UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 26th May 2018 9.00 am to 12.00 noon

PHYSICS A (1)

Attempt **all** *questions from Section A,* **two** *questions from Section B,* **one** *question from Section C, and* **one** *question from Section D.*
Section A will carry approximately 20% of the total marks.
In Sections B, C, and D, each question carries approximately the same number of marks.
The approximate number of marks allocated to each question or part of a question is indicated in the right margin.
The paper contains **7** *sides including this one and is accompanied by a Mathematical Formulae Handbook giving values of constants and containing mathematical formulae, which you may quote without proof.*
Answers from **each** *Section should be written in separate Booklets.*
Write the letter of the Section and your candidate number, not your name, on the cover of **each** *Booklet.*
A separate master (yellow) cover sheet should also be completed, listing all questions attempted in the paper.

STATIONERY REQUIREMENTS
Booklets
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae Handbook
Log-linear graph paper
Approved calculator

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Attempt **all** *questions from this Section. Answers should be concise and relevant formulae may be assumed without proof. Use a separate booklet for the whole of this section.*

A1 Lithium has an atomic number of 3. Are ^{6}Li and ^{7}Li atoms bosons or fermions? Explain your reasoning. [4]

A2 An electron is in the ground state of a one-dimensional infinitely deep quantum well. The maximum wavelength at which it can absorb electromagnetic radiation is 800 nm. Calculate the width of the well. [4]

A3 The eigenfunctions of the spin-half operator $\hat{S}_z$ are α and β with eigenvalues $+\hbar/2$ and $-\hbar/2$ respectively. Find the eigenvalues and eigenfunctions of $\hat{S}_x$. [4]

A4 The moment of inertia, I, of a thin disk is given by $I = \frac{1}{2}\pi\rho s R^4$, where ρ denotes the density of the material, s its thickness and R the radius. A set of disks with $R \approx 1$ m is machined out of a material with $s \approx 0.01$ m and $\rho \approx 8 \times 10^3$ kg m^{-3} both known to an accuracy of 1%. Calculate I. Determine the accuracy needed in R to ensure that I has an accuracy of 2.5%. [4]

A5 An ideal low-pass filter is designed with an input resistance of 10 kΩ and time constant of 1 μs. Sketch the circuit diagram of the filter and determine its capacitance. Sketch the gain as a function of frequency on a Bode plot. [4]

SECTION B

Attempt **two** *questions from this Section. Use a separate booklet for the whole of this section.*

B6 Particles of mass m move in a one-dimensional potential $V(x)$ with a wavefunction $\psi(x,t)$ which satisfies the time-dependent Schrödinger equation.

(a) Show that the probability current is given by [6]

$$J(x,t) = \frac{i\hbar}{2m}\left(\frac{\partial\psi^*}{\partial x}\psi - \psi^*\frac{\partial\psi}{\partial x}\right).$$

(b) Particles of mass m and energy E with the wavefunction [5]

$$\psi(x,t) = A\exp\left(i(kx - \omega t)\right)$$

moving in the positive x-direction are incident on a double potential step defined by

$$\begin{aligned} V(x) &= 0, && x < 0 \\ V(x) &= V_1, && 0 \le x \le a \\ V(x) &= V_2, && x > a \end{aligned}$$

where $E > V_2 > V_1 > 0$. Write down the wavefunctions in the three regions. What boundary conditions apply at the potential steps? Explain the physical significance of these boundary conditions.

(c) Find the relation between E, V_1 and V_2 that is necessary for zero reflection. [5]
You may assume that $\sqrt{2m(E - V_1)/\hbar^2} = \pi/(2a)$.

(d) Find an expression relating the amplitude of the transmitted wave to the incident wave and show that the incident and transmitted waves have equal probability current. [4]

(TURN OVER)

B7

(a) Write down the time-independent Schrödinger equation for a particle of mass m in a potential $V(x)$. Define the terms of the equation. [3]

(b) A harmonic oscillator has a potential $V_1(x) = ax^2$. Show that wavefunctions of the form $\Psi_1 = A\exp(-Cx^2)$ and $\Psi_2 = Bx\exp(-Cx^2)$ are solutions of the Schrödinger equation for this potential. Find values for C and the energies of the two states. What features of the wavefunctions show that these are the two states of lowest energy? [7]

(c) Consider another potential $V_2(x)$ [4]

$$\begin{aligned} V_2(x) &= ax^2, \qquad & x \geq 0 \\ V_2(x) &= \infty, \qquad & x < 0. \end{aligned}$$

What condition must the wavefunction satisfy at $x = 0$? Write down an equation and sketch the wavefunction corresponding to the ground state. What is the energy of this state?

(d) A particle in the potential, $V_2(x)$, is in the ground state. The potential is suddenly changed to $V_1(x)$. What is the probability the particle will be found in the new ground state? [6]

[*You may find the following integrals helpful:*

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2)dx = \sqrt{\frac{\pi}{\alpha}} \qquad \int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}.$$
]

B8 A rigid body rotates in the xy plane. The Hamiltonian describing its motion is

$$\hat{H} = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\phi^2},$$

where ϕ is the angle of orientation of the body and I is its moment of inertia.

(a) Find the eigenfunctions and corresponding energy levels of $\hat{H}$. [4]

(b) The body is prepared in a normalised wavefunction at time $t = 0$ [7]

$$\Psi(\phi, 0) = \sqrt{\frac{4}{3\pi}}\cos^2\phi.$$

Find the possible results of a measurement of the angular momentum and of the energy at time $t = 0$. Obtain their relative probabilities.

(c) Determine an expression for the wavefunction $\Psi(\phi, t)$ at time t. [5]

(d) Sketch the probability distribution $|\Psi(\phi, t)|^2$ as a function of angle and time. [4]

B9

(a) What is the commutator $[\hat{A}, \hat{B}]$ of two operators $\hat{A}$ and $\hat{B}$? Use [3]

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$$

to show that $[\hat{x}, \hat{p}_x] = i\hbar$. What does this imply about the measurement of position and momentum?

(b) Verify the commutation relation $\left[\hat{A}, \hat{B}\hat{C}\right] = \left[\hat{A}, \hat{B}\right]\hat{C} + \hat{B}\left[\hat{A}, \hat{C}\right]$. Using this result or otherwise show that $\left[\hat{x}, \hat{p}_x^2\right] = 2i\hbar\hat{p}_x$. [5]
Show that

$$[\hat{p}_x, V(\hat{x})] = -i\hbar\frac{\partial V(\hat{x})}{\partial \hat{x}}.$$

(c) Write down an expression for the expectation value $\left\langle\hat{Q}\right\rangle$ of the operator $\hat{Q}$ for a system described by a normalised wavefunction ψ. Use the time-dependent Schrödinger equation and the Hermitian property of the Hamiltonian operator $\hat{H}$ to show that [5]

$$\frac{d\left\langle\hat{Q}\right\rangle}{dt} = \frac{1}{i\hbar}\left\langle\left[\hat{Q}, \hat{H}\right]\right\rangle,$$

assuming that $\hat{Q}$ has no explicit time dependence.

(d) A particle of mass m moves in one dimension in the potential $V(x)$. Write down the Hamiltonian operator and use the results above to derive expressions for $d\langle\hat{x}\rangle/dt$ and $d\langle\hat{p}_x\rangle/dt$. Comment on your results. [7]

(TURN OVER)

SECTION C

Attempt **one** *question from this Section. Use a separate booklet for this section.*

C10 Write an essay on the postulates of quantum mechanics and the experiments that gave rise to them. Include discussions of the photoelectric effect, the Davisson and Germer experiment, and Young's double slit experiment. [20]

C11 Write brief notes on **two** of the following:

(a) indistinguishable particles and consequences of their exchange; [10]

(b) the specific heat of diatomic gases due to quantum mechanical vibrations; [10]

(c) quantum tunneling including some experimental examples of it. [10]

SECTION D

Attempt **one** *question from this Section. Use a separate booklet for this section.*

D12

(a) Explain how the χ^2 test can help to determine if a theoretical model fits a set of experimental data. [6]

(b) An experiment measures the radiation emitted by a radioactive source passing through an increasing number of sheets, n, composed of a certain material. Each measurement lasts for 1 s and yields the results listed in the table below. [3]

number of sheets, n	1	2	3	4	5	6	7	8
counts in 1 s	723	244	95	35	8	5	6	5

Plot the counts as a function of n to show that the count rate is proportional to

$$N_0 \exp(-nb),$$

where $N_0 = 2018.5$ counts s^{-1} and the parameter b is a material dependent constant. State over which range of n this model describes the data.

(c) N_0 was determined from a 1000 s long measurement for $n = 0$. Estimate the respective errors in N_0 and in the counts given in the table above. Comment on the accuracy of the two measurements. [2]

(d) The value b of the sheet material is known to be either $b = 1$ or $b = 1.1$. Perform the χ^2 test to determine which value of b provides a better fit to the data. [7]

(e) How should the model be modified to account for the discrepancy between the expected and measured counts? [2]

D13 Write brief notes on **two** of the following:

(a) the use of Bayes' theorem and prior knowledge; [10]

(b) systematic errors and techniques for their removal; [10]

(c) the origin of thermal (Johnson) and shot noise. [10]

END OF PAPER