Math 188 Notes (Professor: Steven Sam)

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Lecture 1 Notes: 9/27/2024

Linear Recurrence Relations:
A sequence $(a_n)_{n\geq 0}$ satisfies a linear recurrence relation of order d if there exists $c_1, \ldots, c_d$ with $c_d \neq 0$ such that for all $n \geq d$:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_d a_{n-d}$$

(For $0 \leq n < d$, we usually explicitely specify a_n.)

To start this course, we're gonna discuss finding explicit (non-recursive) solutions.

Firstly, if $d = 1$, then this problem is easy. We can just plug in previous elements repeatedly to get that:

$$a_n = c_1 a_{n-1} = c_1^2 a_{n-2} = \ldots = c_1^n a_0$$

If $d = 2$, then plugging in previous elements doesn't help us really anymore. So how do we solve this problem now?

Theorem: Consider the characteristic polynomial $t^2 - c_1 t - c_2$ and let r_1, r_2 be the roots of that polynomial. If $r_1 \neq r_2$, then there exists α_1, α_2 such that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for all $n \geq 0$.

To solve for α_1 and α_2, plug in different values of n into our equation. Since $r_1 \neq r_2$, we know the below linear system has a unique solution:

$$a_0 = \alpha_1 + \alpha_2$$
$$a_1 = \alpha_1 r_1 + \alpha_2 r_2$$

Now backing up, why does the above method work?

Approach 1: (Vector Spaces)
The set of sequences $(a_n)_{n\geq 0}$ form a vector space. Furthermore given any constants c_1 and c_2, we know that the set of sequences satisfying $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ for all $n \geq 2$ is a subspace.

Proof:
Suppose (a_n) and (b_n) both satisfy that $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and $b_n = c_1 b_{n-1} + c_2 b_{n-2}$. Then given any constants γ and δ, we have that:

$$(\gamma a_n + \delta b_n) = c_1(\gamma a_{n-1} + \delta b_{n-1}) + c_2(\gamma a_{n-2} + \delta b_{n-2})$$

Hence, all linear combinations of any two sequences satisfying our linear recurrence relation also satisfies our linear recurrence relation.

Now what our above theorem is stating is that the sequences (r_1^n) and (r_2^n) span the subspace of solutions to our linear recurrence relation.

To see this, first note that (r_1^n) and (r_2^n) satisfy our recurrence relation.

If $n \geq 2$, then $r_i^n - c_1 r_i^{n-1} - c_2 r_i^{n-2} = r_i^{n-2}(r_i^2 - c_1 r_i - c_2) = r_i^{n-2}(0)$.
Hence, we know that $r_i^n = c_1 r_i^{n-1} + c_2 r_i^{n-2}$ for all $n \geq 2$.

Also, since we assumed $r_1 \neq r_2$, we know that (r_1^n) is linearly independent to (r_2^n). And finally, as mentioned before, we can solve a linear system of equations to find coffecients for a linear combination of (r_1^n) and (r_2^n) equal to any other sequence satisfying our recurrence relation.

Approach 2: (Formal Power Series)

Define the power series $A(x) = \sum_{n \geq 0} a_n x^n$. We call $A(x)$ a generating function of the sequence (a_n).

(We'll treat the formal power series more rigorously later...)

Now note that:

$$\begin{aligned}
A(x) &= a_0 + a_1 x + \sum_{n \geq 2} a_n x^n \\
&= a_0 + a_1 x + \sum_{n \geq 2} (c_1 a_{n-1} + c_2 a_{n-2}) x^n \\
&= a_0 + a_1 x + c_1 \sum_{n \geq 2} a_{n-1} x^n + c_2 \sum_{n \geq 2} a_{n-2} x^n \\
&= a_0 + a_1 x + c_1 (A(x) - a_0) x + c_2 (A(x)) x^2
\end{aligned}$$

Isolating $A(x)$, we get the equation: $A(x) = \dfrac{a_0 + a_1 x - a_0 c_1 x}{1 - c_1 x - c_2 x^2}$.

Next, let's do fraction decomposition on our equation for $A(x)$.

Issue: We defined r_1 and r_2 as the roots of $t^2 - c_1 t - c_2 = (t - r_1)(t - r_2)$.

Trick: Plug in $t = \frac{1}{x}$. That way, we have that:

$$x^{-2} - c_1 x^{-1} - c_2 = (x^{-1} - r_1)(x^{-1} - r_2).$$

After that, multiply both sides of our equation by x^2 to get that:

$$1 - c_1 x - c_2 x^2 = (1 - r_1 x)(1 - r_2 x)$$

Since we're assuming $r_1 \neq r_2$, we know that for some constants α_1 and α_2, we have that:

$$A(x) = \frac{\alpha_1}{1 - r_1 x} + \frac{\alpha_2}{1 - r_2 x}$$

(If $r_1 = r_2$, then this step is where things will go differently.)

Now finally, we can rewrite $\frac{\alpha_1}{1 - r_1 x}$ as the geometric series $\alpha_1 \sum_{n \geq 0} (r_1 x)^n$. Doing likewise with $\frac{\alpha_2}{1 - r_2 x}$, we get that:

$$A(x) = \sum_{n \geq 0} a_n x^n = \alpha_1 \sum_{n \geq 0} (r_1 x)^n + \alpha_2 \sum_{n \geq 0} (r_2 x)^n = \sum_{n \geq 0} (\alpha_1 r_1^n + \alpha_2 r_2^n) x^n$$

Hence, we have for each n that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$.

Lecture 2: 9/30/2024