UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 2nd June 2018 9.00 to 12:00 noon

PHYSICS B (Paper 1)

*Attempt **all** questions from Section A, **two** questions from Section B, and **two** questions from Section C.*

*Section A as a whole carries **approximately** one fifth of the total marks.*

Each question in Sections B and C carries the same mark.

*The **approximate** number of marks allocated to each part of a question in all Sections is indicated in the right margin.*

*Answers for each Section **must** be written in separate Booklets.*

Write the letter of the Section on the cover of each Booklet.

*Write your candidate number, **not** your name, on the cover of each Booklet.*

A single, separate master (yellow) cover sheet should also to be completed listing all questions attempted.

STATIONERY REQUIREMENTS
12-page Booklets and Treasury Tags
Rough Work Pad
Yellow Cover Sheet

SPECIAL REQUIREMENTS
Physics Mathematical Formulae Handbook (supplied)
Approved Calculators allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Answers should be concise and relevant formulae may be assumed without proof.

A1 An infinite plane with a uniform charge density per unit area $+\sigma$ is positioned in the plane $y = 0$. A second infinite plane with a uniform charge density per unit area $-\sigma$ is positioned in the plane $y = x$. Draw a diagram showing the electric field in the $x{-}y$ plane. [4]

A2 The fluid velocity in an incompressible fluid is rotationally symmetric around the z-axis and has magnitude A/z^3 on this axis away from the origin. Calculate the radial fluid velocity at a radius r, where $r \ll |z|$. [4]

A3 Why might it be dangerous to sail a boat directly over an undersea volcano that is releasing a large volume of gas? [4]

A4 Calculate the plasma frequency for a plasma for which the number densities of electrons and protons are both equal to $10^{25}\ \mathrm{m}^{-3}$. [4]

A5 Prove that the bulk modulus B for an isotropic elastic solid is [4]

$$\frac{E}{3(1-2\sigma)},$$

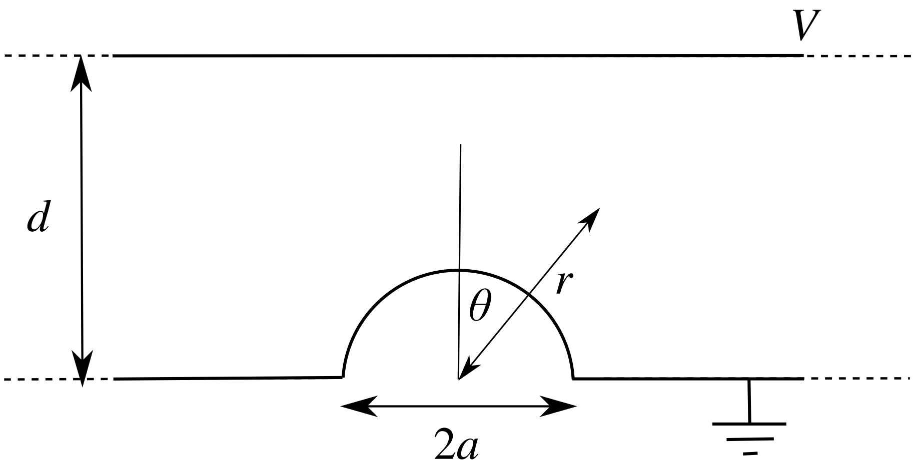
where E is Young's modulus and σ is Poisson's ratio.

SECTION B

B6 Explain how Laplace's equation, the uniqueness theorem and boundary conditions can be used to solve electrostatic problems. [4]

Show that, in spherical polar coordinates, $r\cos\theta$ and $\cos\theta/r^2$ are solutions of Laplace's equation. [3]

A capacitor consists of two infinite parallel plates, separated by a distance d, with one of the plates having a small hemispherical protrusion of radius a ($\ll d$), as shown.



A potential difference of V is applied across the capacitor. Sketch the equipotentials and $\boldsymbol{E}$-field lines in the capacitor. [4]

Show that the charge density per unit area

(a) on the hemisphere is [3]

$$\sigma = -\frac{3\epsilon_0 V}{d}\cos\theta;$$

(b) on the lower plate is [4]

$$\sigma = -\frac{\epsilon_0 V}{d}\left(1 - \frac{a^3}{r^3}\right).$$

The maximum voltage across the capacitor is determined by the breakdown electric field strength. Determine by what factor this voltage is reduced when the hemisphere is present, compared with a parallel plate capacitor, without the hemisphere. [2]

[*In spherical polar coordinates:*

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi^2}.$$
]

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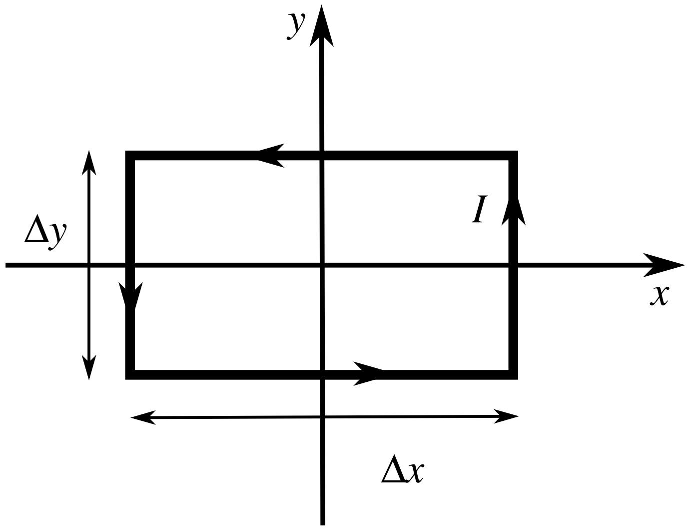
B7 The Biot–Savart law is

$$\mathrm{d}\boldsymbol{B} = \frac{\mu_0 I \mathrm{d}\boldsymbol{\ell} \times \widehat{\boldsymbol{R}}}{4\pi R^2}.$$

Define the symbols that appear in this equation. [2]

A small rectangular wire loop carries a current I. It has sides of length Δx and Δy and lies in the $z = 0$ plane, centred at $x = 0$, $y = 0$ as shown.



What is the magnetic moment $\boldsymbol{m}$ of the loop? [2]

Show, using the Biot-Savart law, that the magnetic flux density far from the loop

(a) on the z-axis is [3]

$$\boldsymbol{B} = \frac{\mu_0 \boldsymbol{m}}{2\pi |z|^3};$$

(b) on the x-axis is [5]

$$\boldsymbol{B} = -\frac{\mu_0 \boldsymbol{m}}{4\pi |x|^3}.$$

Explain how these results can also be applied to a small planar loop of arbitrary shape. [3]

A simple model of the Earth's magnetic field is that of a dipole at the centre of the Earth, directed towards the geographical south pole. Given the magnetic flux density at the Earth's equator is 35 μT, use this model to estimate the magnitude and direction of the magnetic flux density in Cambridge (at latitude of 53°). [5]

B8 A metal has an electrical conductivity σ, relative permittivity ϵ and a relative permeability $\mu = 1$. Starting from Maxwell's equations, show that the effective permittivity for a uniform electric field oscillating at an angular frequency ω is [4]

$$\epsilon' = \epsilon + \frac{\mathrm{i}\sigma}{\omega\epsilon_0},$$

using the convention that fields represented in complex notation have a time dependence $\propto \mathrm{e}^{-\mathrm{i}\omega t}$.

An electromagnetic wave with angular frequency ω is incident normally on a good conductor, for which $\sigma \gg \omega\epsilon\epsilon_0$. Show that if the incident electric field is polarised in the x-direction, the electric field variation with distance z beneath the surface of the conductor can be written, with a suitable choice of origin for time, as [3]

$$E_x(z,t) = E_0 \cos\left(\frac{z}{\delta} - \omega t\right) \mathrm{e}^{-z/\delta},$$

where $\delta = \sqrt{2/(\sigma\mu_0\omega)}$ and E_0 are constants.

Show that the $\boldsymbol{H}$-field in the metal is in the y-direction, and varies with distance z beneath the surface as [2]

$$H_y(z,t) = H_0 \cos\left(\frac{z}{\delta} - \omega t + \phi\right) \mathrm{e}^{-z/\delta},$$

where $\phi = \pi/4$.

Sketch how E_x and H_y inside the metal vary with z, and comment on differences between these fields and those outside the metal. [4]

Show that the time-averaged Poynting flux in the metal is [4]

$$\langle N(z) \rangle = A\mathrm{e}^{-2z/\delta},$$

and find the constant A in terms of σ, ω and E_0.

The time-averaged Poynting flux decreases with depth in the metal. Explain how to reconcile this with the conservation of energy. [3]

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B9 A 'stripline' transmission line consists of two long parallel horizontal metal sheets of width a separated by a distance d, with $d \ll a$. The space between the sheets is filled with a dielectric of relative permittivity ϵ. Show that the capacitance, C, and inductance L, per unit length are [4]

$$C = \frac{\epsilon \epsilon_0 a}{d} \quad \text{and} \quad L = \frac{\mu_0 d}{a},$$

assuming edge effects can be neglected.

Show that the equations relating voltage V across and current I in the transmission line are [4]

$$\frac{\partial I}{\partial x} + C\frac{\partial V}{\partial t} = 0 \quad \text{and} \quad \frac{\partial V}{\partial x} + L\frac{\partial I}{\partial t} = 0,$$

where x is the position along the transmission line.

Derive the speed of electromagnetic signal propagation along the transmission line. [2]

Show that the impedance of the transmission line is $\sqrt{L/C}$. [3]

A transmission line with $a = 20$ mm and $\epsilon = 1.5$ is used to supply power to an antenna which has an impedance of 5 Ω. What separation d should be used to eliminate any reflections from the antenna? [3]

If the power transmitted along the transmission line is 100 kW, what is the force per unit length between the two metal sheets of the transmission line? [4]

SECTION C

C10 Define the terms *moment of inertia tensor* and *principal axes* for a rigid body. [2]

Explain why the rates of change of angular momentum $\boldsymbol{J}$ in the inertial frame S_0 and in the body frame S rotating at an angular velocity $\boldsymbol{\omega}$ with respect to S_0 are related by [2]

$$\left.\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right|_{S_0} = \left.\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right|_{S} + \boldsymbol{\omega} \times \boldsymbol{J}.$$

Show that the Euler equations describing the motion of a rigid body subject to torque $\boldsymbol{G}$ can be written as [2]

$$\begin{aligned} G_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3, \\ G_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1, \\ G_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2. \end{aligned}$$

where 1, 2 and 3 label the principal axes of the rigid body.

A rigid body with moments of inertia, I_1, I_2 and I_3, is freely rotating about the 3 axis with an angular velocity ω_3. It is then subjected to a small instantaneous perturbation, after which the body continues to rotate freely. If the rotation about the 1 and 2 axes immediately following the perturbation can be written in the form $\omega_1 = ae^{pt}$ and $\omega_2 = be^{pt}$, show that [5]

$$p^2 = \omega_3^2 \frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2}.$$

Use this to discuss the stability of the subsequent motion for

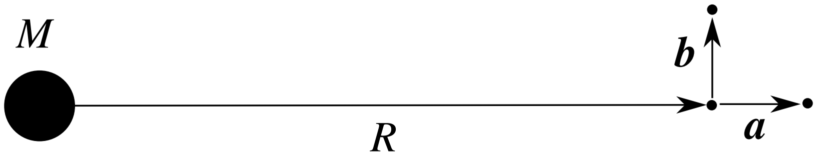
(a) $I_1 \neq I_2 \neq I_3$. [4]

(b) $I_1 = I_2 \neq I_3$, [2]

Provide a brief physical interpretation of the stability condition for the case (a) above. [3]

C11 Define the gravitational *tidal field* and explain its measurable effects on an object of finite size. [2]

Consider small radial and transverse offsets from a reference point a distance R away from a mass M, as shown.



Show that gravitational tidal field $\boldsymbol{T}$ for

(a) a small radial displacement $\boldsymbol{a}$, is [3]

$$\boldsymbol{T}(\boldsymbol{a}) \approx \frac{2GM\boldsymbol{a}}{R^3};$$

(b) a small azimuthal displacement $\boldsymbol{b}$, is [3]

$$\boldsymbol{T}(\boldsymbol{b}) \approx \frac{-GM\boldsymbol{b}}{R^3}.$$

Consider a body in circular orbit about a mass M which is corotating (i.e. it keeps the same orientation towards the main mass M). Show also that because the body is corotating, there is an additional difference in force of GMa/R^3 for a radial displacement of a. [4]

A small asteroid of uniform density ρ, is in a circular corotating orbit of radius R about a body of mass M. If the asteroid is held together by its self-gravitation, by considering radial forces derived above, show that the minimum distance for this asteroid to survive without being torn apart is [4]

$$R_{\rm min} \approx \left(\frac{9M}{4\pi\rho}\right)^{1/3}.$$

Explain why this minimum distance is independent of the size of the asteroid. [2]

For a spherical asteroid of density of 2 g cm^{-3} orbiting around the Earth estimate the minimum distance $R_{\rm min}$. [2]

[*The mass of the Earth is* $\approx 6.0 \times 10^{24}$ kg.]

C12 Explain, including a diagram, how a compression in an isotropic elastic medium in one direction, and a stretch perpendicular to this give rise to shear stress and strain. [4]

A solid cylinder of radius a, length L and density ρ is twisted uniformly about its axis and the small angle of twist between the ends is ϕ. Given the shear modulus G is the shear stress divided by shear angle, show that the couple that the rod exerts is [4]

$$C = \frac{\pi a^4 G}{2L}\phi.$$

The drive shaft of a lorry is a solid cylinder of length 3 m and radius 0.04 m. It is made of steel which has a density of 8,000 kg m^{-3} and a shear modulus of 80 GPa. It rotates at an angular velocity of 300 rad s^{-1} and transmits 40 kW of power to the drive wheels. Calculate

(a) the angle of twist between the ends of the drive shaft; [2]

(b) the elastic strain energy stored in the drive shaft; [2]

(c) the kinetic energy in the drive shaft. [2]

Consider a thin slice of the cylinder, of length Δx. Show that the change of the couple across Δx is [3]

$$\Delta C = \frac{\pi a^4 G}{2}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\Delta x.$$

Hence show that the rotational equation of motion of the thin slice of the cylinder satisfies a wave equation of the form [3]

$$\frac{\partial^2\phi}{\partial t^2} = \frac{G}{\rho}\frac{\partial^2\phi}{\partial x^2}.$$

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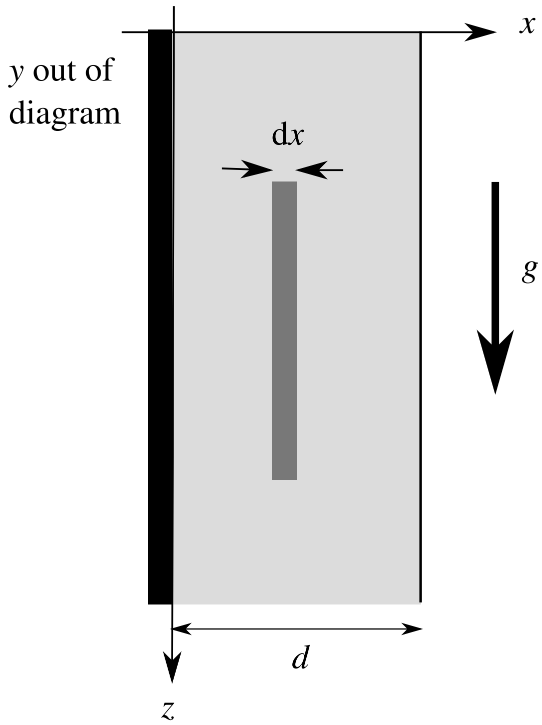
C13 Define *viscosity* and explain how it arises. [3]

An incompressible fluid with viscosity η and density ρ flows down a vertical draining plate, in the $y-z$ plane, under the influence of gravitational acceleration g, as shown. The extent of the fluid in the x-direction is d.



Assume the flow has reached a steady state. From the equation of vertical motion of a thin slice of the fluid of thickness dx, show that the vertical viscous stress across the slice changes by [2]

$$d\tau = -\rho g \, dx.$$

Discuss the boundary conditions on the fluid at $x = 0$ and $x = d$. [2]

Show that the downward speed of a layer at a distance x away from the plate surface is [4]

$$v_z(x) = ax + bx^2,$$

and find the constants a and b.

Calculate the total volume flow rate, per unit length in the y-direction. [2]

The draining plate is now tilted so that it is at an angle θ to the horizontal. The fluid continues to flow along the plate under the influence of gravity. Derive the new equations of motion for a slice of the fluid, and determine how the velocity of the flow varies with distance from the plate when the flow is in a steady state. [5]

Find the total volume flow rate for the tilted draining plate per unit length in the y-direction. [2]

[*You may assume that the extent of the fluid is large in the y-direction compared with d.*]

END OF PAPER