PEKING UNIVERSITY

Homework #1

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Course: *Optimization Methods in Machine Learning* – Professor: *Prof. Zhouchen Lin*
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Question 1

The no-free-lunch (NFL) theorem is actually quite general. Give another instance that you think the NFL theorem holds.

Answer.

- While the airplane is less time-consuming than any other transportations, e.g., cars, high-speed railway, for a long journey, they can outperform airplane for different specific situations.
- All possible travel cases taken into consideration, all transportations are necessary and perform on average equally well.
- There is no universally better transportation exist.

Question 2

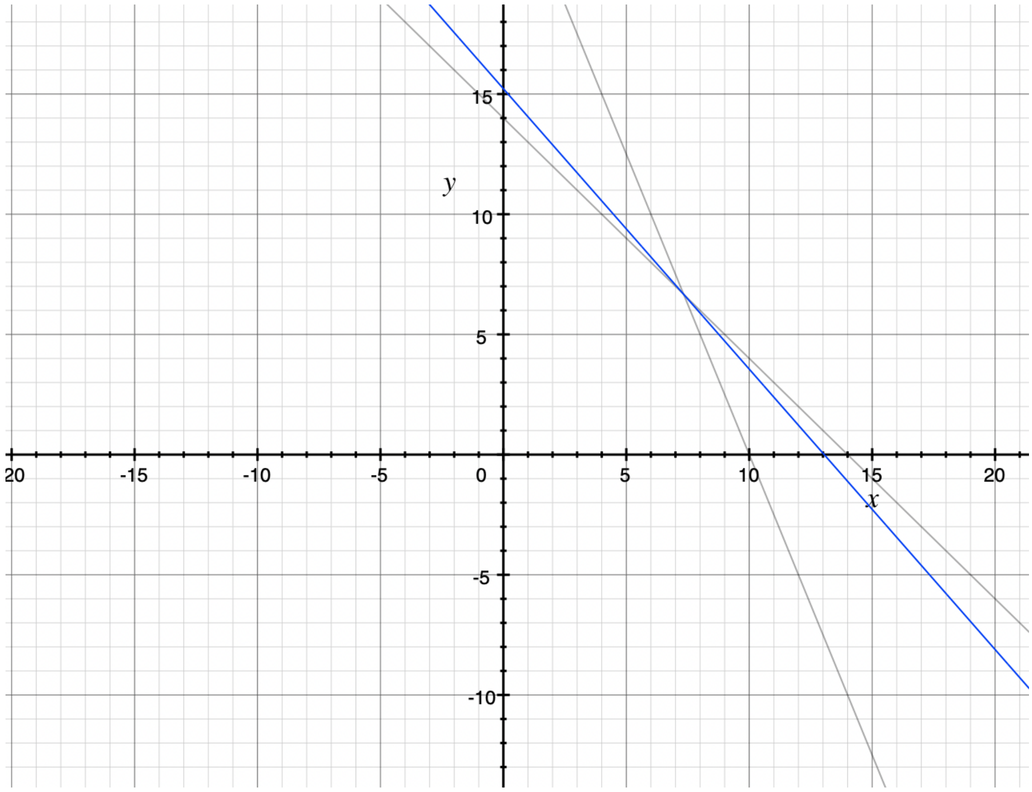
Write down an optimization problem, in the mathematical form, that you encounter in your daily life (do not copy from textbooks). Describe why you formulate in that way.

Answer.
Suppose that I want to take several part-time jobs during my one-week holiday, the working time is expected to be less than 14 hours. There are two coffee houses and these two jobs as a waiter will cost me a deposit of 25/hour and 10/hour, respectively. The deposit will be paid before going to the job. I can afford no more than 300. The salary per hour in two coffee houses are 60/hour and 40/hour, respectively. How can I distribute my time to earn more money?

Sol: Let x and y be the working time at two coffee houses, then

$$min_{x,y} f(x,y) = 35x + 30y$$

$$s.t. x + y \leq 14, 25x + 10y \leq 300$$



By simple linear programming, we can calculate

$$x_m = \frac{22}{3}, y_m = \frac{20}{3}, f_m = \frac{1370}{3}$$

□

Question 3

Judge the properties of the following sets (openess, closeness, boundedness, compactness) and give their interiors, closures, and boundaries:

a. $\mathcal{C}_1 = \emptyset$.

b. $\mathcal{C}_2 = \mathbb{R}^n$.

c. $\mathcal{C}_3 = \{x|0 \leq x < 1\} \cup \{x|2 \leq x \leq 3\} \cup \{x|4 < x \leq 5\}$.

d. $\mathcal{C}_4 = \{(x, y)^T|x \geq 0, y > 0\}$.

e. $\mathcal{C}_5 = \{k|k \in \mathbb{Z}\}$.

f. $\mathcal{C}_6 = \{k^{-1}|k \in \mathbb{Z}, k \neq 0\}$.

g. $\mathcal{C}_7 = \{(1/k, \sin k)^T|k \in \mathbb{Z}, k \neq 0\}$.

Answer.

a. properties: openness, closeness, boundedness, compactness.
interiors: $\emptyset$.
closures: $\emptyset$.
boundaries: $\emptyset$.

b. properties: openness, closeness.
   interiors: $\mathbb{R}^n$.
   closures: $\mathbb{R}^n$.
   boundaries: $\emptyset$.

c. properties: boundedness.
   interiors: $\{x|0 < x < 1\} \cup \{x|2 < x < 3\} \cup \{x|4 < x < 5\}$.
   closures: $\{x|0 \leq x \leq 1\} \cup \{x|2 \leq x \leq 3\} \cup \{x|4 \leq x \leq 5\}$.
   boundaries: $\{0, 1, 2, 3, 4, 5\}$.

d. properties: none.
   interiors: $\{(x, y)^T|x > 0, y > 0\}$.
   closures: $\{(x, y)^T|x \geq 0, y \geq 0\}$.
   boundaries: $\{(x, y)^T|xy \geq 0, x \geq 0, y \geq 0\}$.

e. properties: closeness.
   interiors: $\emptyset$.
   closures: $\{k|k \in \mathbb{Z}\}$.
   boundaries: $\{k|k \in \mathbb{Z}\}$.

f. properties: boundedness.
   interiors: $\emptyset$.
   closures: $\{k^{-1}|k \in \mathbb{Z}, k \neq 0\} \cup \{0\}$.
   boundaries: $\{k^{-1}|k \in \mathbb{Z}, k \neq 0\} \cup \{0\}$.

g. properties: boundedness.
   interiors: $\emptyset$.
   closures: $\{(1/k, \sin k)^T|k \in \mathbb{Z}, k \neq 0\} \cup \{(0, \sin k)^T|k \in \mathbb{Z}, k \neq 0\}$.
   boundaries: $\{(1/k, \sin k)^T|k \in \mathbb{Z}, k \neq 0\} \cup \{(0, \sin k)^T|k \in \mathbb{Z}, k \neq 0\}$.

Question 4

Prove that a point $\mathbf{x}$ is a boundary point of $\mathcal{C} \subseteq \mathbb{R}^n$ iff for $\forall \epsilon > 0$, there exists $\mathbf{y} \in \mathcal{C}$ and $\mathbf{z} \notin \mathcal{C}$ such that

$$\|\mathbf{y} - \mathbf{x}\|_2 \leq \epsilon, \quad \|\mathbf{z} - \mathbf{x}\|_2 \leq \epsilon.$$

Answer.
Proof of Sufficiency:

for $\forall \epsilon > 0$, $B_\epsilon(\mathbf{x}) = \{\mathbf{t} \in \mathbb{R}^n | \|\mathbf{t} - \mathbf{x}\|_2 \leq \epsilon\}$, there exists $\mathbf{y} \in \mathcal{C}$ and $\mathbf{z} \notin \mathcal{C}$ such that $\mathbf{y}, \mathbf{z} \in U(\mathbf{x}, \epsilon)$, so $\mathbf{x}$ is a boundary point of $\mathcal{C} \subseteq \mathbb{R}^n$.

Proof of Necessity:

$\mathbf{x}$ is a boundary point of $\mathcal{C} \subseteq \mathbb{R}^n$, so $\mathbf{x} \notin \mathcal{C}^\circ$, i.e., $\forall \epsilon > 0, B_\epsilon(\mathbf{x}) \not\subset \mathcal{C}$. $\therefore \exists \mathbf{z} \notin \mathcal{C}$ but $\mathbf{z} \in B_\epsilon(\mathbf{x})$. Also, $\mathbf{x} \notin (\mathcal{C}^c)^\circ$, i.e., $\forall \epsilon > 0, B_\epsilon(\mathbf{x}) \not\subset \mathcal{C}^c$. $\therefore \exists \mathbf{y} \in \mathcal{C}$ but $\mathbf{y} \in B_\epsilon(\mathbf{x})$.So there exists $\mathbf{y} \in \mathcal{C}$ and $\mathbf{z} \notin \mathcal{C}$ such that

$$\|\mathbf{y} - \mathbf{x}\|_2 \leq \epsilon, \quad \|\mathbf{z} - \mathbf{x}\|_2 \leq \epsilon.$$

□

Question 5

Prove that $\mathcal{C} \subseteq \mathbb{R}^n$ is closed iff it contains its boundary, and is open iff it contains no boundary points.

Answer.

- $\mathcal{C} \subseteq \mathbb{R}^n$ is closed iff it contains its boundary.
  - Proof of Sufficiency:
    The boundary of $\mathcal{C} \subseteq \mathbf{R}^n$ is defined as $\partial\mathcal{C} = \overline{\mathcal{C}} \backslash \mathcal{C}^\circ$. $\because$ $\mathcal{C}$ is closed, $\therefore$ $\overline{\mathcal{C}} = \mathcal{C}$, $\partial\mathcal{C} = \mathcal{C}\backslash\mathcal{C}^\circ$, $\therefore \partial\mathcal{C} \subset \mathcal{C}$.
  - Proof of Necessity:
    Assume that $\mathcal{C}$ is not closed, then $\mathcal{C}_1 = \overline{\mathcal{C}}\backslash\mathcal{C} \neq \emptyset$. $\because$ $\partial\mathcal{C} = \overline{\mathcal{C}}\backslash\mathcal{C}^\circ \subset \mathcal{C}$, $\therefore$ $(\mathcal{C}_1 \cup \mathcal{C})\backslash\mathcal{C}^\circ \subset \mathcal{C}$. Also, $\mathcal{C}_1 \cap \mathcal{C} = \emptyset$, $\therefore$ $\mathcal{C}_1 \subset \mathcal{C}^\circ$, $\therefore$ $\overline{\mathcal{C}}\backslash\mathcal{C} \subset \mathcal{C}^\circ \subset \mathcal{C}$. $\therefore$ $\overline{\mathcal{C}} \subset \mathcal{C}$, which is contradictory to the definition of $\mathcal{C}_1$. $\therefore$ $\mathcal{C}$ is closed. □
- $\mathcal{C} \subseteq \mathbb{R}^n$ is open iff it contains no boundary points.
  - Proof of Sufficiency:
    The boundary of $\mathcal{C} \subseteq \mathbf{R}^n$ is defined as $\partial\mathcal{C} = \overline{\mathcal{C}}\backslash\mathcal{C}^\circ$. $\because$ $\mathcal{C}$ is open, $\therefore$ $\mathcal{C}^c$ is closed and contains its boundary. $\therefore$ $\mathcal{C}$ contains no boundary points.
  - Proof of Necessity:
    $\because$ $\mathcal{C}$ contains no boundary points, $\therefore$ $\mathcal{C}^c$ contains its boundary. Similarly, $\mathcal{C}^c$ is closed. $\therefore$ $\mathcal{C}$ is open. □

Question 6

Prove the following:
a. $\overline{\mathcal{A}\cup\mathcal{B}} = \overline{\mathcal{A}} \cup \overline{\mathcal{B}}$; $\overline{\mathcal{A}\cap\mathcal{B}} \subseteq \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$. Give an example showing that $\overline{\mathcal{A}\cap\mathcal{B}} \neq \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$.

b. $(\mathcal{A}\cap\mathcal{B})^\circ = \mathcal{A}^\circ \cap \mathcal{B}^\circ$; $(\mathcal{A}\cup\mathcal{B})^\circ \supseteq \mathcal{A}^\circ \cup \mathcal{B}^\circ$. Give an example showing that $(\mathcal{A}\cup\mathcal{B})^\circ \neq \mathcal{A}^\circ \cup \mathcal{B}^\circ$.

Answer.

a. $\overline{\mathcal{A}\cup\mathcal{B}} = (((\mathcal{A}\cup\mathcal{B})^c)^\circ)^c = ((\mathcal{A}^c\cap\mathcal{B}^c)^\circ)^c$. Assume that $(\mathcal{A}\cap\mathcal{B})^\circ = \mathcal{A}^\circ\cap\mathcal{B}^\circ$ (will be proved later in b.), then $\overline{\mathcal{A}\cup\mathcal{B}} = ((\mathcal{A}^c)^\circ \cap (\mathcal{B}^c)^\circ)^c = ((\mathcal{A}^c)^\circ)^c \cup ((\mathcal{B}^c)^\circ)^c = \overline{\mathcal{A}} \cup \overline{\mathcal{B}}$;

$\overline{\mathcal{A}\cap\mathcal{B}} = (((\mathcal{A}\cap\mathcal{B})^c)^\circ)^c = ((\mathcal{A}^c\cup\mathcal{B}^c)^\circ)^c$. Assume that $(\mathcal{A}\cup\mathcal{B})^\circ \supseteq \mathcal{A}^\circ\cup\mathcal{B}^\circ$ (will be proved later in b.), then $\overline{\mathcal{A}\cap\mathcal{B}} \subseteq ((\mathcal{A}^c)^\circ \cup (\mathcal{B}^c)^\circ)^c = ((\mathcal{A}^c)^\circ)^c \cap ((\mathcal{B}^c)^\circ)^c = \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$; □

Example: $\mathcal{A} = \{\mathbf{x}| -\infty < \mathbf{x} \leq 0\} \cup \{2\} \cup \{\mathbf{x}|4 < \mathbf{x} < +\infty\}$, $\mathcal{B} = \{\mathbf{x}| -\infty < \mathbf{x} < 2\} \cup \{\mathbf{x}|2 < \mathbf{x} \leq 4\} \cup \{\mathbf{x}|8 \leq \mathbf{x} < +\infty\}$ then $\overline{\mathcal{A}\cap\mathcal{B}} = \{\mathbf{x}|-\infty < \mathbf{x} \leq 0\} \cup \{\mathbf{x}|8 \leq \mathbf{x} < +\infty\}$, but $\overline{\mathcal{A}} \cap \overline{\mathcal{B}} = \{\mathbf{x}| -\infty < \mathbf{x} \leq 0\} \cup \{2, 4\} \cup \{\mathbf{x}|8 \leq \mathbf{x} < +\infty\}$, they are unequal.

b. $(\mathcal{A}\cap\mathcal{B})^\circ = \{\mathbf{x}|\exists\epsilon > 0 \text{ such that } \mathcal{B}_\epsilon(\mathbf{x}) \subset \mathcal{A}\cap\mathcal{B}\} = \{\mathbf{x}|\exists\epsilon > 0 \text{ such that } \mathcal{B}_\epsilon(\mathbf{x}) \subset \mathcal{A}\} \cap \{\mathbf{x}|\exists\epsilon > 0 \text{ such that } \mathcal{B}_\epsilon(\mathbf{x}) \subset \mathcal{B}\} = \mathcal{A}^\circ \cap \mathcal{B}^\circ$;

$(\mathcal{A}\cup\mathcal{B})^\circ = \{\mathbf{x}|\exists\epsilon > 0 \text{ such that } \mathcal{B}_\epsilon(\mathbf{x}) \subset \mathcal{A}\cup\mathcal{B}\} \supseteq \{\mathbf{x}|\exists\epsilon > 0 \text{ such that } \mathcal{B}_\epsilon(\mathbf{x}) \subset \mathcal{A}\} \cup \{\mathbf{x}|\exists\epsilon > 0 \text{ such that } \mathcal{B}_\epsilon(\mathbf{x}) \subset \mathcal{B}\} = \mathcal{A}^\circ \cup \mathcal{B}^\circ$ (union set will introduce new interior points); □

Example: $\mathcal{A} = \{\mathbf{x}|0 < \mathbf{x} < 2\} \cup \{\mathbf{x}|2 < \mathbf{x} \leq 4\}$, $\mathcal{B} = \{2\} \cup \{\mathbf{x}|4 < \mathbf{x} < 8\}$ then $(\mathcal{A} \cup \mathcal{B})^\circ = \{\mathbf{x}|0 < \mathbf{x} < 8\}$, but $\mathcal{A}^\circ \cup \mathcal{B}^\circ = \{\mathbf{x}|0 < \mathbf{x} < 2\} \cup \{\mathbf{x}|2 < \mathbf{x} < 4\} \cup \{\mathbf{x}|4 < \mathbf{x} < 8\}$, they are unequal.