UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 21st May 2016 9.00 am to 12.00 noon

PHYSICS A (1)

Attempt **all** *questions from Section A,* **two** *questions from Section B,* **one** *question from Section C, and* **one** *question from Section D.*
Section A will carry approximately 20% of the total marks.
In Sections B, C, and D, each question carries approximately the same number of marks.
The approximate number of marks allocated to each question or part of a question is indicated in the right margin.
The paper contains **8** *sides including this one and is accompanied by a Mathematical Formulae Handbook giving values of constants and containing mathematical formulae, which you may quote without proof.*
Answers from **each** *Section should be written in separate Booklets.*
Write the letter of the Section and your candidate number, not your name, on the cover of **each** *Booklet. If several Booklets are used for a given Section, join them together using a Treasury Tag.*
A separate master (yellow) cover sheet should also be completed, listing all questions attempted in the paper.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Booklets and treasury tags	Mathematical Formulae Handbook
Rough workpad	Linear graph paper
Yellow master coversheet	Approved calculator

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Attempt **all** *questions from this Section. Answers should be concise and relevant formulae may be assumed without proof. Use a separate booklet for the whole of this section.*

1 A wavepacket $\psi(x,t)$ describing a free particle at time $t = 0$ is given in coordinate representation by $\psi(x,0) = 1/\sqrt{a}$ for $-a/2 < x < a/2$ and $\psi(x,0) = 0$ for $|x| \geq a/2$. Calculate the wavefunction $\phi(p,0)$ in momentum representation and sketch its form. [4]

2 An electron beam is accelerated through a voltage of 150 V and scatters from an unknown crystalline sample. The first Bragg reflection peak is observed centred at an angle of 22° from the initial direction of the beam. Using Bragg's law, calculate the spacing between the Bragg planes in the sample. [4]

3 Consider a Hermitian operator $\hat{A}$ that has the property $\hat{A}^3 = \hat{I}$, where $\hat{I}$ is the identity operator. By considering the eigenvalues of $\hat{A}$, show that it must be equal to the identity operator. [4]

4 If $z = \sqrt{\tan(x)} + y^2$, with $x = (45 \pm 2)°$ and $y = 0.10 \pm 0.02$, find the value of z and its standard error. [4]

5 An experiment measures a 1 nA current with a 100 kHz bandwidth. Find the r.m.s. shot noise expected in the current. [4]

SECTION B

Attempt **two** *questions from this Section. Use a separate booklet for the whole of this section.*

B6 The probability current associated with the one-dimensional wavefunction $\psi(x,t)$ is given by

$$J(x,t) = \frac{\hbar}{2m\mathrm{i}}\left(\psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x}\psi\right),$$

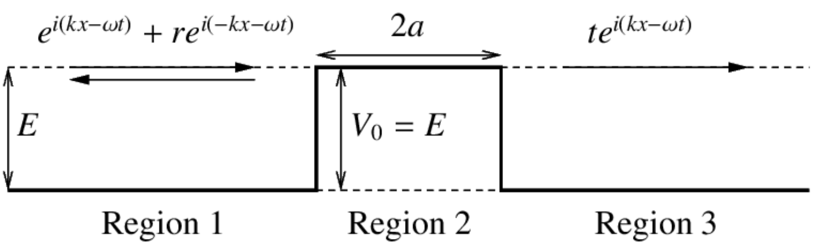
where m is the mass of the particle.

A region of space contains two superimposed plane waves propagating in opposite directions:

$$\psi = c_1 \mathrm{e}^{\mathrm{i}(kx-\omega t)} + c_2 \mathrm{e}^{\mathrm{i}(-kx-\omega t)},$$

where c_1 and c_2 are complex constants. Show that the net probability current is given by $J = \frac{\hbar k}{m}\left(|c_1|^2 - |c_2|^2\right)$. [5]

A particle of kinetic energy E and mass m is incident on a rectangular potential barrier of height $V_0 = E$ that extends from $x = -a$ to $x = a$, as sketched below.



Show that the wavefunction in Region 2 has the form $\psi_2(x,t) = (bx + c)\,\mathrm{e}^{-\mathrm{i}\omega t}$, where b and c are complex constants. [2]

Hence show that the amplitude reflection coefficient r and the amplitude transmission coefficient t are given, respectively, by

$$r = -\frac{ika\mathrm{e}^{-2ika}}{1-ika} \qquad \text{and} \qquad t = \frac{\mathrm{e}^{-2ika}}{1-ika}. \qquad [5]$$

Thus show that the overall probability flux transmission coefficient from Region 1 to Region 3 is given by

$$T = \left(1 + \frac{2mEa^2}{\hbar^2}\right)^{-1},$$

and verify that $R + T = 1$, where R is the probability flux reflection coefficient. [5]

By determining the complex constants b and c in the wavefunction $\psi_2(x,t)$ in Region 2, calculate the probability current in this region and briefly interpret your result. [3]

B7 The time-dependent Schrödinger equation is

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle,$$

where $\hat{H}$ is the Hamiltonian of the system under consideration.

For some general operator $\hat{A}$, show that the expectation value $\langle\hat{A}\rangle = \langle\psi(t)|\hat{A}|\psi(t)\rangle$ satisfies

$$\frac{d\langle\hat{A}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle + \left\langle\frac{d\hat{A}}{dt}\right\rangle.$$ [4]

For general operators $\hat{A}$, $\hat{B}$ and $\hat{C}$, show that $[\hat{A},\hat{B}\hat{C}] = [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]$, where the commutator $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. [2]

Suppose that the operators $\hat{A}$ and $\hat{B}$ each commute with their commutator $[\hat{A},\hat{B}]$. If the relationship $[\hat{A}^n,\hat{B}] = n[\hat{A},\hat{B}]\hat{A}^{n-1}$ holds for some positive integer $n = m$, show that it also holds for $n = m+1$ and hence that it holds where n is any positive integer. [5]

Hence show that for any function $F(x)$ that can be expanded as a power series in x, such that $F(x) = \sum_{n=0}^{\infty} a_n x^n$, one has $[F(\hat{A}),\hat{B}] = [\hat{A},\hat{B}]F'(\hat{A})$, where $F'(x)$ denotes the derivative dF/dx. [3]

For a system described by the time-independent Hamiltonian $\hat{H} = \hat{p}^2/(2m) + V(\hat{x})$, where $V(x)$ can be expanded as a power series in x, show that

$$\frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle\hat{p}\rangle}{m} \qquad \text{and} \qquad \frac{d\langle\hat{p}\rangle}{dt} = -\left\langle\frac{dV(\hat{x})}{d\hat{x}}\right\rangle,$$

and comment briefly on the physical significance of these results. [6]

B8 The operators for the components of orbital angular momentum satisfy the commutation relations $[\hat{L}_x,\hat{L}_y] = i\hbar\hat{L}_z$, $[\hat{L}_y,\hat{L}_z] = i\hbar\hat{L}_x$, $[\hat{L}_z,\hat{L}_x] = i\hbar\hat{L}_y$, and the operator $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ commutes with $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$.

For the operators $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$, show that

$$[\hat{L}_z,\hat{L}_\pm] = \pm\hbar L_\pm \qquad \text{and} \qquad \hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z.$$ [4]

If $|\psi\rangle$ is an eigenstate of $\hat{L}_z$ having eigenvalue $\alpha\hbar$, where α is a real number, show that $\hat{L}_+|\psi\rangle$ and $\hat{L}_-|\psi\rangle$ are also eigenstates of $\hat{L}_z$ having eigenvalues $(\alpha+1)\hbar$ and $(\alpha-1)\hbar$, respectively. [2]

If $|\psi\rangle$ is also an eigenstate of $\hat{L}^2$ having eigenvalue $\Lambda\hbar^2$, where Λ is a real number, show that $\hat{L}_+|\psi\rangle$ and $\hat{L}_-|\psi\rangle$ are also eigenstates of $\hat{L}^2$ having eigenvalue $\Lambda\hbar^2$. [2]

By considering the form of $\hat{L}_z$ in spherical polar coordinates, or otherwise, explain why α must be an integer and hence show that

$$\hat{L}^2|\ell,m_\ell\rangle = \ell(\ell+1)\hbar^2|\ell,m_\ell\rangle \qquad \text{and} \qquad \hat{L}_z|\ell,m_\ell\rangle = m_\ell\hbar|\ell,m_\ell\rangle,$$

where ℓ is an integer and $m_\ell \in \{-\ell,-\ell+1,\ldots,1,0,1,\ldots,\ell-1,\ell\}$. [6]

Consider a particle with wavefunction

$$\psi(x, y, z) = A(x + y + z) \exp\left[-(x^2 + y^2 + z^2)/a^2\right],$$

where A is a normalisation constant and a is a real parameter. Find the probability that a simultaneous measurement of L^2 and L_z yields the values $2\hbar^2$ and $\hbar$, respectively. [6]

[*You may assume that the wavefunctions corresponding to the eigenstates* $|\ell, m_\ell\rangle$ *are the spherical harmonics* $Y_{\ell,m_\ell}(\theta, \phi)$, *and that* $Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta$ *and* $Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta \exp(\pm \mathrm{i}\phi)$.]

B9 For a particle with spin $s = \frac{1}{2}$ the operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ for the spin components along the x-, y- and z-directions, respectively, may be represented by the matrices

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Denoting the eigenstates of $\hat{S}_z$ by $|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, calculate $\hat{S}_j|\uparrow\rangle$ and $\hat{S}_j|\downarrow\rangle$ for $j = x, y, z$ and determine the eigenstates of $\hat{S}_x$ and $\hat{S}_y$. [5]

For a particle in the state $|\chi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$, where a and b are complex numbers, calculate $\langle\hat{S}_z\rangle$ and $\langle\hat{S}^2\rangle$, where $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$. [5]

Suppose one measures the component of spin along some other axis z', inclined at an angle θ to the original z-axis, such that $\hat{S}_{z'} = \hat{S}_x \sin\theta + \hat{S}_z \cos\theta$. Calculate the eigenvalues and eigenstates of $\hat{S}_{z'}$. [5]

If a particle is in the state $|\uparrow\rangle$ referred to the z-axis, calculate the probabilities that a spin measurement along the z'-axis will yield the values $\hbar/2$ or $-\hbar/2$, respectively. [3]

What is the expectation value of a spin measurement along the z'-axis? [2]

(TURN OVER

SECTION C

Attempt **one** *question from this Section. Use a separate booklet for this section.*

C10 Write an essay on systems of identical particles in quantum mechanics. [20]

C11 Write brief notes on **two** of the following:

(a) electron wavefunctions in the hydrogen atom; [10]

(b) experimental evidence for quantum tunnelling; [10]

(c) creation and annihilation (ladder) operators for the harmonic oscillator. [10]

SECTION D

Attempt **one** *question from this Section. Use a separate booklet for this section.*

D12 The mycelium is the vegetative part of a fungus consisting of a mass of branching thread-like filamentous structures growing just below the surface of the soil. Its growth can be modelled by the following simple exponential relationship for the ground area covered by the mycelium $A(t)$ as a function of time, t:

$$A(t) = A_0 e^{\gamma t},$$

where A_0 is the initial area at $t = 0$, and γ is a constant describing how rapidly the mycelium grows.

A laboratory experiment is performed to record the area covered by the mycelium over a period of 5 weeks. The areas are measured with relative errors of 10%.

Time since planting / days	Area covered / cm^2
0	1.5
7	4.0
14	5.0
21	9.5
28	12.5
35	22.0

Transform these data appropriately such that a straight-line relationship is expected, and plot the transformed data on a suitable graph using the graph paper provided. [5]

Show that, since the measured areas have a constant relative error, the logarithms of the areas have a constant absolute error. [2]

Using the method of χ^2 minimisation (least-squares fitting), find the best straight-line fit to the transformed data, and hence find the best-fit values of γ and A_0 . [8]

Draw the best-fit solution you found on the plot made earlier. [3]

Consider what would happen if you took the mycelium grown at the end of the 5-week laboratory experiment and planted it in Jesus Green. Based on your model, estimate how many years it would take for the mycelium to spread under the whole of Cambridge (which you may model as a circle of radius 3 km centred on Jesus Green). [2]

D13 Write brief notes on **two** of the following:

(a) op-amp golden rules and their use in the analysis of feedback circuits; [10]

(b) thermal noise and strategies to reduce it by maintaining constant low temperatures; [10]

(c) sampling and aliasing. [10]

END OF PAPER