Math 140B Lecture Notes (Professor: Brandon Seward)

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Lecture 1: 4/1/2024

Let $f : E \longrightarrow \mathbb{R}$ where $E \subseteq \mathbb{R}$.

Since E is the domain of f, we shall also refer to it as $\mathrm{dom}(f)$.

Fix a point $x \in E \cap E'$. Then consider the function $\frac{f(t)-f(x)}{t-x}$ for $t \in \mathrm{dom}(f) \setminus \{x\}$ and define the derivative of f at x to be $f'(x) = \lim\limits_{t \to x} \left(\frac{f(t)-f(x)}{t-x} \right)$ provided that this limit exists. When the above limit exists, we say f is differentiable at x.

We say f is differentiable on $D \subseteq E$ if f is differentiable at every point in D, and if f is differentiable on its entire domain, then we call f differentiable.

The function $f'(x) = \lim\limits_{t \to x} \left(\frac{f(t)-f(x)}{t-x} \right)$ is called the derivative of f.

Proposition 83: If f is differentiable at x, then f is continuous at x.

Proof:
Note that $\lim\limits_{t \to x} (f(t)) = \lim\limits_{t \to x} \left((t-x)\frac{f(t)-f(x)}{t-x} + f(x) \right)$.

Now $\lim\limits_{t \to x}(t-x) = 0$ and we know $\lim\limits_{t \to x} \frac{f(t)-f(x)}{t-x} = f'(x)$ exists because f is differentiable at x. Also, obviously $\lim\limits_{t \to x} f(x) = f(x)$.

Thus by proposition 66 (check 140A notes), we know that:

$$\begin{aligned}
\lim_{t \to x} \left((t-x)\frac{f(t)-f(x)}{t-x} + f(x) \right) &= \lim_{t \to x}(t-x) \lim_{t \to x} \left(\frac{f(t)-f(x)}{t-x} \right) + \lim_{t \to x} f(x) \\
&= 0 \cdot f'(x) + f(x) \\
&= f(x)
\end{aligned}$$

Thus, f is continuous at x.

Notes:

1. The above proposition says that differentiability is stronger than continuity.
2. The converse of this proposition is false. For example, the function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

Proposition 84: Suppose f and g are real-valued functions with $\mathrm{dom}(f), \mathrm{dom}(g) \subseteq \mathbb{R}$. Also suppose f and g are differentiable at x. Then $f+g$, fg, and (when $g(x) \neq 0$) $\frac{f}{g}$ are differentiable at x with:

(A) $(f+g)'(x) = f'(x) + g'(x)$ (sum rule)

(B) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ (product rule)

(C) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ (quotient rule)

Proof:

(A) Since both f and g are differentiable, we know that both $f'(x) = \lim\limits_{t\to x} \frac{f(t)-f(x)}{t-x}$ and $g'(x) = \lim\limits_{t\to x} \frac{g(t)-g(x)}{t-x}$ exist. So by proposition 66:

$$(f+g)'(x) = \lim_{t\to x} \frac{f(t)+g(t)-f(x)-g(x)}{t-x} = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} + \lim_{t\to x} \frac{g(t)-g(x)}{t-x}$$

This means that $(f+g)'(x) = f'(x) + g'(x)$.

(B) Note that:

$$\begin{aligned}(fg)'(x) &= \lim_{t\to x} \frac{f(t)g(t)-f(x)g(x)}{t-x}\\ &= \lim_{t\to x} \frac{f(t)g(t)-f(x)g(t)+f(x)g(t)-f(x)g(x)}{t-x}\\ &= \lim_{t\to x} \left(g(t)\frac{f(t)-f(x)}{t-x} + f(x)\frac{g(t)-g(x)}{t-x}\right)\end{aligned}$$

By proposition 83, $g(t) \to g(x)$ as $t \to x$. Also, since both f and g are differentiable, we know $f'(x) = \lim\limits_{t\to x} \frac{f(t)-f(x)}{t-x}$ and $g'(x) = \lim\limits_{t\to x} \frac{g(t)-g(x)}{t-x}$ exist. So by proposition 66:

$$\lim_{t\to x} \left(g(t)\frac{f(t)-f(x)}{t-x} + f(x)\frac{g(t)-g(x)}{t-x}\right) = f'(x)g(x) + f(x)g'(x).$$

(C) Note that:

$$\begin{aligned}\left(\tfrac{f}{g}\right)'(x) &= \lim_{t\to x} \frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{t-x}\\ &= \lim_{t\to x} \left(\frac{1}{g(x)g(t)} \frac{f(t)g(x)-f(x)g(t)}{t-x}\right)\\ &= \lim_{t\to x} \left(\frac{1}{g(x)g(t)} \frac{f(t)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(t)}{t-x}\right)\\ &= \lim_{t\to x} \left(\frac{1}{g(x)g(t)} \left(g(x)\frac{f(t)-f(x)}{t-x} - f(x)\frac{g(t)-g(x)}{t-x}\right)\right)\end{aligned}$$

Now, for the same reasons as before, we can use propositions 83 and 66 to separate the parts of the above limit to get that the above limit equals:

$$\frac{1}{(g(x))^2}\left(g(x)f'(x) - f(x)g'(x)\right)$$

If $f(x) = \alpha$ where $\alpha \in \mathbb{R}$ is constant, then trivially $f'(x) = 0$ for all x.
Meanwhile, if $f(x) = x$, then we can trivially find that $f'(x) = 1$.

Claim 1: For all $n \in \mathbb{Z}^+$, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Proof: (we proceed by induction)

Base Case:

If $n = 1$, then for $f(x) = x^1$, we have that $f'(x) = 1 \cdot x^0$.

Induction:

Now assume $n > 1$, and for $f(x) = x^{n-1}$, we have that $f'(x) = (n-1)x^{n-2}$.
For the rest of this proof, I'll abreviate the derivative of x^n as $(x^n)'$ and the derivative of x^{n-1} as $(x^{n-1})'$. Then using product rule, we know that:

$$(x^n)' = x(x^{n-1})' + 1 \cdot x^{n-1} = x \cdot (n-1)x^{n-2} + x^{n-1} = ((n-1)+1)x^{n-1} = nx^{n-1}$$

Claim 2: If f is differentiable at x and $\alpha \in \mathbb{R}$, then $(\alpha f)'(x) = \alpha f'(x)$.

Proof:

By the product rule: $(\alpha f)'(x) = \alpha f' + (\alpha)' f = \alpha f' + 0 \cdot f = \alpha f'$.

These combined with proposition 84 tells us that both polynomials and rational functions are differentiable over their domains.

Proposition 85: (chain rule)

Let f and g be real-valued functions with $\mathrm{dom}(f), \mathrm{dom}(g) \subseteq \mathbb{R}$. Let $x \in \mathbb{R}$. Suppose that f is differentiable at x and that g is differentiable at $f(x)$. Then $g \circ f$ is differentiable at x and $(g \circ f)'(x) = g'(f(x))f'(x)$.

<u>Intuition</u>:

$$\lim_{t \to x} \left(\frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \cdot \frac{f(t) - f(x)}{t - x} \right) = g'(f(t)) \cdot f'(t).$$

That said, the issue with this intuition is that we need to address the possibility that $f(t) - f(x) = 0$.

Proof:

Set $y = f(x)$ and define $v(s) = \begin{cases} \frac{g(s)-g(y)}{s-y} - g'(y) & \text{if } s \neq y \\ 0 & \text{if } s = y \end{cases}$

Note that v is continuous at y. This is because g being differentiable at $f(x) = y$ means that:

$$\lim_{s \to y} v(s) = \lim_{s \to y} \left(\frac{g(s)-g(y)}{s-y} - g'(y) \right) = g'(y) - g'(y) = 0 = v(y).$$

Also, since f is differentiable at x, we know that f is continuous at x. Therefore, $v \circ f$ is continuous at x by proposition 68. Additionally, setting $s = f(t)$, we know that $s \to y$ as $t \to x$ because f is continuous at x. Thus:

$$\lim_{t \to x} v(f(t)) = \lim_{s \to y} v(s) = 0$$

Finally, note that $g(s) - g(y) = (s-y)(g'(y) + v(s))$ for all s. Thus by substituting that into our limit:

$$\begin{aligned}(g \circ f)'(x) &= \lim_{t \to x} \tfrac{g(f(t)) - g(f(x))}{t-x} \\ &= \lim_{t \to x} \tfrac{f(t)-f(x)}{t-x}(g'(f(x)) + v(f(t))) \\ &= f'(x)\,(g'(f(x)) + 0) \qquad \text{(by proposition 66)}\end{aligned}$$

Lecture 2: 4/3/2024

To start off lecture, here is some intuition about the behavior of derivatives. We'll formally define sine and cosine later (on page __) but for this section please take for granted that $(\sin(x))' = \cos(x)$. Additionally, please take for granted that the power rule holds for non-positive integer exponents.

1. Define $f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

   When $x \neq 0$, we have by chain rule that $f'(x) = \sin(\frac{1}{x}) - \frac{1}{x}\cos(\frac{1}{x})$. Meanwhile if $x = 0$, then $\frac{f(t)-f(0)}{t-0} = \frac{t\sin(\frac{1}{t})}{t} = \sin(\frac{1}{t})$ when $t \neq 0$. So $\lim\limits_{t \to 0}\left(\frac{f(t)-f(0)}{t-0}\right)$ does not exist, meaning f is not differentiable at x.

   This shows that $\mathrm{dom}(f')$ can be a proper subset of $\mathrm{dom}(f)$.

2. Define $g(x) = \begin{cases} x^2\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

   When $x \neq 0$, we have by chain rule that $g'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$. Meanwhile when $t \neq 0$:

   $$\left|\tfrac{g(t)-g(0)}{t-0}\right| = \left|\tfrac{t^2\sin(\frac{1}{t})}{t}\right| = \left|t\sin(\tfrac{1}{t})\right| \leq |t|.$$

   Thus $0 = \lim\limits_{t \to 0}(-t) \leq \lim\limits_{t \to 0}\left(\frac{g(t)-g(0)}{t-0}\right) \leq \lim\limits_{t \to 0}(t) = 0$, meaning $g'(0) = 0$. So $\mathrm{dom}(g') = \mathrm{dom}(g)$. That said, note that g' has a discontinuity of the second kind at 0. Therefore, this shows that the derivative of a function does not have to be continuous.

Let X be a metric space. A function $f : X \longrightarrow \mathbb{R}$ has a local maximum at $p \in X$ if $\exists \delta > 0 \ \ s.t. \ \forall x \in B_\delta(p), \ \ f(x) \leq f(p)$. Similarly, f has a local minimum if $\exists \delta > 0 \ \ s.t. \ \forall x \in B_\delta(p), \ \ f(x) \geq f(p)$.

Proposition 86: Let $f : (a,b) \longrightarrow \mathbb{R}$. If f has a local maximum at x and f is differentiable at x, then $f'(x) = 0$.

Proof:
Let $\delta > 0$ so that $\forall t \in B_\delta(x), \ \ f(t) \leq f(x)$. Then for all $t \in (x - \delta, x)$, $\frac{f(t)-f(x)}{t-x} \geq 0$. So $f'(x) \geq 0$. Similarly for all $t \in (x, x+\delta)$, we have $\frac{f(t)-f(x)}{t-x} \leq 0$. Thus $f'(x) \leq 0$.

Hence $f'(x) = 0$.

Note that analogous reasoning can show that if f has a local minimum at x and f is differentiable at x, then $f'(x) = 0$.

Proposition 87: If $f, g : [a,b] \longrightarrow \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b), then there exists $x \in (a,b)$ with:

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x).$$

Proof:
Define $h : [a,b] \longrightarrow \mathbb{R}$ by $h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$.
Then $h(a) = f(b)g(a) - g(b)f(a) = h(b)$.

Notice that h is continuous on $[a,b]$ and differentiable on (a,b) because of propositions 70 and 84. Since $h'(x) = (f(b)-f(a))g'(x)-(g(b)-g(a))f'(x)$, for all $x \in (a,b)$ it now suffices to show that there exists $x \in (a,b)$ with $h'(x) = 0$.

Since h is continuous on a compact set $[a,b]$, we know that h attains a maximum value and a minimum value over the interval $[a,b]$.

Case 1: If h is constant on $[a,b]$, then $h'(x) = 0$ for all $x \in (a,b)$.

Case 2: If there is $t \in (a,b)$ with $h(t) > h(a) = h(b)$, then $h(a)$ and $h(b)$ can't be the max. value that h attains on $[a,b]$. So h has a maximum at some point $x \in (a,b)$. Then by the last theorem, $h'(x) = 0$.

Case 3: If there is $t \in (a,b)$ with $h(t) < h(a) = h(b)$, then $h(a)$ and $h(b)$ can't be the min. value that h attains on $[a,b]$. So h has a minimum at some point $x \in (a,b)$. Then by the last theorem, $h'(x) = 0$.

Proposition 88: (Mean Value Theorem)
If $f : [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then there is $x \in (a,b)$ with $f(b) - f(a) = (b-a)f'(x)$.

To prove this, apply the previous proposition with $g(x) = x$.

Proposition 89: Suppose $f(a,b) \longrightarrow \mathbb{R}$ is differentiable. Then:

- If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is monotone increasing.
- If $f'(x) \leq 0$ for all $x \in (a,b)$, then f is monotone decreasing.
- If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.

Proof:
For all $a < x_1 < x_2 < b$, we know by the mean value theorem that there exists $t \in (x_1, x_2)$ with $f(x_2) - f(x_1) = (x_2 - x_1)f'(t)$. Then since $x_2 - x_1 > 0$, the sign of $f(x_2) - f(x_1)$ depends entirely on $f'(t)$.

Lecture 3: 4/5/2024

Even though derivatives are not necessarily continuous, we can show they always satisfy the conclusion of the intermediate value theorem.

Proposition 90: Suppose $f : [a,b] \to \mathbb{R}$ is differentiable and $\lambda \in \mathbb{R}$ satisfies that $f'(a) < \lambda < f'(b)$. Then there is $x \in (a,b)$ with $f'(x) = \lambda$.

Proof:
Define $g : [a,b] \to \mathbb{R}$ by the rule $g(t) = f(t) - \lambda t$. Then g is differentiable with $g'(t) = f'(t) - \lambda$. So, it suffices to find $x \in (a,b)$ with $g'(x) = 0$

Since g is differentiable, we know that g is continuous. Adding in the fact that $[a,b]$ is compact, we know that g achieves a minimum value. So, let $x \in [a,b]$ be such that $g(x)$ is the minimum value of g.

Now consider that $f'(a) < \lambda < f'(b) \Longrightarrow g'(a) < 0 < g'(b)$. Since $g'(a) < 0$, there is some $t_1 > a$ near a such that $g(x) \leq g(t_1) < g(a)$.

Explanation:
Set $\varepsilon = |g'(a)|$. Then by the definition of limits:

$$\exists \delta > 0 \;\; s.t. \;\; \forall t \in (a, a+\delta), \;\; \left| \frac{g(t)-g(a)}{t-a} - g'(a) \right| < \varepsilon.$$

Then because $g'(a)$ is negative, we must have that $\frac{g(t)-g(a)}{t-a} < 0$.
But as $t - a > 0$, we must have that $g(t) - g(a) < 0$.

This will be a common trick so get used to it.

Similarly, since $g'(b) > 0$, there is some $t_2 < b$ near b such that $g(x) \leq g(t_2) < g(b)$. Hence, we have shown that $x \neq a$ and $x \neq b$, meaning that $x \in (a, b)$. Then, by applying proposition 86 we know that $g'(x) = 0$.

We can prove an analogous theorem for when $f'(b) < \lambda < f'(a)$.

Corollary: If $f : [a, b] \longrightarrow \mathbb{R}$ is differentiable, then f' has no simple discontinuities.

Proof:
Assume that $x \in [a, b)$ and $f'(x+)$ exists. Then let $\varepsilon > 0$. By the definition of $f(x+)$:

$$\exists \delta > 0 \ \ s.t. \ \ \forall t \in (x, x+\delta), \ \ |f'(t) - f'(x+)| < \varepsilon/2.$$

If $f'(t) = f'(x)$ for all $t \in (x, x+\delta)$, then we automatically have that $f'(x+) = f'(x)$. So assume there exists $t \in (x, x+\delta)$ such that $f'(t) \neq f'(x)$. Then by the previous proposition, there exists $s \in (x, t)$ such that $f'(s)$ is between $f'(x)$ and $f'(t)$, and that $|f'(s) - f'(x)| < \varepsilon/2$.
Finally:

$$|f'(x) - f'(x+)| \leq |f'(x) - f'(s)| + |f'(s) - f'(x+)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So $f'(x)$ must equal $f'(x+)$. Similarly, we can show that if $x \in (a, b]$ and $f'(x-)$ exists, then $f'(x) = f'(x-)$. Thus, it is impossible for f' to have a simple discontinuity.

However, we already saw that f' can have discontinuities of the second kind.

Propositon 91: (L'Hôpital's rule)
Suppose $-\infty \leq a \leq b \leq +\infty$, that $f, g : (a, b) \longrightarrow \mathbb{R}$ are differentiable, and that $\forall x \in (a, b), \ \ g'(x) \neq 0$. Then suppose that $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = A \in \mathbb{R} \cup \{-\infty, \infty\}$.
If either:

- both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$
- or either $g(x) \to +\infty$ or $g(x) \to -\infty$ as $x \to a$

then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} \to A$.

(A similar result holds as $x \to b$.)

Proof:
Since $A \in \mathbb{R} \cup \{-\infty, \infty\}$, to show that $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = A$, it suffices to show:

1. If $A \neq +\infty$, then for every $q \in \mathbb{R}$ with $q > A$, there is $c > a$ with $\forall x \in (a, c), \ \ \frac{f(x)}{g(x)} < q$.
2. If $A \neq -\infty$, then for every $q \in \mathbb{R}$ with $q < A$, there is $c > a$ with $\forall x \in (a, c), \ \ \frac{f(x)}{g(x)} > q$

Let's prove requirement 1. Assume $A \neq +\infty$ and fix $q \in \mathbb{R}$ with $q > A$. Next pick $r \in \mathbb{R}$ with $A < r < q$. Since $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$, there is $c_1 > a$ with $\forall x \in (a, c_1),\ \ \frac{f'(x)}{g'(x)} < r$.

Now consider that whenever $a < x < y < c_1$, we have by proposition 87 that there exists $t \in (x, y)$ such that:

$$(f(y) - f(x))g'(t) = (g(y) - g(x))f'(t).$$

By the hypothesis of the theorem, $g'(t)$ can't be zero. Aditionally, because of the mean value theorem, if $g(y) - g(x) = 0$, then there would have to exist $s \in (x, y)$ with $g'(s) = 0$, thus contradicting the hypothesis of the theorem. So, it is safe to rearrange the above expression to get that:

$$\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f'(t)}{g'(t)} < r$$

Case 1: Assume $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$.

Then fixing any $y \in (a, c_1)$, we have that $\lim\limits_{x \to a} \frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f(y)}{g(y)} \leq r < q$.

Case 2: Assume $g(x) \to +\infty$ or $g(x) \to -\infty$ as $x \to a$.

Then fix any $y \in (a, c_1)$ and pick $c_2 \in (a, c_1)$ such that $\forall x \in (a, c_2)$, $g(x)$ and $g(x) - g(y)$ have the same sign. Then, $\forall x \in (a, c_2)$, we have that $\frac{g(x)-g(y)}{g(x)} > 0$. So:

$$\frac{f(y)-f(x)}{g(y)-g(x)} \cdot \frac{g(x)-g(y)}{g(x)} < r \cdot \frac{g(x)-g(y)}{g(x)}$$

Note that $\frac{f(y)-f(x)}{g(y)-g(x)} \cdot \frac{g(x)-g(y)}{g(x)} = \frac{f(x)-f(y)}{g(x)} = \frac{f(x)}{g(x)} - \frac{f(y)}{g(x)}$ and $\frac{g(x)-g(y)}{g(x)} = 1 - \frac{g(y)}{g(x)}$. Thus, we can rearrange terms to get that:

$$\frac{f(x)}{g(x)} < \left(1 - \frac{g(y)}{g(x)}\right) r + \frac{f(y)}{f(x)}$$

Now, $\lim\limits_{x \to a} \left(\left(1 - \frac{g(y)}{g(x)}\right) r + \frac{f(y)}{f(x)}\right) = (1 - 0)r + 0 = r$. So, there is $c_3 \in (a, c_2)$ such that $\forall x \in (a, c_3),\ \ \left(1 - \frac{g(y)}{g(x)}\right) r + \frac{f(y)}{f(x)} < q$.

Hence, $\forall x \in (a, c_3)$, $\frac{f(x)}{g(x)} < q$.

Requirement 2 is proved in a similar fashion. ■

Let f be a real-valued function with $\mathrm{dom}(f) \subseteq \mathbb{R}$. If f' is defined and is itself differentiable, then the derivative of f' is denoted f'' and called the second derivative of f. We similarly define $f''', f^{(4)}, \ldots, f^{(n)}$.

Also, we shall sometimes use $f^{(0)}$ to refer to the original function f.

Lecture 4: 4/8/2024

Proposition 92: (Taylor's Theorem)
Suppose that $f:[a,b]\longrightarrow\mathbb{R}$, that $f^{(n-1)}$ is continuous on $[a,b]$, and that $f^{(n)}$ is defined on (a,b). Then pick $\alpha\in[a,b]$ and define:

$$P(t)=\sum_{k=0}^{n-1}\frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$

Then for every $\beta\in[a,b]\setminus\{\alpha\}$, there is some x between α and β such that $f(\beta)=P(\beta)+\frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n$.

Proof:
Set $M=\frac{f(\beta)-P(\beta)}{(\beta-\alpha)^n}$ so that $f(\beta)=P(\beta)+M(\beta-\alpha)^n$. Having done that, our goal is now to find an x between α and β such that $\frac{f^{(n)}(x)}{n!}=M$.

Define $g(t)=f(t)-P(t)-M(t-\alpha)^n$. Then, since P is a polynomial of degree $n-1$, we have that $P^{(n)}(t)=0$ for all t. So:

$$g^{(n)}(t)=f^{(n)}(t)-Mn!$$

Thus, it suffices to find an x between α and β such that $g^{(n)}(x)=0$.

Importantly, P is the unique polynomial of degree $n-1$ satisfying for all $0\leq k\in\leq k-1$ that $P^{(k)}(\alpha)=f^{(k)}(\alpha)$. Thus, for all $0\leq k\in\leq k-1$, we have that:

$$g^{(k)}(\alpha)=f^{(k)}(\alpha)-P^{(k)}(\alpha)-M\frac{n!}{(n-k)!}(\alpha-\alpha)^{n-k}=0.$$

At the same time, for all $0\leq k\leq n-1$, we know that $g^{(k)}$ is continuous on $[\alpha,\beta]$ and differentiable on (α,β). So, we shall proceed by repeatedly applying the mean value theorem.

- $g(\beta)=0$ and $g(\alpha)=0$. So, there is x_1 between α and β with $g'(x_1)=0$.
- $g'(x_1)=0$ and $g'(\alpha)=0$. So, there is x_2 between α and x_1 with $g''(x_2)=0$.

$$\vdots$$

Eventually, you will get an x_n between α and x_{n-1} with $g^{(n)}(x_n)=0$. ■

Note that this can be interpretted as a higher order analog of the mean value theorem. In fact, if $n=1$ then this is just the mean value theorem.

The limit definition of the derivative still makes sense and can be applied to situations where f is a $\mathbb{C}$-valued or $\mathbb{R}^k$-valued function. Although, because this class is called "real" analysis, we shall always require that $\mathrm{dom}(f) \subseteq \mathbb{R}$.

(We will talk in 140C about when $\mathrm{dom}(f) \subseteq \mathbb{R}^k$)

If f is a $\mathbb{C}$-valued function, then we can write that $f = f_1 + if_2$ where f_1 and f_2 are real-valued. Then, f is differentiable if and only if f_1 and f_2 are differentiable. Also, $f'(x) = f_1'(x) + if_2'(x)$.

Proof:
Firstly consider any sequence (x_n) such that $x_n \to x$ as $n \to \infty$ but $x_n \neq x$ for any n. Then assuming $f'(x)$ exists, we know that:

$$\lim_{n\to\infty} \left|\frac{f(x_n)-f(x)}{x_n-x} - f'(x)\right| = 0$$

Now importantly:

- $0 \leq \left|\frac{f_1(x_n)-f_1(x)}{x_n-x} - \mathrm{Re}(f'(t))\right| = \left|\mathrm{Re}\left(\frac{f(x_n)-f(x)}{x_n-x} - f'(x)\right)\right| \leq \left|\frac{f(x_n)-f(x)}{x_n-x} - f'(x)\right|$
- $0 \leq \left|\frac{f_2(x_n)-f_2(x)}{x_n-x} - \mathrm{Im}(f'(t))\right| = \left|\mathrm{Im}\left(\frac{f(x_n)-f(x)}{x_n-x} - f'(x)\right)\right| \leq \left|\frac{f(x_n)-f(x)}{x_n-x} - f'(x)\right|$

So, $\lim\limits_{n\to\infty} \left|\frac{f_1(x_n)-f_1(x)}{x_n-x} - \mathrm{Re}(f'(x))\right| = 0$ and $\lim\limits_{n\to\infty} \left|\frac{f_2(x_n)-f_2(x)}{x_n-x} - \mathrm{Im}(f'(x))\right| = 0$.
This means $f_1'(x)$ and $f_2'(x)$ exist with $f_1'(x) = \mathrm{Re}(f'(x))$ and $f_2'(x) = \mathrm{Im}(f'(x))$.

Meanwhile, assume that $f_1'(x)$ and $f_2'(x)$ exist. Then:

$$\begin{aligned} f'(x) &= \lim_{t\to x}\left(\frac{f_1(t)+if_2(t)-f_1(x)-if_2(x)}{t-x}\right) \\ &= \lim_{t\to x}\left(\frac{f_1(t)-f_1(x)}{t-x} + i\frac{f_2(t)-f_2(x)}{t-x}\right) = f_1'(x) + if_2'(x). \end{aligned}$$

Similarly, if $\vec{f}$ is $\mathbb{R}^k$-valued, then we can write $\vec{f} = (f_1, f_2, \ldots, f_k)$ where $f_1, f_2, \ldots, f_k$ are real-valued. Then $\vec{f}$ is differentiable if and only if $f_1, f_2, \ldots, f_k$ are all differentiable. Also, $\vec{f}\,'(x) = (f_1'(x), f_2'(x), \ldots, f_k'(x))$.

This follows from the fact that given any sequence (x_n) such that $x_n \to x$ as $n \to \infty$ but $x_n \neq x$ for any n, we have by proposition 34 that: $\left(\frac{\vec{f}(x_n)-\vec{f}(x)}{x_n-x}\right)$ converges if and only if $\left(\frac{f_i(x_n)-f_i(x)}{x_n-x}\right)$ for each i.

For $\mathbb{C}$-valued functions, the addition, product, and quotient rules still hold.
For $\mathbb{R}^k$-valued functions, the addition and (dot) product rules still hold.

But, the mean value theorem and L'hôpital's rule fail in these situations.

For intuition on why this is, if f is $\mathbb{R}^k$ or $\mathbb{C}$-valued, then it is possible for $|f'|$ to be arbitrarily large over some interval of the domain while having f change as little as you want. To do this, make f "spin" in $\mathbb{R}^k$ or $\mathbb{C}$.

At least, we can still make the following theorem which is both similar to the mean value theorem and holds even for vector valued functions.

Proposition 93: Let $\overrightarrow{f} : [a, b] \longrightarrow \mathbb{R}^k$. Assume $\overrightarrow{f}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there is $x \in (a, b)$ such that:

$$\|\overrightarrow{f}(b) - \overrightarrow{f}(a)\| \leq (b - a)\|\overrightarrow{f}'(x)\|$$

A List of How The Proposition Numbering in my Notes Lines up With Our Textbook:

Proposition Number	Label in Textbook	Proposition Number	Label in Textbook
83	5.2	84	5.3
85	5.5	86	5.8
87	5.9	88	5.10
89	5.11	90	5.12
91	5.13	92	5.15
93	5.19	94	
95		96	
97		98	
99		100	
101		102	
103		104	

Our textbook is *Principles of Mathematical Analysis* by Walter Rudin.

Homework 1:

Exercise 5.2: Let $f : (a, b) \longrightarrow \mathbb{R}$ be differentiable with $f'(x) > 0$. Then f is strictly increasing.

For all $a < x_1 < x_2 < b$, we know by the mean value theorem that there exists $t \in (x_1, x_2)$ with $f(x_2) - f(x_1) = (x_2 - x_1)f'(t)$. Since $(x_2 - x_1)$ and $f'(t)$ are positive, we thus have that $f(x_2) - f(x_1) > 0$.

As a consequence of f being strictly increasing, we know f is injective. Thus if we restrict the codomain of f to its image, then f is bijective, meaning there exists a function $g = f^{-1}$ such that $(g \circ f)(x) = x = (f \circ g)(x)$. Now we show that for all $y \in \mathrm{dom}(g), \;\; y \in \mathrm{dom}(g)'$ and $g(z) \to g(y)$ as $z \to y$.

Let $y \in \mathrm{dom}(g)$. Then there exists $x \in \mathrm{dom}(f)$ such that $f(x) = y$ and $g(y) = x$. But, since $\mathrm{dom}(f)$ is an open set, we know that x is an interior point of $\mathrm{dom}(f)$. Hence, there exists r such that $[x - r, x + r] \subseteq \mathrm{dom}(f)$. Let $\widehat{f}$ be the restriction of f whose domain is $[x - r, x + r]$ and whose codomain is the image of $[x - r, x + r]$ with respect to f.

Because f is differentiable, we know f is continuous. Then, note that $\forall x_0 \in \mathrm{dom}\left(\widehat{f}\right) \cap \mathrm{dom}\left(\widehat{f}\right)', \;\; \lim_{t \to x_0} \widehat{f}(t) = \lim_{t \to x_0} f(t) = f(x_0) = f(x_0) = \widehat{f}(x_0)$.
So, $\widehat{f}$ is a continuous function over its domain. Also, note that $\widehat{f}$ is still bijective.

Meanwhile $[x - r, x + r] = \mathrm{dom}\left(\widehat{f}\right)$ is compact and connected. Firstly, this means that $\widehat{f}^{-1}$ is continuous. Secondly, this tells us that the image of $[x - r, x + r]$ is connected. Because f and thus $\widehat{f}$ is strictly increasing, we know that:
$\widehat{f}(x - r) < \widehat{f}(x) = f(x) = y < \widehat{f}(x + r)$. So, $[\widehat{f}(x - r), \widehat{f}(x + r)]$ is a subset of the domain of $\widehat{f}^{-1}$ and y is an interior point of that subset of the domain.

$[\widehat{f}(x - r), \widehat{f}(x + r)]$ is perfect, meaning y is a limit point of $[\widehat{f}(x - r), \widehat{f}(x + r)]$.
In turn, this means y is a limit point of $\mathrm{dom}\left(\widehat{f}\right)$ and $\mathrm{dom}(g)$ because:

$$[\widehat{f}(x - r), \widehat{f}(x + r)] \subseteq \mathrm{dom}\left(\widehat{f}\right) \subseteq \mathrm{dom}(g).$$

Then as $\widehat{f}^{-1}$ is continuous, we know that $\lim_{z \to y} \widehat{f}^{-1}(z) = \widehat{f}^{-1}(y)$. Additionally, because y is an interior point of $[\widehat{f}(x - r), \widehat{f}(x + r)]$, there exists $R > 0$ so that $B_R(y) \subset [\widehat{f}(x - r), \widehat{f}(x + r)]$.

Finally, we show that $g(z) \to g(y)$ as $z \to y$.
Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|z - y| < \delta \Longrightarrow |\widehat{f}^{-1}(z) - \widehat{f}^{-1}(y)| < \varepsilon$.
Set $\delta' = \min(\delta, R)$. Then $|z - y| < \delta' \Longrightarrow |g(z) - g(y)| = |\widehat{f}^{-1}(z) - \widehat{f}^{-1}(y)| \leq \varepsilon$.

Finally, we show that g is differentiable.

Note that $z = f(g(z))$ and $y = f(g(y))$. So, we can write that:

$$\lim_{z\to y} \frac{g(z)-g(y)}{z-y} = \lim_{z\to y} \frac{g(z)-g(y)}{f(g(z))-f(g(y))}.$$

Now let (z_n) be any sequence in the domain of g converging to y such that $z_n \neq y$ for any n. Since $\lim\limits_{z\to y} g(z) = g(y)$ for all $y \in \mathrm{dom}(g)$, we know that $g(z_n) \to g(y)$.

Meanwhile, note that since $f'(x) \neq 0$ for all $x \in \mathrm{dom}(f)$, we can evaluate that:

$$\lim_{t\to g(y)} \frac{t-g(y)}{f(t)-f(g(y))} = \frac{1}{f'(g(y))}.$$

So, given any sequence (t_n) in the domain of f converging to $g(y)$ such that $t_n \neq g(y)$ for any n, we have that $\frac{t_n-g(y)}{f(t_n)-f(g(y))} \to \frac{1}{f'(g(y))}$. Since g is injective and $z_n \neq y$ for any n, $(g(z_n))$ is one such sequence. Hence:

$$\frac{g(z_n)-g(y)}{f(g(z_n))-f(g(y))} \to \frac{1}{f'(g(y))}.$$

Since this is true for all relevant (z_n), we conclude that:

$$\lim_{z\to y} \frac{g(z)-g(y)}{z-y} = \lim_{z\to y} \frac{g(z)-g(y)}{f(g(z))-f(g(y))} = \frac{1}{f'(g(y))}$$

Thus, $g'(y)$ exists and equals $\frac{1}{f'(g(y))}$.

Plugging in $y = f(x)$, we get that $g'(f(x)) = \frac{1}{f'(x)}$.

Exercise 5.4: If $C_0 + \frac{C_1}{2} + \ldots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$ and $C_0, C_1, \ldots, C_n \in \mathbb{R}$, then we shall prove that the equation $C_0 + C_1x + \ldots + C_{n-1}x^{n-1} + C_nx^n = 0$ has at least one real root between 0 and 1.

Define the functions:

$$f(x) = C_0 + C_1x + \ldots + C_{n-1}x^{n-1} + C_nx^n$$
$$F(x) = C_0x + \frac{C_1}{2}x + \ldots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1}$$

Note that $F(0) = 0$ and $F(1) = C_0 + \frac{C_1}{2} + \ldots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$ At the same time, F is differentiable with $F'(x) = f(x)$. Therefore, by the mean value theorem there exists $t \in (0,1)$ such that $0 = F'(t) = f(t)$. Thus, that t is a real root between 0 and 1 for the equation $C_0 + C_1x + \ldots + C_{n-1}x^{n-1} + C_nx^n = 0$.

Exercise 5.6: Suppose the following conditions on f:
(A) f is continuous for $x \geq 0$
(B) f' exists for $x > 0$
(C) $f(0) = 0$
(D) f' is monotonically increasing

Putting $g(x) = \frac{f(x)}{x}$ for $x > 0$, we shall prove that g is monotonically increasing.

Firstly, given any $x > 0$, because of conditions A and B, we can apply the mean value theorem to say that there exists $t \in (0, x)$ such that $f(x) - f(0) = xf'(t)$. Because of condition C, this then simplifies to $f(x) = xf'(t)$. So:

for all $x > 0$, there exists $0 < t < x$ such that $\frac{f(x)}{x} = f'(t)$.

Meanwhile, because of condition B and the quotient rule, g is differentiable when $x > 0$ with $g'(x) = \frac{f'(x)x - f(x)}{x^2}$. So, consider any $b > a > 0$. By the mean value theorem, there exists $s \in (a, b)$ with $g(b) - g(a) = (b - a)g'(s)$. Obviously, $b - a$ is positive. Additionally, consider that:

$$g'(s) = \frac{f'(s)s - f(s)}{s^2} = \frac{1}{s}\left(f'(s) - \frac{f(s)}{s}\right).$$

Pick $t > 0$ such that $t < s$ and $\frac{f(s)}{s} = f'(t)$. Then $g'(s) = \frac{1}{s}(f'(s) - f'(t))$. But, because of condition D, we know that $f'(s) \geq f'(t)$. Hence, $g'(s) \geq 0$.

Therefore, $g(b) - g(a) \geq 0$, meaning g is monotonically increasing.

Exercise 5.8: Consider any real-valued function f which is differentiable on $[a, b]$ with f' being continuous on $[a, b]$. Then we shall prove that:

$$\forall \varepsilon > 0, \;\; \exists \delta > 0 \;\; s.t. \;\; \forall x, t \in [a, b], \;\; 0 < |t - x| < \delta \Longrightarrow \left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| < \varepsilon$$

Because f' is continuous over a compact domain, we know that by theorem 4.19 (proposition 76), f' is uniformly continuous. Thus, let $\varepsilon > 0$ and pick $\delta > 0$ such that for all $x, y \in [a, b]$, we have that $|x - y| < \delta \Longrightarrow |f'(x) - f'(y)| < \varepsilon$.

Since f is differentiable on $[a, b]$, we know by the mean value theorem that for any distinct x and t in $[a, b]$, there exists s between a and b such that:

$$\frac{f(t) - f(x)}{t - x} = f'(s).$$

Hence, $\left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| = |f'(s) - f'(x)|$. And since $|s - x| < |t - x|$, we know that if $0 < |t - x| < \delta$, then $|f'(s) - f'(x)| < \varepsilon$.

An analogous theorem holds for any vector-valued function $\vec{f} : [a, b] \longrightarrow \mathbb{R}^k$ that is differentiable on $[a, b]$ with $\vec{f}\,'$ being continuous on $[a, b]$.

Let $\vec{f}(x) = (f_1(x), f_2(x), \ldots, f_k(x))$. Since $\vec{f}$ is differentiable on $[a, b]$ and $\vec{f}\,'$ is continuous on $[a, b]$, we have for each $i \in \{1, \ldots, k\}$ that f_i is differentiable on $[a, b]$ and f_i' is continuous on $[a, b]$.

Thus, given any $\varepsilon > 0$, we already proved that for each $i \in \{1, \ldots, k\}$, there exists $\delta_i > 0$ such that $\forall t, x \in [a,b], \quad |t-x| < \delta_i \implies \left| \frac{f_i(t)-f_i(x)}{t-x} - f_i'(x) \right| < \frac{1}{\sqrt{k}} \cdot \varepsilon$.
Then setting $\delta = \min(\delta_1, \ldots, \delta_k)$, we have that if $0 < |t-x| < \delta$, then:

$$\left\| \frac{\vec{f}(t)-\vec{f}(x)}{t-x} - \vec{f}'(x) \right\| = \left(\left(\left| \frac{f_1(t)-f_1(x)}{t-x} - f_1'(x) \right| \right)^2 + \ldots + \left(\left| \frac{f_k(t)-f_k(x)}{t-x} - f_k'(x) \right| \right)^2 \right)^{\frac{1}{2}}$$
$$< \left(\left(\frac{1}{\sqrt{k}} \cdot \varepsilon \right)^2 + \ldots + \left(\frac{1}{\sqrt{k}} \cdot \varepsilon \right)^2 \right)^{\frac{1}{2}} = \sqrt{k \left(\frac{1}{k} \cdot \varepsilon^2 \right)} = \varepsilon$$

Exercise 5.9: Let $x_0 \in (a,b)$ and $f : (a,b) \longrightarrow \mathbb{R}$ be continuous at x_0. If $f'(x)$ exists for all $x \in (a,b) \setminus \{x_0\}$ and $\lim\limits_{t \to x_0} f'(t) = L$, then $f'(x_0) = L$.

Since f is continuous at x_0 and x_0 is a limit point of (a,b), we know that $f(x_0)$ exists and that $\lim\limits_{t \to x_0} f(t) = f(x_0)$. So, define $g(x) = f(x) - f(x_0)$. Then $g'(x) = f'(x)$ and $\lim\limits_{t \to x_0} g(t) = 0$. Additionally, define $h(x) = x - x_0$. Then $h'(x) = 1$ and $\lim\limits_{t \to x_0} h(t) = 0$.

Importantly, both g and h are differentiable everywhere on $(a,b) \setminus \{x_0\}$. Also, $h'(t) \neq 0$ for all $t \in (a,b)$. Thus, we can apply L'hôpital's rule to get that:

$$\lim_{t \to x_0} \frac{f(t)-f(x_0)}{t-x_0} = \lim_{t \to x_0} \frac{g(t)}{h(t)} = \lim_{t \to x_0} \frac{g'(t)}{h'(t)} = \lim_{t \to x_0} f'(t) = L$$

Hence $f'(x_0)$ exists and equals L.

To answer what's actually asked in the book,
set $a = -\infty$, $b = +\infty$, $x_0 = 0$ and $L = 3$.

Exercise 5.17: Suppose f is a real, three times differentiable function on $[-1,1]$ such that $f(-1) = 0$, $f(0) = 0$, $f(1) = 1$, and $f'(0) = 0$. Then $f'''(x) \geq 3$ for some $x \in (-1,1)$.

Since f is three times differentiable on $[-1,1]$, we know that f'' is continuous on $[-1,1]$ and that $f'''(t)$ exists for every $t \in (-1,1)$. So define:

$$P(t) = f(0) + f'(0)t + \frac{f''(0)}{2}t^2 = \frac{f''(0)}{2}t^2$$

Then by Taylor's theorem, we know that there exists $s \in (0,1)$ such that $f(1) = P(1) + \frac{f'''(s)}{6}x^3 = \frac{f''(0)}{2} + \frac{f'''(s)}{6}$. Similarly, we know that there exists $t \in (-1,0)$ such that $f(-1) = \frac{f''(0)}{2} - \frac{f'''(t)}{6}$.

Thus, $\frac{f'''(s)}{6} + \frac{f'''(t)}{6} = f(1) - f(-1) = 1$, which in turn means that $f'''(s) + f'''(t) = 6$. If both $f'''(s)$ and $f'''(t)$ are less than 3, then this is impossible. So, either s or t must be greater than or equal to 3.

Exercise 5.26: Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ for $x \in [a, b]$. Then $f(x) = 0$ for all $x \in [a, b]$.

To start off, note that if $A < 0$, then we automatically have that $f'(x) = f(x) = 0$ for all $x \in [a, b]$. Meanwhile, if $A = 0$, then $f'(x) = 0$ for all $x \in [a, b]$, thus forcing f to be a constant function. Then, as $f(a) = 0$, we have that $f(x) = f(a) = 0$ for all $x \in [a, b]$.

Therefore, we now assume $A > 0$ and observe the following:

Assume $\gamma \in [a, b)$ and $f(\gamma) = 0$. Then let $x_0 \in [\gamma, b]$ and set $M = \sup\limits_{\gamma \leq x \leq x_0} |f(x)|$.

Then for any $x \in (\gamma, x_0]$, we know by the mean value theorem that there exists $t \in (\gamma, x)$ such that $f(x) - f(\gamma) = (x - \gamma)f'(t)$. Since $f(\gamma) = 0$ and $x > \gamma$, we thus know that $|f(x)| = (x - \gamma)|f'(t)|$. Hence:

$$|f(x)| = (x - \gamma)|f'(t)| \leq (x - \gamma)A|f(t)| \leq A(x - \gamma)M \leq A(x_0 - \gamma)M$$

Now importantly, since f is continuous on $[\gamma, x_0]$, and $g(x) = |x|$ is continuous on all of $\mathbb{R}$, we know that $(g \circ f)(x) = |f(x)|$ is continuous on $[\gamma, x_0]$. That combined with the fact that $[\gamma, x_0]$ is compact means that we can fix $x \in [\gamma, x_0]$ such that $|f(x)| = M$. Then:

- If $x = \gamma$, then $M = |f(\gamma)| = 0$.
- If $x \neq \gamma$, then $M = |f(x)| \leq A(x_0 - \gamma)M$. Crucially, if $\gamma < x_0 < \gamma + \frac{1}{A}$ then $0 < A(x_0 - \gamma) < 1$. Therefore, the only way for $M \leq A(x_0 - \gamma)M$ is if $M = 0$.

Thus, for $x_0 \in [\gamma, \gamma + \frac{1}{A}) \cap [\gamma, b]$, we have that $\sup\limits_{\gamma \leq x \leq x_0} |f(x)| = 0$.

In other words, $f(x) = 0$ for all $x \in [\gamma, \gamma + \frac{1}{A}) \cap [\gamma, b]$.

Still assuming $A > 0$, we have that $0 < \frac{1}{2A} < \frac{1}{A}$. So for any $\gamma \in [a, b]$, we know that $[\gamma, \gamma + \frac{1}{2A}) \cap [\gamma, b] \subseteq [\gamma, \gamma + \frac{1}{A}) \cap [\gamma, b]$. Hence, we now proceed by the following inductive process:

Start with $\gamma_1 = a$.

Now do this until told to stop.

If $\gamma_i = b$, then stop. Otherwise, use the above reasoning to show that $f(x) = 0$ for all $x \in [\gamma_i, \min(\gamma_i + \frac{1}{2A}, b)]$. Then set $\gamma_{i+1} = \min(\gamma_i + \frac{1}{2A}, b)$ and repeat these steps with γ_{i+1}.

This algorithm will terminate in $\left\lceil \frac{b-a}{\frac{1}{A}} \right\rceil$ iterations, thus showing that $f(x) = 0$ for all $x \in [a, b]$.