UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 27th May 2017 9.00 am to 12.00 noon

PHYSICS A (1)

Attempt **all** *questions from Section A,* **two** *questions from Section B,* **one** *question from Section C, and* **one** *question from Section D.*

Section A will carry approximately 20% of the total marks.

In Sections B, C, and D, each question carries approximately the same number of marks.

The approximate number of marks allocated to each question or part of a question is indicated in the right margin.

The paper contains **8** *sides including this one and is accompanied by a Mathematical Formulae Handbook giving values of constants and containing mathematical formulae, which you may quote without proof.*

Answers from **each** *Section should be written in separate Booklets.*

Write the letter of the Section and your candidate number, not your name, on the cover of **each** *Booklet. If several Booklets are used for a given Section, join them together using a Treasury Tag.*

A separate master (yellow) cover sheet should also be completed, listing all questions attempted in the paper.

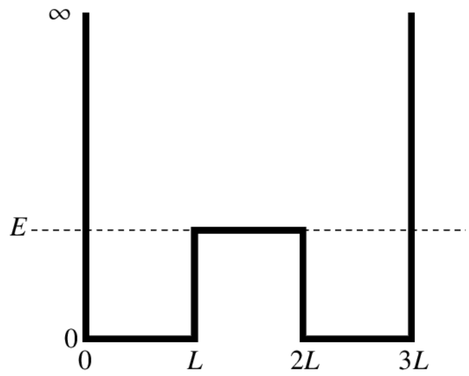
STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Booklets and treasury tags	Mathematical Formulae Handbook
Rough workpad	Linear graph paper
Yellow master coversheet	Approved calculator

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Attempt **all** *questions from this Section. Answers should be concise and relevant formulae may be assumed without proof. Use a separate booklet for the whole of this section.*

1 A particle moves in the potential shown in the diagram below. The energy E of the ground state is indicated. Sketch the corresponding wavefunction.



[4]

2 In classical mechanics the expressions xp^2x and px^2p, where x and p represent position and momentum respectively, are equivalent. Show that they are also equivalent in quantum mechanics. [4]

3 A particle has angular momenum quantum number $\ell = 2$ and spin quantum number $s = 1$. What are the possible total angular momentum states (j, m_j)? [4]

4 Design and sketch an amplifier circuit, based on a single ideal operational amplifier and two resistors, with gain of -10 and input impedance of 10 kΩ. [4]

5 Two students use the same experimental apparatus to perform five measurements each of the magnetic susceptibility of two paramagnetic samples A and B. Student 1 measures sample A and Student 2 measures sample B. They obtain the following sets of values:

Student 1:	2.3	2.1	2.3	2.2	2.4
Student 2:	2.5	2.4	2.3	2.5	2.4

Briefly discuss whether these data are consistent with samples A and B being made of the same material. [4]

SECTION B

Attempt **two** *questions from this Section. Use a separate booklet for the whole of this section.*

B6 A particle of mass m moves in a one-dimensional potential $V(x) = -V_0\delta(x)$, where $V_0 > 0$ and $\delta(x)$ is the Dirac delta function.

Write down the time-independent Schrödinger equation for the wavefunction $\psi(x)$ and explain why $\psi(x)$ should be continuous at the origin, but $\mathrm{d}\psi/\mathrm{d}x$ should be discontinuous there. [2]

By integrating the Schrödinger equation from $x = -\epsilon$ to $x = +\epsilon$ and considering the limit $\epsilon \to 0$, derive the boundary condition

$$\frac{\mathrm{d}\psi}{\mathrm{d}x}(0^+) - \frac{\mathrm{d}\psi}{\mathrm{d}x}(0^-) = -\frac{2m}{\hbar^2}V_0\psi(0),$$

where $x = 0^-$ and $x = 0^+$ are positions immediately to the left and right of the origin, respectively. [3]

For a particle with negative total energy, show that there exists a single bound state, for which the wavefunction decays exponentially away from the origin, and that its energy is

$$E = -\frac{mV_0^2}{2\hbar^2}.$$ [4]

Find the normalised wavefunction for this bound state and sketch its form. [2]

For a particle with positive total energy E incident from $x < 0$ in the above potential, show that the intensity reflection and transmission coefficients, respectively, are given by

$$R = \frac{mV_0^2/\hbar^2}{2E + (mV_0^2/\hbar^2)} \qquad \text{and} \qquad T = \frac{2E}{2E + (mV_0^2/\hbar^2)}.$$ [6]

Sketch R and T as a function of V_0 and give a physical interpretation of your results. [3]

B7 A particle of mass m moves in a one-dimensional potential $V(x) = \frac{1}{2}m\omega^2x^2$. The quantum mechanical ladder operators are defined by

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right) \qquad \text{and} \qquad \hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right),$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators, respectively.

Show that the Hamiltonian for the particle can be written as $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$. [3]

Show further that $[\hat{a}, \hat{a}^\dagger] = 1$ and $[\hat{H}, \hat{a}] = -\hbar\omega\hat{a}$. [4]

If $|\phi_n\rangle$ is a normalised eigenstate of $\hat{H}$, with energy $E_n = (n + \frac{1}{2})\hbar\omega$, show that $\hat{a}|\phi_n\rangle = \sqrt{n}|\phi_{n-1}\rangle$ for integer $n > 0$, and briefly explain why $\hat{a}|\phi_0\rangle = 0$. [5]

(TURN OVER

A certain normalised quantum state of the system, depending on a complex parameter α, has the form

$$|\chi_\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\phi_n\rangle.$$

Show that $\hat{a}|\chi_\alpha\rangle = \alpha|\chi_\alpha\rangle$. [3]

If the system is in the state $|\chi_\alpha\rangle$ at time $t = 0$, show that at a later time t it is in the state $\exp(-\tfrac{1}{2}\mathrm{i}\omega t)|\chi_\beta\rangle$, where $\beta = \alpha \exp(-\mathrm{i}\omega t)$. [5]

B8 For two particles of mass m_1 and m_2 with an interaction potential $V(r)$, where r is the separation of the particles, the time-independent Schrödinger equation describing the relative motion of the two particles may be written as

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{2\mu r^2}\hat{L}^2 \psi + V(r)\psi = E\psi,$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the particles.

By substituting the form $\psi(r, \theta, \phi) = R(r) Y_{\ell m}(\theta, \phi)$ for the wavefunction into the above Schrödinger equation, show that the function $U(r) \equiv rR(r)$ satisfies the equation

$$-\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2 U}{\mathrm{d}r^2} + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} U + V(r) U = EU.$$

[4]

Consider the case in which the potential has the form $V(r) = -V_0$ (where $V_0 > 0$) for $r \leq a$ and $V = 0$ for $r > a$, and for which $-V_0 < E < 0$.

Explain briefly why the function $U(r)$ must satisfy the boundary conditions $U(0) = 0$ and $U(r) \to 0$ as $r \to \infty$. [2]

For the case $\ell = 0$ show that solutions $U(r)$ satisfying the above boundary conditions have the form

$$U(r) = \begin{cases} A \sin kr & \text{for } r \leq a \\ B \exp(-qr) & \text{for } r > a, \end{cases}$$

where A and B are arbitrary constants, and find expressions for k and q. [4]

By imposing appropriate boundary conditions at $r = a$, show that $\tan ka = -k/q$. [2]

Hence show that there exists at least one bound state with $\ell = 0$ provided that $V_0 > \pi^2\hbar^2/(8\mu a^2)$. [2]

What is the condition on V_0 for there to be only one such bound state? [2]

The deuteron is a bound state of a proton and a neutron, which can be modelled using a potential of this form, with $a = 2.1 \times 10^{-15}$ m. Find a lower limit for V_0 using the inequality derived above. [2]

The value of V_0 that accounts for the observed binding energy of the deuteron is 33.5 MeV. Explain whether you would expect any bound excited states of the deuteron to exist. [2]

B9 Write down the time-dependent Schrödinger equation satisfied by the two-particle wavefunction $\Psi(\boldsymbol{r}_1, \boldsymbol{r}_2, t)$ for two non-interacting, distinguishable particles, each of mass m, moving in some potential V, where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the positions of the two particles. [2]

Show that the equation is satisfied by the two-particle wavefunction

$$\Psi(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = \phi_\mathrm{a}(\boldsymbol{r}_1)\,\phi_\mathrm{b}(\boldsymbol{r}_2)\exp(-\mathrm{i}Et/\hbar),$$

where $\phi_\mathrm{a}(\boldsymbol{r}_1)$ and $\phi_\mathrm{b}(\boldsymbol{r}_2)$ are single-particle stationary states and E is the total energy of the particles. [4]

Suppose now that the particles are indistinguishable with spin $\frac{1}{2}$. Discuss how the form of the two-particle wavefunction must be modified and write down the various possibilities that are the product of a spatial and a spin part. [5]

What are the total spin quantum numbers S and m_S corresponding to each of your two-particle wavefunctions? [4]

If there is a small Coulomb repulsion between the particles, which state or states are likely to have the lower energy? [2]

Suppose instead that the particles are non-interacting and indistinguishable with spin 0. Calculate the factor by which the probability of finding two such particles at the same position is enhanced relative to that for two distinguishable particles. [3]

(TURN OVER

SECTION C

Attempt **one** *question from this Section. Use a separate booklet for this section.*

C10 Write an essay on angular momentum in quantum mechanics, including a discussion of orbital and spin angular momentum and the Stern–Gerlach experiment. [20]

C11 Write brief notes on **two** of the following:

(a) Bohr’s model of the atom; [10]

(b) Ehrenfest’s theorem and the correspondence principle; [10]

(c) the properties of Hermitian operators and their role in quantum mechanics. [10]

SECTION D

Attempt **one** *question from this Section. Use a separate booklet for this section.*

D12 A harmonic oscillator at finite temperature is undergoing overdamped Brownian motion. In the frequency domain, the *ideal* spectral density $P_{\rm i}(f)$ of the fluctuations is described by

$$P_{\rm i}(f) \propto \frac{1}{f^2 + f_{\rm c}^2},$$

where f is the frequency of the fluctuations and $f_{\rm c}$ is the characteristic frequency determined by the harmonic potential.

Sketch $P_{\rm i}(f)$ on a log-log plot for two different values of $f_{\rm c}$. [4]

The fluctuations of the oscillator are detected with a position-sensitive detector. An electronic circuit based on an operational amplifier is used to convert the current from the detector into a voltage signal that is digitised with an analogue-to-digital converter. The measured spectral density $P_{\rm m}(f)$ from the detector yields the following results:

Frequency (Hz)	$P_{\rm m}(f)$ ($\mathrm{V^2\,Hz^{-1}}$)
0.01	1×10^{-4}
0.1	1×10^{-5}
1.0	1×10^{-6}
10	1×10^{-6}
100	1×10^{-6}
1,000	5×10^{-7}
10,000	1×10^{-8}
100,000	1×10^{-10}
1,000,000	1×10^{-15}

Plot $P_{\rm m}(f)$ on log-log axes and use your plot over the frequency range 1 Hz to 10^5 Hz to estimate the value of $f_{\rm c}$ graphically. [6]

Describe how $P_{\rm m}(f)$ differs from $P_{\rm i}(f)$ at $f \ll f_{\rm c}$ and comment briefly on the behaviour. What is the minimum recording time required to record the lowest-frequency data point? [3]

What is the minimum sampling rate required to obtain $P_{\rm m}(f)$? [2]

Describe how $P_{\rm m}(f)$ deviates from $P_{\rm i}(f)$ at high frequencies. Comment on why, in general, an amplifier might be designed such that its gain rolls off at high frequencies. [5]

D13 Write brief notes on **two** of the following:

(a) reducing noise in electrical circuits; [10]

(b) Nyquist's theorem and digital sampling; [10]

(c) the importance of Gaussian distributions in experimental physics. [10]

END OF PAPER