UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 24th May 2014 9.00 am to 12.00 noon

PHYSICS A (1)

Attempt **all** *questions from Section A,* **two** *questions from Section B,* **one** *question from Section C, and* **one** *question from Section D.*

Section A will carry approximately 20% of the total marks.

In Sections B, C, and D, each question carries approximately the same number of marks.

The approximate number of marks allocated to each question or part of a question is indicated in the right margin.

The paper contains **7** *sides including this one and is accompanied by a Mathematical Formulae Handbook giving values of constants and containing mathematical formulae, which you may quote without proof.*

Answers from **each** *Section should be written in separate Booklets.*

Write the letter of the Section and your candidate number, not your name, on the cover of **each** *Booklet. If several Booklets are used for a given Section, join them together using a Treasury Tag.*

A separate master (yellow) cover sheet should also be completed, listing all questions attempted in the paper.

STATIONERY REQUIREMENTS
Booklets and Treasury Tags
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae Handbook (supplied)
Students are permitted to bring an approved calculator

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Attempt **all** *questions from this Section. Answers should be concise and relevant formulae may be assumed without proof. Each Section should be answered in a separate booklet.*

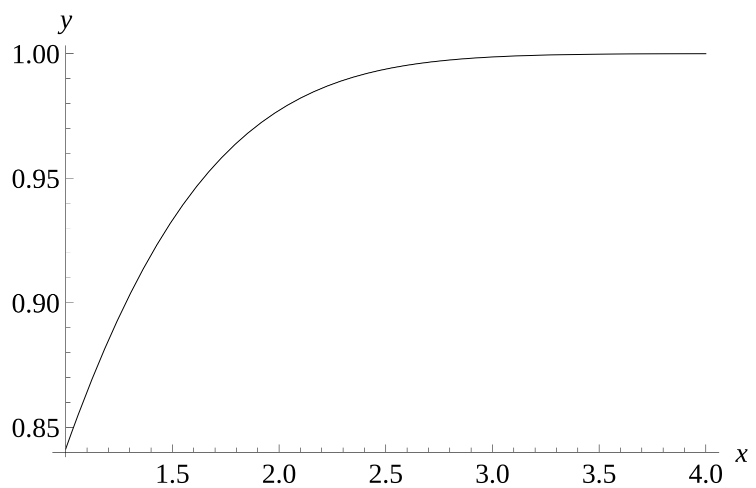
1 Consider a system of nine electrons in a 3D Coulomb potential. Assuming that the electrons do not interact with one another, calculate the ratio of the energy of the first excited state to that of the ground state. [4]

2 Draw an annotated sketch of the wavefunctions of the three lowest energy eigenstates in a quantum-mechanical, 1D, square well of finite depth, assuming that all three states are bound states. [4]

3 Using Heisenberg's uncertainty principle, estimate how many collisions per second an α-particle would have with the walls of a nuclear potential well of size 10^{-14} m. [4]

4 A beam of particles is incident on a detector. The particles are in three energy ranges, labelled Low, Medium, and High; the proportions are 65%, 30%, and 5%, respectively. Due to electronic dead time, there is a probability that a particle in each energy range is not detected, given by 5%, 20%, and 50%, respectively. Using Bayes' theorem, find the probability that a detected particle is in the High energy range. [4]

5 During an experiment, it is found that temperature measurements follow a Gaussian distribution with mean 273 K and r.m.s. 10 K. Using the graph of $y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dt \ \exp(-t^2/2)$ drawn below, estimate the probability that a particular measurement exceeds 293 K. [4]



SECTION B

Attempt **two** *questions from this Section. Each Section should be answered in a separate booklet.*

B6 A quantum-mechanical particle of mass m and energy E is incident on a potential barrier of the form

$$V(x) = \begin{cases} 0, & x < 0 \\ U, & x \geq 0. \end{cases}$$

For $E < U$, show that the particle is totally reflected with phase $\delta = -2\cos^{-1}\sqrt{\frac{E}{U}}$. [7]

Find the wavefunction in the region $x > 0$ and explain how it can be non-zero even though the particle is totally reflected. [3]

Suppose now the potential is modified such that $V(x) = 0$, for $x > a$ (where $a > 0$). Show that, when a is large, the transmission probability depends exponentially on a and find the form of the exponent. [5]

Find the condition for the approximation of large a to be a good one. [2]

Estimate the transmission probability for an electron with energy 50 meV tunnelling through a barrier of height 200 meV and width 10 nm. [3]

B7 The electron in a hydrogen atom satisfies the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi,$$

where the symbols have their usual meanings.

Explain how this equation may be solved by separation of variables with the trial solution

$$\psi_{nlm} = R_{nl}(r)Y_{lm}(\theta, \phi),$$

and explain the meaning of $n, l,$ and m. [4]

Show that there exists a normalized solution of the form $\psi = Are^{-\alpha r}\cos\theta$ and find $A, \alpha, l,$ and the energy eigenvalue. [10]

Explain why ψ is an eigenstate of parity and find its eigenvalue. [4]

Explain how it is possible that the Hamiltonian is spherically symmetric, but that the eigenstate ψ is not. [2]

B8 Write down the commutation relations of the angular momentum operators $\widehat{L}_x$, $\widehat{L}_y$, and $\widehat{L}_z$ and show that $\widehat{L}_z$ commutes with $\widehat{L^2} = \widehat{L}_x^2 + \widehat{L}_y^2 + \widehat{L}_z^2$. [4]

Write down the eigenvalues of $\widehat{L^2}$ and $\widehat{L}_z$. [2]

The Hamiltonian of an asymmetric spinning top is given by

$$\widehat{H} = a_x\widehat{L}_x^2 + a_y\widehat{L}_y^2 + a_z\widehat{L}_z^2,$$

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where a_x, a_y and a_z are constants. Show that $[\widehat{H}, \widehat{L}^2] = 0$ and comment on the implications for the eigenstates of $\widehat{H}$. [3]

Letting $\widehat{L}_\pm = \widehat{L}_x \pm i\widehat{L}_y$, show that $\widehat{H}$ can be written as

$$\widehat{H} = \alpha(\widehat{L}_+^2 + \widehat{L}_-^2) + \beta(\widehat{L}^2 - \widehat{L}_z^2) + \gamma\widehat{L}_z^2,$$

and find expressions for α, β, and γ in terms of a_x, a_y and a_z. [5]

Using the relation

$$\widehat{L}_\pm\,|l,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}\;|l,m\pm 1\rangle,$$

for the eigenstates $|l,m\rangle$ of $\widehat{L}^2$ and $\widehat{L}_z$, show that $|1,0\rangle$ is an eigenstate of $\widehat{H}$ and find the energy eigenvalue. [2]

Find the other two eigenvalues corresponding to $l = 1$ and comment on how all three eigenvalues are related. [4]

B9 Write down the commutation relation for the operators $\widehat{x}$ and $\widehat{p}$ and show that $[f(\widehat{x}), \widehat{p}] = i\hbar f'(\widehat{x})$, where f is any function. [3]

Consider the Hamiltonian $\widehat{H} = \frac{1}{2m}\left(\widehat{p} + if(\widehat{x})\right)\left(\widehat{p} - if(\widehat{x})\right)$. Show that this may be written as

$$\widehat{H} = \frac{\widehat{p}^2}{2m} + V(\widehat{x})$$

and find $V(\widehat{x})$. [2]

By considering $\int dx\, \psi^* \widehat{H}\psi$, show that the energy eigenvalues are non-negative. [3]

The ground state, ψ_0, has vanishing energy. Show that it satisfies the differential equation

$$\hbar\frac{d\psi_0}{dx} + f(x)\psi_0 = 0.$$ [2]

Let $\psi_1 = (\widehat{p} + if(\widehat{x}))\psi_0$. Show $\widehat{H}\psi_1 = \alpha\psi_0 + \beta\psi_1$ and find expressions for α and β in terms of f. [4]

If f is a polynomial of degree n, show that n must be odd for ψ_0 to be normalizable. [3]

By considering $f = kx$, derive the energies and wavefunctions of the ground state and first excited states of the simple harmonic oscillator, with Hamiltonian

$$\widehat{H}_{\mathrm{SHO}} = \frac{\widehat{p}^2}{2m} + \frac{1}{2}m\omega^2\widehat{x}^2.$$ [3]

SECTION C

Attempt **one** *question from this Section. Each Section should be answered in a separate booklet.*

C10 Write an essay on the Planck distribution of black-body radiation, including a discussion of the harmonic oscillator and the probability distribution of vibrational states. You should include, as an example, a sketch of the radiated intensity per unit wavelength *vs.* wavelength for the cosmic microwave background radiation, whose temperature is 2.7 K.

[20]

C11 Write brief notes on **two** of the following:

(a) Heisenberg's uncertainty principle applied to a top-hat wave function; [10]

(b) quantum tunnelling and radioactive alpha decay; [10]

(c) the spin of the electron and why it has no classical counterpart. [10]

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SECTION D

Attempt **one** *question from this Section. Each Section should be answered in a separate booklet.*

D12 The Table below shows a series of five radial velocity measurements, V, with errors σ_V, carried out on a star known to host a transiting planet. The orbit of the planet is circular, with a period $P = 1.6$ days, and the time of the transit (in Julian date) is $T = 2452726.10$ days $\pm nP$, where n is an integer.

Time, t (Julian date)	Radial velocity, V (m s^{-1})	Error, σ_V (m s^{-1})
2452999.20	-15290	±10
2453000.01	-15393	±10
2453001.31	-15348	±10
2453002.18	-15338	±10
2453003.32	-15417	±10

The data are modelled by the equation

$$V(t) = K \sin\left[\frac{2\pi}{P}(t - T) + \pi\right] + V_0,$$

where t is the time of the measurement, and K and V_0 are constants to be determined.

Find the best-fit solution by minimisation of χ^2 and hence find the best-fit values of K and V_0. [8]

Find the value of the minimum χ^2 and comment on how well the model fits the data. [2]

Estimate the error in your best-fit value of K. [4]

The mass of the planet, m_p, is given by the formula

$$m_p = 5 \cdot 10^{-3} \left(\frac{P}{\text{days}}\right)^{1/3} \left(\frac{K}{\text{m s}^{-1}}\right) M_J,$$

where M_J is the mass of Jupiter. Compute the mass of the planet and its error, in units of M_J. [4]

Five additional measurements of the radial velocity are performed, but with errors that are twice as large. Without computation, explain whether the error on m_p obtained using all ten measurements is increased or decreased. [2]

D13 Write brief notes on **two** of the following:

(a) why negative feed-back is useful in op-amp circuits and how it may be achieved; [10]

(b) the principle of phase-sensitive detection; [10]

(c) aliasing. [10]

END OF PAPER