UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 2 June 2012 9.00 to 12.00 noon

PHYSICS B (Paper 1)

*Attempt **all** questions from Section A, **two** questions from Section B, and **two** questions from Section C.*

*Section A as a whole carries **approximately** one fifth of the total marks.*

Each question in Sections B and C carries the same mark.

*The **approximate** number of marks allocated to each part of a question in all sections is indicated in the right margin.*

*Answers for each Section **must** be written in separate Booklets.*

Write the letter of the Section on the cover of each Booklet.

*Write your candidate number, **not** your name, on the cover of each Booklet.*

A single, separate master (yellow) cover sheet should also be completed, listing all questions attempted.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Booklets and Treasury Tags	Physics Mathematical Formulae Handbook (supplied)
Rough Work Pad	Approved Calculators allowed
Yellow Cover Sheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Answers should be concise and relevant formulae may be assumed without proof.

A1 The rate at which a certain chemical reaction proceeds increases from R_0 to $10R_0$ when the temperature is raised from 0°C to 20°C. Determine the rate at 40°C. [4]

A2 A capacitor consists of two parallel circular plates of radius a separated by an air gap of width d, where $d \ll a$. The capacitor is connected to a low frequency voltage source, $V(t) = V_0 \sin \omega t$. Find an expression for the magnetic flux density $\boldsymbol{B}$ as a function of radius in the region between the plates. [4]

A3 What is the lowest frequency of electromagnetic radiation which can propagate in a waveguide with a rectangular cross section of 10 mm × 20 mm? [4]

A4 The radius of the Sun is 7.0×10^5 km and its surface temperature is approximately 6000 K. Treating Mars as a black body in thermal equilibrium with the radiation from the Sun, estimate the mean surface temperature of Mars at its distance of closest approach to the Sun of 2.1×10^8 km. [4]

A5 A uniform solid cube of density ρ and side length $2a$ is rotating about an axis which passes through its centre of mass and two opposite corners. Find an expression for the magnitude of its angular momentum in terms of ρ, a and its angular speed ω. [4]

SECTION B

B6 Explain the use of *Laplace's Equation* and *the Uniqueness Theorem* in the solution of problems in electrostatics. [4]

Show that the potential difference between two points at respective distances r_1 and r_2 from a line charge λ per unit length which is of infinite extent is

$$V_{21} = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_1}{r_2}\right).$$ [4]

Two parallel line charges with charge densities $+\lambda$ and $-\lambda$ per unit length respectively lie parallel to the z-axis and intersect the x-axis at $x = b$ and $x = c$ respectively, with $b > 0,\ c > 0,$ and $b > c$. Show that a cylindrical surface of radius $a = \sqrt{bc}$ with its axis parallel to the z-axis and intersecting the x-axis at $x = 0$ is an equipotential with

$$V = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{c}{b}\right).$$ [7]

A long conducting cylinder of radius a lies with its axis at height $2a$ above an earthed conducting horizontal plane. Find an expression for the electrostatic force per unit length on the cylinder when it carries a charge λ per unit length. [5]

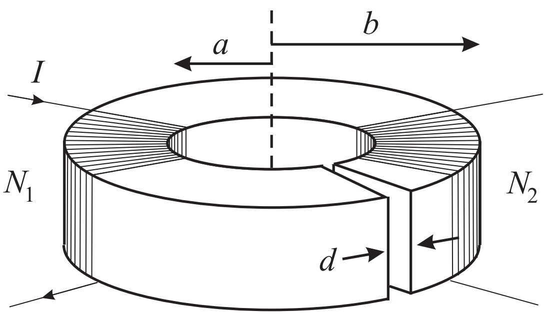
B7 Explain what are meant by *self inductance* and *mutual inductance*. [3]

A ring of material with relative magnetic permeability μ_r has inner and outer radii a and b and a square cross section. Two coils with N_1 and N_2 turns are wound closely onto the ring, and the coil with N_1 turns carries a current I. If $\mu_r \gg 1$, so that the resulting magnetic field is purely azimuthal and is contained entirely within the ring, calculate the magnetic field strength $H_\phi(r)$ in the ring. [3]

Hence, *without* assuming that $b - a \ll a$, calculate the self inductances L_1 and L_2 for the two coils, and confirm that their mutual inductance is given by $M = \sqrt{L_1 L_2}$. [6]

As a result of mechanical damage a crack of uniform width $d \ll a$ opens up across the entire cross section of the ring as shown below. Calculate the resulting fractional change in the mutual inductance M if $a = 1$ cm, $b = 2$ cm, $d = 1$ μm and $\mu_r = 10^5$. [6]

Explain briefly why the non-azimuthal components of the magnetic field are indeed zero. [2]



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B8 State the *Biot-Savart Law* for the magnetic flux density $\boldsymbol{B}$ arising from a current-carrying element. [2]

A coil with radius R has N_1 turns and lies in the xy-plane. If the coil carries a steady current I_1, show that the magnetic flux density at the point A on the vertical axis of the coil a distance z above its centre is

$$B_z(z) = \frac{\mu_0 N_1 I_1 R^2}{2(R^2 + z^2)^{3/2}} \,. \qquad [3]$$

Sketch the field lines of the magnetic flux density. [2]

At a small distance ρ ($\ll R$) off the axis near the point A, the diverging magnetic flux density has a component $B_\rho(z)$ perpendicular to the axis. Assuming that the change in $B_z(z)$ over the small distance ρ is negligible, calculate $B_\rho(z)$. [7]

A second circular coil is mounted coaxially with the first, with their centres a distance d apart. The second coil has radius a ($\ll R$) and N_2 turns, and carries a current I_2. Use the result for $B_\rho(z)$ to find an expression for the force on the second coil. [4]

Discuss your result in relation to the magnetic moment of the second coil. [2]

[*In cylindrical polar coordinates*

$$\nabla \cdot \boldsymbol{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \,.$$

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B9 A plane electromagnetic wave of angular frequency ω, propagating in the x-direction, is linearly polarised with its electric field $\boldsymbol{E}$ in the y-direction, and is incident on a thick slab of non-magnetic material with relative permittivity ε_r and conductivity σ ($\gg \omega\varepsilon_r\varepsilon_0$) which occupies the region $x \geq 0$. Show that within the slab the electric field takes the form

$$E_y = E_0 \exp\left(-\frac{x}{\delta}\right) \exp\left(i\left[\frac{x}{\delta} - \omega t\right]\right),$$

where the skin depth $\delta = \sqrt{\dfrac{2}{\sigma\mu_0\omega}}$. [5]

Find an expression for the wave impedance Z of the medium. [2]

Hence determine the associated magnetic field strength intensity $\boldsymbol{H}$, and show that the time-averaged Poynting flux in the slab has the form

$$\boldsymbol{N} = \left(N_0 e^{-2x/\delta}, 0, 0\right),$$

where N_0 is the time-averaged Poynting flux for $x = 0^+$, just inside the slab. Give an expression for N_0. [4]

The Lorentz force per unit volume is $\boldsymbol{J} \times \boldsymbol{B}$. Show that the wave exerts a time-averaged pressure p on the slab given by

$$p = \frac{\sigma\delta\mu_0}{2} N_0 .$$ [5]

Using your expression for Z, show that the power reflection coefficient of the material for normal incidence is

$$R \approx 1 - 2\sqrt{\frac{2\omega\varepsilon_0}{\sigma}} .$$ [4]

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SECTION C

C10 Explain what is meant by the *a normal mode* and discuss how normal modes can be used to describe the general motion of a system. [4]

A piece of equipment in a playground consists of a rigid beam of length l and mass m. It is raised from the ground by two identical light springs of spring constant k, one at each end of the beam. The beam is free to oscillate in the vertical plane. In equilibrium, both springs are compressed by a length x_0. The motion of the beam can be described by the vertical displacements of its two ends, x_1 and x_2, measured relative to their equilibrium positions. Taking the zero of the gravitational potential energy of the beam to be at the equilibrium position, show that the potential energy of the system is given by

$$V = \tfrac{1}{2}k(x_1^2 + x_2^2) + kx_0^2. \qquad [3]$$

Find an expression for the kinetic energy in terms of $\dot{x}_1$ and $\dot{x}_2$. [3]

Determine the frequencies and corresponding eigenvectors of the normal modes. [5]

A child of mass M sits on one end of the beam. At time $t = 0$ the child instantaneously slides off the beam such that no impulse is imparted. By considering the initial conditions, find an expression for the subsequent position of the centre-of-mass of the beam as a function of time. [5]

C11 Describe the types of orbit possible for a body in a central inverse-square force field for (i) an attractive force and (ii) a repulsive force, making it clear under which conditions the different orbits apply. [5]

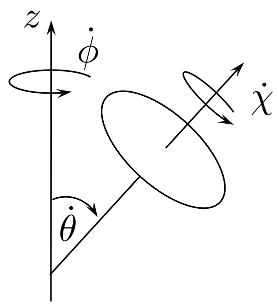
An alpha particle of mass m and charge $+2e$, moves in the electrostatic field of a heavy nucleus of charge $+Ze$ which remains at rest. The alpha particle is initially at a large distance from the nucleus and is travelling with speed $v \ll c$ along a path which if continued without deviation would pass a distance b from the nucleus. Draw a sketch of the trajectory indicating the distance b and the distance of closest approach, d. [2]

Find the distance of closest approach to the nucleus of the alpha particle. [7]

How would the answer change if the force were attractive. [2]

Consider the case where the κ/r^2 force law is replaced by a central κ/r^3 law. Explain why the angular momentum of the alpha particle is still a constant of the motion and derive a new expression for the distance of closest approach. [4]

C12 The motion of a symmetrical top of mass m which is supported at its base and is moving under gravity can be described by the Euler angles ϕ, θ and χ with corresponding angular velocities defined in the diagram below.



Show that the angular velocity $\boldsymbol{\omega}$ and the angular momentum $\boldsymbol{J}$ of the top are given by

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = (\dot{\theta}, \dot{\phi}\sin\theta, \dot{\phi}\cos\theta + \dot{\chi})$$
$$\boldsymbol{J} = (J_1, J_2, J_3) = (I_1\dot{\theta}, I_1\dot{\phi}\sin\theta, I_3\dot{\phi}\cos\theta + I_3\dot{\chi}),$$

where the indices 1, 2 and 3 denote the body axes of the symmetric top and the other symbols have their usual meanings. [4]

Explain why the energy E of the top and the components of its angular momentum J_3 and $J_z = J_3\cos\theta + J_2\sin\theta$ are constants of the motion. [2]

Show that E may be written in the form

$$E = \frac{1}{2}I_1\dot{\theta}^2 + U_{\text{eff}}(\theta),$$

where

$$U_{\text{eff}}(\theta) = \frac{(J_z - J_3\cos\theta)^2}{2I_1\sin^2\theta} + mgh\cos\theta + \frac{J_3^2}{2I_3},$$

and h is the distance from the support to the centre of mass of the top. [6]

Explain why steady precession can occur only when $\mathrm{d}U_{\text{eff}}/\mathrm{d}\theta = 0$ and use this condition to show that

$$I_1\cos\theta\,\dot{\phi}^2 - J_3\dot{\phi} + mgh = 0.$$

Hence find the minimum value of J_3 which will allow steady precession. [8]

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C13 Explaining any assumptions and defining the symbols involved, show that the equation of motion for the deflection $y(x, t)$ of a uniform elastic bar is

$$EI\frac{\partial^4 y}{\partial x^4} = -\rho\frac{\partial^2 y}{\partial t^2}\,.$$ [5]

A uniform elastic bar of length L is pivoted freely at both ends such that it satisfies the boundary conditions

$$y(0, t) = y(L, t) = 0 \quad \text{and} \quad \left.\frac{\partial^2 y}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 y}{\partial x^2}\right|_{x=L} = 0\,.$$

By writing $y(x, t) = X(x)T(t)$, show that if the bar is initially at rest the equation of motion has solutions

$$T(t) = \cos\omega t$$
$$X(x) = A\sin kx + B\cos kx + C\sinh kx + D\cosh kx\,,$$

where A, B, C and D are constants which depend on the initial conditions. Find the expression for the angular frequency ω in terms of k. [5]

Using the boundary conditions, find the normal frequencies ω_n of the bar. [6]

If the bar is initially bent into the form

$$y(x, 0) = ax(x - L)\,,$$

where a is a small constant, find an expression for the subsequent motion and discuss your result. [4]

[*You may find it useful to note that*

$$x(x - L) = -\frac{8L^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n + 1)^3}\sin\frac{(2n + 1)\pi x}{L}\,.$$

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END OF PAPER