Part IB: Complex Analysis
Examples Sheet 1

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1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a real linear map. Regarding T as a map from $\mathbb{C}$ into $\mathbb{C}$ by identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, show that there exist unique complex numbers A, B such that for every $z \in \mathbb{C}$, $T(z) = Az + B\bar{z}$. Show that T is complex differentiable if and only if $B = 0$.

Solution: Recall that we identify $\mathbb{C} = \mathbb{R}^2$ via $x + iy = (x, y)$ for $x, y \in \mathbb{R}$. The action of T is then determined by linearity via its action on a basis of $\mathbb{C} = \mathbb{R}^2$, viewed as a real vector space; choosing the basis $1 = (1, 0)$ and $i = (0, 1)$, we can write:

$$T(x+iy) = xT(1) + yT(i) = \left(\frac{T(1)+T(i)}{2}\right)(x+iy) + \left(\frac{T(1)-T(i)}{2}\right)(x-iy)$$

This completes the proof of existence, with:

$$A = \frac{T(1)+T(i)}{2}, \qquad B = \frac{T(1)-T(i)}{2}.$$

To prove uniqueness, note that if $T(z) = Az + B\bar{z} = Cz + D\bar{z}$ for all z, then we can take $z = 1$ and $z = i$ in turn to establish the equations $A + B = C + D$ and $A - B = C - D$. These simultaneous equations imply $A = C$, $B = D$, and hence uniqueness follows.

For the differentiability part of the question, note that if $B = 0$, then $T(z) = Az$ is complex differentiable, directly from the definition:

$$\lim_{h \to 0}\left[\frac{T(z+h)-T(z)}{h}\right] = \lim_{h \to 0}\left[\frac{A(z+h)-Az}{h}\right] = A.$$

Conversely, suppose that T is complex differentiable. Then $T(z) - Az = B\bar{z}$ is complex differentiable, since it is the difference of complex differentiable functions. But then at any point $(x, y) \in \mathbb{R}^2 = \mathbb{C}$, the Cauchy-Riemann equations give:

$$\frac{\partial}{\partial x}\left(\mathrm{Re}(B\bar{z})\right) = \frac{\partial}{\partial x}(Bx) = B = -B = \frac{\partial}{\partial y}(-By) = \frac{\partial}{\partial y}\left(\mathrm{Im}(B\bar{z})\right),$$

which implies $B = 0$. $\square$

2.

(i) Let $f : D \to \mathbb{C}$ be a holomorphic function defined on a domain D. Show that f is constant if any one of its real part, imaginary part, modulus or argument is constant.

(ii) Find all holomorphic functions on $\mathbb{C}$ of the form $f(x+iy) = u(x) + iv(y)$ where u and v are both real-valued.

(iii) Find all holomorphic functions on $\mathbb{C}$ with real part $x^3 - 3xy^2$.

Solution: (i) To begin with, write $f(x+iy) = u(x,y) + iv(x,y)$ where $u : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $v : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ are real functions (with the appropriate identification of D with a real domain). Since f is holomorphic, we have that u, v are differentiable with partial derivatives satisfying the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y), \qquad \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) \tag{$*$}$$

at all points $(x,y) \in D$. Assuming the real part u is constant, its partial derivatives vanish, so that the Cauchy-Riemann equations $(*)$ reduce to:

$$\frac{\partial v}{\partial x}(x,y) = 0, \qquad \frac{\partial v}{\partial y}(x,y) = 0$$

at all points $(x,y) \in D$. This implies that the derivative of v is zero throughout the domain D, which (since D is path-connected) implies that v is constant throughout the domain D; hence f is constant throughout D. Exactly the same proof applies for the imaginary part being constant.

Now suppose that the modulus is constant. Again, we write $f(x+iy) = u(x,y) + iv(x,y)$ where $u : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $v : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ are real functions. The modulus of f is a function $|f| : D \to \mathbb{R}$ given by:

$$|f|(x+iy) = u(x,y)^2 + v(x,y)^2.$$

Assuming that this function is the constant function, we can take partial derivatives of this equation with respect to x and with respect to y to yield (using the chain rule):

$$\frac{\partial}{\partial x}\left(u(x,y)^2 + v(x,y)^2\right) = 2u(x,y)\frac{\partial u}{\partial x}(x,y) + 2v(x,y)\frac{\partial v}{\partial x}(x,y) = 0,$$

and:

$$\frac{\partial}{\partial y}\left(u(x,y)^2 + v(x,y)^2\right) = 2u(x,y)\frac{\partial u}{\partial x}(x,y) + 2v(x,y)\frac{\partial v}{\partial y}(x,y) = 0.$$

Let's streamline our notation by writing u_x, u_y, v_x and v_y for the partial derivatives evaluated at (x,y), and write u, v for the values of the functions at (x,y). Then the above equations can be packaged as the matrix equation:

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If the matrix on the left hand side is invertible at some fixed point $(x,y) \in D \subseteq \mathbb{R}^2$, then there is a unique solution to this linear equation given by $u(x,y) = v(x,y) = 0$, which implies $f(x+iy) = 0$ at the corresponding point $x+iy \in D \subseteq \mathbb{C}$. But since the modulus of f is constant throughout D, we have $|f| \equiv 0$ everywhere in D, which implies $f \equiv 0$ everywhere in D. Thus f is constant, and zero, if the above matrix is invertible at any point $(x,y) \in D$.

Suppose therefore that the matrix is invertible nowhere in D. The determinant of the matrix must therefore vanish everywhere in the domain D. Using the Cauchy-Riemann equations $(*)$, this is equivalent to the conditions:

$$0 = u_x v_y - v_x u_y = u_x^2 + u_y^2.$$

But this equation implies that $u_x = u_y = 0$ throughout D, so that the real part u is constant throughout D (since D is path connected), and hence f is constant throughout D by the earlier work.

Finally, suppose that the argument is constant (where we define the argument to lie in the range $(-\pi, \pi]$). We consider two possible cases:

- If $\arg(f) \neq \pi$, then it follows that the image of f is contained in the complex plane minus the negative real axis, $\mathrm{im}(f) \subseteq \mathbb{C}\backslash\mathbb{R}_{\leq 0}$ (note zero can also be removed, since $\arg(0)$ is ill-defined, so we assume f is always non-zero). Hence the principal branch of the logarithm $\mathrm{Log} : \mathbb{C}\backslash\mathbb{R}_{\leq 0} \to \mathbb{C}$ is defined on the image of f, so we can construct the function $\mathrm{Log}(f)$. Since both Log and f are holomorphic, it follows that their composition is holomorphic, so $\mathrm{Log}(f)$ is holomorphic.

  Now, we can write f in modulus-argument form as $f(x+iy) = r(x,y)e^{i\theta}$ where $\theta \in (-\pi, \pi)$ is constant and $r : \mathbb{R}^2 \to \mathbb{R}_{>0}$ is the modulus. Using the definition of the principal branch of the logarithm, we have:

  $$\mathrm{Log}(f)(x,y) = \log(r(x,y)) + i\theta,$$

  and hence $\mathrm{Log}(f)$ is a holomorphic function on D with constant imaginary part. It follows by the earlier work that $\mathrm{Log}(f)$ is constant. But Log is a branch of the logarithm, hence $e^{\mathrm{Log}(f)} = f$, and it follows that f is also constant.

- If $\arg(f) = \pi$, we simply choose a different branch of the logarithm, and the above argument still works. For example, choose a branch where the positive real axis is removed.

(ii) Suppose that $f(x+iy) = u(x) + iv(y)$ is holomorphic on $\mathbb{C}$. Then the Cauchy-Riemann equations $(*)$ hold at all points $(x,y) \in \mathbb{R}^2 = \mathbb{C}$. Since u has no y dependence and v has no x dependence, the second of the equations $(*)$ is trivially satisfied, leaving only the equation:

$$\frac{du}{dx}(x) = \frac{dv}{dy}(y).$$

Separating variables, this implies:

$$\frac{du}{dx}(x) = c, \qquad \frac{dv}{dy}(y) = c,$$

for some real constant c. Integrating both equations directly, we have:

$$u(x) = cx + u(0), \qquad v(y) = cy + v(0).$$

This gives the *necessary* form for any such $f(z)$:

$$f(z) = cz + f(0).$$

Furthermore, this is clearly holomorphic, so we have shown that such a form is *sufficient* too. Hence $f(x+iy) = u(x) + iv(y)$ if and only if $f(z) = cz + f(0)$ for some $c \in \mathbb{R}$, $f(0) \in \mathbb{C}$.

(iii) We seek holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ of the form $f(x+iy) = x^3 - 3xy^2 + iv(x,y)$ where $v : \mathbb{R}^2 \to \mathbb{R}$ is a real function. Assuming such a function exists, it must obey the Cauchy-Riemann equations throughout $\mathbb{C}$; at all points $(x,y) \in \mathbb{R}^2$ therefore, we must have:

$$\frac{\partial}{\partial x}(x^3 - 3xy^2) = \frac{\partial v}{\partial y}, \qquad \frac{\partial}{\partial y}(x^3 - 3xy^2) = -\frac{\partial v}{\partial x}.$$

Taking the partial derivatives, we see that we must solve the system of partial differential equations:

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2, \qquad \frac{\partial v}{\partial x} = 6xy.$$

Integrating the first equation, treating x as a constant, we have:

$$v(x, y) = 3x^2y - y^3 + c(x),$$

where $c(x) \in \mathbb{R}$ is an x-dependent constant. Furthermore, viewed as a function of x, the constant $c(x)$ must be differentiable for the partial derivative of v with respect to x to exist. Inserting this into the second equation then, we find:

$$6xy + \frac{dc}{dx} = 6xy \qquad \Rightarrow \qquad \frac{dc}{dx} = 0,$$

and it follows that c is identically constant, independent of both x and y. Hence we have derive the necessary form for $v(x, y)$:

$$v(x, y) = 3x^2y - y^3 + c,$$

from which a necessary form for f follows:

$$f(z) = f(x + iy) = x^3 - 3xy^2 + i(3x^2y - y^3 + c) = z^3 + ic,$$

with $c \in \mathbb{R}$. Furthermore, this form is clearly holomorphic, so this form is sufficient too, and we have indeed found all functions with the given real part.

❊ **Comments:** (i) Note that the use of a *domain* for part (i) is essential. If $f : U \to \mathbb{C}$ is defined on an open set only, the result isn't true. For example, if we define the open set $U = \mathbb{C}\backslash i\mathbb{R}$ to be the complex plane with the imaginary axis deleted, then the function $f : U \to \mathbb{C}$ defined by:

$$f(z) = \begin{cases} 1 & \text{if } \mathrm{Re}(z) < 0, \\ 2 & \text{if } \mathrm{Re}(z) > 0, \end{cases}$$

is clearly holomorphic with constant argument ($\arg(f) = 0$), but is non-constant. The problem arises because U is not path-connected.

3.

(i) Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$, and
$$f(z) = \frac{(1+i)x^3 - (1-i)y^3}{x^2+y^2} \quad \text{for } z = x + iy \neq 0.$$
Show that f satisfies the Cauchy-Riemann equations at 0. Show further that f is continuous everywhere but is not differentiable at 0.

(ii) Define $g : \mathbb{C} \to \mathbb{C}$ by $g(0) = 0$ and $g(z) = e^{-1/z^4}$ for $z \neq 0$. Show that g satisfies the Cauchy-Riemann equations everywhere, but is neither continuous nor differentiable at 0.

Solution: (i) The real and imaginary parts $u : \mathbb{R}^2 \to \mathbb{R}, v : \mathbb{R}^2 \to \mathbb{R}$ of f are given by $u(0,0) = 0, v(0,0) = 0$, and:
$$u(x,y) = \frac{x^3 - y^3}{x^2+y^2}, \qquad v(x,y) = \frac{x^3+y^3}{x^2+y^2}.$$
when $(x,y) \neq (0,0)$. Taking the partial derivatives at zero, we have:
$$\frac{\partial u}{\partial x}(0,0) = \lim_{h\to 0}\left[\frac{u(h,0)-u(0,0)}{h}\right] = \lim_{h\to 0}\left[\frac{h^3}{h^3}\right] = 1,$$
$$\frac{\partial u}{\partial y}(0,0) = \lim_{h\to 0}\left[\frac{u(0,h)-u(0,0)}{h}\right] = -\lim_{h\to 0}\left[\frac{h^3}{h^3}\right] = -1,$$
$$\frac{\partial v}{\partial x}(0,0) = \lim_{h\to 0}\left[\frac{v(h,0)-v(0,0)}{h}\right] = \lim_{h\to 0}\left[\frac{h^3}{h^3}\right] = 1,$$
$$\frac{\partial v}{\partial y}(0,0) = \lim_{h\to 0}\left[\frac{v(0,h)-v(0,0)}{h}\right] = \lim_{h\to 0}\left[\frac{h^3}{h^3}\right] = 1.$$
It follows immediately that the Cauchy-Riemann equations holds at 0.

Obviously the function is continuous at all points except 0, since it is the sum, product and quotient of continuous functions (with non-zero denominator everywhere except 0). To show that the function is continuous at 0 too, let $\epsilon > 0$. Choose $\delta = \epsilon/4$. Then for all $x+iy$ such that $|x+iy| = \sqrt{x^2+y^2} < \delta$, we have must have $|x| < \delta$ and $|y| < \delta$ separately. It follows that:
$$|f(z)| = \left|\frac{x^3 - y^3 + (x^3+y^3)i}{x^2+y^2}\right| \leq \frac{|x|^3}{x^2+y^2} + \frac{|y|^3}{x^2+y^2} + \frac{|x|^3}{x^2+y^2} + \frac{|y|^3}{x^2+y^2},$$
by the triangle inequality. Bounding further, we have:
$$|f(z)| \leq \frac{2|x|^3}{x^2+y^2} + \frac{2|y|^3}{x^2+y^2} \leq 2|x| + 2|y| < 4\delta = \epsilon.$$
Hence f is continuous at the origin.

Finally, we must show that f is not differentiable at 0. It isn't specified in the question whether real or complex differentiability is meant here; however, because the Cauchy-Riemann equations hold at 0, f is real differentiable at 0 in a multivariable sense if and only if it is complex differentiable at 0, so the two notions are actually equivalent here. To show f is not real differentiable at 0 in the sense of multivariable calculus, we first recall that a multivariable function is differentiable if and only if all of its components are differentiable, so it is sufficient to show that one of the real or imaginary parts of f is not differentiable. Choose to look at:
$$u(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

We now recall from Part IB Analysis and Topology that if a function f is real differentiable at 0, then all its directional derivatives $\partial_v f|_0$ at 0 must exist (but importantly, the converse is *not* true). Furthermore, its directional derivatives obey:

$$\partial_v f|_0 = df|_0(v),$$

where $df|_0$ is the derivative of f at 0 (which we recall is a linear map). For the given function u, consider the directional derivatives along the arbitrary direction (v_1, v_2). We have:

$$\partial_{(v_1,v_2)} u|_0 = \lim_{h\to 0}\left[\frac{u(hv_1, hv_2) - u(0,0)}{h}\right] = \lim_{h\to 0}\left[\frac{h^3 v_1^3 - h^3 v_2^3}{h^3 v_1^2 + h^3 v_2^2}\right] = \frac{v_1^3 - v_2^3}{v_1^2 + v_2^2}.$$

We see that all the directional derivatives exist. *However*, if the function is differentiable, these directional derivatives must match up to the derivative $du|_0$ in a linear way. But the proposed derivative:

$$du|_0(v_1, v_2) = \frac{v_1^3 - v_2^3}{v_1^2 + v_2^2}$$

is clearly non-linear in (v_1, v_2). For example, using the computation of the directional (partial) derivatives of u along the x and y-axes from earlier, we observe that:

$$\partial_{(1,0)} u|_0 + 2\partial_{(0,1)} u|_0 = -1 \neq -\frac{7}{5} = \partial_{(1,2)} u|_0.$$

(ii) To show that the Cauchy-Riemann equations hold everywhere, it's useful to rewrite the equations in an equivalent form. We have that if $g(x+iy) = u(x,y) + iv(x,y)$ is the decomposition of g into its real and imaginary parts $u : \mathbb{R}^2 \to \mathbb{R}$, $v : \mathbb{R}^2 \to \mathbb{R}$ respectively, then:

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y), \qquad \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y)$$

are the Cauchy-Riemann equations at the point $(x,y) \in \mathbb{R}^2 = \mathbb{C}$. We can instead combine these equations into a single equivalent complex equation (compare this with Question 4):

$$\frac{\partial(u+iv)}{\partial x}(x,y) = \frac{\partial(v-iu)}{\partial y}(x,y) \qquad \Leftrightarrow \qquad \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\bigg|_{(x,y)} g = 0.$$

This is the best form to apply to this question. For any point $(x,y) \neq (0,0)$, we have:

$$\frac{\partial}{\partial x}\bigg|_{(x,y)}\left(e^{-1/(x+iy)^4}\right) = \frac{4}{(x+iy)^5}e^{-1/(x+iy)^4}, \qquad \frac{\partial}{\partial y}\bigg|_{(x,y)}\left(e^{-1/(x+iy)^4}\right) = \frac{4i}{(x+iy)^5}e^{-1/(x+iy)^4}.$$

Hence we have:

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\bigg|_{(x,y)} g = \frac{4}{(x+iy)^5}e^{-1/(x+iy)^4} - \frac{4}{(x+iy)^5}e^{-1/(x+iy)^4} = 0,$$

for all points $(x,y) \neq (0,0)$, and it follows that the Cauchy-Riemann equations hold everywhere away from zero. When $(x,y) = (0,0)$, we have the partial derivatives:

$$\frac{\partial}{\partial x}\bigg|_{(0,0)} g = \lim_{h\to 0}\left[\frac{e^{-1/h^4} - 0}{h}\right] = 0, \qquad \frac{\partial}{\partial y}\bigg|_{(0,0)} g = \lim_{h\to 0}\left[\frac{e^{-1/(ih)^4} - 0}{h}\right] = 0,$$

where the variable h is a *real* variable in both limits. The limits exist because $e^{-1/h^4} \to 0$ as $h \to 0$, and exponentials beat polynomials. It follows that the Cauchy-Riemann equations hold at 0 too.

However, the function is not continuous at $z = 0$. For example, the sequence $z_n = e^{i\pi/4}/n$ tends to 0 as $n \to \infty$, but:

$$e^{-1/z_n^4} = e^{n^4} \to \infty$$

diverges. It follows that the function is not differentiable at $z = 0$ (in either the real or complex sense) since continuity is necessary for differentiability.

4.

(i) Define the differential operators:

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Prove that a C^1 function f is holomorphic if and only if $\partial f/\partial \bar{z} = 0$. Show that:

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z},$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the usual Laplacian in $\mathbb{R}^2$.

(ii) Let $f : U \to V$ be holomorphic and let $g : V \to \mathbb{C}$ be harmonic. Show that the composition $g \circ f$ is harmonic.

➻ **Solution:** (i) Let us suppose that $f : U \to \mathbb{C}$, viewed as a function from $U \subseteq \mathbb{R}^2$ to $\mathbb{R}^2$, is a continuously differentiable function on the open set U in the sense of *real* multivariable calculus. Write $f(x+iy) = u(x,y) + iv(x,y)$ where $u : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $v : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ are the real and imaginary parts of f respectively. Then the condition of continuous differentiability implies that the partial derivatives:

$$\frac{\partial u}{\partial x}(x,y), \qquad \frac{\partial u}{\partial y}(x,y), \qquad \frac{\partial v}{\partial x}(x,y), \qquad \frac{\partial v}{\partial y}(x,y)$$

exist and are continuous for all points $(x,y) \in U$. In particular, at any point $z = x + iy = (x,y) \in U$, we can construct the derivative:

$$\frac{\partial f}{\partial \bar{z}}(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\bigg|_{(x,y)}(u+iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x}(x,y) - \frac{\partial v}{\partial y}(x,y)\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x}(x,y) + \frac{\partial u}{\partial y}(x,y)\right).$$

We see that this derivative vanishes at the point $(x,y) \in U$ if and only if:

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y), \qquad \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y).$$

But these equations are precisely the Cauchy-Riemann equations; therefore, since a function is complex differentiable at the point $x + iy = (x,y)$ if and only if it is differentiable in the sense of real multivariable calculus there *and* satisfies the Cauchy-Riemann equations, it follows that the C^1 function $f : U \to \mathbb{C}$ is holomorphic on U if and only if:

$$\frac{\partial f}{\partial \bar{z}}(z) = 0$$

holds at every point $z \in U$ (note also this argument applies even if f is merely real differentiable rather than C^1).

Next, we are asked to show how to write the Laplacian in terms of the derivatives $\partial/\partial z$ and $\partial/\partial \bar{z}$. We simply note that at the point $(x,y) \in U$, the Laplacian acting on a C^2 function $f : U \to \mathbb{C}$ (where by C^2, we mean continuously twice-differentiable in the *real* multivariable sense) is given by:

$$\Delta|_{(x,y)} f = \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\bigg|_{(x,y)} f = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\bigg|_{(x,y)} f,$$

where, in particular, we used the symmetry of mixed partial derivatives to construct the two factorisations. Hence we see on C^2 functions, we have:

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z},$$

as required.

(ii) Before embarking on this part of the question, we notice that we will want to take the partial derivatives $\partial/\partial z$, $\partial/\partial \bar{z}$ of compositions and products of functions. Hence, it would be useful to derive some general rules for manipulating these operators; we will address this now. Suppose that $f : U \to \mathbb{C}$, with $U \subseteq \mathbb{C}$ open, is differentiable when viewed as a real multivariable function. By definition, near any point $(x, y) \in U$, we can write:

$$f(x+h_1, y+h_2) = f(x,y) + df|_{(x,y)}(h_1, h_2) + o\left(\sqrt{h_1^2 + h_2^2}\right),$$

where $o\left(\sqrt{h_1^2+h_2^2}\right)$ denotes a function which goes to zero faster than $\sqrt{h_1^2+h_2^2}$ as $h_1, h_2 \to 0$, and $df|_{(x,y)} : \mathbb{R}^2 = \mathbb{C} \to \mathbb{C}$ is a linear operator, the derivative. We know from Part IB Analysis and Topology that we can write this linear operator in the standard basis $\{(1,0),(0,1)\}$ in terms of the partial derivatives of f, yielding the relationship:

$$\begin{aligned} f(x+h_1, y+h_2) &= f(x,y) + h_1 \frac{\partial f}{\partial x}\bigg|_{(x,y)} + h_2 \frac{\partial f}{\partial y}\bigg|_{(x,y)} + o\left(\sqrt{h_1^2+h_2^2}\right) \\ &= f(x,y) + (h_1 + ih_2)\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\bigg|_{(x,y)} f + (h_1 - ih_2)\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\bigg|_{(x,y)} f + o\left(\sqrt{h_1^2+h_2^2}\right) \\ &= f(x,y) + h\frac{\partial f}{\partial z}\bigg|_{(x,y)} + \bar{h}\frac{\partial f}{\partial \bar{z}}\bigg|_{(x,y)} + o\left(\sqrt{h_1^2+h_2^2}\right), \end{aligned}$$

where $h = h_1 + ih_2$ and $\bar{h} = h_1 - ih_2$. Succinctly, we have for all $z \in U$:

$$\boxed{f(z+h) = f(z) + h\frac{\partial f}{\partial z}(z) + \bar{h}\frac{\partial f}{\partial \bar{z}}(z) + o(|h|).}$$

Indeed, one could take this as the *definition* of the derivatives $\partial f/\partial z$ and $\partial f/\partial \bar{z}$. In particular, this expression makes it very clear that the vanishing of the derivative $\partial f/\partial \bar{z}$ gives us the definition of the complex derivative in terms of a limit:

$$f'(z) = \frac{\partial f}{\partial z}(z) = \lim_{h\to 0}\left[\frac{f(z+h) - f(h)}{h}\right].$$

Now consider two functions $f : U \to V$, $g : V \to \mathbb{C}$ which are both differentiable when viewed as real, multivariable functions. Then we have:

$$\begin{aligned} g(f(z+h)) &= g\left(f(z) + h\frac{\partial f}{\partial z}(z) + \bar{h}\frac{\partial f}{\partial \bar{z}}(z) + o(|h|)\right) \\ &= g(f(z)) + \left(h\frac{\partial f}{\partial z}(z) + \bar{h}\frac{\partial f}{\partial \bar{z}}(z)\right)\frac{\partial g}{\partial z}(f(z)) + \left(\bar{h}\overline{\frac{\partial f}{\partial z}}(z) + h\overline{\frac{\partial f}{\partial \bar{z}}}(z)\right)\frac{\partial g}{\partial \bar{z}}(f(z)) + o(|h|) \\ &= g(f(z)) + h\left(\frac{\partial f}{\partial z}(z)\frac{\partial g}{\partial z}(f(z)) + \overline{\frac{\partial f}{\partial \bar{z}}}(z)\frac{\partial g}{\partial \bar{z}}(f(z))\right) + \bar{h}\left(\frac{\partial f}{\partial \bar{z}}(z)\frac{\partial g}{\partial z}(f(z)) + \overline{\frac{\partial f}{\partial z}}(z)\frac{\partial g}{\partial \bar{z}}(f(z))\right) + o(|h|). \end{aligned}$$

Hence we see that we have the following chain rules for the $\partial/\partial z$ and $\partial/\partial \bar{z}$ derivatives:

$$\boxed{\frac{\partial}{\partial z}(g\circ f) = \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\circ f + \overline{\frac{\partial f}{\partial \bar{z}}}\frac{\partial g}{\partial \bar{z}}\circ f, \qquad \frac{\partial}{\partial \bar{z}}(g\circ f) = \frac{\partial f}{\partial \bar{z}}\frac{\partial g}{\partial z}\circ f + \overline{\frac{\partial f}{\partial z}}\frac{\partial g}{\partial \bar{z}}\circ f. \qquad (*)}$$

Similarly, we can derive the product rules:

$$\frac{\partial}{\partial z}(g \cdot f) = \frac{\partial g}{\partial z} \cdot f + g \cdot \frac{\partial f}{\partial z}, \qquad \frac{\partial}{\partial \bar{z}}(g \cdot f) = \frac{\partial g}{\partial \bar{z}} \cdot f + g \cdot \frac{\partial f}{\partial \bar{z}}. \tag{†}$$

and the conjugation rules:

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}}, \qquad \overline{\frac{\partial f}{\partial \bar{z}}} = \frac{\partial \bar{f}}{\partial z}. \tag{$\star$}$$

Now, let us assume that $f : U \to V$ is holomorphic (so f is at least once differentiable in the real sense, and indeed C^∞ as alluded to in Question 10, and which you will prove later in the course) and $g : V \to \mathbb{C}$ is harmonic (recall that a function is *harmonic* if it is C^2 and obeys Laplace's equation $g_{xx} + g_{yy} = 0$). By part (i), we have $\partial f/\partial \bar{z} = 0$ throughout U. Therefore using the chain rule ($*$) and the conjugation rule ($\star$), we have:

$$\frac{\partial}{\partial \bar{z}}(g \circ f) = \overline{\frac{\partial f}{\partial z}} \frac{\partial g}{\partial \bar{z}} = \frac{\partial \bar{f}}{\partial \bar{z}} \frac{\partial g}{\partial \bar{z}}.$$

Taking the next derivative (which exists since both f, g are C^2 - the latter by definition, and the former by a proof we have yet to see), we have by the product rule (†):

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}(g \circ f) = \frac{\partial}{\partial z}\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)\frac{\partial g}{\partial \bar{z}} + \frac{\partial \bar{f}}{\partial \bar{z}}\frac{\partial}{\partial z}\left(\frac{\partial g}{\partial \bar{z}}\right).$$

Now, g is harmonic, so the second term vanishes by the final part of (i). The first term can be evaluated using symmetry of the derivatives $\partial/\partial z$ and $\partial/\partial \bar{z}$ on C^2 functions, together with the conjugation rule ($\star$) and holomorphy:

$$\frac{\partial}{\partial z}\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right) = \frac{\partial}{\partial \bar{z}}\left(\frac{\partial \bar{f}}{\partial z}\right) = \frac{\partial}{\partial \bar{z}}\left(\overline{\frac{\partial f}{\partial \bar{z}}}\right) = 0.$$

Hence we have:

$$\Delta(g \circ f) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}(g \circ f) = 0,$$

and so $g \circ f$ is harmonic as required (note also it is the composition of C^2 functions, so is C^2).

❋ **Comments:** The derivatives $\partial/\partial z$ and $\partial/\partial \bar{z}$ are called *Wirtinger derivatives*, and they can be used for lots of different things in complex analysis. In this question, we have shown that Wirtinger derivatives characterise complex differentiability, and that they provide a particularly simple expression for the Laplacian.

It's possible to develop a theory of complex analysis for functions $\mathbb{R}^2 \to \mathbb{R}^2$ that are real differentiable, and for which the 'anti'-Cauchy-Riemann equations hold, namely $\partial f/\partial z = 0$. Such functions are called *anti-holomorphic* and obey the same theorems we see for holomorphic functions (for example Cauchy's theorem), but involving $\bar{z}$ instead of z.

5.

(i) Denote by Log the principal branch of the logarithm. If $z \in \mathbb{C}$, show that $n\mathrm{Log}(1+z/n)$ is defined if n is sufficiently large, and that it tends to z as n tends to ∞. Deduce that for any $z \in \mathbb{C}$,

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = e^z.$$

(ii) Defining $z^\alpha = \exp(\alpha\mathrm{Log}(z))$, where Log is the principal branch of the logarithm and $z \notin \mathbb{R}_{\leq 0}$, show that $\frac{d}{dz}(z^\alpha) = \alpha z^{\alpha-1}$. Does $(zw)^\alpha = z^\alpha w^\alpha$ always hold?

Solution: (i) Recall that the domain of the principal branch of the logarithm is $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$, namely the complex plane with the negative real axis (together with zero) removed. Provided $n > |z|$ say, we have:

$$\left|1-\left(1+\frac{z}{n}\right)\right| = \frac{|z|}{n} < 1,$$

and thus $1+z/n$ is contained in the open disk centred on 1 of radius 1, safely away from the non-positive real axis. Hence $\mathrm{Log}(1+z/n)$ is always defined for sufficiently large n.

Next, recall from lectures that for $|z/n| < 1$, we have the following Taylor series expansion of the principal branch of the logarithm about $z/n = 0$:

$$n\mathrm{Log}\left(1+\frac{z}{n}\right) = n\sum_{r=1}^{\infty}\frac{(-1)^{r-1}z^r}{n^r r} = z + n\sum_{r=2}^{\infty}\frac{(-1)^{r-1}z^r}{n^r r}.$$

To finish, we bound the sum on the right hand side. We have:

$$\left|\sum_{r=2}^{\infty}\frac{(-1)^{r-1}z^r}{n^r r}\right| \leq \sum_{r=2}^{\infty}\frac{|z|^r}{n^r r} \leq \sum_{r=2}^{\infty}\frac{|z|^r}{n^r} = \frac{|z|^2/n^2}{1-|z|/n},$$

using the sum of a geometric progression. Hence we have:

$$\left|n\sum_{r=2}^{\infty}\frac{(-1)^{r-1}z^r}{n^r r}\right| \leq \frac{|z|^2}{n-|z|} \to 0$$

as $n \to \infty$. It follows that:

$$n\mathrm{Log}\left(1+\frac{z}{n}\right) \to z, \qquad \text{as } n \to \infty.$$

For the deduction, simply note that $\exp : \mathbb{C} \to \mathbb{C}$ is a continuous function, so:

$$\exp\left(n\mathrm{Log}\left(1+\frac{z}{n}\right)\right) = \exp\left(\mathrm{Log}\left(1+\frac{z}{n}\right)\right)^n = \left(1+\frac{z}{n}\right)^n \to \exp(z)$$

as $n \to \infty$, as required. $\square$

(ii) Recall from lectures that the derivative of the principal branch of the logarithm is given by:

$$\frac{d}{dz}\mathrm{Log}(z) = \frac{1}{z} = \exp\left(-\mathrm{Log}(z)\right),$$

everywhere in the domain $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$. Hence by the chain rule, we have:

$$\frac{d}{dz}\left(z^\alpha\right) = \frac{d}{dz}\left(\exp\left(\alpha\mathrm{Log}(z)\right)\right) = \alpha\exp\left(-\mathrm{Log}(z)\right)\exp\left(\alpha\mathrm{Log}(z)\right) = \alpha\exp\left((\alpha-1)\mathrm{Log}(z)\right) = \alpha z^{\alpha-1},$$

as required.

Finally, we asked to determine whether it is always true that $(zw)^\alpha = z^\alpha w^\alpha$. First of all, it could be the case that the expression on the right hand is well-defined whilst the expression on the left hand side is not. For example, take $z = i, w = i$ and $zw = -1$. Then z, w lie in the domain of definition $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$, whilst zw does not. Therefore, let us suppose throughout that z, w and zw all lie in $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$.

Begin by writing out both sides explicitly in terms of the definition of the principle branch of the logarithm:

$$(zw)^\alpha = \exp\left(\alpha \mathrm{Log}(zw)\right) = \exp\left(\alpha \log|zw| + i\alpha \arg(zw)\right) = |zw|^\alpha \exp\left(i\alpha \arg(zw)\right),$$

and:

$$\begin{aligned} z^\alpha w^\alpha &= \exp\left(\alpha \mathrm{Log}(z)\right)\exp\left(\alpha \mathrm{Log}(w)\right) = \exp\left(\alpha \log|z| + i\alpha \arg(z)\right)\exp\left(\alpha \log|w| + i\alpha \arg(w)\right) \\ &= \exp\left(\alpha \log|zw|\right)\exp\left(i\alpha(\arg(z) + \arg(w))\right) = |zw|^\alpha \exp\left(i\alpha(\arg(z) + \arg(w))\right). \end{aligned}$$

Now from lectures we know that two complex exponentials are equal to one another if and only if their exponents differ by an integer multiple of $2\pi i$. Hence $(zw)^\alpha = z^\alpha w^\alpha$ holds if and only if there exists an integer n such that:

$$\alpha \arg(zw) = \alpha \arg(z) + \alpha \arg(w) + 2\pi n \qquad \Leftrightarrow \qquad \arg(zw) - \arg(z) - \arg(w) = \frac{2\pi n}{\alpha}.$$

Note we can assume $\alpha \neq 0$, else the result holds trivially.

Now in general, arguments add when we multiply complex numbers together, up to multiples of 2π required to adjust so that we remain in the interval $(-\pi, \pi)$. Explicitly, we have:

$$\arg(z) + \arg(w) = \arg(zw) + \begin{cases} -2\pi & \text{if } \arg(z) + \arg(w) > \pi, \\ 0 & \text{if } -\pi < \arg(z) + \arg(w) < \pi, \\ 2\pi & \text{if } \arg(z) + \arg(w) < -\pi. \end{cases} \tag{†}$$

It follows that the result $(zw)^\alpha = z^\alpha w^\alpha$ holds for all $z, w, zw \in \mathbb{C}\backslash\mathbb{R}_{\leq 0}$ if and only if there exist integers n_1, n_2, n_3 such that:

$$\frac{2\pi n_1}{\alpha} = 2\pi, \qquad \frac{2\pi n_2}{\alpha} = 0, \qquad \frac{2\pi n_3}{\alpha} = -2\pi.$$

Since the middle condition is trivially satisfied, these conditions can be more succinctly stated as: $n_1 = \alpha, n_3 = -\alpha$. Thus $(zw)^\alpha = z^\alpha w^\alpha$ for all $z, w, zw \in \mathbb{C}\backslash\mathbb{R}_{\leq 0}$ if and only if α is an integer.

As a quick counterexample when α is not an integer, we see that all we need to do is to pick z, w such that we end up in the first or the third cases of (†). Explicitly, we see that we may take $\alpha = 1/2$, $z = w = e^{3\pi i/4}$, $zw = e^{3\pi i/2} = e^{-\pi i/2}$. We have:

$$z^\alpha w^\alpha = \exp\left(\frac{1}{2}\cdot\frac{3\pi i}{4}\right)\exp\left(\frac{1}{2}\cdot\frac{3\pi i}{4}\right) = \exp\left(\frac{3\pi i}{4}\right), \qquad (zw)^\alpha = \exp\left(-\frac{1}{2}\cdot\frac{i\pi}{2}\right) = \exp\left(-\frac{i\pi}{4}\right),$$

which are manifestly unequal.

❊ **Comments:** The general approach we took for the last part of the question has also taught us how to fix things when they go wrong. In general we see that we *do* have the result:

$$(zw)^\alpha = z^\alpha w^\alpha e^{i\alpha(\arg(zw) - \arg(z) - \arg(w))},$$

where $z, w, zw \in \mathbb{C}\backslash\mathbb{R}_{\leq 0}$, and the argument of a complex number lies in $(-\pi, \pi]$.

6. Prove that each of the following series converges uniformly on compact (i.e. closed and bounded) subsets of the given domains in $\mathbb{C}$:

$$\text{(a)} \sum_{n=1}^{\infty} \sqrt{n}e^{-nz} \quad \text{on } \{z : 0 < \mathrm{Re}(z)\}; \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{z^n + z^{-n}} \quad \text{on } \left\{z : |z| < \frac{1}{2}\right\}.$$

➭ **Solution:** In this question, it is useful to recall the *Weierstrass M-test*:

Proposition (The Weierstrass M-test): Let $f_n : A \to \mathbb{C}$ be a set of complex functions on a set $A \subseteq \mathbb{C}$. Suppose there is a non-negative real sequence (M_n) such that $|f_n(z)| \leq M_n$ for all $n \geq 1$ and for all $z \in A$. If the series:

$$\sum_{n=1}^{\infty} M_n$$

is convergent, then the series:

$$\sum_{n=1}^{\infty} f_n(z)$$

is uniformly convergent.

Proof: Since the partial sums of the sequence (M_n) are convergent, they are Cauchy. Hence for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $P \geq Q \geq N$, we have (since all M_n are non-negative real numbers):

$$\left|\sum_{n=1}^{P} M_n - \sum_{n=1}^{Q} M_n\right| = \sum_{n=Q+1}^{P} M_n < \epsilon.$$

Hence we have for all $z \in A$:

$$\left|\sum_{n=1}^{P} f_n(z) - \sum_{n=1}^{Q} f_n(z)\right| = \left|\sum_{n=Q+1}^{P} f_n(z)\right| \leq \sum_{n=Q+1}^{P} |f_n(z)| \leq \sum_{n=Q+1}^{P} M_n < \epsilon,$$

and it follows that the partial sums of the sequence $(f_n(z))$ are Cauchy for each fixed value of z, and hence convergent for each fixed value of z (this is *pointwise convergence*).

Let $S(z)$ be the limit of the partial sums for each fixed value of z. Then for all $Q \geq N$ and for all $z \in A$ (with N and ϵ as above), we have:

$$\left|S(z) - \sum_{n=1}^{Q} f(z)\right| = \left|\lim_{P\to\infty} \sum_{n=1}^{P} f(z) - \sum_{n=1}^{Q} f(z)\right| = \lim_{P\to\infty} \left|\sum_{n=1}^{P} f(z) - \sum_{n=1}^{Q} f(z)\right| \leq \epsilon,$$

where the second from equality follows since the modulus is continuous. The final equality follows since each term in the tail of the limit (i.e. $P \geq Q \geq N$) is bounded above by ϵ from the above argument. Since N was independent of z, the partial sums converge *uniformly* to $S(z)$. $\square$

This test is essentially a uniform version of the comparison test for series. We now apply this test to the examples in the question.

(a) Let $C \subseteq \{z : 0 < \mathrm{Re}(z)\}$ be any compact subset of the given domain. Since $|e^{-z}| : C \to \mathbb{R}$ is a continuous function on a compact set, it is bounded above and attains its maximum. Let $K = \max_{z\in C} |e^{-z}| = \max_{z\in C} e^{-\mathrm{Re}(z)}$. Since $C \subseteq \{z : 0 < \mathrm{Re}(z)\}$, we must have $\max_{z\in C} e^{-\mathrm{Re}(z)} = K < 1$. Defining:

$$M_n = \sqrt{n}K^n,$$

we have $|\sqrt{n}e^{-nz}| \leq M_n$ for all $z \in C$. Furthermore,

$$\frac{M_{n+1}}{M_n} = \frac{\sqrt{n+1}K^{n+1}}{\sqrt{n}K^n} = \sqrt{1+\frac{1}{n}}K \to K < 1$$

as $n \to \infty$. Hence by the ratio test, we have:

$$\sum_{n=1}^{\infty} M_n < \infty$$

which implies the uniform convergence of the original series on C, by the Weierstrass M-test. $\square$

(b) Let $C \subseteq \{z : |z| < 1/2\}$ be any compact subset of the given domain. Then there exists $r < 1/2$ such that $C \subseteq \{z : |z| \leq r < 1/2\}$, else there exists a sequence of points in C with limit point on the boundary $|z| = 1/2$ - but since C is closed (by the Heine-Borel theorem), it contains all its limit points.

Now note that for all $z \in C$, we have:

$$\left|\frac{2^n}{z^n + z^{-n}}\right| = \left|\frac{2^n z^n}{1+z^{2n}}\right| \leq \frac{2^n|z|^n}{1-|z|^{2n}},$$

by the reverse triangle inequality applied to the denominator. Considering $|z|^n/(1-|z|^{2n}) : [0,1) \to \mathbb{R}$ as a function of $|z|$ for each n, we see that these functions are differentiable with derivatives:

$$\frac{d}{d|z|}\left(\frac{|z|^n}{1-|z|^{2n}}\right) = \frac{n|z|^{n-1}}{1-|z|^{2n}} + \frac{2n|z|^{3n-1}}{(1-|z|^{2n})^2} \geq 0.$$

In particular, the functions $|z|^n/(1-|z|^{2n})$ are all non-decreasing for each n. Hence for all $z \in C$ and all n, we have:

$$\frac{|z|^n}{1-|z|^{2n}} \leq \frac{r^n}{1-r^{2n}},$$

where $r < 1/2$ was defined above to be the radius of a disk inside $\{z : |z| < 1/2\}$, but containing C. Defining:

$$M_n = \frac{2^n r^n}{1-r^{2n}},$$

we see that we must have:

$$\left|\frac{2^n}{z^n + z^{-n}}\right| \leq M_n.$$

Furthermore, we have:

$$\frac{M_{n+1}}{M_n} = 2r\left(\frac{1-r^{2n+1}}{1-r^{2n}}\right) \to 2r < 1$$

as $n \to \infty$ (since $r < 1/2$, we have $2r < 1$ and $r^{2n}, r^{2n+1} \to 0$ as $n \to \infty$), which implies by the ratio test:

$$\sum_{n=1}^{\infty} M_n < \infty.$$

Hence the original series is uniformly convergent on C by the Weierstrass M-test. $\square$

7. Find conformal equivalences between the following pairs of domains:

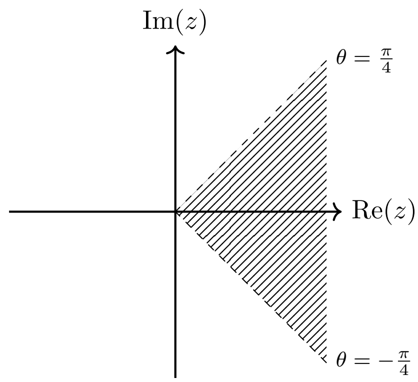
(i) the sector $\{z \in \mathbb{C} : -\pi/4 < \arg(z) < \pi/4\}$ and the open unit disc $D(0,1)$;

(ii) the lune $\{z \in \mathbb{C} : |z-1| < \sqrt{2} \text{ and } |z+1| < \sqrt{2}\}$ and $D(0,1)$;

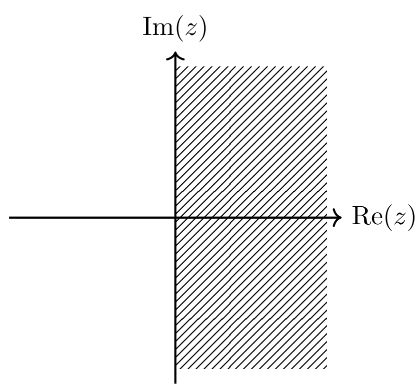
(iii) the strip $S = \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < 1\}$ and the quadrant $Q = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$.

By considering a suitable solution of Laplace's equation $u_{xx} + u_{yy} = 0$ on S, find a non-constant harmonic function ϕ on Q which extends continuously to $\bar{Q}\backslash\{0\}$ with constant values on each of the two components of $\partial Q\backslash\{0\}$. ($\phi$ need not be continuous at the origin. Here $\bar{Q}$ denotes the closure of Q in $\mathbb{R}^2$ and $\partial Q = \bar{Q}\backslash Q$.)

Solution: (i) In the first part of the question, we note that we want to transform a region with some straight segments as a boundary to a region with a circle as a boundary - therefore, we will need to use a Möbius transformation at some point. Let's begin by drawing the sector:



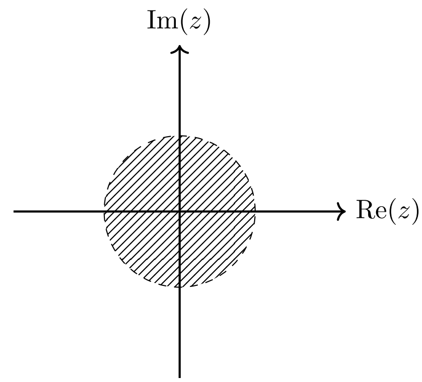
We note that the boundary of this sector isn't a line, so a Möbius transformation won't be very useful immediately (since Möbius transformations map circles to lines). Instead, we first transform the boundary of this sector to a line by applying the transformation $z \mapsto z^2$. This is a obviously a bijection between the right half plane and the sector given above. Furthermore, at all points in the sector, we have $(z^2)' = 2z \neq 0$, so the map is conformal. The result is that we obtain the region:



The boundary of this region is indeed a straight line, so we can now map the boundary to the boundary of the unit circle. To come up with an appropriate Möbius transformation, we know that it is sufficient to check that three points on the imaginary axis map to three points on the unit circle. Let $0 \mapsto -1$, $i \mapsto i$ and $\infty \mapsto 1$ under the transformation (so that the positive imaginary axis gets mapped to the upper semicircular boundary of the disk). Then we see that the appropriate Möbius map is:

$$z \mapsto \frac{z-1}{z+1}$$

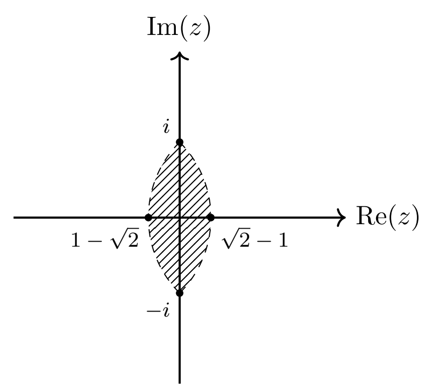
Since Möbius maps correspond to rigid motions of the Riemann sphere, we know that connected components are mapped to connected components under the these transformations. Noting that $1 \mapsto 0$ under this transformation, we see that the right half plane is mapped into the *interior* of the disk. We end up with the region:



Finally, recall from lectures that Möbius maps are conformal. It follows that a conformal equivalence between the regions is given by $f : \{z \in \mathbb{C} : -\pi/4 < \arg(z) < \pi/4\} \to D(0,1)$, with:

$$f(z) = \frac{z^2-1}{z^2+1}.$$

(ii) In the second part of the question, we want to map the following lune to the unit disk:



The boundary of this region is formed of two separate circular arcs (part of circles of radius $\sqrt{2}$), so we suppose that a Möbius transformation might help us out. Such a Möbius transformation could map one of the arcs to a straight segment, and the other arc to another straight segment, and we would then be in the same position we were in at the start of part (i).

A useful Möbius transformation might be one which maps the left arc to the straight line segment emanating from the origin inclined at $\pi/4$ radians to the real axis; this is precisely the straight line segment that we started out with in part (i). To effect this, we choose a Möbius transformation such that $-i \mapsto 0, 1-\sqrt{2} \mapsto \frac{1}{\sqrt{2}}(1+i)$ and $i \mapsto \infty$. Explicitly, we have:

$$z \mapsto -\frac{z+i}{z-i}.$$

Clearly this maps $-i$ to 0 and i to ∞; the overall minus sign is to ensure that $1-\sqrt{2} \mapsto \frac{1}{\sqrt{2}}(1+i)$ - to see why, note that:

$$\frac{1-\sqrt{2}+i}{1-\sqrt{2}-i} = \frac{(1-\sqrt{2})^2 - 1 + 2(1-\sqrt{2})i}{(1-\sqrt{2})^2+1} = \frac{(1-\sqrt{2})(1+i)}{2-\sqrt{2}} = \frac{(1-2)(1+i)}{(2-\sqrt{2})(1+\sqrt{2})} = -\frac{1+i}{\sqrt{2}}.$$

Furthermore, we note that this map takes $\sqrt{2}-1$ to the point $\frac{1}{\sqrt{2}}(1-i)$:

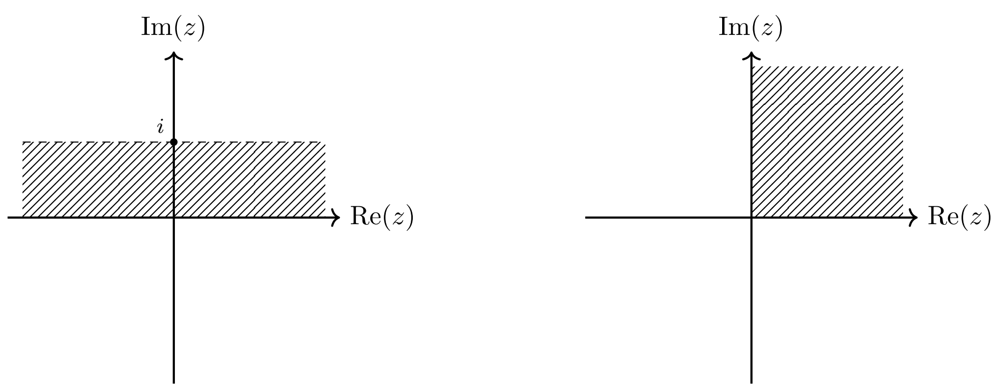
$$-\frac{\sqrt{2}-1+i}{\sqrt{2}-1-i} = -\frac{1-\sqrt{2}-i}{1-\sqrt{2}+i} = -\left(\frac{1-\sqrt{2}+i}{1-\sqrt{2}-i}\right)^* = \frac{1-i}{\sqrt{2}}.$$

So this Möbius transformation maps the left arc of the lune to the straight line segment from 0 to ∞ inclined at $\pi/4$ radians to the real axis, and the right arc of the lune to the straight line segment from 0 to ∞ inclined at $-\pi/4$ radians to the real axis. Noting that $0 \mapsto 1$ under the Möbius transformation, and recalling that connected components must be mapped to connected components, we see that we have constructed a map from the lune to the sector we started with in part (i).

Composing the Möbius map we found at the start of this part with the map we found in part (i), we get the conformal equivalence $f : \{z \in \mathbb{C} : |z-1| < \sqrt{2}$ and $|z+1| < \sqrt{2}\} \to D(0,1)$ given by:

$$f(z) = \frac{\left(-\frac{z+i}{z-i}\right)^2 - 1}{\left(-\frac{z+i}{z-i}\right)^2 + 1} = \frac{(z+i)^2 - (z-i)^2}{(z+i)^2 + (z-i)^2} = \frac{2iz}{z^2-1}.$$

(iii) For the final part, we want to map a strip to a quadrant, both shown below:



We use an exponential; since $\exp(z) = \exp(x)\exp(iy)$, we see that exponentials turn real parts into moduli and imaginary parts into arguments. Here, the strip has imaginary part ranging from 0 to 1, and we want arguments in the range 0 to $\pi/2$, so we consider the exponential:

$$z \mapsto \exp\left(\frac{\pi z}{2}\right).$$

This has non-zero derivative everywhere, since $\exp(\pi z/2)' = (\pi/2)\exp(\pi z/2) \neq 0$ for any $z \in \mathbb{C}$. Furthermore, it is injective, for if:

$$\exp\left(\frac{\pi(x+iy)}{2}\right) = \exp\left(\frac{\pi x}{2}\right)\exp\left(\frac{\pi iy}{2}\right) = \exp\left(\frac{\pi x'}{2}\right)\exp\left(\frac{\pi iy'}{2}\right) = \exp\left(\frac{\pi(x'+iy')}{2}\right),$$

it follows that $x = x'$ and $\pi y/2 = \pi y'/2 + 2\pi n$ for some $n \in \mathbb{Z}$. But then $y = y' + 4n$ for some $n \in \mathbb{Z}$, and for y, y' both in the initial domain this can occur only if $n = 0$. Finally, the map is clearly surjective, since ranging over $x \in (-\infty, \infty)$ and $y \in (0,1)$ in the exponential map $\exp(\pi x/2)\exp(\pi iy/2)$ gives all points in the required quadrant. Thus the map is bijective and has non-zero derivative everywhere, so is a conformal equivalence $f : S \to Q$ given by:

$$f(z) = \exp\left(\frac{\pi z}{2}\right).$$

For the last part of this question, we apply our results to solving Laplace's equation in a quadrant. Consider Laplace's equation:

$$u_{xx} + u_{yy} = 0$$

for the function $u : \bar{S} \to \mathbb{R}$ on the closure of the strip S (viewed as a real subset) with boundary conditions $u(x,0) = 0$ and $u(x,1) = 1$. Clearly a solution of this boundary value problem is:

$$u(x,y) = y.$$

Now, let $f : S \to Q$ be the conformal equivalence we derived in part (iii) above. Then by Question 4 part (ii), we have that:

$$u \circ f^{-1} : Q \to \mathbb{R}$$

is harmonic on Q. The inverse function f^{-1} is given on the region Q explicitly by:

$$f^{-1}(x+iy) = \frac{2}{\pi}\mathrm{Log}(x+iy) = \frac{2}{\pi}\log(\sqrt{x^2+y^2}) + \frac{2i}{\pi}\arctan\left(\frac{y}{x}\right),$$

where Log is the principal branch of the logarithm (note also that the $\arctan$ expression for the argument is valid in the quadrant Q). Hence our final harmonic function on Q is:

$$u \circ f^{-1}(x,y) = \frac{2}{\pi}\arctan\left(\frac{y}{x}\right).$$

We can check that on the x-axis, i.e. $y = 0$, we have the boundary value 0, and on the y-axis, i.e. $x = 0$, we have the boundary value 1.

8.

(i) Show that the most general Möbius transformation which maps the unit disk onto itself has the form:

$$z \mapsto \lambda \frac{z-a}{\bar{a}z-1},$$

with $|a| < 1$ and $|\lambda| = 1$. [*Hint: first show that these maps form a group.*]

(ii) Find a Möbius transformation taking the region between the circles $\{|z| = 1\}$ and $\{|z-1| = 5/2\}$ to an annulus $\{1 < |z| < R\}$. [*Hint : a circle can be described by an equation of the form* $|z-a|/|z-b| = l$.]

(iii) Find a conformal map from an infinite strip onto an annulus. Can such a map ever be a Möbius transformation?

Solution: Begin by noting that all of these Möbius transformations do indeed map the unit disk into itself. Suppose that $|z| = 1$. Then we have, for $|\lambda| = 1$ and $|a| < 1$:

$$\left|\lambda \frac{z-a}{\bar{a}z-1}\right|^2 = \frac{|z-a|^2}{|\bar{a}z-1|^2} = \frac{1-2\operatorname{Re}(\bar{a}z)+|a|^2}{|a|^2-2\operatorname{Re}(\bar{a}z)+1} = 1,$$

and so the boundary of the unit circle is indeed mapped to itself under such Möbius transformations. Noting that $|\lambda(0-a)/(0-1)| = |a| < 1$, we see that 0 maps to the interior of the disk under such transformations, and hence the whole interior of the unit disk is indeed mapped to itself under such transformations.

It remains to prove that *all* Möbius transformations which map the unit disk onto itself are of this form; for this part of the proof, it is useful to observe that the set of all Möbius transformations which map the unit disk to itself form a group:

- Closure. Suppose that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the extended complex plane) are Möbius transformations with the properties:

  $$|z| = 1 \Rightarrow |f(z)| = |g(z)| = 1, \qquad |z| < 1 \Rightarrow |f(z)|, |g(z)| < 1.$$

  Then if $|z| = 1$, we have $|f(g(z))| = 1$ since $|g(z)| = 1$, and if $|z| < 1$, we have $|f(g(z))| < 1$ since $|g(z)| < 1$. Thus we have closure.

- Associativity. Function composition is associative.

- Identity. Clearly $\mathrm{id}(z) = z$ maps the unit disk to itself and is an identity for the group.

- Inverses. Suppose that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius transformation with the properties $|z| = 1 \Rightarrow |f(z)| = 1$ and $|z| < 1 \Rightarrow |f(z)| < 1$. Furthermore, since f is a Möbius transformation we must have $|z| > 1 \Rightarrow |f(z)| > 1$, since this forms a connected component outside of the circle $|z| = |f(z)| = 1$; this implies that all these implications are actually equivalences, and we can replace $\Rightarrow$ with $\Leftrightarrow$ in all of the above. Now, we know that f is invertible because it is a Möbius transformation, and it follows from all the reverse implications taken in the above that its inverse maps the unit disk into itself.

Hence, the set of all Möbius transformations which map the unit disk into itself do indeed form a group.

Now suppose that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius transformation which maps the unit disk onto itself; then $f^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ must similarly map the unit disk onto itself by the existence of group inverses. Let us write $f^{-1}(0) = a$ and $f^{-1}(1) = \lambda(1-a)/(\bar{a}-\lambda)$ for some a, λ (note that we can solve the second equation as a linear equation in λ to derive an explicit expression for λ in terms of $f^{-1}(0)$ and $f^{-1}(1)$ if we so desire). Then we anticipate that f will take the form:

$$f(z) = \lambda \frac{z-a}{\bar{a}z-\lambda}.$$

Therefore, consider the composition of Möbius transformations,

$$g = \left(z \mapsto \lambda \frac{z-a}{\bar{a}z - 1} \right) \circ f^{-1},$$

We note that this transformation maps $0 \mapsto a \mapsto 0$ and $1 \mapsto \lambda(1-a)/(\bar{a}-\lambda) \mapsto 1$, therefore it fixes both 0 and 1. Furthermore, by the group property, we have that this Möbius transformation maps the unit disk to itself. Overall then, we have a Möbius transformation g which fixes 0 and 1, which hence must be of the form:

$$g(z) = \frac{(1-c)z}{1-cz}$$

for some c; further g must map the unit disk to itself. This implies that for $|z| = 1$, we require $|g(z)| = 1$, i.e.

$$1 = |g(z)|^2 = \frac{|(1-c)z|^2}{|1-cz|^2} = \frac{1 - c - \bar{c} + |c|^2}{1 - c\bar{z} - \bar{c}z + |c|^2}.$$

Rearranging, we have:

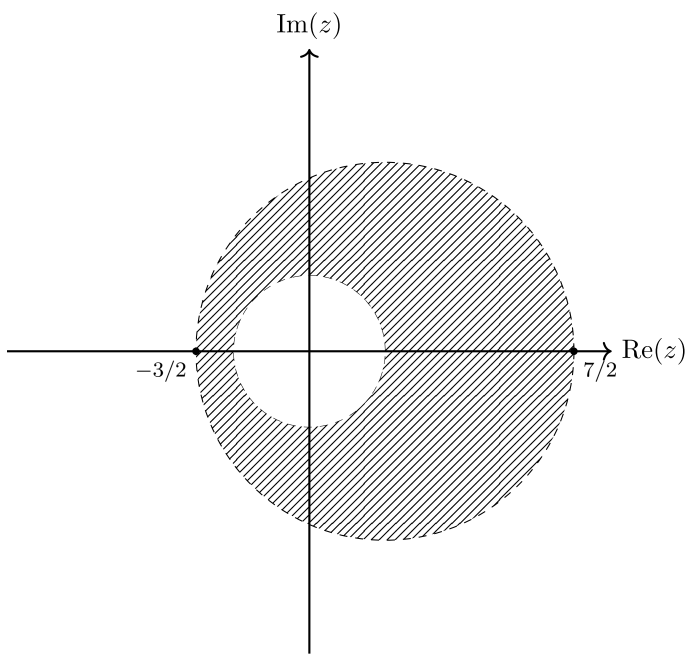
$$c\bar{z} + \bar{c}z = c + \bar{c}.$$

This must hold for all unit modulus complex numbers z; hence we deduce that $c \equiv 0$ and $g \equiv \mathrm{id}$. It follows that:

$$f(z) = \lambda \frac{z-a}{\bar{a}z - 1},$$

as required. □

(ii) In this part of the problem we start with a configuration of circles of the form:



We would like to move the outer circle to a circle centred on the origin, without moving the interior missing disk. Thus we hope that there is a Möbius map of the form in part (i) that does the job.

Consider a Möbius map of the form:

$$f(z) = \lambda \frac{z-a}{\bar{a}z-1}$$

as in part (i). We set $\lambda = -1$, since the only effect of this parameter is to rotate the entire configuration (recall $|\lambda| = 1$; the choice of sign here is merely convenient here as it turns out). Next, we note that the transformation should be symmetric about the real axis, so we expect the preimage of 0, i.e. $f^{-1}(0) = a$ to be a real number. Finally, by symmetry we want this transformation to map the rightmost part of the outer circle and the leftmost part of the inner circle to equal and opposite points, so that:

$$f(-3/2) = -f(7/2).$$

These conditions are enough to fix a. We have:

$$\frac{-\frac{3}{2}-a}{-\frac{3}{2}a-1} = -\left(\frac{\frac{7}{2}-a}{\frac{7}{2}a-1}\right) \qquad \Rightarrow \qquad 4a^2+17a+4=0,$$

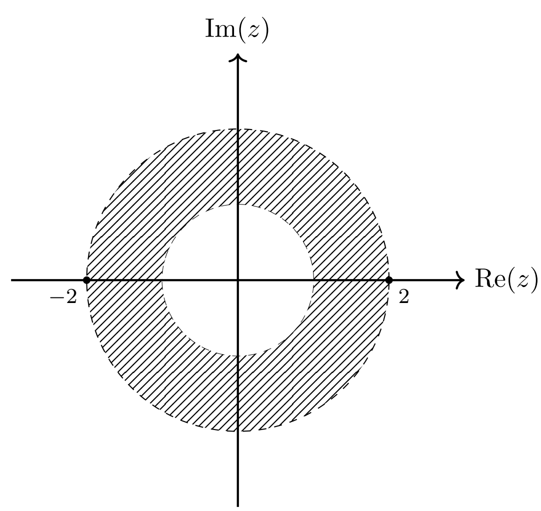
which is a quadratic with roots $a = -4, -1/4$. Clearly $a = -4$ is unacceptable since we require $|a| < 1$ for the transformation to preserve the unit disk. Thus we have $a = -1/4$ and the required initial transformation is:

$$f(z) = \frac{z+1/4}{z/4+1} = \frac{4z+1}{z+4}.$$

This transformation clearly preserves the unit disk since it is a transformation of the form given in part (i). Furthermore, for the points $-3/2, 7/2$ and $1+5i/2$ on the initial outer circle, we have:

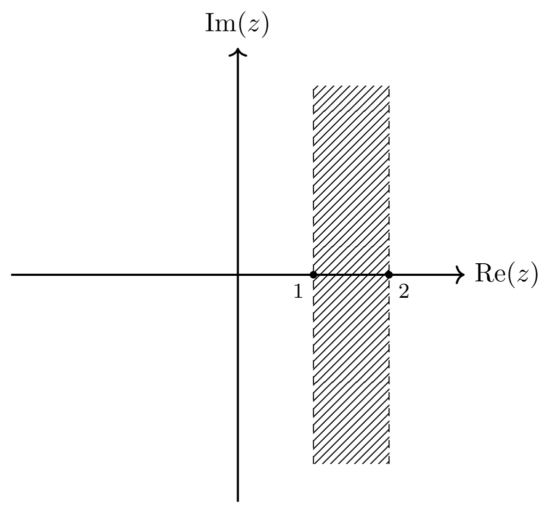
$$f\left(-\frac{3}{2}\right) = -2, \qquad f\left(\frac{7}{2}\right) = 2, \qquad f\left(1+\frac{5i}{2}\right) = \frac{2}{5}\left(4+3i\right),$$

with the final expression lying on the circle centred on 0 and of radius 2. Hence the outer circle is mapped to a circle of radius 2 centred on the origin:

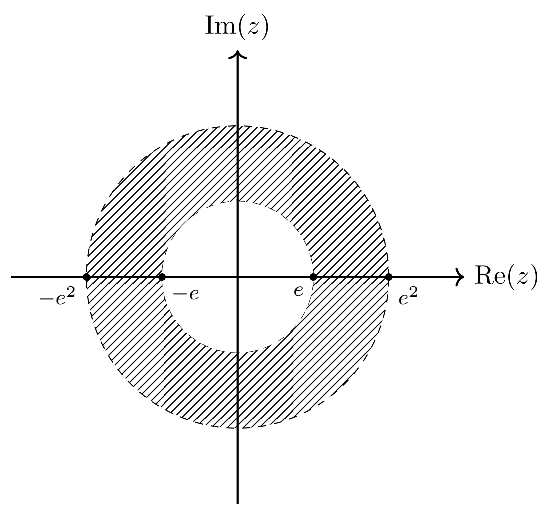


We see that the required radius is $R = 2$.

(iii) Recall that exponentials are good at turning real parts into moduli and imaginary parts into arguments. Consider therefore a strip of the form:



similar to Question 7(iii). We have $\exp(x+iy) = \exp(x)\exp(iy)$ so for $x+iy$ in this strip, the modulus varies between e and e^2. The argument varies from $-\infty$ to ∞, so the result is an annulus of the form:



Note the map is conformal, because $\exp(z)' = \exp(z) \neq 0$, however this is *not* a conformal *equivalence* because the map isn't a bijection.

We note that such a map can never be a Möbius map, because on the Riemann sphere a strip manifests as a *lune*, whilst an annulus is a *band* on the surface of the Riemann sphere - the two cannot be related by a Möbius transformation, which is always a *rigid* transformation of the Riemann sphere.

9. Let $U \subseteq \mathbb{C}$ be open and let $f = u + iv : U \to \mathbb{C}$. Suppose that u and v are C^1 on U as real functions of the real variables x, y, where $x + iy \in U$. Let $w \in U$ and suppose that the map f is angle-preserving at w in the following sense: for any two C^1 curves $\gamma_1, \gamma_2 : (-1, 1) \to U$ with $\gamma_j(0) = w$ and $\gamma_j'(0) \neq 0$ for $j = 1, 2$, the curves $\alpha_j = f \circ \gamma_j = u \circ \gamma_j + iv \circ \gamma_j$ satisfy $\alpha_j'(0) \neq 0$ and

$$\arg \frac{\alpha_1'(0)}{\gamma_1'(0)} = \arg \frac{\alpha_2'(0)}{\gamma_2'(0)}.$$

Show that f is complex differentiable at w with $f'(w) \neq 0$. [You may find it useful to employ the operator $\partial/\partial\bar{z}$ in Q4.]

Solution: Since we are given that f is C^1 viewed as a real multivariable function (its components are both C^1), it is sufficient by the first part of Question 4 to prove that:

$$\frac{\partial f}{\partial \bar{z}}(w) = 0.$$

To show this, begin by noting that $\alpha_j = f \circ \gamma_j$ is C^1 since f and γ_j are C^1. Therefore we can take the derivative of α_j (this is just a derivative with respect to a single real variable for a change!), which is given by the chain rule as:

$$\begin{aligned} \frac{d\alpha_j}{dt}(0) &= \frac{\partial f}{\partial x}\bigg|_w \frac{d\,\mathrm{Re}(\gamma_j)}{dt}(0) + \frac{\partial f}{\partial y}\bigg|_w \frac{d\,\mathrm{Im}(\gamma_j)}{dt}(0) \\ &= \frac{1}{2}\frac{d\gamma_j}{dt}(0)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\bigg|_w f + \frac{1}{2}\frac{d\overline{\gamma_j}}{dt}(0)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\bigg|_w f \\ &= \frac{d\gamma_j}{dt}(0)\frac{\partial f}{\partial z}(w) + \frac{d\overline{\gamma_j}}{dt}(0)\frac{\partial f}{\partial \bar{z}}(w), \end{aligned} \tag{$*$}$$

in terms of the Wirtinger derivatives from Question 4. It follows that:

$$\arg\left(\frac{\alpha_j'(0)}{\gamma_j'(0)}\right) = \arg\left(\frac{\partial f}{\partial z}(w) + \frac{\overline{\gamma_j}'(0)}{\gamma_j'(0)}\frac{\partial f}{\partial \bar{z}}(w)\right),$$

and hence the angle-preserving condition implies that:

$$\arg\left(\frac{\partial f}{\partial z}(w) + \frac{\overline{\gamma_1}'(0)}{\gamma_1'(0)}\frac{\partial f}{\partial \bar{z}}(w)\right) = \arg\left(\frac{\partial f}{\partial z}(w) + \frac{\overline{\gamma_2}'(0)}{\gamma_2'(0)}\frac{\partial f}{\partial \bar{z}}(w)\right).$$

By assumption, this must hold for *all* C^1 curves γ_1, γ_2 passing through w; in other words, the expression:

$$\arg\left(\frac{\partial f}{\partial z}(w) + \frac{\overline{\gamma}'(0)}{\gamma'(0)}\frac{\partial f}{\partial \bar{z}}(w)\right)$$

is independent of the curve γ. Suppose that we take $\gamma : (-1, 1) \to U$ to be $\gamma(t) = w + \epsilon e^{i\theta} t$ where θ is fixed and ϵ is sufficiently small such that $\gamma(t) \in U$ for all t (so we approach w from all possible directions as we vary θ). Then $\overline{\gamma'(0)}/\gamma'(0) = e^{-2i\theta}$, and hence we require:

$$\arg\left(\frac{\partial f}{\partial z}(w) + e^{-2i\theta}\frac{\partial f}{\partial \bar{z}}(w)\right)$$

independent of θ. Geometrically, as θ varies we trace out a circle around the complex number $\partial f/\partial z(w)$ of radius $|\partial f/\partial \bar{z}(w)|$. It follows that this argument is independent of θ if and only if:

$$\frac{\partial f}{\partial \bar{z}}(w) = 0,$$

so that f is indeed complex differentiable at w by Question 4. Note also that $f'(w) = \partial f/\partial z(w) \neq 0$, else by ($*$) we would additionally have have $\alpha_j'(0) = 0$ for all image curves including those with $\gamma_j'(0)$, which contradicts the assumptions.

10. Use the (real) inverse function theorem (from the Analysis & Topology course) to prove the following holomorphic inverse function theorem: if $U \subseteq \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic and $f'(z_0) \neq 0$ for some $z_0 \in U$, then there is an open neighbourhood V of z_0 and an open neighbourhood W of $f(z_0)$ such that $f|_V : V \to W$ is a bijection with holomorphic inverse. [Use the fact that holomorphic functions are C^1, i.e. have C^1 real and imaginary parts; we will prove this - in fact that holomorphic functions are infinitely differentiable - later in the course.]

Solution: We begin by recalling the statement of the inverse function theorem from Part IB Analysis and Topology:

Theorem (The inverse function theorem): Let $f : U \to \mathbb{R}^n$ be a continuously differentiable map on an open subset $U \subseteq \mathbb{R}^m$. Given any point $x = (x_1, ..., x_n) \in U$ such that the derivative $df|_x : \mathbb{R}^m \to \mathbb{R}^n$ is invertible, there exists an open neighbourhood V of x and an open neighbourhood W of $f(x)$ such that the restriction $f|_V : V \to W$ is a bijection with inverse $g : W \to V$, which is itself continuously differentiable at all points in W. Furthermore, the derivative at any point $y \in W$ is given by the explicit formula:

$$dg|_y = df|_{g(y)}^{-1}.$$

Proof: See Part IB Analysis and Topology. □

We now apply this result to the question. Suppose that $f : U \to \mathbb{C}$ is holomorphic, with $f'(z_0) \neq 0$ for some $z_0 \in U$. Consider $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ as a real function, given in terms of components by:

$$f(x, y) = (u(x, y), v(x, y))$$

for real functions $u : \mathbb{R}^2 \to \mathbb{R}$, $v : \mathbb{R}^2 \to \mathbb{R}$. Then the condition that f is complex differentiable at $z_0 = x_0 + iy_0$ is equivalent to f being *real* differentiable at (x_0, y_0), viewed as a real multivariable function, together with the Cauchy-Riemann equations $u_x(x_0, y_0) = v_y(x_0, y_0)$, $u_y(x_0, y_0) = -v_x(x_0, y_0)$. The matrix of the derivative $df|_{(x_0,y_0)}$ is given with respect to the basis $\{(1, 0), (0, 1)\}$ by the Jacobian matrix:

$$df|_{(x_0,y_0)} = \begin{pmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{pmatrix},$$

which has determinant $u_x v_y - u_y v_x = u_x^2 + v_x^2$, making use of the Cauchy-Riemann equations. Hence assuming that $f'(z_0) \neq 0$, we have $u_x^2 + v_x^2 > 0$, and it follows that $df|_{(x_0,y_0)}$ is invertible.

At this point, we use the following fact stated in the question: since f is *holomorphic* on U, it has *continuous* partial derivatives throughout U. As mentioned in the question, we will prove this later in the course (and moreover, we shall prove that holomorphic functions are *infinitely differentiable*). It follows that all of the conditions of the inverse function theorem are satisfied, and hence there exists a neighbourhood V of $z_0 = (x_0, y_0)$ and an open neighbourhood W of $f(z_0) = (u(x_0, y_0), v(x_0, y_0))$ for which the restriction $f|_V : V \to W$ is invertible with a continuously differentiable inverse $g : W \to V$.

It remains to show that g, viewed as a complex function, is holomorphic. Since we already have that g is real differentiable when viewed as a real multivariable function, it remains only to prove that g satisfies the Cauchy-Riemann equations throughout W. To show this, we make use of the explicit form of the derivative of the inverse stated in the inverse function theorem. With respect to the basis $\{(1, 0), (0, 1)\}$, we have that the matrix of $dg|_y$ at any point $y \in W$ is given by:

$$dg|_y = df|_{g(y)}^{-1} = \begin{pmatrix} u_x(g(y)) & u_y(g(y)) \\ v_x(g(y)) & v_y(g(y)) \end{pmatrix}^{-1} = \frac{1}{(u_x v_y - v_x u_y)(g(y))} \begin{pmatrix} v_y(g(y)) & -u_y(g(y)) \\ -v_x(g(y)) & u_x(g(y)) \end{pmatrix}.$$

Note the determinant of this matrix is always non-zero in W, by the statement of the inverse function theorem. Examining the matrix, it is clear that the partial derivatives of the components of g satisfy the Cauchy-Riemann equations at all points $y \in W$ (the top left and bottom right components are equal, and the top right and bottom left components are equal in magnitude but opposite in sign). Hence g is holomorphic, as required.

11. Calculate $\int_\gamma z\sin(z)\,dz$ when γ is the straight line joining 0 to i.

➻ **Solution:** Parametrise the curve $\gamma : [0,1] \to \mathbb{C}$ as $\gamma(t) = it$. Then by the definition of the integral, we have:

$$\int_\gamma z\sin(z)\,dz = \int_0^1 it\sin(it)\,idt = -i\int_0^1 t\sinh(t)\,dt.$$

Integrating by parts, we have:

$$\int_0^1 t\sinh(t)\,dt = [t\cosh(t)]_0^1 - \int_0^1 \cosh(t)\,dt = \cosh(1) - \sinh(1) = \frac{e+e^{-1}}{2} - \frac{e-e^{-1}}{2} = \frac{1}{e}.$$

Hence we have:

$$\int_\gamma z\sin(z)\,dz = -\frac{i}{e}.$$

---

12. Show that the following functions do not have antiderivatives (i.e. functions of which they are the derivatives) on the domains indicated:

(a) $\frac{1}{z} - \frac{1}{z-1} \quad (0 < |z| < 1)$; (b) $\frac{z}{1+z^2} \quad (1 < |z| < \infty)$.

➻ **Solution:** It is sufficient in both cases to produce non-zero integrals of these functions around a closed path in the domain D; for if these functions had antiderivatives, by the fundamental theorem of calculus we would have:

$$\oint_\gamma F'(z)\,dz = \int_a^b F'(\gamma(t))\gamma'(t)\,dt = \int_a^b (F\circ\gamma)'(t)\,dt = F(\gamma(b)) - F(\gamma(a)) = 0,$$

for any path $\gamma : [a,b] \to D$ in the domain D satisfying $\gamma(a) = \gamma(b)$. It will also be useful to use *Cauchy's integral formula for a disk*:

Theorem (Cauchy's integral formula for a disk): Let $D = D(a,r)$ be a disk and let $f : D \to \mathbb{C}$ be holomorphic. For every $w \in D$ and ρ with $|w-a| < \rho < r$, we have:

$$f(w) = \frac{1}{2\pi i}\int_{|z-a|=\rho} \frac{f(z)}{z-w}\,dz.$$

(a) For the first function, note that $1/(z-1)$ has an antiderivative on the domain $\{z : 0 < |z| < 1\}$, since:

$$\frac{d}{dz}\mathrm{Log}(1-z) = -\frac{1}{1-z} = \frac{1}{z-1},$$

where Log denotes the principal branch of the logarithm (and if z is in the unit disk, $-z$ is the unit disk, so $1-z$ is on the open disk centred on 1 of radius 1, safely away from the cut) . Now consider then the closed path $\gamma : [0,2\pi] \to \{z : 0 < |z| < 1\}$ given by $\gamma(\theta) = \frac{1}{2}e^{i\theta}$. If we integrate the relevant function around this path, we have:

$$\oint_\gamma \left(\frac{1}{z} - \frac{1}{z-1}\right)\,dz = 2\pi i \neq 0,$$

using Cauchy's integral formula to evaluate the contribution from the first term, and using the fact that $1/(z-1)$ has an antiderivative on this domain to deduce that the second term doesn't contribute.

(b) For the second function, consider the closed path $\gamma : [0, 2\pi] \to \{z : 1 < |z| < \infty\}$ given by $\gamma(\theta) = 2e^{i\theta}$. Then we have by Cauchy's integral formula:

$$\oint_\gamma \frac{z}{1+z^2}\,dz = \frac{1}{2}\oint_\gamma \left(\frac{1}{z+i} + \frac{1}{z-i}\right)dz = \frac{2\pi i}{2} + \frac{2\pi i}{2} = 2\pi i \neq 0.$$

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13. Does there exist a sequence of polynomials $p_n(z)$ converging uniformly to $1/z$ on: (i) the disk $\{z \in \mathbb{C} : |z-1| < 1/2\}$? (ii) the annulus $\{z \in \mathbb{C} : 1/2 < |z| < 1\}$?

➾ **Solution:** (i) Yes. We consider expanding $1/z$ around $z = 1$. There, we have:

$$\frac{1}{1+(z-1)} = \sum_{r=0}^{\infty}(-1)^r(z-1)^r,$$

using the sum of a geometric progression, for $|z-1| < 1$. Therefore the sequence of polynomials we require is:

$$p_n(z) = \sum_{r=0}^{n}(-1)^r(z-1)^r.$$

That this is *uniformly* convergent follows from a general result for power series: for a power series with radius of convergence R, the series converges uniformly on any disk (centred on the point of expansion) with radius $r < R$; in this case we have $r = 1/2$ and $R = 1$.

(ii) No. For example, we have:

$$\int_{|z|=3/4} \frac{dz}{z} = 2\pi i,$$

but for any polynomial $p_n(z)$ we have:

$$\int_{|z|=3/4} p_n(z)\,dz = 0,$$

since polynomials always have anti-derivatives. If $p_n(z) \to 1/z$ uniformly, we would have by a theorem from lectures (namely we can pass limits through contour integrals, provided that the integrand converges uniformly):

$$\int_{|z|=3/4} p_n(z)\,dz = 0 \qquad \to \qquad 2\pi i = \int_{|z|=3/4} \frac{dz}{z},$$

which is a contradiction.

14. Let $U \subseteq \mathbb{C}$ be a domain, and let $u : U \to \mathbb{R}$ be a C^2 harmonic function. Show that if $z_0 \in U$, then for any disk $D = D(z_0, \rho) \subseteq U$, there is a holomorphic function $f : D \to \mathbb{C}$ such that $u = \mathrm{Re}(f)$ on D. Shown by an example that this need not hold globally, i.e. that there need not exist holomorphic $f : U \to \mathbb{C}$ such that $u = \mathrm{Re}(f)$ on all of U.

➼ **Solution:** We know from lectures that if f is a holomorphic function with real part u, then its derivative can be written in the form:

$$f'(x+iy) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y).$$

Notice that the right hand side defines a holomorphic function, since both $\partial u/\partial x$ and $\partial u/\partial y$ are continuous differentiable (we are given u is C^2) and $u_{xx} = -u_{yy}$, $u_{xy} = u_{yx}$ by the fact that u is harmonic and the symmetry of mixed partial derivatives for C^2 functions, respectively. Now, recall the convex form of Cauchy's theorem:

Cauchy's theorem (for a convex domain): Let U be a convex domain, let $f : U \to \mathbb{C}$ be holomorphic, and let γ be a closed, piecewise C^1 curve in U. Then:

$$\oint_\gamma f(z)\,dz = 0.$$

In particular, since disks $D(z_0, \rho)$ are convex, it follows that:

$$\oint_\gamma \left(\frac{\partial u}{\partial x}(z) - i\frac{\partial u}{\partial y}(z)\right)\,dz = 0$$

for all closed, piecewise C^1 curves γ in U. Recall that this is sufficient for an antiderivative to exist:

Theorem: Let $U \subseteq \mathbb{C}$ be a domain and let $f : U \to \mathbb{C}$ be continuous. If:

$$\oint_\gamma f(z) = 0$$

for every closed, piecewise C^1 curve γ in U, then f has an antiderivative on U.

Hence we can genuinely consider a function $f : D(z_0, \rho) \to \mathbb{C}$ such that:

$$f'(x+iy) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y),$$

throughout the domain $D(z_0, \rho)$.

The function $f : D(z_0, \rho) \to \mathbb{C}$ is clearly holomorphic (it is complex differentiable throughout $D(z_0, \rho)$ by virtue of being an antiderivative). It remains to show that its real part is u. Let $f = U + iV$ be the expression for f in terms of its real and imaginary parts U, V respectively. Then we know its derivative takes the form:

$$f' = U_x - iU_y.$$

Comparing real and imaginary parts, we see that the real multivariable derivative $d(U-u)|_{(x,y)} = 0$ vanishes throughout the disk $D(z_0, \rho)$, hence $U - u$ is constant on the disk (since it is path-connected). It follows that $U = u + \text{constant}$; the constant can be removed simply by a redefinition of f by the constant: $f \mapsto f - \text{constant}$.

The above argument worked because we could apply Cauchy's theorem on a convex domain. The most general type of domain for which this works is a *simply-connected domain* - we will see this later when we study the homotopy form of Cauchy's theorem. We therefore look at a domain which is not simply-connected, for example the unit disk with the origin removed. Here, a harmonic function is:

$$\log(\sqrt{x^2+y^2}),$$

which we anticipate is not the real part of a holomorphic function on this domain, because there is no branch of the logarithm on this domain. We note that if:

$$f(x+iy) = \log(\sqrt{x^2+y^2}) + iv(x,y)$$

is holomorphic on this domain for some v, then:

$$f(x+iy) - \mathrm{Log}(x+iy) = i\left(v(x,y) - \arg(x+iy)\right)$$

is holomorphic on the punctured disk with the negative real axis removed, where Log is taken to be the principal branch of the logarithm. But this is a holomorphic function with constant real part, so must be constant overall: $v(x,y) = \arg(x,y) + c$. It follows that:

$$f(x+iy) = \mathrm{Log}(x+iy) + ic$$

on the punctured disk with the negative real axis removed. But this has no continuous extension to the real axis, and hence f cannot be holomorphic on the entire punctured disk.

Part IB: Complex Analysis
Examples Sheet 2 Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

1. Use the Cauchy integral formula to compute:

$$\oint_{\partial D(0,2)} \frac{dz}{z^2+1} \qquad \text{and} \qquad \oint_{\partial D(0,2)} \frac{dz}{z^2-1}.$$

Are the answers an accident? Formulate and prove a result for a polynomial with n distinct roots.

➭ **Solution:** It's helpful to begin by stating the Cauchy integral formula:

Theorem (Cauchy's integral formula for a disk): Let $D = D(a,r)$ be a disk and let $f : D \to \mathbb{C}$ be holomorphic. For every $w \in D$ and ρ with $|w-a| < \rho < r$, we have:

$$f(w) = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{z-w}\, dz.$$

We can evaluate the given integrals by using partial fractions together with the Cauchy integral formula:

$$\oint_{\partial D(0,2)} \frac{dz}{z^2+1} = \frac{1}{2i} \oint_{\partial D(0,2)} \left(\frac{1}{z-i} - \frac{1}{z+i} \right) dz = \frac{1}{2i} (2\pi i - 2\pi i) = 0.$$

Similarly, we have:

$$\oint_{\partial D(0,2)} \frac{dz}{z^2-1} = \frac{1}{2} \oint_{\partial D(0,2)} \left(\frac{1}{z-1} - \frac{1}{z+1} \right) dz = \frac{1}{2} (2\pi i - 2\pi i) = 0.$$

The answers are not an accident. In general, for polynomials of degree $n = 0, 1$ we can evaluate such integrals immediately using Cauchy's theorem and Cauchy's integral formula:

$$\oint_{\partial D(a,R)} \frac{dz}{A} = 0, \qquad \oint_{\partial D(a,R)} \frac{dz}{A(z-z_1)} = \frac{2\pi i}{A},$$

for constants $A, z_1 \in \mathbb{C}$, with the root $z_1 \in D(a,R)$ inside the open disk in the latter case.

For polynomials of degree $n \geq 2$, like those we worked with in the first part of the question, we find that the following general result holds:

Proposition: Let $P(z)$ be a polynomial with $n \geq 2$ distinct roots $z_1, ..., z_n$. For any open disk $D(a,R) \subseteq \mathbb{C}$ such that $z_1, ..., z_n \in D(a,R)$, we have:

$$\oint_{\partial D(a,R)} \frac{dz}{P(z)} = 0.$$

Proof 1: We can follow the 'partial fraction' method we applied above in the general case. We decompose $1/P(z)$ into partial fractions:

$$\frac{1}{P(z)} = \frac{1}{A(z-z_1)...(z-z_n)} = \frac{1}{A}\left(\frac{a_1}{z-z_1} + \cdots + \frac{a_n}{z-z_n}\right),$$

for some constants $A, a_1, ..., a_n \in \mathbb{C}$ (that this is possible can be proved with some elementary algebra). Combining the partial fractions on the right hand side, we note:

$$\frac{a_1}{z-z_1} + \cdots + \frac{a_n}{z-z_n} = \frac{a_1(z-z_2)(z-z_3)...(z-z_n) + \cdots + a_n(z-z_1)(z-z_2)...(z-z_{n-1})}{(z-z_1)(z-z_2)...(z-z_{n-1})(z-z_n)}.$$

Comparing the coefficient of z^{n-1} in the numerator to 1 yields the relationship: $a_1 + a_2 + \cdots + a_n = 0$. Applying the Cauchy integral formula then, we have:

$$\oint_{\partial D(a,R)} \frac{dz}{P(z)} = \frac{1}{A}\sum_{i=1}^{n} a_i \oint_{\partial D(a,R)} \frac{dz}{z-z_i} = \frac{2\pi i}{A}\sum_{i=1}^{n} a_i = 0. \qquad \square$$

Proof 2: Let us consider bounding the integral. Let $P(z)$ be of degree n, and write $P(z) = a_n z^n + Q(z)$ for a polynomial Q of degree at most z^{n-1}. Then $P(z)/a_n z^n \to 1$ as $z \to \infty$, so there exists R' such that for all $|z| > R'$, we have:

$$\left|\frac{P(z)}{a_n z^n} - 1\right| < \frac{1}{2} \qquad \Rightarrow \qquad ||P(z)| - |a_n||z|^n| < \frac{1}{2}|a_n||z|^n \qquad \Rightarrow \qquad \frac{1}{2}|a_n||z|^n < |P(z)| < \frac{3}{2}|a_n||z|^n,$$

using the reverse triangle inequality in the second implication. Note also that this implies that there are no roots of $P(z)$ in the region $|z| > R'$.

Now for any radius R such that $D(a,R)$ contains all the roots of $P(z)$, the homotopy version of Cauchy's theorem gives us:

$$\oint_{\partial D(a,R')} \frac{dz}{P(z)} = \oint_{\partial D(a,R)} \frac{dz}{P(z)}$$

since $\partial D(a,R')$ and $\partial D(a,R)$ are clearly homotopic curves and $P(z)$ is holomorphic on some annular domain containing $\partial D(a,R)$ and $\partial D(a,R')$. In particular, for all $R'' \geq R'$, we have:

$$\left|\oint_{\partial D(a,R')} \frac{dz}{P(z)}\right| = \left|\oint_{\partial D(a,R'')} \frac{dz}{P(z)}\right| \leq 2\pi R'' \sup_{z\in\partial D(a,R'')} \left|\frac{1}{P(z)}\right| = 2\pi R'' \cdot \frac{2}{|a_n|(R'')^n} = \frac{4\pi}{|a_n|(R'')^{n-1}}.$$

But R'' was arbitrary, so taking the limit $R'' \to \infty$, we see that for $n \geq 2$:

$$\oint_{\partial D(a,R')} \frac{dz}{P(z)} = 0,$$

and hence the result follows. $\square$

Note that the second proof actually implies a stronger result than the question suggests: for any polynomial $P(z)$ of degree $n \geq 2$ with roots $z_1, ..., z_n \in D(a,R)$, *not necessarily distinct*, we have:

$$\oint_{\partial D(a,R)} \frac{1}{P(z)}\, dz = 0.$$

❊ **Comments:** For additional fun, there's another interesting way of proving the result in this question using *integration by substitution*. First, we make this concept rigorous in complex analysis:

Proposition (integration by substitution): Let $f : D \to \mathbb{C}$ be holomorphic on a simply-connected domain D. Let $u : D' \to D$ be a holomorphic map (our 'change of variables') from the domain D' to the domain D. Let $\gamma : [0,1] \to D$ be a piecewise continuously differentiable curve in D, and suppose that $\delta : [0,1] \to D'$ is a piecewise continuously differentiable curve in D' such that $u(\delta(0)) = \gamma(0)$ and $u(\delta(1)) = \gamma(1)$. Then we have:

$$\int_{\gamma} f(z)\,dz = \int_{\delta} f(u(z))u'(z)\,dz.$$

Proof: Since D is simply-connected, by the homotopy version of Cauchy's theorem, the integral of f around any closed contour in D is zero. This implies that f has an antiderivative F on the domain D, so by the fundamental theorem of calculus, we have:

$$\int_{\gamma} f(z)\,dz = F(\gamma(1)) - F(\gamma(0)).$$

On the other hand, $F \circ u$ is holomorphic with derivative $(F \circ u)' = (f \circ u) \cdot u'$. This implies that:

$$\int_{\delta} f(u(z))u'(z)\,dz = F(u(\delta(1))) - F(u(\delta(0))) = F(\gamma(1)) - F(\gamma(0)). \qquad \square$$

In this question, we are inspired by the Riemann sphere to view integrating $1/P(z)$ around a large circular contour containing all roots of $P(z)$ as equivalent to integrating $1/P(z)$ around a large circular contour 'containing infinity'. We can use this idea, together with the substitution $z \mapsto 1/z$, to prove the strong version of the proposition we mentioned earlier:

Proposition: Let $P(z)$ be a polynomial of degree $n \geq 2$. For any open disk $D(a,R) \subseteq \mathbb{C}$ such that all roots of the polynomial are contained in $D(a,R)$, we have:

$$\oint_{\partial D(a,R)} \frac{dz}{P(z)} = 0.$$

Proof 3: Consider making a substitution in the integral, $u(z) = 1/z$. The circle $D(a,R)$ is mapped to some other circle under this transformation (by consideration of the Möbius transformation) which we shall call $D(a',R')$. We thus have:

$$\oint_{\partial D(a,R)} \frac{du}{P(u)} = -\oint_{\partial D(a',R')} \frac{dz}{z^2 P(1/z)},$$

where the substitution was valid because $1/z^2P(1/z) = z^{n-2}/z^nP(1/z)$ is holomorphic (for $n \geq 2$) on the simply-connected domain $D(a',R')$, and u is a holomorphic map (say on an annular domain containing $\mathbb{C}\backslash D(a,R)$) mapping the contour of integration $\partial D(a,R)$ to the contour of integration $\partial D(a',R')$. To show the holomorphy of $z^{n-2}/z^nP(1/z)$ explicitly, we can write $z^{n-2}/z^nP(1/z)$ in terms of the roots of $P(z)$ as:

$$\frac{z^{n-2}}{z^n(1/z - z_1)...(1/z - z_n)} = \frac{z^{n-2}}{(1 - zz_1)...(1 - zz_n)}$$

which shows that the singularities now occur at $z = 1/z_1, ..., z = 1/z_n$, which are on the *exterior* of $D(a',R')$ by the nature of the Möbius transformation. The result now follows immediately by Cauchy's theorem, using the holomorphy of $z^{n-2}/z^nP(1/z)$. $\square$

2.

(i) Use the Cauchy integral formula to compute:

$$\oint_{\partial D(0,1)} \frac{e^{\alpha z}}{2z^2 - 5z + 2}\, dz$$

where $\alpha \in \mathbb{C}$.

(ii) By considering suitable complex integrals, show that if $r \in (0,1)$, we have:

$$\int_0^{\pi} \frac{\cos(n\theta)}{1 - 2r\cos(\theta) + r^2}\, d\theta = \frac{\pi r^n}{1 - r^2} \qquad \text{and} \qquad \int_0^{2\pi} \cos(\cos(\theta))\cosh(\sin(\theta))\, d\theta = 2\pi.$$

➾ **Solution:** (i) We note that the quadratic denominator factorises as $2z^2 - 5z + 2 = (2z-1)(z-2)$, hence we can write the integrand as:

$$\frac{e^{\alpha z}}{2z^2 - 5z + 2} = \frac{e^{\alpha z}/2(z-2)}{z - 1/2}.$$

Note $e^{\alpha z}/2(z-2)$ is holomorphic on the disk $D(0,1)$ and its boundary. So by the Cauchy integral formula, we have:

$$\oint_{\partial D(0,1)} \frac{e^{\alpha z}}{2z^2 - 5z + 2} = 2\pi i \left(\frac{e^{\alpha/2}}{2\left(\frac{1}{2} - 2\right)} \right) = -\frac{2\pi i}{3} e^{\alpha/2}.$$

(ii) For the first integral, the integration range $[0, \pi]$ is more suggestive of a semicircle than a circle, so let's first fix this range so that we can apply the Cauchy integral formula later. Noting that the integrand is an even function of θ, we see that:

$$\int_0^{\pi} \frac{\cos(n\theta)\, d\theta}{1 - 2r\cos(\theta) + r^2} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos(n\theta)\, d\theta}{1 - 2r\cos(\theta) + r^2}.$$

Next, note that we can rewrite the denominator in complex exponential form via:

$$1 - 2r\cos(\theta) + r^2 = 1 - r\left(e^{i\theta} + e^{-i\theta}\right) + r^2 = (1 - re^{i\theta})(1 - re^{-i\theta}).$$

Hence we have:

$$\frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos(n\theta)\, d\theta}{1 - 2r\cos(\theta) + r^2} = \frac{1}{2}\operatorname{Re}\left[\int_{-\pi}^{\pi} \frac{e^{in\theta}\, d\theta}{1 - 2r\cos(\theta) + r^2}\right] = \frac{1}{2}\operatorname{Re}\left[\int_{-\pi}^{\pi} \frac{e^{in\theta}\, d\theta}{(1 - re^{i\theta})(1 - re^{-i\theta})}\right].$$

The argument of the real part is now precisely the contour integral:

$$\int_{-\pi}^{\pi} \frac{e^{in\theta}\, d\theta}{(1 - re^{i\theta})(1 - re^{-i\theta})} = \oint_{\partial D(0,1)} \frac{z^n\, dz}{iz(1 - rz)(1 - r/z)} = \oint_{\partial D(0,1)} \frac{z^n\, dz}{i(1 - rz)(z - r)}.$$

Assume without loss of generality that $n \geq 0$ (note swapping $n \leftrightarrow -n$ is a symmetry of the original integrand). Now for $r \in (0,1)$, we can apply Cauchy's integral formula to this integral to yield:

$$\oint_{\partial D(0,1)} \frac{z^n/(1 - rz)\, dz}{i(z - r)} = \frac{2\pi i}{i}\frac{r^n}{1 - r^2} = \frac{2\pi r^n}{1 - r^2},$$

since $z^n/(1 - rz)$ is holomorphic on the disk $D(0,1)$ and its boundary.

Putting everything together, we have:

$$\int_0^{\pi} \frac{\cos(n\theta)\, d\theta}{1 - 2r\cos(\theta) + r^2} = \frac{\pi r^n}{1 - r^2},$$

as required.

For the second integral, we note the following identity:

$$\begin{aligned}\cos(\cos(\theta) + i\sin(\theta)) &= \cos(\cos(\theta))\cos(i\sin(\theta)) - \sin(\cos(\theta))\sin(i\sin(\theta)) \\ &= \cos(\cos(\theta))\cosh(\sin(\theta)) - i\sin(\cos(\theta))\sinh(\sin(\theta)).\end{aligned}$$

Hence we can write:

$$\cos(\cos(\theta))\cosh(\sin(\theta)) = \mathrm{Re}\left[\cos(\cos(\theta) + i\sin(\theta))\right] = \mathrm{Re}\left[\cos(e^{i\theta})\right].$$

It follows that we can write the given integral as a contour integral around the unit circle:

$$\int_0^{2\pi} \cos(\cos(\theta))\cosh(\sin(\theta))\, d\theta = \mathrm{Re}\left[\int_0^{2\pi} \cos(e^{i\theta})\, d\theta\right] = \mathrm{Re}\left[\oint_{\partial D(0,1)} \frac{\cos(z)}{iz}\, dz\right].$$

Now by Cauchy's integral formula, the contour integral on the right hand side has the value:

$$\oint_{\partial D(0,1)} \frac{\cos(z)}{iz}\, dz = \frac{2\pi i\cos(0)}{i} = 2\pi.$$

Hence putting everything together we have:

$$\int_0^{2\pi} \cos(\cos(\theta))\cosh(\sin(\theta))\, d\theta = 2\pi,$$

as required.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Prove that if any one of the following conditions hold, then f is constant:

(i) $f(z)/z \to 0$ as $|z| \to \infty$.

(ii) There exists $b \in \mathbb{C}$ and $\epsilon > 0$ such that for every $z \in \mathbb{C}$, $|f(z) - b| > \epsilon$.

(iii) $f = u + iv$ and $|u(z)| > |v(z)|$ for all $z \in \mathbb{C}$.

➡ **Solution:** This question is about *Liouville's theorem*. Let's recall the statement of this theorem and its proof:

Liouville's Theorem: Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded, entire function. Then f is constant.

Proof: Suppose the bound is $|f(z)| < M$ for all $z \in \mathbb{C}$. For any $w \in \mathbb{C}$, let $R > |w|$. Then by the Cauchy integral formula, we have:

$$f(w) - f(0) = \frac{1}{2\pi i} \oint\limits_{|z|=R} \frac{f(z)}{z-w}\, dz - \frac{1}{2\pi i} \oint\limits_{|z|=R} \frac{f(z)}{z}\, dz = \frac{1}{2\pi i} \oint\limits_{|z|=R} \frac{wf(z)}{z(z-w)}\, dz.$$

Bounding this expression, we have:

$$|f(w) - f(0)| \le \frac{2\pi R}{2\pi} \sup_{|z|=R} \left| \frac{wf(z)}{z(z-w)} \right| \le M|w| \sup_{|z|=R} \left| \frac{1}{z-w} \right| \le \frac{M|w|}{R-|w|},$$

using the reverse triangle inequality $|z-w| \ge ||z| - |w|| = R - |w|$ for the final bound. But R was arbitrary, so take $R \to \infty$ to deduce $f(w) = f(0)$ for all $w \in \mathbb{C}$. $\square$

(i) We now begin the question proper. There are a couple of ways of doing the first part:

- **1. Use the proof of Liouville's theorem.** Let $w \in \mathbb{C}$ and let $R > |w|$. Then by the Cauchy integral formula, we have:

$$f(w) - f(0) = \frac{1}{2\pi i} \oint\limits_{|z|=R} \frac{f(z)}{z-w}\, dz - \frac{1}{2\pi i} \oint\limits_{|z|=R} \frac{f(z)}{z}\, dz = \frac{1}{2\pi i} \oint\limits_{|z|=R} \frac{wf(z)}{z(z-w)}\, dz.$$

  Bounding this expression, we have:

$$|f(w) - f(0)| \le \frac{2\pi R}{2\pi} \sup_{|z|=R} \left| \frac{wf(z)}{z(z-w)} \right| \le \frac{R|w|}{R-|w|} \sup_{|z|=R} \left| \frac{f(z)}{z} \right|,$$

  where we used the reverse triangle inequality to bound $|z-w| \ge ||z| - |w|| = R - |w|$. This bound holds for arbitrary R, so take the limit as $R \to \infty$. We have $R|w|/(R-|w|) \to |w|$, and $\sup_{|z|=R} |f(z)/z| \to 0$. The result follows. $\square$

- **2. Construct an entire function from $f(z)/z$ and use Liouville's theorem.** We note that the only obstruction to $f(z)/z$ being entire is the possible singularity at the origin $z = 0$. Let's remove this problem by trading up for a new function $g : \mathbb{C} \to \mathbb{C}$, defined by:

$$g(z) = \begin{cases} \dfrac{f(z) - f(0)}{z} & \text{for } z \neq 0, \\ f'(0) & \text{for } z = 0. \end{cases}$$

For $z \neq 0$, g is certainly holomorphic. Furthermore, as $z \to 0$, g is also holomorphic. To see this, we could simply note that the singularity at $z = 0$ of $(f(z) - f(0))/z$ is removable and quote lectures. Alternatively, we can use the power series representation of $f(z)$ about $z = 0$ to write $f(z) = f(0) + f'(0)z + z^2 h(z)$ for some holomorphic h; then:

$$\lim_{z \to 0} \left[\frac{g(z) - g(0)}{z - 0} \right] = \lim_{z \to 0} \left[\frac{z^{-1}(f(0) + f'(0)z + z^2 h(z) - f(0)) - f'(0)}{z} \right] = \lim_{z \to 0} [h(z)] = h(0).$$

Hence $g : \mathbb{C} \to \mathbb{C}$ is entire. Additionally, the function $g : \mathbb{C} \to \mathbb{C}$ retains the nice limiting behaviour at infinity:

$$\lim_{z \to \infty} [g(z)] = \lim_{z \to \infty} \left[\frac{f(z)}{z} - \frac{f(0)}{z} \right] = \lim_{z \to \infty} \left[\frac{f(z)}{z} \right] - \lim_{z \to \infty} \left[\frac{f(0)}{z} \right] = 0.$$

Now, using the definition of a limit, it follows that there exists $R > 0$ such that $|g(z)| < 1$ for all $|z| > R$. Furthermore, g is continuous on the compact domain $\{z : |z| \leq R\}$ so is bounded there. Hence g is bounded everywhere, so is a bounded, entire function, and is thus constant by Liouville's theorem. This constant tends to 0 as $z \to \infty$, so in fact $g \equiv 0$. Consequently,

$$\frac{f(z) - f(0)}{z} = 0 \qquad \Rightarrow \qquad f(z) = f(0)$$

for all $z \neq 0$, and it follows that f is also constant.

(ii) In this part of the question, the modulus of $f(z) - b$ has some lower bound rather than an upper bound: for all $z \in \mathbb{C}$, we have $|f(z) - b| > \epsilon$. We therefore hope that the reciprocal:

$$g(z) = \frac{1}{f(z) - b}$$

will be a bounded, entire function, so that we can apply Liouville's Theorem.

Certainly g is holomorphic for all z such that $f(z) \neq b$. But note that $f(z) = b$ contradicts the assumption that for all $z \in \mathbb{C}$, we have $|f(z) - b| > \epsilon > 0$, so the latter case can never occur. It follows that g is entire. Furthermore, for all $z \in \mathbb{C}$ we have:

$$|g(z)| = \frac{1}{|f(z) - b|} < \frac{1}{\epsilon},$$

so g is bounded. Thus g is constant by Liouville's Theorem; it immediately follows that f is constant.

(iii) In this part of the question, the function $f(z)$ is such that its real part is always greater in magnitude than its imaginary part. Thus the image of f is bounded away from i, say, and hence we can apply part (ii). We note that:

$$|f(z) - i| = \sqrt{u(z)^2 + (v(z) - 1)^2} > \sqrt{v(z)^2 + (v(z) - 1)^2} = \sqrt{2(v(z) - 1/2)^2 + 1/2} > 1/\sqrt{2}.$$

Hence there exist $b = i$ and $\epsilon = 1/\sqrt{2}$ such that $|f(z) - b| > \epsilon$ for all $z \in \mathbb{C}$, so we're done by part (ii).

4. Let $f : D(a, r) \to \mathbb{C}$ be holomorphic, and suppose that $z = a$ is a local maximum for $\mathrm{Re}(f)$. Show that f is constant. What if $z = a$ is a local minimum for $\mathrm{Re}(f)$? What if $z = a$ is a local minimum for $|f|$? Do your answers change if additionally $f(a) \neq 0$?

➼ **Solution:** For this question, it is useful to recall the *local maximum principle* (often called the *maximum modulus principle*) from lectures:

> **The Local Maximum Principle:** Let $f : D(a, r) \to \mathbb{C}$ be a holomorphic function on the open disk $D(a, r)$. If $|f|$ has a maximum at a, i.e. $|f(z)| \leq |f(a)|$ for all $z \in \mathbb{C}$, then f is constant.

In this question, we are asked to show that if just the *real* part of a holomorphic function has an interior maximum, the function is constant. We cannot apply the local maximum principle directly because $\mathrm{Re}(f)$ is *not* in general a holomorphic function - instead, we try to make up a holomorphic function which we *can* apply the local maximum principle to. Recalling that exponentials turn real parts into moduli and imaginary parts into arguments, we are lead to consider the function $g : D(a, r) \to \mathbb{C}$ defined by:

$$g(z) = e^{f(z)}.$$

This is certainly holomorphic on $D(a, r)$ since it is the composition of holomorphic functions. Furthermore, for all $z \in \mathbb{C}$ we have:

$$|g(z)| = |e^{f(z)}| = e^{\mathrm{Re}(f(z))} \leq e^{\mathrm{Re}(f(a))} = |e^{f(a)}| = |g(a)|.$$

Hence g has a maximum at a, and it follows that g is constant. We conclude that at any point $z \in \mathbb{C}$, we have

$$f(z) = f(a) + 2\pi i n(z)$$

where n is an integer-valued function $n : D(a, r) \to \mathbb{Z}$ satisfying $n(a) = 0$ (note that we can't immediately conclude that f is constant!). However, since f is holomorphic, it is continuous, which implies that $n \equiv 0$ everywhere; it follows that f is indeed constant.

In the case that $\mathrm{Re}(f)$ has a local minimum at $z = a$, we have that $-\mathrm{Re}(f) = \mathrm{Re}(-f)$ has a local maximum at $z = a$. It follows by the above work that $-f$ is constant, and hence f is constant.

The local maximum principle does *not* immediately extend to a local minimum principle, however. For example, consider the function $f : D(0, 1) \to \mathbb{C}$ given by $f(z) = z$. Here, we have:

$$|f(z)| = |z|.$$

This clearly has a local minimum $|f(0)| = 0$ at $z = 0$, but f is not constant.

As it turns out, the problem is that the value of the local minimum is zero. If we assume that a local minimum occurs at $z = a$ with value $|f(a)| \neq 0$, then we have $|f(z)| \geq |f(a)| > 0$ for all $z \in D(a, r)$. It follows that the function $g : D(a, r) \to \mathbb{C}$ given by:

$$g(z) = \frac{1}{f(z)}$$

is well-defined. Furthermore, $|g(z)| = 1/|f(z)| \leq 1/|f(a)| = |g(a)|$ for all $z \in D(a, r)$, since $z = a$ was a local minimum of f. It follows that g is constant by the local maximum principle, and hence f is constant too.

5.

(i) Let f be an entire function. Show that f is a polynomial, of degree $\leq k$, if and only if there is a constant M for which $|f(z)| < M(1+|z|)^k$ for all z.

(ii) Show that an entire function f is a polynomial of positive degree if and only if $|f(z)| \to \infty$ as $|z| \to \infty$.

➾ **Solution:** (i) Suppose that $f(z) = a_0 + a_1 z + \cdots + a_k z^k$ is a polynomial of degree k. By the triangle inequality, we have:

$$|f(z)| \leq |a_0| + |a_1||z| + \cdots + |a_k||z|^k \leq \max_i\{|a_i|\}\left(1 + |z| + \cdots + |z|^k\right) \leq \max_i\{|a_i|\}(1+|z|)^k.$$

Hence there exists M with $|f(z)| < M(1+|z|)^k$ (e.g. take $M = \max_i\{|a_i|\} + 1$).

Conversely, suppose that there exists M such that $|f(z)| < M(1+|z|)^k$ for all z. Since f is entire, it is equal to its Taylor series about 0 everywhere, so we can write:

$$f(z) = \sum_{n=0}^\infty a_n z^n = \sum_{n=0}^\infty \left(\frac{1}{2\pi i}\oint_{|z|=R} \frac{f(z)}{z^{n+1}}\,dz\right) z^n,$$

for any $R > 0$. Bounding the coefficients in the Taylor series, we have:

$$|a_n| \leq \frac{2\pi R}{2\pi}\sup_{|z|=R}\left|\frac{f(z)}{z^{n+1}}\right| \leq \frac{1}{R^n}\sup_{|z|=R} M(1+|z|)^k = \frac{M(1+R)^k}{R^n} = \frac{M}{R^{n-k}}\left(1 + \frac{1}{R}\right)^k.$$

Since R is arbitrary, we can take the limit $R \to \infty$ to see that for $n > k$ we have $|a_n| = 0$. Thus f is a polynomial of degree at most k, as required. $\square$

(ii) If f is a polynomial, of degree $k \geq 1$, write:

$$f(z) = a_0 + a_1 z + \cdots + a_k z^k = z^k\left(a_0 z^{-k} + a_1 z^{-k-1} + \cdots + a_k\right),$$

where $a_k \neq 0$. As $|z| \to \infty$, the bracketed term converges to a_k, a finite non-zero number. The modulus of the remaining term z^k obviously tends to infinity. Thus $|f(z)|$ tends to infinity as $|z| \to \infty$.

For the converse, the idea is to use a Möbius transformation to 'put the pole at infinity at the origin'. Since f is entire, we can equate f with its Taylor series about $z = 0$ everywhere:

$$f(z) = \sum_{n=0}^\infty a_n z^n,$$

for coefficients a_n. Now, the condition $|f(z)| \to \infty$ as $|z| \to \infty$ is equivalent to the condition $|f(1/z)| \to \infty$ as $|z| \to 0$. We know from lectures (and the discussion in Question 13) that this condition is equivalent to $f(1/z)$ (viewed as a function on $\mathbb{C}\backslash\{0\}$) having a pole at $z = 0$. But using the Taylor series from above, we have:

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^\infty \frac{a_n}{z^n},$$

and so it follows that f must have only finitely many a_n non-zero (else we would have an essential singularity at 0). $\square$

6. Recall that the (global) maximum principle says that if U is a bounded open subset of $\mathbb{C}$ and if f is continuous on $\overline{U}$ (the closure of U in $\mathbb{C}$) and holomorphic in U, then $\sup_{\overline{U}} |f| = \sup_{\partial U} |f|$. Does this hold if U is not assumed to be bounded?

➾ **Solution:** For completeness, let's begin by recalling a possible statement of the global maximum principle and its proof:

The Global Maximum Principle I: Let $f : D \to \mathbb{C}$ be holomorphic on the domain D. If $|f|$ has a maximum at some interior point $a \in D$, i.e. there exists $\epsilon > 0$ such that $|f(z)| \leq |f(a)|$ for all $z \in D(a, \epsilon) \subseteq D$, then f is constant.

Proof: We immediately apply the local maximum principle to deduce that f is constant on $D(a, \epsilon)$. But then f and the constant function $f(a)$ are both holomorphic functions on the domain D which agree on the subdomain $D(a, \epsilon) \subseteq U$. It follows by the uniqueness of analytic continuations that $f \equiv f(a)$ everywhere (see later in the sheet for more analytic continuation magic!). □

We note that this statement makes no mention of boundedness for U - it merely says that interior maxima are not possible for the moduli of holomorphic functions on domains.

The above statement implies the form of the theorem in the question as follows:

The Global Maximum Principle II: Let U be a bounded open subset of the complex plane, and let $\overline{U}$ be its closure. Suppose that $f : \overline{U} \to \mathbb{C}$ is a continuous function and that f is holomorphic on U. Then $|f|$ attains its maximum value on the boundary of U:

$$\sup_{\overline{U}} |f| = \sup_{\partial U} |f|.$$

Proof: We note that $\overline{U}$ is bounded, since all sequences in U are bounded. Therefore $|f| : \overline{U} \to \mathbb{R}$ is a continuous function on a compact set (by the Heine-Borel theorem), so is bounded above and attains its maximum. If $|f|$ attained its maximum in the interior U, then f would be constant on the connected component of U containing the maximum point by the first statement of the global maximum principle; in particular, f would also attain is maximum on the boundary of that connected component by continuity. Therefore in all cases we find that $|f|$ attains its maximum on ∂U, and the result follows. □

This statement of the global maximum principle only made use of the fact that U was bounded to show that $|f| : \overline{U} \to \mathbb{R}$ was bounded and attained its bounds. Therefore, all we need to do is to consider an *unbounded* function $f : U \to \mathbb{C}$. Here are a couple of examples:

- $f(z) = z$ on the domain $U = \mathbb{C}$. The closure of the domain is $\overline{U} = U$, and hence the boundary is $\partial U = \overline{U} \backslash U = \emptyset$. We have:

  $$\sup_{\overline{U}} |z| = \infty, \qquad \sup_{\partial U} |z| = -\infty,$$

  which are clearly unequal (recall the supremum, i.e. *least upper bound*, over the empty set is $-\infty$).

- $f(z) = e^z$ on the domain $U = \{z : \mathrm{Re}(z) > 0\}$. The closure of the domain is $\overline{U} = \{z : \mathrm{Re}(z) \geq 0\}$, and hence the boundary is $\partial U = \overline{U} \backslash U = \{z : \mathrm{Re}(z) = 0\}$. On the boundary, we have $|e^z| = 1$ for all $z \in \partial U$. But $|e^z| = e^{\mathrm{Re}(z)}$ can become arbitrarily large on the interior, so we have:

  $$\sup_{\overline{U}} |e^z| = \infty, \qquad \sup_{\partial U} |e^z| = 1,$$

  which are again clearly unequal.

❊ **Comments:** It is interesting to consider whether there could be an example of a continuous $f : \overline{U} \to \mathbb{C}$ with f holomorphic on U such that:

$$\sup_{\overline{U}} |f| < \infty, \qquad \text{yet} \qquad \sup_{\overline{U}} |f| > \sup_{\partial U} |f|.$$

There is a trivial case when this can occur. Suppose that $\partial U = \emptyset$, so that $\overline{U} = U$. If the open set $\mathbb{C}\backslash\overline{U} = \mathbb{C}\backslash U$ were non-empty, then $U \cup \mathbb{C}\backslash U$ would disconnect $\mathbb{C}$, which is a contradiction; hence $U = \mathbb{C}$ in this case (technically assuming U non-empty too, but this is really implied as we have a function on U!). Boundedness of $|f|$ on $\overline{U} = \mathbb{C}$ then implies that f is a bounded, entire function, and hence a constant by Liouville's theorem, say $f(z) = c$ everywhere. We then have:

$$\sup_{\overline{U}} |f| = |c| \geq 0 > -\infty = \sup_{\partial U} |f|.$$

In the less trivial case where the boundary is non-empty, we can prove the following generalisation of the global maximum principle, which only requires the very mild assumption of the boundedness of f on the relevant region:

The Global Maximum Principle III: Let $U \subset \mathbb{C}$ be a proper, non-empty open subset of $\mathbb{C}$. Suppose that $f : \overline{U} \to \mathbb{C}$ is continuous and that f is holomorphic on U, with $\sup\limits_{\overline{U}} |f| < \infty$. Then we have:

$$\sup_{\overline{U}} |f| = \sup_{\partial U} |f|.$$

Proof: First note that if U is a proper, non-empty open subset, then $\partial U \neq \emptyset$ by the above discussion. Let's also streamline our notation, writing:

$$M_1 = \sup_{\overline{U}} |f| \qquad \text{and} \qquad M_2 = \sup_{\partial U} |f|.$$

Note that M_1 is a finite real number by assumption (also since ∂U is non-empty, we cannot have $U = \emptyset$, thus $M_1 > -\infty$), and $-\infty < M_2 \leq M_1$ since ∂U is a non-empty subset of $\overline{U}$.

In the case that U is bounded, the previous statement of the global maximum principle immediately gives the result. Thus assume that U is unbounded. Note also that without loss of generality, we may translate our setup so that $0 \in U$ (since U is non-empty), and we can shift the values of f such that $f(0) = 0$ (by replacing $f \mapsto f - f(0)$).

We will now construct a function g which will be used to suppress all powers of $f(z)$ as $|z| \to \infty$. We define $g : \overline{U} \to \mathbb{C}$ via:

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{for } z \neq 0; \\ f'(0) & \text{otherwise.} \end{cases}$$

Clearly g is continuous and it is holomorphic for all $z \in U$ (at $z = 0$, g has a removable singularity). Additionally, $g(z) \to 0$ as $|z| \to \infty$, since for all $\epsilon > 0$, we have for all $z \in \overline{U}$ such that $|z| > M_1/\epsilon > 0$ (here we are using the assumption that ∂U is non-empty):

$$|g(z)| \leq \frac{|f(z)|}{|z|} \leq \frac{M_1}{|z|} < \epsilon.$$

Note also that since U is unbounded, this statement isn't vacuously true, i.e. there is some $z \in \overline{U}$ such that $|z| > M_1/\epsilon$.

In particular, this means that there exists a constant $K > 0$ such that $|g(z)| \leq K$ for all $z \in \overline{U}$; we can see this as follows. Since $g(z) \to 0$ as $|z| \to \infty$, there exists S such that $|g(z)| \leq 1$ for all $z \in \mathbb{C}\backslash\overline{D(0,S)} \cap \overline{U}$. But then on the closed, bounded subset $\overline{D(0,S)} \cap \overline{U}$, we must have that g is bounded. Taking the larger of the bounds, we see there indeed exists K such that $|g(z)| \leq K$ for all $z \in \overline{U}$.

We now construct our 'g-suppressed powers of f'. Fix any positive integer N, and consider the function $h : \overline{U} \to \mathbb{C}$ defined by:

$$h(z) = g(z)f(z)^N.$$

For all $z \in \overline{U}$, h satisfies the bound $|h(z)| \leq K|g(z)|$. In particular, as $|z| \to \infty$, we have that $h(z) \to 0$. Thus there exists $R > 0$ such that for all $z \in \mathbb{C}\backslash D(0,R) \cap \overline{U}$ (i.e. $z \in \overline{U}$ with $|z| \geq R$), we have $|h(z)| \leq KM_2^N$. In particular, this implies that $|h(z)| \leq KM_2^N$ on:

$$\partial D(0,R) \cap \overline{U},$$

since $\partial D(0,R)$ is contained in $\mathbb{C}\backslash D(0,R)$. Note also that on ∂U, we have $|h(z)| = |g(z)| \cdot |f(z)|^N \leq KM_2^N$ too. But:

$$\partial(D(0,R) \cap U) = \partial(D(0,R) \cap U) \cap \overline{U} \subseteq (\partial D(0,R) \cup \partial U) \cap \overline{U} = (\partial D(0,R) \cap \overline{U}) \cup \partial U,$$

so we see that $|h(z)| \leq KM_2^N$ for all $z \in \partial(D(0,R) \cap U)$. Since h is a holomorphic function on the bounded open set $D(0,R) \cap U$ and is continuous on its closure $\overline{D(0,R) \cap U} \subseteq \overline{D(0,R)} \cap \overline{U}$, we can apply the previous statement of the global maximum principle on a bounded open set to deduce that $|h(z)| \leq KM_2^N$ throughout $D(0,R) \cap U$. Since:

$$\overline{U} = U \cup \partial U = (D(0,R) \cap U) \cup ((\mathbb{C}\backslash D(0,R) \cap \overline{U}) \cap U) \cup \partial U,$$

it follows that $|h| \leq KM_2^N$ everywhere in $\overline{U}$.

Hence, for all points $z \in U$ where $g(z) \neq 0$, we have:

$$|f(z)| \leq M_2 \left| \frac{K}{g(z)} \right|^{1/N}.$$

But N was arbitrary, and K was chosen independently of N, so taking the limit as $N \to \infty$, we have $|f(z)| \leq M_2$ for all points where $g(z) \neq 0$. But by the principle of isolated zeroes, g can only be zero at isolated points in U, and hence by continuity of f it follows that $|f(z)| \leq M_2$ everywhere in U. Combined with $|f(z)| \leq M_2$ on ∂U, we have $|f| \leq M_2$ everywhere in $\overline{U}$. $\square$

This style of proof is a special case of a more general technique, the *Phragmén-Linelöf method*, where a multiplicative factor is introduced to suppress the growth of f (in this case, merely to send f to zero at infinity) such that we can split the domain of f into two regions: (i) a bounded region where the maximum modulus principle can be applied to the resulting product function; (ii) an unbounded region where the resulting function can be bounded (with the same bound) using the suppression that our multiplicative factor has introduced. This method can be used to show that very mild conditions on the growth of $|f|$ can be sufficient to deduce that $|f|$ is bounded throughout its domain of definition.

7.

(i) (Schwarz's Lemma) Let f be holomorphic on $D(0,1)$, satisfying $|f(z)| \leq 1$ and $f(0) = 0$. By applying the maximum principle to $f(z)/z$, show that $|f(z)| \leq |z|$. Show also that if $|f(w)| = |w|$ for some $w \neq 0$, then $f(z) = cz$ for some constant c.

(ii) Use Schwarz's Lemma to prove that any conformal equivalence from $D(0,1)$ to itself is given by a Möbius transformation.

➭ **Solution:** (i) To fix notation, let's write $g : D(0,1) \to \mathbb{C}$ for the function defined by:

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } z \neq 0; \\ f'(0) & \text{if } z = 0. \end{cases}$$

Clearly g is holomorphic everywhere in $D(0,1)$ except possibly 0. At 0, either note the singularity is removable, or write $f(z)$ as its Taylor series $f(z) = f'(0)z + z^2 h(z)$ for some holomorphic function $h(z)$ to note that g is complex differentiable:

$$\lim_{z \to 0} \left[\frac{g(z) - g(0)}{z - 0} \right] = \lim_{z \to 0} \left[\frac{f(z)/z - f'(0)}{z} \right] = \lim_{z \to 0} \left[\frac{zh(z)}{z} \right] = h(0).$$

It follows by the maximum principle that for any $0 < r < 1$, the function $|g| : \overline{D(0,r)} \to \mathbb{R}$ on the closed disc of radius r attains its maximum on the boundary, $\partial D(0,r)$. Hence for all $z \in \overline{D(0,r)}$, we have:

$$|g(z)| = \left| \frac{f(z)}{z} \right| \leq \left| \frac{f(z_0)}{z_0} \right| = \frac{|f(z_0)|}{r}$$

for some point on the boundary $z_0 \in \partial D(0,r)$. Recalling that $|f(z_0)| \leq 1$, we have:

$$\left| \frac{f(z)}{z} \right| \leq \frac{1}{r} \qquad \Rightarrow \qquad |f(z)| \leq \frac{|z|}{r},$$

for all $z \in \overline{D(0,r)}$. This was true for arbitrary $r \in (0,1)$, so take the limit as $r \to 1^-$ to deduce that $|f(z)| \leq |z|$ for all $z \in D(0,r)$ as required.

If $|f(w)| = |w|$ for some $w \neq 0$, then the inequality $|f(z)| \leq |z|$ implies that:

$$|g(z)| = \left| \frac{f(z)}{z} \right| \leq \left| \frac{f(w)}{w} \right| = |g(w)|$$

for all $z \in D(0,r)$, so that g has a local maximum at w. Thus g is constant, and hence $f(z) = cz$ for some $c \in \mathbb{C}$.

(ii) Let $f : D(0,1) \to D(0,1)$ be conformal with $f(0) = a$. Inspired by the form of the Möbius transformation on Sheet 1, consider:

$$h(z) = \frac{z - a}{\bar{a}z - 1}$$

as something potentially related to the inverse of f. Then $h \circ f$ is a holomorphic function on $D(0,1)$ with $(h \circ f)(0) = 0$ and $|(h \circ f)(z)| \leq 1$ for all $z \in D(0,1)$ (since both maps preserve the unit disc). Thus by Schwarz's Lemma we have:

$$|(h \circ f)(z)| \leq |z|.$$

But f, h are both invertible and their inverses both map $D(0,1)$ to $D(0,1)$. Thus we have $(h \circ f)^{-1}(0) = 0$ and $|(h \circ f)^{-1}(z)| \leq 1$ for all $z \in D(0,1)$. Thus by Schwarz's Lemma again we have:

$$|(h \circ f)^{-1}(z)| \leq |z| \qquad \Rightarrow \qquad |z| \leq |(h \circ f)(z)|$$

Hence $|(h \circ f)(z)| = |z|$ for all $z \in \mathbb{C}$. It follows by part (i) that $(h \circ f)(z) = cz$ for some (non-zero, since $h \circ f$ invertible) constant c, and hence $f(z) = h^{-1}(cz)$ is a Möbius map.

8. Let U be a bounded open subset of $\mathbb{C}$ and let (f_n) be a sequence of continuous functions on $\overline{U}$ such that f_n is holomorphic on U for each n. If (f_n) converges uniformly on $\partial U = \overline{U} \backslash U$, show that (f_n) converges uniformly on $\overline{U}$.

➾ **Solution:** Let $\epsilon > 0$. Since (f_n) is uniformly convergent on the boundary, it is uniformly Cauchy there. Hence for all $z \in \partial U$, there exists an N such that for all $p \geq q \geq N$ and for all $z \in \partial U$ we have:

$$|f_p(z) - f_q(z)| < \epsilon.$$

But by the global maximum principle, the maximum of f on the closed bounded set $\overline{U}$ (note the set is bounded, since all sequences in U are bounded) occurs on ∂U, and hence there exists $z_0 \in \partial U$ such that for all $z \in \overline{U}$ we have:

$$|f_p(z) - f_q(z)| \leq |f_p(z_0) - f_q(z_0)| < \epsilon.$$

This implies that for each $z \in \overline{U}$, $f_n(z)$ is uniformly Cauchy and hence uniformly convergent. $\square$

9.

(i) Let f be an entire function such that for every positive integer n, $f(1/n) = 1/n$. Show that $f(z) = z$.

(ii) Let f be an entire function with $f(n) = n^2$ for every $n \in \mathbb{Z}$. Must $f(z) = z^2$?

(iii) Let f be holomorphic on $D(0, 2)$. Show that for some integer $n > 0$, $f(1/n) \neq 1/(n+1)$.

➾ **Solution:** Before starting this question, it is useful to recall the *identity theorem*:

Definition: Let $S \subseteq \mathbb{C}$ be any subset of the complex plane. We say that $s \in S$ is *isolated* in S if there exists $\epsilon > 0$ such that $D(s, \epsilon) \cap S = \{s\}$. We say that s is *non-isolated* in S otherwise.

The identity theorem: Suppose that $f, g : D \to \mathbb{C}$ are holomorphic on the domain D. Suppose that f, g agree on a subset $S \subseteq D$ which contains a non-isolated point. Then $f \equiv g$ everywhere in D.

(i) Note that since f is entire, it is continuous, so in particular we have

$$f(0) = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Now let $S = \{0\} \cup \{1, 1/2, 1/3, ...\}$. Then $0 \in S$ is a non-isolated point, since for any $\epsilon > 0$, there exists $n \in \mathbb{Z}^+$ such that $1/n \in D(0, \epsilon)$; it follows that $D(a, \epsilon) \cap S \neq \{0\}$. But $g : \mathbb{C} \to \mathbb{C}$ given by $g(z) = z$ is entire and agrees with f on the set S, and hence by the identity theorem we have $f \equiv g$ everywhere in $\mathbb{C}$. $\square$

(ii) No. For example:

$$f(z) = z^2 \cos(2\pi z)$$

is entire, unequal to z^2 (e.g. $f(1/2) = (1/4)^2 \cos(\pi) = -(1/4)^2 \neq (1/4)^2$), and obeys $f(n) = n^2$ for all $n \in \mathbb{Z}$. This does not contradict the identity theorem because the set $\mathbb{Z}$ does not contain a non-isolated point.

(iii) Suppose for a contradiction that $f(1/n) = 1/(n+1)$ for all positive integers n. By continuity:

$$f(0) = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

As we saw in (i), the set $S = \{0\} \cup \{1, 1/2, 1/3, ...\}$ contains a non-isolated point. Furthermore, $g : D(0, 2)\backslash\{-1\} \to \mathbb{C}$ given by $g(z) = 1/(1 + (1/z)) = z/(z+1)$ is holomorphic on $D(0, 2)\backslash\{-1\}$ and agrees with f on the set S. Hence by the identity theorem, we have $f \equiv g$ throughout $D(0, 2)\backslash\{-1\}$.

But then for $x \in \mathbb{R}$, we can approach the point -1 along the real axis from above. We know that:

$$\lim_{x\to -1^+} f(x)$$

exists, since f is holomorphic on $D(0, 2)$, and hence continuous on $D(0, 2)$. However, we also have:

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} g(x) = \lim_{x\to -1^+} \frac{x}{x+1},$$

which diverges. Hence we cannot have had $f(1/n) = 1/(n+1)$ for all $n > 0$, and it follows that there exists some positive integer n such that $f(1/n) \neq 1/(n+1)$. $\square$

10.

(i) Show that the power series:

$$\sum_{n=1}^{\infty} z^{n!}$$

defines an analytic function f on $D(0,1)$. Show that f cannot be analytically continued to any domain which properly contains $D(0,1)$. [Hint: consider $z = \exp(2\pi i p/q)$ with p/q rational.]

(ii) It is true, but takes some effort to show, that: *if $U \subseteq \mathbb{C}$ is a domain and (a_n) is a sequence of distinct points in U such that the set $\{a_n\}$ has no limit point in U, then there is a holomorphic function on U whose only zeros are the point a_n, $n = 1, 2, 3, \ldots$.* Use this fact to show that if $U \subseteq \mathbb{C}$ is any bounded domain, then there is a holomorphic function $f : U \to \mathbb{C}$ such that f has no analytic continuation to any domain which properly contains U.

Solution: (i) We should really begin by showing that the series is convergent on $D(0,1)$. Taking the ratio of the $(n+1)$th term and the nth term in the series, we have:

$$\left| \frac{z^{(n+1)!}}{z^{n!}} \right| = |z|^{(n+1)!-n!} = |z|^{n!\cdot n}.$$

We note that this converges to 0 for $|z| < 1$ and diverges if $|z| > 1$. Hence by the ratio test the radius of convergence of the given series is precisely 1. To see that the resulting function is an analytic function on $D(0,1)$, just note it is a power series where the ith term $a_i z^i$ is non-zero if and only if $i = n!$ for some n; hence, it is differentiable everywhere in $D(0,1)$, and thus an analytic function there.

Suppose now that f has some analytic continuation g on the domain $U \supset D(0,1)$. Note that U must contain at least one point on the boundary $\partial D(0,1)$; if not, $U \backslash D(0,1) \cup D(0,1)$ would disconnect U, but U is path-connected since it is a domain. Choosing any point on the boundary $e^{i\theta} \in \partial D(0,1)$ for fixed $\theta \in \mathbb{R}$, there exists $\epsilon > 0$ such that the open disc $D(e^{i\theta}, \epsilon)$ is contained in U. But then $\partial D(0,1) \cap D(e^{i\theta}, \epsilon) \subset U$ contains an arc of the circle $\partial D(0,1)$, and hence takes the form $\{e^{i\phi} : \phi \in (\alpha, \beta)\}$ for some real angles $\alpha < \beta$. In particular, since the rationals are dense in the reals, $\partial D(0,1) \cap D(e^{i\theta}, \epsilon)$ contains a point of the form $\exp(2\pi i p/q)$, where p/q is rational, and hence U contains a point of this form too.

Now the analytic continuation g on U must be continuous at the point $\exp(2\pi i p/q)$, since it must be holomorphic there. It follows that we must have:

$$g(\exp(2\pi i p/q)) = \lim_{r\to 1^-} g\left(r \exp\left(2\pi i p/q\right)\right)$$

where r is a real variable, and the limit is defined as r approaches 1 from below. But g agrees with f on $D(0,1)$, and hence it follows that:

$$g(\exp(2\pi i p/q)) = \lim_{r\to 1^-} g\left(r \exp\left(2\pi i p/q\right)\right) = \lim_{r\to 1^-} \left[\sum_{n=0}^{\infty} \left(r \exp(2\pi i p/q)\right)^{n!} \right].$$

We aim to show that the right hand side doesn't exist. First, note that the limit exists if and only if the limit:

$$\lim_{r\to 1^-} \left[\sum_{n=q}^{\infty} \left(r \exp(2\pi i p/q)\right)^{n!} \right] = \lim_{r\to 1^-} \left[\sum_{n=q}^{\infty} r^{n!} \right]$$

exists, simply removing the first few finite terms in the series. This equivalent series has a real, positive summand, and hence it is much easier to tell whether it converges or diverges.

We wish to show this limit diverges. First note that for every term in the series, $r^{n!}$, we have $r^{n!} \to 1$ as $r \to 1^{-1}$. This implies that for all $\epsilon > 0$ there exists a δ such that if $1 - r < \delta$, we have $1 - r^{n!} < \epsilon$, or equivalently $1 - \epsilon < r^{n!}$.

Now, for any $M > 0$, choose $\delta_1, \delta_2, ..., \delta_{2M}$ such that:

$$1 - r < \delta_n \qquad \Rightarrow \qquad \frac{1}{2} < r^{(q+n-1)!}.$$

Then we have for $1 - r < \min\{\delta_1, ..., \delta_{2M}\}$:

$$\sum_{n=q}^{\infty} r^{n!} = \sum_{n=q+2M}^{\infty} r^{n!} + r^{q!} + r^{(q+1)!} + \cdots + r^{(q+2M-1)!} > r^{q!} + r^{(q+1)!} + \cdots + r^{(q+2M-1)!} > \frac{2M}{2} = M.$$

It follows that:

$$\lim_{r \to 1^-} \left[\sum_{n=q}^{\infty} r^{n!} \right] = \infty. \qquad \square$$

(ii) Before starting this part of the question, it's important to remember what a *limit point* is:

Definition: Let X be a topological space and let (x_n) be a sequence in X. We say that x is a *limit point* (or *accumulation point*) of the sequence (x_n) if for every open neighbourhood U of x, there are infinitely many x_n such that $x_n \in U$.

A useful example to bear in mind is $x_n = (-1)^n$ in $\mathbb{R}$. This sequence has no limit, but has two limit points, namely 1 and -1. It's quite easy to prove that in metric spaces, 'limit point' and 'non-isolated point' (see Question 9) mean the same thing.

It will also be useful to know the following rather technical result:

Proposition: Any subspace of a separable metric space is separable (recall, a space is *separable* if it contains a countable, dense subset).

Proof: Let (X, d) be a separable metric space. By definition, it contains a countable, dense subset $S_X \subseteq X$. We can use this set to construct a basis for the topology on X. We define a potential basis $\mathcal{B}_X$ by taking all balls with rational radii around points in S:

$$\mathcal{B}_X = \{B(s, p/q) : s \in S_X, p/q \in \mathbb{Q}\},$$

where $B(x, r) = \{x' \in X : d(x, x') < r\}$. To prove this indeed works, suppose that $U \subseteq X$ is open. Then for any $x \in U$, there exists a ball $B(x, \epsilon) \subseteq U$. Since S_X is a dense subset, its closure is X by definition, and since we are in a metric space this implies that there exists a sequence (s_n) in S_X such that $s_n \to x$ as $n \to \infty$. Then by definition, there exists N such that for all $n \geq N$ we have $s_n \in B(x, \epsilon/3)$. But then $x \in B(s_{N+1}, p/q) \subseteq B(x, \epsilon) \subseteq U$ for some rational p/q such that $\epsilon/3 < p/q < 2\epsilon/3$. It follows that we can write U as the union of balls of the form $B(s_{N+1}, p/q)$ constructed in this way, and so $\mathcal{B}_X$ is a basis for the topology on X.

Next, let $Y \subseteq X$ be any subspace of X. Then $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}_X\}$ is clearly a countable basis for the topology on the subspace Y. To show that Y is separable, we must construct a countable dense subset in Y; we claim that this is provided by the countable basis $\mathcal{B}_Y$. For each element $B \in \mathcal{B}_Y$, fix an element $y_B \in B$, to construct a countable set $S_Y = \{y_B\}$.

Now let $y \in Y$ be any point in the subspace Y. For any open ball $B(y, 1/n) \cap Y$, we have that $B(y, 1/n) \cap Y$ is the union of balls drawn from $\mathcal{B}_Y$ since it is a basis. Thus $B(y, 1/n) \cap Y$ contains at least one of the y_B; choose one and call it $y_{B,n}$. Then by construction $y_{B,n} \to y$ as $n \to \infty$, so the closure of S_Y is indeed Y, and S_Y is thus dense. $\square$

Now let's begin the question. Let U be a bounded domain, and let $\partial U = \overline{U} \backslash U$ be its boundary. Since the boundary is a subspace of a separable metric space (obviously $\mathbb{C} = \mathbb{R}^2$ is separable - just choose all rational points $\mathbb{Q}^2$ as our countable dense subset), it is itself separable. Let S be some countable dense subset of ∂U. Since S is countable, we may write its members as $s_1, s_2, ...$ with some ordering.

We define sequences in U converging to the points $s_i \in \overline{U} \backslash U$ as follows. Since $\overline{U}$ is equal to the set of all limits of sequences in U, we might naïvely say 'pick a sequence in U tending to s_i' - but this can be problematic, because the convergence isn't necessarily controlled, and we don't know where individual sequence elements are relative to the boundary.

We instead take a constructive approach. First note that for any point $z \in \partial U = \overline{U} \backslash U$ and for any $\epsilon > 0$, we have $B(z, \epsilon) \cap U \neq \emptyset$. To see this, note that since z is in the closure $\overline{U}$, there exists a sequence (z_n) in U such that $z_n \to z$ as $n \to \infty$. Hence for any $B(z, \epsilon)$, there exists N such that for all $n \geq N$, we have $z_n \in B(z, \epsilon)$; thus $B(z, \epsilon) \cap U$ is non-empty for all $\epsilon > 0$, since it must at least contain the tail of our sequence in U. In particular, given any $s_i \in S$, for all positive integers k we have that $B(s_i, 1/ik) \cap U \neq \emptyset$. Let us choose any point:

$$s_{i,k} \in B(s_i, 1/ik) \cap U.$$

Then by construction, $s_{i,k} \to s_i$ as $k \to \infty$; these are the 'controlled' sequences we are after.

We can define a new sequence (a_n) by pasting these convergent sequences together:

$$a_1 = s_{1,1}, \quad a_2 = s_{1,2}, \quad a_3 = s_{2,1}, \quad a_4 = s_{2,2}, \quad a_5 = s_{1,3}, \quad a_6 = s_{2,3}, \quad a_7 = s_{3,1}, \quad a_8 = s_{3,2}, \quad \dots$$

The ordering we have written above is based on writing down all terms $s_{i,k}$ with $i, k \in \{1\}$, then all remaining terms $s_{i,k}$ where $i, k \in \{1, 2\}$ (which must involve either $i = 2$ or $k = 2$ for the term to not have already been), then all remaining terms $s_{i,k}$ where $i, k \in \{1, 2, 3\}$, etc. In particular, this implies that when we are drawing indices from $i, k \in \{1, 2, 3, ..., n\}$, the points in the sequence are all at least $(1/n)$-close to the boundary (since $s_{1,n} \in B(s_1, 1/n)$, $s_{2,n} \in B(s_2, 1/2n)$, etc).

By construction, every s_i is a limit point of the sequence (a_n) since $s_{i,k} \to s_i$; hence given any neighbourhood of s_i, the subsequence $s_{i,k}$ of (a_n) always contains infinitely many points in that neighbourhood.

Also by construction, every point $z \in U$ is not a limit point of (a_n). To see this, let us define the sets of all points that are ϵ-close to the boundary ∂U:

$$V_\epsilon = \{z \in U : \text{there exists } w \in \partial U \text{ with } |z - w| < \epsilon\}.$$

Now fix some $z \in U$. If $z \in V_\epsilon$ for all $\epsilon > 0$, then for any positive integer n there exists $w_n \in \partial U$ such that $|z - w_n| < 1/n$. It follows by construction that $w_n \to z$ as $n \to \infty$. But $\partial U = \overline{U} \backslash U = \overline{U} \cap \mathbb{C} \backslash U$ is the intersection of closed sets, so is closed, and hence contains all limits of convergent sequences in ∂U; thus $z \in \partial U$, which is a contradiction. It follows that there exists $\epsilon > 0$ such that $z \notin V_{\epsilon/2}$. Since the tail of the sequence (a_n) is eventually $\epsilon/2$-close to the boundary by construction, it follows that $D(z, \epsilon/2)$ does not contain infinitely many sequence terms from (a_n), and hence z is not a limit point of (a_n).

We are now finally in a position where we can apply the result given in the question:

> **The Weierstrass product theorem:** Let D be a domain and let (a_n) be a sequence with no limit point in D. Then there exists a holomorphic function $f : D \to \mathbb{C}$ whose zeroes are precisely the points $a_n \in D$.

Hence, there exists a function $f : U \to \mathbb{C}$ whose zeroes are precisely the a_n as constructed above. Now suppose that $V \supset U$ is a domain properly containing U, and $g : V \to \mathbb{C}$ is the analytic continuation of f to this domain.

We note V must contain at least one point w on ∂U, else $V = U \cup V \backslash U$ would disconnect V. Since V is open, there exists $\epsilon > 0$ such that $D(w, \epsilon) \subseteq V$. Recalling our countable dense subset $S \subset \partial U$, we know that there is some $s_i \in S$ such that $s_i \in D(w, \epsilon)$. But then s_i is a non-isolated zero of g, since it is the limit of zeroes $s_{i,k}$ of f. This contradicts the principle of isolated zeroes for holomorphic functions (since f, and hence g, is not identically zero), so g cannot exist. □

11.

(i) Let $w \in \mathbb{C}$, and let $\gamma, \delta : [0,1] \to \mathbb{C}$ be closed curves such that for all $t \in [0,1]$, $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$. By computing the winding number of the closed curve:

$$\sigma(t) = \frac{\delta(t) - w}{\gamma(t) - w}$$

about the origin, show that $I(\gamma; w) = I(\delta; w)$.

(ii) If $w \in \mathbb{C}$, $r > 0$, and γ is a closed curve which does not meet $D(w,r)$, show that $I(\gamma; w) = I(\gamma; z)$ for every $z \in D(w,r)$.

(iii) Deduce that if γ is a closed curve in $\mathbb{C}$ and U is the complement of (the image of) γ, then the function $w \mapsto I(\gamma; w)$ is a locally constant function on U.

➾ **Solution:** Let's begin by recalling the definition of the winding number:

Definition: Let $\gamma : [0,1] \to \mathbb{C}$ be a continuous curve in the complex plane. Let $w \in \mathbb{C}\backslash\gamma([a,b])$ be a point in the complement of the image of the curve. If we can write $\gamma(t) = w + |\gamma(t) - w|e^{i\theta(t)}$ for some continuous function $\theta : [0,1] \to \mathbb{R}$, we say that θ is a *continuous choice of argument* for γ about w. We define the *winding number of γ about w* to be:

$$I(\gamma; w) = \frac{\theta(1) - \theta(0)}{2\pi}.$$

For a closed curve γ, we have $\gamma(0) = \gamma(1)$, and hence for a continuous choice of argument we have $\theta(1) = \theta(0) + 2n\pi$ for some integer $n \in \mathbb{Z}$; it follows that the winding number of a closed curve is always an integer.

We showed in lectures that: (i) there always exists a continuous choice of argument about any point in the complement of a curve's image; (ii) all continuous choices of argument give rise to the same winding number, so it is well-defined. We also proved an integral formula for the winding number, provided a curve is piecewise C^1:

Proposition: If $\gamma : [0,1] \to \mathbb{C}$ is a piecewise continuously differentiable curve, then for all $w \in \mathbb{C}\backslash\gamma([0,1])$, we have:

$$I(\gamma; w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - w}.$$

Note, however, that in this question we are not given that γ is piecewise C^1, so this formula can't be used!

(i) In the first part of this question, we consider two curves which are a 'sufficiently small' perturbation of one another (where in this context, 'sufficiently small' intuitively means that either both curves encircle w, or neither curve encircles w). Rearranging the expression for $\sigma(t)$ to make use of the given inequality, we have:

$$\sigma(t) = \frac{\delta(t) - \gamma(t) + \gamma(t) - w}{\gamma(t) - w} = 1 + \frac{\delta(t) - \gamma(t)}{\gamma(t) - w}.$$

In particular, this implies:

$$|\sigma(t) - 1| = \frac{|\delta(t) - \gamma(t)|}{|\gamma(t) - w|} < 1.$$

This means that the curve $\sigma(t)$ is entirely contained in the disc $D(1,1)$, so intuitively can never encircle the point 0, and hence has winding number 0 about 0. To make this rigorous, note that the argument $\arg:\mathbb{C}\backslash\mathbb{R}_{\leq 0}\to(-\pi,\pi)$ is a continuous function defined on the range of σ, hence $\arg(\sigma(t))$ is well-defined everywhere and is a continuous function of t. In particular, we note that we can write $\sigma(t)=|\sigma(t)|e^{i\arg(\sigma(t))}$ everywhere, and hence $\arg\circ\sigma$ is a continuous choice of the argument for the curve σ. It follows that the winding number is given by:

$$I(\sigma;0)=\frac{\arg(\sigma(1))-\arg(\sigma(0))}{2\pi}=0,$$

as required.

To make the deduction $I(\gamma;w)=I(\delta;w)$, let's write $\gamma(t)=w+r(t)e^{i\theta(t)}$ and $\delta(t)=w+s(t)e^{i\phi(t)}$ for continuous choices of argument θ,ϕ for each of the curves γ and δ. Then:

$$\sigma(t)(\gamma(t)-w)=\delta(t)-w\qquad\Rightarrow\qquad|\sigma(t)|r(t)e^{i\arg(\sigma(t))}e^{i\theta(t)}=s(t)e^{i\phi(t)}.$$

Comparing arguments, we must have $\arg(\sigma(t))+\theta(t)=\phi(t)+2\pi n(t)$ for some integer-valued function $n:[0,1]\to\mathbb{Z}$. Continuity of $\arg(\sigma(t))+\theta(t)-\phi(t)$ forces $n(t)\equiv n$ to be constant. It follows that:

$$I(\delta;w)=\frac{\phi(1)-\phi(0)}{2\pi}=\frac{\arg(\sigma(1))+\theta(1)-2\pi n-\arg(\sigma(0))-\theta(0)+2\pi n}{2\pi}=\frac{\theta(1)-\theta(0)}{2\pi}=I(\gamma;w),$$

as required. □

(ii) We compare the two curves $\delta_w(t)=\gamma(t)-w$ and $\delta_z(t)=\gamma(t)-z$, and use part (i). We have:

$$|\delta_w(t)-\delta_z(t)|=|(\gamma(t)-w)-(\gamma(t)-z)|=|w-z|<r\leq|\gamma(t)-w|=|\delta_w(t)|,$$

where in the last inequality we used the fact that $\gamma(t)$ is outside $D(w,r)$ for all $t\in[0,1]$. It follows that δ_w,δ_z are closed curves such that $|\delta_w(t)-\delta_z(t)|<|\delta_w(t)|$, and hence it follows from (i) that $I(\delta_w;0)=I(\delta_z;0)$.

Since we now have that the curves δ_w,δ_z can be expressed in terms of continuous choices of the argument θ,ϕ such that $\gamma(t)-w=|\gamma(t)-w|e^{i\theta(t)}$ and $\gamma(t)-z=|\gamma(t)-z|e^{i\phi(t)}$ and $\theta(1)-\theta(0)=\phi(1)-\phi(0)$, it follows immediately that $I(\gamma;w)=I(\gamma;z)$, as required. □

(iii) For the last part, note that $\gamma([0,1])$ is the continuous image of a compact set, hence it is compact. It follows by the Heine-Borel theorem that $\gamma([0,1])$ is closed and bounded, hence in particular $U=\mathbb{C}\backslash\gamma([0,1])$ must be open. Now for any $w\in U$, there exists $\epsilon>0$ such that $D(w,\epsilon)\subseteq U$. But $\gamma([0,1])\cap D(w,\epsilon)=0$, hence γ is a closed curve which does not meet $D(w,\epsilon)$; thus by part (ii) we have that $z\mapsto I(\gamma;z)$ is constant on $D(w,\epsilon)$, and hence the winding number is indeed a locally constant function on U. □

❊ **Comments:** In fact, the result of this question can be strengthened, using the fact that locally constant functions on topological spaces are actually constant on their connected components. We find the result:

Proposition: Let $\gamma:[0,1]\to\mathbb{C}$ be a closed curve. The function which takes a point to the winding number of γ about that point, $z\mapsto I(\gamma;z)$, is constant on the connected components of $\mathbb{C}\backslash\gamma([a,b])$.

Proof: We already know that $z\mapsto I(\gamma;z)$ is locally constant on $\mathbb{C}\backslash\gamma([a,b])$. Let C be a connected component of $\mathbb{C}\backslash\gamma([a,b])$, and let $w\in C$ be a fixed point. Define $U=\{z\in C:I(\gamma;z)=I(\gamma;w)\}$ and define $V=\{z\in C:I(\gamma;z)\neq I(\gamma;w)\}$. Then U and V are open by the locally constant nature of $I(\gamma;z)$; furthermore U is non-empty. But then if V is non-empty, $C=U\cup V$ disconnects C, which is a contradiction. □

12. Find the Laurent expansion (in powers of z) of $1/(z^2-3z+2)$ in each of the regions:

$$\{z : |z| < 1\}; \qquad \{z : 1 < |z| < 2\}; \qquad \{z : |z| > 2\}.$$

➾ **Solution:** It is useful to begin by expression $1/(z^2-3z+2)$ in partial fractions. We have:

$$\frac{1}{z^2-3z+2} = \frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

For $|z| < 1$, we have:

$$\frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2}\left(\frac{1}{1-\frac{1}{2}z}\right) + \frac{1}{1-z} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}z\right)^n + \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right) z^n,$$

using the sum of geometric progressions.

For $1 < |z| < 2$, we can again use the sum of geometric progressions, but this time we should expand $1/(z-1)$ in powers of $1/z$, since $|1/z| < 1$. We have:

$$\frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2}\left(\frac{1}{1-\frac{1}{2}z}\right) - \frac{1}{z}\left(\frac{1}{1-1/z}\right) = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}z\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = -\sum_{n=-\infty}^{-1} z^n - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}.$$

For $|z| > 2$, once again we use the sum of geometric progressions, but we additionally expand $1/(z-2)$ in powers of $2/z$, since $|2/z| < 1$. We have:

$$\frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{z}\left(\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right) = \sum_{n=0}^{\infty}\frac{2^n-1}{z^{n+1}} = \sum_{n=-\infty}^{-1}\left(2^{-n-1}-1\right) z^n = \sum_{n=-\infty}^{-1}\left(\frac{1}{2^{n+1}} - 1\right) z^n.$$

Note that all of these expressions are Laurent series, and hence are the unique such Laurent series on each of the domains, by lectures.

13. Classify the singularities of each of the following functions:

$$\frac{z}{\sin(z)}, \qquad \sin\left(\frac{\pi}{z^2}\right), \qquad \frac{1}{z^2}+\frac{1}{z^2+1}, \qquad \frac{1}{z^2}\cos\left(\frac{\pi z}{z+1}\right).$$

➡ **Solution:** Before starting this question, let's summarise the results from lectures on the nature of isolated singularities:

Proposition: (Classification of singularities) Let $f : D(a,r)\backslash\{a\} \to \mathbb{C}$ be holomorphic. We know from lectures that f has a unique Laurent series on $D(a,r)\backslash\{a\}$ of the form:

$$\sum_{n=-\infty}^{\infty} a_n(z-a)^n.$$

(i) By definition, f has a removable singularity at a if and only if $\min\{n : a_n \neq 0\} = 0$. Equivalently, we have:

- (Limit characterisation I) f has a removable singularity at a if and only if $\lim_{z\to a}(z-a)f(z) = 0$.
- (Limit characterisation II) f has a removable singularity at a if and only if $\lim_{z\to a} f(z)$ exists in $\mathbb{C}$.
- (Boundedness characterisation) f has a removable singularity at a if and only if there exists $\epsilon > 0$ such that $|f|$ is bounded on $D(a,\epsilon)\backslash\{a\}$.

(ii) By definition, f has a pole of order $k > 0$ at a if and only if $\min\{n : a_n \neq 0\} = -k$. Equivalently, we have:

- (Limit characterisation I) f has a pole of order $k > 0$ at a if and only if $\lim_{z\to a}(z-a)^k f(z)$ exists in $\mathbb{C}$ *and* is non-zero.
- (Limit characterisation II) f has a pole at a if and only if $\lim_{z\to a}|f(z)| = \infty$.
- (Poles are reciprocals of zeroes) f has a pole of $k > 0$ at a if and only if there exists some holomorphic function $h : D(a,r) \to \mathbb{C}$ with a zero of order k at a such that $f(z) = 1/h(z)$.

(iii) By definition, f has an essential singularity at a if and only if $\min\{n : a_n \neq 0\}$ does not exist. Equivalently, we have:

- (Limit characterisation I) f has an essential singularity at a if and only if $\lim_{z\to a} f(z)$ does not exist in $\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$.
- (Limit characterisation II) f has an essential singularity at a if and only if $\lim_{z\to a}|f(z)|$ does not exist in $[0,\infty]$.
- (Denseness characterisation) f has an essential singularity at a if and only if for all $b \in \mathbb{C}$, there exists a sequence $z_n \in D(a,r)$ such that $z_n \to a$ and $f(z_n) \to b$ as $n \to \infty$. This result is called the *Casorati-Weierstrass Theorem*, and we will prove this result in Question 14.

A nice consequence of this classification is that we can often simply count zeroes in the numerator and denominator of a function, and see whether they 'cancel out'. This is formalised in the following result:

Proposition: (Counting zeroes and poles) Suppose that $f : D(a,r) \to \mathbb{C}$, $g : D(a,r) \to \mathbb{C}$ are holomorphic functions which are non-zero for $z \neq a$. If f has a zero of order m at a and g has a zero of order n at a, then at $z = a$ the quotient $f/g : D(a,r)\backslash\{a\} \to \mathbb{C}$ has:

(i) a removable singularity if $m \geq n$;

(ii) a pole of order $n - m$ if $m < n$.

Proof: Using the order of the zeroes of f, g, we can write $f(z) = (z-a)^m \hat{f}(z)$ and $g(z) = (z-a)^n \hat{g}(z)$ where $\hat{f}, \hat{g} : D(a,r) \to \mathbb{C}$ are holomorphic and non-zero everywhere in $D(a,r)$. Then we have:

$$\frac{f(z)}{g(z)} = (z-a)^{m-n}\frac{\hat{f}(z)}{\hat{g}(z)}.$$

Note that if $m \geq n$, we can take the limit $z \to a$ safely to get a finite result, hence we have a removable singularity. On the other hand, if $m < n$, we have:

$$\lim_{z\to a}\left[(z-a)^{n-m}\frac{f(z)}{g(z)}\right] = \lim_{z\to a}\left[\frac{\hat{f}(z)}{\hat{g}(z)}\right] = \frac{\hat{f}(a)}{\hat{g}(a)} \neq 0,$$

so that this limit exists and is non-zero (since $\hat{f}(a), \hat{g}(a) \neq 0$). Thus we have a pole of order $n - m$ as required. $\square$

We now begin the question proper. For the first function, we note that $z/\sin(z)$ is singular if and only if $\sin(z) = 0$. We can solve this equation over the complex numbers by setting:

$$\frac{e^{iz} - e^{-iz}}{2i} = 0 \qquad \Leftrightarrow \qquad e^{2iz} = 1 \qquad \Leftrightarrow \qquad z = n\pi,\ n \in \mathbb{Z}.$$

Hence the singularities occur precisely at $z = n\pi$ for $n \in \mathbb{Z}$.

Near $z = 0$, the numerator has a simple zero and the denominator has a simple zero (since $\sin'(0) = \cos(0) = 1 \neq 0$). Hence we have a removable singularity at $z = 0$. On the other hand, near $z = n\pi$, the numerator is non-zero whilst the denominator has a simple zero (since $\sin'(n\pi) = \cos(n\pi) = (-1)^n \neq 0$). Hence we have a simple pole at $z = n\pi$ for $n \neq 0$.

For the second function, for all $z \neq 0$ we have (using the Taylor series for $\sin$ about 0, which is convergent everywhere):

$$\sin\left(\frac{\pi}{z^2}\right) = \frac{\pi}{z^2} - \frac{\pi^3}{z^6 3!} + \frac{\pi^5}{z^{10}5!} + \cdots,$$

so the singularity at $z = 0$ is essential. Alternatively, simply observe that the limit of the modulus as $z \to 0$, for z positive and real, does not exist in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (we have infinite oscillation between 1 and -1, so the limit is not finite or infinite).

For the third function, provided the singularities of the two terms occur at different locations, we can consider the terms separately because the singularities will be isolated from one another. Examining the function:

$$\frac{1}{z^2} + \frac{1}{z^2+1}$$

we see that the singularities of the first term occur at $z = 0$ and the singularities of the second occur at $z = \pm i$. Near $z = 0$, the numerator of the first term is non-zero whilst the denominator has a zero of order 2. Hence we have a double pole at $z = 0$. Near $z = \pm i$, the numerator of the second term is non-zero whilst the denominator has a simple zero. Hence we have simple poles at $z = \pm i$.

For the fourth function,

$$\frac{1}{z^2}\cos\left(\frac{\pi z}{z+1}\right),$$

the singularities clearly occur at $z = 0$ and $z = -1$. Near $z = 0$, we have $\cos(\pi \cdot 0/(0+1)) = 1 \neq 0$, so the numerator is non-zero whilst the denominator has a zero of order 2. Hence we have a double pole at $z = 0$.

Near $z = -1$, for z real we have:

$$\lim_{z \to -1^+} \frac{1}{z^2}\cos\left(\frac{\pi z}{z+1}\right) = \lim_{z \to -1^+} \cos\left(\pi - \frac{\pi}{z+1}\right),$$

which does not converge to a limit in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, it just oscillates infinitely. Hence $z = -1$ is an essential singularity of this function.

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14. (The Casorati-Weierstrass theorem) Let f be holomorphic on $D(a,R)\backslash\{a\}$ with an essential singularity at $z = a$. Show that for any $b \in \mathbb{C}$, there exists a sequence of points $z_n \in D(a,R)$ with $z_n \neq a$ such that $z_n \to a$ and $f(z_n) \to b$ as $n \to \infty$. Find such a sequence when $f(z) = e^{1/z}$, $a = 0$ and $b = 2$.

[A much harder theorem of Picard says that in any neighbourhood of an essential singularity, an analytic function takes *every* complex value except possible one.]

➻ **Solution:** It is actually slightly easier to prove the following version of the Casorati-Weierstrass theorem:

The Casorati-Weierstrass theorem: Let $f : D(a,R)\backslash\{a\} \to \mathbb{C}$ be holomorphic with an essential singularity at $z = a$. For all $b \in \mathbb{C}$ and for all $r, \epsilon > 0$, there exists z such that $0 < |z-a| < r$ and $|f(z) - b| < \epsilon$.

Proof: Suppose not. Then there exists $b \in \mathbb{C}$, $r, \epsilon > 0$ such that for all z with $0 < |z-a| < r$, we have $|f(z) - b| > \epsilon$. This implies that:

$$g(z) = \frac{1}{f(z) - b}$$

is holomorphic on $D(a,r)\backslash\{a\}$, and since $|g(z)| < 1/\epsilon$, we have that g is also bounded on this domain. It follows from the classification of singularities (see Question 13) that g has a removable singularity at $z = a$, so let's view g as a function on $D(a,r)$, supplemented at $z = a$ with the value of $\lim_{z \to a} g(z)$.

To finish, note that: $f(z) = (1 + bg(z))/g(z)$. From this form, we see that if g is non-zero at $z = a$, then f has a removable singularity at $z = a$. On the other hand if g is zero at $z = a$, then f has a pole at $z = a$. Either case is a contradiction since f has an essential singularity at $z = a$, and the result follows. □

This implies the version of the Casorati-Weierstrass theorem given in the question. Suppose that $b \in \mathbb{C}$. Then for any $n > 0$, there exists z_n such that $0 < |z_n - a| < 1/n$ and $|f(z_n) - b| < 1/n$. This implies that $f(z_n) \to b$ as $n \to \infty$, and we're done. □

In the case that $a = 0$ and $b = 2$, such a sequence is:

$$z_n = \frac{1}{\log(2) + 2\pi i n}.$$

We have $z_n \to 0$ as $n \to \infty$, and:

$$e^{1/z_n} = e^{\log(2)+2\pi i n} = 2e^{2\pi i n} = 2 \to 2$$

as $n \to \infty$.

---

15. Let f be a holomorphic on $D(a, R)\backslash\{a\}$. Show that if f has a non-removable singularity at $z = a$, then the function $\exp(f(z))$ has an essential singularity at $z = a$. Deduce that if there exists M such that $\mathrm{Re}(f(z)) < M$ for $z \in D(a, R)\backslash\{a\}$, then f has a removable singularity at $z = a$.

Solution: First note that if $z = a$ is a removable singularity of f, then $\lim\limits_{z\to a} f(z)$ exists and is finite, and hence $\lim_{z\to a} \exp(f(z))$ exists and is finite, which in turn implies that $\exp(f(z))$ has a removable singularity at $z = a$. Thus the question is asking us about the interesting cases.

Next, let's suppose that $z = a$ is a kth order pole of f. Then $\lim\limits_{z\to a}(z - a)^k f(z)$ exists, is finite and non-zero. Thus:

$$\lim_{z\to a} |\exp(f(z))| = \lim_{z\to a} \left|\exp\left((z - a)^k f(z)\right)\right| \left|\exp\left(\frac{1}{(z - a)^k}\right)\right|.$$

The first factor, $|\exp\left((z - a)^k f(z)\right)|$ converges to a finite, non-zero constant. The remaining factor has no limit in $[0, \infty]$ (consider z approaching a from different complex directions). It follows this limit does not exist and the singularity is essential.

Finally, suppose that $z = a$ is an essential singularity of f. Then by the Casorati-Weierstrass theorem, there exist sequences z_n, w_n such that $z_n \to a$, $w_n \to a$ and $f(z_n) \to 1$, $f(w_n) \to 2$ as $n \to \infty$. It follows that:

$$\lim_{z\to a} |\exp(f(z))|$$

cannot exist in $[0, \infty]$, else it would have to equal both e^1 and e^2. Thus the singularity is essential. □

To make the deduction, simply note that:

$$|e^{f(z)}| = e^{\mathrm{Re}(f)} < e^M$$

Hence $|e^{f(z)}|$ is bounded on $D(a, R)\backslash\{a\}$. This implies that e^f has a removable singularity at $z = a$, which in turn implies that f has a removable singularity at $z = a$ by the above work.

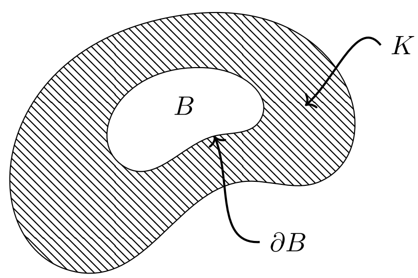
Part IB: Complex Analysis
Examples Sheet 3 Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

1. The Weierstrass approximation theorem in real analysis says that every continuous function $f : I \to \mathbb{R}$ on a compact interval $I \subset \mathbb{R}$ is the uniform limit of a sequence of polynomials. The direct analogue of this to the complex setting (obtained by replacing $\mathbb{R}$ with $\mathbb{C}$, I with a compact set $K \subset \mathbb{C}$ and real polynomials with complex polynomials) is false, even if we make a suitable holomorphicity assumption on f. Construct, for any given compact set $K \subset \mathbb{C}$ with $\mathbb{C}\backslash K$ not connected, a function f that is holomorphic on an open set containing K such that f is not the uniform limit on K of a sequence of complex polynomials. [Hint: your may wish to generalise the idea of Question 13(ii) on Sheet 1 for the construction, and use the global maximum principle to prove it works.] Look up, on the other hand, Runge's theorem and Mergelyan's theorem!

Solution: In Examples Sheet 1, Question 13(ii), we showed that there was no sequence of polynomials $p_n(z)$ on the annulus $\{1/2 < |z| < 1\}$ which converged uniformly to $1/z$ as $n \to \infty$. The idea was to have a singularity inside the 'hole' of the annulus.

In this more general case, we are given that the complement $\mathbb{C}\backslash K$ is disconnected. First, we would like to find a 'hole' inside K where we can place our singularity. Since there is a *unique* unbounded component of $\mathbb{C}\backslash K$ (all unbounded components can be linked up by a sufficiently large circle, with K lying inside), there must be at least one bounded component of $\mathbb{C}\backslash K$ in order for $\mathbb{C}\backslash K$ to be disconnected. Let z_0 be a point in any bounded component B of $\mathbb{C}\backslash K$, and let $D(z_0, r) \subset B$ be an open disk contained in B centred on z_0.



Now let U be any open neighbourhood of K. Then $U' = U\backslash\overline{D(z_0, r/2)}$ is an open neighbourhood of K on which the function $f : U' \to \mathbb{C}$ given by:

$$f(z) = \frac{1}{z - z_0}$$

is holomorphic. Suppose that there exists a sequence of polynomials $p_n : K \to \mathbb{C}$ which converges uniformly to f on K as $n \to \infty$.

In Examples Sheet 1 Question 13(ii), we derived a contradiction from here by integrating $p_n(z)$ around the singularity. However, K can be arbitrarily complicated in this case, so constructing a curve that lies entirely in K might be difficult in this instance. Instead, we follow the hint in the question, and aim to use the global maximum principle instead. Since $p_n \to f$ uniformly on K as $n \to \infty$, for all $\epsilon > 0$, there exists N such that for all $m \geq N$ we have:

$$|p_m(z) - f(z)| < \epsilon$$

for all $z \in K$. Now, note that $\partial(\mathbb{C}\backslash K) = (\overline{\mathbb{C}\backslash K})\backslash(\mathbb{C}\backslash K) = (\mathbb{C}\backslash \mathrm{Int}(K))\backslash(\mathbb{C}\backslash K) = K\backslash\mathrm{Int}(K) = \partial K$ (using the fact that the closure of the complement is the complement of the interior), so ∂B is contained completely in ∂K. In particular, we can make the difference $|p_m(z) - f(z)|$ arbitrarily small across all of ∂B uniformly - this will causes problems inside of B due to the global maximum principle.

Let us define:

$$d = \max_{z \in K} |z - z_0|,$$

which exists because $|z - z_0|$ is a continuous function of z on the compact set K (hence it is bounded and attains its bounds). Choosing $\epsilon = 1/2d$ in the above, it follows that there exists N such that for all $m \geq N$ we have $|p_m(z) - f(z)| < 1/2d$ for all $z \in \partial B \subset K$ (extending the polynomials to K is possible since they are polynomials). It follows that:

$$|(z - z_0)p_m(z) - 1| < \frac{|z - z_0|}{2d} \leq \frac{1}{2} < \frac{3}{4}$$

for all $z \in \partial B \subset K$. But $(z - z_0)p_m(z) - 1$ is a holomorphic function on B (it's a polynomial), which is a bounded domain, thus it must attain its maximum on the boundary by the global maximum principle. Yet at $z = z_0$, it has modulus $1 > 3/4$; contradiction.

In fact, this is the only way that uniform approximation by polynomials of a holomorphic function on a compact set K can fail - if the complement $\mathbb{C} \backslash K$ is connected, there are no 'holes' in K, and this argument fails. We find instead the result:

Mergelyan's theorem: Let K be a compact subset of $\mathbb{C}$ with connected complement $\mathbb{C} \backslash K$. Then if $f : K \to \mathbb{C}$ is continuous and the restriction $f : \mathrm{Int}(K) \to \mathbb{C}$ is holomorphic, then f is the uniform limit of a sequence of polynomials on K.

Note the conditions on Mergelyan's theorem are slightly weaker than you might expect from the above (in particular we do not need f to be holomorphic on an open neighbourhood of K, only continuous on K and holomorphic on the interior).

Furthermore, the case when the complement *is* disconnected (and K contains 'holes') can still be saved, provided we approximate via rational functions instead of polynomials:

Runge's theorem: Let K be a compact subset of $\mathbb{C}$ and let A be a set containing at least one complex number from each of the bounded components of $\mathbb{C} \backslash K$. Then the restriction $f : K \to \mathbb{C}$ of any function which is holomorphic on a neighbourhood of K is the uniform limit of a sequence of rational functions, and whose poles are completely contained in A.

Remarkably, the set A in Runge's theorem is *completely arbitrary*. For example, consider the function $f(z) = 1/z$ on the annulus $\{1/2 < |z| < 1\}$. Here, we might naturally choose the set $A = \{0\}$, and then the following trivial sequence of rational functions works:

$$f_n(z) = \frac{1}{z} \to \frac{1}{z} = f(z)$$

as $n \to \infty$. *However*, Runge's theorem tells us that we are well within our rights to choose $A = \{1/4\}$, for instance. To construct a sequence of rational approximants in this case, we note that:

$$\frac{1}{z} = \frac{1}{(z - 1/4) + 1/4} = \frac{1}{z - 1/4} \cdot \frac{1}{1 + \left(\dfrac{1/4}{z - 1/4}\right)} = \frac{1}{z - 1/4}\left(1 - \frac{1/4}{z - 1/4} + \frac{(1/4)^2}{(z - 1/4)^2} - \cdots\right).$$

on the domain $1/4 < |z - 1/4|$, i.e. points more than a $1/4$ away from $1/4$. This includes all points in the original annulus $\{1/2 < |z| < 1\}$, so a uniform sequence in this case is:

$$f_1(z) = \frac{1}{z - 1/4}, \qquad f_2(z) = \frac{1}{z - 1/4} - \frac{1/4}{(z - 1/4)^2}, \qquad f_3(z) = \frac{1}{z - 1/4} - \frac{1/4}{(z - 1/4)^2} + \frac{(1/4)^2}{(z - 1/4)^3}, \qquad \ldots$$

The technique of replacing the poles as above is sometimes referred to as *pole pushing*.

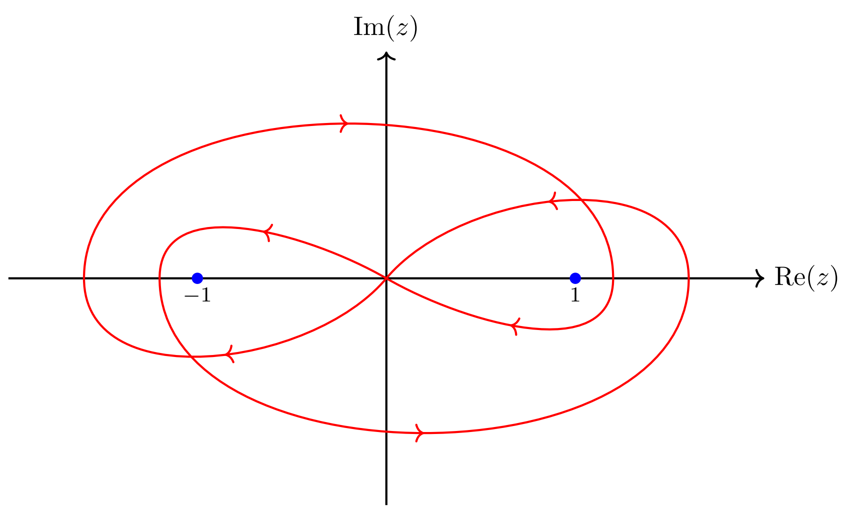
2. (a) Draw a (convincing!) picture of a domain Ω and a closed curve γ in Ω such that γ is homologous to zero in Ω but is not null-homotopic in Ω. (The reverse direction, as proved in lectures, is always true, i.e. if γ is null-homotopic in Ω, then it is homologous to zero in Ω).

(b) Let U be a domain with the property that *every* closed piecewise C^1 curve in U is homologous to zero in U.

(i) Use Cauchy's theorem to show that if f is a nowhere-vanishing holomorphic function on U, then f admits a holomorphic square-root (i.e. there is a holomorphic function h such that $h(z)^2 = f(z)$ for every $z \in U$).

(ii) The key ingredient of a standard proof of the Riemann mapping theorem is to show that whenever a domain has the property that every nowhere zero holomorphic function on it admits a holomorphic square-root, then it is homeomorphic to the open unit disk (the non-trivial case of this being when the domain is not equal to $\mathbb{C}$, in which case the homeomorphism is in fact shown to be a conformal map). Assuming this, deduce that U is simply connected, i.e. has the property that every closed curve in U is null-homotopic in U. Thus a domain U is simply-connected if and only if every closed curve in U is homologous to zero in U.

Solution: (a) This is a standard example called the *Pochhammer contour*.[1] We draw the contour on the domain $\mathbb{C}\backslash\{-1,1\}$:



Staring at the diagram carefully, we see that the winding numbers of the curve around -1 and 1 are both 0. In particular, this implies that the Pochhammer contour is null-homologous on this domain, $\mathbb{C}\backslash\{-1,1\}$. On the other hand, if we try to deform the contour to a constant curve via a homotopy, we will get 'caught' on one of the missing points $-1, 1$, so this curve is not null-homotopic on this domain.

The winding number computation can be made fairly rigorous if we write down some messy expressions for the Pochhammer contour (though the results of Examples Sheet 2, Question 11, essentially mean we can always compute winding numbers 'by eye'). Showing that the Pochhammer contour is *not* null-homotopic is considerably more difficult, however.[2]

[1] Which remarkably is actually useful outside of this counterexample, in the integral representation of Euler's beta function!

[2] If you decide to take Part II Algebraic Topology, you will learn techniques that allow you to classify curves into homotopy equivalence classes - after taking that course, you should be able to prove (using the Siefert-Van Kampen theorem) that the first homotopy group (fundamental group) of the doubly-punctured plane about any base point is $\mathbb{Z} * \mathbb{Z}$, the free group with two generators, and you should be able to identify the Pochhammer contour with one of the non-trivial group elements.

(b) (i) We *construct* a holomorphic square root as follows. First of all, we motivate the argument by some non-rigorous comments. If we were in the best of all possible worlds, we would just be able to write an answer down such as:

$$h(z) = \sqrt{|f(z)|}\exp\left(\frac{1}{2}i\arg(f(z))\right).$$

However, it may be the case that this isn't a continuous function, so we have to be a bit more careful. The question tells us that we should assume that all piecewise C^1 closed curves in U are null-homologous, which suggests the use of the homology form of Cauchy's theorem on some integrals at some point. Therefore, we are inspired to recast our equation in terms of some integrals instead.

The key 'heuristic' integral we are guided by here is:

$$\log\frac{|f(\gamma(1))|}{|f(\gamma(0))|} + i\left(\arg(f(\gamma(1))) - \arg(f(\gamma(0)))\right) = \log(f(\gamma(1)))) - \log(f(\gamma(0))) = \int_\gamma \frac{f'(z)}{f(z)}\,dz,$$

for some curve $\gamma : [0,1] \to \mathbb{C}$; note the right hand side is an antiderivative of f'/f. This suggests writing h in the form:

$$h(z) = \sqrt{|f(z)|}\exp\left(\frac{1}{2}i\arg(f(z))\right) = \exp\left(\frac{1}{2}\log|f(z)| + \frac{1}{2}i\arg(f(z))\right)$$

$$= \exp\left(\frac{1}{2}\log(f(z_0))\right)\exp\left(\frac{1}{2}\int_\gamma \frac{f'(z)}{f(z)}\,dz\right)$$

$$= \sqrt{|f(z_0)|}\exp\left(\frac{1}{2}i\arg(f(z_0))\right)\exp\left(\frac{1}{2}\int_\gamma \frac{f'(z)}{f(z)}\,dz\right)$$

where $\gamma : [0,1] \to \mathbb{C}$ is a curve satisfying $\gamma(0) = z_0$ and $\gamma(1) = z$, for some fixed $z_0 \in U$. This will be our guess for the correct square root function, which we shall now prove rigorously.

Claim: Let U be a domain on which every piecewise C^1 closed curve is null-homologous. Given any holomorphic function $f : U \to \mathbb{C}\backslash\{0\}$, there exists a holomorphic square root of f, say $h : U \to \mathbb{C}$, given explicitly by:

$$h(z) = \sqrt{|f(z_0)|}\exp\left(\frac{1}{2}i\arg(f(z_0))\right)\exp\left(\frac{1}{2}(F(z) - F(z_0))\right),$$

where $z_0 \in U$ is a fixed point in U (which can be chosen arbitrarily), and $F(z)$ is an antiderivative of f'/f.

Proof: First, we must check that h is well-defined. Certainly $\sqrt{|f(z_0)|}$ exists, and $\arg(f(z_0)) \in (-\pi,\pi]$ exists since f is never zero. Since f is never zero, f'/f is holomorphic on U and hence the homology form of Cauchy's theorem tells us that the integral of f'/f around any null-homologous closed piecewise C^1 curve in U is zero. Since all piecewise C^1 closed curves in U are null-homologous, the 'antiderivative theorem' ensures that f'/f has an antiderivative, hence an F described in the claim exists. Note since F is holomorphic, h is holomorphic.

All that remains is to show that h squares to f. Taking the derivative of $h(z)$, we see that:

$$h'(z) = h(z)f'(z)/2f(z),$$

since $F'(z) = f'(z)/f(z)$ by definition. This implies that:

$$\frac{d}{dz}\left(\frac{h(z)^2}{f(z)}\right) = \frac{2h(z)h'(z)}{f(z)} - \frac{h(z)^2 f'(z)}{f(z)^2} = 0,$$

and hence $h(z)^2 = (\text{constant}) \cdot f(z)$. But $h(z_0)^2 = f(z_0)$, so the constant is one. $\square$

(ii) In part (i), we showed that on any domain U where every piecewise C^1 closed curve is null-homologous, any holomorphic function $f : U \to \mathbb{C}\backslash\{0\}$ has a holomorphic square root. In part (ii), we are given that such a domain is homeomorphic to the open unit disk. Therefore, since the open unit disk is simply-connected, to show that U is simply-connected we need only show that simple-connectedness is preserved by homeomorphism.

Proposition: Simple-connectedness of domains in $\mathbb{C}$ is preserved by homeomorphism.

Proof: Suppose that U, V are homeomorphic domains, via the homeomorphism $f : U \to V$, and suppose that U is simply-connected. Let $\gamma : [0,1] \to V$ be a closed curve in V. Then $f^{-1} \circ \gamma : [0,1] \to U$ is a curve in U (since it is the composition of continuous maps) and $f^{-1} \circ \gamma(0) = f^{-1} \circ \gamma(1)$ since $\gamma(0) = \gamma(1)$, so $f^{-1} \circ \gamma$ is closed in U. It follows that $f^{-1} \circ \gamma$ is null-homotopic in U, since U is simply-connected.

Let the homotopy be given by $H : [0,1] \times [0,1] \to U$, with $H(0,t) = f^{-1} \circ \gamma(t)$ and $H(1,t) = c(t)$ for some constant curve $c(t) = c \in U$. This homotopy induces a homotopy $\tilde{H} : [0,1] \times [0,1] \to V$, given by:

$$\tilde{H} = f \circ H$$

This is a continuous function since it is the composition of continuous functions. Furthermore, $\tilde{H}(0,t) = \gamma(t)$ and $\tilde{H}(1,t) = f(c(t)) = f(c) \in V$, so γ is homotopic to a constant curve, and hence is null-homotopic. It follows that V is simply-connected. $\square$

In fact, this proof clearly extends to any topological space (replace 'domain' with 'topological space' everywhere).

We conclude that if every piecewise C^1 curve in the domain U is null-homologous, then U is simply-connected. The converse was proved in lectures, so this is an alternative characterisation of simply-connected domains.

❊ **Comments:** In part (b)(i) of this question, we handled one of the key steps of the proof of the *Riemann mapping theorem*, which recall states:

The Riemann mapping theorem: Let $U \subsetneq \mathbb{C}$ be a simply-connected domain which is not all of $\mathbb{C}$. Then U is conformally equivalent to the unit disk $D(0,1)$.

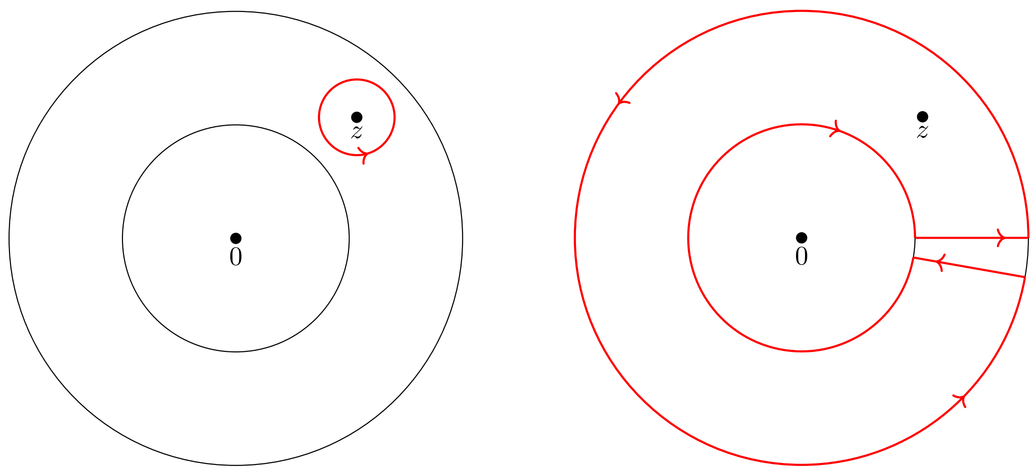
3. (a) Let $0 \leq r < R \leq \infty$, $A = \{r < |z| < R\}$ and let f be holomorphic on A. Show that there is a *unique* decomposition $f = f_1 + f_2$ such that f_1 is holomorphic on $\{|z| < R\}$, f_2 is holomorphic on $\{|z| > r\}$ and $f_2(z) \to 0$ as $z \to \infty$.

(b) How does this extend to the case when A is a (bounded) domain between two non-concentric disjoint circles? What about a domain bounded by three disjoint circles?

Solution: (a) This part is really just an alternative proof of Laurent's theorem. To begin with, we note that for any r_1, r_2 such that $r < r_2 < r_1 < R$, by the Cauchy integral formula for a disk we have for all $z \in \{r_2 < |z| < r_1\}$:

$$f(z) = \frac{1}{2\pi i} \oint\limits_{D(z,\epsilon)} \frac{f(w)}{w - z}\, dw,$$

where $D(z,\epsilon)$ is a disk of sufficiently small radius ϵ. We now 'blow up' the disk via a homotopy so that it fills the entire annulus $\{r_2 < |z| < r_1\}$, as in the image below.



Left: the initial circular contour around the point z, sandwiched between the circles $|z| = r_2$ and $|z| = r_1$. Right: the deformation of the circular contour, via some homotopy, to meet the annulus.

Writing down the homotopy explicitly is technical and complicated, but is certainly possible. We see that the circular contour can be deformed such that it is composed of two circular contours on the surface of the annulus, plus some small straight segments whose integrals will cancel out if they are taken close enough to one another. The homotopy form of Cauchy's theorem then tells us that:

$$f(z) = \frac{1}{2\pi i} \oint\limits_{D(z,\epsilon)} \frac{f(w)}{w - z}\, dw = \frac{1}{2\pi i} \oint\limits_{|w|=r_1} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \oint\limits_{|w|=r_2} \frac{f(w)}{w - z}\, dw,$$

which gives a natural decomposition of $f(z)$ into two functions. We claim that:

$$f_1(z) = \frac{1}{2\pi i} \oint\limits_{|w|=r_1} \frac{f(w)}{w - z}\, dw, \qquad f_2(z) = -\frac{1}{2\pi i} \oint\limits_{|w|=r_2} \frac{f(w)}{w - z}\, dw,$$

are the functions that we need. It remains to show that they have the required properties.

First, note that we can reduce r_2 to be arbitrarily close to r, and we can increase r_1 to be arbitrarily close to R, hence f_1, f_2 are well defined at any point $z \in A$. It remains to show that they define holomorphic functions on the given regions, and they are unique given $f_2(z) \to 0$ as $z \to \infty$.

Holomorphicity follows from the fact that we can expand f_1, f_2 in power series. For $f_1(z)$, we note that for $|z| < r_1$, we have:

$$f_1(z) = \frac{1}{2\pi i} \oint\limits_{|w|=r_1} \frac{f(w)}{w(1-z/w)}\, dw = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left(\oint\limits_{|w|=r_1} \frac{f(w)}{w^{n+1}}\, dw \right) z^n,$$

using the uniform convergence of the series to switch the order of the sum and integral. This equality holds on $|z| < r_1$, so $f_1(z)$ indeed defines a holomorphic function there. On the other hand, for $f_2(z)$, we note that for $|z| > r_2$, and hence:

$$f_2(z) = \frac{1}{2\pi i} \oint\limits_{|w|=r_2} \frac{f(w)}{z(1-w/z)}\, dw = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left(\oint\limits_{|w|=r_2} w^n f(w)\, dw \right) \frac{1}{z^{n+1}},$$

again using the uniform convergence to switch the order of the sum and integral. In particular, we can view the right hand side as the composition of a power series (convergent in the variable $1/z$, with $|1/z| < r_2$) with the holomorphic function $1/z$ (which is indeed holomorphic on $|z| > r_2$), so overall $f_2(z)$ is holomorphic on the specified region. Note additionally that this choice of $f_2(z)$ has yielded $f_2(z) \to 0$ as $z \to \infty$.

Uniqueness can be proved using Liouville's theorem. Suppose that $f(z) = f_1(z) + f_2(z) = g_1(z) + g_2(z)$, with both f_1, f_2 and g_1, g_2 satisfying the properties given in the question. Then we see that:

$$f_1(z) - g_1(z) = f_2(z) - g_2(z)$$

for all $z \in A$, the overlap of the regions of definition of f_1, f_2 and g_1, g_2. In particular, we see that we can define an entire function $h : \mathbb{C} \to \mathbb{C}$ via:

$$h(z) = \begin{cases} f_1(z) - g_1(z) & \text{for } |z| < R \\ f_2(z) - g_2(z) & \text{for } |z| > r. \end{cases}$$

But $f_2(z), g_2(z) \to 0$ as $z \to \infty$, so $h(z)$ is bounded. Hence $h(z)$ is a bounded, entire function, which tends to 0 at infinity, hence must be identically zero by Liouville's theorem. It follows $f_1 = g_1$ and $f_2 = g_2$ throughout their respective regions of definition.

(b) It should be clear that the 'Cauchy integral formula' proof in (a) extends immediately to the case of non-concentric circles, with no significant differences. If the circles are $|z-a| = r$ and $|z-b| = R$, with $R > r$ and $|a-b| < R-r$ (this condition is necessary for one circle to lie inside the other), then we can use the argument above to write f in the region bounded by the circles as:

$$f(z) = \frac{1}{2\pi i} \oint\limits_{|w-a|=R} \frac{f(w)}{w-z}\, dz - \frac{1}{2\pi i} \oint\limits_{|w-b|=r} \frac{f(w)}{w-z}\, dz,$$

where this time we have ignored the subtlety that we actually need to integrate around circles close to the initial circles but contained within the region they bound.

Again defining $f_1(z)$ to be the first term, and $f_2(z)$ to be the second term, we see that for $|z-a| < R$ we can expand $f_1(z)$ via:

$$f_1(z) = \frac{1}{2\pi i} \oint\limits_{|w-a|=R} \frac{f(w)}{(w-a)(1-(z-a)/(w-a))}\, dw = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-a)^n \left(\oint\limits_{|w-a|=R} \frac{f(w)}{(w-a)^{n+1}}\, dw \right).$$

Hence we see that $f_1(z)$ is holomorphic on $|z-a| < R$.

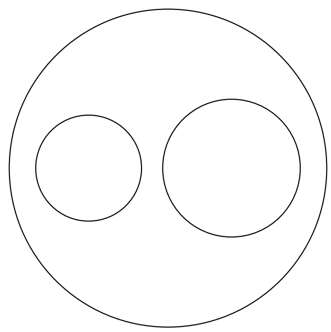
For $|z-b|>r$, we can expand $f_2(z)$ via:

$$f_2(z) = \frac{1}{2\pi i} \oint\limits_{|w-b|=r} \frac{f(w)}{(z-b)(1-(w-b)/(z-b))}\,dw = \frac{1}{2\pi i}\sum_{n=0}^{\infty} \frac{1}{(z-b)^{n+1}} \left(\oint\limits_{|z-b|=r} (w-b)^n f(w)\,dw \right).$$

We see that this is a holomorphic function of $1/(z-b)$ for $1/|z-b|<r$, and since $1/(z-b)$ is holomorphic for $|z-b|>r$, we have that f is the composition of holomorphic functions and is thus itself holomorphic on $|z-b|>r$. Furthermore, $f_2(z)\to 0$ as $|z|\to\infty$. Uniqueness follows by the Liouville argument in exactly the same way as above.

Therefore, we again have a unique decomposition $f = f_1 + f_2$, where f_1 is holomorphic inside of the outer circle, and f_2 is holomorphic outside of the inner circle, satisfying $f_2(z)\to 0$ as $z\to\infty$.

In the case of three disjoint circles, in order to have a non-zero region bounded by the circles, the question must be referring to a setup of the form:



That is, we should consider two smaller circles contained within a larger one. Here, the argument changes, but only slightly. Let C_1 be an anticlockwise contour around the outer circle and let C_2, C_3 be anticlockwise contours around the inner circles. The standard Cauchy's integral formula deformation step then yields:

$$f(z) = \frac{1}{2\pi i}\oint\limits_{C_1} \frac{f(w)}{w-z}\,dz - \frac{1}{2\pi i}\oint\limits_{C_2} \frac{f(w)}{w-z}\,dz - \frac{1}{2\pi i}\oint\limits_{C_3} \frac{f(w)}{w-z}\,dz.$$

for all z in the region bounded by the circles. Let us call each of the terms f_1, f_2, f_3 respectively. As usual, we have that f_1 defines an analytic function on the interior of the circle C_1, and f_2, f_3 define analytic functions on the exterior of the circles C_2, C_3 respectively, with $f_2(z), f_3(z)\to 0$ as $z\to\infty$. Therefore we have a decomposition $f(z) = f_1(z)+f_2(z)+f_3(z)$ of the type discussed above.

For uniqueness, suppose that $f(z) = f_1(z)+f_2(z)+f_3(z) = g_1(z)+g_2(z)+g_3(z)$ for two sets of appropriate functions f_1, f_2, f_3 and g_1, g_2, g_3 satisfying all above conditions. Consider $g_2 - f_2 = f_1 + f_3 - g_1 - g_3$. The left hand side is well-defined outside of the circle C_2. The right hand side is well-defined inside the circle C_1 and outside the circle C_3. In particular, we can construct an entire function $h:\mathbb{C}\to\mathbb{C}$ via:

$$h(z) = \begin{cases} g_2(z) - f_2(z) & \text{if } z \text{ outside } C_2; \\ f_1(z)+f_3(z)-g_1(z)-g_3(z) & \text{if } z \text{ inside } C_2. \end{cases}$$

In particular, this function satisfies $h(z)\to 0$ as $|z|\to\infty$, so it follows that h is entire, and bounded by zero at infinity, so is identically zero everywhere. In particular $g_2 = f_2$. Similarly, $g_3 = f_3$, and hence $g_1 = f_1$. Thus we have uniqueness as before.

4. Use the residue theorem to give a proof of Cauchy's derivative formula: if f is holomorphic on $D(a,R)$ and $|w-a| < r < R$, then:

$$f^{(n)}(w) = \frac{n!}{2\pi i} \oint\limits_{\partial D(a,r)} \frac{f(z)}{(z-w)^{n+1}}\,dz.$$

➼ **Solution:** Let's begin by recalling a clear statement of the residue theorem:

Cauchy's residue theorem: Let U be a simply-connected domain, let $z_1, ..., z_n \in U$ and let $g : U\backslash\{z_1, ..., z_n\} \to \mathbb{C}$ be holomorphic. For any piecewise C^1 closed curve $\gamma : [0,1] \to U$ whose image does not contain any of the points $z_1, ..., z_n$, we have:

$$\frac{1}{2\pi i}\oint\limits_{\gamma} g(z)\,dz = \sum_{i=1}^{n} I(\gamma; z_i)\mathrm{Res}(g; z_i),$$

where $\mathrm{Res}(g; z_i)$ denotes the *residue* of g at the point z_i (i.e. the coefficient of $1/(z-z_i)$ in its Laurent expansion about $z = z_i$) and $I(\gamma; z_i)$ denotes the winding number of the curve γ about z_i.

We can now apply the residue theorem to give a fairly direct proof of the derivative formula. We consider the simply-connected region $U = D(a,R)$, and the function:

$$g(z) = \frac{f(z)}{(z-w)^{n+1}}$$

which is holomorphic on U except for a single singularity at $z = w$. We let $\gamma = \partial D(a,r)$, with the boundary traversed anticlockwise, be a piecewise closed curve in U; since $|w-a| < r < R$, the image of γ does not contain w. Furthermore, the winding number of γ about w is clearly 1 since it is encircled anticlockwise once.

Hence, all the conditions are satisfied for us to apply the residue theorem to the integral on the right hand side of the question. We get:

$$\frac{n!}{2\pi i}\oint\limits_{\partial D(a,r)} \frac{f(z)}{(z-w)^{n+1}}\,dz = n! \cdot \mathrm{Res}\left(\frac{f(z)}{(z-w)^{n+1}}; w\right).$$

It remains to compute the residue. There is a standard formula which allows us to quickly achieve this:

Proposition: Let $f : U \to \mathbb{C}$ be holomorphic except for a pole at $z = z_0$ of order N. The residue of f at the pole $z = z_0$ is given by:

$$\mathrm{Res}\left(f(z); z_0\right) = \frac{1}{(N-1)!}\lim_{z\to z_0}\frac{d^{N-1}}{dz^{N-1}}\left[(z-z_0)^N f(z)\right].$$

Proof: We just manipulate the Laurent series of f about $z = z_0$. The Laurent series takes the form:

$$f(z) = \sum_{n=-N}^{\infty} a_n(z-z_0)^n,$$

so applying the operations on the right hand side of the proposition, we have:

$$\frac{d^{N-1}}{dz^{N-1}}(z-z_0)^N f(z) = \frac{d^{N-1}}{dz^{N-1}}\sum_{n=-N}^{\infty} a_n(z-z_0)^{n+N} = \frac{d^{N-1}}{dz^{N-1}}\sum_{n=0}^{\infty} a_{n-N}(z-z_0)^n$$

$$= \sum_{n=0}^{\infty} a_{n-N}n(n-1)...(n-N+2)(z-z_0)^{n-N+1}.$$

Taking the limit as $z \to z_0$ in this expression, we see that only the $n = N-1$ term survives, leaving $a_{-1}(N-1)!$. Thus the formula stated in the proposition gives the coefficient a_{-1}, which by definition is the required residue. □

It remains to reply this result to the question. We wish to compute the residue:

$$\operatorname{Res}\left(\frac{f(z)}{(z-w)^{n+1}}; w\right).$$

Since f is holomorphic, we can write it as $f(z) = (z-w)^k g(z)$ where g is holomorphic and $g(w) \neq 0$. If $k \geq n+1$, then f had the function has a removable singularity at $z = w$, and it follows by Cauchy's theorem that the residue is zero; this is fine, since if f has a zero at w of at least order $n+1$, then its first n derivatives must vanish at w.

On the other hand, if $k < n+1$, the relevant function has a pole of order $n+1-k$ at $z = w$. Thus using the formula from above, we have:

$$\begin{aligned}\operatorname{Res}\left(\frac{f(z)}{(z-w)^{n+1}}; w\right) &= \frac{1}{(n-k)!} \lim_{z \to w} \frac{d^{n-k}}{dz^{n-k}} \left[(z-w)^{n+1-k} \frac{(z-w)^k g(z)}{(z-w)^{n+1}}\right] \\ &= \frac{g^{(n-k)}(w)}{(n-k)!}.\end{aligned}$$

To finish, note (by the generalised Leibniz rule):

$$\frac{f^{(n)}(w)}{n!} = \frac{d^n}{dz^n}\left[(z-w)^k g(z)\right]\Big|_{z=w} = \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\frac{d^k}{dz^k}\left[(z-w)^k\right]\Big|_{z=w} g^{(n-k)}(w) = \frac{k!}{n!}\binom{n}{k} g^{(n-k)}(w) = \frac{g^{(n-k)}(w)}{(n-k)!}.$$

Hence we have in all cases:

$$f^{(n)}(w) = \frac{n!}{2\pi i} \oint\limits_{\partial D(a,r)} \frac{f(z)}{(z-w)^{n+1}}\, dz.$$

as required.

5. Let U be a domain in $\mathbb{C} = \mathbb{R}^2$ and let $u : U \to \mathbb{R}$ be a C^2 harmonic function.

(a) If U is simply-connected, show that there is a holomorphic function $f : U \to \mathbb{C}$ such that $u = \mathrm{Re}(f)$. [Hint: consider $g = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$.]

(b) If $U = D(a, r)$, show that:

$$\sup_{z \in D(a,r/2)} |Du(z)| \leq \frac{C}{r} \sup_{z \in D(a,r)} |u|$$

where C is a fixed constant independent of U, u, r and a. (Here, $Du = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ and $|Du| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}$.)

(c) Now suppose $U = D(0,1)\backslash\{0\}$. If u has a continuous extension to $D(0,1)$, show that the point $z = 0$ is a removable singularity of u, i.e. show that the extended function is C^2 and harmonic in $D(0,1)$.

(d) In (c), it suffices to assume that u is bounded on U. Does your proof generalise to this case?

(e) In (c), does it suffice to assume $|z||u(z)| \to 0$ as $z \to 0$? Compare with the case of holomorphic $f : D(0,1)\backslash\{0\} \to \mathbb{C}$.

Solution: (a) The proof is identical to the one we gave for Examples Sheet 1, Question 13; we include it again here for completeness. If there were such an $f : U \to \mathbb{C}$, then we would be able to write its derivative $g : U \to \mathbb{C}$ in the form:

$$g(z) = \frac{\partial u}{\partial x}(z) - i\frac{\partial u}{\partial y}(z).$$

Notice that this defines a holomorphic function g, since both $\partial u/\partial x$ and $\partial u/\partial y$ are continuously differentiable (we are given u is C^2) and $u_{xx} = -u_{yy}$, $u_{xy} = u_{yx}$ by the fact that u is harmonic and the symmetry of mixed partial derivatives for C^2 functions, respectively. Now, recall Cauchy's theorem for a simply-connected domain:

Cauchy's theorem (for a simply-connected domain): Let U be a simply-connected domain, let $g : U \to \mathbb{C}$ be holomorphic, and let γ be a closed, piecewise C^1 curve in U. Then:

$$\oint_\gamma g(z)\, dz = 0.$$

In particular, since U is simply-connected, it follows that:

$$\oint_\gamma \left(\frac{\partial u}{\partial x}(z) - i\frac{\partial u}{\partial y}(z)\right) dz = 0$$

for all closed, piecewise C^1 curves γ in U. Recall that this is sufficient for an antiderivative to exist:

Theorem: Let $U \subseteq \mathbb{C}$ be a domain and let $g : U \to \mathbb{C}$ be continuous. If:

$$\oint_\gamma g(z) = 0$$

for every closed, piecewise C^1 curve γ in U, then g has an antiderivative on U.

Hence we can genuinely consider a function $f : U \to \mathbb{C}$ such that:

$$g(z) = f'(z) = \frac{\partial u}{\partial x}(z) - i\frac{\partial u}{\partial y}(z),$$

throughout the domain U.

The function $f : U \to \mathbb{C}$ is clearly holomorphic (it is complex differentiable throughout U by virtue of being an antiderivative). It remains to show that its real part is u. Let $f = U + iV$ be the expression for f in terms of its real and imaginary parts U, V respectively. Then we know its derivative takes the form:

$$f' = U_x - iU_y.$$

Comparing real and imaginary parts, we see that the real multivariable derivative $d(U - u)|_z = 0$ vanishes throughout the domain U, hence $U - u$ is constant on U (since it is simply-connected, hence path-connected). It follows that $U = u +$ constant; the constant can be removed simply by a redefinition of f by the constant: $f \mapsto f -$ constant.

(b) There is quite a fast way of doing this part of the question using methods from real, multivariable calculus. Begin by fixing $z \in D(a, r/2)$, letting $\rho < r/2$ and recalling the mean value property for harmonic functions (in this instance, this property follows by taking the real part of the mean value property for a holomorphic function f satisfying $u = \mathrm{Re}(f)$):

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + \rho e^{i\theta})\, d\theta,$$

for some $z \in D(a, r/2)$. Let us multiply both sides by ρ, then integrate both sides over $\rho \in [0, r/2]$ to yield:

$$\frac{1}{8}r^2 u(z) = \frac{1}{2\pi}\int_0^{r/2}\int_0^{2\pi} u(z + \rho e^{i\theta})\, d\theta\, \rho d\rho.$$

In particular, we have established another mean value property for harmonic functions, namely they equal their mean values on disks:

$$u(z) = \frac{4}{\pi r^2}\iint_{D(z,r/2)} u(x, y)\, dx\, dy.$$

Taking a linear combination of derivatives of this formula, and using Green's theorem in the plane, we have:

$$|Du(z)| = \left|\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right| = \frac{4}{\pi r^2}\left|\iint_{D(z,r/2)} \left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) dx\, dy\right| = \frac{4}{\pi r^2}\left|-i\oint_{\partial D(z,r/2)} u(z)\, dz\right|.$$

We can now apply the standard integral estimates to yield:

$$|Du(z)| \le \frac{4}{r}\sup_{\partial D(z,r/2)} |u| \le \frac{4}{r}\sup_{D(a,r)} |u|,$$

with the final inequality following by the maximum principle for harmonic functions. Taking the supremum over all $z \in D(a, r/2)$ yields the result:

$$\sup_{D(a,r/2)} |Du| \le \frac{4}{r}\sup_{D(a,r)} |u|.$$

(c) For parts (c), (d) and (e), we will prove a general result and use it in each instance.

Proposition: Suppose that $u : D(0,1)\backslash\{0\} \to \mathbb{R}$ is a harmonic function on the punctured unit disk. Then u is of the form:

$$u(z) = c\log|z| + \mathrm{Re}(f(z))$$

for some constant $c \in \mathbb{R}$ and $f : D(0,1)\backslash\{0\} \to \mathbb{C}$ holomorphic.

Proof: Recall from Examples Sheet 1, Question 4(ii) that if $g : U \to V$ is holomorphic and $u : V \to \mathbb{C}$ is harmonic, then the composition $u \circ g : U \to \mathbb{C}$. Now, note that the punctured unit disk is the image of the exponential map acting on the left half plane, since:

$$e^{x+iy} = e^x e^{iy}$$

parametrises $D(0,1)\backslash\{0\}$ for $x \in (-\infty, 0)$ and $y \in (-\infty, \infty)$. However, the left half plane is simply-connected, so $u \circ \exp : \{\mathrm{Re}(z) < 0\} \to \mathbb{R}$ is a harmonic function on a simply-connected domain, hence we can write:

$$u \circ \exp(z) = U(z)$$

for a holomorphic function $U + iV : \{\mathrm{Re}(z) < 0\} \to \mathbb{C}$. Since u is single-valued on the punctured disk, we require $U(z + 2\pi i) = U(z)$ whenever $z, z + 2\pi i$ are in the left half plane; that is, U is $2\pi i$-periodic. On the other hand, we have by the Cauchy-Riemann equations:

$$\frac{\partial}{\partial x}\left(V(z+2\pi i) - V(z)\right) = \frac{\partial V}{\partial x}(z + 2\pi i) - \frac{\partial V}{\partial x}(z) = -\frac{\partial U}{\partial y}(z + 2\pi i) + \frac{\partial U}{\partial y}(z + 2\pi i) = 0,$$

and similarly the $\partial/\partial y$ derivative of $V(z + 2\pi i) - V(z)$ is zero. Since the left half plane is connected, this implies that $c = 2\pi(V(z + 2\pi i) - V(z))$ is constant everywhere. It follows that the function:

$$U(z) + iV(z) - cz$$

is $2\pi i$-periodic in the left half plane. It follows that:

$$f(z) = U(\log(z)) + iV(\log(z)) - c\log(z)$$

is a holomorphic function on the punctured unit disk, where $\log(z) = \log(|z|) + i\arg(z)$ (i.e. the complex logarithm *without* the branch cut). Taking the real part, we have:

$$\mathrm{Re}(f(z)) = \mathrm{Re}(U(\log(z))) - c\log|z| \qquad \Rightarrow \qquad u(z) = c\log|z| + \mathrm{Re}(f(z)),$$

as required. □

We can now begin this part of the question. Suppose $u : D(0,1)\backslash\{0\} \to \mathbb{R}$ is a harmonic function. Then by the above proposition, we have:

$$u(z) = c\log|z| + \mathrm{Re}(f(z))$$

for some holomorphic $f : D(0,1)\backslash\{0\} \to \mathbb{C}$ and some real constant c. Assuming that u has a continuous extension to the unit disk $D(0,1)$, we must have that the limit of $u(z)$ as $z \to 0$ exists, which implies $c = 0$ in the above representation and:

$$\lim_{z\to 0} \mathrm{Re}(f(z))$$

exists. We expect this condition to imply that $z = 0$ is a removable singularity for $f(z)$ too; to see this, note that the limit as $z \to 0$ of $|e^f| = e^{\mathrm{Re}(f)}$ exists, hence by Examples Sheet 2, Question 15, the singularity at $z = 0$ is removable. Thus $u(z) = \mathrm{Re}(f(z))$ for a holomorphic function $f : D(0,1) \to \mathbb{C}$ and indeed the singularity in u was removable.

For part (d), we note immediately that our proof immediately generalises to the case where $u(z)$ is merely bounded on $D(0,1)\backslash\{0\}$, since this implies $c = 0$, and $\mathrm{Re}(f(z))$ is bounded in a neighbourhood of $z = 0$ (so we can apply Examples Sheet 2, Question 15 to conclude $z = 0$ is removable again).

For part (e), note that it does not suffice to assume that $|z||u(z)| \to 0$ to conclude that u has a removable singularity. Indeed, $u(z) = \log|z|$ satisfies this condition but does not have a removable singularity at $z = 0$. On the other hand, for holomorphic f, the condition:

$$\lim_{z\to 0} zf(z) = 0$$

precisely states that f has a removable singularity at $z = 0$. This highlights a difference between the treatment of singularities in the theory of harmonic functions versus the theory of holomorphic functions.

6. Evaluate the following integrals:

(a) $\displaystyle\int_0^{\pi} \frac{d\theta}{4+\sin^2(\theta)}$; (b) $\displaystyle\int_0^{\infty} \sin(x^2)\,dx$; (c) $\displaystyle\int_0^{\infty} \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx$; (d) $\displaystyle\int_0^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx$.

➾ **Solution:** (a) This trigonometric integral can be tackled in a similar way to the method used for Examples Sheet 2, Question 2. First, we will replace $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$ in the denominator in the hope that this will make the algebra slightly simpler later on:

$$\int_0^{\pi} \frac{d\theta}{4+\sin^2(\theta)} = \int_0^{\pi} \frac{d\theta}{4+\frac{1}{2}(1-\cos(2\theta))} = 2\int_0^{\pi} \frac{d\theta}{9-\cos(2\theta)}.$$

Next, make the substitution $u = 2\theta$, so that the integration variable is the full range $[0, 2\pi]$. We have:

$$2\int_0^{\pi} \frac{d\theta}{9-\cos(2\theta)} = \int_0^{2\pi} \frac{du}{9-\cos(u)}.$$

We notice that this integral can now be recast as a contour integral:

$$\int_0^{2\pi} \frac{du}{9-\cos(u)} = \int_0^{2\pi} \frac{du}{9-\frac{1}{2}(e^{iu}+e^{-iu})} = -i\oint_{|z|=1} \frac{dz}{z(9-\frac{1}{2}(z+z^{-1}))} = 2i\oint_{|z|=1} \frac{dz}{z^2-18z+1}.$$

The singularities of the integral occur when $z^2 - 18z + 1 = 0$. Solving this quadratic, we have:

$$z = 9 \pm 4\sqrt{5}.$$

Note $9+4\sqrt{5}$ lies outside of the circle $|z| = 1$, but $9-4\sqrt{5} \approx 0.056$ lies inside the circle. In particular, we can write:

$$2i\oint_{|z|=1} \frac{dz}{z^2-18z+1} = 2i\oint_{|z|=1} \frac{1/(z-9-4\sqrt{5})}{z-9+4\sqrt{5}}\,dz = \frac{(2\pi i)\cdot 2i}{((9-4\sqrt{5})-9-4\sqrt{5})} = \frac{\pi}{2\sqrt{5}},$$

by the Cauchy integral formula. Hence we have:

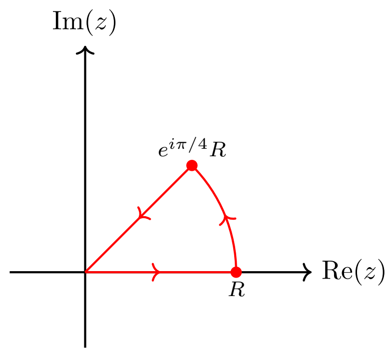
$$\int_0^{\pi} \frac{1}{4+\sin^2(\theta)}\,d\theta = \frac{\pi}{2\sqrt{5}}.$$

(b) Let $R > 0$, and consider the contour integral:

$$\oint_{C_R} e^{iz^2}\,dz,$$

where C_R consists of a straight line segment along the real axis from 0 to R, then an eighth of a circular arc in the upper half plane, followed by a straight line segment from $e^{i\pi/4}R$ to 0 along the imaginary axis (see the diagram). Note this contour encloses no singularities, so by Cauchy's theorem we have:

$$\oint_{C_R} e^{iz^2}\,dz = 0.$$



Now consider parametrising the integral along each section of the contour. We see that the integral can be rewritten in the form:

$$0 = \oint_{C_R} e^{iz^2}\,dz = \int_0^R e^{ix^2}\,dx + iR\int_0^{\pi/4} e^{iR^2(\cos(2\theta)+i\sin(2\theta))}e^{i\theta}\,d\theta + e^{i\pi/4}\int_R^0 e^{i(e^{i\pi/4}x)^2}\,dx$$

Let us analyse each of these terms separately:

- The first term's imaginary part is precisely the integral we want as $R\to\infty$. We have:

$$\int_0^\infty e^{ix^2}\,dx = \int_0^\infty \cos(x^2)\,dx + i\int_0^\infty \sin(x^2)\,dx.$$

- The third term is related to a Gaussian integral in the limit as $R\to\infty$, which we know how to do. We have:

$$e^{i\pi/4}\int_\infty^0 e^{i(e^{i\pi/4}x)^2}\,dx = -e^{i\pi/4}\int_0^\infty e^{-x^2}\,dx = -\frac{e^{i\pi/4}\sqrt{\pi}}{2},$$

  using the standard result for the Gaussian integral.

- Finally, the second term can be bounded as $R\to\infty$, though the argument is a little subtle. We have:

$$\left| iR\int_0^{\pi/4} e^{iR^2(\cos(2\theta)+i\sin(2\theta))}e^{i\theta}\,d\theta\right| \le R\int_0^{\pi/4} e^{-R^2\sin(2\theta)}\,d\theta.$$

  On the interval $[0,\pi/4]$, we have *Jordan's inequality* for $\sin(2\theta)$, namely $\sin(2\theta)\ge 4\theta/\pi$ (this can be proved by drawing the graphs of both these functions). Using this inequality, the integral is bounded above as:

$$R\int_0^{\pi/4} e^{-R^2\sin(2\theta)}\,d\theta \le R\int_0^{\pi/4} e^{-4R^2\theta/\pi}\,d\theta = \frac{\pi}{4R}\left[-e^{-4R^2\theta/\pi}\right]_0^{\pi/4}.$$

  As $R\to\infty$, this converges to zero. Thus as $R\to\infty$, the second term vanishes.

It follows that:

$$0 = \int_0^\infty \cos(x^2)\,dx + i\int_0^\infty \sin(x^2)\,dx - \frac{\sqrt{\pi}}{2}e^{i\pi/4}.$$

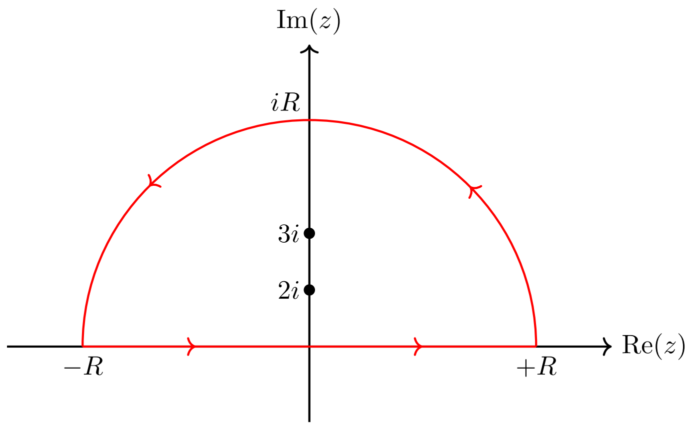
Comparing imaginary parts, we have:

$$\int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

(c) This is a more standard application of the residue theorem. We consider the contour integral:

$$\oint_{C_R} \frac{z^2}{(z^2+4)^2(z^2+9)}\,dz$$

where C_R is the contour shown in the figure below.



The contour of integration encloses two singularities of the integrand, at $z = 2i$ and $z = 3i$. Hence by the residue theorem, we have:

$$\oint_{C_R} \frac{z^2}{(z^2+4)^2(z^2+9)}\,dz = 2\pi i\left(\operatorname{Res}\left(\frac{z^2}{(z^2+4)^2(z^2+9)};2i\right) + \operatorname{Res}\left(\frac{z^2}{(z^2+4)^2(z^2+9)};3i\right)\right).$$

The singularity at $z = 3i$ is a simple pole, so we can use the standard simple pole formula for the residue:

$$\lim_{z\to 3i}\left[\frac{z^2(z-3i)}{(z^2+4)^2(z^2+9)}\right] = \lim_{z\to 3i}\left[\frac{z^2}{(z^2+4)^2(z+3i)}\right] = -\frac{9}{(-9+4)^2(3i+3i)} = \frac{3i}{50}.$$

The singularity at $z = 2i$ is a double pole, so we instead use the formula we proved in Question 4:

$$\lim_{z\to 2i}\frac{d}{dz}\left[(z-2i)^2\frac{z^2}{(z^2+4)^2(z^2+9)}\right] = \lim_{z\to 2i}\frac{d}{dz}\left[\frac{z^2}{(z+2i)^2(z^2+9)}\right].$$

The derivative is given by:

$$\frac{2z}{(z+2i)^2(z^2+9)} - \frac{2z^2\left(z^2+9+z(z+2i)\right)}{(z+2i)^3(z^2+9)^2}.$$

Taking the limit as $z \to 2i$, we arrive at the value:

$$-\frac{4i}{16\cdot 5} - \frac{8(-4+9-8)}{64i(9-4)^2} = -\frac{13i}{200}.$$

Thus we have:

$$\oint_{C_R} \frac{z^2}{(z^2+4)^2(z^2+9)}\,dz = 2\pi i\left(-\frac{13i}{200} + \frac{3i}{50}\right) = \frac{\pi}{100}.$$

We now consider evaluating the integral on each section of the contour separately. We have:

$$\frac{\pi}{100} = \oint_{C_R} \frac{z^2}{(z^2+4)^2(z^2+9)}\,dz = \int_{-R}^{R} \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx + \int_{\gamma} \frac{z^2}{(z^2+4)^2(z^2+9)}\,dz,$$

where γ is the large semicircular arc of radius R. As $R \to \infty$, the first term converges to the *Cauchy principal value integral*:

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx = \mathcal{P}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx,$$

where the $\mathcal{P}$ indicates the fact we are taking symmetric infinite limits. Since the integral without the $\mathcal{P}$ is convergent however, this privileged limit is actually equal to the standard integral anyway, so we can ignore this subtlety. Note we have:

$$\int_{-R}^{R} \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx = 2\int_{0}^{\infty} \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx.$$

using the fact that the integrand is even.

The remaining term is bounded, using the standard integral estimate:

$$\left|\int_{\gamma} \frac{z^2}{(z^2+4)^2(z^2+9)}\,dz\right| \leq \pi R^3 \sup_{z\in\gamma} \frac{1}{|(z^2+4)^2(z^2+9)|}.$$

By the reverse triangle inequality, we have $|(z^2+4)^2| \geq (|z|^2-4)^2 = (R^2-4)^2$ on γ, and similarly $|z^2+9| \geq R^2-9$ on γ. Thus we have:

$$\left|\int_{\gamma} \frac{z^2}{(z^2+4)^2(z^2+9)}\,dz\right| \leq \frac{\pi R^2}{(R^2-4)^2(R^2-9)} \to 0$$

as $R\to\infty$. Putting everything together, we have:

$$\int_{0}^{\infty} \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx = \frac{\pi}{200}.$$

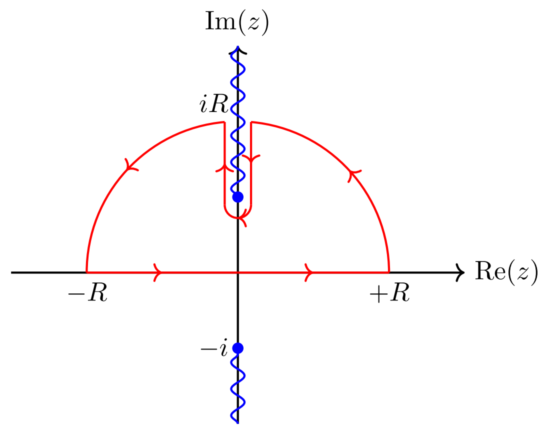
(d) The final integral needs a bit of care in defining our branches. In particular, $\log(z^2+1)$ as a complex function has two 'branch points' at $z = \pm i$ respectively; to define a single-valued function, we are inspired by the heuristic formula:

$$\log(z^2+1) = \log((z-i)(z+i)) = \log(z-i) + \log(z+i).$$

Given a point z then, we define $r_1, \theta_1, r_2, \theta_2$ to be constants obeying $z - i = r_1 e^{i\theta_1}$ and $z + i = r_2 e^{i\theta_2}$, where θ_1, θ_2 are chosen to be in the ranges $\theta_1 \in (-3\pi/2, \pi/2)$ and $\theta_2 \in (-\pi/2, 3\pi/2)$. Explicit formulae for r_1, r_2 are given by $r_1 = |z-i|$ and $r_2 = |z+i|$. We then define:

$$\log(z^2+1) = \log\left|z^2+1\right| + i(\theta_1+\theta_2).$$

This definition makes $\log(z^2+1)$ into a holomorphic function on $\mathbb{C}\backslash(i(-\infty,-1] \cup i[1,\infty))$; this is shown in the figure below.



Now with our definition of the function $\log(z^2+1)$ out of the way, we can start to do the integral. Consider the contour integral:

$$\oint_{C_{R,\epsilon}} \frac{\log(z^2+1)}{z^2+1}\,dz$$

around the contour shown above (the parameter ϵ is the distance of the contour from the upper branch cut at each point). The contour encloses no singularities of the integrand, so by Cauchy's theorem we have:

$$\oint_{C_{R,\epsilon}} \frac{\log(z^2+1)}{z^2+1}\,dz = 0.$$

Now, as usual, we evaluate the integral on each section of the contour separately. We have the following contributions:

- On the straight section of the contour, along the real axis, the parameters θ_1, θ_2 which parametrise $z-1$ and $z+1$ are such that $\theta_1+\theta_2=0$. In particular, we find that we have the contribution to the integral:

$$\int_{-R}^{R} \frac{\log(x^2+1)}{x^2+1}\,dx.$$

In the limit as $R\to\infty$, this tends to the Cauchy principal value integral:

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx,$$

but since the integral that the $\mathcal{P}$ acts on is itself convergent, we can ignore the $\mathcal{P}$. Since the integrand is even, the result is twice the integral we are interested in:

$$\int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx = 2\int_{0}^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx.$$

- Along the large circular arcs of radius R, the contribution to the integral tends to zero as $R\to\infty$. We can see this using the standard integral estimate; on either of the large circular arcs, say γ, we have:

$$\left|\int_{\gamma} \frac{\log(z^2+1)}{z^2+1}\,dz\right| \leq \text{length}(\gamma)\sup_{z\in\gamma}\left|\frac{\log(z^2+1)}{z^2+1}\right| \leq 2\pi R\sup_{|z|=R}\left|\frac{\log(z^2+1)}{z^2+1}\right|$$

The denominator is bounded by the reverse triangle inequality, $|z^2+1| \geq R^2-1$ everywhere. The numerator is bounded using the explicit definition of $\log(z^2+1)$ above:

$$|\log(z^2+1)| = |\log|z^2+1| + i(\theta_1+\theta_2)| \leq \sqrt{\log^2|z^2+1| + (\theta_1+\theta_2)^2} \leq \sqrt{\log^2|2R^2| + (2\pi)^2}.$$

Therefore, the integral is bounded above by:

$$\frac{2\pi R\sqrt{\log^2(2R^2)+(2\pi)^2}}{R^2-1} \to 0,$$

as $R\to\infty$. Hence the contribution is zero as promised.

- On the straight segments next door to the branch cut, the contributions are quite complicated. We suppose that we are working in the limit as $\epsilon\to 0$, so that the straight segments have been moved extremely close to the branch cut. However, we also suppose that the arc at $z=i$ is in fact a fully circular arc, so that the straight parts of the contour have imaginary parts in the range $[1+\epsilon, R]$ (this is an important technical point required to regulate some of the integrals we get). Essentially, the point is that we can move the straight segments close to the branch cut, but we can't move them that close to the branch point (yet, anyway).

Now, on the right hand side, z has parameters $\theta_1 = \pi/2$ and $\theta_2=\pi/2$. In particular, this means that the logarithm takes the form:

$$\log(z^2+1) = \log|z^2+1| + i\pi.$$

Parametrising the right hand contour as $z=iy$ for $y\in[1+\epsilon, R]$, we have $|z^2+1| = |-y^2+1| = y^2-1$. It follows that the contribution from the integral on the right hand side of the cut is:

$$i\int_R^{1+\epsilon} \frac{\log(y^2-1)+i\pi}{1-y^2}\,dy = i\int_{1+\epsilon}^R \frac{\log(y^2-1)+i\pi}{y^2-1}\,dy.$$

On the left hand side, z has parameters $\theta_1 = -3\pi/2$ and $\theta_2 = \pi/2$. In particular, this means that the logarithm takes the form:

$$\log(z^2+1) = \log|z^2+1| - i\pi.$$

Parametrising the left hand contour as $z=iy$ for $y\in[1+\epsilon,R]$ in the same way, it follows that the contribution from the integral on the left hand side of the cut is:

$$i\int_{1+\epsilon}^R \frac{\log(y^2-1)-i\pi}{1-y^2}\,dy = -i\int_{1+\epsilon}^R \frac{\log(y^2-1)-i\pi}{y^2-1}\,dy.$$

We see that the horrible logarithmic terms cancel when we add these contributions, and overall we are left with:

$$2\pi\int_{1+\epsilon}^R \frac{1}{y^2-1}\,dy = \pi\int_{1+\epsilon}^R\left[\frac{1}{y-1}-\frac{1}{y+1}\right]dy = \pi\left[\log\left(\frac{y-1}{y+1}\right)\right]_{1+\epsilon}^R = \pi\left[\log\left(\frac{R-1}{R+1}\right) - \log\left(\frac{\epsilon}{\epsilon+2}\right)\right].$$

As $R\to\infty$, the first term drops out. However, the remaining term is singular as $\epsilon\to 0$, and we cannot drop it just yet; we hope that the term we found above cancels with something when we perform the integration over the circular arc about $z=i$.

We will only care about the behaviour as $\epsilon\to 0$ eventually however, so we can ignore any contributions which drop out in this respect. Since:

$$\log\left(\frac{\epsilon}{\epsilon+2}\right) = \log(\epsilon) - \log(2) - \log\left(1+\frac{1}{2}\epsilon\right) = \log(\epsilon) - \log(2) + O(\epsilon),$$

we can treat the contribution from the straight segments of the contour as $\pi\left(\log(\epsilon)-\log(2)\right)$.

- Finally, consider the integration around the circular arc of radius ϵ centred on $z = i$. We parametrise the contour by $z = i + \epsilon e^{i\theta}$, where θ lies in the range $\theta \in (-3\pi/2, \pi/2)$. This parametrisation means that the logarithm parameters are given by $\theta_1 = \theta$, but $\theta_2 \sim \pi/2$ throughout (up to terms of order ϵ which we will neglect, because we don't want to do the necessary geometry); in particular, the logarithm looks like:

$$\log(z^2+1) = \log|(i+\epsilon e^{i\theta})^2+1| + i\left(\theta + \frac{\pi}{2}\right) = \log(\epsilon) + \log|2i+\epsilon e^{i\theta}| + i\left(\theta+\frac{\pi}{2}\right),$$

The resulting contribution is:

$$i\int_{\pi/2}^{-3\pi/2} \frac{\log(\epsilon)+\log|2i+\epsilon e^{i\theta}| + i(\theta+\pi/2)}{2i\epsilon e^{i\theta}+\epsilon^2 e^{2i\theta}}\epsilon e^{i\theta}\, d\theta.$$

Simplifying slightly, this reduces to:

$$i\int_{\pi/2}^{-3\pi/2} \frac{\log(\epsilon)+\log|2i+\epsilon e^{i\theta}| + i(\theta+\pi/2)}{2i+\epsilon e^{i\theta}}\, d\theta.$$

We are only interested in the result in the limit $\epsilon \to 0$, so (performing necessary series expansion, etc, all of which are valid for ϵ sufficiently small) in fact this integral effectively reduces to:

$$\frac{1}{2}(\log(\epsilon)+\log(2))\int_{\pi/2}^{-3\pi/2} d\theta + \frac{1}{2}i\int_{\pi/2}^{-3\pi/2}\left(\theta+\frac{\pi}{2}\right)\, d\theta.$$

The second term vanishes (shift $\phi = \theta + \pi/2$ to see that it is an odd integral over a symmetric range), leaving us with:

$$-\pi\left(\log(\epsilon)+\log(2)\right).$$

Hooray! The $\epsilon \to 0$ singularity from the straight sides cancels precisely with the contribution from the circular part. In particular, the total contribution from both the straight segments and the small circular part about $z = i$ is:

$$\pi\left(\log(\epsilon)-\log(2)\right) - \pi\left(\log(\epsilon)+\log(2)\right) = -2\pi\log(2),$$

up to terms of order $O(\epsilon)$, which tend to zero as $\epsilon \to 0$.

Putting everything together, we (finally) conclude that:

$$2\int_0^\infty \frac{\log(x^2+1)}{x^2+1}\, dx - 2\pi\log(2) = 0,$$

which gives the value of the integral as:

$$\int_0^\infty \frac{\log(x^2+1)}{x^2+1}\, dx = \pi\log(2).$$

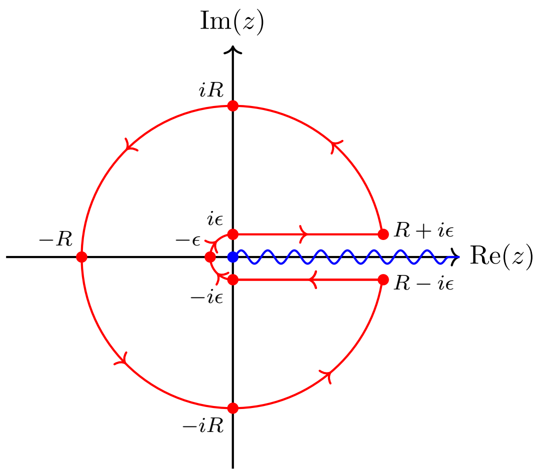
7. For $\alpha \in (-1,1)$ with $\alpha \neq 0$, compute:

$$\int_0^\infty \frac{x^\alpha}{x^2+x+1}\,dx.$$

➾ **Solution:** Let $R > 0$ and consider the contour integral:

$$\oint_{C_{R,\epsilon}} \frac{z^\alpha}{z^2+z+1}\,dz,$$

where we define the symbol z^α to mean $\exp(\alpha\log(z))$, where $\log(z)$ is a branch of the logarithm along the positive real axis (in particular, with a resulting choice of angles in the range $\theta \in (0, 2\pi)$). We choose the contour $C_{R,\epsilon}$ to be a *keyhole contour* of radius R (and bounded away from the branch cut by a distance ϵ), as depicted below.



We note that this contour encloses all the singularities of the integrand, occurring at $z^2+z+1=0$ (for sufficiently large R and sufficiently small ϵ), which are of course the two non-unity third roots of unity:

$$\omega = e^{2i\pi/3}, \qquad \omega^2 = e^{4i\pi/3}.$$

Now, by Cauchy's residue theorem we have for all sufficiently large R and sufficiently small ϵ:

$$\oint_{C_{R,\epsilon}} \frac{z^\alpha}{z^2+z+1}\,dz = 2\pi i\left(\mathrm{Res}\left(\frac{z^\alpha}{z^2+z+1};\omega\right) + \mathrm{Res}\left(\frac{z^\alpha}{z^2+z+1};\omega^2\right)\right).$$

Since $z=\omega$, $z=\omega^2$ are obviously simply poles of the integrand, we can use the standard formula for the residue at a simple pole to obtain the values on the right hand side:

$$\mathrm{Res}\left(\frac{z^\alpha}{z^2+z+1};\omega\right) = \mathrm{Res}\left(\frac{z^\alpha}{(z-\omega)(z-\omega^2)};\omega\right) = \lim_{z\to\omega}\left[\frac{(z-\omega)z^\alpha}{(z-\omega)(z-\omega^2)}\right] = \frac{\omega^\alpha}{\omega-\omega^2} = \frac{e^{2\pi i\alpha/3}}{2i\sin(2\pi/3)} = \frac{e^{2\pi i\alpha/3}}{i\sqrt{3}},$$

and similarly:

$$\mathrm{Res}\left(\frac{z^\alpha}{z^2+z+1};\omega\right) = \mathrm{Res}\left(\frac{z^\alpha}{(z-\omega)(z-\omega^2)};\omega^2\right) = \lim_{z\to\omega^2}\left[\frac{(z-\omega^2)z^\alpha}{(z-\omega)(z-\omega^2)}\right] = \frac{(\omega^2)^\alpha}{\omega^2-\omega} = -\frac{e^{4\pi i\alpha/3}}{i\sqrt{3}},$$

Overall then, we have:

$$\oint\limits_{C_{R,\epsilon}} \frac{z^\alpha}{z^2+z+1}\,dz = 2\pi i\left(\frac{e^{2\pi i\alpha/3}}{i\sqrt{3}} - \frac{e^{4\pi i\alpha/3}}{i\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}\left(e^{2\pi i\alpha/3} - e^{4\pi i\alpha/3}\right).$$

On the other hand, we can evaluate the contour integral on each section of the contour separately:

- The contribution from the large circular section of the contour, say γ_1, can be bounded using the standard estimate:

$$\left|\int\limits_{\gamma_1} \frac{z^\alpha}{z^2+z+1}\,dz\right| \leq \text{length}(\gamma_1)\cdot\sup_{\gamma_1}\left|\frac{z^\alpha}{z^2+z+1}\right| \leq 2\pi R\cdot\sup_{|z|=R}\left|\frac{z^\alpha}{z^2+z+1}\right|$$

  We note that the denominator can be bounded using the reverse triangle inequality, $|z^2+z+1| = |z-\omega||z-\omega^2| \geq (R-1)^2$, and hence we have:

$$\left|\int\limits_{\gamma_1} \frac{z^\alpha}{z^2+z+1}\,dz\right| \leq \frac{2\pi\cdot R^{\alpha+1}}{(R-1)^2}.$$

  In particular, the contribution from the large circular section of the contour tends to zero as $R\to\infty$, provided $\alpha<1$.

- The contribution from the small semicircular arc, say γ_2, can similarly be bounded using the standard estimate:

$$\left|\int\limits_{\gamma_2} \frac{z^\alpha}{z^2+z+1}\,dz\right| \leq \text{length}(\gamma_2)\cdot\sup_{\gamma_2}\left|\frac{z^\alpha}{z^2+z+1}\right| = \pi\epsilon\cdot\sup_{\theta\in(\pi/2,3\pi/2)}\left|\frac{\epsilon^\alpha e^{i\alpha\theta}}{\epsilon^2 e^{2i\theta}+\epsilon e^{i\theta}+1}\right|.$$

  Again using the reverse triangle inequality, we have $|\epsilon^2e^{2i\theta}+\epsilon e^{i\theta}+1| = |\epsilon e^{i\theta}-\omega||\epsilon e^{i\theta}-\omega^2| \geq (\epsilon-1)^2$. Thus we have:

$$\left|\int\limits_{\gamma_2} \frac{z^\alpha}{z^2+z+1}\,dz\right| \leq \frac{\pi\epsilon^{\alpha+1}}{(\epsilon-1)^2}.$$

  This tends to zero as $\epsilon\to 0$, provided $\alpha>-1$. Hence the contribution from the small semicircular arc is zero in the appropriate limit.

- Finally, we must compute the contributions from the straight segments on either side of the branch cut. Just above the cut, the function z^α takes the form x^α, while just below the cut it takes the form $x^\alpha e^{2i\pi\alpha}$ instead. In particular, this implies that the contribution from the straight segments in the limit $R\to\infty$ is given by:

$$\left(1-e^{2\pi i\alpha}\right)\int\limits_0^\infty \frac{x^\alpha}{x^2+x+1}dx.$$

Putting everything together then, we have (using $\alpha\neq 0$ to perform the division):

$$\int\limits_0^\infty \frac{x^\alpha}{x^2+x+1}dx = \frac{2\pi}{\sqrt{3}}\frac{e^{2\pi i\alpha/3}-e^{4\pi i\alpha/3}}{1-e^{2\pi i\alpha}} = \frac{2\pi}{\sqrt{3}}\frac{e^{-\pi i\alpha/3}-e^{\pi i\alpha/3}}{e^{-i\pi\alpha}-e^{i\pi\alpha}} = \frac{2\pi}{\sqrt{3}}\frac{\sin(2\pi\alpha/3)}{\sin(\pi\alpha)}.$$

8. (i) For a positive integer N, let γ_N be the square contour with vertices $(\pm 1 \pm i)(N + 1/2)$. Show that there exists $C > 0$ such that for every N, $|\cot(\pi z)| < C$ on γ_N.

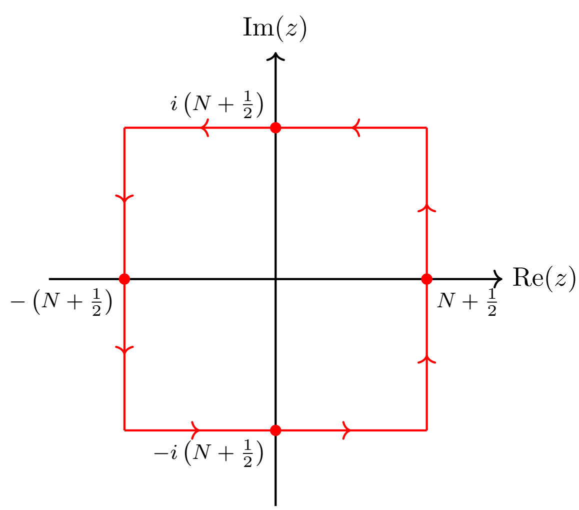
(ii) By integrating $\pi \cot(\pi z)/(z^2 + 1)$ around γ_N, show that:

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1 + \pi \coth(\pi)}{2}.$$

(iii) Evaluate:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}.$$

Solution: (i) The relevant contour γ_N looks like:



We wish to bound $\cot(\pi z)$ along the sides of this contour. There are two cases:

- Along the vertical sides of the contour, we have $z = \pm(N + \frac{1}{2}) + iy$ for $-(N+1/2) \leq y \leq (N+1/2)$. Hence, by the compound angle formula, $|\cot(\pi z)|$ takes the form:

$$\left|\cot\left(\pm\pi\left(N+\frac{1}{2}\right)+\pi i y\right)\right| = \frac{|\cot(\pm\pi(N+1/2))\cot(i\pi y) - 1|}{|\cot(\pm\pi(N+1/2)) + \cot(i\pi y)|} = |\tan(i\pi y)|,$$

  since $\cot(\pi(N+1/2)) = 0$. But $|\tan(i\pi y)| = |\tanh(\pi y)| \leq 1$, which gives the bound.

- Along the horizontal sides of the contour, we have $z = x \pm (N+\frac{1}{2})i$ for $-(N+1/2) \leq x \leq (N+1/2)$. Hence:

$$\left|\cot\left(\pi x \pm \left(N+\frac{1}{2}\right)i\right)\right| = \frac{|e^{i\pi x \mp \pi(N+1/2)} + e^{-i\pi x \pm \pi(N+1/2)}|}{|e^{i\pi x \mp \pi(N+1/2)} - e^{-i\pi x \pm \pi(N+1/2)}|} \leq \frac{e^{\mp\pi(N+1/2)} + e^{\pm\pi(N+1/2)}}{|e^{\mp\pi(N+1/2)} - e^{\pm\pi(N+1/2)}|} = \coth\left(\pi\left(N+\frac{1}{2}\right)\right),$$

  using the triangle inequality in the numerator and the reverse triangle inequality in the denominator. But $\coth$ is a decreasing function on $(0,\infty)$, thus the right hand side is bounded by $\coth(3\pi/2)$.

Hence there exists $C > 0$ such that for every N, $|\cot(\pi z)| < C$ for all $z \in \gamma_N$ (in fact we have shown $C = \max\{1, \coth(3\pi/2)\} = \coth(3\pi/2)$ suffices).

(ii) As suggested in the question, we consider the contour integral:

$$\oint_{\gamma_N} \frac{\pi\cot(\pi z)}{z^2+1}.$$

Using the bound from (i), we have:

$$\left|\oint_{\gamma_N} \frac{\pi\cot(\pi z)}{z^2+1}\right| \leq \pi \mathrm{length}(\gamma_N)\cdot \sup_{z\in\gamma_N}\left|\frac{\cot(\pi z)}{z^2+1}\right| \leq (4N+2)\pi C \sup_{z\in\gamma}\frac{1}{|z^2+1|}.$$

The denominator can be bounded with the reverse triangle inequality, $|z^2+1|\geq |z|^2-1$. The least the modulus of z can be along the contour is $(N+1/2)$, so in particular we have $|z^2+1|\geq (N+1/2)^2-1$. Thus overall we have:

$$\left|\oint_{\gamma_N} \frac{\pi\cot(\pi z)}{z^2+1}\right| \leq \frac{(4N+2)\pi C}{(N+1/2)^2-1}.$$

As $N\to\infty$, this tends to zero, and hence we have:

$$\lim_{N\to\infty}\oint_{\gamma_N} \frac{\pi\cot(\pi z)}{z^2+1} = 0.$$

On the other hand, we can evaluate the integral using Cauchy's residue theorem. There are simple poles at $z=\pm i$, and at all points where $\sin(\pi z)=0$, i.e. $z\in\{-N,-(N-1),...,N-1,N\}$ (for N fixed). Hence by Cauchy's residue theorem we have:

$$\oint_{\gamma_N} \frac{\pi\cot(\pi z)}{z^2+1} = 2\pi i\left(\sum_{n=-N}^{N}\mathrm{Res}\left(\frac{\pi\cot(\pi z)}{z^2+1};n\right)+\mathrm{Res}\left(\frac{\pi\cot(\pi z)}{z^2+1};i\right)+\mathrm{Res}\left(\frac{\pi\cot(\pi z)}{z^2+1};-i\right)\right).$$

At $z=i$, the residue is given by:

$$\mathrm{Res}\left(\frac{\pi\cot(\pi z)}{z^2+1};i\right)=\lim_{z\to i}\left[\frac{\pi\cot(\pi z)}{z+i}\right]=\frac{\pi\cot(\pi i)}{2i}=-\frac{\pi\coth(\pi)}{2},$$

using the fact that $\cot(ix)=-i\coth(x)$. At $z=-i$, the residue is given by:

$$\mathrm{Res}\left(\frac{\pi\cot(\pi z)}{z^2+1};-i\right)=\lim_{z\to -i}\left[\frac{\pi\cot(\pi z)}{z-i}\right]=-\frac{\pi\cot(-\pi i)}{2i}=-\frac{\pi\coth(\pi)}{2}.$$

At each integer n, the residue is given by:

$$\mathrm{Res}\left(\frac{\pi\cot(\pi z)}{z^2+1};n\right)=\frac{\pi}{n^2+1}\lim_{z\to n}\left[(z-n)\cot(\pi z)\right]=\frac{\pi\cos(\pi n)}{(n^2+1)\pi\cos(\pi n)}=\frac{1}{n^2+1},$$

using $\sin'(\pi z)=\pi\cos(\pi z)$ to evaluate the limit of $(z-n)/\sin(\pi z)$ as $z\to n$. Overall then, we have:

$$\oint_{\gamma_N} \frac{\pi\cot(\pi z)}{z^2+1} = 2\pi i\left(\sum_{n=-N}^{N}\frac{1}{n^2+1}-\pi\coth(\pi)\right)=2\pi i\left(2\sum_{n=0}^{N}\frac{1}{n^2+1}-1-\pi\coth(\pi)\right).$$

Hence in the limit as $N\to\infty$, we find that:

$$\sum_{n=0}^{\infty}\frac{1}{n^2+1}=\frac{1+\pi\coth(\pi)}{2},$$

as required.

(iii) Comparing with the argument in part (ii), we see that we should replace $\cot(\pi z)$ with a function that has residue $(-1)^n$ at each integer $z = n$; studying the way the residue is computed in (i), we see that an appropriate function would be $\operatorname{cosec}(\pi z)$. Hence we consider the contour integral:

$$\oint_{\gamma_N} \frac{\pi \operatorname{cosec}(\pi z)}{z^2+1}\, dz.$$

Since $|\operatorname{cosec}(\pi z)| = \sqrt{|\operatorname{cosec}^2(\pi z)|} = \sqrt{|1+\cot^2(\pi z)|} \leq \sqrt{1+C^2}$ for all N, and for all $z \in \gamma_N$, we have that $\operatorname{cosec}(\pi z)$ is bounded on the edges of the contour. Thus this contour integral tends to zero as $N \to \infty$, just as in part (ii).

On the other hand, the residues change slightly (indeed this is what we wanted to happen!). The residue at $z = i$ is now given by:

$$\operatorname{Res}\left(\frac{\pi \operatorname{cosec}(\pi z)}{z^2+1}; i\right) = \lim_{z \to i}\left[\frac{\pi \operatorname{cosec}(\pi z)}{z+i}\right] = \frac{\pi \operatorname{cosec}(\pi i)}{2i} = -\frac{\pi \operatorname{cosech}(\pi)}{2},$$

using $\operatorname{cosec}(ix) = -i\operatorname{cosech}(x)$. Similarly the residue at $z = -i$ is $-\pi \operatorname{cosech}(\pi)/2$.

At each integer $z = n$, the residue is now given by:

$$\operatorname{Res}\left(\frac{\pi \operatorname{cosec}(\pi z)}{z^2+1}; n\right) = \frac{\pi}{n^2+1} \lim_{z \to n}\left[\frac{z-n}{\sin(\pi z)}\right] = \frac{\pi}{(n^2+1)\pi \cos(\pi n)} = \frac{(-1)^n}{n^2+1},$$

precisely the residues we wanted. Putting everything together, we find that:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{1+\pi \operatorname{cosech}(\pi)}{2}.$$

9. Let f be holomorphic in an open set U except at a point $a \in U$ and at a sequence of points $a_n \in U$ with $a_n \neq a$ and $a_n \to a$. Suppose that each a_n is a pole of f. Note that a is then a non-isolated singularity.

(i) Given an explicit example of such a function f, points a_n, and a.

(ii) What can you say (in general) about the image $f(U \backslash \{a, a_1, a_2, ...\})$?

➾ **Solution:** (i) A simple example is $f : \mathbb{C}\backslash\{a_1, a_2, ...\} \to \mathbb{C}$ with:

$$f(z) = \operatorname{cosec}\left(\frac{1}{z}\right),$$

and the $a_{2n} = 1/n\pi$, $a_{2n+1} = -1/(n+1)\pi$ at all of the zeroes of sine (except zero itself), and $a = 0$. The points a_i in this case are simple zeroes of $\sin(1/z)$, so must be simple poles of the function $\operatorname{cosec}(1/z)$. However, the singularity at 0 is clearly non-isolated, as there is no neighbourhood of zero which contains no other singularities of f.

(ii) We claim that the behaviour near a *non-isolated singularity* that is the limit of poles is the same as that of an *isolated essential singularity*; we can prove this using the same method as we used for the Casorati-Weierstrass theorem.

The Casorati-Weierstrass theorem (for a singularity that is the limit of poles): Let $f : D(a, R)\backslash\{a, a_1, a_2, ...\} \to \mathbb{C}$ be holomorphic with poles, and $a_n \to a$ as $n \to \infty$. For all $b \in \mathbb{C}$, and for all $r, \epsilon > 0$, there exists z such that $0 < |z - a| < r$ and $|f(z) - b| < \epsilon$.

Proof: First note that the singularity a is non-isolated, so in particular the Casorati-Weierstrass theorem (for an essential singularity) does not apply. However, the method of proof is essentially identical. We suppose that the theorem is false; that is, there exists $b \in \mathbb{C}, r, \epsilon > 0$ such that for all z with $0 < |z - a| < r$, we have $|f(z) - b| > \epsilon$. We now define a function:

$$g(z) = \frac{1}{f(z) - b}$$

which is holomorphic on $D(a, r)$ except at a and at any of the a_n contained in $D(a, r)$. Since the a_n are poles of $f(z)$ however, in a neighbourhood $z = a_n$ we have $f(z) = h(z)/(z - a_n)^p$ for holomorphic h, satisfying $h(a_n) \neq 0$, and some integer $p > 1$. This implies that $g(z) = (z - a_n)^p/(h(z) - b(z - a_n)^p)$ in a neighbourhood of $z = a_n$, so that $g(a_n) \to 0$ as $z \to a_n$. Thus all the a_n are removable singularities of g, and we can extend g to a function which is zero at all the a_n. In particular, $z = a$ is an *isolated* singularity for g.

Next, we note that $|g(z)| < 1/\epsilon$ on this domain by assumption, so $z = a$ is a removable singularity for g. But then we can extend g to a holomorphic function on all of $D(a, r)$, such that $0 = g(a_n) \to g(a)$, implying that $g(a) = 0$. In particular, g has a non-isolated zero and is holomorphic, thus is identically zero by the principle of isolated zeroes, $g \equiv 0$. This contradicts the assumption that $|f(z) - b| > \epsilon$ everywhere in $D(a, r)$. □

Hence we can say that $f(U\backslash\{a, a_1, ..., \})$ is dense in $\mathbb{C}$.

❊ **Comments:** In part (ii) of this question, we demonstrated that the behaviour of a holomorphic function near a non-isolated singularity which is the limit of a sequence of poles is the same as that of a holomorphic function near an isolated, essential singularity. This is *not* necessarily the case for other types of non-isolated singularities (e.g. compare with the non-isolated singularities which populate the unit disk on Examples Sheet 2, Question 10(i), or the non-isolated 'branch-point' singularity of the logarithm $\log(z)$ at $z = 0$).

10. Let f_n be a sequence of holomorphic functions on a domain U converging locally uniformly to a function $f : U \to \mathbb{C}$. If $f_n(z) \neq 0$ for each n and each $z \in U$, show that either $f(z) = 0$ for all $z \in U$ or $f(z) \neq 0$ for all $z \in U$. What if we allow each f_n to have at most k zeroes in U for some fixed positive integer k independent of n?

➾ **Solution:** This question can be done very nicely using *Rouché's theorem*, which gives us information about the roots of a holomorphic function, so let's begin with a clear statement:

Rouché's theorem: Let U be a simply-connected domain and let γ be a simple (i.e. non-self-intersecting) closed curve in U which bounds some domain. Let f, g be holomorphic on U and suppose that $|f| > |g|$ everywhere on the image of γ. Then f and $f + g$ have the same number of roots, counted with multiplicity, in the domain bounded by γ.

Now, let's apply this to the question; we aim to prove the contrapositive. Suppose that f is not identically zero, yet $f(w) = 0$ for some point $w \in U$. Then there exists a disk $D(w, 2r)$ on which we have both: (i) $f_n(z) \to f(z)$ uniformly; (ii) $f(z) = 0$ only at $z = w$ on this disk (since f is not identically zero, the fact we can choose this follows from the principle of isolated zeroes). The uniform convergence implies that for all $\epsilon > 0$ there exists N such that for all $n \geq N$:

$$|f_n(z) - f(z)| < \epsilon$$

for all $z \in \partial D(w, r)$ (i.e. the convergence is uniform on this circle). This is almost of Rouché form; to make this so, simply choose $\epsilon = \frac{1}{2}\min\{|f(z)| : z \in \partial D(w, r)\}$, which is positive since $f(z) \neq 0$ on $\partial D(w, r)$, and exists since $\partial D(w, r)$ is compact. Then:

$$|f_n(z) - f(z)| < \frac{1}{2}\min\{|f(z)| : z \in \partial D(w, r)\} < |f(z)|$$

for all $z \in \partial D(w, r)$. It follows by Rouché's theorem that for all $n \geq N$, $f_n(z)$ and $f(z)$ have the same number of roots in $D(w, r)$, so $f_n(z)$ has at least one zero for $n \geq N$.

The contrapositive is the statement in the question: if $f_n(z) \neq 0$ for any $z \in U$, and for any n, it follows that either $f \equiv 0$, or $f(z) \neq 0$ for all $z \in U$.

For general domains U, we can't say very much about the *exact* number of roots that f should have, given that the f_n have at most k roots; the problem is that the roots can move towards the edges of U. For example, consider $f_n : D(0, 1) \to \mathbb{C}$ given by:

$$f_n(z) = z - 1 + \frac{1}{n}.$$

Each f_n has exactly one root on the unit disk, namely $z = 1 - 1/n$. However, the limit of the f_n on the unit disk is given by:

$$f(z) = z - 1.$$

The limit is uniform, yet f has no zeroes on the unit disk $D(0, 1)$ (and is not identically zero).

On the other hand, we can *bound* the number of roots f must have. We note that if $z \in U$ is not a limit point of the set of zeroes of the f_n, we can simply rerun the argument above on on a sufficiently small neighbourhood of z to find $f(z) \neq 0$. It follows that f may take zeroes only at limit points of the set of zeroes of the f_n; however, this set must be of size at most k, since any element of the set must be visited infinitely often as $n \to \infty$, but the sequence itself converges. It follows that f can have at most k zeroes on U.

11. Establish the following refinement of the Fundamental Theorem of Algebra. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial of degree n, and let $A = \max\{|a_i| : 0 \leq i \leq n-1\}$. Then $p(z)$ has n roots (counted with multiplicity) in the disk $\{z \in \mathbb{C} : |z| < A+1\}$.

➾ **Solution:** In this question, we would like to prove that the polynomial $p(z)$ has n roots in $\{z \in \mathbb{C} : |z| < A+1\}$, where $A = \max\{|a_i| : 0 \leq i \leq n-1\}$. We aim to use Rouché's theorem; the statement suggests that we should consider $p(z)$ to be the sum of two simpler functions, one of which we know the roots of, on the curve $|z| = A+1$. A natural splitting is $f(z) = z^n$ and $g(z) = a_{n-1}z^{n-1} + \cdots + a_0$.

With these definitions, along $|z| = A+1$, we have:

$$|f(z)| = |z|^n = (A+1)^n.$$

On the other hand, we have:

$$\begin{aligned}
|g(z)| &= |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \\
&\leq |a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0| \\
&\leq \max\{|a_0|, |a_1|, ..., |a_{n-1}|\} \left(1 + |z| + \cdots + |z|^{n-1}\right) \\
&= A\left(1 + (1+A) + \cdots + (1+A)^{n-1}\right) \\
&= A\left(\frac{(1+A)^n - 1}{(1+A) - 1}\right) && \text{(sum of geometric progression)} \\
&= (1+A)^n - 1 < (1+A)^n.
\end{aligned}$$

It follows that $|g(z)| \leq |f(z)|$ everywhere on the curve $|z| = A+1$, and hence in the domain $\{z \in \mathbb{C} : |z| < A+1\}$ it follows that $f(z) = z^n$ and $p(z) = f(z) + g(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z^n + a_0$ have the same number of roots. The first function, $f(z) = z^n$, clearly has n roots (when counted with multiplicity) at 0, and so it follows that $p(z)$ indeed has n roots within the required domain, when counted with multiplicity.

❊ **Comments:** This question is not just about establishing a *refinement* of the fundamental theorem of algebra - we don't need to assume the theorem to get this result, so this is a fully-fledged *proof* of the fundamental theorem of algebra. Indeed, it is probably one of the slickest proofs of the fundamental theorem of algebra you will have seen!

12. If $f : U \to \mathbb{C}$ is holomorphic and one-to-one, show that $f'(z) \neq 0$ for all $z \in U$.

➾ **Solution:** Note the term 'one-to-one' means *injective* in this question. There are numerous ways of proving this result, but we will make use of Rouché's theorem, because again it is particularly slick.

Since Rouché's theorem deals with zeroes, we suppose for a contradiction that $f'(z_0) = 0$ for some $z_0 \in U$. Since f is one-to-one, it must be non-constant, and hence f' is not identically zero. Thus the zeroes of f' are isolated by the principle of isolated zeroes; it follows that there exists a disk $D(z_0, r)$ on which there are no other zeroes of f'; on this disk, let's expand f about z_0:

$$f(z) = f(z_0) + (z - z_0)^p g(z)$$

with $p \geq 2$ and g a holomorphic function satisfying $g(z_0) \neq 0$. Since $(z - z_0)^p g(z) \to 0$ as $z \to z_0$, there exists $\epsilon < r$ such that $|(z - z_0)^p g(z)| < |f(z_0)|$ for all $z \in D(z_0, \epsilon)$. In particular, $|(z - z_0)^p g(z)| < |f(z_0)|$ for all $z \in \partial D(z_0, \epsilon/2)$, so by Rouché's theorem we have that $f(z) = f(z_0) + (z - z_0)^p g(z)$ and $(z - z_0)^p g(z)$ have the same number of zeroes on $D(z_0, \epsilon/2)$.

It follows that $f(z)$ has at least two zeroes , say z_1, z_2 on $D(z_0, r)$. But $f'(z) \neq 0$ on $D(z_0, r)$, so the roots must be distinct. We conclude that $f(z_1) = f(z_2)$ for $z_1 \neq z_2$, and thus f is non-injective; contradiction.

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13. (i) Show that $z^4 + 12z + 1 = 0$ has exactly three zeroes with $1 < |z| < 4$.

(ii) Prove that $z^5 + 2 + e^z$ has exactly three zeroes in the half-plane $\{z : \mathrm{Re}(z) < 0\}$.

(iii) Show that the equation $z^4 + z + 1 = 0$ has one solution in each quadrant. Prove that all solutions lie inside the circle $\{z : |z| = 3/2\}$.

➾ **Solution:** This question is about some more direct applications of Rouché's theorem.

(i) Let's begin by comparing $f(z) = z^4$ and $g(z) = 12z + 1$ on the boundary $|z| = 4$. Here, we have:

$$|f(z)| = |z|^4 = 4^4 = 256, \qquad \text{and} \qquad |g(z)| = |12z + 1| \leq 12|z| + 1 = 49.$$

Hence $|f(z)| < |g(z)|$ everywhere on the curve $|z| = 4$, so in the domain $|z| < 4$ the functions $f(z) = z^4$ and $(f+g)(z) = z^4 + 12z + 1$ have an equal number of roots, i.e. four (since f has exactly four roots at 0).

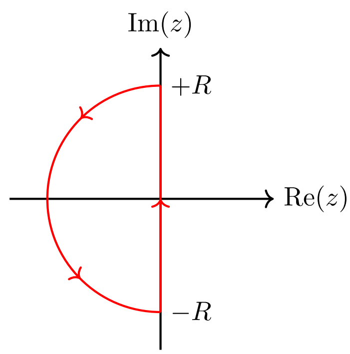
On the boundary $|z| = 1$, we can instead compare the functions $\tilde{f}(z) = 12z$ and $\tilde{g}(z) = z^4 + 1$. We now have:

$$|\tilde{f}(z)| = 12|z| = 12, \qquad \text{and} \qquad |\tilde{g}(z)| = |z^4 + 1| \leq |z|^4 + 1 = 2.$$

Hence $|\tilde{f}(z)| < |\tilde{g}(z)|$ everywhere on the curve $|z| = 1$, so in the domain $|z| < 1$ the functions $\tilde{f}(z) = 12z$ and $(\tilde{f} + \tilde{g})(z) = z^4 + 12z + 1$ have an equal number of roots, i.e. one (since $\tilde{f}$ has exactly one root at 0).

It remains to show that there are no roots on the boundary $|z| = 1$ itself. We simply note that any root must obey $z^4 = -12z - 1$, and for $|z| = 1$, the left hand side has modulus $|z^4| = 1$, whilst the right hand side has modulus $|12z + 1| \geq |12|z| - 1| = 11$. It follows that indeed all three roots lie in $1 < |z| < 4$.

(ii) In this case, there isn't a curve which 'bounds' the left half plane directly, so we choose to only encircle a bit of it at a time. Consider the semicircular contour C_R shown in the diagram below.



Let's compare $f(z) = z^5 + 2$ and $g(z) = e^z$ on the contour C_R. Along the straight segment, we parametrise the contour as $z = iy$. Then:

$$|f(z)| \geq |iy^5 + 2| = \sqrt{y^{10} + 4} \geq 2 > 1 = |g(z)|.$$

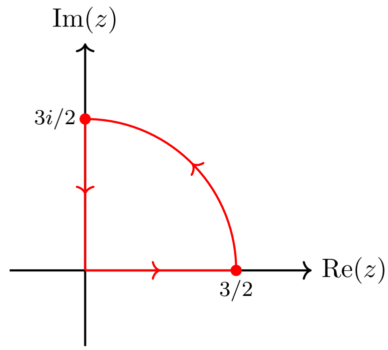
Hence we have $|f(z)| > |g(z)|$ along the straight segment. Along the arc, we parametrise the contour as $z = Re^{i\theta}$, leading to:

$$|f(z)| = |R^5 e^{5i\theta} + 2| \geq R^5 - 2 > 1 \geq e^{R\cos(\theta)} = |\exp(Re^{i\theta})| = |g(z)|,$$

since $\theta \in [\pi/2, 3\pi/2]$ so $-1 \leq \cos(\theta) \leq 0$ along this section of the curve. Hence for sufficiently large R, we have $|f(z)| > |g(z)|$ along the arc. It follows by Rouché's theorem, that for sufficiently large R, $f(z) = z^5+2$ and $f(z)+g(z) = z^5+2+e^z$ have the same number of roots inside C_R.

Note that the polynomial $z^5 + 2$ has three roots inside C_R for sufficiently large R (the roots are just $2^{1/5}e^{i\pi/5}$, $2^{1/5}e^{3i\pi/5}$, $2^{1/5}e^{i\pi}$, $2^{1/5}e^{7i\pi/5}$ and $2^{1/5}e^{9i\pi/5}$). Since this holds for arbitrary R, and C_R covers the whole left half plane as $R \to \infty$, it follows that $z^5 + 2 + e^z$ has exactly three roots in the left half plane, as required.

(iii) Consider putting a contour consisting of a quarter circle of radius $3/2$ and two straight segments in each quadrant; for example, in the first quadrant consider the contour:



We split $z^4 + z + 1$ as $f(z) = z^4 + 1$ and $g(z) = z$, and analyse the behaviour of these functions along each of these contours.

- Along the circular parts of all the contours, we have $|z| = 3/2$, hence:

$$|f(z)| = |z^4 + 1| \geq |z|^4 - 1 = \frac{3^4}{2^4} - 1 = \frac{65}{16} > \frac{3}{2} = |z| = |g(z)|.$$

  In particular, $|f(z)| > |g(z)|$ along all the circular segments.

- Along the horizontal straight segments of the contours, we have $z = x$ for $x \in [-3/2, 3/2]$ which yields:

$$|f(z)| = \sqrt{x^4 + 1} > |x| = |g(z)|,$$

  since $\sqrt{x^4 + 1} > \sqrt{x^2}$ if and only if $x^4 + 1 > x^2$ if and only if $(x^2 - 1)^2 > -x^2$. Hence $|f(z)| > |g(z)|$ along all the horizontal segments.

- Finally, along the vertical straight segments of the contours, we have $z = iy$ for $y \in [-3/2, 3/2]$ which yields:

$$|f(z)| = \sqrt{y^4 + 1} > |y| = |g(z)|$$

  similar to the horizontal segments. Thus $|f(z)| > |g(z)|$ along all the vertical segments.

Therefore $|f(z)| > |g(z)|$ in all possible cases. It follows by Rouché's theorem that for each of our four 'quadrant contours', $f(z) = z^4 + 1$ and $(f + g)(z) = z^4 + z + 1$ have the same number of roots inside each of the contours. Since f has roots $e^{i\pi/4}$, $e^{3i\pi/4}$, $e^{5i\pi/4}$ and $e^{7i\pi/4}$, it follows that there is exactly one root of $z^4 + z + 1$ in each of the quadrants. Since there are no more roots of $z^4 + z + 1$ (it's a quartic), it also follows that all roots lie in $\{|z| < 3/2\}$.

14. Let f be a function which is analytic on $\mathbb{C}$ apart from a finite number of poles. Show that if there exists k such that $|f(z)| \leq |z|^k$ for all z with $|z|$ sufficiently large, then f is a rational function (i.e. a quotient of two polynomials).

Solution: This mirrors the result we proved for Examples Sheet 2, Question 5(i): an entire function $g : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree at most k if and only if there exists M such that $|g(z)| \leq M(1+|z|)^k$ everywhere. Indeed, we can apply this result exactly here.

Let $z_1, ..., z_n$ be the poles and let $p_1, ..., p_n$ be the orders of the poles. Consider the function:

$$g(z) = (z - z_1)^{p_1} ... (z - z_n)^{p_n} f(z).$$

This function is bounded as $z \to z_i$ for any i, hence g has removable singularities at each of the poles of f, and we can extend g to an entire function $g : \mathbb{C} \to \mathbb{C}$. Now, since $(z - z_1)^{p_1} ... (z - z_n)^{p_n}$ is a polynomial of degree $p_1 + \cdots + p_n$, there exists M such that:

$$|(z - z_1)^{p_1} ... (z - z_n)^{p_n}| \leq M(1 + |z|)^{p_1 + \cdots + p_n}.$$

everywhere in $\mathbb{C}$. On the other hand, suppose that we are given k such that $|f(z)| \leq |z|^k$ for $|z| > R$. Then combining the inequalities, we have for all $|z| > R$:

$$|g(z)| \leq M(1 + |z|)^{p_1 + \cdots + p_n} |z|^k \leq M(1 + |z|)^{p_1 + \cdots + p_n + k}.$$

In the region $|z| \leq R$, since $g(z)$ is continuous, it is bounded and attains its bounds on the compact set $|z| \leq R$; say $|g(z)| \leq K$ on this region. Since $|z| > 0$, we can further write:

$$|g(z)| \leq K \leq K(1 + |z|)^{p_1 + \cdots + p_n + k}$$

throughout $|z| \leq R$. Taking the larger of the two constants M, K, we see that g satisfies precisely the polynomial condition from Examples Sheet 2, Question 5(i). It follows that g is a polynomial, and hence f is a rational function, as required.

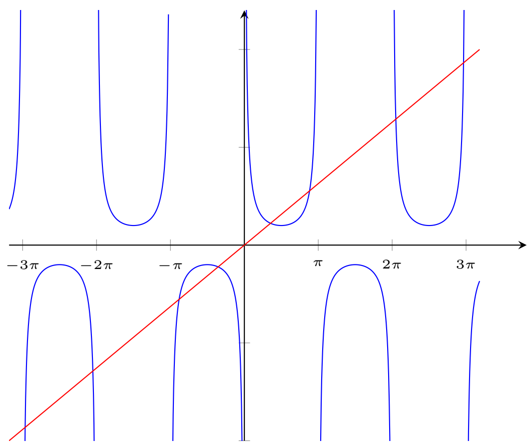
15. Show that the equation $z\sin(z) = 1$ has only real solutions. [*Hint:* Find the number of real roots in the interval $[-(n+1/2)\pi, (n+1/2)\pi]$ and compare with the number of zeroes of $z\sin(z) - 1$ in a square box $\{|\operatorname{Re}(z)|, |\operatorname{Im}(z)| < (n+1/2)\pi\}$.]

➻ **Solution:** Again, this question is a natural candidate for an application of Rouché's theorem. We follow the advice of the question, and first consider the number of roots of the equation $x\sin(x) = 1$ on $[-(n+1/2)\pi, (n+1/2)\pi]$.

We note immediately that $x = m\pi$ is never a root for $m \in \mathbb{Z}$. In particular, we can safely recast the equation as:

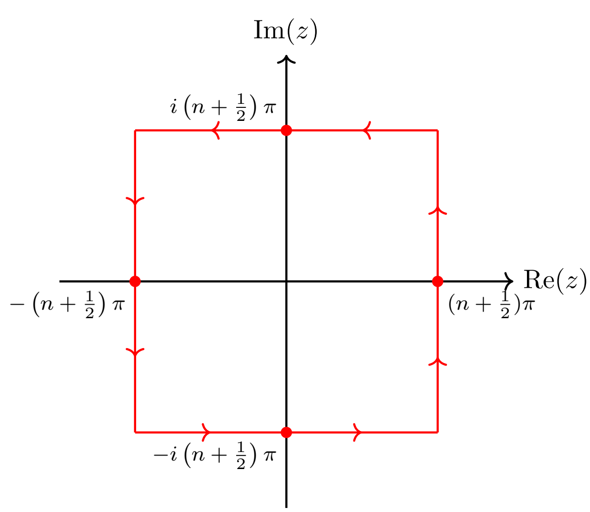
$$x = \operatorname{cosec}(x).$$

The problem then comes down to finding where x and $\operatorname{cosec}(x)$ intersect. Plotting their respective graphs, we have:



We see that there are precisely two intersections on every interval $[0,\pi]$, $[2\pi, 3\pi]$, $[5\pi, 7\pi]$, etc, with the first always occurring on the first half of the interval, hence there are precisely $n+1$ solutions to $x\sin(x) = 1$ on the interval $[0, (n+1/2)\pi]$. Since the function is even, there are precisely $2n+2$ solutions to $x\sin(x) = 1$ on the interval $[-(n+1/2)\pi, (n+1/2)\pi]$.

We now consider the closed contour suggested in the question, call it γ_n, pictured below:



We now break down the function $z\sin(z) - 1$ into two functions $f(z) = z\sin(z)$ and $g(z) = -1$ in an attempt to apply Rouché's theorem. On γ_n, we obviously have $|g(z)| = 1$ everywhere. On the other hand, for $f(z)$ on the boundary we have:

- On the vertical segments, $z = \pm(n+1/2)\pi + iy$ for $-(n+1/2)\pi \leq y \leq (n+1/2)\pi$. We therefore have:

  $$|\sin(z)| = |\sin(\pm(n+1/2)\pi + iy)| = |\sin(\pm(n+1/2)\pi)\cosh(y) + i\cos((n+1/2)\pi)\sinh(y)| = \cosh(y).$$

  This implies that $|z\sin(z)| = \sqrt{(n+1/2)^2\pi^2 + y^2}\cosh(y) \geq \frac{1}{2}\pi$ throughout the vertical segment. In particular, $|f(z)| > |g(z)|$ along the vertical segment.

- On the horizontal segments, $z = x \pm i(n+1/2)\pi$ for $-(n+1/2)\pi \leq x \leq (n+1/2)\pi$. We therefore have:

  $$\begin{aligned}|\sin(z)|^2 = |\sin(x \pm i(n+1/2)\pi)|^2 &= |\sin(x)\cosh((n+1/2)\pi) + i\sinh(\pm(n+1/2)\pi)\cos(x)|^2 \\ &= \sin^2(x)\cosh^2((n+1/2)\pi) + \cos^2(x)\sinh^2((n+1/2)\pi).\end{aligned}$$

  Since $\cosh^2((n+1/2)\pi) = 1 + \sinh^2((n+1/2)\pi)$, we have:

  $$|\sin(z)|^2 = \sin^2(x) + \sinh^2((n+1/2)\pi) \geq \sinh^2(\frac{1}{2}\pi).$$

  In particular, we have $|\sin(z)| \geq \sinh(\frac{1}{2}\pi)$. This implies that $|z\sin(z)| \geq \sqrt{x^2 + (n+1/2)^2\pi^2}\sinh(\frac{1}{2}\pi) \geq \frac{1}{2}\pi\sinh(\frac{1}{2}\pi) > 1$. Thus $|f(z)| > |g(z)|$ on the horizontal segments of the contour too.

It follows by Rouché's theorem that $f(z) = z\sin(z)$ and $(f+g)(z) = z\sin(z) - 1$ have the same number of roots inside γ_n. Since $f(z) = 0$ if and only if $z = 0$ or $\sin(z) = 0$, the latter of which occur at all integer multiples of π, we see that f has precisely $2n+2$ roots inside γ_n, counted with multiplicity (there is a double root at $z = 0$). In particular, since we already proved that all of these roots occur on the real line, it follows that indeed all roots of $z\sin(z) = 1$ are real.

16. Let U be a domain, let $f : U \to \mathbb{C}$ be holomorphic and suppose $a \in U$ with $f'(a) \neq 0$. Show that for $r > 0$ sufficiently small,

$$g(w) = \frac{1}{2\pi i} \int\limits_{\partial D(a,r)} \frac{zf'(z)}{f(z) - w}\, dz$$

defines a holomorphic function g in a neighbourhood of $f(a)$ which is inverse to f.

➼ **Solution:** The result we prove in this question is a famous formula, called the *Lagrange inversion formula*, though Lagrange originally derived it without the use of complex analysis. Here we see the power of complex analysis in deriving this amazing result.

First, let us examine the integrand on the right hand side. This integrand defines a holomorphic function of z at everywhere point such that $f(z) \neq w$; we deal with this exception as follows. Since $f'(a) \neq 0$, by the holomorphic inverse function theorem (Examples Sheet 1, Question 10) there exists $r > 0$ such that $f : D(a, 2r) \to f(D(a, 2r))$ is a bijection. In particular, for any point $w \in f(D(a, r))$ there exists a single $f^{-1}(w) \in D(a, r)$ such that $f(f^{-1}(w)) = w$. Therefore given such a w, $f(z) \neq w$ throughout $D(a, 2r)$ except at a single point in $D(a, r)$. It follows that for a fixed $w \in f(D(a, r))$, the integrand has a single singularity at the unique $z = f^{-1}(w)$.

Applying Cauchy's residue theorem then, we have:

$$\frac{1}{2\pi i} \int\limits_{\partial D(a,r)} \frac{zf'(z)}{f(z) - w}\, dz = \operatorname{Res}\left(\frac{zf'(z)}{f(z) - w}; f^{-1}(w)\right).$$

We now compute the residue. Let's guess that the singularity is a simple pole, and then test that assertion by computing:

$$\lim_{z \to f^{-1}(w)} \left[\frac{z(z - f^{-1}(w))f'(z)}{f(z) - w}\right] = f^{-1}(w)f'(f^{-1}(w)) \lim_{z \to f^{-1}(w)} \left[\frac{z - f^{-1}(w)}{f(z) - f(f^{-1}(w))}\right] = \frac{f^{-1}(w)f'(f^{-1}(w))}{f'(f^{-1}(w))} = f^{-1}(w).$$

Note that $f'(f^{-1}(w)) \neq 0$ since f is bijective on the relevant domain, which implies by Question 12 that f' is everywhere non-zero on this domain. This residue is certainly finite; if $f^{-1}(w) \neq 0$ our initial guess that the residue was a simple pole was justified and we're done, otherwise the singularity was removable and we could have applied Cauchy's theorem in the first place to an analytic extension of the integrand to see the result would have been zero anyway. It follows that for $w \in f(D(a, r))$ we have:

$$g(w) = f^{-1}(w),$$

as required.

Bonus integrals

1. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(mx)}{x(a^2+x^2)}\,dx$, where $a, m \in \mathbb{R}^+$.

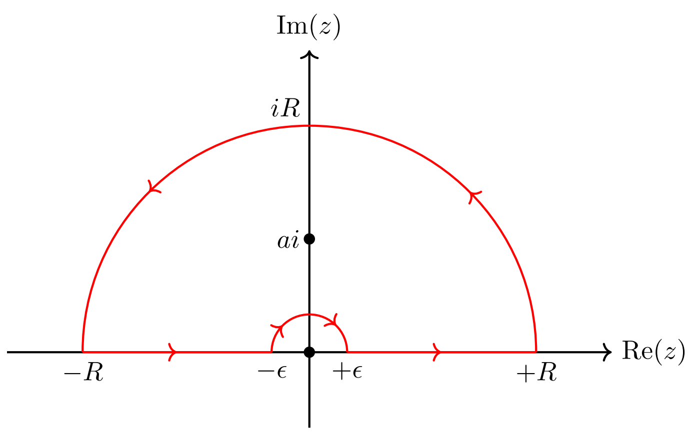
Solution: Note that:

$$\int_{-\infty}^{\infty} \frac{\sin(mx)}{x(a^2+x^2)}\,dx = \mathrm{Im}\int_{-\infty}^{\infty} \frac{e^{imx}}{x(a^2+x^2)}\,dx$$

which suggests considering the contour integral:

$$\oint_{C_{R,\epsilon}} \frac{e^{imz}}{z(a^2+z^2)}\,dz,$$

around the contour $C_{R,\epsilon}$ shown below.



Note that the contour encloses a single singularity of the integrand at $z = ai$ (since $a > 0$), which is obviously a simple pole. By the residue theorem, for R sufficiently large and ϵ sufficiently small, we have:

$$\oint_{C_{R,\epsilon}} \frac{e^{imz}}{z(a^2+z^2)}\,dz = 2\pi i \mathrm{Res}\left(\frac{e^{imz}}{z(a^2+z^2)}; ai\right) = 2\pi i \lim_{z\to ai}\left[\frac{e^{imz}}{z(z+ai)}\right] = \frac{2\pi i e^{-ma}}{ai(2ai)} = -\frac{i\pi e^{-ma}}{a^2}.$$

On the other hand, we can evaluate the integral on each section of the contour separately. The integral along the large semi-circular section of the contour vanishes as $R\to\infty$, by Jordan's lemma, since $1/|z(z^2+a^2)| \to 0$ as $|z|\to\infty$ and $m > 0$. The integrals along the straight segments of the contour give the required integral as $\epsilon \to 0$ and $R \to \infty$ (technically its principal value, but we don't really care about such things in this course). Finally, the integral on the small arc gives:

$$i\int_{\pi}^{0} \frac{e^{im\epsilon e^{i\theta}}}{\epsilon e^{i\theta}(\epsilon^2 e^{2i\theta}+a^2)}\,\epsilon e^{i\theta}\,d\theta \to -\frac{i\pi}{a^2}$$

as $\epsilon \to 0$. Putting everything together, we have:

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x(a^2+x^2)}\,dx - \frac{i\pi}{a^2} = -\frac{i\pi e^{-ma}}{a^2} \qquad \Rightarrow \qquad \int_{-\infty}^{\infty} \frac{\sin(mx)}{x(x^2+a^2)}\,dx = \frac{\pi}{a^2}(1-e^{-ma}).$$

2. $\displaystyle\int_0^{2\pi} \frac{\cos^3(3t)}{1-2a\cos(t)+a^2}\,dt$, where $a \in (0,1)$.

➡ **Solution:** First, let's deal with the nasty cosine function on the top of the integrand. We note that:

$$\cos^3(3t) = \frac{1}{8}(e^{3it}+e^{-3it})^3 = \frac{1}{8}\left(e^{9it}+3e^{3it}+3e^{-3it}+e^{-9it}\right) = \frac{1}{4}\cos(9t)+\frac{3}{4}\cos(3t).$$

Hence, we can rewrite the integral as:

$$\int_0^{2\pi} \frac{\cos^3(3t)}{1-2a\cos(t)+a^2} = \operatorname{Re}\int_0^{2\pi} \frac{e^{9it}/4+3e^{3it}/4}{1-2a\cos(t)+a^2}\,dt,$$

which looks much more approachable. Now let $z = e^{it}$ so that $\cos(t) = (z+1/z)/2$. Then:

$$1-2a\cos(t)+a^2 = 1-a(z+1/z)+a^2 = -\frac{a}{z}(z-a)(z-1/a).$$

Thus we can recast the given integral as an integral around the unit circle:

$$\int_0^{2\pi} \frac{e^{9it}/4+3e^{3it}/4}{1-2a\cos(t)+a^2}\,dt = -i\oint_{|z|=1} \frac{z^9/4+3z^3/4}{-a(z-a)(z-1/a)/z}\frac{dz}{z} = i\oint_{|z|=1} \frac{z^9/4+3z^3/4}{a(z-a)(z-1/a)}\,dz.$$

There is a single singularity enclosed in the contour, namely a simple pole at $z=a$ (since $0<a$ the pole is simple and since $a<1$ there is no pole at $1/a$). It follows by the residue theorem (and the formula for the residue at a simple pole) that:

$$i\oint_{|z|=1} \frac{z^9/4+3z^3/4}{a(z-a)(z-1/a)}\,dz = -2\pi\lim_{z\to a}\left[\frac{z^9/4+3z^3/4}{a(z-1/a)}\right] = \frac{\pi a^3\left(a^6+3\right)}{2(1-a^2)}.$$

3. $\displaystyle\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx$. [*Hint: 'dog-bone contour'*].

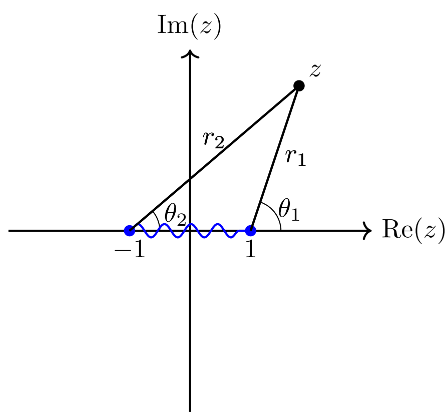
➾ **Solution:** We consider integrating the complex function:

$$\frac{\sqrt{1-z^2}}{1+z^2}.$$

Before we can do so however, we need to explain what we mean by the function $\sqrt{1-z^2}$ by choosing a branch. The branch points are at $z = \pm 1$, so we consider defining:

$$\sqrt{1-z^2} = i|z^2-1|^{1/2}e^{i\theta_1/2}e^{i\theta_2/2},$$

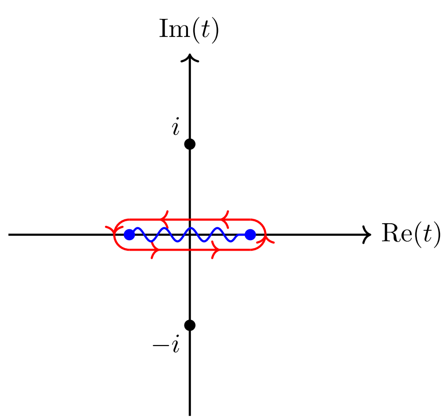
where $z-1 = r_1 e^{i\theta_1/2}$ and $z+1 = r_2 e^{i\theta_2/2}$, with $r_1 = |z-1|$, $r_2 = |z+1|$, $\theta_1 \in [0, 2\pi)$ and $\theta_2 \in [0, 2\pi)$. This ensures that $\sqrt{1-(\sqrt{2})^2} = i$ for example, and corresponds to the choice of branch cut shown below:



We now consider the contour integral:

$$\oint_C \frac{\sqrt{1-z^2}}{1+z^2}\,dz$$

around the contour shown below (some people put circles rather than semicircles at the end of this contour when there are possible singularities of the integrand at ± 1, hence the name 'dog-bone' given in the question), where the semicircular parts of the contour have radius ϵ.



Just above the contour, working in the limit as the straight segment approaches the branch cut from above, the arguments θ_1, θ_2 are given by $\theta_1 = \pi$ and $\theta_2 = 0$. Hence parametrising with $z = x$ for $x \in [-1, 1]$, we have the contribution:

$$ie^{i\pi/2} \int_1^{-1} \frac{\sqrt{1-x^2}}{1+x^2}\, dx = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\, dx,$$

which is precisely the integral we want.

Just below the contour, working in the limit as the straight segment approaches the branch cut from below, the arguments θ_1, θ_2 are given by $\theta_1 = \pi$ and $\theta_2 = 2\pi$. Hence parametrising with $z = x$ for $x \in [-1, 1]$, we have the contribution:

$$ie^{i\pi/2} e^{i\pi} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\, dx = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\, dx,$$

which is just another copy of the integral we want.

On the left semicircular parts of the contour, we can parametrise using $z = -1 + \epsilon e^{i\theta}$, where $\theta \in (\pi/2, 3\pi/2)$. In particular, the arguments θ_1, θ_2 are given by $\theta_1 = \pi$ (up to order ϵ corrections - the argument barely changes as we move around the semicircle) and $\theta_2 = \theta$. We can also take $|z - 1| = 2$ up to order ϵ corrections. It follows that the contribution is:

$$i\sqrt{2} \int_{\pi/2}^{3\pi/2} \frac{\epsilon^{1/2} e^{i\theta/2} e^{i\pi/2}}{1 + (\epsilon e^{i\theta} - 1)^2} i\epsilon e^{i\theta}\, d\theta = -i\sqrt{2}\epsilon^{3/2} \int_{\pi/2}^{3\pi/2} \frac{e^{3i\theta/2}}{2 - 2\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}}\, d\theta.$$

Bounding, we have:

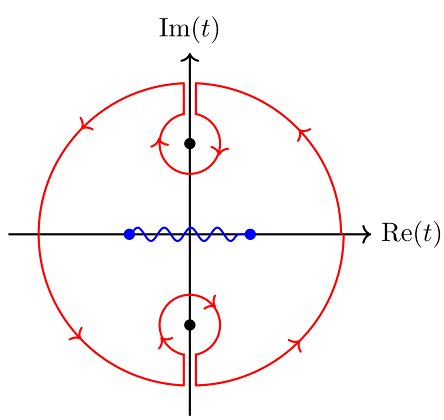
$$\left| -i\sqrt{2}\epsilon^{3/2} \int_{\pi/2}^{3\pi/2} \frac{e^{3i\theta/2}}{2 - 2\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}}\, d\theta \right| \leq \sqrt{2}\pi\epsilon^{3/2} \sup_{\theta \in [\pi/2, 3\pi/2]} \frac{1}{|2 - 2\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}|}.$$

The supremum is a continuous function of ϵ for ϵ sufficiently close to 0, so we can exchange the supremum and the limit as $\epsilon \to 0$ to see:

$$\left| -i\sqrt{2}\epsilon^{3/2} \int_{\pi/2}^{3\pi/2} \frac{e^{3i\theta/2}}{2 - 2\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}}\, d\theta \right| \to 0$$

as $\epsilon \to 0$. In particular, the contribution from the left semicircle goes to zero as $\epsilon \to 0$; similarly we can show that the contribution from the right semicircle vanishes as $\epsilon \to 0$.

We now deform the contour to a large circle of radius $R \gg 1$, encircling the poles at $z = \pm i$ along the way:



As we push the vertical segments at the top and the bottom closer together, their contributions will cancel. By the residue theorem then (note we have some clockwise contours, so we pick up minus signs from the winding numbers), we will be left with:

$$\oint_C \frac{\sqrt{1-z^2}}{1+z^2}\,dz = -2\pi i\left(\operatorname{Res}\left(\frac{\sqrt{1-z^2}}{1+z^2}; i\right) + \operatorname{Res}\left(\frac{\sqrt{1-z^2}}{1+z^2}; -i\right)\right) + \oint_{|z|=R} \frac{\sqrt{1-z^2}}{1+z^2}\,dz.$$

The singularities at $z = \pm i$ are simple poles, so the residues can be easily computed using the formula for the residue at a simple pole:

$$\operatorname{Res}\left(\frac{\sqrt{1-z^2}}{1+z^2}; i\right) = \lim_{z\to i}\left[\frac{\sqrt{1-z^2}}{z+i}\right] = \frac{i\sqrt{2}e^{3i\pi/8}e^{i\pi/8}}{2i} = \frac{i\sqrt{2}}{2}.$$

Similarly:

$$\operatorname{Res}\left(\frac{\sqrt{1-z^2}}{1+z^2}; -i\right) = \lim_{z\to -i}\left[\frac{\sqrt{1-z^2}}{z-i}\right] = -\frac{i\sqrt{2}e^{7i\pi/8}e^{5i\pi/8}}{2i} = \frac{i\sqrt{2}}{2}.$$

It remains to compute the integral around the large circle of radius R. But note that the integrand $\sqrt{1-z^2}/(1+z^2)$ is an analytic function on the (infinite) annulus $|z| > R/2$ (for R sufficiently large), so has a unique Laurent series on this annulus. For now, take $z \in [R/2, \infty)$ and consider expanding the integrand:

$$\frac{\sqrt{1-z^2}}{1+z^2} := \frac{i\sqrt{z^2-1}}{1+z^2} = \frac{iz}{z^2}\frac{\sqrt{1-1/z^2}}{1+1/z^2} = \frac{i}{z}\left(1-\frac{1}{z^2}\right)^{-1/2}\left(1+\frac{1}{z^2}\right)^{-1}.$$

By the binomial theorem, we can expand to give:

$$\frac{i}{z}\left(1+\frac{1}{2z^2}+\cdots\right)\left(1-\frac{1}{z^2}+\cdots\right) = \frac{i}{z} + O\left(\frac{1}{z^3}\right).$$

Whilst we expanded for $z \in [R/2, \infty)$, uniqueness of the Laurent series implies this holds everywhere in $R/2 < |z|$. This implies that the integral is simply given by:

$$\oint_{|z|=R} \frac{\sqrt{1-z^2}}{1+z^2}\,dz = \oint_{|z|=R}\left(\frac{i}{z} + O\left(\frac{1}{z^3}\right)\right)dz = 2\pi i\cdot i = -2\pi.$$

Putting everything together, we are left with:

$$2\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx = -2\pi i\left(\frac{i\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right) - 2\pi = 2\pi\left(\sqrt{2}-1\right).$$

Thus we conclude that:

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx = \pi(\sqrt{2}-1).$$

4. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)}{x} e^{-itx}\, dx$, where $t \in \mathbb{R}$.

Solution: This question is asking us to obtain the *Fourier transform* of the *sinc function*,

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x},$$

used in signal processing theory. First we expand $\sin(x)$ in terms of exponentials to get:

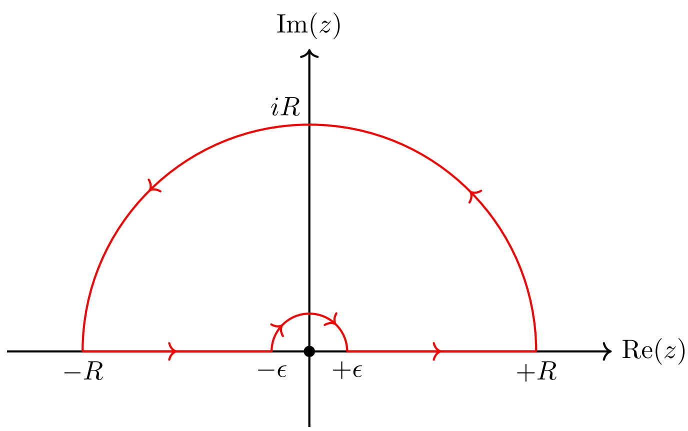
$$\frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{i(1-t)x} - e^{-i(1+t)x}}{x}\, dx.$$

We see that whether we can apply Jordan's lemma or not will depend strongly on the value of t. Furthermore, it is clear that we will not be able to apply Jordan's lemma to both exponentials simultaneously.

Instead, we suggest the following approach. Consider the integral:

$$\oint_{C_{R,\epsilon}} \frac{e^{i\lambda x}}{x}\, dx$$

for $\lambda \geq 0$, around the contour shown below.



The integral is zero by the residue theorem, since it encloses no singularities of the integrand. On the other hand, we can evaluate the integral on each section of the contour separately. We have:

$$0 = \oint_{C_{R,\epsilon}} \frac{e^{i\lambda z}}{z}\, dz = \int_{-R}^{-\epsilon} \frac{e^{i\lambda x}}{x}\, dx + \int_{\epsilon}^{R} \frac{e^{i\lambda x}}{x}\, dx + \int_{\pi}^{0} \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}\, d\theta + \int_{\gamma_R} \frac{e^{i\lambda z}}{z}\, dz,$$

where γ_R is the large semicircular arc shown above. As $\epsilon \to 0$, the contribution from the small semicircular arc evidently tends to $-i\pi$, hence we can write:

$$0 = \lim_{\epsilon \to 0} \oint_{C_{R,\epsilon}} \frac{e^{i\lambda z}}{z}\, dz = \lim_{\epsilon \to 0} \left[\int_{-R}^{-\epsilon} \frac{e^{i\lambda x}}{x}\, dx + \int_{\epsilon}^{R} \frac{e^{i\lambda x}}{x}\, dx \right] - i\pi + \int_{\gamma_R} \frac{e^{i\lambda z}}{z}\, dz.$$

As $R \to \infty$, if $\lambda > 0$ the large semicircular arc's contribution tends to zero by Jordan's lemma. On the other hand, if $\lambda = 0$, we can parametrise to see:

$$\int_{\gamma_R} \frac{1}{z}\, dz = \int_0^{\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta}\, d\theta = i\pi$$

which is constant as $R \to \infty$. Thus overall we have:

$$0 = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \oint_{C_{R,\epsilon}} \frac{e^{i\lambda z}}{z}\, dz = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left[\int_{-R}^{-\epsilon} \frac{e^{i\lambda x}}{x}\, dx + \int_{\epsilon}^{R} \frac{e^{i\lambda x}}{x}\, dx \right] - i\pi + \begin{cases} 0 & \text{if } \lambda > 0; \\ i\pi & \text{if } \lambda = 0. \end{cases}$$

Similarly, we can evaluate the integral when $\lambda < 0$. In this case, we close in the lower half plane and indent just below the singularity at $z = 0$; overall we get for all $\lambda \in \mathbb{R}$:

$$0 = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \oint_{C_{R,\epsilon}} \frac{e^{i\lambda z}}{z}\, dz = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left[\int_{-R}^{-\epsilon} \frac{e^{i\lambda x}}{x}\, dx + \int_{\epsilon}^{R} \frac{e^{i\lambda x}}{x}\, dx \right] - i\operatorname{sign}(\lambda)\pi + \begin{cases} 0 & \text{if } \lambda \neq 0; \\ i\pi & \text{if } \lambda = 0, \end{cases}$$

with the convention $\operatorname{sign}(0) = 0$. Rearranging and simplifying, we can rewrite this result as:

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left[\int_{-R}^{-\epsilon} \frac{e^{i\lambda x}}{x}\, dx + \int_{\epsilon}^{R} \frac{e^{i\lambda x}}{x}\, dx \right] = \begin{cases} -i\pi & \text{if } \lambda < 0; \\ 0 & \text{if } \lambda = 0; \\ i\pi & \text{if } \lambda > 0. \end{cases}$$

We can apply this to the problem at hand. Assuming the integral is convergent as an improper integral, it is equal to its principal value, so we have:

$$\frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{i(1-t)x} - e^{-i(1+t)x}}{x}\, dx = \frac{1}{2i} \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left[\int_{-R}^{-\epsilon} \left(\frac{e^{i(1-t)x}}{x} - \frac{e^{-i(1+t)x}}{x} \right) dx + \int_{\epsilon}^{R} \left(\frac{e^{i(1-t)x}}{x} - \frac{e^{-i(1+t)x}}{x} \right) dx \right]$$

Using the above work, this can be evaluated to give:

$$\frac{1}{2i} \begin{cases} -i\pi & \text{if } 1-t < 0; \\ 0 & \text{if } 1-t = 0; \\ i\pi & \text{if } 1-t > 0. \end{cases} - \frac{1}{2i} \begin{cases} -i\pi & \text{if } -(1+t) < 0; \\ 0 & \text{if } -(1+t) = 0; \\ i\pi & \text{if } -(1+t) > 0. \end{cases}$$

Simplifying all the cases, we have:

$$\begin{cases} -\pi/2 & \text{if } t > 1; \\ 0 & \text{if } t = 1; \\ \pi/2 & \text{if } t < 1. \end{cases} - \begin{cases} -\pi/2 & \text{if } t > -1; \\ 0 & \text{if } t = -1; \\ \pi/2 & \text{if } t < -1. \end{cases}$$

Thus separating everything out into five separate cases, we have the final answer:

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} e^{-itx}\, dx = \begin{cases} 0 & \text{if } t < -1; \\ \pi/2 & \text{if } t = -1; \\ \pi & \text{if } -1 < t < 1; \\ \pi/2 & \text{if } t = 1; \\ 0 & \text{if } t > 1. \end{cases}$$

This is called the *rectangular function* in signal processing.

5. By integrating $z/(a-e^{-iz})$ round the rectangle with vertices $\pm\pi$, $\pm\pi+iR$, prove that:

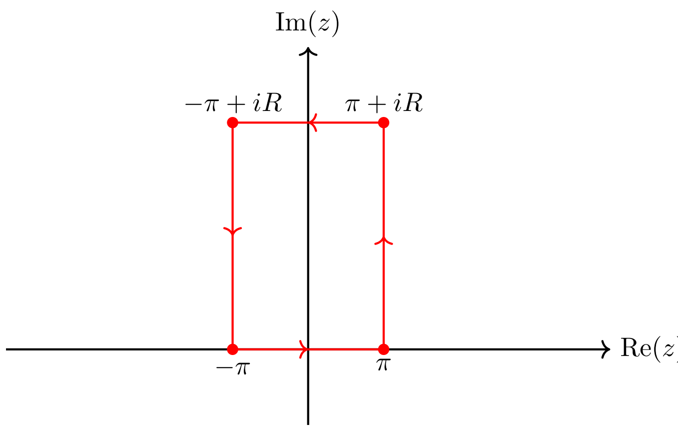
$$\int_0^\pi \frac{x\sin(x)}{1-2a\cos(x)+a^2}\,dx = \frac{\pi}{a}\log(1+a)$$

for every $a\in(0,1)$.

➡ **Solution:** We consider the integral:

$$\oint_{C_R} \frac{z}{a-e^{-iz}}\,dz$$

where C_R is the contour shown below.



Singularities of the integrand occur when $a=e^{-iz}$, which has solutions:

$$z = i\log(a)+2n\pi$$

for $n\in\mathbb{Z}$. There is precisely one of these solutions whose real part lies in $[-\pi,\pi]$, namely $z=i\log(a)$, but $\log(a)<0$ for $a\in(0,1)$, and hence there are no singularities enclosed by the contour. It follows by the residue theorem that:

$$\oint_{C_R} \frac{z}{a-e^{-iz}}\,dz = 0.$$

On the other hand, we can evaluate the integral along each section of the contour separately. Along the straight segment from $-\pi$ to π, we have the contribution:

$$\int_{-\pi}^{\pi} \frac{x}{a-e^{-ix}}\,dx = \int_{-\pi}^{\pi} \frac{x}{a-\cos(x)+i\sin(x)}\,dx.$$

Along the straight segment from $\pi+iR$ to $-\pi+iR$, we can bound the contribution:

$$\left|\int_{\pi}^{-\pi} \frac{x+iR}{a-e^{-i(x+iR)}}\,dx\right| \le 2\pi \sup_{x\in[-\pi,\pi]} \frac{\sqrt{x^2+R^2}}{|e^{-i(x+iR)}-a|} \le 2\pi \sup_{x\in[-\pi,\pi]} \frac{\sqrt{x^2+R^2}}{e^R-a} \to 0$$

as $R\to\infty$.

Along the vertical segments, we have the contributions:

$$i \int_0^R \frac{\pi + iy}{a - e^{-i(\pi+iy)}}\, dy + i \int_R^0 \frac{-\pi + iy}{a - e^{-i(-\pi+iy)}}\, dy.$$

Tidying things up, we're left with:

$$i \int_0^R \frac{\pi + iy}{a + e^y}\, dy - i \int_0^R \frac{iy - \pi}{a + e^y}\, dy = 2\pi i \int_0^R \frac{dy}{a + e^y}.$$

As $R \to \infty$, this contribution approaches:

$$2\pi i \int_0^\infty \frac{dy}{a + e^y} = \frac{2\pi i}{a} \int_0^\infty \frac{ae^{-y}}{ae^{-y} + 1}\, dy = \frac{2\pi i}{a} \left[-\log(ae^{-y} + 1)\right]_0^\infty = \frac{2\pi}{a} \log(1 + a).$$

Putting everything together, we have shown that:

$$\int_{-\pi}^{\pi} \frac{x}{a - \cos(x) + i\sin(x)}\, dx = -\frac{2\pi i}{a} \log(1 + a).$$

Emboldened by seeing the answer on the right hand side, we take imaginary parts and see what happens. The integrand on the left hand side has imaginary part:

$$\begin{aligned} \frac{1}{2i}\left[\frac{x}{a - \cos(x) + i\sin(x)} - \frac{x}{a - \cos(x) - i\sin(x)}\right] &= \frac{1}{2i}\left[\frac{x(a - \cos(x) - i\sin(x)) - x(a - \cos(x) + i\sin(x))}{(a - \cos(x))^2 + \sin^2(x)}\right] \\ &= -\frac{x\sin(x)}{a^2 - 2a\cos(x) + 1}. \end{aligned}$$

Thus we have:

$$\int_{-\pi}^{\pi} \frac{x\sin(x)}{1 - 2a\cos(x) + a^2}\, dx = \frac{2\pi}{a} \log(1 + a)$$

Noticing the left hand side's integrand is an even function of x, we get the required answer:

$$\int_0^{\pi} \frac{x\sin(x)}{1 - 2a\cos(x) + a^2}\, dx = \frac{\pi}{a} \log(1 + a).$$

6. Assuming $\alpha \geq 0$ and $\beta \geq 0$, prove that:

$$\int_0^\infty \frac{\cos(\alpha x) - \cos(\beta x)}{x^2}\, dx = \frac{\pi}{2}(\beta - \alpha),$$

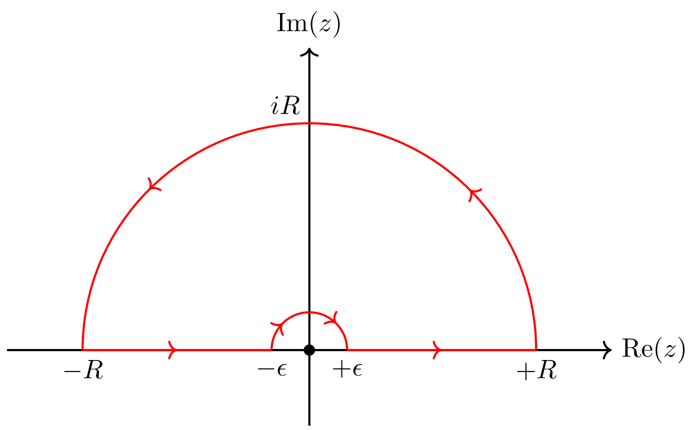
and deduce the value of:

$$\int_0^\infty \left(\frac{\sin(x)}{x}\right)^2 dx.$$

Solution: Consider the integral:

$$\oint_{C_{R,\epsilon}} \frac{e^{i\alpha z} - e^{i\beta z}}{z^2}\, dz.$$

around the contour shown below.



The contour encloses no singularities of the integrand, hence the integral is zero by the residue theorem.

On the other hand, we can evaluate the integral on each section of the contour separately. Along the large semicircular arc, the integral tends to zero as $R \to \infty$ (by Jordan's lemma in the case $\alpha, \beta > 0$, and if one of α, β is zero the bound $\pi R \cdot \sup |1/|z|^2| = O(1/R)$ works). Along the straight segments of the contour, the integral converges to twice the required integral (since the required integral is even). Finally, parametrising the small semicircular arc as $z = \epsilon e^{i\theta}$, we have the contribution:

$$i\int_\pi^0 \frac{e^{i\alpha\epsilon e^{i\theta}} - e^{i\beta\epsilon e^{i\theta}}}{\epsilon^2 e^{2i\theta}} \epsilon e^{i\theta}\, d\theta = \frac{i}{\epsilon}\int_\pi^0 \left(e^{-i\theta} + i\alpha\epsilon - e^{-i\theta} - i\beta\epsilon + O(\epsilon^2)\right) d\theta = \pi(\alpha - \beta) + O(\epsilon).$$

Slightly more careful bounding can shown that indeed as $\epsilon \to 0$, we get the contribution $\pi(\alpha - \beta)$. Putting everything together, we see that:

$$2\int_0^\infty \frac{\cos(\alpha x) - \cos(\beta x)}{x^2}\, dx + \pi(\alpha - \beta) = 0 \qquad \Rightarrow \qquad \int_0^\infty \frac{\cos(\alpha x) - \cos(\beta x)}{x^2}\, dx = \frac{\pi}{2}(\beta - \alpha).$$

To make the deduction, simply recall the sum to product formula for cosines: $\cos(\alpha x) - \cos(\beta x) = -2\sin((\alpha+\beta)x/2)\sin((\alpha - \beta)x/2)$. Choose $\alpha = 0$ and $\beta = 2$ to get the required integral on the left hand side; this leaves us with:

$$\int_0^\infty \left(\frac{\sin(x)}{x}\right)^2 dx = \frac{\pi}{2}.$$

Part IB: Complex Analysis
Past Paper Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

2016, Paper 1, Section I, 2A

Classify the singularities of the following functions at both $z = 0$ and at the point at infinity on the extended complex plane:

$$f_1(z) = \frac{e^z}{z\sin^2(z)}, \qquad f_2(z) = \frac{1}{z^2(1-\cos(z))}, \qquad f_3(z) = z^2\sin\left(\frac{1}{z}\right).$$

Solution: For the first function, note that the numerator e^z is never zero, whilst the denominator has a triple zero at $z = 0$ since the first non-vanishing term in its Taylor series about $z = 0$ is the third term,

$$z\sin^2(z) = z\left(z - \frac{z^3}{3!} + \cdots\right)^2 = z^3 + O(z^4).$$

This implies that $f_1(z))$ has a triple pole at $z = 0$.

To examine the singularity at $z = \infty$, make the substitution $z = 1/t$ in the function. We then have:

$$f_1(1/t) = \frac{te^{1/t}}{\sin^2(1/t)}.$$

As $t \to 0, \sin^2(1/t)$ is zero infinitely often (namely at $t = 1/n\pi$). Thus the singularity at $t = \infty$ is non-isolated.

For the second function, note that the numerator 1 is never zero, whilst the denominator has a quadruple zero at $z = 0$ since the first non-vanishing term in its Taylor series about $z = 0$ is the fourth term,

$$z^2(1-\cos(z)) = \frac{z^4}{2} + O(z^5).$$

This implies that $z = 0$ is a fourth-order pole of the function $f_2(z)$.

Again, to examine the singularity at $z = \infty$, make the substitution $z = 1/t$. We then have:

$$f_2(1/t) = \frac{t^2}{1-\cos(1/t)}.$$

As $t \to 0$, the denominator is zero infinitely often (namely at $t = 1/2n\pi$). Thus the singularity at $t = \infty$ is non-isolated.

Finally, for the third function we can expand the sine in a Taylor series as a function of $1/z$ for $|z| > 0$, yielding:

$$f_3(z) = z^2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n!z^n}.$$

This is the Laurent series of $f_3(z)$ on the domain $|z| > 1$. It has infinitely many negative powers of z, so it follows $z = 0$ is an essential singularity of $f_3(z)$. On the other hand, making the substitution $z = 1/t$, we have:

$$f_3(t) = \frac{\sin(t)}{t^2}.$$

Since t^2 has a double zero at $t = 0$, and $\sin(t)$ has a simple zero at $t = 0$, it follows that $f_3(1/t)$ has a simple pole at $t = 0$, i.e. $f_3(z)$ has a simple pole at $z = \infty$.

2016, Paper 1, Section II, 13A

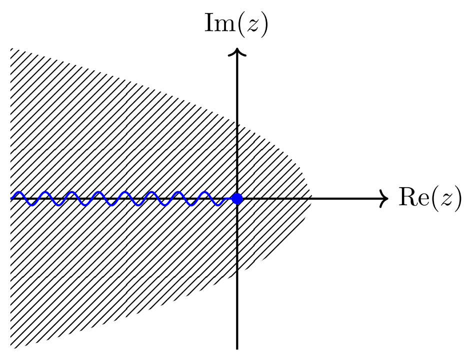
Let $w = u + iv$ and let $z = x + iy$, for u, v, x, y real.

(a) Let A be the map defined by $w = \sqrt{z}$ using the principal branch. Show that A maps the region to the left of the parabola $y^2 = 4(1-x)$ on the z-plane, with the negative real axis $x \in (-\infty, 0]$ removed, into the vertical strip of the w-plane between the lines $u = 0$ and $u = 1$.

(b) Let B be the map defined by $w = \tan^2(z/2)$. Show that B maps the vertical strip of the z-plane between the lines $x = 0$ and $x = \pi/2$ into the region inside the unit circle on the w-plane, with the part $u \in (-1, 0]$ of the negative real axis removed.

(c) Using the result of parts (a) and (b), show that the map C, defined by $w = \tan^2(\pi\sqrt{z}/4)$, maps the region to the left of the parabola $y^2 = 4(1-x)$ on the z-plane, *including* the negative real axis, onto the unit disc on the w-plane.

Solution: (a) We begin by sketching the parabola $y^2 = 4(1-x)$ with the negative real axis removed: We see that the region under consideration is given by $y^2 < 4(1-x)$, with the negative real axis removed.



Note that if $x+iy \mapsto u+iv$, we must have $(u+iv)^2 = x+iy$, which implies that $y = 2uv$ and $x = u^2 - v^2$ by comparing real and imaginary parts. In particular if x, y obey $y^2 < 4(1-x)$, we must have:

$$4u^2v^2 < 4(1-u^2-v^2) \qquad \Leftrightarrow \qquad 0 < (1-u^2)(1+v^2).$$

Since $1 + v^2 > 0$, we deduce that $-1 < u < 1$. Thus the image is contained in $-1 < u < 1$.

On the other hand, if we write $x = r\cos(\theta)$, $y = r\sin(\theta)$ for polar coordinates such that $r > 0$ and $\theta \in (-\pi, \pi)$ (dodging the negative real axis as a result), we have $w = \sqrt{r}\cos(\theta/2) + i\sqrt{r}\sin(\theta/2)$ by the definition of the principal branch of the square root. In particular, $u > 0$, so the image is also contained in $u > 0$. Overall we have that the image is contained in $0 < u < 1$.

Finally, note that for any point $u + iv$ such that $0 < u < 1$, we need to show that there exists a point $x + iy$ mapping to it. In other words, we need to find $r > 0$ and $\theta \in (-\pi, \pi)$ such that $u = \sqrt{r}\cos(\theta/2)$ and $v = \sqrt{r}\sin(\theta/2)$. Solving these equations simultaneously, we see that $r = u^2 + v^2 > 0$ works, and θ satisfying:

$$\cos(\theta/2) = \frac{u}{u^2+v^2}, \qquad \sin(\theta/2) = \frac{v}{u^2+v^2},$$

a solution of which can always be obtained in $(-\pi, \pi)$ since $u \neq 0, u > 0$.

We conclude that $w = \sqrt{z}$ indeed maps the region $y^2 < 4(1-x)$, with the negative real axis omitted, precisely to the region $0 < u < 1$.

(b) We note that:

$$\frac{d}{dz}\tan^2(z/2) = \tan(z/2)\sec^2(z/2),$$

so this map is conformal away from poles of $\sec$ and zeroes of $\tan$. The zeroes occur at $z/2 = n\pi$, i.e. $z = 2n\pi$, so these are unimportant for our region. The poles occur at $z/2 = (2n+1)\pi/2$, i.e. $z = (2n+1)\pi$, so again these are unimportant in our region. Therefore it is sufficient to check where the boundaries of the region map to, and where an interior point maps to.

Note that by the addition formula for tangent, we have:

$$\tan\left(\frac{x+iy}{2}\right) = \frac{\tan(x/2)+\tan(iy/2)}{1-\tan(x/2)\tan(iy/2)} = \frac{\tan(x/2)+i\tanh(y/2)}{1-i\tan(x/2)\tanh(y/2)}.$$

Squaring, we have:

$$\tan^2\left(\frac{x+iy}{2}\right) = \frac{\tan^2(x/2)-\tanh^2(y/2)+2i\tan(x/2)\tanh(y/2)}{1-\tan^2(x/2)\tanh^2(y/2)-2i\tan(x/2)\tanh(y/2)}.$$

In particular, for $x = 0$, we have:

$$\tan^2(iy/2) = -\tanh^2(y/2)$$

As y varies, $\tanh$ varies from -1 to 1, so we see that the boundary $x = 0$ is mapped to the line segment $(-1, 0]$.

On the other hand, for $x = \pi/2$, we have:

$$\begin{aligned}\tan^2(\pi/4+iy/2) &= \frac{1-\tanh^2(y/2)+2i\tanh(y/2)}{1-\tanh^2(y/2)-2i\tanh(y/2)}\\ &= \frac{(1-\tanh^2(y/2))^2-4\tanh^2(y/2)}{(1-\tanh^2(y/2))^2+4\tanh^2(y/2)} + \frac{4i(1-\tanh^2(y/2))\tanh(y/2)}{(1-\tanh^2(y/2))^2+4\tanh^2(y/2)}.\end{aligned}$$

The modulus of this expression is evidently 1, so the image lies on the unit circle. As $y \to \infty$, the expression tends to -1, with imaginary part greater than zero as we approach. On the other hand as $y \to -\infty$, the expression tends to -1, with imaginary part greater than zero as we approach. Note also that the imaginary part is zero only at $y = 0$; it follows that $x = \pi/2$ is mapped to the unit circle minus -1.

Finally, note that $\tan^2(\pi/8) < \tan^2(\pi/4) = 1$, so the interior point $(x, y) = (\pi/4, 0)$ is mapped to a point on the interior of the region in question. The result follows.

(c) Excluding the negative real axis, we have $z \mapsto \sqrt{z}$ maps $y^2 < 4(1-x)$ to $0 < u < 1$, then $z \mapsto \pi/4$ maps $0 < x < 1$ to $0 < u < \pi/2$ (simply by scaling all real parts appropriately), then $z \mapsto \tan^2(z/2)$ maps $0 < x < \pi/2$ to the unit disk with $(-1, 0]$ removed. Hence we have:

$$\tan^2\left(\frac{\pi\sqrt{z}}{4}\right)$$

maps $y^2 < 4(1-x)$ excluding the negative real axis to the unit disk with $(-1, 0]$ removed.

It remains to consider what happens to $(-\infty, 0]$; it suffices to show it maps to $(-1, 0]$. Technically, considering $\sqrt{z}$ for $z \in (-\infty, 0]$ is meaningless, so we will just say that $\sqrt{z}$ is the value of *some* square root of z on $(-\infty, 0]$. The possible square roots of numbers of the form $-y$ for $y \in [0, \infty)$ are precisely $\pm i\sqrt{y}$, i.e. they completely populate the imaginary axis as y is varied. Under the map $z \mapsto \pi z/4$ the imaginary axis is fixed. Finally, we saw in part (b) that the imaginary axis is mapped to the line segment $(-1, 0]$ under $\tan^2(\pi z/4)$; it follows that $(-\infty, 0]$ is mapped to $(-1, 0]$ under the complete transformation, and it is evident that varying y will produce the whole of that line segment.

2016, Paper 2, Section II, 13A

Let $a = N + 1/2$ for a positive integer N. Let C_N be the clockwise contour defined by the square with its four vertices at $a \pm ia$ and $-a \pm ia$. Let:

$$I_N = \oint\limits_{C_N} \frac{dz}{z^2 \sin(\pi z)}.$$

Show that $1/\sin(\pi z)$ is uniformly bounded on the contours C_N as $N \to \infty$, and hence that $I_N \to 0$ as $N \to \infty$.

Using this result, establish that:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

Solution: We showed $1/\sin(\pi z) = \operatorname{cosec}(\pi z)$ is uniformly bounded on C_N as part of Examples Sheet 3, Question 8(iii). Here is a reminder of the proof:

Proposition: $\operatorname{cosec}(\pi z)$ is uniformly bounded on C_N.

Proof: Using the addition formula for sine, we notice that:

$$\sin(\pi(x+iy)) = \sin(\pi x)\cos(\pi i y) + \sin(\pi i y)\cos(\pi x) = \sin(\pi x)\cosh(\pi y) + i\sinh(\pi y)\cos(\pi x).$$

Along the vertical segments of the contour, we have $x = \pm(N + 1/2)$. Hence:

$$\begin{aligned} |\sin(\pi(\pm(N+1/2)+iy))|^2 &= \sin^2(\pi(N+1/2))\cosh^2(\pi y) + \sinh^2(\pi y)\cos^2(\pi(N+1/2)) \\ &= \cosh^2(\pi y). \end{aligned}$$

It follows that $|\operatorname{cosec}(\pi(\pm(N+1/2)+iy))| = |\operatorname{sech}(\pi y)| \leq 1$ for all y, N.

Along the horizontal segments of the contour, we have $y = \pm(N+1/2)$. Hence:

$$\begin{aligned} |\sin(\pi(x \pm i(N+1/2)))|^2 &= \sin^2(\pi x)\cosh^2(\pi(N+1/2)) + \sinh^2(\pi(N+1/2))\cos^2(\pi x) \\ &= \sin^2(\pi x)(1+\sinh^2(\pi(N+1/2))) + \sinh^2(\pi(N+1/2))(1-\sin^2(\pi x)) \\ &= \sin^2(\pi x) + \sinh^2(\pi(N+1/2)) \\ &\geq \sinh^2(\pi(N+1/2)) \\ &\geq \sinh^2\left(\frac{\pi}{2}\right) \end{aligned}$$

for $N \geq 0$. It follows that $|\operatorname{cosec}(\pi(x \pm i(N+1/2)))| \leq \operatorname{cosech}(\pi/2)$ for all x, N. Thus we can uniformly bound $\operatorname{cosec}(\pi z)$ along the given contour by $\max(\operatorname{cosech}(\pi/2), 1)$. $\square$

We can use this to show $I_N \to 0$ as $N \to \infty$:

$$|I_N| = \left| \oint\limits_{C_N} \frac{dz}{z^2\sin(\pi z)} \right| \leq \text{length}(C_N) \sup_{z \in C_N} \left| \frac{1}{z^2 \sin(\pi z)} \right| \leq \frac{(8N+4)M}{(N+1/2)^2} \to 0,$$

where M is the uniform bound on $\operatorname{cosec}(\pi z)$ along the contour, $8N+4$ is the length of the contour, and $N + 1/2$ is the closest point on the contour to the origin (hence maximising $1/|z|^2$).

We can also evaluate the integral using Cauchy's residue theorem (and then compare the result to zero as $N \to \infty$). There is a triple pole at $z = 0$, and simple poles at all points where $\sin(\pi z) = 0$, i.e. $z \in \{-N, -(N-1), ..., N-1, N\}$ and $z \neq 0$ (for N fixed). Hence by the residue theorem we have:

$$\oint_{\gamma_N} \frac{\operatorname{cosec}(\pi z)}{z^2} = 2\pi i \left(\sum_{\substack{n=-N \\ n\neq 0}}^{N} \operatorname{Res}\left(\frac{\operatorname{cosec}(\pi z)}{z^2}; n\right) + \operatorname{Res}\left(\frac{\operatorname{cosec}(\pi z)}{z^2}; 0\right) \right).$$

At $z = 0$, we can find the residue by performing a series expansion. We have:

$$\begin{aligned}\frac{\operatorname{cosec}(\pi z)}{z^2} &= \frac{1}{z^2}\left(\pi z - \frac{\pi^3 z^3}{3!} + \frac{\pi^5 z^5}{5!} + \cdots\right)^{-1} = \frac{1}{\pi z^3}\left(1 - \frac{\pi^2 z^2}{3!} + \frac{\pi^4 z^4}{5!} + \cdots\right)^{-1} \\ &= \frac{1}{\pi z^3}\left(1 + \frac{\pi^2 z^2}{3!} + O(z^4)\right),\end{aligned}$$

using the Taylor series for sine, and the binomial expansion. It follows that the residue is $\pi/6$.

At $z = n \neq 0$, the residue can be obtained using the standard formula for the residue at a simple pole: At each integer n, the residue is given by:

$$\operatorname{Res}\left(\frac{\operatorname{cosec}(\pi z)}{z^2}; n\right) = \frac{1}{n^2}\lim_{z\to n}\left[\frac{z-n}{\sin(\pi z)}\right] = \frac{1}{n^2}\lim_{z\to n}\left[\frac{1}{\pi\cos(\pi z)}\right] = \frac{(-1)^n}{\pi n^2},$$

using $\sin'(\pi z) = \pi\cos(\pi z)$ to evaluate the limit of $(z-n)/\sin(\pi z)$ as $z \to n$. Overall then, we have:

$$\oint_{\gamma_N} \frac{\operatorname{cosec}(\pi z)}{z^2} = 2\pi i \left(\sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{(-1)^n}{\pi n^2} + \frac{\pi}{6} \right) = 2\pi i \left(2\sum_{n=1}^{N} \frac{(-1)^n}{\pi n^2} + \frac{\pi}{6} \right).$$

Hence in the limit as $N \to \infty$, we find that:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12},$$

as required.

2016, Paper 3, Section II, 13G

(a) Prove Cauchy's theorem for a triangle.

(b) Write down an expression for the winding number $I(\gamma, a)$ of a closed, piecewise continuously differentiable curve γ about a point $a \in \mathbb{C}$ which does not lie on γ.

(c) Let $U \subset \mathbb{C}$ be a domain, and $f : U \to \mathbb{C}$ a holomorphic function with no zeroes in U. Suppose that for infinitely many positive integers k the function f has a holomorphic kth root. Show that there exists a holomorphic function $F : U \to \mathbb{C}$ such that $f = \exp(F)$.

Solution: (a) This is a standard proof from lectures:

Cauchy's theorem for a triangle: Let U be a domain and let $T \subseteq U$ be a triangle. If $f : U \to \mathbb{C}$ is a holomorphic function, then:

$$\oint_{\partial T} f(z)\,dz = 0.$$

Proof: Let:

$$\eta = \left| \oint_{\partial T} f(z)\,dz \right|,$$

and let $l = \mathrm{length}(\partial T)$. Let $T^0 = T$ and subdivide the triangle T into 4 equal subtriangles by taking midpoints, $T = T_1 \cup T_2 \cup T_3 \cup T_4$, all with the same orientation. Then:

$$\oint_{\partial T} f(z)\,dz = \sum_{i=1}^{4} \oint_{\partial T_i} f(z)\,dz.$$

So there exists i, $1 \le i \le 4$, such that:

$$\left| \oint_{\partial T_i} f(z)\,dz \right| \ge \frac{\eta}{4},$$

since it is not possible for each of them to have a modulus strictly less than $\eta/4$, else overall we would have a modulus strictly less than η. We now iterate the process. Let $T' = T_i$ for the above i and repeat. We produce in this way a sequence of nested triangles such that:

$$\left| \oint_{\partial T^i} f(z)\,dz \right| \ge \frac{\eta}{4^i}, \qquad \mathrm{length}(\partial T^i) = \frac{l}{2^i}.$$

We now observe that:

$$\bigcap_{i=0}^{\infty} T^i \neq \emptyset$$

since all T^i are compact sets (see Part IB Metric and Topological Spaces). Thus there is some:

$$z_0 \in \bigcap_{i=0}^{\infty} T^i.$$

Since f is differentiable at z_0, given $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|w - z_0| < \delta$ we have:

$$|f(w) - f(z_0) - (w - z_0)f'(z_0)| < \epsilon |w - z_0|.$$

Pick any n such that $T^n \subseteq D(z_0, \delta)$. Then:

$$\frac{\eta}{4^n} \leq \left| \oint_{\partial T^n} f(z)\, dz \right| = \left| \oint_{\partial T^n} (f(z) - f(z_0) - (z - z_0)f'(z_0))\; dz \right| \leq \text{length}(\partial T^n)\epsilon \sup_{z \in \partial T^n} |z - z_0|$$

$$\leq \text{length}(\partial T^n)^2 \epsilon = \frac{l^2 \epsilon}{4^n}.$$

Note that the second equality follows since

$$\oint_{\partial T^n} dz = \oint_{\partial T^n} z\, dz = 0,$$

since the functions $1, z$ have antiderivatives $z, z^2/2$. Hence we have shown $\eta \leq \epsilon l^2$ for arbitrary $\epsilon > 0$; so $\eta = 0$. $\square$

(b) Given a closed piecewise C^1 curve γ, its winding number about a point $a \in \mathbb{C}$ not on the curve γ is given by:

$$I(\gamma, a) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - a}.$$

(c) We are given that f is everywhere non-zero and that the kth roots $f^{1/k} : U \to \mathbb{C}$ exist for all $k = 1, 2, ...$.

Suppose first that $f^{1/k}$ has a zero at $a \in U$; then on a neighbourhood of a we can write:

$$f^{1/k}(z) = (z - a)^m h(z)$$

for some holomorphic h which vanishes nowhere on this neighbourhood, and $m \geq 1$. Then $f(z) = (z - a)^{mk} h(z)^m$, so f has a zero at $z = a$. This is a contradiction, hence $f^{1/k}$ is non-zero everywhere in U.

In particular, the quotients $(f^{1/k})'/f^{1/k}$ exist throughout U for all $k = 1, 2, ...$, and the quotient f'/f exists throughout U too. We now observe that the winding number of $f \circ \gamma$ (for any curve γ in U) about 0 is given by:

$$I(f \circ \gamma, 0) = \frac{1}{2\pi i} \oint_{f \circ \gamma} \frac{dz}{z} = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\, dz.$$

In particular, this quantity is an integer, since it is a winding number. Now for each $k = 1, 2, ...$ we note that:

$$I(f \circ \gamma, 0) = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \oint_\gamma \frac{(f^{1/k}(z)^k)'}{f^{1/k}(z)^k}\, dz = \frac{1}{2\pi i} \oint_\gamma \frac{k(f^{1/k}(z))'}{f^{1/k}(z)}\, dz = kI(f^{1/k} \circ \gamma, 0).$$

This implies that k divides the left hand side for all $k = 1, 2, ...$. In particular, since the left hand side is an integer, it follows that $I(f \circ \gamma, 0)$ for all curves γ.

This is sufficient to construct a logarithm. By the antiderivative theorem, the fact that the integral of f'/f around any curve γ in the domain U vanishes implies that f'/f has an antiderivative on U; call this antiderivative h. We have:

$$h' = \frac{f'}{f}.$$

To finish, note that:

$$\left(\frac{e^h}{f}\right)' = \frac{h'e^h}{f} - \frac{f'e^h}{f^2} = \frac{f'e^h}{f^2} - \frac{f'e^h}{f^2} = 0,$$

so $e^{h(z)} = cf(z)$ everywhere, for some constant $c \neq 0$ (since $e^h \neq 0$). A logarithm is then given by:

$$f(z) = e^{h(z)-\log(c)},$$

where $\log(c)$ is any choice of logarithm for c.

2016, Paper 4, Section I, 4G

State carefully Rouché's theorem. Use it to show that the function $z^4 + 3 + e^{iz}$ has exactly one zero $z = z_0$ in the quadrant:

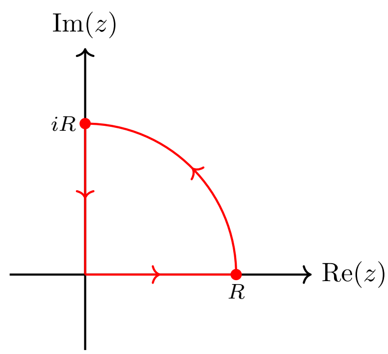
$$\{z \in \mathbb{C} \,|\, 0 < \arg(z) < \pi/2\}$$

and that $|z_0| \leq \sqrt{2}$.

Solution: We recall from lectures:

> **Rouché's theorem:** Let U be a simply-connected domain and let γ be a simple (i.e. non-self-intersecting) closed curve in U which bounds some domain. Let f, g be holomorphic on U and suppose that $|f| > |g|$ everywhere on the image of γ. Then f and $f + g$ have the same number of roots, counted with multiplicity, in the domain bounded by γ.

We now apply this to the question. Consider the quarter-circle contour shown below, of radius R, shown below.



On this quarter circle, we split z^4+3+e^{iz} into two functions, $f(z) = z^4+3$ and $g(z) = e^{iz}$. We note that $f(z)$ has zeroes given by $z^4 = -3$, i.e.

$$z = \sqrt[4]{3}e^{i\pi/4}, \sqrt[4]{3}e^{3i\pi/4}, \sqrt[4]{3}e^{5i\pi/4}, \sqrt[4]{3}e^{7i\pi/4}.$$

so in particular there is exactly one root contained in the contour above (for R sufficiently large).

We now prove that $|f(z)| > |g(z)|$ everywhere on the contour. On the circular part of the contour, setting $z = Re^{i\theta}$, we have:

$$|f(z)| = |z^4+3| = |R^4e^{4i\theta}+3| = \sqrt{(R^4\cos(4\theta)+3)^2 + R^8\sin^2(4\theta)} = \sqrt{R^8+9+6R^4\cos(4\theta)}$$

$$\geq \sqrt{R^8 - 6R^4 + 9} = R^4 - 3.$$

On the other hand, we have $|g(z)| = |e^{iz}| = e^{-R\sin(\theta)} \leq 1$, again since $\theta \in [0, \pi/2]$ on the contour. Hence provided $R^4 - 3 > 1$, i.e. $R > \sqrt{2}$, we have $|g| < |f|$ strictly on the contour.

On the vertical segment, we have $z = iy$ for $y \in [0, R]$. Hence $|f(z)| = y^4 + 3 \geq 3$, whilst $|g(z)| = e^{-y} \leq 1$. Thus $|f| > |g|$ on the vertical segment of the contour. Similarly on the horizontal segment, we have $z = x$ for $x \in [0, R]$. Hence $|f(z)| = x^4 + 3 \geq 3$, whilst $|g(z)| = |e^{ix}| = 1$. Thus $|f| > |g|$ on the horizontal segment of the contour too.

It follows by Rouché's theorem that $(f+g)(z) = z^4 + 3 + e^{iz}$ has the same number of zeroes inside the contour as $f(z)$ for all $R > \sqrt{2}$. In particular, z^4+3+e^{iz} has precisely one zero in the first quadrant, and it can have modulus at most $\sqrt{2}$ as required.

2017, Paper 1, Section I, 2A

Let $F(z) = u(x,y) + iv(x,y)$ where $z = x + iy$. Suppose that $F(z)$ is an analytic function of z in a domain $\mathcal{D}$ of the complex plane.

Derive the Cauchy-Riemann equations satisfied by u and v.

For $u = x/(x^2+y^2)$, find a suitable function v and a domain $\mathcal{D}$ such that $F = u + iv$ is analytic in $\mathcal{D}$.

Solution: Recall that F is complex differentiable at the point $z \in \mathcal{D}$ if:

$$F'(z) = \lim_{\Delta z \to 0}\left[\frac{F(z+\Delta z) - F(z)}{\Delta z}\right] = \lim_{\Delta z \to 0}\left[\frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y}\right]$$

exists, where $\Delta z = \Delta x + i\Delta y$. In particular, this limit must exist for all directions of approach $\Delta z \to 0$. Suppose therefore that we take $\Delta z = \Delta x$ purely real; then we require:

$$\begin{aligned} F'(z) &= \lim_{\Delta x \to 0}\left[\frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}\right] \\ &= \lim_{\Delta x \to 0}\left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}\right] + i\lim_{\Delta x \to 0}\left[\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right] \\ &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y). \end{aligned}$$

exists. On the other hand if we take $\Delta z = i\Delta y$ purely imaginary, then we require:

$$\begin{aligned} F'(z) &= \lim_{\Delta y \to 0}\left[\frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u(x,y) - iv(x,y)}{i\Delta y}\right] \\ &= -i\lim_{\Delta y \to 0}\left[\frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}\right] + \lim_{\Delta y \to 0}\left[\frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}\right] \\ &= -i\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y). \end{aligned}$$

exists. Both limits must agree, hence we deduce that the Cauchy-Riemann equations must hold:

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y), \qquad \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y).$$

Suppose now that $u = x/(x^2+y^2)$. For u to be the real part of an analytic function F on a domain $\mathcal{D}$, the imaginary part of the function v must necessarily obey the Cauchy-Riemann equations throughout this domain:

$$\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y) = \frac{2xy}{(x^2+y^2)^2}, \qquad \frac{\partial v}{\partial y}(x,y) = \frac{\partial u}{\partial x}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Note that the first equation can be directly integrated for constant y:

$$v(x,y) = -\frac{y}{x^2+y^2} + f(y),$$

where f is a function of y.

Differentiating with respect to y and inserting into the second equation, we obtain:

$$\frac{y^2 - x^2}{(x^2+y^2)^2} + f'(y) = \frac{y^2 - x^2}{(x^2+y^2)^2} \qquad \Rightarrow \qquad f = \text{constant}.$$

Hence we have $v = -y/(x^2+y^2) + \text{constant}$, which implies that $F = (x - iy)/(x^2+y^2) + i \cdot \text{constant}$; rewriting in terms of z we have $F(z) = 1/z + i \cdot \text{constant}$. On the domain $\mathbb{C}\backslash\{0\}$, F is an analytic function.

❊ **Comments from the Examiner:** *This seemed to be a straight-forward question, with most attempts being successful. Some candidates 'guessed' a solution but failed to show it then satisfied the C-R equations, resulting in a loss of 2 marks.*

2017, Paper 1, Section II, 13A

(a) Let $f(z)$ be defined on the complex plane such that $zf(z) \to 0$ as $|z| \to \infty$ and $f(z)$ is analytic on an open set containing $\mathrm{Im}(z) \geq -c$, where c is a positive real constant.

Let C_1 be the horizontal contour running from $-\infty - ic$ to $+\infty - ic$ and let:

$$F(\lambda) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-\lambda}dz.$$

By evaluating the integral, show that $F(\lambda)$ is analytic for $\mathrm{Im}(\lambda) > -c$.

(b) Let $g(z)$ be defined on the complex plane such that $zg(z) \to 0$ as $|z| \to \infty$ with $\mathrm{Im}(z) \geq -c$. Suppose that $g(z)$ is analytic at all points except $z = \alpha_+$ and $z = \alpha_-$ which are simple poles with $\mathrm{Im}(\alpha_+) > c$ and $\mathrm{Im}(\alpha_-) < -c$.

Let C_2 be the horizontal contour running from $-\infty + ic$ to $+\infty + ic$, and let:

$$H(\lambda) = \frac{1}{2\pi i}\int_{C_1} \frac{g(z)}{z-\lambda}dz, \qquad J(\lambda) = -\frac{1}{2\pi i}\int_{C_2} \frac{g(z)}{z-\lambda}dz.$$

(i) Show that $H(\lambda)$ is analytic for $\mathrm{Im}(\lambda) > -c$.

(ii) Show that $J(\lambda)$ is analytic for $\mathrm{Im}(\lambda) < c$.

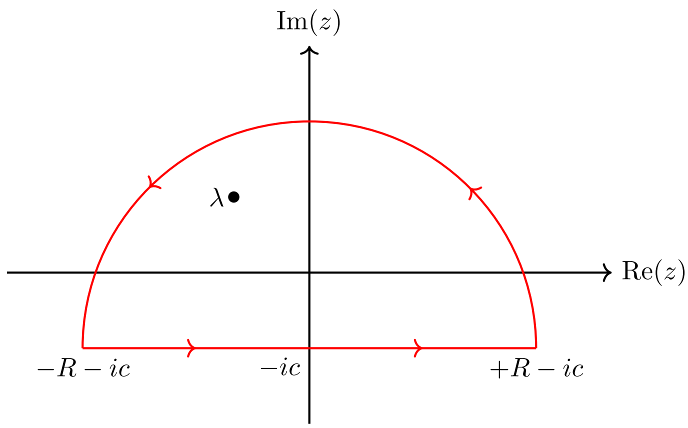
(iii) Show that if $-c < \mathrm{Im}(\lambda) < c$, then $H(\lambda) + J(\lambda) = g(\lambda)$.

[You should be careful to make sure you consider all points in the required regions.]

➭ **Solution:** (a) We consider instead the integral:

$$F_R(\lambda) = \frac{1}{2\pi i}\int_{C_{1,R}} \frac{f(z)}{z-\lambda}\,dz,$$

where $C_{1,R}$ is the semicircular contour shown in the figure below (with R chosen to be sufficiently large such that the contour encloses λ).



By the residue theorem, we can find an explicit expression for $F_R(\lambda)$:

$$F_R(\lambda) = \frac{1}{2\pi i} \cdot 2\pi i \mathrm{Res}\left(\frac{f(z)}{z-\lambda}; \lambda\right) = \lim_{z\to\lambda}\left[\frac{(z-\lambda)f(z)}{z-\lambda}\right] = f(\lambda),$$

where we used the formula for a simple pole since f is analytic on an open set containing $\mathrm{Im}(z) \geq -c$.

On the other hand, we could parametrise $F_R(\lambda)$ on each section of the contour separately. We have:

$$F_R(\lambda) = \frac{1}{2\pi i}\int_{\gamma_{1,R}} \frac{f(z)}{z-\lambda}\,dz + \frac{1}{2\pi i}\int_{\gamma_{2,R}} \frac{f(z)}{z-\lambda}\,dz,$$

where $\gamma_{1,R}$ is the straight segment of the contour and $\gamma_{2,R}$ is the circular arc segment of the contour. As $R \to \infty$, we have:

$$\frac{1}{2\pi i}\int_{\gamma_{1,R}} \frac{f(z)}{z-\lambda}\,dz = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-\lambda}\,dz = F(\lambda),$$

(at least in the principal value sense). On the hand, we can bound the integral over $\gamma_{2,R}$. For any $\epsilon > 0$, there exists an R such that $|zf(z)| < \epsilon$ for $|z| \geq R$, since we are given $zf(z) \to 0$ as $|z| \to \infty$; it follows that:

$$\left|\int_{\gamma_{2,R}} \frac{f(z)}{z-\lambda}\,dz\right| = \left|\int_0^{2\pi} \frac{Re^{i\theta}f(Re^{i\theta})}{Re^{i\theta}-\lambda}\,d\theta\right| \leq \int_0^{2\pi} \frac{|Re^{i\theta}f(Re^{i\theta})|}{|Re^{i\theta}-\lambda|}\,d\theta < \epsilon\int_0^{2\pi}\frac{1}{|Re^{i\theta}-\lambda|}\,d\theta.$$

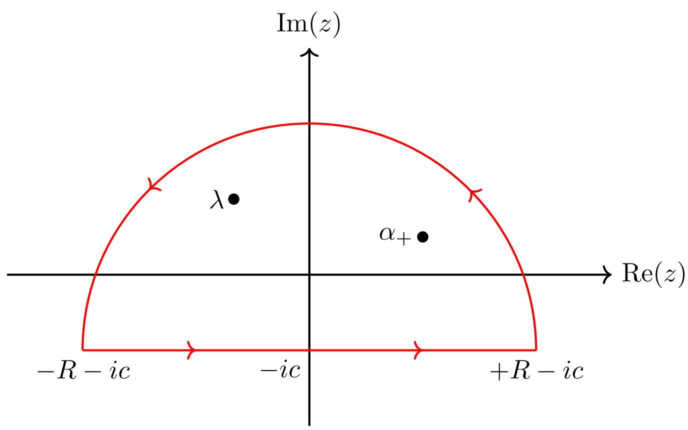
For λ fixed, we can choose R large enough such that $|Re^{i\theta}-\lambda| > 1$ for all θ. Then the integral is bounded by $2\pi\epsilon$ for sufficiently large R; it follows that the integral over $\gamma_{2,R}$ tends to zero as $R \to \infty$.

Hence we have:

$$F_R(\lambda) = f(\lambda) \to F(\lambda)$$

as $R \to \infty$ for each fixed λ in $\mathrm{Im}(\lambda) > -c$. It follows that $F(\lambda) = f(\lambda)$ for each fixed λ, and thus F is an analytic function of λ in the region $\mathrm{Im}(\lambda) > -c$.

(b) (i) This part is the same as part (a), except we now generalise the analytic function f to a function with a pole α_+ in the region which we encircle, as in the below figure.



By the residue theorem, we get:

$$H_R(\lambda) = \begin{cases} \mathrm{Res}\left(\dfrac{g(z)}{z-\lambda};\lambda\right) + \mathrm{Res}\left(\dfrac{g(z)}{z-\lambda};\alpha_+\right) & \text{when } \lambda \neq \alpha_+, \\ \mathrm{Res}\left(\dfrac{g(z)}{z-\alpha_+};\alpha_+\right) & \text{when } \lambda = \alpha_+. \end{cases}$$

where H_R has the same meaning as the analogous function we defined in part (a). In the first case, we have:

$$\mathrm{Res}\left(\frac{g(z)}{z-\lambda};\lambda\right) + \mathrm{Res}\left(\frac{g(z)}{z-\lambda};\alpha_+\right) = g(\lambda) - \frac{\mathrm{Res}\,(g(z);\alpha_+)}{\lambda-\alpha_+},$$

and it is evident that this function is analytic on the region $\mathrm{Im}(\lambda) > -c$, except possibly at the point $\lambda = \alpha_+$. However, we note that if we expanded $g(\lambda)$ in a Laurent series about $\lambda = \alpha_+$, the pole term would cancel with the second term in the above; it follows that the limit $\lambda \to \alpha_+$ is well-defined for this function and gives the constant term in the Laurent series of $g(\lambda)$ about $\lambda = \alpha_+$. It follows that the apparent singularity at $\lambda = \alpha_+$ is removable.

Comparing with $H_R(\lambda)$ at $\lambda = \alpha_+$, we immediately note that the residue expression is simply the constant term in the Laurent series of $g(\lambda)$ about $\lambda = \alpha_+$. Hence $H_R(\lambda)$ supplies the correct value to remove the singularity at $\lambda = \alpha_+$ to create a smooth analytic function overall.

Finally, exactly as in part (a), we can show that $H_R(\lambda) \to H(\lambda)$ as $R \to \infty$. It follows that for each fixed λ:

$$H_R(\lambda) = \begin{cases} g(\lambda) - \dfrac{\mathrm{Res}(g(z);\alpha_+)}{\lambda-\alpha_+} & \text{when } \lambda \neq \alpha_+ \\ \mathrm{Res}\left(\dfrac{g(z)}{z-\alpha_+};\alpha_+\right) & \text{when } \lambda = \alpha_+. \end{cases} \to H(\lambda)$$

as $R \to \infty$, and hence:

$$H(\lambda) = \begin{cases} g(\lambda) - \dfrac{\mathrm{Res}(g(z);\alpha_+)}{\lambda-\alpha_+} & \text{when } \lambda \neq \alpha_+ \\ \mathrm{Res}\left(\dfrac{g(z)}{z-\alpha_+};\alpha_+\right) & \text{when } \lambda = \alpha_+. \end{cases},$$

for all λ such that $\mathrm{Im}(\lambda) > -c$. By the above discussion, $H(\lambda)$ is indeed analytic on $\mathrm{Im}(\lambda) > -c$.

(ii) By the same argument as (b)(i), this time using a semicircular contour beneath the initial infinite contour rather than above, we can show that:

$$J(\lambda) = \begin{cases} g(\lambda) - \dfrac{\mathrm{Res}(g(z);\alpha_-)}{\lambda - \alpha_-} & \text{when } \lambda \neq \alpha_- \\ \mathrm{Res}\left(\dfrac{g(z)}{z-\alpha_-};\alpha_-\right) & \text{when } \lambda = \alpha_-. \end{cases}.$$

Just as in the $H(\lambda)$ case, it is analytic everywhere in $\mathrm{Im}(\lambda) < c$.

(iii) Finally, we consider λ in the region $-c < \mathrm{Im}(\lambda) < c$. In this case, $\lambda \neq \alpha_+, \alpha_-$, so from the above expressions for $H(\lambda), J(\lambda)$, we have:

$$H(\lambda) + J(\lambda) = 2g(\lambda) - \frac{\mathrm{Res}(g(z);\alpha_+)}{\lambda - \alpha_+} - \frac{\mathrm{Res}(g(z);\alpha_-)}{\lambda - \alpha_-}. \qquad (*)$$

To finish, we establish a quick relation between the residues. Consider the integral:

$$\oint_{|z|=R} \frac{g(z)}{z-\lambda}\,dz$$

around a very large circle $|z| = R$, with $-c < \mathrm{Im}(\lambda) < c$ so that $\lambda \neq \alpha_+, \alpha_-$. By the residue theorem, this integral is given by:

$$\oint_{|z|=R} \frac{g(z)}{z-\lambda}\,dz = g(\lambda) - \frac{\mathrm{Res}(g(z);\alpha_+)}{\lambda - \alpha_+} - \frac{\mathrm{Res}(g(z);\alpha_-)}{\lambda - \alpha_-}.$$

However, we also know from the bounding work in part (a) that the integral vanishes in the limit $R \to \infty$. It follows that there is a relationship between the residues:

$$g(\lambda) - \frac{\mathrm{Res}(g(z);\alpha_+)}{\lambda - \alpha_+} - \frac{\mathrm{Res}(g(z);\alpha_-)}{\lambda - \alpha_-} = 0.$$

Inserting this into ($*$) above, we have:

$$H(\lambda) + J(\lambda) = g(\lambda)$$

for all $-c < \mathrm{Im}(\lambda) < c$ as required.

❋ **Comments from the Examiner:** *This was not a popular question. Of those attempts that got through to the end it was unfortunate to see a majority of them closing in the LHP for part (b)(ii) which is incorrect. Many solutions also lacked proper justification of why (b)(i) was analytic specifically at $\lambda = \alpha_+$, despite the note in brackets at the bottom of the question. Part (b)(iii) was answered well, with candidates either summing their two previous solutions or closing in a rectangular contour.*

2017, Paper 2, Section II, 13A

State the residue theorem.

By considering:

$$\oint_C \frac{z^{1/2}\log(z)}{1+z^2}\,dz$$

with C a suitably chosen contour in the upper half plane or otherwise, evaluate the real integrals:

$$\int_0^\infty \frac{x^{1/2}\log(x)}{1+x^2}\,dx, \qquad \text{and} \qquad \int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx,$$

where $x^{1/2}$ is taken to be the positive square root.

➡ **Solution:** We recall the statement of the residue theorem from lectures:

Cauchy's residue theorem: Let U be a simply-connected domain, let $z_1, ..., z_n \in U$ and let $g : U\backslash\{z_1, ..., z_n\} \to \mathbb{C}$ be holomorphic. For any piecewise C^1 closed curve $\gamma : [0,1] \to U$ whose image does not contain any of the points $z_1, ..., z_n$, we have:

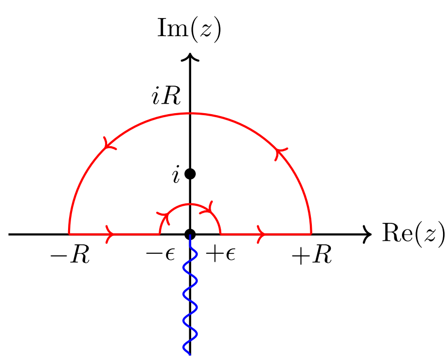
$$\frac{1}{2\pi i}\oint_\gamma g(z)\,dz = \sum_{i=1}^n I(\gamma; z_i)\mathrm{Res}(g; z_i),$$

where $\mathrm{Res}(g; z_i)$ denotes the *residue* of g at the point z_i (i.e. the coefficient of $1/(z-z_i)$ in its Laurent expansion about $z = z_i$) and $I(\gamma; z_i)$ denotes the winding number of the curve γ about z_i.

We now apply this to the given problem. Consider the contour integral

$$\oint_{C_{R,\epsilon}} \frac{z^{1/2}\log(z)}{1+z^2}\,dz, \qquad (*)$$

around a contour $C_{R,\epsilon}$ which we will momentarily describe. Notice that the integrand of this function is 'multi-valued' with a branch point at $z = 0$; we choose a branch by inserting a cut along the negative imaginary axis and choosing arguments in the range $(-\pi/2, 3\pi/2)$. We choose $C_{R,\epsilon}$ to be an arch contour shown below:



For R sufficiently large and ϵ sufficiently small, the contour $C_{R,\epsilon}$ encircles exactly one singularity of the integrand at $z = i$. Hence, by the residue theorem, the integral in (∗) is given by:

$$\oint_{C_{R,\epsilon}} \frac{z^{1/2}\log(z)}{1+z^2}\,dz = 2\pi i \mathrm{Res}\left(\frac{z^{1/2}\log(z)}{1+z^2}; i\right) = 2\pi i\left(\frac{e^{i\pi/4}(i\pi/2)}{2i}\right) = \frac{i\pi^2 e^{i\pi/4}}{2}.$$

But we can also evaluate the integral on each section of the contour separately, giving:

$$\frac{i\pi^2 e^{i\pi/4}}{2} = \oint_{C_{R,\epsilon}} \frac{z^{1/2}\log(z)}{1+z^2}\,dz = \int_0^\pi \frac{R^{1/2}e^{i\theta/2}\left(\log(R)+i\theta\right)}{1+R^2e^{2i\theta}} iRe^{i\theta}\,d\theta + \int_R^\epsilon \frac{e^{i\pi/2}x^{1/2}\left(\log(x)+i\pi\right)}{1+x^2}\,e^{i\pi}dx$$
$$+ \int_\pi^0 \frac{\epsilon^{1/2}e^{i\theta/2}\left(\log(\epsilon)+i\theta\right)}{1+\epsilon^2e^{2i\theta}} i\epsilon e^{i\theta}\,d\theta + \int_\epsilon^R \frac{x^{1/2}\log(x)}{1+x^2}\,dx.$$

It remains to bound each of these terms appropriately. We note:

- On the large semicircular section of the contour, we have:

$$\left|\int_0^\pi \frac{R^{3/2}e^{i\theta/2}(\log(R)+i\theta)}{1+R^2e^{2i\theta}}\,d\theta\right| \le \int_0^\pi \frac{R^{3/2}(\log^2(R)+\theta^2)}{|1+R^2e^{2i\theta}|}\,d\theta \le \frac{R^{3/2}\log^2(R)\pi}{R^2-1} + \frac{R^{3/2}}{R^2-1}\int_0^\pi \theta^2\,d\theta,$$

  using the reverse triangle inequality in the denominator in the final step. As $R \to \infty$, this converges to zero, since power-law growth beats logarithmic growth.

- On the small semicircular section of the contour, we similarly have:

$$\left|\int_\pi^0 \frac{\epsilon^{1/2}e^{i\theta/2}\left(\log(\epsilon)+i\theta\right)}{1+\epsilon^2e^{2i\theta}} i\epsilon e^{i\theta}\,d\theta\right| \le \frac{\epsilon^{3/2}\log^2(\epsilon)\pi}{1-\epsilon^2} + \frac{\epsilon^{3/2}}{1-\epsilon^2}\int_0^\pi \theta^2\,d\theta.$$

  Again, we see that as $\epsilon \to 0$, this converges to zero.

Putting everything together, we see that as $\epsilon \to 0$ and $R \to \infty$, we're left with the result:

$$i\int_0^\infty \frac{x^{1/2}(\log(x)+i\pi)}{1+x^2}\,dx + \int_0^\infty \frac{x^{1/2}\log(x)}{1+x^2}\,dx = \frac{i\pi^2 e^{i\pi/4}}{2}.$$

Comparing real and imaginary parts, we find the results:

$$\int_0^\infty \frac{x^{1/2}\log(x)}{1+x^2}\,dx = \frac{\pi^2\sqrt{2}}{4}, \qquad -\pi\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx + \int_0^\infty \frac{x^{1/2}\log(x)}{1+x^2}\,dx = -\frac{\pi^2\sqrt{2}}{4} \qquad \Rightarrow \qquad \int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx = \frac{\pi\sqrt{2}}{2}.$$

❊ **Comments from the Examiner:** *This proved to be a bit too simple for a long question but there were a remarkable number of algebraic mistakes - these were penalised harshly given the question was relatively easy.*

2017, Paper 3, Section II, 13F

Let f be an entire function. Prove Taylor's theorem, that there exist complex numbers $c_0, c_1, \ldots$ such that:

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

for all z. *[You may assume Cauchy's Integral Formula.]*

For a positive real r, let $M_r = \sup\{|f(z)| : |z| = r\}$. Explain why we have:

$$|c_n| \leq \frac{M_r}{r^n}$$

for all n.

Now let n and r be fixed. For which entire functions f do we have $|c_n| = M_r/r^n$?

Solution: Recall Cauchy's integral formula for a disk states:

> **Cauchy's integral formula for a disk:** Let $D = D(a, r)$ be a disk and let $f : D \to \mathbb{C}$ be holomorphic. For every $w \in D$ and ρ with $|w - a| < \rho < r$, we have:
>
> $$f(w) = \frac{1}{2\pi i} \int\limits_{|z-a|=\rho} \frac{f(z)}{z - w}\, dz.$$

To prove Taylor's theorem, let us take $a = 0$ and fix any radius $r > 0$ in Cauchy's integral formula. Then for all w and ρ such that $|w| < \rho < r$ we can write:

$$f(w) = \frac{1}{2\pi i} \int\limits_{|z|=\rho} \frac{f(z)}{z - w}\, dz.$$

The denominator of the integrand can be expanded in a geometric series:

$$\frac{1}{z-w} = \frac{1}{z(1 - w/z)} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{w^n}{z^n},$$

which is convergent since $|w| < \rho = |z|$. Furthermore, this is a power series in $1/z$, and hence is uniformly convergent on $\rho = |z|$. It follows that we may exchange the order of summation and integration to yield:

$$f(w) = \sum_{n=0}^{\infty} w^n \cdot \frac{1}{2\pi i} \oint\limits_{|z|=\rho} \frac{f(z)}{z^{n+1}}\, dz.$$

Since r was arbitrary, this holds for each $w \in \mathbb{C}$, proving Taylor's theorem. We see that the coefficients are given by:

$$c_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho} \frac{f(z)}{z^{n+1}}\, dz,$$

where we may now choose any $\rho > 0$ to evaluate them (they are independent of ρ e.g. by the homotopy form of Cauchy's theorem).

Given $M_r = \sup\{|f(z)| : |z| = r\}$, we note that we can bound the coefficients in the expansion using the standard integral estimate (here using circles of radius r to evaluate the coefficients):

$$|c_n| = \frac{1}{2\pi}\left|\oint\limits_{|z|=r} \frac{f(z)}{z^{n+1}}\,dz\right| \leq \frac{\text{length}(\{|z|=r\})}{2\pi}\sup_{|z|=r}\left|\frac{f(z)}{z^{n+1}}\right| = r\cdot\frac{M_r}{r^{n+1}} = \frac{M_r}{r^n}.$$

The integral estimate here is a simple consequence of the triangle inequality for integrals of complex functions over real intervals, which is saturated if and only if the integrand is constant. In particular, we require:

$$\left|\int\limits_0^{2\pi} \frac{f(re^{i\theta})}{r^n e^{in\theta}}\,d\theta\right| = \int\limits_0^{2\pi}\left|\frac{f(re^{i\theta})}{r^n e^{in\theta}}\right|\,d\theta,$$

which occurs if and only if $f(z)/z^n$ is constant on $|z| = r$. This implies that the functions:

$$\frac{f(z)}{z^n}, \qquad \frac{f(r)}{r^n}$$

are holomorphic on $\mathbb{C}\backslash\{0\}$ and agree on the circle $|z| = r$, which in particular contains a non-isolated point. It follows by the identity theorem that:

$$f(z) = \frac{f(r)z^n}{r^n}$$

everywhere in $\mathbb{C}\backslash\{0\}$. Since f is entire, by continuity we have $f(0) = 0$. It follows that if one of the Cauchy estimates is saturated, f must be an appropriate monomial.

Conversely, to demonstrate sufficiency, note that for this function we have:

$$|c_n| = \left|\frac{f(r)}{r^n}\right| = \frac{M_r}{r^n},$$

as required.

❊ **Comments from the Examiner:** *The bookwork on Taylor's theorem was well done, as was the inequality coming from Cauchy's formula. But the case of equality was fudged by most candidates, who generally applied 'wishful thinking' to the integrals.*

2017, Paper 4, Section I, 4F

Let D be a star-domain, and let f be a continuous complex-valued function on D. Suppose that for every triangle T contained in D we have:

$$\oint_{\partial T} f(z)\,dz = 0.$$

Show that f has an antiderivative on D.

If we assume that D is a domain (not necessarily a star-domain), does this conclusion still hold? Briefly justify your answer.

Solution: It is useful to begin by recalling the definition of a star-domain:

Definition: A domain D is called a *star-domain* if there exists a point $p \in D$ such that for any $a \in D$, the straight line segment from a to p is completely contained in D. We say that D is *star-shaped about* p.

We now prove the proposition directly by constructing an antiderivative.

Proposition: Let $f : D \to \mathbb{C}$ be continuous on the star-domain D, and suppose that for every triangle T contained in D we have:

$$\oint_{\partial T} f(z)\,dz = 0.$$

Then f has an antiderivative on D.

Proof: Let D be star-shaped about p. We propose an antiderivative:

$$F(w) = \int_p^w f(z)\,dz,$$

where the contour of integration is a straight line segment from p to w. Note we do not need to show that F is 'well-defined' as in the proof of the standard antiderivative theorem, because we have specified precisely the contour we are integrating along. All that remains is to show F is differentiable with derivative f.

Now given any $w \in D$, and any $h \in \mathbb{C}$ such that $D(w, |h|) \subseteq D$, we have:

$$\int_p^w f(z)\,dz + \int_w^{w+h} f(z)\,dz - \int_p^{w+h} f(z)\,dz = 0,$$

by hypothesis, since the left hand side corresponds to integration around a triangle. It follows that:

$$F(w+h) = \int_p^{w+h} f(z)\,dz = F(w) + \int_w^{w+h} f(z)\,dz = F(w) + hf(w) + \int_w^{w+h} (f(z) - f(w))\,dz.$$

Rearranging and estimating, we have:

$$\left| \frac{F(w+h) - F(w)}{h} - f(w) \right| = |h| \left| \int_w^{w+h} (f(z) - f(w))\,dz \right| \leq \frac{|h|}{|h|} \sup_{z \in [w, w+h]} |f(z) - f(w)| \to 0$$

as $h \to 0$, since f is continuous (the notation $[w, w+h]$ means the straight segment from w to $w+h$). $\square$

For the final part, it is sufficient to provide a counterexample. Consider an annulus of the form $A = \{z \in \mathbb{C} : 1 < |z| < r\}$ for some radius r. Notice that such an annulus is never a star-domain, since for any point $p \in A$, the straight line segment from p to $-p$ is not contained in A; thus A cannot be star-shaped about any $p \in A$.

We begin by observing that any triangle T in A which encircles 0 must have perimeter at least $|\partial T| > 2\pi$, i.e. the perimeter of the inner circle. On the other hand, any triangle which lies completely in the circle $\{z \in \mathbb{C} : |z| < r\}$ has its perimeter bounded above; an extremely crude bound is given by $|\partial T| < 6r$ (this follows by inserting lines from the centre to each of the vertices of the triangle, subdividing the triangle in three - the interior lines all have length at most r, so by the triangle inequality each of the outer legs has length at most $2r$). In particular, taking $r = \pi/3$ (note this is greater than 1) we find that $|\partial T| < 2\pi$ for all triangles in A.

This implies that there are no triangles T in A which encircle 0. In particular, by the residue theorem (viewing the integrand as a function on $\mathbb{C}\backslash\{0\}$ now for example) this implies that:

$$\oint_{\partial T} \frac{1}{z}\, dz = 0$$

for all triangles T in A. On the other hand, choosing a radius R such that $1 < R < \pi/3$, we have:

$$\oint_{|z|=R} \frac{1}{z}\, dz = 2\pi i$$

by the residue theorem, which is sufficient for $1/z$ to have no antiderivative on A.

❊ **Comments from the Examiner:** *The first part of this question, which was bookwork about antiderivatives, was well done, but the second part was very poorly done. Most students quoted false statements about antiderivatives, with no regard to whether the domain was a star domain or not - despite the fact that the question clearly suggests that this distinction is key.*

2018, Paper 1, Section I, 2A

(a) Show that $w = \log(z)$ is a conformal mapping from the right half z-plane, $\mathrm{Re}(z) > 0$, to the strip:

$$S = \left\{ w : -\frac{\pi}{2} < \mathrm{Im}(w) < \frac{\pi}{2} \right\},$$

for a suitably chosen branch of $\log(z)$ that you should specify.

(b) Show that:

$$w = \frac{z-1}{z+1}$$

is a conformal mapping from the right half z-plane, $\mathrm{Re}(z) > 0$, to the unit disc $D = \{w : |w| < 1\}$.

(c) Deduce a conformal mapping from the strip S to the disc D.

Solution: (a) We choose the principal branch of the logarithm, defined by $\mathrm{Log}(z) = \log|z| + i\arg(z)$, where $\arg(z) \in (-\pi, \pi)$ and $z \in \mathbb{C}\backslash(-\infty, 0]$. If we write $z = re^{i\theta}$ in polar coordinates, we see that:

$$\mathrm{Log}(re^{i\theta}) = \log(r) + i\theta.$$

The right half-plane can be described in polar coordinates by $r > 0$ and $\theta \in (-\pi/2, \pi/2)$. Thus the image of the logarithm is the horizontal strip S as anticipated. On the other hand, given any $x + iy \in S$, there exists a point $z = e^{x+iy}$ in the right half plane which maps to $x + iy$ via the logarithm. It follows that the logarithm is a bijection between $\mathrm{Re}(z) > 0$ and the strip S.

The map is conformal since its derivative is $\log'(z) = 1/z \neq 0$, for $\mathrm{Re}(z) > 0$. Hence in fact the logarithm provides a conformal equivalence between the two regions in this instance.

(b) The map is conformal because it is a Möbius map. More explicitly, we can compute the derivative:

$$\frac{d}{dz}\left(\frac{z-1}{z+1}\right) = \frac{1}{z+1} - \frac{z-1}{(z+1)^2} = \frac{z+1-(z-1)}{(z+1)^2} = \frac{2}{(z+1)^2} \neq 0$$

for $\mathrm{Re}(z) > 0$. Now, we know that Möbius maps take circles/lines to circles/lines, so let us compute the image of the boundary $\mathrm{Re}(z) = 0$ under this Möbius map. Taking three points $z = -i, 0, i$ on the boundary, we have:

$$\left|\frac{i-1}{i+1}\right| = 1, \qquad \left|\frac{-i-1}{-i+1}\right| = 1, \qquad \frac{0-1}{0+1} = -1,$$

so we see that the image of the boundary is the unit circle. Since the map is continuous, it is a homeomorphism, so maps connected regions to connected regions. In particular, we note that $1 \mapsto 0$ under this map so $\mathrm{Re}(z) > 0$ is mapped into the interior of the unit disk.

Since the map is a Möbius map, it is a bijection, so this is in particular a conformal equivalence.

(c) We can just use the conformal equivalences we developed in (a) and (b). Using the inverse of the logarithm (the exponential), we see the required map is simply:

$$w = \frac{e^z - 1}{e^z + 1}.$$

Comments from the Examiner: *This question was done very well by nearly all candidates. The common approach to (b) was to consider just the mapping of three points. Candidates who did not explain why this was sufficient (e.g. Mobius maps send circlines to circlines) lost marks, but these were typically not sufficient to lose out on the beta.*

2018, Paper 1, Section II, 13A

(a) Let C be a rectangular contour with vertices at $\pm R+\pi i$ and $\pm R-\pi i$ for some $R>0$ taken in the anticlockwise direction. By considering:

$$\lim_{R\to\infty}\oint_C \frac{e^{iz^2/4\pi}}{e^{z/2}-e^{-z/2}}\,dz,$$

show that:

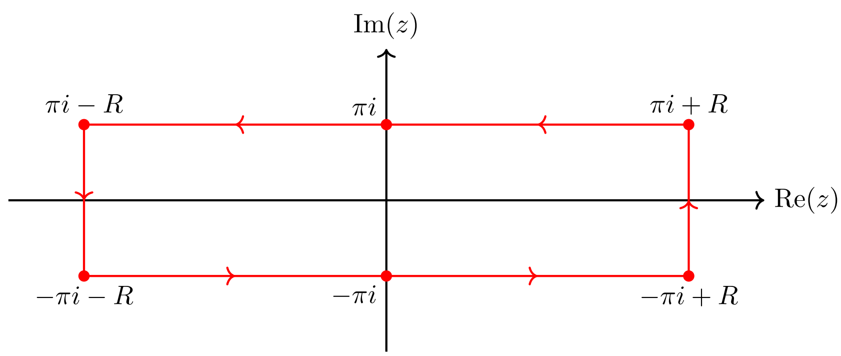
$$\lim_{R\to\infty}\int_{-R}^{R} e^{ix^2/4\pi}\,dx = 2\pi e^{\pi i/4}.$$

(b) By using a semi-circular contour in the upper half plane, calculate:

$$\int_0^\infty \frac{x\sin(\pi x)}{x^2+a^2}\,dx,$$

for $a>0$. *[You may use Jordan's Lemma without proof.]*

Solution: A sketch of the contour is given below:



Consider the given contour integral:

$$\oint_C \frac{e^{iz^2/4\pi}}{e^{z/2}-e^{-z/2}}\,dz.$$

The integrand is singular if and only if $e^z=1$, which occurs if and only if $z=2n\pi i$. In particular, there is a single singularity enclosed by the contour at $z=0$; since the denominator of the integrand is proportional to $\sinh(z/2)$ and the numerator is everywhere non-zero, the nature of this singularity is a simple pole. Therefore, by Cauchy's residue theorem, we have:

$$\oint_C \frac{e^{iz^2/4\pi}}{e^{z/2}-e^{-z/2}}\,dz = 2\pi i \mathrm{Res}\left(\frac{e^{iz^2/4\pi}}{e^{z/2}-e^{-z/2}};0\right).$$

We can obtain the residue using the standard formula for the residue at a simple pole:

$$\lim_{z\to 0}\left[\frac{ze^{iz^2/4\pi}}{e^{z/2}-e^{-z/2}}\right] = \lim_{z\to 0}\left[\frac{e^{iz^2/4\pi}+iz^2e^{iz^2/4\pi}/2\pi}{\cosh(z)}\right] = 1,$$

using L'Hôpital's rule in the second step.

It follows that:

$$\oint_C \frac{e^{iz^2/4\pi}}{e^{z/2}-e^{-z/2}}\,dz = 2\pi i,$$

for all R.

We now evaluate the contour integral in a different way, by parametrising each section of the contour separately. On the vertical segments of the contour, we have the contributions:

$$i\int_{-\pi}^{\pi} \frac{e^{i(ix+R)^2/4\pi}}{e^{(ix+R)/2}-e^{-(ix+R)/2}}\,dx + i\int_{\pi}^{-\pi} \frac{e^{i(ix-R)^2/4\pi}}{e^{(ix-R)/2}-e^{-(ix-R)/2}}\,dx.$$

We claim that we can bound each of these terms. For the first, we have:

$$\left| i\int_{-\pi}^{\pi} \frac{e^{i(ix+R)^2/4\pi}}{e^{(ix+R)/2}-e^{-(ix+R)/2}}\,dx\right| \leq \int_{-\pi}^{\pi} \frac{e^{-Rx/2\pi}}{e^{R/2}-e^{-R/2}}\,dx = \frac{1}{2\sinh(R/2)}\int_{-\pi}^{\pi} e^{-Rx/2\pi}\,dx,$$

using the reverse triangle inequality in the denominator (and the triangle inequality for integrals). Evaluating the integral, we establish the bound:

$$\frac{1}{2\sinh(R/2)}\int_{-\pi}^{\pi} e^{-Rx/2\pi}\,dx = \frac{\pi}{R\sinh(R/2)}\left[-e^{-Rx/2\pi}\right]_{-\pi}^{\pi} = \frac{2\pi}{R} \to 0,$$

as $R\to\infty$. Similarly, we have that the other vertical segment goes to zero as $R\to\infty$ (just send $R\mapsto -R$ in the argument, and insert modulus signs as appropriate).

We now consider the horizontal segments of the contour, which give the contributions:

$$\int_{-R}^{R} \frac{e^{i(x-i\pi)^2/4\pi}}{e^{(x-i\pi)/2}-e^{-(x-i\pi)/2}}\,dx + \int_{R}^{-R} \frac{e^{i(x+i\pi)^2/4\pi}}{e^{(x+i\pi)/2}-e^{-(x+i\pi)/2}}\,dx.$$

We note that the integrands can be simplified to:

$$\frac{e^{i(x\pm i\pi)^2/4\pi}}{e^{(x\pm i\pi)/2}-e^{-(x\pm i\pi)/2}} = \frac{e^{ix^2/4\pi}e^{\mp x/2}e^{-i\pi/4}}{\pm i(e^{x/2}+e^{-x/2})} = \mp ie^{-i\pi/4}\frac{e^{ix^2/4\pi}e^{\mp x/2}}{e^{x/2}+e^{-x/2}}.$$

In particular, adding the two integrals together we get:

$$ie^{-i\pi/4}\int_{-R}^{R} e^{ix^2/4\pi}\left(\frac{e^{x/2}+e^{-x/2}}{e^{x/2}+e^{-x/2}}\right)\,dx = ie^{-i\pi/4}\int_{-R}^{R} e^{ix^2/4\pi}\,dx.$$

Thus in the limit as $R\to\infty$, we have established:

$$\lim_{R\to\infty}\int_{-R}^{R} e^{ix^2/4\pi}\,dx = 2\pi e^{i\pi/4},$$

as required.

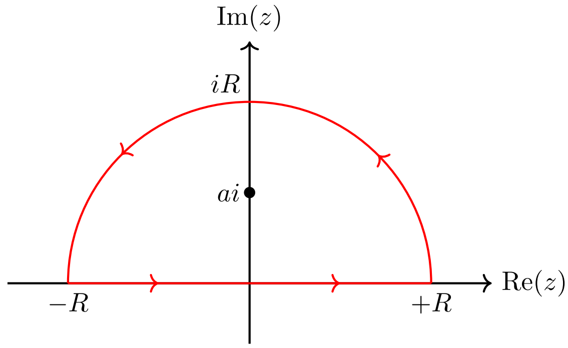
(b) We note that:

$$\int_0^\infty \frac{x\sin(\pi x)}{x^2+a^2}\,dx = \operatorname{Im}\int_0^\infty \frac{xe^{i\pi x}}{x^2+a^2}\,dx,$$

which suggests we should consider the contour integral:

$$\oint_{C_R} \frac{ze^{i\pi z}}{z^2+a^2}\,dz,$$

around the contour shown below.



There is a single singularity enclosed by the contour at $z = ai$, which is a simple pole. By the residue theorem then, and the formula for the residue at a simple pole, we have:

$$\oint_{C_R} \frac{ze^{i\pi z}}{z^2+a^2}\,dz = 2\pi i\lim_{z\to ai}\left[\frac{(z-ai)ze^{i\pi z}}{(z+ai)(z-ai)}\right] = 2\pi i\lim_{z\to ai}\left[\frac{ze^{i\pi z}}{z+ai}\right] = \frac{2\pi i\cdot aie^{-\pi a}}{2ai} = \pi i e^{-\pi a}.$$

On the other hand, we can evaluate the integral on each section of the contour separately. On the large semicircular part of the contour, we note that since $\pi > 0$, and:

$$\left|\frac{z}{z^2+a^2}\right| \to 0$$

as $|z|\to\infty$, we can apply Jordan's lemma to deduce that the integral tends to zero as $R\to\infty$. On the straight segment of the contour, we get the integral:

$$\int_{-\infty}^{\infty} \frac{xe^{i\pi x}}{x^2+a^2}\,dx$$

in the limit as $R\to\infty$. Putting everything together then, we find that:

$$\int_{-\infty}^{\infty} \frac{xe^{i\pi x}}{x^2+a^2}\,dx = \pi i e^{-\pi a}.$$

Taking imaginary parts, and noting that the resulting integrand is even, we have:

$$\int_0^\infty \frac{x\sin(\pi x)}{x^2+a^2}\,dx = \frac{1}{2}\pi e^{-\pi a}.$$

❊ **Comments from the Examiner:** *Candidates struggled with justifying why the integration along the vertical contours was negligible, but otherwise tackled this question as expected. There was abundant but inappropriate quoting of Jordan's Lemma in section (b) if candidates considered both $e^{i\pi z}$ and $e^{-i\pi z}$ terms.*

2018, Paper 2, Section II, 13A

(a) Let $f(z)$ be a complex function. Define the *Laurent series* of $f(z)$ about $z = z_0$ and give suitable formulae in terms of integrals for calculating the coefficients of the series.

(b) Calculate, by any means, the first 3 terms in the Laurent series about $z = 0$ for:

$$f(z) = \frac{1}{e^{2z} - 1}.$$

Indicate the range of values of $|z|$ for which your series is valid.

(c) Let:

$$g(z) = \frac{1}{2z} + \sum_{k=1}^{m} \frac{z}{z^2 + \pi^2 k^2}.$$

Classify the singularities of $F(z) = f(z) - g(z)$ for $|z| < (m+1)\pi$.

(d) By considering:

$$\oint_{C_R} \frac{F(z)}{z^2}\, dz,$$

where $C_R = \{|z| = R\}$ for some suitable chosen $R > 0$, show that:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Solution: (a) Let $f : A \to \mathbb{C}$ be an analytic function on the annulus $\{r < |z - z_0| < R\}$. The *Laurent series* of f about $z = z_0$ is the unique series representation of f on A of the form:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where the coefficients are given by:

$$a_n = \frac{1}{2\pi i} \oint_{|w - w_0| = \rho} \frac{f(w)}{(z - w)^{n+1}}\, dw,$$

with $\rho \in (r, R)$. Such a representation exists and is unique by *Laurent's theorem*, proved in the course.

(b) An efficient method is just to use series expansions. We have:

$$\left(e^{2z} - 1\right)^{-1} = \left(2z + \frac{4z^2}{2!} + \frac{8z^3}{3!} + \cdots\right)^{-1} = \frac{1}{2z}\left(1 + \frac{2z}{2!} + \frac{4z^2}{3!} + \cdots\right)^{-1}.$$

Using the binomial theorem, we have for z sufficiently small:

$$\frac{1}{2z}\left(1 - \left(\frac{2z}{2!} + \frac{4z^2}{3!} + \cdots\right) + \left(\frac{2z}{2!} + \frac{4z^2}{3!} + \cdots\right)^2 + \cdots\right)$$

$$= \frac{1}{2z} - \frac{1}{2} + \frac{1}{2}\left(-\frac{4}{3!} + \left(\frac{2}{2!}\right)^2\right) z + \cdots,$$

so that the first three terms are:
$$\frac{1}{2z} - \frac{1}{2} + \frac{z}{6} + \cdots.$$
Since f is analytic on the annulus $0 < |z| < \pi$, but is not analytic on a larger annulus (since $e^{2\pi i} = 1$), this is the radius of convergence of the series (by the uniqueness of Laurent series on a given annulus).

(c) First, let's classify the singularities individually:

- The singularities of $g(z)$ occur precisely at $z = 0$ and $z = \pm i\pi n$ for $n = 1, ..., m$; all of these and the singularities are simple poles. The residue at $z = 0$ is evidently $1/2$ and the residue at $z = \pm i\pi n$ is given by:
$$\lim_{z \to \pm i\pi n} \left[\frac{(z \mp \pi i k)}{2z} + \sum_{k=1}^{m} \frac{z(z \mp \pi i n)}{(z \mp \pi i k)(z \pm \pi i k)} \right] = \frac{\pm i\pi n}{\pm 2\pi i n} = \frac{1}{2}.$$
- The singularities of $f(z)$ occurs precisely at $z = n\pi i$ for $n \in \mathbb{Z}$; again, all of these singularities are simple poles. We saw above that the residue at $z = 0$ is $1/2$. Similarly, we have:
$$\lim_{z \to \pi i n} \left[\frac{z - \pi i n}{e^{2z} - 1} \right] = \lim_{z \to \pi i n} \left[\frac{1}{2e^{2z}} \right] = \frac{1}{2}.$$

We now consider:
$$F(z) = f(z) - g(z).$$
We have just seen that for $|z| < (m+1)\pi$, the singularities of f, g coincide at $z = \pi i n$ for $n = 0, 1, ..., m$. Furthermore they are of the same nature for both f, g (they are all simple poles) and all have the same residues $(1/2)$. In particular, in a Laurent expansion of both f and g about $z = \pi i n$, the terms involving $1/(z - \pi i n)$ will necessarily all cancel. It follows that *all singularities of the function F in the region $|z| < (m+1)\pi$ are removable singularities*.

(d) Choose $R = (m + 1/2)\pi$. We have just seen that we may treat the integrand as analytic at all singularities except $z = 0$. By the residue theorem then, we have:
$$\oint_{|z|=(m+1/2)\pi} \frac{F(z)}{z^2}\, dz = 2\pi i \mathrm{Res}\left(\frac{F(z)}{z^2}; 0 \right).$$

To compute the residue, note that the Laurent series of $F(z)/z^2$ about $z = 0$ is given by:
$$\begin{aligned}
\frac{f(z) - g(z)}{z^2} &= \left(\frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{6z} + \cdots \right) - \left(\frac{1}{2z^3} + \sum_{k=1}^{m} \frac{1}{z(z^2 + \pi^2 k^2)} + \cdots \right) \\
&= \cdots + \left(\frac{1}{6z} - \frac{1}{z\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} \right) + \cdots
\end{aligned}$$

Hence we have:
$$\oint_{|z|=(m+1/2)\pi} \frac{F(z)}{z^2}\, dz = 2\pi i \left(\frac{1}{6} - \frac{1}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} \right).$$

On the other hand, we can bound the integral as $m \to \infty$ as follows. We note that:

$$\left| \oint_{|z|=(m+1/2)\pi} \frac{F(z)}{z^2}\, dz \right| \leq 2\pi^2(m+1/2) \sup_{|z|=(m+1/2)\pi} \left| \frac{F(z)}{z^2} \right| = \frac{2}{(m+1/2)} \sup_{|z|=(m+1/2)\pi} |F(z)|\,.$$

It remains to bound $|F(z)|$; this turns out to be rather involved. On $|z| = (m+1/2)\pi$, we have:

$$|F(z)| = |f(z) - g(z)| \leq |f(z)| + |g(z)| \leq \left| \frac{1}{e^{2z}-1} \right| + \frac{1}{2|z|} + \sum_{k=1}^{m} \frac{|z|}{|z^2+\pi^2 k^2|}.$$

We can further bound this by using the reverse triangle inequality, yielding:

$$|F(z)| \leq \frac{1}{|e^{2z}-1|} + \frac{1}{2|z|} + \sum_{k=1}^{m} \frac{|z|}{|z|^2 - \pi^2 k^2},$$

though we leave the exponential term for further work. We now bound each of these terms individually:

- The term $1/2|z| = 1/2(m+1/2)\pi$ obviously tends to zero as $m \to \infty$, so we are not concerned with this term.
- The term $1/|e^{2z}-1|$ can be bounded above by the following analysis. Let $z = x + iy$. Then:

$$\begin{aligned} |e^{2z}-1|^2 &= |e^{2(x+iy)}-1|^2 = |e^{2x}\cos(2y) - 1 + ie^{2x}\sin(2y)|^2 = (e^{2x}\cos(2y)-1)^2 + e^{4x}\sin^2(2y) \\ &= e^{4x} - 2e^{2x}\cos(2y) + 1. \end{aligned}$$

  Now recall that $|z| = (m+1/2)\pi$. In particular, this implies that $|y| \leq (m+1/2)\pi$, and we see that we can split into two cases depending on whether $\cos(2y)$ is positive or negative:

  (i) In the case $(m+1/4)\pi \leq |y| \leq (m+1/2)\pi$, we have $\cos(2y) \leq 0$, so $-\cos(2y) \geq 0$. Thus we have:

  $$|e^{2z}-1|^2 \geq e^{4x} + 1 \geq 1,$$

  on these arcs of the circle $|z| = (m+1/2)\pi$.

  (ii) In the case $|y| \leq (m+1/4)\pi$, we have $0 \leq \cos(2y) \leq 1$, so $-\cos(2y) \geq -1$. It follows that $|e^{2z}-1|^2 \geq e^{4x} - 2e^{2x} + 1 = (e^{2x}-1)^2$.

  Since $|y| \leq (m+1/4)\pi$, we have that $|x|$ must be bounded below. Explicitly, we have:

  $$|x| = \sqrt{|z|^2 - |y|^2} \geq \sqrt{(m+1/2)^2\pi^2 - (m+1/4)^2\pi^2} = \pi\sqrt{m/2 + 3/16}.$$

  If $x > \pi\sqrt{m/2+3/16}$, we have:

  $$|e^{2z}-1|^2 \geq (e^{2x}-1)^2 \geq (e^{2\pi\sqrt{m/2+3/16}} - 1)^2 \to \infty$$

  as $m \to \infty$. On the other hand if $x < -\pi\sqrt{m/2+3/16}$, we have:

  $$|e^{2z}-1|^2 \geq (e^{2x}-1)^2 \geq (e^{-2\pi\sqrt{m/2+3/16}} - 1)^2 \to 1$$

  as $m \to \infty$.

  Putting all the cases together, we see that $1/|e^{2z}-1| \leq 1$ as $m \to \infty$.

- Finally, we must bound the complicated term:

$$\sum_{k=1}^{m} \frac{|z|}{|z|^2 - \pi^2 k^2} = \frac{(m+1/2)}{\pi} \sum_{k=1}^{m} \frac{1}{(m+1/2)^2 - k^2}.$$

This is particularly problematic since m appears both in the limits of the sum and in the summand itself. To obtain a bound, we consider an integral bound for the sum. We use:

$$\sum_{k=1}^{m} \frac{1}{(m+1/2)^2 - k^2} \leq \frac{1}{(m+1/2)^2 - m^2} + \int_0^m \frac{dk}{(m+1/2)^2 - k^2},$$

which can be obtained from a well-drawn diagram (the lower limit is chosen to make the integral easier, but technically it could start at 1). To perform the integral, make the substitution $k = (m+1/2)\tanh(u)$, reducing the right hand side to:

$$\frac{1}{m+1/4} + \frac{1}{m+1/2} \int_0^{\operatorname{artanh}(m/(m+1/2))} \frac{\operatorname{sech}^2(u)du}{1-\tanh^2(u)} = \frac{1}{m+1/4} + \frac{1}{m+1/2} \operatorname{artanh}\left(\frac{m}{m+1/2}\right)$$

Therefore overall we have:

$$\frac{(m+1/2)}{\pi} \sum_{k=1}^{m} \frac{1}{(m+1/2)^2 - k^2} \leq 1 + \operatorname{artanh}(1),$$

as $m \to \infty$.

Putting everything together, we see that $|F(z)|$ is uniformly bounded as $m \to \infty$, for $|z| = (m+1/2)\pi$. This is sufficient to conclude that the integral converges to zero as $m \to \infty$ - phew!

❊ **Comments from the Examiner:** *I expected this question to be answered better - a significant number of candidates could not answer the bookwork part (a). Determining that all singularities of F were removable was also found tricky, which made completing the rest of the question hard going.*

2018, Paper 3, Section II, 13F

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and let $f : D \to \mathbb{C}$ be analytic.

(i) If there is a point $a \in D$ such that $|f(z)| \leq |f(a)|$ for all $z \in D$, prove that f is constant.

(ii) If $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in D$, prove that $|f(z)| \leq |z|$ for all $z \in D$.

(iii) Show that there is a constant C independent of f such that if $f(0) = 1$ and $f(z) \notin (-\infty, 0]$ for all $z \in D$, then $|f(z)| \leq C$ whenever $|z| \leq 1/2$. *[Hint: you may find it useful to consider the principal branch of the map* $z \mapsto z^{1/2}$.*]*

(iv) Does the conclusion in (c) hold if we replace the hypothesis $f(z) \notin (-\infty, 0]$ for $z \in D$ with the hypothesis that $f(z) \neq 0$ for $z \in D$, and keep all other hypotheses? Justify your answer.

➾ **Solution:** (i) This part is bookwork: it is just the maximum modulus principle. We recall the proof:

The local maximum principle: Let $f : D(a,r) \to \mathbb{C}$ be holomorphic. If for every $z \in D(a,r)$, we have $|f(z)| \leq |f(a)|$, then f is constant. In other words, a non-constant function cannot achieve an interior local maximum.

Proof: By the mean value property (which can be proved swiftly by simply applying the Cauchy integral formula for a disk, and parametrising) we have for $0 < \rho < r$:

$$|f(a)| = \left| \int_0^1 f(a + \rho e^{2\pi i t})\, dt \right| \leq \int_0^1 |f(a + \rho e^{2\pi i t})|\, dt \leq \sup_{|z-a|=\rho} |f(z)| \leq |f(a)|,$$

by hypothesis. Hence all of these equalities are in fact equalities, and it follows that:

$$\int_0^1 \left(|f(a)| - |f(a + \rho e^{2\pi i t})| \right) dt = 0.$$

Further, the integrand is non-negative by hypothesis, since $z = a$ is a local maximum. Thus $|f(a)| = |f(a + \rho e^{2\pi i t})|$ for all t. Further, this holds for all $\rho \in (0, r)$ since ρ was arbitrary. Thus $|f|$ is constant throughout $D(a,r)$ and equal to $|f(a)|$. It follows from Examples Sheet 1 that f itself is constant. □

The result applies to any $a \in D$, not just the centre 0 of the disk D, since about any $a \in D$ we can find a disk contained in D to which the above theorem applies; it follows that the function is constant on a disk centred on $a \in D$. Then the identity theorem for holomorphic functions implies that the function is constant everywhere in D.

(ii) This part is just *Schwarz's lemma for a disk*, from Examples Sheet 2, Question 7(i); the proof is as follows. Consider the function $g : D \to \mathbb{C}$ defined by:

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } z \neq 0; \\ f'(0) & \text{if } z = 0. \end{cases}$$

Clearly g is holomorphic everywhere in D except possibly 0. At 0, either note the singularity is removable, or write $f(z)$ as its Taylor series $f(z) = f'(0)z + z^2 h(z)$ for some holomorphic function $h(z)$ to note that g is complex differentiable:

$$\lim_{z\to 0} \left[\frac{g(z) - g(0)}{z - 0} \right] = \lim_{z\to 0} \left[\frac{f(z)/z - f'(0)}{z} \right] = \lim_{z\to 0} \left[\frac{zh(z)}{z} \right] = h(0).$$

It follows by the maximum principle that for any $0 < r < 1$, the function $|g| : \overline{D(0,r)} \to \mathbb{R}$ on the closed disc of radius r attains its maximum on the boundary, $\partial D(0,r)$. Hence for all $z \in \overline{D(0,r)}$, we have:

$$|g(z)| = \left|\frac{f(z)}{z}\right| \leq \left|\frac{f(z_0)}{z_0}\right| = \frac{|f(z_0)|}{r}$$

for some point on the boundary $z_0 \in \partial D(0,r)$. Recalling that $|f(z_0)| \leq 1$, we have:

$$\left|\frac{f(z)}{z}\right| \leq \frac{1}{r} \qquad \Rightarrow \qquad |f(z)| \leq \frac{|z|}{r},$$

for all $z \in \overline{D(0,r)}$. This was true for arbitrary $r \in (0,1)$, so take the limit as $r \to 1^-$ to deduce that $|f(z)| \leq |z|$ for all $z \in D$ as required.

(iii) Since $f(z) \notin (-\infty, 0]$, we have that f is an analytic map $f : D \to \mathbb{C}\backslash(-\infty, 0]$. In particular, the image of f lies in the domain of the principal branch of the square root $z \mapsto z^{1/2}$, so we may consider $f^{1/2} : D \to \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ as a map to the right half plane.

Ideally, we would like to apply Schwarz's lemma from part (ii). In order to do so, we map the right half plane to the open unit disk using a Möbius map; in particular we will choose a Möbius map such that $1 \mapsto 0$, since then the composition of all our transformations will send 0 to 0 overall. Such a Möbius map is given by:

$$z \mapsto \frac{z-1}{z+1},$$

as we saw on Examples Sheet 1, Question 7(i). It follows that:

$$\frac{\sqrt{f(z)}-1}{\sqrt{f(z)}+1}$$

is a holomorphic map from D to D, sending 0 to 0. By Schwarz's lemma, we have:

$$\left|\frac{\sqrt{f(z)}-1}{\sqrt{f(z)}+1}\right| \leq |z|.$$

for all $z \in D$. We now perform some manipulation with this inequality; first note that it is equivalent to:

$$\left|\sqrt{f(z)}-1\right| \leq |z| \left|\sqrt{f(z)}+1\right|$$

Using the reverse triangle inequality on the left, and the triangle inequality on the right, we have:

$$\left|\sqrt{f(z)}\right| - 1 \leq |z| \left(\left|\sqrt{f(z)}\right| + 1\right).$$

Rearranging, we have:

$$\left|\sqrt{f(z)}\right|(1-|z|) \leq 1+|z| \qquad \Rightarrow \qquad \left|\sqrt{f(z)}\right| \leq \frac{1+|z|}{1-|z|} \qquad \Rightarrow \qquad |f(z)| \leq \left(\frac{1+|z|}{1-|z|}\right)^2,$$

where we use the fact that $|z| < 1$ to perform the division and get the direction of the inequality correct. This inequality holds for all $z \in D$; to finish, note that for $|z| \leq 1/2$, we have $1+|z| \leq 3/2$ and $1-|z| \geq 1/2$. It follows that:

$$|f(z)| \leq \left(\frac{3/2}{2}\right)^2 = \frac{9}{16}.$$

(iv) We can come up with a counterexample fairly quickly; consider $f_n(z) = e^{2nz}$ for $n = 1, 2, \ldots$. We see that $f_n(0) = 1$ and $f_n(z) \neq 0$ throughout the unit disk. But for $z = 1/2$, we have:

$$|f_n(z)| = e^{2n/2} = e^n,$$

so that no constant independent of f can bound f on $|z| \leq 1/2$.

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(a) Let $\Omega \subset \mathbb{C}$ be open, $a \in \Omega$, and suppose that $D_\rho(a) = \{z \in \mathbb{C} : |z-a| \leq \rho\} \subset \Omega$. Let $f : \Omega \to \mathbb{C}$ be analytic.

State the Cauchy integral formula expressing $f(a)$ as a contour integral over $C = \partial D_\rho(a)$. Give, without proof, a similar expression for $f'(a)$.

If additionally $\Omega = \mathbb{C}$ and f is bounded, deduce that f must be constant.

(b) If $g = u + iv : \mathbb{C} \to \mathbb{C}$ is analytic where u, v are real, and if $u^2(z) - u(z) \geq v^2(z)$ for all $z \in \mathbb{C}$, show that g is constant.

➾ **Solution:** (a) We recall Cauchy's integral formula for a disk:

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz.$$

Cauchy's derivative formula tells us that:

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\, dz.$$

We now apply these formulae to a standard proof of Liouville's theorem. For any $a \in \mathbb{C}$ and any $R > 0$, we have:

$$|f'(a)| \leq \frac{2\pi R}{2\pi} \sup_{|z|=R} \left| \frac{f(z)}{(z-a)^2} \right|$$

by the standard integral estimate. Assuming $|f(z)| \leq M$ everywhere, i.e. f is bounded, we have by the reverse triangle inequality:

$$|f'(a)| \leq \frac{2\pi R}{2\pi} \frac{M}{R^2 - |a|^2} \to 0$$

as $R \to \infty$ (the inequality holds for all R). Hence $f'(a) = 0$ for all $a \in \mathbb{C}$, which is sufficient to deduce that f is constant everywhere, since $\mathbb{C}$ is connected.

(b) As a figure in the (u, v) plane, $v^2 = u^2 - u$ has no graph in the region $0 < u < 1$, since $u^2 - u < 0$ there. In particular, this implies that $|f(z) - 1/2| \geq 1/2$ for all z, which can be checked explicitly from the given inequality:

$$|f(z) - 1/2| = |u + iv - 1/2| = \sqrt{(u-1/2)^2 + v^2} = \sqrt{u^2 - u + 1/4 + v^2} \geq \sqrt{2v^2 + 1/4} \geq 1/2.$$

It follows that $g(z) = 1/(f(z) - 1/2)$ is entire, since f is entire and never takes the value $1/2$. Furthermore, g is bounded since:

$$|g(z)| = \frac{1}{|f(z) - 1/2|} \leq 2.$$

Thus g is a constant by part (a) (Liouville's theorem). It follows that $f(z) - 1/2$ is constant, and hence $f(z)$ is constant.