UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 25th May 2013 9.00 am to 12.00 noon

PHYSICS A (1)

Attempt **all** *questions from Section A,* **two** *questions from Section B,* **one** *question from Section C and* **one** *question from Section D.*
Section A will carry approximately 20% of the total marks.
In Sections B, C and D each question carries approximately the same number of marks.
The approximate number of marks allocated to each question or part of a question is indicated in the right margin.
The paper contains 7 sides including this one and is accompanied by a Mathematical Formulae Handbook giving values of constants and containing mathematical formulae which you may quote without proof.
Answers from **each** *Section must be written in separate Booklets.*
Write the letter of the Section and your candidate number, not your name, on the cover of **each** *Booklet. If several Booklets are used for a given Section join them together using a Treasury Tag.*
A separate master (yellow) cover sheet should also be completed, listing all questions attempted in the paper.

STATIONERY REQUIREMENTS
Booklets and Treasury tags
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Physics Mathematical Formulae Handbook (supplied)
Students are permitted to bring an approved calculator

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Attempt **all** *questions from this Section. Answers should be concise and relevant formulae may be assumed without proof; each Section to be answered in a separate booklet.*

1 The pendulum of a grandfather clock has a period of 1 s and makes excursions of 3 cm either side of its equilibrium position. Given that the bob weighs 0.2 kg, around what value of the quantum number n would you expect its quantum amplitudes to cluster? [4]

2 Write down the phase and group velocities of the wavefunction of a free, non-relativistic particle in terms of its mass, wavevector and $\hbar$. Describe the physical significance of each velocity. [4]

3 Explain why the tunneling probability of an alpha particle through a barrier is much lower than that of an electron through the same barrier. [4]

4 The ten digits (0-9) are expected to appear with equal frequency in the real number π. The table below provides the fraction f_0 of the occurrences of the digit zero, as a function of the number N of digits considered:

N	30	100	300	1000	3000	10000
f_0	0.000	0.080	0.087	0.093	0.094	0.097

Sketch a plot, on log-log scales, of the square deviations of f_0 relative to the expected value, as a function of N, and comment on what function you would expect to fit your plotted data. [4]

5 Light incident from air at an angle θ_i is refracted into glass at an angle θ_r such that $n \sin\theta_r = \sin\theta_i$, where n is the refractive index of glass. If the angles are measured to be $\theta_i = (20 \pm 1)°$ and $\theta_r = (13 \pm 1)°$, estimate the error on n and state your assumption(s). [4]

SECTION B

Attempt **two** *questions from this Section. Each Section to be answered in a separate booklet.*

B6 The Schrödinger equation for energy eigenstates of the hydrogen atom may be written as $\widehat{H}|\psi\rangle = E|\psi\rangle$, where the Hamiltonian operator is given by

$$\widehat{H} = \frac{\widehat{p}^2}{2m} - \frac{e^2}{4\pi\epsilon_0 r}$$

and where r is the radial spherical coordinate.

The operator $\widehat{p}_r$ is defined by $\widehat{p}_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$. Show that: [3]

$$(i) \quad [r, \widehat{p}_r] = i\hbar\,,$$
$$(ii) \quad \left[\frac{1}{r}, \widehat{p}_r\right] = -\frac{i\hbar}{r^2}\,,$$
$$(iii) \quad \widehat{p}_r^2 = -\frac{\hbar^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right).$$

Without detailed derivation, explain why the Hamiltonian can be reduced to

$$\widehat{H}_l = \frac{\widehat{p}_r^2}{2m} + \frac{l(l+1)\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}$$

when acting on energy eigenstates, and explain the significance of l. [3]

The operator $\widehat{A}_l$ is defined by

$$\widehat{A}_l = \frac{a}{\sqrt{2}}\left(\frac{i}{\hbar}\widehat{p}_r - \frac{l+1}{r} + \frac{1}{(l+1)a}\right),$$

where a is a real constant. Show that for a suitable value of a, one may write the Hamiltonian as

$$\widehat{H}_l = \frac{\hbar^2}{ma^2}\left(\widehat{A}_l^\dagger \widehat{A}_l - \frac{1}{2(l+1)^2}\right)$$

and give the corresponding value of a. [4]

Show that the following relations hold: [5]

$$(i) \quad \left[\widehat{A}_l, \widehat{A}_l^\dagger\right] = \frac{a^2(l+1)}{r^2} = \frac{ma^2}{\hbar^2}\left(\widehat{H}_{l+1} - \widehat{H}_l\right);$$
$$(ii) \quad \left[\widehat{A}_l, \widehat{H}_l\right] = \left(\widehat{H}_{l+1} - \widehat{H}_l\right)\widehat{A}_l.$$

Given an eigenstate $|\psi\rangle \equiv |E, l\rangle$ of $\widehat{H}_l$, with energy eigenvalue E, show that $\widehat{H}_{l+1}\widehat{A}_l|E, l\rangle = E\widehat{A}_l|E, l\rangle$, and explain why there must exist a particular value L such that $\widehat{A}_L|E, L\rangle = 0$. By considering $|\widehat{A}_L|E, L\rangle|^2$, derive the energy eigenvalues of the hydrogen atom. [5]

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B7 The components $\widehat{S}_x$, $\widehat{S}_y$, and $\widehat{S}_z$ of the spin angular momentum operator satisfy the commutation relation $\left[\widehat{S}_x, \widehat{S}_y\right] = i\hbar\widehat{S}_z$. Write down the other commutation relations and state the possible eigenvalues of $\widehat{S}_z$ and $\widehat{S}_x^2 + \widehat{S}_y^2 + \widehat{S}_z^2$. [5]

Give a set of 2×2 matrices that satisfy these commutation relations and describe what sort of particle they could be used to represent. [2]

Suppose that such a particle has a spin-dependent Hamiltonian

$$\widehat{H} = A\widehat{S}_z^2 + B\left(\widehat{S}_x^2 - \widehat{S}_y^2\right),$$

where A and B are constants. Find the normalized energy eigenstates and their eigenvalues. [3]

A new particle has $\widehat{S}_x$ given by the 3×3 matrix

$$\widehat{S}_x = i\hbar\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Find corresponding matrices $\widehat{S}_y$ and $\widehat{S}_z$ that satisfy the commutation relations and describe what sort of particle they could be used to represent. [3]

Assuming the same Hamiltonian as above, find the new normalized energy eigenstates and their eigenvalues. [3]

At time $t = 0$, the new particle is in a state with its spin aligned parallel to the x-axis. When is the particle's spin anti-parallel to the x-axis? [4]

[*The Pauli matrices are* $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, *and* $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.]

B8 A single particle of mass m moves in a one-dimensional potential given by $V(x) = -U\delta(x)$, where $U > 0$. Write down the time-independent Schrödinger equation and explain the origin of the terms in it. [2]

Starting from the Schrödinger equation, explain why $d\psi/dx$ should be discontinuous at the origin and ψ should be continuous there. Hence derive the boundary condition

$$\frac{d\psi}{dx}(0^-) - \frac{d\psi}{dx}(0^+) = \frac{2m}{\hbar^2}U\psi(0),$$

where $x = 0^-$ and $x = 0^+$ are positions immediately to the left and right of the origin. [2]

Find general solutions to the Schrödinger equation with *negative* energy in the regions $x < 0$ and $x > 0$. [2]

By using these solutions, together with the necessary boundary conditions, find the energy eigenvalue and the normalized eigenfunction of the negative-energy bound state. [4]

Find general solutions to the Schrödinger equation with *positive* energy in the regions $x < 0$ and $x > 0$. [3]

Using these solutions, together with the necessary boundary conditions, find the intensity reflection and transmission coefficients R and T, respectively, for a beam of particles incident from $x < 0$ on the above potential and show that $R + T = 1$. [4]

Sketch R and T as a function of U and give a physical interpretation of your results. [3]

B9 Show how the method of separation of variables can be used to solve the time-independent Schrödinger equation for the three-dimensional potential [3]

$$V = \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right).$$

Explain what is meant by a conserved quantity and give conditions on the constants ω_x, ω_y, and ω_z such that the operators

$$(i)\quad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

and $$(ii)\quad \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)^2 + \left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)^2 + \left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)^2$$

are conserved. Explain your reasoning. [4]

From now on, you may assume that $\omega_x = \omega_y = \omega_z = \omega$.

Write down the energy levels for a single electron moving in the potential and show that a suitable wavefunction for the first excited state takes the form $\psi = Axe^{-\alpha(x^2+y^2+z^2)}$. Find the value of α. [5]

Calculate the uncertainty, Δx, in the x-coordinate in the first excited state and compare your answer with the amplitude of a classical oscillator with the same energy. [4]
[*Hint: Use* $\int_{-\infty}^{\infty} x^4 e^{-2x^2} dx = \frac{3}{4}\int_{-\infty}^{\infty} x^2 e^{-2x^2} dx$].

Now consider a system of nine non-interacting electrons moving in the potential. What are the energies of the ground state and of the first excited state? [4]

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SECTION C

Attempt **one** *question from this Section. Each Section to be answered in a separate booklet.*

C10 Write an essay on the postulates of quantum mechanics and the experiments that gave rise to them. Include discussions of the photoelectric effect, and the Stern-Gerlach and Davisson-Germer experiments. [20]

C11 Write brief notes on **two** of the following:

(a) symmetries and conservation laws in quantum mechanics; [10]

(b) the vibrational and rotational specific heat capacities of gases; [10]

(c) black-body radiation. [10]

SECTION D

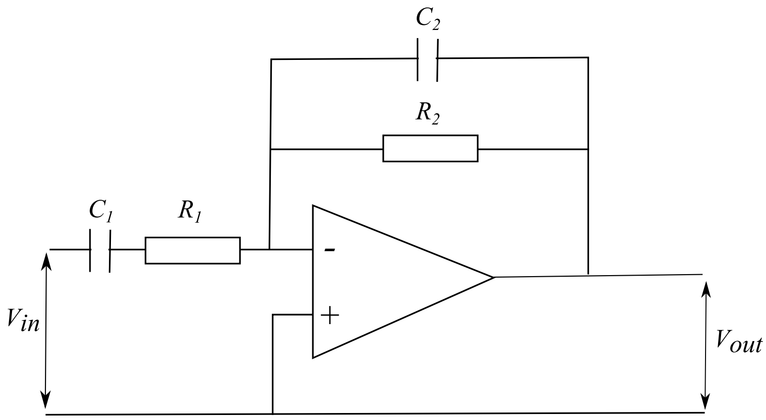
Attempt **one** *question from this Section. Each Section to be answered in a separate booklet.*

D12 What does it mean for an operational amplifier (op-amp) to be "ideal"? When analyzing the response of a circuit containing an op-amp with negative feedback, what rules follow from approximating the op-amp as ideal? [5]

Consider the circuit shown below. An AC voltage of angular frequency ω is applied at V_{in}. Assuming an ideal op-amp, show that

$$\left|\frac{V_{out}}{V_{in}}\right|^2 = \left(\frac{R_2}{R_1}\right)^2 \frac{\omega^2\tau_1^2}{(1-\omega^2\tau_1\tau_2)^2 + \omega^2(\tau_1+\tau_2)^2},$$

where $\tau_1 = R_1C_1,\ \tau_2 = R_2C_2$. [5]



Sketch $|V_{out}/V_{in}|^2$ as a function of ω, quantifying characteristic features of the function, and highlighting the behavior at low and high ω in the cases $\tau_1 \gg \tau_2$ and $\tau_2 \gg \tau_1$. [5]

This circuit could be used as a band-pass filter. How would you choose the components to achieve a sharp pass-band? In a circuit with a very sharp band, what ratio of which components determines the maximum amplification? [3]

With real components, explain what differences, especially at high ω, you would expect compared with the ideal behavior above. [2]

D13 Write brief notes on **two** of the following:

(a) causes of systematic errors and methods to eliminate them; [10]

(b) "likelihood" and Bayes' theorem, including how they are applied to experimental data analysis; [10]

(c) reducing unwanted environmental thermal effects in an experiment. [10]

END OF PAPER