Analog Computer for Solving Second-Order Nonhomogeneous Differential Equations

Jason Li

***Abstract*—In this project, we built an analog computer using Operational Amplifier circuits, which can be built to perform mathematical functions, including plus-minus and integration, required for solving second-order linear nonhomogeneous differential equations for specific nonhomogeneous terms. Practically, such an equation resembles damped driven oscillations in the vertical direction experienced by automobiles traveling over a certain road surface that we describe as the non-homogeneous term. While designing the analog computer, we first found the analytic solution of sinusoidal driving term, using it to confirm the LTspice test simulation, which is then compared to oscilloscope readings of the actual device. Finally, a comparison was made between the three measurements.**

I. INTRODUCTION

OPERATIONIAL Amplifier (op-amp) circuits are high-gain voltage amplifiers with differential inputs that, when used with appropriate negative feedback, can perform addition, subtraction, multiplication by a constant, and integration of voltage signals over time.

In this project, we made an analog computer with a circuit connecting multiple inverting amplifiers, an inverting integrator, inverting differentiators, and an inverting summing amplifier. Chaining multiple op-amps together allows for the computation of nonhomogeneous differential equations. By inputting nonhomogeneous terms of different forms, such as input voltage, the output signal theoretically resembles the general solution of this differential equation. Given the wide applicability for this form of differential equation, in the context of automobile dynamics, the solution of such resembles the vertical motion after traveling through a speed bump. Yet, given the limitation of actual circuit components, such as parasitic effects and tolerance of resistors, the output voltage from LTspice simulation and analog computer differs from the analytic solution, and such deviation is quantified and discussed.

II. THEORY

A. Second Order Nonhomogeneous Differential Equations

GIVEN the configuration of the mass-spring damper system for a automobile, as illustrated in Figure 1, we derive the differential equation for motion in the vertical direction,

$$m\ddot{y} = -b(\dot{y} - \dot{x}) - k(y - x) \tag{1}$$

,where m is one-quarter of the car's total mass, b is damping constant, k is spring constant, y is car displacement from equilibrium height at when travelling on horizon, and x is the height of bump.

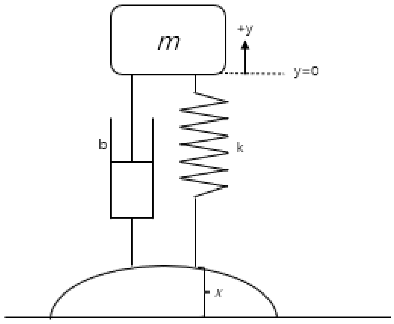

Fig. 1. Illustration of Mass-Spring-Damper System when the vehicle is passing over a bump. Here y=0 is defined as the vertical equilibrium position of mass when the spring is connected to ground level, and x the displacement of bump's tip from horizon. Developed from figure 1 in [1].

Rearranging variables, we have,

$$\ddot{y} = -\gamma(\dot{y} - \dot{x}) - \omega_0^2(y - x) \tag{2}$$

where $\gamma = b/m, \omega_0^2 = k/m$. This form of differential equation could be solved by an analog computer, and the circuit design could be further simplified to fewer op-amps and mathematical operators, by generalizing Equation (2) to the following form,

$$\ddot{a} = \frac{f(t)}{m} - \gamma\dot{a} - \omega_0^2 a \tag{3}$$

, where a is variable, f(t) is a arbitrary function, represented by voltage signal input in analog circuit.

For a sinusoidal non-homogeneous term f(t)=F_0cos(ωt), where F_0 is amplitude, being constant,and ω is the angular frequency of the sinusoidal term, the analytic solution for Equation (3) is given for under-damped oscillation as:

$$a = Acos(\omega t + \phi) + Be^{-t\frac{\gamma}{2}}cos(\omega_d t + \theta) \tag{4}$$

$$\omega_d = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2} \tag{5}$$

$$A = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}} \tag{6}$$

$$\phi = tan^{-1}(-\frac{\omega\gamma}{\omega_0^2 - \omega^2}) \tag{7}$$

, where B and θ are determined by initial conditions.

B. Analog Computer

Equation (3) indicates that $\ddot{a}$ can be determined by summing the terms on the right hand side, where $\dot{a}$ and a are determined by integration, and scaled by $-\gamma$ and $-\omega_0^2$ respectively. This mathematical operation could be achieved by connecting op

Op-Amp	Output-Input Relation
Inverting Amplifier	$V_{out} = -\frac{R_f}{R} V_{in}$
Inverting Integrator	$V_{out} = -\frac{1}{RC} \int V_{in}\, dt$
Inverting Differentiator	$V_{out} = -R_f C \frac{d}{dt} V_{in}$
Inverting Summing Amplifier	$V_{out} = -\frac{R_f}{R}(V_1 + V_2 + V_3)$

TABLE I

DIFFERENT OP-AMPS AND THEIR RESPECTIVE $V_{out} - V_{in}$ RELATIONS. HERE, R_f IS THE RESISTANCE ON FEEDBACK CIRCUIT, R IS THE INPUT RESISTANCE, V_1, V_2, V_3 ARE THE THREE INPUT VOLTAGES AND C IS CAPACITANCE, WHICH COULD BE ON EITHER THE FEEDBACK CIRCUIT OR NEGATIVE INPUT CIRCUIT

amps, which relates the output voltage V_{out} with input voltage V_{in} as given in Table 1.

Connecting the op-amps to a specific signal source, the output signal theoretically resembles the solution of Equation (3), which could be confirmed by cross-referring to the general analytic solution in Equation (4) by setting sinusoidal input signal voltage in analog circuit.

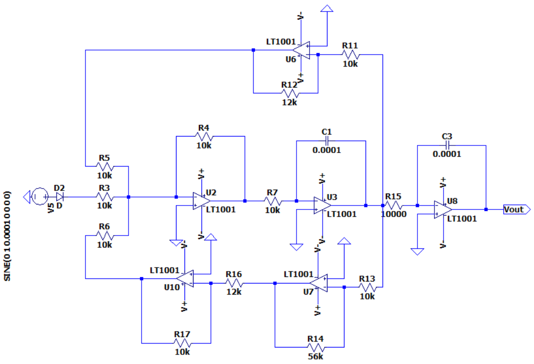

Fig. 2. LTspice circuit illustration of the analog circuit. Non-inverting op-amps are not applied, as they are restricted to a gain with magnitude above 1, limiting the range of applicable values for ω_0 and γ. The input signal resembles $\frac{f(t)}{m}$ rather than f(t) to reduce circuit complexity but avoiding the use of one additional scaling op-amp, and the value of RC for inverting integrator is set to 1 by using R=10kΩ and C=100μF. The resistance tolerance were set to 5%.

III. EXPERIMENTAL METHOD

A. Solving Differential Equation via analog Circuit

Setting a sinusoidal input signal with angular frequency at ω_0 and amplitude 2 V in the LTspice circuit setup in figure 2, we compared simulation results with the analytic solution given in Equation (4), as shown in figure 3.

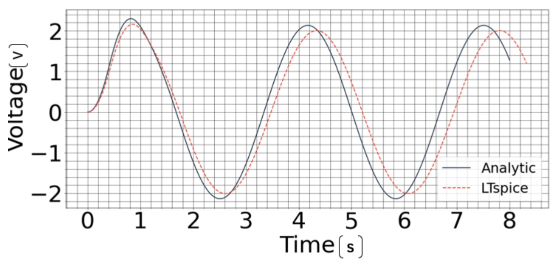

Fig. 3. Comparison between analytic solution and LTspice simulation. The green and blue curves resemble the analytic solution and LTspice simulation, respectively. Referring to [1], we set ω_0=7.4Hz, $\gamma = 5.93Hz$, the analytic solution is $y = 0.021cos(7.4t - \frac{\pi}{2}) + e^{-3t}cos(6.79t - \frac{\pi}{2})$ while the fit for the LTspice simulation is $y = 0.0205cos(7.4t - 1.55) + e^{-3t}cos(6.76t - 1.55)$, where each parameter differs from the analytic solution by less than 1%.

Confirming the analog circuit configuration, the breadboard analog circuit is setup, as shown in Figure 4. With such a device, Equation (3) is solved for various nonhomogeneous terms, including square wave pulse and triangle (sawtooth) waveform.

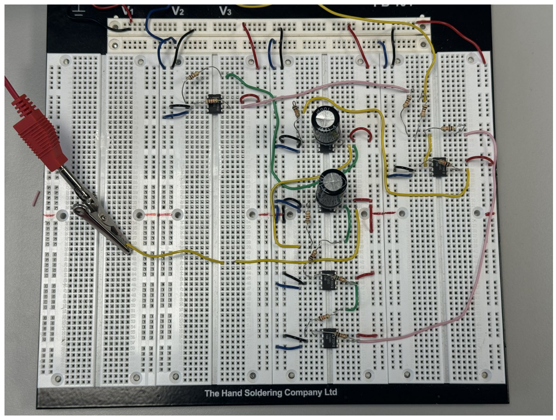

Fig. 4. Analog circuit designed on breadboard, resembling the spice circuit in Figure 2. Analog circuit designed on breadboard. This circuit is connected to an oscilloscope by low-capacitance coaxial cables.

B. Extension: Car oscillation

Defining the relation of the top of the hill in Figure 1 as a road function of time, x(t), the analog circuit in Figure 2 is redesigned to solve Equation (2), x(t) being the input signal. Figure 5 illustrates the setup of this analog circuit.

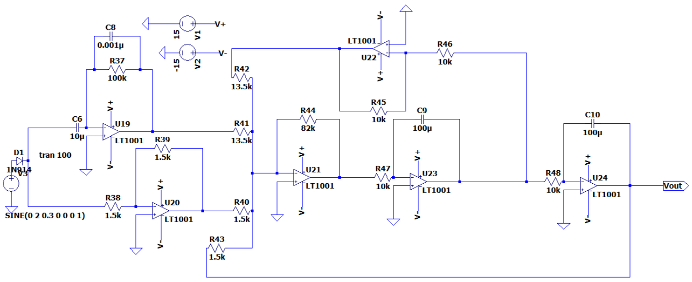

Fig. 5. Analog circuit designed to solve road function differential equations, given by Equation (2). Additionally, a high pass filter was made by connecting a capacitor with nano-scale capacitance in parallel with the inverting differentiator, which yields noise that covers data of output voltage at low frequencies

Considering the common curvature of road bumps, the input function in the redesigned analog circuit is considered to have a half-wave rectified sinusoid waveform, which was produced by connecting a diode in series with the signal generator.

IV. RESULT, ERRORS AND DISCUSSION

A. Results: General analog Circuit Characteristics

With the setup of Figure 2, the general features of the actual analog circuits are investigated and compared to the analytic solution.

While the analytic solution for its sinusoidal nonhomogeneous term for the underdamped system is provided in Equation (4), analytically solving for other waveforms, such as square or triangular wave, is complicated. Instead, setting these waveforms in the analog circuit, the output voltage resembles the general solution, shown in Figure 6.

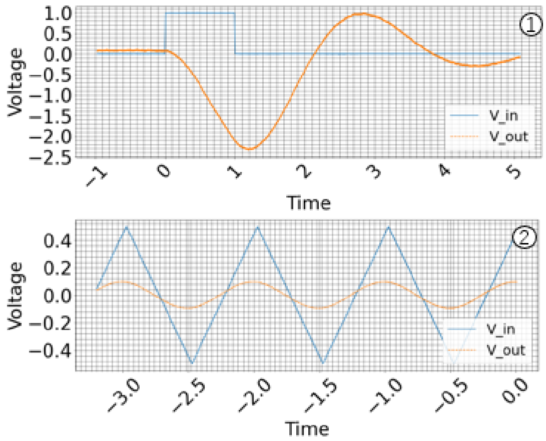

Fig. 6. Here the blue and orange plots resemble the input and output signals, respectively. ① Solution for a differential equation with a nonhomogeneous term of the square pulse waveform, which could be generalized to the case of instantaneous forcing, where f(t) resembles the approximation of a constant force over the instant. ② Solution for the nonhomogeneous term of sawtooth input.

Likewise, V_{out}'s amplitude response with sinusoidal nonhomogeneous term's frequency is investigated. Since the transient solution decays to be negligible(contributes to less than 1% of V_{out}) within the first period, the amplitude of the general solution may be represented by A, which is related to ω, as stated in Equation (6). Setting a sweep input signal, the resulting output voltage is shown in figure 7.

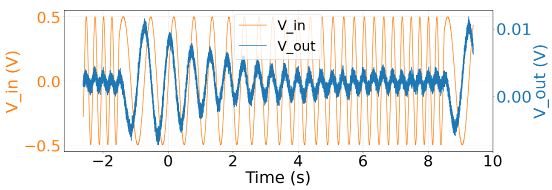

Fig. 7. Here the blue and orange plots resembles the input and output signals, respectively. The sweep input function serves as a sinusoidal function with changing frequency, driving the orange function's amplitude to change accordingly.

In Figure 7, the input angular frequency corresponding to maximum output is $\omega = 7.6Hz$. As Equation (6) indicates that the amplitude of steady-state solution is maximized for ω= ω_o, where $\omega_o = 7.4Hz$ in the circuit setup, the natural frequency for the analytic solution differs from that for the analog output by less than 5%, indicating alignment. The noise of the output function will be discussed further in the following section.

B. Extension: Car Oscillation

Keeping the original parameters ω_0 and γ, and assuming an average bump height at 20 dm and a passing time of 3 seconds, we have designed the parameters as in Figure 5, where the input signal is characterized as V_{in}=2sin(2π(0.3)t) for 2 reasons. (1) The input voltage is scaled down by a factor of 10, as it should be controlled below the power supply voltages to avoid saturation in the measurement of the output signal. (2) High-speed automobiles, passing through half-wave sinusoid-shaped speed bumps by less than 3 seconds; yet, the differentiator op-amp is susceptible to noise at low frequencies[2], as the capacitor, a high-frequency-pass filter is connected in series to the op-amp. Thus, to reduce noise, an additional high-pass filter (a capacitor) is connected in parallel to the feedback circuit for the differentiator, yet the signal is subject to noise below 1Hz.

Setting these parameters, we have obtained the following output signals from the analog circuit and LTspice simulation, which is further compared with the analytic solution, as shown in Figure 8.

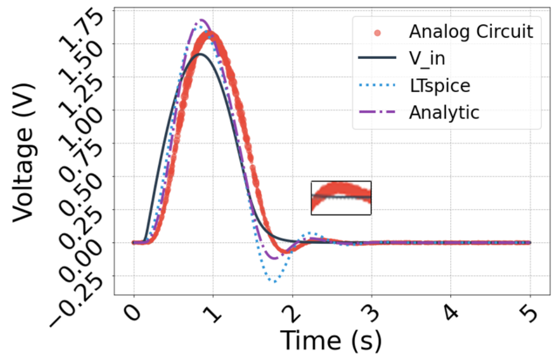

Fig. 8. Plot illustration of the solutions for the half-wave rectified voltage $V_{in}(t)$=2sin(2π(0.3)t), which are determined by three means: analog Circuit, LTspice Simulation, and Analytic calculation through python ODE directory. The half-wave sinusoid V_{in} is produced from one period of sinusoidal voltage input, and the plot within the black box magnifies the displacement of the second peak from the corresponding value of input voltage.

The three waves are shown to oscillate after reaching the first peak, characterizing the trend of under-damped oscillation. Thus, we assume that the analog circuit's output voltage could be described by Equation (4).

C. Discussion: Disagreement from Analytic Solution

Figure 8 indicates that, the solutions, while sharing under-damped waveform, differ in the rate of decay, which is quantified by measuring decay ratio and settling time[3]. The decay ratio is defined as the ratio of the amplitude of largest peak to the amplitude of the second peak, while settling time is the time between the two peaks.

For the analog computer, its deviation from the analytic solution (shown in Table 2) is caused by systematic errors such as parasitic effects and resistors' tolerance values. In relation

Form of Solution	Decay Ratio	Settling Time
analog	0.05	2.34
Analytic	0.08	2.23
LTspice	0.09	2.17

TABLE II
THE DECAY RATIO AND SETTLING TIME FOR SOLUTIONS DETERMINED BY DIFFERENT METHODS.

to Equation (4), the analog computer could be described to have a larger value of γ, than the simulation of LTspice or the analytic solution; the remaining parameters in Equation (4) are assumed to be unchanged, with identical input voltage. The two terms in Equation (4) are the transient (with ω_d) and steady-state(with ω) solution, respectively. In the analog circuit, $\omega \approx 5\%\omega_0$, according to Equation (6), the $\omega\gamma$ term at the denominator's effect is negligible for $\omega_0 \gg \omega$, thus we may neglect the parasitic effect on the steady-state solution. Due to Equation (5), ω_d for the analog circuit is larger, resulting in a longer settling Time, as shown in Table 2.

Likewise, the quality factor Q could be defined as[4]:

$$Q = \frac{\omega_0^2 + \omega^2}{2\omega\gamma} \tag{8}$$

Since Q resembles the energy loss in every cycle, the larger γ due to parasitic effects results in a greater magnitude of energy loss from the first to second peak for the analog circuit compared to analytic solution, and hence a smaller decay ratio.

Similarly, LTspice simulation, while regulating parasitic effects, includes tolerance values for circuit components such as resistors, rendering the value of γ to deviate from the planned γ.

D. Extension:Road Function

Aside from half-wave sinusoid road surfaces, common speed bumps are often triangular or rectangular, which are either simulated as a sawtooth or rectanglar function. The motion in vertical direction after hitting single crests are discussed in this section.

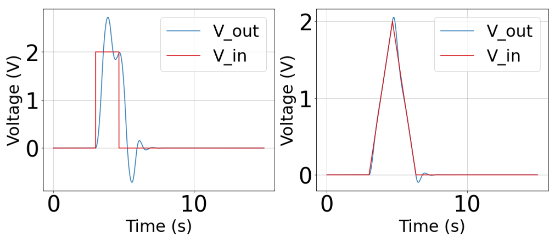

Fig. 9. Input and Output signals from the redesigned analog computer. The rectangular and triangular input voltage signals, like the sinusoidal input in the previous section are set to have amplitude at 2V and period of 3.33 seconds.

Figure 9 shows the motion response with the respective input signal, where the 2 features provides insight for automobile's motion in response to speed bumps in reality.

1. At the tip of the sawtooth function, the car's motion overshoots by 1 ms, while it follows a strict triangular form at other other moments. This feature is due to the motion of the automobile on a large gradient surface(such as sawtooth) its inertia maintains the motion in previous trend in a short time. Since the square road surface has zero gradient, the motion remains in x-direction, hence such feature is absent in the plot for rectangular input waveform.

2. The output voltage corresponding to the rectangular waveform input damps out over a longer time. In reality, this feature could be explained by the sharper drop at the edge of rectangular speed bump, resulting in greater kinetic energy and thus a larger initial spring displacement, which is damped out over a longer period. On contrary, the automobile gradually drops to the ground on triangular road surface, thus acquiring lower initial vertical displacement and hence less damping time.

The alignment of voltage's output with reality further confirms the validity of the analog circuit's setupand provides test prediction for oscillation-related events.

E. Discussion: Op-Amp Limitation

To reduce experimental noise, we mainly attempted three changes. (1) Connect a high-pass filter(capacitor) in parallel with the inverting differentiator. (2) Set the input frequency above 0.1 Hz. (3) Use probe cable to reduce parasitic capacitance.

However, the LT1001 op-amp's systematic uncertainty limits the accuracy of data, which exhibits a input noise voltage density of 10.3 nV/$\sqrt{Hz}$ to 18 nV/$\sqrt{Hz}$ at a frequency of 10 Hz. Comparatively, the input noise voltage density for the OPA211-HT series op-amps has 1.0nV/$\sqrt{Hz}$ at 10Hz[5-6]. To improve the accuracy of the measured output voltage and increase the range of input signal frequencies(with negligible noise), the LT1001 op-amps in analog circuit could be replaced to OPA211-HT, which reduces the systematic uncertainty by a factor of ten.

V. Conclusion

In this experiment, we successfully demonstrated the utility of an analog computer in solving second-order linear non-homogeneous differential equations, where the non-homogeneous term is represented by various waveforms of voltage signal input. Further adapting the differential equation to include the road function, we determined the under-damped dynamics of a mass-spring-damper system in response to various road surfaces, including sinusoidal, square, and triangular. In both cases, the output of the analog computer is compared to the analytic solution and LTspice simulations, observing a reasonable, yet imperfect agreement among the three methods.

The deviation in results are attributed to systematic errors such as parasitic effects and resistor tolerance in the analog circuit. The performance of the analog computer can be improved by using op-amps with lower input noise voltage density to reduce the impact of noise on the measurements.

VI. Reference

[1] Y. Uchida (Ed.) Advanced Electronics 2023-2024: "Project D: analog Computer" Imperial College London.2023.

[2] Fiore, J. M. (Year). Operational Amplifiers and Linear Integrated Circuits: Theory and Application. LibreTexts.

[3] Tom (n.d.). Class notes: Second-order systems. [PDF file]. Retrieved from https://www.et.byu.edu/ tom/classes/436/ClassNotes/Class20(Second-Order).pdf

[4] Tarbutt, M. (2023-2024). Oscillations and Waves. First Year Physics. Imperial College London.

[5] ”OPA211-HT High Temperature 1.1nV/Hz Noise, Low Power, Precision Operational Amplifier.” Texas Instruments. Retrieved from Texas Instruments website.

[6] ”LT1001 Precision Operational Amplifier Datasheet.” Analog Devices. PDF document.