Exercise 3

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1. Derivation of LDA

1.1 Compute $\hat{b}$

$$0 = \frac{\partial}{\partial b} \sum_{i=1}^{N} (w^T x_i + b - y_i)^2 = \sum_{i=1}^{N} \frac{\partial}{\partial b} (w^T x_i + b - y_i)^2$$

$$= \sum_{i=1}^{N} \frac{\partial}{\partial b} \left((w^T x_i)^2 + b^2 + y_i^2 + 2(w^T x_i)(b - y_i) - 2 b y_i \right)$$

$$= \sum_{i=1}^{N} 0 + 2b + 0 + 2w^T x_i - 0 - 2y_i$$

$$= 2N \cdot b + 2 \sum_{i=1}^{N} w^T x_i - y_i = 2Nb + 2 \sum_{i=1}^{N} w^T x_i - \underbrace{\sum_{i=1}^{N} y_i}_{=0}$$

$$\hat{b} = -\frac{2}{N} b w^T \sum_{i=1}^{N} x_i$$

1.2

$$0 = \frac{\partial}{\partial w} \sum_{i=1}^{N} \left(w^T x_i - y_i - \frac{w^T}{N} \sum_{i=1}^{N} x_i \right)^2$$

$$= \sum_{i=1}^{N} 2 \left(w^T x_i - y_i - \frac{w^T}{N} \sum_{j}^{N} x_j \right) \left(x_i - 0 - \frac{1}{N} \sum_{k}^{N} x_k \right)$$

$$= 2 \sum_{i=1}^{N} \left[w^T x_i^2 - \underline{y_i x_i} - \frac{w^T}{N} \left(\sum_{j=1}^{N} x_j \right) x_i - \frac{w^T x_i}{N} \sum_{k=1}^{N} x_k + \underline{\frac{y_i}{N} \sum_{k=1}^{N} x_k} + \left(\frac{w^T}{N} \sum_{j=1}^{N} x_j \right) \left(\frac{1}{N} \sum_{k=1}^{N} x_k \right) \right]$$

$$= 2 w^T \sum_{i=1}^{N} x_i^2 - \frac{2}{N} \left(\sum_{j=1}^{N} x_j \right) x_i + \left(\frac{1}{N} \sum_{j=1}^{N} x_j \right)^2 + 2 \sum_{i=1}^{N} \frac{y_i}{N} \sum_{j=1}^{N} x_j - 2 \sum_{i=1}^{N} y_i x_i$$

$$= 2 \frac{w^T}{N} \sum_{i=1}^{N} \left(x_i - \frac{1}{N} \sum_{j=1}^{N} x_j \right)^2 + \frac{2\mu}{N} \underbrace{\sum_{i=1}^{N} y_i}_{=0} - \frac{2}{N} \sum_{i \in \{i \mid y_i = 1\}} y_i x_i - \frac{2}{N} \sum_{i \in \{i \mid y_i = -1\}} y_i x_i$$

$$= 2 \frac{w^T}{N} \sum_{i=1}^{N} \left(x_i - \frac{1}{N} \sum_{j=1}^{N} x_j \right)^2 - \frac{2}{2N_1} \sum_{i \in \{i \mid y_i = 1\}} x_i + \frac{2}{2N_2} \sum_{i \in \{i \mid y_i = -1\}} x_i$$

$$= 2 \frac{w^T}{N} \sum_{i=1}^{N} \left(x_i - \frac{1}{2} (\mu_1 + \mu_{-1}) \right)^2 - \mu_1 + \mu_{-1}$$

$$\Rightarrow \frac{w^T}{N} \sum_{i=1}^{N} \left(x_i - \frac{1}{2} (\mu_1 + \mu_{-1}) \right)^2 = \frac{\mu_1 - \mu_{-1}}{2}$$

$$\vdots$$

$$\left(S_w + \frac{1}{n} S_B \right) w^T = \frac{\mu_1 - \mu_{-1}}{2}$$