Theorem 19.6. Barycentric weights for Legendre points. *The barycentric weights of Theorem 5.1 for polynomial interpolation in Legendre points can be written as*

$$\lambda_k = C(-1)^k \sqrt{(1-x_k^2)w_k}, \tag{19.13}$$

where x_k and w_k are the nodes and weights for $(n+1)$-point Gauss quadrature.

Proof. Both the barycentric weights $\{\lambda_k\}$ and the quadrature weights $\{w_k\}$ are related to the node polynomial ℓ. In particular, it is shown in [Winston 1934] that the weights can be written as $w_k = C(1-x_k^2)^{-1}[\ell'(x_k)]^{-2}$. ∎

This theorem implies that polynomial interpolants in Legendre points, just as with Chebyshev points, can be evaluated in $O(n)$ operations. The formulas are implemented in Chebfun and accessed when one calls `legpts`, `jacpts`, `hermpts`, or `lagpts` with three output arguments [Hale & Trefethen 2012].

SUMMARY OF CHAPTER 19. *Clenshaw–Curtis quadrature is derived by integrating a polynomial interpolant in Chebyshev points, and Gauss quadrature from Legendre points. The nodes and weights for both families can be computed quickly and accurately, even for millions of points. Though Gauss has twice the polynomial order of accuracy of Clenshaw–Curtis, their rates of convergence are approximately the same for nonanalytic integrands.*

Exercise 19.1. Riesz Representation Theorem. (a) Look up the Riesz Representation Theorem and write down a careful mathematical statement of it. (b) Show that the computation of an approximate integral I_n from $n+1$ samples of a function $f \in C([-1,1])$ by integrating the degree n polynomial interpolant through a fixed set of $n+1$ nodes in $[-1,1]$ is an example of the kind of linear functional to which this theorem applies, provided we work in a finite-dimensional space rather than all of $C([-1,1])$. (c) In what sense is the Riesz Representation Theorem significantly more general than is needed for this particular application to quadrature?

Exercise 19.2. `integral`. Evaluate (19.11) with the MATLAB `integral` command. As a function of the specified precision, what is the actual accuracy obtained and how long does the computation take? How do these results compare with Chebfun `sum`?

Exercise 19.3. Quadrature weights. (a) Use Chebfun to illustrate the identity (19.4) for Clenshaw–Curtis quadrature in the case $n = 20$, $k = 7$. (b) Do the same for Gauss quadrature.

Exercise 19.4. Accuracy of Clenshaw–Curtis quadrature. Using theorems of Chapters 4 and 19, derive an exact expression for the error $I - I_n$ in Clenshaw–Curtis quadrature applied to the function $f(x) = T_k(x)$ for $k > n$ [Gentleman 1972A].

Exercise 19.5. Sharpening Theorem 19.3. Suppose we assume $n \geq 2$ instead o[f] $n \geq 1$ in the Gauss quadrature bound of Theorem 19.3. Show why the constant 64/1[…] improves to 144/35. What is this actual "constant" as a function of n?

Exercise 19.6. Integral of a Chebyshev polynomial. Derive the formula (19.6) fo[r] the integral of $T_k(x)$ with k even. (Hint: Following the proof of Theorem 3.1, repla[ce] $T_k(x)dx$ by $(z^k + z^{-k})(dx/dz)dz$.)

Exercise 19.7. Symmetrization in the Golub–Welsch algorithm. The nodes $\{x$[…] of the $(n+1)$-point Gauss quadrature rule are the zeros of the Legendre polynom[ial] P_{n+1}. From the recurrence relation (17.6), it follows as in Theorem 18.1 that they are t[he] eigenvalues of a certain $(n+1) \times (n+1)$ tridiagonal matrix A. (a) Give formulas for t[he] entries of A. (b) Find the unique diagonal matrix $D = \mathrm{diag}(d_0, \dots, d_n)$ with $d_0 = 1$ and $d_j > 0$ for $j \geq 1$ such that $B = DAD^{-1}$, which has the same eigenvalues as A, is real symmetric. What are the entries of B? (This symmetrized matrix is the Jacobi matrix that is the basis of the Golub–Welsch algorithm.)

Exercise 19.8. Integrating the Bernstein polynomial. Given $f \in C([-1,1])$, let $B_n(x)$ be the Bernstein polynomial defined by (6.2) and let I_n be the approximation to $\int_{-1}^{1} f(x)dx$ defined by $I_n = \int_{-1}^{1} B_n(x)dx$. (a) Show that $I_n = (n+1)^{-1}\sum_{k=0}^{n} f(k/n)$. (b) Is this an interpolatory quadrature formula? (c) What is its order of accuracy α for smooth integrands as defined by the condition $I - I_n = O(n^{-\alpha})$?

Exercise 19.9. Nonnegative weights and convergence. Suppose the weights w_k of an interpolatory quadrature formula (19.3) are all nonnegative. Show using the Weierstrass approximation theorem that $I_n \to I$ for any continuous integrand f. (Pólya [1933] showed that uniform boundedness of the sums of the absolute values of the weights is necessary and sufficient for convergence for all f.)

Exercise 10.1. A function with spikes. Compute numerically the degree 10 polynomial best approximation to $\operatorname{sech}^2(5(x+0.6))+\operatorname{sech}^4(50(x+0.2))+\operatorname{sech}^6(500(x-0.2))$ on $[-1,1]$ and plot f together with p^* as well as the error curve. What is the error? How does this compare with the error in 11-point Chebyshev interpolation?

Exercise 10.2. Best approximation of $|x|$. (a) Use Chebfun to determine the errors $E_n = \|f-p_n\|$ in the degree n best approximation of $f(x)=|x|$ on $[-1,1]$ for $n = 2,4,8,\dots,256$, and make a table of the values $\beta_n = nE_n$ as a function of n. (b) Use Richardson extrapolation to improve your data. How many digits can you estimate for the limiting number $\beta = \lim_{n\to\infty}\beta_n$? (We shall discuss this problem in detail in Chapter 25.)

Exercise 10.3. de la Vallée Poussin lower bound. Suppose an approximation $p\in\mathcal{P}_n$ to $f\in C([-1,1])$ approximately equioscillates in the sense that there are points $-1\le s_0 < s_1 < \cdots < s_{n+1}\le 1$ at which $f-p$ alternates in sign with $|f(s_j)-p(s_j)|\ge\varepsilon$ for some $\varepsilon>0$. Show that $\|f-p^*\|\ge\varepsilon$. (This estimate originates in [de la Vallée Poussin 1910].)

Exercise 10.4. Best approximation of even functions. Let $f\in C([-1,1])$ be an even function, i.e., $f(-x)=f(x)$ for all x. (a) Prove as a corollary of Theorem 10.1 that for any $n\ge 0$, the best approximation p_n^* is even. (b) Prove that for any $n\ge 0$, $p_{2n}^* = p_{2n+1}^*$. (c) Conversely, suppose $f\in C([-1,1])$ is not even. Prove that for all sufficiently large n, its best approximations p_n^* are not even.

Exercise 10.5. An invalid theorem. The first two figures of this chapter suggest the following "theorem": *if f is an even function on $[-1,1]$ and p^* is its best approximation of some degree n, then one of the extreme points of $|(f-p^*)(x)|$ occurs at $x=0$.* Pinpoint the flaw in the following "proof." By the argument of Exercise 10.4(b), p^* is the best approximation to f for all n in some range of the form even $\le n\le$ odd, such as $4\le n\le 5$ or $10\le n\le 13$. By Theorem 10.1, the maximal number of equioscillation points of $f-p^*$ must accordingly be of the form odd $+\,2$, that is, odd. By symmetry, 0 must be one of these points. Exactly which of these three logical steps is invalid, and why?

Exercise 10.6. Nonlinearity of best approximation operator. We have mentioned that for given n, the operator that maps a function $f\in C([-1,1])$ to its best degree n approximation p_n^* is nonlinear. Prove this (on paper, not numerically) by finding two functions f_1 and f_2 and an integer $n\ge 0$ such that the best approximation of the sum in $\mathcal{P}_n$ is not the sum of the best approximations.

Exercise 10.7. Bernstein's lethargy theorem. Exercise 6.1 considered a function of Weierstrass, continuous but nowhere differentiable. A variant of the same function based on Chebyshev polynomials would be

$$f(x) = \sum_{k=0}^{\infty} 2^{-k} T_{3^k}(x). \tag{10.1}$$

(a) Show that the polynomial f_{3^k} obtained by truncating (10.1) to degree 3^k is the best approximation to f in the spaces $\mathcal{P}_n$ for certain n. What is the complete set of n for which this is true? What is the error? (b) Let $\{\varepsilon_n\}$ be a sequence decreasing monotonically to 0. Prove that there is a function $f\in C([-1,1])$ such that $\|f-p_n^*\|\ge\varepsilon_n$ for all n. (Hint: Change the coefficients 2^{-k} of (10.1) to values related to $\{\varepsilon_n\}$.)

Exercise 10.8. Continuity of best approximation operator. For any $n\ge 0$, the mapping from functions $f\in C([-1,1])$ to their best approximants $p^*\in\mathcal{P}_n$ is continuous with respect to the ∞-norm in $C([-1,1])$. Prove this by an argument combining the uniqueness of best approximations with compactness. (This continuity result appears in Section I.5 of [Kirchberger 1902]. In fact, the mapping is not just continuous but Lipschitz continuous, a property known as *strong uniqueness*, but this is harder to prove.)

Exercise 10.9. Approximation of e^x. Truncating the Taylor series for e^x gives polynomial approximations with maximum error $E_n \sim 1/(n+1)!$ on $[-1,1]$, but the best approximations do better by a factor of 2^n:

$$E_n \sim \frac{1}{2^n(n+1)!}, \quad n \to \infty. \tag{10.2}$$

(a) Derive (10.2) by combining Exercises 3.15 and 10.3 with the asymptotic formula $I_k(1) \sim 1/(2^n n!)$. (b) Make a table comparing this estimate with the actual values E_n computed numerically for $0 \le n \le 10$.

Exercise 10.10. Alternative proof of uniqueness. Prove uniqueness of the degree n best approximant to a real continuous function f by a simpler argument than the one given in the proof of Theorem 10.1: suppose p and q are best approximants, and apply the equioscillation characterization to $r = (p+q)/2$.

Exercise 10.11. Chebyshev polynomials and best approximations. (a) What is the best degree n polynomial approximation to x^{n+1} on $[-1,1]$? What is the error? Derive the answers from Theorem 10.1, using the fact that T_{n+1} oscillates between values ± 1 at $n+2$ points in $[-1,1]$. (b) What is the best approximation to 0 among monic polynomials of degree $n+1$? What is the error?

Exercise 10.12. Every best approximant is an interpolant. Let p be the best approximation in $\mathcal{P}_n$ to a real function $f \in C([-1,1])$. Show that there exist $n+1$ distinct points $-1 < x_0 < x_1 < \cdots < x_n < 1$ such that p is the interpolant in $\mathcal{P}_n$ to f in the points $\{x_j\}$.

Exercise 10.13. A contrast to Faber's theorem. Although Faber showed that there does not exist an array of nodes in $[-1,1]$ whose polynomial interpolants converge for every $f \in C[-1,1]$, for any *fixed* f there exists an array whose interpolants converge to f [Marcinkiewicz 1936–37]. Prove this by combining the Weierstrass approximation theorem with the result of the previous exercise.

Exercise 10.14. Asymptotics of the leading coefficient. Let $\{p_n^*\}$ be the sequence of best approximations of a function $f \in C([-1,1])$, and let p_n^* have leading Chebyshev coefficient a_n^*. It is known that $\limsup_{n\to\infty} |a_n^*|^{1/n} \le 1$, with strict inequality if and only if f is analytic on $[-1,1]$ [Blatt & Saff 1986, Thm. 2.1]. Verify this result numerically by estimating $\limsup_{n\to\infty} |a_n^*|^{1/n}$ for $f(x) = |x|$ and $f(x) = 1/(1+25x^2)$.