Computing with Chebyshev ~~polynom~~ series

Evaluation:

$$\underbrace{\begin{bmatrix} 1 & & & & \\ -x & 1 & & & \\ 1 & -2x & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2x & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} T_0(x) \\ \vdots \\ T_n(x) \end{bmatrix}}_{\mathbf{T}_n(x)} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{e_1}$$

Then $f(x_*) \approx p_n(x_*) = \sum_{k=0}^{n}{}' c_k T_k(x_*) = \mathbf{T}_n(x_*)^T \underline{c} = e_1^T(L^{-T}\underline{c})$

solve this.

The cost of $p_n(x_*)$ is $O(n)$ operations.

Integration:

$$\int_{-1}^{1} f(x)\,dx \approx \int_{-1}^{1} p_n(x)\,dx = \sum_{k=0}^{n}{}' c_k \int_{-1}^{1} T_k(x)\,dx = 2c_0 + \sum_{k=0}^{n}{}' c_{2k}\frac{2}{1-(2k)^2}$$

$$\int_{-1}^{1} T_k(x)\,dx = \begin{cases} 2, & \text{if } k=0, \\ \frac{2}{1-k^2}, & \text{if } k=\text{even}, k>0, \\ 0, & \text{if } k=\text{odd}, \end{cases}$$

Integration costs $O(n)$ operations.

Differentiation:

$$f'(x) \approx p_n'(x) = \sum_{k=0}^{n}{}' c_k T_k'(x) = \sum_{k=0}^{n-1}{}' d_k T_k(x)$$

find the d_k's by writing $T_k'(x)$ as a Chebyshev series.

Calculating d_k's from c_k's costs $O(n)$ operations.

<u>Recap</u>: let $f:[-1,1] \to \mathbb{C}$ be Lipschitz cont.

$$f(x) = \sum_{k=0}^{\infty}{}' a_k T_k(x) .$$ CHEBYSHEV SERIES

INTERPOLANT : $$p_n(x) = \sum_{k=0}^{n}{}' c_k T_k(x)$$ c_k's fast to compute, direct to analyze directly.

PROJECTION : $$f_n(x) = \sum_{k=0}^{n}{}' a_k T_k(x)$$ a_k's slow to compute, easier to analyze.

We related c_k's to a_k's by aliasing.

Moreover, for any cont. func.

$$\max_{x\in[-1,1]} |f(x)-p_n(x)| \le \left(2+\frac{2}{\pi}\log(n+1)\right) \max_{x\in[-1,1]} |f(x)-p_n^{\text{best}}(x)|$$

Best poly. you could select.

CONCLUSION:
In practice and in theory, $p_n(x)$ rules!

Weierstrass approximation theorem

1885.

<u>Thm 6.1</u>: Given $f:[-1,1] \to \mathbb{C}$ continuous and $\varepsilon > 0$, there exists a polynomial p such that $\|f-p\|_\infty < \varepsilon$.

Here, $$\|f\|_\infty = \max_{x\in[-1,1]} |f(x)|.$$

(Aside: There is a famous theorem of Faber that says <u>any</u> interpolation scheme can fail to converge.)

Weierstrass' proof (1885) : "smooth f and the approximate"

Three steps:

1. Extend f to $F: \mathbb{R} \to \mathbb{R}$ continuous with compact support.
2. Convolve F with a sufficiently narrow Gaussian to get an entire G with $\|F - G\|_\infty < \varepsilon/2$.
   (i.e. F is initial data for $\partial F/\partial t = \dfrac{\partial^2 F}{\partial x^2}$.
3. Define p = Taylor expansion of G with $\|p - G\|_\infty < \varepsilon/2$.

Fejer's proof (1900)

"Cesaro means" : $f(x) \overset{!}{=} \sum_{k=0}^{\infty} a_k T_k(x)$ doesn't necessarily make sense because f only continuous.

But, $S_n(x) \to f(x)$ for each $x \in [-1,1]$ if

$$S_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} f_k(x), \quad f_k(x) = \sum_{j=0}^{k} a_k T_k(x).$$

"Cesaro means are very popular (an analysis trick) to remove annoying pointwise convergence issues with approximants".

For about 80 years, researchers focused on schemes other than interpolation because interpolants ~~were~~ did not convergence to EVERY continuous function.

Convergence of Chebyshev series

The general message: [Jackson-type theorem]

"The smoother your function, the faster Chebyshev series converges."

- f has k derivatives (+ a little bit) $\Rightarrow$ $|a_n| = O(n^{-k})$
- f is analytic $\Rightarrow$ $|a_n| = O(\rho^{-n})$

Variation of functions:

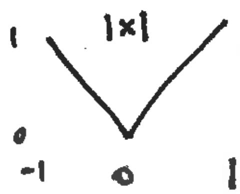
$$V(f) = \sup_{\substack{-1 \le x_1 \le \cdots x_m \le 1 \\ \text{for any } m}} \sum_{i=1}^{m} |f(x_{i+1}) - f(x_i)| = \int_{-1}^{1} |f'(x)|\, dx$$

(Distributional sense.)

Ex. $f(x) = \text{sgn}(x)$

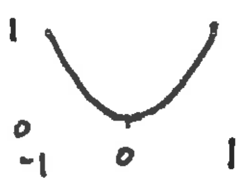
$V(f) = 2$

$f(x) = |x|$



$V(f) = 2$

$f(x) = |x|^3$



$V(f') = 2$

$f(x) = \dfrac{\sin(100x)}{1+x^2}$ $V(f) \approx 100.$

Assumption for Thms 7.1 & 7.2.

There is an integer $\nu \geq 0$ such that $f, f', \dots, f^{(\nu-1)}$ are continuous

and $f^{(\nu)}$ has variation $V < \infty$.

Thm 7.1:

$$|a_k| \leq \frac{2V}{\pi(k-\nu)^{\nu+1}}, \quad k \geq \nu+1.$$

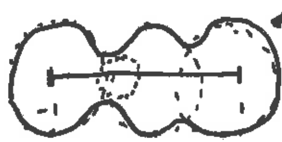
Thm 7.2: If $\nu \geq 1$, then

$$\|f - f_n\|_\infty \leq \frac{2V}{\pi\nu(n-\nu)^{\nu}}, \quad \|f - p_n\|_\infty \leq \frac{4V}{\pi\nu(n-\nu)^{\nu}}, \quad n > \nu$$

"A νth derivative of bounded variation $\Rightarrow$ $\|f - p_n\|_\infty = O(n^{-\nu})$".

Convergence for analytic functions

We say that $f: [-1,1] \to \mathbb{R}$ is analytic if f has a Taylor series about s that converges to f in a neighborhood of s.



region of analyticity.