UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 31 May 2014 9.00 to 12.00 noon

PHYSICS B (Paper 1)

*Attempt **all** questions from Section A, **two** questions from Section B, and **two** questions from Section C.*

*Section A as a whole carries **approximately** one fifth of the total marks.*

Each question in Sections B and C carries the same mark.

*The **approximate** number of marks allocated to each part of a question in all Sections is indicated in the right margin.*

*Answers for each Section **must** be written in separate Booklets.*

Write the letter of the Section on the cover of each Booklet.

*Write your candidate number, **not** your name, on the cover of each Booklet.*

A single, separate master (yellow) cover sheet should also be completed, listing all questions attempted.

STATIONERY REQUIREMENTS
20-Page Booklets and Treasury Tags
Rough Work Pad
Yellow Cover Sheet

SPECIAL REQUIREMENTS
Physics Mathematical Formulae Handbook (supplied)
Approved Calculators allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Answers should be concise and relevant formulae may be assumed without proof.

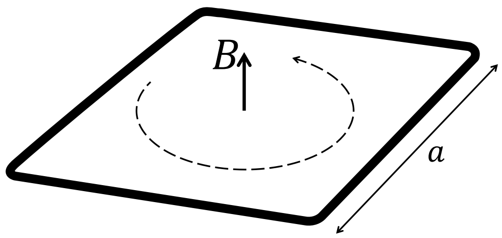
A1 A ball, a solid cylinder and a hollow pipe, all with the same mass and outer radius, are rolled down an inclined plane. Explain which attains the greatest speed and why. [4]

A2 What is the most *probable* speed of oxygen molecules at a temperature of 300 K? [4]

A3 Find the ratio between the electric and magnetic forces between two parallel beams of electrons moving at velocity v. [4]

A4 A large number of small spherical metal balls are placed in a circular tray and set into constant random motion. A small gap in the outer rim of the tray (much larger than the ball diameter) allows balls to escape. If their average energy and number is unchanged, how does the escape rate change if the mass of each ball is halved? [4]

A5 A charged insulating square loop of side a is rotated around an axis through its centre perpendicular to the loop plane as shown below. By what factor is the magnetic field in the centre of the square smaller than if a circular loop of the same diameter a and total charge is used?



[*You may assume* $\int_0^{\pi/4} \sec x \, dx = \ln(\sqrt{2} + 1)$.] [4]

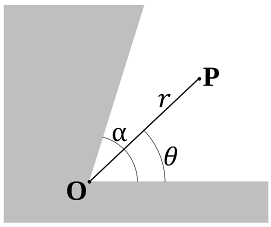
SECTION B

B6 The Laplace equation for the electrostatic potential Φ of a two-dimensional system expressed in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0$$

Discuss what type of boundary conditions can be used to uniquely determine its solutions. What are the boundary conditions imposed by a metal surface ? [5]

Consider a metallic body from which a wedge-shaped segment with opening angle α has been cut out as shown in the figure. The system can be treated as two-dimensional without making reference to the third dimension. The metal is at potential V_0.



By trying the separated solution $\Phi(r,\theta) = R(r)\Psi(\theta)$ with $R(r) \propto r^\nu$ and applying the boundary conditions, find the general solution for Φ at any point outside the metal. [8]

Show that in the vicinity of $r = 0$ the electrical field components outside the metal are approximately proportional to

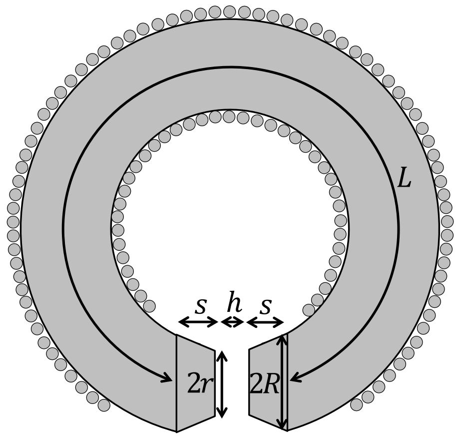
$$E_r(r,\theta) \propto r^{(\frac{\pi}{\alpha}-1)} \sin\left(\frac{\pi\theta}{\alpha}\right)$$
$$E_\theta(r,\theta) \propto r^{(\frac{\pi}{\alpha}-1)} \cos\left(\frac{\pi\theta}{\alpha}\right)$$

[4]

Sketch the electric field lines in the vicinity of $r = 0$ for the case of $\alpha = \frac{\pi}{2}$. Discuss qualitatively to what extent the behavior for $\alpha = \frac{3\pi}{2}$ is likely to be observed for a *real* metallic wedge. [3]

B7 Derive the boundary conditions for the magnetic flux density $\boldsymbol{B}$ and the magnetic field strength $\boldsymbol{H}$ between two magnetic materials with different magnetic permeabilities. [6]

Consider a toroidal electromagnet filled with a magnetic material of large permeability μ. The torus contains a small vacuum gap of length h. Over most of its length the torus has a circular cross section of radius R, but towards the gap the torus is tapered on both of its ends, i.e., its radius is decreased from R to r over a distance s towards the gap. The electromagnet has N windings through which a current of I is flowing.



Explain why the magnetic flux across the cross section of the torus is conserved along the total length of the torus and within the gap. [3]

Determine the magnetic field strength inside the gap. [8]

Calculate the ratio of the magnetic field strength inside the gap of an electromagnet with tapered ends to that of an untapered, but otherwise identical electromagnet. Explain the benefit of the tapered ends in the limit in which $\frac{s}{L} \ll \frac{r}{R}$, and what might limit it in practise. [3]

B8 State Maxwell's equations in a medium with current density $\boldsymbol{J}$. [4]

A transverse electromagnetic wave with electric field amplitude A polarised in the y-direction is travelling in the $+x$-direction through vacuum in the left half-space ($x < 0$) and impinges at normal incidence onto a body which fills the right half-space ($x > 0$). Inside the body there are no free charges, and its relative dielectric permittivity and relative magnetic permeability can be assumed to be one, although the body can sustain a current density $\boldsymbol{J}$ which is not necessarily equal to $\sigma\boldsymbol{E}$. The electrical field has the form $\boldsymbol{E} = (0, E_y(x,t), 0)$ everywhere. Find the direction of the accompanying field $\boldsymbol{H}$ inside the body. [2]

Use Maxwell's equations to express $\boldsymbol{J}$ in terms of derivatives of $\boldsymbol{E}$ and $\boldsymbol{H}$. [3]

$\boldsymbol{H}$ produces a Lorentz force on a unit volume element of the body $\boldsymbol{f} = \mu_0\boldsymbol{J} \times \boldsymbol{H}$. Show that the force per unit volume is in the x-direction and has a time average

$$\overline{f}_x = -\frac{\partial}{\partial x}\overline{\left(\frac{1}{2}\mu_0 H_z^2 + \frac{1}{2}\epsilon_0 E_y^2\right)}$$

[5]

Hence by integrating, express the total time-averaged pressure on the body in terms of A for the cases of (a) a perfectly absorbing body (in which no reflected wave is present) and (b) a perfectly conducting body. [4]

How are your results for the two cases related to the time-averaged Poynting vector of the incident electromagnetic wave ? [2]

B9 A transmission line has input impedance

$$Z_{in} = Z_0 \frac{Z\cos kl + i Z_0 \sin kl}{Z_0 \cos kl + i Z \sin kl}$$

Explain the meaning of Z, Z_0 and k and discuss the behaviour of Z_{in} for very large, very small, and matched load impedance. [6]

A ladder transmission line comprises an alternating sequence of segments of two different transmission lines both of length l with characteristic impedance Z_1 and Z_2. The line is constructed such that its input impedance remains unchanged when another pair of Z_1 and Z_2 segments is added. Show that the input impedance of the ladder transmission line obeys the following relationship:

$$i Z_{in}^2(Z_1 + Z_2) + Z_{in}(Z_1^2 - Z_2^2)\tan kl - i Z_1 Z_2 (Z_1 + Z_2) = 0$$

[8]

Derive an expression for the input impedance. [2]

In what frequency range is a wave incident on this ladder totally reflected ? [4]

SECTION C

C10 What is meant by a *fictitious force*? [3]

Show that the acceleration $\boldsymbol{a}$ of a particle with mass m and velocity $\boldsymbol{v}$ as measured in a frame that is rotating with constant angular velocity $\boldsymbol{\Omega}$ is given by

$$m\boldsymbol{a} = \boldsymbol{F} - 2m(\boldsymbol{\Omega} \times \boldsymbol{v}) - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$$

where $\boldsymbol{F}$ is the force experienced by the particle. [4]

The motion of a particle on a smooth saddle shaped surface is modeled by a potential

$$V(x, y) = \frac{1}{2}m\omega^2(x^2 - y^2).$$

When the potential is rotated at an angular frequency $\boldsymbol{\Omega} = (0, 0, \Omega)$, show that in the rotating frame the equations of motion are

$$\ddot{x} = (\Omega^2 - \omega^2)x + 2\Omega\dot{y}$$
$$\ddot{y} = (\Omega^2 + \omega^2)y - 2\Omega\dot{x}$$

[4]

By using a trial solution of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{i\epsilon t},$$

find the condition on Ω necessary for a particle initially placed near the origin to remain in the vicinity of the origin for all subsequent times. [5]

Describe and explain the motion in the rotating frame and the lab frame when $\Omega \gg \omega$. [4]

C11 Explain what is meant by the *normal modes* of a mechanical system. [4]

Three ions of identical charge q and mass m are constrained to move in one dimension in a harmonic potential well

$$V(x) = \frac{1}{2}m\omega^2 x^2.$$

In equilibrium, one ion is located at the centre of the trap with the other two a distance x_0 away on either side. Find the distance x_0. [2]

Treating the ions classically, confirm that small oscillations about the equilibrium positions are governed by the equations

$$\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{pmatrix} = -\frac{\omega^2}{5} \begin{pmatrix} 14 & -8 & -1 \\ -8 & 21 & -8 \\ -1 & -8 & 14 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where y_i are the deviations from the equilibrium positions. [7]

Explain why two of the normal modes are $(1, 1, 1)$ and $(1, 0, -1)$, and find their frequencies. Find also the frequency and eigenvector for the remaining normal mode. [4]

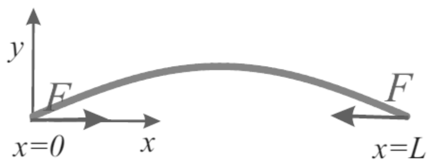
With the ions initially at rest in equilibrium, the natural frequency of the confining potential is abruptly changed to $\omega + \delta\omega$. Show that only one normal mode is excited, and find the subsequent motion of the ions, assuming $\delta\omega/\omega \ll 1$. [3]

C12 Consider an elastic beam of Young's modulus E.
Show that the *bending moment* B at a point on the beam is given by

$$B = \frac{EI}{R},$$

where the radius of curvature is R. Define the quantity I. [4]

Consider a light (i.e. massless) beam loaded between fixed supports.



Show that the angle θ that the beam makes to the horizontal obeys the equation

$$EI\frac{d^2\theta}{ds^2} + F\sin\theta = 0$$

where s is the distance along the beam. [6]
[*Hint:* $R^{-1} = \frac{d\theta}{ds}$. *DO NOT assume the deflection is small.*]

Show that

$$\frac{EI}{2}\left(\frac{d\theta}{ds}\right)^2 - F\cos\theta = \text{const.} \qquad (\star)$$

[2]

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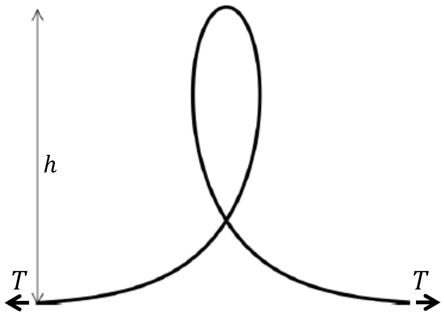

A long beam under *tension* T can take on the configuration shown above with θ and $d\theta/ds$ going to zero at both ends. By solving the equation ($\star$), show that θ is

$$\theta(s) = 4\tan^{-1}[\exp(\alpha s)],$$

where s is measured from halfway along the beam. Identify the coefficient α. [4]

[*You may find the following integral useful:* $\int \operatorname{cosec}(\theta)\, d\theta = \ln[\tan(\theta/2)]$]

By considering the bending moment halfway along, find the height h of the loop in the beam in terms of E, I, and T. [4]

C13 For irrotational flow of an incompressible fluid, explain why it is possible to define a velocity potential such that $\mathbf{v} = \nabla\Phi$ and show that Φ satisfies Laplace's equation. [3]

A sphere of radius a moves at velocity $\mathbf{v}$ through an incompressible fluid of zero viscosity. Show that in spherical polar coordinates with a moving origin at the centre of sphere, Φ has the form

$$\Phi = \frac{\beta\cos\theta}{r^2},$$

where you should determine the coefficient β. [5]

By considering the total kinetic energy of the fluid, and using

$$\mathbf{v}^2 = \Phi(\nabla\cdot\mathbf{v}) - \nabla\cdot(\Phi\mathbf{v}),$$

show that the sphere's effective mass is increased by $2\pi a^3\rho_0/3$, where ρ_0 is the density of the fluid. [6]

A spherical mass of density ρ oscillates on a spring at a frequency ω_s when the spring moves in vacuum. What is the frequency of oscillation when the mass is immersed in the fluid? [2]

The same mass on a pendulum oscillates at a frequency ω_p when the pendulum moves in vacuum. What is the frequency of oscillation when the mass is immersed in the fluid? [4]

END OF PAPER