UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 01 June 2019 9.00 to 12:00 noon

PHYSICS B (Paper 1)

Attempt ***all*** *questions from Section A,* ***two*** *questions from Section B, and* ***two*** *questions from Section C.*

Section A as a whole carries ***approximately*** *one fifth of the total marks.*

Each question in Sections B and C carries the same mark.

The ***approximate*** *number of marks allocated to each part of a question in all Sections is indicated in the right margin.*

Answers for each Section ***must*** *be written in separate Booklets.*

Write the letter of the Section on the cover of each Booklet.

Write your candidate number, ***not*** *your name, on the cover of each Booklet.*

A single, separate master (yellow) cover sheet should also to be completed listing all questions attempted.

STATIONERY REQUIREMENTS
12-page Booklets
Rough Work Pad
Yellow Cover Sheet

SPECIAL REQUIREMENTS
Physics Mathematical Formulae Handbook (supplied)
Approved Calculators allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

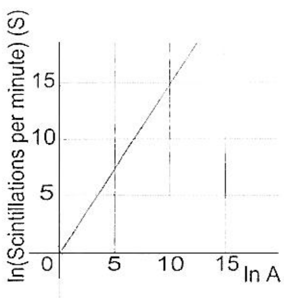
Answers should be concise and relevant formulae may be assumed without proof.

A1 A satellite orbits the Earth at an altitude of 120 km with a period of about 109 minutes. Estimate the radius of the orbit of a geostationary satellite. [4]
[*Take the radius of the Earth to be 6400 km.*]

A2 An object with potential energy Ar^3 at radius r (where A is a constant) moves in a circular orbit. Calculate the ratio of its kinetic to potential energy. [4]

A3 What pressure, in units of atmospheres, needs to be isotropically exerted on a 1 m^3 block of rubber with Young's modulus 1×10^{-3} GPa and Poisson ratio 1.47 in order to reduce the volume of the block by 30%? What is the shear modulus of the rubber? [4]

A4 In an experiment a 5.51 MeV alpha source was placed in front of a thin foil of lead (atomic weight $A = 207$ and atomic number $Z = 82$). Calculate the upper limit to the size of the lead nucleus in metres. [4]



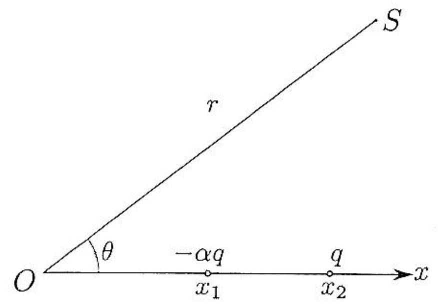
60 alpha particles per minute were observed to backscatter from the lead. Use the Figure and Rutherford's theory to estimate how many particles we would expect to backscatter from a thin foil of silver (atomic weight $A = 108$ and atomic number $Z = 47$).

A5 An ideal monatomic gas at temperature $T = 320$ K is adiabatically expanded so that the final volume is nine times larger than the original. Calculate the final temperature of the gas. [4]

SECTION B

Describe the method of images in electrostatics. Give two examples of its use. [5]

Two point charges, $-\alpha q$ and q, are placed at distances x_1 and x_2 from the origin on the x-axis as shown in the figure below.



Show that the electrical potential at a point S is given by [2]

$$\phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(r^2 + x_2^2 - 2x_2 r\cos\theta)^{1/2}} - \frac{\alpha}{(r^2 + x_1^2 - 2x_1 r\cos\theta)^{1/2}}\right),$$

where r is the distance from the origin O to S, θ is the angle between the x-axis and the line OS and ϵ_0 is the permittivity of free space.

A point charge q is held at a distance $x_2 > a$ from the centre of a spherical conductor of radius a. Show that, if the spherical conductor is earthed, the potential outside the sphere may be found by placing an image charge a distance $x_1 = a^2/x_2$ from the centre of the sphere, and determine the charge stored on the spherical conductor. [6]

Instead of the conductor being earthed, the net charge on the spherical conductor is now set to zero. Explain why, in order to represent the electric field outside the conductor, an additional image charge must now be included at the centre of the sphere, and determine the potential of the conductor. [4]

The distance x_2 is varied while adjusting q to maintain a constant value E_0 for the contribution to the electric field at the centre of the sphere from the external point charge. Show that the dipole moment of the image charge distribution is independent of x_2. [2]

Comment on what happens as $x_2 \rightarrow \infty$. [1]

B6 Starting from Maxwell's equations, demonstrate that Faraday's law and Lenz's law are obeyed in the case of a fixed loop of wire in a time-varying magnetic field. [3]

A rectangular loop made of wire with negligible resistance encloses an area A and has a self-inductance L. The loop is held fixed in a vertical plane and placed in a horizontal oscillating magnetic field $B = B_0 \sin \omega t$. The field makes an angle θ with the normal to the rectangle. Show that the current in the wire is [4]

$$I = -\frac{B_0 A}{L} \cos\theta \, \sin\omega t.$$

Draw one or more sketches showing, at a given instant, the directions of the magnetic field, the current flowing in the loop and the forces acting on the loop due to the field. [2]

Find the time-averaged torque acting on the loop from the magnetic field. [4]

The loop is now allowed to rotate freely about a vertical axis through its centre. Describe qualitatively the motion of the loop if it is released from rest at a value of θ such that $0 < \theta < \pi/2$. You may assume that the loop has a large moment of inertia. [4]

Describe qualitatively the motion of the loop in the case where the loop now has negligible self-inductance L but finite resistance R. [3]

B7 Starting from Maxwell's equations show that Ampère's law generalises to [3]

$$\oint \boldsymbol{H} \cdot \mathbf{d}\boldsymbol{l} = \int \boldsymbol{J} \cdot \mathbf{d}\boldsymbol{S} + \int \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathbf{d}\boldsymbol{S},$$

when the fields are time-varying.

A parallel-plate capacitor is formed from two large circular coaxial metal discs of radius a and separation d (where $d \ll a$). The capacitor is initially charged. A thin straight wire of high resistance R is connected between the centres of the discs and the capacitor discharges slowly.

Draw a labelled diagram showing the configuration of the electric and magnetic fields and the Poynting vector in the space between the discs as the capacitor discharges. [4]

Show that, when the potential difference between the plates is V, the magnetic field strength between the plates at a distance r from the wire (where $r \ll a$) is given by [6]

$$H = \frac{V}{2\pi r R}\left(1 - \frac{r^2}{a^2}\right).$$

Calculate the Poynting vector and the total power crossing a cylindrical surface of radius r between the plates. [4]

Discuss the energy flow and dissipation in this situation. [3]

B8 An electromagnetic wave is incident from a medium of refractive index n_1 to one of refractive index n_2 at an angle of incidence of θ_i. State what happens when $\sin\theta_i > n_2/n_1$ (assuming $n_2 < n_1$) and when $\tan\theta_i = n_2/n_1$, and give an example in each case of how these phenomena have practical application. [4]

By considering the motions of the electrons and ions in a plasma when an electromagnetic wave is passing through it, show that the relative permittivity at angular frequency ω of a plasma with electron number density n is [5]

$$\epsilon_r = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2},$$

where

$$\omega_\mathrm{p}^2 = \frac{ne^2}{m_\mathrm{e}\epsilon_0},$$

and where e and m_e are the charge and mass of an electron respectively.

The ionosphere can be modelled as a uniform layer of plasma a few hundred km thick, with the bottom of the layer 200 km above the Earth. A radio transmitter on the ground is used to transmit a signal to a satellite orbiting the Earth at an altitude of 1000 km. It is found that no signal is received by the satellite when it is directly above the transmitter and the transmitter is operating at frequencies below 5 MHz. Explain this observation and calculate the electron number density in the ionosphere. [4]

Calculate the minimum angle above the horizontal at which the transmitter must be pointed to successfully transmit a signal to the satellite when the transmitter is operating at a frequency of 8 MHz. [3]

[*The satellite can be assumed to be located at the appropriate angle above the horizon to receive the radiation.*]

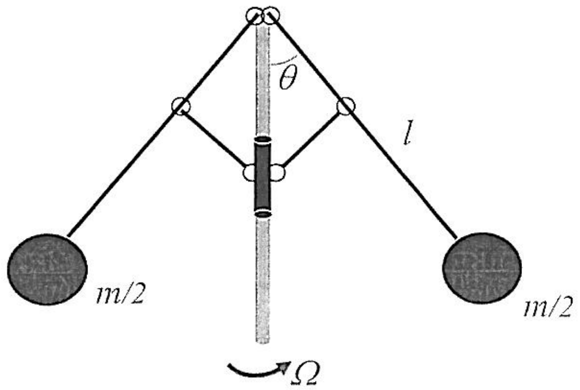
The transmitter is now used to transmit a signal to a distant receiver on the ground but the direct view between the two is blocked by a mountain range. When the transmitter frequency is kept tuned at 8 MHz, radio waves can be "bounced" off the ionosphere and detected by the receiver, but only if the radio waves are polarised with the electric field in the horizontal plane and not if the electric field is polarised in the vertical plane. Explain why this occurs and calculate how far away the receiver is. [4]

[*In all cases you may neglect the curvature of the Earth and the ionosphere, and any reflections off the top of the ionosphere.*]

SECTION C

C9 Briefly describe the Lagrangian approach in classical mechanics, including the conditions necessary for its validity. [3]

Consider the assembly illustrated below, where a vertical shaft has two arms of length l, connected at the apex with hinges, making an angle θ; the arms are connected to a sliding ring at the shaft by hinged rods. You may assume all the connections are frictionless and the rods and ring are massless.



The shaft rotates at an angular velocity Ω. Show that the Lagrangian for the system is

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\sin^2\theta\,\Omega^2 + mgl\cos\theta\ .$$

[2]

Show how this Lagrangian can be written to highlight an effective potential

$$V_{\text{eff}} \propto -\frac{1}{2}\Omega^2\sin^2\theta - \omega_0^2\cos\theta.$$

[2]

Sketch $V_{\text{eff}}(\theta)$ for small values of θ at steady state after Ω is varied, and use the graphs to explain the behaviour of the system when the shaft is rotating. Describe qualitatively how the equilibrium θ depends on Ω. [5]

Find analytical expressions for the equilibrium points. [4]

Find the frequency of small oscillations around the equilibrium points. [4]

C10 Describe the types of orbit possible for a body in a central inverse-square force field for an attractive force, making it clear under which conditions the different orbits occur. [5]

A small asteroid of mass m and with an initial velocity v_∞ approaches a stationary atmosphere-free planet of radius R and mass M. The asteroid is initially very far away and on a path that would pass a distance b from the planet's centre if it continued without deviation. Draw a sketch of the trajectory indicating b and also the distance of closest approach, r_{min}. [2]

Show that the minimum value of the distance b such that the asteroid just misses the planet is [7]

$$b_{min} = R\left(1 + \frac{2GM}{Rv_\infty^2}\right)^{\frac{1}{2}},$$

where G is the gravitational constant. Comment on the dependence of b_{min} on v_∞.

At large distances from the planet, the approaching and leaving trajectories form a 'scattering angle' χ. By considering the change in momentum along the axis of symmetry of this trajectory (or otherwise) show that, if the asteroid barely avoids the collision, the angle χ satisfies [5]

$$\tan\frac{\chi}{2} = \frac{GM}{b_{min}v_\infty^2}.$$

What range of angular deflection can be achieved by changing v_∞? [1]

C11 What is meant by "normal modes" of a dynamical system? Include the concepts of stability, the number of modes, and the role of symmetry in the system. [5]

Consider two identical pendula, each of mass m and length L, their attachment points separated by l and their masses coupled to each other by a spring of rest length l and stiffness k. If these pendula are free to swing in any direction, what are the normal modes of this system? [4]

Now consider adding another pendulum, arranged on the same line as the two above, and also coupled to one of them by a spring, as above. What are the normal modes of this system, for motion in the plane of the pendula? [7]

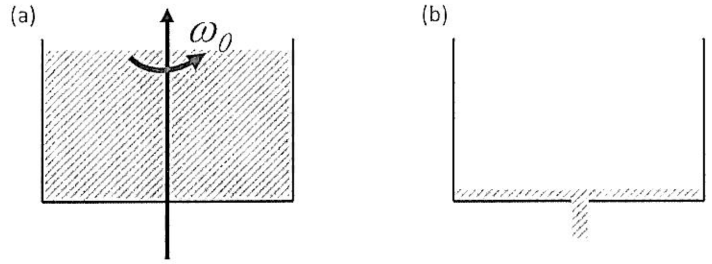
Now consider a system with a large number N of such pendula, arranged with equal spacing. Their attachments are arranged in a closed circular horizontal ring, and each pendulum oscillates in a direction tangential to the circle.

Illustrate the symmetries of the highest and lowest frequency normal modes, for an even N with a sketch of the amplitudes. What are the lowest and highest normal mode frequencies for large N? [3]

Qualitatively, what would change in the behaviour of the system above if N were odd? [1]

(TURN OVER

C12 Define the terms "streamlines" and "streaklines". Under what conditions is it appropriate to consider a fluid as "ideal"? [4]



Cross sections of the trough at initial and late times.

Consider a cylindrical trough of radius R, with a small hole at the centre of its base. In the initial state the hole is plugged, and the trough contains an ideal fluid that is rotating with uniform angular velocity ω_0 about a vertical axis through the hole, as shown in Figure (a). The plug is removed, and the liquid starts to drain very slowly away. After some time, when most of the liquid has drained, the remaining liquid layer is shallow, as shown in Figure (b). Considering the nature of the fluid, draw plausible streaklines for fluid initially near the top surface, and justify why you can now expect the angular velocity at a distance r from the rotation axis to be approximately [4]

$$\omega \simeq \omega_0 \frac{R^2}{r^2}.$$

Considering the forces that will balance the centripetal acceleration, show that there is a depth difference between the fluid at the rim and the fluid at r near the hole, of approximately [5]

$$h \simeq \frac{\omega_0^2 R^4}{2r^2 g}.$$

Estimate h based on the ω_0 you expect from the Earth's rotation, if this experiment is carried out in a well isolated trough of $r = 1$ cm and $R = 1$ m, here in Cambridge. [2]

Show that the vorticity is in the vertical direction, and that its magnitude is [2]

$$\Omega = 2\omega + r\frac{\mathrm{d}\omega}{\mathrm{d}r},$$

.

Evaluate Ω for the two states (a) and (b) shown in the figure above. [1]

Do you expect the vorticity to be conserved between these states? Explain your answer. [2]

END OF PAPER