Math 240A Notes (Professor: Luca Spolaor)

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October 3, 2024

Lecture 1 Notes: 9/26/2024

Given an indexed family of sets $\{X_\alpha\}_{\alpha \in A}$, we define its Cartesian Product to be:

$$\prod_{\alpha \in A} X_\alpha = \{f : A \longrightarrow \bigcup_{\alpha \in A} X_\alpha \mid f(\alpha \in X_\alpha)\}$$

A projection is a function $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \longrightarrow X_\alpha$ satisfying that $f \mapsto f(\alpha)$.

If X, Y are sets, we define:

- $\operatorname{card}(X) \leq \operatorname{card}(Y)$ if there exists an injection $f : X \longrightarrow Y$.
- $\operatorname{card}(X) \geq \operatorname{card}(Y)$ if there exists a surjection $f : X \longrightarrow Y$.
- $\operatorname{card}(X) = \operatorname{card}(Y)$ if there exists a bijection $f : X \longrightarrow Y$.

Note that $\operatorname{card}(X) \leq \operatorname{card}(Y) \iff \operatorname{card}(Y) \geq \operatorname{card}(X)$. After all, given an injection in one direction, we can easily make a surjection in the other direction. Or given a surjection in one direction, we can (using A.O.C (axiom of choice)) easily make an injection in the other direction.

Also, if $\operatorname{card}(X) \leq \operatorname{card}(Y)$ and $\operatorname{card}(Y) \leq \operatorname{card}(X)$, then we know that $\operatorname{card}(Y) = \operatorname{card}(X)$.

Proof:
We know there exists $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ which are both injective. Hence, $g \circ f$ is an injection from X to $g(Y) \subseteq X$. By an exercise done in my math journal on page 8, we thus there exists a bijection h from X to $g(Y)$. And letting g^{-1} be any left-inverse of g, we then have that $g^{-1} \circ h$ is a bijection from X to Y.

We say X has the cardinality of the continuum if $\operatorname{card}(X) = \operatorname{card}(\mathbb{R})$.

Proposition: $\operatorname{card}(\mathcal{P}(\mathbb{N})) = \operatorname{card}(\mathbb{R})$.

Our textbook goes about proving this by constructing two functions: an injection and a surjection, from $\mathcal{P}(\mathbb{N})$ to $\mathbb{R}$ based on the binary expansion of any real number. That way, we know that $\operatorname{card}(\mathcal{P}(\mathbb{N})) \leq \operatorname{card}(\mathbb{R})$ and $\operatorname{card}(\mathcal{P}(\mathbb{N})) \geq \operatorname{card}(\mathbb{R})$.

Given a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ we know there exists: $\limsup x_n = \inf_{k \geq 1}(\sup_{n \geq k} x_n)$ and $\liminf x_n = \sup_{k \geq 1}(\inf_{n \geq k} x_n)$.

Also, given a function $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, we can define:

$$\limsup_{x \to a} f(x) = \inf_{\delta > 0}\left(\sup_{0 < |x-a| < \delta} f(x)\right).$$

If X is an arbitrary set and $f : X \longrightarrow [0, \infty]$, we define:

$$\sum_{x \in X} f(x) = \sup \left\{ \sum_{x \in F} f(x) \mid F \subseteq X \ s.t. \ F \text{ is finite} \right\}.$$

Cool Proposition from textbook (not covered in lecture):

Let $A = \{x \in X \mid f(x) > 0\}$. If A is uncountable, then $\sum_{x \in X} f(x) = \infty$.

If A is countably infinite and $g : \mathbb{N} \longrightarrow A$ is a bijection, then $\sum_{x \in X} f(x) = \sum_{n=1}^{\infty} f(g(n))$.

Proof of first statement:
$A = \bigcup_{n \in \mathbb{N}} A_n$ where $A_n = \{x \in X \mid f(x) > \frac{1}{n}\}$.

If A is uncountable, we must have that some A_n is uncountable. But then for any finite set $F \subseteq X$, we have that $\sum_{x \in F} f(x) > \frac{\text{card}(F)}{n}$. So $\sum_{x \in X} f(x)$ is unbounded.

A metric space (X, ρ) is a set X equipped with a distance function $\rho : X \times X \longrightarrow [0, \infty)$. We denote the open ball of radius r about x to be $B(r, x) = \{y \in X \mid \rho(x, y) < r\}$. And you remember our definitions from 140A... right?

Proposition 0.21: Every open set in $\mathbb{R}$ is a countable union of disjoint open intervals.

We proved this as part of a homework exercise in Math 140A.

Given a metric space (X, ρ), an element $x \in X$, and sets $F, E \subseteq X$, we can define:

- $\rho(x, E) = \rho_E(x) = \inf\{\rho(x, y) \mid y \in E\}$.
- $\rho(F, E) = \inf\{\rho_E(y) \mid y \in F\}$.

Exercise: $g(x, E) = 0 \iff x \in \overline{E}$.

Proof:
If $\inf\{\rho(x, y) \mid y \in E\} = 0$, then there exists a sequence $\{y_n\}$ in E such that $\rho(x, y_n) \to 0$. This implies $x \in \overline{E}$. Similarly, if $x \in \overline{E}$, we can construct a sequence $\{y_n\}$ such that $\rho(x, y_n) < \frac{1}{n}$ for all n. Then:

$$0 \leq \inf\{\rho(x, y) \mid y \in E\} \leq \inf\{\rho(x, y_n) \mid n \in \mathbb{N}\} = 0.$$

Given a subset E of a metric space (X, ρ), we define:

$$\text{diam}(E) = \sup\{\rho(x, y) \mid x, y \in E\}.$$

If $\text{diam}(E) < \infty$, we say E is bounded. If $\forall \varepsilon > 0$, E can be covered by finitely many balls of radius ε, then we say E is totally bounded.

Exercise: E being totally bounded implies E is bounded.

Pick $\varepsilon > 0$ and let $\{z_1, \ldots, z_n\}$ be the set of points such that $E \subseteq \bigcup_{k=1}^{n} B(\varepsilon, z_n)$.

Then given any $x, y \in E$, we can assume that $x \in B(\varepsilon, z_i)$ and $y \in B(\varepsilon, z_j)$. So, $\rho(x,y) \leq \rho(x,z_i) + \rho(z_i,z_j) + \rho(z_j,y) < 2\varepsilon + \max\{\rho(z_i,z_j) \mid 1 \leq i,j \leq n\}$.

The converse is not generally true. For instance, if you use the discrete metric, then any set with more than one element will have a diameter of 1. But if $0 < \varepsilon < 1$, then it will be impossible to cover an infinite set with finitely many balls.

Lecture 2 Notes: 10/1/2024

Proposition: Suppose E is a subset of a metric space (X, ρ). Then the following are equivalent.

1. E is complete and totally bounded
2. All sequences $(x_n) \subseteq E$, have a convergent subsequence.
3. For all open covers $\{V_\alpha\}_{\alpha \in A}$ of E, there exists $V_{\alpha_1}, \ldots, V_{\alpha_n}$ such that $E \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$.

Proof:
(1) $\Longrightarrow$ (2):

Lemma:
If E is totally bounded and $F \subseteq E$, then F is totally bounded.

Given any $\varepsilon > 0$, let $\{x_1, \ldots, x_n\}$ be a subset of E such that $E \subseteq \bigcup_{i=1}^{n} B(\varepsilon/2, x_i)$. Then consider the collection of sets:

$$\{F \cap B(\varepsilon/2, x_i)\} - \{\emptyset\}.$$

We know the diameter of each $F \cap B(\varepsilon/2, x_i)$ is at most ε. So in each set, pick $y_i \in F \cap B(\varepsilon/2, x_i)$. Then for some $m \leq n$:

$$F \subseteq \bigcup_{i=1}^{m} B(\varepsilon, y_i)$$

Let $A_1 = E$. Then for $k \geq 2$ we recursively define A_k as follows:

Assuming $A_{k-1} \cap (x_n)_{n \in \mathbb{N}}$ is infinite and A_{k-1} is totally bounded, choose $\{y_1, \ldots, y_m\}$ in A_k such that $A_k \subseteq \bigcup_{i=1}^{m} B(2^{-n}, y_i)$. Importantly, since $(x_n)_{n \in \mathbb{N}} \cap A_{k-1}$ is infinite, we know one of those open balls contains infinitely many points in our sequence. So set A_k equal to that ball intersected with E. Note that by our lemma, A_k is totally bounded.

Now pick any x_{n_1} and then for all $k \geq 2$ pick $x_{n_k} \in A_k$ such that $n_k > n_{k-1}$. That way, $(x_{n_k})_{k\in\mathbb{Z}_+}$ is a subsequence of $(x_n)_{n\in\mathbb{Z}_+}$. Also, we know that $(x_{n_k})_{k\in\mathbb{Z}_+}$ is Cauchy. Hence, since E is complete, we know that it converges to some x in E.

(2) $\Longrightarrow$ (1):

Firstly, suppose E is not complete. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ that is Cauchy but does not converge in E. Importantly, because $(x_n)_{n\in\mathbb{N}}$ is Cauchy, if there was a convergent subsequence, we know the limit of that subsequence would have to be the limit of the whole sequence. But that doesn't exist. So, we know (2) can't be true.

Secondly, suppose E is not totally bounded. Then there exists $\varepsilon > 0$ such that it is impossible to cover E in balls of radius ε. So, we can recursively define a sequence $(x_n)_{n\in\mathbb{N}}$ in E satisfying that:

$$x_n \in E - \bigcup_{i=1}^{n-1} B(\varepsilon, x_i).$$

Importantly, for all natural numbers $n \neq m$, we have that $\rho(x_n, x_m) \geq \varepsilon$. So, it is impossible to find a convergent subsequence of (x_n), meaning (2) is false.

(1) and (2) $\Longrightarrow$ (3):

Let $\{V_\alpha\}_{\alpha\in A}$ be an open cover of E.

Suppose for the sake of contradiction that for all $n \in \mathbb{N}$, there is a ball B_n of radius 2^{-n} centered in E such that $B_n \cap E \neq \emptyset$ but $B_n \not\subseteq V_\alpha$ for all $\alpha \in A$. Then we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $x_n \in B_n \cap E$ for all $n \in \mathbb{N}$. By (2), we know there is a subsequence that converges to some $x \in E$. Importantly, we know $x \in V_\alpha$ for some $\alpha \in A$, and because V_α is open, there is $\varepsilon > 0$ such that $B(\varepsilon, x) \subseteq V_\alpha$. But now we get a contradiction because by picking n such that $2^{-n} < \varepsilon/3$ and $\rho(x, x_n) < \varepsilon/3$, we have for all $y \in B_n$ that:

$$\rho(x, y) \leq \rho(x, x_n) + \rho(x_n, y) < 2^{-n} + 2^{-n+1} < \varepsilon$$

So $B_n \subseteq B(\varepsilon, x) \subseteq V_\alpha$.

We've thus shown that for some $n \in N$, all balls of radius 2^{-n} centered in E are contained by some V_α. And assuming (1), we can cover E with finitely many balls of radius 2^{-n} It follows that by picking a V_α containing a ball for each ball covering E, we've found a finite covering E using the sets in $\{V_\alpha\}_{\alpha\in A}$.

(3) $\Longrightarrow$ (2):

Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in E with no convergent subsequence. Then for each $x \in E$, there must exist $\varepsilon_x > 0$ such that $B(\varepsilon_x, x) \cap (x_n)_{n\in\mathbb{N}}$ is finite. (If ε_x didn't exist, we could construct a Cauchy subsequence converging to x).

But now $\{B(\varepsilon_x, x)\}_{x\in E}$ is an open cover of E with no finite subcover of E because it will take an infinite cover to cover all of $(x_n)_{n\in\mathbb{N}}$.

If E satisfies all three of the above properties, we say E is compact.

Corollary: $K \subseteq \mathbb{R}^n$ is compact iff it's closed and bounded.

Roughly speaking, we want a measure to be a function $\mu : \mathcal{P}(\mathbb{R}^n) \longrightarrow [0, \infty)$ such that $E \mapsto \mu(E) =$ "the area of E". Also, we would like it if:

(i) $\mu([0,1)^n) = 1$

(ii) $\mu($rotation, translation, or reflection of $A) = \mu(A)$

(iii) $\mu(\bigcup\limits_{i=1}^{\infty} A_i) = \sum\limits_{i=1}^{\infty} \mu(A_i)$ if $A_i \cap A_j \neq \emptyset \Longrightarrow i = j$.

Unfortunately, the properties as written above are inconsistent.

Vitali Sets:
Defining $x \sim y$ iff $x - y \in \mathbb{Q}$, let $N \subseteq [0,1)$ be a set such that $N \cap [x]$ has precisely one element for all $x \in \mathbb{R}$. Next let $R = [0,1) \cap \mathbb{Q}$, and for all $r \in R$ define:

$$N_r = \{x + r \mid x \in N \cap [0, 1-r)\} \cup \{x + r - 1 \mid x \in N \cap [1-r, 1)\}.$$

Importantly, note that $N_r \subseteq [0,1)$. Plus, the two sets being unioned over to make N_r are both disjoint and can be translated around so that they are still disjoint but their union forms N. Hence assuming $\mu : \mathcal{P}(\mathbb{R}^n) \longrightarrow [0, \infty)$ satisfying (ii) and (iii), we know $\mu(N_r) = \mu(N)$.

Also, for all $y \in [0,1)$, if $x \in N \cap [y]$, we know that $y \in N_r$ where $r = x - y$ if $x \geq y$, or $r = x - y + 1$ if $x < y$. Hence, $[0,1) = \bigcup\limits_{r\in R} N_r$.

Also, given any N_r and N_s, if $x \in N_r \cap N_s$, then we'd be able to show that both $x - r$ or $x - r + 1$ and $x - s$ or $x - s + 1$ are distinct elements of N in the same equivalence class, which contradicts how we defined N.

You work through the scratch work of the different cases on your own! :P

So supposing μ satisfies (i) and (iii) and because R is countable, we have that:

$$1 = \sum_{r\in R} \mu(N_r) = \sum_{r\in R} (N) = 0 \text{ or } \infty.$$

This is a contradiction.