UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 28 May 2016 9.00 to 12.00 noon

PHYSICS B (Paper 1)

*Attempt **all** questions from Section A, **two** questions from Section B, and **two** questions from Section C.*

*Section A as a whole carries **approximately** one fifth of the total marks.*

*Each question in Sections B and C carries the same mark.*

*The **approximate** number of marks allocated to each part of a question in all Sections is indicated in the right margin.*

*Answers for each Section **must** be written in separate Booklets.*

*Write the letter of the Section on the cover of each Booklet.*

*Write your candidate number, **not** your name, on the cover of each Booklet.*

*A single, separate master (yellow) cover sheet should also be completed, listing all questions attempted.*

| STATIONERY REQUIREMENTS | SPECIAL REQUIREMENTS |
|---|---|
| 20-Page Booklets and Treasury Tags | Physics Mathematical Formulae Handbook (supplied) |
| Rough Work Pad | Approved Calculators allowed |
| Yellow Cover Sheet | |

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Answers should be concise and relevant formulae may be assumed without proof.*

A1 A thin conducting strip of width $w$ carries a current $I$ along its length. Sketch the magnetic field lines around it, and find the magnitude of the flux density (i) at distances $\gg w$, and (ii) at distances $\ll w$ from the centre. [4]

A2 A thin sheet of material with Young's modulus 50 MPa and Poisson ratio 0.48 is compressed between two smooth rigid plates that apply a force of 200 N. By how much does its surface area increase? [4]

A3 For what range of angles of incidence is a 12 MHz radio wave totally reflected at the surface of a plasma for which the plasma frequency is 10 MHz? [4]

A4 How much work can be obtained from an isolated system consisting of two bodies, one with heat capacity 1250 J/K and initially at temperature 450 K, and the other with heat capacity 750 J/K and initially at temperature 273 K? [4]

A5 A space station has the form of a large wheel of radius $a$ rotating about its axis with an angular velocity $\omega$ chosen so that people standing on the inside of the rim experience Earth gravity $g$. What is the effective gravity experienced by people walking around the rim at speeds of $a\omega$ and $-a\omega$? [4]

SECTION B

B6 Describe the method of images in electromagnetism, giving a simple example. [6]

Consider a thin ring, radius $R$, of superconducting wire lying flat on a non-magnetic insulating layer of thickness $h \gg R$ above a plane surface of a superconductor. Sketch the lines of magnetic flux density $\boldsymbol{B}$ if a constant current $I$ is flowing through the ring. [4]

[*Note that* $B = 0$ *inside superconductors.*]

Find $\boldsymbol{B}$ at the surface of the plane superconductor at distance $a$ from the axis of the ring. [4]

Also show that the force on the ring is

$$\frac{3\pi\mu_0 I^2 R^4}{32h^4}. \quad [4]$$

Suggest sensible experimental parameters that would require the least current to levitate the ring above the surface, and estimate this current. [2]

[*In spherical polar coordinates* $(r, \theta, \phi)$, *the magnetic scalar potential of a magnetic dipole m directed along the* $\theta = 0$ *axis is*

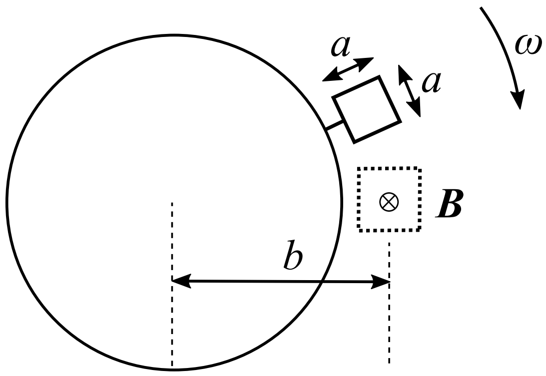
$$\phi_{\mathrm{m}} = \frac{m\cos\theta}{4\pi r^2},$$

*and for a scalar* $V(r, \theta, \phi)$,

$$\nabla V = \left(\frac{\partial V}{\partial r}, \frac{1}{r}\frac{\partial V}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\right).$$
]

B7 State *Faraday's Law* and *Lenz's Law*. [3]

In a magnetic braking system, a small, square wire loop of side $a$ and resistance $R$ is attached to the outer edge of an insulating cylinder rotating about its own axis with angular velocity $\omega$, as shown in the diagram. On each rotation, the wire loop passes through a region containing a uniform magnetic flux density $B$ directed into the paper. The field region has a square cross-section slightly larger than the size of the loop. The angular velocity of the loop decreases by only a small amount during each pass through the magnetic field.




Draw a diagram showing the direction of the current flowing around the loop and the forces acting on each side of the loop at a time when half of the loop has entered the magnetic field region. Draw similar diagrams for times when the loop is completely contained within the field region and when half of the loop has left the field region. [6]

Sketch the time dependence of the total force $F$ acting on the loop during each pass through the magnetic field. Show that an impulse $2B^2a^3/R$ is exerted on the loop on each pass. [5]

Show that the reduction in the total rotational kinetic energy of the system on each pass due to the impulse exerted on the loop is equal to the energy dissipated by the current flowing around the loop. [5]

Suggest a practical way in which the degree of braking could be varied by the user. [1]

[*The self-inductance of the wire loop can be neglected.*]

B8 A metal has electrical conductivity $\sigma$, relative permittivity $\epsilon$ and relative permeability $\mu$. Starting from Maxwell's equations, show that the effective dielectric constant for a uniform electric field oscillating as $e^{-i\omega t}$ is

$$\epsilon' = \epsilon + \frac{i\sigma}{\omega\epsilon_0}.$$ [4]

Electromagnetic waves with angular frequency $\omega$ impinge at normal incidence on the surface of a good conductor, for which $\sigma \gg \omega\epsilon\epsilon_0$. Show that, if the incident wave is linearly polarised along the $x$-direction, the $x$-component of the electric field varies with distance $z$ into the conductor according to

$$E_x(z,t) = \mathrm{Re}\left\{E_0 \exp(i(z/\delta - \omega t) - z/\delta)\right\},$$

where $\delta \equiv \sqrt{2/\sigma\mu\mu_0\omega}$ and $E_0$ are constants. [5]

Calculate the corresponding magnetic field strength $\boldsymbol{H}(z,t)$ and find the time-averaged Poynting vector $\langle \boldsymbol{N}(z)\rangle$. Use this to calculate the power lost per unit volume in the region between $z$ and $z + \mathrm{d}z$. [6]

Show that the mean power dissipated per unit volume by a current density $\boldsymbol{J}(z)$ is $\langle \boldsymbol{J}^2/\sigma\rangle$, and confirm that this power is the same as that lost by the electromagnetic wave. [3]

Qualitatively, how would the behaviour differ for a plasma? [2]

B9 Describe the flow of energy carried by electromagnetic waves, making clear the role of the energy density and the Poynting vector. [5]

A lossless transmission line consists of two coaxial conducting cylinders with radii $a$ and $b$, with $a < b$; the medium between them is a uniform dielectric with relative permittivity $\epsilon$ and relative permeability 1. Find the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ when the inner conductor is at a fixed potential $V$ relative to the outer and carries a fixed current $I$. Show that the impedance $\sqrt{L/C}$ (where $L$ and $C$ are the inductance and capacitance per unit length) is

$$\frac{\ln(b/a)}{2\pi}\sqrt{\frac{\mu_0}{\epsilon\epsilon_0}}.$$ [6]

Show that oscillating fields obtained by multiplying the static fields by wave factors $e^{i(kz-\omega t)}$, with $z$ taken along the line, satisfy Maxwell's equations. [4]

By calculating the Poynting vector, find the total power carried by such a wave and comment on your result. [3]

How is the Poynting vector affected if the conductors are not lossless? [2]

[*In cylindrical polar coordinates $(r, \theta, z)$, for a vector $\boldsymbol{A} = (A_r, A_\theta, A_z)$,*

$$\nabla \times \boldsymbol{A} = \left(\frac{1}{r}\left(\frac{\partial A_z}{\partial \theta} - \frac{\partial (rA_\theta)}{\partial z}\right), \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right), \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\right)$$

*and*

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}.$$
]

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SECTION C

C10 Explain the concept of an *effective potential* in studying orbits in a central potential, and how it can be used to understand bound and unbound orbits. [6]

Suppose that the radial motion $r(t)$ of an orbiting planet and its angular momentum $L$ are known. Show that the angle $\theta$ through which the planet has moved in time $t$ is

$$\theta(t) = \frac{L}{m}\int_0^t \frac{\mathrm{d}t'}{r(t')^2}.$$ [2]

Newton examined the behaviour of orbits in the central potential

$$V(r) = -\frac{A}{r} + \frac{B}{r^2},$$

with $B > 0$, giving rise to both inverse-square attractive and inverse-cube repulsive radial forces on a planet. Show that the radial motions $r(t)$ possible for $L = L_0$ and $B = B_0$ are the same as those for $L = \sqrt{L_0^2 + 2mB_0}$ and $B = 0$. [4]

Given that the orbits for $B = 0$ have the general form

$$r(\theta) = \frac{r_0}{1 + e\cos\theta}, \qquad (\star)$$

show that for non-zero $B$ they have the form

$$r(\theta) = \frac{r_0}{1 + e\cos(\theta/k)},$$

where

$$k = \frac{L_0}{\sqrt{L_0^2 + 2mB_0}}.$$ [4]

By considering the equation equivalent to ($\star$) in the case $A = 0$, $B = 0$, find the equation for orbits under a pure inverse-cube force. Sketch the possible orbits, considering both positive and negative $B$. [4]

C11 Any vector $\boldsymbol{A}$ fixed in the body frame of a rigid body has time derivative

$$\dot{\boldsymbol{A}} = \boldsymbol{\omega} \times \boldsymbol{A},$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the body. Use this relation to outline how the angular momentum is related to the angular velocity. Explain the concepts of principal axes and principal moments of inertia. [5]

A symmetric top has principal moments of inertial $I_1 = I_2 \neq I_3$ about its tip, which is on the 3-axis. Show that in this case the angular momentum can be written

$$\boldsymbol{L} = I_1 \left( \hat{\mathbf{e}}_3 \times \frac{\mathrm{d}\hat{\mathbf{e}}_3}{\mathrm{d}t} \right) + I_3 \hat{\mathbf{e}}_3 \omega_3,$$

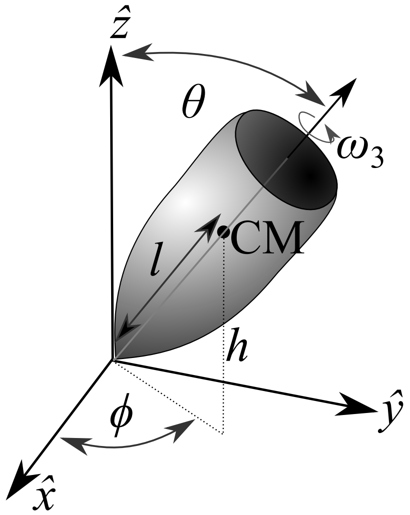
where $\hat{\mathbf{e}}_3$ is the unit vector along the 3-axis, and $\omega_3$ is the angular velocity about it. [4]

Show that, when the top spins on its tip, $\hat{\mathbf{e}}_3$ satisfies the equation of motion

$$I_3 \omega_3 \dot{\hat{\mathbf{e}}}_3 + I_1 \hat{\mathbf{e}}_3 \times \ddot{\hat{\mathbf{e}}}_3 = -mg\ell\, \hat{\mathbf{e}}_3 \times \hat{\mathbf{z}},$$

given that the centre of mass is also on the 3-axis a distance $\ell$ from the tip, as shown in the diagram; $g$ is the acceleration due to gravity and $m$ is the top's mass. [3]

[*You may assume that $\omega_3$ is constant.*]




The top precesses steadily, at fixed $\theta$ and constant $\dot{\phi} = \Omega_{\mathrm{P}}$. By considering the $y$ component of the equation of motion at a time when $\phi = 0$ (or otherwise), show that the possible values of $\Omega_{\mathrm{P}}$ are

$$\Omega_{\mathrm{P}} = \frac{I_3 \omega_3 \pm \sqrt{I_3^2 \omega_3^2 - 4mg\ell I_1 \cos\theta}}{2 I_1 \cos\theta}.$$ [5]

Discuss the limits of the precession frequency in the two cases $I_3^2 \omega_3^2 \gg 4mg\ell I_1 \cos\theta$, and $g = 0$. [3]

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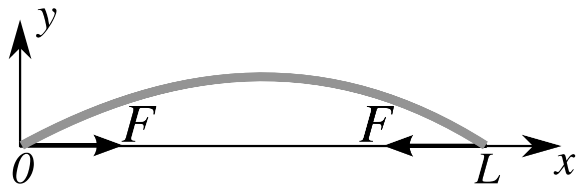
C12 A thin elastic beam has Young's modulus $E$. Show that if the beam is straight when unstressed, the *bending moment* $B$ at a point on the beam where the radius of curvature is $R$ is given by

$$B = \frac{EI}{R},$$

where you should define the quantity $I$. [5]

A beam of length $L$ with fixed endpoints is compressed longitudinally under forces $F$ as shown:




Find a differential equation satisfied by the transverse deviation of the beam $y(x)$ if $y(x) \ll L$. Discuss what happens as $F$ is increased from zero. [4]

A different beam of the same material, length and cross-section has an *unstressed* configuration

$$y_0(x) = Y \sin\left(\frac{\pi x}{L}\right),$$

where $Y \ll L$. Show that the *additional* transverse deviation of the beam $\eta(x)$, when compressed by the forces $F$, satisfies

$$EI\eta'' = -F(y_0 + \eta). \qquad [5]$$

Solve this equation to find how $\eta(x)$ depends on $F$, sketching the maximum displacement of the beam as a function of $F$. [6]

C13 For irrotational flow of an incompressible fluid, explain why it is possible to define a velocity potential such that $\mathbf{v} = \nabla \Phi$, where $\Phi$ satisfies Laplace's equation. [4]

An incompressible fluid of density $\rho$ fills the region $z < Z(x, y, t)$, where the surface profile

$$Z(x, y, t) = h_0 \sin(\omega t) \sin(kx)$$

describes a standing wave of small amplitude. Show that the velocity potential

$$\Phi(x, y, z, t) = \frac{h_0 \omega}{k} \cos(\omega t) \sin(kx) e^{kz}$$

describes incompressible flow, with $v_z$ matching the value of $\frac{\partial Z}{\partial t}$ at $z = 0$. [4]

Calculate the kinetic energy of the fluid per unit surface area

$$T = \frac{\rho}{2} \int_{-\infty}^{0} |\mathbf{v}|^2 dz,$$

where the upper limit can be taken as 0 rather than $Z(x, y, t)$ in the small-amplitude approximation. [5]

Show that the spatial average of the excess gravitational potential energy per unit surface area due to the wave is $\frac{1}{4}\rho g h_0^2 \sin^2(\omega t)$. [4]

Find the relationship between $\omega$ and $k$. [3]

END OF PAPER