10/17/2025

So for functional analysis I need some fixed point theorems that would normally have been taught in a topology class. But I've never taken an actual topology class. Hence, my goal for today is to prove those theorems. First, I will be taking notes from the paper *Brouwer's Fixed Point Theorem* by Jasmine Katz. (See the *17th entry* of the bibliography...)

Katz defines a convex body in $\mathbb{R}^n$ to be a set $X \subseteq \mathbb{R}^n$ that is compact, convex, and has a nonempty interior (relative to the Euclidean topology of all of $\mathbb{R}^n$).

Also, for $m \geq n \geq 1$ suppose we are given any $n+1$ points $p_0, p_1 \ldots, p_n \in \mathbb{R}^m$ in general linear position (see my math 190b notes). Then we say the convex hull of those points (i.e. $\operatorname{conv}(p_0, p_1 \ldots, p_n)$) is an n-simplex. Recall from the notes I took before I dropped math 190b last Spring that all n-simplices are compact and homeomorphic to each other.

Katz in particular denotes $\Delta_0^n := \operatorname{conv}(e_1, \ldots, e_n, -\sum_{i=1}^n e_i)$ where $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$. Also, Katz denotes Δ^n to be the standard n-simplex $\operatorname{conv}(u_1, \ldots, u_{n+1})$ where $u_1, \ldots, u_{n+1}$ are the standard basis vectors of $\mathbb{R}^{n+1}$.

Proposition 3.2: Δ_0^n is a convex body in $\mathbb{R}^n$.

Proof:

The only thing not trivial from the definition of Δ_0^n is that Δ_0^n has a nonempty interior. So to prove this, first define for each $k \in \mathbb{N}$ the $(k+1)$ by $(k+1)$ matrices:

$$A_k := \left(\begin{array}{cc} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ \begin{matrix} 1 & \cdots & 1 \end{matrix} & 1 \end{array}\right) \text{ and } B_k := \left(\begin{array}{cc} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \\ 1 & \begin{matrix} 1 & \cdots & 1 \end{matrix} \end{array}\right)$$

We can calculate the determinant of A_k as follows:

Clearly $\det(A_1) = \det(\left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]) = 2$ and $\det(B_1) = \det(\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]) = -1$.

Meanwhile, by considering the Laplace expansions going along the top rows of A_k and B_k where $k > 1$, we get that:

$$\det(A_k) = \det(A_{k-1}) + (-1)^{k+3}\det(B_{k-1}) \text{ and } \det(B_k) = -\det(B_{k-1})$$

From there it is easy to see that $\det(B_k) = (-1)^k$. And hence:

$$\begin{aligned} \det(A_k) = \det(A_{k-1}) + (-1)^{(k+3)+(k-1)} &= \det(A_{k-1}) + (-1)^{2(k+1)} \\ &= \det(A_{k-1}) + 1 \\ &= \det(A_{k-2}) + 2 \\ &\vdots \\ &= \det(A_1) + (k-1) = k+1 \end{aligned}$$

In particular, we now know that A_n is invertible since it has nonzero determinant. And this is important because we can now say that $x = (x_1, \ldots, x_n) \in \Delta_0^n$ if and only if:

$$f(x_1, \ldots, x_n) := (A_n)^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{bmatrix} \in [0, \infty)^{n+1}$$

But f is a continuous injection and $f(0) = (\frac{1}{n+1}, \dots, \frac{1}{n+1})$ since:

$$0 = \sum_{i=1}^{n} \frac{1}{n+1} e_i - \frac{1}{n+1}(\sum_{i=1}^{n} e_i).$$

Hence, there must be some $\delta > 0$ such that $\|f(x) - f(0)\|_2 < \frac{1}{n+1}$ when $\|x\|_2 < \delta$. And in turn, the open ball of radius δ about 0 is contained in Δ_0^n. ■

Katz defines a ray from $x_0 \in \mathbb{R}^n$ to be a set $\{x_0 + ty : t \geq 0\}$ where $y \in \mathbb{R}^n$ with $\|y\|_2 = 1$.

Here is a basic topology fact I somehow haven't proved before:

Proposition: Suppose X is a topological space, $E \subseteq X$ is connected, and $A \subseteq X$ satisfies that $E \cap A \neq \emptyset$ and $E \cap A^{\mathsf{C}} \neq \emptyset$. Then $E \cap \partial A \neq \emptyset$.

Proof:
Note that $\partial A = \overline{A} \cap \overline{A^{\mathsf{C}}}$. So if $E \cap \partial A = \emptyset$, then this implies that $\overline{A} \cap E$ and $\overline{A^{\mathsf{C}}} \cap E$ are two disjoint nonempty closed sets (in the subspace topology of E) whose union is all of E. But this contradicts that E is connected. ■

Lemma 3.6: Suppose $X \subseteq \mathbb{R}^n$ is a convex body with $0 \in X^\circ$. Then every ray from 0 intersects ∂X exactly once.

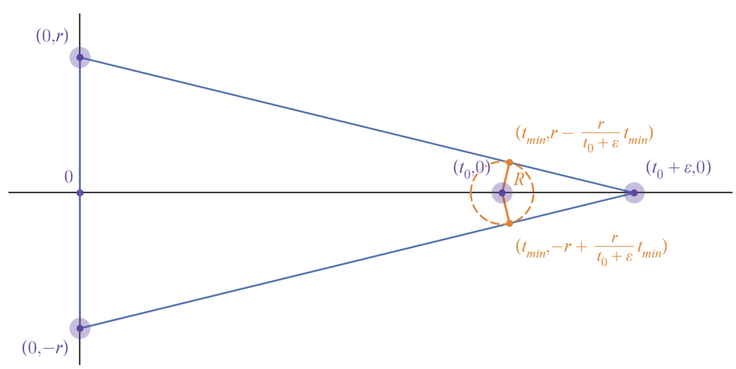
Proof:
Fix $y \in \mathbb{R}^n$ such that $\|y\| = 1$ and let $f(t) = ty$. Then we want to show there is a unique $t_0 \geq 0$ such that $f(t_0) \in \partial X$. Fortunately, it's easy to see that $f(t)$ intercepts ∂X at least once. After all, $f(0) \in X$. But at the same time, since X is compact and thus bounded, we know that there is some $t' > 0$ such that $f(t') \in X^{\mathsf{C}}$. Since the ray traced by f is connected, it now follows that there exists some $t_0 > 0$ such that $f(t_0) \in \partial X$.

We now show that our t_0 is unique. Suppose for the sake of contradiction that there exists $\varepsilon > 0$ with $(t_0 + \varepsilon)y \in X$. Then pick $r > 0$ such that the ball $\overline{B_r(0)} \subseteq X$. Since X is convex, we know that any line segment from $(t_0 + \varepsilon)y$ to a point in $\overline{B_r(0)}$ is contained in X. And thus, while it is fucking awful to calculate, we have that the open ball of radius $R = \min(t_0, \sqrt{Q(t_{\min})})$ about $t_0 y$ is contained in X where:

$$Q(t) = (r - \tfrac{r}{t_0+\varepsilon}t)^2 + (t - t_0)^2 \text{ and } t_{\min} = \frac{\frac{r^2}{t_0+\varepsilon} + t_0}{\frac{r^2}{(t_0+\varepsilon)^2} + 1}$$

For a hint at how I got that fucked up radius R, observe the diagram to the right:

However, the fact we were able to find such an R contradicts $t_0 y \in \partial X$. Hence, our supposed ε can't exist.



Since X is closed (since it's compact) and thus includes it's boundary, we've now proven that if $f(t_0) \in \partial X$ then $f(t) \notin \partial X$ for all $t > t_0$. And hence, our ray can intercept ∂X at most once. ■

As a side note: something even more fucked up is that the radius proposed by this paper doesn't work. It's too large. Anyways, because the paper handwaves this next bit, I'm going to deviate from the paper a bit.

Lemma: Let $\mathcal{X}$ be a topological K-vector space where $K = \mathbb{R}$ or $\mathbb{C}$. Also suppose $E \subseteq X$ and $c \in K$.

- If E is convex, then so is cE.

  Proof:
  If E is empty or $c = 0$, this is obvious. Otherwise, suppose $x, y \in cE$ and then note that $c^{-1}x, c^{-1}y \in E$. It follows that $c^{-1}(tx+(t-1)y) \in E$. So, $tx+(1-t)y \in cE$.

- $c(\overline{E}) = \overline{cE}$.

  Proof:
  If $c = 0$ then it must be the case that both sets are empty or are $\{0\}$. Meanwhile, note that as a general fact if $f : X \to Y$ is a continuous map and $A \subseteq X$, then $f^{-1}(\overline{f(A)})$ is a closed set containing A. Hence $\overline{A} \subseteq f^{-1}(\overline{f(A)})$ and this proves that $f(\overline{A}) \subseteq \overline{f(A)}$. Since scalar multiplication by c is a homeomorphism of $\mathcal{X}$ when $c \neq 0$, we thus have that $c\overline{E} = \overline{cE}$. ■

- $c(\partial E) = \partial(cE)$.

  Proof:
  By the last bullet point plus the fact that $f(A \cap B) = f(A) \cap f(B)$ for any function, we know that $c(\partial E) = c(\overline{E} \cap \overline{E^{\mathsf{C}}}) = \overline{cE} \cap \overline{c(E^{\mathsf{C}})}$. Now if $c = 0$, then we know that $c(\partial E)$ and $\partial(cE)$ are either both empty or both $\{0\}$. So, we may assume that $c \neq 0$. But in that case scalar multiplication is a homeomorphism on $\mathcal{X}$. So $c(E^{\mathsf{C}}) = (cE)^{\mathsf{C}}$ and we've shown that $c(\partial E) = \overline{cE} \cap \overline{(cE)^{\mathsf{C}}} = \partial(cE)$.

Lemma: Suppose $f : X \to Y$ is a homeomorphism and $A, B \subseteq X$. (This would include the case that $X = Y = \mathcal{X}$ and f is scalar multiplication by a nonzero scalar). Then $f(E^\circ) = (f(E))^\circ$.

Proof:
Since f is continuous and bijective, $f^{-1}(f(A)^\circ)$ is an open set contained in A. So, $f^{-1}(f(A)^\circ) \subseteq A^\circ$ and hence $f(A)^\circ \subseteq f(A^\circ)$. On the other hand, since f^{-1} is continuous, we know that $f(A^\circ)$ is an open subset of $f(A)$. So, $f(A^\circ) \subseteq f(A)^\circ$.

Theorem: Suppose $X \subseteq \mathbb{R}^n$ is a convex body and contains 0. Then X is homeomorphic to the closed unit ball $\overline{B_1(0)}$.

Proof:

Let p_{X° be the Minkowski functional associated with X°. In other words, $p_{X^\circ}(x) = \inf\{t \geq 0 : x \in tX\}$. Unfortunately, since X isn't a balanced set necessarily, we can't directly apply the proof on *page 233* to get that $P_{X^\circ}(cx) = |c|x$ for all $c \in \mathbb{R}$ and $x \in \mathbb{R}^n$. That said, since X is convex and a neighborhood of 0, we are able to copy the reasoning that shows that p_{X° is well-defined and satisfies the triangle inequality.

Claim 1: Suppose y is any unit vector and t_0 is the unique nonnegative number satisfying that $t_0 y \in \partial X$. Then $p_{X^\circ}(ty) = {}^t/_{t_0}$.

Proof:

If $t = 0$, then it is obvious that $p_{X^\circ}(ty) = 0 = {}^t/_{t_0}$. As for when $t \neq 0$, first note that $t_0 y \in \partial X \iff bt_0 y \in \partial(bX)$ for all $b \geq 0$. Also note that bX is easily checked to be convex body when $b \neq 0$ (and so we can apply theorem 3.6 to bX).

Suppose $b < {}^t/_{t_0}$. Then $bt_0 < t$ and thus $t \notin X$ since $bt_0 y \in \partial(bX)$. On the other hand, suppose $b > {}^t/_{t_0}$. Then $bt_0 > t$ and thus $t \in X - \partial X = X^\circ$ since again $bt_0 y \in \partial(bX)$. It follows that $p_{X^\circ}(ty) = {}^t/_{t_0}$.

Corollary: Suppose $x \in X$ and $t_0 > 0$ is the unique nonnegative number such that $t_0 \frac{x}{\|x\|_2} \in \partial X$. Then $p_{X^\circ}(x) = {}^{\|x\|_2}/_{t_0}$.

Claim 2: p_{X° is continuous.

Proof:

While we don't necessarily have reverse triangle inequality since $p_{X^\circ}(x) \neq p_{X^\circ}(-x)$, we can at least prove using just triangle inequality that:

$$|p_{X^\circ}(x) - p_{X^\circ}(y)| \leq \max(p_{X^\circ}(x-y), p_{X^\circ}(y-x))$$

Then, you just need to follow almost identical reasoning to that of *page 233* to show that $p_{X^\circ}(x)$ is continuous.

At last we are ready to write our homeomorphism from X to B. Define $g : X \to \overline{B_1(0)}$ by $g(x) = p_{X^\circ}(x)\frac{x}{\|x\|_2}$ when $x \neq 0$ and $g(0) = 0$. And similarly, define $h : \overline{B_1(0)} \to X$ by $h(y) = ty$ such that $ty \in \partial(\|y\|X)$.

- g is continuous when $x \neq 0$ by virtue of being a scalar product of continuous functions. Also, given any sequence $(x_n)_{n\in\mathbb{N}}$ in X converging to 0 we have that $\|g(x_n)\|_2 = p_{X^\circ}(x_n) \cdot 1 \to 0$ as $n \to \infty$. So g is also continuous at 0.

- $h(g(0)) = 0$. Meanwhile, suppose $x \in X - \{0\}$. Then let t_0 be the unique nonnegative number satisfying that $t_0 \frac{x}{\|x\|_2} \in \partial X$. Now $g(x) = p_{X^\circ}(x)\frac{x}{\|x\|_2} = \frac{x}{t_0}$. Thus $h(g(x)) = t\frac{x}{t_0}$ where $t \geq 0$ satisfies that $t\frac{x}{t_0} \in \partial(\frac{\|x\|}{t_0}X)$. Or in other words, $t\frac{x}{\|x\|_2} \in \partial X$. It follows by theorem 3.6 that $t = t_0$ and so $h(g(x)) = x$.

- $g(h(0)) = g(0) = 0$. Meanwhile, if $y \in \overline{B_1(0)} - \{0\}$ then let t_0 be the unique nonnegative number such that $t_0\frac{y}{\|y\|_2} \in \partial X$. Like before, we can show that $h(y) = t_0 y$. And now $p_{X^\circ}(t_0 y) = \frac{t_0\|y\|_2}{t_0} = \|y\|_2$ since $t_0 y = t_0\|y\|_2\frac{y}{\|y\|_2}$. Thus $g(t_0 y) = g(t_0 y) = \|y\|_2\frac{t_0 y}{\|t_0 y\|_2} = y$ and we've proven that $g(h(y)) = y$.

- Since $h = g^{-1}$, g is continuous, X is compact, and $\overline{B_1(0)}$ is Hausdorff, we have that h is continuous. ■

Corollary: All convex bodies in $\mathbb{R}^n$ are homeomorphic to the closed unit ball $\overline{B_1(0)}$.
Why? We can just translate them so that they contain 0 and then apply the last theorem.

Corollary 2: All n-simplices are homeomorphic to the closed unit ball $\overline{B_1(0)}$ in $\mathbb{R}^n$.

10/18/2025

I need to first do the rest of my math 200a homework. Then I think there is a different paper I want to try to following to prove the fixed point theorems from before.

Math 200a Homework:

Set 3 Problem 3: Suppose G is a finite group and $h < G$. Suppose also for all $x \in H - \{1\}$ that $C_G(x) \subseteq H$. Then $\gcd(|H|, [G : H]) = 1$.

Proof:
If $\gcd(|H|, [G : H]) \neq 1$, then we know there is some prime number p dividing both $|H|$ and $[G : H]$. And as a result, $\nu_p(|H|) < \nu_p(|G|)$. So, by choosing $P \in \mathrm{Syl}_p(H)$ and $Q \in \mathrm{Syl}_p(G)$, we know that $|P| < |Q|$. Also, by Sylow's second theorem, we can conjugate Q in order to say without loss of generality that $P \subseteq Q$. We'll need two observations:

- Since the order of Q doesn't divide H, we know that Q isn't a subgroup of H. Hence, $Q \cap (G - H) \neq \emptyset$.
- At the same time though, we know that $P < Q \cap H < H$. And since the only prime factors of $|Q \cap H|$ are p, this tells us by Lagrange's theorem that $|P| = p^{\nu_p(|H|)}$ divides $|Q \cap H|$ which itself divides $p^{\nu_p(|H|)}$. Hence, $|P| = |Q \cap H|$ and this proves that $Q \cap H = P$.

Now recall from the 2nd proposition on *page 272* that if $\{1\} \neq N \lhd P$ and P is a p-group, then $Z(P) \cap N \neq \{1\}$.

Side note: Whenever $H < G$, we define $Z(H) = \bigcap_{h \in H} C_H(h)$. Just wanted to make that's clear since I didn't know better before.

As a special case, if we set $N = P$ (where P is nontrivial), then this proposition says that the center of a p-group is always nontrivial. Hence, there exists $y \in Z(P) - \{1\}$.

Side note: How the hell hadn't I processed that consequence of the theorem we proved before?

Now note that $Z(Q) \subseteq C_Q(y) \subseteq C_G(y) \subseteq H$ where the last inclusion is by the assumption of the problem. Hence $Z(Q) \subseteq H \cap Q = P$. At the same time, we know that $Z(Q)$ is nontrivial. So, there exists $x \in Z(Q) - \{1\}$ which we know from before is in $P \subseteq H$. Finally, we now have that $Q \subseteq C_Q(x) \subseteq C_G(x) \subseteq H$. But this contradicts that $Q \cap (G - H) \neq \emptyset$. ■

Set 3 Problem 8: Suppose G is a finite group.

(a) Prove that a normal Sylow p-subgroup is a characteristic subgroup.

A Sylow p-subgroup P is normal if and only if it is the only Sylow p-subgroup. And since automorphisms preserve the order of subgroups, we have that if P is the only subgroup of G with order $p^{\nu_p(|G|)}$ then $\theta(P) = P$ for all $\theta \in \mathrm{Aut}(G)$. So, P is a characteristic subgroup.

(b) Suppose $H \lhd G$ and $\gcd(|H|, [G : H]) = 1$. Prove that H is a characteristic subgroup.

Suppose $\theta \in \mathrm{Aut}(G)$. Then since H is normal, we know that $\theta(H)H$ is a subgroup of G and $H \lhd \theta(H)H$. So by the correspondance theorem, we know that $|\frac{\theta(H)H}{H}|$ divides $|G/H| = [G : H]$. At the same time, $|\theta(H)H| = \frac{|H||\theta(H)|}{|H \cap \theta(H)|}$. So, $|\frac{\theta(H)H}{H}| = \frac{|\theta(H)|}{|H \cap \theta(H)}$ divides $|\theta(H)| = |H|$.

This shows that $|\frac{\theta(H)H}{H}| \mid \gcd(|H|, [G : H]) = 1$. And hence, $\theta(H)H = H$. Since $|\theta(H)| = |H|$ and $\theta(H) \subseteq \theta(H)H$, we can thus conclude that $\theta(H) = H$. ■

Set 3 Problem 4: Suppose G is a finite group, $N \lhd G$, and p is a prime factor of $|N|$.

(a) Suppose $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_p(N)$. Prove there exists $g \in G$ such that $Q = (gPg^{-1}) \cap N$.

Uhhhh, I think I already basically proved this fact when doing problem 3.

By Sylow's second theorem, we know there is some $g \in G$ such that $Q \subseteq gPg^{-1}$. Then since $(gPg^{-1}) \cap N$ is a subgroup of gPg^{-1} where the latter is a p-group, we know that $(gPg^{-1}) \cap N$ is also a p-group. Also, since $(gPg^{-1}) \cap N$ is a subgroup of N, we know that $|(gPg^{-1}) \cap N|$ divides $|N|$. It follows that $|(gPg^{-1}) \cap N|$ divides $p^{\nu_p(|N|)}$. At the same time, since $|Q| = p^{\nu_p(|N|)}$ and $Q \subseteq gPg^{-1} \cap N$, we have that $p^{\nu_p(|N|)} \leq |(gPg^{-1}) \cap N|$.

So, $Q \subseteq (gPg^{-1}) \cap N$ and $|Q| = p^{\nu_p(|N|)} = |(gPg^{-1}) \cap N|$. It follows that $Q = (gPg^{-1}) \cap N$.

Side note: none of the reasoning in this part requires having N be normal.

(b) Prove that the following is a well-defined surjective function:

$$\Phi : \mathrm{Syl}_p(G) \to \mathrm{Syl}_p(N) \text{ where } \Phi(P) = P \cap N.$$

Part (a) guarentees that the above map will be surjective provided that it is well-defined (since $gPg^{-1} \in \mathrm{Syl}_p(G)$ for any $P \in \mathrm{Syl}_p(G)$). Hence, all we need to show is that $P \cap N \in \mathrm{Syl}_p(N)$ whenever $P \in \mathrm{Syl}_p(G)$.

Fortunately since $P \cap N < P$, we know that $P \cap N$ is a p-group. But suppose for the sake contradiction that $|P \cap N| \neq p^{\nu_p(|N|)}$ (meaning $P \cap N \notin \mathrm{Syl}_p(G)$). Then, by Sylow's first and second theorems there exists a group $Q \in \mathrm{Syl}_p(N)$ such that $P \cap N \lneqq Q < N$. Also, by part (a) there exists $g \in G$ such that $Q = gPg^{-1} \cap N$. But since N is normal, $gPg^{-1} \cap N = gPg^{-1} \cap gNg^{-1} = g(P \cap N)g^{-1}$.

The last equality is obvious if you think about it. So I'd rather not write out a proof since I want to go on an outing soon.

So $P \cap N \lneqq Q = g(P \cap N)g^{-1}$. Since $|P \cap N| = |g(P \cap N)g^{-1}|$ this is a contradiction.

(c) For $P \in \mathrm{Syl}_p(G)$, prove that $N_G(P) \subseteq N_G(\Phi(P))$ and:
$$|\Phi^{-1}(\Phi(P))| = [N_G(\Phi(P)) : N_G(P)].$$

To start off, note that:
$$\begin{aligned}\Phi(xPx^{-1}) = xPx^{-1} \cap N &= xPx^{-1} \cap xNx^{-1} \\ &= x(P \cap N)x^{-1} = x\Phi(P)x^{-1}.\end{aligned}$$

Therefore, if $x \in N_G(P)$ then $\Phi(P) = \Phi(xPx^{-1}) = x\Phi(P)x^{-1}$. And this proves the first claim that $N_G(P) \subseteq N_G(\Phi(P))$.

Next, note by Sylow's second theorem that:
$$\Phi^{-1}(\Phi(P)) = \{gPg^{-1} : g \in G \text{ and } \Phi(P) \subseteq gPg^{-1}\}.$$

But $\Phi(P) \subseteq gPg^{-1}$ if and only if:
$$\begin{aligned}\Phi(P) = P \cap N \subseteq gPg^{-1} \cap N &= (gPg^{-1}) \cap (gNg^{-1}) \\ &= g(P \cap N)g^{-1} = g\Phi(P)g^{-1}.\end{aligned}$$

And since $|\Phi(P)| = |g\Phi(P)g^{-1}|$, this is the same as saying that $\Phi(P) = g\Phi(P)g^{-1}$. So, we really have that:
$$\Phi^{-1}(\Phi(P)) = \{gPg^{-1} : g \in N_G(\Phi(G))\}.$$

Hence, if we consider the action $N_G(\Phi(P)) \curvearrowright \mathrm{Syl}_p(G)$ by conjugation, then $\Phi^{-1}(\Phi(P))$ is precisely the orbit of P with respect to this action. Also, x is in the stabilizer of P with respect to this action precisely when $xPx^{-1} = P$. Or in other words, $x \in (N_G(\Phi(P)))_P$ when $x \in N_G(\Phi(P)) \cap N_G(P) = N_G(P)$. By the orbit-stabilizer theorem, we thus can conclude that:
$$|\Phi^{-1}(\Phi(P))| = [N_G(\Phi(P)) : N_G(P)].$$

(d) Prove that $|\mathrm{Syl}_p(N)|$ divides $|\mathrm{Syl}_p(G)|$.

We know that $\{\Phi^{-1}(Q) : Q \in \mathrm{Syl}_p(N)\}$ forms a partion of $\mathrm{Syl}_p(G)$. Hence, we now seek to prove that there exists a single integer $r \in \mathbb{N}$ such that $|\Phi^{-1}(Q)| = r$ for all $Q \in \mathrm{Syl}_p(N)$. After all, once we know that we will then be able to say that $\mathrm{Syl}_p(N) = r\mathrm{Syl}_p(G)$.

Luckily, note that since conjugation is an automorphism of G we have that:
- $|N_G(P)| = |g(N_G(P))g^{-1}| = |(N_G(gPg^{-1}))|$,
- $|N_G(\Phi(P))| = |g(N_G(\Phi(P)))g^{-1}| = |N_G(g\Phi(P)g^{-1})| = |N_G(\Phi(gPg^{-1}))|$.

It follows for all $g \in G$ and $P \in \mathrm{Syl}_p(G)$ that:
$$\begin{aligned}|\Phi^{-1}(\Phi(P))| &= [N_G(\Phi(P)) : N_G(P)] \\ &= \tfrac{|N_G(\Phi(P))|}{|N_G(P)|} = \tfrac{|N_G(\Phi(gPg^{-1}))|}{|N_G(gPg^{-1})|} = [N_G(\Phi(gPg^{-1})) : N_G(gPg^{-1})] \\ &= |\Phi^{-1}(\Phi(gPg^{-1}))|\end{aligned}$$

And since G acts transitively on $\mathrm{Syl}_p(G)$ by conjugation, this means that there exists $r \in \mathbb{N}$ such that $r = |\Phi^{-1}(\Phi(P))|$ for all $P \in \mathrm{Syl}_p(G)$. Then since Φ is surjective, we have that $\Phi^{-1}(Q) = r$ for all $Q \in \mathrm{Syl}_p(G)$.