UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Thursday 30th May 2013 1.30 to 4.30 pm

PHYSICS A (2)

Attempt **all** *questions from Section A,* **two** *questions from Section B,* **one** *question from Section C and* **one** *question from Section D.*
Section A will carry approximately 20% of the total marks.
In Sections B, C and D each question carries approximately the same number of marks.
The approximate number of marks allocated to each question or part of a question is indicated in the right margin.
The paper contains 7 sides including this one and is accompanied by a Mathematical Formulae Handbook giving values of constants and containing mathematical formulae which you may quote without proof.
Answers from **each** *Section must be written in separate Booklets.*
Write the letter of the Section and your candidate number, not your name, on the cover of **each** *Booklet. If several Booklets are used for a given Section join them together using a Treasury Tag.*
A separate master (yellow) cover sheet should also be completed, listing all questions attempted in the paper.

STATIONERY REQUIREMENTS
Booklets and Treasury tags
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Physics Mathematical Formulae Handbook (supplied)
Students are permitted to bring an approved calculator

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Attempt **all** *questions from this Section. Answers should be concise and relevant formulae may be assumed without proof; each Section to be answered in a separate booklet.*

1 An interferometric array of radio telescopes operates at a frequency of 1.5×10^{10} Hz and has a longest baseline of 100 m and a shortest baseline of 5 m. Estimate the angular resolution of the array and the largest angular scale to which it is sensitive. [4]

2 Explain why many spectacle lenses appear slightly purple. [4]

3 The speed of sound through air is measured to be $320\,\mathrm{ms}^{-1}$. Estimate the air temperature.

[*Hint: You may assume that air pressure p is given by $p = nk_BT$, where n is the number density of molecules, T the temperature, and k_B Boltzmann's constant.*]

[4]

4 Explain the difference between fcc and bcc crystal lattices. How do these lattices look in reciprocal space? [4]

5 Explain the Hall effect, and show how, by measuring the Hall coefficient, one can deduce the average number of free electrons per atom. [4]

SECTION B

Attempt **two** *questions from this Section. Each Section to be answered in a separate booklet.*

B6 Given that the product Re$\{A\}$Re$\{B\}$ can be written as $\frac{1}{2}(A + A^*)\frac{1}{2}(B + B^*)$, where A and B are complex numbers and * denotes the complex conjugate, show that the time-averaged power for an oscillator subject to a force Re$\{F\}$ and having velocity Re$\{u\}$, where $F = F_0 e^{i\omega t}$ and $u = u_0 e^{i(\omega t+\phi)}$, is $\frac{1}{2}$Re$\{Fu^*\}$. [4]

A light string, of mass ρ per unit length, is stretched along the x-axis under tension T. Show that small *transverse* displacements $\psi(x, t)$ satisfy the wave equation

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2},$$

and give an expression for v in terms of ρ and T. [5]

Using the representation $\psi(x, t) = A_0 e^{i(\omega t - kx)}$ for a transverse wave, show that the average power carried along the string by the wave is $\frac{1}{2}Z\omega^2 A_0^2$ and briefly describe the meaning of Z. [4]

State the conditions satisfied by ψ and $\partial\psi/\partial x$ at any point on the string. [2]

State the general expressions for the reflected and transmitted powers at an impedance boundary. [2]

What are these reflected and transmitted powers if: (a) the string density changes from ρ_1 at $x < 0$ to ρ_2 at $x > 0$; (b) the string is constrained in such a way that $\partial\psi/\partial x = 0$ at $x = 0$ but ψ is otherwise unconstrained? [3]

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B7 The equation for a damped, driven, harmonic oscillator can be represented by $m\ddot{x} + b\dot{x} + kx = \mathrm{Re}\{F_0 e^{i\omega t}\}$. Explain the terms in this equation. [2]

By substituting the trial solution $x = \mathrm{Re}\{Ae^{i\omega t}\}$ into the above equation, show that $A = R(\omega)F_0$, where $R(\omega)$ is the response function [2]

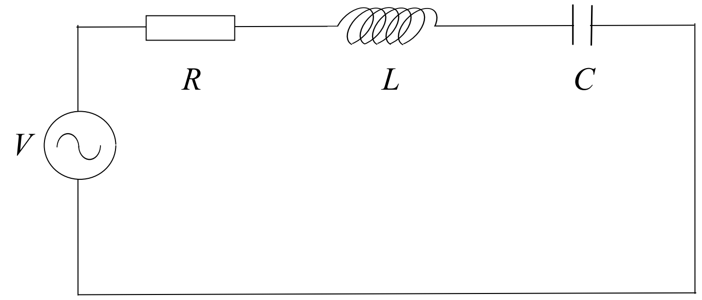
$$R(\omega) = \frac{1}{k - m\omega^2 + i\omega b}.$$

Show that the angular frequency at which amplitude resonance occurs is [4]

$$\omega_{\mathrm{res}} = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}.$$

Draw an annotated sketch of the magnitude of $R(\omega)$ as a function of ω, and indicate clearly how $R(\omega)$ changes for different levels of damping. [4]

For the circuit shown below, in which $V = \mathrm{Re}\{V_0 e^{i\omega t}\}$, show that the equation for



the charge on the capacitor can be represented by [3]

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = V.$$

If ω_{max} is the angular frequency at which the root-mean-square voltage across the resistor R is a maximum, show that, for light damping, [5]

$$\omega_{\mathrm{max}} - \omega_{\mathrm{res}} = \frac{R^2}{4L}\sqrt{\frac{C}{L}}.$$

B8 Explain the term *wavepacket*, and define *phase velocity* and *group velocity*. [3]

For electromagnetic waves of frequency f in a dispersive medium of refractive index $n(f)$, the phase velocity is $c/n(f)$, where c is the speed of light in vacuum. Show that the group velocity u_g is given by [4]

$$\frac{c}{u_g} = n(f) + f\frac{\mathrm{d}n(f)}{\mathrm{d}f}.$$

A communications transmitter sends pulses of radiation of centre frequency $f_0 = 2 \times 10^{14}$ Hz and duration 10^{-10} s at a rate of 10^9 per second into an optical fibre. The fibre has refractive index $n(f) = 1.5 + \tau(f - f_0)$ where $\tau = 3 \times 10^{-16}$ s. A receiver is connected to the other end of the fibre.

Justify the fact that the range of frequencies present in each transmitted pulse covers around 10^{10} Hz. [1]

Find the group velocity at the centre frequency, and estimate the range of group velocities present in each transmitted pulse. [4]

Explain what limits the useable length of this optical fibre, and estimate the maximum useable length in kilometres. [6]

A longer useable length could be obtained by fitting repeaters at intervals along the fibre. What must each of these repeaters do? [2]

B9 A thin opaque sheet in the x-y plane contains a transparent aperture. The aperture is described by an aperture function $h(x)$, where x is the horizontal direction. The aperture is infinitely long in the vertical y direction. If this aperture is illuminated by plane-wave, monochromatic light of wavelength λ travelling in the z-direction, state the condition that the resulting diffraction is Fraunhofer rather than Fresnel. [2]

Starting from the conditions for the *Fraunhofer* regime, show that the light intensity that appears on a screen placed parallel to and after the aperture plane depends on a Fourier transform of $h(x)$. [5]

Consider a particular system illuminated as described, where the aperture is a slit of width b in the x-direction. A very thin layer of glass, which reverses the phase of the light passing through it, is placed over the left-hand half of the slit. Show that the resulting intensity diffraction pattern on the screen in the Fraunhofer regime has an angular dependence proportional to

$$\frac{1}{\sin^2\theta}\sin^4\left(\frac{kb\sin\theta}{4}\right),$$

where $k = 2\pi/\lambda$ and θ is the angle, at the aperture centre and in the x, z plane, between the z axis and a point on the screen. [7]

Draw an annotated sketch of this diffraction pattern. [4]

How would the diffraction pattern change if the thickness of the glass layer were doubled? [2]

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SECTION C

Attempt **one** *question from this Section. Each Section to be answered in a separate booklet.*

C10 Outline the assumptions of the free-electron model of a solid. [3]

Assuming that electrons are confined to a 2D plane (as in graphene, for example), show that the density of states for the number of these electrons at each energy value is given by $g(\varepsilon) = (A/2\pi)(m/\hbar^2)$ where A is the area. [5]

Electrons are fermions and obey the Pauli exclusion principle. Calculate the maximum energy ε_F for N electrons on a plane of area A at $T = 0$ and sketch the diagram of occupied states. [6]

By considering the probability that a state of energy ε is occupied, outline the derivation of the Fermi-Dirac probability distribution

$$p(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/k_BT} + 1},$$

and demonstrate how the chemical potential μ is related to the Fermi energy ε_F at very low temperatures. [6]

C11 A string is composed of identical masses, each of mass m, each separated by a distance a, and connected by elastic springs of spring constant α. Show that the frequency of longitudinal waves in this string is given by

$$\omega(q) = \sqrt{\frac{4\alpha}{m}}\left|\sin\left(\frac{qa}{2}\right)\right| ,$$

where q is the wavevector. [5]

Sketch the dispersion relation of these waves and explain the meaning of the “first Brillouin zone”. [4]

Show how this analysis extends to the situation in which every second mass in this 1D lattice is replaced by a larger mass M. [*Hint: This can be considered as a 1D diatomic lattice with a period* $2a$.] In particular, show that the dispersion relation is

$$\omega^2 = \frac{\alpha}{Mm}\left[M + m \pm \sqrt{(M+m)^2 - 4Mm\sin^2(qa)}\right] ,$$

demonstrating how the optical and acoustic phonon modes emerge, and explaining the difference in atomic motion between these two modes. [6]

Sketch the dispersion relation and show that the gap at the edge of the first Brillouin zone is equal to $\Delta\omega = \sqrt{2\alpha/m} - \sqrt{2\alpha/M}$. [3]

What would change in the spectra of phonon modes if the diatomic lattice were three-dimensional? [2]

SECTION D

Attempt **one** *question from this Section. Each Section to be answered in a separate booklet.*

D12 Explain the key principles and approximations underlying the Debye theory of heat capacity in solids. In particular, show why the heat capacity according to this theory is proportional to T^3 at low temperature. [20]

D13 Write brief notes on **two** of the following:

(a) Fresnel diffraction from circular structures; [10]

(b) the Fabry-Perot interferometer; [10]

(c) diffraction patterns from diffraction gratings. [10]

END OF PAPER