M.Sc. Simulation Sciences, Summer Semester 2023

Fast Iterative Solvers

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Project 2

Due: July 25, 2023, 11.59pm

Summary

We implement a multigrid solver for the Poisson equation

$$\begin{aligned} -\nabla^2 u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\Omega = (0,1) \times (0,1)$, using a finite difference discretization on a Cartesian grid,

$$\mathcal{G}_h := \{(ih, jh) : i, j = 0, \ldots, N;\ hN = 1\}.$$

This means, find $u_{i,j} \approx u(x_i, y_j) = u(ih, jh)$, such that

$$\begin{aligned} -f_{i,j} &= \frac{1}{h^2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) + \frac{1}{h^2}\left(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right), && i, j = 1, \ldots, N-1 \\ u_{i,j} &= 0, && \text{otherwise.} \end{aligned}$$

Use $f(x,y) = 8\pi^2 \sin(2\pi x)\sin(2\pi y)$. For this choice, the solution is $u(x,y) = \sin(2\pi x)\sin(2\pi y)$.

Instructions

- Use meshes with $N = 2^n$ for the fine mesh. For the next coarser mesh, use $N^c := N/2$. This means that points in the coarse mesh will also be points in the fine mesh, while every other point in the fine mesh is deleted.
- Mandatory: Implement the Gauss-Seidel Smoother, restriction, and prolongation as defined in the tutorials
- Optional: you may implement and test other choices for these operators
- Use W-cycles (i.e. $\gamma = 2$, as discussed in class).
- You should use as many multigrid levels as possible. Use the same iterative solver on each mesh level. (Recall: One should solve exactly on the coarsest mesh. If there is only one interior grid point on the coarsest mesh, Gauss-Seidel becomes exact in one step!)
- Plot the convergence using the measure $||\mathbf{r}^{(m)}||_\infty / ||\mathbf{r}^{(0)}||_\infty$ against multigrid iterations m for meshes with $n = 4$, $n = 7$ (resulting in $N = 16$, and $N = 128$). Use a semi-log scale, and do this for

1. $\nu_1 = \nu_2 = 1$
2. $\nu_1 = 2,\ \nu_2 = 1$

where the ν_i are the Gauss-Seidel pre- and post smoothing iterations, $\mathbf{r}^{(m)}$ is the residual evaluated at the m^{th} iteration. For the initial guess, $m = 0$, you may use =0. We define $||\mathbf{r}||_\infty := \max_{i,j} |r_{i,j}|$, where (i,j) ranges over all interior points.

- You may optionally want to do more numerical experiments:
  - For instance, you may want to verify the claim that it doesn't make sense to do too many smoothing iterations, by measuring the convergence against *run-time* (instead of iteration), and increase the number of smoothing steps ν.
  - You may also want to compare the run-times obtained from using different values of γ, such as $\gamma = 1$ (V-cycle), or higher values of γ.