Recap:

- Cheb. pts: $x_j = \cos(j\pi/n)$, $0 \le j \le n$.
- Cheb. interpolants: polynomial p so that $p(x_j) = f_j$.
- Cheb. polys. $T_k(x) = \cos(k \cos^{-1} x) = \mathrm{Re}\left(e^{ik\theta}\right)$, $x = \cos(\theta)$.
- Cheb. series. $$f(x) = \sum_{k=0}^{\infty} a_k T_k(x), \quad a_k = \begin{cases} \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\,dx, & k \ge 1, \\ \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)T_0(x)}{\sqrt{1-x^2}}\,dx, & k = 0, \end{cases}$$

  If $f : [-1,1] \to \mathbb{R}$ is Lipschitz continuous, then series converges absolutely and uniformly.

Two ways to form a polynomial approximation

INTERPOLATION, PROJECTION, AND ALIASING

Given Lip. cont. f on $[-1,1]$, $n \ge 0$: $f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$.

Chebyshev interpolant

$$p_n(x) = \sum_{k=0}^{n} c_k T_k(x)$$

where c_k's selected so that $p_n(x_j) = f(x_j)$, $0 \le j \le n$

Chebyshev projection

$$f_n(x) = \sum_{k=0}^{n} a_k T_k(x)$$

where

$$a_k = \begin{cases} \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\,dx, & k \ge 1 \\ \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)T_0(x)}{\sqrt{1-x^2}}\,dx, & k = 0 \end{cases}$$

Interpolant: Easy to compute, in the sense of $O(n\log n)$ operations via FFT. But, difficult to immediately ~~correct~~ understand error: $\max_{x\in[-1,1]} |f(x) - p_n(x)|$??

Projection: Hard to compute (involves integrals). But, simpler to understand the decay of a_k's.

Aliasing:

Theorem 4.1: Given $n \geq 1$ and $x_j = \cos(j\pi/n)$ $0 \leq j \leq n$. For any $0 \leq m \leq n$, the following take the same values on the grid:

$$T_m, \quad T_{2n-m}, \quad T_{2n+m}, \quad T_{4n-m}, \quad T_{4n+m}, \ldots$$

Proof:

$$T_{2n+m}(x_j) = \mathrm{Re}\left(e^{i(2n+m)\frac{j\pi}{n}}\right) = \mathrm{Re}\left(e^{i2\pi j} e^{im\frac{j\pi}{n}}\right)$$
$$= \mathrm{Re}\left(e^{im\frac{j\pi}{n}}\right) = T_m(x_j).$$

Real-life versions: Wagon wheels in movies
strobe lights

Theorem 4.2: If f is Lip. cont., then

$$c_0 = a_0 + a_{2n} + a_{4n} + a_{6n} + \cdots; \quad c_n = a_n + a_{3n} + a_{5n} + \cdots$$

and

$$c_k = a_k + (a_{k+2n} + a_{k+4n} + \cdots) + (a_{-k+2n} + a_{-k+4n} + \cdots)$$

Proof:

By absolute convergence, $c_0, c_n, c_k \cdots$ is well-defined by the above series.

Hence,
$$q(x) = \sum_{k=0}^{n} c_k T_k(x) \quad \in \mathcal{P}_n .$$

To verify $q(x)$ interpolates:

$$f(x_j) = \sum_{k=0}^{\infty} a_k T_k(x_j) ,$$
$$= (a_0 + a_{2n} + a_{4n} + \cdots) T_0(x_j)$$
$$+ \left[a_1 + (a_{2n+1} + a_{4n+1} + \cdots) + (a_{2n-1} + a_{4n-1} + \cdots)\right] T_1(x_j)$$
$$\vdots$$
$$+ \left[a_n + a_{3n} + a_{5n} + \cdots\right] T_n(x_j) .$$

<u>Corollary</u>:

$$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} a_k T_k(x) , \quad f(x) - p_n(x) = \sum_{k=n+1}^{\infty} a_k \left(T_k(x) - T_m(x)\right)$$
$$m = \left| (k+n-1) \bmod 2n - (n-1) \right|$$

This corollary explains why interpolants ~~are generally~~ can be $\times 2$ worse than projections.

Suppose

$$f(x) = \sum_{k=0}^{\infty} 2^{-k} T_k(x), \qquad f_n(x) = \sum_{k=0}^{n} 2^{-k} T_k(x).$$

Then, $$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} 2^{-k} T_k(x)$$

~~1st negligec~~ 1st drop term is of size 2^{-n-1}.

so $$\max_{x\in[-1,1]} |f(x) - f_n(x)| \le \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n-1} \sum_{k=0}^{\infty} 2^{-k}$$

$$= 2^{-n-1} \cdot \frac{1}{1-\frac{1}{2}} = 2 \cdot \left(2^{-n-1}\right)$$

The error $|f(x) - f_n(x)|$ is at most a small factor times the first neglected term if $|a_k| \to 0$ geometrically as $k \to \infty$.

In general, for functions with decaying chebyshev coefficients, the 1st neglected chebyshev coefficient, $|a_{n+1}|$, is a rough estimate for the error in $\max_{x\in[-1,1]} |f(x) - p_n(x)|$.