Hilbert space: "Usually infinite-dimensional func. space".

- Generalizes ~~Eu~~ Euclidean space $\mathbb{R}^d$, $d \geq 1$. [Pythagorean thm, parallelogram law, etc.]
- A vector space with an inner-product. ← Can measure lengths and angles.
- Complete ← Enough limits in the space to allow calculus.

Motivating example: Euclidean space. $\mathbb{R}^3$.

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$ vector space!

Inner-product:
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3 \qquad \|\underline{x}\|_2^2 = x_1^2 + x_2^2 + x_3^2.$$

Complete: Consider $\sum_{n=0}^{\infty} x_n$ s.t $\sum_{n=0}^{\infty} \|x_n\|_2 < \infty$

then $\sum_{n=0}^{\infty} x_n \in \mathbb{R}^3$.

Definition: A Hilbert space $\mathcal{H}$ is a real (or complex) inner vector space, with an inner-product $\langle x, y \rangle$ for $x, y \in \mathcal{H}$:

i) $\langle y, x \rangle = \overline{\langle x, y \rangle}$

ii) $\langle a x_1 + b x_2, y \rangle = a \langle x_1, y \rangle + b \langle x_2, y \rangle$.

iii) $\langle x_1, x_1 \rangle \geq 0$ with equilality iff $x = 0$.

that is a complete metric space.

Everytime you take a series $\sum_{k=0}^{\infty} x_k$ s.t $\sum_{k=0}^{\infty} \|x_k\| < \infty$

then $\lim_{N \to \infty} \sum_{k=0}^{N} x_k \in \mathcal{H}$.

Example : $k(x, y) = \underline{x}^T y$, $\quad K : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$.

Consider $f : \mathbb{R}^d \to \mathbb{R}$. Given $y_1, \dots, y_n \in \mathbb{R}^d$.

We want $\quad f(\underline{x}) \overset{?}{=} \sum_{i=1}^{n} a_i \, \underline{x}^T y_i$.

we construct $\quad p_n(\underline{x}) = \sum_{i=1}^{n} a_i \, \underline{x}^T y_i \quad$ s.t. $\; p_n(y_i) = f(x_i)$.

Solve : $$Y^T Y \underline{a} = \begin{bmatrix} f(y_1) \\ \vdots \\ f(y_n) \end{bmatrix}$$

Example. (Square integrable functions) $L^2([0,1])$

$$L^2([0,1]) = \left\{ f : \int_0^1 |f(x)|^2 dx < \infty \right\}$$

① Vector space;

② Inner-product: $\langle f, g \rangle = \int_0^1 \overline{f(x)}\, g(x)\, dx$

③ Complete: ~~Let $\{f_0, f_1, \ldots\}$ be a sequence of $L^2([0,1])$ func.~~

Riesz-Fischer theorem.

Other classical examples are [illegible] [illegible] ℓ^p [illegible]

$\mathcal{H}^s$ = Sobolev spaces

H^2 = Hardy spaces.

Def$^{\underline{n}}$ We say K is a reproducing kernel for Hilbert space if for every $f \in \mathcal{H}$ we have $f(x) = \langle f, K(\cdot, x) \rangle_{\mathcal{H}}$.

Ex. ~~Suppose $X : \mathbb{R}^d \to \mathbb{R}$~~

~~$\mathcal{H} = \mathbb{R}^d$. $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ $\quad K(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$~~

$$f \in \ell^2(X) = \left\{ f : X \to \mathbb{R} : \sum_{x \in X} |f(x)|^2 < \infty \right\}$$

(X ↑ discrete set)

$$f, g \in \ell^2(X) \qquad \langle f, g \rangle = \sum_{x \in X} f(x)\, g(x).$$

Note that $\langle f, e_y \rangle = \sum_{x \in X} f(x)\, e_y(x) = f(y)$

$$e_y(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

$$K(x,y) = \langle e_x, e_y \rangle = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Moore – Aronszajn theorem

Theorem: Suppose K is a symm. positive definite kernel ~~and~~ on a set Ω. Then, there is a unique Hilbert space of functions on Ω for which K is a reproducing kernel.

Proof:

For any $x \in \Omega$, define $K_x = K(\cdot, x)$. Let $H_0 = \text{Span}\{K(\cdot, x) : x \in \Omega\}$

Define the inner-product on H_0 by:

$$\left\langle \sum_{j=1}^{n} b_i K(\cdot, x_i), \sum_{s=1}^{n} a_s K(\cdot, y_s) \right\rangle_{H_0} = \sum_{j=1}^{n} \sum_{s=1}^{n} a_s b_j K(x_i, y_s).$$

which implies that $K(x, y) = \langle K_x, K_y \rangle_{H_0}$.

Now, complete H_0 to a Hilbert space by taking limit of the form

$$\lim_{n \to \infty} \sum_{j=1}^{n} b_j K(\cdot, x_i)$$

[series needs to abs. converge]

If limit not in H_0, then throw it into H_0.

Example: ~~Let $K(x,y) = x^T y$ for $x, y \in \mathbb{R}^d$.~~

~~Hence, $f(v) = \sum_{i=1}^{n} b_i v^T y_i$~~

~~Let $K(x,y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases}$~~