UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Thursday 28th May 2015 1.30 to 4.30 pm

PHYSICS A (2)

Attempt **all** *questions from Section A,* **two** *questions from Section B,* **one** *question from Section C, and* **one** *question from Section D.*
Section A will carry approximately 20% of the total marks.
In Sections B, C, and D, each question carries approximately the same number of marks.
The approximate number of marks allocated to each question or part of a question is indicated in the right margin.
The paper contains **6** *sides including this one and is accompanied by a Mathematical Formulae Handbook giving values of constants and containing mathematical formulae, which you may quote without proof.*
Answers from **each** *Section should be written in separate Booklets.*
Write the letter of the Section and your candidate number, not your name, on the cover of **each** *Booklet. If several Booklets are used for a given Section, join them together using a Treasury Tag.*
A separate master (yellow) cover sheet should also be completed, listing all questions attempted in the paper.

STATIONERY REQUIREMENTS
Booklets and treasury tags
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae Handbook
Approved calculator

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Attempt **all** *questions from this Section. Answers should be concise and relevant formulae may be assumed without proof. Use a separate booklet for the whole of this section.*

1 A telescope in space images γ-rays of energy 500 keV and has an effective diameter of 0.1 m. Estimate (a) its angular resolution in radians, and (b) its resolution in metres when it views a source of γ-rays that is at a distance of 10^{21}m. [4]

2 Calculate the percentage reflected power when a transverse wave travelling along a stretched string encounters a junction beyond which the density of the string increases by a factor of 3. [4]

3 Using a diagram, show that the optical path difference between consecutive beams emerging from an air-spaced parallel-plate Fabry-Perot interferometer is $2d\cos\theta$, where d is the plate separation and θ is the angle between the incident beam and the normal to the plates. [4]

4 Assuming that each atom contributes one electron to the conduction band, estimate the Fermi energy of Cu within the free-electron approximation. [*Metallic Cu has a free electron density* $n = 8.5 \times 10^{28}$ m^{-3}.] [4]

5 Find the volume occupied in k-space by the first Brillouin zone, for a primitive cubic lattice in real space, with a unit cell volume V_c. [4]

SECTION B

Attempt **two** *questions from this Section. Use a separate booklet for the whole of this section.*

B6 The equation for damped simple-harmonic motion is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

where m, k and $b > 0$. Solve this equation for $t > 0$, and discuss your solution for the three cases:

(a) $b^2 - 4mk > 0$;

(b) $b^2 - 4mk = 0$;

(c) $b^2 - 4mk < 0$. [6]

The behaviour of the suspension of a motor car can be modelled by this equation, with m the mass of the car plus passengers, k the force constant of the suspension, b the damping of the suspension, and x the vertical displacement of the car body. Which of the three cases (a), (b) or (c) from the first part of the question would be the best choice for passenger comfort? Justify your answer. [2]

A car of mass $m = 1000$ kg including passengers has a suspension with $b = 500$ kg s^{-1} and $k = 300$ kg s^{-2}. The car is driven over a long undulating road at constant speed u so that x satisfies the equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = kA\cos\frac{2\pi u}{d}t,$$

where $d = 10$ m is the distance between undulation maxima with amplitude $A = 0.05$ m. Derive the steady-state solution to this equation. [6]

Find the car speed u at which the amplitude of x is maximised, and find the value of this amplitude. [6]

B7 Define *phase velocity* v_p and *group velocity* v_g, and explain what they mean. [4]

Show that

$$v_g = v_p - \lambda\frac{dv_p}{d\lambda},$$

where λ is the wavelength. [4]

Waves in deep water obey the dispersion relation

$$\omega^2 = gk + (\sigma/\rho)k^3\ ,$$

where g is the acceleration due to gravity, σ the surface tension per unit length, and ρ the water density. Show that

$$\frac{v_g}{v_p} = 1 - \frac{g/k^2 - \sigma/\rho}{2(g/k^2 + \sigma/\rho)}.$$ [6]

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Find the wavelength λ^* for which the ratio $v_g/v_p = 1$, and comment on the nature of the wave propagation at this wavelength. [4]

Determine the value of λ^* on a planet with a deep ocean for which $\sigma = 0.11$ N m^{-1}, $\rho = 1040$ kg m^{-3}, and $g = 5.0$ m s^{-2}. [2]

B8 A *plane* wave of wavelength λ is normally incident on a planar aperture with aperture function $h(x, y)$. With the aid of a diagram, show that the amplitude measured on the optic axis beyond the aperture is given by

$$\psi \propto \int h(x, y)K(\theta)\exp\left[i\pi\frac{x^2 + y^2}{\lambda b}\right] \mathrm{d}x\,\mathrm{d}y\,,$$

explaining the assumptions you make, and explaining the meanings of $K(\theta)$ and b. [6]

If the aperture is a circle of radius a, and using the substitution $s = x^2 + y^2$, show that

$$\psi \propto \int_0^{a^2} K(\theta)\exp\left[i\frac{\pi s}{\lambda b}\right] \mathrm{d}s\,.$$

[3]

With the aid of a phasor diagram, explain the operation of a Fresnel zone plate. [4]

Calculate the outer radii of the first three open zones of a zone plate that is designed to focus x-rays of wavelength $\lambda = 10$ nm at a distance of 1 m. [3]

For a zone plate of radius 0.7 mm, estimate the ratio of the intensity of light at the focus of the zone plate to the intensity at this position without the zone plate. [4]

B9 A Michelson interferometer has two perpendicular mirrors. Draw a labelled sketch of the optical arrangement and explain how it is used. [5]

By considering a wave that is the real part of the superposition of the waves $a\exp(i\omega t)$ and $a\exp(i\omega t + \phi)$, show that the intensity measured by the Michelson detector when fed with monochromatic light of wavevector k is $I(x) = I_0(1 + \cos kx)$, where x is the difference in path travelled by the two beams. [5]

The light source is then changed from monochromatic to having an intensity spectrum that has a rectangular top-hat shape from $k - \Delta k/2$ to $k + \Delta k/2$. Show that the intensity measured by the Michelson detector is now

$$I(x) \propto 1 + \cos(kx)\,\mathrm{sinc}(x\Delta k/2).$$

[5]

Draw an annotated sketch of $I(x)$ for light with a top-hat spectrum extending from 4.95×10^{14} Hz to 5.05×10^{14} Hz. [5]

SECTION C

Attempt **one** *question from this Section. Use a separate booklet for this section.*

C10 Explain the principles and approximations that form the free-electron model in metals and state what key predictions it is able to make. [4]

The equation governing the transport of electrons in an electric field is

$$m^* \frac{\mathrm{d}\langle v\rangle}{\mathrm{d}t} = -\gamma\langle v\rangle + eE \ ,$$

where e is the electron charge, m^* their effective mass, and $\tau = m^*/\gamma$ is the mean time between collision of electrons with lattice defects or phonons. Explain the origin of each term in this equation and show that the free-electron estimate for the DC electric conductivity is

$$\sigma_0 = \frac{n\, e^2 \tau}{m^*} \ , \qquad [6]$$

where n is the number of free electrons per unit volume.

Consider the case of AC current and the possible contribution of the inertial term in the equation for electron motion. Derive the corresponding frequency-dependent expression for complex electric conductivity, $\sigma(\omega)$. [6]

Sketch the plots of real and imaginary parts of $\sigma(\omega)$, and estimate the characteristic frequency at which the inertial effects become relevant. [4]

[*Metallic Cu has a free electron density* $n = 8.5 \times 10^{28}\ \mathrm{m}^{-3}$ *and an effective mass* $m^* \approx m_e$. *The DC conductivity of copper is* $\sigma_0 = 5.96 \times 10^7\ \mathrm{S\ m}^{-1}$.]

C11 Consider a longitudinal wave of displacements $u_n = u \exp(i\omega t - iqna)$, which propagates in a monoatomic linear lattice of atoms of mass M, with spacing a, bonded by harmonic springs with a force constant α. Draw a sketch of the system and show that the total energy of the wave is $E = \frac{1}{2}M \sum_n (\mathrm{d}u_n/\mathrm{d}t)^2 + \frac{1}{2}\alpha \sum_n (u_n - u_{n+1})^2$, where n runs over all atoms. [4]

By writing down the equation of motion for an atom n, show that $\omega^2 = 4(\alpha/M)\sin^2(qa/2)$. [3]

Show that the time-average total energy per atom is equal to $\frac{1}{4}M\omega^2 u^2 + \frac{1}{2}\alpha(1 - \cos qa)u^2 = \frac{1}{2}M\omega^2 u^2$. [3]

If this 1-dimensional line has a total number N of equivalent atoms, show that the density of phonon states in q-space is $g(q) = Na/\pi$. [3]

Therefore show that the density of states $g(\omega)$ takes the form

$$g(\omega) = \frac{2N}{\pi} \frac{1}{(\omega_m^2 - \omega^2)^{1/2}}$$

and write down the maximum frequency ω_m. [7]

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SECTION D

Attempt **one** *question from this Section. Use a separate booklet for this section.*

D12 Explain the concept of holes and the mechanism of hole conductivity in semiconductors, including a discussion of how the characteristics of holes can be measured experimentally and how the holes can be generated by doping. [20]

D13 Write brief notes on **two** of the following:

(a) the propagation of sound waves in gases; [10]

(b) the Q-factor and how it can be measured for free and for driven oscillators; [10]

(c) the polarisation of transverse waves, including linear, circular and elliptical polarisation. [10]

END OF PAPER