Math 100A Notes (Professor: Aaron Pollack)

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Lecture 1 Notes: 9/27/2024

Motivation for this class:
Let $\mathcal{F}$ be any figure in $\mathbb{R}^2$. We want some way of talking about the symmetries of $\mathcal{F}$.

Letting d be the standard metric for $\mathbb{R}^2$, we say $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is distance preserving if $d(P, Q) = d(f(P), f(Q))$ for all $P, Q \in \mathbb{R}^2$. If f is distance-preserving and $f(\mathcal{F}) = \mathcal{F}$, then we call f a symmetry of $\mathcal{F}$.

We define $\mathrm{Sym}(\mathcal{F})$ to be the set of symmetries of $\mathcal{F}$.

Lemma 2: The set $\mathrm{Sym}(\mathcal{F})$ has the following properties:

1. The identity map Id is in $\mathrm{Sym}(\mathcal{F})$
2. If $f \in \mathrm{Sym}(\mathcal{F})$, then $f^{-1} \in \mathrm{Sym}(\mathcal{F})$.

   I realize we haven't yet shown that every $f \in \mathrm{Sym}(\mathcal{F})$ is a bijection. Given such an f, it's easy to see that f must be injective. After all, the distance preserving property of f means that $f(P) = f(Q) \implies P = Q$. Showing that f is surjective is harder. By assumption, we know that f is surjective when restricted to $\mathcal{F}$. More complicatedly, we can show that f must have a certain form which happens to be surjective. Perhaps I'll prove that later.

   Once, you've accepted that f^{-1} exists, then it's clearly true that f^{-1} is also order-preserving with $f^{-1}(\mathcal{F}) = \mathcal{F}$.
3. If $f_1, f_2 \in \mathrm{Sym}(\mathcal{F})$, then $f_1 \circ f_2 \in \mathrm{Sym}(\mathcal{F})$ and $f_2 \circ f_1 \in \mathrm{Sym}(\mathcal{F})$.

   This is pretty trivial to show.

Now while it's all good that we have a concrete way of describing the symmetries of a figure, our current terminology is not the most useful. After all, suppose $\mathcal{S}$ and $\mathcal{S}'$ are two squares such that $\mathcal{S}$ is centered at the origin and $\mathcal{S}'$ is centered at the point $(5, 5)$. Then even though we know both $\mathcal{S}$ and $\mathcal{S}'$ have symmetries in the form of rotating and reflecting, the particular functions in $\mathrm{Sym}(\mathcal{S})$ and $\mathrm{Sym}(\mathcal{S})$ will be different (except for Id). So, how do we compare the symmetries of those two squares?

Proof that all symmetries are surjective (from our textbook):

Note:

- Our textbook calls a distance-preserving function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ an isometry.
- Rather than writing $f_1 \circ f_2$ to represent function composition, our textbook just writes $f_1 f_2$.

Some Facts:

(a) Orthogonal linear operators are isometries.

Let φ be n orthogonal linear map. φ being linear means that $\varphi(u) - \varphi(v) = \varphi(u - v)$. Meanwhile, φ being orthogonal means that $|\varphi(u - v)| = \sqrt{\varphi(u - v) \cdot \varphi(u - v)} = \sqrt{(u - v) \cdot (u - v)} = |u - v|$. So, for any $u, v \in \mathbb{R}^n$, we have that $|\varphi(u) - \varphi(v)| = |u - v|$.

(b) The translation t_a by a vector a defined by $t_a(x) = x + a$ is an isometry.

For any $u, v \in \mathbb{R}^n$, we have $|t_a(u) - t_a(v)| = |u + a - v - a| = |u - v|$.

(c) The composition of isometries is an isometry.

If f_1, f_2 are isometries, then for all $u, v \in \mathbb{R}^n$, we have that $|f_1(f_2(u)) - f_1(f_2(v))| = |f_2(u) - f_2(v)| = |u - v|$.

Our textbook is *Algebra, Second Edition* by Michael Artin.