Approximation theory

Two goals for part I of this course:

Goal 1: Bring approx. theory into 21st century

Goal 2: Sweep away misconceptions about polynomials.

Chebyshev points and interpolants

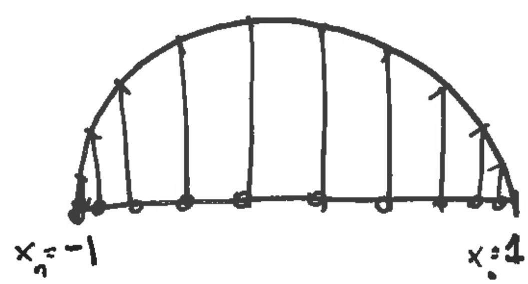
Let $n \geq 0$. $\quad P_n = \{$ poly. of degree $\leq n\}$.

$\{x_0, \dots, x_n\}$ = $n+1$ distinct points in $[-1, 1]$.

$\{f_0, \dots, f_n\}$ = real or complex samples.

There exists a unique $p \in P_n$ such that $p(x_j) = f(x_j)$, $0 \leq j \leq n$.

Chebyshev points: $\quad x_j = \cos\left(\frac{j\pi}{n}\right), \quad 0 \leq j \leq n$.



$n = 10$ $\quad$ (chebpts.)

Chebyshev interpolant:

p = unique interpolant in chebpts

Clustering at boundary:

Interpolation pts as point charges: imagine x_j is a pt charge repelling all others with force $\frac{1}{n+1} \cdot \frac{1}{|x_j - x_k|}$.

{Chebpts} $\approx$ equilibrum configuration

= minimal-energy configuration.

Each chebpt has a distance $\approx \frac{1}{2}$ to the others on average, measured by geometric mean.

(capacity of $[-1, 1]$ is $\frac{1}{2}$)

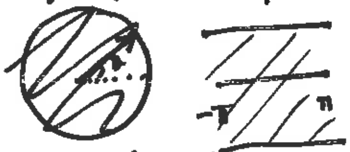
Chebyshev polynomials and series

Fourier

$\theta \in [-\pi, \pi]$

$F(\theta)$ with $F(\theta) = F(-\theta)$

$2n$ equispaced pts



F = analytic in a strip.

$$\frac{1}{2} \sum_{k=0}^{n}{}' a_k \left(e^{i\theta k} + e^{-i\theta k}\right)$$

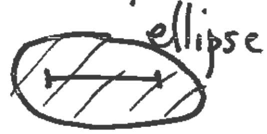
Fourier series

Chebyshev

$x \in [-1, 1]$ $f(x)$

analytic in an ellipse

$n+1$ chebpts.



polynomial

$$\sum_{k=0}^{n}{}' a_k T_k(x)$$

Chebyshev series

Laurent

z = unit circle

$F(z)$ with $F(z) = F(z^{-1})$

analytic in an annulus

Laurent polynomial

$$\frac{1}{2} \sum_{k=0}^{n}{}' a_k \left(z^k + z^{-k}\right)$$

Laurent series.

Chebyshev polynomials

$$z = e^{i\theta} \qquad x = \mathrm{Re}(z) = \cos(\theta) = \tfrac{1}{2}(z + z^{-1})$$

$$T_k(x) = \mathrm{Re}(z^k) = \tfrac{1}{2}(z^k + z^{-k}) = \cos(k\theta)$$

$$T_k(x) = \cos(k \cos^{-1}(x))$$

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x.$$

Chebyshev polynomials satisfy 3-term recurrence

$$T_{k+1}(x) = 2x T_k(x) - T_{k-1}(x), \quad k \geq 1.$$

Verify: $T_{k+1}(x) = \frac{1}{2}(z^{k+1} + z^{-k-1}) = \frac{z^k + z^{-k}}{2}(z + z^{-1}) - \frac{z^{k-1} + z^{-k+1}}{2}$

$$= 2x T_k(x) - T_{k-1}(x).$$

Theorem: Given $f : [-1,1] \to \mathbb{R}$ Lipschitz continuous,

- f has unique representation $f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$

where
$$a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_k(x)}{\sqrt{1-x^2}}\, dx, \quad k \geq 1$$

$$a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx$$

Moreover, series converges absolutely and uniformly.

Pf: Transplant to z. Use Laurent series.

Signal processing for non-periodic functions

- Sample on chebpt grids of size 17, 33, 65, ... pts.
- On each, find coeffs $a_0, \dots, a_n$ of the chebyshev interpolant (using FFT).
- Stop when coeffs fall below 10^{-16}.
- Trim to appropriate n.

Why formula for a_k?

Note that
$$\frac{2}{\pi}\int_{-1}^{1} \frac{T_j(x)T_k(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} \frac{1}{2}, & j=k=0, \\ 1, & j=k>0 \\ 0, & j \neq k. \end{cases}$$

Example:

$$|x| = \sum_{k=0}^{\infty} a_k T_k(x),$$

$$a_0 = \frac{2}{\pi}, \quad a_2 = \frac{4}{3\pi}, \quad a_4 = -\frac{4}{15\pi}$$

$$a_6 = \frac{4}{63\pi}, \quad a_8 = -\frac{4}{255\pi}.$$

Example:

$$e^x = \sum_{k=0}^{\infty} a_k T_k(x),$$

$$a_k = 2 I_k(1)$$

$$a_0 = I_0(1)$$

modified Bessel function.