UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Thursday 26th May 2016 1.30 to 4.30 pm

PHYSICS A (2)

Attempt **all** *questions from Section A,* **two** *questions from Section B,* **one** *question from Section C, and* **one** *question from Section D.*
Section A will carry approximately 20% of the total marks.
In Sections B, C, and D, each question carries approximately the same number of marks.
The approximate number of marks allocated to each question or part of a question is indicated in the right margin.
The paper contains **8** *sides including this one and is accompanied by a Mathematical Formulae Handbook giving values of constants and containing mathematical formulae, which you may quote without proof.*
Answers from **each** *Section should be written in separate Booklets.*
Write the letter of the Section and your candidate number, not your name, on the cover of **each** *Booklet. If several Booklets are used for a given Section, join them together using a Treasury Tag.*
A separate master (yellow) cover sheet should also be completed, listing all questions attempted in the paper.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Booklets and treasury tags	Mathematical Formulae Handbook
Rough workpad	Linear graph paper
Yellow master coversheet	Approved calculator

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Attempt **all** *questions from this Section. Answers should be concise and relevant formulae may be assumed without proof. Use a separate booklet for the whole of this section.*

1 A material with refractive index 1.4 is joined to a material of refractive index 1.6 by a thin planar layer of refractive index n and thickness d. Find the values of n and d required in order to minimise reflection of light with a wavelength in free space of 500 nm incident normally on the interface. [*You are not expected to derive a general expression for the reflection coefficient.*] [4]

2 Describe the structure and dimensions of a Fresnel zone plate suitable for focussing X-rays of energy 30 keV with a primary focal length of 1 cm. [4]

3 Light with a spectrum containing two closely-spaced wavelength components is incident on a Michelson interferometer. Sketch how the output intensity varies with mirror displacement, and annotate your sketch to explain qualitatively how the observed features depend on the wavelength and wavelength spacing. [4]

4 Derive an expression for the phonon density of states, $g(\omega)$, in a three-dimensional monatomic crystal of volume V, assuming a linear dispersion relation $\omega/k = v_s$. [4]

5 Electrons in a 1-D lattice of period a may be considered to have energy $E = E_0 - (\hbar^2/m_e a^2)\cos(ka)$, where k is the electron wavevector and m_e is the free electron mass. Determine the effective mass, $m_e{}^*$, of electrons at the first Brillouin zone boundary, and comment on your result. [4]

$$\left[\textit{The electron effective mass is given by } \frac{1}{m_e{}^*} = \frac{1}{\hbar^2}\frac{\mathrm{d}^2E}{\mathrm{d}k^2}.\right]$$

SECTION B

Attempt **two** *questions from this Section. Use a separate booklet for the whole of this section.*

B6 A damped simple harmonic oscillator has the equation of motion

$$m\ddot{x} + b\dot{x} + kx = 0,$$

where x is the displacement from equilibrium.

Explain the terms "heavy damping", "critical damping" and "light damping", and sketch the displacement as a function of time for an oscillator displaced and released from rest at $t = 0$ for each of the three damping regimes. [4]

Consider a critically damped oscillator released at $t = 0$ from a positive displacement x_0 with a speed v_0 towards the origin (i.e. $\dot{x}(0) = -v_0$). How large must v_0 be in order for the oscillator to pass through $x = 0$ before returning to equilibrium? [5]

A different oscillator has $m = 1\,\mathrm{kg}$, $k = 1\,\mathrm{N\,m^{-1}}$ and $b = 4/\sqrt{3}\,\mathrm{N\,m^{-1}\,s}$. Verify that this oscillator is heavily damped. [1]

The oscillator is displaced to $x = 0.01$ m and is given an initial speed of $0.03\,\mathrm{m\,s^{-1}}$ towards the origin (i.e. $\dot{x}(0) = -0.03\,\mathrm{m\,s^{-1}}$). Find an expression for $x(t)$ for $t \geq 0$. [5]

Evaluate the expression at $t = 0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$ s and plot it using the graph paper provided. [4]

Hence estimate the time at which the oscillator passes through $x = 0$. [1]

B7 A waveguide is formed by clamping a rubber sheet along the lines $y = 0$ and $y = b$. Waves on the unconstrained sheet travel at a speed v that is independent of frequency.

Show that waves propagating in the waveguide have the form

$$\psi(x, y) = \mathrm{Re}\left[A \sin\left(\frac{n\pi}{b} y\right) \exp\left(\mathrm{i}\omega t - \mathrm{i}k_x x\right)\right], \qquad [4]$$

where n is a positive integer. Find the relationship between ω and k_x, and sketch it for $n = 1, 2, 3$. [5]

Derive an expression for the phase velocity, v_p, of guided waves. [2]

Calculate k_x and v_p for waves of frequency 200 Hz propagating in the $n = 1$ and $n = 2$ modes for $b = 0.1$ m and $v = 10\,\mathrm{m\,s^{-1}}$. [3]

A superposition of waves is launched into the waveguide by oscillating the edge of the sheet at $x = 0$ with a profile

$$\psi(0, y) = A\,(\sin 10\pi y + \sin 20\pi y) \cos(400\pi t),$$

where here y and t are respectively the values of distance across the waveguide measured in metres and time measured in seconds. Find the first distance, x_1, along the waveguide where the profile of the wave is given by

$$\psi(x_1, y) = A\,(\sin 10\pi y - \sin 20\pi y) \cos(400\pi t + \phi). \qquad [4]$$

Why would propagation in different modes present a problem in optical fibre waveguides for telecommunications systems, and how is this problem avoided in practice? [2]

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B8 The Fraunhofer diffraction pattern of a two-dimensional aperture defined by $h(x, y)$ illuminated at normal incidence by light of wavevector k is given by

$$\psi(p, q) \propto \iint h(x, y)\, e^{-i(px+qy)}\, \mathrm{d}x\, \mathrm{d}y,$$

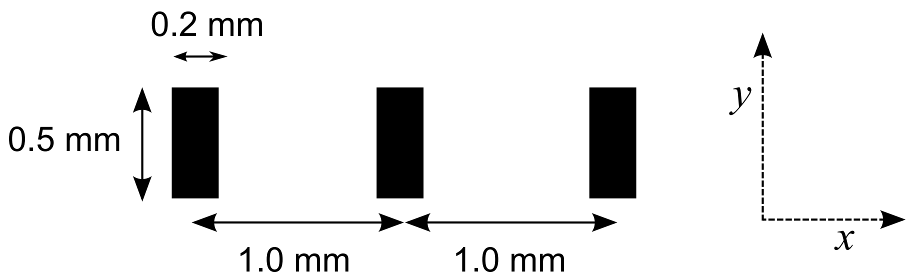
where $q = k \sin\theta$ and $p = k \sin\xi$, and θ and ξ are the diffraction angles in the x and y directions respectively.

Under what conditions is this expression valid? [2]

Derive and explain *Babinet's principle* relating to complementary apertures. [5]

Collimated light of wavelength 500 nm is incident normally on a rectangular obstruction of size 0.2 mm × 0.5 mm. The diffracted light is collected with a lens of focal length 20 cm and the diffraction pattern is recorded on a screen placed 20 cm behind the lens. The planes of the obstruction, lens and screen are all parallel. Make a sketch, annotated with quantitative detail, of the intensity on the screen. [6]

Find the diffraction pattern of the more complicated obstruction comprising three rectangles as shown below, and sketch a graph of the intensity distribution on the screen along the direction parallel to x. [7]



B9 A Fabry–Pérot etalon comprises two semi-transparent mirrors separated by an air gap of thickness d. Each mirror has (intensity) reflection and transmission coefficients R and T respectively. Consider the simple case where the etalon is illuminated with collimated light of wavelength λ at normal incidence, and the transmission of the etalon is measured in the same direction. Show that the transmission of the etalon is given by

$$B = \frac{T^2}{1 + R^2 - 2R\cos\delta},$$

where $\delta = 4\pi\, d/\lambda$. [5]

Show that for highly reflective mirrors the half-width at half maximum of each peak is given by

$$\delta_{1/2} \approx \frac{1 - R}{R^{1/2}}.$$ [3]

[*You may find it convenient to note, and may assume without proof, that the expression above for B can be rearranged as*

$$B = \frac{T^2}{(1-R)^2} \frac{1}{1 + [4R/(1-R)^2] \sin^2(\delta/2)}.$$
]

Calculate $\delta_{1/2}$ for $R = 0.80$ and sketch B as a function of δ for an etalon with $R = 0.80$ and $T = 0.20$. [3]

The etalon described above is illuminated with light containing wavelengths of 1000.0 nm and 1000.2 nm with equal intensity. Sketch the output intensity of the etalon as a function of d as d is increased from 0.4998 mm to 0.5003 mm. Indicate the values of d at which any peaks occur, and comment on whether it is possible to resolve the two wavelength components in the spectrum. [6]

A third wavelength component at 1001.1 nm is added to the input spectrum. Add any new peaks that arise in your sketch above, and comment on the implications for spectroscopy using a Fabry-Pérot etalon. [3]

SECTION C

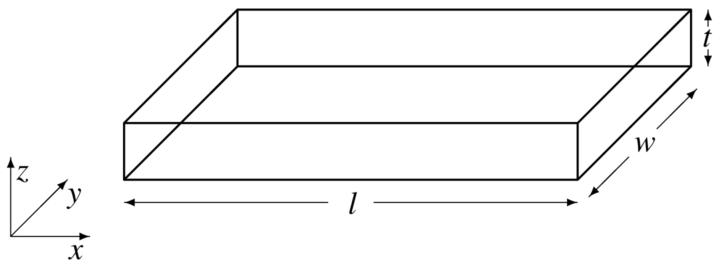
Attempt **one** *question from this Section. Use a separate booklet for this section.*

C10 Briefly explain, with the aid of a diagram, the origin of the transverse voltage which arises when a magnetic field is applied perpendicularly to a current flow in a conducting bar (the Hall effect). State one property of a periodic solid that may be determined from a measurement of the Hall effect. [5]

The semi-classical equation governing the transport of electrons in a periodic lattice under the application of an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ is

$$m^* \frac{\mathrm{d}\langle \mathbf{v} \rangle}{\mathrm{d}t} = -\frac{m^*}{\tau}\langle \mathbf{v} \rangle + e\mathbf{E} + e\langle \mathbf{v} \rangle \times \mathbf{B}\,, \qquad (\star)$$

where e is the electron charge, m^* their effective mass, and τ is the mean time between collision of electrons with lattice defects or phonons. Explain the origin of each term in this equation. [2]



A Hall bar has length l, width w, and thickness t, and is oriented as shown in the diagram. A current $\mathbf{I} = (I_x, 0, 0)$ flows through the bar and $\mathbf{B} = (0, 0, B_z)$. Using equation $(\star)$ above, derive expressions for the steady state electric field components within the bar, $\mathbf{E} = (E_x, E_y, E_z)$. [3]

Determine expressions for the conductivity, σ, and Hall coefficient, $R_H = E_y/j_xB_z$, where j_x is the current density in the bar. [2]

The measured potential difference across the length of the bar is V_x, and across the width is V_y. Show that

$$\frac{V_y}{V_x} = -\frac{eB_z}{m^*}\tau\frac{w}{l}\,.$$ [1]

A Hall bar made from high-purity Cu, with $t = 5\,\mu$m and $l/w = 10$, has a measured resistance of $R = 11\,\mathrm{m}\Omega$ across its length. Determine the free electron density in Cu, given that $V_y/V_x = 0.013$ when $B_z = 10\,\mathrm{T}$. [3]

Assuming $m^* = m_e$ for Cu, estimate the mean electron scattering time τ. Justify why this a good assumption in the case of Cu, which may be considered as a monovalent metal. [4]

C11 What is meant by the term *reciprocal lattice*? [2]

A plane wave of wavevector $\mathbf{k}_i$ is incident on a crystal having identical atoms at points $\mathbf{r}_n$ relative to the origin, O. Diffraction from the periodic structure produces an outgoing plane wave of wavevector $\mathbf{k}_f$, which is measured at detector, D, at a distance from the crystal $\gg |\mathbf{r}_n|$.

Consider the scattering of the incident plane wave from two atoms, one at O and one at $\mathbf{r}_n$. Sketch this geometry, including the vectors $\mathbf{k}_i$, $\mathbf{k}_f$, $\mathbf{r}_n$. [2]

Show that the phase difference between the scattered waves arriving at D from these two atoms is given by $\mathbf{k}_s \cdot \mathbf{r}_n$, where $\mathbf{k}_s = \mathbf{k}_i - \mathbf{k}_f$, and hence that the total amplitude, S, of the outgoing wave measured at D may be written

$$S \propto \sum_n \mathrm{e}^{\mathrm{i}\mathbf{k}_s \cdot \mathbf{r}_n} .$$

[5]

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the primitive unit cell vectors of the crystal lattice. A general reciprocal lattice vector takes the form:

$$\mathbf{G}_{hkl} = h\mathbf{A} + k\mathbf{B} + l\mathbf{C} ,$$

where h, k and l are integers. Write down values for $\mathbf{A} \cdot \mathbf{a}$, $\mathbf{A} \cdot \mathbf{b}$ and $\mathbf{A} \cdot \mathbf{c}$ and show that diffraction maxima will be observed at D when $\mathbf{k}_s$ is equal to a reciprocal lattice vector. [4]

A two-dimensional crystal contains one type of atom. The atomic spacing is a along each axis of the primitive unit cell, and the angle between the axes is $\pi/3$ rad. Find the general reciprocal lattice vector $\mathbf{G}_{hk}$ for this crystal and draw labelled sketches of both the primitive unit cell and reciprocal lattice. Calculate the smallest value of $|\mathbf{k}_s|$ for which a diffraction maximum will be measured at D. (You may ignore contributions from lattice vibrations.) [7]

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SECTION D

Attempt **one** *question from this Section. Use a separate booklet for this section.*

D12 Write an essay on Fresnel diffraction, including detailed discussion of the Fresnel diffraction patterns of edges and rectangular apertures. [20]

D13 Write brief notes on **two** of the following:

(a) successes and failures of the free electron model; [10]

(b) the thermal conductivity of an insulator, including a discussion of its temperature dependence; [10]

(c) the use of *p*-*n* junctions in semiconductor devices. [10]

END OF PAPER