B$ and $A \notin \mathcal{M}$ but $B \in \mathcal{M}$, then we immediately get a contradiction since that would suggest $A = A \cap B \in \mathcal{M}$. As a result, supposing $(E_j)_{j\in\mathbb{N}}$ is a sequence of disjoint sets in $\widetilde{\mathcal{M}}$, we have that if any $E_j \notin \mathcal{M}$, then:

$$\widetilde{\mu}(\bigcup_{j\in\mathbb{N}} E_j) = \infty = \sum_{j=1}^{\infty} \widetilde{\mu}(E_j).$$

Meanwhile, if all sets of $(E_j)_{j\in\mathbb{N}}$ are in $\mathcal{M}$, then:

$$\widetilde{\mu}(\bigcup_{j\in\mathbb{N}} E_j) = \mu(\bigcup_{j\in\mathbb{N}} E_j) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \widetilde{\mu}(E_j).$$

(d) If $\mu$ is complete, then so is $\widetilde{\mu}$.

This fact is obvious because by the way we defined $\widetilde{\mu}$, we know a set is $\widetilde{\mu}$-null if and only if it is $\mu$-null.

(e) Suppose that $\mu$ is semifinite. For $E \in \widetilde{\mathcal{M}}$, define:

$$\underline{\mu}(E) = \sup\{\mu(A) \mid A \in \mathcal{M} \text{ and } A \subseteq E\}.$$

This is well defined because $\mu(\emptyset)$ is always in the above set and $\mu(X)$ is an upper-bound. Then $\underline{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ that extends $\mu$.

Firstly, we show $\underline{\mu}$ is a measure. To start, it's trivial to see that $\underline{\mu}(\emptyset) = \mu(\emptyset) = 0$.

Lemma: If $E \in \widetilde{\mathcal{M}}$ and $\underline{\mu}(E) = \infty$, then there exists a set $A \in \mathcal{M}$ such that $A \subseteq E$ and $\mu(A) = \infty$.

To show this, construct a sequence of "increasing" sets $(A_j)_{j\in\mathbb{N}}$ satisfying that $A_j \subseteq E$ and $\mu(A_j) \geq j$. Then the union $A$ of that sequence will satisfy that $A \in \mathcal{M}$, that $A \subseteq E$, and that $\mu(A) = \infty$.

Because of that lemma, we don't need to deal with the edge case that a least upper bound equaling infinity doesn't mean a set contains infinity. So, let $(E_j)_{j\in\mathbb{N}}$ be a sequence of disjoint sets in $\widetilde{\mathcal{M}}$ with $E = \bigcup_{j\in\mathbb{N}} E_j$. Then let $\varepsilon > 0$.

To show one inequality, pick a sequence $(A_j)_{j\in\mathbb{N}}$ of sets in $\mathcal{M}$ satisfying that $A_j \subseteq E_j$ and $\underline{\mu}(E_j) - \varepsilon/2^j \leq \mu(A_j)$. Since $A = \bigcup_{j\in\mathbb{N}} A_j \subseteq E$, and each $A_j$ is disjoint, we thus have:

$$-\varepsilon + \sum_{j=1}^{\infty} \underline{\mu}(E_j) \leq \sum_{j=1}^{\infty} \mu(A_j) = \mu(A) \leq \underline{\mu}(E)$$

To show the other inequality, pick $B \in \mathcal{M}$ satisfying that $B \subseteq E$ and $\underline{\mu}(E) - \varepsilon < \mu(B)$. Because $E, E_j \in \widetilde{\mathcal{M}}$ for each $j$, we know that $B \cap E$ and $B \cap E_j$ are in $\mathcal{M}$ for each $j$. So:

$$\underline{\mu}(E) - \varepsilon < \mu(B) = \mu(B \cap E) = \sum_{j=1}^{\infty} \mu(B \cap E_j) \leq \sum_{j=1}^{\infty} \underline{\mu}(E_j)$$

Taking $\varepsilon \to 0$, we thus get that $\underline{\mu}(E) = \sum\limits_{j=1}^{\infty} \underline{\mu}(E_j)$.

Proving that $\mu(E) = \underline{\mu}(E)$ when $E \in \mathcal{M}$ is trivial. Obviously, $\mu(E) \leq \underline{\mu}(E)$. Meanwhile for any $F \in \mathcal{M}$ satisfying that $F \subseteq E$, we know that $\mu(F) \leq \mu(E)$. So, there does not exists a subset of $E$ in $\mathcal{M}$ with greater measure than $\mu(E)$.

Note that $\widetilde{\mu}$ and $\underline{\mu}$ are not necessarily equal. Part (f) of this problem gives a relatively simple counterexample.

---

**Exercise 1.17:** If $\mu^*$ is an outer measure on $X$ and $(A_j)_{j \in \mathbb{N}}$ is a sequence of disjoint $\mu^*$-measurable sets, then:

$$\mu^*(E \cap \bigcup_{j \in \mathbb{N}} A_j) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \text{ for any } E \subseteq X.$$

Note that by induction, we can show that for any $n \in \mathbb{N}$:

$$\begin{aligned}
\mu^*(E \cap \bigcup_{j=1}^{\infty} A_j) &= \mu^*(E \cap \bigcup_{j=1}^{\infty} A_j \cap A_1) + \mu^*(E \cap \bigcup_{j=1}^{\infty} A_j - A_1) \\
&= \mu^*(E \cap A_1) + \mu^*(E \cap \bigcup_{j=2}^{\infty} A_j) \\
&= \sum_{j=1}^{2} \mu^*(E \cap A_j) + \mu^*(E \cap \bigcup_{j=3}^{\infty} A_j) \\
&= \cdots = \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap \bigcup_{j=n+1}^{\infty} A_j)
\end{aligned}$$

Thus, we clearly have for all $n$ that $\sum\limits_{j=1}^{n} \mu^*(E \cap A_j) \leq \mu^*(E \cap \bigcup\limits_{j=1}^{\infty} A_j)$.

Taking the limit as $n \to \infty$, we thus know $\sum\limits_{j=1}^{\infty} \mu^*(E \cap A_j) \leq \mu^*(E \cap \bigcup\limits_{j \in \mathbb{N}} A_j)$.

The other inequality is obvious from the subadditivity property of outer measures.

---

**Exercise 1.18:** Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ be be the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ be the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let $\mu_0$ be a premeasure on $\mathcal{A}$ and $\mu^*$ the induced outer measure.

(a) For any $E \subseteq X$ and $\varepsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ with $E \subseteq A$ and $\mu^*(A) \leq \mu^*(E) + \varepsilon$.

Let $(A_j)_{j\in\mathbb{N}}$ be a sequence of sets in $\mathcal{A}$ which cover $E$ and satisfy that $\sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \varepsilon$. In turn, by the subadditivity of outer measures, and the fact that $\mu^*(A) = \mu_0(A)$ for all $A \in \mathcal{A}$, we know that:

$$E \subseteq \bigcup_{j\in\mathbb{N}} A_j \in \mathcal{A}_\sigma \text{ and } \mu^*(\bigcup_{j\in\mathbb{N}} A_j) \leq \sum_{j=1}^{\infty} \mu^*(A_j) = \sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \varepsilon$$

(b) If $\mu^*(E) < \infty$, then $E$ is $\mu^*$-measurable if and only if there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B - E) = 0$.

Suppose $E$ is $\mu^*$-measurable. Then for all $j \in \mathbb{N}$, pick $A_j \in \mathcal{A}_\sigma$ satisfying that $E \subseteq A_j$ and $\mu^*(E) \leq \mu^*(A_j) \leq \mu^*(E) + 1/j$. Since $E$ is $\mu^*$-measurable, we know that for all $j \in \mathbb{N}$:

$$\mu^*(A_j) = \mu^*(A_j \cap E) + \mu^*(A_j - E) = \mu^*(E) + \mu^*(A_j - E)$$

In turn, because $\mu^*(E) < \infty$, this tell us that $\mu^*(A_j - E) \leq 1/j$. Also, because $\bigcap_{j\in\mathbb{N}} A_j \subseteq A_n$ for all $n \in \mathbb{N}$, we know that $\mu^*(\bigcap_{j\in\mathbb{N}} A_j - E) \leq 1/n$ for all $n \in \mathbb{N}$.

As a result, we know that $\bigcap_{j\in\mathbb{N}} A_j \in \mathcal{A}_{\sigma\delta}$, $E \subseteq \bigcap_{j\in\mathbb{N}} A_j$, and $\mu^*(\bigcap_{j\in\mathbb{N}} A_j - E) = 0$.

To prove the reverse implication, suppose there exists a $\mu^*$-separable set $B$ satisfying that $E \subseteq B$ and $\mu^*(B - E) = 0$ (any set in $\mathcal{A}_{\sigma\delta}$ will be $\mu^*$-separable because $\mathcal{A}_{\sigma\delta} \subseteq \mathcal{M}(\mathcal{A})$). Then given any set $F$, we have that:

$$\begin{aligned}\mu^*(F - E) &= \mu^*(F \cap E^\mathsf{C} \cap B) + \mu^*(F \cap E^\mathsf{C} \cap B^\mathsf{C}) \\ &= \mu^*(F \cap (B - E)) + \mu^*(F - B) = \mu^*(F - B)\end{aligned}$$

Also, since $F \cap E \subseteq F \cap B$, we know that $\mu^*(F \cap E) \leq \mu^*(F \cap B)$.

So, $\mu^*(F \cap E) + \mu^*(F - E) \leq \mu^*(F \cap B) + \mu^*(F - B) = \mu^*(F)$. Hence, $E$ is $\mu^*$-measurable.

(c) If $\mu_0$ is $\sigma$-finite, we can remove the requirement in part (b) that $\mu^*(E) < \infty$.

Because the backwards implication proof never required $\mu^*(E)$ to be finite, it suffices to show that $E$ being $\mu^*$-measurable implies there exists $B \in A_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B - E) = 0$.

To start, let $(C_i)_{i\in\mathbb{N}}$ satisfy that $\mu_0(C_i) < \infty$ and $\bigcup_{i=1}^{\infty} C_i = X$.

Next for all $j, i \in \mathbb{N}$, pick $A_j^{(i)} \in \mathcal{A}_\sigma$ satisfying that $E \cap C_i \subseteq A_j^{(i)}$ and $\mu^*(E) \leq \mu^*(A_j^{(i)}) \leq \mu^*(E) + {}^1/_{j2^i}$. Since $\mu^*(E \cap C_i)$ is finite, we can use the same reasoning as in part (b) to say that $\mu^*(A_j^{(i)} - (E \cap C_i)) \leq {}^1/_{j2^i}$.

Importantly, $A_j^{(i)} - E \subseteq A_j^{(i)} - (E \cap C_i)$. for all $i$. Therefore:

$$\begin{aligned}\mu^*((\bigcup_{i\in\mathbb{N}} A_j^{(i)}) - E) &= \mu^*(\bigcup_{i\in\mathbb{N}}(A_j^{(i)} - E)) \\ &\leq \mu^*(\bigcup_{i\in\mathbb{N}}(A_j^{(i)} - (E \cap C_i))) \\ &\leq \sum_{i\in\mathbb{N}} \mu^*(A_j^{(i)} - (E \cap C_i)) \leq \sum_{i\in\mathbb{N}} \frac{1}{j2^i} = \frac{1}{j}\end{aligned}$$

Since $E \subseteq \bigcup_{i\in\mathbb{N}} A_j^{(i)} \in \mathcal{A}_\sigma$, we've thus shown for all $j \in \mathbb{N}$ that there exists a set $A_j \in \mathcal{A}_\sigma$ satisfying that $\mu^*(A_j - E) \leq {}^1/_j$ and $E \subseteq A_j$.

Finally, intersecting all those $A_j$ like in part (b), we get our set satisfying the right-side of the implication.

---

**Exercise 1.19:** Let $\mu^*$ be an outer measure on $X$ induced from a finite premeasure $\mu_0$ defined on an algebra $\mathcal{A}$. If $E \subseteq X$, define the inner measure of $E$ to be $\mu_*(E) = \mu_0(X) - \mu^*(E^{\mathsf{C}})$. Then $E$ is $\mu^*$-measurable if and only if $\mu^*(E) = \mu_*(E)$.

($\Longrightarrow$)
If $E$ is $\mu^*$-measurable, then we have $\mu^*(X \cap E) + \mu^*(X - E) = \mu^*(X)$. Because $\mu^*(X) = \mu_0(X)$, we thus have that $\mu_*(E) = \mu_0(X) - \mu^*(X - E) = \mu_*(E)$.

($\Longleftarrow$)
By part (a) of the previous exercise, we know there exists $A_j \in \mathcal{A}_\sigma$ satisfying that $E \subseteq A_j$ and $\mu^*(A_j) \leq \mu^*(E) + {}^1/_j$.

Note that $A_j$ is $\mu^*$-measurable because $\mathcal{A}_\sigma \subseteq \mathcal{M}(\mathcal{A})$. This means that:

- $\mu_0(X) = \mu^*(X) = \mu^*(X \cap A_j) + \mu^*(X - A_j) = \mu^*(A_j) + \mu^*(A_j^{\mathsf{C}})$.
- $\mu^*(E^{\mathsf{C}}) = \mu^*(E^{\mathsf{C}} \cap A_j) + \mu^*(A_j^{\mathsf{C}} \cap E^{\mathsf{C}}) = \mu^*(A_j - E) + \mu^*(A_j^{\mathsf{C}})$.

Supposing that $\mu^*(E) = \mu_*(E) = \mu_0(X) - \mu^*(X - E)$ and plugging in the first bullet-pointed identity, we get that:

$$\mu^*(E^{\mathsf{C}}) = \mu^*(A_j) + \mu^*(A_j^{\mathsf{C}}) - \mu^*(E).$$

Substituting that into the second bullet-pointed identity, we have:

$$\mu^*(A_j) - \mu^*(E) = \mu^*(A_j - E).$$

And finally, using the inequality: $\mu^*(A_j) \leq \mu^*(E) + 1/j$, we get $\mu^*(A_j - E) \leq 1/j$.

Hence, we've shown that there exists a set $A_j \in \mathcal{A}_\sigma$ such that $E \subseteq A_j$ and $\mu^*(A_j - E) < 1/j$ for all $j \in \mathbb{N}$. From there, we can proceed exactly like in part (b) of exercise 1.18. Pick such an $A_j$ for all $j \in \mathbb{N}$ and then intersect them together. The result will be a set $B \in \mathcal{A}_{\sigma\delta}$ satisfying that $E \subseteq B$ and $\mu^*(B - E) = 0$. Since such a set exists, we know by the conclusion of part (b) of exercise 1.18 that $E$ is $\mu^*$-measurable.

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**Exercise 1.21:** Let $\mu^*$ be an outer measure induced from a premeasure defined on an algebra $\mathcal{A}$ and $\overline{\mu}$ be the restriction of $\mu^*$ to the collection $\mathcal{M}$ of $\mu^*$-measurable sets. Then $\overline{\mu}$ is saturated.

Let $E$ be a locally $\overline{\mu}$-measurable set and choose any $F \subseteq X$. Given any $\varepsilon > 0$, by part (a) of exercise 1.18, we know that there exists a $\mu^*$-measurable set $A \in \mathcal{A}_\sigma \subseteq \mathcal{M}$ such that $F \subseteq A$ and $\mu^*(A) \leq \mu^*(F) + \varepsilon$.

Assuming without loss of generality that $\mu^*(F)$ is finite, we thus know that $\mu^*(A) < \infty$. So, since $E$ is locally $\overline{\mu}$-measurable, we have that $E \cap A \in \mathcal{M}$.

Now, we first note that because $F \cap E \subseteq A \cap E$ and $F - E \subseteq A - E$, we have that:

$$\mu^*(F \cap E) + \mu^*(F - E) \leq \mu^*(A \cap E) + \mu^*(A - E)$$

Next we note that:

- $A \cap (A \cap E) = A \cap E$
- $A \cap (A \cap E)^{\mathsf{C}} = (A \cap A^{\mathsf{C}}) \cup (A \cap E^{\mathsf{C}}) = A - E$

So: $\mu^*(A \cap E) + \mu^*(A - E) = \mu^*(A \cap (A \cap E)) + \mu^*(A \cap (A \cap E)^{\mathsf{C}}) = \mu^*(A)$.

And finally, since $\mu^*(A) \leq \mu^*(F) + \varepsilon$, we can thus conclude that:

$$\mu^*(F \cap E) + \mu^*(F - E) \leq \mu^*(F) + \varepsilon.$$

Taking $\varepsilon \to 0$, we get that $\mu^*(F \cap E) + \mu^*(F - E) \leq \mu^*(F)$ So, $E$ is $\mu^*$-measurable, meaning $E \in \mathcal{M}$.

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Consider the collection $H = \{\emptyset, (a, b], (a, \infty) \mid -\infty \leq a < b < \infty\}$ of "half-open-intervals" of $\mathbb{R}$.

This forms an elementary family.

- We specified in the definition that $\emptyset \in H$.
- If $x \in (a, b] \cap (c, d] \neq \emptyset$, then we know $a < x < d$ and $c < x < b$. So $(a, b] \cap (c, d] = (\max(a, c), \min(b, d)] \in H$.

- Given $(a,b] \in H$, we have that $(a,b]^{\mathsf{C}} = (-\infty, a] \cup (b, \infty)$.

  For the sake of time, I'm ignoring edge cases of a right bound of infinity since they are still trivial.

By exercise 1.2, we know that $\mathcal{M}(H) = \mathcal{B}_{\mathbb{R}}$. And, by a previous proposition, we know that $\mathcal{A}$ equal to the collection of finite disjoint unions of $H$ is an algebra.

**Proposition:** Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a monotonically increasing and right continuous function (meaning $\lim\limits_{t \to x^+} F(t) = F(x)$ for all $x \in \mathbb{R}$). Also, define:

$$F(-\infty) = \lim_{t \to -\infty} F(t) \text{ and } F(\infty) = \lim_{t \to \infty} F(t).$$

If $(a_j, b_j]$ for $j = 1, \ldots, n$ are disjoint intervals in $H$, define:

$$\mu_0(\bigcup_{j=1}^{n} (a_j, b_j]) = \sum_{j=1}^{n} F(b_j) - F(a_j)$$

Also let $\mu_0(\emptyset) = 0$. And if $F(\infty) = \infty$ and $F(-\infty) = -\infty$, define $\mu_0(\mathbb{R}) = \infty$. Then this is a premeasure.

Proof:

1. $\mu_0$ is well defined.

   Suppose $(a_j, b_j]$ for $j = 1, \ldots, n$ are disjoint intervals in $H$ satisfying that $(a,b] = \bigcup\limits_{j=1}^{n} (a_j b_j] = (a,b]$. Then after indexing those half intervals in a certain way, we must have that:

   $$a = a_1 < b_1 = a_2 < \ldots < b_{n-1} = a_n < b_{n-1} = b$$

   It follows that:

   $$F(b) - F(a) = \mu_0(\bigcup_{j=1}^{n} (a_j, b_j]) = \sum_{j=1}^{n} \mu_0((a_j, b_j]) = \sum_{j=1}^{n} F(b_j) - F(a_j)$$

   In other words, $\mu_0$ is well defined for individual intervals.

   Now suppose $I_i$ and $J_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ are disjoint intervals in $H$ satisfying that $\bigcup\limits_{i=1}^{m} I_i = \bigcup\limits_{j=1}^{n} J_j$.

   For each $I_i$, we can repeat the same reasoning as above with the collection of sets $I_i \cap J_j$ for $j = 1, \ldots, n$ in order to get that:

   $$\mu_0(I_i) = \sum_{j=1}^{n} \mu_0(I_i \cap J_j)$$

   Similarly, we can show for each $J_j$ that:

   $$\mu_0(J_j) = \sum_{i=1}^{m} \mu_0(I_i \cap J_j)$$

   Thus: $\sum\limits_{i=1}^{m} \mu_0(I_i) = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \mu_0(I_i \cap J_j) = \sum\limits_{j=1}^{n} \mu(J_j)$.

2. If $(I_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ is a disjoint sequence satisfying that $\bigcup_{j\in\mathbb{N}} I_j \in \mathcal{A}$, then:
$$\mu_0(\bigcup_{j\in\mathbb{N}} I_j) = \sum_{j=1}^{\infty} \mu_0(I_j).$$

Since $\bigcup_{j\in\mathbb{N}} I_j \in \mathcal{A}$, we know it is equal to a finite union of disjoint intervals of $H$. By considering those intervals separately, we can thus assume without loss of generality that $\bigcup_{j\in\mathbb{N}} I_j = (a,b]$ where not both $a=-\infty$ and $b=\infty$.

Also, without loss of generality we can assume each $I_j$ is one interval.

Now it's obvious from the construction of $\mu_0$ that $\mu_0$ is additive. And since $(a,b] - \bigcup_{j=1}^{m} I_j \in \mathcal{A}$ for all $m$, we thus know that:
$$\mu_0((a,b]) = \mu_0(\bigcup_{j=1}^{m} I_j) + \mu_0((a,b] - \bigcup_{j=1}^{m} I_j) \geq \mu_0(\bigcup_{j=1}^{m} I_j) = \sum_{j=1}^{m} \mu_0(I_j)$$
Taking the limit as $m\to\infty$, we get that: $\sum_{j=1}^{\infty} \mu_0(I_j) \leq \mu_0(\bigcup_{j\in\mathbb{N}} I_j)$.

# Lecture 6 Notes: 10/15/2024

To show the reverse inequality, suppose $a,b\in\mathbb{R}$ (a.k.a. finite), and let $\varepsilon>0$. Since $F$ is right-continuous, there exists $\delta>0$ such that $F(a+\delta)-F(a)<\varepsilon$. Similarly, given that $I_j=(a_j,b_j]$, there exists $\delta_j>0$ such that $F(b_j+\delta_j)-F(b_j)<\frac{\varepsilon}{2^j}$ for all $j\in\mathbb{N}$.

Next, note that the collection $\{(a_j,b_j+\delta_j)\}_{j\in\mathbb{N}}$ of open intervals covers the set $[a+\delta,b]$. Thus by compactness, there is a finite subcover. In other words,
$$(a_1,b_1+\delta_1),\dots,(a_N,b_N+\delta_N) \text{ cover } [a+\delta,b]$$

Furthermore, by removing intervals in that finite subcover which are subsets of other intervals and by reindexing, we can assume that:
$$b_j+\delta_j \in (a_{j+1}, b_{j+1}+\delta_{j+1}) \text{ for all } j=1,\dots,N-1.$$

Then:
$$\begin{aligned}
\mu_0((a,b]) &= F(b)-F(a)\\
&< F(b)-F(a+\delta)+\varepsilon\\
&\leq F(b_N+\delta_N)-F(a_1)+\varepsilon && \text{(since } F \text{ is monotone increasing)}\\
&= F(b_N+\delta_N)-f(a_N)+\sum_{j=1}^{N-1}(F(a_{j+1})-F(a_j))+\varepsilon\\
&\leq F(b_N+\delta_N)-f(a_N)+\sum_{j=1}^{N-1}(F(b_j+\delta_j)-F(a_j))+\varepsilon && \text{(again since } F \text{ is monotone increasing)}\\
&= \sum_{j=1}^{N}(F(b_j+\delta_j)-F(a_j))+\varepsilon\\
&< \sum_{j=1}^{N}(F(b_j)+\tfrac{\varepsilon}{2^j}-F(a_j))+\varepsilon < \sum_{j=1}^{\infty}\mu(I_j)+2\varepsilon
\end{aligned}$$

Since $\varepsilon$ is arbitrary, we've now shown the reverse inequality when $a$ and $b$ are finite. To extend this result to when $a = -\infty$ or $b = \infty$, note that the invervals $(a_j, b_j + \delta_j)$ cover $[-M + \delta, b]$ or $[a + \delta, M]$ for all $M$ in either $(-\infty, b]$ or $(a, \infty)$.

So, doing the same manipulations as before, since $\sum_{j=1}^{\infty} \mu(I_j) + 2\varepsilon$ is an upper bound of $\mu_0((-M, b])$ or $\mu_0((a, M])$, we know that the limit of $\mu_0((-M, b])$ or $\mu_0((a, M])$ as $M \to \infty$ will not exceed that upper bound. Then taking $\varepsilon \to 0$, we get the same result as before.

Plus, based on how we defined $F(\infty)$ and $F(-\infty)$, we know that $\lim_{M\to\infty} \mu_0((-M, b]) = \mu_0((-\infty, b])$ and $\lim_{M\to\infty} \mu_0((a, M]) = \mu_0((a, \infty))$.

**Theorem:**

1. If $F$ is a monotone increasing and right-continuous like above, there is a unique Borel measure $\mu_F$ on $\mathcal{B}_{\mathbb{R}}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all $a < b$.
2. If $G$ is another such monotone increasing and right-continuous, then $\mu_G = \mu_F$ if and only if $F - G$ is a constant.
3. If $\mu$ is a $\mathcal{B}_{\mathbb{R}}$ measure that is finite on all bounded sets, then we can define:

$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu((x, 0]) & \text{if } x < 0 \end{cases}$$

Then $F$ is a monotone increasing and right-continuous like above with $\mu = \mu_F$.

Proof:

1. By the previous proposition, we know $F$ induces a premeasure $\mu_F$. Also, $\mu_F$ is a $\sigma$-finite premeasure. Thus, by the theorem on page 24, we know $\mu_F$ induces a unique measure on $\mathcal{M}(H) = \mathcal{B}_{\mathbb{R}}$.

2. Clearly, if $G(b) - G(a) = F(b) - F(a)$ for all $-\infty \leq a < b \leq \infty$, then we must have that $G - F$ is constant.

3. Finally, by the theorem at the bottom of page 16, we can fairly easily show that $F$ is right-continuous and monotone increasing.

Given a monotone increasing and right-continuous function $F$, we call $\mu_F$ the Lesbesgue-Stieltjes measure. If $F(x) = x$, we just write $m$ and call it the Lesbesgue measure.

Also, we'll almost always use $\mu_F$ to refer to the completion of $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_F)$. So, this measure is actually defined on more than just $\mathcal{B}_{\mathbb{R}}$.

Let $\mu$ be a Lesbesgue-Stieltjes measure associated with a monotone increasing, right-continuous function $F$, and let $\mathcal{M}_\mu$ be the set of $\mu$-measureable sets. Here are some nice properties of $\mu$:

**Lemma:** Suppose $\nu(E) = \inf(\{\sum_{j=1}^{\infty} \mu((a_j, b_j)) \mid E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j)\})$. Then $\nu(E) = \mu(E)$ for all $E \in \mathcal{M}_\mu$.

Proof:
Fix $\varepsilon > 0$. Then, there exists $((a_j, b_j])_{j \in \mathbb{N}}$ such that $E \subseteq \bigcup_{j \in \mathbb{N}} (a_j, b_j]$ and $\sum_{j=1}^{\infty} \mu((a_j, b_j]) \leq \mu(E) + \varepsilon$.

Then for all $j \in \mathbb{N}$, pick $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) \leq \varepsilon/2^j$. Thus:

$$\begin{aligned}\nu(E) \leq \sum_{j=1}^{\infty} \mu((a_j, b_j + \delta_j)) &\leq \sum_{j=1}^{\infty} \mu((a_j, b_j + \delta_j]) \\ &\leq \sum_{j=1}^{\infty} \mu((a_j, b_j]) + \varepsilon \leq \mu(E) + 2\varepsilon\end{aligned}$$

Taking $\varepsilon \to 0$, we get that $\nu(E) \leq \mu(E)$.

To get the reverse inequality, note that we can write any open interval as the countable union of a sequence of disjoint half-open-intervals. It follows that $\mu(E) \leq \nu(E)$.

**Theorem:** If $E \subseteq \mathcal{M}_\mu$, then:

- $\mu(E) = \inf(\{\mu(U) \mid E \subseteq U$ and $U$ is open$\})$
- $\mu(E) = \sup(\{\mu(K) \mid K \subseteq E$ and $K$ is compact$\})$

Proof:
The first bullet point follows almost immediately from the previous lemma because unions of open sets are open.

To show the second bullet point, first suppose $E$ is bounded. If $E$ is closed, then the equality is trivial. So suppose $\overline{E} - E \neq \emptyset$. Then given any $\varepsilon > 0$, let $U$ be an open set such that $\overline{E} - E \subseteq U$ and $\mu(U) \leq \mu(\overline{E} - E) + \varepsilon$. (Note that $\overline{E} \in \mathcal{B}_\mathbb{R}$ and so $\overline{E} - E \in \mathcal{M}_\mu$.)

We have that $K = \overline{E} - U$ is a closed bounded set satisfying that $K \subseteq E$ and:

$$\begin{aligned}\mu(K) = \mu(E) - \mu(E \cap U) &= \mu(E) - (\mu(U) - \mu(U - E)) \\ &\geq \mu(E) - \mu(U) + \mu(\overline{E} - E) \geq \mu(E) - \varepsilon\end{aligned}$$

If $E$ is unbounded, then consider $E_j \cap (j, j+1]$ for all $j \in \mathbb{Z}$. By the previous reasoning, given any $\varepsilon > 0$ there exists a compact set $K_j$ such that $K_j \subseteq E_j$ and $\mu(K_j) \geq \mu(E) - \varepsilon/2^{|j|}$.

Now define $H_n = \bigcup_{j=-n}^{n} K_j$. That way $H_n$ is a compact subset of $E$ and:

$$\mu(H_n) \geq \mu(\bigcup_{j=-n}^{n} E_j) - 3\varepsilon.$$

Since $\mu(E) = \lim_{n\to\infty} (\mu(\bigcup_{j=-n}^{n} E_j)) \leq \lim_{n\to\infty} (\mu(H_n)) + 3\varepsilon$, the bulllet pointed claim follows.

**Theorem:** If $E \subseteq \mathbb{R}$, then the following are equivalent:

(a) $E \in \mathcal{M}_\mu$.

(b) $E = V - N_1$ where $V$ is a $G_\delta$ set and $\mu(N_1) = 0$.

(c) $E = H \cup N_2$ where $H$ is an $F_\sigma$ set and $\mu(N_2) = 0$.

Proof:
We trivially have that (b) $\Longrightarrow$ (a) and (c) $\Longrightarrow$ (a) because $V, H \in \mathcal{B}_\mathbb{R}$ and $N_1, N_2 \in \mathcal{M}_\mu$.

To show the other direction, suppose $E \subseteq \mathcal{M}_\mu$. Firstly, we'll suppose $\mu(E) < \infty$. Then for all $j \in \mathbb{N}$, we can choose an open set $U_j \supseteq E$ and a compact set $K_j \subseteq E$ such that:

$$\mu(U_j) - {}^1\!/_{2^j} \leq \mu(E) \leq \mu(K_j) + {}^1\!/_{2^j}$$

Define $V = \bigcap_{j\in\mathbb{N}} U_j$ and $H = \bigcup_{j\in\mathbb{N}} K_j$.

Then $H \subseteq E \subseteq V$ and $\mu(H) = \mu(E) = \mu(V)$. It follows since $\mu(E) < \infty$ that:

- Letting $N_1 = V - E$, we have that $E = V - N_1$ and
$$\mu(V) = \mu(N_1) + \mu(E) \Longrightarrow 0 = \mu(V) - \mu(E) = \mu(N_1).$$
- Letting $N_2 = E - H$, we have that $E = H \cup N_2$ and
$$\mu(E) = \mu(H) + \mu(N_2) \Longrightarrow 0 = \mu(E) - \mu(H) = \mu(N_2).$$

The rest is **Exercise 1.25**.
Now suppose $\mu(E) = \infty$.

For all $j \in \mathbb{Z}$, define $E_j = E \cap (j, j+1]$. That way $E$ is the disjoint union of all $E_j$. Importantly, each $\mu(E_j) \leq \mu((j, j+1]) < \infty$. So, using the previous logic, for all $j$ we can find an $F_\sigma$ set $H_j$ satisfying that $H_j \subseteq E_j$ and:

- Letting $N_2^{(j)} = E_j - H_j$, we have that $E_j = H_j \cup N_2^{(j)}$ and $\mu(N_2^{(j)}) = 0$.

Setting $H = \bigcup_{j\in\mathbb{Z}} H_j$, we have that $H$ is a countable union of countable unions of closed sets. Thus, $H$ is an $F_\sigma$ set.

Also $N_2 = \bigcup_{j\in\mathbb{Z}} N_2^{(j)}$ is still a null set since the set of null sets is closed under countable union. And finally, $H \cup N_2 = E$ becuase each $E_j = H_j \cup N_2^{(j)}$. Thus, we've shown (c).

Meanwhile, to show (b), consider that if $E \in \mathcal{M}_\mu$, then $E^{\mathsf{C}} \in \mathcal{M}_\mu$. So, we can find an $F_\sigma$ set $H$ and null set $N_1$ such that $E^{\mathsf{C}} = H \cup N_1$. Taking complements of both sides, we get that $E = H^{\mathsf{C}} - N_1$. And importantly, the complement of a union of sets is equivalent to the intersection of the complements of those sets. So, since the complement of a closed set is open, $H^{\mathsf{C}}$ is a $G_\delta$ set.

Note that we call measures with the property described in the above two theorems regular.

**Proposition:** If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for all $\varepsilon > 0$, there is a set $A$ that is a finite union of open intervals such that $\mu(E \,\Delta\, A) < \varepsilon$.

Note that $E\,(\,\Delta)A = (E - A) \cup (A - E) = (E \cup A) - (E \cap A)$.

The proof is **Exercise 1.26**

Let $U$ be an open set satisfying that $E \subseteq U$ and $\mu(U) \leq \mu(E) + \varepsilon/2$. Then, a fact about the standard metric space of $\mathbb{R}$ is that any open set is a countable union of disjoint open intervals.

(writing this so I have it in my notes)

To see this, given the open set $U \subseteq \mathbb{R}$, and any $x \in U$, define $I_x = (a, b)$ where $a = \inf(\alpha \in \overline{\mathbb{R}} \mid (\alpha, x) \subset U)$ and $b = \sup(\beta \in \overline{\mathbb{R}} \mid (x, \beta) \subset U)$.

Then consider the collection $\{I_x\}_{x \in U}$. Given $I_x = (a_x, b_x)$ and $I_y = (a_y, b_y)$, supposing $t \in I_x \cap I_y$, we must have that:

$$(\min(a_x, a_y), t] \cup [t, \max(b_x, b_y)) = (\min(a_x, a_y), \max(b_x, b_y)) \subseteq U.$$

Because of how we defined $a_x, b_x, a_y, b_y$, and the fact that obviously $I_x, I_y \subseteq (\min(a_x, a_y), \max(b_x, b_y))$, we must have that $a_x = \min(a_x, a_y) = a_y$ and $b_x = \min(b_x, b_y) = b_y$. So, $I_x = I_y$.

Hence, $\{I_x\}_{x \in U}$ is a disjoint collection of intervals. Also, because each interval can be injectively mapped to a rational number inside that interval, we know this collection is countable.

So, let $(I_j)_{j \in \mathbb{N}}$ be a sequence of disjoint open intervals whose union is $U$. Then observe that $\lim\limits_{n \to \infty} (\mu(\bigcup\limits_{j=1}^{n})I_j) = \mu(U)$. So, we can pick $n$ large enough that for $A = \bigcup\limits_{j=1}^{n} I_j$, we have that $A$ is a finite union of open intervals and:

$$\mu(U) - \varepsilon/2 \leq \mu(A) \leq \mu(U) + \varepsilon/2$$

Now since $A \subseteq U$ and $E \subseteq U$, we know that $E - A \subseteq U - A$ and $A - E \subseteq U - A$. So:

- $\mu(E) + \mu(A - E) \leq \mu(E) + \mu(U - E) = \mu(U) \leq \mu(E) + \varepsilon/2$.
- $\mu(E - A) \leq \mu(U - A) = \mu(U) - \mu(A) \leq \mu(U) - \mu(U) + \varepsilon/2 = \varepsilon/2$

(because $\mu(U) \leq \mu(E) + \varepsilon/2$ is finite)

Since $\mu(E)$ is finite, we can subtract it out of the first bulleted inequality and add it to the second to get:

$$\mu(A \,\Delta\, E) = \mu(A - E) + \mu(E - A) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

**Exercise 1.28:** Let $F$ be increasing and right continuous, and let $\mu_F$ be the associated measure. Then:

(a) $\mu_F(a) = F(a) - F(a-)$ (Note that this is a special case of (c))

(b) $\mu_F([a, b)) = F(b-) - F(a-)$

(c) $\mu_F([a, b]) = F(b) - F(a-)$

(d) $\mu_F((a, b)) = F(b-) - F(a)$

To start let's show (c).
Note that $[a, b] = \bigcap_{n\in\mathbb{N}} (a - \frac{1}{n}, b]$ and $((a - \frac{1}{n}, b])_{n\in\mathbb{N}}$ is a decreasing sequence of sets.
So:

$$\mu_F([a, b]) = \lim_{n\to\infty} (\mu_F((a - \tfrac{1}{n}, b])) = \lim_{n\to\infty} (F(b) - F(a - \tfrac{1}{n})) = F(b) - F(a-)$$

Next, let's show (b).
Note that $[a, b) = \bigcup_{n\in\mathbb{N}} [a, b - \frac{1}{n}]$ and $([a, b - \frac{1}{n}])_{n\in\mathbb{N}}$ is an increasing sequence of sets.
So:

$$\mu_F([a, b)) = \lim_{n\to\infty} (\mu_F([a, b - \tfrac{1}{n}])) = \lim_{n\to\infty} (F(b - \tfrac{1}{n}) - F(a-)) = F(b-) - F(a-)$$

Finally, let's show (d).
Note that $(a, b) = \bigcup_{n\in\mathbb{N}} (a, b - \frac{1}{n}]$ and $((a, b - \frac{1}{n}])_{n\in\mathbb{N}}$ is an increasing sequence of sets.
So:

$$\mu_F((a, b)) = \lim_{n\to\infty} (\mu_F((a, b - \tfrac{1}{n}])) = \lim_{n\to\infty} (F(b - \tfrac{1}{n}) - F(a)) = F(b-) - F(a)$$

---

If $\mu = m$ is the Lesbesgue measure (a.k.a when $F(x) = x$), we denote the collection of Lesbesgue measurable sets $\mathcal{L}$ to be the closure of $\mathcal{B}_\mathbb{R}$.

**Theorem:** If $E \in \mathcal{L}$ and $s, r \in \mathbb{R}$, we define:

$$E + s = \{x + s \mid x \in E\}, \qquad rE = \{rx \mid x \in E\}.$$

Then $E + s, rE \in \mathcal{L}$, $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$.

Proof: (the professor skipped doing this proof and the book doesn't seem to prove this well. So I'm going on my own here.)

Supposing $E + s \in \mathcal{L}$, define $m_s(E) = m(E + s)$. Similarly define $m^r(E) = m(rE)$ if $rE \in \mathcal{L}$.

Let $\mathcal{A}$ be the collection of disjoint unions of half-open-intervals, and define $m^*$ as the outer measure induced by $m|_\mathcal{A}$. Then you can fairly trivially (albeit with a lot of notation) show that $m^*(E + s) = m^*(E)$ and $m^*(rE) = |r|m^*(E)$. Also, we know that $m^*(E)$ must equal $m(E)$ for all $E \in \mathcal{B}_\mathbb{R}$ since any measure on $\mathcal{B}_\mathbb{R}$ is uniquely determined by it's restriction to $\mathcal{A}$ and $m^*$ is a measure on $\mathcal{B}_\mathbb{R}$ (see the theorem on the bottom of page 24).

Next, note that $E \in \mathcal{B}_{\mathbb{R}} \implies E+s, rE \in \mathcal{B}_{\mathbb{R}}$. The reason why is that as seen in a bonus proposition below, we have that a set $E \in \mathcal{B}_{\mathbb{R}}$ if and only if it can be gained by taking a countable amount of countable unions, countable intersections, and complements of sets of $\mathcal{A}$. But note that given any $A \in \mathbb{R}$:

- $(A+s)^{\mathsf{C}} = A^{\mathsf{C}} + s$ and $(rA)^{\mathsf{C}} = rA^{\mathsf{C}}$
- $\bigcup\limits_{j\in\mathbb{N}}(A_j+s) = (\bigcup\limits_{j\in\mathbb{N}} A_j)+s$ and $\bigcup\limits_{j\in\mathbb{N}}(rA_j) = r(\bigcup\limits_{j\in\mathbb{N}} A_j)$
- $\bigcap\limits_{j\in\mathbb{N}}(A_j+s) = (\bigcap\limits_{j\in\mathbb{N}} A_j)+s$ and $\bigcap\limits_{j\in\mathbb{N}}(rA_j) = r(\bigcap\limits_{j\in\mathbb{N}} A_j)$

So, by replacing all the sets from $\mathcal{A}$ in our expression for $E$ with their translated or scaled sets which are also in $\mathcal{A}$, we can show that $E+s$ or $rE$ is also in $\mathcal{B}_{\mathbb{R}}$.

Combining that with the reasoning on the previous page, we know that $m(E) = m(E+s)$ and $m(rE) = |r|m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}}$. What's left to show now is that for all sets in the completion $\mathcal{L}$ of $\mathcal{B}_{\mathbb{R}}$, we have that $E \in \mathcal{L} \implies E+s, rE \in \mathcal{L}$, $m(E+s) = m(E)$, and $m(rE) = |r|m(E)$.

This is not hard to show and I'm bored of this. So do it yourself.

Some fairly trivial observations:

1. $m(\{x\}) = 0$ for all $x \in \mathbb{R}$.
2. All countable sets have a Lesbesgue measure of zero (this includes $\mathbb{Q}$).
3. There exists uncountable sets with a Lesbesgue measure of zero (example: the Cantor set).

---

Given a collection of sets: $\mathcal{A}$, here is how to actually construct $\mathcal{M}(\mathcal{A})$.

**Bonus Proposition:** Let $\Omega$ be a minimal uncountable well-ordered set. Denoting $0$ as the least element of $\Omega$, define $\mathcal{E}_0 = \mathcal{A}$. Then for all $\alpha \in \Omega - \{0\}$, if $\alpha$ has an immediate predecessor $\beta$, define $\mathcal{E}_\alpha$ as the collection of all countable unions of sets and complements of sets in $\mathcal{E}_\beta$. Otherwise, define $\mathcal{E}_\alpha = \bigcup\limits_{\beta\in S_\alpha} \mathcal{E}_\beta$ (where $S_\alpha$ is the set $\{\beta \in \Omega \mid \beta < \alpha\}$).

Then $\mathcal{M} = \mathcal{M}(\mathcal{A}) = \bigcup\limits_{\alpha\in\Omega} \mathcal{E}_\alpha$.

Proof:
Suppose for all $\beta \in S_\alpha$, we have that $\mathcal{E}_\beta \subseteq \mathcal{M}$.

- Suppose $\alpha$ has no immediate predecessor. If $S_\alpha = \emptyset$, then we know that $\mathcal{E} = \mathcal{A} \subseteq \mathcal{M}$. Otherwise, we trivially have that:
$$\mathcal{E}_\alpha = \bigcup_{\beta\in S_\alpha} \mathcal{E}_\beta \subseteq \mathcal{M}.$$
- If $\alpha$ has a predecessor $\beta'$, then because $\mathcal{M}$ is closed under countable unions and complements and $\mathcal{E}_{\beta'} \subseteq \mathcal{M}$, we know that $\mathcal{E}_\alpha \subseteq \mathcal{M}$.

It follows by transfinite induction that $\mathcal{E}_\alpha \subseteq \mathcal{M}$ for all $\alpha \in \Omega$. So, we've shown that $\bigcup\limits_{\alpha\in\Omega} \mathcal{E}_\alpha \subseteq \mathcal{M}$.

To show the other inclusion, it suffices to show that $\bigcup_{\alpha\in\Omega} \mathcal{E}_\alpha$ is a $\sigma$-algebra (we know it contains $\mathcal{A}$ since $\mathcal{E}_0 = \mathcal{A}$).

Firstly, note that if $A \in \mathcal{E}_\alpha$ for any $\alpha \in \Omega$, then denoting $\beta$ be the immediate successor of $\alpha$, we have that $A^{\mathsf{C}} \in \mathcal{E}_\beta$.

Next, suppose $(A_j)_{j\in\mathbb{N}}$ is a sequence of sets in $\bigcup_{\alpha\in\Omega} \mathcal{E}_\alpha$ satisfying that $A_j \in \mathcal{E}_{\alpha_j}$ and $\alpha_j \in \Omega$ for all $j \in \mathbb{N}$.

Now every countable sequence in $\Omega$ has a supremum. So let $\alpha'$ be the supremum of the sequence $(\alpha_j)_{j\in\mathbb{N}}$. Next, let $\beta'$ be the least element of $\Omega$ which is both greater than $\alpha'$ and has no immediate successor. That way $\mathcal{E}_{\alpha_j} \subseteq \mathcal{E}_{\beta'}$ for all $j \in \mathbb{N}$. Finally, let $\gamma'$ be the successor of $\beta'$. Then:

$$\bigcup_{j\in\mathbb{N}} A_j \in \mathcal{E}_{\gamma'} \subseteq \bigcup_{\alpha\in\Omega} \mathcal{E}_\alpha$$

So, we've shown that $\bigcup_{\alpha\in\Omega} \mathcal{E}_\alpha$ is a $\sigma$-algebra containing $\mathcal{A}$.

It follows from this proposition that if a set is in $\mathcal{M}(\mathcal{A})$, then it can be gained through a procedure of countably many steps of taking countable unions, intersections, and complements of sets in $\mathcal{A}$.

---

**Measurable Functions:**

$f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ is called $(\mathcal{M}, \mathcal{N})$ measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

**Proposition:** If $\mathcal{N}$ is generated by $\mathcal{E}$, then $f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{M})$ is $(\mathcal{M}, \mathcal{N})$ measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Proof:
The rightward implication is trivial. Meanwhile suppose the right statement holds. Then consider the collection $\{E \subseteq Y \mid f^{-1}(E) \in \mathcal{M}\}$. By assumption we know it contains $\mathcal{E}$.

Firstly, this collection is closed under countable unions because:

$$f^{-1}(\bigcup_{j\in\mathbb{N}} A_j) = \bigcup_{j\in\mathbb{N}} f^{-1}(A_j).$$

Secondly, this collection is closed under complements. To prove this, first note that since $\emptyset \in \mathcal{E}$, we have that $Y - \emptyset = Y$ is a finite union of sets in $\mathcal{E}$. It follows then that $F^{-1}(Y) \in \mathcal{M}$. Next consider any $E \subseteq Y$ satisfying that $f^{-1}(E) \in \mathcal{M}$. Then $f^{-1}(E^{\mathsf{C}}) = f^{-1}(Y) - f^{-1}(E) \in \mathcal{M}$.

So, we've shown that the collection $\{E \subseteq Y \mid f^{-1}(E) \in \mathcal{M}\}$ is a $\sigma$-algebra containing $\mathcal{E}$. It follows that it contains $\mathcal{N} = \mathcal{M}(\mathcal{E})$.

**Corollary:** If $X, Y$ are topological spaces, then every continuous function $f : X \longrightarrow Y$ is $(\mathcal{B}_X, \mathcal{B}_y)$ measurable.

We say a function $f : (X, \mathcal{M}) \longrightarrow \mathbb{R}$ (or $\mathbb{C}$ or $\mathbb{R}^n$) is $\mathcal{M}$-measurable if it is $(\mathcal{M}, \mathcal{B}_\mathbb{R}$ (or $\mathcal{B}_\mathbb{C}$ or $\mathcal{B}_{\mathbb{R}^n}$)) measurable. If it's obvious what $\mathcal{M}$ is, we can omit saying it.

In other words, the measurable sets of the range of a real or complex function is assumed to be the collection of Borel sets.

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Lesbesgue measurable if $f$ is $(\mathcal{L}, \mathcal{B}_\mathbb{R})$ measurable.

Meanwhile, $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Borel measurable if $f$ is $(\mathcal{B}_\mathbb{R}, \mathcal{B}_\mathbb{R})$ measurable.

# Lecture 7 Notes: 10/17/2024

**Exercise:** Suppose $f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ is $(\mathcal{M}, \mathcal{N})$ measurable and $g : (Y, \mathcal{N}) \longrightarrow (Z, \mathcal{O})$ is $(\mathcal{N}, \mathcal{O})$ measurable. Then $g \circ f$ is $(\mathcal{M}, \mathcal{O})$ measurable.

Proof:
Given any $E \in \mathcal{O}$, because $g$ is measurable, we know that $g^{-1}(E) \in \mathcal{N}$. Therefore, because $f$ is measurable, we know $f^{-1}(g^{-1}(E)) = (g \circ f)^{-1}(E) \in \mathcal{M}$.

**Proposition:** Suppose $(X, \mathcal{M})$ is a measurable space. Given $f : X \longrightarrow \mathbb{R}$, the following are equivalent:

- $f$ is measurable.
- $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
- $f^{-1}([a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
- $f^{-1}((-\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
- $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

Proof:
It's trivial that the first bullet point implies the others. Meanwhile, to go from any other bullet point back to the first, recall exercise 1.2 and the proposition on the bottom of page 39.

Suppose $(Y_\alpha, \mathcal{N}_\alpha)_{\alpha \in A}$ is a collection of measurable spaces. Then given a set $X$ and functions $f_\alpha : X \longrightarrow Y_\alpha$ for all $\alpha \in A$, there exists a smallest $\sigma$-algebra $\mathcal{M}$ on $X$ such that each $f_\alpha$ is measurable. Specifically:

$$\mathcal{M} = \mathcal{M}(\{f_\alpha^{-1}(E_\alpha) \mid \alpha \in A \text{ and } E_\alpha \in \mathcal{N}_\alpha\}).$$

Note that if $X = \prod_{\alpha \in A} Y_\alpha$ and $f_\alpha = \pi_\alpha$ for all $\alpha \in A$, then the $\mathcal{M}$ defined above is equal to $\bigotimes_{\alpha \in A} \mathcal{N}_\alpha$.

**Proposition:** Suppose $(Y_\alpha, \mathcal{N}_\alpha)_{\alpha\in A}$ is a collection of measurable spaces. Then define $Y = \prod_{\alpha\in A} Y_\alpha$ and $\mathcal{N} = \bigotimes_{\alpha\in A} \mathcal{N}_\alpha$.

We claim $f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ is $(\mathcal{M}, \mathcal{N})$ measurable if and only if $f_\alpha := \pi_\alpha \circ f$ is $(\mathcal{M}, \mathcal{N}_\alpha)$ measurable for all $\alpha \in A$.

Proof:

We already showed at the beginning of lecture that the composition of measurable functions is measurable. So, the rightward implication is obvious. Conversely, if $f_\alpha := \pi_\alpha \circ f$ is $(\mathcal{M}, \mathcal{N}_\alpha)$ measurable for all $\alpha \in A$, then given any $E_\alpha \in \mathcal{N}_\alpha$ for any $\alpha \in A$, we know that $f_\alpha^{-1}(E_\alpha) = f^{-1}(\pi_\alpha^{-1}(E_\alpha)) \in \mathcal{M}$. Now $\mathcal{N}$ is generated by the collection of all such $\pi_\alpha^{-1}(E_\alpha)$. So by the proposition on the bottom of page 39, we know that $f$ is $(\mathcal{M}, \mathcal{N})$ measurable.

**Corollary:** $f : X \longrightarrow \mathbb{C}$ is measurable if and only if $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ are measurable.

In the extended real numbers $\overline{\mathbb{R}}$, we define $\mathcal{B}_{\overline{\mathbb{R}}} = \{E \in \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}_\mathbb{R}\}$. In other words, $\mathcal{B}_{\overline{\mathbb{R}}} = \{A, A \cup \{\infty\}, A \cup \{-\infty\}, A \cup \{\infty, -\infty\} \mid A \in \mathcal{B}_\mathbb{R}\}$.

We define $f : (X, \mathcal{M}) \longrightarrow \overline{\mathbb{R}}$ to be $\mathcal{M}$-measurable if $f$ is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$ measurable.

Given a measurable space $(X, \mathcal{M})$ and set $E \in \mathcal{M}$, we say $f : X \longrightarrow \mathbb{R}$ (or $\mathbb{C}$ or $\mathbb{R}^n$) is measurable on $E$ if $f^{-1}(B) \cap E \in \mathcal{M}$ for each Borel set $B$.

**Exercise 2.1:** Suppose $(X, \mathcal{M})$ is a measurable space. Let $f : X \longrightarrow \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then $f$ is measurable if and only if $f^{-1}(\{-\infty\}) \in \mathcal{M}$, $f^{-1}(\{\infty\}) \in M$, and $f$ is measurable on $Y$.

The forward implication is trivial. After all, $\{\infty\}$ and $\{-\infty\}$ are in $\mathcal{B}_{\overline{\mathbb{R}}}$. So, $f$ being measurable implies $f^{-1}(\{-\infty\}) \in \mathcal{M}$ and $f^{-1}(\{\infty\}) \in \mathcal{M}$. Also, we know $\mathbb{R} \in \mathcal{B}_{\overline{\mathbb{R}}}$. So if $f^{-1}(B) \in \mathcal{M}$ for all $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, then:

$$f^{-1}(B) \cap Y = f^{-1}(B) \cap f^{-1}(\mathbb{R}) \in \mathcal{M} \text{ for all } B \in \mathcal{B}_{\overline{\mathbb{R}}}.$$

Now suppose $f^{-1}(\{-\infty\}), f^{-1}(\{\infty\}) \in \mathcal{M}$ and $f$ is measurable on $Y$. Then given any set $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, we have that:

$$f^{-1}(B) = f^{-1}(B \cap \{-\infty\}) \cup f^{-1}(B \cap \{\infty\}) \cup (f^{-1}(B \cap \mathbb{R}))$$

Note that: $f^{-1}(B \cap \mathbb{R}) = f^{-1}(B) \cap Y$. So, since we know that $f^{-1}(B \cap \{-\infty\})$, $f^{-1}(B \cap \{\infty\})$, and $(f^{-1}(B \cap \mathbb{R}))$ are in $\mathcal{M}$ for all $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, we thus know $f^{-1}(B) \in \mathcal{M}$ for all $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, meaning $f$ is measurable.

---

Now it's a bit frustrating but I don't think I have enough knowledge of topology right now to show this all fully rigorously (I'm running into the problem that $\overline{\mathbb{R}}$ is not a metric space). So right now I don't have a working definition of what it means for a subset of $\overline{\mathbb{R}}$ to be open in this context. I'll just leave these as notes and maybe in the future I'll come back and prove them.

Also neither the professor nor Folland seem to care...

**Lemma 1:** The collections $\{(a, \infty] \mid a \in \mathbb{R}\}$ and $\{[\infty, a) \mid a \in \mathbb{R}\}$ are both bases of the $\sigma$-algebra $\mathcal{B}_{\overline{\mathbb{R}}}$.

**Lemma 2:** Suppose $(X, \mathcal{M})$ is a measurable space. Given $f : X \longrightarrow \overline{\mathbb{R}}$, the following are equivalent:

- $f$ is measurable.
- $f^{-1}((a, \infty]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
- $f^{-1}([-\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

---

**Proposition:** Suppose $f, g : X \longrightarrow \mathbb{C}$ are measurable. Then $f + g$ and $fg$ are measurable.

Proof:
Define $F(x) = (f(x), g(x))$. Then $F$ is $(X, \mathcal{B}_{\mathbb{C}^2} = \bigotimes_{i=1}^{2} \mathcal{B}_{\mathbb{C}})$ measurable by the above proposition.

Also, defining $G(z, w) = z + w$ and $H(z, w) = zw$ for all $z, w \in \mathbb{C}$, we have that $G$ and $H$ are continuous functions and thus $(\mathcal{B}_{\mathbb{C}^2}, \mathcal{B}_{\mathbb{C}})$ measurable. Since the composition of measurable functions is also measurable, this proposition follows.

**Proposition:** Suppose $(f_j)_{j \in \mathbb{N}}$ is a sequence of measurable functions from $(X, \mathcal{M})$ to $\overline{\mathbb{R}}$. Then the following functions are measurable.

$$g_1(x) = \sup_{j \in \mathbb{N}}(f_j(x)), \qquad g_2(x) = \inf_{j \in \mathbb{N}}(f_j(x)),$$
$$g_3(x) = \limsup_{j \to \infty}(f_j(x)), \quad g_4(x) = \liminf_{j \to \infty}(f_j(x)).$$

Proof:
Note that $g_1^{-1}((a, \infty]) = \bigcup_{j=1}^{\infty} f_j^{-1}((a, \infty])$ and $g_2^{-2}([-\infty, a)) = \bigcup_{j=1}^{\infty} f_j^{-1}([-\infty, a))$.

I'll show where the first of those equalities come from. Showing the other would be basically the same.

Suppose $x \in \bigcup_{j=1}^{\infty} f_j^{-1}((a,\infty])$. Then there exists $f_j$ with $f_j(x) > a$. It follows that $g_1(x) \geq f_j(x) > a$ and thus $x \in g_1^{-1}((a,\infty])$. So:

$$\bigcup_{j=1}^{\infty} f_j^{-1}((a,\infty]) \subseteq g_1^{-1}((a,\infty]).$$

On the other hand, suppose $x \in g_1^{-1}((a,\infty])$. We'll first address if $x \neq \infty$. Then letting $0 < \varepsilon < x - a$, we know there exists $f_j$ in the sequence such that $f_j(x) \geq g(x) - \varepsilon$. It follows that $x \in \bigcup_{j=1}^{\infty} f_j^{-1}((a,\infty])$. On the other hand, if $x = \infty$, then we know there exists $f_j$ satisfying $f_j(x) > M$ for any $M \in \mathbb{R}$. Setting $M = a$, we get that $x \in \bigcup_{j=1}^{\infty} f_j^{-1}((a,\infty])$. In both cases, we've thus shown that:

$$g_1^{-1}((a,\infty]) \subseteq \bigcup_{j=1}^{\infty} f_j^{-1}((a,\infty]).$$

Since all $f_j$ are measurable, it follows that $g_1^{-1}((a,\infty])$ and $g_2^{-1}([\infty,a))$ are in $\mathcal{M}$ for all $a \in \mathbb{R}$. By our lemma on the previous page, we thus have that $g_1$ and $g_2$ are measurable.

To show that $g_3$ is measurable, define $h_k(x) = \sup_{n \geq k}(f_n(x))$ for all $k \in \mathbb{N}$. Then note that $g_3(x) = \inf_{k \in \mathbb{N}}(h_k(x))$.

By our previous reasoning about $g_1$, we know all $h_k$ are measurable functions. Hence, it follows that $g_3$ is measurable by the same reasoning that $g_2$ is measurable. An analogous argument shows that $g_4$ is measurable.

Also if $f(x) = \lim_{j \to \infty}(f_j(x))$ exists for all $x \in X$, then $f(x)$ is measurable. This is because $f(x)$ existing for all $x \in X$ means that $f(x) = g_3(x) = g_4(x)$.

**Corollary 1:** If $f, g : X \longrightarrow \overline{\mathbb{R}}$ are measurable, then $\max(f,g)$ and $\min(f,g)$ are measurable.

**Corollary 2:** If $(f_j)_{j \in \mathbb{N}}$ is a sequence of measurable functions from $X$ to $\mathbb{C}$, and $f(x) = \lim_{j \to \infty}(f_j(x))$ for all $x \in X$, then $f$ is measurable.

To show this, apply the corollary on page 41 to consider the components of $f$ and of each $f_j$. Then use the previous proposition.

Here are two important decompositions of functions:

1. Given a function $f : X \longrightarrow \overline{\mathbb{R}}$, define $f^+(x) := \max(f(x), 0)$ and $f^-(x) := \max(-f(x), 0)$. Then $f = f^+(x) - f^-(x)$ and $|f| = f^+(x) + f^-(x)$.

2. Given $f : X \longrightarrow \mathbb{C}$, we define it's polar decomposition:

$$f(z) = \operatorname{sgn}(f(z))|f(z)| \text{ where } \operatorname{sgn}(z) = \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

A function $f : X \longrightarrow \mathbb{C}$ is called simple if it is measurable and has a finite image. A standard representation of $f$ is given by:

$$\sum_{j=1}^{n} z_j \chi_{E_j}$$

...where $E_j = f^{-1}(\{z_j\})$ and $\chi_E = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$

**Observation:** If $f, g : X \longrightarrow \mathbb{C}$ are simple functions, then so are $f + g$ and $fg$.

**Theorem:** Let $(X, \mathcal{M})$ be a measurable space.

1. If $f : X \longrightarrow [0, \infty]$ be a measurable function, then there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ of simple functions such that $0 \leq \phi_1 \leq \ldots \leq f$ and $\phi_n \to f$ pointwise as $n \to \infty$. If $f$ is bounded, then $\phi_n \to f$ uniformly.

2. If $f : X \longrightarrow \mathbb{C}$ is measurable, then there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ of simple functions such that $0 \leq |\phi_1| \leq \ldots \leq |f|$ and $\phi_n \to f$ pointwise. If $f$ is bounded, then $\phi_n \to f$ uniformly.

Proof:

1. For all $n \in \mathbb{N}$, define $\phi_n$ as follows:

   Define $E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-n}])$ for all $k \in \{0, \ldots, 2^{2n} - 1\}$. Also define $F_n = f^{-1}((2^n, \infty])$. Then finally, set:

$$\phi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_n^k} + 2^m \chi_{F_m}$$

   To check that $\phi_n \leq \phi_{n+1}$, first consider the case that $f(x) \leq 2^n$. Note that we can rewrite the expression $\phi_n(x) = k2^{-n} < f(x) \leq (k+1)2^{-n}$ as:

$$(2k)2^{-(n+1)} < f(x) \leq (2k+2)2^{-(n+1)}.$$

   But then note that $\phi_{n+1}(x)$ is either $(2k)2^{-(n+1)}$ or $(2k+1)2^{-(n+1)}$. So, $\phi_{n+1}(x) \geq \phi_n(x)$. As for if $f(x) > 2^n$, then note that $\phi_{n+1}(x) \geq 2^n = \phi_n(x)$ because $E_{n+1}^{(2^{2n+1})} = f^{-1}(((2^{2n+1})2^{-(n+1)}, (2^{2n+1}+1)2^{-(n+1)}])$

   Additionally, note that $\phi_n(x) \leq f(x)$ for all $x \in X$. And, when $f(x) < 2^n$, we clearly have that $|f(x) - \phi_n(x)| < 2^{-n}$. So it's clear that $(\phi_n)_{n\in\mathbb{N}}$ meets our convergence requirements.

2. Given $f(x) = g(x) + ih(x)$, apply part 1 of this theorem to $g^+$, $g^-$, $h^+$, and $h^-$ to get the sequences $(\psi_n^+)$, $(\psi_n^-)$, $(\zeta_n^+)$, and $(\zeta_n^-)$ respectively of simple functions. Then letting $\phi_n = (\psi_n^+ - \psi_n^-) + i(\zeta_n^+ - \zeta_n^-)$ for all $n$ satisfies our theorem.

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We've mostly been working without regard to a particular measure. Here are some theorems which do require that we are working with a complete measure space $(X, \mathcal{M}, \mu)$:

**Exercise 2.10:** Prove the following are true if and only if $(X, \mathcal{M}, \mu)$ is complete:

(a) Suppose $f$ is measurable and $f = g$ $\mu$-a.e., then $g$ is measurable.

Note that if we say $f$ is "measurable", it's implied that $f$ is a function to $\mathbb{R}$, $\overline{\mathbb{R}}$, $\mathbb{C}$, or $\mathbb{R}^n$. We'll work here in $\mathbb{R}$ (and $\overline{\mathbb{R}}$) since this can all be extended to $\mathbb{C}$ and $\mathbb{R}^n$ by working component wise.

Proof:
($\Longrightarrow$)
Suppose $(X, \mathcal{M}, \mu)$ is complete. Then given such an $f$ and $g$ above, we know $\{x \mid f(x) \neq g(x)\}$ is a subset of a null set, meaning that all it's subsets are measurable. Next note that given any $a \in \mathbb{R}$, we have that:

$$\{x \mid g(x) > a\} = \{x \mid g(x) > a, f(x) = g(x)\} \cup \{x \mid g(x) > a, f(x) \neq g(x)\}.$$

Importantly, $\{x \mid g(x) > a, f(x) \neq g(x)\}$ is measurable since it's a subset of $\{x \mid f(x) \neq g(x)\}$. On the other hand, note that:

$$\{x \mid g(x) > a, f(x) = g(x)\} = \{x \mid f(x) > a\} - \{x \mid f(x) > a, f(x) \neq g(x)\}$$

Thus $\{x \mid g(x) > a, f(x) = g(x)\}$ is measurable because $f$ is a measurable function and $\{x \mid f(x) > a, f(x) \neq g(x)\}$ is a subset of $\{x \mid f(x) \neq g(x)\}$. Hence, we've shown that $\{x \mid g(x) > a\}$ is measurable for all $a \in \mathbb{R}$, meaning $g$ is measurable.

($\Longleftarrow$)
Let $N$ be a null set of $(X, \mathcal{M}, \mu)$ and $F \subseteq N$. Then, let $f = \chi_N$ and $g = \chi_F$. Clearly, $f(x) = g(x)$ for all $x \in X - N$. And looking back at the definition of something being true $\mu$-a.e., that means that $f(x) = g(x)$ $\mu$-a.e. So by assumption, we know $g$ is measurable, meaning that $g^{-1}(1) = F \in \mathcal{M}$.

(b) If $f_n$ is measurable for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} f_n(x) = f(x)$ $\mu$-a.e., then $f$ is measurable.

($\Longrightarrow$)
Given $a \in \mathbb{R}$, note that:

$$\{x \mid f(x) > a\} = \{x \mid f(x) > a, f_n(x) \to f(x)\} \cup \{x \mid f(x) > a, f_n(x) \not\to f(x)\}.$$

The first of those two sets is measurable by the proposition on pages 42 and 43. The second is a subset of a null set. So since $(X, \mathcal{M}, \mu)$ is complete, it follows that $\{x \mid f(x) > a\} \in \mathcal{M}$.

$(\Longleftarrow)$
Let $N$ be a null set of $(X, \mathcal{M}, \mu)$ and $F \subseteq N$. Then define $f_n = 0$ for all $n \in \mathbb{N}$ and $f = \chi_F$. Then $f_n(x) \to f(x)$ for all $x \in N^{\mathsf{C}}$, meaning $f_n(x) \to f(x)$ $\mu$-a.e. So by assumption, $f$ is measurable, meaning that $f^{-1}(1) = F \in \mathcal{M}$.

**Proposition:** If $(X, \mathcal{M}, \mu)$ is a measure space and $(X, \overline{\mathcal{M}}, \overline{\mu})$ is its completion. If $f$ is $\overline{\mathcal{M}}$-measurable, then there exists $g$ that is $\mathcal{M}$-measurable such that $f = g$ $\overline{\mu}$-a.e.

If $f = \chi_E$ for $E \in \mathcal{M}$, then this is obvious.

$E = E' \cup F'$ where $E' \in \mathcal{M}$ and $F' \subseteq N' \in \mathcal{M}$ with $\mu(N') = 0$. So define $g = \chi_{E'}$. Then $g$ is $\mathcal{M}$-measurable and $f(x) = g(x)$ for all $x \in (N')^{\mathsf{C}}$. So $f(x) = g(x)$ $\mu$-a.e. and thus also $\overline{\mu}$-a.e.

Now let $f$ be any $\overline{\mathcal{M}}$-measurable function and let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of $\overline{\mathcal{M}}$-measurable simple functions converging pointwise to $f$. Next, for each $n$, let $\psi_n$ be an $\mathcal{M}$-simple function satisfying that $\psi_n(x) = \phi_n(x)$ except on a null set $E_n$ of $\overline{\mathcal{M}}$.

Remember above definition for why we can do that.

Choose a $\mu$-null set $N \in \mathcal{M}$ such that $\bigcup_{n\in\mathbb{N}} E_n \subseteq N$. Then define $g = \lim_{n\to\infty} \chi_{N^{\mathsf{C}}} \psi_n(x)$.

By the second corollary on page 43, we thus know that $g$ is $\mathcal{M}$-measurable with $g(x) = f(x)$ for all $x \in N^{\mathsf{C}}$.

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# Lecture 8 Notes: 10/22/2024

**Proposition:** If $f \in L^+$ and $\int f < \infty$, then $\{x \mid f(x) = \infty\}$ is a null set and $\{x \mid f(x) > 0\}$ is $\sigma$-finite.

The proof is **Exercise 2.12**:
Firstly, suppose $\mu(\{x \mid f(x) = \infty\}) = \alpha > 0$. Then for all $n \in \mathbb{N}$, we can define: $\phi_n = n\chi_{\{x|f(x)=\infty\}}$. Thus, $(\phi_n)$ is an increasing sequence of simple functions less than $f$ satisfying that $\int f \geq \int \phi_n = n\alpha$ for all $n$. It follows that $\int f \geq \lim_{n\to\infty}(n\alpha) = \infty$, meaning that $\int f = \infty$.

Hence, we've shown that $\int f < \infty \Longrightarrow \mu(\{x \mid f(x) = \infty\}) = 0$.

Next, suppose that for some $\beta > 0$, we have that $\mu(\{x \mid f(x) > \beta\}) = \infty$. Then $\phi = \beta\chi_{\{x|f(x)>\beta\}}$ is a simple function less than $f$ satisfying that:

$$\infty = \beta\infty = \int \phi \leq \int f.$$

It follows that $\int f < \infty \Longrightarrow \mu(\{x \mid f(x) > \beta\}) < \infty$ for all $\beta > 0$.

Finally, note that $\{x \mid f(x) > 0\} = \bigcup_{n\in\mathbb{N}} \{x \mid f(x) > \frac{1}{n}\}$ where each $\mu(\{x \mid f(x) > \frac{1}{n}\}) < \infty$.

So, $\{x \mid f(x) > 0\}$ is $\sigma$-finite.

**Exercise 2.13:** Suppose $(f_n)_{n\in\mathbb{N}} \subset L^+$, $f_n \to f$ pointwise, and $\int f = \lim_{n\to\infty} \int f_n < \infty$. Then $\int_E f = \lim_{n\to\infty} \int_E f_n$ for all $E \in \mathcal{M}$.

Firstly let $E \in \mathcal{M}$. Then $\int f\chi_E + \int f\chi_{E^\mathsf{C}} = \int f$ and $\int f_n\chi_E + \int f_n\chi_{E^\mathsf{C}} = \int f_n$ for all $n \in \mathbb{N}$. Based on that, we can show:

$$\limsup_{n\to\infty} \int f_n\chi_E + \liminf_{n\to\infty} \int f_n\chi_{E^\mathsf{C}} = \lim_{n\to\infty} \int f_n = \int f$$

To see this, consider any subsequence $(\int f_{n_k}\chi_E)_{k\in\mathbb{N}}$ converging to $\limsup_{n\to\infty} \int f_n\chi_E$.

Because $\int f$ is finite, we know there is $N_1 \in \mathbb{N}$ such that $\int f_n < \infty$ for all $n > N_1$. In turn, we know that $\int f_n\chi_E < \infty$ for all $n > N_1$ and thus $\limsup_{n\to\infty} \int f_n\chi_E < \infty$.

Now since $(\int f_{n_k}\chi_E)_{k\in\mathbb{N}}$ and $(\int f_n)_{n\in\mathbb{N}}$ have finite limits, given any $\varepsilon > 0$, we can pick $N_2 \in \mathbb{N}$ greater than $N_1$ such that for all $k \geq N_2$, we have that:

$$|\int f_{n_k}\chi_E - \limsup_{n\to\infty} \int f_n\chi_E| < {}^{\varepsilon}/_{2} \text{ and } |\int f_{n_k} - \int f| < {}^{\varepsilon}/_{2}$$

It follows that because $\int f_{n_k} - \int f_{n_k}\chi_E = \int f_{n_k}\chi_{E^\mathsf{C}}$, we have that:

$$(\int f - \limsup_{n\to\infty} \int f_n\chi_E) - \varepsilon < \int f_{n_k}\chi_{E^\mathsf{C}} < (\int f - \limsup_{n\to\infty} \int f_n\chi_E) + \varepsilon$$

So $\lim_{k\to\infty} \int f_{n_k}\chi_{E^\mathsf{C}}$ exists and:

$$\liminf_{n\to\infty} \int f_n\chi_{E^\mathsf{C}} \leq \lim_{k\to\infty} \int f_{n_k}\chi_{E^\mathsf{C}} = \int f - \limsup_{n\to\infty} \int f_n\chi_E.$$

Meanwhile, consider any subsequence $(\int f_{n_j}\chi_{E^{\mathsf{C}}})_{j\in\mathbb{N}}$ converging to $\liminf\limits_{n\to\infty}\int f_n\chi_{E^{\mathsf{C}}}$.

By analogous reasoning to before, we know that $\liminf\limits_{n\to\infty}\int f_n\chi_{E^{\mathsf{C}}}$ is finte.

Also similarly to before, we can show that $\lim\limits_{j\to\infty}\int f_{n_j}\chi_E$ exists and:

$$\int f - \liminf_{n\to\infty}\int f_n\chi_{E^{\mathsf{C}}} = \lim_{j\to\infty}\int f_{n_j}\chi_E \leq \limsup_{n\to\infty}\int f_n\chi_E$$

Finally, by rearranging terms, we get that:

$$\int f \leq \limsup_{n\to\infty}\int f_n\chi_E + \liminf_{n\to\infty}\int f_n\chi_{E^{\mathsf{C}}} \leq \int f.$$

Next, note that $f_n\chi_E \to f\chi_E$ and $f_n\chi_{E^{\mathsf{C}}} \to f\chi_{E^{\mathsf{C}}}$ pointwise as $n\to\infty$. Thus by Fatou's lemma, we immediately have that:

$$\int f\chi_E = \int \lim_{n\to\infty}(f_n\chi_E) \leq \liminf_{n\to\infty}\int f_n\chi_E$$

Also, more round-aboutly we get that:

$$\int f\chi_E = \int f - \int f\chi_{E^{\mathsf{C}}} = \int f - \int \lim_{n\to\infty} f_n\chi_{E^{\mathsf{C}}}$$
$$\geq \int f - \liminf_{n\to\infty}\int f_n\chi_{E^{\mathsf{C}}} = \limsup_{n\to\infty}\int f_n\chi_E$$

Thus, $\int f\chi_E \leq \liminf\limits_{n\to\infty} f_n\chi_E \leq \limsup\limits_{n\to\infty} f_n\chi_E \leq \int f\chi_E$.

And so, we have shown that $\lim\limits_{n\to\infty}\int f_n\chi_E$ exists and that $\int f\chi_E = \lim\limits_{n\to\infty}\int f_n\chi_E$

However, this need not be true if $\int f = \lim\limits_{n\to\infty}\int f_n = \infty$.

Suppose $(X,\mathcal{M},\mu) = (\mathbb{R},\mathcal{L},m)$ and then define $f_n = \chi_{(-\infty,0)} + \chi_{(n,n+1]}$ for all $n\in\mathbb{N}$ and $f = \chi_{(-\infty,0)}$. Then note that $f_n \to f$ pointwise as $n\to\infty$. Also, for all $n\in\mathbb{N}$ we have that $\int f = \infty = \int f_n$. However, $E = [0,\infty)\in\mathcal{L}$ and $\int_E f = 0$ while $\int_E f_n = 1$ for all $n$. So, $\int_E f \neq \lim\limits_{n\to\infty}\int_E f_n$.

**Exercise 2.14:** If $f\in L^+$, let $\lambda(E) = \int_E f d\mu$ for $E\in\mathcal{M}$. Then $\lambda$ is a measure on $\mathcal{M}$ and for any $g\in L^+$, $\int g d\lambda = \int fg d\mu$.

To start, we show $\lambda$ is measure. Clearly, $\lambda(\emptyset) = \int_\emptyset f d\mu = \int f\chi_\emptyset d\mu = \int 0 d\mu = 0$. Meanwhile, suppose $(E_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{M}$ whose union is $E$.

Note that $\chi_E = \sum\limits_{n=0}^{\infty}\chi_{E_n}$. So:

$$\lambda(E) = \int_E f d\mu = \int f\chi_E d\mu = \int\left(\sum_{n=0}^{\infty} f\chi_{E_n}\right)d\mu = \sum_{n=0}^{\infty}\int f\chi_{E_n}d\mu = \sum_{n=0}^{\infty}\lambda(E_n)$$

Hence, we've shown that $\lambda$ is a measure on $\mathcal{M}$.

Now, in order to show that $\int g d\lambda = \int f g d\mu$ for all $g \in L^+$, let's first consider a simple function $\phi \in L^+$.

Suppose $\phi = \sum_{k=1}^{N} x_k \chi_{E_k}$ where each $E_k$ is disjoint. Then:

$$\int \sum_{k=1}^{N} \chi_{E_k} d\lambda = \sum_{k=1}^{N} x_k \lambda(E_k)$$
$$= \sum_{k=1}^{N} x_k \int f \chi_{E_k} d\mu = \sum_{k=1}^{N} \int x_k f \chi_{E_k} d\mu$$
$$= \int \sum_{k=1}^{N} x_k f \chi_{E_k} d\mu = \int f \sum_{k=1}^{N} x_k \chi_{E_k} d\mu$$

Hence, $\int \phi d\lambda = \int f \phi d\mu$.

To extend this to any function $g \in L^+$, let $(\phi_n)_{n \in \mathbb{N}}$ be an increasing sequence of simple functions such that $\phi_n \to g$ pointwise as $n \to \infty$. Importantly, we also have that $f\phi_n \to fg$ pointwise as $n \to \infty$ with $f\phi_n \leq f\phi_{n+1}$ for all $n \in \mathbb{N}$. So, applying the monotone convergence theorem, we have that:

$$\int g d\lambda = \lim_{n \to \infty} \left( \int \phi_n d\lambda \right) = \lim_{n \to \infty} \left( \int \phi_n f d\mu \right) = \int f g d\mu$$

**Exercise 2.25:** Let $f(x) = x^{-1/2}$ if $0 < x < 1$ and $f(x) = 0$ otherwise. Let $(r_n)_{n\in\mathbb{N}}$ be an enumeration of the rational numbers, and set $g(x) = \sum\limits_{n=1}^{\infty} 2^{-n} f(x - r_n)$.

(a) $g \in L^1(m)$ and in particular $g < \infty$ a.e.

Note that $\int |2^{-n} f(x - r_n)| dm = 2^{-n} \int |f(x - r_n)| dm$ since $2^{-n} > 0$. Next, note that by the monotone convergence theorem, if $(a_k)_{k\in\mathbb{N}}$ is any decreasing sequence of numbers converging to $r_n$, then:

$$\begin{aligned}\int |f(x-r_n)|dm &= \lim_{k\to\infty} \int |f(x-r_n)|\chi_{\{x|x>a_k\}}dm \\ &= \lim_{k\to\infty} \int_{a_k}^{r_n+1} |f(x-r_n)|dx \\ &= \lim_{a\to r_n+} \int_a^{r_n+1} f(x-r_n)dx \\ &= \lim_{a\to 0+} \int_a^1 f(x)dx = \lim_{a\to 0+} \int_a^1 x^{-1/2}dx = \lim_{a\to 0+} (2x|_a^1) = 2\end{aligned}$$

It follows that:

$$\int \sum_{n=1}^{\infty} |2^{-n} f(x-r_n)|dm = \sum_{n=1}^{\infty} \int |2^{-n} f(x-r_n)|dm = \sum_{n=1}^{\infty} 2^{-n}\cdot 2 = 1.$$

So by theorem 2.25, we know that $\sum\limits_{n=1}^{\infty} 2^{-n} f(x-r_n) \in L^1(m)$.

By an earlier exercise, since $\sum\limits_{n=1}^{\infty} 2^{-n} f(x-r_n) \geq 0$ for all $x$, we know that it has a finite integral if and only if it equals infinity on a null set. Hence, it follows that:

$$g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x-r_n) < \infty \text{ almost everywhere.}$$

(b) $g$ is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lesbesgue null set.

To start, let's fix $x \in \mathbb{R}$ and $\delta \in (0,1)$. Then consider any $N > 0$.

Note that there exists $r_n \in (r_n)_{n\in\mathbb{N}}$ such that $x < r_n < x + \delta$. Then, note that if $r_n < x < \min(r_n + \frac{1}{N^2}2^{-2n}, r_n + 1)$, we have that $2^{-n} f(x - r_n) > N$. Since $x + \delta < r_n + 1$, we thus know that $g(s) > N$ for all $s$ in

$$(r_n, \min(x+\delta, r_n + \tfrac{1}{N^2}2^{-2n})) \subseteq (x, x+\delta).$$

Also note that we can pick $s$ in that interval such that $g(s)$ is finite. Furthermore, if we change $g$ on a null set $N$, then we can also select $s$ to be not in $N$.

To see this, first note that $m(r_n, \min(r_n + N^2 2^{2n}, x + \delta))$ is an open interval and thus has positive measure. Also, by the completeness of $m$, we know that $\{x \mid g(x) = \infty\}$ is a subset of a null set and thus itself a null set.

Finally, suppose we have another null set $N$. Then taking the difference of the interval with the two null sets sets must give a set with the positive measure of the original interval. And thus the difference can't be the empty set which has a measure of zero.

It immediately follows that even after modifying $g$ on a null set, we still have that $g$ is unbounded on all non-singleton intervals since we can fix $x$ to be any interior point of the interval and $\delta$ such that $(x, x+\delta)$ is a subset of that interval.

Also, if $g(x)$ is finite and $\varepsilon > 0$. Then we've shown that even if we modify $g$, we still have that for all $\delta > 0$, there exists $s \in (x-\delta, x+\delta)$ with $g(x) + \varepsilon < g(s) < \infty$. So $g$ is not continuous at $x$.

We haven't defined continuity yet for when $g$ can equal $\infty$ on an interval. But here's my current intuition on what being continuous at $x$ when $g(x) = \infty$ should mean.

Finally if $g(x) = \infty$, then by the previous reasoning, even if we modify $g$ on a null set, we can still find $s$ in any neighborhood of $x$ such that $g(s) < \infty$. Thus, $g$ is not continuous at $x$.

(c) $g^2 < \infty$ a.e. but $g^2$ is not integrable on any interval.

We know $g^2 < \infty$ a.e. because $g < \infty$ a.e.

Now note that since $2^{-n} f(x - r_n) \geq 0$ for all $n \in \mathbb{N}$, we can say that for all $N \in \mathbb{N}$:

$$\sum_{n=1}^{N} (2^{-n} f(x-r_n))^2 \leq \left( \sum_{n=1}^{N} 2^{-n} f(x-r_n) \right)^2$$

Taking $N \to \infty$, we thus get that $\sum\limits_{n=1}^{\infty} (2^{-n} f(x-r_n))^2 \leq (g(x))^2$.

Also, given any interval $I$, we can multiply our above inequality by $\chi_I$ and it won't flip because $\chi_I \geq 0$. Then as $(2^{-n} f(x-r_n))^2 \chi_I \geq 0$ for all $n \in \mathbb{N}$, we can thus conclude that:

$$\sum_{n=1}^{\infty} \int_I (2^{-n} f(x-r_n))^2 dm = \int_I \sum_{n=1}^{\infty} (2^{-n} f(x-r_n))^2 dm \leq \int_I (g(x))^2 dm$$

But now fix $n$ such that there exists $\varepsilon \in (0,1)$ with $[r_n - \varepsilon, r_n + \varepsilon] \subseteq I$. Then note that:

$$\begin{aligned} \int_I (2^{-n} f(x-r_n))^2 dm &\geq 2^{-2n} \cdot \lim_{a \to r_n +} \int_a^{r_n+\varepsilon} (f(x-r_n))^2 dx \\ &= 2^{-2n} \cdot \lim_{a \to r_n +} \int_a^{r_n+\varepsilon} \tfrac{1}{x-r_n} dx \\ &= 2^{-2n} \cdot \lim_{a \to r_n^+} (\log(\varepsilon) - \log(a - r_n)) = \infty \end{aligned}$$

So $\int_I (2^{-n} f(x - r_n))^2 dm = \infty$. Hence, so does $\int_I (g(x))^2 dm = \infty$. And thus, $(g(x))^2$ is not integrable on the interval $I$.

**Exercise 2.26:** If $f \in L^1(m)$ and $F(x) = \int_{-\infty}^{x} f(t)dt$, then $F$ is continuous on $\mathbb{R}$.

Let $(x_k)_{k\in\mathbb{N}}$ be any sequence of points converging to a given point $x$. Importantly, $f \in L^1(m)$ implies that $\int |f| < \infty$. Also, we have that $|f\chi_{(-\infty,x_k)}| \leq |f|$ for all $k \in \mathbb{N}$. Plus, $f\chi_{(-\infty,x_k)} \to f\chi_{(-\infty,x)}$ as $k \to \infty$.

So, by applying the dominated convergence theorem, we have that:

$$\lim_{k\to\infty} (F(x_k)) = \lim_{k\to\infty} (\int f\chi_{(-\infty,x_k)}) = \int f\chi_{(-\infty,x)} = F(x)$$

Hence, $\lim\limits_{t\to x}(F(t)) = F(x)$ and thus $F$ is continuous at $x$.

**Exercise 2.20: Generalized Dominated Convergence Theorem** If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \leq g_n$, and $\int g_n \to \int g$. Then $\int f_n \to \int f$.

The most murky part of this theorem was saying that we could assume $f$ is measurable. Luckily, the problem statement tells me out right that all the functions are measurable so I don't need to struggle to justify that.

Now by seperating the functions into their real and imaginary parts, we can without loss of generality assume $f_n$ and $f$ are real-valued. Also, the problem statement doesn't make sense if the $g_n$ aren't real-valued.

Since $|f_n| < g_n$ for all $n$, we know that $g_n + f_n > 0$ and $g_n - f_n > 0$ for all $n$. Hence, applying Fatou's lemma we know that:

- $\int g + \int f = \int \lim\limits_{n\to\infty}(g_n + f_n) = \int \liminf\limits_{n\to\infty}(g_n + f_n) \leq \liminf\limits_{n\to\infty} \int (g_n + f_n)$
- $\int g - \int f = \int \lim\limits_{n\to\infty}(g_n - f_n) = \int \liminf\limits_{n\to\infty}(g_n - f_n) \leq \liminf\limits_{n\to\infty} \int (g_n - f_n)$

Next, here's a lemma. If $(a_n), (b_n)$ are sequences of real numbers satisfying that $\lim\limits_{n\to\infty} a_n \to a \in \mathbb{R}$ and $\liminf\limits_{n\to\infty} b_n = \beta \in \overline{\mathbb{R}}$, then $\liminf\limits_{n\to\infty}(a_n + b_n) = a + \beta$.

Proof:
Let $(a_{n_k} + b_{n_k})$ be a subsequence of $(a_n + b_n)$ converging to $\liminf\limits_{n\to\infty}(a_n + b_n)$. Then because $a_{n_k} \to a$ as $n \to \infty$, we must have that $b_{n_k} \to \liminf\limits_{n\to\infty}(a_n + b_n) - a \geq \beta$. In other words, $\liminf\limits_{n\to\infty}(a_n + b_n) \geq a + \beta$

On the other hand, let $(b_{n_j})$ be a subsequence of $(b_n)$ converging to $\beta$. Then $a_{n_j} + b_{n_j} \to a + \beta \geq \liminf\limits_{n\to\infty}(a_n + b_n)$ as $n \to \infty$.

So:

- $\liminf\limits_{n\to\infty} \int (g_n + f_n) = \liminf\limits_{n\to\infty}(\int g_n + \int f_n) = \int g + \liminf\limits_{n\to\infty} \int f_n$
- $\liminf\limits_{n\to\infty} \int (g_n - f_n) = \liminf\limits_{n\to\infty}(\int g_n - \int f_n) = \int g + \liminf\limits_{n\to\infty}(-\int f_n)$

Finally, $\liminf\limits_{n\to\infty}(-\int f_n) = -\limsup\limits_{n\to\infty} \int f_n$.

Therefore subtracting out $\int g$ which we can do because $g \in L^1$ and thus $\int g$ is finite:

- $\int g + \int f \leq \int g + \liminf\limits_{n\to\infty} \int f_n \implies \int f \leq \liminf\limits_{n\to\infty} \int f_n$
- $\int g - \int f \leq \int g + \liminf\limits_{n\to\infty}(-\int f_n) \implies \limsup\limits_{n\to\infty} \int f_n \leq \int f$

Hence $\lim\limits_{n\to\infty} \int f_n = \int f$.

**Exercise 2.21**: Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.

($\Longrightarrow$)
Note that $|f_n| \leq |f_n - f| + |f|$ for all $n$, $|f_n - f| + |f| \to 0 + |f| = |f|$ a.e., and $\int(|f_n - f| + |f|) \to \int 0 + \int |f| = \int |f|$ by assumption. Then by exercise 2.20, we know that $\lim\limits_{n\to\infty} \int |f_n| = \int \lim\limits_{n\to\infty} |f_n| = \int |f|$.

($\Longleftarrow$)
Note that $|f_n - f| \leq |f_n| + |f|$ for all $n$, $|f_n| + |f| \to 2|f|$ a.e., and by assumption $\int(|f_n| + |f|) \to \int 2|f| = \int \lim\limits_{n\to\infty}(|f_n| + |f|)$. So by exercise 2.20, we know that $\lim\limits_{n\to\infty} \int |f_n - f| = \int \lim\limits_{n\to\infty} |f_n - f| = \int 0 = 0$.

**Exercise 2.28**: Compute the following limits:

(a) $\lim\limits_{n\to\infty} \int_0^\infty (1 + {}^x\!/\!_n)^{-n} \sin({}^x\!/\!_n)dx$

Note that $|(1 + {}^x\!/\!_n)^{-n} \sin({}^x\!/\!_n)| \leq (1 + {}^x\!/\!_n)^{-n}$ for all $n \geq 1$ and $x \geq 0$ because $|\sin({}^x\!/\!_n)| \leq 1$ and $(1 + {}^x\!/\!_n)^{-n} \geq 0$ when $n$ and $x$ satisfy the above inequalities.

Next, note that $(1 + {}^x\!/\!_n)^n$ is a strictly increasing sequence for all $x > 0$.

To see this, note that because $\frac{x}{n+1} < \frac{x}{n}$, we have that:

$$(1 + \tfrac{x}{n+1})^{n+1} - (1 + \tfrac{x}{n})^n \geq (1 + \tfrac{x}{n+1})^{n+1} - (1 + \tfrac{x}{n+1})^n = (1 + \tfrac{x}{n+1})^n \cdot \tfrac{x}{n+1} > 0$$

As a result, $(1 + {}^x\!/\!_n)^{-n}$ is a strictly decreasing sequence for all $x > 0$.

Now, while $\int_0^\infty (1+x)^{-1}dx = \infty$ and so we can't use that as our upperbound, note that $(1 + {}^x\!/\!_n)^{-n} \leq (1 + {}^x\!/\!_2)^{-2}$ for all $n \geq 2$ by our above reasoning and:

$$\int_0^\infty (1 + {}^x\!/\!_2)^{-2}dx = 2\int_1^\infty u^{-2}du = 2 \lim_{N\to\infty}(\tfrac{-1}{N} + 1) = 2$$

Hence, $g(x) = (1 + \frac{x}{2})^{-2} \in L^1$ with $|f_n| \leq g$ for all $n \geq 2$.

Clearly the subsequence leaving out when $n = 1$ will still have identical limiting behavior as the full sequence including $n = 1$. So by applying D.C.T (technically to the subsequence without $n = 1$), we get that:

$$\lim_{n\to\infty} \int_0^\infty (1 + {}^x\!/\!_n)^{-n} \sin({}^x\!/\!_n)dx = \int_0^\infty (\lim_{n\to\infty} (1 + {}^x\!/\!_n)^{-n} \sin({}^x\!/\!_n))dx$$

Now $\sin$ is continuous and ${}^x\!/\!_n \to 0$ pointwise. Thus $\sin({}^x\!/\!_n) \to \sin(0) = 0$ pointwise as $n \to \infty$. At the same time, note that $(1 + {}^x\!/\!_n)^n \to e^x$ as $n \to \infty$.

Small proof:
By L'Hôpital's rule: $\lim\limits_{y\to 0} \frac{\log(1+ay)}{y} = \lim\limits_{y\to 0} \frac{a}{1+ay} = a$.
So, since $\exp$ is continuous, we have that:

$$\lim_{y\to 0}(1+ay)^{1/y} = \lim_{y\to 0} \exp(\tfrac{\log(1+ay)}{y}) = \exp(\lim_{y\to 0} \tfrac{\log(1+ay)}{y}) = \exp(a)$$

Thus $(1+{}^{x}\!/_{n})^{-n}\sin({}^{x}\!/_{n}) \to \frac{0}{e^{-x}} = 0$. And so:

$$\int_0^\infty (\lim_{n\to\infty}(1+{}^{x}\!/_{n})^{-n}\sin({}^{x}\!/_{n}))dx = \int_0^\infty 0dx = 0.$$

(b) $\lim\limits_{n\to\infty}\int_0^1 (1+nx^2)(1+x^2)^{-n}dx$

Note that $(1+nx^2)(1+x^2)^{-n}$ is a decreasing sequence of functions.

Proof: $\frac{1+(n+1)x^2}{(1+x^2)^{n+1}} - \frac{1+nx^2}{(1+x^2)^n} = \frac{1+nx^2+x^2-1-x^2-nx^2-nx^4}{(1+x^2)^{n+1}} = \frac{-nx^4}{(1+x^2)^{n+1}} \leq 0$

Also, when $n=1$, we have that $(1+nx^2)(1+x^2)^{-n} = 1$ and $\int_0^1 1dx = 1$. So, by applying D.C.T, we get that:

$$\lim_{n\to\infty}\int_0^1 (1+nx^2)(1+x^2)^{-n}dx = \int_0^1 \lim_{n\to\infty}((1+nx^2)(1+x^2)^{-n})dx$$

If $x=0$, then $(1+nx^2)(1+x^2)^{-n} = 1$ for all $n$. Meanwhile if $x \neq 0$, then $1+x>0$ and thus:

$$\frac{1+nx^2}{(1+x^2)^n} = \frac{1}{(1+x^2)^n} + x^2\frac{n}{(1+x^2)^n} \to 0 + x^2\cdot 0 = 0$$

It follows that $\int_0^1 \lim\limits_{n\to\infty}((1+nx^2)(1+x^2)^{-n})dx = \int 0dx = 0$ (we can change the inside function at the point $x=0$ without changing the value of the integral).

**Exercise 2.29**: Show that $\int_0^\infty x^n e^{-x}dx = n!$ by differentiating the equation $\int_0^\infty e^{-tx}dx = {}^{1}\!/_{t}$. Similarly show that $\int_{-\infty}^\infty x^{2n}e^{-x^2}dx = \frac{(2n)!\sqrt{\pi}}{4^n n!}$ by differentiating the equation $\int_{-\infty}^\infty e^{-tx^2}dx = \sqrt{{}^{\pi}\!/_{t}}$.

Note that for all $x\geq 0$ and $t\geq 1$, $x^k e^{-tx} \leq x^k e^{-x} \leq \frac{x^k}{\frac{x^k}{k!}+\frac{x^{k+2}}{(k+2)!}} \leq \frac{(k+2)!}{(k+2)(k+1)+x^2}$.

Similarly, for all $x\geq 0$ and $t \geq 1$, we have:

$$x^{2k}e^{-tx^2} \leq x^{2k}e^{-x^2} \leq \frac{x^{2k}}{\frac{x^{2k}}{k!}+\frac{x^{2k+2}}{(k+1)!}} = \frac{(k+1)!}{1+k+x^2}.$$

Note that these upper bounds are gotten by considering the taylor series of $e^x$.

Now importantly $\int_0^\infty \frac{1}{a^2+x^2}dx = \frac{1}{a}(\arctan({}^{x}\!/_{a}))|_0^\infty = \frac{\pi}{2a}$ for all real nonzero $a$.
Similarly, $\int_{-\infty}^\infty \frac{1}{a^2+x^2}dx = \frac{1}{a}(\arctan({}^{x}\!/_{a}))|_{-\infty}^\infty = \frac{\pi}{a}$

Thus, $\frac{(k+2)!}{(k+2)(k+1)+x^2}dx$ is integrable on $[0,\infty)$ and $\int_{-\infty}^\infty \frac{(k+1)!}{1+k+x^2}dx$ is integrable on $(-\infty,\infty)$ for all $k$.

So, we've shown that for all $t\geq 1$:

- $\frac{d}{dt}\int_0^\infty x^{k-1}e^{-tx}dx = \int_0^\infty \frac{\partial}{\partial t}x^{k-1}e^{-tx}dx = \int_0^\infty -x^k e^{-tx}dx$
- $\frac{d}{dt}\int_0^\infty x^{2k-2}e^{-tx^2}dx = \int_0^\infty \frac{\partial}{\partial t}x^{2k-2}e^{-tx^2}dx = \int_0^\infty -x^{2k}e^{-tx^2}dx$

And now the rest of the problem is just repeatedly differentiating the equations the problem tells you to.

Firstly, note that:

$$\int_0^\infty e^{-tx}dx = t^{-1} \xRightarrow{\partial/\partial t} \int_0^\infty -xe^{-tx}dx = -t^{-2} \implies \int_0^\infty -1(x)e^{-tx}dx = 1t^{-2}$$
$$\xRightarrow{\partial/\partial t} \int_0^\infty -x^2e^{-tx}dx = -2!t^{-3} \implies \int_0^\infty x^2e^{-tx}dx = 2!t^{-2}$$
$$\cdots \xRightarrow{\partial/\partial t} \int_0^\infty -x^ne^{-tx}dx = -n!t^{-n-1} \implies \int_0^\infty x^ne^{-tx}dx = n!t^{-n-1}$$

Plugging in $t = 1$, we then get $\int_0^\infty x^n e^{-x}dx = n!$.

Secondly, note that:

$$\int_{-\infty}^\infty e^{-tx^2}dx = \sqrt{\pi/t} \xRightarrow{\partial/\partial t} \int_{-\infty}^\infty -x^2e^{-tx^2}dx = -\sqrt{\pi}(\tfrac{1}{2})t^{-3/2}$$
$$\implies \int_{-\infty}^\infty x^2e^{-tx^2}dx = \sqrt{\pi}(\tfrac{1}{2})t^{-3/2}$$
$$\xRightarrow{\partial/\partial t} \int_{-\infty}^\infty -x^{2(2)}e^{-tx^2}dx = -\sqrt{\pi}(\tfrac{1}{2})(\tfrac{3}{2})t^{-5/2}$$
$$\implies \int_{-\infty}^\infty x^{2(2)}e^{-tx^2}dx = \sqrt{\pi}(\tfrac{1}{2})(\tfrac{3}{2})t^{-5/2}$$
$$\cdots \xRightarrow{\partial/\partial t} \int_{-\infty}^\infty -x^{2(n)}e^{-tx^2}dx = -\sqrt{\pi}(\tfrac{1}{2})(\tfrac{3}{2})\cdots(\tfrac{2n-1}{2})t^{-(2n+1)/2}$$
$$\implies \int_{-\infty}^\infty x^{2(n)}e^{-tx^2}dx = \sqrt{\pi}(\tfrac{1}{2})(\tfrac{3}{2})\cdots(\tfrac{2n-1}{2})t^{-(2n+1)/2}$$

Plugging in $t = 1$, we get:

$$\int_{-\infty}^\infty x^{2n}e^{-x^2}dx = \sqrt{\pi}(\tfrac{1}{2})(\tfrac{3}{2})\cdots(\tfrac{2n-1}{2})$$
$$= \tfrac{\sqrt{\pi}}{2^n}(2n-1)(2n-3)\cdots(3)(1) = \tfrac{\sqrt{\pi}}{2^n}\cdot\tfrac{(2n)!}{2^n n!} = \tfrac{\sqrt{\pi}(2n)!}{4^n n!}$$

**Exercise 2.31.a**: Derive the following formula by expanding part of the integrand into an infinite series and justifying the term-by term integration:

For $a > 0$, $\int_{-\infty}^\infty e^{-x^2}\cos(ax)dx = \sqrt{\pi}e^{-a^2/4}$

Note that:

$$\int_{-\infty}^\infty e^{-x^2}\cos(ax)dx = \int_{-\infty}^\infty e^{-x^2}\sum_{i=0}^\infty \frac{(-1)^i a^{2i}x^{2i}}{(2i)!}dx = \int_{-\infty}^\infty \sum_{i=0}^\infty \frac{a^{2i}(-1)^i}{(2i)!}e^{-x^2}x^{2i}dx$$

Next, consider the sequence of functions: $\left(\frac{a^{2i}(-1)^i}{(2i)!}e^{-x^2}x^{2i}\right)_i$.

By exercise 29, we know:

$$\sum_{i=0}^\infty \int_{-\infty}^\infty \left|\frac{a^{2i}(-1)^i}{(2i)!}e^{-x^2}x^{2i}\right| dx = \sum_{i=0}^\infty \frac{a^{2i}}{(2i)!}\cdot\frac{\sqrt{\pi}(2i)!}{4^i i!} = \sqrt{\pi}\sum_{i=0}^\infty \frac{a^{2i}}{4^i i!} = \sqrt{\pi}e^{a^2/4} < \infty$$

Therefore:

$$\int_{-\infty}^\infty \sum_{i=0}^\infty \frac{a^{2i}(-1)^i}{(2i)!}e^{-x^2}x^{2i}dx = \sum_{i=0}^\infty \frac{a^{2i}(-1)^i}{(2i)!}\int_{-\infty}^\infty e^{-x^2}x^{2i}dx$$
$$= \sum_{i=0}^\infty \frac{a^{2i}(-1)^i}{4^i i!} = \sum_{i=0}^\infty \frac{1}{i!}(\tfrac{-a^2}{4})^i = e^{-a^2/4}$$

**Exercise 2.32**: Suppose $\mu(X) < \infty$. If $f$ and $g$ are complex-valued measurable functions on $X$, define

$$\rho(f,g) = \int \frac{|f-g|}{1+|f-g|} d\mu$$

Then $\rho$ is a metric on the space of measurable functions if we identify functions are equal a.e., and $f_n \to f$ with respect to this metric iff $f_n \to f$ in measure.

First we show $\rho$ is a metric. To start, a reoccuring relevant fact is that $\frac{|f-g|}{1+|f-g|} \geq 0$. So:

- $\rho(f,g) \geq 0$
- $\rho(f,g) = 0 \Longleftrightarrow \frac{|f-g|}{1+|f-g|} = 0$ a.e. $\Longleftrightarrow |f-g| = 0$ a.e. $\Longleftrightarrow f = g$ a.e.

It's also trivial that $\rho(f,g) = \rho(g,f)$ since $|f-g| = |g-f|$.

Finally, we show the triangle inequality:

Consider any $x, y, z \in \mathbb{C}$.

- If $|x-y| \geq |x-z|$, then we have that $\frac{|x-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-y|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}$.
- If $|y-z| \geq |x-z|$, then we have that $\frac{|x-z|}{1+|x-z|} \leq \frac{|y-z|}{1+|y-z|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}$.
- If both $|x-y| < |x-z|$ and $|y-z| < |x-z|$, then by triangle inequality:
$$\frac{|x-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-z|} + \frac{|y-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}$$

Thus in all cases, we have that $\frac{|x-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}$.

It follows that $\frac{|f-h|}{1+|f-h|} \leq \frac{|f-g|}{1+|f-g|} + \frac{|g-h|}{1+|g-h|}$. And since all these terms are nonnegative:

$$\rho(f,h) = \int \frac{|f-h|}{1+|f-h|} d\mu \leq \int \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} d\mu$$
$$= \int \frac{|x-y|}{1+|x-y|} d\mu + \int \frac{|y-z|}{1+|y-z|} d\mu = \rho(f,g) + \rho(g,h)$$

Having shown $\rho$ is a metric, we now prove the two implications.

($\Longleftarrow$)
Fix $\varepsilon > 0$ and define for some $\gamma > 0$ the sets $E_n = \{x \mid |f_n(x) - f(x)| > \gamma\}$ for all $n$. Then note that for all $n$:

$$\int \frac{|f_n - f|}{1+|f_n - f|} d\mu = \int_{E_n} \frac{|f_n - f|}{1+|f_n - f|} d\mu + \int_{E_n^{\complement}} \frac{|f_n - f|}{1+|f_n - f|} d\mu$$
$$\leq \int_{E_n} \frac{|f_n - f|}{1+|f_n - f|} d\mu + \int_{E_n^{\complement}} \frac{\gamma}{1+\gamma} d\mu$$
$$\leq \int_{E_n} 1 d\mu + \int_{E_n^{\complement}} \frac{\gamma}{1+\gamma} d\mu = \mu(E_n) + \mu(E_n^{\complement}) \frac{\gamma}{1+\gamma}$$
$$\leq \mu(E_n) + \mu(X) \frac{\gamma}{1+\gamma}$$

Note that if $\gamma < \frac{\varepsilon}{1-\varepsilon}$ (assuming that $\varepsilon < 1$), then $\frac{\gamma}{1+\gamma} < \varepsilon$. So by fixing $\gamma$ sufficiently small (and because $\mu(X)$ is finite), we can force $\mu(X)\frac{\gamma}{1+\gamma} < \mu(X)\varepsilon$. Then, because $f_n \to f$ in measure, we know that $\mu(E_n) \to 0$ as $n \to \infty$. So, we can pick $N$ such that $\forall n \geq N$, $\mu(E_n) < \varepsilon$.

Therefore, for all $n \geq N$, we have that:

$$\int \frac{|f_n - f|}{1+|f_n - f|} d\mu \leq \varepsilon(1 + \mu(X))$$

$1 + \mu(X)$ is a finite constant and $\varepsilon$ was arbitrary. So, we've shown that $\int \frac{|f_n - f|}{1+|f_n - f|} d\mu \to 0$ as $n \to \infty$.

($\Longrightarrow$)
Fix $\varepsilon > 0$ and then for all $n$ define $E_n = \{x \mid |f_n(x) - f(x)| > \varepsilon\}$. Then note that:

$$\begin{aligned}\int \frac{|f_n - f|}{1+|f_n - f|} d\mu &= \int_{E_n} \frac{|f_n - f|}{1+|f_n - f|} d\mu + \int_{E_n^{\mathsf{C}}} \frac{|f_n - f|}{1+|f_n - f|} d\mu \\ &\geq \int_{E_n} \frac{|f_n - f|}{1+|f_n - f|} d\mu \geq \int_{E_n} \frac{\varepsilon}{1+\varepsilon} = \mu(E_n)\frac{\varepsilon}{1+\varepsilon}\end{aligned}$$

Hence, we have that $0 \leq \mu(E_n) \leq \frac{1+\varepsilon}{\varepsilon} \int \frac{|f_n - f|}{1+|f_n - f|} d\mu$.

And now we are done because by our hypothesis, $\frac{1+\varepsilon}{\varepsilon} \int \frac{|f_n - f|}{1+|f_n - f|} d\mu \to 0$ as $n \to \infty$. Therefore, so must $\mu(E_n)$.

Side note: only the backward direction required the fact that $\mu(X)$ is finite.

**Exercise 2.34**: Suppose $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure.

(a) $\int f = \lim\limits_{n \to \infty} \int f_n$.

A hopefully clear fact is that $f_n \to f$ in measure only if $(f_n)$ is Cauchy in measure.

To prove this, fix any $\varepsilon > 0$ and note that:

$$\{x \mid |f_n - f_m| \geq \varepsilon\} \subseteq \{x \mid |f_n - f| \geq \varepsilon/2\} \cup \{x \mid |f_m - f| \geq \varepsilon/2\}$$

Since $f_n \to f$ in measure, for all $\delta > 0$ there exists $N$ such that when $n, m \geq N$, then:

$$\begin{aligned}&\mu(\{x \mid |f_n - f| \geq \varepsilon/2\} \cup \{x \mid |f_m - f| \geq \varepsilon/2\}) \\ &\quad \leq \mu(\{x \mid |f_n - f| \geq \varepsilon/2\}) + \mu(\{x \mid |f_m - f| \geq \varepsilon/2\}) < \tfrac{\delta}{2} + \tfrac{\delta}{2} = \delta\end{aligned}$$

Another relavent fact is that if $f_n \to f$ in measure, then we must also have that that $f_{n_j} \to f$ in measure for any subsequence $(f_{n_j})$ of $(f_n)$. As a result, we know that given any subsequence $(f_{n_j})$ of $f_n$, it must be Cauchy and thus it must have subsequence $(f_{n_{j_k}})$ such that there exists a function $g$ such that $f_{n_{j_k}} \to h$ pointwise and $h = f$ a.e.. Importantly, by D.C.T we then know that $f \in L^1$ and $\int f_{n_{j_k}} \to \int h = \int f$.

Now consider the sequence $(\int f_n)_{n \in \mathbb{N}}$. Then consider two subsequences $(\int f_{n_j})_{j \in \mathbb{N}}$ and $(\int f_{n_J})_{J \in \mathbb{N}}$ such that $\int f_{n_j} \to \liminf \int f_n$ and $\int f_{n_J} \to \limsup \int f_n$.

By prior reasoning, we know there exists a subsequence of $(f_{n_{j_k}})$ such that $\int f_{n_{j_k}} \to \int f$. But at the same time, since $\int f_{n_{j_k}}$ is a subsequence of an already convergent sequence, we know that $\int f_{n_{j_k}} \to \lim_{j\to\infty} \int f_{n_j} = \liminf \int f_n$. So, we must have that $\int f = \liminf \int f_n$.

By analogous reasoning to $(\int f_{n_J})$, we can show that $\int f = \limsup \int f_n$. So, $\liminf \int f_n = \int f = \limsup \int f_n \implies \lim_{n\to\infty} \int f_n = \int f$.

(b) $f_n \to f$ in $L^1$.

Note that $|f_n - f| \leq |f_n| + |f| \leq g + |f|$. In the previous part, we showed that $f \in L^1$. Also $g$ is assumed to be nonnegative and in $L^1$. So

$$\int (g + |f|) = \int g + \int |f| < \infty.$$

Now recall that given any subsequence $(f_{n_j})$ of $(f_n)$ we can find a sub-subsequence $(f_{n_{j_k}})$ such that $f_{n_{j_k}}$ converges to a function $h$ pointwise a.e and $h = f$ a.e. In simpler terms we can just say that $f_{n_{j_k}} \to f$ pointwise a.e.. But then we have that $|f_{n_{j_k}} - f| \to 0$ pointwise a.e., and by applying D.C.T (which we can do because of the previous paragraph), we know that $\int |f_{n_{j_k}} - f| \to \int 0 = 0$.

Finally, consider the sequence $(\int |f_n - f|)_n$ and let $(\int |f_{n_j} - f|)_j$ and $(\int |f_{n_J} - f|)_J$ be subsequences converging to $\liminf \int |f_n - f|$ and $\limsup \int |f_n - f|$ respectively.

From before, we know there exists sub-subsequences $(\int |f_{n_{j_k}} - f|)_k$ and $(\int |f_{n_{J_K}} - f|)_K$ which both converge to $0$. But those subsequences must also converge to $\liminf \int |f_n - f|$ and $\limsup \int |f_n - f|$. So:

$$\liminf \int |f_n - f| = 0 = \limsup \int |f_n - f| \implies \lim_{n\to\infty} \int |f_n - f| = 0$$

**Exercise 2.38**: Suppose $f_n \to f$ in measure and $g_n \to g$ in measure.

(a) $f_n + g_n \to f + g$ in measure.

Fix $\varepsilon > 0$ and note that:

$$\begin{aligned} \{x \mid |f_n + g_n - f - g| \geq \varepsilon\} &\subseteq \{x \mid |f_n - f| + |g_n - g| \geq \varepsilon\} \\ &\subseteq \{x \mid |f_n - f| \geq \varepsilon/2\} \cup \{x \mid |g_n - g| \geq \varepsilon/2\} \end{aligned}$$

Therefore, we have that:

$$\begin{aligned} \mu(\{x \mid |f_n + g_n - f - g| \geq \varepsilon\}) &\leq \mu(\{x \mid |f_n - f| \geq \varepsilon/2\} \cup \{x \mid |g_n - g| \geq \varepsilon/2\}) \\ &\leq \mu(\{x \mid |f_n - f| \geq \varepsilon/2\}) + \mu(\{x \mid |g_n - g| \geq \varepsilon/2\}) \end{aligned}$$

Since $f_n \to f$ in measure and $g_n \to g$ in measure, the two terms on the right go to $0$ as $n \to \infty$. Thus, we also have that $\mu(\{x \mid |f_n + g_n - f - g| > \varepsilon\}) \to 0$ as $n \to \infty$.

(b) $f_n g_n \to fg$ in measure if $\mu(X) < \infty$ but not necessarily if $\mu(X) = \infty$.

Note that:

$$\begin{aligned}|f_n g_n - fg| &= |f_n g_n + f_n g + f g_n + fg - f_n g - f g_n - fg - fg| \\ &\leq |f_n g_n + fg - f_n g - f g_n| + |f_n g - fg| + |f g_n - fg| \\ &\leq |f_n(g_n - g) - f(g_n - g)| + |g(f_n - f)| + |f(g_n - g)| \\ &\leq |f - f_n||g - g_n| + |g||f_n - f| + |f||g_n - g|\end{aligned}$$

Thus, fixing $\varepsilon > 0$ we have that:

$$\begin{aligned}\mu(\{x : |f_n g_n - fg| > \varepsilon\}) \leq \mu(\{x : |f - f_n||g - g_n| \geq {}^{\varepsilon}\!/3\}) \\ + \mu(\{x : |g||f_n - f| \geq {}^{\varepsilon}\!/3\}) + \mu(\{x : |f||g_n - g| \geq {}^{\varepsilon}\!/3\})\end{aligned}$$

Now, define $E_n := \{|f| > n\}$ and note that $E_n$ is a decreasing sequence of sets. Also $\mu(E_1) < \mu(X) < \infty$ and $\bigcap_{n \geq 1} E_n = \emptyset$ It follows that $\lim_{n \to \infty} \mu(E_n) = \mu(\bigcap_{n \geq 1} E_n) = 0$.

Doing the same reasoning with $F_n := \{|g| > n\}$, we get that $\mu(F_n) \to 0$ as $n \to \infty$. So given any $\delta > 0$, there exists integers $M_1$ and $M_2$ such that for all $n \geq M_1$ and $m \geq M_2$, $\mu(E_n) < \delta$ and $\mu(F_n) < \delta$. Set $N_1 = \max(M_1, M_2)$. Thus $\mu(E_{N_1}) < \delta$, $\mu(F_{N_2}) < \delta$, and outside $E_{N_1}$ and $F_{N_1}$ we have that $|f| \leq N_1$ and $|g| \leq N_1$ respectively.

In rapid succession, use the fact that $f_n \to f$ and $g_n \to g$ in measure in order to find integers $M_3, M_4, M_5, M_6$ such that:

- $n \geq M_3 \implies \mu(\{x : |f_n - f| \geq \frac{\varepsilon}{3N_1}\}) < \delta$
- $n \geq M_4 \implies \mu(\{x : |g_n - g| \geq \frac{\varepsilon}{3N_1}\}) < \delta$
- $n \geq M_5 \implies \mu(\{x : |f_n - f| \geq \sqrt{{}^{\varepsilon}\!/3}\}) < \delta$
- $n \geq M_6 \implies \mu(\{x : |g_n - g| \geq \sqrt{{}^{\varepsilon}\!/3}\}) < \delta$

Take $N_2 = \max(M_3, M_4, M_5, M_6)$. Then finally, note that:

- $$\begin{aligned}\mu(\{x : |f - f_n||g - g_n| \geq {}^{\varepsilon}\!/3\}) &\leq \mu(\{x : |f - f_n| \geq \sqrt{{}^{\varepsilon}\!/3}\} \cup \{x : |g - g_n| \geq \sqrt{{}^{\varepsilon}\!/3}\}) \\ &\leq \mu(\{x : |f - f_n| \geq \sqrt{{}^{\varepsilon}\!/3}\}) + \mu(\{x : |g - g_n| \geq \sqrt{{}^{\varepsilon}\!/3}\}) \\ &< \delta + \delta = 2\delta \text{ when } n \geq N_2.\end{aligned}$$

- $$\begin{aligned}\mu(\{x : |g||f - f_n| \geq {}^{\varepsilon}\!/3\}) &\leq \mu(\{x : |g| \geq N_1\} \cup \{x : |f - f_n| \geq \tfrac{\varepsilon}{3N_1}\}) \\ &\leq \mu(\{x : |g| \geq N_1\}) + \mu(\{x : |f - f_n| \geq \tfrac{\varepsilon}{3N_1}\}) \\ &< \delta + \delta = 2\delta \text{ when } n \geq N_2.\end{aligned}$$

- $\mu(\{x : |f||g - g_n| \geq \varepsilon/3\}) \leq \mu(\{x : |f| \geq N_1\} \cup \{x : |g - g_n| \geq \frac{\varepsilon}{3N_1}\})$
$\leq \mu(\{x : |f| \geq N_1\}) + \mu(\{x : |g - g_n| \geq \frac{\varepsilon}{3N_1}\})$
$< \delta + \delta = 2\delta$ when $n \geq N_2$.

So for $n \geq N_2$, we have that $\mu(\{x : |f_n g_n - fg| > \varepsilon\}) < 6\delta$. And since $\delta$ is arbitrary, we thus know that $\mu(\{x : |f_n g_n - fg| > \varepsilon\}) \to 0$ as $n \to \infty$.

Now in our work above we assumed $\mu(X) < \infty$. To see that this isn't necessarily true if $\mu(X) = \infty$, let us consider the real Lesbesgue measure space restricted to $(0, \infty)$.

Define $f_n(x) = x + \frac{1}{n}\chi_{(n,n+1)} = g_n(X)$. Then clearly $f_n \to x$ and $g_n \to x$ in measure. However, $f_n g_n = x^2 + \frac{2x}{n}\chi_{(n,n+1)} + \frac{1}{n^2}\chi_{n,n+1}$.

For all $x \in (n, n+1)$, we have that

$$x^2 + \frac{2x}{n}\chi_{(n,n+1)} + \frac{1}{n^2}\chi_{n,n+1} - x^2 \geq \frac{2x}{n}\chi_{(n,n+1)} > 2.$$

So $\mu(|f_n g_n - x^2| \geq 2) > \mu((n, n+1)) = 1$ for all $n \in \mathbb{N}$.

**Exercise 51:** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be arbitrary measure spaces (not necessarily $\sigma$-finite).

(a) If $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable, $g : Y \to \mathbb{C}$ is $\mathcal{N}$-measurable, and $h(x, y) = f(x)g(y)$, then $h$ is $\mathcal{M} \otimes \mathcal{N}$-measurable.

Define $F(x, y) = f(x)$ and $G(x, y) = g(y)$. Then given any set $B \in \mathcal{B}_{\mathbb{C}}$, we have that $F^{-1}(B) = f^{-1}(B) \times Y$ and $G^{-1}(B) = X \times g^{-1}(B)$. And since both $f$ and $g$ are measurable, we know that $f^{-1}(B) \in \mathcal{M}$ and $g^{-1}(B) \in \mathcal{N}$. So, $F^{-1}(B)$ and $G^{-1}(B)$ are both rectangles and thus in $\mathcal{M} \otimes \mathcal{N}$.

Next, note that $h(x, y) = F(x, y)G(x, y)$. Thus, $h$ is the product of two $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}_{\mathbb{C}})$-measurable functions and thus itself also a $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}_{\mathbb{C}})$-measurable function.

(b) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and $\int h d(\mu \times \nu) = [\int f d\mu][\int g d\nu]$.

The simplest case is when $f$ and $g$ are both nonnegative real-valued functions. Then we can find increasing sequences of simple functions $(\phi_n)$ and $(\varphi_n)$ such that $\phi_n \to f$ and $\varphi_n \to g$.

Now for all $n$, define $\psi_n(x, y) = \phi_n(x)\varphi_n(y)$. It's clear that $\psi_n$ can only equal a finite amount of values (at most the number of values $\phi_n$ can take on times the number of values $\varphi_n$ can take on). Also by part (a), we know that $\psi_n$ is measurable for all $n$.

Thirdly, note that $(\psi_n)$ is an increasing sequence of functions converging pointwise to $h$.