UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 4th June 2022 (3 hours)

PHYSICS A (2)

Attempt **all** *questions.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. Section A will carry approximately 20% of the total marks. In Sections B, C, and D, each question carries approximately the same number of marks. The paper contains **5** *sides including this one.*

You may refer to the Mathematical Formulae Handbook supplied, which gives values of constants and contains mathematical formulae which you may quote without proof. You may also use an approved calculator.

If you are taking the exam in person, answers from **each** *Section should be written in separate Booklets. Write the letter of the Section and your candidate number, not your name, on the cover of* **each** *Booklet. If several Booklets are used for a given Section, join them together using a Treasury Tag. A separate master (yellow) cover sheet should also be completed, listing all questions attempted in the paper.*

If you are taking the exam online, **each** *Section should be scanned or photographed after the Examination and uploaded in a* **separate** *file according to the instructions provided. Before submitting your answers, ensure that all pages are of sufficient image quality to be readable.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Attempt **all** *questions from this Section. Answers should be concise and relevant formulae may be assumed without proof.*

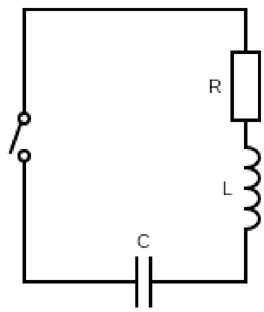
1 An aperture consisting of a circular opening of radius r is illuminated with plane-wave monochromatic light with $\lambda = 450$nm. As a detector is moved away from the aperture along the optic axis, one of the maxima in the diffracted intensity is observed at a distance of 1m from the aperture plane, with the next minimum occurring at 1.25m. Calculate the radius of the opening. [4]

2 A transverse wave propagating in the $+z$ direction has polarisation vector in the $x - y$ plane given by

$$\begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix} = \mathcal{R}\left[\begin{pmatrix} 1 \\ A \end{pmatrix} e^{i\omega t}\right].$$

Find A for a right-elliptically polarised wave where the semi-major axis is in the y-direction and is twice the length of the semi-minor axis. Draw a diagram showing the polarisation vector at $\omega t = 0$ and $\omega t = \frac{\pi}{2}$. [4]

3 A circuit consists of a 10Ω resistor, a 0.15H inductor, a 10μF capacitor and a switch which is initially open, all connected in series as shown below. The capacitor is initially charged to 15V and at $t = 0$ the switch is closed. By calculating the Q-factor, or otherwise, draw an annotated sketch of the current in the circuit as a function of time. [4]



4 Draw a detailed, annotated diagram of a p-n junction with no applied voltage showing the energy bands, the impurity levels, the chemical potential and the depletion region. [4]

5 How does the thermal conductivity of an insulator vary with temperature T at low temperatures and at high temperatures? Briefly explain these dependencies. [4]

SECTION B
Attempt **all** *questions from this Section.*

B6 (a) Explain the difference between dispersive and non-dispersive waves. [2]

(b) (i) Non-dispersive transverse waves on a rubber membrane in the $x - y$ plane have a phase-velocity v. If the membrane is clamped along the lines $y = 0$ and $y = b$ to create a waveguide, show that $z(x, y, t) = A \sin(k_y y) \cos(\omega t - k_x x)$ is a solution to the 2D wave equation for the z-displacement of the membrane:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$$

and write down the allowed values of k_x and k_y. [5]

(ii) Calculate the phase and group velocities associated with these solutions and define the cut-off frequency. What happens below this frequency? [5]

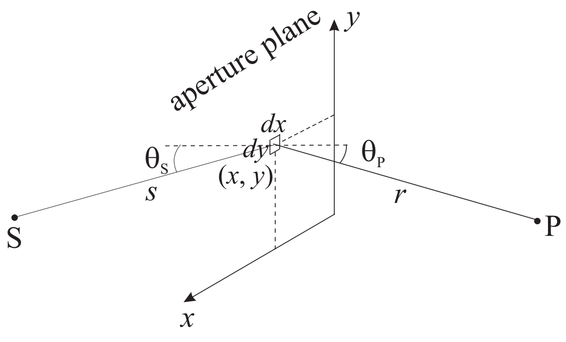
(c) The frequency spectrum of a wavegroup propagating in a waveguide with $v = 2$m/s and $b = 0.1$m can be modelled by a gaussian centred on $\omega_0 = 100$ rad/s with width $\sigma_\omega = 5$ rad/s. By considering the range of group velocities present estimate the amount by which the wavegroup will have spread after travelling 10m. [6]

(d) What are the challenges associated with multiple modes propagating in a waveguide and how can they be mitigated in practice? [2]

B7 The Fresnel-Kirchoff diffraction integral is.

$$\psi_P = \int\int_{\Sigma} -\frac{i}{\lambda} h(x, y) K(\theta_S, \theta_P) \frac{a_S\, e^{ik(s+r)}}{s\, r} dxdy$$

where K is the obliquity factor, $h(x, y)$ is the aperture function and other quantities are defined in the diagram below.



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(a) Assuming plane-wave monochromatic light at normal incidence to the aperture plane, list the key assumptions that allow this integral to be solved in the Fraunhofer regime. [4]

(b) Explain how Fourier transforms can be used to simplify the calculation of diffraction patterns for complex apertures. [2]

(c) An aperture consisting of three long narrow slits separated by a distance of 50μm in the x-direction is illuminated by monochromatic light of wavelength $\lambda = 500$nm and the resulting diffraction pattern is observed on a screen 1m from the aperture plane.

(i) Sketch the diffracted intensity on the screen as a function of distance from the optical axis. [3]

(ii) Describe quantitatively how this pattern would change if each slit had a finite width of 25μm? [3]

(d) A diffraction grating with total width W and line-spacing D is illuminated with plane-wave monochromatic light of wavelength λ. Sketch the diffracted intensity as a function of the diffraction angle θ and use your result to derive an expression for the resolving power of the diffraction grating $\frac{\delta\lambda}{\lambda}$. [5]

(e) Fraunhofer's original diffraction grating consisted of 260 parallel slits and was used to measure dark absorption lines in the light spectrum of the sun. Discuss quantitatively whether this grating would have been able to distinguish the sodium D-line, which consists of a narrow doublet at $\lambda = 589.3$nm with $\Delta\lambda = 0.6$nm. [3]

SECTION C

Attempt the question from this Section.

C8 (a) Use cyclic boundary conditions to show that, ignoring spin, the density of states in k-space of a one-dimensional system of length L is $\frac{L}{2\pi}$. [2]

The dispersion relation for free-electrons is $\epsilon_k = \frac{\hbar^2 k^2}{2m}$, where m is the electronic mass.

(b) Calculate the density of states in energy for free electrons in the following systems:

(i) a linear system of length L;

(ii) a square of side L;

(iii) a cube of side L

to show that these densities of states vary as $\epsilon^{-\frac{1}{2}}$, ϵ^0 and $\epsilon^{\frac{1}{2}}$, respectively. [5]

Crystals of dimension 1, 2 and 3 contain linear, square and cubic unit cells, respectively. The length of the edge of each unit cell is a and each unit cell contains a single monovalent atom.

(c) Starting from the density of states in k-space, for each crystal calculate the magnitude of the Fermi wavevector and, hence, determine the Fermi energy. [5]

(d) For each crystal, calculate the ratio of the magnitude of the Fermi wavevector to the distance from the centre of the Brillouin zone to the nearest Brillouin zone boundary and comment on the results. [3]

(e) For each crystal, calculate the length (area or volume) in k-space enclosed by the Fermi surface in terms of the length (area or volume) of a Brillouin zone and comment on your answer. [3]

At finite temperature T the probability of occupation of an electronic state of energy ϵ is given by the Fermi-Dirac distribution

$$p_F(\epsilon) \;=\; \frac{1}{\exp\left(\frac{\epsilon-\mu}{k_B T}\right) \;+\; 1},$$

where μ is the chemical potential.

(f) Without detailed analysis, discuss how the chemical potential varies with temperature in each system. [2]

SECTION D

Attempt the question from this Section.

D9 Write brief notes on the following:

(a) Damped oscillators (you should discuss both free and driven systems); [10]

(b) Experimental methods for determining crystal structures and measuring phonon dispersion curves. [10]

END OF PAPER