UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Saturday 26th May 2012 9.00 am to 12.00 noon

PHYSICS A (1)

Attempt **all** *questions from Section A,* **two** *questions from Section B,* **one** *question from Section C and* **one** *question from Section D.*
Section A will carry approximately 20% of the total marks.
In Sections B, C and D each question carries approximately the same number of marks.
The approximate number of marks allocated to each question or part of a question is indicated in the right margin.
The paper contains 6 sides including this one and is accompanied by a Mathematical Formulae Handbook giving values of constants and containing mathematical formulae which you may quote without proof.
Answers from **each** *Section must be written in separate Booklets.*
Write the letter of the Section and your candidate number, not your name, on the cover of **each** *Booklet. If several Booklets are used for a given Section join them together using a Treasury Tag.*
A separate master (yellow) cover sheet should also be completed, listing all questions attempted in the paper.

STATIONERY REQUIREMENTS
Booklets and Treasury tags
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Physics Mathematical formulae handbook (supplied)
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Answers should be concise and relevant formulae may be assumed without proof

1 Photoelectrons are produced from a metal with maximum energy 2.3 eV for incident radiation of wavelength 200 nm and 0.9 eV for incident radiation of wavelength 258 nm. Hence calculate Planck's constant. [4]

2 For the one-dimensional potential $V(x) = kx^2/2$, draw an annotated sketch of the wavefunctions corresponding to the three lowest energy eigenstates, indicating the points of inflection of the wavefunctions. [4]

3 Estimate the order of magnitude of the distance scale that can be probed at the Large Hadron Collider using a beam of protons with an energy of 4 TeV. [4]

4 The power P radiated by a body is proportional to T^4, where T is its temperature. If $P = P_{\text{total}} - P_{\text{back}}$ and P_{total} is $\approx 10P_{\text{back}}$, what is the uncertainty in T due to uncertainties in P_{total} and P_{back} of 1% and 10% respectively? [4]

5 What is a typical uncertainty in the amount of charge flowing in 10^{-6} s for a current of 1 nA? [4]

SECTION B

B6 The Hamiltonian of a 2-dimensional isotropic harmonic oscillator is given by

$$\widehat{H} = \frac{\widehat{p}_x^2 + \widehat{p}_y^2}{2m} + \frac{m\omega^2}{2}(\widehat{x}^2 + \widehat{y}^2)$$

where $\widehat{x}$ and $\widehat{y}$ denote position operators and $\widehat{p}_x$ and $\widehat{p}_y$ the corresponding momentum operators.

State (without proof) the commutation relations between the operators $\widehat{x}$, $\widehat{y}$, $\widehat{p}_x$, $\widehat{p}_y$. [2]

Now consider the operators defined by

$$\widehat{J}_1 = \frac{\widehat{p}_x\widehat{p}_y}{2m\omega} + \frac{m\omega\widehat{x}\,\widehat{y}}{2}, \quad \widehat{J}_2 = \frac{(\widehat{p}_y^2 - \widehat{p}_x^2)}{4m\omega} + \frac{m\omega(\widehat{y}^2 - \widehat{x}^2)}{4}, \quad \widehat{J}_3 = \frac{\widehat{x}\,\widehat{p}_y - \widehat{y}\,\widehat{p}_x}{2}.$$

Show that $\widehat{J}_1$ is an observable. [1]

$\widehat{H}^2$ can be written in the form $\widehat{H}^2 = \alpha(\widehat{J}_1^2 + \widehat{J}_2^2 + \widehat{J}_3^2) + \beta$. Determine the constants α and β. [7]

Given further that the operators $\widehat{J}_i$ satisfy the same commutation relations as angular momentum operators in *three* dimensions, namely $[\widehat{J}_i, \widehat{J}_j] = i\hbar\epsilon_{ijk}\widehat{J}_k$, show that $\widehat{J}_1$ is conserved and that the system cannot, in general, be in a simultaneous eigenstate of $\widehat{J}_1$ and $\widehat{J}_2$. [5]

Use standard results (which you may quote without proof) about the eigenvalues of $\widehat{J}_1^2 + \widehat{J}_2^2 + \widehat{J}_3^2$ to find the energies and degeneracies of eigenstates of the *two* dimensional harmonic oscillator. [5]

B7 Discuss Heisenberg's uncertainty principle and describe the spin-statistics governing the behaviour of bosons and fermions. [5]

Consider a single particle of mass *m* in the potential

$$V(x) = \begin{cases} 0, & \text{if } 0 < x < L, \\ \infty, & \text{elsewhere.} \end{cases}$$

Find the energy eigenstate wavefunctions and their corresponding eigenvalues. [5]

Evaluate Δx and Δp of the particle in the ground state and compare with the bound set by the uncertainty principle. [7]

Four identical, non-interacting particles are now put into the well. Find the total ground state energy for the cases when the particles each have spin $s = 0$, $1/2$ and $3/2$. [3]

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B8 Show that the Pauli spin matrices, $\widehat{\sigma}_i$, defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfy the commutation relations $[\widehat{\sigma}_i, \widehat{\sigma}_j] = 2\mathrm{i}\epsilon_{ijk}\widehat{\sigma}_k$, and briefly discuss the relevance of the Pauli spin matrices for angular momentum in quantum mechanics. [2]

A neutron in a magnetic field $\boldsymbol{B}$ has Hamiltonian given by

$$\widehat{H} = \alpha \boldsymbol{B} \cdot \widehat{\boldsymbol{S}},$$

where $\widehat{\boldsymbol{S}} \equiv S_i \widehat{\sigma}_i$ is the spin operator. What are the possible eigenvalues of the spin measured along some axis? [1]

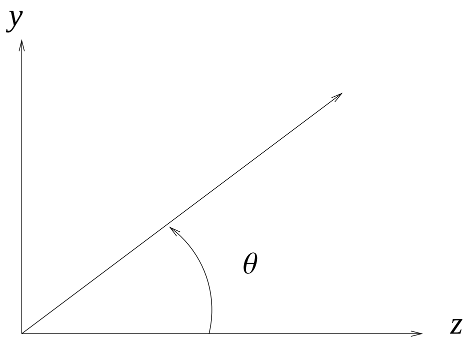
Suppose that initially the neutron is in an eigenstate with spin up in the $+z$ direction, with the magnetic field in the $+y$ direction. Write down the time-dependent Schrödinger equation. Solve it to obtain an expression for the wavefunction, $\psi(t)$, at a later time t, of the form

$$\psi(t) = \begin{pmatrix} \cos\phi(t) \\ \sin\phi(t) \end{pmatrix}$$

and determine the function $\phi(t)$. [6]

If the spin is measured along the x-axis, find the two normalised spin eigenstates. [2]

Suppose now that the spin is measured about an axis in the $z-y$ plane making an angle θ with the $+z$ direction, as shown.



The two spin eigenstates with respect to this new measurement direction are

$$\begin{pmatrix} \cos(\theta/2) \\ \mathrm{i}\sin(\theta/2) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{i}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}.$$

Show that these are orthonormal and comment on their values for $\theta = 0, \pi/2, \pi$ and 2π. [3]

Derive the probability of measuring spin up or down in this direction at time t, again commenting on the values for $\theta = 0, \pi/2, \pi$ and 2π. [6]

B9 For an electron of mass m in a central potential $V(r)$, the time-independent Schrödinger equation may be written as

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{2mr^2}\widehat{L}^2\psi+V(r)\psi=E\psi.$$

The energy eigenstates may be written as

$$\psi_{n\ell m}(r,\theta,\phi)=R_n(r)Y_{\ell m}(\theta,\phi).$$

State (without proof) the corresponding eigenvalues of

(a) the z component of the angular momentum operator, $\widehat{L}_z$,

(b) the square of the angular momentum operator, $\widehat{L}^2$. [4]

For the hydrogen atom, with

$$V(r)=-\frac{e^2}{4\pi\epsilon_0 r},$$

show that

$$R_n(r)=Ar^\ell\exp\left(-\frac{r}{a_0(\ell+1)}\right)$$

is an energy eigenstate and find an expression for a_0. Determine the energy E and the normalisation constant A. [6]

For the ground state, estimate the probability for the electron to be found within the proton (of radius r_p), and give reasons why this is only an estimate. [4]

Also for the ground state, show that classically the electron must be at radii $\leq 2a_0$, and find the probability that the electron lies outside this region. [6]

[*You may assume that* $\int_0^\infty t^\rho e^{-t}\,\mathrm{d}t=\rho!$ *and that the* $Y_{\ell m}$ *functions are normalised with* $\int |Y_{\ell m}|^2\,\mathrm{d}\Omega=1$.]

SECTION C

C10 Write an essay on angular momentum in quantum mechanics, including a discussion of orbital and spin angular momentum, the Stern–Gerlach experiment and singlet and triplet spin states. [20]

C11 Write brief notes on **two** of the following:

(a) tunnelling in quantum mechanics and its applications; [10]

(b) the treatment of time-dependence in quantum mechanics; [10]

(c) Compton scattering. [10]

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SECTION D

D12 Write brief notes on **two** of the following:

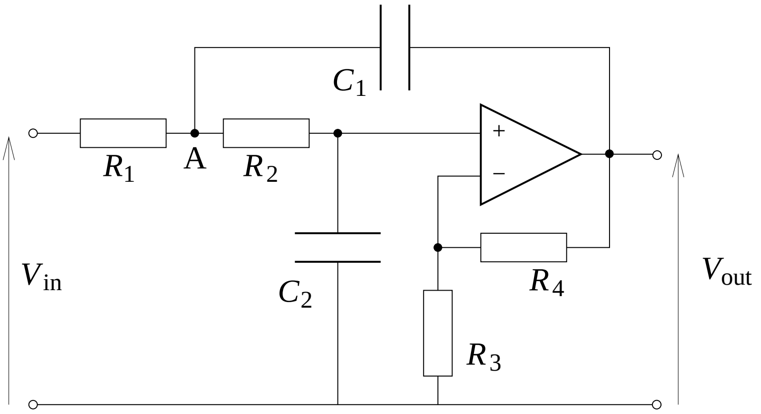
(a) the χ^2 test; [10]

(b) Nyquist's theorem; [10]

(c) shielding experimental equipment from electric and magnetic fields. [10]

D13 State the rules (sometimes called the 'golden rules') that apply to an idealised operational amplifier. [2]

An ideal operational amplifier is connected with four resistors and two capacitors to an a.c. voltage source $V = \mathrm{Re}(V_{\mathrm{in}} e^{i\omega t})$ as shown below.



For this circuit:

(a) in the special case that the values of C_1 and C_2 are zero, show that the voltage gain of the circuit is [3]

$$\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} = \left(1 + \frac{R_4}{R_3}\right) = K,$$

where K is a constant;

(b) in the general case, with C_1 and C_2 both non-zero, by considering the currents through R_2 and C_2, or otherwise, show that the voltage at point A is [3]

$$V_A = \frac{V_{\mathrm{out}}}{K}\,(1 + i\omega R_2 C_2)\,;$$

(c) hence show that the complex voltage gain of the circuit in the general case is [4]

$$\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} = \frac{K}{1 - \omega^2 C_1 C_2 R_1 R_2 + i\omega[C_1 R_1 (1 - K) + C_2 (R_1 + R_2)]}.$$

In the general case, if the values of resistors and capacitors are chosen such that $R_1 = R_2 = R_3 = 10\ \mathrm{k\Omega}$, $R_4 = 40\ \mathrm{k\Omega}$ and $C_1 = C_2 = 10\ \mathrm{nF}$, sketch the amplitude and phase of the voltage gain. [6]

Comment on how this circuit might be useful. [2]

END OF PAPER