The results are similar for the other example, $f(x) = |\sin(5x)|^3$, whose third derivative, we saw, has variation $V \approx 21028$. Equation (7.5) implies that the Chebyshev interpolants satisfy $\|f - p_n\| < 7020/(n-1)^3$, whereas in fact we have $\|f - p_n\| \approx 309/n^3$ for large odd n and $\|f - p_n^*\| \approx 80/n^3$.

We close with a comment about Theorem 7.2. We have assumed in this theorem that $f^{(\nu)}$ is of bounded variation. A similar but weaker condition would be that $f^{(\nu-1)}$ is Lipschitz continuous (Exercise 7.2). This weaker assumption is enough to ensure $\|f - p_n^*\| = O(n^{-\nu})$ for the best approximations $\{p_n^*\}$; this is one of the Jackson theorems. On the other hand, it is not enough to ensure $O(n^{-\nu})$ convergence of Chebyshev projections and interpolants. The reason we emphasize the stronger assumption with the stronger conclusion is that in practice, one rarely deals with a function that is Lipschitz continuous while lacking a derivative of bounded variation, whereas one constantly deals with projections and interpolants rather than best approximations.

Incidentally, it was de la Vallée Poussin [1908] who first showed that the strong hypothesis is enough to reach the weak conclusion: if $f^{(\nu)}$ is of bounded variation, then $\|f - p_n^*\| = O(n^{-\nu})$ for the best approximation p_n^*. Three years later Jackson [1911] sharpened the result by weakening the hypothesis as just indicated.

SUMMARY OF CHAPTER 7. *The smoother a function f defined on $[-1,1]$ is, the faster its approximants converge. In particular, if the νth derivative of f is of bounded variation V, then the Chebyshev coefficients $\{a_k\}$ of f satisfy $|a_k| \le 2\pi^{-1}V(k-\nu)^{-\nu-1}$. For $\nu \ge 1$, it follows that the degree n Chebyshev projection and interpolant of f both have accuracy $O(Vn^{-\nu})$.*

Exercise 7.1. Total variation. (a) Determine numerically the total variation of $f(x) = \sin(100x)/(1+x^2)$ on $[-1,1]$. (b) It is no coincidence that the answer is ≈ 100; the total variation of $\sin(Mx)/(1+x^2)$ on $[-1,1]$ is asymptotic to M as $M \to \infty$. Explain why.

Exercise 7.2. Lipschitz continuous vs. derivative of bounded variation. (a) Prove that if the derivative f' of a function f has bounded variation, then f is Lipschitz continuous. (b) Give an example to show that the converse does not hold.

Exercise 7.3. Convergence for Weierstrass's function. Exercise 6.1 considered a "pathological function" $w(x)$ that is continuous but nowhere differentiable on $[-1,1]$. (a) Make an anonymous function in MATLAB that evaluates w(**xx**) for a vector **xx** to machine precision by taking the sum (6.1) to 53 terms. (b) Use Chebfun to produce a plot of $\|w - p_n\|$ accurate enough and for high enough values of n to confirm that convergence appears to take place as $n \to \infty$. Thus w is not one of the functions for which interpolants fail to converge, a fact we can prove with the techniques of Chapter 15 (Exercise 15.9).

Exercise 7.4. Sharpness of Theorem 7.2. Consider the functions (a) $f(x) = |x|$, (b) $f(x) = |x|^5$, and (c) $f(x) = \sin(100x)$. In each case plot, as functions of n, the error $\|f - p_n\|$ in Chebyshev interpolation on $[-1,1]$ and the bound on this quantity from (7.5). How close is the bound to the actuality? In cases (a) and (b), take ν as large as possible, and in case (c), take $\nu = 2, 4$, and 8.

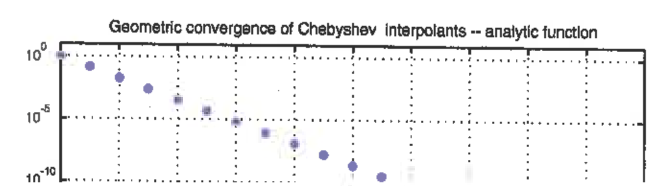
Exercise 7.5. Total variation. Let f be a smooth function defined on $[0,1]$ and let $t(x)$ be its total variation over the interval $[0,x]$. What is the total variation of t over $[0,1]$?

Chapter 8. Convergence for Analytic Func

Suppose f is not just k times differentiable but infinitely differentiab analytic on $[-1,1]$. (Recall that this means that for any $s \in [-1,1]$, f series about s that converges to f in a neighborhood of s.) Then witho assumptions we may conclude that the Chebyshev projections anc converge *geometrically*, that is, at the rate $O(C^{-n})$ for some constan means the errors will look like straight lines (or better) on a semilo than a loglog scale. This kind of connection was first announced in 1 stein, who showed that the best approximations to a function f on [– geometrically as $n \to \infty$ if and only if f is analytic [Bernstein 1911 &

For example, for Chebyshev interpolants of the function (1 + 25 as the *Runge function* (Chapter 13), we get steady geometric converg the level of rounding errors:

```
x = chebfun('x'); f = 1/(1+25*x^2); nn = 0:10:200; ee = 0*
for j = 1:length(nn)
    n = nn(j); fn = chebfun(f,n+1); ee(j) = norm(f-fn,inf)
end
semilogy(nn,ee,'.')
```



function in any complex neighborhood. Find another formula for f that does, and use it to explain what singularities f has in the complex plane.

Exercise 8.8. Chebyshev polynomials on the Bernstein ellipse. Show that for any $\rho > 1$ and any z on the boundary of the ellipse E_ρ in the complex x-plane, $\lim_{n\to\infty} |T_n(x)|^{1/n} = \rho$.

Exercise 8.9. You can't judge smoothness by eye. Define $f(x) = 2 + \sin(50x)$ and $g(x) = f(x)^{1.0001}$ and construct chebfuns for these functions on $[-1, 1]$. What are their lengths? Explain this effect quantitatively using the theorems of this chapter.

Exercise 8.10. Convergence of conjugate gradient iteration. Suppose we wish to approximate $f(x) = x^{-1}$ on the interval $[m, M]$ with $0 < m < M$. Show that for any $\kappa > M/m$, there exist polynomials $p_n \in \mathcal{P}_n$ such that $\|f - p_n\| = O((1 + 2/\sqrt{\kappa})^{-n})$ as $n \to \infty$, where $\|\cdot\|$ is the ∞-norm on $[m, M]$. This result is famous in numerical linear algebra as providing an upper bound for the convergence of the conjugate gradient iteration applied to a symmetric positive definite system of equations $Ax = b$ with condition number κ. See Theorem 38.5 of [Trefethen & Bau 1997].

Exercise 8.11. Bernstein's theorem. Show that the conclusion of Theorem 8.3 also holds if the hypothesis is weakened to $\limsup_{n\to\infty} \|f - q_n\|^{1/n} \le \rho^{-1}$.

Exercise 8.12. Resolution power of Chebyshev interpolants. The function $f_M(x) = \exp(-M^2x^2/2)$ has a spike of width $O(1/M)$ at $x = 0$. Let $n(M)$ be the degree of a chebfun for f_M. (a) Determine the asymptotic behavior of $n(M)$ as $M \to \infty$ by numerical experiments. (b) Explain this result based on the theorems of this chapter.

Exercise 8.13. Resolution power of Bernstein polynomials. Continuing the last exercise, now let $n(M)$ be the degree of a Bernstein polynomial (6.2) needed to approximate f_M to machine precision. (For this discussion rescale (6.2) from $[0, 1]$ to $[-1, 1]$.) (a) Determine the asymptotic behavior of $n(M)$ as $M \to \infty$ by numerical experiments. (b) Explain this result, not necessarily rigorously.

Exercise 8.14. Formulas for ellipse parameter. Derive (8.4) and (8.5).

Exercise 8.15. Simple poles on the Bernstein ellipse. (a) Explain how (3.16) illustrates that Theorem 8.3 is not an exact converse of Theorem 8.2. (b) Let f be analytic in the open Bernstein ellipse region E_ρ for some $\rho > 1$ with the only singularities on the ellipse itself being simple poles. Show that $\|f - f_n\|$ and $\|f - p_n\|$ are of size $O(\rho^{-n})$ as $n \to \infty$.

Chapter 9. Gibbs Phenomenon

Polynomial interpolants and projections oscillate and overshoot ne
ities. We have observed this *Gibbs phenomenon* already in Chapter
shall look at it more carefully. We shall see that the Gibbs effect fo
can be regarded as a consequence of the oscillating inverse-linear tail
polynomials, i.e., interpolants of Kronecker delta functions. Chapte
that these same tails, combined together in a different manner, are
of Lebesgue constants of size $O(\log n)$, with implications throughout
theory.

To start, consider `sign(x)`, interpolated in $n+1 = 10$ and 20 Che
We take n odd to avoid a grid point at the middle of the step.

```
x = chebfun('x'); f = sign(x); subplot(1,2,1) plot(f,'k')
hold on f9 = chebfun(f,10); plot(f9,'.-') subplot(1,2,2)
plot(f,'k'), hold on, f19 = chebfun(f,20); plot(f19,'.-')
```