UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 3rd June 2017 9.00 to 12:00 noon

PHYSICS B (Paper 1)

*Attempt **all** questions from Section A, **two** questions from Section B, and **two** questions from Section C.*

*Section A as a whole carries **approximately** one fifth of the total marks.*

Each question in Sections B and C carries the same mark.

*The **approximate** number of marks allocated to each part of question in all Sections is indicated in the right margin.*

*Answers for each Section **must** be written in separate Booklets.*

Write the letter of the Section on the cover of each Booklet.

*Write your candidate number, **not** your name, on the cover of each Booklet.*

A single, separate master (yellow) cover sheet should also to be completed listing all questions attempted.

STATIONERY REQUIREMENTS
12-page Booklets and Treasury Tags
Rough Work Pad
Blue Cover Sheets
Yellow Cover Sheet

SPECIAL REQUIREMENTS
Physics Mathematical Formulae Handbook (supplied)
Approved Calculators allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

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SECTION A

Answers should be concise and relevant formulae may be assumed without proof.

A1 A parallel-plate capacitor has capacitance C. Half the volume between the plates is filled with a dielectric that has a relative permittivity $\epsilon > 1$. What are the largest and smallest possible values of the capacitance of the resulting component? [4]

A2 Paraboloid mirrors can be made by spin casting, which involves rotating molten glass and its container about a vertical axis as the glass solidifies. What angular velocity is required to create a mirror with focal length 1 m? [4]
[*The equation of a parabola of focal length f is $z = r^2/4f$.*]

A3 An electrostatic potential in cylindrical polar coordinates (ρ, ϕ, z) is given by $V = A\rho^2 \cos 2\phi$, where A is a constant. Sketch the form of the electric field $\boldsymbol{E}$ corresponding to this potential. [4]

A4 The density of ice is 0.92×10^3 kg m^{-3} and its latent heat of fusion is 3.3×10^5 J kg^{-1}. Estimate the melting point of ice under the skates of an ice skater of mass 50 kg assuming that the total contact area of their skates with the ice is 10^{-4} m^2. [4]

A5 Estimate the time taken for the cold air to fall out of a domestic refrigerator when the door is opened. [4]

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SECTION B

B6 Discuss the *method of images* in electrostatics, and give one example of its use. [4]

Using Gauss's theorem, show that the potential at a radius r due to an infinitely long line charge λ per unit length is given by [3]

$$V = -\frac{\lambda}{2\pi\epsilon_0}\ln r + \text{constant}.$$

An infinitely long conducting cylinder of radius a, with a charge λ per unit length, is placed with its axis a distance d from an infinite conducting plane, which is kept at zero potential. Show that two infinitely long line charges λ and $-\lambda$ per unit length, positioned at a distance of $\sqrt{d^2 - a^2}$ on either side of the plane, produce the appropriate potential in the region between the cylinder and the plane. [6]

Hence show that the potential of the cylinder is [4]

$$V = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{d + \sqrt{d^2 - a^2}}{a}\right).$$

A power cable of radius 20 mm is at a potential of 11 kV and at a height of 10 m above the ground. Estimate the electrostatic force on the cable, and specify its direction. [3]

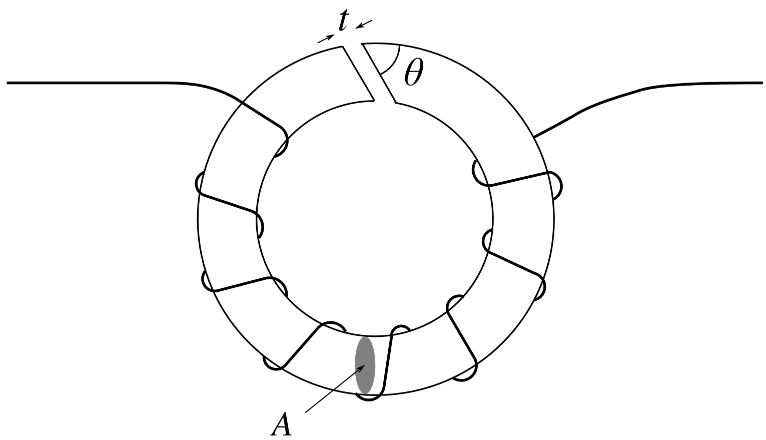
B7 Define *self-inductance* and state the boundary conditions for magnetic flux density $\boldsymbol{B}$ and magnetic field strength $\boldsymbol{H}$ at the boundary between two different materials. [4]

A complete iron torus of cross-sectional area A and relative permeability μ ($\gg 1$) is wound with N turns of insulated wire. Show that the self-inductance of the wire coil is [5]

$$L = \frac{N^2 A \mu \mu_0}{\ell}$$

where ℓ is the length around the torus. You may assume that $\ell \gg$ the width of the torus.

A slot of width t is now cut in the torus at an angle θ as shown. The width of the torus is $\gg t$, but note that μt may be comparable to ℓ.



Explain, without detailed calculation, why the self-inductance of the coil is now less than when the torus was complete. [2]

Find the new self-inductance of the coil when $\theta = 45°$. [6]

Find the magnitude of $\boldsymbol{H}$ in the slot when a current I flows in the wire, and comment on its direction. [3]

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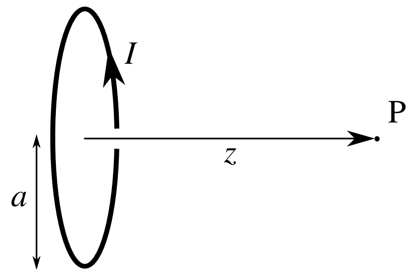
B8 State the *Biot–Savart law*, and use it to derive the magnetic flux density $\boldsymbol{B}$ at the centre of a circular loop of radius a that carries a current I. [4]

Calculate $\boldsymbol{B}$ on the axis of the loop, at point P a distance z from the loop, as shown. [4]

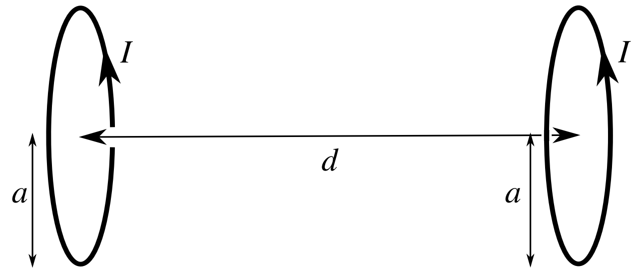


Explain, including a sketch, the directions of the $\boldsymbol{B}$-field in the region around the point P. [3]

Show that the radial component of $\boldsymbol{B}$ at a small distance ρ from the axis of the loop is given by [4]

$$B_\rho \approx \frac{3\mu_0 I a^2 z\rho}{4(z^2+a^2)^{5/2}}.$$

A second loop, identical to the first, and also carrying a current I in the same direction, is placed with its axis coincident with the axis of the first loop, as shown.



The loops are separated by a distance $d \gg a$. By considering the components of the magnetic flux density due to the first loop, show that the magnitude of the force on the second loop is [3]

$$F \approx \frac{3\pi\mu_0 I^2 a^4}{2d^4}.$$

In which direction is the force? [2]

[*In cylindrical polar coordinates* (ρ, ϕ, z)*:*

$$\nabla \cdot \boldsymbol{B} = \frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}.$$
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B9 Write down Maxwell's equations in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$ for a non-magnetic dielectric medium; i.e. one with a relative permittivity ϵ, and with a relative permeability $\mu = 1$. [4]

Show that an electromagnetic wave of the form

$$\boldsymbol{H} = (0, H_y, 0)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\mathrm{e}^{-Kz},$$

$$\boldsymbol{E} = (E_x, 0, E_z)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\mathrm{e}^{-Kz},$$

where H_y, E_x and E_z are constants, satisfies all of Maxwell's equations for a non-magnetic dielectric medium provided that [6]

$$k^2 - K^2 = \frac{\omega^2 \epsilon}{c^2}.$$

The region $z < 0$ is filled with a material with a relative permittivity ϵ, and the region $z > 0$ is free space (i.e. $\epsilon = 1$). An electromagnetic 'surface' mode has $\boldsymbol{H}$ and $\boldsymbol{E}$ fields of the forms given above, with $K = B_1$ (> 0) for $z > 0$ and $K = \beta_2$ (< 0) for $z < 0$. By considering the boundary conditions for $\boldsymbol{E}$ and $\boldsymbol{H}$ at $z = 0$, show that [3]

$$B_1 \epsilon = \beta_2.$$

Hence show that the surface mode can only exist if $\epsilon < -1$, and that its dispersion relation is [3]

$$k^2 = \frac{\omega^2}{c^2}\frac{\epsilon}{\epsilon + 1}.$$

If $\epsilon = 1 - \omega_{\mathrm{p}}^2/\omega^2$, where ω_{p} is a constant, find the maximum possible ω for which a surface mode exists, and determine the relationship between k and ω when $\omega \ll \omega_{\mathrm{p}}$. [4]

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SECTION C

C10 Show with the aid of a diagram that the rates of change of a vector $\boldsymbol{A}$ in a frame S rotating at angular velocity $\boldsymbol{\omega}$ and in an inertial frame S_0 differ by $\boldsymbol{\omega} \times \boldsymbol{A}$. Hence show that the acceleration of an object in S_0 is given by

$$\ddot{\boldsymbol{r}}_0 = \boldsymbol{a} + 2\omega \times \boldsymbol{v} + \omega \times (\omega \times \boldsymbol{r}),$$

where $\boldsymbol{r}$, $\boldsymbol{v}$ and $\boldsymbol{a}$ are the apparent position, velocity and acceleration, respectively, in S. [6]

What Coriolis force acts on an object moving due south on the Earth's surface at speed v? [2]

A plane sets out to fly at 800 km hr $^{-1}$ from London to a destination due south at latitude 16°S. Show that, to compensate for the Coriolis force, the pilot points the nose of the plane east of south at an angle ϕ, given by [4]

$$\phi = \sin^{-1}\left(\frac{2m\Omega v \sin\lambda}{T}\right),$$

where m, v and T are the mass, velocity and thrust of the plane, Ω is the Earth's angular velocity, and λ is the latitude. Assume that the plane is travelling in still air and that the drag is anti-parallel to the plane's velocity.

At the equator the co-pilot takes over, keeps the velocity of the plane constant and remembers to correct for the Coriolis force, but gets the sign of the correction wrong. By considering the horizontal forces on the plane, calculate by what distance the plane will miss the destination. [8]

[*The radius of the Earth is 6400* km.]

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C11 Write down and explain the effective potential for a particle moving under the influence of a central force. [3]

A planet of mass m moves in a bound orbit about a fixed star of mass M according to the central force law

$$f(r) = -\frac{GMm}{r^2}\left(1 + \frac{\gamma}{r}\right),$$

where γ is a constant. Show that the *shape* of the orbit can be described by the differential equation [4]

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 - \left(\frac{2mE}{J^2}\right)r^4 - \left(\frac{2mA}{J^2}\right)r^3 + \left(1 - \frac{mA\gamma}{J^2}\right)r^2 = 0,$$

where $r(\phi)$ is the radius at angle ϕ, $A = GMm$, E is the energy and J is the angular momentum.

An attempt was made to explain the precession of Mercury's orbit by 43 seconds of arc per century by a modification of the Newtonian law of gravity of the above form. By means of the substitution $u = 1/r$, or otherwise, show that [6]

$$\frac{\alpha_1}{r} - \alpha_2 = \cos(\alpha_3\phi),$$

where α_1, α_2 and α_3 are particular constants, is a solution of the equation for the shape of the orbit.

Show that α_3 is given by

$$\gamma = \frac{J^2(1-\alpha_3^2)}{mA},$$

and hence, if α_3 deviates slightly from 1, find approximately the value of γ that could explain the observed precession. [7]

[*Assume that the eccentricity of Mercury's orbit is small, that the period of the orbit is 88 days and that the mass of the Sun is* 2×10^{30} kg.]

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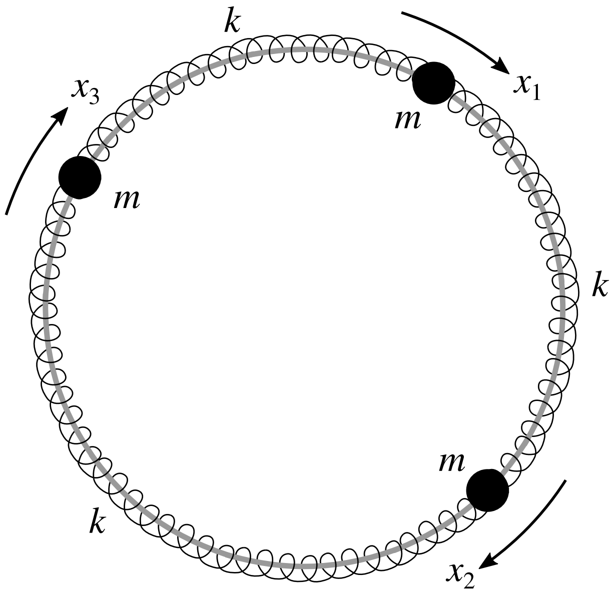
C12 Briefly discuss normal modes in mechanical systems and explain why these are applicable to many real systems. [3]

A simple model of a molecule consists of three beads of equal mass, m, on a circular wire, connected by three light springs of equal spring constant k coiled around the wire. The beads and springs can move without friction along the wire, which is sufficiently heavy that it may be assumed to remain stationary. The positions of the beads are given by their displacements x_1, x_2, x_3 along the wire from an equilibrium configuration.



Write down the Lagrangian for the system. Using the Euler–Lagrange equations, find equations of motion for each of the beads. [5]

Find the normal-mode frequencies for the system. [3]

Show that a general expression for the motion of the system is

$$\boldsymbol{x}(t) \equiv \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} A\cos(\omega t + \phi) + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} B\cos(\omega t + \psi) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (Ct + D),$$

where ω, A, B, C, D, ϕ and ψ are constants. [3]

Sketch and describe the form of the motion corresponding to each of the terms on the right-hand side of this expression. Are the eigenvectors unique? [2]

Find the values of the constants if $\boldsymbol{x}(0) = (a, 0, 0)$ and $\dot{\boldsymbol{x}}(0) = (0, 0, 0)$ for some constant a. [4]

C13 A uniform elastic isotropic medium with Young's modulus E and Poisson's ratio σ is subject to stresses τ_1, τ_2, and τ_3 along a set of orthogonal axes. Write down the strain e_1 in terms of τ_1, τ_2, τ_3, E and σ, and explain the physical significance of σ. [4]

A long pipe of internal radius a and external radius b is made of an elastic material with Young's modulus E and Poisson's ratio σ. The pipe contains gas at high pressure p_1, and the external pressure is p_0. There is no longitudinal stress in the pipe.

Show that the radial strain e_r and the tangential strain e_ϕ are [2]

$$e_r = \frac{\partial R}{\partial r} \quad \text{and} \quad e_\phi = \frac{R}{r},$$

where $R(r)$ is the radial displacement at radius r.

Also show that the tangential stress τ_ϕ is related to the radial stress τ_{r} by [3]

$$\tau_\phi = \frac{\mathrm{d}}{\mathrm{d}r}(r\tau_r).$$

Show that the radial displacement satisfies [4]

$$r^2\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} - R = 0.$$

Hence, by looking for power-law solutions, find the radial pressure $p(r)$ in the wall of the pipe. [4]

What is the maximum absolute value of the tangential stress? [3]

END OF PAPER

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