Logistic Growth Models

Logistic growth models have both exponential growth and decline within the same model.

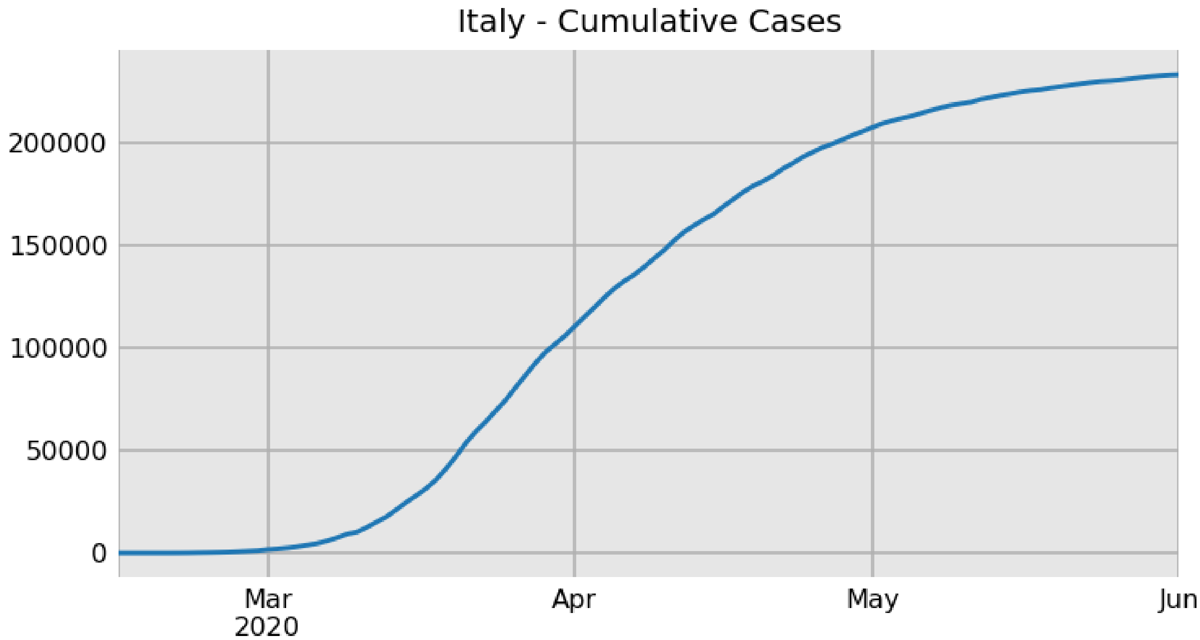
In [1]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
plt.style.use('dashboard.mplstyle')
```

In [2]:
```
def logistic_func(x, L, x0, k):
    """
    Logistic function for modeling data.

    Parameters
    ----------
    x : array
    L, x0, k : float

    Returns
    -------
    array
    """
    return L / (1 + np.exp(-k * (x - x0)))
```

In [3]:
```
from functions import logistic_func
logistic_func(np.arange(20), 5000, 12, 0.1).round(-1)
```

Out[3]:
```
array([1160., 1250., 1340., 1450., 1550., 1660., 1770., 1890., 2010.,
       2130., 2250., 2380., 2500., 2620., 2750., 2870., 2990., 3110.,
       3230., 3340.])
```

In [4]:
```
from prepare import PrepareData
data = PrepareData(download_new=False).run()
italyc = data['world_cases']['Italy']
italyc = italyc.loc["2020-02-15":"2020-06-01"]
italyc.plot(title="Italy - Cumulative Cases");
```



In [5]:
```
def logistic_guess_plot(s, L, x0, k):
    """
    Plots data with a logistic function estimate.

    Parameters
    ----------
    s : Series
    L, x0, k : float

    Returns
    -------
    None
    """
    x = np.arange(len(s))
    y = logistic_func(x, L, x0, k)
    s_guess = pd.Series(y, index=s.index)
    s.plot()
    s_guess.plot()
```

Estimating logistic function parameters

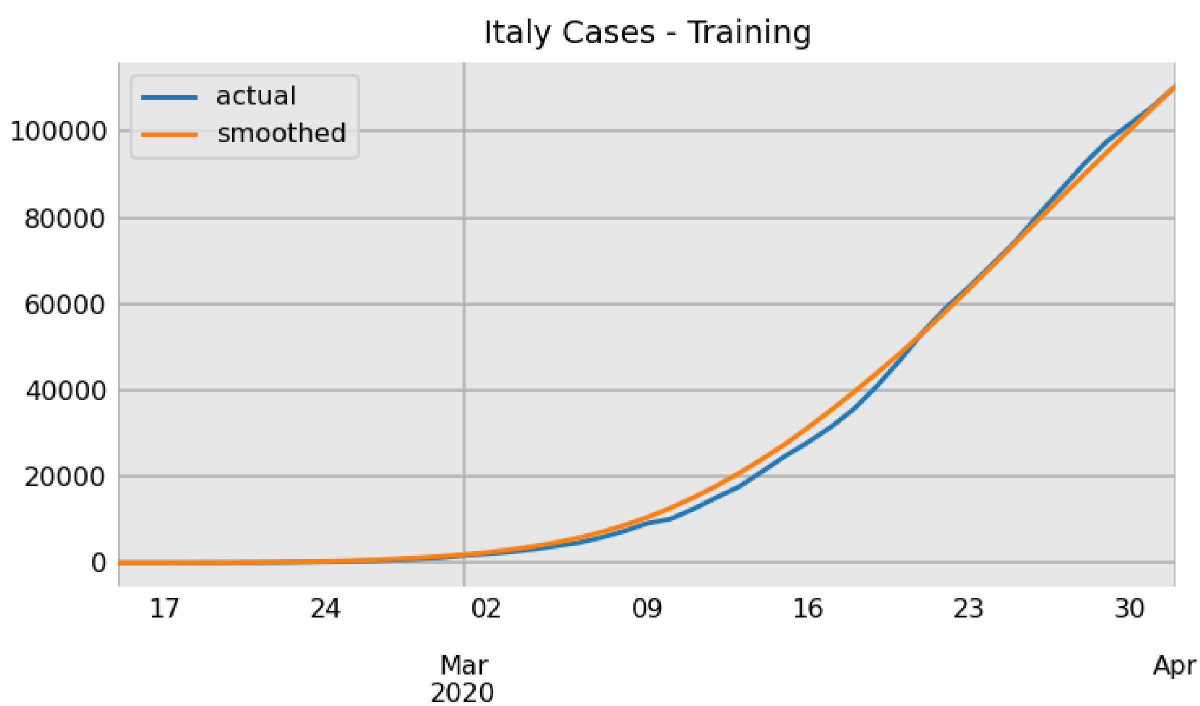
In [6]:
```
from functions import smooth
last_date = '2020-04-01'
y = italyc.loc[:last_date]
y_smooth = smooth(y, n=15)
y_smooth.head()
```

Out[6]:
```
2020-02-15     0.000000
2020-02-16     0.000000
2020-02-17     0.000000
2020-02-18     8.144961
2020-02-19    27.319120
dtype: float64
```

In [7]:
```
y_smooth.tail()
```

Out[7]:
```
2020-03-28     89992.495694
2020-03-29     95198.975637
2020-03-30    100365.643598
2020-03-31    105491.416174
2020-04-01    110574.000000
dtype: float64
```

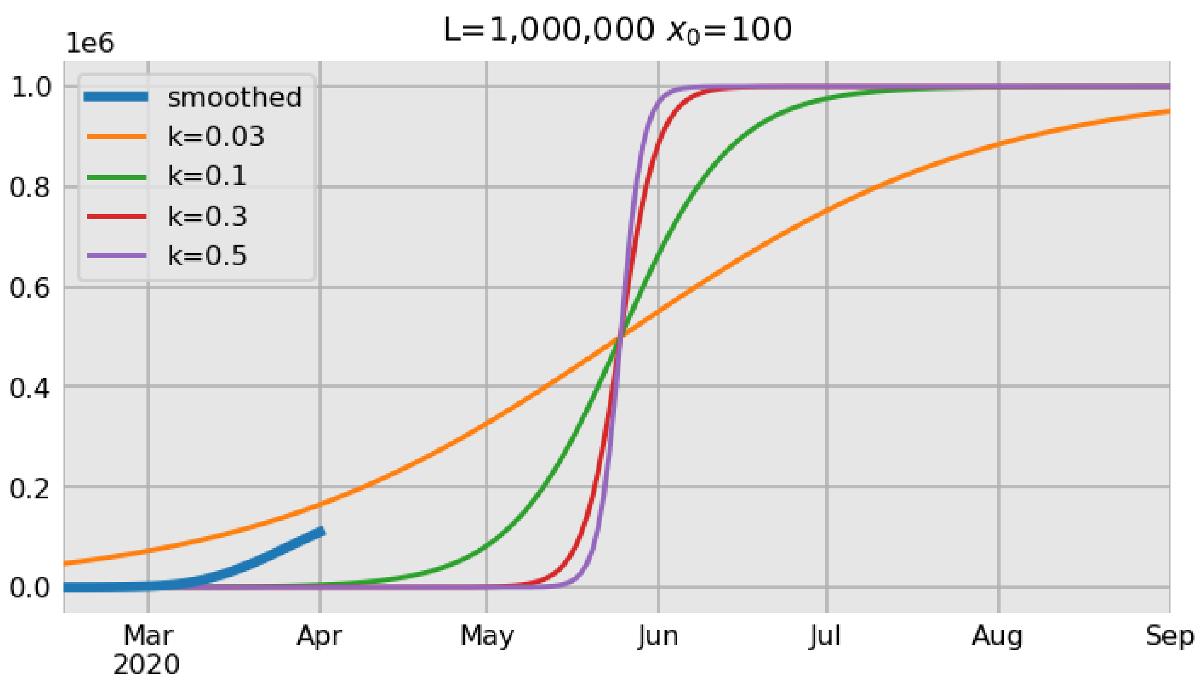
In [8]:
```
y.plot(label='actual', title="Italy Cases - Training");
y_smooth.plot(label='smoothed').legend();
```



Parameter bounds

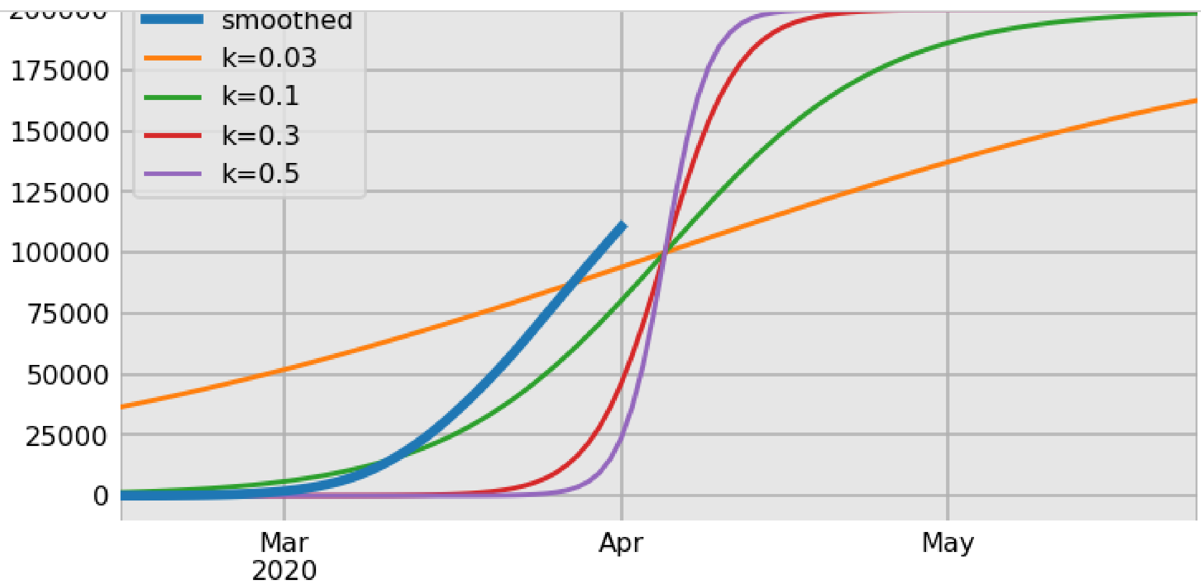
In [9]:
```
def plot_ks(s, ks, L, x0):
    """
    Visualizes various logistic curves to assist in parameter tuning.

    Parameters
    ----------
    s : Series
    ks : list of floats
    L, x0 : float

    Returns
    -------
    None
    """
    start = s.index[0]
    index = pd.date_range(start, periods=2 * x0)
    x = np.arange(len(index))
    s.plot(label="smoothed", lw=3, title=f"L={L:,} $x_0$={x0}", zorder=3)
    for k in ks:
        y = logistic_func(x, L, x0, k)
        y = pd.Series(y, index=index)
        y.plot(label=f"k={k}").legend()
```

In [10]:
```
from functions import plot_ks
ks = [0.03, 0.1, 0.3, 0.5]
plot_ks(y_smooth, ks, 1_000_000, 100)
```



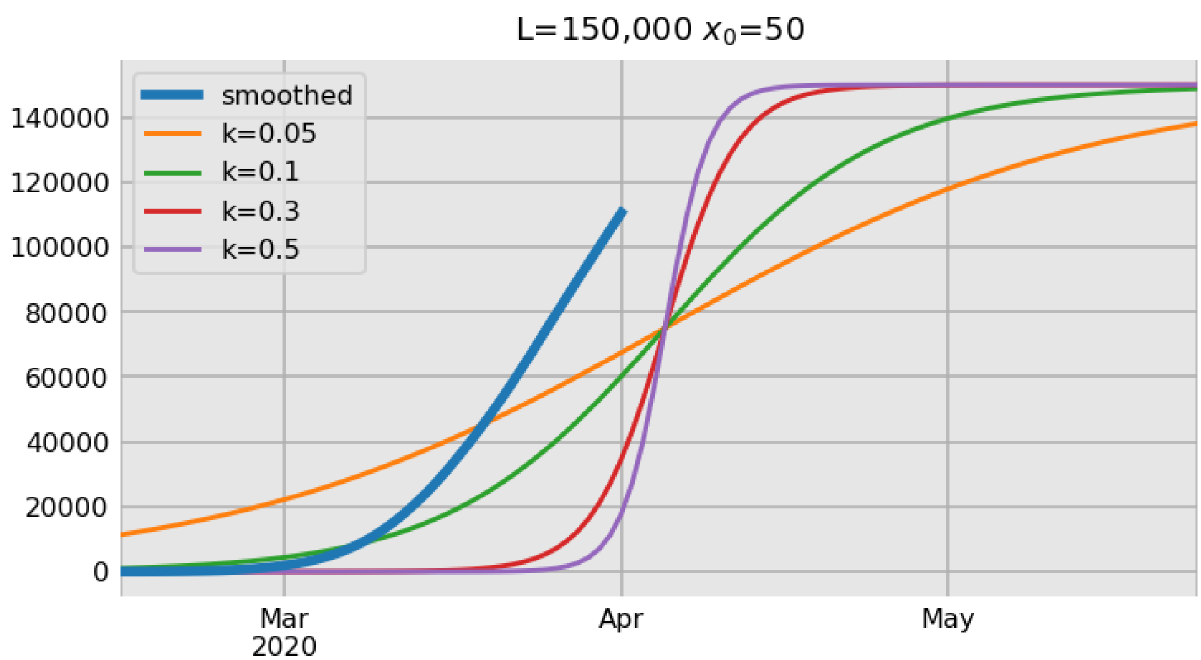
When using a shorter amount of time to the mid-point and an L of 200,000, the shape of the smoothed curve is closer to 0.1.

In [11]:
```
ks = [0.03, 0.1, 0.3, 0.5]
plot_ks(y_smooth, ks, 200_000, 50)
```



If we assume just a small future increase in total cases, the smoothed slope is closer to 0.3.

In [12]:
```
ks = [.05, 0.1, 0.3, 0.5]
plot_ks(y_smooth, ks, 150_000, 50)
```



We use `least_squares` just as before to get the fitted parameter values.

In [13]:
```python
from functions import logistic_func, optimize_func
from scipy.optimize import least_squares
y = y_smooth
x = np.arange(len(y))
res = least_squares(optimize_func, p0, args=(x, y, logistic_func), bounds=bounds)
L, x0, k = res.x
print(f'L = {L:,.0f}\nx0 = {x0:.0f}\nk = {k:.3f}')
```

```
---------------------------------------------------------------------------
NameError                                 Traceback (most recent call last)
Cell In[13], line 5
      3 y = y_smooth
      4 x = np.arange(len(y))
----> 5 res = least_squares(optimize_func, p0, args=(x, y, logistic_func), bounds
=bounds)
      6 L, x0, k = res.x
      7 print(f'L = {L:,.0f}\nx0 = {x0:.0f}\nk = {k:.3f}')

NameError: name 'p0' is not defined
```

Let's use our handy function `predict_all` to plot the next 50 days with this model.

In [14]:
```python
from functions import predict_all
predict_all(italyc, start_date=None, last_date="2020-04-01", n_pred=50, n_smooth=1
            model=logistic_func, bounds=bounds, p0=p0, title="Italy");
```

```
---------------------------------------------------------------------------
NameError                                 Traceback (most recent call last)
Cell In[14], line 3
      1 from functions import predict_all
      2 predict_all(italyc, start_date=None, last_date="2020-04-01", n_pred=50, n
_smooth=15,
----> 3             model=logistic_func, bounds=bounds, p0=p0, title="Italy")

NameError: name 'bounds' is not defined
```

In [15]:
```python
predict_all(italyc, start_date=None, last_date="2020-05-01", n_pred=30, n_smooth=1
            model=logistic_func, bounds=bounds, p0=p0, title="Italy");
```

```
---------------------------------------------------------------------------
NameError                                 Traceback (most recent call last)
Cell In[15], line 2
      1 predict_all(italyc, start_date=None, last_date="2020-05-01", n_pred=30, n
_smooth=15,
----> 2             model=logistic_func, bounds=bounds, p0=p0, title="Italy")

NameError: name 'bounds' is not defined
```

Generalized Logistic Function

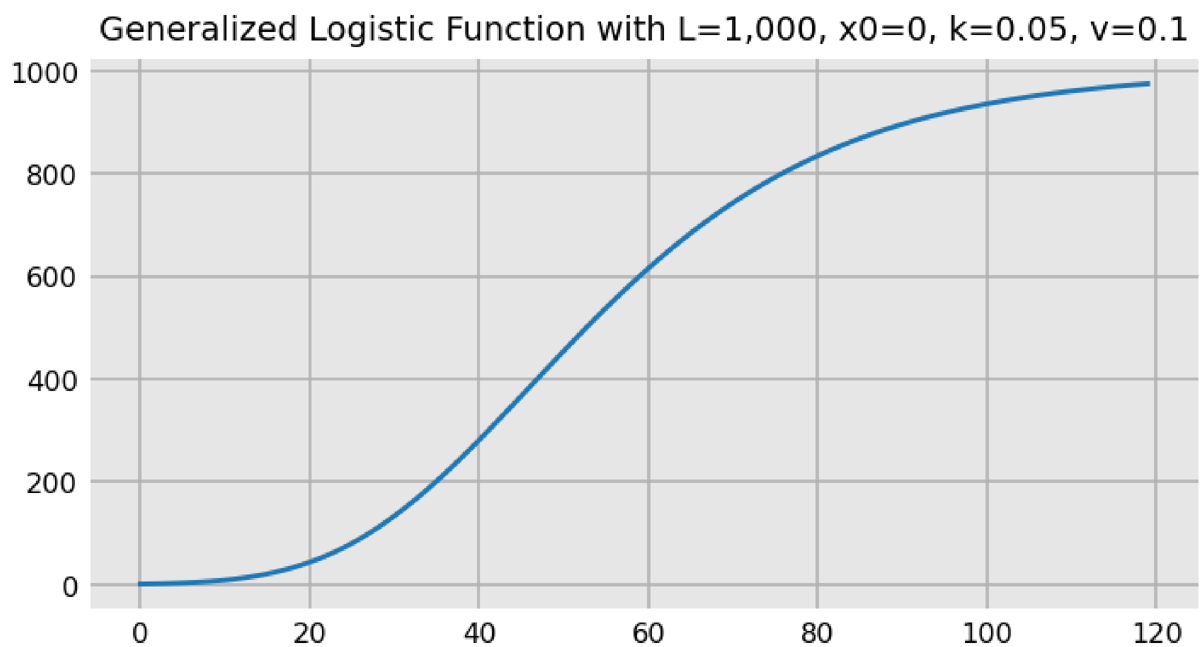
$$f(x) = \frac{L}{(1 + e^{-k(x-x_0)})^{\frac{1}{v}}}$$

The parameters L and k maintain their previous meanings, representing the curve's maximum and the growth rate, respectively. A new parameter, v, is introduced to modify the curve's symmetry and effectuate a horizontal shift. It's important to note that v should always be positive. The impact of varying v is as follows:

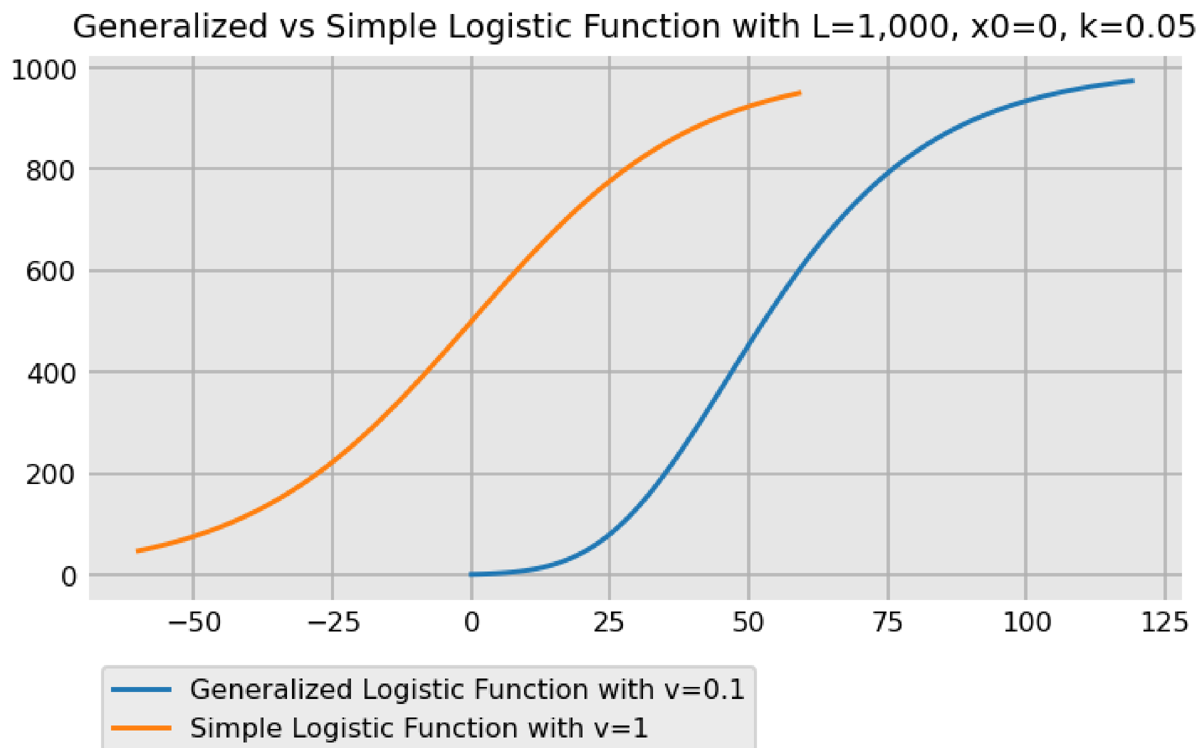
- For `v < 1`:
  - The growth rate is accelerated before reaching the midpoint.
  - This results in the curve being shifted towards the right.
- For `v > 1`:
  - The growth rate is quicker post-midpoint.
  - Consequently, the curve is shifted towards the left.

In [16]:
```
def general_logistic(x, L, x0, k, v):
    return L / ((1 + np.exp(-k * (x - x0))) ** (1 / v))
```

In [17]:
```
L, x0, k, v = 1_000, 0, 0.05, 0.1
x = np.arange(120)
y = general_logistic(x, L, x0, k, v)
fig, ax = plt.subplots()
ax.plot(x, y)
ax.set_title(f"Generalized Logistic Function with L={L:,}, x0={x0}, k={k}, v={v}")
```



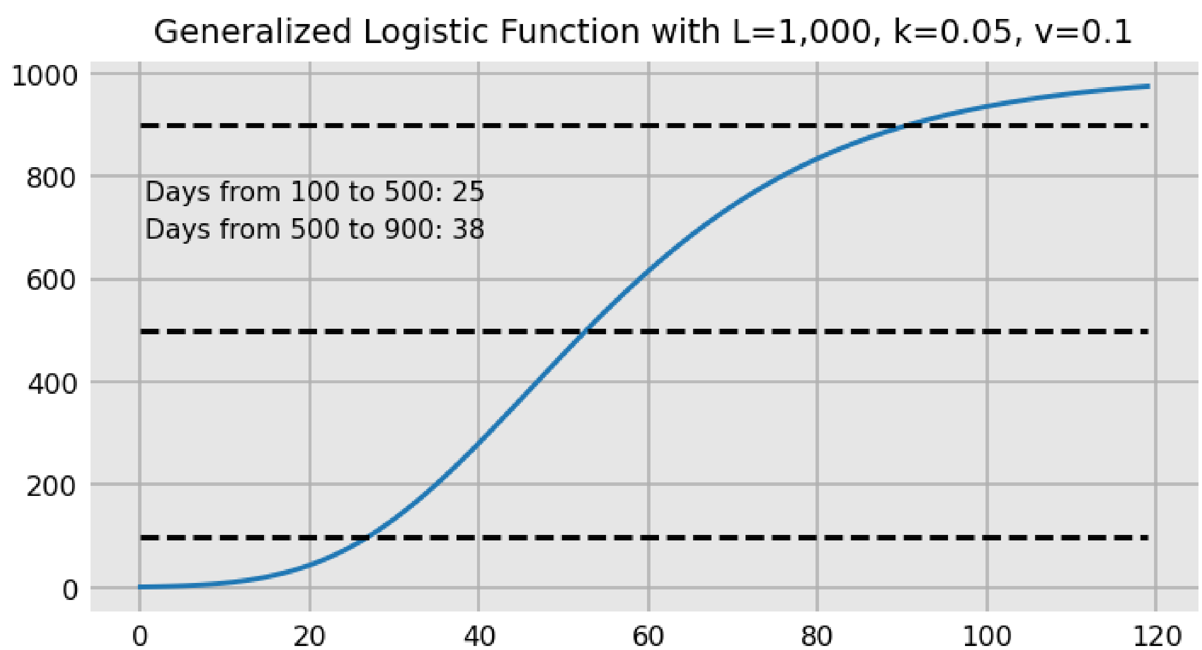
In [18]:
```python
# same as above
L, x0, k, v = 1_000, 0, 0.05, 0.1
x = np.arange(120)
y = general_logistic(x, L, x0, k, v)

# v set to 1 - simple logistic function
L, x0, k, v = 1_000, 0, 0.05, 1
x1 = np.arange(-60, 60)
y1 = general_logistic(x1, L, x0, k, v)

fig, ax = plt.subplots()
ax.plot(x, y, label='Generalized Logistic Function with v=0.1')
ax.plot(x1, y1, label='Simple Logistic Function with v=1')
ax.legend(bbox_to_anchor=(0, -.1), loc='upper left')
ax.set_title(f"Generalized vs Simple Logistic Function with L={L:,}, x0={x0}, k={k
```



Plotting Asymmetry

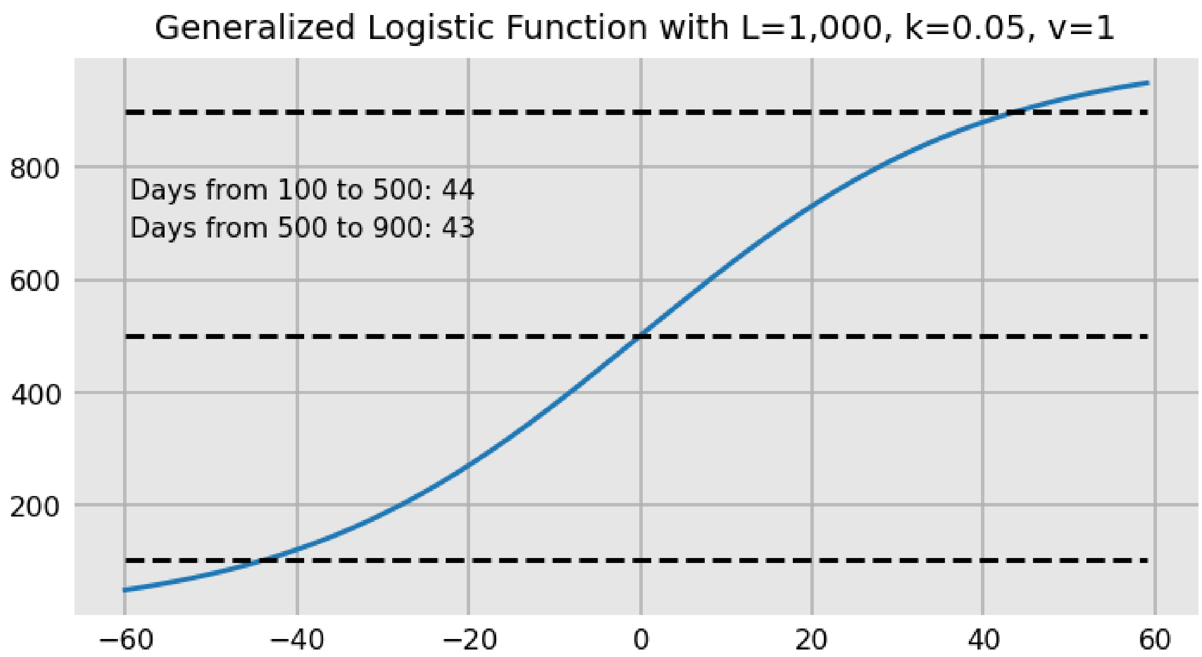
In [19]:
```
def plot_asymmetry(x, L, x0, k, v):
    y = general_logistic(x, L, x0, k, v)
    fig, ax = plt.subplots()
    ax.plot(x, y)
    low, mid, high = int(0.1 * L), int(0.5 * L), int(0.9 * L)
    ax.hlines([low, mid, high], x[0], x[-1], color='black', ls='--')

    days_to_10 = np.argmax(y > low)
    days_to_50 = np.argmax(y > mid)
    days_to_90 = np.argmax(y > high)
    days_10_to_50 = days_to_50 - days_to_10
    days_50_to_90 = days_to_90 - days_to_50
    ax.set_title(f"Generalized Logistic Function with L={L:,}, k={k}, v={v}")
    ax.text(0.05, 0.75, f'Days from {low} to {mid}: {days_10_to_50}', transform=ax
    ax.text(0.05, 0.68, f'Days from {mid} to {high}: {days_50_to_90}', transform=a
```

In [20]:
```
x = np.arange(120)
plot_asymmetry(x, L=1000, x0=0, k=0.05, v=0.1)
```



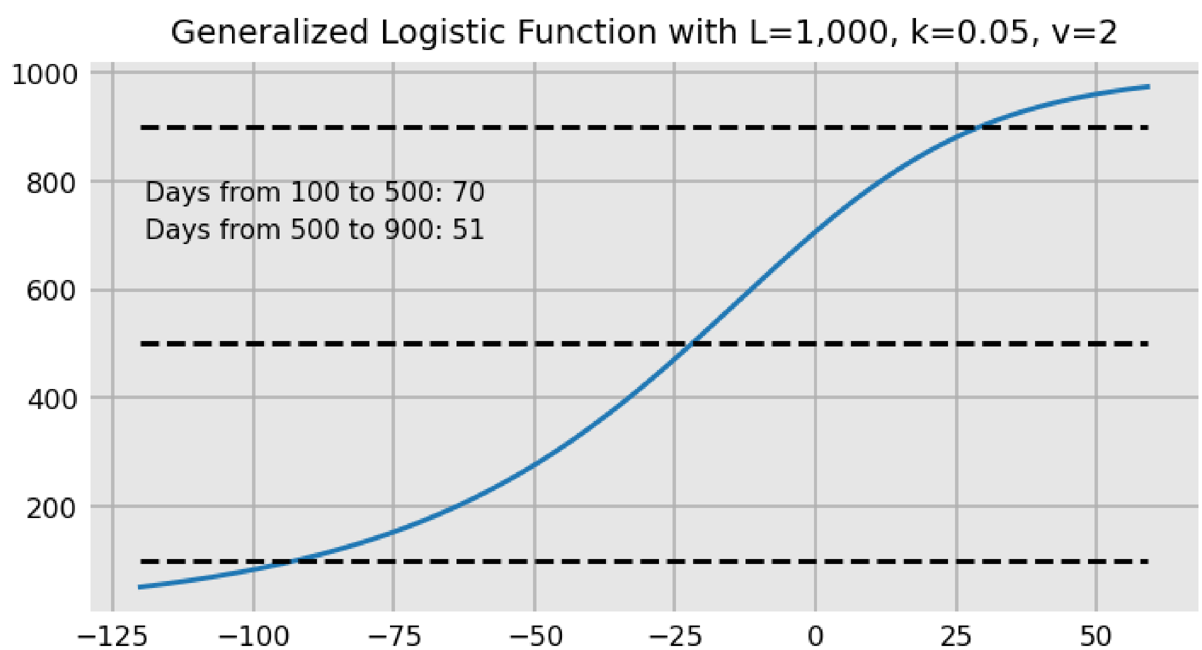
Setting v to 1 simplifies the logistic growth function by making it symmetric.

In [21]:
```
x = np.arange(-60, 60)
plot_asymmetry(x, L=1000, x0=0, k=0.05, v=1)
```



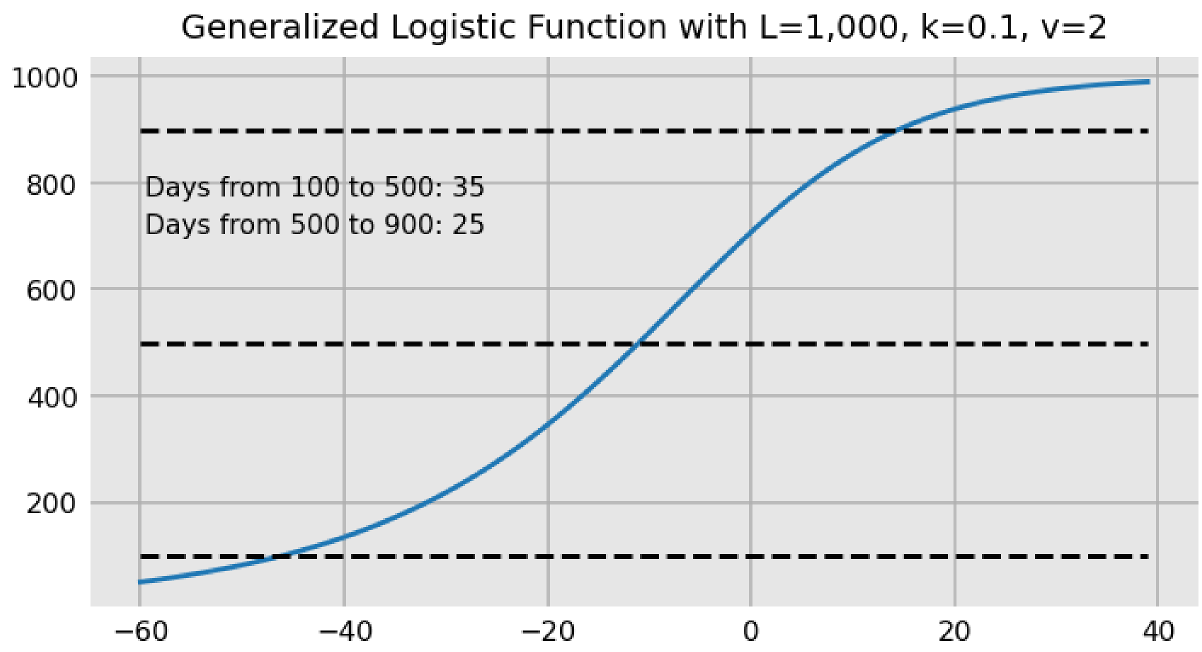
Setting the v value to 2 allows the curve to reach the right asymptote faster.

In [22]:
```
x = np.arange(-120, 60)
plot_asymmetry(x, L=1000, x0=0, k=0.05, v=2)
```

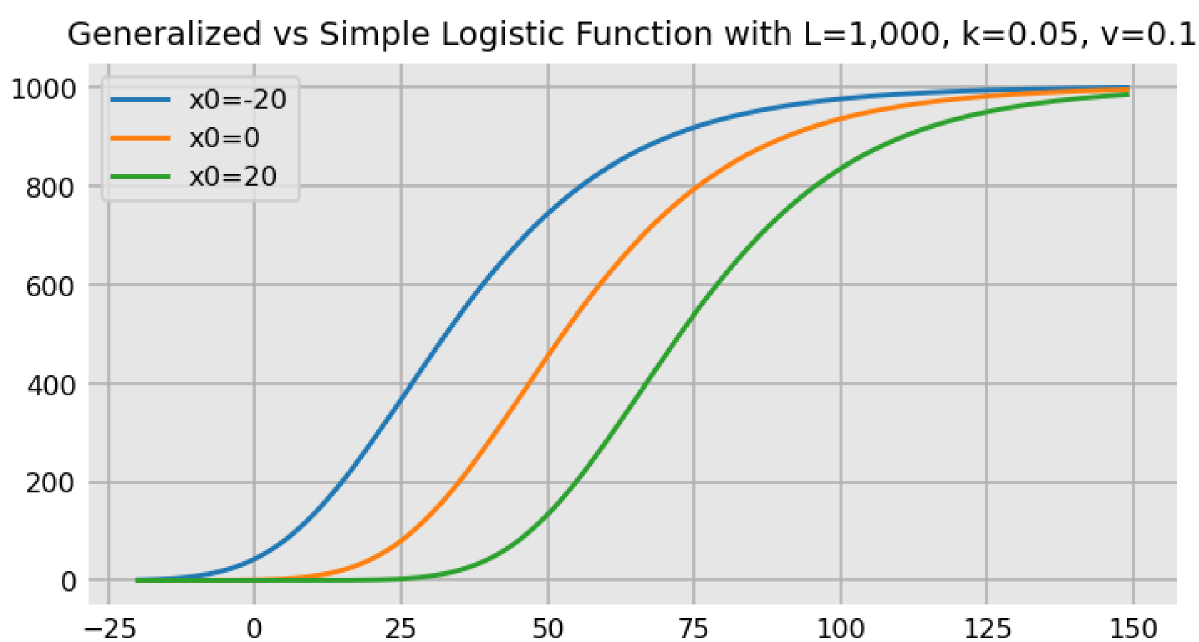


Increasing k as v increases helps decrease the time between the 10th and 90th percentile.

In [23]:
```
x = np.arange(-60, 40)
plot_asymmetry(x, L=1000, x0=0, k=0.1, v=2)
```

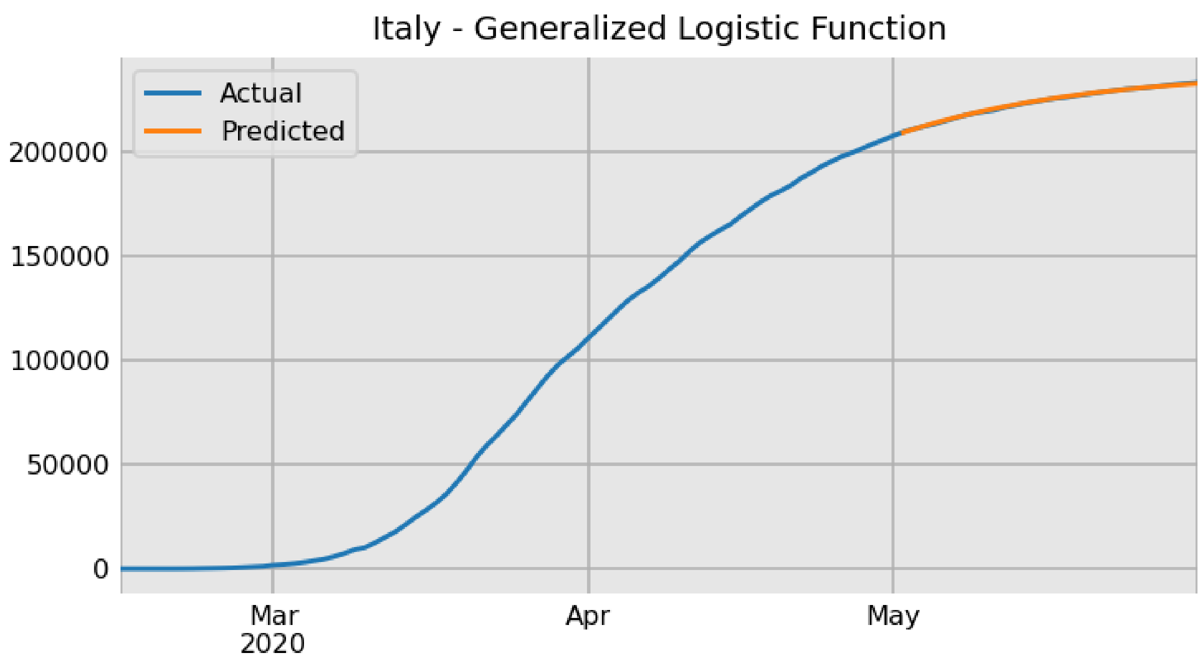


In [24]:
```
L, k, v = 1_000, 0.05, 0.1
x = np.arange(-20, 150)

fig, ax = plt.subplots()
for x0 in [-20, 0, 20]:
    y = general_logistic(x, L, x0, k, v)
    ax.plot(x, y, label=f'x0={x0}')
ax.legend()
ax.set_title(f"Generalized vs Simple Logistic Function with L={L:,}, k={k}, v={v}"
```



Predictions with the generalized logistic function

In [25]:
```
last_date = "2020-04-01"
L_min, L_max = italyc[last_date], 1_000_000
x0_min, x0_max = -50, 50
k_min, k_max = 0.01, 0.5
v_min, v_max = 0.01, 2
lower = L_min, x0_min, k_min, v_min
upper = L_max, x0_max, k_max, v_max
bounds = lower, upper
p0 = L_max, 0, 0.1, 0.1

params, y_pred = predict_all(italyc, start_date=None, last_date="2020-05-01", n_sm
                             n_pred=30, model=general_logistic, bounds=bounds, p0=
                             title="Italy - Generalized Logistic Function");
```

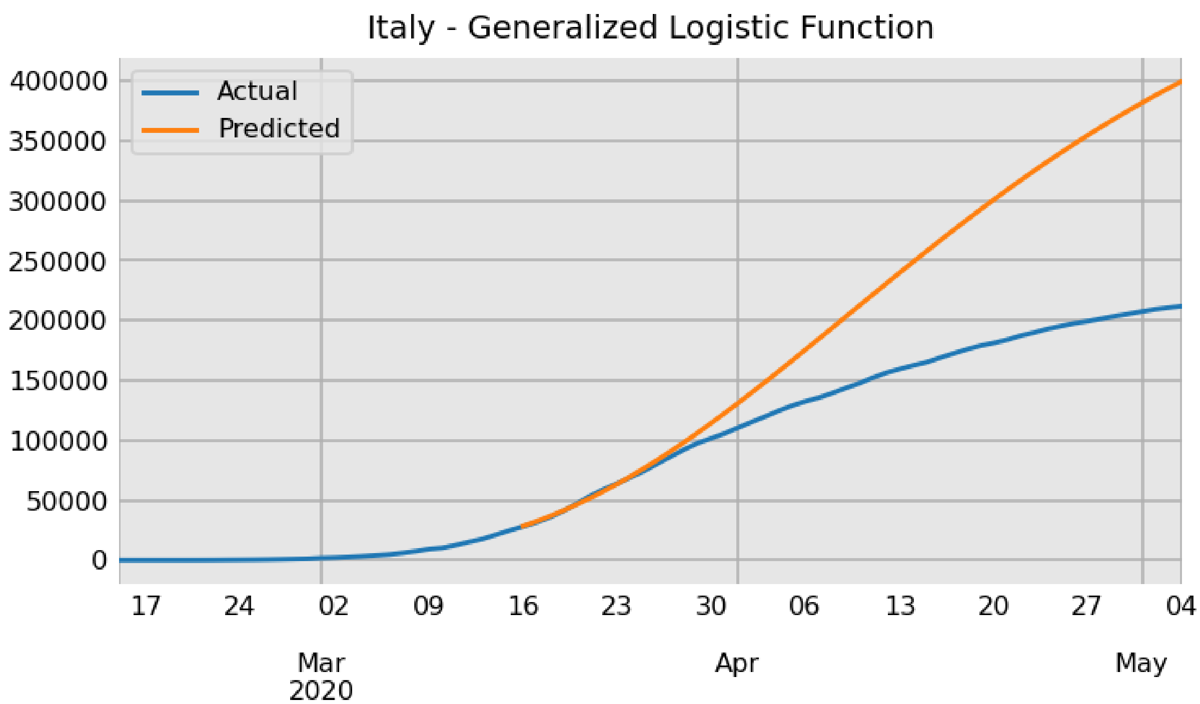


The predictive model now aligns with the actual values.

In [26]:
```
params
```

Out[26]:
```
array([ 2.36841824e+05, -3.61763199e+01,  5.86413407e-02,  1.00000000e-02])
```

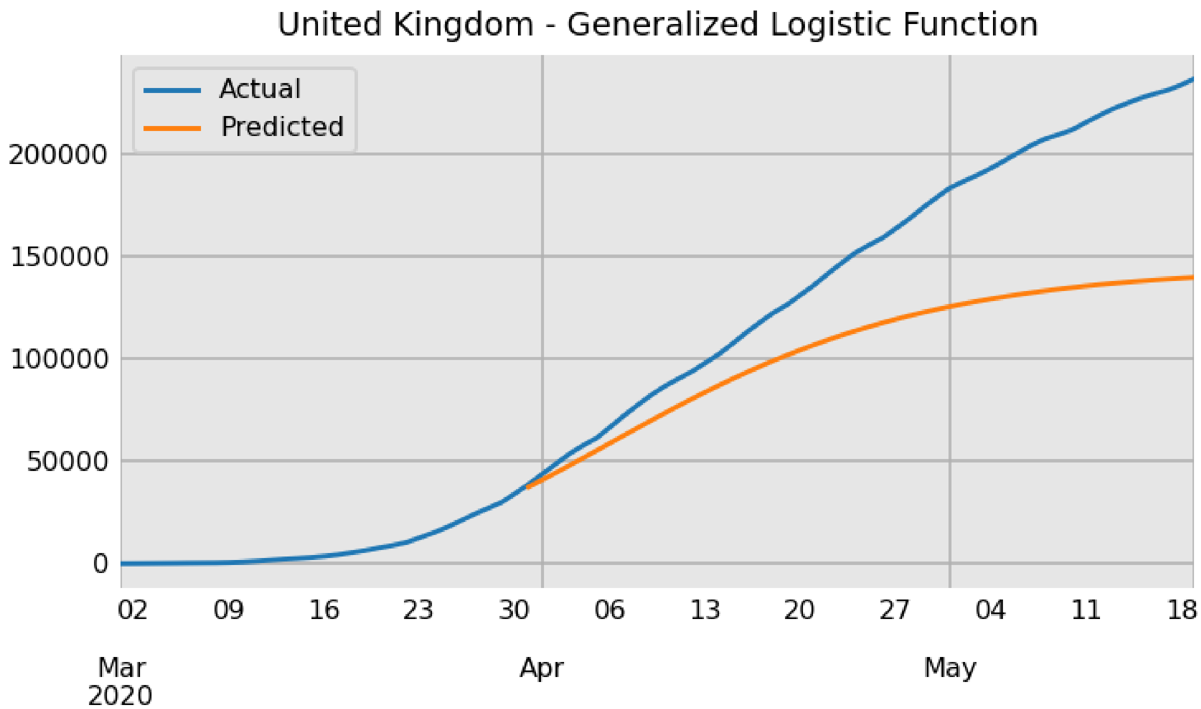
In [27]:
```
predict_all(italyc, start_date=None, last_date="2020-03-15", n_smooth=15,
            n_pred=50, model=general_logistic, bounds=bounds, p0=p0,
            title="Italy - Generalized Logistic Function");
```



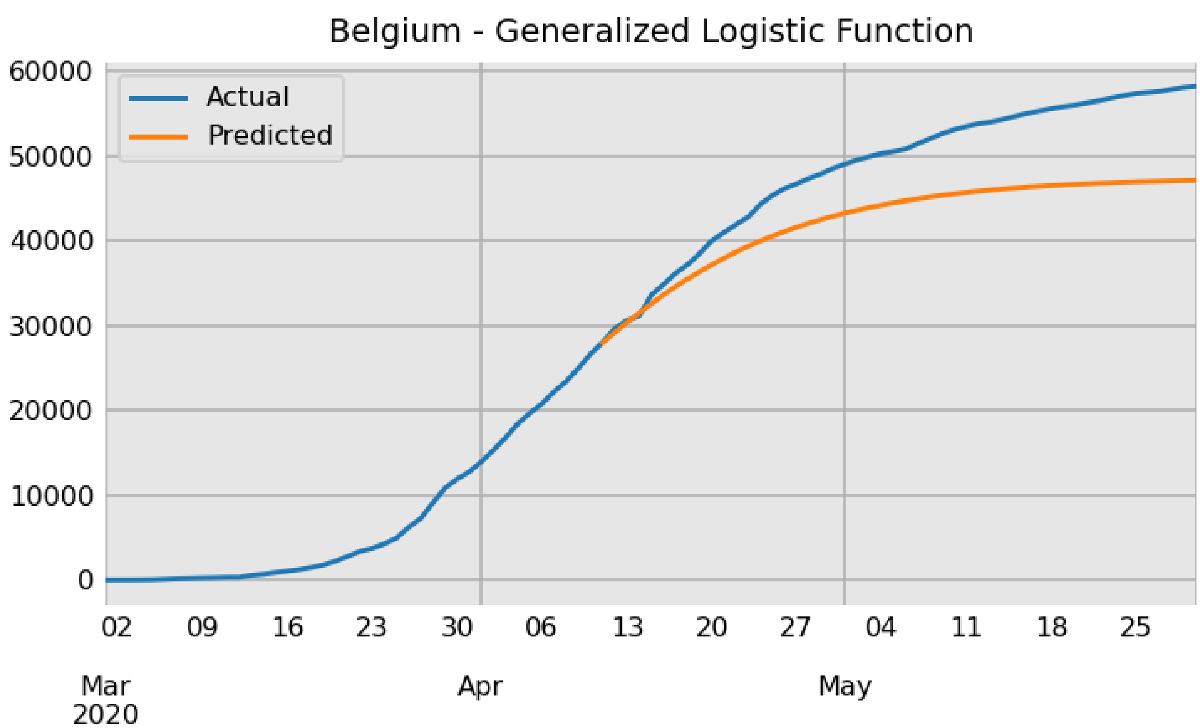
Modeling other countries

In [28]:
```
def model_country(data, name, start_date, last_date):
    s = data['world_cases'][name]
    L_min, L_max = s.iloc[0], s.iloc[-1] * 1000
    x0_min, x0_max = -50, 50
    k_min, k_max = 0.01, 0.5
    v_min, v_max = 0.01, 2
    lower = L_min, x0_min, k_min, v_min
    upper = L_max, x0_max, k_max, v_max
    bounds = lower, upper
    p0 = L_min * 5, 0, 0.1, 0.1
    predict_all(s, start_date=start_date, last_date=last_date, n_smooth=15,
                n_pred=50, model=general_logistic, bounds=bounds, p0=p0,
                title=f"{name} - Generalized Logistic Function");
```

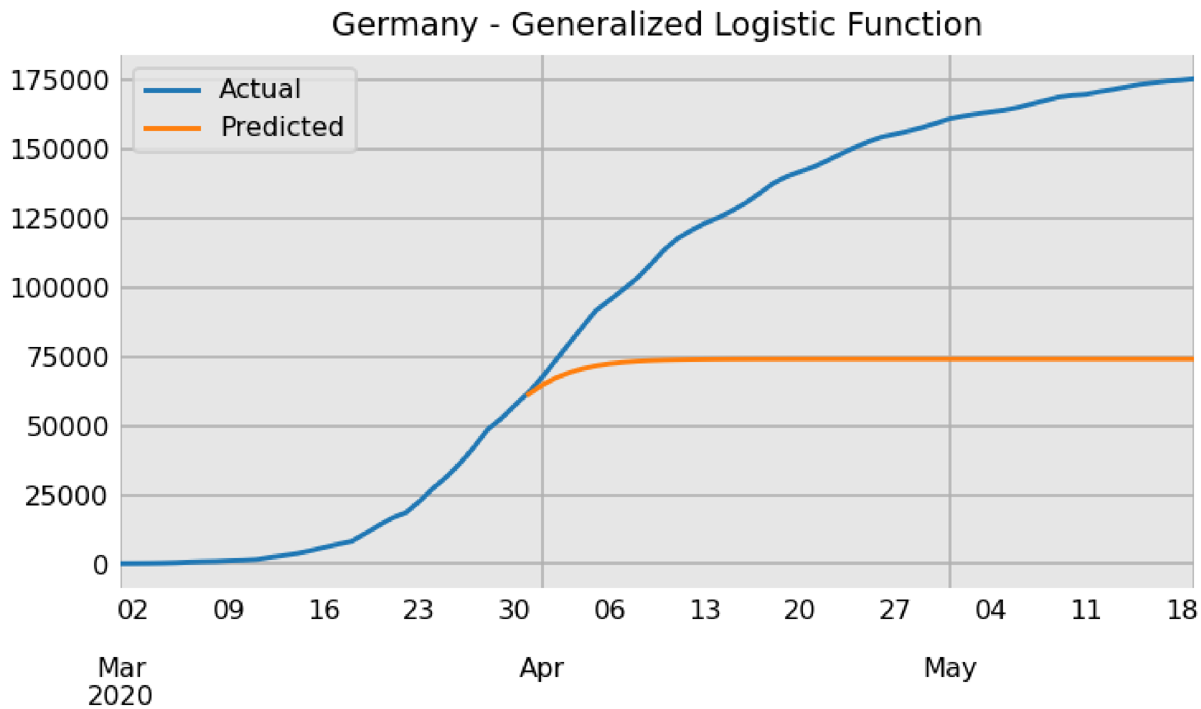
In [29]:
```
model_country(data, "United Kingdom", "2020-03-01", "2020-03-30")
```



In [30]:
```
model_country(data, "Belgium", "2020-03-01", "2020-04-10")
```



```
In [31]: model_country(data, "Germany","2020-03-01", "2020-03-30")
```




```
In [32]: model_country(data, "South Korea","2020-02-15", "2020-03-1")
```