Hankel operators

Let ~~k~~ $\{\alpha_j\}_{j=0}^{\infty}$ be a sequence of complex numbers. We define

$$\Gamma_\alpha = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \alpha_1 & \alpha_2 & \iddots & & \\ \alpha_2 & \iddots & & & \\ \vdots & & & & \end{bmatrix}$$

to be a Hankel operator.

Associated to Γ_α is the <u>Hankel function</u>:

$$k(z) = \frac{1}{z}\sum_{j=0}^{\infty} \alpha_j z^{-j}.$$

<u>Three examples</u>:

① $\Gamma_\alpha = \begin{bmatrix} 1 & \frac{1}{2} & 1/4 & \cdots \\ 1/2 & 1/4 & \iddots & \\ 1/4 & \iddots & & \\ \vdots & & & \end{bmatrix}$, $\alpha_j = 2^{-j}$, $k(z) = \frac{1}{z}\sum_{j=0}^{\infty} 2^{-j} z^{-j} = \frac{1}{z}\cdot\frac{1}{1-\frac{1}{2z}}$

$$= \frac{1}{z - 1/2}.$$

② $\Gamma_\alpha = \begin{bmatrix} 1 & 1 & \frac{1}{2} & 1/4 & \cdots \\ 1 & 1/2 & 1/4 & & \\ 1/2 & 1/4 & \vdots & \ddots & \\ 1/4 & \vdots & & & \\ \vdots & & & & \end{bmatrix}$, $\alpha_0 = 1$, $\alpha_j = 2^{-j+1}$, $j \geq 1$. $k(z) = \frac{1}{z} + \frac{1}{z^2}\sum_{j=0}^{\infty} 2^{-j} z^{-j}$

$$= \frac{1}{z} + \frac{1}{z(z - 1/2)}$$

$$= \frac{z + 1/2}{z(z - 1/2)}$$

③ $\Gamma_\alpha = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/3 & 1/4 & & \\ 1/3 & 1/4 & \vdots & \ddots & \\ 1/4 & \vdots & & & \\ \vdots & & & & \end{bmatrix}$ $\alpha_j = \frac{1}{j+1}$, $j \geq 1$, $k(z) = \sum_{j=0}^{\infty} \frac{1}{j+1} z^{-j}$.

(Baby) AAK theory

Let me start with 3 basic facts.

Fact 1: Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. ~~~~ Then, the non-zero eigenvalues of AB and BA are the same.

Proof:

$$\begin{bmatrix} I_m & \lambda A \\ 0 & I_n \end{bmatrix}^{-1} \begin{bmatrix} I_m - \lambda AB & 0 \\ -B & I_n \end{bmatrix} \begin{bmatrix} I_m & \lambda A \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ -B & I_n - \lambda BA \end{bmatrix}$$

This is a similarity transformation.

Fact 2: Let $A \in \mathbb{C}^{n\times n}$. Then,

$$(zI - A)^{-1} = \sum_{j=1}^{\infty} A^{j-1} z^{-j}$$

Proof:

$$(zI - A)\sum_{j=1}^{\infty} A^{j-1} z^{-j} = \sum_{j=1}^{\infty} A^{j-1} z^{-j+1} - \sum_{j=1}^{\infty} A^{j} z^{-j}$$

$$= I.$$

Fact 3: The solution to

$$X - AXA^* = BB^*$$

$A \in \mathbb{C}^{n\times n}$

$B \in \mathbb{C}^{n\times k}$

X = unknown.

is

$$X = \sum_{k=0}^{\infty} A^k BB^* (A^*)^k.$$

Proof: Substitute in expression and check.

<u>Assumption</u>! The Hankel function is a rational function with poles in the unit disk.

(Includes Examples 1 and 2, but not example 3.)

If $k(z)$ = rational function, then we can write it in partial fraction form. That is, if all poles of $k(z)$ are distinct,

$$k(z) = \sum_{i=1}^{n} \frac{b_i}{z - a_i} \quad \text{for some } n. \qquad (b_i \neq 0)$$

Here, a_i's are the poles of k. We can write this as

$$k(z) = C(zI - A)^{-1}B, \qquad A = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad C = [1 \cdots 1].$$

<u>Kronecker's theorem</u>: The rank of Γ_α is ~~the~~ equal to the number of poles of $k(z)$.

<u>Proof</u>: Let n be the number of poles of $k(z)$.

$$k(z) = C(zI-A)^{-1}B \overset{\text{Fact 1}}{=} \sum_{j=1}^{\infty} CA^{j-1}Bz^{-j} \quad \text{so} \quad \alpha_j = CA^{j-1}B.$$

Therefore,
$$(\Gamma_\alpha)_{j,k} = \alpha_{j+k} = CA^{j+k}B = (CA^j)(A^kB).$$

Hence,
$$\Gamma_\alpha = \underset{\infty \times n}{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}} \underset{n \times \infty}{\begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}}$$

We conclude that $\mathrm{rank}(\Gamma_\alpha) \leq n$.

Can you show that $n \leq \mathrm{rank}(\Gamma_\alpha)$?

AAK theory (special case)

Theorem:

$$\sigma_k(\Gamma_\alpha)^2 = \lambda_k(PQ), \qquad 1 \le k \le n.$$

singular values of Γ_α — $\sigma_k(\Gamma_\alpha)$

Eigenvalues of PQ — $\lambda_k(PQ)$

where

$$P = \overset{n\times n}{\Xi\,\Xi^*}, \qquad \Xi = \overset{n\times\infty}{\begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}}$$

$$Q = \overset{n\times n}{\Omega^*\Omega}, \qquad \Omega = \overset{\infty\times n}{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}}$$

Proof:

$$\sigma_k(\Gamma_\alpha)^2 = \sigma_k(\overset{\infty\times\infty}{\Omega\,\Xi})^2 = \lambda_k(\overset{\infty\times\infty}{\Xi^*\Omega^*\Omega\,\Xi})$$

$$= \lambda_k(\overset{n\times n}{\Xi\,\Xi^*\Omega^*\Omega})$$

$$= \lambda_k(PQ).$$

Note that $P = \Xi\,\Xi^* = BB^* + ABB^*A^* + A^2BB^*(A^2)^* + \cdots$

$$= \sum_{k=0}^{\infty} A^k BB^* (A^k)^*$$

By Fact 3, P is the solution to

$$\overset{n\times n}{X} - \overset{n\times n}{A}\,\overset{n\times n}{X}\,\overset{n\times n}{A^*} = \overset{n\times n}{BB^*}.$$

Similarly,

$$Q = \Omega^* \Omega = C^*C + (A^*)^* C^* C A + (A^2)^* C^* C A^2 + \cdots$$
$$= \sum_{k=0}^{\infty} (A^k)^* C^* C A^k.$$

By Fact 3, Q is the solution to

$$X - A^* X A = C^* C$$

Algorithm for calculating singular values of Γ_α

- Suppose associated Hankel function is $k(z) = C(zI - A)^{-1}B$.
- Solve $X - AXA^* = BB^*$ for P
- Solve $X - A^*XA = C^*C$ for Q
- Compute the eigenvalues of PQ.
- Set $\sigma_k(\Gamma_\alpha)^2 = \lambda_k(PQ)$ for $1 \le k \le n$

  $\sigma_k(\Gamma_\alpha) = 0$ for $k > n$.

Baby AAK theory II

Recap: Let $\{\alpha\}_{j=0}^{\infty}$ be a complex sequence.

$$\Gamma_\alpha = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots \\ \alpha_1 & \alpha_2 & \cdots & \\ \alpha_2 & \cdots & & \\ \vdots & & & \end{bmatrix} \quad \text{is called a Hankel operator.}$$

$$k(z) = \sum_{j=0}^{\infty} \alpha_j z^{-j-1} \quad \text{is called a } \underline{\text{symbol}} \text{ of } \Gamma_\alpha.$$

Kronecker's theorem: $\operatorname{rank}(\Gamma_\alpha) = \#$ poles $k(z)$ has ~~on~~ inside the unit disk.

Last time: If $k(z) = \cancel{B8}\, C(zI-A)^{-1}B$, then

$$\sigma_k(\Gamma_\alpha) = \lambda_k(PQ)$$

where P satisfies $X - AXA^* = BB^*$ and Q satisfies $X - A^*XA = C^*C$.

Today: Connecting the singular values of Γ_α to a rational approximation problem.

~~Again assume: $k(z) = C(zI-A)^{-1}B$ where all the poles of $k(z)$ are inside the unit disk, $C \in \mathbb{C}^{1\times n}$, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times 1}$.~~

This time: No assumption on $k(z)$: Only that it rational and $\|k\|_\infty = \sup_{\theta\in[0,2\pi]} |k(e^{i\theta})| < \infty$.

Nehari's theorem :

$$\|\Gamma_\alpha\|_2 = \inf\{\, \|k\|_\infty : \hat{k}(-m) = \alpha_{m-1},\ m \ge 1 \}.$$

Proof: $\|\Gamma_\alpha\|_2 \le \|k\|_\infty$ if $\hat{k}(-m) = \alpha_{m-1},\ m \ge 1$.

$$\tilde{a}^T \Gamma_\alpha \tilde{a} = \sum_{j,k=0}^{\infty} \rho^j a_j \alpha_{j+k} a_k \rho^k = \sum_{m=0}^{\infty} \alpha_m \rho^m \sum_{j=0}^{m} a_j a_{m-j} \qquad 0<\rho<1$$

Claim: $\tilde{a}^T \Gamma_\alpha \tilde{a} = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta} k(e^{i\theta}) \left(\tilde{f}(e^{i\theta})\right)^2 d\theta$

where $\tilde{f}(e^{i\theta}) = \sum_{\ell=0}^{\infty} a_\ell e^{-i\ell\theta} \rho^\ell$

$$\cancel{e^{i\theta}k(e^{i\theta})|f(e^{i\theta})|^2 = \sum_{m=-\infty}^{\infty} \alpha_m e^{-im\theta} \sum_{j=0} a_j}$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta} k(e^{i\theta}) \left(f(e^{i\theta})\right)^2 d\theta = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} a_\nu a_\mu \rho^{\nu+\mu} \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta} k(e^{i\theta}) e^{-i(\nu+\mu)\theta} d\theta$$

$$= \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} a_\nu a_\mu \alpha_{\nu+\mu} \rho^{\nu+\mu}$$

Hence,

$$|\tilde{a}^T \Gamma_\alpha \tilde{a}| \le \left| \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta} k(e^{i\theta}) \left(\tilde{f}(e^{i\theta})\right)^2 d\theta \right|$$

$$\le \frac{\|k\|_\infty}{2\pi}\int_0^{2\pi} |\tilde{f}(e^{i\theta})|^2 d\theta$$

$$\le \|k\|_\infty \sum_{\nu=0}^{\infty} |a_\nu|^2 \rho^{2\nu}$$

$$\le \|k\|_\infty \sum_{\nu=0}^{\infty} |a_\nu|^2 \qquad \text{(Now let } \rho \to 1.)$$

Hence,

$$|\underline{a}^* \Gamma_\alpha \underline{b}| \le \frac{1}{2\pi} \sup_{\theta \in [0,2\pi]} |e^{i\theta} K(e^{i\theta})| \cdot \left| \int_0^{2\pi} \overline{f(e^{i\theta})}\, g(e^{i\theta})\, d\theta \right|$$

$$\le \frac{1}{2\pi} \|K\|_\infty \sqrt{\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta} \sqrt{\int_0^{2\pi} |g(e^{i\theta})|^2 d\theta}$$

$$= \frac{1}{2\pi} \|K\|_\infty \sqrt{2\pi}\, \|\underline{a}\|_2 \cdot \sqrt{2\pi}\, \|b\|_2$$

$$= \|K\|_\infty \|a\|_2 \|b\|_2 .$$

We find that $\dfrac{|\underline{a}^* \Gamma_\alpha \underline{b}|}{\|\underline{a}\|_2 \|\underline{b}\|_2} \le \|K\|_\infty$, i.e., $\|\Gamma_\alpha\|_2 \le \|K\|_\infty$.

Stronger version:

$$\|\Gamma_\alpha\|_2 = \inf \left\{ \|K\|_\infty : K(z) = \sum_{m=-\infty}^{\infty} \alpha_m z^{-m-1} \right\}$$

Note: $\alpha_{-1}, \alpha_{-2}, \ldots$ are free choices.

AAK theory:

$$\sigma_{k+1}(\Gamma_\alpha) \le \|K - r\|_\infty \quad \text{for any } r \in R_k(\mathbb{D})$$

where $R_k(\mathbb{D}) = \{$ set of rational func. with $\le k$ poles in $\mathbb{D} \}$.

Let $f = K - r$ for $r \in R_k(\mathbb{D})$,

$r = r_1 + r_2$ $\quad r_1$ analytic in $\mathbb{D}$, r_2 all poles are in $\mathbb{D}$. $r_2(\infty) = 0$.

$$r_2 = \tilde{C}(zI - \tilde{A})^{-1} \tilde{B}, \quad \tilde{A} \in \mathbb{C}^{k \times k}.$$

Let $\beta_j = \vec{C}\tilde{A}^j\tilde{B}$. Then, Γ_β has a symbol r_2.

$$\|\Gamma_\alpha - \underset{\substack{\uparrow \\ \text{rank} \leq k.}}{\Gamma_\beta}\|_2 \leq \|f\|_\infty \quad \text{as} \quad \hat{f}(-m) = \alpha_{m-1} - \beta_{m-1}.$$

Therefore, $\sigma_{k+1}(\Gamma_\alpha) \leq \|f\|_\infty$.

<u>Stronger version</u>:

$$\sigma_{k+1}(\Gamma_\alpha) = \inf\{ \|K - r\|_\infty : r \in R_k(\mathbb{D}) \}$$

<u>Warning</u>: Every rank n Hankel operator <u>cannot</u> be expressed as a sum of rank-1 Hankel operators.

Every rank-1 Hankel is

$$\Gamma_\alpha = \alpha_0 \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \alpha_1^3 & \cdots \\ \alpha_1 & \alpha_1^2 & \alpha_1^3 & & \\ \alpha_1^2 & \alpha_1^3 & & \ddots & \\ \alpha_1^3 & \vdots & & & \\ \vdots & & & & \end{bmatrix}$$

$$K(z) = \alpha_0 \sum_{m=0}^{\infty} \alpha_1^m z^{-m-1}$$
$$= \frac{\alpha_0}{z} \sum_{m=0}^{\infty} \left(\frac{\alpha_1}{z}\right)^m = \frac{\alpha_0}{z} \cdot \frac{1}{1 - \alpha_1/z}$$
$$= \frac{\alpha_0}{z - \alpha_1}$$

Only possible if $\text{rank}(\Gamma_\alpha)$ is finite and a symbol has distinct poles.