UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 25th May 2019 1.30 pm to 4.30 pm

PHYSICS A (1)

Attempt **all** *questions from Section A,* **two** *questions from Section B,* **one** *question from Section C, and* **one** *question from Section D.*

Section A will carry approximately 20% of the total marks.

In Sections B, C, and D, each question carries approximately the same number of marks.

The approximate number of marks allocated to each question or part of a question is indicated in the right margin.

The paper contains **7** *sides including this one and is accompanied by a Mathematical Formulae Handbook giving values of constants and containing mathematical formulae, which you may quote without proof.*

Answers from **each** *Section should be written in separate Booklets.*

Write the letter of the Section and your candidate number, not your name, on the cover of **each** *Booklet.*

A separate master (yellow) cover sheet should also be completed, listing all questions attempted in the paper.

STATIONERY REQUIREMENTS

Booklets

Rough workpad

Yellow master coversheet

SPECIAL REQUIREMENTS

Mathematical Formulae Handbook

Approved calculator

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Attempt **all** *questions from this Section. Answers should be concise and relevant formulae may be assumed without proof. Use a separate booklet for the whole of this section.*

1 An electron is confined to a cube of edge 1 nm. Use the uncertainty principle to estimate the minimum kinetic energy of the electron. [4]

2 Write down the kinetic energy of a rotating object in terms of its angular momentum L and moment of inertia I. Hence find the energies, in eV, of the three lowest energy levels for an object with $I = 10^{-47}$ kg m^2. [4]

3 An electron of kinetic energy 8 eV and zero potential energy approaches a square potential barrier of height 10 eV and width 1 nm. Estimate the probability that the electron will tunnel through the barrier. [4]

4 The current through a resistor $R = 1$ GΩ at an applied voltage $V = 1$ mV is measured for short periods lasting 1 s. Calculate the expected current and estimate how many measurements are necessary to determine the average current to a precision of $\leq 10^{-5}$. [4]

5 The numerical aperture N of a microscope objective is given by $N = n \sin \alpha$, where n is the refractive index of the immersion fluid and α the opening angle. n is known to be 1.35 ± 0.075 while α is known to be around 60°. To what precision do you need to determine α so that N is known to within 6%. [4]

SECTION B

Attempt **two** *questions from this Section. Use a separate booklet for the whole of this section.*

B6 The matrices for the three components of the spin of a spin-half particle are

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that $\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{3}{4}\hbar^2 I$ where I is the identity matrix and that $\hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x = i\hbar\hat{S}_z$. [4]

(b) Find the normalised eigenvectors, and corresponding eigenvalues of $\hat{S}_x$ written in terms of the eigenvectors of $\hat{S}_z$. [4]

(c) A spin-half system is in a magnetic field B oriented in the x-direction. The corresponding part of the Hamiltonian is $\hat{H} = \lambda B\hat{S}_x$ where λ is a constant. At time $t = 0$, the spin wavefunction and corresponding eigenvalue are given by [6]

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{\hbar}{2}.$$

Determine the spin wavefunction at subsequent times. If a measurement of the z-component of the spin is made, find the probability that the result $\frac{1}{2}\hbar$ is obtained.

(d) Describe the Stern-Gerlach experiment and how it could be used to demonstrate the rotation of spins in a magnetic field. [6]

B7 The ground state wavefunction of a single electron atom with nuclear charge Ze is given by $\psi(r) = \alpha\exp(-\beta r)$ where α and β are constants and r is the distance between the electron and the nucleus.

(a) Show that $\psi(r)$ is an eigenstate of the time-independent Schrödinger equation and that $\alpha^2 = \beta^3/\pi$ and $\beta = Z/a_0$ where $a_0 = 4\pi\varepsilon_0\hbar^2/m_e e^2$. Find the energy E of the wavefunction $\psi(r)$. [6]

(b) Show that the expectation values for the potential and kinetic energies of ψ are $2E$ and $-E$ respectively. [4]

(c) Find expressions for the mean value and most probable value of r. [4]

(d) An atom of tritium (^{3}H) in its ground state decays to ^{3}He$^+$. By considering overlap between the initial and final wavefunctions calculate the probability that the resulting helium atom will be left in the 1s state. Explain why a transition to the 2p state is unlikely. [6]

[*The following equations may be helpful*

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\hat{L}^2}{\hbar^2 r^2}, \qquad \int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}} \quad (a > 0; n > 0)$$
]

B8 A particle of mass m is in a one-dimensional potential well with infinite walls described by $V(x) = 0$ for $0 \le x \le L$ and $V(x) = \infty$ for both $x < 0$ and $x > L$.

(a) Use the Schrödinger equation to find the wavefunction ψ_n and energy E_n of the quantum state n. [4]

(b) Sketch the probability densities for the lowest two energy levels as a function of position. How will this picture change if the walls of the well are finite? [4]

(c) The wavefunction of the particle is a linear sum of the ground state and the first excited state wavefunctions, with equal amplitudes. Find the expectation value of the energy. Show that the expectation value of the position of the particle oscillates. Find the oscillation frequency and amplitude. [7]

(d) What is the force exerted by the oscillating particle in (c) on the walls of the well? How does this compare with the time-averaged force exerted by a classical particle of the same energy that bounces from wall to wall? [5]

B9 The Hamiltonian for a particle of mass m in a one-dimensional harmonic oscillator potential is

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{x}^2.$$

Two operators are defined

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{1}{\sqrt{2m\hbar\omega}}\hat{p}, \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\frac{1}{\sqrt{2m\hbar\omega}}\hat{p}.$$

(a) Show that $\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$. Evaluate the commutators $\left[\hat{a}, \hat{a}^\dagger\right]$, $\left[\hat{H}, \hat{a}\right]$ and $\left[\hat{H}, \hat{a}^\dagger\right]$. [5]

(b) If $|n\rangle$ is a normalised eigenstate of $\hat{H}$ with eigenvalue $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$ for $n = 0, 1, 2, 3...$, find expressions for $\hat{a}\,|n\rangle$ and $\hat{a}^\dagger\,|n\rangle$ in terms of energy eigenstates. [5]

(c) If $\hat{a}\,|0\rangle = 0$ find wavefunctions for the ground state and first excited state and show that they are orthogonal. [6]

(d) If the particle is charged, an oscillating electric field in the x-direction can induce a transition $|n_1\rangle \to |n_2\rangle$. The probability of this transition is proportional to $|\langle n_2|\,\hat{x}\,|n_1\rangle|^2$. Find the relationship required between n_1 and n_2 for a transition to take place. [4]

SECTION C

Attempt **one** *question from this Section. Use a separate booklet for this section.*

C10 Write an essay on the time evolution of (Gaussian) wave packets in quantum mechanics. [20]

C11 Write brief notes on **two** of the following:

(a) orbital angular momentum in quantum mechanics; [10]

(b) the correspondence principle and time evolution of operators in quantum mechanics; [10]

(c) measurements in quantum mechanics and the consequences of the non-commutation of operators. [10]

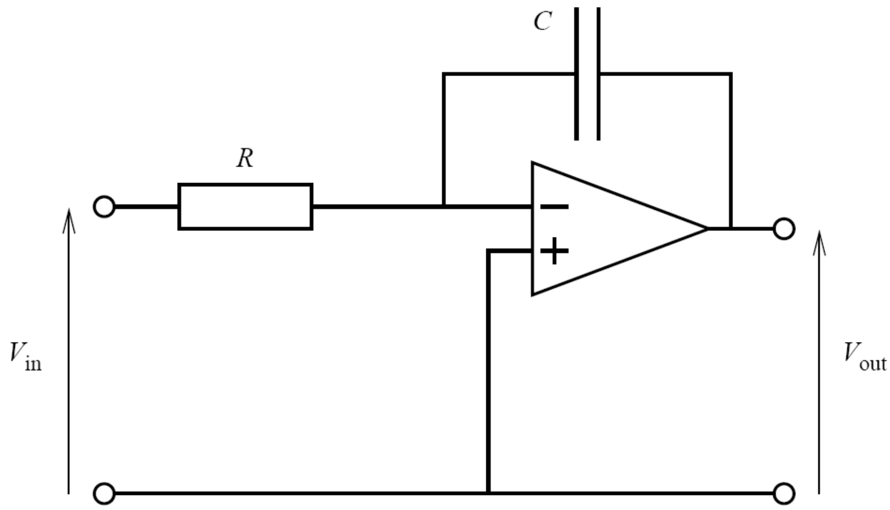
SECTION D

Attempt **one** *question from this Section. Use a separate booklet for this section.*

D12 Operational amplifiers are important in measuring AC and DC voltages.

(a) State the characteristics of ideal operational amplifiers including the so-called golden rules. [3]

(b) A circuit based on an ideal operational amplifier is connected with a resistor R and a capacitor C to a voltage source V_{in} as shown below. Derive the relationship between V_{out} and V_{in} and indicate which mathematical operation is performed by this circuit. [6]



(c) Comment on possible problems with this circuit and methods to alleviate them. [2]

(d) The ideal operational amplifier is replaced with a non-ideal version. State the differences between ideal and non-ideal operational amplifiers. [4]

(e) Sketch the circuit shown above with a non-ideal amplifier. If the non-ideal circuit is provided with a voltage of angular frequency ω and amplitude V_{in}, derive the ratio V_{out}/V_{in} as a function of ω for the non-ideal amplifier. You may assume that the internal resistance at the inputs of the amplifier is infinite. Sketch your result for V_{out}/V_{in} and the corresponding phase as a function of ω indicating relevant regimes. [5]

D13 Write brief notes on **two** of the following:

(a) the χ^2-test; [10]

(b) methods for shielding from or accounting for thermal fluctuations in experiments; [10]

(c) digital recording of analogue data. [10]

END OF PAPER