Part II: Further Complex Methods
Examples Sheet 1 Solutions

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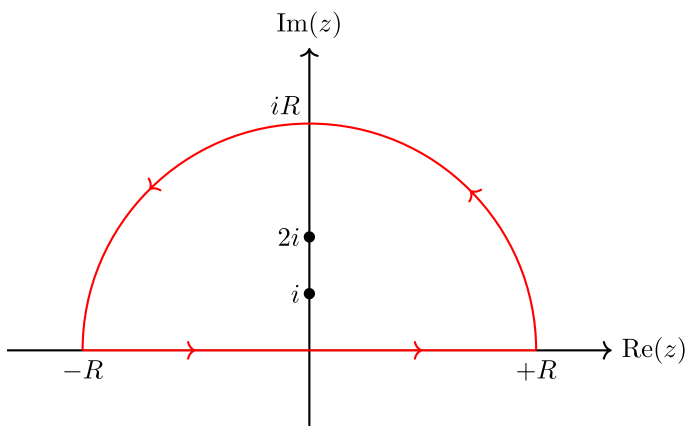
1. Show that:

(i) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2(x^2+4)} = \frac{\pi}{18}$, (ii) $\displaystyle\int_{-\infty}^\infty \frac{\cos(x)\,dx}{x^2+a^2} = \frac{\pi}{a}e^{-a}$, where $a>0$, (iii) $\displaystyle\int_{-\infty}^\infty \frac{x-\sin(x)}{x^3}\,dx = \frac{\pi}{2}$.

Solution: (i) Consider the contour integral

$$\oint_C \frac{dz}{(z^2+1)^2(z^2+4)},$$

where the contour C is defined by a line segment from $-R$ to R, followed by a semicircular arc of radius R centred on 0 from R to $-R$ in the upper half plane. This contour is usually referred to as 'closing in the upper half plane'.



The integrand has a simple pole at $z=2i$ and a double pole at $z=i$, which are enclosed by the contour C as shown. To find the residues at these poles, we recall the formula:

Theorem: The residue of a meromorphic function $f(z)$ at an Nth order pole $z=z_0$ is given by

$$\mathrm{Res}(f(z);z_0) = \lim_{z\to z_0}\frac{1}{(N-1)!}\left[\frac{d^{N-1}}{dz^{N-1}}\left((z-z_0)^N f(z)\right)\right].$$

Hence the residues we need are:

$$\mathrm{Res}\left(\frac{1}{(z^2+1)^2(z^2+4)};2i\right) = \lim_{z\to 2i}\left[\frac{z-2i}{(z^2+1)^2(z^2+4)}\right] = -\frac{i}{36},$$

$$\mathrm{Res}\left(\frac{1}{(z^2+1)^2(z^2+4)};i\right) = \lim_{z\to i}\left[\frac{d}{dz}\left(\frac{(z-i)^2}{(z^2+1)^2(z^2+4)}\right)\right] = \lim_{z\to i}\left[\frac{-2(z+i)(z^2+4)-2z(z+i)^2}{(z+i)^4(z^2+4)^2}\right] = -\frac{i}{36}.$$

It follows, by the residue theorem, that the value of the contour integral is:

$$\oint_C \frac{dz}{(z^2+1)^2(z^2+4)} = 2\pi i\left(-\frac{i}{36}-\frac{i}{36}\right) = \frac{\pi}{9}.$$

We now evaluate the contour integral in a different way, by separating the integral into integrals over each part of the contour. Let C_1 be the straight line segment from $-R$ to R and let C_2 be the semicircular arc from R to $-R$. Then

$$\oint_C \frac{dz}{(z^2+1)^2(z^2+4)} = \int_{C_1} \frac{dz}{(z^2+1)^2(z^2+4)} + \int_{C_2} \frac{dz}{(z^2+1)^2(z^2+4)}.$$

We can parametrise the contour C_1 as $z = x$, for $x \in [-R, R]$. In particular the measure becomes: $dz = dx$. Similarly, we can parametrise the contour C_2 as $z = Re^{i\theta}$, for $\theta \in [0, \pi]$. The measure becomes $dz = iRe^{i\theta}d\theta$.

Therefore, we have:

$$\oint_C \frac{dz}{(z^2+1)^2(z^2+4)} = \int_{-R}^{R} \frac{dx}{(x^2+1)^2(x^2+4)} + \int_0^{\pi} \frac{iRe^{i\theta}d\theta}{(R^2e^{2i\theta}+1)^2(R^2e^{2i\theta}+4)}.$$

In the limit as $R \to \infty$, the first integral on the right hand side becomes something close to the real integral we want. The second integral is of order $O(1/R^5)$, and hence vanishes as $R \to \infty$. It follows that we have:

$$\lim_{R\to\infty} \oint_C \frac{dz}{(z^2+1)^2(z^2+4)} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)}.$$

But we also know that the contour integral around the whole of C is independent of R; we calculated it to be $\pi/9$ by the residue theorem. Substituting this into the left hand side, we have the result:

$$\frac{\pi}{9} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)}.$$

To finish, notice that the integrand on the right hand side is even. Therefore, we have:

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)} = 2\int_0^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)}.$$

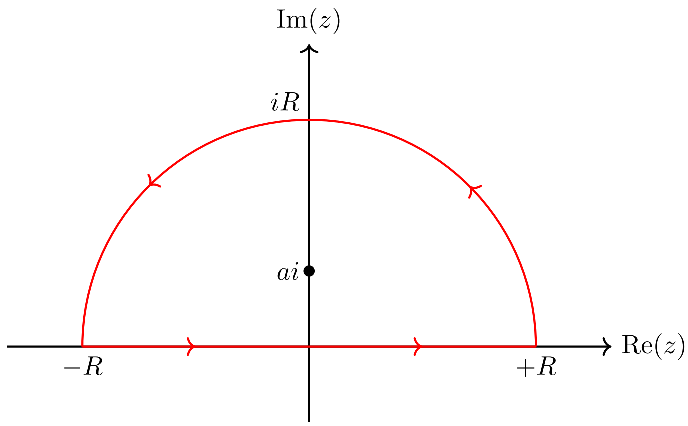
Hence we have the required integral:

$$\int_0^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)} = \frac{\pi}{18}.$$

(ii) Consider the contour integral

$$\oint_C \frac{e^{iz}}{z^2+a^2}\,dz$$

where we have closed the contour in the upper half plane, as in (i).



The integrand has a simple pole at $z = ai$, which is enclosed by the contour C as shown (we are given that $a > 0$). The residue at the pole is given by:

$$\operatorname{Res}\left(\frac{e^{iz}}{z^2+a^2}; i\right) = \lim_{z\to ai}\left[\frac{(z-ai)e^{iz}}{z^2+a^2}\right] = -\frac{ie^{-a}}{2a}.$$

Hence by the residue theorem, we have

$$\oint_C \frac{e^{iz}\,dz}{z^2+a^2} = 2\pi i\left(-\frac{ie^{-a}}{2a}\right) = \frac{\pi}{a}e^{-a}.$$

Now consider evaluating the contour integral over each part of the contour separately. Let C_1 be the straight line segment from $-R$ to R, and let C_2 be the semicircular arc from R to $-R$. On C_1 we have the usual parametrisation $z = x$, $x \in [-R, R]$, with measure $dz = dx$. Therefore we can write the contour integral as:

$$\oint_C \frac{e^{iz}\,dz}{z^2+a^2} = \int_{-R}^{R} \frac{e^{ix}\,dx}{x^2+a^2} + \int_{C_2} \frac{e^{iz}\,dz}{z^2+a^2}.$$

We now take the limit as $R \to \infty$. The first integral on the right hand side becomes something close to the real integral we want (we simply have to take a real part). The second integral tends to 0 as $R \to \infty$ by a result from Part IB Complex Methods - *Jordan's Lemma*:

> **Jordan's Lemma:** Suppose that $\lambda > 0$, and let S_R be a semicircular contour from R to $-R$ centred on 0 and of radius R in the upper half plane. Suppose that $f(z)$ is a complex-valued function, and let $\max_{z\in S_R}|f(z)| = M_R$. Then
>
> $$\left|\int_{S_R} e^{i\lambda z} f(z)\,dz\right| \le \frac{\pi}{\lambda}M_R.$$
>
> In particular, the integral of $e^{i\lambda z}f(z)$ over S_R converges to zero as $R \to \infty$ if $M_R \to 0$ as $R \to \infty$.

Proof: Parametrise the integral using $z = Re^{i\theta}$, $\theta \in [0, \pi]$ so that the measure becomes $dz = iRe^{i\theta}d\theta$. Then the integral we need to take the limit of is:

$$\int_0^\pi e^{i\lambda Re^{i\theta}} f(Re^{i\theta}) iRe^{i\theta}\, d\theta.$$

Using Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ we can expand the exponent of the exponential to get:

$$\int_0^\pi e^{iR\lambda\cos(\theta) - R\lambda\sin(\theta)} f(Re^{i\theta}) iRe^{i\theta}\, d\theta.$$

We can bound this as follows. First, using a result from Part IB Analysis, we have:

$$\left| \int_0^\pi e^{iR\lambda\cos(\theta) - R\lambda\sin(\theta)} f(Re^{i\theta}) iRe^{i\theta}\, d\theta \right| \leq R \int_0^\pi e^{-R\lambda\sin(\theta)} |f(Re^{i\theta})|\, d\theta.$$

Using $M_R = \max\limits_{z \in R} |f(z)|$, we can further bound this integral as:

$$\int_0^\pi e^{-R\lambda\sin(\theta)} |f(Re^{i\theta})| R\, d\theta \leq RM_R \int_0^\pi e^{-R\lambda\sin(\theta)}\, d\theta.$$

It remains to approximate the θ integral, which no longer depends on f. To do so, notice that $\sin(\theta)$ is symmetric about $\pi/2$, so we have

$$\int_0^\pi e^{-R\lambda\sin(\theta)}\, d\theta = 2\int_0^{\pi/2} e^{-R\lambda\sin(\theta)}\, d\theta.$$

Now notice (e.g. via a quick sketch) that $\sin(\theta) \geq 2\theta/\pi$ on $[0, \pi/2]$, and hence $-\sin(\theta) \leq -2\theta/\pi$ on $[0, \pi/2$. It follows that we can bound the final integral as:

$$2\int_0^{\pi/2} e^{-R\lambda\sin(\theta)}\, d\theta \leq 2\int_0^{\pi/2} e^{-2R\lambda\theta/\pi}\, d\theta = \frac{\pi(1 - e^{-R\lambda})}{R\lambda} \leq \frac{\pi}{R\lambda}.$$

Putting together all our inequalities, we see that

$$\left| \int_{S_R} e^{i\lambda z} f(z)\, dz \right| \leq RM_R \cdot \frac{\pi}{R\lambda} = \frac{\pi}{\lambda} M_R,$$

as required. □

In our particular integral, we have

$$f(z) = \frac{1}{z^2 + a^2} \qquad \Rightarrow \qquad |f(z)| = \frac{1}{|z^2 + a^2|},$$

which certainly converges to zero as $|z| \to \infty$. Therefore, Jordan's Lemma applies.

We have therefore shown that:

$$\lim_{R\to\infty} \oint\limits_C \frac{e^{iz}\,dz}{z^2+a^2} = \int\limits_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x^2+a^2}.$$

But of course, our earlier work using the residue theorem showed that the integral on the left hand side was constant and equal to $\pi e^{-a}/a$. Substituting this result in, we see that

$$\frac{\pi}{a}e^{-a} = \int\limits_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x^2+a^2}.$$

To get the integral required in the question, simply take the real part of this equation:

$$\int\limits_{-\infty}^{\infty} \frac{\cos(x)\,dx}{x^2+a^2} = \frac{\pi}{a}e^{-a}.$$

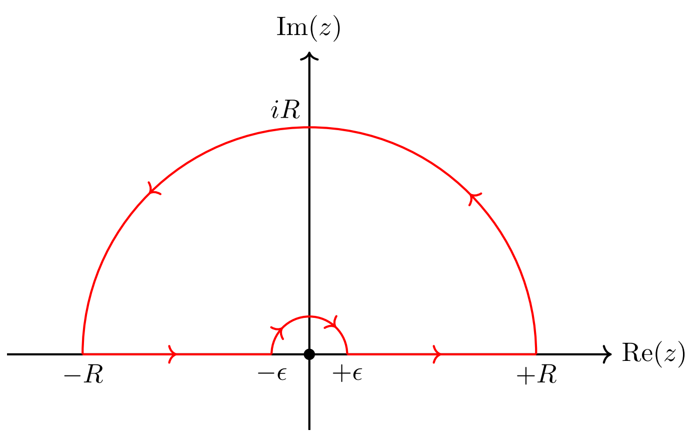
(iii) With a view to taking the imaginary part at the end, consider the contour integral

$$\oint\limits_C \frac{iz-e^{iz}}{z^3}\,dz.$$

Here, the integrand has a triple pole at $z=0$, since

$$\frac{iz-e^{iz}}{z^3} = \frac{iz-1-iz-\frac{1}{2}(iz)^2-\frac{1}{6}(iz)^3-\ldots}{z^3} = -\frac{1}{z^3}+\frac{1}{2z}+\textit{finite terms}.$$

Therefore, we must pick our contour C to avoid $z=0$. We do so by defining C to be a straight line segment from $-R$ to $-\epsilon$ (for ϵ small), followed by a semicircular arc centred on 0 and of radius ϵ in the upper half plane from $-\epsilon$ to ϵ, then another straight line segment from ϵ to R, followed by a final semicircular arc centred on 0 and of radius R in the upper half plane from R to $-R$.



There are no singularities inside the contour of integration, so by the residue theorem we have

$$\oint\limits_C \frac{iz-e^{iz}}{z^3}\,dz = 0.$$

Now consider performing the integral on each part of the contour separately. With the appropriate parametrisation on each piece, we find that the integral can be written as:

$$\int_C \frac{iz - e^{iz}}{z^3}\, dz = \int_{-R}^{-\epsilon} \frac{ix - e^{ix}}{x^3}\, dx + \int_{\pi}^{0} \frac{i\epsilon e^{i\theta} - e^{i\epsilon e^{i\theta}}}{\epsilon^3 e^{3i\theta}}\, i\epsilon e^{i\theta}\, d\theta + \int_{\epsilon}^{R} \frac{ix - e^{ix}}{x^3}\, dx + \int_0^{\pi} \frac{iRe^{i\theta} - e^{iRe^{i\theta}}}{R^3 e^{3i\theta}}\, iRe^{i\theta}\, d\theta.$$

The first and the third integrals are related to the one we want to find in the limit as $R \to \infty$ and $\epsilon \to 0$. The fourth integral goes to zero as $R \to \infty$, since

$$\int_0^{\pi} \frac{iRe^{i\theta} - e^{iRe^{i\theta}}}{R^3 e^{3i\theta}}\, iRe^{i\theta}\, d\theta = \int_0^{\pi} -\frac{1}{Re^{i\theta}} - \frac{ie^{iR\cos(\theta)}e^{-R\sin(\theta)}}{R^2 e^{2i\theta}}\, d\theta = O\left(\frac{1}{R}\right) + O\left(\frac{1}{R^2}\right),$$

where in particular we used the fact that $\sin(\theta) \geq 0$ on the range of integration $\theta \in [0, \pi]$.

Finally, we need to evaluate the second integral (i.e. the one over the small arc). We have:

$$\int_{\pi}^{0} \frac{i\epsilon e^{i\theta} - e^{i\epsilon e^{i\theta}}}{\epsilon^3 e^{3i\theta}}\, i\epsilon e^{i\theta}\, d\theta = \int_0^{\pi} d\theta \left(\frac{e^{-i\theta}}{\epsilon} + \frac{ie^{i\epsilon e^{i\theta}}}{\epsilon^2} e^{-2i\theta}\right).$$

The exponential containing an ϵ in the exponent can be expanded:

$$e^{i\epsilon e^{i\theta}} = 1 + i\epsilon e^{i\theta} - \frac{1}{2}\epsilon^2 e^{2i\theta} - \frac{i}{6}\epsilon^3 e^{3i\theta} + \cdots$$

and since we are going to take the limit as $\epsilon \to 0$, we ignore terms of order $O(\epsilon)$. We're left with:

$$\int_0^{\pi} d\theta \left(\frac{ie^{-2i\theta}}{\epsilon^2} - \frac{i}{2}\right).$$

Performing the integral, we have

$$\int_0^{\pi} d\theta \left(\frac{ie^{-2i\theta}}{\epsilon^2} - \frac{i}{2}\right) = \frac{i}{\epsilon^2}\left[\frac{e^{-2i\theta}}{-2i}\right]_0^{\pi} - \frac{\pi i}{2} = -\frac{\pi i}{2}.$$

This is the finite limit as $\epsilon \to 0$. Hence we see that:

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_C \frac{iz - e^{iz}}{z^3}\, dz = \int_{-\infty}^{\infty} \frac{ix - e^{ix}}{x^3}\, dx - \frac{\pi i}{2}.$$

But we know from the residue theorem that the integral on the left is independent of R and ϵ, and is always equal to 0. Hence substituting this in, we have:

$$\int_{-\infty}^{\infty} \frac{ix - e^{ix}}{x^3}\, dx = \frac{\pi i}{2}.$$

Taking the imaginary part of both sides, we get

$$\int_{-\infty}^{\infty} \frac{x - \sin(x)}{x^3}\, dx = \frac{\pi}{2},$$

as required.

2. Show that:

(a) $\displaystyle\int_0^\infty \frac{x^\beta}{1+x}\,dx$, where $\beta \in \mathbb{C}$, converges for $-1 < \mathrm{Re}(\beta) < 0$.

(b) $\displaystyle\int_\gamma (1+\tanh(z))\,dz$, where γ is the path $\{z = se^{i\alpha} : 0 \leq s < \infty\}$ and $\alpha \in \mathbb{R}$, converges for

$$\alpha \in (-\pi, -\pi/2) \cup (\pi/2, \pi).$$

Does the integral converge at $\alpha = \pi$?

Solution: (a) In these sorts of problems, it's useful to think about what could stop the integral from converging. There are really only two reasons why an integral might diverge: (i) the integrand is singular somewhere in the integration range; (ii) the integration range is infinitely large.

For this integral, the integration range is from 0 to ∞ so we immediately notice that we could get a divergence from the infinite range. Furthermore, we notice that the integrand

$$\frac{x^\beta}{1+x}$$

is finite for all values of x except $x = -1$ and possibly $x = 0$ (say if $\beta = -2$). Since $x = -1$ is outside of the range of integration, only $x = 0$ could cause problems with convergence.

What we do next depends on how rigorous we want to be. Here's two arguments with both ends of the spectrum of rigour:

- 1. Further Complex Methods rigour. Let's look at the problem regions of integration. Near $x = 0$, we have $1 + x \approx 1$, and hence the lower limit of integration gives us an approximate contribution

$$\int_0 \frac{x^\beta}{1+x}\,dx \approx \int_0 x^\beta\,dx = \begin{cases} \left.\dfrac{x^{\beta+1}}{\beta+1}\right|_{x=0} & \text{if } \beta \neq -1; \\ \left.\log(x)\right|_{x=0} & \text{if } \beta = -1. \end{cases}$$

We're only interested in the case $\mathrm{Re}(\beta) > -1$, so we can ignore the logarithmic case. Notice that when $\mathrm{Re}(\beta) > -1$, we have:

$$\lim_{x\to 0} \frac{x^{\beta+1}}{\beta+1} = \lim_{x\to 0} \frac{x^{\mathrm{Re}(\beta)+1+i\,\mathrm{Im}(\beta)}}{\beta+1} = 0,$$

since $x^{\mathrm{Re}(\beta)+1} \to 0$ as $x \to 0$ when $\mathrm{Re}(\beta) > -1$. Hence we have a finite contribution provided $\mathrm{Re}(\beta) > -1$.

Similarly, near $x = \infty$, we have $1 + x \approx x$. Therefore the upper limit of integration gives us an approximate contribution

$$\int^\infty \frac{x^\beta}{1+x}\,dx \approx \int^\infty x^{\beta-1}\,dx = \left.\frac{x^\beta}{\beta}\right|_{x=\infty}.$$

We need this to be finite as $x \to \infty$. We notice that in the case $\mathrm{Re}(\beta) < 0$, we have:

$$\lim_{x\to\infty} \frac{x^\beta}{\beta} = \lim_{x\to\infty} \frac{x^{\mathrm{Re}(\beta)+i\,\mathrm{Im}(\beta)}}{\beta} = 0,$$

since $x^{\mathrm{Re}(\beta)} \to 0$ as $x \to \infty$ for $\mathrm{Re}(\beta) < 0$, and the phase $x^{i\,\mathrm{Im}(\beta)}$ is bounded as $x \to \infty$.

It follows that the integral converges for $-1 < \mathrm{Re}(\beta) < 0$, as required.

- 2. More rigorous method. Let's bound the integral as follows:

$$\left|\int_\epsilon^R \frac{x^\beta}{1+x}\,dx\right| \leq \int_\epsilon^R \left|\frac{x^\beta}{1+x}\right|\,dx = \int_\epsilon^s \left|\frac{x^\beta}{1+x}\right|\,dx + \int_s^R \left|\frac{x^\beta}{1+x}\right|\,dx,$$

where s is some fixed, finite number. In the region $x \in [\epsilon, s]$, we have $1 + x \geq 1$, and hence

$$\frac{1}{1+x} \leq 1$$

in this region. For sufficiently large s, we have throughout the region $x \in [s, R]$ that $1 + x \geq \frac{1}{2}x$, and hence

$$\frac{1}{1+x} \leq \frac{2}{x}.$$

It follows that we can bound our integrals as:

$$\int_\epsilon^s \left|\frac{x^\beta}{1+x}\right|\,dx + \int_s^R \left|\frac{x^\beta}{1+x}\right|\,dx \leq \int_\epsilon^s |x^\beta|\,dx + 2\int_s^R |x^{\beta-1}|\,dx = \int_\epsilon^s x^{\mathrm{Re}(\beta)}\,dx + 2\int_s^R x^{\mathrm{Re}(\beta)-1}\,dx.$$

Assuming that $-1 < \mathrm{Re}(\beta) < 0$, we can perform the integrals on the right hand side to give:

$$\int_\epsilon^s x^{\mathrm{Re}(\beta)}\,dx + 2\int_s^R x^{\mathrm{Re}(\beta)-1}\,dx = \frac{s^{\mathrm{Re}(\beta)+1}}{\mathrm{Re}(\beta)+1} - \frac{\epsilon^{\mathrm{Re}(\beta)+1}}{\mathrm{Re}(\beta)+1} + 2\left(\frac{R^{\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)} - \frac{s^{\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\right).$$

This is completely finite as $\epsilon \to 0$, and as $R \to \infty$. Hence we've bounded the integral by a finite number, and it follows that the integral is convergent if $-1 < \mathrm{Re}(\beta) < 0$.

(b) There's a nice trick for the second part of this question. We notice that

$$1 + \tanh(z) = 1 + \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{2e^z}{e^z + e^{-z}} = \frac{2e^{2z}}{e^{2z}+1} = \frac{d}{dz}\left(\log\left(1 + e^{2z}\right)\right).$$

Hence we can perform the integral in question exactly. We have:

$$\int_\gamma (1 + \tanh(z))\,dz = \lim_{s\to\infty}\left[\log\left(1 + e^{2se^{i\alpha}}\right) - \log(1 + e^0)\right] = \lim_{s\to\infty}\left[\log\left(1 + e^{2se^{i\alpha}}\right) - \log(2)\right].$$

Let's examine the exponential $e^{2se^{i\alpha}}$ carefully. First, notice that we can expand $e^{i\alpha}$ using Euler's formula, and then separate out the exponential to get:

$$e^{2se^{i\alpha}} = e^{2s(\cos(\alpha)+i\sin(\alpha))} = e^{2s\cos(\alpha)}e^{2si\sin(\alpha)}.$$

If $\alpha \in (-\pi, -\pi/2) \cup (\pi/2, \pi)$, then $\cos(\alpha) < 0$. It follows that $e^{2s\cos(\alpha)} \to 0$. Since $e^{2si\sin(\alpha)}$ is a phase, it is bounded and hence

$$e^{2se^{i\alpha}} \to 0 \qquad \text{as } s \to \infty.$$

Hence the integral converges in the case $\alpha \in (-\pi, -\pi/2) \cup (\pi/2, \pi)$, with value:

$$\int_\gamma (1 + \tanh(z))\,dz = -\log(2).$$

When $\alpha = \pi$, we find $e^{2se^{i\alpha}} = e^{2se^{i\pi}} = e^{-2s} \to 0$ as $s \to \infty$ also, and hence the integral also converges to $-\log(2)$.

3. Let $f(t)$ be analytic at $t = 0$ with $f(0) = 0$ and $f'(0) \neq 0$. Let C be a circle centred on the origin, with interior D, such that f is analytic in D and the inverse of f exists on $f(D)$.

For a fixed point z within C, let $w = f(z)$. Assuming that w is small, show (using the residue theorem) that

$$z = \frac{1}{2\pi i} \oint_C \frac{tf'(t)}{f(t) - w}\, dt,$$

and hence that

$$z = \sum_{n=1}^{\infty} b_n w^n,$$

where

$$b_n = \frac{1}{2\pi i} \oint_C \frac{tf'(t)}{f(t)^{n+1}}\, dt = \frac{1}{2\pi i n} \oint_C \frac{1}{f(t)^n}\, dt = \frac{1}{n!} \lim_{t \to 0} \frac{d^{n-1}}{dt^{n-1}} \left(\frac{t}{f(t)} \right)^n.$$

Show that the equation $w = ze^{-z}$ has a solution for sufficiently small w (how small?),

$$z = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} w^n.$$

Find also one solution of $w = 2z - z^2$.

➻ **Solution:** The integral we are trying to compute is:

$$\oint_C \frac{tf'(t)}{f(t) - w}\, dt.$$

Since $f(t)$ is analytic in D, it follows that $tf'(t)$ is analytic in D. The integrand is therefore analytic everywhere except $f(t) = w$. Since f is one-to-one, we see that $t = z$ is the only solution to $f(t) = w$ on the disc D.

Furthermore, we have that $t = z$ is a simple pole. We can prove this as follows. First note that the finiteness of the limit

$$\lim_{t \to z} \left[\frac{tf'(t)(t-z)}{f(t) - w} \right] = zf'(z) \lim_{t \to z} \left[\frac{t - z}{f(t) - f(z)} \right] = \frac{zf'(z)}{f'(z)} = z$$

shows that $t = z$ is at most a simple pole. Indeed, we see that the only case in which $t = z$ is *not* a simple pole is if $z = 0$; in this case, the integrand has a *removable singularity* at $t = z$, and hence may be viewed as analytic on D - it follows that the integral is 0 and the result is true anyway.

In the case that $z \neq 0$, we have

$$\frac{1}{2\pi i} \oint_C \frac{tf'(t)}{f(t) - w}\, dt = \frac{1}{2\pi i} \cdot 2\pi i \mathrm{Res}(f(t); z) = z,$$

by the residue theorem. The result follows for all z.

To get the required series, we simply expand $1/(f(t) - w)$ as a geometric series:

$$\frac{1}{f(t) - w} = \frac{1}{f(t)} \cdot \frac{1}{1 - (w/f(t))} = \frac{1}{f(t)} \sum_{n=0}^{\infty} \left(\frac{w}{f(t)} \right)^n = \sum_{n=0}^{\infty} \frac{w^n}{f(t)^{n+1}}.$$

Hence we have:

$$z = \frac{1}{2\pi i}\oint\limits_C \frac{tf'(t)}{f(t)-w}\,dt = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint\limits_C \frac{tf'(t)}{f(t)^{n+1}}\,dt\right)w^n.$$

Thus we have derived the first form of the b_n.

We are asked to manipulate the b_n into some alternative forms. We have:

$$b_n = \frac{1}{2\pi i}\oint\limits_C \frac{tf'(t)}{f(t)^{n+1}}\,dt = \frac{1}{2\pi i}\oint\limits_C \frac{t}{n}\frac{d}{dt}\left(-\frac{1}{f(t)^n}\right)\,dt = \frac{1}{2\pi i n}\oint\limits_C \frac{1}{f(t)^n}\,dt,$$

where in the last equality we used integration by parts (note there is no boundary term since we are integrating round a circle). The only exception to this argument is when $n=0$; there, we need the integral of $tf'(t)/f(t)$ which we saw above was given by z such that $f(z)=0$. But $f(0)=0$ and f is a bijection on D, and it follows that $b_0=0$.

The final form of the b_n can be obtained by noting that we have an nth order pole at the origin (we are given that $f(0)=0$, and $f'(0)\neq 0$). It follows that we can use the reside theorem to evaluate the integral of $1/f(t)^n$ around C. Thus using the standard formula for the residue of a function at an nth order pole, we immediately find the final form for the b_n:

$$b_n = \frac{1}{2\pi i n}\oint\limits_C \frac{1}{f(t)^n}\,dt = \frac{1}{n!}\lim_{t\to 0}\left[\frac{d^{n-1}}{dt^{n-1}}\left(\frac{t^n}{f(t)^n}\right)\right],$$

which is the required formula for the b_n.

In particular, we have established *Lagrange's inversion formula*:

$$z = \sum_{n=1}^{\infty}\frac{1}{n!}\lim_{t\to 0}\left[\frac{d^{n-1}}{dt^{n-1}}\left(\frac{t}{f(t)}\right)^n\right]w^n.$$

The last part of the question asks to apply the formula to a couple of examples.

- In the case that $w=ze^{-z}$, we have $f(z)=ze^{-z}$. It follows from the inversion formula we have derived that, for w sufficiently small, we have

  $$z = \sum_{n=1}^{\infty}\frac{1}{n!}\lim_{t\to 0}\left[\frac{d^{n-1}}{dt^{n-1}}\left(\frac{t}{te^{-t}}\right)^n\right]w^n.$$

  Evaluating the relevant limit, we have

  $$\lim_{t\to 0}\left[\frac{d^{n-1}}{dt^{n-1}}\left(\frac{t}{te^{-t}}\right)^n\right] = \lim_{t\to 0}\left[\frac{d^{n-1}}{dt^{n-1}}(e^{tn})\right] = n^{n-1}\lim_{t\to 0}e^{tn} = n^{n-1}.$$

  Thus we have:

  $$z = \sum_{n=1}^{\infty}\frac{n^{n-1}}{n!}w^n$$

  as required.

  We are asked how small w must be for this relationship to hold. The easiest way to find out is simply to find the radius of convergence of the series on the right hand side. By the ratio test, the series is convergent if and only if

  $$1 > \lim_{n\to\infty}\left(\frac{(n+1)^n n!}{n^{n-1}(n+1)!}\right)|w| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n-1}|w| = \lim_{n\to\infty}\frac{n}{n+1}\left(1+\frac{1}{n}\right)^n|w| = e|w|.$$

  Hence we see that the radius of convergence of the series is $1/e$, i.e. we want $|w|<1/e$.

- The second example we are asked to apply our formula to is $w = 2z - z^2$. Using the inversion formula, we have

$$z = \sum_{n=1}^{\infty} \frac{1}{n!} \lim_{t\to 0} \frac{d^{n-1}}{dt^{n-1}} \left(\frac{t}{2t - t^2} \right)^n w^n = \sum_{n=1}^{\infty} \frac{1}{n!} \lim_{t\to 0} \frac{d^{n-1}}{dt^{n-1}} \left(\frac{1}{2-t} \right)^n w^n.$$

The derivative in the formula is given by (one can check this by trying a few simple cases, then using induction):

$$\frac{d^{n-1}}{dt^{n-1}} \left(\frac{1}{(2-t)^n} \right) = \frac{n(n+1)...(2n-2)}{(2-t)^{2n-1}} = \frac{(2n-2)!}{(n-1)!(2-t)^{2n-1}},$$

hence we have

$$z = \sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!2^{2n-1}} w^n. \qquad (*)$$

Now let's go back the equation we were trying to solve: $w = 2z - z^2$. This is a quadratic, so has two roots given by the quadratic formula as:

$$z = 1 \pm \sqrt{1-w}.$$

For $|w| < 1$, we can expand the square root using the binomial theorem:

$$\sqrt{1-w} = 1 + \frac{\left(\frac{1}{2}\right)}{1!}(-w) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-w)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-w)^3 + \cdots .$$

The nth term in this sequence can be written in the form:

$$\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)...\left(-\frac{(2n-3)}{2}\right)}{n!}(-w)^n = -\frac{(2n-3)(2n-5)...(3)(1)}{n!2^n} w^n.$$

Notice that the numerator can be written as

$$(2n-3)(2n-5)...(3)(1) = \frac{(2n-2)!}{(2n-2)(2n-4)(2n-6)...(4)(2)} = \frac{(2n-2)!}{2^{n-1}(n-1)!}.$$

Thus we see that:

$$\sqrt{1-w} = 1 - \sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!2^{2n-1}} w^n.$$

Hence comparing with $(*)$, we see that the solution the inversion formula produced was precisely $z = 1 - \sqrt{1-w}$.

4. Let $\phi(x,y)$ be a harmonic function. Show that ϕ is the real part of any analytic function $f(z)$ of the form

$$f(z) = 2\phi\left(\frac{z+1}{2}, \frac{z-1}{2i}\right) - \phi(1,0) + ic$$

where c is a real constant (provided ϕ is such that the right hand side exists). Use this formula to find analytic functions whose real parts are (i) $x/(x^2+y^2)$; (ii) $\arctan(y/x)$.

[*Hint: You might like to start by considering a harmonic conjugate $\psi(x,y)$, where ϕ and ψ obey the Cauchy-Riemann equations, and then write*

$$f(z) = \phi(x,y) + i\psi(x,y) = \sum_{n=0}^{\infty} a_n (z-1)^n.]$$

➾ **Solution:** Essentially all approaches to this question require two steps:

- First, given a harmonic function $\phi(x,y)$, we must show that there exists an analytic function $f(z) = f(x+iy)$ with real part $\phi(x,y)$.
- Once we have the first result, it is possible to write $f(z) = \phi(x,y) + i\psi(x,y)$ for some harmonic conjugate $\psi(x,y)$. From this expression, we should try to deduce the formula given in the question.

Let's start by proving that if ϕ is harmonic, then it is the real part of some analytic function f. This isn't too difficult; given harmonic $\phi(x,y)$, let's define a *harmonic conjugate* $\psi(x,y)$ using the Cauchy-Riemann equations:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Since these are linear, first order equations they always have a solution; hence $\psi(x,y)$ exists. Furthermore, since ϕ is harmonic it is twice-differentiable and it follows that its partial derivatives are continuous. Thus the partial derivatives of ψ are continuous, which we can see from the Cauchy-Riemann equations. From Analysis II, we then know that ψ is differentiable.

It follows that the real and imaginary parts of the function

$$f(z) = f(x+iy) = \phi(x,y) + i\psi(x,y)$$

obey the Cauchy-Riemann equations, and are both differentiable in a real sense. This is the necessary and sufficient condition for $f(z)$ to be analytic. Hence we have constructed an analytic function with real part $\phi(x,y)$.

We now need to derive the formula for $f(z)$ in terms of ϕ. Here are some possible approaches:

1. Using $\partial f/\partial \overline{z} = 0$ for analytic functions.
First, note that we can write any analytic function in terms of its real and imaginary parts as:

$$f(z) = \phi(x,y) + i\psi(x,y) = \phi\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) + i\psi\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right).$$

Since ϕ, ψ are both real-valued functions, we also have that

$$\overline{f(z)} = \phi\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) - i\psi\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right).$$

Adding these two equations, we see that

$$f(z) = 2\phi\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) - \overline{f(z)}.$$

We now use the following useful lemma:

Lemma: If $f(z)$ is analytic, then $\overline{f(\overline{z})}$ is analytic.

Proof: Just use the definition of complex differentiability.

$$\frac{d}{dz}\overline{f(\overline{z})} = \lim_{\Delta z \to 0}\left[\frac{\overline{f(\overline{z}+\overline{\Delta z})} - \overline{f(\overline{z})}}{\Delta z}\right] = \overline{\lim_{\Delta z \to 0}\left[\frac{f(\overline{z}+\overline{\Delta z}) - f(\overline{z})}{\overline{\Delta z}}\right]} = \overline{f'(\overline{z})},$$

where in the last step we used the fact that f is complex differentiable, and the limit $\Delta z \to 0$ is equivalent to the limit $\overline{\Delta z} \to 0$, since they both involve approaching 0 from all possible complex directions. $\square$

In particular, returning to our formula for $f(z)$:

$$f(z) = 2\phi\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) - \overline{f(z)},$$

we see that the subtraction on the right hand side is $\overline{f(\overline{\overline{z}})}$; since $\overline{f(\overline{z})}$ is an analytic function, it must be a function only of z, and hence the subtraction $\overline{f(\overline{\overline{z}})}$ is a function only of $\overline{z}$.

But $f(z)$ is analytic, so is independent of $\overline{z}$ by the Cauchy-Riemann equations $\partial f/\partial \overline{z} = 0$. Thus we can simply set $\overline{z} = 1$ on the right hand side, giving:

$$f(z) = 2\phi\left(\frac{z+1}{2}, \frac{z-1}{2i}\right) - \overline{f(1)}.$$

Noting that $f(1) = \phi(1,0) + i\psi(1,0)$, and writing $c = \psi(1,0)$, we get the result.

---

2. Using series expansions.
This is the approach suggested in the question. To begin with, note that since $f(z)$ is analytic, it can be expanded in a series about 1. Hence we can write:

$$f(z) = \phi(x,y) + i\psi(x,y) = \sum_{n=0}^{\infty} a_n(z-1)^n = \sum_{n=0}^{\infty} a_n(x+iy-1)^n.$$

Now notice that since $\phi(x,y), \psi(x,y)$ are real, we have

$$\overline{f(z)} = \phi(x,y) - i\psi(x,y) = \sum_{n=0}^{\infty} \overline{a}_n(\overline{z}-1)^n = \sum_{n=0}^{\infty} \overline{a}_n(x-iy-1)^n.$$

Adding these equations, we find

$$2\phi(x,y) = \sum_{n=0}^{\infty}\left(a_n(x+iy-1)^n + \overline{a}_n(x-iy-1)^n\right).$$

Letting $x = (z+1)/2$ and $y = (z-1)/(2i)$, we see that $x+iy = z$ and $x - iy = 1$. Hence substituting into the above, we have (note the $n=0$ term does not vanish in the second term in the sum!):

$$2\phi\left(\frac{z+1}{2}, \frac{z-1}{2i}\right) = \sum_{n=0}^{\infty} a_n(z-1)^n + \overline{a}_0 = f(z) + \overline{f(1)}.$$

Rearranging, it follows that

$$f(z) = 2\phi\left(\frac{z+1}{2}, \frac{z-1}{2i}\right) - \overline{f(1)}.$$

Writing $f(1) = \phi(1,0) + ic$, for $c = \psi(1,0)$, the result follows.

The final part of this question asks us to apply our results to two examples.

- (i) First, we seek a function with real part $\phi(x, y) = x/(x^2 + y^2)$. Notice that we have:

$$\phi\left(\frac{1}{2}(z+1), \frac{1}{2i}(z-1)\right) = \frac{\frac{1}{2}(z+1)}{\frac{1}{4}(z+1)^2 - \frac{1}{4}(z-1)^2} = \frac{2(z+1)}{4z} = \frac{1}{2} + \frac{1}{2z}.$$

  We also have $\phi(1, 0) = 1$. Hence using our formula from above, we have:

$$f(z) = 2\phi\left(\frac{1}{2}(z+1), \frac{1}{2i}(z-1)\right) - \phi(1,0) + ic = 1 + \frac{1}{z} - 1 + ic = \frac{1}{z} + ic.$$

- (ii) Finally, we want a function with real part $\phi(x, y) = \arctan(y/x)$. Noting we have:

$$\phi\left(\frac{1}{2}(z+1), \frac{1}{2i}(z-1)\right) = \arctan\left(\frac{i(1-z)}{1+z}\right).$$

  Recall that we can write $\arctan(w)$ in terms of a logarithm:

$$\arctan(w) = \frac{1}{2i}\log\left(\frac{i-w}{i+w}\right);$$

  hence we can simplify the above to:

$$\phi\left(\frac{1}{2}(z+1), \frac{1}{2i}(z-1)\right) = \arctan\left(\frac{i(1-z)}{1+z}\right) = \frac{1}{2i}\log(z).$$

  We also have $\phi(1, 0) = 0$, thus the analytic function we seek is:

$$f(z) = 2\phi\left(\frac{1}{2}(z+1), \frac{1}{2i}(z-1)\right) - \phi(1,0) + ic = -i\log(z) + ic.$$

5. Let $P(z)$ be a polynomial of degree n, with n roots, none of which lie on a simple closed contour L. Show that

$$\frac{1}{2\pi i}\oint_L \frac{P'(z)}{P(z)}\,dz = \text{number of roots lying within } L,$$

where the roots should be counted according to their multiplicity. Try to do this question without assuming the fundamental theorem of algebra.

Solution: We can do this question simply using the residue theorem:

$$\frac{1}{2\pi i}\oint_L \frac{P'(z)}{P(z)}\,dz = \sum_{\substack{\text{zeroes } z_0\\ \text{of } P(z) \text{ in } L}} \mathrm{Res}\left(\frac{P'(z)}{P(z)};z_0\right).$$

Note that we have not assumed the fundamental theorem of algebra here, because we have not assumed that any zeroes of $P(z)$ actually do exist - it could well be the case that the right hand side is just zero, which would occur if $P(z)$ had no zeroes in the complex numbers (the integrand would then be analytic, and we could invoke Cauchy's theorem).

Near a zero of $P(z)$, we have that $P(z) = (z-z_0)^{n(z_0)}g(z)$ where $n(z_0)$ is the multiplicity of the zero and $g(z)$ is a holomorphic function with $g(z_0)\neq 0$. Thus:

$$\frac{P'(z)}{P(z)} = \frac{n(z_0)(z-z_0)^{n(z_0)-1}g(z)+(z-z_0)^{n(z_0)}g'(z)}{(z-z_0)^{n(z_0)}g(z)} = \frac{n(z_0)}{z-z_0}+\frac{g'(z)}{g(z)}.$$

Hence we have:

$$\mathrm{Res}\left(\frac{P'(z)}{P(z)};z_0\right) = \lim_{z\to z_0}\left[n(z_0)+(z-z_0)\frac{g'(z)}{g(z)}\right] = n(z_0).$$

The result follows:

$$\frac{1}{2\pi i}\oint_L \frac{P'(z)}{P(z)}\,dz = \sum_{\substack{\text{zeroes } z_0\\ \text{of } P(z) \text{ in } L}} n(z_0) = \text{number of roots of } P(z) \text{ in } L\text{, counted with multiplicity.}$$

Comments: This question is then a special case of the *argument principle*, which states that for any meromorphic function $f(z)$, we have (counting zeroes and poles *with* multiplicity/orders):

$$\oint_C \frac{f'(z)}{f(z)}\,dz = 2\pi i\left((\text{number of zeroes of } f(z) \text{ in } C) - (\text{number of poles of } f(z) \text{ in } C)\right),$$

We can also use this question to prove the fundamental theorem of algebra. Suppose that $P(z)$ has no zeroes in the complex plane $\mathbb{C}$. Then we have shown

$$\oint_C \frac{P'(z)}{P(z)}\,dz = 0$$

for all closed contours C in $\mathbb{C}$. But also note that the integral can be evaluated explicitly:

$$\oint_C \frac{P'(z)}{P(z)}\,dz = \oint_C d(\log(P(z))) = \log(P(C(\text{end}))) - \log(P(C(\text{start}))).$$

This is proportional to the change in argument of $P(z)$ as we move z around the contour C once. For an extremely large circular contour $z = Re^{i\theta}$ say, $P(Re^{i\theta}) \approx a_n R^n e^{i\theta n}$ for a degree n polynomial and some $a_n \neq 0$. From this we see the change in argument of $P(z)$ as we move around the contour is $\arg\left(a_n R^n e^{i(\theta+2\pi)n}\right) - \arg\left(a_n R^n e^{i\theta n}\right) = 2\pi i n$. This is non-zero if $n>0$, i.e. the degree of the polynomial is greater than zero, hence we have a contradiction, and so $P(z)$ does have at least one zero in the complex plane.

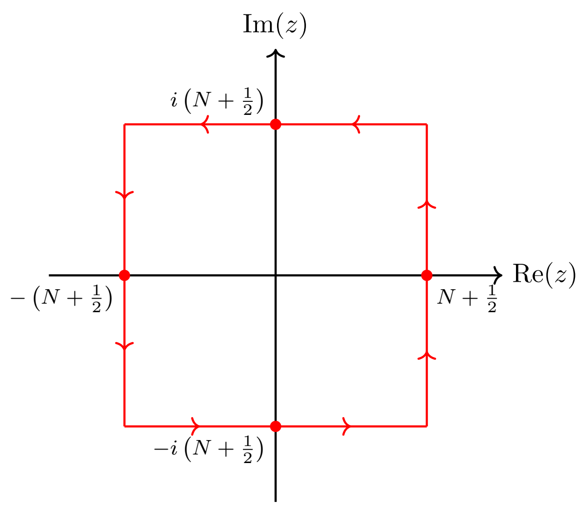
6. Consider a rectangular contour C, with corners at $(N+\frac{1}{2})(\pm 1 \pm i)$ to evaluate

$$\frac{1}{2\pi i}\oint_C \frac{\pi\cot(\pi z)\coth(\pi z)}{z^3}\,dz.$$

In the limit as $N\to\infty$, show that

$$\sum_{n=1}^{\infty}\frac{\coth(n\pi)}{n^3}=\frac{7\pi^3}{180}.$$

➼ **Solution:** We are told to use the contour shown:



In order to apply the residue theorem, we need to know what singularities are inside the contour. The integrand:

$$\frac{\pi\coth(\pi z)\cot(\pi z)}{z^3}=\frac{\pi\cosh(\pi z)\cos(\pi z)}{z^3\sinh(\pi z)\sin(\pi z)}$$

is singular only when $z=0, \sin(\pi z)=0$ or $\sinh(\pi z)=0$. We know that $\sin(\pi z)=0$ if and only if $z=n$ for some integer n, and since $\sinh(\pi z)=i\sin(\pi i z)$, we see that $\sinh(\pi z)=0$ if and only if $z=in$ for some integer n. Hence the residue theorem tells us that

$$\frac{1}{2\pi i}\oint_C \frac{\pi\cot(\pi z)\coth(\pi z)}{z^3}\,dz=\mathrm{Res}\left(\frac{\pi\cot(\pi z)\coth(\pi z)}{z^3},0\right)+\sum_{\substack{n=-N\\n\neq 0}}^{N}\mathrm{Res}\left(\frac{\pi\cot(\pi z)\coth(\pi z)}{z^3},n\right)$$
$$+\sum_{\substack{n=-N\\n\neq 0}}^{N}\mathrm{Res}\left(\frac{\pi\cot(\pi z)\coth(\pi z)}{z^3},in\right).$$

We must now find these residues. At zero, things are very complicated, as we have singularities in $\cot(\pi z)$, $\coth(\pi z)$ and the denominator z^3. So let's go back to first principles and just series expand things. Note that:

$$\begin{aligned}\cos(\pi z)\cosh(\pi z)&=\left(1-\frac{\pi^2z^2}{2}+\frac{\pi^4z^4}{4!}+O(z^6)\right)\left(1+\frac{\pi^2z^2}{2}+\frac{\pi^4z^4}{4!}+O(z^6)\right)\\&=1-\frac{\pi^4z^4}{4}+2\frac{\pi^4z^4}{4!}+O(z^6)\\&=1-\frac{\pi^4z^4}{6}+O(z^6).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sin(\pi z)\sinh(\pi z) &= \left(\pi z - \frac{\pi^3 z^3}{3!} + \frac{\pi^5 z^5}{5!} + O(z^7)\right)\left(\pi z + \frac{\pi^3 z^3}{3!} + \frac{\pi^5 z^5}{5!} + O(z^7)\right) \\ &= \pi^2 z^2 - \frac{\pi^6 z^6}{36} + 2\frac{\pi^6 z^6}{5!} + O(z^7) \\ &= \pi^2 z^2 - \frac{\pi^6 z^6}{90} + O(z^7).\end{aligned}$$

Using these formulae, we find

$$\begin{aligned}\frac{\pi\cot(\pi z)\coth(\pi z)}{z^3} &= \frac{\pi\cosh(\pi z)\cos(\pi z)}{z^3\sinh(\pi z)\sin(\pi z)} \\ &= \frac{\pi}{z^3}\left(1 - \frac{\pi^4 z^4}{6} + O(z^6)\right)\left(\pi^2 z^2 - \frac{\pi^6 z^6}{90} + O(z^7)\right)^{-1} \\ &= \frac{\pi}{z^3}\left(1 - \frac{\pi^4 z^4}{6} + O(z^6)\right)\cdot\frac{1}{\pi^2 z^2}\left(1 - \frac{\pi^4 z^4}{90} + O(z^5)\right)^{-1} \\ &= \frac{1}{\pi z^5}\left(1 - \frac{\pi^4 z^4}{6} + O(z^6)\right)\left(1 + \frac{\pi^4 z^4}{90} + O(z^5)\right) \qquad \text{(using the binomial theorem)} \\ &= \frac{1}{\pi z^5}\left(1 - \frac{7\pi^4 z^4}{45} + O(z^5)\right).\end{aligned}$$

We see that the singularity at zero was actually a fifth order pole, so it would have been very difficult to apply the derivative formula for residues! Reading off the coefficient of $1/z$, which is exactly what we *mean* by the residue of the integrand at $z = 0$, we see that

$$\operatorname{Res}\left(\frac{\pi\cot(\pi z)\coth(\pi z)}{z^3}, 0\right) = -\frac{7\pi^3}{45}.$$

We must now find the residues at $z = n$ and $z = in$ for all $n \in \mathbb{Z}\backslash\{0\}$. Since the denominator of the integrand is non-singular at all these poles, they are simple, and hence we can apply the standard formula:

$$\operatorname{Res}\left(\frac{\pi\cot(\pi z)\coth(\pi z)}{z^3}, n\right) = \lim_{z\to n}\left[\frac{\pi\cos(\pi z)\coth(\pi z)}{z^3}\cdot\frac{z-n}{\sin(\pi z)}\right] = \frac{\pi\cos(\pi n)\coth(\pi n)}{n^3}\lim_{z\to n}\left[\frac{z-n}{\sin(\pi z)}\right]$$

$$\operatorname{Res}\left(\frac{\pi\cot(\pi z)\coth(\pi z)}{z^3}, in\right) = \lim_{z\to in}\left[\frac{\pi\cosh(\pi z)\cot(\pi z)}{z^3}\cdot\frac{z-in}{\sinh(\pi z)}\right] = \frac{\pi\cosh(i\pi n)\cot(i\pi n)}{i^3 n^3}\lim_{z\to in}\left[\frac{z-in}{\sinh(i\pi z)}\right].$$

To evaluate the limits on the right hand side, we Taylor expand $\sin(\pi z)$ around $z = n$:

$$\sin(\pi z) = \sin(\pi n) + \pi(z-n)\cos(\pi n) + O((z-n)^2) \qquad \Rightarrow \qquad \lim_{z\to n}\left[\frac{\sin(\pi z)}{z-n}\right] = \pi\cos(\pi n).$$

Using this limit, we have:

$$\lim_{z\to n}\left[\frac{z-n}{\sin(\pi z)}\right] = \frac{1}{\pi\cos(\pi n)}, \qquad i\lim_{z\to in}\left[\frac{z-in}{\sinh(\pi z)}\right] = i\lim_{w\to n}\left[\frac{w-n}{\sinh(\pi i w)}\right] = \lim_{w\to n}\left[\frac{w-n}{\sin(\pi w)}\right] = \frac{1}{\pi\cos(\pi n)},$$

where to evaluate the hyperbolic sine version of the limit, we made the substitution $z = iw$, used the fact that $\sinh(\pi i w) = i\sin(\pi w)$, and then used the previous limit for the normal sine.

Substituting these results into our formula for the residues, we see that:

$$\operatorname{Res}\left(\frac{\pi\cot(\pi z)\coth(\pi z)}{z^3}, n\right) = \frac{\coth(\pi n)}{n^3},$$

$$\operatorname{Res}\left(\frac{\pi\cot(\pi z)\coth(\pi z)}{z^3}, in\right) = \frac{\coth(\pi n)}{n^3},$$

where we substituted in the previous results and used the facts that $\sinh(\pi i n) = i\sin(\pi n)$, $\cosh(\pi i n) = \cos(\pi n)$ to clean up the right hand sides.

Putting all our residues together, we have by the residue theorem that:

$$\oint_C \frac{\pi\coth(\pi z)\cot(\pi z)}{z^3}\,dz = 2\pi i\left(2\sum_{\substack{n=-N\\ n\neq 0}}^{N}\frac{\coth(\pi n)}{n^3} - \frac{7\pi^3}{45}\right).$$

Notice that the terms of the sum are all even, so we can rewrite the sum on the right hand side as:

$$2\sum_{\substack{n=-N\\ n\neq 0}}^{N}\frac{\coth(\pi n)}{n^3} = 4\sum_{n=1}^{N}\frac{\coth(\pi n)}{n^3}.$$

It follows that:

$$\oint_C \frac{\pi\coth(\pi z)\cot(\pi z)}{z^3}\,dz = 8\pi i\left(\sum_{n=1}^{N}\frac{\coth(\pi n)}{n^3} - \frac{7\pi^3}{180}\right).$$

In particular, we notice that we will get the required value for the series if in the limit as $N\to\infty$, the left hand side converges to zero. That is, we must show that

$$\lim_{N\to\infty}\oint_C \frac{\pi\coth(\pi z)\cot(\pi z)}{z^3}\,dz = 0,$$

from which we may deduce

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^3} - \frac{7\pi^3}{180}$$

as required.

In order to prove that the contour integral converges to zero as the side length of the square N converges to infinity, $N \to \infty$, we compute the contour on each of its straight line segments separately, and then bound them appropriately.

On the right hand side of the square, we parametrise the contour as $z = \frac{1}{2} + N + iy$ with $y \in (-\frac{1}{2} - N, \frac{1}{2} + N)$. Then the contribution to the integral is:

$$\int_{-\frac{1}{2}-N}^{\frac{1}{2}+N} \frac{\pi \coth\left(\pi\left(\frac{1}{2}+N+iy\right)\right)\cot\left(\pi\left(\frac{1}{2}+N+iy\right)\right)}{(\frac{1}{2}+N+iy)^3}\, idy$$

We now use the following trigonometric/hyperbolic bounds:

- To bound $\cot\left(\pi\left(\frac{1}{2}+N+iy\right)\right)$, we note

  $$\left|\cot\left(\pi\left(\frac{1}{2}+N+iy\right)\right)\right| = \frac{\left|\cos\left(\pi\left(\frac{1}{2}+N+iy\right)\right)\right|}{\left|\sin\left(\pi\left(\frac{1}{2}+N+iy\right)\right)\right|} = \frac{|\sin(i\pi y)|}{|\cos(i\pi y)|} = \frac{|\sinh(\pi y)|}{|\cosh(\pi y)|} = |\tanh(\pi y)| \leq 1,$$

  where we used the fact that $\sin\left(z + \frac{(2n+1)\pi}{2}\right) = \pm\cos(z)$ and $\cos\left(z + \frac{(2n+1)\pi}{2}\right) = \pm\sin(z)$.

- To bound $\coth\left(\pi\left(\frac{1}{2}+N+iy\right)\right)$, we first write

  $$\left|\coth\left(\pi\left(\frac{1}{2}+N+iy\right)\right)\right| = \frac{\left|\cosh\left(\pi\left(\frac{1}{2}+N+iy\right)\right)\right|}{\left|\sinh\left(\pi\left(\frac{1}{2}+N+iy\right)\right)\right|}.$$

  We then note that $\cosh(x+iy) = \cosh(x)\cosh(iy) + \sinh(x)\sinh(iy) = \cosh(x)\cos(y) + i\sinh(x)\sin(y)$, and hence

  $$|\cosh(x+iy)| = \sqrt{\cosh^2(x)\cos^2(y) + \sinh^2(x)\sin^2(y)} = \sqrt{\cosh^2(x)\cos^2(y) + (\cosh^2(x)-1)\sin^2(y)}$$

  $$= \sqrt{\cosh^2(x) - \sin^2(y)} \leq \cosh(x).$$

  Similarly, $\sinh(x+iy) = \sinh(x)\cosh(iy) + \sinh(iy)\cosh(x) = \sinh(x)\cos(y) + i\sin(y)\cosh(x)$, which implies

  $$|\sinh(x+iy)| = \sqrt{\sinh^2(x)\cos^2(y) + \sin^2(y)\cosh^2(x)} = \sqrt{\sinh^2(x)\cos^2(y) + \sin^2(y)(1+\sinh^2(x))}$$

  $$= \sqrt{\sinh^2(x) + \sin^2(y)} \geq \sinh(x).$$

  Using these results to bound $\coth(\pi(\frac{1}{2}+N+iy))$, we have

  $$\frac{\left|\cosh\left(\pi\left(\frac{1}{2}+N+iy\right)\right)\right|}{\left|\sinh\left(\pi\left(\frac{1}{2}+N+iy\right)\right)\right|} \leq \frac{\left|\cosh\left(\pi\left(\frac{1}{2}+N\right)\right)\right|}{\left|\sinh\left(\pi\left(\frac{1}{2}+N\right)\right)\right|} = \left|\coth\left(\pi\left(\frac{1}{2}+N\right)\right)\right| \leq \coth(\pi/2).$$

Combining these results, we see that

$$\left|\int_{-\frac{1}{2}-N}^{\frac{1}{2}+N} \frac{\pi \coth\left(\pi\left(\frac{1}{2}+N+iy\right)\right)\cot\left(\pi\left(\frac{1}{2}+N+iy\right)\right)}{(\frac{1}{2}+N+iy)^3}\, idy\right| \leq \int_{-\infty}^{\infty} \left|\frac{\pi\coth(\pi/2)}{\left(\frac{1}{2}+N+iy\right)^3}\right| dy \to 0$$

as $N \to \infty$. Thus the contribution from the right side of the square goes to zero as $N \to \infty$. Similarly, the contribution from the left side of the square goes to zero as $N \to \infty$ (just exchange $\frac{1}{2} + N \mapsto -\frac{1}{2} - N$ in the above argument).

We still need to show the contributions from the top and bottom of the square go to zero as $N \to \infty$. On the top side of the square, we parametrise the contour as $z = x + \left(\frac{1}{2} + N\right) i$ with $x \in (-\frac{1}{2} - N, \frac{1}{2} + N)$. Then the contribution to the integral is:

$$\int_{-\frac{1}{2}-N}^{\frac{1}{2}+N} \frac{\pi \coth\left(\pi\left(x + \left(\frac{1}{2}+N\right)i\right)\right)\cot\left(\pi\left(x + \left(\frac{1}{2}+N\right)i\right)\right)}{(x + \left(\frac{1}{2}+N\right)i)^3}\, dx$$

Similar to the right and left sides of the square contour, we use trigonometric/hyperbolic bounds to show the contributions goes to zero as $N \to \infty$. We have:

- To bound $\cot\left(\pi\left(x + \left(\frac{1}{2}+N\right)i\right)\right)$, we note that since $\cos(iz) = \cosh(z)$ and $-i\sin(iz) = \sinh(z)$, we have:

  $$\left|\cot\left(\pi\left(x + \left(\frac{1}{2}+N\right)i\right)\right)\right| = \left|\coth\left(\pi\left(ix - \left(\frac{1}{2}+N\right)\right)\right)\right| \leq \coth(\pi/2),$$

  using the bound we derived on the hyperbolic cotangent for the left and right sides of the contour.

- To bound $\coth\left(\pi\left(x + \left(\frac{1}{2}+N\right)i\right)\right)$, we use the same trick:

  $$\left|\coth\left(\pi\left(x + \left(\frac{1}{2}+N\right)i\right)\right)\right| = \left|\cot\left(\pi\left(ix - \left(\frac{1}{2}+N\right)\right)\right)\right| \leq 1,$$

  using the same bound we derived on the cotangent for the left and right sides of the contour.

Therefore, we have:

$$\left|\int_{-\frac{1}{2}-N}^{\frac{1}{2}+N} \frac{\pi \coth\left(\pi\left(x + \left(\frac{1}{2}+N\right)i\right)\right)\cot\left(\pi\left(x + \left(\frac{1}{2}+N\right)i\right)\right)}{(x + \left(\frac{1}{2}+N\right)i)^3}\, dx\right| \leq \int_{-\infty}^{\infty} \left|\frac{\pi\coth(\pi/2)}{(x + \left(\frac{1}{2}+N\right)i)^3}\right| dx \to 0,$$

as $N \to \infty$. Thus we have shown that the contribution from the top of the square contour vanishes as $N \to \infty$; again, by symmetry, the contribution from the bottom of the square contour also vanishes as $N \to \infty$.

Hence, we have established the result:

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^3} = \frac{7\pi^3}{180}$$

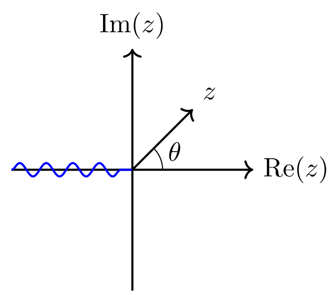
as required.

7. Evaluate

$$\int_0^\infty \frac{x^{m-1}}{x^2+1}\,dx$$

where $0 < m < 2$. Why is it necessary for m to satisfy the restrictions?

➻ **Solution:** Since it's possible for m to be a non-integer, in order to use complex methods here we must first define what we mean by z^{m-1} in the complex plane. To do so, we define a branch of z^{m-1} described by (i) z^{m-1} is real for z on the positive real axis; (ii) there is a branch cut along the negative real axis. The branch is pictured below.

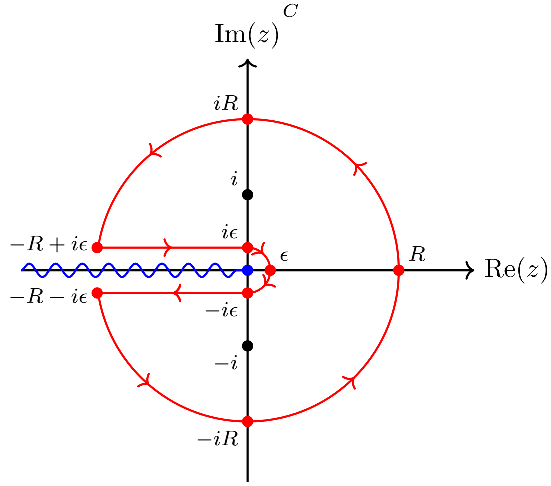


We can realise this branch by always choosing the argument of $z = re^{i\theta}$ to be in the range $\theta \in [-\pi, \pi]$. Then z^{m-1} is given by:

$$r^{m-1}e^{i\theta(m-1)}.$$

This function is continuous in θ as we move around $\mathbb{C}$, because the branch cut prevents us from immediately crossing from $\theta = -\pi$ to $\theta = \pi$.

Now let's begin our calculation. Consider the contour integral $\oint\limits_C \frac{z^{m-1}}{1+z^2}\,dz$ around the *keyhole contour* C shown.



The contour C encloses two simple poles at $z = \pm i$ respectively. The residues at these poles are given by:

$$\mathrm{Res}\left(\frac{z^{m-1}}{z^2+1}; i\right) = \lim_{z\to i}\left[\frac{z^{m-1}}{z+i}\right] = \frac{e^{i\pi(m-1)/2}}{2i}, \qquad \mathrm{Res}\left(\frac{z^{m-1}}{z^2+1}; -i\right) = \lim_{z\to -i}\left[\frac{z^{m-1}}{z-i}\right] = -\frac{e^{-i\pi(m-1)/2}}{2i}.$$

Hence, by the residue theorem, we have $\oint\limits_C \frac{z^{m-1}}{z^2+1}\,dz = 2\pi i\left(\frac{e^{i\pi(m-1)/2}}{2i} - \frac{e^{-i\pi(m-1)/2}}{2i}\right) = 2\pi i \sin\left(\frac{\pi(m-1)}{2}\right).$

Now consider evaluating the integral on each part of the contour separately. Considering each part one at a time, we have:

- We can parametrise the large circular arc of radius R as $z = Re^{i\theta}$. We want the end points of this section of the contour to be $-R + i\epsilon$ and $-R - i\epsilon$, so the appropriate range of θ is actually:
$$\theta \in \left(-\pi + \arctan\left(\frac{\epsilon}{R}\right), \pi - \arctan\left(\frac{\epsilon}{R}\right)\right);$$
however, in eventually taking the limit $\epsilon \to 0$, $R \to \infty$, we see that this interval tends to $(-\pi, \pi)$. So let's work with $\theta \in (-\pi, \pi)$ as the limits are much easier to write then. It follows that the contribution from this section of the contour is:
$$\int_{-\pi}^{\pi} \frac{R^{m-1}e^{i\theta(m-1)}}{R^2e^{2i\theta}+1} iRe^{i\theta}\, d\theta = O(R^{m-2}) \to 0$$
as $R \to \infty$, since $m < 2$.
- We can parametrise the small semicircular contour of radius ϵ as $z = \epsilon e^{i\theta}$, with $\theta \in (-\pi/2, \pi/2)$. We see that the contribution from this section of the contour is:
$$\int_{\pi/2}^{-\pi/2} \frac{\epsilon^{m-1}e^{i\theta(m-1)}}{\epsilon^2 e^{2i\theta}+1} i\epsilon e^{i\theta}\, d\theta.$$
For ϵ small, we can use the binomial theorem to expand the denominator, giving:
$$\int_{\pi/2}^{-\pi/2} \epsilon^{m-1}e^{i\theta(m-1)} i\epsilon e^{i\theta}(1 - \epsilon^2 e^{2i\theta} + O(\epsilon^4))\, d\theta = O(\epsilon^m) \to 0$$
as $\epsilon \to 0$, since $m > 0$.
- On the straight section of the contour in the upper half plane, we can parametrise using $z = xe^{i\pi}$ for $x \in (0, R)$ (we have ignored the fact that the straight segment is $i\epsilon$ away from the real axis, since we will take the limit as $\epsilon \to 0$ eventually). This gives a contribution
$$e^{i\pi}\int_R^0 \frac{(xe^{i\pi})^{m-1}}{x^2+1}\, dx = e^{i\pi(m-1)}\int_0^R \frac{x^{m-1}}{x^2+1}\, dx.$$
Similarly, on the straight section of the contour in the lower half plane, we can parametrise using $z = xe^{-i\pi}$ for $x \in (0, R)$ (again ignoring the fact the contour is $-i\epsilon$ away from the real axis). This gives a contribution
$$e^{-i\pi}\int_0^R \frac{(xe^{-i\pi})^{m-1}}{x^2+1}\, dx = -e^{-i\pi(m-1)}\int_0^R \frac{x^{m-1}}{x^2+1}\, dx.$$

Putting everything together, we see that in the limit as $\epsilon \to 0$ and $R \to \infty$, we have
$$2\pi i \sin\left(\frac{\pi(m-1)}{2}\right) = \oint_C \frac{z^{m-1}}{z^2+1}\, dz = (e^{i\pi(m-1)} - e^{-i\pi(m-1)})\int_0^\infty \frac{x^{m-1}}{x^2+1}\, dx = 2i\sin(\pi(m-1))\int_0^\infty \frac{x^{m-1}}{x^2+1}\, dx.$$
Rearranging, we see that
$$\int_0^\infty \frac{x^{m-1}}{x^2+1}\, dx = \frac{\pi\sin(\frac{\pi}{2}(m-1))}{\sin(\pi(m-1))} = \frac{\pi\sin(\frac{\pi}{2}(m-1))}{2\sin(\frac{\pi}{2}(m-1))\cos(\frac{\pi}{2}(m-1))} = \frac{\pi}{2}\operatorname{cosec}\left(\frac{\pi m}{2}\right).$$
We saw that the conditions $0 < m < 2$ were necessary for the convergence of the integrals on the large circular contour and the small circular contour as $\epsilon \to 0, R \to \infty$.

8. Let

$$F(z) = \int_{-\infty}^{\infty} \frac{e^{uz}}{1+e^u}\, du.$$

For what region of the z-plane does $F(z)$ define an analytic function? Show by closing the contour (use a rectangle) in the upper half plane that

$$F(z) = \pi \operatorname{cosec}(\pi z).$$

Explain how this result provides the analytic continuation of $F(z)$.

Solution: First we are asked where $F(z)$ is an analytic function. We use the theorem we saw in lectures; namely, a function $F(z)$ defined in terms of an integral over u with integrand $f(z,u)$ is analytic with domain of analyticity D if:

(i) $f(z,u)$ is continuous in z and u;

(ii) $f(z,u)$ is analytic in z for each fixed u;

(iii) the integral defining $F(z)$ is uniformly convergent on each compact subset $K \subseteq D$.

In the case that

$$F(z) = \int_{-\infty}^{\infty} \frac{e^{uz}}{1+e^u}\, du,$$

the integrand clearly satisfies conditions (i) and (ii). The third condition (iii) is rather technical, and in this course we'll suppose that it's sufficient to check that the integral is convergent.

We can check for convergence of the integral using the Further Complex Methods method. We first note that the integrand contains no singularities, so the only possible divergence comes from the infinite range of the integral. At $u = -\infty$, the contribution to the integral is given by:

$$\int_{-\infty} \frac{e^{uz}}{1+e^u}\, du \approx \int_{-\infty} e^{uz}\, du = \left[\frac{e^{uz}}{z}\right]_{-\infty}.$$

Hence for a finite contribution from the lower limit, we need $\operatorname{Re}(z) > 0$. At $u = \infty$, the contribution to the integral is given by:

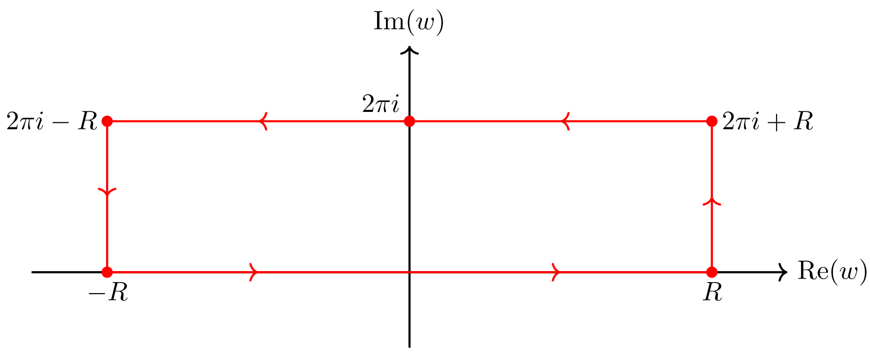
$$\int^{\infty} \frac{e^{uz}}{1+e^u}\, du \approx \int^{\infty} \frac{e^{uz}}{e^u}\, du = \int^{\infty} e^{u(z-1)}\, du = \left[\frac{e^{u(z-1)}}{z-1}\right]^{\infty}.$$

Hence for a finite contribution from the upper limit, we need $\operatorname{Re}(z) < 1$. It follows that $F(z)$, as given by the above integral, is analytic in the domain $0 < \operatorname{Re}(z) < 1$.

To perform the integral, consider instead the contour integral

$$\oint_C \frac{e^{wz}}{1+e^w}\, dw,$$

where C is the rectangular contour shown below.



We chose to position the upper part of the contour so that its imaginary part was 2π so that the integrand along this path will look like:

$$\frac{e^{(2\pi i+u)z}}{1+e^{(2\pi i+u)}} = e^{2\pi iz} \cdot \frac{e^{uz}}{1+e^{u}},$$

i.e. proportional to the original integral.

The integrand has a simple pole when $w = i\pi$, since $1 + e^{i\pi} = 0$. This is the the only singularity enclosed by the contour. The residue at this pole is given by:

$$\lim_{w\to i\pi}\left[\frac{(w-i\pi)e^{wz}}{1+e^{w}}\right] = \lim_{w\to i\pi}\left[\frac{z(w-i\pi)e^{wz}+e^{wz}}{e^{w}}\right] = -e^{i\pi z},$$

where we used L'Hôpital's rule in the first step. It follows by the residue theorem that

$$\oint_C \frac{e^{wz}}{1+e^{w}}\,dw = -2\pi i e^{i\pi z}.$$

Now consider evaluating the contour integral on each part of the contour separately. We find:

$$-2\pi i e^{i\pi z} = \oint_C \frac{e^{wz}}{1+e^{w}}\,dw = \int_{-R}^{R} \frac{e^{uz}}{1+e^{u}}\,du + \int_{R}^{-R} \frac{e^{(2\pi i+u)z}}{1+e^{u}}\,du + \int_{0}^{2\pi} \frac{e^{(iu+R)z}}{1+e^{iu+R}}\,idu + \int_{2\pi}^{0} \frac{e^{(iu-R)z}}{1+e^{iu-R}}\,idu.$$

The last two terms tend to zero as $R\to\infty$, since

$$\int_{0}^{2\pi} \frac{e^{(iu+R)z}}{1+e^{iu+R}}\,idu = O(e^{(z-1)R}) \to 0, \text{ as } \mathrm{Re}(z) < 1, \quad \text{and} \quad \int_{2\pi}^{0} \frac{e^{(iu-R)z}}{1+e^{iu-R}}\,idu = O(e^{-Rz}) \to 0, \text{ as } \mathrm{Re}(z) > 0.$$

It follows that in the limit as $R\to\infty$, we have

$$(1-e^{2\pi iz})\int_{-\infty}^{\infty} \frac{e^{uz}}{1+e^{u}}\,du = -2\pi i e^{\pi iz} \qquad \Rightarrow \qquad F(z) = \int_{-\infty}^{\infty} \frac{e^{uz}}{1+e^{u}}\,du = \pi\mathrm{cosec}(\pi z),$$

as required.

In the above work, we showed that $F(z) = \pi\mathrm{cosec}(\pi z)$ in the region $0 < \mathrm{Re}(z) < 1$. Notice that $\pi\mathrm{cosec}(\pi z)$ is actually meromorphic on the entire complex plane, with simple poles at $z \in \mathbb{Z}$; we also just proved that it agrees with $F(z)$ on the domain $0 < \mathrm{Re}(z) < 1$. Thus it provides the meromorphic continuation of $F(z)$ to the rest of the complex plane (excluding the integers).

9. Evaluate the following integrals, where $|f(z)/z| \to 0$ as $|z| \to \infty$ and $f(z)$ is analytic in the upper half plane (including the real axis):

$$\text{(i) } \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx, \qquad \text{(ii) } \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x(x-i)}\,dx, \qquad \text{(iii) } \mathcal{P}\int_{-\infty}^{\infty} \frac{dx}{x-i}, \qquad \text{(iv) } \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{-x^2}}{x}\,dx.$$

Which of the integrals (i), (iii) and (iv) is real?

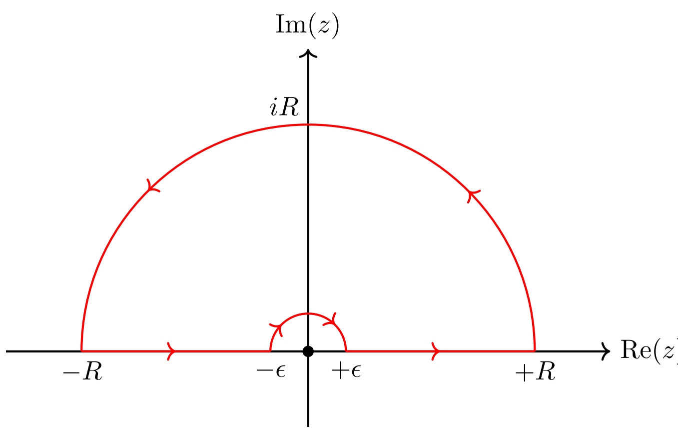
Solution: (i) Recall from the definition of the principal value of an integral that

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left[\int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\,dx + \int_{\epsilon}^{R} \frac{e^{ix}}{x}\,dx\right].$$

Hence we consider the contour integral

$$\oint_C \frac{e^{iz}}{z}\,dz$$

where the contour C is defined to be a straight line segment from $-R$ to $-\epsilon$, followed by a semicircular arc in the upper half plane of radius ϵ from $-\epsilon$ to ϵ, then another straight line segment from ϵ to R, followed by closing the contour in the upper half plane with a semicircle of radius R.



By the residue theorem (or Cauchy's theorem in this case), we have

$$\oint_C \frac{e^{iz}}{z}\,dz = 0.$$

Let's now consider evaluating the integral on each section of the contour separately. We find:

$$0 = \oint_C \frac{e^{iz}}{z}\,dz = \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\,dx + \int_{\pi}^{0} \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}\,d\theta + \int_{\epsilon}^{R} \frac{e^{ix}}{x}\,dx + \int_{C'} \frac{e^{iz}}{z}\,dz,$$

where C' is the large semicircular part of the contour. Notice that the integral over C' vanishes in the limit as $R \to \infty$, by Jordan's Lemma. Hence taking the limit as $\epsilon \to 0$ and $R \to 0$ of both sides, we find that

$$0 = \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx + \lim_{\epsilon\to 0}\int_{\pi}^{0} \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}\,d\theta = \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx - \int_{0}^{\pi} i\,d\theta = \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx - i\pi.$$

Hence we have:

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx = i\pi.$$

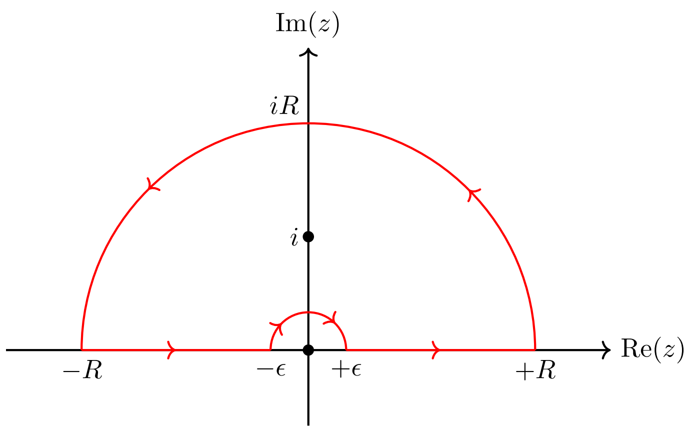
(ii) Again, the definition of the principal value of an integral gives:

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)}{x(x-i)}\,dx = \lim_{\substack{R\to\infty\\ \epsilon\to 0}}\left[\int_{-R}^{-\epsilon}\frac{f(x)}{x(x-i)}\,dx + \int_{\epsilon}^{R}\frac{f(x)}{x(x-i)}\,dx\right].$$

Hence we consider the contour integral

$$\oint_C \frac{f(z)}{z(z-i)}\,dz$$

where the contour C is defined to be a straight line segment from $-R$ to $-\epsilon$, followed by a semicircular arc in the upper half plane of radius ϵ from $-\epsilon$ to ϵ, then another straight line segment from ϵ to R, followed by closing the contour in the upper half plane with a semicircle of radius R.



This time, we have a singularity in the contour at $z = i$. Since $f(z)$ is analytic in the upper half plane, we see that $z = i$ is a simple pole. Thus the residue at $z = i$ is given by

$$\mathrm{Res}\left(\frac{f(z)}{z(z-i)};i\right) = \lim_{z\to i}\left[\frac{f(z)}{z}\right] = \frac{f(i)}{i}.$$

Hence, by the residue theorem, we see that

$$\oint_C \frac{f(z)}{z(z-i)}\,dz = 2\pi f(i).$$

Now evaluate the integral on each section of the contour separately. We have:

$$2\pi f(i) = \oint_C \frac{f(z)}{z(z-i)}\,dz = \int_{-R}^{-\epsilon}\frac{f(x)}{x(x-i)}\,dx + \int_{\pi}^{0}\frac{f(\epsilon e^{i\theta})}{\epsilon e^{i\theta}(\epsilon e^{i\theta}-1)}i\epsilon e^{i\theta}\,d\theta + \int_{\epsilon}^{R}\frac{f(x)}{x(x-i)}\,dx + \int_{0}^{\pi}\frac{f(Re^{i\theta})}{Re^{i\theta}(Re^{i\theta}-1)}iRe^{i\theta}\,d\theta. \tag{$*$}$$

We can bound the final integral as $R \to \infty$, since we are given that $|f(z)/z| \to 0$ as $|z| \to \infty$. This gives us:

$$\left|\int_0^\pi \frac{f(Re^{i\theta})}{Re^{i\theta}(Re^{i\theta}-1)} iRe^{i\theta}\, d\theta\right| \leq \int_0^\pi \left|\frac{f(Re^{i\theta})}{(Re^{i\theta}-1)}\right| d\theta \leq \int_0^\pi \frac{|f(Re^{i\theta})|}{||Re^{i\theta}|-1|}\, d\theta = \int_0^\pi \frac{|f(Re^{i\theta})|/|Re^{i\theta}|}{|1-1/|Re^{i\theta}||}\, d\theta \to 0$$

as $|Re^{i\theta}| \to \infty$, i.e. as $R \to \infty$. Note that in the second inequality we used the *reverse triangle inequality*, i.e. $||z|-|w|| \leq |z+w|$.

Hence, taking the limit in (∗) as $R \to \infty$ and $\epsilon \to 0$, we get:

$$2\pi f(i) = \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x(x-i)}\, dx + i \lim_{\epsilon\to 0}\int_\pi^0 \frac{f(\epsilon e^{i\theta})}{(\epsilon e^{i\theta}-i)}\, d\theta = \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x(x-i)}\, dx + \pi f(0).$$

Hence we have:

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x(x-i)}\, dx = 2\pi f(i) - \pi f(0).$$

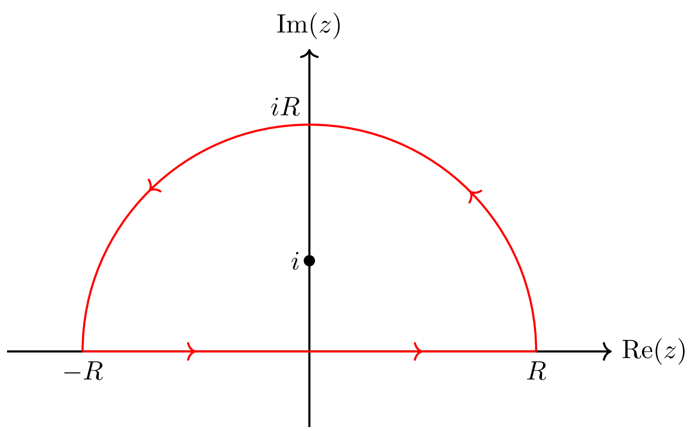
(iii) This is similar to (ii). The definition of the principal value of an integral gives:

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{dx}{x-i} = \lim_{R\to\infty}\left[\int_{-R}^{R} \frac{dx}{x-i}\right],$$

where in this case we note that there is no need to indent the contour at 0, because there is no singularity at zero. Hence we consider the contour integral

$$\oint_C \frac{dz}{z-i}$$

where the contour C is defined to be a straight line segment from $-R$ to R, followed by closing the contour in the upper half plane with a semicircle of radius R.



The contour encloses a simple pole at $z = i$, which has residue

$$\operatorname{Res}\left(\frac{1}{z-i}; i\right) = 1.$$

Hence by the residue theorem, we have:

$$\oint_C \frac{dz}{z-i} = 2\pi i.$$

We now evaluate the integral on each section of the contour separately. We have

$$2\pi i = \oint_C \frac{dz}{z-1} = \int_{-R}^{R} \frac{dx}{x-i} + \int_0^{\pi} \frac{Rie^{i\theta}}{Re^{i\theta}-1} d\theta.$$

In the limit as $R \to \infty$, the RHS converges to:

$$2\pi i = \mathcal{P}\int_{-\infty}^{\infty} \frac{dx}{x-i} + \int_0^{\pi} i\, d\theta.$$

Hence we find that

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{dx}{x-i} = i\pi.$$

(iv) Finally, for the last integral recall from the definition of the principal value that we can write:

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{-x^2}}{x}\, dx = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left[\int_{-R}^{-\epsilon} \frac{e^{-x^2}}{x}\, dx + \int_{\epsilon}^{R} \frac{e^{-x^2}}{x}\, dx\right].$$

In the first integral inside this limit, make the substitution $x \mapsto -x$. Then we see that:

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{-x^2}}{x}\, dx = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left[-\int_{\epsilon}^{R} \frac{e^{-x^2}}{x}\, dx + \int_{\epsilon}^{R} \frac{e^{-x^2}}{x}\, dx\right] = 0.$$

Hence the principal value is zero. This is really just a consequence of the odd parity of the integrand.

At the very end of the question, we are asked to comment on which of the integrals (i), (iii) and (iv) is real. We have evaluated all three, and we see that only (iv) is real (it has the value 0).

10. Use the principal value technique to evaluate

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}\,dx.$$

Solution: In terms of limits, the definition of this integral is:

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}\,dx = \lim_{\substack{R\to\infty\\ \epsilon\to 0}} \int_\epsilon^R \frac{\sin(x)}{x(x^2+1)}\,dx.$$

We can split the integral inside this limit into two identical copies of itself:

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}\,dx = \lim_{\substack{R\to\infty\\ \epsilon\to 0}} \left[\frac{1}{2}\int_\epsilon^R \frac{\sin(x)}{x(x^2+1)}\,dx + \frac{1}{2}\int_\epsilon^R \frac{\sin(x)}{x(x^2+1)}\right],$$

and then make the substitution $x \mapsto -x$ in the second copy of the integral. We then see that:

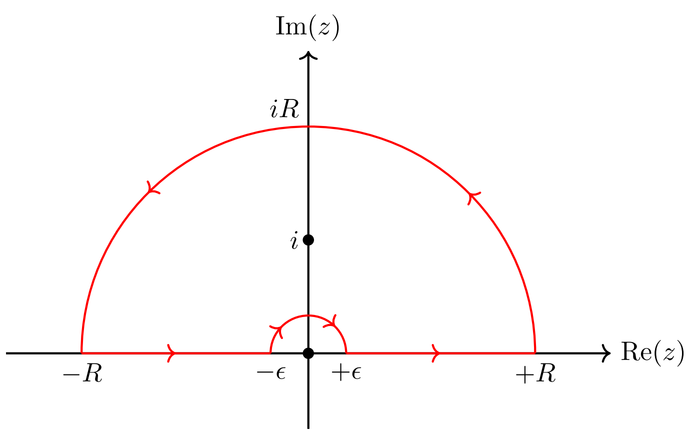
$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}\,dx = \lim_{\substack{R\to\infty\\ \epsilon\to 0}} \left[\frac{1}{2}\int_\epsilon^R \frac{\sin(x)}{x(x^2+1)}\,dx + \frac{1}{2}\int_{-R}^{-\epsilon} \frac{\sin(x)}{x(x^2+1)}\,dx\right] = \frac{1}{2}\mathcal{P}\int_{-\infty}^\infty \frac{\sin(x)}{x(x^2+1)},$$

and it is now possible to apply the principal value technique to evaluate this integral.

With a view to taking the imaginary part at the end of our calculation, consider the contour integral

$$\oint_C \frac{e^{iz}}{z(z^2+1)}\,dz$$

where C is the contour defined by a straight line segment from $-R$ to $-\epsilon$, followed by a semicircular arc in the upper half plane of radius ϵ from $-\epsilon$ to ϵ, then another straight line segment from ϵ to R, finally followed by a semicircular arc in the upper half plane of radius R from R to $-R$.



There is a simple pole of the function at $z = i$ enclosed by the contour C, but no other singularities enclosed by C. The residue at the pole $z = i$ is given by

$$\mathrm{Res}\left(\frac{e^{iz}}{z(z^2+1)}; i\right) = \lim_{z\to i}\left[\frac{e^{iz}}{z(z+i)}\right] = \frac{e^{-1}}{i\cdot 2i} = -\frac{1}{2e}.$$

By the residue theorem then, we have

$$\oint_C \frac{e^{iz}}{z(z^2+1)}\,dz = 2\pi i\left(-\frac{1}{2e}\right) = -\frac{i\pi}{e}.$$

Now consider evaluating the integral on each part of the contour separately. We have

$$-\frac{i\pi}{e} = \oint_C \frac{e^{iz}}{z(z^2+1)}\,dz = \int_{-R}^{-\epsilon} \frac{e^{ix}}{x(x^2+1)}\,dx + \int_{\pi}^{0} \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}(\epsilon^2 e^{2i\theta}+1)} i\epsilon e^{i\theta} d\theta + \int_{\epsilon}^{R} \frac{e^{ix}}{x(x^2+1)}\,dx + \int_{C'} \frac{e^{iz}}{z(z^2+1)}\,dz,$$

where C' is the large semicircular arc part of the contour. We know that the integral on C' converges to zero as $R \to \infty$ by Jordan's Lemma, hence we have

$$-\frac{i\pi}{e} = \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}\,dx + \lim_{\epsilon\to 0}\int_{\pi}^{0} \frac{e^{i\epsilon e^{i\theta}}}{\epsilon^2 e^{2i\theta}+1}\, i d\theta = \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}\,dx - i\pi.$$

It follows that the principal value integral we want is:

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}\,dx = i\pi\left(1-\frac{1}{e}\right).$$

Taking the imaginary part, and dividing by two, we have

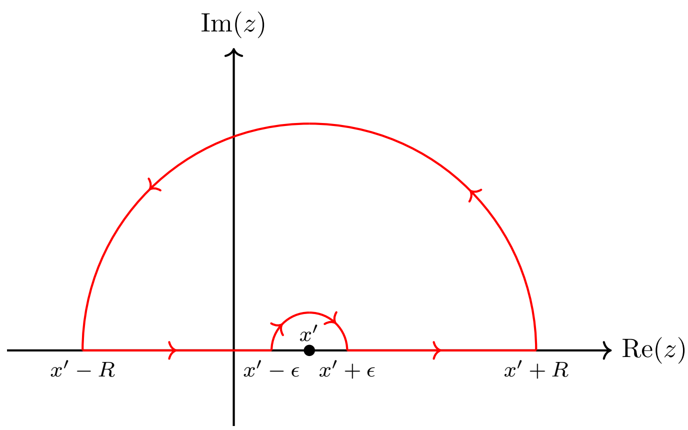
$$\int_{0}^{\infty} \frac{\sin(x)}{x(x^2+1)}\,dx = \frac{1}{2}\operatorname{Im}\left(\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}\,dx\right) = \frac{\pi}{2}\left(1-\frac{1}{e}\right).$$

11. By considering the function
$$\frac{e^{iz}-1}{z},$$
compute the Hilbert transforms of
$$\frac{\cos(x)-1}{x} \quad \text{and} \quad \frac{\sin(x)}{x}.$$

Solution: The easiest way to do this question is to recall the *Kramers-Kronig relations*:

Theorem: Let $f(z)=f(x+iy)=\phi(x,y)+i\psi(x,y)$ be an analytic function with real part $\phi(x,y)$ and imaginary part $\psi(x,y)$ such that $|f(z)|\to 0$ as $|z|\to\infty$ in the upper half plane. Let $u(x)=\phi(x,0)$ and $v(x)=\psi(x,0)$ be the real and imaginary parts of f along the real line. Then the *Kramers-Kronig relations* hold:
$$\mathcal{H}u(x')=-v(x'), \qquad \mathcal{H}v(x')=u(x').$$

Proof: Consider the contour integral
$$\oint_C \frac{f(z)}{z-x'}\,dz,$$
where C is the standard indented contour closed in the upper half plane as shown. Note we indent around the point $z=x'$ on the real axis.



The contour contains no singularities, since we assume $f(z)$ is analytic, and hence by Cauchy's theorem we have
$$\oint_C \frac{f(z)}{z-x'}\,dz=0.$$

Now consider evaluating the integral on each section of the contour separately. We find that:
$$0=\oint_C \frac{f(z)}{z-x'}\,dz=\int_{x'-R}^{x'-\epsilon}\frac{f(x)}{x-x'}\,dx+\int_{\pi}^{0}\frac{f(x'+\epsilon e^{i\theta})}{x'+\epsilon e^{i\theta}-x'}i\epsilon e^{i\theta}\,d\theta+\int_{x'+\epsilon}^{x'+R}\frac{f(x)}{x-x'}\,dx+\int_{0}^{\pi}\frac{f(x'+Re^{i\theta})}{x'+Re^{i\theta}-x'}iRe^{i\theta}\,d\theta.$$

The final integral vanishes as $R\to\infty$, since it is of order $O(f(x'+Re^{i\theta}))$, and we are given that $|f(z)|\to 0$ as $|z|\to\infty$ in the upper half plane.

The integral over the small semicircular arc is given by:

$$\int_{\pi}^{0} \frac{f(x'+\epsilon e^{i\theta})}{x'+\epsilon e^{i\theta}-x'} i\epsilon e^{i\theta}\, d\theta = -i\int_{0}^{\pi} f(x'+\epsilon e^{i\theta})\, d\theta \to -i\pi f(x')$$

as $\epsilon \to 0$.

Finally, notice that the remaining integrals over the straight line parts of the contour converge to a principal value integral. Hence putting everything together we get:

$$0 = \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x-x'}\, dx - i\pi f(x').$$

Rearranging and writing things in terms of real and imaginary parts we have:

$$\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{u(x,0)+iv(x,0)}{x-x'}\, dx = iu(x',0) - v(x',0).$$

Taking real and imaginary parts, and recalling the definition of the Hilbert transform:

$$\mathcal{H}f(x') = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x-x'}\, dx,$$

we see that the Kramers-Kronig relations follow. □

In this question, we set

$$f(z) = \frac{e^{iz}-1}{z}.$$

Notice first that $f(z)$ looks like it has a singularity at $z=0$, but expanding e^{iz}, we see that this is in fact removable; therefore we can treat $f(z)$ as if it is analytic.

We also notice that the condition $|f(z)| \to 0$ as $|z| \to \infty$ in the upper half plane is satisfied, since

$$|f(z)| = \frac{|e^{iz}-1|}{|z|} \leq \frac{|e^{iz}|+1}{|z|} = \frac{|e^{ix}|e^{-y}+1}{|z|} = \frac{e^{-y}+1}{|z|}.$$

This converges to zero as $|z| \to \infty$ in the upper half plane, since the imaginary part y satisfies $y \geq 0$ in the upper half plane, and hence the numerator is always bounded. Hence we can apply the Kramers-Kronig relations to the real and imaginary parts of this $f(z)$.

Along the real line, we have

$$f(x) = \frac{e^{ix}-1}{x} = \frac{\cos(x)-1}{x} + i\frac{\sin(x)}{x}.$$

Hence the Kramers-Kronig relations immediately give us the Hilbert transforms:

$$\mathcal{H}\left(\frac{\cos(x)-1}{x}\right)(x') = -\frac{\sin(x')}{x'}, \qquad \mathcal{H}\left(\frac{\sin(x)}{x}\right)(x') = \frac{\cos(x')-1}{x'}.$$

12. Let $f_1(z)$ be the branch of $(z^2-1)^{1/2}$ defined by branch cuts in the z-plane along the real axis from -1 to $-\infty$ and from 1 to ∞ with $f_1(z)$ real and positive just above the latter cut. Let $f_2(z)$ be the branch of $(z^2-1)^{1/2}$ defined by a cut along the real axis from -1 to $+1$ with $f_2(x)$ real and positive for $x-1$ real and positive. Show that $f_1(z) = f_1(-z)$, but $f_2(z) = -f_2(-z)$.

Solution: Let $f_i(z)$ be a branch of $(z^2-1)^{1/2}$. Then we must have

$$(f_i(z))^2 = z^2 - 1 = (-z)^2 - 1 = (f_i(-z))^2,$$

from which it follows that either $f_i(z) = f_i(-z)$ or $f_i(z) = -f_i(-z)$. Thus all branches of $(z^2-1)^{1/2}$ are either completely even or completely odd functions.

Now we just need to deal with the branches $f_1(z)$, $f_2(z)$ given in the question and decide whether each is odd or even. Since we have proven that the whole branch is either even or odd, we can just verify things for a couple of values, e.g. on the real axis.

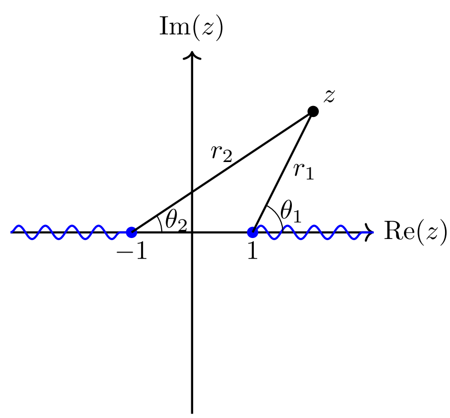
- For the first branch, we are given that $f_1(x)$ is real and non-negative just above the branch cut $(1,\infty)$. In order to find an explicit expression for this branch of $(z^2-1)^{1/2}$, let's write $z-1 = r_1 e^{i\theta_1}$ and $z+1 = r_2 e^{i\theta_2}$. Then

  $$z^2 - 1 = (z-1)(z+1) = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

  It follows that on this branch, we have

  $$(z^2-1)^{1/2} = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}.$$

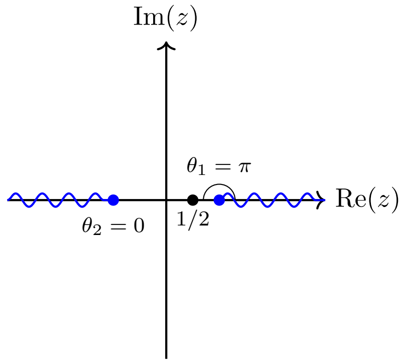
  We then note that the choices $0 \leq \theta_1 \leq 2\pi$ and $-\pi \leq \theta_2 \leq \pi$ make the function continuous everywhere except from across the branches $(1,\infty)$, $(-\infty,-1)$ as we can see from the following diagram:

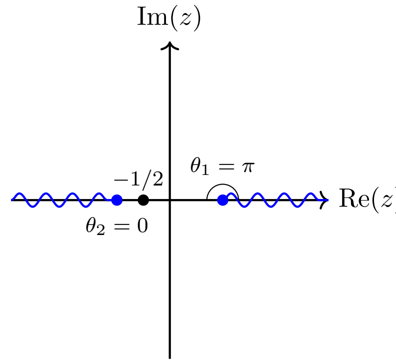
  If we imagine moving the point z around, all of r_1, r_2, θ_1 and θ_2 change continuously, *except* if we cross the branch cuts, where θ_1, θ_2 suddenly jump and we get a discontinuity.

  Now we have a practical description of the branch, we can do some calculations. Recall we only need to evaluate the branch at two points z' and $-z'$ and compare to see whether the whole branch is even or odd. Let's choose to evaluate $f_1(1/2)$ and $f_1(-1/2)$.

The points $z = 1/2$ and $z = -1/2$ are represented in the diagrams below, showing the choice of arguments and moduli that give rise to these points.



(a) $z = 1/2$



(b) $z = -1/2$

We see that $z = 1/2$ has $r_1 = 1/2$, $r_2 = 3/2$, $\theta_1 = \pi$, $\theta_2 = 0$, and $z = -1/2$ has $r_1 = 3/2$, $r_2 = 1/2$, $\theta_1 = \pi$, $\theta_2 = 0$. Hence we have:

$$f_1\left(\frac{1}{2}\right) = \sqrt{\frac{1}{2}\cdot\frac{3}{2}}e^{i(\pi+0)/2} = i\frac{\sqrt{3}}{2}, \qquad f_1\left(-\frac{1}{2}\right) = \sqrt{\frac{3}{2}\cdot\frac{1}{2}}e^{i(0+\pi)/2} = i\frac{\sqrt{3}}{2}.$$

Hence $f_1(1/2) = f_1(-1/2)$. It follows that the branch f_1 of $(z^2-1)^{1/2}$ is *even* everywhere, as required.

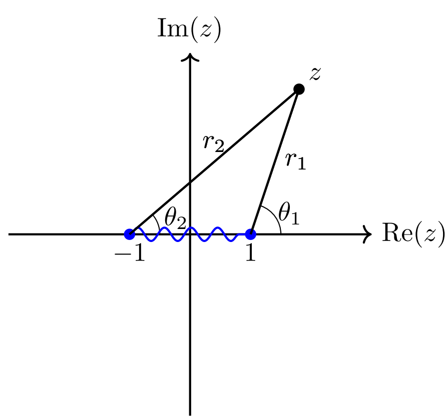
- The second branch is similar. We are given that $f_2(x)$ is real and positive for $x-1$ real and positive, and we are also given that there is a single branch cut from -1 to 1. Again, we aim to find an explicit expression for this branch of $(z^2-1)^{1/2}$, so we write $z-1 = r_1e^{i\theta_1}$ and $z+1 = r_2e^{i\theta_2}$. Then

$$z^2 - 1 = (z-1)(z+1) = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}.$$

It follows that we have:

$$(z^2-1)^{1/2} = \sqrt{r_1r_2}e^{i(\theta_1+\theta_2)/2}.$$

We then note that the choices $0 \le \theta_1 \le 2\pi$ and $0 \le \theta_2 \le 2\pi$ make the function continuous everywhere except from across the branch $(-1,1)$ as we can see from the following diagram:



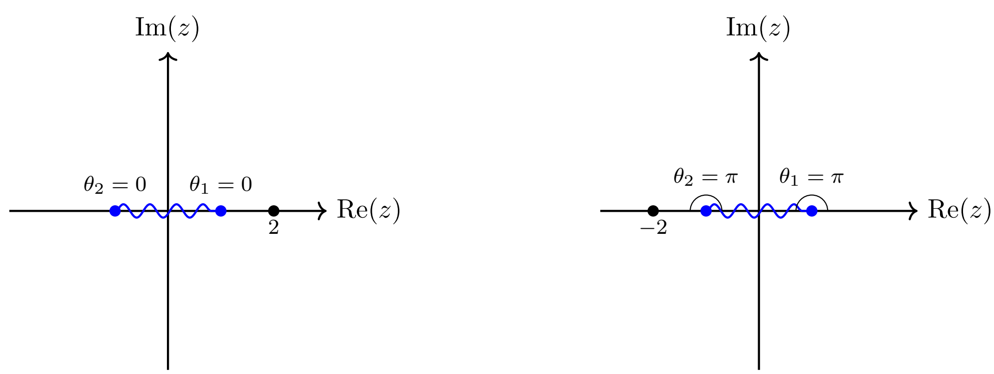
Again, if we imagine moving the point z around, all of r_1, r_2, θ_1 and θ_2 change continuously, *except* (i) if we cross the branch cut, where θ_1, θ_2 suddenly jump and we get a discontinuity; (ii) if we cross the positive real axis excluded from the branch cut $[1, \infty)$.

The reason we get the case (ii) is that if we cross $[1, \infty)$, we have that θ_1 goes from 2π to 0 discontinuously and θ_2 goes from 2π to 0 discontinuously too. *However*, we are lucky because the two discontinuities cancel each other out at the level of the function:

$$\sqrt{r_1 r_2}e^{i(2\pi+2\pi)/2} = \sqrt{r_1 r_2} = \sqrt{r_1 r_2}e^{i(0+0)/2}.$$

Thus there isn't really a discontinuity across $[1, \infty)$. Hence our description of the function indeed gives the required branch.

Now we just evaluate the function at two points z' and $-z'$ again to see whether the branch is odd or even. Let's choose 2 and -2 this time. We can plot the values of r_1, r_2, θ_1, θ_2 for each of 2, -2 on diagrams as follows:



(a) $z = 2$ (b) $z = -2$

Hence we see that $z = 2$ is described by $r_1 = 1, r_2 = 3, \theta_1 = 0, \theta_2 = 0$ and $z = -2$ is described by $r_1 = 3, r_2 = 1$, $\theta_1 = \pi$ and $\theta_2 = \pi$. Hence we have:

$$f_2(2) = \sqrt{1 \cdot 3}e^{i(0+0)/2} = \sqrt{3}, \qquad f_2(-2) = \sqrt{3 \cdot 1}e^{i(\pi+\pi)/2} = -\sqrt{3}.$$

Hence $f_2(2) = f_2(-2)$. It follows that the branch f_2 of $(z^2-1)^{1/2}$ is *odd* everywhere, as required.

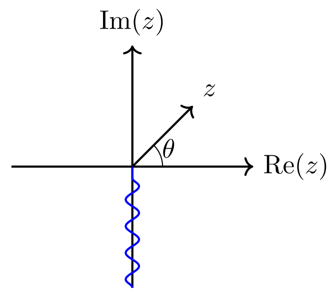
13. $(*)$ By integrating the function

$$\frac{\log^2(z)}{z^2+1}$$

around an appropriate contour, compute the following integrals:

$$\int_0^\infty \frac{\log^m(x)}{x^2+1}\,dx, \qquad \text{for } m = 1, 2.$$

Solution: First, we must choose a branch of the complex logarithm. Let's choose a branch of the logarithm defined such that (i) $\log(z)$ is real for z real and in the interval $(0,\infty)$; (ii) there is a branch cut along the negative imaginary axis $i(-\infty, 0]$. We can picture this as:



In practical terms, this choice of branch can be realised by stating that for an argument $z = re^{i\theta}$ of the logarithm, we always choose $\theta \in [-\pi/2, 3\pi/2)$. Then we have

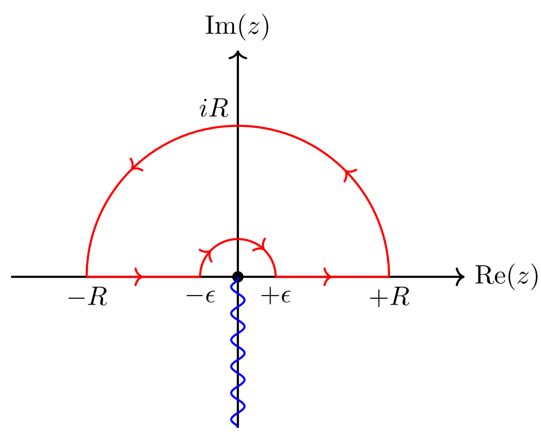
$$\log(z) = \log(r) + i\theta, \qquad \text{with } \theta \in [-\pi/2, 3\pi/2).$$

This gives a discontinuity as we pass through the branch cut, and also gives $\log(x)$ real just above the branch cut.

Now we have appropriately defined our complex logarithm, we can begin our calculation. Consider the contour integral

$$\oint_C \frac{\log^2(z)}{z^2+1},$$

where the contour C is defined as a standard arch contour that we've used plenty of times already on this examples sheet; see the figure below.



This contour encloses one pole of the integrand at $z = i$, which is a simple pole. The residue at this pole is given by:

$$\operatorname{Res}\left(\frac{\log^2(z)}{z^2+1}; i\right) = \lim_{z\to i}\left[\frac{\log^2(z)}{z+i}\right] = \frac{(\log(1) + \frac{i\pi}{2})^2}{2i} = -\frac{\pi^2}{8i}.$$

Using the residue theorem, we immediately have:

$$\oint_C \frac{\log^2(z)}{z^2+1} = -\frac{\pi^3}{4}.$$

Now consider evaluating the integral on each section of the contour at a time. On the line segment $-R$ to $-\epsilon$, we can parametrise the contour as $z = xe^{i\pi}$; note we are being very careful to make our choice of argument clear so that we can take the complex logarithm properly. On the line segment ϵ to R, the contour can be parametrise as $z = xe^{i0}$. Finally, the semicircular parts of the contour can be parametrised as usual: $z = \epsilon e^{i\theta}$, $z = Re^{i\theta}$ respectively.

Hence our contour integral can be written as:

$$-\frac{\pi^3}{4} = \oint_C \frac{\log^2(z)}{z^2+1} = -\int_R^{\epsilon} \frac{(\log(x)+i\pi)^2}{x^2+1}\,dx + \int_{\pi}^{0} \frac{(\log(\epsilon)+i\theta)^2}{\epsilon^2 e^{2i\theta}+1} i\epsilon e^{i\theta}\,d\theta + \int_{\epsilon}^{R} \frac{\log^2(x)}{x^2+1}\,dx + \int_0^{\pi} \frac{(\log(R)+i\theta)^2}{R^2e^{2i\theta}+1} iRe^{i\theta}\,d\theta.$$

Notice that in the limit $R \to \infty$, the final integral contains terms of order:

$$O\left(\frac{\log^2(R)}{R}\right), \qquad O\left(\frac{\log(R)}{R}\right), \qquad O\left(\frac{1}{R}\right).$$

These terms all converge to zero; in particular, the first two converge to zero because polynomial growth is always faster than logarithmic growth.

As $\epsilon \to 0$, the integral on the small semicircle gives a contribution:

$$\lim_{\epsilon\to 0} \int_{\pi}^{0} \frac{(\log(\epsilon)+i\theta)^2}{\epsilon^2 e^{2i\theta}+1} i\epsilon e^{i\theta}\,d\theta = \lim_{\epsilon\to 0} \int_{\pi}^{0} (\log(\epsilon)+i\theta)^2 i\epsilon e^{i\theta}\left(1-\epsilon^2 e^{2i\theta}+O(\epsilon^4)\right)\,d\theta = 0,$$

where in the first step we expanded the denominator using the binomial theorem. The final equality follows because all terms are of order $O(\epsilon)$, $O(\epsilon\log(\epsilon))$ or $O(\epsilon\log^2(\epsilon))$; terms of this type all converge to zero as $\epsilon \to 0$. We can see this by transforming to exponentials; for example, when considering the limit of $\epsilon\log(\epsilon)$ as $\epsilon \to 0$, we can set $\epsilon = e^s$ and instead consider $s \to -\infty$, transforming our limit to:

$$\lim_{\epsilon\to 0}\left[\epsilon\log(\epsilon)\right] = \lim_{s\to-\infty}\left[se^s\right] = 0,$$

where in the last step, we've used the fact that exponential decay is faster than polynomial growth.

The upshot of all this limit analysis is the equality:

$$-\frac{\pi^3}{4} = \int_0^{\infty} \frac{(\log(x)+i\pi)^2}{x^2+1}\,dx + \int_0^{\infty} \frac{\log^2(x)}{x^2+1}\,dx.$$

Expanding $(\log(x)+i\pi)^2$ and comparing real and imaginary parts, we have

$$\int_0^{\infty} \frac{\log(x)}{x^2+1}\,dx = 0, \qquad \int_0^{\infty} \frac{\log^2(x)}{x^2+1}\,dx = \frac{\pi^2}{2}\int_0^{\infty} \frac{1}{x^2+1}\,dx - \frac{\pi^3}{8}.$$

Thus we've found both the integrals we need. The value of the second integral requires a little simplification; to do so, notice that $1/(x^2+1)$ integrates to $\arctan(x)$, and $\arctan(\infty) - \arctan(0) = \pi/2$. Thus we see:

$$\int_0^{\infty} \frac{\log(x)}{x^2+1}\,dx = 0, \qquad \int_0^{\infty} \frac{\log^2(x)}{x^2+1}\,dx = \frac{\pi^3}{8}.$$

Part II: Further Complex Methods
Examples Sheet 2 Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

1. Define the branch of $f(z) = (1-z^2)^{1/2}$ by the branch cut along the real axis from -1 to $-\infty$ and from 1 to ∞, with $f(0) = 1$. Use this branch and a suitably chosen semi-circular contour (with finite radius R greater than 1) in the upper half plane to evaluate:

$$\int_{-1}^{1} (1-x^2)^{1/2}\, dx.$$

Solution: Let's begin by writing down an explicit expression for the branch $f(z) = (1-z^2)^{1/2}$. We note that this function has branch points at $z = \pm 1$, so let's define:

$$z - 1 = r_1 e^{i\theta_1}, \qquad z + 1 = r_2 e^{i\theta_2},$$

where $\theta_1 \in (0, 2\pi)$ and $\theta_2 \in (-\pi, \pi)$. Then we can explicitly write the branch as:

$$f(z) = (1-z^2)^{1/2} = ((1-z)(1+z))^{1/2} = (-r_1 r_2 e^{i(\theta_1+\theta_2)})^{1/2} = \pm i\sqrt{r_1 r_2} e^{i(\theta_1+\theta_2)/2}.$$

At $z = 0$, we have $f(0) = 1$. The coordinates of z in the above parametrisation are $r_1, r_2 = 1$, $\theta_1 = \pi$ and $\theta_2 = 0$. Hence we have:

$$f(0) = 1 = \pm i e^{i\pi/2} = \pm i^2 = \mp 1.$$

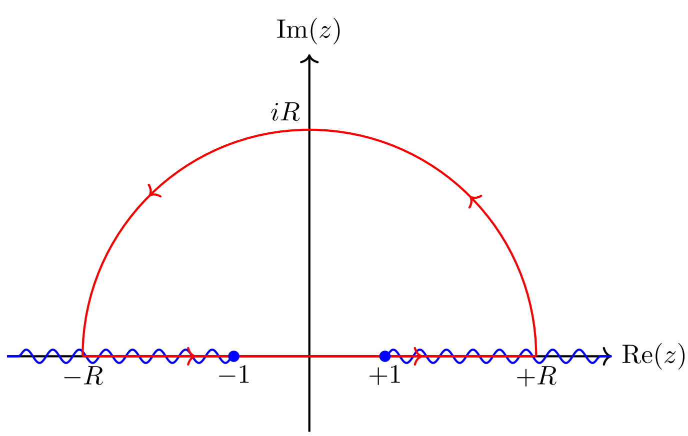
Therefore, we should use the minus sign, and the branch can be explicitly realised as:

$$f(z) = -i\sqrt{r_1 r_2} e^{i(\theta_1+\theta_2)/2} = -i|z^2 - 1|^{1/2} e^{i(\theta_1+\theta_2)/2}.$$

Now consider the integral:

$$\oint_C (1-z^2)^{1/2}\, dz$$

around a semicircular contour C, as shown below:



Technically, we should avoid the branch points and branch cuts by letting the semicircle be a height $\epsilon > 0$ above the real axis, but we can just assume that the diagram above represents the limit $\epsilon \to 0$, as the contour approaches the real axis from above. This prescription then fixes the values of the integrand on the parts of the contour which lie on the branch cut.

First, note that by Cauchy's theorem, since the contour contains no singularities we have:

$$\oint_C (1-z^2)^{1/2}\,dz = 0.$$

We now instead evaluate the integral along each section of the contour in turn. Along the section $[-R,-1]$ of the contour, we use the parametrisation $z=-x$, and we note that $\theta_1=\pi$, $\theta_2=\pi$. Therefore the contribution from this section is:

$$\int_R^1 -i(x^2-1)^{1/2}e^{i(\pi+\pi)/2}\,d(-x) = i\int_1^R (x^2-1)^{1/2}\,dx.$$

Along the section $[-1,1]$ of the contour, we use the parametrisation $z=x$, and we note that $\theta_1=\pi$, $\theta_2=0$. Therefore the contribution from this section is:

$$\int_{-1}^1 -i(1-x^2)^{1/2}e^{i\pi/2}\,dx = \int_{-1}^1 (1-x^2)^{1/2}\,dx.$$

Along the section $[1,R]$ of the contour, we use the parametrisation $z=x$ again, but we instead note that $\theta_1=0$, $\theta_2=0$. Therefore the contribution from this section is:

$$\int_1^R -i(x^2-1)^{1/2}\,dx.$$

Note that this precisely cancels the contribution along the section $[-R,-1]$. Putting these results together, we have:

$$0=\oint_C (1-z^2)^{1/2} = \int_{-1}^1 (1-x^2)^{1/2}\,dx + \int_{C_R}(1-z^2)^{1/2}\,dz \qquad \Rightarrow \qquad \int_{-1}^1 (1-x^2)^{1/2} = -\int_{C_R}(1-z^2)^{1/2}\,dz, \quad (*)$$

where C_R is the large semicircular segment of the contour. It remains the evaluate the integral along the large semicircular part of the contour.

We note that the integrand can be rewritten in the form:

$$(1-z^2)^{1/2} = \left(-z^2\left(1-1/z^2\right)\right)^{1/2} = \pm iz\left(1-\frac{1}{z^2}\right)^{1/2}.$$

To determine the correct sign here, we can evaluate at a point. For example, $(1-(1/2)^2)^{1/2}=\sqrt{3}/2$ on this branch, and as we approach 2 from above, we have $(1/2)\cdot(1-2^2)^{1/2}=-i\sqrt{3}/2$ on this branch. For things to match up, we need to take the minus sign. Now for $R\gg 1$, we can expand the integrand in a Laurent series as:

$$(1-z^2)^{1/2} = -iz\left(1-\frac{1}{z^2}\right)^{1/2} = -iz\left(1+\sum_{r=1}^{\infty}\frac{(\frac{1}{2})(\frac{1}{2}-1)...(\frac{1}{2}-r+1)}{r!}\left(-\frac{1}{z^2}\right)^r\right),$$

by the binomial theorem. Parametrising C_R via $z=Re^{i\theta}$, for $0\le\theta\le\pi$, we find that the contour integral is given by:

$$\int_{C_R}(1-z^2)^{1/2}\,dz = \int_0^{\pi} R^2e^{2i\theta}\left(1+\sum_{r=1}^{\infty}\frac{(\frac{1}{2})(\frac{1}{2}-1)...(\frac{1}{2}-r+1)}{r!}\frac{(-1)^r}{R^{2r}e^{2i\theta r}}+\cdots\right)d\theta.$$

The general term vanishes by periodicity, since we'll integrate things of the form $e^{2i\theta n}$, which cancel out when evaluated at the upper and lower limits. The only exception is the term $r = 1$ in the sum, which gives a contribution:

$$-\frac{1}{2}\int_0^\pi d\theta = -\frac{\pi}{2}.$$

Inserting this into $(*)$, we deduce the required result:

$$\int_{-1}^{1} (1-x^2)^{1/2}\,dx = \frac{\pi}{2}.$$

---

2. The function $\sin^{-1}(z)$ is defined, for $0 \leq \arg(z) < 2\pi$, by

$$\sin^{-1}(z) = \int_0^z \frac{dt}{\sqrt{1-t^2}},$$

where the integrand has a branch cut along the real axis from -1 to $+1$ and takes the value $+1$ at the origin on the upper side of the cut. The path of integration is a straight line for $0 \leq \arg(z) \leq \pi$ and is curved in a positive sense round the branch cut for $\pi < \arg(z) < 2\pi$. Express $\sin^{-1}(e^{i\pi}z)$ ($0 < \arg(z) < \pi$) in terms of $\sin^{-1}(z)$ and deduce that $\sin(\phi - \pi) = -\sin(\phi)$.

[Hint: $\sin^{-1}(e^{i\pi}z) = -\pi + \sin^{-1}(z)$, as can be derived by calculating the integral half way around the cut and remembering that the integrand is an odd function.]

➻ **Solution:** As always with a question involving branch cuts, we should make it completely clear what branch we are working with. The branch of the integrand is specified in this question through the conditions (i) there is a cut from -1 to $+1$; (ii) the value of the integrand is $+1$ at the origin on the upper side of the cut.

As usual, to realise the branch we try to write things in terms of the r_1, r_2, θ_1 and θ_2 variables. Here we expand:

$$(1-t^2)^{-1/2} = ((1-t)(1+t))^{-1/2}.$$

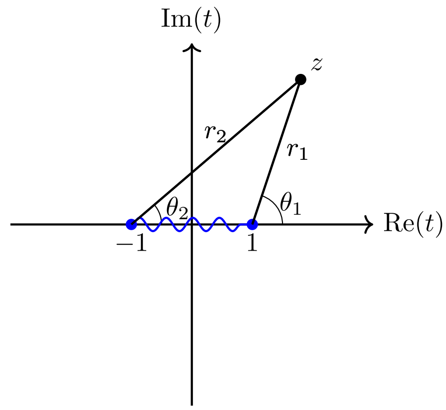
Now we might naïvely set $1 - t = r_1 e^{i\theta_1}$ and $1 + t = r_2 e^{i\theta_2}$, but then the geometry becomes quite fiddly. This is because the first equation gives us a formula for $-t$, relative to 1, rather than our more usual expectation, which is a formula for t relative to -1.

Instead, we can swap the sign for $1 - t$ in the original bracket, but we have to be a bit careful here. We can expand:

$$((1-t)(1+t))^{-1/2} = ((-1)(t-1)(t+1))^{-1/2} = \pm(-1)^{-1/2}((t-1)(t+1))^{-1/2} = \mp i((t-1)(t+1))^{-1/2},$$

where the sign *depends on the branch of the square root that we are using*. We will use the value of the branch at 0 just above the branch cut to fix the sign later on.

Now things are more normal, and we set $t-1=r_1e^{i\theta_1}$, $t+1=r_2e^{i\theta_2}$. We then note that the choices $0\le\theta_1\le 2\pi$ and $0\le\theta_2\le 2\pi$ make the function continuous everywhere except from across the branch $(-1,1)$ as we can see from the following diagram:



From the figure, it's clear that the choice of arguments $0\le\theta_1<2\pi$, $0\le\theta_2<2\pi$ makes the function continuous everywhere except across the branch cut. In particular, the fact that both arguments jump by 2π when we cross the positive real axis prevents a discontinuity.

Therefore, we can write our function explicitly as:

$$(1-t^2)^{-1/2}=\mp i(r_1r_2)^{-1/2}e^{-\frac{1}{2}i(\theta_1+\theta_2)},\qquad 0\le\theta_1,\theta_2<2\pi.$$

At the point $t=0^+$, evaluated just above the branch cut, we require the chosen branch of the function to be equal to $+1$. At $t=0^+$, our parameters are equal to the values $r_1=1$, $r_2=1$, $\theta_1=\pi$ and $\theta_2=0$, from which it follows that the value of the branch there is:

$$\mp i(1\cdot 1)^{-1/2}e^{-\frac{1}{2}i(\pi+0)}=\mp i\cdot(-i)=\mp 1.$$

Thus we choose the bottom sign. It follows that an explicit representation of the integrand, with the correct branch, is:

$$\frac{1}{\sqrt{1-t^2}}=\frac{i}{\sqrt{r_1r_2}}e^{-\frac{1}{2}i(\theta_1+\theta_2)},\qquad 0\le\theta_1,\theta_2<2\pi.$$

A useful thing we notice about this branch is that it is *odd*, which is suggested by the hint. Notice that any branch $f_i(t)$ is such that

$$(f_i(t))^2=\frac{1}{1-t^2}=\frac{1}{1-(-t)^2}=(f_i(-t))^2\qquad\Rightarrow\qquad f_i(t)=\pm f_i(-t).$$

Thus, by continuity, any branch of the function is either odd or even.

It's sufficient, as in Examples Sheet 1, Question 12, to check the values of the function at two points in order to determine its parity everywhere. For our branch, consider evaluating things at 0^+ and 0^-.

- At $t=0^+$, the branch has the value $+1$ by definition.
- At $t=0^-$, the parameters take the values $r_1=r_2=1$, $\theta_1=\pi$ and $\theta_2=2\pi$. Thus we find the branch has the value:

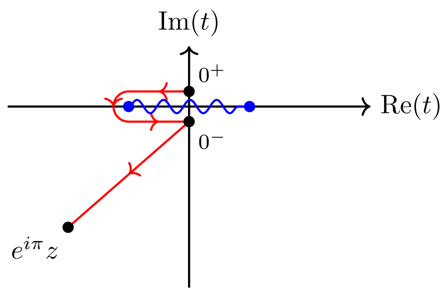
$$ie^{-\frac{1}{2}i(\pi+2\pi)}=-1.$$

It follows that our branch is everywhere *odd*; this will simplify our work a little bit later on.

Now we've got all the technical issues out of the way, we can go ahead and start the question. We are given that z is such that $0 < \arg(z) < \pi$, and we are asked to evaluate

$$\sin^{-1}(e^{i\pi}z).$$

Notice that because $\arg(z) \in (0,\pi)$, we must have that $\arg(e^{i\pi}z) \in (\pi, 2\pi)$. Thus to evaluate the inverse sine of $e^{i\pi}z$, we must use the second definition given in the question - namely, we must wrap halfway around the cut, and then proceed via a straight line segment to $e^{i\pi}z$. This is pictured below:



We now perform this integral. Parametrising on each section of the contour separately, we have:

- On the straight segment on top of the contour, we write $t = x$, for $x \in (-1, 0)$. It follows that $r_1 r_2 = |t-1|\cdot|t+1| = |t^2-1| = 1-x^2$ and $\theta_1 = \pi$, $\theta_2 = 0$ on this section of the contour. Hence we have the integrand:

$$\frac{1}{\sqrt{1-t^2}} = \frac{i}{\sqrt{1-x^2}} \cdot e^{-\frac{1}{2}i(\pi+0)} = \frac{1}{\sqrt{1-x^2}}.$$

  The measure changes as $dt = dx$, and hence the contribution from this section of the contour is simply:

$$\int_0^{-1} \frac{1}{\sqrt{1-x^2}}\,dx.$$

  This is a standard real integral; we can integrate it by using the real version of inverse sine. We get the result:

$$\int_0^{-1} \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(-1) - \arcsin(0) = -\frac{\pi}{2}.$$

- On the semicircular arc part of the contour, we parametrise things as $t = -1 + \epsilon e^{i\theta}$, i.e. $t + 1 = \epsilon e^{i\theta}$, for $\theta \in (\pi/2, 3\pi/2)$. This corresponds to the parameter choice $r_1 = 2$, $r_2 = \epsilon$, $\theta_1 = \pi$, $\theta_2 = \theta$. Hence we have the integrand:

$$\frac{1}{\sqrt{1-t^2}} = \frac{i}{\sqrt{2\epsilon}} \cdot e^{-\frac{1}{2}i(\pi+\theta)} = \frac{e^{i\theta}}{\sqrt{2\epsilon}}.$$

  The measure changes as $dt = i\epsilon e^{i\theta}$, hence we get the contribution from this section of the contour:

$$\int_{\pi/2}^{3\pi/2} \frac{e^{i\theta}}{\sqrt{2\epsilon}} i\epsilon e^{i\theta}\,d\theta = O(\epsilon^{1/2}) \to 0, \qquad \text{as } \epsilon \to 0.$$

- On the lower straight segment of the contour, we write $t = x$ for $x \in (-1, 0)$ again. The parameter values here are such that $r_1 r_2 = |t^2 - 1| = 1 - x^2$, and $\theta_1 = \pi$, $\theta_2 = 2\pi$. Hence the integrand on this section of the contour is:

$$\frac{1}{\sqrt{1-t^2}} = \frac{i}{\sqrt{1-x^2}} \cdot e^{-\frac{1}{2}i(\pi+2\pi)} = -\frac{1}{\sqrt{1-x^2}}.$$

The measure changes as $dt = dx$, and hence the contribution from this section of the contour is:

$$-\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(-1) - \arcsin(0) = -\frac{\pi}{2}.$$

- Finally, on the straight segment joining 0^- to $e^{i\pi}z$, we have the contribution to the integral:

$$\int_{0}^{e^{i\pi}z} \frac{dt}{\sqrt{1-t^2}}.$$

  Now the integrand is an odd function, so making the substitution $t \mapsto -t$, we see that the limits change as $[0, e^{i\pi}z] \mapsto [0, z]$ and the measure changes as $dt = -dt$. Thus we get:

$$\int_{0}^{z} \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(z).$$

  since the integrand is mapped to minus itself, because this branch is odd.

Putting everything together, we see that $\sin^{-1}(e^{i\pi}z) = -\pi + \sin^{-1}(z)$, as suggested in the hint. To get the deduction at the end of this question, take the sine of this equation (which is a single-valued function thankfully) to get $e^{i\pi}z = \sin(\sin^{-1}(z) - \pi)$. Write $z = \sin(\phi)$, and the result follows: $-\sin(\phi) = \sin(\phi - \pi)$.

3. Let $\omega_{m,n} = m\omega_1 + n\omega_2$, where (m, n) are integers not both zero, and let

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \left[\frac{1}{(z-\omega_{m,n})^2} - \frac{1}{\omega_{m,n}^2}\right]$$

be the Weierstrass elliptic function with period (ω_1, ω_2) such that ω_1/ω_2 is not real. Show that, in a neighbourhood of $z = 0$,

$$\wp(z) = \frac{1}{z^2} + \frac{1}{20}g_2 z^2 + \frac{1}{28}g_3 z^4 + O(z^6),$$

where

$$g_2 = 60 \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{\omega_{m,n}^4}, \qquad g_3 = 140 \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{\omega_{m,n}^6}.$$

Deduce that $\wp$ satisfies a first order non-linear ODE:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

➻ **Solution:** Near $z = 0$, we can expand each term in the sum using the binomial theorem:

$$\frac{1}{(z-\omega_{m,n})^2} = \frac{1}{\omega_{m,n}^2}\left(1 - \frac{z}{\omega_{m,n}}\right)^{-2} = \frac{1}{\omega_{m,n}^2}\left(1 + \frac{2z}{\omega_{m,n}} + \frac{3z^2}{\omega_{m,n}^2} + \frac{4z^3}{\omega_{m,n}^3} + \frac{5z^4}{\omega_{m,n}^4} + \cdots\right),$$

by the binomial theorem. Hence we have:

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \left[\frac{2z}{\omega_{m,n}^3} + \frac{3z^2}{\omega_{m,n}^4} + \frac{4z^3}{\omega_{m,n}^5} + \frac{5z^4}{\omega_{m,n}^6} + \cdots\right]$$

Now notice that all terms with odd powers of $\omega_{m,n}$ must vanish, since in the sum, when we sum over the point (m, n), we also sum over the point $(-m, -n)$. Hence we're left with:

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \left[\frac{3z^2}{\omega_{m,n}^4} + \frac{5z^4}{\omega_{m,n}^6} + \cdots\right] = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + O(z^6),$$

using the definitions of g_2, g_3 given in the question. This is the required expression for $\wp(z)$ near $z = 0$.

We are now asked to show that $\wp(z)$ satisfies $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$. Let's start by working out the series expansion of each term in this equation near $z = 0$ for some inspiration. Using the expression we derived above for the series expansion of $\wp(z)$ near $z = 0$, we have:

- $(\wp'(z))^2 = \left(-\frac{2}{z^3} + \frac{g_2}{10}z + \frac{g_3}{7}z^3 + O(z^5)\right)^2 = \frac{4}{z^6} - \frac{2g_2}{5z^2} - \frac{4g_3}{7} + O(z^2).$
- $(\wp(z))^3 = \left(\frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + O(z^6)\right)^3 = \frac{1}{z^6} + \frac{3g_2}{20z^2} + \frac{3g_3}{28} + O(z^2).$

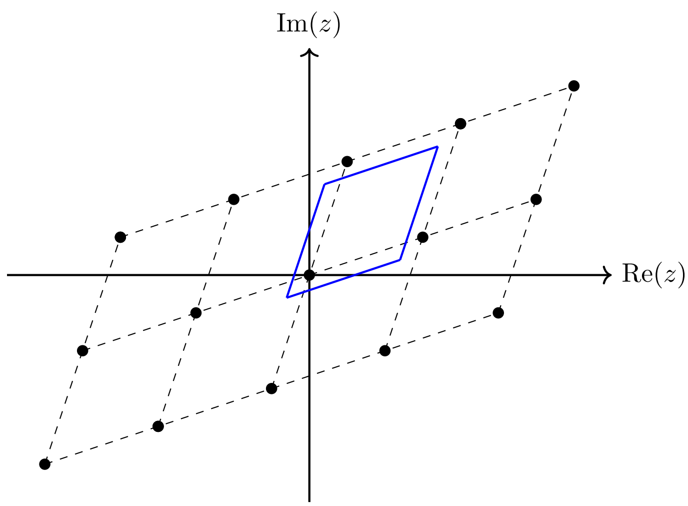
Hence, using these series expansions, we find that the series expansion of $(\wp')^2 - 4\wp^3 + g_2\wp + g_3$ near $z = 0$ is given by:

$$\begin{aligned}(\wp')^2 - 4\wp^3 + g_2\wp + g_3 &= \frac{4}{z^6} - \frac{2g_2}{5z^2} - \frac{4g_3}{7} + O(z^2) - \frac{4}{z^6} - \frac{12g_2}{20z^2} - \frac{12g_3}{28} + O(z^2) + \frac{g_2}{z^2} + O(z^2) + g_3 \\ &= O(z^2).\end{aligned}$$

We can use this series expansion to deduce that the left hand side is actually zero, using a clever argument involving *Liouville's theorem*. Let's first recall the theorem:

Liouville's Theorem: Any bounded, entire function is constant.

To apply the theorem to our problem, let's first note that $\wp(z)$ is an elliptic function with periods ω_1, ω_2. Furthermore, note that we can divide the complex plane $\mathbb{C}$ into parallelogram 'cells' of $\wp(z)$ which each contain exactly one double pole, and are such that $\wp(z)$ is analytic throughout the cell otherwise. Let's focus on the particular cell that contains the double pole at $z = 0$ (the figure shows the parallelograms created by the grid of lattice points $m\omega_1 + n\omega_2$, plus the slightly displaced parallelogram cell that we will focus on):



In this cell, it is certainly the case that $\wp$ and $\wp'$ are both analytic functions everywhere except at $z = 0$. Therefore

$$(\wp')^2 - 4\wp^3 + g_2\wp + g_3$$

is an analytic function throughout the cell, with the possible exception of a singularity at $z = 0$. But we showed from our series expansions above that this combination of Weierstrass functions is actually such that the singularity at $z = 0$ is cancelled out; it follows that this function is analytic throughout the entire cell.

But then by periodicity of elliptic functions, $(\wp')^2 - 4\wp^3 + g_2\wp + g_3$ is an entire function. Furthermore, since it is periodic, it must be bounded by the maximum and minimum values that it takes in a particular cell. It follows, by Liouville's theorem, that it is constant:

$$(\wp')^2 - 4\wp^3 + g_2\wp + g_3 = O(z^2) = C.$$

Taking the limit as $z \to 0$, we see that the constant must be zero. It follows that:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

as required.

4. (a) Show that:

$$4\wp(2z) - \left(\frac{\wp''(z)}{\wp'(z)}\right)^2 + 8\wp(z) = 0.$$

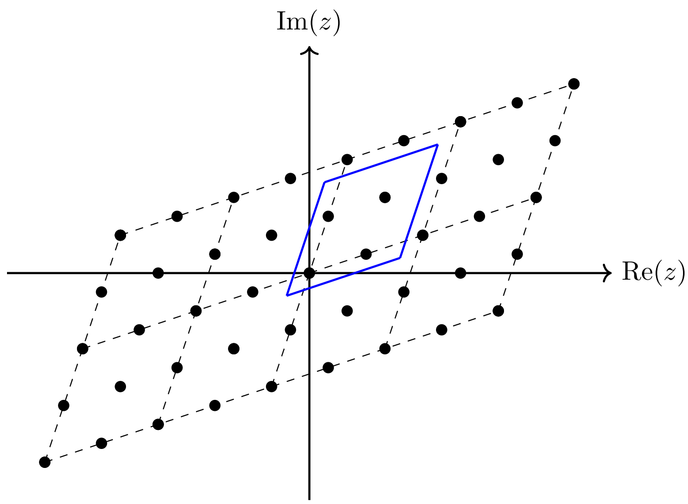
(b) ($*$) Show that:

$$\wp(w+z) = \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}\right)^2 - \wp(z) - \wp(w).$$

The result in (a) is a special case of this.

Solution: (a) We will begin by presenting a solution to part (a) based on the method of Question 3. Note, however, that this is a highly suboptimal argument; it is much better to do this question by proving the more general result in part (b) first, then deducing (a) by taking the limit as $w \to z$.

In part (a), we have to be a bit more careful than we were in Question 3 because the argument of the first Weierstrass elliptic function on the left hand side is $2z$, so its periods are all 'squished', but the arguments of the remaining Weierstrass elliptic functions are all just z, so the periods are not squished. It follows that the left hand side is still doubly periodic with periods ω_1, ω_2, *but*, there could be additional poles in our fundamental cell at the half-points: $\omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2$.



This is shown in the figure above: we now have to check the expansion of the left hand side at each of the points $z = 0$, $z = \omega_1/2$, $z = \omega_2/2$ and $z = (\omega_1+\omega_2)/2$.

In addition, there could *also* be poles in the function on the left hand side arising due to zeroes of $\wp'(z)$. However, during the course of investigating the expansions of $z = 0$, $z = \omega_1/2$, $z = \omega_2/2$ and $z = (\omega_1+\omega_2)/2$ we will discover the locations of the zeroes of $\wp'(z)$, and find that they don't matter in this case.

Recall that the Weierstrass $\wp$ function is given by:

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \left[\frac{1}{(z-\omega_{m,n})^2} - \frac{1}{\omega_{m,n}^2}\right]$$

We will expand near each of the points $z = 0$, $z = \omega_1/2$, $z = \omega_2/2$, $z = (\omega_1 + \omega_2)/2$. Near $z = 0$, we have already found the expansion:

$$\wp(z) = \frac{1}{z^2} + O(z^2),$$

although originally we retained more terms - we'll see that we won't need them in this case. It follows that:

$$\wp'(z) = -\frac{2}{z^3} + O(z), \qquad \wp''(z) = \frac{6}{z^4} + O(1).$$

Hence we have:

$$\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 = \left(\frac{\frac{6}{z^4} + O(1)}{-\frac{2}{z^3} + O(z)}\right)^2 = \left(\frac{\frac{6}{z} + O(z^3)}{-2 + O(z^4)}\right)^2 = \frac{1}{4}\left(\frac{\frac{6}{z} + O(z^3)}{1 + O(z^4)}\right)^2.$$

We can expand the denominator here using the binomial theorem giving $(1 + O(z^4))^{-1} = 1 + O(z^4)$. Thus we're left with:

$$\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 = \frac{1}{4}\left(\left(\frac{6}{z} + O(z^3)\right)(1 + O(z^4))\right)^2 = \frac{1}{4}\left(\frac{6}{z} + O(z^3)\right)^2 = \frac{9}{z^2} + O(z^2).$$

Putting everything together, we see that the function:

$$4\wp(2z) - \left(\frac{\wp''(z)}{\wp'(z)}\right)^2 + 8\wp(z) = 4\left(\frac{1}{4z^2} + O(z^2)\right) - \frac{9}{z^2} + O(z^2) + \frac{8}{z^2} + O(z^2) = O(z^2)$$

is analytic at $z = 0$.

Now let's consider expanding about a general half-point $\zeta_{k,l} = \frac{1}{2}\omega_{k,l}$. Let's consider the problem term $\wp''(z)/\wp'(z)$ first, and thus first consider the expansion of the derivative of the Weierstrass $\wp$ function:

$$\wp'(z) = -\frac{2}{z^3} - \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \left[\frac{2}{(z - \omega_{m,n})^3}\right] = -\sum_{(m,n)\in\mathbb{Z}^2} \frac{2}{(z - \omega_{m,n})^3}$$

Expanding around a half-point, we have

$$\wp'(z) = -\sum_{(m,n)\in\mathbb{Z}^2} \frac{2}{((z - \zeta_{k,l}) - \omega_{m,n} + \zeta_{k,l})^3} = -\sum_{(m,n)\in\mathbb{Z}^2} \frac{2}{(-\omega_{m,n} + \zeta_{k,l})^3}\left(1 + \frac{z - \zeta_{k,l}}{\zeta_{k,l} - \omega_{m,n}}\right)^{-3}.$$

Expanding each term using the binomial theorem, we have:

$$\wp'(z) = \sum_{(m,n)\in\mathbb{Z}^2} \frac{2}{(\omega_{m,n} - \zeta_{k,l})^3}\left(1 + \frac{3(z - \zeta_{k,l})}{(\omega_{m,n} - \zeta_{k,l})} + O((z - \zeta_{k,l})^2)\right)$$

We can argue that the terms with odd powers of $\omega_{m,n} - \zeta_{k,l}$ will all cancel; this is most easily seen by imagining shifting the grid in the figure on page 10 by one of the half periods - we still see that the grid is symmetric on the grid indices (m, n). Note this is unique to the fact we have shifted by a half period! Hence we are left simply with:

$$\wp'(z) = 6\sum_{(m,n)\in\mathbb{Z}^2} \left(\frac{(z - \zeta_{k,l})}{(\omega_{m,n} - \zeta_{k,l})^4} + O((z - \zeta_{k,l})^3)\right).$$

Note that the cube appears in the final big-'O' because all the terms with odd powers of $\omega_{m,n} - \zeta_{k,l}$ disappear. Notice that in particular, we have:

$$\wp'(\zeta_{k,l}) = 0.$$

That is, we have found the zeroes of the derivative of the Weierstrass $\wp$ function! They occur at all half-periods (note, not at full periods, because then our expansion breaks down - it's singular when $\zeta_{k,l} = \omega_{m,n}$ for some m, n).

How do we know that we have found all the zeroes? We use a general result about elliptic functions.

Theorem: The following properties of elliptic functions hold:

(i) The sum of the residues of the poles of an elliptic function in any fundamental cell is zero.

(ii) In any fundamental cell of an elliptic function, there are an equal number of zeroes and poles (counted with multiplicity).

Proof: (i) Let $g(z)$ be an elliptic function. By the residue theorem, the sum of the residues of the poles in a fundamental cell is given by

$$\sum_{\text{poles } z} \text{Res}\,(g; z) = \frac{1}{2\pi i} \oint_C g(z)\, dz$$

where C is a contour going around the edges of a fundamental cell, assumed to be appropriately shifted to avoid poles on the boundary C if necessary. Consider evaluating the right hand side over each section of the fundamental cell:

$$\oint_C g(z)\, dz = \int_{C_1} g(z)\, dz + \int_{C_2} g(z)\, dz + \int_{C_3} g(z)\, dz + \int_{C_4} g(z)\, dz$$

where C_1, C_2, C_3, C_4 are the edges of the fundamental cell, traversed anticlockwise. But now by the double periodicity, it must be the case that

$$\int_{C_1} g(z)\, dz = -\int_{C_3} g(z)\, dz, \qquad \int_{C_2} g(z)\, dz = -\int_{C_4} g(z)\, dz,$$

because we are integrating the same values of the function, just in opposite directions on each contour. The result follows.

(ii) We recall from Examples Sheet 1, the *argument principle*: for any meromorphic function $f(z)$, we have

$$\oint_C \frac{f'(z)}{f(z)}\, dz = 2\pi i \left((\text{number of zeroes of } f(z) \text{ in } C) - (\text{number of poles of } f(z) \text{ in } C)\right),$$

where the number of zeroes and poles is counted *with multiplicity*.

Elliptic functions are certainly meromorphic, so this principle applies to them. But now note that $f'(z)/f(z)$ is an elliptic function with the same periodicity properties as $f(z)$. It follows by (i) that the integral on the left hand side is zero. The result follows. $\square$

Now, from the explicit formula for $\wp(z)$, we know that $\wp'(z)$ only has a pole at the origin. From the expression

$$\wp'(z) = -\frac{2}{z^3} + O(z)$$

we found when we expanded around $z = 0$, we see that this pole is a triple pole. But we have seen that $\wp'(\frac{1}{2}\omega_1) = \wp'(\frac{1}{2}\omega_2) = \wp'(\frac{1}{2}(\omega_1 + \omega_2)) = 0$, so that $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$, $\frac{1}{2}(\omega_1 + \omega_2)$ are three zeroes of $\wp'(z)$. It follows by the above Theorem that these are the *only* zeroes in a fundamental cell.

We can now continue to expand near a generic half-point, safe in the knowledge that we won't be troubled by zeroes of $\wp'(z)$ making our function even more singular (because we already expanding around the zeroes!). Using the expansion we found for $\wp'(z)$, we can take the derivative to find:

$$\wp''(z) = 6 \sum_{(m,n)\in\mathbb{Z}^2} \left(\frac{1}{(\omega_{m,n} - \zeta_{k,l})^4} + O((z-\zeta_{k,l})^2) \right).$$

Hence

$$\frac{\wp''(z)}{\wp'(z)} = \left(6 \sum_{(m,n)\in\mathbb{Z}^2} \left(\frac{1}{(\omega_{m,n} - \zeta_{k,l})^4} + O((z-\zeta_{k,l})^2) \right) \right) \cdot \left(6 \sum_{(m,n)\in\mathbb{Z}^2} \left(\frac{(z-\zeta_{k,l})}{(\omega_{m,n} - \zeta_{k,l})^4} + O((z-\zeta_{k,l})^3) \right) \right)^{-1}.$$

We can expand the denominator using the binomial theorem. We have:

$$\left(6 \sum_{(m,n)\in\mathbb{Z}^2} \left(\frac{(z-\zeta_{k,l})}{(\omega_{m,n} - \zeta_{k,l})^4} + O((z-\zeta_{k,l})^3) \right) \right)^{-1}$$

$$= \frac{1}{6(z-\zeta_{k,l})} \left(\sum_{(m,n)\in\mathbb{Z}^2} \frac{1}{(\omega_{m,n} - \zeta_{k,l})^4} \right)^{-1} \left(1 + O((z-\zeta_{k,l})^2)\right).$$

Hence we find:

$$\frac{\wp''(z)}{\wp'(z)} = \frac{1}{z-\zeta_{k,l}} + O(z-\zeta_{k,l}),$$

which is surprisingly simple given all the work we had to go through! Squaring, we have

$$\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 = \frac{1}{(z-\zeta_{k,l})^2} + O(1).$$

We now try to construct the expansion of the whole final expression

$$4\wp(2z) - \left(\frac{\wp''(z)}{\wp'(z)}\right)^2 + 8\wp(z)$$

around the half-periods $\zeta_{k,l}$. Notice that near a half-period, the expansion of $\wp(2z)$ will be just like the expansion of $\wp(z)$ near a full-period point. Recall that at the full-period point $z = 0$, we have:

$$\wp(2z) = \frac{1}{4z^2} + O(z^2),$$

hence

$$\wp(2z) = \frac{1}{4(z-\zeta_{k,l})^2} + O((z-\zeta k,l)^2)$$

near a half-period point $\zeta k, l$. We also know that $\wp(z)$ itself is finite at the half-periods, so $\wp(z) = \wp(\zeta_{k,l}) + O(z-\zeta_{k,l}) = O(1)$ as $z \to \zeta_{k,l}$. Thus, putting everything together, we have the full expansion:

$$4\wp(2z) - \left(\frac{\wp''(z)}{\wp'(z)}\right)^2 + 8\wp(z) = \frac{4}{4(z-\zeta k,l)^2} + O((z-\zeta k,l)^2) - \frac{1}{(z-\zeta k,l)^2} + O(1) + O(1) = O(1).$$

Hooray! We find that the left hand side is analytic at the half-periods too.

We now put everything together just as in Question 3. We have showed that the elliptic function

$$4\wp(2z) - \left(\frac{\wp''(z)}{\wp'(z)}\right)^2 + 8\wp(z)$$

is analytic, and hence bounded, everywhere inside a fundamental cell. Thus it is entire and bounded, and hence constant by Liouville's Theorem. Recalling the expansion near $z = 0$, we have

$$4\wp(2z) - \left(\frac{\wp''(z)}{\wp'(z)}\right)^2 + 8\wp(z) = O(z^2),$$

so taking the limit as $z \to 0$, we discover that this constant is zero, as required.

(b) ($*$) It is much better to approach this question by starting from (b), the more general result. It is rather difficult to prove this result using the style of argument we used above, because there are now two independent complex variables, z and w, featuring in the equation. Instead, we use a clever trick, described in Section 13.5 of Copson's book, *An Introduction to the Theory of Functions of a Complex Variable*.

Before beginning the proof, it is useful to prove the following general property of elliptic functions:

Theorem: Let $z_1, ..., z_n$ be the zeroes of an elliptic function f in the fundamental cell, with multiplicities $m_1, ..., m_n$, and let $p_1, ..., p_n$ be the poles of f in the fundamental cell, with orders $k_1, ..., k_n$. Then:

$$\sum_{i=1}^{n}(m_i z_i - k_i p_i)$$

is a period of f.

Proof: Let C be a contour running around the edge of the fundamental cell (shifted by t to avoid poles and zeroes on C if necessary). Then we have, by the residue theorem:

$$\oint_C \frac{zf'(z)}{f(z)}\,dz = 2\pi i\left(\sum_{i=1}^{n}\operatorname{Res}\left(\frac{zf'(z)}{f(z)};z_i\right)+\sum_{i=1}^{n}\operatorname{Res}\left(\frac{zf'(z)}{f(z)};p_i\right)\right),$$

since the singularities of $zf'(z)/f(z)$ occur precisely when $f(z)=0$, and when $f'(z)$ has a pole (which occurs when $f(z)$ has a pole). Near a zero of $f(z)$, we have $f(z)=(z-z_i)^{m_i}g(z)$ for g holomorphic, hence:

$$\frac{zf'(z)}{f(z)} = \frac{m_i z(z-z_i)^{m_i-1}g(z)+z(z-z_i)^{m_i}g'(z)}{(z-z_i)^{m_i}g(z)} = \frac{m_i z}{z-z_i}+\frac{zg'(z)}{g(z)}.$$

This shows that the residue at $z=z_i$ is $m_i z_i$. Similarly, near a pole of $f(z)$, we have $f(z)=(z-p_i)^{-k_i}g(z)$ for g holomorphic, hence:

$$\frac{zf'(z)}{f(z)} = -\frac{k_i z(z-p_i)^{-k_i-1}g(z)}{(z-p_i)^{-k_i}g(z)}+\frac{z(z-p_i)^{-k_i}g'(z)}{(z-p_i)^{-k_i}g(z)} = -\frac{k_i z}{z-p_i}+\frac{zg'(z)}{g(z)}.$$

This shows that the residue at $z=p_i$ is $k_i p_i$. Hence we see that:

$$\frac{1}{2\pi i}\oint_C \frac{zf'(z)}{f(z)}\,dz = \sum_{i=1}^{n}(m_i z_i - k_i p_i).$$

On the other hand, we have:

$$\oint_C \frac{zf'(z)}{f(z)}\,dz = \int_{C_1}\frac{zf'(z)}{f(z)}\,dz+\int_{C_2}\frac{zf'(z)}{f(z)}\,dz+\int_{C_3}\frac{zf'(z)}{f(z)}\,dz+\int_{C_4}\frac{zf'(z)}{f(z)}\,dz,$$

where C_1, C_2, C_3, C_4 are the edges of the fundamental cell, traversed anticlockwise. Letting ω_1, ω_2 be the fundamental periods, we have:

$$\oint_C \frac{zf'(z)}{f(z)}\,dz = \int_{t}^{t+\omega_1}\frac{zf'(z)}{f(z)}\,dz+\int_{t+\omega_1}^{t+\omega_1+\omega_2}\frac{zf'(z)}{f(z)}\,dz+\int_{t+\omega_1+\omega_2}^{t+\omega_2}\frac{zf'(z)}{f(z)}\,dz+\int_{t+\omega_2}^{t}\frac{zf'(z)}{f(z)}\,dz.$$

We can use periodicity to combine some of these integrals together. We have:

$$\oint_C \frac{zf'(z)}{f(z)}\,dz = \int_t^{t+\omega_1}\left(\frac{zf'(z)}{f(z)} - \frac{(z+\omega_2)f'(z)}{f(z)}\right)dz - \int_t^{t+\omega_2}\left(\frac{zf'(z)}{f(z)} - \frac{(z+\omega_1)f'(z)}{f(z)}\right)dz$$

$$= \omega_1 \int_t^{t+\omega_2} \frac{f'(z)}{f(z)}\,dz - \omega_2 \int_t^{t+\omega_1} \frac{f'(z)}{f(z)}\,dz$$

$$= \omega_1\left(\log(f(t+\omega_2)) - \log(f(t))\right) - \omega_2\left(\log(f(t+\omega_1)) - \log(f(t))\right).$$

Note that $\log(f)$ can be multi-valued, but $f(t+\omega_2) = f(t)$ so the difference in arguments between $f(t+\omega_2)$, $f(t)$ is an integer multiple of 2π. Thus we have:

$$\oint_C \frac{zf'(z)}{f(z)}\,dz = 2\pi i\left(n\omega_1 + m\omega_2\right).$$

for some integers n, m. The result follows. $\square$

We now begin the main proof. We consider the function:

$$F(z) := \wp'(z)^2 - (A\wp(z) + B)^2 = (\wp'(z) - A\wp(z) - B)(\wp'(z) + A\wp(z) + B),$$

where A, B are complex constants. This function is elliptic, since $\wp(z)$ and $\wp'(z)$ are elliptic. The function also contains a single pole in the fundamental cell, of order six, since $\wp'(z)^2$ features in the definition on the right hand side. In particular, by the Theorem we proved in part (a), we know that there must be six *zeroes* of F in the fundamental cell too.

We can arrange for this function to have some convenient zeroes by a clever choice of the constants A, B. Let $u, v \in \mathbb{C}$ be such that u, v and $u \pm v$ are not poles of $\wp(z)$ (and hence not poles of $\wp'(z)$). Then the simultaneous equations:

$$A\wp(u) + B = -\wp'(u)$$

$$B\wp(v) + B = -\wp'(v)$$

can be solved to determine A, B,[1] yielding the solution:

$$A = -\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}, \qquad B =$$

Fixing these values of A, B, we see that:

$$G(z) = \wp'(z) + A\wp(z) + B$$

is an elliptic function with a single pole in the fundamental cell, of order three. Hence it has three zeroes in the fundamental cell. Two zeroes are given by appropriate translates of u, v. In particular, by the above result, we see that the third zero of $G(z)$ in the fundamental zero is given by an appropriate translate of $-u - v$.

[1] To see why, note that the determinant of the system is given by $\wp(u) - \wp(v)$. Viewing this determinant as a function of u, it is an elliptic function with a single pole, of order 2, in its fundamental cell. Hence it has precisely two zeroes in its fundamental cell. One is given by an appropriate translate of v, and the other is given by an appropriate translate of $-v$ since $\wp$ is an even function. Thus $\wp(u) = \wp(v)$ if and only if $u = v + n\omega_1 + m\omega_2$ or $u = -v + n\omega_1 + m\omega_2$, for $n, m \in \mathbb{Z}$ and ω_1, ω_2 fundamental periods of $\wp$. But we assumed that $u \pm v$ are not poles of $\wp(z)$, hence they are not periods.

Now, since F is an even function, we see that the zeroes of F are given by $\pm u, \pm v, \pm(u+v)$, modulo periods of $\wp$. Recalling from Question 3, we see that $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, we see that we can rewrite F as:

$$F(z) = 4\wp(z)^3 - g_2\wp(z) - g_3 - (A\wp(z) + B)^2,$$

so that the cubic equation:

$$p^3 - \frac{1}{4}g_2 p - \frac{1}{4}g_3 - \frac{1}{4}(Ap + B)^2 = 0$$

has roots $\wp(u), \wp(v), \wp(u+v)$. Reading off the coefficient of p^2, we see that the sum of the roots is given by:

$$\wp(u) + \wp(v) + \wp(u+v) = \frac{1}{4}A^2.$$

Recalling the value of A, we have:

$$\wp(u) + \wp(v) + \wp(u+v) = \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2,$$

as required.

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5. (a) ($*$) Show that there is no such thing as a non-constant triply periodic meromorphic function (with 'triply periodic' being some sensible extension of 'doubly periodic').

(b) ($*$) Show, by considering contour integrals around a suitably defined cell, that a doubly-periodic function cannot have a single pole of order 1 within the cell. Show also that the number of poles and the number of zeroes within the cell are equal (with appropriate counting for repeated roots, poles of order greater than 1, etc).

➾ **Solution:** (a) We will use a proof that is originally due to Jacobi. First, we make a definition:

Definition: Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function. We say that f has a *period* $\omega \in \mathbb{C}\backslash\{0\}$ if:

$$f(z) = f(z + \omega)$$

for all $z \in \mathbb{C}$. We will write the set of all periods of f as $\Omega(f)$.

A basic consequence of this definition that will prove useful is the following:

Proposition: Let $f : \mathbb{C} \to \mathbb{C}$ be meromorphic, and let $\omega, \mu \in \Omega(f)$. Then:

(i) For any non-zero integer n, we have $n\omega \in \Omega(f)$.

(ii) If ω, μ are distinct, we have $\omega - \mu \in \Omega(f)$.

Proof: (i) If n is positive, we have $f(z + n\omega) = f(z + (n-1)\omega + \omega) = f(z + (n-1)\omega) = ... = f(z)$. If n is negative, we have $f(z + n\omega) = f(z + n\omega + \omega) = f(z + (n+1)\omega) = ... = f(z)$.

(ii) By (i), we have that $-\mu$ is a period. Thus $f(z + \omega - \mu) = f(z - \mu) = f(z)$. $\square$

In order to prove the main result, we prove some useful Lemmas first:

Lemma 1: Let $f : \mathbb{C} \to \mathbb{C}$ be meromorphic, and suppose that $\inf\{|\omega| : \omega \in \Omega(f)\} = 0$; that is, the function possesses periods of arbitrarily small modulus. Then f is constant.

Proof: Since f is meromorphic, it possesses only poles (which are *isolated* by definition), and is analytic everywhere else. In particular, we can fix some point $z_0 \in \mathbb{C}$ at which f is analytic. Away from poles, define:

$$g(z) = f(z) - f(z_0).$$

In particular, we have $g(z_0) = 0$ and $g(z_0 + \omega) = 0$ for all $\omega \in \Omega(f)$.

Now, since $\inf\{|\omega| : \omega \in \Omega(f)\} = 0$, there exists a sequence $\omega_n \in \Omega(f)$ such that $\omega_n \to 0$ as $n \to \infty$. Thus the set:

$$\{z_0\} \cup \{z_0 + \omega : \omega \in \Omega(f)\}$$

contains a non-isolated point, z_0. Furthermore, g is identically zero on this set. Thus by the uniqueness of meromorphic continuation, g is identically zero everywhere. It follows that f is constant. $\square$

Lemma 2: Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant meromorphic function. Given any period $\omega \in \Omega(f)$, there exists a unique (up to a sign) period $\tilde{\omega}$ such that $|\tilde{\omega}|$ is minimal and $\omega \in \tilde{\omega}\mathbb{Z} \subseteq \Omega(f)$.

Proof: Let $S = \Omega(f) \cap \{\alpha\omega : \alpha \in (0, 1]\}$ be the set of periods of the form $\alpha\omega$ where $\alpha \in (0, 1]$. Note S is non-empty since $\omega \in S$. We begin by proving that S is finite, so that there is a least $\tilde{\alpha} \in (0, 1]$ such that $\tilde{\alpha}\omega$ is a period.

We note that S is a bounded subset of the complex plane, so by the Bolzano-Weierstrass theorem any sequence in S must possess a convergent subsequence. If S were infinite, we would thus be able to construct a convergent sequence $(\alpha_n\omega)$ where all α_n were distinct. In particular, $(\alpha_{n+1}\omega - \alpha_n\omega)$ would be a convergent sequence of periods (since $\alpha_{n+1}\omega, \alpha_n\omega$ are distinct, so their difference is a period), with limit 0. This contradicts Lemma 1, which tells us that periods cannot have arbitrarily small modulus.

Thus S is finite. Write $\tilde{\omega} = \tilde{\alpha}\omega$ where $\tilde{\alpha}$ is the least element of $(0, 1]$ such that $\tilde{\alpha}\omega$ is a period. In particular, $\pm\tilde{\omega}$ are the unique periods of minimal modulus which are parallel to ω. It remains to show that $\omega \in \tilde{\omega}\mathbb{Z}$.

Suppose not. Then since $\omega, \tilde{\omega}$ are parallel, there exist $n \in \mathbb{Z}, \beta \in (0, 1)$ such that:

$$\omega = n\tilde{\omega} + \beta\tilde{\omega} = n\tilde{\omega} + \beta\tilde{\alpha}\omega.$$

Rearranging, we have $\beta\tilde{\alpha}\omega = \omega - n\tilde{\omega}$. We do not have $\omega = n\tilde{\omega}$, since $\beta\tilde{\omega} \neq 0$. Thus $\beta\tilde{\alpha}\omega = \omega - n\tilde{\omega}$ is a period. But this is a contradiction, since $\tilde{\alpha}$ is the smallest $\alpha \in (0, 1]$ such that $\alpha\omega$ is a period. It follows that $\omega \in \tilde{\omega}\mathbb{Z}$ as required. $\square$

It follows that given any particular non-zero complex number $\omega \in \mathbb{C}$, all periods parallel to this direction are integer multiplies of a single 'fundamental' period. Thus if ω, μ are periods such that $\omega/\mu \in \mathbb{R}$, we must have $\omega, \mu \in \tilde{\omega}\mathbb{Z}$ for some fundamental period $\tilde{\omega}$.

We are now ready to prove the main result:

Jacobi's Theorem: Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant meromorphic function. Then there exist $\omega_1, \omega_2 \in \mathbb{C}$ such that:

$$\Omega(f) = \{n\omega_1 + m\omega_2 : n, m \in \mathbb{Z}\}\backslash\{0\}.$$

In particular, there does not exist a non-constant meromorphic function with three independent 'fundamental' periods, so triply-periodic meromorphic functions do not exist.

Proof: If $\Omega(f) = \emptyset$, then $\omega_1, \omega_2 = 0$ work (the function is totally aperiodic in this case). If $\Omega(f)$ is non-empty, but given any two periods $\omega, \mu \in \Omega(f)$ we have $\omega/\mu \in \mathbb{R}$, then all periods are parallel in $\Omega(f)$, so by Lemma 2 we have $\Omega(f) = \omega_1\mathbb{Z}$ for some $\omega_1 \in \Omega(f)$ (the function is singly-periodic in this case).

Hence, without loss of generality, we may assume that $\Omega(f)$ is non-empty and there exist two periods $\mu_1, \mu_2 \in \Omega(f)$ such that their ratio is not real, $\mu_1/\mu_2 \notin \mathbb{R}$. By Lemma 2, we may also assume that these periods are fundamental; that is, the set of all periods parallel to μ_i is given by $\mu_i\mathbb{Z}$ for $i = 1, 2$. Since the ratio of μ_1, μ_2 is not real, they are linearly independent over $\mathbb{R}$, hence given any period $\omega \in \Omega(f)$ we may write $\omega = \lambda_1\mu_1 + \lambda_2\mu_2$ for real $\lambda_1, \lambda_2 \in \mathbb{R}$.

We now ask if there are any periods in the interior of the fundamental period parallelogram P, with vertices at $0, \mu_1, \mu_2, \mu_1 + \mu_2$. Let $S = \Omega(f) \cap P$. First, we note that if S is empty, we're done; we can take $\omega_1 = \mu_1, \omega_2 = \mu_2$, proving the result (any other period would take the form $m\mu_1 + n\mu_2 + \alpha\mu_1 + \beta\mu_2$ with $m, n \in \mathbb{Z}$, $\alpha, \beta \in [0, 1)$, and hence could be translated to the interior of P, or to an edge of P contradicting the assumption that μ_1, μ_2 were fundamental). On the other hand, if S is infinite, we can repeat the argument of Lemma 1 to show that f is constant, which contradicts our assumptions. Thus we may assume that S is finite and non-empty.

We note that given $\alpha\mu_1 + \beta\mu_2, \alpha'\mu_1 + \beta'\mu_2 \in S$, we cannot have $\beta = \beta'$. For in this case, we would have $|\alpha - \alpha'|\mu_1$ a period, which contradicts our assumption that μ_1 is fundamental. Therefore, there exists a unique element $\tilde{\alpha}\mu_1 + \tilde{\beta}\mu_2 \in S$ with $\tilde{\beta}$ minimal, and $\tilde{\alpha}, \tilde{\beta} \in (0, 1)$.

We now claim that setting $\omega_1 = \mu_1$ and $\omega_2 = \tilde{\alpha}\mu_1 + \tilde{\beta}\mu_2$ works. Certainly ω_1, ω_2 are linearly independent over $\mathbb{R}$, so we can write any period in the form $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}$. Now suppose for a contradiction that there exists some period $\omega \in \Omega(f)$ such that λ_1, λ_2 are not integral. In this case, there exist integers m, n and reals $\alpha, \beta \in [0, 1)$ (not both zero) such that:

$$\omega = m\omega_1 + n\omega_2 + \alpha\omega_1 + \beta\omega_2.$$

But now note that $\alpha\omega_1 + \beta\omega_2$ is a period, given by:

$$\alpha\omega_1 + \beta\omega_2 = (\alpha + \beta\tilde{\alpha})\mu_1 + \beta\tilde{\beta}\mu_2.$$

If $\alpha + \beta\tilde{\alpha} < 1$, then this period lies in P, contradicting the minimality of $\tilde{\beta}$. If $\alpha + \beta\tilde{\alpha} = 1$, this contradicts the assumption that μ_2 was fundamental. If $\alpha + \beta\tilde{\alpha} > 1$, we can translate by multiples of $-\mu_1$ and again deduce a contradiction to the minimality of $\tilde{\beta}$. Thus in all cases, we get a contradiction. The result follows. $\square$

(b) This part of the question follows trivially the results we needed in the first approach to Question 4(a). There, we proved:

- *The sum of the residues of the poles of an elliptic function in any fundamental cell is zero.* In particular, it follows that if there was a single pole of order 1 within a fundamental cell, its residue would have to be zero, so it would be removable.
- *In any fundamental cell of an elliptic function, there are an equal number of zeroes and poles (counted with multiplicity).* This is precisely the required result.

6. By using a contour consisting of the boundary of a quadrant, indented at the origin, show that (for a range of z to be specified):

$$\int_0^\infty t^{z-1}e^{-it}\,dt = e^{-\frac{1}{2}\pi i z}\Gamma(z).$$

Hence evaluate (again for ranges of z to be stated):

$$\int_0^\infty t^{z-1}\cos(t)\,dt \qquad \text{and} \qquad \int_0^\infty t^{z-1}\sin(t)\,dt.$$

Use your results to evaluate

$$\int_0^\infty \frac{\cos(t)}{t^{1/2}}\,dt, \qquad \int_0^\infty \frac{\sin(t)}{t}\,dt, \qquad \text{and} \qquad \int_0^\infty \frac{\sin(t)}{t^{3/2}}\,dt.$$

Solution: Before we begin, we notice that the function t^{z-1} has a branch point at $t=0$; hence, we must first choose a branch of this function. We choose a branch such that (i) there is a cut along the negative real axis and (ii) for $t = x \in \mathbb{R}$, the function is given by x^{z-1}.

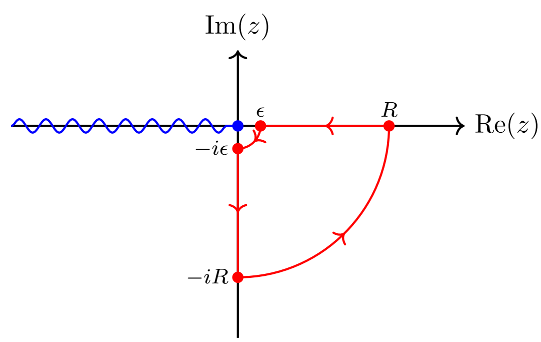
We can realise this branch explicitly by letting $t = re^{i\theta}$, and defining:

$$t^{z-1} = r^{z-1}e^{i\theta(z-1)}, \qquad \text{for } \theta \in (-\pi, \pi).$$

We now proceed to evaluate the first integral. Consider the integral:

$$\oint_C w^{z-1}e^{-iw}\,dw$$

around the contour C shown:



We choose this contour because it will give an integral containing an exponential e^{-t} along the negative imaginary axis - this should give us the gamma function we expect in the answer.

Since the contour encloses no singularities of the integrand, we have by Cauchy's Theorem:

$$\oint_C w^{z-1}e^{-iw}\,dw = 0.$$

Instead, evaluating the integral on each section of the contour separately, we have:

$$0 = \oint_C w^{z-1}e^{-iw}\,dw = \int_R^{\epsilon} t^{z-1}e^{-it}\,dt + \int_0^{-\frac{\pi}{2}} (\epsilon e^{i\theta})^{z-1}e^{-i\epsilon e^{i\theta}}\, i\epsilon e^{i\theta}\,d\theta + \int_{\epsilon}^{R} (te^{-i\pi/2})^{z-1}e^{-t}e^{-i\pi/2}\,dt + \int_{C_R} w^{z-1}e^{-iw}\,dw,$$

where C_R is the large quarter circle part of the contour. The integral over the small quarter circle is of the order:

$$\int_0^{-\frac{\pi}{2}} (\epsilon e^{i\theta})^{z-1}e^{-i\epsilon e^{i\theta}}\, i\epsilon e^{i\theta}\,d\theta = O(\epsilon^z) \to 0 \qquad \text{as } \epsilon \to 0, \text{ for } \mathrm{Re}(z) > 0.$$

The integral over the large quarter circle vanishes in the limit as $R \to \infty$, provided $|w^{z-1}| \to 0$ as $|z| \to \infty$ (by the proof of Jordan's Lemma) - this requirement is fulfilled only when $\mathrm{Re}(z-1) < 0$, i.e. $\mathrm{Re}(z) < 1$.

Thus as $\epsilon \to 0$ and $R \to \infty$, we find for $0 < \mathrm{Re}(z) < 1$:

$$\int_0^{\infty} t^{z-1}e^{-it}\,dt = e^{-\frac{1}{2}\pi iz}\int_0^{\infty} t^{z-1}e^{-t}\,dt = e^{-\frac{1}{2}\pi iz}\Gamma(z),$$

as required. Our derivation was valid in the range $0 < \mathrm{Re}(z) < 1$; indeed, we see that the integral on the left hand side is convergent only when $0 < \mathrm{Re}(z) < 1$, so this is the range of validity of the result.

We now use the result derived above to get the trigonometric integrals. First, we notice that by a completely analogous proof to the above (but instead integrating round a quarter circle in the upper half plane), we immediately have:

$$\int_0^{\infty} t^{z-1}e^{it}\,dt = e^{\frac{1}{2}\pi iz}\Gamma(z).$$

If we add our previous result to this, or subtract our previous result from this, we find:

$$\int_0^{\infty} t^{z-1}\cos(t)\,dt = \cos\left(\frac{1}{2}\pi z\right)\Gamma(z), \qquad \int_0^{\infty} t^{z-1}\sin(t)\,dt = \sin\left(\frac{1}{2}\pi z\right)\Gamma(z).$$

For some strange reason, we are now asked to state the range of z for which these relationships are valid - but surely we already found from earlier that things only work for $0 < \mathrm{Re}(z) < 1$? In fact, the ranges of validity are now slightly different because of analytic continuation! We will now carefully discuss this.

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1. Cosine relationship. First, we don't get any change for the range of validity of the cosine relationship. This is because the cosine integral only converges for $0 < \mathrm{Re}(z) < 1$; we can see this using a short Further Complex Methods style argument:

- Near $z = \infty$, we need the integrand $t^{z-1}\cos(t)$ to converge to zero in order for the contribution to the integral vanish. This occurs only when $\mathrm{Re}(z-1) < 0$, i.e. $\mathrm{Re}(z) < 1$; this condition is also sufficient for convergence by virtue of the fact that we can actually perform the integral using contour integration.

- Near zero, the contribution to the integral is given by

$$\int_0 t^{z-1}\cos(t)\,dt \approx \int_0 t^{z-1} = \left[\frac{t^z}{z}\right]_0 .$$

  We see that we get a finite contribution only when $\mathrm{Re}(z) > 0$. Again, this is also sufficient since we do the integral using contour integration.

Things start to get very complicated on the boundaries, $\mathrm{Re}(z) = 1$ and $\mathrm{Re}(z) = 0$ (we get *oscillatory integrals*, where it is is famously difficult to prove convergence or divergence) and hence we'll simply ignore them in this course. Thus we can only hope that the cosine relationship holds in the region $0 < \mathrm{Re}(z) < 1$; there is no possibility that the relationship could hold in a larger region.

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2. Sine relationship. Things are different for the sine relationship. We note that the sine integral actually converges in the larger region $-1 < \mathrm{Re}(z) < 1$:

- Near $z = \infty$, we get convergence if $\mathrm{Re}(z) < 1$, exactly as for the cosine integral.
- Near zero, the contribution to the integral is given by

$$\int_0 t^{z-1}\sin(t)\,dt \approx \int_0 t^{z-1}\cdot t\,dt = \left[\frac{t^{z+1}}{z+1}\right]_0 .$$

  We see that in this case we get a finite contribution only when $\mathrm{Re}(z+1) > 0$, i.e. when $\mathrm{Re}(z) > -1$.

It follows that

$$\int_0^\infty t^{z-1}\sin(t)\,dt$$

converges in $-1 < \mathrm{Re}(z) < 1$. Furthermore, the integrand is continuous in t and z, and for fixed t the integrand is analytic in z. It follows from the theorem in lectures about functions defined in terms of integrals that this integral defines an analytic function on $-1 < \mathrm{Re}(z) < 1$.

We now note that $\sin(\frac{1}{2}\pi z)\Gamma(z)$ also defines an analytic function on $-1 < \mathrm{Re}(z) < 1$. The only thing that could stop this function being analytic in this range is the point $z = 0$, where $\Gamma(z)$ has a simple pole. But $\sin(\frac{1}{2}\pi z)$ has a zero there too, which actually cancels out the pole; we can see this by trying to calculate the residue at $z = 0$:

$$\lim_{z\to 0}\left[z\sin\left(\frac{1}{2}\pi z\right)\Gamma(z)\right] = \lim_{z\to 0}\left[\frac{z\sin(\frac{1}{2}\pi z)\Gamma(z+1)}{z}\right] = 0.$$

It follows that any singularity at $z = 0$ is removable; thus indeed $\sin(\frac{1}{2}\pi z)\Gamma(z)$ is analytic on $-1 < \mathrm{Re}(z) < 1$.

But we proved that

$$\int_0^\infty t^{z-1}\sin(t)\,dt = \sin\left(\frac{1}{2}\pi z\right)\Gamma(z)$$

in the region $0 < \mathrm{Re}(z) < 1$. We have now seen that both sides are analytic functions on the domain $-1 < \mathrm{Re}(z) < 1$. Hence we have two analytic functions that agree on a subdomain; it follows that they are equal throughout the larger domain by the principle of analytic continuation. *Thus the sine relationship actually holds for all z such that $-1 < \mathrm{Re}(z) < 1$.*

We are now asked to apply our work to the evaluation of three real integrals. We have:

- The first integral is:

$$\int_0^\infty \frac{\cos(t)}{\sqrt{t}}\,dt = \int_0^\infty t^{\frac{1}{2}-1}\cos(t)\,dt = \cos\left(\frac{1}{4}\pi\right)\Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{2}},$$

  using $\cos(\pi/4) = 1/\sqrt{2}$ and $\Gamma(1/2) = \sqrt{\pi}$. Here we have taken $z = 1/2$ which is within the range of validity for the cosine integral.

- The second integral is:

$$\int_0^\infty \frac{\sin(t)}{t}\,dt = \int_0^\infty t^{0-1}\sin(t)\,dt = \lim_{z\to 0}\left[\sin\left(\frac{1}{2}\pi z\right)\Gamma(z)\right] = \lim_{z\to 0}\left[\left(\frac{1}{2}\pi z + O(z^3)\right)\cdot\frac{\Gamma(z+1)}{z}\right] = \frac{\pi}{2}.$$

  Here, we have taken $z = 0$, which is the location of the removable singularity of $\sin\left(\frac{1}{2}\pi z\right)\Gamma(z)$; this is why we used a limit on the right hand side. This is still within the range of validity for the sine integral, though.

- The third and final integral is:

$$\int_0^\infty \frac{\sin(t)}{t^{3/2}}\,dt = \int_0^\infty t^{-1/2-1}\sin(t)\,dt = \sin\left(-\frac{1}{4}\pi\right)\Gamma\left(-\frac{1}{2}\right).$$

  Here, we have used $z = -1/2$, which is in the region which we applied analytic continuation to, hence is within the region of validity of the sine integral. We can compute $\Gamma(-1/2)$ using a standard property of the gamma function:

$$\Gamma(z+1) = z\Gamma(z) \qquad \Rightarrow \qquad \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right) \qquad \Rightarrow \qquad \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}.$$

  Thus we have shown:

$$\int_0^\infty \frac{\sin(t)}{t^{3/2}}\,dt = -\frac{1}{\sqrt{2}}\cdot(-2\sqrt{\pi}) = \sqrt{2\pi}.$$

7. Starting with the Weierstrass canonical product representation of the gamma function, and using the definition of γ, derive Euler's product formula for the gamma function, i.e.

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(1+z)(2+z)...(n+z)}.$$

Solution: Recall that the Weierstrass canonical product formula for the gamma function is given by:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \frac{e^{z/k}}{1+z/k},$$

and the definition of the Euler-Mascheroni constant is:

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log(n)\right).$$

We derive the Euler product formula by reversing the proof we saw in lectures:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \frac{e^{z/k}}{1+z/k}$$

$$= \frac{1}{z} \lim_{n\to\infty} \left[\left(\prod_{k=1}^{n} \frac{1}{1+z/k}\right) \cdot e^{\frac{z}{1}+\frac{z}{2}+...+\frac{z}{n}} e^{-\gamma z}\right] \qquad \text{(converting to limit of partial products)}$$

$$= \frac{1}{z} \lim_{n\to\infty} \left[\left(\prod_{k=1}^{n} \frac{1}{1+z/k}\right) \cdot e^{z\left(\frac{1}{1}+...+\frac{1}{n}\right)} e^{-z\left(\frac{1}{1}+...+\frac{1}{n}-\log(n)\right)}\right] \qquad \text{(using definition of } \gamma\text{)}$$

$$= \frac{1}{z} \lim_{n\to\infty} \left[\left(\prod_{k=1}^{n} \frac{1}{1+z/k}\right) \cdot e^{z\log(n)}\right] \qquad \text{(cancelling harmonic sums)}$$

$$= \lim_{n\to\infty} \left[\frac{n^z}{z(1+z/1)(1+z/2)...(1+z/n)}\right] \qquad \text{(writing out product, and using } e^{z\log(n)} = n^z\text{)}$$

$$= \lim_{n\to\infty} \left[\frac{n^z n!}{z(z+1)(z+2)...(z+n)}\right] \qquad \text{(multiplying top and bottom by } n!\text{)}$$

This is the Euler product formula for the gamma function, as required.

8.

(a) Use Stirling's approximation $\sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}/n! \to 1$ as $n \to \infty$ and Euler's product formula to show that

$$\Gamma_n(z) := \frac{\sqrt{2\pi}e^{-n}n^{z+n+\frac{1}{2}}}{z(z+1)...(z+n)} \to \Gamma(z)$$

as $n \to \infty$. Hence, prove that

$$\frac{2^{2z}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}{\Gamma(2z)}$$

is a constant independent of z. Then, by letting $z \to \frac{1}{2}$ evaluate the relevant constant and thus establish the following identity:

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2z).$$

(b) (*) Furthermore, by constructing

$$\frac{\Gamma_n(z)\Gamma_n\left(z+\frac{1}{m}\right)...\Gamma_n\left(z+\frac{m-1}{m}\right)}{\Gamma_{mn}(mz)},$$

prove the Gauss multiplication formula

$$\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\Gamma\left(z+\frac{2}{m}\right)...\Gamma\left(z+\frac{m-1}{m}\right) = m^{\frac{1}{2}-mz}(2\pi)^{\frac{m-1}{2}}\Gamma(mz)$$

for $m = 1, 2, ...$ and $mz \neq 0, -1, -2, ...$.

Solution: (a) We are first asked to show that $\Gamma_n(z) \to \Gamma(z)$ as $n \to \infty$. This can be achieved by rewriting $\Gamma_n(z)$ slightly:

$$\Gamma_n(z) = \frac{\sqrt{2\pi}e^{-n}n^{z+n+\frac{1}{2}}}{z(z+1)...(z+n)} = \underbrace{\frac{\sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}}{n!}}_{\to 1 \text{ by Stirling's formula}} \cdot \underbrace{\frac{n!\cdot n^z}{z(z+1)...(z+n)}}_{\to\Gamma(z) \text{ by Euler's product formula}}.$$

It follows that we indeed have $\Gamma_n(z) \to \Gamma(z)$ as $n \to \infty$.

Now consider the quotient

$$\frac{2^{2z}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}{\Gamma(2z)}.$$

Using the result we just proved, this can be written in the form:

$$\frac{2^{2z}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}{\Gamma(2z)} = \lim_{n\to\infty}\left[\frac{2^{2z}\Gamma_n(z)\Gamma_n\left(z+\frac{1}{2}\right)}{\Gamma_{2n}(2z)}\right].$$

We choose to write $\Gamma_{2n}(2z)$ (which must also converge to $\Gamma(2z)$ as $n \to \infty$ since $2n$ is a subsequence of n), since these means that the e^{-n}'s in the numerator will cancel with a e^{-2n} in the denominator, and similarly we will have (almost) the same number of $(z+\cdots)$ factors in the numerator as in the denominator, etc. Hence we hope that writing things in this way will aid in cancellation between the numerator and denominator.

Writing out the argument of the limit carefully, we have:

$$\frac{2^{2z}\Gamma_n(z)\Gamma_n\left(z+\frac{1}{2}\right)}{\Gamma_{2n}(2z)} = \frac{2^{2z}\sqrt{2\pi}e^{-n}n^{z+n+1/2}\cdot\sqrt{2\pi}e^{-n}n^{z+n+1}\cdot(2z)(2z+1)...(2z+2n)}{z(z+1)...(z+n)\cdot(z+1/2)(z+3/2)...(z+(2n+1)/2)\cdot\sqrt{2\pi}e^{-2n}(2n)^{2z+2n+1/2}}.$$

Cancelling the factors of $\sqrt{2\pi}$, the exponentials, the powers of 2 and the powers of n, we're left with:

$$\frac{\sqrt{\pi}n}{2^{2n}} \cdot \frac{(2z)(2z+1)...(2z+2n)}{z(z+1)...(z+n)\cdot(z+1/2)(z+3/2)...(z+(2n+1)/2)}.$$

Drawing $2n+2$ factors of 2 into each of the factors in the denominator, we're left with:

$$4\sqrt{\pi}n \cdot \frac{(2z)(2z+1)...(2z+2n)}{(2z)(2z+2)...(2z+2n)\cdot(2z+1)(2z+3)...(2z+2n+1)} = \sqrt{\pi}\left(\frac{4n}{2z+2n+1}\right).$$

Hence we have:

$$\frac{2^{2z}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}{\Gamma(2z)} = \lim_{n\to\infty}\left[\frac{2^{2z}\Gamma_n(z)\Gamma_n\left(z+\frac{1}{2}\right)}{\Gamma_{2n}(2z)}\right] = \lim_{n\to\infty}\left[\frac{4\sqrt{\pi}n}{2z+2n+1}\right] = 2\sqrt{\pi}.$$

This is indeed a constant, independent of z. In particular, we see that the identity

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2z),$$

immediately follows, as required. *[Note in particular that in this derivation, we didn't need to take $z \to 1/2$ as suggested in the question. This is because we were more sensible than the question expected us to be and wrote the denominator as $\Gamma_{2n}(2z)$ as $\Gamma_n(2z)$ - if we naïvely write the denominator as $\Gamma_n(2z)$, then we see that we get something independent of z, but the limit becomes very difficult to evaluate.]*

(b) This is a simple extension of part (a), where we need to be a bit more careful with cancelling everything out. We are asked to consider the quotient:

$$\frac{\Gamma_n(z)\Gamma_n\left(z+\frac{1}{m}\right)...\Gamma_n\left(z+\frac{m-1}{m}\right)}{\Gamma_{nm}(mz)}.$$

Writing everything out in terms of the formula in (a), and gathering up like terms in the product, we have:

$$\frac{(\sqrt{2\pi}e^{-n})^m n^{z+n+\frac{1}{2}} n^{z+\frac{1}{m}+n+\frac{1}{2}}...n^{z+n+\frac{m-1}{m}+\frac{1}{2}}}{\sqrt{2\pi}e^{-nm}(nm)^{mz+nm+\frac{1}{2}}} \cdot \frac{mz(mz+1)...(mz+mn)}{z...(z+n)\left(z+\frac{1}{m}\right)...(z+\frac{1}{m}+n)...\left(z+\frac{m-1}{m}\right)...\left(z+\frac{m-1}{m}+n\right)}.$$

First, let's simplify the factors of $\sqrt{2\pi}$, e, and the powers of n. We have:

$$\frac{(\sqrt{2\pi}e^{-n})^m n^{z+n+\frac{1}{2}} n^{z+\frac{1}{m}+n+\frac{1}{2}}...n^{z+n+\frac{m-1}{m}+\frac{1}{2}}}{\sqrt{2\pi}e^{-nm}(nm)^{mz+nm+\frac{1}{2}}} = \frac{(2\pi)^{\frac{m-1}{2}} n^{mz} n^{nm} n^{m/2} n^{\frac{m(m-1)}{2m}}}{n^{mz+nm+1/2} m^{mz+nm+1/2}} = \frac{(2\pi)^{\frac{m-1}{2}} n^{m-1}}{m^{mz+nm+1/2}},$$

where we summed $1/m + 2/m + ... + (m-1)/m = m(m-1)/2m$ in the exponents of the n's.

Now considering the $(z+\cdots)$ factors, we have:

$$\frac{mz(mz+1)...(mz+mn)}{z...(z+n)\left(z+\frac{1}{m}\right)...(z+\frac{1}{m}+n)...\left(z+\frac{m-1}{m}\right)...\left(z+\frac{m-1}{m}+n\right)}$$

$$= \frac{m^{mn+1}z\left(z+\frac{1}{m}\right)...(z+n)}{z...(z+n)\left(z+\frac{1}{m}\right)...\left(z+\frac{1}{m}+n\right)...\left(z+\frac{m-1}{m}\right)...\left(z+\frac{m-1}{m}+n\right)} = \frac{m^{mn+1}}{\left(z+\frac{1}{m}+n\right)\left(z+\frac{2}{m}+n\right)...\left(z+\frac{m-1}{m}+n\right)},$$

where we drew out $mn+1$ factors of m from the factors in the numerator, and then cancelled off all the factors that matched between the numerator and the denominator. Thus we have:

$$\frac{\Gamma_n(z)...\Gamma_n\left(z+\frac{m-1}{m}\right)}{\Gamma_{nm}(mz)} = \frac{(2\pi)^{\frac{m-1}{2}} m^{nm+1}}{m^{mz+nm+1/2}} \frac{n^{m-1}}{\left(z+\frac{1}{m}+n\right)\left(z+\frac{2}{m}+n\right)...\left(z+\frac{m-1}{m}+n\right)} \to (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-mz},$$

as $n \to \infty$. This establishes the Gauss multiplication formula, as required.

9. Using $t = s\tau$, $s > 0$, it follows that

$$\frac{\Gamma(z)}{s^z} = \int_0^\infty e^{-s\tau}\tau^{z-1}\,d\tau.$$

Letting $z = 1$ and integrating the resulting formula with respect to s from 1 to t, show that

$$\log(t) = \int_0^\infty \left(e^{-\tau} - e^{-t\tau}\right)\frac{d\tau}{\tau}.$$

Using this formula in the expression for $\Gamma'(z)$, prove that

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left(e^{-\tau} - \frac{1}{(1+\tau)^z}\right)\frac{d\tau}{\tau}.$$

Hence, deduce that

$$\gamma = -\int_0^\infty \left(e^{-\tau} - \frac{1}{1+\tau}\right)\frac{d\tau}{\tau}.$$

Solution: The definition of the gamma function is

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$$

in the region $\mathrm{Re}(z) > 0$, plus any meromorphic continuation to $\mathrm{Re}(z) \leq 0$. Letting $t = s\tau$ in the integral, we see that

$$\Gamma(z) = \int_0^\infty s^{z-1}\tau^{z-1}e^{-s\tau}\,sd\tau = s^z\int_0^\infty \tau^{z-1}e^{-s\tau}\,d\tau,$$

and so it follows that

$$\frac{\Gamma(z)}{s^z} = \int_0^\infty \tau^{z-1}e^{-s\tau}\,d\tau$$

as claimed. Following the question, we let $z = 1$ and integrate from $s = 1$ to $s = t$ to give:

$$\int_1^t \frac{\Gamma(1)}{s}\,ds = \int_0^\infty\int_1^t e^{-s\tau}\,dsd\tau.$$

Now $\Gamma(1) = 1$, the integral of $1/s$ is $\log(s)$, and the integral of $e^{-s\tau}$ is $-e^{-s\tau}/\tau$. Hence we have

$$\log(t) = \int_0^\infty \left(e^{-\tau} - e^{-t\tau}\right)\frac{d\tau}{\tau}, \qquad (*)$$

as required.

Taking the derivative of the original integral formula for the gamma function, we find

$$\frac{d}{dz}\Gamma(z) = \int_0^\infty \frac{d(t^{z-1})}{dz}e^{-t}\,dt = \int_0^\infty \frac{d(e^{(z-1)\log(t)})}{dz}e^{-t}\,dt = \int_0^\infty \log(t)t^{z-1}e^{-t}\,dt.$$

Inserting the formula (∗) for $\log(t)$ we derived above, we have:

$$\Gamma'(z) = \int_0^\infty \left(\int_0^\infty \left(e^{-\tau} - e^{-t\tau}\right) \frac{d\tau}{\tau} \right) t^{z-1} e^{-t}\, dt = \int_0^\infty e^{-\tau} \underbrace{\left(\int_0^\infty t^{z-1} e^{-t}\, dt \right)}_{\Gamma(z)} \frac{d\tau}{\tau} - \int_0^\infty \left(\int_0^\infty t^{z-1} e^{-t(1+\tau)}\, dt \right) \frac{d\tau}{\tau}.$$

We see that we can easily pull a $\Gamma(z)$ out of the first term. The inner integral in the second term can be reduced to gamma function form with a substitution $u = t(1+\tau)$. Then we have:

$$\int_0^\infty t^{z-1} e^{-t(1+\tau)}\, dt = \int_0^\infty \frac{u^{z-1}}{(1+\tau)^{z-1}} e^{-u} \frac{du}{1+\tau} = \frac{1}{(1+\tau)^z}\Gamma(z).$$

Substituting this into the formula we derived for $\Gamma'(z)$, we have $\Gamma'(z) = \Gamma(z) \int\limits_0^\infty \left(e^{-\tau} - \frac{1}{(1+\tau)^z}\right) \frac{d\tau}{\tau}$ as required.

In the final part of the question, we are asked to show

$$\gamma = -\int_0^\infty \left(e^{-\tau} - \frac{1}{1+\tau}\right) \frac{d\tau}{\tau}.$$

The integral on the right hand side of this formula suggests we should take $z = 1$ in our previous result; thus we see to establish this formula, it's sufficient to show that $\Gamma'(1)/\Gamma(1) = \Gamma'(1) = -\gamma$. It's easiest to find $\Gamma'(1)$ using the Weierstrass canonical product formula for the gamma function. Recall that this product formula gives the gamma function as:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^\infty \frac{e^{z/k}}{1+z/k}.$$

Taking the logarithm of this product, we can convert it into a sum:

$$\log\left(\Gamma(z)\right) = -\gamma z - \log(z) + \sum_{k=1}^\infty \left(\frac{z}{k} - \log\left(1 + \frac{z}{k}\right)\right)$$

Now taking the derivative with respect to z, we have

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{k=1}^\infty \left(\frac{1}{k} - \frac{1/k}{1+z/k}\right) = -\gamma - \frac{1}{z} + \sum_{k=1}^\infty \left(\frac{1}{k} - \frac{1}{k+z}\right).$$

When $z = 1$, the sum becomes a telescoping sum with value 1, which cancels the -1. Hence we see that $\Gamma'(1)/\Gamma(1) = -\gamma$, which establishes the result, as required.

❇ **Comments:** The Euler-Mascheroni constant γ satisfies lots of interesting relations and often seemingly crops up out of nowhere (see for example the list at Wikipedia: https://en.wikipedia.org/wiki/Euler%E2%80%93Mascheroni_constant#Appearances - my personal favourite is its use in dimensional regularisation of divergent integrals in quantum field theory). We'll see an example of γ cropping up out of nowhere in the next question, where we find a series expansion of the exponential integral.

Despite its ubiquity in mathematics, very little is actually known about the constant γ - in fact, it is not even known whether it is irrational or not!

10. Show that

$$E_1(k) = \int_k^\infty \frac{e^{-t}}{t}\,dt = -\gamma - \log(k) + k - \frac{k^2}{4} + O(k^3) \quad \text{as } k \to 0^+.$$

[Hint: $E_1(k) = \int_k^\infty \frac{dt}{t(t+1)} + \int_0^\infty \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t} - \int_0^k \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t}.$*]*

➼ **Solution:** We can write $E_1(k)$ in the form suggested by the hint as follows:

$$\begin{aligned}
E_1(k) &= \int_k^\infty \frac{e^{-t}}{t}\,dt \\
&= \int_k^\infty \left(\frac{e^{-t}}{t} + \frac{1}{t(t+1)} - \frac{1}{t(t+1)}\right)dt \\
&= \int_k^\infty \frac{1}{t(t+1)}\,dt + \int_k^\infty \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t} \\
&= \int_k^\infty \frac{1}{t(t+1)}\,dt + \int_0^\infty \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t} - \int_0^k \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t},
\end{aligned}$$

which is the form suggested in the hint.

By the result of the previous question, we know that

$$\int_0^\infty \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t} = -\gamma,$$

which gives the value of one of the terms in the hint's expansion of $E_1(k)$. We now evaluate the other terms:

- The first term, i.e. the rational function integral, can be evaluated directly in terms of logarithms:

$$\int_k^\infty \frac{dt}{t(t+1)} = \int_k^\infty \left(\frac{1}{t} - \frac{1}{t+1}\right)dt = \Big[\log(t) - \log(t+1)\Big]_k^\infty = \left[\log\left(\frac{t}{t+1}\right)\right]_k^\infty = \log(1+k) - \log(k).$$

  As $k \to 0^+$, we can expand $\log(1+k)$ using its Taylor series, and so we find the contribution

$$\int_k^\infty \frac{dt}{t(t+1)} = \log(1+k) - \log(k) \sim k - \frac{k^2}{2} - \log(k) + O(k^3) \qquad \text{as } k \to 0^+.$$

- The remaining term we need to analyse is:

$$\int_0^k \left(e^{-t} - \frac{1}{t+1}\right) \frac{dt}{t}.$$

In the limit as $k \to 0^+$, it becomes valid to approximate $1/(t+1)$ by its binomial expansion. Expanding e^{-t} using its Taylor series also, we obtain the expansion of the integral:

$$\int_0^k \left(e^{-t} - \frac{1}{t+1}\right) \frac{dt}{t} = \int_0^k \left(\left(1 - t + \frac{t^2}{2!} + O(t^3)\right) - \left(1 - t + t^2 + O(t^3)\right)\right) \frac{dt}{t} = \int_0^k \left(-\frac{t}{2} + O(t^2)\right) dt.$$

Performing this integral, we have:

$$\int_0^k \left(e^{-t} - \frac{1}{t+1}\right) \frac{dt}{t} = -\frac{k^2}{4} + O(k^3) \qquad \text{as } k \to 0^+.$$

Putting all our results together, we have:

$$\begin{aligned} E_1(k) &= \int_k^\infty \frac{dt}{t(t+1)} + \int_0^\infty \left(e^{-t} - \frac{1}{t+1}\right) \frac{dt}{t} - \int_0^k \left(e^{-t} - \frac{1}{t+1}\right) \frac{dt}{t} \\ &= \left(k - \frac{k^2}{2} - \log(k) + O(k^3)\right) + (-\gamma) + \left(\frac{k^2}{4} + O(k^3)\right) \\ &= -\gamma - \log(k) + k - \frac{k^2}{4} + O(k^3) \qquad \text{as } k \to 0^+, \end{aligned}$$

as required.

❊ **Comments:** The function

$$E_1(k) = \int_k^\infty \frac{e^{-t}}{t}\, dt$$

is called the *exponential integral*. In this question, we are asked to derive the first few terms in a series for the exponential integral as $k \to 0^+$.

The method of this question actually extends easily to deriving a full series expansion for $E_1(k)$ near $k = 0$. We find the result:

$$E_1(k) = -\gamma - \log(k) - \sum_{r=1}^{\infty} \frac{(-k)^r}{r \cdot r!}.$$

We can obtain this series simply by retaining more terms in the expansions of $\log(1+k)$ and $1/(1+t)$ in the above.

11. Derive the formula

$$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}(\theta)\cos^{2q-1}(\theta)\,d\theta$$

and prove that

$$B(z,z) = 2^{1-2z}B\left(z,\frac{1}{2}\right).$$

For which values of z does this result hold?

Solution: Recall that the beta function is defined by

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt,$$

for $\mathrm{Re}(p) > 0, \mathrm{Re}(q) > 0$, and any suitable meromorphic continuation from these ranges. To get the trigonometric form in the question, we let $t = \sin^2(\theta)$. Then the measure changes as $dt = 2\cos(\theta)\sin(\theta)d\theta$, and the limits change as $[0,1] \mapsto [0,\pi/2]$. Thus we find:

$$B(p,q) = 2\int_0^{\pi/2} (\sin^2(\theta))^{p-1}(1-\sin^2(\theta))^{q-1}\sin(\theta)\cos(\theta)\,d\theta = 2\int_0^{\pi/2} \sin^{2p-1}(\theta)\cos^{2q-1}(\theta)\,d\theta,$$

for $\mathrm{Re}(p), \mathrm{Re}(q) > 0$, as required.

Now let $p = q = z$ in the trigonometric result above. Using the double angle formula $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, we find:

$$B(z,z) = 2\int_0^{\pi/2} (\sin(\theta)\cos(\theta))^{2z-1}\,d\theta = \frac{2}{2^{2z-1}}\int_0^{\pi/2} \sin^{2z-1}(2\theta)\,d\theta$$

We want to reduce this to beta function form, hence we make the substitution $u = 2\theta$ and see what happens. The measure changes as $du = 2d\theta$ and the limits change as $[0,\pi/2] \mapsto [0,\pi]$. We find that:

$$B(z,z) = \frac{1}{2^{2z-1}}\int_0^{\pi} \sin^{2z-1}(u)\,du.$$

Now notice that $\sin(u)$ is symmetric in the line $u = \pi/2$; hence we can write the integral on the right hand side as:

$$\int_0^{\pi} \sin^{2z-1}(u)\,du = 2\int_0^{\pi/2} \sin^{2z-1}(u)\,du = 2\int_0^{\pi/2} \sin^{2z-1}(u)\cos^{2(1/2)-1}(u)\,du = B\left(z,\frac{1}{2}\right).$$

It follows that $B(z,z) = 2^{1-2z}B(z,1/2)$ as required.

We used the trigonometric form of the beta function in the derivation of this result, so superficially it looks like we require $\mathrm{Re}(z) > 0$ for the result to be true. However, we can view $B(z,z)$ and $2^{1-2z}B(z,1/2)$ as separate meromorphic functions of z. We have shown, using the trigonometric form of the beta function, that these meromorphic functions agree on the domain $\mathrm{Re}(z) > 0$. Hence by uniqueness of meromorphic continuation, any meromorphic continuations of these functions from this domain must agree.

From lectures, we know that the beta functions $B(z,z)$ and $B(z,1/2)$ have meromorphic continuations to the whole complex plane $\mathbb{C}$, except for poles at $z = 0, -1, -2, \ldots$ (we can see this from the identity $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ for example). Hence the result is valid for all $z \in \mathbb{C}\backslash\{0,-1,-2,\ldots,\}$.

12. Show, using properties of the B function, that

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{\sqrt{32\pi}} \left(\Gamma\left(\frac{1}{4}\right)\right)^2.$$

Using the change of variable $x = t(2-t^2)^{-1/2}$, deduce that

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{\pi}}\left(\Gamma\left(\frac{5}{4}\right)\right)^2, \quad \text{where } K(k) \text{ is the complete elliptic integral} \quad K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Solution: There are lots of substitutions that convert the first integral into beta function form. We will choose to let $t = x^4$. Then the new measure is given by $dt = 4x^3 dx = 4t^{3/4} dx$, and the limits change as $[0,1] \mapsto [0,1]$. Thus we have:

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^1 \frac{dt}{4t^{3/4}\sqrt{1-t}} = \frac{1}{4}\int_0^1 t^{-3/4}(1-t)^{-1/2}\,dt = \frac{1}{4}B\left(\frac{1}{4},\frac{1}{2}\right).$$

Using the result from Question 7, namely $B(z,z) = 2^{1-2z}B(z,1/2) \qquad \Rightarrow \qquad B(z,1/2) = 2^{2z-1}B(z,z)$, we find

$$\frac{1}{4}B\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{4}\cdot 2^{-1/2}B\left(\frac{1}{4},\frac{1}{4}\right) = \frac{1}{\sqrt{32}}B\left(\frac{1}{4},\frac{1}{4}\right).$$

Finally, we use the result from lectures: $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$, which gives us:

$$\frac{1}{\sqrt{32}}B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{\sqrt{32}}\frac{\Gamma\left(\frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{32\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2,$$

where in the last equality we recalled from lectures that $\Gamma(1/2) = \sqrt{\pi}$. Hence we have the required value of the integral.

We are now asked to make a change of variables $x = t(2-t^2)^{-1/2}$ to find the value of another, related integral. Under this change of variables, the measaure changes as

$$dx = \left((2-t^2)^{-1/2} + t^2(2-t^2)^{-3/2}\right)dt = (2-t^2)^{-3/2}(2-t^2+t^2)dt = 2(2-t^2)^{-3/2}\,dt.$$

The limits transform as $[0,1] \mapsto [0,1]$. Hence our integral becomes:

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 2\int_0^1 \frac{(2-t^2)^{-3/2}}{\sqrt{1-t^4(2-t^2)^{-2}}}\,dt = 2\int_0^1 \frac{dt}{\sqrt{(2-t^2)((2-t^2)^2-t^4)}}$$
$$= 2\int_0^1 \frac{dt}{\sqrt{(2-t^2)(4-4t^2)}} = \frac{1}{\sqrt{2}}\int_0^1 \frac{dt}{\sqrt{\left(1-\frac{1}{2}t^2\right)(1-t^2)}} = \frac{1}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right).$$

Thus we have related the initial integral to the 'complete elliptic integral' given in the question. We have already evaluated the initial integral, and hence we immediately have:

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}\Gamma\left(\frac{1}{4}\right)^2}{\sqrt{32\pi}} = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}}.$$

To finish, we use the property $\Gamma(z+1) = z\Gamma(z)$, which gives $\Gamma\left(\frac{5}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$. Hence we have

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{\pi}}\left(\Gamma\left(\frac{5}{4}\right)\right)^2,$$

as required.

Part II: Further Complex Methods
Examples Sheet 3 Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

1. (a) Prove that for $\mathrm{Re}(z) > 1$,

$$\frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1}\, dt = \frac{\Gamma(1-z)}{2\pi i} \int_\gamma \frac{t^{z-1}}{e^{-t} - 1}\, dt,$$

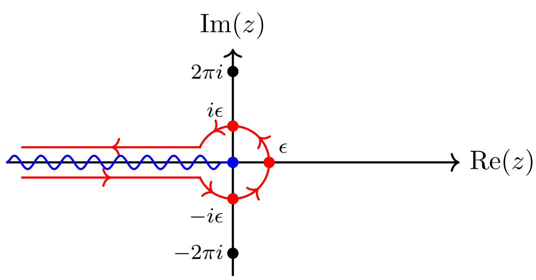
where γ denotes the Hankel contour. Hence deduce that the RHS of the above equation provides the analytic continuation of Riemann's zeta function.

(b) The Bernoulli numbers B_n are defined by

$$\frac{1}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^{m-1}}{m!},$$

and $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_{2m+1} = 0$ for $m = 0, 1, 2, \dots$. Use (a) and the residue theorem to compute $\zeta(-n)$ for $n = 0, 1, 2, \dots$ in terms of B_n. Hence, deduce that the negative even integers are zeroes of $\zeta(z)$.

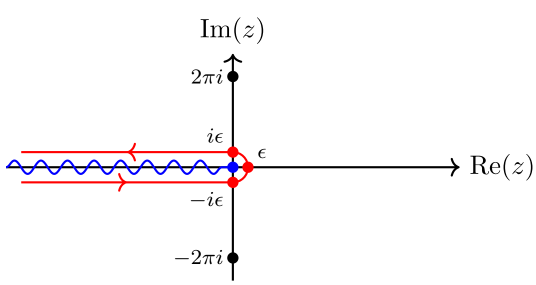
Solution: (a) To prove the first equality, we do the contour integral on the right hand side. Recall that the *Hankel contour* looks like:



The branch cut pictured is for the power function t^{z-1}. The choice of arguments is such that $t = re^{i\pi}$ just above the branch cut, and $t = re^{-i\pi}$ just below the branch cut. The contour consists of a straight line segment from $-\infty$ to $-\epsilon$ just beneath the branch cut, followed by a circle of radius ϵ centred on the origin traverse anticlockwise, and finally a straight line segment from $-\epsilon$ to $-\infty$ just above the branch cut.

Note that the integrand $t^{z-1}/(e^{-t} - 1)$ has singularities whenever $e^{-t} = 1$, i.e. whenever $t = 2\pi i n, n \in \mathbb{Z}$. The Hankel contour is defined to only encircle the one at $t = 0$ (which coincides with the branch point at $t = 0$).

This realisation of the Hankel contour is not unique. By Cauchy's Theorem, it's always possible to deform the above contour into lots of other shapes; in particular, the small circular part can be squished into lots of other interesting shapes, e.g.



However, some shapes of the Hankel contour can be more *convenient* to use than others. In particular, if we use a semicircular arc near the origin rather than a circular arc near the origin, then the integrals on the upper and lower straight segments are not regulated near zero. Compare the contribution

$$\int_{\epsilon}^{\infty} f(xe^{i\pi})\,dx$$

from the upper straight segment in the case the Hankel contour has a circular arc around zero, and the contribution

$$\int_{0}^{\infty} f(xe^{i\pi})\,dx$$

from the upper straight segment in the case the Hankel contour has a semicircular arc around zero.

In the circular case, the integral *could* diverge as $\epsilon \to 0$; then, we would hope that the divergence would cancel with the contribution as $\epsilon \to 0$ from the lower straight segment of the contour to give something finite. This will happen explicitly on Sheet 3.

In the semicircular case we'd get rubbish for our answers, and wouldn't be able to get a meaningful cancellation. Hence we conclude: *it is better to work with a circular arc near the origin when picking our realisation of the Hankel contour*.

Now we've established the branch of the function we're using and the contour we're integrating over, let's actually do the integral. We let $t = xe^{i\pi}$ on the upper straight segment of the contour, $t = xe^{-i\pi}$ on the lower straight segment of the contour, and $t = \epsilon e^{i\theta}$ on the circular part of the contour. Then we can write the integral as:

$$\int_{\gamma} \frac{t^{z-1}}{e^{-t}-1}\,dt = e^{-i\pi z}\int_{\infty}^{\epsilon} \frac{x^{z-1}}{e^{x}-1}\,dx + e^{i\pi z}\int_{\epsilon}^{\infty} \frac{x^{z-1}}{e^{x}-1}\,dx + \int_{-\pi}^{\pi} \frac{\epsilon^{z-1}e^{i\theta(z-1)}}{e^{-\epsilon e^{i\theta}}-1}\,\epsilon i e^{i\theta}\,d\theta.$$

We can now deform the Hankel contour in such a way that $\epsilon \to 0$. As $\epsilon \to 0$, the third term on the right hand side behaves as:

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{\epsilon^{z-1}e^{i\theta(z-1)}}{e^{-\epsilon e^{i\theta}}-1}\,\epsilon i e^{i\theta}\,d\theta &= \int_{-\pi}^{\pi} \frac{\epsilon^{z-1}e^{i\theta(z-1)}}{-\epsilon e^{i\theta}+\frac{1}{2}\epsilon^2 e^{2i\theta}+O(\epsilon^3)}\,\epsilon i e^{i\theta}\,d\theta && \text{(expanding exponential)}\\
&= -i\int_{-\pi}^{\pi} \frac{\epsilon^{z-1}e^{i\theta(z-1)}}{1-\frac{1}{2}\epsilon e^{i\theta}+O(\epsilon^2)}\,d\theta && \text{(cancelling } \epsilon e^{i\theta}\text{)}\\
&= -i\epsilon^{z-1}\int_{-\pi}^{\pi} e^{i\theta(z-1)}\left(1+\frac{1}{2}\epsilon e^{i\theta}+O(\epsilon^2)\right)d\theta && \text{(binomial theorem on denominator)}\\
&= O(\epsilon^{z-1})
\end{aligned}$$

Thus this term converges to zero as $\epsilon \to 0$, provided $\mathrm{Re}(z) > 1$. Hence in the limit as $\epsilon \to 0$, we obtain:

$$\int_{\gamma} \frac{t^{z-1}}{e^{-t}-1}\,dt = (e^{i\pi z}-e^{-i\pi z})\int_{0}^{\infty} \frac{x^{z-1}}{e^{x}-1}\,dx = 2i\sin(\pi z)\int_{0}^{\infty} \frac{x^{z-1}}{e^{x}-1}\,dx.$$

Hence we have shown:

$$\frac{\Gamma(1-z)}{2\pi i}\int\limits_{\gamma}\frac{t^{z-1}}{e^{-t}-1}\,dt = \frac{\Gamma(1-z)}{2\pi i}\cdot 2i\sin(\pi z)\int\limits_{0}^{\infty}\frac{x^{z-1}}{e^{x}-1}\,dx = \frac{\sin(\pi z)\Gamma(1-z)}{\pi}\int\limits_{0}^{\infty}\frac{x^{z-1}}{e^{x}-1}\,dx.$$

To finish, we use the *reflection formula* for the gamma function: $\Gamma(z)\Gamma(1-z) = \pi\mathrm{cosec}(\pi z)$. Thus we have established, for $\mathrm{Re}(z) > 1$

$$\frac{\Gamma(1-z)}{2\pi i}\int\limits_{\gamma}\frac{t^{z-1}}{e^{-t}-1}\,dt = \frac{1}{\Gamma(z)}\int\limits_{0}^{\infty}\frac{x^{z-1}}{e^{x}-1}\,dx,$$

as required.

We are now asked to explain why this provides the analytic continuation of the zeta function. We recognise that the right hand side is equal to the Riemann zeta function $\zeta(z)$ for $\mathrm{Re}(z) > 1$, as we saw in lectures (and is proved again in these solutions as part of Question 9). Thus it follows that:

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i}\int\limits_{\gamma}\frac{t^{z-1}}{e^{-t}-1}\,dt,$$

for $\mathrm{Re}(z) > 1$. But now notice that the right hand side is in fact analytic for all $z \in \mathbb{C}\backslash\{1\}$. We can prove this as follows:

Theorem: The integral

$$\frac{\Gamma(1-z)}{2\pi i}\int\limits_{\gamma}\frac{t^{z-1}}{e^{-t}-1}\,dt$$

is an analytic function of z for all $z \in \mathbb{C}\backslash\{1\}$.

Proof: Notice that the integrand is a continuous function of t and z, for all t on the contour γ (which in particular does not include any locations where $e^{-t}-1=0$ or $t=0$), and all $z \in \mathbb{C}$. Notice also that the integrand is analytic in z for each fixed t.

Finally, note that the integral is convergent; we can prove this using the Further Complex Methods technique as follows. There are no singularities in the integrand for this contour, and hence the only thing that could stop the integral converging is the infinite range. At the start of the contour, we get contribution:

$$\int\limits_{\text{start of }\gamma}\frac{t^{z-1}}{e^{-t}-1}\,dt = e^{-i\pi z}\int\limits_{\infty}\frac{x^{z-1}}{e^{x}-1}\,dx \sim 0,$$

since the integrand is exponentially suppressed and goes to zero for $x \sim \infty$, i.e. $x^{z-1}/(e^{x}-1) \approx 0$. Similarly we get a zero contribution from the end of the contour. This occurs for all $z \in \mathbb{C}$, hence we have convergence of the integral for all $z \in \mathbb{C}$.

It follows by the Theorem in lectures for a function defined in terms of an integral that

$$\int\limits_{\gamma}\frac{t^{z-1}}{e^{-t}-1}\,dt$$

is an analytic function of z for all $z \in \mathbb{C}$.

To finish, we must consider the effect of multiplying by $\Gamma(1-z)/2\pi i$. Notice $\Gamma(1-z)$ is an analytic function for all $z \notin \{1, 2, 3, ...\}$ and hence

$$\frac{\Gamma(1-z)}{2\pi i} \int_\gamma \frac{t^{z-1}}{e^{-t}-1}\, dt$$

is analytic for all $z \in \mathbb{C} \backslash \{1, 2, 3, ...\}$.

To get analyticity at the points $z \in \{2, 3, ...\}$, we consider the residues at $z = 2, 3....$ If these residues are zero, it follows that there aren't poles at $z = 2, 3, ...$ and hence we have removable singularities at $z = 2, 3, ...$; we can then view the function as analytic at these points anyway.

Checking the residue at $z = n \in \{2, 3, ...\}$, we must consider:

$$\lim_{z \to n} \left[(z-n) \cdot \frac{\Gamma(1-z)}{2\pi i} \int_\gamma \frac{t^{z-1}}{e^{-t}-1}\, dt \right].$$

Notice that $(z-n)\Gamma(1-z)$ tends to a finite, non-zero number as $z \to n$ for $n \in \{1, 2, 3, ...\}$ since $\Gamma(1-z)$ has poles at $z = 1, 2, 3, ...$. Note also that

$$\frac{t^{n-1}}{e^{-t}-1} = \frac{t^{n-1}}{-t + \frac{1}{2}t^2 + O(t^3)} = \frac{t^{n-2}}{-1 + \frac{1}{2}t + O(t^2)}.$$

So provided $n \in \{2, 3, ...\}$ (note $n \neq 1$), the integrand loses its singularity at $t = 0$ that we naïvely expect from $e^{-t} - 1$. It follows by Cauchy's theorem that

$$\int_\gamma \frac{t^{n-1}}{e^{-t}-1}\, dt = 0$$

for $n \in \{2, 3, ...\}$ since γ encloses no singularities. If $n = 1$, the integrand has a simple pole at $t = 0$, and hence we get a non-zero answer for the integral. Thus we can deduce:

$$\begin{aligned} \lim_{z \to n} \left[(z-n) \cdot \frac{\Gamma(1-z)}{2\pi i} \int_\gamma \frac{t^{z-1}}{e^{-t}-1}\, dt \right] &= \begin{cases} \text{finite non-zero} \times \text{zero} & \text{for } n \in \{2, 3, ...\} \\ \text{finite non-zero} \times \text{finite non-zero} & \text{for } n = 1 \end{cases} \\ &= \begin{cases} 0 & \text{if } n \in \{2, 3, ...\} \\ \text{non-zero} & \text{for } n = 1. \end{cases} \end{aligned}$$

Thus we see that the function is completely analytic for $z \in \mathbb{C} \backslash \{1\}$, and has a simple pole when $z = 1$ as required. □

We now argue using analytic continuation. We have shown that

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i} \int_\gamma \frac{t^{z-1}}{e^{-t}-1}\, dt$$

for $\mathrm{Re}(z) > 1$. We have also show that the right hand side is analytic for $z \in \mathbb{C} \backslash \{1\}$. The zeta function is defined by a series in $\mathrm{Re}(z) > 1$ plus *any analytic continuation from this domain*. Hence we have two analytic functions defined on all of $z \in \mathbb{C} \backslash \{1\}$ agreeing on $\mathrm{Re}(z) > 1$; it follows by the principle of analytic continuation that they agree everywhere in $\mathbb{C} \backslash \{1\}$. Thus we have indeed constructed the analytic continuation of the zeta function.

(b) We are now asked to use the Hankel representation of $\zeta(z)$, proved in part (a), to find some special values of the zeta function, namely those at the negative integers, $\zeta(-n)$.

The Hankel representation gives:

$$\begin{aligned}\zeta(-n) &= \frac{\Gamma(1+n)}{2\pi i}\int\limits_{\gamma}\frac{t^{-n-1}}{e^{-t}-1}\,dt\\ &= \frac{n!}{2\pi i}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}}{m!}B_m\int\limits_{\gamma}\frac{t^{m-1}}{t^{n+1}}\,dt \qquad \text{(definition of } B_m\text{, and } \Gamma(n+1)=n!)\\ &= \frac{n!}{2\pi i}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}}{m!}B_m\int\limits_{\gamma}\frac{1}{t^{n-m+2}}\,dt.\end{aligned}$$

Now notice that the integral of $1/t^{n-m+2}$ around γ gives $2\pi i$ times the residue of $1/t^{n-m+2}$ at the only pole $t=0$. But $1/t^{n-m+2}$ is its own Laurent series, hence by inspection we have zero residue unless $n-m+2=1 \Rightarrow m=n+1$; in the case that $m=n+1$ we have a residue 1 at the pole Hence:

$$\int\limits_{\gamma}\frac{1}{t^{n-m+2}}\,dt = \begin{cases}0 & \text{if } m\neq n+1\\ 1 & \text{otherwise.}\end{cases}$$

Hence we deduce that:

$$\zeta(-n) = \frac{n!}{2\pi i}\frac{(-1)^n}{(n+1)!}B_{n+1}\cdot 2\pi i = \frac{(-1)^n B_{n+1}}{n+1},$$

as required.

We can deduce that the negative even integers are zeroes of the zeta function since the odd Bernoulli numbers, apart from B_1, vanish:

$$\zeta(-2n) = \frac{(-1)^{2n}B_{2n+1}}{2n+1} = 0.$$

We can see that the odd Bernoulli numbers, apart from B_1, must be zero because

$$\sum_{m=0}^{\infty}\frac{B_m t^{m-1}}{m!} - B_1 = \frac{1}{e^t-1} - B_1 = \frac{1}{e^t-1}+\frac{1}{2}$$

is an odd function. This follows by simply noting that its even part vanishes:

$$\frac{1}{2}\left(\frac{1}{e^t-1}+\frac{1}{2}+\frac{1}{e^{-t}-1}+\frac{1}{2}\right) = \frac{1}{2}\left(\frac{e^{-t}-1+e^t-1}{(e^t-1)(e^{-t}-1)}+1\right) = \frac{1}{2}\left(\frac{e^{-t}+e^t-2}{2-e^t-e^{-t}}+1\right) = 0.$$

❊ **Comments:** The zeroes of the zeta function that you find in this question are called the *trivial zeroes* of the Riemann zeta function. The famous *Riemann hypothesis* states that all non-trivial zeroes of the Riemann zeta function lie on the line with real part $\frac{1}{2}$.

2. Show that for $\mathrm{Re}(z) > 1$,

$$(1-2^{1-z})\zeta(z) = (1^{-z} - 2^{-z} + 3^{-z} - 4^{-z} + \cdots) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t+1}\, dt.$$

[Note: This result is actually valid for $\mathrm{Re}(z) > 0$*.]*

Solution: Recall that the Riemann zeta function is defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

for $\mathrm{Re}(z) > 1$, and any analytic continuation to $\mathrm{Re}(z) \leq 1$. We are asked to prove the results in the region $\mathrm{Re}(z) > 1$, so it's sufficient to use the series definition of the zeta function.

Multiplying the series by $1 - 2^{1-z}$, we immediately get the first equality:

$$\begin{aligned}
(1-2^{1-z})\zeta(z) &= (1-2^{1-z})\sum_{n=1}^{\infty} \frac{1}{n^z} && \text{(for } \mathrm{Re}(z) > 1\text{)} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^z} - \frac{2}{2^z}\sum_{n=1}^{\infty} \frac{1}{n^z} && \text{(multiplying out, and using } 2^{1-z} = 2/2^z\text{)} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^z} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^z} && \text{(taking } 2^z \text{ into sum)} \\
&= 1^{-z} - 2^{-z} + 3^{-z} - 4^{-z} + \ldots && \text{(writing out the terms of each the sums).}
\end{aligned}$$

Note that the final result can be written as:

$$(1-2^{1-z})\zeta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z}. \qquad (*)$$

To obtain the second equality, we use a slightly modified version of the method from lectures that we used to show

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^{-t}-1}\, dt.$$

First, let's recall how to show this result:

Theorem: The Riemann zeta function can be written as:

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t-1}\, dt$$

in the region $\mathrm{Re}(z) > 1$.

Proof: The proof is by a clever trick. We start by recalling the definition of the gamma function in $\mathrm{Re}(z) > 0$:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

Make the substitution $t = ns$ in this integral, for generic $n \in \mathbb{N}$. Then the measure changes as $dt = n\,ds$ and the limits change as $[0,\infty] \mapsto [0,\infty]$. The result is:

$$\Gamma(z) = n^z \int_0^\infty s^{z-1}e^{-ns}\,ds \qquad \Rightarrow \qquad \frac{\Gamma(z)}{n^z} = \int_0^\infty s^{z-1}e^{-ns}\,ds.$$

Now sum both sides from $n=1$ to $n=\infty$. We have:

$$\sum_{n=1}^\infty \frac{\Gamma(z)}{n^z} = \int_0^\infty s^{z-1}\left(\sum_{n=1}^\infty e^{-ns}\right)ds. \tag{†}$$

On the left hand side, we can pull out the $\Gamma(z)$ since it is constant. This leaves the definition of the Riemann zeta function on the left hand side (for $\mathrm{Re}(z) > 1$). On the right hand side, we have the sum of a geometric progression e^{-ns}, with first term e^{-s} and common ratio e^{-s}, to infinity (since $e^{-s} < 1$ for $s \in (0,\infty)$, this geometric progression converges). This is simply given by:

$$\sum_{n=1}^\infty e^{-ns} = \frac{e^{-s}}{1-e^{-s}} = \frac{1}{e^s-1}.$$

It follows that (†) can be written as:

$$\Gamma(z)\zeta(z) = \int_0^\infty \frac{s^{z-1}}{e^s-1}\,ds \qquad \Rightarrow \qquad \zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{s^{z-1}}{e^s-1}\,ds,$$

as required. We only needed the series expression for the zeta function to converge to derive this result (and also $\mathrm{Re}(z) > 0$ for the gamma integral to exist), and hence it must be valid in the region $\mathrm{Re}(z) > 1$. $\square$

We now explain the modified version of this method used to derive the result in this question. Again, we start by making the substitution $t = ns$ in the definition of the gamma function (for $\mathrm{Re}(z) > 0$) for generic $n \in \mathbb{N}$:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt = n^z\int_0^\infty s^{z-1}e^{-ns}\,ds \qquad \Rightarrow \qquad \frac{\Gamma(z)}{n^z} = \int_0^\infty s^{z-1}e^{-ns}\,ds$$

as before. Now before summing from $n=1$ to $n=\infty$, we instead multiply by $(-1)^{n+1}$ first in order to replicate the sum $(*)$ we obtained above. Hence we find:

$$\Gamma(z)\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^z} = \int_0^\infty s^{z-1}\left(\sum_{n=1}^\infty (-1)^{n+1}e^{-ns}\right)ds.$$

The sum on the left hand side is given by $(*)$ above (from the first equality). The geometric progression on the right hand side now has first term e^{-s} and common ratio $-e^{-s}$, hence has sum:

$$\sum_{n=1}^\infty (-1)^{n+1}e^{-ns} = \frac{e^{-s}}{1+e^{-s}} = \frac{1}{e^s+1}.$$

Putting everything together, we have

$$(1-2^{1-z})\zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{s^{z-1}}{e^s+1}\,ds$$

as required. Our derivation required only $\mathrm{Re}(z) > 1$ for the convergence of the sums, and hence we have proved the result in this region.

<u>An alternative method.</u> Instead of copying the derivation of the theorem expressing $\zeta(z)$ as a real integral, we can actually use this result directly to get the integral expression for $(1-2^{1-z})\zeta(z)$. We have:

$$(1-2^{1-z})\zeta(z) = \frac{(1-2^{1-z})}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t-1}\,dt = \frac{1}{\Gamma(z)}\int_0^\infty \left(\frac{t^{z-1}}{e^t-1} - \frac{(t/2)^{z-1}}{e^t-1}\right)\,dt.$$

Let $t/2 \mapsto t$ in the second term in the final integral. The limits change as $[0,\infty) \mapsto [0,\infty)$, and the measure changes as $dt \mapsto 2dt$. Thus we're left with:

$$(1-2^{1-z})\zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty \left(\frac{t^{z-1}}{e^t-1} - \frac{2t^{z-1}}{e^{2t}-1}\right)\,dt = \frac{1}{\Gamma(z)}\int_0^\infty \left(\frac{t^{z-1}}{e^t-1} - \frac{2t^{z-1}}{(e^t+1)(e^t-1)}\right)\,dt.$$

Simplifying the integrand, we have:

$$\frac{t^{z-1}}{e^t-1} - \frac{2t^{z-1}}{(e^t+1)(e^t-1)} = \frac{t^{z-1}}{e^t-1}\left(1-\frac{2}{e^t+1}\right) = \frac{t^{z-1}}{e^t-1}\left(\frac{e^t-1}{e^t+1}\right) = \frac{t^{z-1}}{e^t+1}.$$

Hence we find:

$$(1-2^{1-z})\zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t+1}\,dt,$$

as required.

❊ **Comments:** The question hints that the results we prove in this question are actually valid in the region $\mathrm{Re}(z) > 0$. We can prove this claim using analytic continuation as follows. First, we shall prove:

Theorem: For all z such that $\mathrm{Re}(z) > 0$, we have

$$(1-2^{1-z})\zeta(z) = 1^{-z} - 2^{-z} + 3^{-z} - \ldots$$

Proof: We have already shown this result is true for $\mathrm{Re}(z) > 1$. Notice that the left hand side is defined and analytic for $\mathrm{Re}(z) > 0$, provided $z \neq 1$ where there is a pole in the ζ function.

Notice also that the right hand side is convergent for $\mathrm{Re}(z) > 0$. To show this, start by grouping the terms two by two:

$$1^{-z} - 2^{-z} + 3^{-z} - \ldots = \sum_{n=1}^{\infty}\left((2n-1)^{-z} - (2n)^{-z}\right).$$

Now note that the general term is given by

$$(2n-1)^{-z} - (2n)^{-z} = (2n)^{-z}\left(\left(1-\frac{1}{2n}\right)^{-z} - 1\right) = O((2n)^{-z-1}),$$

by the binomial theorem. The big-O notation means that there exists N such that for all $n \geq N$, we have $|(2n-1)^{-z} - (2n)^{-z}| \leq M|(2n)^{-z-1}|$ for some constant M. So our series is eventually bounded in modulus by the series $(2n)^{-z-1}$, which is convergent for $\mathrm{Re}(z) > 0$ (note $\mathrm{Re}(z) = 0$ gives a harmonic series). Furthermore, the series is a convergent sum of analytic functions, and hence is also analytic. This is *Weierstrass' theorem*, which you might have seen in Part IB Complex Analysis.

Hence both the left and right hand sides are defined and analytic on the domain $\mathrm{Re}(z) > 0$, and they agree on the domain $\mathrm{Re}(z) > 1$. Hence by the principle of analytic continuation, they agree on the whole domain $\mathrm{Re}(z) > 0$ as required. $\square$

Now that we have the series result, we notice immediately that our derivation of the integral result extends to $\mathrm{Re}(z) > 0$, because the only thing we used in the proof was the radius of convergence of the series.

It's also possible to show the integral result holds in $\mathrm{Re}(z) > 0$ directly using analytic continuation (rather than going via the series). First, we need a Lemma showing that $\Gamma(z) \neq 0$ anywhere (though we'll only use it for $\mathrm{Re}(z) > 0$):

Lemma: The gamma function is everywhere non-zero.

Proof: Certainly $\Gamma(n) \neq 0$ for any positive or negative integer n, since $\Gamma(n) = (n-1)!$ for $n \geq 1$ and $\Gamma(n) = \infty$ for $n \leq 0$. So suppose that $\Gamma(z_0) = 0$, with $z_0 \notin \mathbb{Z}$. Since $z_0 \notin \mathbb{Z}$, the gamma function obeys the *reflection formula* at this point:

$$\Gamma(z_0)\Gamma(1-z_0) = \frac{\pi}{\sin(\pi z_0)},$$

which you should have seen in lectures. In particular, note that since $z_0 \notin \mathbb{Z}$, the right hand side is non-zero and finite. But since $\Gamma(z_0) = 0$, the fact the right hand side is non-zero implies $\Gamma(1-z_0) = \infty$. It follows that $1 - z_0$ is a pole of the gamma function. But z_0 was not an integer, hence we have a contradiction. □

The main result then follows from a simple application of analytic continuation:

Theorem: The relation

$$(1-2^{1-z})\zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t+1}\,dt$$

holds in the region $\mathrm{Re}(z) > 0$.

Proof: $\zeta(z)$ is defined by a series in $\mathrm{Re}(z) > 1$, and any analytic continuation to $\mathrm{Re}(z) \leq 1$. We proved in lectures (and earlier on this sheet) using a Hankel contour that $\zeta(z)$ extends to an analytic function on $\mathbb{C}\backslash\{-1\}$. In particular, the left hand side $(1-2^{1-z})\zeta(z)$ is defined and analytic for $\mathrm{Re}(z) > 0$.

We now show that the right hand side is also analytic for $\mathrm{Re}(z) > 0$. First, note that $\Gamma(z)$ is analytic and non-zero for all $z \in \mathbb{C}\backslash\{0,-1,-2,...,\}$, and hence $1/\Gamma(z)$ is well-defined an analytic on $\mathrm{Re}(z) > 0$.

Now notice that the integral on the right hand side is convergent for $\mathrm{Re}(z) > 0$. We can prove this by looking at the contributions from the possibly divergent parts of the integral, near $t = 0$ and $t = \infty$. Near $t = \infty$, the integrand is exponentially suppressed and guaranteed to go to zero quickly. Near $t = 0$, we can approximate $e^t \approx 1$, and hence get a contribution to the integral

$$\int_0 \frac{t^{z-1}}{e^t+1}\,dt \approx \int_0 \frac{t^{z-1}}{2}\,dt = \left[\frac{t^z}{2z}\right]_0.$$

This converges to something finite if $\mathrm{Re}(z) > 0$. Hence the integral is convergent for $\mathrm{Re}(z) > 0$. Furthermore, its integrand is continuous in both t and z, and is analytic in z for each fixed t. It follows by the theorem on functions defined by integrals in lectures that the integral is an analytic function of z in $\mathrm{Re}(z) > 0$.

Hence both the left and right hand sides of our relation are analytic functions on $\mathrm{Re}(z) > 0$, and they agree on $\mathrm{Re}(z) > 1$ as we showed above. It follows by the principle of analytic continuation that they agree on $\mathrm{Re}(z) > 0$. □

3. Show that

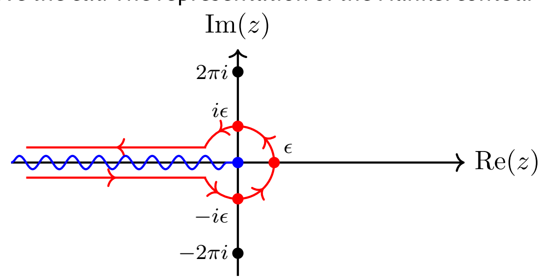
$$\int_{-\infty}^{(0+)} \frac{\log(t)}{e^{-t}-1}\,dt = 0.$$

Hence show that

$$\lim_{z\to 1}\left(\zeta(z) - \frac{1}{z-1}\right) = \gamma,$$

and $\zeta'(0) = -\log(\sqrt{2\pi})$.

Solution: For the first, part we use a branch of the logarithm defined by a branch cut along the negative real axis, with $\log(re^{i\pi}) = \log(r) + i\pi$ just above the cut. The representation of the Hankel contour we use is as in Sheet 2, Question 12:



Evaluating the integral on each part of the contour separately, we have

$$\int_{-\infty}^{(0+)} \frac{\log(t)}{e^{-t}-1}\,dt = e^{-i\pi}\int_{\infty}^{\epsilon} \frac{\log(x)-i\pi}{e^{x}-1}\,dx + e^{i\pi}\int_{\epsilon}^{\infty} \frac{\log(x)+i\pi}{e^{x}-1}\,dx + \int_{-\pi}^{\pi} \frac{\log(\epsilon)+i\theta}{e^{-\epsilon e^{i\theta}}-1}\epsilon i e^{i\theta}\,d\theta.$$

Notice that in the first two terms, the two $\log(x)/(e^x-1)$ integrals cancel. This leaves us with:

$$\int_{-\infty}^{(0+)} \frac{\log(t)}{e^{-t}-1}\,dt = -2i\pi\int_{\epsilon}^{\infty} \frac{1}{e^{x}-1}\,dx + \int_{-\pi}^{\pi} \frac{\log(\epsilon)+i\theta}{e^{-\epsilon e^{i\theta}}-1}\epsilon i e^{i\theta}\,d\theta.$$

Let's evaluate each of these terms separately. The first is an integral we can do exactly:

$$\int_{\epsilon}^{\infty} \frac{1}{e^{x}-1}\,dx = \int_{\epsilon}^{\infty} \frac{e^{-x}}{1-e^{-x}}\,dx = \left[\log(1-e^{-x})\right]_{\epsilon}^{\infty} = \log(1) - \log(1-e^{-\epsilon}) = -\log(1-e^{-\epsilon}).$$

Note the need for $\epsilon > 0$ here to regulate the integral; else, we would get something divergent. Expanding in ϵ small, we find that the first term gives a contribution:

$$\begin{aligned}
-\log(1-e^{-\epsilon}) &= -\log\left(1-\left(1-\epsilon+\frac{1}{2}\epsilon^2+O(\epsilon^3)\right)\right) && \text{(expanding exponential)}\\
&= -\log\left(\epsilon-\frac{1}{2}\epsilon^2+O(\epsilon^3)\right)\\
&= -\log(\epsilon) - \log\left(1-\frac{1}{2}\epsilon+O(\epsilon^2)\right) && (\log(ab)=\log(a)+\log(b))\\
&= -\log(\epsilon) + O(\epsilon) && \text{(expanding } \log(1+O(\epsilon))\text{)}.
\end{aligned}$$

The second term we need to consider is:

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{\log(\epsilon)+i\theta}{e^{-\epsilon e^{i\theta}}-1}\epsilon i e^{i\theta}\,d\theta &= \int_{-\pi}^{\pi} \frac{\log(\epsilon)+i\theta}{-\epsilon e^{i\theta}+\frac{1}{2}\epsilon^2 e^{2i\theta}+O(\epsilon^3)}\epsilon i e^{i\theta}\,d\theta && \text{(expanding exponential in denominator)}\\
&= -i\int_{-\pi}^{\pi} \frac{\log(\epsilon)+i\theta}{1-\frac{1}{2}\epsilon e^{i\theta}+O(\epsilon^2)}\,d\theta && \text{(dividing through by } \epsilon e^{i\theta}\text{)}\\
&= -i\int_{-\pi}^{\pi} (\log(\epsilon)+i\theta)\left(1+\frac{1}{2}\epsilon e^{i\theta}+O(\epsilon^2)\right)d\theta && \text{(binomial theorem)}\\
&= -2\pi i\log(\epsilon)+O(\epsilon) && \text{(integral of } \theta \text{ vanishes as odd).}
\end{aligned}$$

Putting everything together, we see that

$$\int_{-\infty}^{(0+)} \frac{\log(t)}{e^{-t}-1}\,dt = -2\pi i\left(-\log(\epsilon+O(\epsilon^2)\right)-2\pi i\log(\epsilon)+O(\epsilon)=O(\epsilon).$$

In particular, we see that the divergence arising from the straight sections of the contour cancels exactly with the divergence coming from the circular part of the contour. Deforming the Hankel contour such that $\epsilon \to 0$, we find:

$$\int_{-\infty}^{(0+)} \frac{\log(t)}{e^{-t}-1}\,dt = 0,$$

as required.

The nicest way to do the next part of the question is to think about the Laurent series for the zeta function around $z=1$. We know from lectures (or from Sheet 2, Question 12) that the ζ function has a simple pole at $z=1$ with residue

$$\operatorname{Res}(\zeta(z);1)=1.$$

Hence the Laurent series of the ζ function around $z=1$ is of the form:

$$\zeta(z)=\frac{1}{z-1}+a_0+O(z-1).$$

The question asks us to find:

$$\lim_{z\to1}\left[\zeta(z)-\frac{1}{z-1}\right]=a_0,$$

i.e. the constant term in the Laurent series of $\zeta(z)$ around $z=1$. To find this, we use the Hankel contour representation of the zeta function:

$$\zeta(z)=\frac{\Gamma(1-z)}{2\pi i}\int_{-\infty}^{(0+)}\frac{t^{z-1}}{e^{-t}-1}\,dt.$$

On the right hand side, we need to expand $\Gamma(1-z)$ and the integral around $z=1$.

- Expansion of integral. The expansion of the integral is straightforward, and is greatly aided by the result from the first part of this question. Rewriting t^{z-1} as an exponential, we find:

$$\int_{-\infty}^{(0+)} \frac{t^{z-1}}{e^{-t}-1}\,dt = \int_{-\infty}^{(0+)} \frac{e^{\log(t)(z-1)}}{e^{-t}-1}\,dt = \int_{-\infty}^{(0+)} \left(\frac{1}{e^{-t}-1} + \frac{\log(t)(z-1)}{e^{-t}-1}\right)\,dt + O((z-1)^2)$$

up to order $O(z-1)$. The integral of the logarithm on the right hand side vanishes by our earlier work in this question, and hence we only need to find:

$$\int_{-\infty}^{(0+)} \frac{1}{e^{-t}-1}\,dt.$$

This Hankel contour integral encircles a single simple pole at $t = 0$, with residue given by:

$$\lim_{t\to 0}\left[\frac{t}{e^{-t}-1}\right] = \lim_{t\to 0}\left[\frac{1}{-e^{-t}}\right] = -1,$$

using L'Hopital's Rule. Hence by the residue theorem, we have

$$\int_{-\infty}^{(0+)} \frac{1}{e^{-t}-1}\,dt = -2\pi i.$$

It follows that:

$$\int_{-\infty}^{(0+)} \frac{t^{z-1}}{e^{-t}-1}\,dt = -2\pi i + O((z-1)^2).$$

- Expansion of $\Gamma(1-z)$. For this expansion, it's easiest to first consider the expansion of $\Gamma(w)$ near $w = 0$, and then let $w = 1-z$ later on. Using the equation $\Gamma(w+1) = w\Gamma(w)$ for the gamma function, we have

$$\Gamma(w) = \frac{\Gamma(w+1)}{w} = \frac{1}{w}\left(\Gamma(1) + w\Gamma'(1) + O(w^2)\right) = \frac{\Gamma(1)}{w} + \Gamma'(1) + O(w).$$

Recall that $\Gamma(1) = 1$, and we found that $\Gamma'(1) = -\gamma$ at the end of Sheet 2, Question 11. It follows that:

$$\Gamma(w) = \frac{1}{w} - \gamma + O(w).$$

Inserting $w = 1-z$, we find:

$$\Gamma(1-z) = -\frac{1}{z-1} - \gamma + O(z-1).$$

We now have everything we need to find the Laurent series of $\zeta(z)$ around $z = 1$. Putting our expansions into the Hankel representation of the zeta function, we find:

$$\zeta(z) = \frac{1}{2\pi i}\left(-\frac{1}{z-1} - \gamma + O(z-1)\right)\left(-2\pi i + O((z-1)^2)\right) = \frac{1}{z-1} + \gamma + O(z-1).$$

Hence we have the result:

$$\lim_{z\to 1}\left[\zeta(z) - \frac{1}{z-1}\right] = \gamma,$$

as required.

Finally, we are asked to find $\zeta'(0)$. Our previous work was mainly concerned with expanding ζ near $z=1$, and we now want information about ζ near $z=0$, hence the relation we should use is the *reflection formula* for the ζ function:

$$\zeta(z) = 2^z\pi^{z-1}\Gamma(1-z)\sin\left(\frac{1}{2}\pi z\right)\zeta(1-z).$$

Notice that the right hand side of this equation consists entirely of things we can estimate when z is near 0, since

(i) $2^z\pi^{z-1} = \pi^{-1}e^{\log(2\pi)z}$ has a nice expansion near $z=0$;

(ii) $\Gamma(1-z)$ has no pole at $z=0$, hence we can Taylor expand this function around $z=0$;

(iii) similarly, $\sin(\frac{1}{2}\pi z)$ can be expanded near $z=0$;

(iv) we previously approximated $\zeta(z)$ near $z=1$, which is equivalent to approximating $\zeta(1-z)$ near $z=0$.

Thus we spend the last part of this question doing lots of series expansions! First of all, we have the easy series:

$$2^z\pi^{z-1} = \frac{1}{\pi}(2\pi)^z = \frac{1}{\pi}e^{z\log(2\pi)} = \frac{1}{\pi}\left(1+z\log(2\pi)+O(z^2)\right),$$

$$\sin\left(\frac{1}{2}\pi z\right) = \frac{1}{2}\pi z + O(z^3).$$

The gamma function $\Gamma(1-z)$ can be Taylor expanded near $z=0$ to give:

$$\Gamma(1-z) = \Gamma(1) - \frac{\Gamma'(1)}{\Gamma(1)}z + O(z^2) = 1+\gamma z+O(z^2),$$

using the fact that $\Gamma'(1) = -\gamma$ and $\Gamma(1) = 1$. Finally, the zeta function $\zeta(1-z)$ can be expanded using the result we proved earlier in the question:

$$\zeta(z) = \frac{1}{z-1}+\gamma+O(z-1) \qquad \Rightarrow \qquad \zeta(1-z) = -\frac{1}{z}+\gamma+O(z).$$

Putting everything together, we see that near $z=0$, the reflection formula looks like:

$$\begin{aligned}
\zeta(z) &= \frac{1}{\pi}\left(1+z\log(2\pi)+O(z^2)\right)\left(1+\gamma z+O(z^2)\right)\left(\frac{1}{2}\pi z+O(z^3)\right)\left(-\frac{1}{z}+\gamma+O(z)\right)\\
&= \frac{1}{\pi}\left(1+z(\log(2\pi)+\gamma)+O(z^2)\right)\left(-\frac{1}{2}\pi+\frac{1}{2}\pi\gamma z+O(z^2)\right)\\
&= \frac{1}{\pi}\left(-\frac{1}{2}\pi-\frac{1}{2}\pi\log(2\pi)z+O(z^2)\right)\\
&= -\frac{1}{2}-\frac{1}{2}\log(2\pi)z+O(z^2).
\end{aligned}$$

Hence we can read off $\zeta'(0) = -\frac{1}{2}\log(2\pi) = -\log(\sqrt{2\pi})$ as required.

❊ **Comments:** Furthermore, we can read off the value $\zeta(0) = -\frac{1}{2}$ for free from the expansion of ζ around $z=0$. Written in terms of the $\mathrm{Re}(z)>1$ definition of the zeta function, this result seems to be telling us that:

$$1+1+1+\cdots = -\frac{1}{2}.$$

4. The psi-function is defined to be

$$\psi(z) = \frac{d}{dz}\log(\Gamma(z)).$$

Show that

$$\psi'(z) = \sum_{s=0}^{\infty} \frac{1}{(s+z)^2}, \qquad (z \neq 0, -1, -2, ...).$$

Then show that when z is real and positive, $\Gamma(z)$ has a single minimum which lies between $z = 1$ and $z = 2$.

Show also that

$$\log(\Gamma(z)) = -\gamma(z-1) + \sum_{s=2}^{\infty}(-1)^s \frac{\zeta(s)}{s}(z-1)^s.$$

➻ **Solution:** The *psi-function* in this question is more commonly called the *digamma function*; hence we'll call it the digamma function in these solutions.

To derive the series representation of the derivative of the digamma function in the question, we start by using the Weierstrass canonical product representation of the gamma function. Recall the Weierstrass canonical product is given by:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \frac{e^{z/k}}{1+z/k}.$$

Taking the logarithm of this product, we can convert it into a sum:

$$\log\left(\Gamma(z)\right) = -\gamma z - \log(z) + \sum_{k=1}^{\infty}\left(\frac{z}{k} - \log\left(1 + \frac{z}{k}\right)\right)$$

Now taking the derivative with respect to z, we have

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1/k}{1+z/k}\right) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+z}\right). \qquad (*)$$

Taking a second derivative, we have

$$\psi'(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}\frac{1}{(z+k)^2} = \sum_{k=0}^{\infty}\frac{1}{(z+k)^2},$$

as required. The Weierstrass canonical product converges for all $z \in \mathbb{C}\backslash\{0, -1, -2, ...\}$, and hence our result is valid throughout this region too.

We are now asked to use this result about the derivative of the digamma function to show that $\Gamma(z)$ has a single minimum lying in $z \in (1,2)$ for z real and positive. First, we note that the definition of $\psi(z)$ is such that

$$\psi(z) = \frac{d}{dz}\left(\log(\Gamma(z))\right) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Hence $\psi(z)\Gamma(z) = \Gamma'(z)$. Thus at extrema of the gamma function on the real line, we must have $\Gamma'(z) = 0$, and so either $\psi(z) = 0$ or $\Gamma(z) = 0$. However, we have the following handy result that allows us to eliminate the case $\Gamma(z) = 0$:

Theorem: The gamma function is everywhere non-zero.

Proof: Certainly $\Gamma(n) \neq 0$ for any positive or negative integer n, since $\Gamma(n) = (n-1)!$ for $n \geq 1$ and $\Gamma(n) = \infty$ for $n \leq 0$. So suppose that $\Gamma(z_0) = 0$, with $z_0 \notin \mathbb{Z}$. Since $z_0 \notin \mathbb{Z}$, the gamma function obeys the *reflection formula* at this point:

$$\Gamma(z_0)\Gamma(1-z_0) = \frac{\pi}{\sin(\pi z_0)},$$

which you should have seen in lectures. In particular, note that since $z_0 \notin \mathbb{Z}$, the right hand side is non-zero and finite. But since $\Gamma(z_0) = 0$, the fact the right hand side is non-zero implies $\Gamma(1-z_0) = \infty$. It follows that $1 - z_0$ is a pole of the gamma function. But z_0 was not an integer, hence we have a contradiction. $\square$

It follows from this result that $\Gamma(z)$ is extremised on $[0,\infty)$ if and only if $\psi(z) = 0$.

In the first part of this question, we showed that

$$\psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}.$$

This is positive for $z \in [0,\infty)$, since it is the sum of positive terms, and it follows that $\psi'(z) > 0$ everywhere on $[0,\infty)$. Thus $\psi(z)$ is a monotonic function on the real line, and hence $\psi(z)$ can be zero at most once. Thus $\Gamma(z)$ has at most one extremum.

To show that the extremum lies between 1 and 2, we evaluate $\psi(1)$ and $\psi(2)$ to check for a change of sign; if we get one, this will indicate $\psi(z) = 0$ for some $z \in (1,2)$ and hence $\Gamma'(z) = 0$ for some $z \in (1,2)$. We can evaluate these special values of the digamma function using the formula $(*)$ for $\psi(z) = \Gamma'(z)/\Gamma(z)$ we derived at the start of this question (and at the end of Question 6):

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+z}\right).$$

We see that

$$\psi(1) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = -\gamma - 1 + 1 = -\gamma,$$

since the series telescopes. Similarly,

$$\psi(2) = -\gamma - \frac{1}{2} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+2}\right) = -\gamma - \frac{1}{2} + 1 + \frac{1}{2} = 1 - \gamma.$$

Thus we see that $\psi(1) = -\gamma < 0 < 1 - \gamma < \psi(2)$ (recall that $\gamma \approx 0.577$). So it follows that $\psi(z)$ indeed has a change of sign in $z \in (1,2)$, and thus $\Gamma(z)$ has its unique extremum on $[0,\infty)$ in the interval $(1,2)$.

Checking a few values, we have $\Gamma(1) = 1, \Gamma(3/2) = (1/2)\Gamma(1/2) = (1/2)\cdot\sqrt{\pi} \approx 0.886$, and $\Gamma(2) = 1$. This shows that the gamma function must decrease in this interval, and then increase again. Thus $\Gamma(z)$ has a unique *minimum* on $[0,\infty)$ somewhere in the interval $(1,2)$, as required.

Finally, we are asked to prove the result

$$\log(\Gamma(z)) = -\gamma(z-1) + \sum_{s=2}^{\infty}(-1)^s \frac{\zeta(s)}{s}(z-1)^s.$$

The series on the right hand side looks like a Taylor expansion around $z = 1$; thus we simply expand $\log(\Gamma(z))$ around $z = 1$:

- The first term has coefficient $\log(\Gamma(1)) = \log(1) = 0$.
- The second term has coefficient
$$\frac{d}{dz}\left(\log(\Gamma(z))\right)\Big|_{z=1} = \psi(1) = -\gamma,$$
using our calculation of $\psi(1)$ from above.
- The nth term, for $n \geq 2$, has coefficient
$$\frac{1}{n!}\frac{d^n}{dz^n}\left(\log(\Gamma(z))\right)\Big|_{z=1} = \frac{1}{n!}\frac{d^{n-2}}{dz^{n-2}}\left(\psi'(z)\right)\Big|_{z=1}.$$
Inserting the series expansion we derived for $\psi'(z)$ in the very first part of this question, we see this is equal to:
$$\frac{1}{n!}\frac{d^{n-2}}{dz^{n-2}}\left(\sum_{s=0}^{\infty}\frac{1}{(s+z)^2}\right)\Big|_{z=1} = \frac{1}{n!}\left(\sum_{s=0}^{\infty}\frac{(-1)^{n-2}(n-1)!}{(s+z)^n}\right)\Big|_{z=1} = \frac{(-1)^n}{n}\sum_{s=0}^{\infty}\frac{1}{(s+1)^n}.$$
To finish, recognise that the sum is given by:
$$\sum_{s=0}^{\infty}\frac{1}{(s+1)^n} = \sum_{s=1}^{\infty}\frac{1}{s^n} = \zeta(n).$$
Thus the coefficient of the nth term for $n \geq 2$ is:
$$\frac{(-1)^n\zeta(n)}{n}.$$

Putting everything together, we establish that $\log(\Gamma(z))$ has the Taylor series:

$$\log(\Gamma(z)) = -\gamma(z-1) + \sum_{n=2}^{\infty}(-1)^{n-2}\frac{\zeta(n)}{n}(z-1)^n,$$

about $z = 1$, as required.

❊ **Comments:** The digamma function has lots of applications, as with all the other special functions you've learned about. A particularly nice application of the digamma function is in evaluating a wide class of series; this is hinted at in this question where you find a series representation for $\psi(z)$, $\psi'(z)$. In particular, all convergent series of the form

$$\sum_{n=1}^{\infty}\frac{a(n)}{b(n)}$$

where $a(n)$ and $b(n)$ are polynomials can be expressed in terms of the digamma function and its derivatives (the derivatives of the digamma function are often called the *polygamma functions*). We can see this by thinking about how $a(n)/b(n)$ would decompose into partial fractions, and how the derivatives of $\psi(z)$ allow us to sum any set of terms that could arise in this partial fraction decomposition.

5. Find two independent solutions of the Airy equation $w'' - zw = 0$ in the form

$$w(z) = \int_\gamma e^{zt} f(t)\, dt,$$

where γ is to be specified in each case. Show that there is a solution for which γ can be chosen to consist of two straight line segments in the left half t-plane ($\mathrm{Re}(t) \leq 0$).

For this solution, show that, if $w(z)$ is normalised so that $w(0) = iA3^{-1/6}\Gamma(1/3)$, where A is a constant, then $w'(0) = -iA3^{1/6}\Gamma(2/3)$.

➾ **Solution:** To find solutions of Airy's equation of the given integral form, we substitute the integral into Airy's equation and see what constraints we find on $f(t)$ and γ. First, note that

$$w''(z) = \int_\gamma t^2 e^{zt} f(t)\, dt,$$

so that Airy's equation becomes:

$$w'' - zw = \int_\gamma (t^2 - z) e^{zt} f(t)\, dt = 0.$$

We need this condition to hold for all z. In order to enforce this, we manipulate the integrand such that all z dependence is completely factored out. This is achieved by integration by parts:

$$0 = \int_\gamma (t^2 - z)e^{zt} f(t)\, dt = \int_\gamma \left(t^2 e^{zt} f(t) - f(t)\frac{d}{dt}(e^{zt}) \right) dt = -\left[e^{zt} f(t)\right]_\gamma + \int_\gamma \left(t^2 f(t) + f'(t)\right) e^{zt}\, dt.$$

Thus we see that the equation is solved for all z if we choose $f(t)$ and γ such that

$$\left[e^{zt} f(t)\right]_\gamma = 0, \qquad \text{and} \qquad t^2 f(t) + f'(t) = 0.$$

The second condition is a separable differential equation with solution:

$$f'(t) = -t^2 f(t) \qquad \Rightarrow \qquad f(t) = Ce^{-t^3/3}.$$

Hence our integral representation of the solution to Airy's equation is

$$w(z) = C\int_\gamma e^{zt - t^3/3}\, dt, \quad \text{where the contour } \gamma \text{ obeys} \quad \left[e^{zt-t^3/3}\right]_\gamma = 0.$$

To find two independent solutions, we must pick γ. If we think about it, there are only a few ways in which we can pick γ:

Choosing a contour in an integral representation: Suppose that we have the integral representation:

$$w(z) = \int_\gamma f(z,t)\, dt, \qquad \text{where the contour } \gamma \text{ obeys} \quad [g(z,t)]_\gamma = 0.$$

Then the possible choices for γ are as follows:

- γ could be closed. In this case, the boundary condition is trivially satisfied, since the contour starts and ends at the same place. This is true *provided we do not pass through a branch cut of the function $g(z,t)$*, since then we'd get a jump.

  By Cauchy's theorem, a closed contour gives a non-zero $w(z)$, and hence a solution, if and only if the contour γ is chosen such that it encircles a singularity of the function $f(z,t)$ (else we'd get $w(z) = 0$).
- γ could be an open contour (that is, a contour that does not start and end in the same place). In this case, we try to choose γ such that $g(z,t)$ is zero at the ends of the contour γ. There are two ways this can be achieved: (i) an end of the contour γ is located at a zero of the function $g(z,t)$; (ii) an end of the contour γ disappears off to infinity in such a way that $g(z,t) \to 0$ as we travel towards the infinite end of the contour *and* the integral expression for $w(z)$ gives a finite contribution from the ends of the contour at infinity.

Let's see what our options are in the case of Airy's equation. Recall that our integral representation is

$$w(z) = C \int\limits_\gamma e^{zt-t^3/3}\,dt, \quad \text{where the contour } \gamma \text{ obeys} \quad \left[e^{zt-t^3/3}\right]_\gamma = 0.$$

Applying the argument for choosing a contour we described above, we note:

- The integrand $e^{zt-t^3/3}$ has no poles, so γ cannot be a closed contour. It follows that γ must be an open contour.
- The boundary term $e^{zt-t^3/3}$ has no zeroes, so the ends of γ cannot be located at finite points in the t plane. Thus the only option for where we can put both ends of the contour γ is at infinity.

Now we must decide on the directions in which the contour will travel to infinity. In order to fulfil the boundary condition, we must choose directions such that

$$e^{zt-t^3/3} \to 0 \qquad \text{as } t \to \infty \text{ in the chosen direction.}$$

Our contour must also be such that

$$w(z) = C \int\limits_\gamma e^{zt-t^3/3}\,dt$$

converges. Fortunately the integrand and the boundary term are the same in this question, so our analysis of one will also apply to the other.

For very large t, the exponential will be dominated by the t^3 term in its argument; hence we have $e^{zt-t^3/3} \approx e^{-t^3/3}$ as $t \to \infty$. Writing $t = |t|(\cos(\theta) + i\sin(\theta))$, we see that the exponential looks like

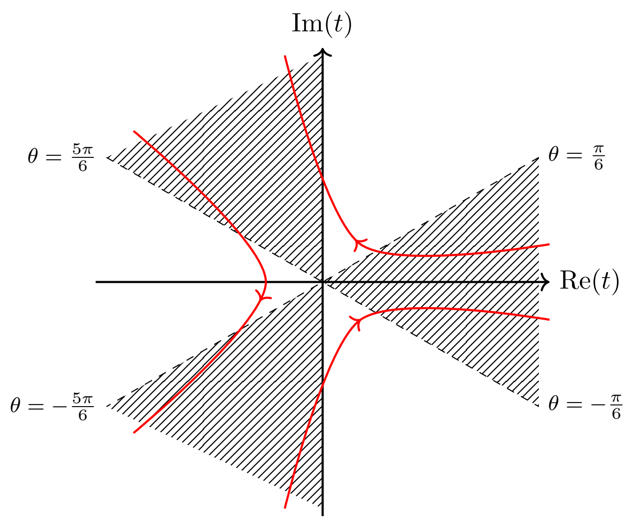
$$\exp\left(-\frac{|t|^3}{3}(\cos(3\theta) + i\sin(3\theta))\right)$$

as $t \to \infty$. Thus the condition that this exponential decays to zero at infinity is precisely $\cos(3\theta) > 0$. This simultaneously ensures that the boundary condition is satisfied, and the integral expression for $w(z)$ is convergent at the end points of the contour (since the integrand will be exponentially suppressed).

Restricting θ such that $-\pi < \theta \leq \pi$, the condition $\cos(3\theta) > 0$ corresponds to the regions of the complex plane described by:

$$-\frac{\pi}{6} < \theta < \frac{\pi}{6}, \qquad \frac{\pi}{2} < \theta < \frac{5\pi}{6}, \qquad -\frac{5\pi}{6} < \theta < -\frac{\pi}{2}.$$

Drawing these regions in the complex plane, we see that our contour must look like one of the following three types (up to any deformation such that the ends of the contour still tend to infinity in the appropriate directions):



We are asked to briefly comment on the independence of the solutions generated by each of these contours. If $\gamma_1, \gamma_2, \gamma_3$ are each of the contours, then we must have

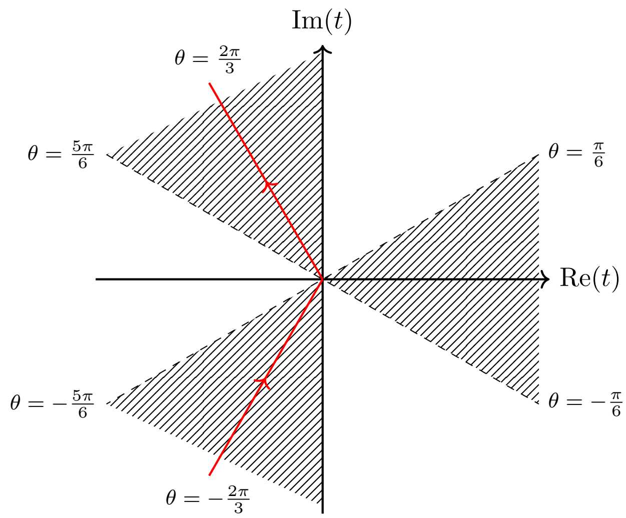
$$C\left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}\right) e^{zt-t^3/3}\,dt = C\oint_{\gamma_1\cup\gamma_2\cup\gamma_3} e^{zt-t^3/3}\,dt = 0,$$

since the concatenation of the three contours creates a closed contour $\gamma_1 \cup \gamma_2 \cup \gamma_3$ containing no singularities - hence we get the result by Cauchy's theorem. It follows that the solutions induced by the contours γ_1, γ_2 and γ_3 cannot all be linearly independent; at most two of the three can be linearly independent.

Can we go further and say that if we choose any two of the contours, then we'll get linearly independent solutions? The answer in this case is 'yes', but proving this to be the case is quite difficult. In this course, we'll just assume that if two contours don't obviously give linearly *dependent* solutions, then they give linearly *independent* solutions. If you are really worried about this, a nice paper on the subject is available here.[2]

[2] https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160140204

We are now asked to consider a particular choice of contour consisting of two straight line segments in the left half plane, $\mathrm{Re}(t) \leq 0$. All choices are equivalent by Cauchy's theorem, but a sensible choice might be:



On the straight segment in the upper half plane we use the parametrisation $t = xe^{2\pi i/3}$ and on the straight segment in the lower half plane we use the parametrisation $t = xe^{-2\pi i/3}$. Thus we have the solution:

$$w(z) = C\left(e^{2\pi i/3}\int_0^\infty \exp\left(zxe^{2\pi i/3} - \frac{x^3}{3}\right)\,dx + e^{-2\pi i/3}\int_\infty^0 \exp\left(zxe^{-2\pi i/3} - \frac{x^3}{3}\right)\,dx\right).$$

We are asked to compare $w(0)$ and $w'(0)$ for this solution. First, we note that $w(0)$ is given by:

$$w(0) = C\left(e^{2\pi i/3} - e^{-2\pi i/3}\right)\int_0^\infty e^{-x^3/3}\,dx = 2iC\sin\left(\frac{2\pi}{3}\right)\int_0^\infty e^{-x^3/3}\,dx = iC\sqrt{3}\int_0^\infty e^{-x^3/3}\,dx.$$

Making the substitution $u = x^3/3$ in the integral changes the limits via $[0,\infty) \mapsto [0,\infty)$ and the measure via

$$x = 3^{1/3}u^{1/3} \qquad \Rightarrow \qquad dx = 3^{-2/3}u^{-2/3}\,du.$$

Hence we find:

$$w(0) = iC3^{1/2}\cdot 3^{-2/3}\int_0^\infty u^{-2/3}e^{-u}\,du = iC3^{-1/6}\Gamma\left(\frac{1}{3}\right).$$

Hence we identify $C = A$, where A is the normalisation constant given in the question.

Computing the derivative, we have:

$$w'(0) = A\left(e^{4\pi i/3} - e^{-4\pi i/3}\right)\int_0^\infty xe^{-x^3/3}\,dx = -iA3^{1/2}3^{-2/3}\int_0^\infty (3^{1/3}u^{1/3})\cdot u^{-2/3}e^{-u}\,du = -iA3^{1/6}\Gamma\left(\frac{2}{3}\right),$$

using the same substitution as above and using $C = A$. This is the required expression for $w'(0)$, given the correct normalisation for $w(0)$.

6. By writing $w(z)$ in the form of an integral representation with the Laplace kernel show that the confluent hypergeometric equation $zw'' + (c-z)w' - aw = 0$ has solutions of the form

$$w(z) = \int_\gamma t^{a-1}(1-t)^{c-a-1}e^{tz}\,dt,$$

provided the path γ is chosen such that $[t^a(1-t)^{c-a}e^{tz}]_\gamma = 0$.

In the case $\mathrm{Re}(z) > 0$, find paths which provide two independent solutions in each of the following cases (where m is a positive integer):

(i) $a = -m, c = 0$;

(ii) $\mathrm{Re}(a) < 0, c = 0, a$ is not an integer;

(iii) $a = 0, c = m$;

(iv) $\mathrm{Re}(c) > \mathrm{Re}(a) > 0, a$ and $c - a$ are not integers.

Solution: We write the integral solution as:

$$w(z) = \int_\gamma f(t)e^{tz}\,dt,$$

where the Laplace kernel is e^{tz}. As usual, we restrict $f(t)$ and γ by substituting into the equation. We find:

$$0 = zw''(z) + (c-z)w'(z) - aw = \int_\gamma (zt^2 + (c-z)t - a)e^{zt}f(t)\,dt.$$

We factor out the z dependence using integration by parts:

$$\begin{aligned} 0 &= \int_\gamma \left(\left(t^2 f(t) - tf(t)\right)\frac{d}{dt}\left(e^{zt}\right) + (ct-a)e^{zt}f(t)\right)\,dt \\ &= \int_\gamma \left(-(2tf(t) + t^2 f'(t) - f(t) - tf'(t))e^{zt} + (ct-a)e^{zt}f(t)\right)\,dt + \left[(t^2 f(t) - tf(t))e^{zt}\right]_\gamma. \end{aligned}$$

Collecting everything together neatly, we have:

$$0 = -\int_\gamma \left((t^2-t)f'(t) + ((2-c)t + a - 1)f(t)\right)e^{zt}\,dt + \left[(t^2-t)f(t)e^{zt}\right]_\gamma$$

Hence we see that this equation is satisfied for all z if we choose $f(t)$ and γ such that:

$$(t-t^2)f'(t) = ((2-c)t + a - 1)f(t), \qquad \left[(t^2-t)f(t)e^{zt}\right]_\gamma = 0.$$

We can solve the differential equation, since it is separable. We have

$$\int \frac{f'(t)}{f(t)}\,df = \int \frac{(2-c)t + (a-1)}{t(1-t)}\,dt = \int \left(\frac{2-c}{1-t} + (a-1)\left(\frac{1}{1-t} + \frac{1}{t}\right)\right)\,dt.$$

Performing the integrals, all of which are logarithmic, we get:

$$\log(f) = -(2-c)\log(1-t) - (a-1)\log(1-t) + (a-1)\log(t) + \text{constant}$$

$$\Rightarrow \qquad f(t) = Ct^{a-1}(1-t)^{c-2+1-a} = Ct^{a-1}(t-1)^{c-a-1}.$$

Putting everything together (in particular, substituting this form of $f(t)$ into the boundary condition), we indeed see that the equation has solutions of the form:

$$w(z) = \int_\gamma t^{a-1}(1-t)^{c-a-1}e^{zt}\,dt, \qquad \text{provided } \gamma \text{ is chosen such that} \qquad \left[t^a(1-t)^{c-a}e^{zt}\right]_\gamma = 0.$$

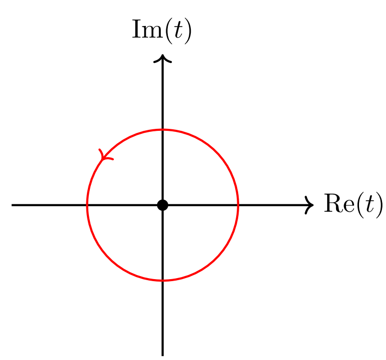
We now examine the possible contours γ for four special cases of the confluent hypergeometric equation, when we are interested in solutions in the region $\mathrm{Re}(z) > 0$:

- (i) For $a = -m$ a negative integer, and $c = 0$, the solution and boundary condition reduce to:

  $$w(z) = \int_\gamma \frac{(1-t)^{m-1}}{t^{m+1}}e^{zt}\,dt, \qquad \text{with } \gamma \text{ chosen such that} \qquad \left[\frac{(1-t)^m}{t^m}e^{zt}\right]_\gamma = 0.$$

  We now work through the possible choices for γ, as we described in the previous question.

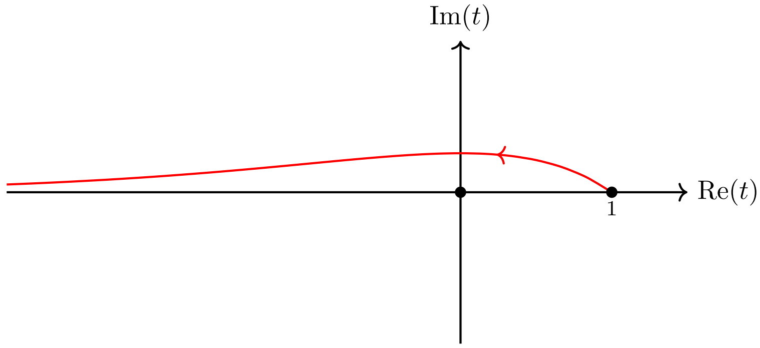
  The integrand of $w(z)$ has a single singularity at $t = 0$. This means it is possible to pick γ to be a closed contour encircling the singularity at $t = 0$; the fact that γ is closed means the boundary condition is automatically satisfied. Thus a possible choice of γ is:



  Now suppose that γ is not closed. It follows that the end points of γ must be either at zeroes of the boundary function, or at infinity in a direction where the boundary function tends to zero and the integral expression for $w(z)$ converges. In our case, we note:

  - The boundary function has a single zero at $t = 1$, so we can put one end of our contour there, and we are essentially forced to put the other end at infinity to avoid getting something that can be deformed to our previous contour.
  - Since we are assuming that $\mathrm{Re}(z) > 0$, if $t \to -\infty$ along the real axis, we find that $e^{zt} \to 0$. The boundary term and the integrand are thus exponentially suppressed, and everything behaves as it should. Hence a suitable place for the contour γ to go to infinity is along the negative real axis.

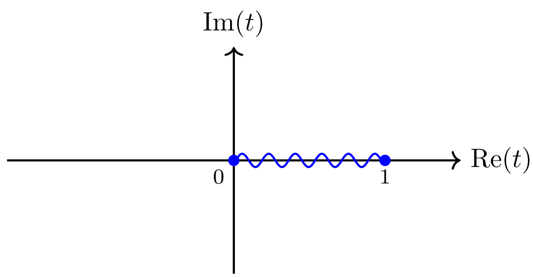
Ensuring that we avoid the singularity in the integrand at $z = 0$, we see that another possible choice of the contour γ is:



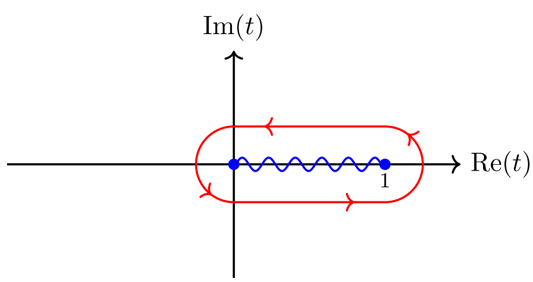
- (ii) The second example we are asked to consider is the case when $\mathrm{Re}(a) < 0$, with a not an integer, and $c = 0$. In this case, the integral solutions looks like:

$$w(z) = \int_\gamma t^{a-1}(1-t)^{-a-1}e^{zt}\,dt, \qquad \text{provided } \gamma \text{ is chosen such that} \qquad \left[t^a(1-t)^{-a}e^{zt}\right]_\gamma = 0.$$

Since a is no longer an integer, we notice that we are now working with a multivalued function, so we need to insert a branch cut. The branch points are at $t = 0$ and $t = 1$ (in principle, we need to check $t = \infty$ too - see part (iv) - but in this case, it is not a branch point). We choose our branch cut to range from 0 to 1 along the real axis:

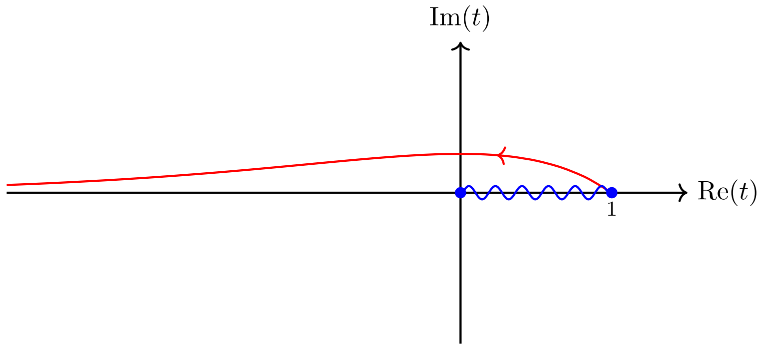


We now think about how we can pick the contour again. The integrand is now a complicated multivalued function and doesn't have a nice Laurent expansion, so it looks like we can't rely on poles and hence can't use closed contours γ. *However*, we know that if we integrate around the branch cut we will get something non-zero for $w(z)$, e.g. by choosing a branch where the function is odd. Therefore a suitable choice of γ is:



This is really the only possibility for closed γ; if we close anywhere apart from around the branch cut, we'll get zero by Cauchy's theorem because the function is analytic everywhere except from across the branch cut.

Thus we instead turn to open contours γ. We note that the same contour we used in part (i) works. This is because $\mathrm{Re}(a) < 0$, the boundary term has a zero at $t = 1$, and as $t \to \infty$ along the negative real axis both the integrand and the boundary term are exponentially suppressed so behave as expected. Thus another appropriate contour is:



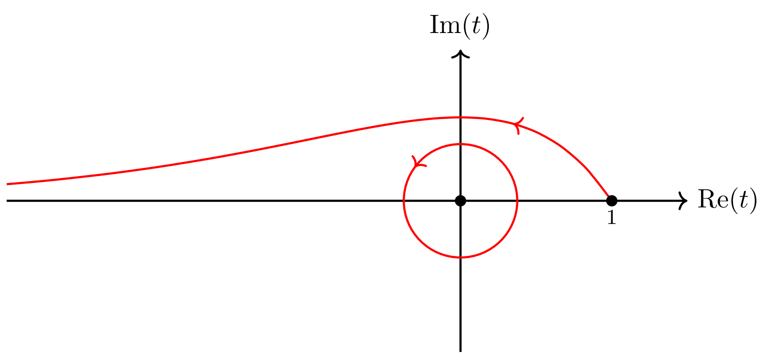
- (iii) In the case that $a = 0$ and $c = m$ is a positive integer, our integral representation becomes:

$$w(z) = \int_\gamma \frac{(1-t)^{m-1}}{t} e^{zt}\, dt, \qquad \text{provided } \gamma \text{ is chosen such that} \qquad \left[(1-t)^m e^{zt}\right]_\gamma = 0.$$

We're now getting the hang of the method for identifying suitable contours γ, so we can do this part quite quickly. Note that just as in part (i), we can choose γ closed and encircling the origin since there is a pole in the integrand at $t = 0$.

We can also choose the other contour from part (i), i.e. choose γ to start at $t = 1$ and finish at $t = \infty$ along the negative real axis. This is because $t = 1$ is a zero of the boundary function and both the integrand and the boundary function are exponentially suppressed as $t \to \infty$ along the negative real axis.

Therefore, suitable contours are:



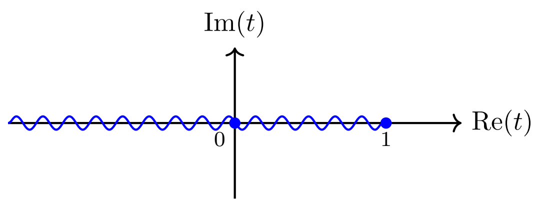
- (iv) For the final part, we have $\mathrm{Re}(a) > 0, \mathrm{Re}(c-a) > 0$, with a and $c-a$ both non-integers. In this case, our integral representation doesn't really simplify at all:

$$w(z) = \int\limits_{\gamma} t^{a-1}(1-t)^{c-a-1}e^{zt}\,dt, \qquad \text{provided } \gamma \text{ is chosen such that} \qquad \left[t^a(1-t)^{c-a}e^{zt}\right]_\gamma = 0.$$

We notice that since a and $c-a$ are non-integers, there are branch points at $t=0$ and $t=1$. In this case, there is also a branch point at $t=\infty$, since if we replace $t=1/s$ in the function $t^a(1-t)^{c-a}$ (common to both the integrand and the boundary function), we have

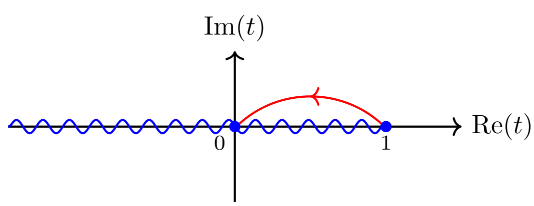
$$\frac{1}{s^a}\left(1-\frac{1}{s}\right)^{c-a} = \frac{1}{s^a}\cdot\frac{1}{s^{c-a}}\left(s-1\right)^{c-a} = s^{-c}\left(s-1\right)^{c-a}.$$

We see this has a branch point at $s=0$, so the original function has a branch point at $t=\infty$. Thus we must choose our branch cut to go through $0, 1$ and ∞. This can be achieved by choosing the cut along the real axis from 1 to $-\infty$:

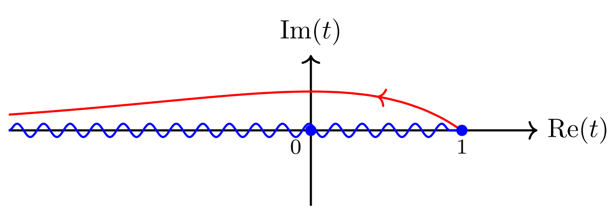


In this case then, we cannot choose γ closed, because there is no reasonable way of encircling the branch cut. If we try to encircle it, we will inevitably cross it, which results in a discontinuity and the boundary condition failing even though γ is closed.

Thus we must choose γ to be open. We notice that the boundary function has zeroes at $t=0$ and $t=1$ since $\mathrm{Re}(a)>0$ and $\mathrm{Re}(c-a)>0$, hence we can start and end a contour at those points. It follows that a possible choice of γ is:



To get a linearly independent solution, we must think of another place to send the contour. Naturally, we send an end of the contour to infinity; due to the exponential suppression in the boundary term and the integrand, we send the contour to infinity along the negative real axis. An example of another linearly independent contour is thus:



There are many other possibilities for contours here. For example, we could start at 0 and tend to $-\infty$. However, it's possible to argue that all other contours will lead to linearly dependent solutions when considered with the two we have found already.

7. Use the Laplace transform to solve the ordinary differential equation

$$\frac{d^2y}{dt^2} - k^2 y = f(t), \qquad k > 0, \qquad y(0) = y_0, \qquad y'(0) = y_0'.$$

Let $f(t) = e^{-k_0 t}$, $k_0 \neq k$, $k_0 > 0$, so that the Laplace transform of f is

$$\hat{f}(s) = \frac{1}{s + k_0}.$$

Show that

$$y(t) = y_0 \cosh(kt) + \frac{y_0'}{k}\sinh(kt) + \frac{e^{-k_0 t}}{k_0^2 - k^2} - \frac{\cosh(kt)}{k_0^2 - k^2} + \frac{k_0/k}{k_0^2 - k^2}\sinh(kt). \tag{$*$}$$

Now suppose that $f(t)$ is an arbitrary continuous function that possesses a Laplace transform. Use the convolution theorem for Laplace transforms, or otherwise, to show that

$$y(t) = y_0 \cosh(kt) + \frac{y_0'}{k}\sinh(kt) + \int_0^t f(t')\frac{\sinh(k(t-t'))}{k}\,dt'.$$

Put $f(t) = e^{-k_0 t}$ and re-obtain your answer to the first part of this question. Suppose now that $k_0 = k$. What is $y(t)$? Could you have found this solution by taking the limit in $(*)$ as $k_0 \to k$?

➾ **Solution:** We are first asked to solve the equation $\ddot{y} - k^2 y = f(t)$ using Laplace transforms, given some initial data. We'll first need some basic facts about Laplace transforms:

Theorem: We have the following facts about Laplace transforms:

(i) The Laplace transform of $y(t) = e^{kt}$ is given by:

$$\hat{y}(s) = \frac{1}{s-k}.$$

(ii) The Laplace transform of the nth derivative of a function is given by:

$$\mathcal{L}\left[\frac{d^n y}{dt^n}\right](s) = s^n \hat{y}(s) - s^{n-1}y(0) - s^{n-2}\frac{dy}{dt}(0) - ... - \frac{d^{n-1}y}{dt^{n-1}}(0) = s^n\hat{y}(s) - \sum_{r=1}^{n} s^{n-r}\frac{d^{r-1}y}{dt^{r-1}}(0).$$

(iii) The Laplace transform of a product is given by the *convolution* of the Laplace transforms:

$$\mathcal{L}[f(t)g(t)](s) = \hat{f}(s) * \hat{g}(s) := \int_{-\infty}^{\infty} ds\, \hat{f}(u)\hat{g}(s-u).$$

Similarly, the Laplace transform of a convolution is given by the product of the Laplace transforms:

$$\mathcal{L}[f(t) * g(t)](s) = \hat{f}(s)\hat{g}(s).$$

Proof: (i) We have:

$$\hat{y}(s) = \int_0^\infty e^{kt}e^{-st}\,dt = \left[\frac{e^{(k-s)t}}{k-s}\right]_0^\infty = \frac{1}{k-s},$$

assuming that $\mathrm{Re}(k - s) < 0$. However, note that the final Laplace transform is a meromorphic function on all of $s \in \mathbb{C}\backslash\{k\}$, and hence we can consider the Laplace transform to be defined on all of $\mathbb{C}\backslash\{k\}$ by meromorphic continuation.

(ii) It's sufficient to prove this for the first derivative, and then the result follows by induction. We have:

$$\mathcal{L}\left[\frac{dy}{dt}\right](s) = \int_0^\infty \frac{dy}{dt} e^{-st}\,dt = \left[y(t)e^{-st}\right]_0^\infty + s\int_0^\infty ye^{-st}\,dt = s\hat{y}(s) - y(0),$$

using integration by parts. Iterating this process for the higher derivatives gives the result.

(iii) The proof of the convolution theorem is rather technical, and isn't really part of this course; see the notes for Part IB Complex Methods to remind you if you're interested. □

We now solve the equation given in the question:

$$\frac{d^2y}{dt^2} - k^2 y = f(t), \qquad k > 0, \qquad y(0) = y_0, \qquad y'(0) = y_0'.$$

Taking the Laplace transform of the equation, we find:

$$s^2\hat{y}(s) - y_0' - sy_0 - k^2\hat{y}(s) = \hat{f}(s) \qquad \Rightarrow \qquad \hat{y}(s) = \frac{y_0 s}{s^2 - k^2} + \frac{y_0'}{s^2 - k^2} + \frac{\hat{f}(s)}{s^2 - k^2},$$

where we used property (ii) in the Theorem above to find the Laplace transform of the derivative.

We now wish to invert the Laplace transform. You may reach keenly for the Bromwich inversion contour integral - but wait! It's usually much easier to invert Laplace transforms by breaking things down into *partial fractions* and then remembering the Laplace transform of an exponential (property (i) in the Theorem above). In our case, we note that we have the partial fraction decompositions:

$$\frac{s}{s^2 - k^2} = \frac{1}{2}\left(\frac{1}{s+k} + \frac{1}{s-k}\right), \qquad \frac{1}{s^2 - k^2} = \frac{1}{2k}\left(\frac{1}{s-k} - \frac{1}{s+k}\right).$$

Hence the Laplace transform in this question can be written as:

$$\begin{aligned}\hat{y}(s) &= \frac{y_0}{2}\left(\frac{1}{s+k} + \frac{1}{s-k}\right) + \frac{y_0'}{2k}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) + \frac{\hat{f}(s)}{2k}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) \\ &= \frac{y_0}{2}\left(\mathcal{L}[e^{-kt}](s) + \mathcal{L}[e^{kt}](s)\right) + \frac{y_0'}{2k}\left(\mathcal{L}[e^{kt}](s) - \mathcal{L}[e^{-kt}](s)\right) + \frac{\mathcal{L}[f](s)}{2k}\left(\mathcal{L}[e^{kt}](s) - \mathcal{L}[e^{-kt}](s)\right).\end{aligned}$$

We can now take the inverse Laplace transform directly. Notice that the last term is a product of Laplace transforms, so by the convolution theorem its inverse is a convolution. Hence we have:

$$\begin{aligned}y(t) &= y_0\left(\frac{e^{-kt} + e^{kt}}{2}\right) + \frac{1}{k}y_0'\left(\frac{e^{kt} - e^{-kt}}{2}\right) + \frac{1}{k}f(t) * \left(\frac{e^{kt} - e^{-kt}}{2}\right) \\ &= y_0\cosh(kt) + y_0'\frac{\sinh(kt)}{k} + \int_{-\infty}^{\infty} dt'\, f(t')\frac{\sinh(k(t-t'))}{k}.\end{aligned}$$

Note that we can simplify the convolution integral, since when we are working with Laplace transformable functions $g(t)$, we assume they are zero whenever $t < 0$. Thus we're left with the final result:

$$y(t) = y_0\cosh(kt) + y_0'\frac{\sinh(kt)}{k} + \int_0^t dt'\, f(t')\frac{\sinh(k(t-t'))}{k}.$$

We are now asked to specialise to the case where $f(t) = e^{-k_0 t}$ where $k_0 \neq k$. Inserting this into the formula we derived earlier, we see that we simply need to evaluate the integral:

$$\begin{aligned}
\frac{1}{k}\int_0^t dt'\, e^{-k_0 t'} \sinh(k(t-t')) &= \frac{1}{2k}\int_0^t dt'\, e^{-k_0 t'} \left(e^{k(t-t')} - e^{-k(t-t')}\right) && \text{(definition of } \sinh) \\
&= \frac{1}{2k}\int_0^t dt' \left(e^{kt-(k+k_0)t'} - e^{-kt+(k-k_0)t'}\right) \\
&= \frac{1}{2k}\left[-\frac{e^{kt-(k+k_0)t'}}{k+k_0} - \frac{e^{-kt+(k-k_0)t'}}{k-k_0}\right]_0^t \\
&= \frac{1}{2k}\left[-\frac{e^{-k_0 t}}{k+k_0} - \frac{e^{-k_0 t}}{k-k_0} + \frac{e^{kt}}{k_0+k} + \frac{e^{-kt}}{k-k_0}\right] \\
&= \frac{e^{-k_0 t}}{k_0^2-k^2} - \frac{1}{2k}\left(\frac{e^{kt}(k-k_0) + e^{-kt}(k+k_0)}{k_0^2-k^2}\right) && \text{(combine similar exponentials)} \\
&= \frac{e^{-k_0 t}}{k_0^2-k^2} - \frac{\cosh(kt)}{k_0^2-k^2} + \frac{k_0/k}{k_0^2-k^2}\sinh(kt) && \text{(definitions of } \sinh, \cosh)
\end{aligned}$$

Thus putting everything together, we have:

$$y(t) = y_0\cosh(kt) + \frac{y_0'}{k}\sinh(kt) + \frac{e^{-k_0 t}}{k_0^2-k^2} - \frac{\cosh(kt)}{k_0^2-k^2} + \frac{k_0/k}{k_0^2-k^2}\sinh(kt)$$

as required.

The next part of the question asks us to solve the equation for an arbitrary, continuous, Laplace-transformable function $f(t)$, by establishing an integral formula for $f(t)$. We've already done this.

We are then asked to look at the special case $f(t) = e^{-k_0 t}$ using our integral formula. We have also done this already. I believe that this question needs some restructuring.

Finally, we are asked to look at the case $k_0 = k$, i.e. look at $f(t) = e^{-kt}$. This will give different results because the system will resonate, as e^{-kt} is one of the solutions to the homogeneous equation.

We can find $y(t)$ using our integral formula as usual. The integral we need to evaluate is in this case:

$$\begin{aligned}
\frac{1}{k}\int_0^t dt'\, e^{-kt'}\sinh(k(t-t')) &= \frac{1}{2k}\int_0^t dt'\, e^{-kt'}\left(e^{k(t-t')} - e^{-k(t-t')}\right) && (\text{definition of } \sinh) \\
&= \frac{1}{2k}\int_0^t dt'\,\left(e^{kt-2kt'} - e^{-kt}\right) \\
&= \frac{1}{2k}\left[-\frac{e^{kt-2kt'}}{2k} - t'e^{-kt}\right]_0^t \\
&= \frac{1}{2k}\left[-\frac{e^{-kt}}{2k} - te^{-kt} + \frac{e^{kt}}{2k}\right] \\
&= \frac{\sinh(kt)}{2k^2} - \frac{te^{-kt}}{2k}.
\end{aligned}$$

Hence we see that

$$y(t) = y_0\cosh(kt) + \frac{y_0'}{k}\sinh(kt) + \frac{\sinh(kt)}{2k^2} - \frac{te^{-kt}}{2k}.$$

We can also easily derive this result by taking the limit $k_0 \to k$ in our earlier solution. We have:

$$\lim_{k_0\to k}[y(t)] = y_0\cosh(kt) + \frac{y_0'}{k}\sinh(kt) + \lim_{k_0\to k}\left[\frac{e^{-k_0t}}{k_0^2-k^2} - \frac{\cosh(kt)}{k_0^2-k^2} + \frac{k_0/k}{k_0^2-k^2}\sinh(kt)\right].$$

To take the limit, we can use L'Hôpital's rule:

$$\lim_{k_0\to k}\left[\frac{e^{-k_0t} - \cosh(kt) + (k_0/k)\sinh(kt)}{k_0^2-k^2}\right] = \lim_{k_0\to k}\left[\frac{-te^{-k_0t} + (1/k)\sinh(kt)}{2k_0}\right] = -\frac{te^{-kt}}{2k} + \frac{\sinh(kt)}{2k^2}.$$

Hence we get the solution:

$$y(t) = y_0\cosh(kt) + \frac{y_0'}{k}\sinh(kt) + \frac{\sinh(kt)}{2k^2} - \frac{te^{-kt}}{2k}$$

as before.

8. The Schrödinger equation is

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0.$$

Suppose that $u(x,0) = f(x)$. Fourier transform this equation with respect to x to find

$$u(x,t) = \frac{e^{-i\pi/4}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{i\frac{(x-x')^2}{4t}} f(x')\,dx'.$$

[Hint: You may find it useful to recall that $\int_{-\infty}^{\infty} e^{iu^2}\,du = e^{i\pi/4}\sqrt{\pi}$*.]*

Now use Laplace transform methods to find the same solution to this problem.

➻ **Solution:** For the Fourier transform part of this question, it's useful to remember some properties of the Fourier transform:

Theorem: The Fourier transform has the following properties:

(i) The Fourier transform of the derivative of a function is given by

$$\mathcal{F}\left[\frac{df}{dx}\right](k) = ik\tilde{f}(k)$$

where $\tilde{f}(k)$ is the Fourier transform of the function f.

(ii) The Fourier transform of a product is given by the convolution:

$$\mathcal{F}[f(x)g(x)](k) = \tilde{f}(k) * \tilde{g}(k).$$

Similarly, the Fourier transform of a convolution is given by the product:

$$\mathcal{F}[f(x) * g(x)](k) = \frac{1}{2\pi}\tilde{f}(k)\tilde{g}(k).$$

Proof: (i) is very similar to the Laplace transform proof in Question 7, and (ii) is deferred to another course such as Part IB Complex Methods. □

We can apply these properties to solve the equation in the question, namely

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0, \qquad u(x,0) = f(x).$$

Taking the Fourier transform with respect to x of the equation and the boundary condition, we have

$$i\frac{\partial \tilde{u}(k,t)}{\partial t} - k^2\tilde{u}(k,t) = 0, \qquad \tilde{u}(k,0) = \tilde{f}(k).$$

Here, we used property (i) from the Theorem above to evaluate the Fourier transform of the derivative.

Solving this first order, separable differential equation, we have

$$\tilde{u}(k,t) = \tilde{f}(k)e^{-ik^2 t}.$$

To invert, we use property (ii) (i.e. the convolution theorem) from the Theorem above:

$$u(x,t) = f(x) * \mathcal{F}^{-1}[e^{-ik^2t}](x) = \int_{-\infty}^{\infty} dx'\, f(x')\mathcal{F}^{-1}[e^{-ik^2t}](x-x').$$

We see that to finish, we see that we require the inverse Fourier transform of a Gaussian. This is given by:

$$\mathcal{F}^{-1}[e^{-ik^2t}](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik^2t+ikx}\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix^2/4t}e^{-it(k-x/2t)^2}\, dk,$$

where in the last step, we completed the square. Making the substitution $u = \sqrt{t}(k - x/2t)$ in the final integral, we get:

$$\mathcal{F}^{-1}[e^{-ik^2t}](x) = \frac{e^{ix^2/4t}}{2\pi\sqrt{t}}\int_{-\infty}^{\infty} e^{-iu^2}\, du = \frac{e^{ix^2/4t}}{2\pi\sqrt{t}}\cdot e^{-i\pi/4}\sqrt{\pi},$$

evaluating the final integral using the hint (or more precisely, the complex conjugate of the hint). Thus we find:

$$u(x,t) = \frac{e^{-i\pi/4}}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} dx'\, \exp\left(\frac{i(x-x')^2}{4t}\right) f(x')\, dx',$$

as required.

We are now asked to derive the same result using Laplace transforms. Things get very messy from here on out, so you should consider this part of the question non-examinable.

If we are being honest with ourselves, we have to take the Laplace transform of the Schrödinger equation with respect to time. This is because the Laplace transform is applicable to initial value problems; in particular, we can only take the Laplace transform of something if it is zero sufficiently far in the past. This will only be okay for our initial data $u(x,0) = f(x)$ if we take the transform with respect to time t.

Going ahead and taking the Laplace transform, we have:

$$\mathcal{L}\left[i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2}\right](s) = 0 \qquad \Rightarrow \qquad is\hat{u}(x,s) - iu(x,0) + \frac{\partial^2 \hat{u}(x,s)}{\partial x^2} = 0$$

$$\Rightarrow \qquad \frac{\partial^2 \hat{u}(x,s)}{\partial x^2} + is\hat{u}(x,s) = if(x).$$

The nicest way to solve this inhomogeneous second order differential equation is using a Green's function. The Green's function $G(x,\xi)$ for this problem must satisfy:

$$\frac{\partial^2 G}{\partial x^2} + isG = \delta(x-\xi).$$

This is simple enough to solve, since for $x \neq \xi$, the equation is just that of a simple harmonic oscillator. But let's be extra careful cautious, because things are not as simple as they appear. Suppose that a solution for $x \neq \xi$ is of the form $e^{\lambda x}$. Then we have:

$$\lambda^2 e^{\lambda x} + ise^{\lambda x} = 0 \qquad \Rightarrow \qquad \lambda^2 = -is.$$

This equation is called the *auxiliary equation* of the differential equation. Now recall that s is a *complex* number (especially as we will eventually want to use the Bromwich inversion contour to invert our Laplace transform). Thus we need to be careful when taking the square root of the right hand side - in particular, we will need to specify a suitable branch.

For now, let's just write down the solution though, bearing in mind we need to sort the branch out:

$$G(x,\xi)=\begin{cases}Ae^{x\sqrt{-is}}+Be^{-x\sqrt{-is}} & \text{for } x>\xi,\\ Ce^{x\sqrt{-is}}+De^{-x\sqrt{-is}} & \text{for } x<\xi.\end{cases}$$

Here, A, B, C and D are constants which are fixed by the boundary conditions and the requirement of continuity and the jump condition of the derivative at $x=\xi$. Let's summarise these conditions:

- The standard boundary condition we must assume on $u(x,t)$ is that it decays to zero as $|x|\to\infty$. This is the condition that the wavefunction in quantum mechanics is *normalisable*. Therefore, the Laplace transformed u must also obey this condition, $\hat{u}(x,t)\to 0$ as $|x|\to\infty$, and it follows that the Green's function also obeys this boundary condition:

$$G(x,\xi)\to 0 \qquad \text{as } |x|\to\infty.$$

- Continuity at $x=\xi$ of the Green's function comes from the fact that if it were discontinuous, its derivative would look like a delta function at $x=\xi$ and thus its second derivative would look like a derivative of the delta function - it follows our equation would not be satisfied. Hence we require:

$$\lim_{x\to\xi^+}G(x,\xi)=\lim_{x\to\xi^-}G(x,\xi).$$

- The *jump condition* at $x=\xi$ in the derivative of the Green's function comes from the integral of the equation over a region including $x=\xi$:

$$\int\limits_{\xi-\epsilon}^{\xi+\epsilon}\frac{\partial^2 G}{\partial x^2}\,dx+is\int\limits_{\xi-\epsilon}^{\xi+\epsilon}G(x,\xi)\,dx=1.$$

  As $\epsilon\to 0$, the second integral must give zero because $G(x,\xi)$ is a continuous function of x. Hence, evaluating the first integral using the fundamental theorem of calculus, we are left only with:

$$\lim_{x\to\xi^+}\frac{\partial G}{\partial x}-\lim_{x\to\xi^-}\frac{\partial G}{\partial x}=1,$$

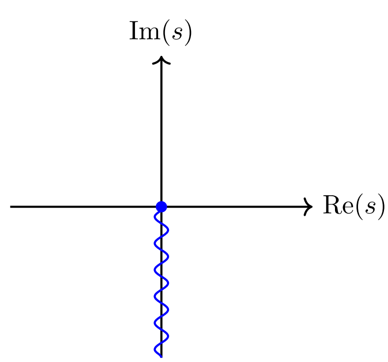
  which is the *jump condition*.

It turns out that the first condition, $G(x,\xi)\to 0$ as $|x|\to\infty$, is a bit tricky to enforce. For example, consider enforcing this condition in the region $x>\xi$ where $x\to\infty$:

$$G(x,\xi)=Ae^{x\sqrt{-is}}+Be^{-x\sqrt{-is}}.$$

Depending on the value of the real part of $\sqrt{-is}$, we can find that either the constant A or the constant B should be zero because of this condition. In particular, if our choice of branch is such that $\sqrt{-is}$ can have both negative and positive real parts as s varies, we will find that we will need *two different types of Green's functions depending on the value of s!*

This is clearly undesirable. Thus we carefully choose a branch of $\sqrt{-is}$ such that its real part is either strictly positive everywhere or strictly negative everywhere. One such branch is described as follows: choose a branch cut along the negative imaginary axis, and choose $\sqrt{-is}\in\mathbb{R}^+$ on the positive imaginary axis.



Let's realise this branch explicitly. Write $s = re^{i\theta}$ with a choice of arguments $-\pi/2 < \theta < 3\pi/2$, then our function can be written as

$$\sqrt{-is} = \left(e^{-i\pi/2}re^{i\theta}\right)^{1/2} = r^{1/2}e^{i(\theta/2-\pi/4)} = r^{1/2}\left(\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)+i\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\right).$$

This is indeed the correct realisation of the branch, since we notice that on the positive imaginary axis $\theta = \pi/2$, we have $\sqrt{-is} = r^{1/2}$, which is indeed real and positive.

Notice also that for this branch, we have that $\mathrm{Re}(\sqrt{-is}) = \cos\left(\frac{1}{2}\theta - \frac{1}{4}\pi\right)$ is *positive* for all $\theta \in (-\frac{1}{2}\pi, \frac{3}{2}\pi)$, and hence we only need one Green's function for the whole s plane - phew!

Let's start to fix our constants A, B, C and D then. Recall that the Green's function had the general form:

$$G(x,\xi) = \begin{cases} Ae^{x\sqrt{-is}} + Be^{-x\sqrt{-is}} & \text{for } x > \xi, \\ Ce^{x\sqrt{-is}} + De^{-x\sqrt{-is}} & \text{for } x < \xi. \end{cases}$$

With our choice of branch for $\sqrt{-is}$, we have that $\mathrm{Re}(\sqrt{-is}) > 0$. Hence as $x \to \infty$, we discover that $A = 0$ is required for $G(x,\xi) \to 0$. Similarly as $x \to -\infty$, we find $D = 0$. Changing our constants around slightly to impose continuity at $x = \xi$, we find that the general form of the Green's function is reduced to:

$$G(x,\xi) = \begin{cases} Ke^{-(x-\xi)\sqrt{-is}} & \text{for } x > \xi, \\ Ke^{(x-\xi)\sqrt{-is}} & \text{for } x < \xi. \end{cases}$$

Finally, imposing the jump condition, we have:

$$1 = \lim_{x\to\xi^+}\frac{\partial G}{\partial x} - \lim_{x\to\xi^-}\frac{\partial G}{\partial x} = -K\sqrt{-is} - K\sqrt{-is} \qquad \Rightarrow \qquad K = -\frac{1}{2\sqrt{-is}}.$$

It follows that we can write the Green's function as:

$$G(x,\xi) = -\frac{1}{2\sqrt{-is}}\exp\left(-\sqrt{-is}|x-\xi|\right).$$

We now recall why we were finding this Green's function: to determine the solution $\hat{u}(x,s)$ of the second order inhomogeneous equation we were faced with after taking the Laplace transform. Using the Green's function, the solution is given by:

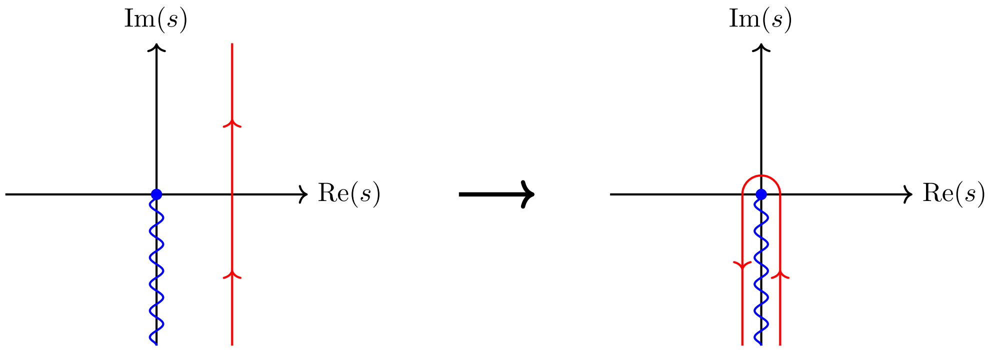
$$\hat{u}(x,s) = -\frac{i}{2\sqrt{-is}}\int_{-\infty}^{\infty} f(\xi)\exp\left(-\sqrt{-is}|x-\xi|\right)\,d\xi.$$

It remains to perform the Bromwich inversion integral. We find:

$$u(x,t) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} f(\xi)\left(\int_{c-i\infty}^{c+i\infty}\frac{1}{\sqrt{-is}}\exp\left(-\sqrt{-is}|x-\xi|\right)e^{st}\,ds\right)\,d\xi,$$

where c is a constant chosen such that the contour of integration is to the right of all singularities of the integrand. But note that in fact there are no singularities of the integrand for our problem, so we are free to deform the Bromwich contour however we want.

In this case, there is a particularly useful deformation of the Bromwich contour, which is suggested by the fact that we have a branch cut. Usually whenever we have a branch cut, it is useful to deform the contour close to it; thus, consider deforming the Bromwich contour as shown:



The arc at the top of the deformed contour is a small semicircular arc of radius ϵ; we will eventually take the limit as $\epsilon \to 0$.

We now evaluate our Bromwich inversion integral on each section of the deformed contour separately. On the first straight segment of the contour, we parametrise via

$$s = ue^{-i\pi/2}, \qquad u \in (0, \infty),$$

and on the second straight segment of the contour, we parametrise via

$$s = ue^{3i\pi/2}, \qquad u \in (0, \infty).$$

We are also careful to remember that our choice of branch defines $\sqrt{-is}$ for $s = re^{i\theta}$ as:

$$\sqrt{-is} = r^{1/2}e^{i(\theta/2-\pi/4)}.$$

Going ahead and evaluating the integral then, we have:

$$\begin{aligned}\int\limits_C \frac{1}{\sqrt{-is}} \exp\left(-\sqrt{-is}|x-\xi|\right) e^{st}\, ds &= \int\limits_\infty^0 \frac{1}{u^{1/2}e^{-i\pi/2}} \exp\left(-u^{1/2}e^{-i\pi/2}|x-\xi|\right) \exp\left(ue^{-i\pi/2}t\right) e^{-i\pi/2}\, du \\ &+ \int\limits_0^\infty \frac{1}{u^{1/2}e^{i\pi/2}} \exp\left(-u^{1/2}e^{i\pi/2}|x-\xi|\right) \exp\left(ue^{3i\pi/2}t\right) e^{3i\pi/2}\, du \\ &+ \int\limits_0^\pi \frac{1}{\epsilon^{1/2}e^{i(\theta/2-\pi/4)}} \exp\left(-\epsilon^{1/2}e^{i(\theta/2-\pi/4)}|x-\xi|\right) \exp\left(\epsilon e^{i\theta}t\right) i\epsilon e^{i\theta}\, d\theta.\end{aligned}$$

Clearly as $\epsilon \to 0$, the term from the angular integration vanishes, since it is of order $O(\epsilon^{1/2})$. Tidying up the remaining integrals, we're left with:

$$\int\limits_C \exp\left(-\sqrt{-is}|x-\xi|\right) e^{st}\, ds = -\int\limits_0^\infty \frac{1}{\sqrt{u}} \left(e^{-iut-i\sqrt{u}|x-\xi|} + e^{-iut+i\sqrt{u}|x-\xi|}\right) du.$$

To make further progress, let's make the substitution $u = v^2$. Then $du = 2vdv$ and the limits change as $[0,\infty) \mapsto [0,\infty)$. Thus we get:

$$-2\int\limits_0^\infty \left(e^{-iv^2t-iv|x-\xi|} + e^{-iv^2t+iv|x-\xi|}\right) dv.$$

From this expression, it's clear that we can combine the terms by extending the range of integration:

$$-2\int_{0}^{\infty}\left(e^{-iv^2t-iv|x-\xi|}+e^{-iv^2t+iv|x-\xi|}\right)\,dv=-2\int_{-\infty}^{\infty}e^{-iv^2t-iv|x-\xi|}\,dv.$$

This integral is now of a standard form that can be evaluated. Completing the square in the exponent, we have

$$-2\int_{-\infty}^{\infty}e^{-iv^2t-iv|x-\xi|}\,dv=-2\exp\left(\frac{i|x-\xi|^2}{4t^2}\right)\int_{-\infty}^{\infty}\exp\left(-it\left(v+\frac{|x-\xi|}{2t}\right)^2\right)\,dv.$$

We're left with exactly the same integral as we had to tackle in the Fourier case. Making the same substitution and using the hint, we get the final contribution from the Bromwich inversion integral:

$$-\frac{2}{\sqrt{t}}e^{-i\pi/4}\sqrt{\pi}\exp\left(\frac{i|x-\xi|^2}{4t^2}\right).$$

Putting everything together, we see that:

$$u(x,t)=\frac{e^{-i\pi/4}}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}f(\xi)\exp\left(\frac{i|x-\xi|^2}{4t^2}\right)\,d\xi,$$

which is exactly the same solution, as required.

9. ($*$) A simple version of the Klein-Gordon equation is:

$$\psi_{tt} = \psi_{xx} - \psi.$$

This equation describes, amongst other things, the propagation of large-scale variations in the height of the sea surface in the presence of rotation.

(a) Solve this equation subject to the initial conditions $\psi(x,0) = 0, \psi_t(x,0) = \delta(x)$ using Laplace transform methods. Show that, for $t < |x|$, $\psi(x,t) = 0$, and, for $t > |x|$,

$$\psi(x,t) = \frac{1}{2\pi i} \int_\gamma e^{st} \exp\left(-(1+s^2)^{1/2}|x|\right) \frac{ds}{2(1+s^2)^{1/2}},$$

where γ, followed anticlockwise, encloses a branch cut along the imaginary axis from $s = -i$ to $s = i$.

(b) Show that, defining the variable w by $(t^2 - x^2)^{1/2} w = st - (1+s^2)^{1/2}|x|$, the above integral may be transformed to give:

$$\psi(x,t) = \frac{1}{2\pi i} \int_\gamma \exp\left((t^2 - x^2)^{1/2} w\right) \frac{dw}{2(1+w^2)^{1/2}},$$

with γ defined in the w-plane as in the s-plane.

(c) Show using Laplace's method that $J_0(z)$, which is the solution of $zy'' + y' + zy = 0$ such that $y(0) = 1$ and $y'(0) = 0$, can be represented as:

$$J_0(z) = \frac{1}{2\pi i} \int_\gamma \frac{e^{zs}}{(1+s^2)^{1/2}}\, ds,$$

with γ again as defined above.

(d) Deduce that the solution of the Klein-Gordon equation specified above for $t > |x|$ may be written as:

$$\psi(x,t) = \frac{1}{2} J_0\left((t^2 - x^2)^{1/2}\right).$$

Draw a sketch of the solution.

Solution: (a) Taking the Laplace transform of the given equation, we have:

$$s^2\hat{\psi} - s\psi(x,0) - \frac{\partial \psi}{\partial t}(x,0) = \hat{\psi}_{xx} - \hat{\psi}.$$

Substituting the given initial conditions, we see that:

$$\hat{\psi}_{xx} - (1+s^2)\hat{\psi} = -\delta(x).$$

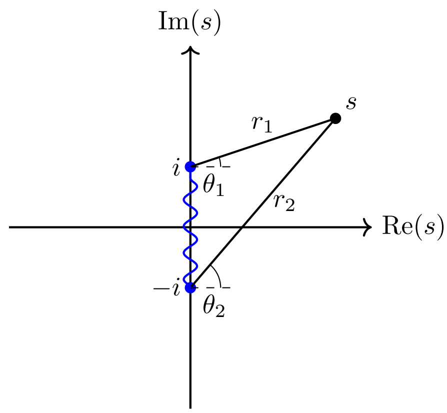
Now for $x \neq 0$, the equation becomes a standard second order differential equation with constant coefficients:

$$\hat{\psi}_{xx} - (1+s^2)\hat{\psi} = 0.$$

The auxiliary equation is $\lambda^2 = 1 + s^2$, so the solution to this equation is given by:

$$\hat{\psi}(x,s) = \begin{cases} A(s)\exp\left((1+s^2)^{1/2}x\right) + B(s)\exp\left(-(1+s^2)^{1/2}x\right), & x > 0; \\ C(s)\exp\left((1+s^2)^{1/2}x\right) + D(s)\exp\left(-(1+s^2)^{1/2}x\right), & x < 0. \end{cases}.$$

Since $s \in \mathbb{C}$, $(1+s^2)^{1/2}$ is multi-valued so we will need to define it. As in Question 8, the choice of branch matters, and if we make a poor choice we will end up with much more complicated expressions for $A(s), B(s)$ than is desirable. Fortunately, this question instructs us to take a particular branch, which we display in the following figure.



With this choice of branch cut, we define:

$$s - i = r_1 e^{i\theta_1}, \qquad s + i = r_2 e^{i\theta_2},$$

with $\theta_1, \theta_2 \in (-\frac{3\pi}{2}, \frac{\pi}{2})$, and hence:

$$(1+s^2)^{1/2} := \sqrt{r_1 r_2} e^{i(\theta_1+\theta_2)/2} = \sqrt{|1+s^2|} e^{i(\theta_1+\theta_2)/2}.$$

In particular, the real part of this function is given by:

$$\mathrm{Re}\left((1+s^2)^{1/2}\right) = \sqrt{|1+s^2|}\cos\left(\frac{1}{2}(\theta_1+\theta_2)\right).$$

We see that:

- In the right half plane, $\theta_1, \theta_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, hence $\frac{1}{2}(\theta_1+\theta_2) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. It follows that the real part of $(1+s^2)^{1/2}$ is always positive in the right half plane.
- In the left half plane, $\theta_1, \theta_2 \in (\frac{\pi}{2}, \frac{3\pi}{2})$, hence $\frac{1}{2}(\theta_1+\theta_2) \in (\frac{\pi}{2}, \frac{3\pi}{2})$. It follows that the real part of $(1+s^2)^{1/2}$ is always negative in the left half plane.

The Bromwich inversion contour is initially restricted to the right half plane, so we will assume that $\mathrm{Re}((1+s^2)^{1/2}) > 0$ until we need to deform the contour. There are now cases depending on whether $x > 0$ or $x < 0$:

- We require that $\hat{\psi}(x,s) \to 0$ as $|x| \to \infty$, which is an 'implicit' boundary condition of the original problem. Applying this condition, we see that:

$$\hat{\psi}(x,s) = \begin{cases} B(s)\exp\left(-(1+s^2)^{1/2}x\right), & x > 0; \\ C(s)\exp\left((1+s^2)^{1/2}x\right), & x < 0. \end{cases}$$

- Continuity at $x = 0$ implies that $B(s) = C(s)$.
- The jump condition at $x = 0$ implies that (note the minus sign in front of the delta function in the original equation):

$$-(1+s^2)^{1/2}B(s) - (1+s^2)^{1/2}C(s) = -1.$$

Putting everything together, we see that:

$$\hat{\psi}(x,s) = \frac{\exp\left(-(1+s^2)^{1/2}|x|\right)}{2(1+s^2)^{1/2}}.$$

Now, the Bromwich inversion formula tells us that:

$$\psi(x,t) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} e^{st}\exp\left(-(1+s^2)^{1/2}|x|\right)\frac{ds}{2(1+s^2)^{1/2}}.$$

Since there are no singularities of the integrand apart from on the branch cut, we can take c infinitesimally small and positive. We now consider deforming the contour in two separate cases:

- **Case 1.** When $t < |x|$, we consider $\mathrm{Re}(s) > 0$, where $\mathrm{Re}((1+s^2)^{1/2}) > 0$. In this case, we have:

  $$\mathrm{Re}(st - (1+s^2)^{1/2}|x|) = t\,\mathrm{Re}(s) - |x|\,\mathrm{Re}((1+s^2)^{1/2}) < |x|\,\mathrm{Re}\left(s - (1+s^2)^{1/2}\right).$$

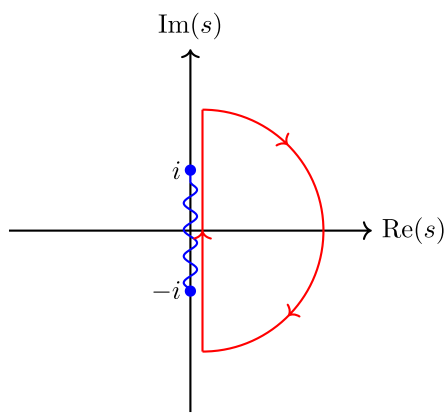
  We now note that (by rationalising the denominator), we have:

  $$s - (1+s^2)^{1/2} = -\frac{1}{s+(1+s^2)^{1/2}}.$$

  The denominator on the right hand side must have positive real part, so overall the right hand side has negative real part. It follows that $\mathrm{Re}\left(s-(1+s^2)^{1/2}\right) < 0$, and hence:

  $$\mathrm{Re}(st - (1+s^2)^{1/2}|x|) < 0.$$

  In particular, consider integrating around the contour shown in the figure below. In the limit as the radius of the semi-circle approaches infinity, the integrand is exponentially suppressed on the large arc, and we obtain the Bromwich inversion contour again.

  On the other hand, this contour contains no singularities of the integrand. Hence by Cauchy's theorem, we have:

  $$\psi(x,t) = 0, \qquad \text{for } t < |x|,$$

  as required.

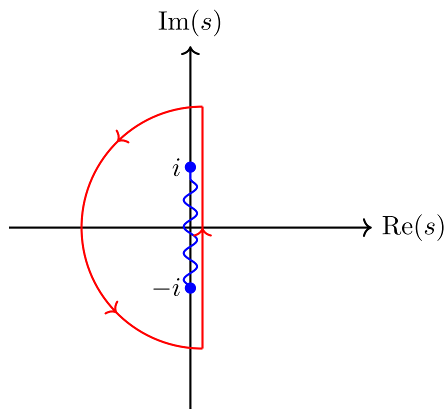
· **Case 2.** When $t > |x|$, we consider $\mathrm{Re}(s) < 0$, where $\mathrm{Re}((1+s^2)^{1/2}) < 0$. In this case, we have $t\,\mathrm{Re}(s) < t(-\mathrm{Re}(s)) < |x|(-\mathrm{Re}(s)) < -|x|\,\mathrm{Re}(s)$, thus:

$$\mathrm{Re}(st - (1+s^2)^{1/2}|x|) = t\,\mathrm{Re}(s) - |x|\,\mathrm{Re}((1+s^2)^{1/2}) < -|x|\,\mathrm{Re}\left(s - (1+s^2)^{1/2}\right).$$

Here, we have $\mathrm{Re}\left(s - (1+s^2)^{1/2}\right) > 0$ by the same argument as above. So overall, we see that:

$$\mathrm{Re}(st - (1+s^2)^{1/2}|x|) < 0.$$

Therefore, this time we can consider integrating around the contour shown in the figure below. Again, in the limit as the radius of the semicircle approaches infinity, the integrand is exponentially suppressed on the large arc, and we obtain the Bromwich inversion contour.



On the other hand, we can deform this contour onto the branch, yielding the formula:

$$\psi(x,t) = \frac{1}{2\pi i}\int\limits_{\gamma} e^{st}\exp\left(-(1+s^2)^{1/2}|x|\right)\frac{ds}{2(1+s^2)^{1/2}},$$

for $t > |x|$, as required.

(b) The suggested change of variables is:

$$(t^2-x^2)^{1/2}w = st - (1+s^2)^{1/2}|x|.$$

This implicitly defines s as a function of w, which is required for the change of variables. In particular, we must solve this equation for s in terms of w. We have:

$$x^2(1+s^2) = ((t^2-x^2)^{1/2}w - st)^2 = (t^2-x^2)w^2 - 2wst(t^2-x^2)^{1/2} + s^2t^2.$$

Rearranging, we have:

$$x^2(1+w^2) = (t^2-x^2)s^2 - 2wst(t^2-x^2)^{1/2} + t^2w^2,$$

which is of exactly the same form, hence:

$$(t^2-x^2)^{1/2}s = wt - (1+w^2)^{1/2}|x|.$$

This shows that as a function of w, s has branch points at $w = \pm i$. In particular in w-space, we can insert a branch cut between $-i$ and i as in s-space.

Next, note for very large s, we have:

$$(1+s^2)^{1/2} = \pm s\left(1+\frac{1}{s^2}\right)^{1/2} = \pm s\left(1+\frac{1}{2s^2}+\cdots\right).$$

We should take the $+$ sign since $(1+s^2)^{1/2}$ is positive on the positive real axis. Therefore we have:

$$(t^2-x^2)^{1/2}w = st-(1+s^2)^{1/2}|x| = (t-|x|)Re^{i\theta}+O(1/R).$$

For $t>|x|$, the range we are interested in, we see that the contour in s-space is mapped to another large circle in w-space. We can deform this contour in w-space back to an integral around the branch cut, yielding exactly the same contour that we started with, but now in w-space.

It remains to deal with the measure. We see that it transform as:

$$(t^2-x^2)^{1/2}dw = \left(t-\frac{s|x|}{(1+s^2)^{1/2}}\right)ds = \left(t(1+s^2)^{1/2}-s|x|\right)\frac{ds}{(1+s^2)^{1/2}}.$$

To simplify this, we note that:

$$\begin{aligned}(t^2-x^2)(1+w^2) &= t^2-x^2+s^2t^2-2st|x|(1+s^2)^{1/2}+x^2(1+s^2)\\ &= t^2(1+s^2)-2st|x|(1+s^2)^{1/2}+x^2s^2\\ &= \left(t(1+s^2)^{1/2}-s|x|\right)^2.\end{aligned}$$

Hence overall, we have $t(1+s^2)^{1/2}-s|x| = (t^2-x^2)^{1/2}(1+w^2)^{1/2}$, and so:

$$\frac{dw}{(1+w^2)^{1/2}} = \frac{ds}{(1+s^2)^{1/2}}.$$

It follows that we can indeed rewrite the integral in the required form:

$$\psi(x,t) = \frac{1}{2\pi i}\int\limits_{\gamma}\exp\left((t^2-x^2)^{1/2}w\right)\frac{dw}{2(1+w^2)^{1/2}}.$$

(c) We are asked to determine an integral representation for the solution of $zy''+y'+zy=0$, subject to the conditions $y(0)=1$ and $y'(0)=0$. We pose a solution of the form:

$$y(z) = \int\limits_C e^{zt}f(t)\,dt,$$

where we use the Laplace kernel e^{zt}, and $f(t)$ is some function to be determined. We also have freedom to choose the integration contour C. Substituting into the equation, we derive:

$$\int\limits_C \left(zt^2+t+z\right)e^{zt}f(t)\,dt = 0.$$

Now using integration by parts, we see that:

$$\int\limits_C \left(-(t^2+1)f'(t)-tf(t)\right)e^{zt}\,dt + \left[f(t)(t^2+1)e^{zt}\right]_C = 0.$$

To ensure that this equation holds for all z, we choose $f(t)$ and C to satisfy:

$$\frac{f'(t)}{f(t)} = -\frac{t}{t^2+1}, \qquad \left[f(t)(t^2+1)e^{zt}\right]_C = 0.$$

Solving the first equation, we see that:

$$\log(f(t)) = A - \frac{1}{2}\log(t^2+1) \qquad \Rightarrow \qquad f(t) = K(t^2+1)^{-1/2}.$$

This function is multi-valued; we choose to take a branch cut from $-i$ to i as above. We see that the contour C must be chosen to satisfy:

$$\left[K\frac{e^{zt}}{(t^2+1)^{1/2}}\right]_C = 0.$$

Hence, we can integrate around the branch cut using a closed contour to guarantee this condition. Overall, we see that we have derived a solution of the form:

$$y(z) = K\int_\gamma \frac{e^{zt}}{(t^2+1)^{1/2}}\,dt.$$

We can fix the value of K using the initial data. First, we note that:

$$1 = K\int_\gamma \frac{1}{(t^2+1)^{1/2}}\,dt.$$

Deforming γ to a very large circular contour, we can expand the integrand to yield:

$$(t^2+1)^{-1/2} = \frac{1}{t}\left(1+\frac{1}{t^2}\right)^{-1/2} = \frac{1}{t}\left(1 - \frac{1}{2t^2} + \cdots\right).$$

In particular, by the residue theorem we have:

$$1 = K \cdot 2\pi i \qquad \Rightarrow \qquad K = \frac{1}{2\pi i}.$$

We must also check that we have found the right solution by determining $y'(0)$. We note that (again, by deforming γ to a large circular contour and using the Laurent series):

$$y'(0) = \frac{1}{2\pi i}\int_\gamma \frac{t}{(t^2+1)^{1/2}}\,dt = \frac{1}{2\pi i}\int_\gamma \left(1 - \frac{1}{2t^2} + \cdots\right)dt = 0,$$

by the residue theorem. Hence this is indeed the solution to the equation. We identify:

$$J_0(z) = \frac{1}{2\pi i}\int_\gamma \frac{e^{zt}}{(t^2+1)^{1/2}}\,dt,$$

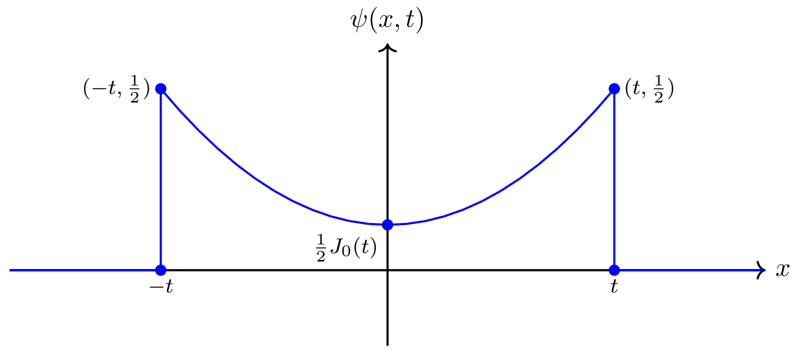
as required.

(d) Combining parts (b) and (c), we immediately conclude that the solution of the given Klein-Gordon equation is:

$$\psi(x,t) = \begin{cases} 0, & t < |x|; \\ \frac{1}{2}J_0((t^2-x^2)^{1/2}), & t > |x|. \end{cases}$$

as required.

Finally, we are asked to sketch the solution. Searching online for the shape of the Bessel function J_0, we see that an appropriate sketch for some generic time $t > 0$ is given by:



In particular, we have two moving wave-fronts, one moving to the right, and one moving to the left. The value of $J_0(t)$ decreases as t increases, so the amount of 'material' at the origin decreases in time. The heights of the wave-fronts remain constant, and take the value $\frac{1}{2}$ for all time.

Part II: Further Complex Methods
Examples Sheet 4 Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

1. Show that the most general linear second order ordinary differential equation whose only singularities are regular singular points at $z = a$ and $z = b$ can be written in the form

$$w'' + \left[\frac{1-A}{z-a} + \frac{1+A}{z-b}\right] w' + \frac{B(a-b)^2}{(z-a)^2(z-b)^2} w = 0, \qquad (\dagger)$$

where A and B are arbitrary constants.

Write down and solve the equation when the two singular points are at 0 and ∞, in the two cases $A^2 \neq 4B$ and $A^2 = 4B$. Use a Möbius transformation to deduce the general solution of (†). What is the significance of the two constants α and α' such that $\alpha + \alpha' = A$ and $\alpha\alpha' = B$?

➾ **Solution:** Let's begin by recalling some definitions. The general linear, homogeneous, second-order ODE in the complex plane takes the form

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0, \qquad (*)$$

where $p(z)$ and $q(z)$ are complex functions with at worst isolated singularities (i.e. no branch points). Recall that a point $z = z_0$ is classified as either *ordinary*, *regular singular* or *irregular singular* according to the following:

Definition: Consider the ODE $(*)$ at the point $z = z_0$ in the complex plane $\mathbb{C}$.

- If $p(z)$ and $q(z)$ are analytic at $z = z_0$, the point is called an *ordinary point* of the ODE.
- If the point $z = z_0$ is not ordinary, but both $(z - z_0)p(z)$, $(z - z_0)^2 q(z)$ are analytic at $z = z_0$, the point is called a *regular singular point* of the ODE.
- Otherwise, the point is called an *irregular singular point* of the ODE.

Something else we need throughout this sheet is a condition for the point at infinity, $z = \infty$, to be *ordinary*, *regular singular* or *irregular*. We have:

Theorem: The point at infinity, $z = \infty$, is an ordinary point if the functions $p(z)$, $q(z)$ obey:

$$p(z) = \frac{2}{z} + O\left(\frac{1}{z^2}\right), \qquad q(z) = O\left(\frac{1}{z^4}\right),$$

as $z \to \infty$. Similarly, $z = \infty$ is a regular singular point if it is not an ordinary point and the functions $p(z)$, $q(z)$ obey:

$$p(z) = O\left(\frac{1}{z}\right), \qquad q(z) = O\left(\frac{1}{z^2}\right),$$

as $z \to \infty$. Else, $z = \infty$ is an irregular singular point.

Proof: To investigate the point at infinity, we set $z = 1/t$ in the ODE $(*)$ and see what happens as $t \to 0$. The derivatives transform as:

$$\frac{d}{dz} = \frac{dt}{dz}\frac{d}{dt} = -\frac{1}{z^2}\frac{d}{dt} = -t^2\frac{d}{dt},$$

and hence the ODE $(*)$ transforms to:

$$t^2\frac{d}{dt}\left(t^2\frac{dw}{dt}\right) - p\left(\frac{1}{t}\right)t^2\frac{dw}{dt} + q\left(\frac{1}{t}\right)w = 0.$$

Recasting this in the standard form, we find:

$$\frac{d^2w}{dt^2} + \left(\frac{2}{t} - \frac{p(1/t)}{t^2}\right)\frac{dw}{dt} + \frac{q(1/t)}{t^4}w = 0.$$

We have an ordinary point at $t = 0$, i.e. $z = \infty$, if the limits of the new coefficient functions exist as $t \to 0$, i.e.

$$\frac{2}{t} - \frac{p(1/t)}{t^2} = O(1), \quad \frac{q(1/t)}{t^4} = O(1) \qquad \Leftrightarrow \qquad p(z) = \frac{2}{z} + O\left(\frac{1}{z^2}\right), \quad q(z) = O\left(\frac{1}{z^4}\right).$$

We have a regular singular point at $t = 0$, i.e. $z = \infty$, if the limits of t and t^2 times the new coefficient functions exist as $t \to 0$ respectively, i.e.

$$2 - \frac{p(1/t)}{t} = O(1), \quad \frac{q(1/t)}{t^2} = O(1) \qquad \Leftrightarrow \qquad p(z) = O\left(\frac{1}{z}\right), \quad q(z) = O\left(\frac{1}{z^2}\right),$$

as required. □

These conditions are really very useful, and it's worth memorising them. One of the most useful facts that we get from these conditions is: *if $z = \infty$ is an ordinary or regular singular point, then $p(z)$ and $q(z)$ must decay to zero as $z \to \infty$.*

In this particular question, we are given that $z = a$ and $z = b$ are the only regular singular points of the ODE $(*)$; we'll assume that a and b are finite points in the complex plane as we're not told otherwise. First, we can use the definition of a regular singular point to pare down the possible options for the functional forms of $p(z)$, $q(z)$:

- Since $z = a$ is a regular singular point, we have that $(z-a)p(z)$ is analytic at $z = a$. Thus we can write:

  $$p(z) = \frac{1-A}{z-a} + P(z),$$

  where $P(z)$ is a function that is analytic at $z = a$ and A is a constant. But we also have that $(z-b)p(z)$ is analytic at $z = b$, hence

  $$(z-b)p(z) = \frac{(1-A)(z-b)}{z-a} + (z-b)P(z)$$

  is an analytic function at $z = b$. The first term on the right hand side is clearly analytic near $z = b$ (provided $a \neq b$, which is guaranteed, since we assume exactly two regular singular points). Therefore, it follows that $(z-b)P(z)$ must be analytic at $z = b$ and hence we can write

  $$P(z) = \frac{1+A'}{z-b} + \tilde{P}(z),$$

  where $\tilde{P}(z)$ is a function that is analytic at $z = b$ and A' is another constant. Note furthermore that $\tilde{P}(z)$ is analytic at $z = a$, because $P(z)$ is analytic at $z = a$ and $(1+A')/(z-b)$ is analytic at $z = a$, so $\tilde{P}(z)$ is the difference of two analytic functions at $z = a$.

It follows that $p(z)$ can be written in the form:

$$p(z) = \frac{1-A}{z-a} + \frac{1+A'}{z-b} + \tilde{P}(z),$$

where $\tilde{P}(z)$ is a function which is analytic at $z = a$ and $z = b$, and A, A' are constants.

Now, $p(z)$ is analytic at all points except $z = a$ and $z = b$, which implies that $\tilde{P}(z)$ is actually entire. Furthermore, since $z = \infty$ is an ordinary point, we must have $p(z) \to 0$ as $z \to \infty$, and so $\tilde{P}(z) \to 0$ as $z \to \infty$. Thus $\tilde{P}(z)$ is bounded at infinity, so is constant by Liouville's Theorem. We thus have $\tilde{P}(z) \equiv 0$, with the value of the constant coming from $\tilde{P}(z) \to 0$ as $z \to \infty$.

Hence we have the form:

$$p(z) = \frac{1-A}{z-a} + \frac{1+A'}{z-b}.$$

Moreover, since $z = \infty$ is an *ordinary point*, the coefficient of the $1/z$ term in the Laurent series of $p(z)$ around $z = \infty$ must be 2 by the above conditions. Expanding near $z = \infty$, we have:

$$p(z) = \frac{1-A}{z-a} + \frac{1+A'}{z-b} = \frac{1}{z}\left(\frac{1-A}{1-a/z} + \frac{1+A'}{1-b/z}\right) = \frac{1}{z}\left(2 - A + A' + O\left(\frac{1}{z}\right)\right) = \frac{2-A+A'}{z} + O\left(\frac{1}{z^2}\right),$$

and hence we see that $A' = A$. So we have the required form for $p(z)$:

$$p(z) = \frac{1-A}{z-a} + \frac{1+A}{z-b}.$$

- We now turn to $q(z)$. We know that $(z-a)^2 q(z)$ is an analytic function near $z = a$, and hence

$$q(z) = \frac{Cz+D}{(z-a)^2} + Q(z),$$

where $Q(z)$ is analytic near $z = a$, and C, D are constants. Similarly, $(z-b)^2 q(z)$ is an analytic function near $z = b$, so using similar arguments to those we used for $p(z)$, we have

$$q(z) = \frac{Cz+D}{(z-a)^2} + \frac{Ez+F}{(z-b)^2} + \tilde{Q}(z),$$

where $\tilde{Q}(z)$ is an entire function, and E, F are constants. Now note that since $z = \infty$ is ordinary, we must have $q(z) \to 0$ as $z \to \infty$, and so $\tilde{Q}(z) \to 0$ as $z \to \infty$. It follows by Liouville's Theorem that $\tilde{Q}(z) \equiv 0$, similar to before.

At this point, it's useful to combine the fractions into a special form. Cross-multiplying, we can add them to get:

$$q(z) = \frac{Cz+D}{(z-a)^2} + \frac{Ez+F}{(z-b)^2} = \frac{Q_3(z)}{(z-a)^2(z-b)^2},$$

for some cubic polynomial $Q_3(z)$. Now using partial fractions, we can write:

$$q(z) = \frac{Q_3(z)}{(z-a)^2(z-b)^2} = \frac{1}{(z-a)(z-b)}\left(\frac{G}{z-a} + \frac{H}{z-b} + Iz\right),$$

for constants G, H and I (here, we divided the cubic $Q_3(z)$ by the quadratic denominator $(z-a)(z-b)$ to get the partial fractions). We now recall that $z = \infty$ is ordinary, so that we must have $q(z) = O(1/z^4)$ near $z = \infty$. Expanding near $z = \infty$ then, we find:

$$q(z) = \frac{1}{z^2}\left(1 + O\left(\frac{1}{z^3}\right)\right)\left(\frac{G}{z} + \frac{Ga}{z^2} + \cdots + \frac{H}{z} + \frac{Hb}{z^2} + \cdots + Iz\right)$$

Hence we see we need $I = 0$, and $G = -H$. Thus our final form for $q(z)$ is:

$$q(z) = \frac{1}{(z-a)(z-b)}\left(\frac{G}{z-a} - \frac{G}{z-b}\right) = \frac{G(a-b)}{(z-a)(z-b)} =: \frac{B(a-b)^2}{(z-a)(z-b)},$$

where we define the constant B via $B(a-b) = G$. This is the required form for $q(z)$.

We have thus established the required equation with exactly two regular singular points at $z = a$, $z = b$ in the extended complex plane:

$$\frac{d^2w}{dz^2} + \left(\frac{1-A}{z-a} + \frac{1+A}{z-b}\right)\frac{dw}{dz} + \frac{B(a-b)^2}{(z-a)^2(z-b)^2}w = 0.$$

Next, we are asked to construct the equation with regular singular points at $z = 0$ and $z = \infty$. We can do this by setting $a = 0$ in the above equation, and letting $b \to \infty$ in the above expressions for $p(z)$ and $q(z)$; though, a little care is required with this limit. For $p(z)$, we get:

$$p(z) = \frac{1-A}{z} + \lim_{b\to\infty}\left[\frac{1+A}{z-b}\right] = \frac{1-A}{z}.$$

For $q(z)$, we get:

$$q(z) = \lim_{b\to\infty}\left[\frac{Bb^2}{z^2(z-b)^2}\right] = \frac{B}{z^2}\lim_{b\to\infty}\left[\frac{b^2}{b^2 - 2zb + z^2}\right] = \frac{B}{z^2}.$$

Hence we are left with the equation:

$$\frac{d^2w}{dz^2} + \frac{1-A}{z}\frac{dw}{dz} + \frac{B}{z^2}w = 0$$

which has regular singular points at $z = 0$ and $z = \infty$ only.

To solve this equation, we note that it is a *linear equidimensional equation*, of the type studied in Part IA Differential Equations. Hence it has solutions of the form $w = z^k$ for some k. Trialling this solution, we find that k must obey:

$$k(k-1) + (1-A)k + B = 0 \qquad \Rightarrow \qquad k^2 - Ak + B = 0.$$

There are two cases that we must consider:

- In the case $A^2 \neq 4B$, this equation has distinct roots, given by:

  $$k_\pm = \frac{A \pm \sqrt{A^2 - 4B}}{2}.$$

  Hence $w_1(z) = z^{k_+}$ and $w_2(z) = z^{k_-}$ provide independent solutions of the equation. The general solution is:

  $$w(z) = C_1 z^{k_+} + C_2 z^{k_-},$$

  for arbitrary constants C_1 and C_2.

- In the case $A^2 = 4B$, the equation has equal roots, given by:

  $$k = \frac{A}{2}.$$

  Thus we get only one independent solution of the equation, $w_1(z) = z^k$. To obtain another, we trial $w_2(z) = z^k\log(z)$; we find that this indeed solves the equation. Hence the general solution in this case is:

  $$w(z) = C_1 z^k + C_2 z^k \log(z)$$

  for arbitrary constants C_1 and C_2.

We are now asked to use a Möbius transformation to deduce the general solution of the equation with regular singular points at $z = a$ and $z = b$. Consider the transformation:

$$\tilde{z} = \frac{bz + a}{z + 1},$$

with inverse

$$z = \frac{\tilde{z} - a}{b - \tilde{z}}.$$

We hope that this Möbius transformation maps the equation

$$\frac{d^2w}{dz^2} + \frac{1 - A}{z}\frac{dw}{dz} + \frac{B}{z^2}w = 0$$

to the equation:

$$\frac{d^2w}{d\tilde{z}^2} + \left(\frac{1 - A}{\tilde{z} - a} + \frac{1 + A}{\tilde{z} - b}\right)\frac{dw}{d\tilde{z}} + \frac{B(a - b)^2}{(\tilde{z} - a)^2(\tilde{z} - b)^2}w = 0.$$

We will show in Question 2 that this is indeed completely true in general (this will justify one of the main manipulations of the Riemann P-symbol you saw in lectures), but here we will check it directly. The derivative transforms as:

$$\frac{d}{dz} = \frac{d\tilde{z}}{dz}\frac{d}{d\tilde{z}} = \left(\frac{dz}{d\tilde{z}}\right)^{-1}\frac{d}{d\tilde{z}} = \left(\frac{b - a}{(b - \tilde{z})^2}\right)^{-1}\frac{d}{d\tilde{z}} = \frac{(\tilde{z} - b)^2}{b - a}\frac{d}{d\tilde{z}}.$$

Hence the equation transforms as:

$$\frac{(\tilde{z} - b)^2}{b - a}\frac{d}{d\tilde{z}}\left(\frac{(\tilde{z} - b)^2}{b - a}\frac{dw}{d\tilde{z}}\right) + \left(\frac{(1 - A)(b - \tilde{z})}{\tilde{z} - a}\right)\frac{(\tilde{z} - b)^2}{b - a}\frac{dw}{d\tilde{z}} + \frac{B(\tilde{z} - b)^2}{(\tilde{z} - a)^2}w = 0.$$

Simplifying we have:

$$\frac{d^2w}{d\tilde{z}^2} + \left(\frac{1 - A}{\tilde{z} - a} + \frac{1 + A}{\tilde{z} - b}\right)\frac{dw}{d\tilde{z}} + \frac{B(a - b)^2}{(\tilde{z} - a)^2(\tilde{z} - b)^2}w = 0,$$

which is indeed the required equation.

Now, we know the solution w to the equation with regular singular points at $z = 0$, $z = \infty$ is given by:

$$w = \begin{cases} C_1 z^{k_+} + C_2 z^{k_-} & \text{if } A^2 \neq 4B, \\ C_1 z^k + C_2 z^k \log(z) & \text{otherwise.} \end{cases}$$

Hence rewriting the right hand side in terms of $\tilde{z}$, the transformed variable, we have that:

$$w = \begin{cases} C_1 \left(\frac{\tilde{z} - a}{b - \tilde{z}}\right)^{k_+} + C_2 \left(\frac{\tilde{z} - a}{b - \tilde{z}}\right)^{k_-} & \text{if } A^2 \neq 4B, \\ C_1 \left(\frac{\tilde{z} - a}{b - \tilde{z}}\right)^{k} + C_2 \left(\frac{\tilde{z} - a}{b - \tilde{z}}\right)^{k} \log\left(\frac{\tilde{z} - a}{b - \tilde{z}}\right) & \text{otherwise,} \end{cases}$$

solves the equation with regular singular points at $\tilde{z} = a$, $\tilde{z} = b$. The constants k_+, k_-, k are the same as those defined above.

Finally, we are asked the significance of the constants α, α' such that $\alpha + \alpha' = A$ and $\alpha\alpha' = B$. The notation is pretty suggestive of the fact that they are *exponents*. We recall from the course that the exponents at a regular singular point in the finite complex plane $z = z_0$ obey the quadratic equation:

$$\sigma(\sigma - 1) + p_{-1}\sigma + q_{-2} = 0,$$

where p_{-1}, q_{-2} are the coefficients of $(z-z_0)^{-1}$, $(z-z_0)^{-2}$ in the Laurent series of $p(z)$, $q(z)$ respectively in the expansion around the point $z = z_0$. If $z = \infty$, the exponents obey the equation

$$(-\sigma)(-\sigma - 1) + p_1(-\sigma) + q_2 = 0,$$

where p_1, q_2 are the coefficients of z, z^2 in the Laurent series of $p(z)$, $q(z)$ respectively in the expansion around $z = \infty$. These facts can be derived by considering a solution of the equation of the form $(z - z_0)^\sigma$ around $z = z_0$ or $1/z^\sigma$ around $z = \infty$.

In our question, the initial equation has regular singular points at $z = a$, $z = b$ in the finite complex plane. At $z = a$, the coefficients p_{-1}, q_{-2} are clearly given by:

$$p_{-1} = 1 - A, \qquad q_{-2} = B,$$

and hence the indicial equation near $z = a$ is $\sigma(\sigma - 1) + (1 - A)\sigma + B = 0$. This can be rewritten in the form $\sigma^2 - A\sigma + B = 0$. Now recalling that the roots of this quadratic obey $\alpha + \alpha' = A$ and $\alpha\alpha' = B$, we see that indeed such α, α' should be interpreted as exponents. We have seen that they are exponents near $z = a$.

The exponents at $z = b$ are $-\alpha$, $-\alpha'$. We can see this since at $z = b$, the coefficients p_{-1}, q_{-2} are given by $p_{-1} = 1 + A$, q_{-2}, and the argument goes through just as before, with indicial equation $\sigma^2 + A\sigma + B = 0$ instead now.

The fact that the exponents at the two regular singular points are equal but opposite is actually just an accident in the case of two regular singular points - this kind of phenomenon does not continue for larger numbers of regular singular points, and generically there is only relation that holds between all the exponents: their sum must be a constant called the *Fuchsian invariant* of the differential equation.

❊ **Comments:** There are lots of interesting aspects to this question that are worth commenting on.

1. Fuchsian equations simplify if we move one of our regular singular points to $z = \infty$.

The initial equation we derive in this question with regular singular points at $z = a$, $z = b$ looks horrible and is very difficult to solve! However, by moving one of our regular singular points to ∞, the whole equation simplifies in form to a very nice linear equidimensional equation of the type studied in Part IA Differential Equations.

This is actually a completely general phenomenon; if we put a regular singular point of a Fuchsian equation at infinity, the coefficient functions $p(z)$, $q(z)$ will generically simplify. This is why the *hypergeometric equation*, which we study later in the course is so nice compared to the Papperitz equation (the general Fuchsian equation with three regular singular points).

2. Fuchsian equations are essentially uninteresting when they have $N \leq 2$ regular singular points.

In lectures, you saw that (i) there are no equations with zero regular singular points; (ii) the solution to the equation with exactly one regular singular point is trivial. You have now seen that you can also solve, in a closed form, all Fuchsian equations with two regular singular points. The next natural generalisation is to consider three regular singular points, and as you know from the course, the theory instantly becomes much more exciting from there on out!

3. There is an easy general method to classify *all* Fuchsian equations.
Notice that the arguments we have outlined above generalise completely to the case of having N regular singular points (don't worry about remembering this formula, it's just cool that exists!):

Theorem: Suppose that a Fuchsian equation has $N \geq 1$ regular singular points at the locations $z = z_1, ..., z_N$ in the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. Let α_i, α_i' denote the exponents at the point z_i, and let α_∞, α_∞' be the exponents at $z = \infty$ (if $z = \infty$ is ordinary, then the exponents are equal to 0 and 1). Then we can write the coefficient functions $p(z)$ and $q(z)$ in the following forms:

$$p(z) = \sum_{i:z_i \neq \infty} \frac{1 - \alpha_i - \alpha_i'}{z - z_i},$$

$$q(z) = \frac{1}{\prod\limits_{i:z_i \neq \infty} (z - z_i)} \left(\sum_{i:z_i \neq \infty} \frac{\alpha_i \alpha_i' \pi_i}{z - z_i} + C_0 + C_1 z + \cdots + C_{N-4} z^{N-4} + \alpha_\infty \alpha_\infty' z^{N-3} \right)$$

where the π_i are constants, designed to give the correct behaviour of $q(z)$ near each of the finite regular singular points, given by

$$\pi_i = \prod_{\substack{j:j\neq i, \\ z_j \neq \infty}} (z_i - z_j).$$

The constants C_0, ..., C_{N-4} are arbitrary, and are called *accessory parameters*. Note that they do not appear in the course, because we only consider the case $N \leq 3$. Note also that the only bearing they can have on series solutions is to change the coefficients; they do not affect the exponents of the terms in a series solution.

We also have the following conditions on the exponents:

- The sum of the exponents at all the regular singular points in the extended complex plane satisfies:

$$\sum_{i=1}^{N} (\alpha_i + \alpha_i') = N - 2.$$

  This invariant is called the *Fuchsian invariant*.

- For small N, there are some bonus constraints on the exponents:
  (i) If $N = 1$, then the exponents obey $\alpha_1 \alpha_1' = 0$.
  (ii) If $N = 2$, then the exponents obey $\alpha_1 \alpha_1' = \alpha_2 \alpha_2'$.

Proof: By exactly the same method we use in this question! □

Notice that this general Theorem encapsulates some of the points we mentioned earlier: $p(z)$, $q(z)$ are genuinely simpler when one of the regular singular points is at infinity (since we only sum over finite regular singular points in the sums), and the cases $N = 1, 2$ are highly constrained by extra conditions on the exponents so are uninteresting. There are no general constraints on the exponents when $N \geq 3$, apart from the Fuchsian invariant constraint, so the theory becomes exciting at this point!

2. Consider the second order linear differential equation with three, and only three, regular singular points in the finite complex plane, all other points being ordinary points,

$$\frac{d^2y}{dz^2} + \left(\frac{A}{z-z_1} + \frac{B}{z-z_2} + \frac{C}{z-z_3}\right)\frac{dy}{dz} + \frac{1}{(z-z_1)(z-z_2)(z-z_3)}\left(\frac{D}{z-z_1} + \frac{E}{z-z_2} + \frac{F}{z-z_3}\right)y = 0.$$

Show that by making a fractional linear transformation of the form

$$z \mapsto z' = \frac{(z_3-z_2)(z-z_1)}{(z_3-z_1)(z-z_2)}$$

that the new differential equation is of the same form but now with regular singular points at $0, 1, \infty$ with the same roots of the indicial equation as in the original equation.

Can a similar transformation be made to map $0, 1, \infty$ into any three points in the finite complex plane?

Solution: This question is about how the Papperitz equation transforms under a Möbius transformation (called a fractional linear transformation in the question). There is a terribly long way of doing this question - just carry out the transformation and see what happens. But this is long, technical and fiddly, so we're going to use a quicker method.

First, let's relate A, B, C, D, E and F to the roots of the indicial equation at the locations $z = z_1$, z_2 and z_3, so it's clearer what happens to them under the transformation. Near $z = z_1$, the indicial equation is

$$\sigma^2 + (p_{-1} - 1)\sigma + q_{-2} = 0,$$

where p_{-1} is the coefficient of $(z-z_1)^{-1}$ in the Laurent expansion of $p(z)$ near $z = z_1$, and q_{-2} is the coefficient of $(z-z_1)^{-2}$ in the Laurent expansion of $q(z)$ near $z = z_1$. Looking at $p(z), q(z)$ in our case, we can write down:

$$p_{-1} = A, \qquad q_{-2} = \frac{D}{(z_1-z_2)(z_1-z_3)}.$$

Hence the indicial equation near $z = z_1$ is given by:

$$\sigma^2 + (A-1)\sigma + \frac{D}{(z_1-z_2)(z_1-z_3)} = 0.$$

The roots of the indicial equation, α_1 and α_1' must then obey:

$$A - 1 = -(\alpha_1 + \alpha_1'), \qquad \frac{D}{(z_1-z_2)(z_1-z_3)} = \alpha_1\alpha_1'.$$

It follows that $A = 1 - \alpha_1 - \alpha_1'$ and $D = \alpha_1\alpha_1'(z_1-z_2)(z_1-z_3)$.

Exactly the same argument goes through for the regular singular points at $z = z_2$ and $z = z_3$. Thus we see that we can write the equation as:

$$\begin{aligned}
&\frac{d^2y}{dz^2} + \left(\frac{1-\alpha_1-\alpha_1'}{z-z_1} + \frac{1-\alpha_2-\alpha_2'}{z-z_2} + \frac{1-\alpha_3-\alpha_3'}{z-z_3}\right)\frac{dy}{dz} \\
&\quad + \frac{1}{(z-z_1)(z-z_2)(z-z_3)}\left(\frac{\alpha_1\alpha_1'(z_1-z_2)(z_1-z_3)}{z-z_1} + \frac{\alpha_2\alpha_2'(z_2-z_1)(z_2-z_3)}{z-z_2} + \frac{\alpha_3\alpha_3'(z_3-z_1)(z_3-z_2)}{z-z_3}\right)y \\
&\quad = 0.
\end{aligned}$$

This makes manifest the dependence of the the coefficients A, B, C, D, E and F on the roots of the indicial equations at each of the regular singular points. This will make it easier to see how the exponents transform under Möbius transformations.

One thing that is useful to bear in mind about the exponents is that they sum to 1, which you saw in lectures (this is called the *Fuchsian invariant*):

$$\alpha_1 + \alpha_1' + \alpha_2 + \alpha_2' + \alpha_3 + \alpha_3' = 1.$$

This follows simply from the condition that $z = \infty$ must be an ordinary point of the above equation; in particular, from the general conditions of $p(z), q(z)$, we require the coefficient of $1/z$ in the Laurent expansion of $p(z)$ near $z = \infty$ to be 2.

Now, let's note that the general Möbius transformation can be generated by some sequence of *translations*, *scalings* and *inversion*:

$$z' = z + \mu, \qquad z' = \lambda z, \qquad z' = \frac{1}{z}.$$

Therefore, it's sufficient to see what happens to the equation under these transformations separately (this is much, much easier than doing them all at once as the question suggests!). We have:

- 1. Translations. The translation $z' = z+\mu$ has inverse $z = z'-\mu$. Under the transformation, the derivatives transform as:

$$\frac{d}{dz} = \frac{dz'}{dz}\frac{d}{dz'} = \frac{d}{dz'}.$$

Thus the equation simply transforms to:

$$\begin{aligned}&\frac{d^2y}{dz'^2} + \left(\frac{1-\alpha_1-\alpha_1'}{z'-\mu-z_1} + \frac{1-\alpha_2-\alpha_2'}{z'-\mu-z_2} + \frac{1-\alpha_3-\alpha_3'}{z'-\mu-z_3}\right)\frac{dy}{dz'}\\ &\quad + \frac{1}{(z'-\mu-z_1)(z'-\mu-z_2)(z'-\mu-z_3)}\left(\frac{\alpha_1\alpha_1'((z_1+\mu)-(z_2+\mu))((z_1+\mu)-(z_3+\mu))}{z'-\mu-z_1}\right.\\ &\quad \left. + \frac{\alpha_2\alpha_2'((z_2+\mu)-(z_1+\mu))((z_2+\mu)-(z_3+\mu))}{z'-\mu-z_2} + \frac{\alpha_3\alpha_3'((z_3+\mu)-(z_1+\mu))((z_3+\mu)-(z_2+\mu))}{z'-\mu-z_3}\right)y\\ &\quad = 0.\end{aligned}$$

We see that the regular singular points are mapped via $z_1 \mapsto z_1 + \mu$, $z_2 \mapsto z_2 + \mu$ and $z_3 \mapsto z_3 + \mu$. Thus they transform according to the Möbius transformation. We note that the exponents α_i, α_i' are unchanged however.

- 2. Scalings. The scaling $z' = \lambda z$ has inverse $z = z'/\lambda$. Under the transformation, the derivatives transform as:

$$\frac{d}{dz} = \frac{dz'}{dz}\frac{d}{dz'} = \lambda\frac{d}{dz'}.$$

Thus, after some straightforward simplification, we see that the equation transforms to:

$$\begin{aligned}&\frac{d^2y}{dz'^2} + \left(\frac{1-\alpha_1-\alpha_1'}{z'-\lambda z_1} + \frac{1-\alpha_2-\alpha_2'}{z'-\lambda z_2} + \frac{1-\alpha_3-\alpha_3'}{z'-\lambda z_3}\right)\frac{dy}{dz'}\\ &\quad + \frac{1}{(z'-\lambda z_1)(z'-\lambda z_2)(z'-\lambda z_3)}\left(\frac{\alpha_1\alpha_1'(\lambda z_1-\lambda z_2)(\lambda z_1-\lambda z_3)}{z'-\lambda z_1}\right.\\ &\quad \left. + \frac{\alpha_2\alpha_2'(\lambda z_2-\lambda z_1)(\lambda z_2-\lambda z_3)}{z'-\lambda z_2} + \frac{\alpha_3\alpha_3'(\lambda z_3-\lambda z_1)(\lambda z_3-\lambda z_2)}{z'-\lambda z_3}\right)y\\ &\quad = 0.\end{aligned}$$

We see that the regular singular points are mapped according to the Möbius transformation again, i.e. $z_1 \mapsto \lambda z_1$, $z_2 \mapsto \lambda z_2$ and $z_3 \mapsto \lambda z_3$. We also note that the exponents α_i, α_i' are again unchanged.

- 3. Inversion. Finally, we deal with the transformation $z' = 1/z$, which has inverse $z = 1/z'$. The derivative transforms as:
$$\frac{d}{dz} = \frac{dz'}{dz}\frac{d}{dz'} = -\frac{1}{z^2}\frac{d}{dz'} = -(z')^2\frac{d}{dz'}.$$
Hence the second derivative term transforms as:
$$\frac{d^2y}{dz^2} = (z')^2\frac{d}{dz'}\left((z')^2\frac{dy}{dz'}\right) = (z')^4\frac{d^2y}{dz'^2} + 2(z')^3\frac{dy}{dz'}.$$
After some small manipulations then, the equation becomes:
$$\begin{aligned}&\frac{d^2y}{dz'^2} + \left(\frac{2}{z'} - \frac{1}{z'^2}\left(\frac{1-\alpha_1-\alpha_1'}{1/z'-z_1} + \frac{1-\alpha_2-\alpha_2'}{1/z'-z_2} + \frac{1-\alpha_3-\alpha_3'}{1/z'-z_3}\right)\right)\frac{dy}{dz'}\\ &\quad + \frac{1}{z'^4(1/z'-z_1)(1/z'-z_2)(1/z'-z_3)}\bigg(\frac{\alpha_1\alpha_1'(z_1-z_2)(z_1-z_3)}{1/z'-z_1}\\ &\quad + \frac{\alpha_2\alpha_2'(z_2-z_1)(z_2-z_3)}{1/z'-z_2} + \frac{\alpha_3\alpha_3'(z_3-z_1)(z_3-z_2)}{1/z'-z_3}\bigg)y\\ &\quad = 0.\end{aligned}$$
Manipulating this into a recognisable form requires a little more thought. Let's look at the new $p(z), q(z)$ separately. We have:
$$p(z) = \frac{2}{z'} - \left(\frac{1-\alpha_1-\alpha_1'}{z'-z_1(z')^2} + \frac{1-\alpha_2-\alpha_2'}{z'-z_2(z')^2} + \frac{1-\alpha_3-\alpha_3'}{z'-z_3(z')^2}\right)$$
Notice that we can manipulate each of the $1-\alpha_i-\alpha_i'$ terms using partial fractions as follows:
$$-\frac{1-\alpha_1-\alpha_1'}{z'-z_1(z')^2} = \frac{1-\alpha_1-\alpha_1'}{z'z_1(z'-1/z_1)} = \frac{1-\alpha_1-\alpha_1'}{z'-1/z_1} - \frac{1-\alpha_1-\alpha_1'}{z'}.$$
Therefore, we can write:
$$\begin{aligned}p(z) &= \frac{2}{z'} + \frac{1-\alpha_1-\alpha_1'}{z'-1/z_1} + \frac{1-\alpha_2-\alpha_2'}{z'-1/z_2} + \frac{1-\alpha_3-\alpha_3'}{z'-1/z_3} - \frac{1-\alpha_1-\alpha_1'}{z'} - \frac{1-\alpha_2-\alpha_2'}{z'} - \frac{1-\alpha_3-\alpha_3'}{z'}\\ &= \frac{1-\alpha_1-\alpha_1'}{z'-1/z_1} + \frac{1-\alpha_2-\alpha_2'}{z'-1/z_2} + \frac{1-\alpha_3-\alpha_3'}{z'-1/z_3},\end{aligned}$$
where the coefficient of $1/z'$ vanishes since $\alpha_1+\alpha_1'+\alpha_2+\alpha_2'+\alpha_3+\alpha_3' = 1$.

The new $q(z)$ is a little easier to simplify. Through a series of manipulations, the generic term can be reduced as:
$$\begin{aligned}&\frac{1}{z'^4(1/z'-z_1)(1/z'-z_2)(1/z'-z_3)}\cdot\frac{\alpha_1\alpha_1'(z_1-z_2)(z_1-z_3)}{1/z'-z_1}\\ &= \frac{1}{(z'-1/z_1)(z'-1/z_2)(z'-1/z_3)}\cdot\frac{\alpha_1\alpha_1'(1/z_1-1/z_2)(1/z_1-1/z_3)}{z'-1/z_1},\end{aligned}$$
which is exactly of the form we want.

The final equation is of the form:

$$\begin{aligned}&\frac{d^2y}{dz'^2}+\left(\frac{1-\alpha_1-\alpha_1'}{z'-1/z_1}+\frac{1-\alpha_2-\alpha_2'}{z'-1/z_2}+\frac{1-\alpha_3-\alpha_3'}{z'-1/z_3}\right)\frac{dy}{dz'}\\&\quad+\frac{1}{(z'-1/z_1)(z'-1/z_2)(z'-1/z_3)}\left(\frac{\alpha_1\alpha_1'(1/z_1-1/z_2)(1/z_1-1/z_3)}{z'-1/z_1}\right.\\&\quad\left.+\frac{\alpha_2\alpha_2'(1/z_2-1/z_1)(1/z_2-1/z_3)}{z'-1/z_2}+\frac{\alpha_3\alpha_3'(1/z_3-1/z_1)(1/z_3-1/z_2)}{z'-1/z_3}\right)y\\&\quad=0.\end{aligned}$$

This shows that the exponents α_i, α_i' are unchanged as usual, but the regular singular points are mapped via $z_1 \mapsto 1/z_1$, $z_2 \mapsto 1/z_2$ and $z_3 \mapsto 1/z_3$. If one of z_i is zero, we can just consider $z_i = \epsilon$ and then take the limit $\epsilon \to 0$ at the end to see that the result still holds.

The above arguments have demonstrated that under any Möbius transformation (which is the composition of translations, scalings and inversion), the form of the equation is unchanged, though with the regular singular points transformed via the Möbius transformation. The exponents of the regular singular points are also unchanged.

In particular, we can return to the transformation given in the question:

$$z \mapsto z' = \frac{(z_3-z_2)(z-z_1)}{(z_3-z_1)(z-z_2)}.$$

Under this transformation, the regular singular points are mapped via $z_1 \mapsto 0$, $z_2 \mapsto \infty$ and $z_3 \mapsto 1$. Hence, by our above work, the transformed equation is of the same form, but with regular singular points at $0, \infty$ and 1 now. The exponents are unchanged; so we're done.

If we want to explicitly find the transformed equation, we note that $p(z)$ is mapped as:

$$p(z)=\frac{1-\alpha_1-\alpha_1'}{z-z_1}+\frac{1-\alpha_2-\alpha_2'}{z-z_2}+\frac{1-\alpha_3-\alpha_3'}{z-z_3}\quad\mapsto\quad\frac{1-\alpha_1-\alpha_1'}{z'}+\frac{1-\alpha_2-\alpha_2'}{z'-1},$$

and, carefully taking the limit when we perform the transformation $z_2 \mapsto \infty$, we find that $q(z)$ is mapped as:

$$\begin{aligned}q(z)&=\frac{1}{(z-z_1)(z-z_2)(z-z_3)}\left(\frac{\alpha_1\alpha_1'(z_1-z_2)(z_1-z_3)}{z-z_1}+\frac{\alpha_2\alpha_2'(z_2-z_1)(z_2-z_3)}{z-z_2}+\frac{\alpha_3\alpha_3'(z_3-z_1)(z_3-z_2)}{z-z_3}\right)\\&\mapsto\quad\frac{1}{z'(z'-1)}\left(-\frac{\alpha_1\alpha_1'}{z'}+\alpha_2\alpha_2'+\frac{\alpha_3\alpha_3'}{z'-1}\right).\end{aligned}$$

Relating things back to the original parameters A, B, C, D, E and F, we have the final transformed equation:

$$\begin{aligned}&\frac{d^2y}{dz'^2}+\left(\frac{A}{z'}+\frac{C}{z'-1}\right)\frac{dy}{dz'}\\&\quad+\frac{1}{z'(z'-1)}\left(-\frac{D}{z'(z_1-z_2)(z_1-z_3)}+\frac{E}{(z_2-z_1)(z_2-z_3)}+\frac{F}{(z'-1)(z_3-z_1)(z_3-z_2)}\right)y=0,\end{aligned}$$

where z_1, z_2, z_3 were the original locations of the regular singular points before the transformation.

Finally, we are asked whether a similar transformation can be made to map the regular singular points $0, 1, \infty$ into any other three points in the complex plane. The answer is yes, provided the new three points are distinct - simply take the inverse of the map we used in this question.

❊ **Comments:** In this question, we proved an important property of Fuchsian equations: *under a Möbius transformation, the regular singular points of a Fuchsian equation are moved according to the transformation, but the exponents remain unchanged.*

As you saw in lectures, this can be translated into a transformation of the *Riemann P-symbol*. Let's recall what this means:

Definition: Recall that a Fuchsian equation with three regular singular points is uniquely determined by (i) the locations of its regular singular points and (ii) the exponents at its regular singular points. Hence we can represent a general Fuchsian equation in the variable z by the symbol:

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix},$$

where a, b and c are the locations of the regular singular points of the equation, and $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ are the exponents at the respective regular singular points. This uniquely determines the required Fuchsian equation and vice-versa.

Furthermore, since a Fuchsian equation uniquely determines a two-dimensional solution space, and vice-versa, we could *also* interpret the Riemann P-symbol as representing the *solution space* of the corresponding Fuchsian equation.

In this question, we showed that for any Möbius transformation M, the general relation between Riemann P-symbols holds:

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} M(a) & M(b) & M(c) & \\ \alpha & \beta & \gamma & M(z) \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

Using the 'solution space' interpretation of the Riemann P-symbol, we see that if $w = w(z)$ solves the Fuchsian equation represented by the P-symbol on the left hand side, then $w = w(z) = w(M^{-1}(z'))$, viewed as a function of $z' = M(z)$ solves the Fuchsian equation represented by the P-symbol in the right hand side.

A good check of this result is to look at Question 1, where we transformed a Fuchsian equation using a Möbius transformation (the equation had two regular singular points, but this just corresponds to taking one of the three regular singular points in the Papperitz equation to be ordinary, e.g. by setting $\alpha = 0, \alpha' = 1$).

3. Consider the equation for $y(z)$ with P-symbol

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix}.$$

By transforming $y \to \tilde{y}(z) = z^{-\alpha_1}(1-z)^{-\beta_1}y(z)$, show that the resulting equation for $\tilde{y}(z)$ is the hypergeometric equation.

➾ **Solution:** In this question, we will derive another very important property of the Riemann P-symbol: the *exponent shifting property*. Instead of doing what the question tells us to do, let's do something very general (and hence more useful) instead.

Recall that the general type of equation we are trying to solve is:

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0.$$

Let's see what happens when we make the general transformation $w = hv$ for some function $h(z)$ and some new dependent variable $v(z)$. We find:

$$w' = h'v + hv', \qquad w'' = h''v + 2h'v' + hv'',$$

and hence the equation transforms to:

$$\frac{d^2v}{dz^2} + \left(p + \frac{2h'}{h}\right)\frac{dv}{dz} + \left(q + \frac{ph'}{h} + \frac{h''}{h}\right)v = 0.$$

Now let's suppose that the equation is Fuchsian, with three regular singular points as in Question 2. The generic form of $p(z), q(z)$ is then known, and hence we can write down the transformed equation as:

$$\begin{aligned}\frac{d^2v}{dz^2} &+ \left(\frac{1-\alpha_1-\alpha_2}{z-z_1} + \frac{1-\beta_1-\beta_2}{z-z_2} + \frac{1-\gamma_1-\gamma_2}{z-z_3} + \frac{2h'(z)}{h(z)}\right)\frac{dv}{dz} \\ &+ \left[\frac{1}{(z-z_1)(z-z_2)(z-z_3)}\left(\frac{\alpha_1\alpha_2(z_1-z_2)(z_1-z_3)}{z-z_1} + \frac{\beta_1\beta_2(z_2-z_1)(z_2-z_3)}{z-z_2} + \frac{\gamma_1\gamma_2(z_3-z_1)(z_3-z_2)}{z-z_3}\right)\right. \\ &+ \left.\frac{h'(z)}{h(z)}\left(\frac{1-\alpha_1-\alpha_2}{z-z_1} + \frac{1-\beta_1-\beta_2}{z-z_2} + \frac{1-\gamma_1-\gamma_2}{z-z_3}\right) + \frac{h''(z)}{h(z)}\right]v \\ &= 0,\end{aligned}$$

where z_1, z_2, z_3 are the regular singular points and $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are their respective exponents.

Now let's restrict to a specific form of h. Suppose that $h(z)$ is given by:

$$h(z) = \left(\frac{z-z_1}{z-z_2}\right)^k\left(\frac{z-z_2}{z-z_3}\right)^l$$

Now, you may rightfully ask, why are we doing this? That is, why are we considering transformations of the form $w(z) = h(z)v(z)$? And why the weird form of $h(z)$? The reason is that we are inspired by the fact that the series solutions to our equations are generically of the form:

$$w(z) = (z-z_i)^{\sigma_i}\sum_{n=0}^{\infty} a_n(z-z_i)^n,$$

i.e. of Frobenius form, near a regular singular point $z = z_i$.

Hence, we might hope and pray that multiplying $w(z)$ by something that looks like $(z-z_i)^k$ will give a solution of a Fuchsian equation with a regular singular point at $z = z_i$ still, but with exponents of that regular singular point shifted by $\sigma_i \mapsto \sigma_i + k$. Near any other regular singular point $z = z_i'$ (in the *finite* complex plane), it's clear that multiplying by $(z - z_i)^k$ does nothing to the exponents:

$$\begin{aligned}
(z - z_i)^k w(z) &= (z - z_i)^k (z - z_i')^{\sigma_i'} \sum_{n=0}^{\infty} a_n (z - z_i')^n \\
&= (z - z_i' + z_i' - z_i)^k (z - z_i')^{\sigma_i'} \sum_{n=0}^{\infty} a_n (z - z_i')^n \\
&= \underbrace{(z_i' - z_i)^k}_{\text{constant}} (z - z_i')^{\sigma_i'} \left[\underbrace{\left(1 + \frac{k(z - z_i')}{z_i' - z_i} + \cdots \right)}_{\text{analytic function of } z - z_i'} \sum_{n=0}^{\infty} a_n (z - z_i')^n \right],
\end{aligned}$$

where in the last step we used the binomial theorem.

Now, the above manipulations are *not* a proof that $(z - z_i)^k w(z)$ solves the Fuchsian equation with regular singular points at the same locations, but with exponents of z_i shifted as $\sigma_i \mapsto \sigma_i + k$. In fact, this naïve guess *cannot be true*. If we only shifted exponents at *some* of the regular singular points, then the Fuchsian invariant

$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = 1$$

might not remain invariant! Hence the naïve transformation $(z-z_i)^k w(z)$ must include extra multiplicative factors to shift some exponents by a bit, and other exponents by the opposite amount, in order to preserve the Fuchsian invariant.

In general then, if $w(z)$ solves the Fuchsian equation with regular singular points at $z = z_1$, $z = z_2$ and $z = z_3$, and exponents $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$, we hope that

$$v(z) = \frac{w(z)}{h(z)} = \left(\frac{z - z_3}{z - z_2}\right)^l \left(\frac{z - z_2}{z - z_1}\right)^k w(z)$$

will solve the Fuchsian equation with regular singular points at $z = z_1, z = z_2, z = z_3$, but with exponents $\alpha_1 - k, \alpha_2 - k$, $\beta_1 + k - l, \beta_2 + k - l$ and $\gamma_1 + l, \gamma_2 + l$ respectively. Hopefully which exponents shift should now be clear from the series solutions! Notice also that this choice of multiplicative factor preserves the Fuchsian invariant, so isn't obviously wrong either.

Now that we have motivated why we should consider $h(z)$ of this form, let's go ahead and prove the result we want:

Theorem: Suppose that $w(z)$ satisfies the Fuchsian equation with regular singular points at $z = z_1, z_2$ and z_3, with respective exponents $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$ and γ_2. Then

$$v(z) = \left(\frac{z - z_3}{z - z_2}\right)^l \left(\frac{z - z_2}{z - z_1}\right)^k w(z)$$

solves the Fuchsian equation with regular singular points at $z = z_1, z_2$ and z_3, with respective exponents $\alpha_1 - k$, $\alpha_2 - k, \beta_1 + k - l, \beta_2 + k - l$ and $\gamma_1 + l, \gamma_2 + l$. This property is called the *exponent shifting property* of Fuchsian equations.

Proof: We substitute

$$h(z) = \left(\frac{z - z_1}{z - z_2}\right)^k \left(\frac{z - z_2}{z - z_3}\right)^l$$

into the general form of the equation for $v = w/h$ we derived above.

Notice that:

$$\frac{2h'}{h} = 2\frac{d}{dz}\log(h) = 2\frac{d}{dz}\left(k\log(z-z_1) + (l-k)\log(z-z_2) - l\log(z-z_3)\right) = \frac{2k}{z-z_1} + \frac{2(l-k)}{z-z_2} - \frac{2l}{z-z_3}.$$

Hence the coefficient of dv/dz in the $v(z)$ equation is given by:

$$\frac{1-(\alpha_1-k)-(\alpha_2-k)}{z-z_1} + \frac{1-(\beta_1+k-l)-(\beta_2+k-l)}{z-z_2} + \frac{1-(\gamma_1+l)-(\gamma_2+l)}{z-z_3}.$$

So far so good! The exponents are correctly shifted, and the regular singular points remain in their initial locations.

The coefficient of v in the $v(z)$ equation is a little bit more fiddly to deal with. The general idea is that we want to combine

$$\frac{ph'}{h} + \frac{h''}{h} = \frac{h'}{h}\left(\frac{1-\alpha_1-\alpha_2}{z-z_1} + \frac{1-\beta_1-\beta_2}{z-z_2} + \frac{1-\gamma_1-\gamma_2}{z-z_3}\right) + \frac{h''}{h}$$

with our original $q(z)$:

$$\frac{1}{(z-z_1)(z-z_2)(z-z_3)}\left(\frac{\alpha_1\alpha_2(z_1-z_2)(z_1-z_3)}{z-z_1} + \frac{\beta_1\beta_2(z_2-z_1)(z_2-z_3)}{z-z_2} + \frac{\gamma_1\gamma_2(z_3-z_1)(z_3-z_2)}{z-z_3}\right).$$

The strategy will be to evaluate ph'/h, h''/h and then take out a factor of $1/(z-z_1)(z-z_2)(z-z_3)$ from $ph'/h + h''/h$ somehow, hopefully leaving something that we can easily understand in terms of exponent shifts.

Let's begin by finding h''/h. Recall that we have already:

$$h' = \left(\frac{k}{z-z_1} + \frac{l-k}{z-z_2} - \frac{l}{z-z_3}\right)h.$$

Hence the second derivative is:

$$\begin{aligned} h'' &= \left(-\frac{k}{(z-z_1)^2} - \frac{(l-k)}{(z-z_2)^2} + \frac{l}{(z-z_3)^2}\right)h + \left(\frac{k}{z-z_1} + \frac{l-k}{z-z_2} - \frac{l}{z-z_3}\right)h' \\ &= \left(-\frac{k}{(z-z_1)^2} - \frac{(l-k)}{(z-z_2)^2} + \frac{l}{(z-z_3)^2}\right)h + \left(\frac{k}{z-z_1} + \frac{l-k}{z-z_2} - \frac{l}{z-z_3}\right)^2 h \end{aligned}$$

Therefore, we're left with:

$$\begin{aligned} \frac{h''}{h} &= -\frac{k}{(z-z_1)^2} - \frac{(l-k)}{(z-z_2)^2} + \frac{l}{(z-z_3)^2} + \left(\frac{k}{z-z_1} + \frac{l-k}{z-z_2} - \frac{l}{z-z_3}\right)^2 \\ &= \frac{k(k-1)}{(z-z_1)^2} + \frac{(l-k)(l-k-1)}{(z-z_2)^2} + \frac{(-l)(-l-1)}{(z-z_3)^2} + \frac{2k(l-k)}{(z-z_1)(z-z_2)} \\ &\quad - \frac{2kl}{(z-z_1)(z-z_3)} - \frac{2(l-k)l}{(z-z_2)(z-z_3)}. \end{aligned}$$

Now let's evaluate ph'/h. We have p and h'/h, so we just need to multiply them! We have:

$$\begin{aligned}\frac{ph'}{h} &= \left(\frac{1-\alpha_1-\alpha_2}{z-z_1}+\frac{1-\beta_1-\beta_2}{z-z_2}+\frac{1-\gamma_1-\gamma_2}{z-z_3}\right)\left(\frac{k}{z-z_1}+\frac{l-k}{z-z_2}-\frac{l}{z-z_3}\right)\\
&= \frac{k(1-\alpha_1-\alpha_2)}{(z-z_1)^2}+\frac{(l-k)(1-\alpha_1-\alpha_2)}{(z-z_1)(z-z_2)}-\frac{l(1-\alpha_1-\alpha_2)}{(z-z_1)(z-z_3)}+\frac{k(1-\beta_1-\beta_2)}{(z-z_2)(z-z_1)}\\
&+\frac{(l-k)(1-\beta_1-\beta_2)}{(z-z_2)^2}-\frac{l(1-\beta_1-\beta_2)}{(z-z_2)(z-z_3)}+\frac{k(1-\gamma_1-\gamma_2)}{(z-z_1)(z-z_3)}+\frac{(l-k)(1-\gamma_1-\gamma_2)}{(z-z_2)(z-z_3)}-\frac{l(1-\gamma_1-\gamma_2)}{(z-z_3)^2}.\end{aligned}$$

Putting everything together then, and factorising out $1/(z-z_1)(z-z_2)(z-z_3)$ from $ph'/h+h''/h$, we see that we must simplify:

$$\begin{gathered}\frac{\alpha_1\alpha_2(z_1-z_2)(z_1-z_3)}{z-z_1}+\frac{\beta_1\beta_2(z_2-z_1)(z_2-z_3)}{z-z_2}+\frac{\gamma_1\gamma_2(z_3-z_1)(z_3-z_2)}{z-z_3}\\
+\frac{k(k-1)(z-z_2)(z-z_3)}{z-z_1}+\frac{(l-k)(l-k-1)(z-z_1)(z-z_3)}{z-z_2}+\frac{(-l)(-l-1)(z-z_1)(z-z_3)}{z-z_3}\\
+2k(l-k)(z-z_3)-2kl(z-z_2)-2(l-k)l(z-z_2)\\
+\frac{k(1-\alpha_1-\alpha_2)(z-z_2)(z-z_3)}{z-z_1}+(l-k)(1-\alpha_1-\alpha_2)(z-z_3)-l(1-\alpha_1-\alpha_2)(z-z_2)\\
+k(1-\beta_1-\beta_2)(z-z_3)+\frac{(l-k)(1-\beta_1-\beta_2)(z-z_1)(z-z_3)}{z-z_2}-l(1-\beta_1-\beta_2)(z-z_1)\\
+k(1-\gamma_1-\gamma_2)(z-z_2)+(l-k)(1-\gamma_1-\gamma_2)(z-z_1)-\frac{l(1-\gamma_1-\gamma_2)(z-z_1)(z-z_2)}{z-z_3},\end{gathered}$$

which in fairness is rather complicated! We can quickly gather up the singular terms, by writing the numerators in the forms $(z-z_2)(z-z_3)=(z-z_1+z_1-z_2)(z-z_1+z_1-z_3)$ and dividing out using polynomial division for example, and then (by some very careful algebra), we find that all other terms actually cancel out. Thus we're left with:

$$\begin{gathered}\frac{(\alpha_1-k)(\alpha_2-k)(z_1-z_2)(z_1-z_3)}{z-z_1}+\frac{(\beta_1+k-l)(\beta_2+k-l)(z_2-z_1)(z_2-z_3)}{z-z_2}\\
+\frac{(\gamma_1+l)(\gamma_2+l)(z_3-z_1)(z_3-z_2)}{z-z_3},\end{gathered}$$

which is indeed exactly of the required form. $\square$

Note you will never, ever be expected to reproduce this proof in an exam! It is much too complicated and long. If asked why exponent shifting is true in an exam, I would simply use the motivational argument about multiplying series solutions in such a way that the Fuchsian invariant is kept unchanged.

We can restate this important result as a manipulation of Riemann P-symbols:

$$\left(\frac{z-z_3}{z-z_2}\right)^l\left(\frac{z-z_2}{z-z_1}\right)^k P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix} = P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha_1-k & \beta_1+k-l & \gamma_1+l & z \\ \alpha_2-k & \beta_2+k-l & \gamma_2+l & \end{Bmatrix}.$$

Here, in order for this notation to make sense, we should interpret the P-symbol as being the 'solution space' of the relevant Fuchsian equation.

Now you might rightly be concerned that one of our regular singular points might be the point at infinity, in which case things are more complicated. Let's see what happens when this is the case. We have just stated the general result:

$$\left(\frac{z-z_3}{z-z_2}\right)^l \left(\frac{z-z_2}{z-z_1}\right)^k P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix} = P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha_1 - k & \beta_1 + k - l & \gamma_1 + l & z \\ \alpha_2 - k & \beta_2 + k - l & \gamma_2 + l & \end{Bmatrix}. \qquad (*)$$

and hence there are three possibilities for where the point at infinity might enter: $z_1 = \infty$, $z_2 = \infty$ or $z_3 = \infty$. Let's see what happens for $z_1 = \infty$; similar considerations apply for $z_2 = \infty$, $z_3 = \infty$.

Consider taking $z_1 \to \infty$. Then the multiplicative factor has the limit:

$$\left(\frac{z-z_3}{z-z_2}\right)^l \left(\frac{z-z_2}{z-z_1}\right)^k \to 0.$$

Uh oh! Looks like our solution vanishes, which is not very good at all.

To fix this, notice that we can always add a constant factor to our multiplicative factor without changing anything. This arises because if we scale a solution of a linear ODE, then it remains a solution of the ODE. In particular then, we can instead rewrite $(*)$ in the form:

$$z_1^k\left(\frac{z-z_3}{z-z_2}\right)^l \left(\frac{z-z_2}{z-z_1}\right)^k P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix} = P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha_1 - k & \beta_1 + k - l & \gamma_1 + l & z \\ \alpha_2 - k & \beta_2 + k - l & \gamma_2 + l & \end{Bmatrix},$$

where we have 'drawn out' a handy factor of z_1^k from the solution space represented by the P-symbol on the left hand side. Now taking the limit as $z_1 \to \infty$ is fine, and we just end up with the final result:

$$(z-z_3)^l (z-z_2)^{k-l} P\begin{Bmatrix} \infty & z_2 & z_3 & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix} = P\begin{Bmatrix} \infty & z_2 & z_3 & \\ \alpha_1 - k & \beta_1 + k - l & \gamma_1 + l & z \\ \alpha_2 - k & \beta_2 + k - l & \gamma_2 + l & \end{Bmatrix}$$

One can remember this result by thinking about the series solution near $z = \infty$.

In general, the same sort of thing happens for $z_2 = \infty$, $z_3 = \infty$. We find the general result: *if we have a regular singular point at infinity, exponent shifting is permitted for any multiplication $(z-z_i)^\mu (z-z_j)^\nu$ involving the two other regular singular points at $z = z_i$, $z = z_j$, and arbitrary μ, ν, provided we remember to move the exponents at infinity appropriately too.*

Now, let's actually do Question 3! We are given that $y(z)$ lies in the solution space

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix}.$$

We are asked to consider the function $\tilde{y}(z) = z^{-\alpha_1}(1-z)^{-\beta_1} y(z)$. By the exponent shifting property, this function lies in the solution space:

$$z^{-\alpha_1}(1-z)^{-\beta_1} P\begin{Bmatrix} 0 & 1 & \infty & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \gamma_1 + \alpha_1 + \beta_1 & z \\ \alpha_2 - \alpha_1 & \beta_2 - \beta_1 & \gamma_2 + \alpha_1 + \beta_1 & \end{Bmatrix}.$$

This exponent shifting is allowed because one of the regular singular points is at the point at infinity, as we discussed, so we are allowed to multiply by $z^\mu(z-1)^\nu$ for generic μ,ν, provided we remember to shift the exponents at infinity. We can find out how the exponents must shift by considering series solutions as usual:

- Near $z=0$, we have a series solution $w(z)\sim z^\sigma$, so $z^{-\alpha_1}(1-z)^{-\beta_1}w(z)\sim z^{\sigma-\alpha_1}(1-z)^{-\beta_1}\sim z^{\sigma-\alpha_1}$ near $z=0$. So the exponents at $z=0$ are affected by subtracting α_1.
- Near $z=1$, we have a series solution $w(z)\sim(z-1)^\sigma$, so $z^{-\alpha_1}(1-z)^{-\beta_1}w(z)\sim(z-1)^{\sigma-\beta_1}(1+(z-1))^{-\alpha_1}\sim(z-1)^{\sigma-\beta_1}$ near $z=1$. So the exponents at $z=1$ are affected by subtracting β_1.
- Near $z=\infty$, we have a series solution $w(z)\sim 1/z^\sigma$, so $z^{-\alpha_1}(1-z)^{-\beta_1}w(z)\sim 1/z^{\sigma+\alpha_1+\beta_1}(1+\cdots)$, near $z=\infty$. So the exponents at $z=\infty$ are affect by adding $\alpha_1+\beta_1$.

Hence $\tilde{y}(z)$ lies in the solution space of the P-symbol on the right hand side. It follows that $\tilde{y}(z)$ satisfies the *hypergeometric equation*, which, as you saw in lectures, is defined to be the equation with the P-symbol:

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix}\right\}.$$

As a consequence of Questions 2 and 3, we have seen that: *any Fuchsian equation with three regular singular points can be reduced by a Möbius transformation and exponent shifting to the hypergeometric equation.* Hence it makes sense to study the hypergeometric equation!

4. By expanding $(1-tz)^{-a}$, show that

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt = \frac{\Gamma(c-b)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!},$$

where $(1-tz)^{-a}$ takes its principal value, provided $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $|z| < 1$. You should explain the reason for these conditions.

State the regions of the complex plane where (i) the integral defines an analytic function and (ii) the sum defines an analytic function. Explain how the integral provides an analytic continuation in z of the function defined by the sum.

Given that the above integral is a multiple of $F(a,b,c;z)$, show that

$$F(a,b,c;1) = \frac{\Gamma(c-b-a)\Gamma(c)}{\Gamma(c-b)\Gamma(c-a)},$$

when a, b and c satisfy a condition that you should give.

Solution: First, we note that $t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}$ typically has branch points at $t=0$, $t=1$ and $t=1/z$, hence we require lots of branch cuts. Fortunately, since we are integrating from $t=0$ to $t=1$, we can always arrange our branch cuts to be out of the way, so we can essentially ignore any issues of multi-valuedness in this question.

We can now begin the question properly. We notice that an expansion of $(1-tz)^{-a}$ using the binomial theorem is valid if and only if $|tz| < 1$ for all $t \in (0,1)$. Taking $t=1$, we see we require $|z| < 1$, and we see that this is also sufficient. Expanding then, we have:

$$(1-tz)^{-a} = 1 + atz + \frac{a(a+1)t^2z^2}{2!} + \frac{a(a+1)(a+2)t^3z^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(a)}\frac{t^nz^n}{n!}.$$

The final equality follows since the recurrence property of the gamma function gives $\Gamma(a+n) = (a+n)\Gamma(a+n-1) = \cdots = (a+n)(a+n-1)\cdots(a)\Gamma(a)$. Substituting into the integral, we see that we must evaluate:

$$\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(a)}\frac{z^n}{n!}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,dt.$$

We recognise each of the integrals as beta function integrals. We have:

$$B(b+n,c-b) = \int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,dt,$$

for each $n = 0,1,2,\dots$, provided $\mathrm{Re}(b+n) > 0$ and $\mathrm{Re}(c-b) > 0$. The first condition must, in particular, hold for $n=0$ - and note that if it holds for $n=0$, then it holds for all larger n too. Hence we get the two conditions $\mathrm{Re}(b) > 0$, $\mathrm{Re}(c-b) > 0$ which can be combined into $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ as required.

To finish, recall that the beta function can be expressed in terms of the gamma function via the formula:

$$B(b+n,c-b) = \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)}.$$

Hence we have the final result as required:

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt = \frac{\Gamma(c-b)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}.$$

We are now asked to comment on where the sum and the integral define analytic functions respectively. In the case of the sum, things are quite straightforward; we simply get an analytic function wherever the sum converges. Using the ratio test, we need to check the limit:

$$\lim_{n\to\infty}\left|\frac{\Gamma(a+n+1)\Gamma(b+n+1)\Gamma(c+n)}{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n+1)}\frac{z^{n+1}}{z^n}\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{(a+n)(b+n)}{(c+n)(n+1)}\right|\cdot|z| = |z|.$$

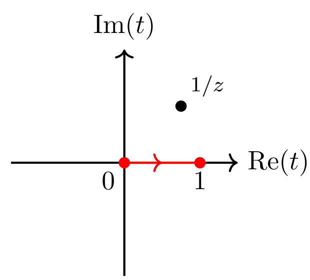
Hence the sum's radius of convergence is $|z|<1$. So the sum defines an analytic function on the domain $|z|<1$ only.

Things are a bit more complicated when we consider the integral:

$$f(z) := \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt,$$

because the integral $f(z)$ itself is multi-valued function of z; let's explain why, and hence decide on how we must restrict the values of z to construct a single-valued analytic function.

Let's draw the path of integration involved in the integral $f(z)$ in the complex t-plane. This is straightforward, it's just a straight line segment from $t=0$ to $t=1$ along the real axis:

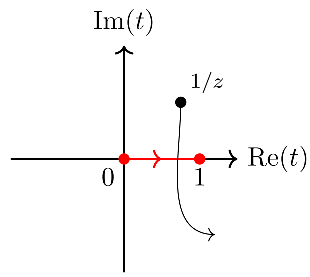


There are singularities of the integrand at $t=0$, $t=1$ and $t=1/z$. The singularities at $t=0$, $t=1$ are at the end-points of the integral, so there's nothing we can do about them - we *must* choose b,c such that we get convergence at these end-points. In particular, we have the contributions:

- Near $t=0$, we get:
$$\int_0 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt \approx \int_0 t^{b-1}\,dt = \left[\frac{t^b}{b}\right]_0,$$
and hence to get a finite contribution we need $\mathrm{Re}(b)>0$.

- Near $t=1$, we get:
$$\int^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt \approx \int^1 (1-t)^{c-b-1}\,dt = -\left[\frac{(1-t)^{c-b}}{c-b}\right]^1,$$
and hence to get a finite contribution we need $\mathrm{Re}(c-b)>0$, i.e. $\mathrm{Re}(c)>\mathrm{Re}(b)>0$.

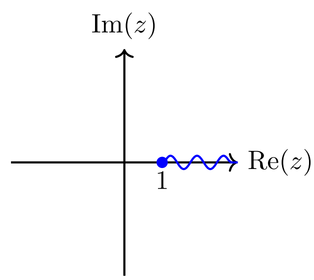
Let's now consider what happens as z varies. We notice that the singularity of the integrand $t=1/z$ moves around the complex plane t-plane. Nothing much happens in general, *however*, there is a possibility that $1/z$ will *cross the path of integration*, as indicated in the next figure. This can occur if and only if $1/z$ cross $[0,1]$, i.e. z crosses $[1,\infty]$.



What happens when $1/z$ crosses the path of integration? You saw in lectures that contour integrals are independent of the path they take, provided that we don't deform the path through a singularity - this is a consequence of *Cauchy's Theorem*. It is completely possible to generalise the method used in the proof of this result to include the possibility that we deform the path through a singularity (or equivalent, a singularity passes through our contour as it is moved); we find that we get a *discontinuity* in the value of the integral.

This is what happens here. As $1/z$ passes over the contour of integration along $t \in [0,1]$, the final integral $f(z)$ suffers a discontinuity. It's not too hard to visualise the fact that encircling $t=0$, $t=1$ with $1/z$ again and again, we find that we can get more and more discontinuities in $f(z)$. Thus $f(z)$ is a seriously multi-valued function!

We see that the points that we encircle using $1/z$ to give the multi-valuedness are $1/z = 1$ and $1/z = 0$, which translate to $z = 1$ and $z = \infty$ in the complex z-plane. Thus we see that $f(z)$ is a multi-valued function with branch points at $z = 1$ and $z = \infty$ only. Thus we insert a branch cut along $[1,\infty)$ in the complex z-plane:



We conclude that the integral

$$f(z) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

is an analytic function everywhere in the complex z-plane except the branch cut $[1,\infty)$, provided $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$.

We can now discuss analytic continuation. We saw that the sum is an analytic function in $|z| < 1$ and the integral is an analytic function on $\mathbb{C}\backslash[1,\infty)$. Furthermore, we proved that the sum and integral agree on $|z| < 1$. Therefore, the integral can indeed be considered an analytic continuation of the sum to $\mathbb{C}\backslash[1,\infty)$.

The last part of the question relates our work to the hypergeometric function. We are given that the integral

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

is proportional to the hypergeometric function, i.e.

$$F(a,b,c;z) = \lambda(a,b,c)\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt,$$

for some λ. In order to find the constant $\lambda(a,b,c)$, let's recall that the hypergeometric function is defined in this course as follows:

> **Definition:** The hypergeometric function $F(a,b,c;z)$ is the *unique* solution to the hypergeometric function that is analytic at $z=0$ and obeys $F(a,b,c;0)=1$. It is defined whenever $c \notin \{0,-1,\cdots\}$.
>
> If you're not happy with this definition, we will verify in Question 5 that the the hypergeometric defined in this way exists and is indeed unique.

First we must find the constant $\lambda(a,b,c)$. We can do so by taking $z=0$, which gives:

$$1 = F(a,b,c;0) = \lambda(a,b,c)\int_0^1 t^{b-1}(1-t)^{c-b-1}\,dt = \lambda(a,b,c)B(b,c-b) = \frac{\lambda(a,b,c)\Gamma(b)\Gamma(c-b)}{\Gamma(c)},$$

provided that $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, so that we can use the integral form of the beta function. Hence we see that

$$\lambda(a,b,c) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}.$$

It follows that:

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt,$$

for $z \in \mathbb{C}\backslash[1,\infty)$ and $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$.

We can now use this expression to find $F(a,b,c;1)$. We have:

$$F(a,b,c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-a-1}\,dt = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\cdot B(b,c-b-a) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)},$$

provided $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, and $\mathrm{Re}(c-b-a) > 0$, which are the conditions we used in the derivation to ensure that the beta function was equal to its integral representation. To summarise, we have found:

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)} \qquad (*)$$

provided $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $\mathrm{Re}(c-b-a) > 0$.

However, as usual we can do better, using analytic continuation! First note that the right hand side of the above result ($*$) is an analytic function of a, b and c, with poles only at isolated points (namely, where c and $c-b-a$ are non-positive integers).

To use analytic continuation, we must also ask where the left hand side $F(a,b,c;1)$ is an analytic function of the parameters a, b and c. This is a little more tricky than it first appears:

Theorem: $F(a,b,c;1)$ is an analytic function of the parameters a, b, c in the domain $\mathrm{Re}(c-b-a)>0$.

Proof: Recall that in this question we are given the hypergeometric function as a series:

$$F(a,b,c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!},$$

which we saw converged for $|z|<1$. Let's take the limit as $z\to 1$ and see what happens. We get:

$$F(a,b,c;1)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!}$$

where we can view the right hand side as an analytic function of a, b and c, *provided the sum converges*.

Let's check that the sum indeed converges under only the assumption that $\mathrm{Re}(c-b-a)>0$. To do so, we can use Stirling's formula for the gamma function, which tells us that $\Gamma(a+n)\sim(n+a)^{n+a-1/2}\sim n^{n+a-1/2}$. Therefore, the general term in this sum is of the form:

$$\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!}=\frac{n^{n+a-1/2}n^{n+b-1/2}}{n^{n+c-1/2}n^{n-1/2}}\sim\frac{1}{n^{c-a-b}}.$$

Hence this series is convergent provided $\mathrm{Re}(c-a-b)>0$. Thus we conclude that $F(a,b,c;1)$ is an analytic function of a, b and c if $\mathrm{Re}(c-b-a)>0$. $\square$

Thus we proved the result

$$F(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)},$$

under the assumption $\mathrm{Re}(c)>\mathrm{Re}(b)>0$ and $\mathrm{Re}(c-b-a)>0$, but now realise that both the left and right hand sides are meromorphic functions on the domain $\mathrm{Re}(c-b-a)>0$. Hence by the principle of analytic continuation, the result actually holds in the larger domain $\mathrm{Re}(c-b-a)>0$ (that is, we can ignore the conditions $\mathrm{Re}(c)>\mathrm{Re}(b)>0$), except for at poles. Explicitly, we have:

$$F(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)},\qquad \text{provided } \mathrm{Re}(c-b-a)>0 \text{ only, and we are away from poles.}$$

You may still wonder: can we extend this result to the region where $\mathrm{Re}(c-b-a)\leq 0$? The answer is in fact *no*, as we shall see carefully in the answer to Question 8.

5. What can be said about the nature of the solutions of a second order linear ordinary differential equation in the neighbourhood of a regular singular point?

Explain carefully why the hypergeometric equation with the usual parameters a, b and c (so that the exponents at $z = 0$ are 0 and $1-c$), has, for any given value of the parameter c, a solution $w(z)$ that satisfies $w(0) = 1$. Is $w(z)$ uniquely determined?

Is it the case that, for any given value of the parameter c, there is a solution $w(z)$ that is analytic at $z = 0$ and satisfies $w(0) = 1$? If such a solution exists, is it unique?

Solution: This question involves a very careful application *Fuchs' Theorem* used in order to define the hypergeometric function properly. Let's begin by recalling Fuchs' Theorem:

Fuchs' Theorem: Let $z = z_0$ be a regular singular point of a second order linear ODE, with exponents σ_1, σ_2 such that $\mathrm{Re}(\sigma_1) \geq \mathrm{Re}(\sigma_2)$. Then:

(i) If σ_1, σ_2 do not differ by an integer, i.e. $\sigma_1 - \sigma_2 \notin \mathbb{Z}$, then there exist two linearly independent Frobenius series solutions of the ODE near $z = z_0$ of the forms:

$$w_1(z) = (z-z_0)^{\sigma_1} \sum_{n=0}^{\infty} a_n (z-z_0)^n, \qquad w_2(z) = (z-z_0)^{\sigma_2} \sum_{n=0}^{\infty} b_n (z-z_0)^n,$$

where a_0, b_0 are arbitrary constants which uniquely determine all other a_n, b_n respectively by some recurrence relation.

(ii) If σ_1, σ_2 differ by an integer, i.e. $\sigma_1 - \sigma_2 \in \mathbb{Z}$, then there exist two linearly independent solutions (one a Frobenius series, the other 'series-like') of the ODE near $z = z_0$, of the forms:

$$w_1(z) = (z-z_0)^{\sigma_1} \sum_{n=0}^{\infty} a_n (z-z_0)^n, \qquad w_2(z) = C w_1(z) \log(z-z_0) + (z-z_0)^{\sigma_2} \sum_{n=0}^{\infty} b_n (z-z_0)^n,$$

where a_0 is arbitrary and determines all subsequent a_n via a recurrence relation.

Note also that b_0 is arbitrary and determines the constant C (which may be zero), and b_0 determines all the b_n up to $b_{\sigma_1 - \sigma_2}$ by the same recurrence relation. We have that $b_{\sigma_1-\sigma_2}$ is again arbitrary, but this doesn't create an extra degree of freedom in the solutions, since it just corresponds to a constant multiple of $w_1(z)$.

Let's apply Fuchs' Theorem to this question. Recall that the hypergeometric equation is the equation with the Riemann P-symbol

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix}.$$

We are asked to explore the solutions to the hypergeometric equation near $z = 0$. The exponents at this point are 0 and $1-c$, and hence we must carefully consider two cases:

(1) The first case of Fuchs' Theorem occurs if the difference of the exponents is not an integer. In particular, we must have $1 - c \notin \mathbb{Z}$, which is equivalent to $c \notin \mathbb{Z}$.

(2) The second case of Fuchs' Theorem occurs if c is an integer, $c \in \mathbb{Z}$.

In both cases, we must consider whether there exists (a) a solution obeying $w(0) = 1$; (b) an *analytic* solution obeying $w(0) = 1$. Let's begin:

- Case 1: c is not an integer.
  (a) In the case that c is not an integer, Fuchs' Theorem tells us that there must be two linearly independent Frobenius series solutions of the hypergeometric equation near $z = 0$ of the forms:

$$w_1(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad w_2(z) = z^{1-c} \sum_{n=0}^{\infty} b_n z^n,$$

  where a_0, b_0 are arbitrary and determine all other a_n, b_n uniquely via a recurrence relation. It follows that the general solution to the equation can be written as:

$$w(z) = Aw_1(z) + Bw_2(z),$$

  where $w_1(z)$, $w_2(z)$ are the Frobenius solutions above with $a_0 = b_0 = 1$ (we have essentially just turned the arbitrary constants a_0, b_0 into A,B instead).

  We would like to impose the condition $w(0) = 1$. We notice that $w_1(0) = 1$ if $a_0 = 1$, but the value of $w_2(0)$ depends on c:

  - If $\mathrm{Re}(1-c) < 0$, then $w_2(z)$ is singular at $z = 0$. Therefore to enforce $w(0) = 1$, we see that we need $A = 1$, $B = 0$. Hence there is a unique solution with $w(0) = 1$.
  - If $\mathrm{Re}(1-c) > 0$, then $w_2(0) = 0$. Hence to enforce $w(0) = 1$, we need $A = 1$, but B can be arbitrary. Any solution of the form $w(z) = w_1(z) + Bw_2(z)$ works. The solution is clearly not unique.
  - Finally, if $\mathrm{Re}(c) = 1$ (and $c \neq 1$, as we assume $c \notin \mathbb{Z}$), the solution $w_2(z)$ looks like z^{ai} for some $a \in \mathbb{R}$. But 0^{ai} is *undefined*[3], so in this case there is again a unique solution obeying $w(0) = 1$: $w(z) = w_1(z)$.

  To summarise: if $c \notin \mathbb{Z}$, there is a solution obeying $w(0) = 1$. This solution is unique if and only if $\mathrm{Re}(1-c) \leq 0$.

  (b) We now go further and ask: when is there an *analytic* solution obeying $w(0) = 1$? We know that the solution $w_2(z)$ is *not* analytic at $z = 0$ for any values of $c \notin \mathbb{Z}$, because we have a branch point at $z = 0$ due to the factor of z^{1-c} in $w_2(z)$. Thus in any open neighbourhood of $z = 0$, we will have to intersect a branch cut, and hence our neighbourhood will contain some discontinuity of $w_2(z)$.

  It follows that if we want an *analytic* solution near $z = 0$, we must set $B = 0$ in the general solution. Thus in all cases we get the unique solution $w(z) = w_1(z)$, obeying $w(0) = 1$.

  To summarise: if $c \notin \mathbb{Z}$, there is a unique solution that is analytic and obeys $w(0) = 1$.

- Case 2: c is an integer.
  (a) In the case that c is an integer, we must divide into sub-cases, depending on which of the exponents 0 and $1-c$ is larger.

  If $1 - c > 0$, the two solutions look like:

$$w_1(z) = z^{1-c} \sum_{n=0}^{\infty} a_n z^n, \qquad w_2(z) = Cw_1(z)\log(z) + \sum_{n=0}^{\infty} b_n z^n,$$

  where a_0 determines all subsequent a_n, and b_0 determines C and all b_n up to but not including b_{1-c}; the coefficient b_{1-c} is also arbitrary and determines all subsequent b_n.

[3] To define it, we attempt to write $z^i = e^{i\log(z)} = \cos(\log(z)) + i\sin(\log(z))$. But this makes no sense as $z \to 0$ (it doesn't explode and go to infinity, but it certainly doesn't converge either).

Let's choose $a_0 = b_0 = 1$ as usual, and choose $b_{1-c} = 1$ without loss of generality too. Then the general solution takes the form:

$$w(z) = Aw_1(z) + Bw_2(z),$$

for some constants A and B. Now note that $w_1(0) = 0$, and we can calculate $w_2(0)$ by considering the limit:

$$w_2(0) = \lim_{z\to 0}\left[Cw_1(z)\log(z) + \sum_{n=0}^{\infty} b_n z^n\right].$$

Since $w_1(z) \sim z^{1-c}$, and polynomial growth beats logarithmic growth in all cases, we have $z^{1-c}\log(z) \to 0$ as $z \to 0$. Thus we're left with $w_2(0) = b_0 = 1$. It follows that there exists a solution with $w(0)$ given by $w(z) = Aw_1(z) + w_2(z)$, but this solution is clearly not unique.

On the other hand, if $0 \geq 1 - c$, i.e. $c \geq 1$, the two solutions look like:

$$w_1(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad w_2(z) = Cw_1(z)\log(z) + z^{1-c}\sum_{n=0}^{\infty} b_n z^n.$$

Choosing $a_0 = b_0 = 1$ determines all the coefficients in this case; then our general solution looks like $w(z) = Aw_1(z) + Bw_2(z)$ as usual. We note that $w_1(0) = 1$, so there certainly exists a solution with $w(0) = 1$.

We must also consider $w_2(0)$. This doesn't exist if $c \geq 2$, because we have a singularity in $w_2(z)$. Thus our solution *is* unique; we are forced to choose $A = 1, B = 0$.

If $c = 1$ (the only remaining case), then we see that $C \neq 0$, else $w_1(z)$ and $w_2(z)$ would be linearly dependent (recall that a_n, b_n obey the *same* recurrence relation). Thus $w_2(z) \sim \log(z)$ (since $a_0 = 1$), and it follows that $w_2(0)$ again doesn't exist, because $w_2(z)$ is singular at $z = 0$. Hence we require $A = 1, B = 0$ and our solution is unique again.

To summarise: if $c \in \mathbb{Z}$, there is always a solution obeying $w(0) = 1$. This is solution is unique if and only if $c \geq 1$.

(b) We must now consider the analyticity of the solutions we described above. We recall that in the case $c \geq 1$, the solution obeying $w(0) = 1$ is always given by

$$w(z) = w_1(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which is trivially analytic. Thus for $c \geq 1$, there is a unique analytic solution obeying $w(0) = 1$.

In the case $1 - c > 0$, i.e. $c \leq 0$, things are a lot more complicated. We know that any solution obeying $w(0) = 1$ must take the form from above:

$$w(z) = Aw_1(z) + w_2(z) = Az^{1-c}\sum_{n=0}^{\infty} a_n z^n + Cz^{1-c}\log(z)\sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n.$$

If the constant $C \neq 0$, which depends on the equation we are trying to solve, then $w_2(z)$ is not analytic at $z = 0$, and it follows that there is no analytic solution near $z = 0$.

If the constant $C = 0$, then $w(z) = Aw_1(z) + w_2(z)$ *is* analytic and satisfies $w(0) = 1$. *However*, it is clearly not unique.

To summarise: if $c \in \{1, 2, 3, \dots\}$ there always exists a unique analytic solution obeying $w(0) = 1$. If $c \in \{0, -1, -2, \dots\}$ there sometimes exists an analytic solution obeying $w(0) = 1$, *but it will never be unique*.

Putting everything together we see:

- For all values of c, there exists a solution of the hypergeometric equation obeying $w(0) = 1$. This solution is unique if and only if $\mathrm{Re}(c) \geq 1$.
- For all values of $c \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$, there exists a unique analytic solution of the hypergeometric equation obeying $w(0) = 1$.
- If $c \in \{0, -1, -2, \ldots\}$, then there sometimes exists an analytic solution of the hypergeometric equation obeying $w(0) = 1$, but it is never unique.

All of our hard work shows that we can unambiguously define:

Definition: The *hypergeometric function* $F(a, b, c; z)$ is the unique analytic solution of the hypergeometric equation which satisfies $F(a, b, c; 0) = 1$. It is defined whenever $c \notin \{0, -1, -2, \ldots\}$.

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6. Is $(1-z)^{c-a-b}F(c-a, c-b, c; z)$ analytic at $z = 0$? The branch of $(1-z)^{c-a-b}$ is defined by $|\arg(1-z)| < \pi$ (which implies a branch cut from 1 to ∞ along the positive real axis). Assume that $c \neq 0, -1, -2, \ldots$.

Show, by considering transformation of P-symbols, that

$$(1-z)^{c-a-b}F(c-a, c-b, c; z) = F(a, b, c; z).$$

Show also that

$$(1-z)^{-a}F\left(a, c-b, c; \frac{z}{z-1}\right) = F(a, b, c; z).$$

Solution: In the first part of this question, we have to decide whether $(1-z)^{c-a-b}F(c-a, c-b, c; z)$ is analytic at $z = 0$. Since the hypergeometric function is *defined* to be analytic at $z = 0$ (for c not a non-positive integer), we must have that $F(c-a, c-b, c; z)$ is an analytic function at $z = 0$. Also notice that with the branch cut given in the question, $(1-z)^{c-a-b}$ is also clearly at $z = 0$. Thus $(1-z)^{c-a-b}F(c-a, c-b, c; z)$ is analytic at $z = 0$, provided c is not a non-positive integer.

To show the identities in this question, we use the fact that $F(a, b, c; z)$ is the *unique* solution of the hypergeometric equation that is analytic at $z = 0$ and obeys $F(a, b, c; 0) = 1$, provided c is not a non-positive integer. We proved this fact in Question 5. In particular, we see that to prove the identities in this question, it is sufficient to show that:

$$(1-z)^{c-a-b}F(c-a, c-b, c; z) \qquad \text{and} \qquad (1-z)^{-a}F\left(a, c-b, c; \frac{z}{z-1}\right)$$

are analytic at $z = 0$, take the value 1 at $z = 0$ and solve the hypergeometric equation. The identities then follow by uniqueness of the hypergeometric function.

Analyticity at $z = 0$ is obvious for both of these functions (we proved it carefully for the first), and clearly both take the value 1 at $z = 0$ since both the hypergeometric functions and the $(1-z)$ factors take the value 1 at $z = 0$. Therefore it remains to show that both these functions satisfy the hypergeometric equation. We can show this by manipulating P-symbols.

Recall that the hypergeometric equation has the Riemann P-symbol

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix}.$$

We can interpret this as the 'solution space' of the hypergeometric equation. Notice that by exponent shifting (from Question 3), we have:

$$(1-z)^{a+b-c}P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & a+b-c & c-b & z \\ 1-c & 0 & c-a & \end{Bmatrix},$$

since the factor of $(1-z)^{a+b-c}$ will shift the exponents of the solutions near $z=1$ and also the exponents of the solutions near $z=\infty$ (by writing $(1-z)^{a+b-c} \sim (-1)^{a+b-c}z^{a+b-c}(1-1/z)^{a+b-c}$, with potential factors from the branch choice, and expanding in powers of $1/z$). Note there is no shift of the exponents near $z=0$, as that would require an additional multiplicative factor of z^{μ}, say.

The final P-symbol is a hypergeometric P-symbol, with parameters $a' = c-a$, $b' = c-b$ and $c' = c$ (notice that the order of the exponents in the columns is immaterial). Hence an element of the solution space on the right hand side is $F(c-a, c-b, c; z)$. Multiplying through by $(1-z)^{c-a-b}$, we see that $(1-z)^{c-a-b}F(c-a, c-b, c; z)$ is an element of the solution space on the left hand side, and hence solves the hypergeometric equation!

For the final identity, we need to use Möbius transformations to transform the P-symbol too, since the argument of the hypergeometric function on the left hand side is $z/(z-1)$ rather than z.

First, let's notice that by exponent shifting, we have:

$$(1-z)^{a}P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & a & 0 & z \\ 1-c & c-b & b-a & \end{Bmatrix}$$

By using the Möbius transformation $z \mapsto z/(z-1)$, we also have:

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & a & 0 & z \\ 1-c & c-b & b-a & \end{Bmatrix} = P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & \dfrac{z}{z-1} \\ 1-c & c-b & b-a & \end{Bmatrix},$$

hence we deduce that:

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix} = (1-z)^{-a}P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & \dfrac{z}{z-1} \\ 1-c & c-b & b-a & \end{Bmatrix}.$$

The right hand side's P-symbol is of hypergeometric form, hence an element of the solution space on the right hand side is given by:

$$(1-z)^{-a}F\left(a, c-b, c; \frac{z}{z-1}\right).$$

The left hand side *it* the P-symbol of the hypergeometric equation, and hence this function also solves the hypergeometric equation. We deduce the required identity.

7. The hypergeometric function $F(a, b, c; z)$ may be defined for $|z| < 1$ (and as usual $c \neq 0, -1, -2 \ldots$) by

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}.$$

Let

$$g(z) = \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+t)\Gamma(b+t)}{\Gamma(c+t)} \Gamma(-t)(-z)^t \, dt$$

where the contour runs to the left of all the poles of $\Gamma(-t)$ and to the right of all poles of $\Gamma(a+t)$ and $\Gamma(b+t)$, $|\arg(-z)| < \pi$ and $a \neq 0, -1, -2, \ldots, b \neq 0, -1, -2, \ldots$.

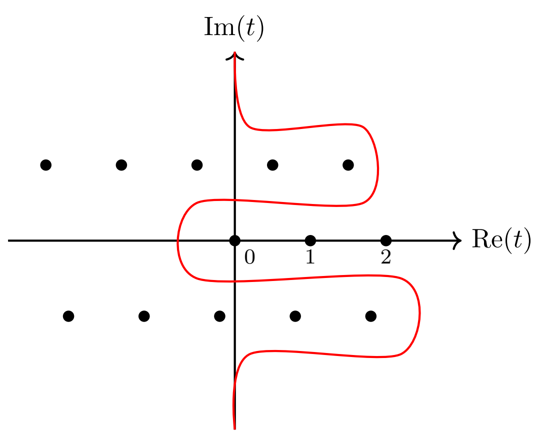
(i) Give a sketch of the complex t-plane showing the positions of the singularities of the integrand and the curve. By closing the contour with a large semi-circle in the right half complex plane (which maybe assumed to make a negligible contribution to the integral), show that $g(z) = F(a, b, c; z)$.

(ii) By closing the contour instead with a large semi-circle in the left half complex plane (which may be assumed to make a negligible contribution to the integral), show that the analytic continuation to $|z| > 1$ of the series for $F(a, b, c; z)$ in the case when a and b do not differ by an integer is provided by

$$\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a} F(a, 1-c+a, 1-b+a; z^{-1}) + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(a)}(-z)^{-b} F(b, 1-c+b, 1-a+b; z^{-1}).$$

Solution: (i) The integrand has singularities whenever $\Gamma(a+t)$, $\Gamma(b+t)$ and $\Gamma(-t)$ have singularities (note that $\Gamma(c+t)$ doesn't give us a problem, because the gamma function is never zero, as we saw on Sheet 2). The singularities of $\Gamma(-t)$ occur at $t = 0, 1, 2, \ldots$, the singularities of $\Gamma(a+t)$ occur at $t = -a, -a-1, \ldots$, and finally the singularities of $\Gamma(b+t)$ occur at $t = -b, -b-1, \ldots$.

Generically a and b will not be real, therefore, we have the sketch:



The black dots denote the singularities, and the contour of integration is the red line. Closing the contour to the right, and ignoring any convergence issues, we find by the residue theorem that:

$$g(z) = -\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \text{Res}\left(\frac{\Gamma(a+t)\Gamma(b+t)}{\Gamma(c+t)} \Gamma(-t)(-z)^t, n\right),$$

where we get the minus sign from the fact that our contour is traversed clockwise. Evaluating the residues, we have:

$$\begin{aligned}\operatorname{Res}\left(\frac{\Gamma(a+t)\Gamma(b+t)}{\Gamma(c+t)}\Gamma(-t)(-z)^t, n\right) &= \lim_{t\to n}\left[\frac{(t-n)\Gamma(a+t)\Gamma(b+t)}{\Gamma(c+t)}\Gamma(-t)(-z)^t\right] \\ &= \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}(-z)^n \cdot \lim_{t\to n}\left[(t-n)\Gamma(-t)\right].\end{aligned}$$

The last limit can be obtained by recalling that the residue of the gamma function at $-n$ for n a non-negative integer is given by:

$$\lim_{t\to -n}\left[(t+n)\Gamma(t)\right] = \frac{(-1)^n}{n!}.$$

Substituting $t \mapsto -t$, we find:

$$\lim_{t\to n}\left[(-t+n)\Gamma(-t)\right] = \frac{(-1)^n}{n!} \qquad \Rightarrow \qquad \lim_{t\to n}\left[(t-n)\Gamma(-t)\right] = \frac{(-1)^{n+1}}{n!},$$

and hence

$$\begin{aligned} g(z) &= -\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}(-z)^n\cdot\frac{(-1)^{n+1}}{n!} \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\cdot\frac{z^n}{n!}.\end{aligned}$$

This is equal to the hypergeometric function, provided $|z| < 1$, so we have $g(z) = F(a,b,c;z)$ whenever $|z| < 1$.

(ii) Closing to the left, and again assuming the integral on the arc vanishes, we again have by the residue theorem:

$$\begin{aligned} g(z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c)}\left[\sum_{n=0}^{\infty}\operatorname{Res}\left(\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)}{\Gamma(c+t)}(-z)^t, -a-n\right) + \sum_{n=0}^{\infty}\operatorname{Res}\left(\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)}{\Gamma(c+t)}(-z)^t, -b-n\right)\right] \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(b-a-n)\Gamma(a+n)}{\Gamma(c-a-n)}(-z)^{-a-n}\cdot\lim_{t\to -a-n}\left[\Gamma(a+t)(t+a+n)\right]\right. \\ &\qquad\qquad \left. + \sum_{n=0}^{\infty}\frac{\Gamma(a-b-n)\Gamma(b+n)}{\Gamma(c-b-n)}(-z)^{-b-n}\cdot\lim_{t\to -b-n}\left[\Gamma(b+t)(t+b+n)\right]\right] \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\left[(-z)^{-a}\sum_{n=0}^{\infty}\frac{\Gamma(b-a-n)\Gamma(a+n)}{\Gamma(c-a-n)}\frac{z^{-n}}{n!} + (-z)^{-b}\sum_{n=0}^{\infty}\frac{\Gamma(a-b-n)\Gamma(b+n)}{\Gamma(c-b-n)}\frac{z^{-n}}{n!}\right],\end{aligned}$$

where in the last line we used the residues of the gamma function we calculated carefully in part (a).

There's still a little bit of work needed to get everything in the correct form to be interpreted as hypergeometric functions. We have to manipulate the gamma functions a little bit; this can be achieved using the *reflection formula* from the special functions part of the course. We notice that:

$$\frac{\Gamma(b-a-n)}{\Gamma(c-a-n)} = \frac{\pi\operatorname{cosec}(\pi(b-a))}{\pi\operatorname{cosec}(\pi(c-a))}\cdot\frac{\Gamma(1-c+a+n)}{\Gamma(1-b+a+n)},$$

and

$$\frac{\Gamma(a-b-n)}{\Gamma(c-b-n)} = \frac{\pi\operatorname{cosec}(\pi(a-b))}{\pi\operatorname{cosec}(\pi(c-b))}\cdot\frac{\Gamma(1-c+b+n)}{\Gamma(1-a+b+n)}.$$

Therefore, we can write:

$$\begin{aligned} g(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}(-z)^{-a}\frac{\pi\mathrm{cosec}(\pi(b-a))}{\pi\mathrm{cosec}(\pi(c-a))}\sum_{n=0}^{\infty}\frac{\Gamma(1-c+a+n)\Gamma(a+n)}{\Gamma(1-b+a+n)}\frac{z^{-n}}{n!} \\ + \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}(-z)^{-b}\frac{\pi\mathrm{cosec}(\pi(a-b))}{\pi\mathrm{cosec}(\pi(c-b))}\sum_{n=0}^{\infty}\frac{\Gamma(b+n)\Gamma(1-c+b+n)}{\Gamma(1-a+b+n)}\frac{z^{-n}}{n!} \\ = \frac{\Gamma(c)}{\Gamma(b)}(-z)^{-a}\cdot\frac{\pi\mathrm{cosec}(\pi(b-a))}{\pi\mathrm{cosec}(\pi(c-a))}\cdot\frac{\Gamma(1-c+a)}{\Gamma(1-b+a)}F(a,1-c+a,1-b+a;z^{-1}) \\ + \frac{\Gamma(c)}{\Gamma(a)}(-z)^{-b}\frac{\pi\mathrm{cosec}(\pi(a-b))}{\pi\mathrm{cosec}(\pi(c-b))}\cdot\frac{\Gamma(1-c+b)}{\Gamma(1-a+b)}F(b,1-c+b,1-a+b;z^{-1}), \end{aligned}$$

where in the last step, we used the definition of the hypergeometric function. The equality holds only when $|1/z| < 1$, i.e. $|z| > 1$, for the hypergeometric functions to agree with their series definitions.

It remains to simplify the coefficients. Again, use the reflection formula to write:

$$\frac{\Gamma(1-c+a)}{\Gamma(1-b+a)} = \frac{\pi\mathrm{cosec}(\pi(c-a))}{\pi\mathrm{cosec}(\pi(b-a))}\frac{\Gamma(b-a)}{\Gamma(c-a)},$$

and

$$\frac{\Gamma(1-c+b)}{\Gamma(1-a+b)} = \frac{\pi\mathrm{cosec}(\pi(c-b))}{\pi\mathrm{cosec}(\pi(a-b))}\frac{\Gamma(c-b)}{\Gamma(a-b)}.$$

Hence we deduce that for $|z| > 1$, we have

$$g(z) = \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a}F(a,1-c+a,1-b+a;z^{-1}) + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(a)}(-z)^{-b}F(b,1-c+b,1-a+b;z^{-1}),$$

as required.

Now $F(a,b,c;z)$ agrees with $g(z)$ on $|z| < 1$, and hence agrees with any analytic continuation of $g(z)$ to $|z| > 1$. But now we have found such an analytic continuation, hence we have the continuation of the hypergeometric function to $|z| > 1$ as required.

8. By changing variable in the hypergeometric equation, deduce that

$$F(a, b, 1 + a + b - c; 1 - z) \equiv y_1(z)$$

is a solution of the hypergeometric equation near $z = 1$, and deduce that another solution is

$$(1 - z)^{c-a-b} F(c - a, c - b, 1 + c - a - b; 1 - z) \equiv y_2(z).$$

Assume that $c \neq 0, -1, -2, \cdots$.

Use the integral representation

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(c - b)\Gamma(b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - zt)^{-a}\, dt,$$

which is valid if $0 < \arg(z - 1) < 2\pi$ and $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, to show that

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - b)\Gamma(c - a)} \tag{††}$$

provided the above conditions hold, and $\mathrm{Re}(c - a - b) > 0$.

Find constants A and B such that (under conditions to be specified)

$$F(a, b, c; z) = Ay_1(z) + By_2(z).$$

Solution: To show that $y_1(z)$ is a solution of the hypergeometric equation, we use the standard P-symbol arguments that we learned in Question 6. Using the Möbius transformation $z \mapsto 1 - z$ we can transform the hypergeometric equation's P-symbol via:

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix} = P\begin{Bmatrix} 1 & 0 & \infty & \\ 0 & 0 & a & 1-z \\ 1-c & c-a-b & b & \end{Bmatrix}.$$

Note that the P-symbol on the right hand side is of hypergeometric form, with parameters $a' = a$, $b' = b$ and $c' = 1 + a + b - c$. Hence an element of the solution space on the right hand side is

$$y_1(z) = F(a, b, 1 + a + b - c; 1 - c),$$

which must also be an element of the solution space on the left hand side, and hence solves the hypergeometric equation as required.

Similarly, we can use exponent shifting and Möbius transformations to show that $y_2(z)$ satisfies the hypergeometric equation. Notice that we have:

$$(1 - z)^{a+b-c} P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & a+b-c & c-b & z \\ 1-c & 0 & c-a & \end{Bmatrix},$$

by exponent shifting, and then

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & a+b-c & c-b & z \\ 1-c & 0 & c-a & \end{Bmatrix} = P\begin{Bmatrix} 1 & 0 & \infty & \\ 0 & a+b-c & c-b & 1-z \\ 1-c & 0 & c-a & \end{Bmatrix}$$

by a Möbius transformation. It follows that

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix}\right\} = (1-z)^{c-a-b} P\left\{\begin{matrix} 1 & 0 & \infty & \\ 0 & a+b-c & c-b & 1-z \\ 1-c & 0 & c-a & \end{matrix}\right\}.$$

Again, the right hand side is of hypergeometric form with $a' = c-a$, $b' = c-b$ and $c' = 1+c-a-b$ (note the order of the exponents in the columns is of course immaterial). Hence an element of the solution space on the right hand side is:

$$(1-z)^{c-a-b} F(c-a, c-b, 1+c-a-b; 1-z),$$

which must also be an element of the solution space on the left hand side, and hence must solve the hypergeometric equation, as required.

We already did the second part of this question earlier on in the sheet, at the end of Question 4! We just set $z = 1$ in the integral representation, and evaluate using the beta function:

$$F(a, b, c; 1) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1}\, dt = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} B(b, c-b-a).$$

The fact that we used the integral representation of the beta function required us to have $\mathrm{Re}(b) > 0$, which is given, and $\mathrm{Re}(c-b-a) > 0$. Finally, we can use the relationship between the beta function and the gamma function to write down:

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)},$$

as required. We were asked only to prove this result for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $\mathrm{Re}(c-b-a) > 0$, and we have done so using the integral representations given. However, we also saw in Question 4 that by analytic continuation in a, b and c, we can extend this result to hold in the larger domain $\mathrm{Re}(c-b-a) > 0$.

To find the constants A and B, we simply set $z = 0$, $z = 1$ on the left hand side in turn, and solve the resulting system of equations. We notice that:

$$y_1(z) = F(a, b, 1+a+b-c; 1-z) \qquad \Rightarrow \qquad y_1(1) = 1,$$
$$y_2(z) = (1-z)^{c-a-b} F(c-a, c-b, 1+c-a-b; 1-z) \qquad \Rightarrow \qquad y_2(1) = 0,$$

provided that $\mathrm{Re}(c-a-b) > 0$, since the value of the hypergeometric function at 0 is just 1, and the value of $(1-z)^{c-a-b}$ at $z = 1$ is 0 for $\mathrm{Re}(c-a-b) > 0$.

In particular, if $F(a, b, c; z) = Ay_1(z) + By_2(z)$, we see that $F(a, b, c; 1) = A$, and so

$$A = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)},$$

under the condition that $\mathrm{Re}(c-b-a) > 0$.

Finding B is significantly trickier if we just try to use simultaneous equations as normal. However, there is a trick that speeds things up a lot. Notice that we are writing:

$$F(a,b,c;z) = A(a,b,c)F(a,b,1+a+b-c;1-z) + B(a,b,c)(1-z)^{c-a-b}F(c-a,c-b,1+c-a-b;1-z) \quad (*)$$

for some a, b and c dependent constants A and B. Since there is a symmetry between the arguments on the two hypergeometric functions on the right hand side (namely relabelling $a \mapsto c-a$, $b \mapsto c-b$ and $c \mapsto c$), we should expect there to be a similar symmetry on the coefficient functions A, B. Set $z=0$ on both sides to get:

$$1 = A(a,b,c)F(a,b,1+a+b-c;1) + B(a,b,c)F(c-a,c-b,1+c-a-b;1).$$

Looking at $(*)$, we notice that $F(a,b,c;1)$ is finite if and only if $\mathrm{Re}(c-a-b) > 0$. Thus for the two hypergeometric functions appearing in this expression to be finite, we require

$$\mathrm{Re}(1+a+b-c-a-b) > 0, \qquad \mathrm{Re}(1+c-a-b-(c-a)-(c-b)) > 0,$$

i.e. precisely $\mathrm{Re}(1-c) > 0$. Now consider making the transformation $a \mapsto c-a$, $b \mapsto c-b$ and $c \mapsto c$:

$$1 = A(c-a,c-b,c)F(c-a,c-b,1+c-a-b;1) + B(c-a,c-b,c)F(a,b,1+a+b-c;1).$$

Comparing coefficients of hypergeometric functions, we find that $B(a,b,c) = A(c-a,c-b,c)$. Hence we have:

$$B = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(b)\Gamma(a)},$$

provided $\mathrm{Re}(1-c) > 0$ (which is needed for the hypergeometric functions to be finite at $z=1$) and $\mathrm{Re}(c-a-b) > 0$, which is needed for A to be well-defined. thus we have the final expression:

$$F(a,b,c;z) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}F(a,b,1+a+b-c;1-z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(b)\Gamma(a)}(1-z)^{c-a-b}F(c-a,c-b,1+c-a-b;1-z).$$

In our derivation, we used the facts that $\mathrm{Re}(c-a-b) > 0$ and $\mathrm{Re}(1-c) > 0$. However, it is no clear that we can just relax these constraints by analytic continuation, and just say that this relationship holds whenever the two sides are well-defined and analytic.

❊ **Comments:** This question also fixes our problem from Question 4: can we analytically continue the relation we found into the domain $\mathrm{Re}(c-a-b) \leq 0$? As we have seen we can express the hypergeometric function in the form:

$$F(a,b,c;z) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}F(a,b,1+a+b-c;1-z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(b)\Gamma(a)}(1-z)^{c-a-b}F(c-a,c-b,1+c-a-b;1-z).$$

for all values of a, b and c such that F can be appropriately analytically continued. But if $\mathrm{Re}(c-a-b) \leq 0$, there will *definitely* be a singularity at $z=1$ (since the multiplying hypergeometric function is 1 at $z=1$, and the multiplying gamma term is zero only at the poles of $\Gamma(a)$ and $\Gamma(b)$), which cannot be gotten rid of. So there is a natural barrier at $\mathrm{Re}(c-a-b) = 0$, and the relationship in Question 4 cannot be continued past it via analytic continuation.

Part II: Further Complex Methods
Past Paper Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

2014, Paper 1, Section I, 8B

Show that the Cauchy-Riemann equations for $f : \mathbb{C} \to \mathbb{C}$ are equivalent to

$$\frac{\partial f}{\partial \bar{z}} = 0,$$

where $z = x+iy$, and $\partial/\partial\bar{z}$ should be defined in terms of $\partial/\partial x$ and $\partial/\partial y$. Use Green's Theorem, together with the formula $dzd\bar{z} = -2idxdy$, to establish the generalised Cauchy formula

$$\oint_\gamma f(z,\bar{z})\,dz = -\iint_D \frac{\partial f}{\partial \bar{z}} dz d\bar{z}$$

where γ is a contour in the complex plane enclosing the region D and f is sufficiently differentiable.

Solution: If $z = x+iy$, then it follows that $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$. Hence, using the chain rule, we can obtain the definition of $\partial/\partial\bar{z}$:

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial x}{\partial \bar{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar{z}}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Now write $f(z) = u(x,y) + iv(x,y)$ in terms of its real and imaginary parts. Then we have:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u(x,y)+iv(x,y)) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

We see that this is equal to zero if and only if the real and imaginary parts on the right hand side are separately equal to zero, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are precisely the Cauchy-Riemann equations, as required.

For the last part of the question, we simply write out the left hand side of the generalised Cauchy formula in terms of x,y and the real and imaginary parts of f, i.e. u, v. We have:

$$\oint_\gamma f(z,\bar{z})\,dz = \oint_\gamma (u+iv)(dx+idy) = \oint_\gamma (udx - vdy) + i\oint_\gamma (vdx+udy).$$

Recall Green's Theorem in the plane states that

$$\oint_\gamma (g_1 dx + g_2 dy) = \iint_D \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right) dxdy.$$

Hence using Green's Theorem on both the integrals above, and also using the formula we derived for $\partial f/\partial\bar{z}$ above, and the given measure $dzd\bar{z} = -2idxdy$, we have

$$\oint_\gamma f(z,\bar{z})\,dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy = -\iint_D \frac{\partial f}{\partial \bar{z}}\,dzd\bar{z},$$

as required.

2014, Paper 2, Section I, 8B

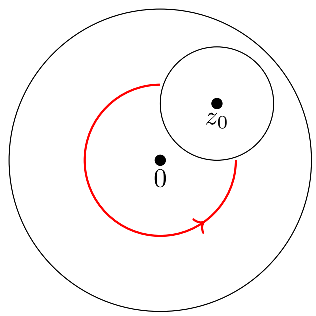
Suppose $z = 0$ is a regular singular point of a linear second-order homogeneous ordinary differential equation in the complex plane. Define the *monodromy matrix* M around $z = 0$.

Demonstrate that if

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then the differential equation admits a solution of the form $a(z) + b(z)\log(z)$, where $a(z)$ and $b(z)$ are single-valued functions.

Solution: Let $\mathcal{D} = \{z : 0 < |z| < R\}$ be the largest punctured disk centred on $z = 0$ such that $\mathcal{D}$ does not contain any other regular singular points of our equation. Then for any $z_0 \in \mathcal{D}$, we have that z_0 is an *ordinary point* of the equation, and hence by Fuchs' Theorem there is an open disk $\mathcal{D}_0 \subset \mathcal{D}$ centred on z_0 on which there are two analytic solutions $w_1(z)$, $w_2(z)$.



With a view of translating our *local* solution into a *global* solution on $\mathcal{D}$ in its entirety, we consider analytically continuing $w_1(z)$, $w_2(z)$ around the point $z = 0$ using a series of disks, following the arrow in the figure.

After performing the analytic continuation, the functions $w_1(z)$, $w_2(z)$ are mapped to new functions $\hat{w}_1(z) = w_1(e^{2\pi i}z)$, $\hat{w}_2(z) = w_2(e^{2\pi i}z)$. Since the derivatives of these functions are $\hat{w}_1'(z) = e^{2\pi i}w_1'(e^{2\pi i}z) = w_1'(e^{2\pi i}z)$, etc, we see that $\hat{w}_1(z)$, $\hat{w}_2(z)$ satisfy the original equation on $\mathcal{D}_0$, and hence we must be able to express $\hat{w}_1(z)$, $\hat{w}_2(z)$ as linear combinations of $w_1(z)$ and $w_2(z)$. The *monodromy matrix* is defined to be the matrix M relating the two systems of solutions:

$$\begin{pmatrix} \hat{w}_1(z) \\ \hat{w}_2(z) \end{pmatrix} = M \begin{pmatrix} w_1(z) \\ w_2(z) \end{pmatrix}.$$

Given the monodromy matrix in the equation, we see that our solutions are in fact related via:

$$w_1(e^{2\pi i}z) = w_1(z) + w_2(z), \qquad w_2(e^{2\pi i}z) = w_2(z).$$

Notice that $w_2(z)$ returns to itself under the analytic continuation, and hence $w_2(z)$ is a single-valued function. Consider the function

$$a(z) = w_1(z) - \frac{\log(z)}{2\pi i}w_2(z).$$

Under the analytic continuation, this maps to

$$a(e^{2\pi i}z) = w_1(z) + w_2(z) - \frac{\log(z)}{2\pi i}w_2(z) - w_2(z) = a(z),$$

and hence it is single-valued. Rearranging, we see that $w_1(z) = a(z) + \log(z)w_2(z)/2\pi i = a(z) + \log(z)b(z)$, where $a(z), b(z) = w_2(z)/2\pi i$ are indeed single-valued functions.

2014, Paper 3, Section I, 8B

State the conditions for a point $z = z_0$ to be a *regular singular point* of a linear second-order homogeneous ordinary differential equation in the complex plane.

Find all singular points of the Airy equation

$$w''(z) - zw(z) = 0,$$

and determine whether they are regular or irregular.

Solution: The general linear second-order homogeneous ODE in the complex plane takes the form:

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0.$$

A point $z = z_0$ in the finite complex plane $\mathbb{C}$ is called an *ordinary point* of this equation if $p(z)$ and $q(z)$ are both analytic at $z = z_0$. A point $z = z_0$ in the finite complex plane $\mathbb{C}$ is called a *regular singular point* of this equation if it is not ordinary, and both $(z - z_0)p(z)$, $(z - z_0)^2 q(z)$ are analytic functions at $z = z_0$.

Slightly different conditions hold at the point $z = \infty$. To find them, we substitute $z = 1/t$ into the ODE and look at the behaviour near $t = 0$. The derivatives transform as:

$$\frac{d}{dz} = \frac{dt}{dz}\frac{d}{dt} = -\frac{1}{z^2}\frac{d}{dt} = -t^2\frac{d}{dt}.$$

Hence the equation transforms to:

$$t^2\frac{d}{dt}\left(t^2\frac{dw}{dt}\right) - p(1/t)t^2\frac{dw}{dt} + q(1/t) = 0 \qquad \Rightarrow \qquad \frac{d^2w}{dt^2} + \left(\frac{2}{t} - \frac{p(1/t)}{t^2}\right)\frac{dw}{dt} + \frac{q(1/t)}{t^4} = 0.$$

It follows that $z = \infty$ is an *ordinary point* if these are analytic functions as $t \to 0$, i.e.

$$\frac{2}{t} - \frac{p(1/t)}{t^2} = O(1), \quad \frac{q(1/t)}{t^4} = O(1) \qquad \Rightarrow \qquad p(1/t) = 2t + O(t^2), \quad q(1/t) = O(t^4).$$

Writing these conditions in terms of z, we have

$$p(z) = \frac{2}{z} + O\left(\frac{1}{z^2}\right), \qquad q(z) = O\left(\frac{1}{z^4}\right)$$

as $z \to \infty$. We also find that $z = \infty$ is a *regular singular point* if it is not an ordinary point, and as $t \to 0$, we have

$$t\left(\frac{2}{t} - \frac{p(1/t)}{t^2}\right) = O(1), \quad t^2 \cdot \frac{q(1/t)}{t^4} = O(1) \qquad \Rightarrow \qquad p(1/t) = O(t), \quad q(1/t) = O(t^2).$$

Writing this in terms of z, we have as $z \to \infty$:

$$p(z) = O\left(\frac{1}{z}\right), \qquad q(z) = O\left(\frac{1}{z^2}\right).$$

We are now asked to apply this to Airy's equation, which has $p(z) = 0$, $q(z) = -z$. Clearly all points in the finite complex plane are ordinary. We just need to check the point at infinity.

Since $q(z) = -z$ does not converge to zero as $z \to \infty$, we see from the above conditions that $z = \infty$ cannot be an ordinary point, and also cannot be a regular singular point. It follows that the point at infinity is an irregular singular point of Airy's equation.

2014, Paper 4, Section I, 8B

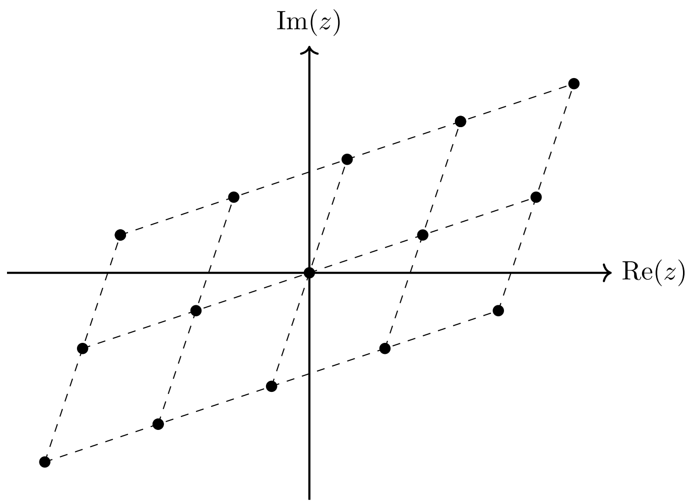
Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that

$$f(z+\omega_1) = f(z), \qquad f(z+\omega_2) = f(z), \tag{1}$$

where $\omega_1, \omega_2 \in \mathbb{C}\backslash\{0\}$ and ω_2/ω_1 is not real. Show that if f is analytic on $\mathbb{C}$ then it is constant. [Liouville's Theorem may be used if stated.] Given an example of a non-constant meromorphic function which satisfies (1).

➻ **Solution:** Liouville's Theorem states that any bounded entire function $f : \mathbb{C} \to \mathbb{C}$ is constant.

In this question, we are given that this doubly periodic function is analytic on $\mathbb{C}$, that is, it is entire. To apply Liouville's Theorem, we just need to prove that f is bounded. To do so, recall that the points $m\omega_1 + n\omega_2$ for $m, n \in \mathbb{Z}$ define a lattice (since $\omega_2/\omega_1 \notin \mathbb{R}$) in the complex plane:



The values that $f(z)$ takes on one cell are exactly the same as the values that $f(z)$ takes on another cell, by the periodicity properties (1). Since any specific cell is a closed, bounded set, we know from analysis courses that $f(z)$ must be bounded on that cell. It follows by the periodicity that $f(z)$ is everywhere bounded, and we conclude by Liouville's Theorem that $f(z)$ is constant.

Finally, we are asked for a non-constant meromorphic function satisfying (1). The obvious example (and in fact the only one mentioned in the course) is the Weierstrass $\wp$ function:

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z-\omega_{m,n})^2} - \frac{1}{\omega_{m,n}^2} \right],$$

where $\omega_{m,n} = m\omega_1 + n\omega_2$.

2014, Paper 1, Section II, 14B

Obtain solutions of the second-order ordinary differential equation

$$zw'' - w = 0$$

in the form

$$w(z) = \int_\gamma f(t)e^{-zt}\,dt,$$

where the function f and the choice of contour γ should be determined from the differential equation.

Show that non-trivial solution can be obtained by choosing γ to be a suitable closed contour, and find the resulting solution in this case, expressing your answer in the form of a power series.

Describe a contour γ that would provide a second linearly independent solution for the case $\mathrm{Re}(z) > 0$.

➬ **Solution:** We insert the integral form of $w(z)$ into the given equation to find:

$$\int_\gamma \left(zt^2 f - f\right) e^{-zt}\,dt = 0.$$

As always, we try to put all the z dependence back into the kernel e^{-zt}. We achieve this by using integration by parts:

$$0 = \int_\gamma \left(\frac{d}{dt}\left(t^2 f\right) - f\right) e^{-zt}\,dt - \left[t^2 f(t)e^{-zt}\right]_\gamma = \int_\gamma \left(t^2 f' + (2t-1)f\right) e^{-zt}\,dt - \left[t^2 f(t)e^{-zt}\right]_\gamma.$$

Hence we choose the function $f(t)$ such that $t^2 f' = (1-2t)f$, which is a separable equation with solution:

$$\int \frac{df}{f} = \int \frac{1-2t}{t^2}\,dt \qquad \Rightarrow \qquad \log(f) = A - \frac{1}{t} - 2\log(t) \qquad \Rightarrow \qquad f(t) = \frac{Ce^{-1/t}}{t^2}.$$

Hence solutions of this equation are of the form:

$$w(z) \propto \int_\gamma \frac{e^{-1/t-zt}}{t^2}\,dt, \qquad \text{where the contour } \gamma \text{ obeys} \qquad \left[e^{-zt-1/t}\right]_\gamma = 0.$$

Since the integrand for $w(z)$ contains an essential singularity at $z = 0$, we can pick γ to be a small circle around the origin (this trivially satisfies the boundary condition, since γ is closed). Then, by the residue theorem, the solution will be proportional to the residue of the integrand at $t = 0$. To find this residue, we expand the integrand in series:

$$\frac{e^{-1/t-zt}}{t^2} = \frac{1}{t^2}\left(\sum_{m=0}^{\infty} \frac{(-zt)^m}{m!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!t^n}\right) = \sum_{m,n=0}^{\infty} \frac{z^m(-1)^{m+n}}{m!n!t^{n+2-m}}.$$

We want the coefficient of t^{-1} to get the residue, hence we need $n+2-m = 1$, i.e. $m = n+1$. Hence a possible solution is:

$$w_1(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n!(n+1)!},$$

where we removed some constants to tidy things up, because any constant multiple of this expression is a solution.

To get a second solution, consider putting the contour γ along the positive real axis from $t = 0$ to $t = \infty$. At the start of the contour at $t = 0$, we have $e^{-1/t} \to 0$ exponentially quickly as $t \to 0$ along the real axis. At the opposite end of the contour, $e^{-zt} \to 0$ exponentially quickly as $t \to \infty$ since, $\mathrm{Re}(z) > 0$. So the boundary condition is satisfied, and the integral is convergent; hence we get another (linearly independent) solution.

2014, Paper 2, Section II, 14B

Use the Euler product formula

$$\Gamma(z) = \lim_{n\to\infty}\left[\frac{n!n^z}{z(z+1)...(z+n)}\right]$$

to show that:

(i) $\Gamma(z+1) = z\Gamma(z)$;

(ii) $\dfrac{1}{\Gamma(z)} = ze^{\gamma z}\displaystyle\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k}$ where $\gamma = \displaystyle\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log(n)\right)$.

Deduce that

$$\frac{d}{dz}\log(\Gamma(z)) = -\gamma - \frac{1}{z} + z\sum_{k=1}^{\infty}\frac{1}{k(z+k)}.$$

Solution: (i) Note that:

$$\Gamma(z+1) = \lim_{n\to\infty}\left[\frac{n!n^{z+1}}{(z+1)(z+2)...(z+1+n)}\right] = z\lim_{n\to\infty}\left[\frac{n!n^z}{z(z+1)...(z+n)}\cdot\frac{n}{z+1+n}\right].$$

In the final step, we've split up the product inside the bracket into a factor resembling an Euler product for $\Gamma(z)$, and a factor that just tends to 1 as $n\to\infty$. Since each of these factors converges separately, we can split up the limit to give:

$$\Gamma(z+1) = z\lim_{n\to\infty}\left[\frac{n!n^z}{z(z+1)...(z+n)}\right]\cdot\lim_{n\to\infty}\left[\frac{n}{z+1+n}\right] = z\Gamma(z),$$

as required.

(ii) This is standard proof from lectures. We have:

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= \lim_{n\to\infty}\left[\frac{z(z+1)...(z+n)}{n!n^z}\right] && \text{(given in question)}\\
&= z\lim_{n\to\infty}\left[\left(\frac{1+z}{1}\right)\left(\frac{2+z}{2}\right)\cdots\left(\frac{n+z}{n}\right)e^{-z\log(n)}\right] && \text{(split } n!\text{, use } n^z = e^{z\log(n)}\text{)}\\
&= z\lim_{n\to\infty}\left[\left(1+\frac{z}{1}\right)e^{-z}\left(1+\frac{z}{2}\right)e^{-z/2}\cdots\left(1+\frac{z}{n}\right)e^{-z/n}e^{z(1+1/2\cdots+1/n-\log(n))}\right]\\
&= ze^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k} && \text{(given definition of } \gamma\text{)}
\end{aligned}$$

where in the third step we inserted $1 = e^{z(1+...+1/n)}e^{-z(1+...+1/n)}$. This is the required form of $1/\Gamma(z)$, which is the *Weiestrass canonical product formula*.

To get the final result, take the logarithm of the product formula in (ii) and then take the derivative. We have:

$$\begin{aligned}
\frac{d}{dz}\left(\log(\Gamma(z))\right) &= -\frac{d}{dz}\left(\log\left(\frac{1}{\Gamma(z)}\right)\right) = -\frac{d}{dz}\left(\log(z)+\gamma+\sum_{k=1}^{\infty}\left(\log\left(1+\frac{z}{k}\right)-\frac{z}{k}\right)\right)\\
&= -\frac{1}{z}-\gamma-\sum_{k=1}^{\infty}\left(\frac{1/k}{1+z/k}-\frac{1}{k}\right) = -\frac{1}{z}-\gamma+z\sum_{k=1}^{\infty}\frac{1}{k(z+k)},
\end{aligned}$$

as required.

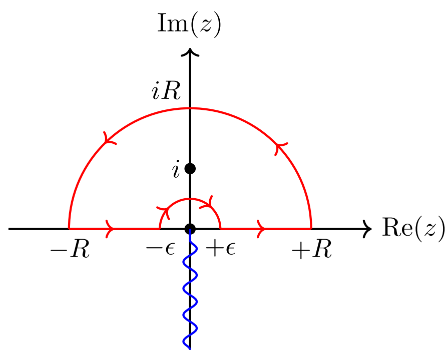
2015, Paper 1, Section I, 6B

Evaluate the real integral $\int_0^\infty \frac{x^{1/2}\log(x)}{1+x^2}\,dx$, where $x^{1/2}$ is taken to be the positive square root. What is the value of $\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx$?

Solution: We instead consider the contour integral

$$\oint_C \frac{z^{1/2}\log(z)}{1+z^2}\,dz, \tag{$*$}$$

around a contour C. Notice that the integrand of this function is multi-valued with a branch point at $z=0$; we choose a branch by inserting a cut along the negative imaginary axis and choosing arguments in the range $(-\pi/2, 3\pi/2)$. We choose C to be an arch contour shown below:



The contour C encloses one singularity of the integrand at $z=i$. Hence, by the residue theorem, the integral in ($*$) is given by:

$$\oint_C \frac{z^{1/2}\log(z)}{1+z^2}\,dz = 2\pi i \mathrm{Res}\left(\frac{z^{1/2}\log(z)}{1+z^2}; i\right) = 2\pi i\left(\frac{e^{i\pi/4}(i\pi/2)}{2i}\right) = \frac{i\pi^2 e^{i\pi/4}}{2}.$$

But we can also evaluate the integral on each section of the contour separately, giving:

$$\frac{i\pi^2 e^{i\pi/4}}{2} = \oint_C \frac{z^{1/2}\log(z)}{1+z^2}\,dz = \int_0^\pi \frac{R^{1/2}e^{i\theta/2}\left(\log(R)+i\theta\right)}{1+R^2e^{2i\theta}} iRe^{i\theta}\,d\theta + \int_R^\epsilon \frac{e^{i\pi/2}x^{1/2}\left(\log(x)+i\pi\right)}{1+x^2}e^{i\pi}dx$$

$$+ \int_\pi^0 \frac{\epsilon^{1/2}e^{i\theta/2}\left(\log(\epsilon)+i\theta\right)}{1+\epsilon^2 e^{2i\theta}} i\epsilon e^{i\theta}\,d\theta + \int_\epsilon^R \frac{x^{1/2}\log(x)}{1+x^2}\,dx.$$

The powers of R in the R integral are of the form $O(\log(R)R^{-1/2})$ and $O(R^{-1/2})$, so tend to zero as $R\to\infty$. After using the binomial expansion on the denominator, it's clear that the powers of ϵ in the ϵ integral are $O(\epsilon^{3/2}\log(\epsilon))$ and $O(\epsilon^{3/2})$, which both tend to zero as $\epsilon\to\infty$. Thus as $\epsilon\to 0$ and $R\to\infty$, we're left with the result:

$$i\int_0^\infty \frac{x^{1/2}(\log(x)+i\pi)}{1+x^2}\,dx + \int_0^\infty \frac{x^{1/2}\log(x)}{1+x^2}\,dx = \frac{i\pi^2 e^{i\pi/4}}{2}.$$

Comparing real and imaginary parts, we find the results:

$$\int_0^\infty \frac{x^{1/2}\log(x)}{1+x^2}\,dx = \frac{\pi^2\sqrt{2}}{4}, \qquad -\pi\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx + \int_0^\infty \frac{x^{1/2}\log(x)}{1+x^2}\,dx = -\frac{\pi^2\sqrt{2}}{4} \qquad \Rightarrow \qquad \int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx = \frac{\pi\sqrt{2}}{2}.$$

2015, Paper 2, Section I, 6B

Give a brief description of what is mean by *analytic continuation*.

The dilogarithm function is defined by

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \qquad |z| < 1.$$

Let

$$f(z) = -\int_C \frac{\log(1-u)}{u}\, du,$$

where C is a contour that runs from the origin to the point z. Show that $f(z)$ provides an analytic continuation of $\mathrm{Li}_2(z)$ and describe its domain of definition in the complex plane, given a suitable branch cut.

➻ **Solution:** Let $g(z)$ be an analytic function on a domain U. Suppose that $h(z)$ is an analytic function on a domain U' such that $U \cap U' \neq \emptyset$ and $g(z) = h(z)$ on the intersection $U \cap U'$. Then $h(z)$ is called an *analytic continuation* of the function $g(z)$ to the domain U'.

Let's consider evaluating the integral $f(z)$ when $|z| < 1$. Since the integrand has a branch point at $u = 1$, we must specify the branch of the logarithm we are using. We choose to insert a branch cut from 1 to ∞ along the positive real axis, and choose the integrand to be real just above the branch cut. Notice also that by Cauchy's Theorem, we can deform the contour of integration arbitrarily, provided we don't pass through singularities of the integrand. So deform C such that it is a straight line segment from the origin to the point z.

If $|z| < 1$, then on each point of our straight-line contour C, the integration variable u obeys $|u| < 1$; it follows that we can expand the integrand in a Taylor series:

$$f(z) = -\int_C \frac{\log(1-u)}{u}\, du = \int_C -\frac{1}{u}\sum_{n=1}^{\infty} \frac{(-u)^n(-1)^{n-1}}{n}\, du = \sum_{n=1}^{\infty} \int_C \frac{u^{n-1}}{n}\, du,$$

where in the second step we used the Taylor series for $\log(1-u)$, which is valid when $|u| < 1$. Now just integrate term by term:

$$\int_C \frac{u^{n-1}}{n}\, du = \int_C \frac{d}{du}\left(\frac{u^n}{n^2}\right) du = \left[\frac{u^n}{n^2}\right]_C = \frac{z^n}{n^2}.$$

Hence we see that for $|z| < 1$, we have

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = \mathrm{Li}_2(z).$$

To see that $f(z)$ provides the analytic continuation of $\mathrm{Li}_2(z)$ to a larger domain than $|z| < 1$, we just need to ask where $f(z)$ is analytic. But we note:

$$\frac{df}{dz} = -\frac{d}{dz}\int_0^z \frac{\log(1-u)}{u}\, du = -\frac{\log(1-z)}{z}.$$

Hence $f(z)$ is differentiable everywhere in $\mathbb{C}$, except for along the branch cut we specified earlier (note there is a removable singularity of this function at $z = 0$, so $f(z)$ can be viewed as differentiable there). Thus $f(z)$ is analytic everywhere in $\mathbb{C}$ minus the branch cut, and it follows that $f(z)$ provides the analytic continuation of $\mathrm{Li}_2(z)$ to the complex plane $\mathbb{C}$, minus the branch cut.

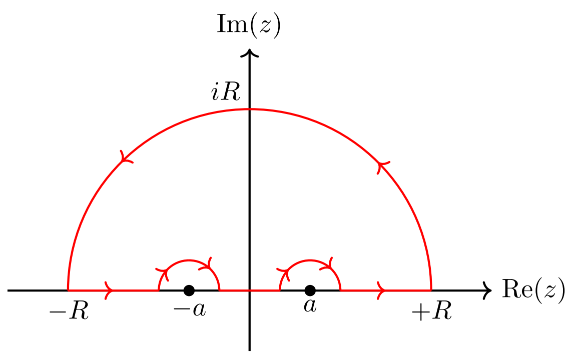
2015, Paper 3, Section I, 6B

Define what is meant by the *Cauchy principal value* in the particular case: $\mathcal{P}\int\limits_{-\infty}^{\infty} \frac{\cos(x)}{x^2-a^2}\,dx$, where the constant a is real and strictly positive. Evaluate this expression explicitly, stating clearly any standard results involving contour integrals that you use.

Solution: In this case, the Cauchy principal value is given by:

$$\mathcal{P}\int\limits_{-\infty}^{\infty} \frac{\cos(x)}{x^2-a^2}\,dx = \lim_{\substack{R\to\infty\\ \epsilon_1\to 0\\ \epsilon_2\to 0}} \left[\int\limits_{-R}^{-a-\epsilon_1} \frac{\cos(x)}{x^2-a^2}\,dx + \int\limits_{-a+\epsilon_1}^{a-\epsilon_2} \frac{\cos(x)}{x^2-a^2}\,dx + \int\limits_{a+\epsilon_2}^{R} \frac{\cos(x)}{x^2-a^2}\,dx \right].$$

To evaluate this integral, consider instead the contour integral $\oint\limits_C \frac{e^{iz}}{z^2-a^2}\,dz$ around the contour C shown.



Since the contour encloses no singularities of the integrand, we have by Cauchy's Theorem: $\oint\limits_C \frac{e^{iz}}{z^2-a^2}\,dz = 0.$

Now consider evaluating the integral on each section of the contour separately. Recall Jordan's Lemma states that if $f(z)$ is a function such that $|f(z)| \to 0$ uniformly on the large semicircular contour C_R as $R\to\infty$, then the integral of $e^{i\lambda z}f(z)$ over C_R converges to zero as $R\to\infty$, for $\lambda>0$. Hence in this case we can ignore the contribution from the large semicircular part of the contour.

The straight segments give the Cauchy principal value we are looking for. The remaining small semicircular arcs give the contribution:

$$\int\limits_{\pi}^{0} \frac{e^{i(-a+\epsilon_1 e^{i\theta})} i\epsilon_1 e^{i\theta}}{(-a+\epsilon_1 e^{i\theta})^2-a^2}\,d\theta + \int\limits_{\pi}^{0} \frac{e^{i(a+\epsilon_2 e^{i\theta})} i\epsilon_2 e^{i\theta}}{(a+\epsilon_2 e^{i\theta})^2-a^2}\,d\theta = e^{-ia}\int\limits_{\pi}^{0} \frac{(1+O(\epsilon_1))}{-2ae^{i\theta}+O(\epsilon_1)} ie^{i\theta}\,d\theta + e^{ia}\int\limits_{\pi}^{0} \frac{(1+O(\epsilon_2))}{2ae^{i\theta}+O(\epsilon_2)} ie^{i\theta}\,d\theta.$$

Taking the limit as $\epsilon_1\to 0$, $\epsilon_2\to 0$, we're left with the contribution:

$$-\frac{ie^{-ia}}{2a}\cdot(-\pi) + \frac{ie^{ia}}{2a}\cdot(-\pi) = \frac{\pi}{a}\left(\frac{e^{ia}-e^{-ia}}{2i}\right) = \frac{\pi\sin(a)}{a}.$$

so putting everything together, and taking real parts, we find:

$$\mathcal{P}\int\limits_{-\infty}^{\infty} \frac{e^{ix}}{x^2-a^2}\,dx + \frac{\pi\sin(a)}{a} = 0 \qquad \Rightarrow \qquad \mathcal{P}\int\limits_{-\infty}^{\infty} \frac{\cos(x)}{x^2-a^2}\,dx = -\frac{\pi\sin(a)}{a}.$$

2015, Paper 4, Section I, 6B

Explain how the Papperitz symbol

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix}$$

represents a differential equation with certain properties. [You need not write down the differential equation explicitly.]

The hypergeometric function $F(a, b, c; z)$ is defined to be the solution of the equation with Papperitz symbol

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix}$$

that is analytic at $z = 0$ and such that $F(a, b, c; 0) = 1$. Show that

$$F(a, b, c; z) = (1-z)^{-a} F\left(a, c-b, c; \frac{z}{z-1}\right),$$

indicating clearly any general results for manipulating Papperitz symbols that you use.

➼ **Solution:** The given Papperitz symbol describes the Fuchsian equation in the variable z with exactly three regular singular points at $z = z_1$, $z = z_2$, $z = z_3$, with exponents α_1, α_2 at $z = z_1$, exponents β_1, β_2 at $z = z_2$ and exponents γ_1, γ_2 at $z = z_3$.

To do this question, we use the following results for manipulating Papperitz symbols, which you should just quote:

- For any Möbius transformation M, we have:

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix} = P\begin{Bmatrix} M(z_1) & M(z_2) & M(z_3) & \\ \alpha_1 & \beta_1 & \gamma_1 & M(z) \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix}$$

  Thus we notice that under a Möbius transformation of the independent variable in the Papperitz equation, the regular singular points are mapped according to the Möbius transformation, but the exponents are left unchanged.

- We have the *exponent-shifting property* of the Papperitz symbol:

$$\left(\frac{z-z_1}{z-z_2}\right)^k \left(\frac{z-z_2}{z-z_3}\right)^l P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix} = P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha_1+k & \beta_1+l-k & \gamma_1-l & z \\ \alpha_2+k & \beta_2+l-k & \gamma_2-l & \end{Bmatrix}.$$

  In our particular case of interest, we will need the limit of this as $z_3 \to \infty$. By introducing a constant z_3^l on the LHS, we do not affect the solutions of the equation with the P-symbol on the left hand side, so we can take the limit as $z_3 \to \infty$ just to get:

$$(z-z_1)^k (z-z_2)^{l-k} P\begin{Bmatrix} z_1 & z_2 & \infty & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix} = P\begin{Bmatrix} z_1 & z_2 & \infty & \\ \alpha_1+k & \beta_1+l-k & \gamma_1-l & z \\ \alpha_2+k & \beta_2+l-k & \gamma_2-l & \end{Bmatrix}.$$

We now prove the hypergeometric function identity. Notice that the function

$$(1-z)^{-a} F\left(a, c-b, c; \frac{z}{z-1}\right) \tag{$*$}$$

takes the value 1 at $z = 0$, since $(1-0)^{-a} = 1$ and $F(a, c-b, c; 0) = 1$ by definition of the hypergeometric function. Notice also that this function is analytic near $z = 0$, since $(1-z)^{-a}$ is analytic near $z = 0$, and $F\left(a, c-b, c; z/(z-1)\right)$ is analytic near $z = 0$ (since the hypergeometric function itself is analytic near $z = 0$ and $z/(z-1)$ is analytic near $z = 0$, so we have the composition of analytic functions).

Finally, we must show that this function satisfies the hypergeometric equation. We can achieve this via manipulation of P-symbols. Using the definition of the hypergeometric equation, we notice that the function ($*$) solves the equation with P-symbol:

$$(1-z)^{-a} P \left\{ \begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & \dfrac{z}{z-1} \\ 1-c & b-a & c-b & \end{matrix} \right\}.$$

Now using exponent-shifting, and the Möbius transformation $w = z/(z-1)$ with inverse $z = w/(w-1)$, we have

$$(1-z)^{-a} P \left\{ \begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & \dfrac{z}{z-1} \\ 1-c & b-a & c-b & \end{matrix} \right\} = (1-z)^{-a} P \left\{ \begin{matrix} 0 & \infty & 1 & \\ 0 & 0 & a & z \\ 1-c & b-a & c-b & \end{matrix} \right\}$$

$$= P \left\{ \begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-b-a & \end{matrix} \right\}.$$

After reordering columns, this is the P-symbol of the hypergeometric equation. It follows that indeed the function ($*$) solves the hypergeometric equation. Now since $F(a, b, c; z)$ is the *unique* analytic function that solves the hypergeometric function and takes the value 1 at $z = 0$, we see that

$$F(a, b, c; z) = (1-z)^{-a} F\left(a, c-b, c; \frac{z}{z-1}\right)$$

as required.

2015, Paper 1, Section II, 11B

Consider the differential equation

$$xy'' + (a-x)y' - by = 0 \qquad (*)$$

where a and b are constants with $\mathrm{Re}(b) > 0$ and $\mathrm{Re}(a-b) > 0$. Laplace's method for finding solutions involves writing

$$y(x) = \int_C e^{xt} f(t)\, dt$$

for some suitable contour C and some suitable function $f(t)$. Determine $f(t)$ for the equation $(*)$ and use a clearly labelled diagram to specify contours C giving two independent solution when x is real in each of the cases $x > 0$ and $x < 0$.

Now let $a = 3$ and $b = 1$. Find explicit expressions for two independent solutions to $(*)$. Find, in addition, a solution $y(x)$ with $y(0) = 1$.

➾ **Solution:** Substituting the given form of $y(x)$ into the equation, we have

$$\int_C \left(xt^2 f + (a-x)tf - bf\right) e^{xt}\, dt = 0 \qquad \Rightarrow \qquad \int_C \left(-\frac{d}{dt}\left((t^2-t)f\right) + (at-b)f\right) e^{xt}\, dt + \left[(t^2-t)fe^{xt}\right]_C = 0,$$

using integration by parts to put all the x-dependence into the Laplace kernel function e^{xt}. Hence we need both:

$$\frac{d}{dt}\left((t^2-t)f\right) = (at-b)f, \qquad \left[(t^2-t)fe^{xt}\right]_C = 0.$$

We use separation of variables to solve the first equation for $f(t)$. We have:

$$(2t-1)f + (t^2-t)f' = (at-b)f \qquad \Rightarrow \qquad \frac{f'}{f} = \frac{(a-2)t-(b-1)}{t(t-1)} \qquad \Rightarrow \qquad \log(f) = \int \frac{(a-2)t-(b-1)}{t(t-1)}\, dt.$$

Performing the integral using partial fractions, we have

$$\int \left(\frac{a-2}{t-1} + (b-1)\left(\frac{1}{t} - \frac{1}{t-1}\right)\right) dt = (a-2)\log(t-1) + (b-1)\left[\log(t) - \log(t-1)\right] + C,$$

Hence we see that:

$$f(t) = At^{b-1}(t-1)^{((a-2)-(b-1))} = At^{b-1}(t-1)^{a-b-1}.$$

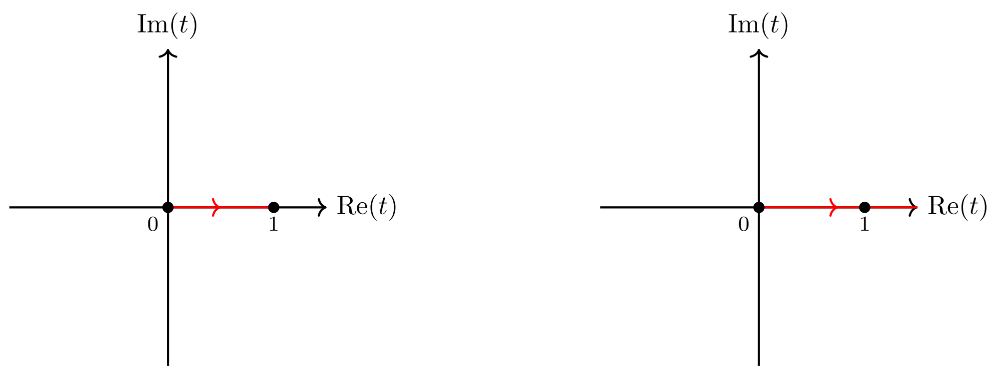
for a constant A. It follows that we can construct a solution of the given equation with:

$$y(x) = A\int_C t^{b-1}(t-1)^{a-b-1}e^{xt}\, dt, \qquad \text{provided } C \text{ is chosen such that } \left[t^b(t-1)^{a-b}e^{xt}\right]_C = 0.$$

We notice that since the integrand contains no poles (since $\mathrm{Re}(b) > 0$ and $\mathrm{Re}(a-b) > 0$), we cannot use a closed contour C. Thus we must use an open contour C. The zeroes of the boundary function occurs at $t = 0$ and $t = 1$. There could also be a zero at infinity, provided we choose the direction to go to infinity in appropriately; in particular, if $x > 0$, we could choose to take $t \to \infty$ along the negative real axis, and if $x < 0$, we could choose to take $t \to \infty$ along the positive real axis.

With this information, we can construct suitable contours. Sketches are given on the following page.

Above: *Two contours generating linearly independent solutions in the case $x > 0$. One goes from 0 to 1, and the other from 0 to $-\infty$.*



Above: *Two contours generating linearly independent solutions in the case $x < 0$. One goes from 0 to 1, and the other from 0 to ∞.*

Let's evaluate the integral solution in the cases $x > 0, x < 0$ separately. We have:

- For $x > 0$, two solutions are given by:

$$y_1(x) = \int_0^1 t^{b-1}(t-1)^{a-b-1}e^{xt}\,dt, \qquad y_2(x) = \int_0^\infty t^{b-1}(t+1)^{a-b-1}e^{-xt}\,dt,$$

  where we have removed some arbitrary multiplicative constants (namely a factor of $-(-1)^{b-1}(-1)^{a-b-1}$, in some branch choice) from the second solution to make it look a little tidier.

In the case that $a = 3$ and $b = 1$, these solutions reduce to:

$$y_1(x) = \int_0^1 (t-1)e^{xt}\,dt = \left[\frac{(t-1)e^{xt}}{x}\right]_0^1 - \int_0^1 \frac{e^{xt}}{x}\,dt = \frac{1}{x} - \frac{e^x}{x^2} + \frac{1}{x^2},$$

$$y_2(x) = \int_0^\infty (t+1)e^{-xt}\,dt = \left[-\frac{(t+1)e^{-xt}}{x}\right]_0^\infty + \int_0^\infty \frac{e^{-xt}}{x}\,dt = \frac{1}{x} + \frac{1}{x^2}$$

using integration by parts. Thinking about combining these solutions in a suitable way, we see that the general solution for $x > 0$ can be written in the form:

$$y(x) = A\left(\frac{1}{x} + \frac{1}{x^2}\right) + \frac{Be^x}{x^2}.$$

for constants A and B.

- For $x < 0$, two solutions are given by:

$$y_1(x) = \int_0^1 t^{b-1}(t-1)^{a-b-1}e^{xt}\,dt, \qquad y_2(x) = \int_0^\infty t^{b-1}(t-1)^{a-b-1}e^{xt}\,dt$$

In the case that $a = 3$ and $b = 1$, these solutions reduce to:

$$y_1(x) = \int_0^1 (t-1)e^{xt}\,dt = \frac{1}{x} - \frac{e^x}{x^2} + \frac{1}{x^2},$$

$$y_2(x) = \int_0^\infty (t-1)e^{xt}\,dt = \left[\frac{(t-1)e^{xt}}{x}\right]_0^\infty - \int_0^\infty \frac{e^{xt}}{x}\,dt = \frac{1}{x} + \frac{1}{x^2},$$

again using integration by parts (and noticing that our integrals are essentially the same as those we've already done). Thus the general solution for $x < 0$ can be written in the form:

$$y(x) = C\left(\frac{1}{x} + \frac{1}{x^2}\right) + \frac{De^x}{x^2},$$

exactly as for $x > 0$.

We now wish to find a solution that has $y(0) = 1$. There may be issues with continuity at $x = 0$, so the coefficients A, B could jump as they turn into C, D, but fortunately it turns out we don't actually have to worry about that here. Simply expanding the exponential e^x near $x = 0$, we see that:

$$y(x) = A\left(\frac{1}{x} + \frac{1}{x^2}\right) + \frac{Be^x}{x^2} = \frac{A}{x} + \frac{A}{x^2} + \frac{B}{x^2} + \frac{B}{x} + \frac{1}{2}B + O(x).$$

So choosing $B = 2$ and $A = -2$ gives $y(0) = 1$ as required. Thus we have the required solution:

$$y(x) = \frac{2(e^x - x - 1)}{x^2}.$$

2015, Paper 2, Section II, 11B

The Riemann zeta function is defined by the sum

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

which converges for $\mathrm{Re}(s) > 1$. Show that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt, \qquad \mathrm{Re}(s) > 1. \tag{$*$}$$

The analytic continuation of $\zeta(s)$ is given by the Hankel contour integral

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{(0+)} \frac{t^{s-1}}{e^{-t} - 1}\, dt.$$

Verify that this agrees with the integral $(*)$ above when $\mathrm{Re}(s) > 1$ and s is not an integer. [You may assume $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$.] What happens when $s = 2, 3, 4, ...$?

Evaluate $\zeta(0)$. Show that $(e^{-t} - 1)^{-1} + \frac{1}{2}$ is an odd function of t and hence, or otherwise, show that $\zeta(-2n) = 0$ for any positive integer n.

Solution: The first part of this question is bookwork from lectures (the derivation is reproduced in Question 9 of Examples Sheet 2 if you need a reminder). We recall that the gamma function has an integral representation for $\mathrm{Re}(s) > 0$ given by:

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t}\, dt.$$

Make the substitution $t = nx$ in the integral. The measure changes as $dt = ndx$ and the limits change as $[0, \infty] \mapsto [0, \infty]$, so we're left with:

$$\Gamma(s) = n^s \int_0^{\infty} x^{s-1} e^{-nx}\, dx \qquad \Rightarrow \qquad \frac{\Gamma(s)}{n^s} = \int_0^{\infty} x^{s-1} e^{-nx}\, dx.$$

Now sum both sides from $n = 1$ to $n = \infty$. On the left hand side, we have:

$$\sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} = \Gamma(s)\zeta(s),$$

provided $\mathrm{Re}(s) > 1$, which is where the series expression for the zeta function is valid. On the right hand side, we have:

$$\sum_{n=1}^{\infty} \int_0^{\infty} x^{s-1} e^{-nx}\, dx = \int_0^{\infty} x^{s-1} \sum_{n=1}^{\infty} e^{-nx}\, dx = \int_0^{\infty} \frac{x^{s-1} e^{-x}}{1 - e^{-x}}\, dx = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx,$$

using the sum of a geometric progression; the sum to infinity of the progression exists since $e^{-x} < 1$ for all $x > 0$, which is where the integration is taking place. Thus we're left with the identity:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx,$$

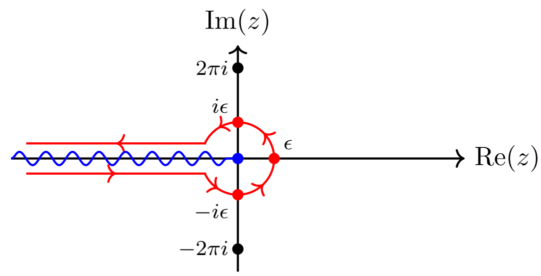
which holds for all $\mathrm{Re}(s) > 1$, as required.

We are now asked to verify that this expression agrees with the Hankel contour integral

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{(0+)} \frac{t^{s-1}}{e^{-t}-1}\, dt.$$

when $\mathrm{Re}(s) > 1$. To do so, we simply evaluate the Hankel contour integral. This whole derivation is discussed in much more detail in my solution to Question 7 on Examples Sheet 2.

Recall that the Hankel contour can be deformed to look like:



With this picture of the contour, we have the contributions to $\zeta(s)$ from the straight segments:

$$\frac{\Gamma(1-s)}{2\pi i} \int_{\infty}^{\epsilon} \frac{e^{-i\pi(s-1)}x^{s-1}}{e^x - 1} e^{-i\pi}\, dx + \frac{\Gamma(1-s)}{2\pi i} \int_{\epsilon}^{\infty} \frac{e^{i\pi(s-1)}x^{s-1}}{e^x-1} e^{i\pi}\, dx = \frac{\Gamma(1-s)}{2\pi i} \left(e^{i\pi s} - e^{-i\pi s}\right) \int_{\epsilon}^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx$$

$$= \frac{\Gamma(1-s)\sin(\pi s)}{\pi} \int_{\epsilon}^{\infty} \frac{x^{s-1}}{e^x-1}\, dx = \frac{1}{\Gamma(s)} \int_{\epsilon}^{\infty} \frac{x^{s-1}}{e^x-1}\, dx,$$

where in the last step, we used the reflection formula for the gamma function, $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$.

From the small circular part of the Hankel contour, we have the contribution:

$$\frac{\Gamma(1-s)}{2\pi i} \int_{-\pi}^{\pi} \frac{\epsilon^{s-1}e^{i\theta(s-1)}}{e^{\epsilon e^{i\theta}} - 1} i\epsilon e^{i\theta}\, d\theta = \frac{\Gamma(1-s)}{2\pi} \int_{-\pi}^{\pi} \frac{\epsilon^s e^{i\theta s}}{1 + \epsilon e^{i\theta} + O(\epsilon^2) - 1}\, d\theta = \frac{\Gamma(1-s)}{2\pi} \int_{-\pi}^{\pi} \frac{\epsilon^{s-1}e^{i\theta s}}{1 + O(\epsilon)}\, d\theta = O(\epsilon^{s-1}).$$

Hence this contribution tends to zero as $\epsilon \to 0$, provided that $\mathrm{Re}(s) > 1$. It follows that if we deform the Hankel contour via $\epsilon \to 0$, we arrive at the formula:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x-1}\, dx$$

for the zeta function, provided $\mathrm{Re}(s) > 1$, as required.

At $s = 2, 3, 4, \cdots$, the Hankel contour representation

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{(0+)} \frac{t^{s-1}}{e^{-t}-1}\, dt$$

looks like it fails, since $\Gamma(1-s)$ diverges at these points. But actually there is no problem - these singularities are actually *removable*. The reason is that $\Gamma(1-s)$ suffers only a simple pole at the points, and the integral has *zeroes* at these points which cancel it out. We can see this by using the residue theorem when $s = 2, 3, 4, \ldots$:

$$\int_{-\infty}^{(0+)} \frac{t^{s-1}}{e^{-t}-1}\, dt = 2\pi i \mathrm{Res}\left(\frac{t^{s-1}}{e^{-t}-1}; 0\right).$$

Examining the integrand closely near $t = 0$, we have

$$\frac{t^{s-1}}{e^{-t}-1} = \frac{t^{s-1}}{1 - t + O(t^2) - 1} = -t^{s-2} + O(t^{s-1}),$$

so we get zero residue for $s = 2, 3, \ldots$, since the pole term $1/t$ vanishes from the expansion.

We are now asked to evaluate $\zeta(0)$. We use the Hankel contour expression and the residue theorem to write:

$$\zeta(0) = \frac{\Gamma(1)}{2\pi i} \int_{-\infty}^{(0+)} \frac{t^{-1}}{e^{-t}-1}\, dt = \mathrm{Res}\left(\frac{1}{t(e^{-t}-1)}; 0\right)$$

To find the residue, note that

$$\frac{1}{t(e^{-t}-1)} = \frac{1}{t}\left(\frac{1}{1 - t + \frac{1}{2}t^2 + O(t^3) - 1}\right) = \frac{1}{t}\left(-t + \frac{1}{2}t^2 + O(t^3)\right)^{-1}$$

$$= -\frac{1}{t^2}\left(1 - \frac{1}{2}t + O(t^2)\right)^{-1} = -\frac{1}{t^2}\left(1 + \frac{1}{2}t + O(t^2)\right),$$

using the binomial theorem. Reading off the coefficient of $1/t$, we find it is $\zeta(0) = -1/2$.

Finally, we are asked to show that $\zeta(-2n) = 0$ for all positive integers n. Again, using the Hankel contour representation, we have:

$$\zeta(-2n) = \frac{\Gamma(2n+1)}{2\pi i} \int_{-\infty}^{(0+)} \frac{t^{-2n-1}}{e^{-t}-1}\, dt = (2n)!\,\mathrm{Res}\left(\frac{1}{t^{2n+1}(e^{-t}-1)}; 0\right).$$

The residue is the same as the coefficient of $1/t$ in the expansion of

$$\frac{1}{t^{2n+1}(e^{-t}-1)} = \frac{1}{t^{2n+1}}\left(\frac{1}{e^{-t}-1} + \frac{1}{2}\right) - \frac{1}{2t^{2n+1}}.$$

For $n \geq 1$, we can forget about the last term $-1/2t^{2n+1}$ as it doesn't contribute to the coefficient of $1/t$. In the remaining term, we notice that $1/t^{2n+1}$ multiplies an odd function:

$$\frac{1}{e^t - 1} + \frac{1}{2} = \frac{1 + e^t}{2(e^t - 1)} = \frac{e^{-t} + 1}{2(1 - e^{-t})} = \frac{e^{-t} - 1 + 2}{2(1 - e^{-t})} = -\frac{1}{2} - \frac{1}{e^{-t} - 1}.$$

This function can only have odd powers of t in its Laurent series expansion, and hence when multiplied by $1/t^{2n+1}$ gives only even powers of t in the total Laurent expansion. Hence the coefficient of $1/t$ in the expansion is zero, and it follows that $\zeta(-2n) = 0$ as required.

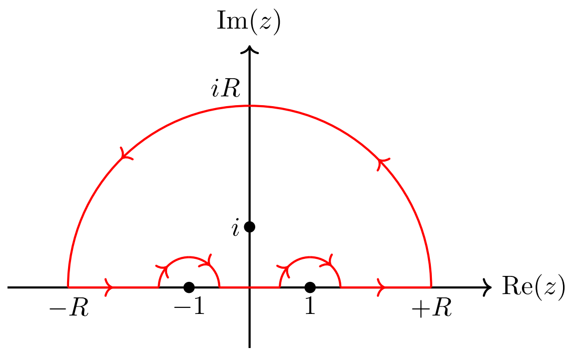
2016, Paper 1, Section I, 7A

Evaluate the integral

$$f(p) = \mathcal{P} \int_{-\infty}^{\infty} \frac{e^{ipx}}{x^4 - 1}\, dx,$$

where p is a real number, for (i) $p > 0$ and (ii) $p < 0$.

➠ **Solution:** (i) Consider the contour integral $\oint_C \frac{e^{ipz}}{z^4 - 1}\, dz$ around the contour C shown:



We consider the small semicircular parts of the contour to be of radius ϵ. The contour contains a single singularity of the integrand at $z = i$, hence by the residue theorem we have:

$$\oint_C \frac{e^{ipz}}{z^4 - 1}\, dz = 2\pi i \text{Res}\left(\frac{e^{ipz}}{z^4 - 1}; i\right) = 2\pi i \lim_{z \to i}\left[\frac{e^{ipz}}{(z^2 - 1)(z + i)}\right] = -\frac{\pi e^{-p}}{2}.$$

Now considering evaluating the contour integral on each section of the contour separately. From the straight segments, as $\epsilon \to 0$ and $R \to \infty$, we recover the principle value integral. From the large semicircular arc, we get zero contribution as $R \to \infty$ by Jordan's lemma. This leaves the small semicircular arcs, which give the contributions:

$$\int_{\pi}^{0} \frac{e^{ip(-1+\epsilon e^{i\theta})}}{(-1 + \epsilon e^{i\theta})^4 - 1} i\epsilon e^{i\theta}\, d\theta + \int_{\pi}^{0} \frac{e^{ip(1+\epsilon e^{i\theta})}}{(1 + \epsilon e^{i\theta})^4 - 1} i\epsilon e^{i\theta}\, d\theta.$$

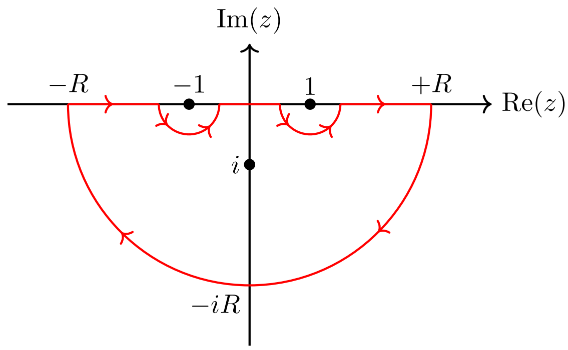
Expanding near $\epsilon = 0$, we arrive at the contributions:

$$-e^{-ip}\int_{0}^{\pi} \frac{e^{ip\epsilon e^{i\theta}}}{-4\epsilon e^{i\theta} + O(\epsilon^2)} i\epsilon e^{i\theta}\, d\theta - e^{ip}\int_{0}^{\pi} \frac{e^{ip\epsilon e^{i\theta}}}{4\epsilon e^{i\theta} + O(\epsilon^2)} i\epsilon e^{i\theta}\, d\theta \to \frac{\pi i e^{-ip}}{4} - \frac{\pi i e^{ip}}{4} = \frac{\pi}{2}\sin(p),$$

as $\epsilon \to 0$. Putting everything together then, we have in the limit as $\epsilon \to 0$ and $R \to \infty$,

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ipx}}{x^4 - 1}\, dx = -\frac{\pi}{2}\sin(p) - \frac{\pi e^{-p}}{2}.$$

(ii) The argument is essentially the same in the case $p < 0$, but we have to flip the contour upside down to use Jordan's lemma this time. Hence we are considering the integral $\oint\limits_C \frac{e^{ipz}}{z^4-1}\,dz$ where the contour C is this time given by:



By the residue theorem, we have (note the contour is traverse clockwise):

$$\oint\limits_C \frac{e^{ipz}}{z^4-1}\,dz = -2\pi i \mathrm{Res}\left(\frac{e^{ipz}}{z^4-1}; -i\right) = -2\pi i \lim_{z\to -i}\left[\frac{e^{ipz}}{(z^2-1)(z-i)}\right] = -\frac{\pi e^p}{2}.$$

Consider evaluating the integral on each section of the contour separately. On the large semicircular part, we get zero contribution as $R \to \infty$ by Jordan's lemma. From the straight segments, we get the principal value integral. Finally, on the small semicircular parts we get:

$$\int\limits_{-\pi}^{0} \frac{e^{ip(-1+\epsilon e^{i\theta})}}{(-1+\epsilon e^{i\theta})^4-1} i\epsilon e^{i\theta}\,d\theta + \int\limits_{-\pi}^{0} \frac{e^{ip(1+\epsilon e^{i\theta})}}{(1+\epsilon e^{i\theta})^4-1} i\epsilon e^{i\theta}\,d\theta.$$

So we see that we just need to map $\pi \mapsto -\pi$ in the previous result we had for the contribution from the small semicircular parts of the contour. The result is:

$$\mathcal{P}\int\limits_{-\infty}^{\infty} \frac{e^{ipx}}{x^4-1}\,dx = \frac{\pi}{2}\sin(p) - \frac{\pi e^p}{2}.$$

Putting everything together, we have the general result for all $p > 0$ or $p < 0$:

$$\mathcal{P}\int\limits_{-\infty}^{\infty} \frac{e^{ipx}}{x^4-1}\,dx = -\frac{\pi}{2}\sin(|p|) - \frac{\pi e^{-|p|}}{2}.$$

2016, Paper 2, Section I, 7A

The Euler product formula for the gamma function is

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)...(z+n)}.$$

Use this to show that

$$\frac{\Gamma(2z)}{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})} = c,$$

where c is a constant, independent of z. Find the value of c.

Solution: Recall from the Examples Sheet that it's useful to use *Stirling's formula*, which states

$$\frac{\sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}}{n!} \to 1.$$

With this formula, we can 'cancel off' the $n!$ in the Euler product formula, which helps to clear things up a bit later on. We do so by defining *partial products* for the gamma function by:

$$\Gamma_n(z) = \underbrace{\frac{\sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}}{n!}}_{\to 1 \text{ as } n\to\infty} \cdot \underbrace{\frac{n!n^z}{z(z+1)...(z+n)}}_{\to\Gamma(z) \text{ as } n\to\infty} = \frac{\sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}n^z}{z(z+1)...(z+n)}.$$

We note that $\Gamma_n(z) \to \Gamma(z)$ as $n \to \infty$. Hence to evaluate the given quotient, we consider:

$$\begin{aligned}
&\frac{\Gamma_{2n}(2z)}{2^{2z}\Gamma_n(z)\Gamma_n(z+\frac{1}{2})} \\
&= \frac{\sqrt{2\pi}e^{-2n}(2n)^{2n+\frac{1}{2}}(2n)^{2z}}{2^{2z}(\sqrt{2\pi})^2 e^{-n}n^{n+\frac{1}{2}}n^z \cdot e^{-n}n^{n+\frac{1}{2}}n^{z+1/2}} \cdot \frac{z(z+1)...(z+n)\cdot(z+1/2)(z+3/2)...(z+(2n+1)/2)}{(2z)(2z+1)...(2z+2n)} \\
&= \frac{2^{2n}}{\sqrt{\pi}} \cdot \frac{1}{n} \cdot \frac{z(z+1)...(z+n)\cdot(z+1/2)(z+3/2)...(z+n+1/2)}{2^{2n+1}z(z+1/2)(z+1)...(z+n-1/2)(z+n)} \\
&= \frac{1}{2\sqrt{\pi}} \cdot \frac{z+n+1/2}{n}.
\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\frac{\Gamma(2z)}{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})} = \frac{1}{2\sqrt{\pi}},$$

which is a constant independent of z. We have also found the value of the constant.

2016, Paper 3, Section I, 7A

The functions $f(x)$ and $g(x)$ have Laplace transforms $F(p)$ and $G(p)$ respectively, and $f(x) = g(x) = 0$ for $x \le 0$. The *convolution* $h(x)$ of $f(x)$ and $g(x)$ is defined by

$$h(x) = \int_0^x f(y)g(x-y)\,dy \quad \text{for } x < 0, \quad \text{and } h(x) = 0 \text{ for } x \le 0.$$

Express the Laplace transform $H(p)$ of $h(x)$ in terms of $F(p)$ and $G(p)$.

Now suppose that $f(x) = x^\alpha$ and $g(x) = x^\beta$ for $x > 0$, where $\alpha, \beta > -1$. Find expressions for $F(p)$ and $G(p)$ using a standard integral formula for the gamma function. Find an expression for $h(x)$ by using a standard integral formula for the beta function. Hence deduce that

$$\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = B(z,w)$$

for all $\mathrm{Re}(z) > 0, \mathrm{Re}(w) > 0$.

➫ **Solution:** The Laplace transform of the convolution is given by

$$H(p) = \int_0^\infty \left(\int_0^x f(y)g(x-y)\,dy \right) e^{-px}\,dx.$$

We now change the order of integration. The original region of integration is $0 \le y \le x$ and $0 \le x \le \infty$; if we integrate with respect to x first, the relevant limits of the inner integral will be $y \le x \le \infty$, and on the outer limit they will be $0 \le y \le \infty$. Thus we have:

$$H(p) = \int_0^\infty \left(\int_y^\infty e^{-px} g(x-y)\,dx \right) f(y)\,dy.$$

Changing variables to $x = y + u$ in the inner integral, the measure changes as $dx = du$ and the limits change as $[y,\infty) \mapsto [0,\infty)$. Hence we obtain:

$$H(p) = \int_0^\infty \left(\int_0^\infty e^{-p(u+y)} g(u)\,du \right) f(y)\,dy = \left(\int_0^\infty e^{-py} f(y)\,dy \right) \left(\int_0^\infty e^{-pu} g(u)\,du \right) = F(p)G(p).$$

That is, the Laplace transform of a convolution is the product of the Laplace transforms.

If $f(x) = x^\alpha$, then the Laplace transform $F(p)$ is given by:

$$F(p) = \int_0^\infty x^\alpha e^{-px}\,dx.$$

Make the substitution $u = px$. Then the measure changes as $dx = du/p$ and the limits change as $[0,\infty) \mapsto [0,\infty)$. Thus we're left with:

$$F(p) = \frac{1}{p^{\alpha+1}} \int_0^\infty u^{(\alpha+1)-1} e^{-u}\,du = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}},$$

where we could use the integral formula for the gamma function in the final step since we are given $\alpha > -1 \Rightarrow \alpha + 1 > 0$.

Similarly, we have

$$G(p) = \frac{\Gamma(\beta+1)}{p^{\beta+1}}.$$

The Laplace transform $H(p)$ can be evaluated in two ways - using the product $F(p)G(p)$, or directly from the definition. If we use the definition directly then, we obtain:

$$H(p) = \int_0^\infty e^{-px}\left(\int_0^x y^\alpha (x-y)^\beta \, dy\right) dx.$$

We can rescale the inner integral to get something with limits $[0,1]$ (like the beta function integral) by letting $u = y/x$. Then the measure changes as $du = dy/x$ and the limits change as $[0,x] \mapsto [0,1]$. Thus we're left with:

$$H(p) = \int_0^\infty e^{-px}\left(x^{\alpha+\beta+1}\int_0^1 u^\alpha (1-u)^\beta \, du\right) dx$$

$$= \left(\int_0^1 u^{(\alpha+1)-1}(1-u)^{(\beta+1)-1} \, du\right)\left(\int_0^\infty x^{(\alpha+1)+(\beta+1)-1}e^{-px} \, dx\right) = \frac{B(\alpha+1,\beta+1)\Gamma(\alpha+1+\beta+1)}{p^{\alpha+\beta+2}}.$$

where in the last step we could use the integral formulae for the beta and gamma functions because $\alpha+1 > 0$ and $\beta+1 > 0$, so in particular for the gamma function integral, $\alpha+\beta+2 > 0$.

Alternatively, we know that $H(p) = F(p)G(p)$ by the result from the first part of this question. Thus we have:

$$B(\alpha+1,\beta+1)\Gamma(\alpha+1+\beta+1) = \Gamma(\alpha+1)\Gamma(\beta+1) \qquad \Rightarrow \qquad B(\alpha+1,\beta+1) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+1+\beta+1)}.$$

for all $\alpha+1 > 0$ and $\beta+1 > 0$.

Now notice that in fact our proof did not rely on α, β being real. The exact same manipulations go through if $\mathrm{Re}(\alpha+1) > 0$, $\mathrm{Re}(\beta+1) > 0$ are assumed instead. Replacing $z = \alpha+1$ and $w = \beta+1$, we see that

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},$$

for all $z, w \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0, \mathrm{Re}(w) > 0$ as required.

2016, Paper 4, Section I, 7A

Consider the equation for $w(z)$:

$$w'' + p(z)w' + q(z)w = 0. \tag{$*$}$$

State necessary and sufficient conditions on $p(z)$ and $q(z)$ for $z = 0$ to be (i) an *ordinary point* or (ii) a *regular singular point*. Derive the corresponding conditions for the point $z = \infty$.

Determine the most general equation of the form $(*)$ that has regular singular points at $z = 0$ and $z = \infty$, with all other points being ordinary.

Solution: The point $z = 0$ is called an *ordinary point* of the given differential equation if $p(z)$ and $q(z)$ are analytic functions at $z = 0$. The point $z = 0$ is called a *regular singular point* of the given differential equation if it is not an ordinary point and if $zp(z)$, $z^2 q(z)$ are both analytic functions at $z = 0$.

To derive the conditions for the point at $z = \infty$, we consider making the transformation $z = 1/t$ in the differential equation and investigating the behaviour near $t = 0$. The derivative transforms as:

$$\frac{d}{dz} = \frac{dt}{dz}\frac{d}{dt} = -\frac{1}{z^2}\frac{d}{dt} = -t^2\frac{d}{dt}.$$

Substituting into the differential equation, we have

$$t^2\frac{d}{dt}\left(t^2\frac{dw}{dt}\right) - t^2 p\left(\frac{1}{t}\right)\frac{dw}{dt} + q\left(\frac{1}{t}\right)w = 0.$$

Simplifying and putting the equation into the canonical form, we have

$$\frac{d^2w}{dt^2} + \left(\frac{2}{t} - \frac{p(1/t)}{t^2}\right)\frac{dw}{dt} + \frac{q(1/t)w}{t^4} = 0.$$

We see that:

- $t = 0$ is an ordinary point of this equation if and only if

  $$\frac{2}{t} - \frac{p(1/t)}{t^2} = O(1), \qquad \frac{q(1/t)}{t^4} = O(1)$$

  as $t \to 0$ (i.e. the coefficient functions are analytic as $t \to 0$). Rearranging we see that these conditions can be written as:

  $$p\left(\frac{1}{t}\right) = 2t + O\left(t^2\right), \qquad q\left(\frac{1}{t}\right) = O\left(t^4\right),$$

  as $t \to 0$. Recasting these conditions in terms of z, we see that $z = \infty$ is an ordinary point of the original equation if and only if we have

  $$p(z) = \frac{2}{z} + O\left(\frac{1}{z^2}\right), \qquad q(z) = O\left(\frac{1}{z^4}\right)$$

  as $z \to \infty$.

- Similarly, $t = 0$ is a regular singular point of the equation if and only if it is not ordinary and

  $$2 - \frac{p(1/t)}{t} = O(1), \qquad \frac{q(1/t)}{t^2} = O(1)$$

  as $t \to 0$. Similarly, we can rearrange and recast in terms of z to get the condition that $z = \infty$ is a regular singular point of the original equation if and only if it is not ordinary and we have:

  $$p(z) = O\left(\frac{1}{z}\right), \qquad q(z) = O\left(\frac{1}{z^2}\right)$$

  as $z \to \infty$.

We are now asked to determine the most general equation with exactly two regular singular points at $z = 0$ and $z = \infty$. We note that since $z = 0$ is a regular singular point, we must have:

$$P(z) = zp(z), \qquad Q(z) = z^2 q(z)$$

analytic at $z = 0$. Furthermore, these functions are actually *entire*, since all other finite points in the complex plane are ordinary points of the equation, which implies $p(z)$, $q(z)$ are analytic everywhere except $z = 0$.

Since $z = \infty$ is a regular singular point, we have

$$p(z) = O\left(\frac{1}{z}\right), \qquad q(z) = O\left(\frac{1}{z^2}\right)$$

as $z \to \infty$. In particular, we see that $P(z) \to P$ and $Q(z) \to Q$, where P and Q are constants, as $z \to \infty$. It follows that $P(z)$ and $Q(z)$ are bounded entire functions and hence are constant. Thus we have:

$$p(z) = \frac{P}{z}, \qquad q(z) = \frac{Q}{z^2}.$$

It follows that the most general equation with regular singular points at $z = 0$ and $z = \infty$ is given by:

$$\frac{d^2 w}{dz^2} + \frac{P}{z}\frac{dw}{dz} + \frac{Q}{z^2} w = 0. \tag{$*$}$$

This form of the equation is *necessary* for the equation to have regular singular points at $z = 0$ and $z = \infty$ with all other points ordinary, but is it *sufficient*? Looking at the conditions for $z = 0$ and $z = \infty$ to be ordinary, we see that for this equation to have *precisely* two regular singular points at $z = 0$ and $z = \infty$, we must choose P, Q such that:

- We cannot simultaneously have $P = Q = 0$, else both P/z and Q/z^2 become analytic at $z = 0$, and hence $z = 0$ becomes ordinary.
- We cannot simultaneously have $P = 2, Q = 0$, else the point at $z = \infty$ becomes ordinary.

Thus the equation ($*$) for arbitrary constants P, Q is the most general form of the required equation, provided we do not choose $P = Q = 0$ simultaneously, or $P = 2, Q = 0$ simultaneously.

2016, Paper 1, Section II, 13A

(a) Legendre's equation for $w(z)$ is

$$(z^2-1)w''+2zw'-l(l+1)w=0, \qquad \text{where } l=0,1,2,\dots .$$

Let C be a closed contour. Show by direct substitution that for z within C,

$$\int_C \frac{(t^2-1)^l}{(t-z)^{l+1}}\,dt$$

is a non-trivial solution of Legendre's equation.

(b) Now consider

$$Q_\nu(z)=\frac{1}{4i\sin(\nu\pi)}\int_{C'}\frac{(t^2-1)^\nu}{(t-z)^{\nu+1}}\,dt$$

for real $\nu > -1$ and $\nu \neq 0,1,2,\dots$. The closed contour C' is defined to start at the origin, wind around $t=1$ in an anticlockwise direction, then wind around $t=-1$ in a clockwise direction, then return to the origin, without encircling the point z. Assuming that z does not lie in the real interval $-1\le x\le 1$, show by deforming C' onto this interval that functions $Q_l(z)$ may be defined as limits of $Q_\nu(z)$ with $\nu\to l=0,1,2,\dots$.

Find an explicit expression for $Q_0(z)$ and verify that it satisfies Legendre's equation with $l=0$.

Solution: (a) Let's first try to make sense of the form we've been given. We've been asked to trial the solution

$$w(z)=\int_C \frac{(t^2-1)^l}{(t-z)^{l+1}}\,dt,$$

where C is a closed contour. If we compare this to our notes for the course, we see that this is a solution of the form

$$w(z)=\int_C f(t)(t-z)^{-\alpha}\,dt,$$

so that we are using the *Euler kernel* method to solve the ODE. The Euler kernel comes with an extra parameter α, which here has been set to $\alpha=l+1$. The function that we normally get from solving an ODE has been given to us as $f(t)=(t^2-1)^l$. To check that this works then, we must keep a cool head, and do the normal things we try to do when using this transform method; namely, *we should somehow try to move all our z dependence back into the kernel using integration by parts.*

Let's go ahead and start. First, note we have the derivatives:

$$w(z)=\int_C \frac{(t^2-1)^l}{(t-z)^{l+1}}\,dt, \qquad w'(z)=(l+1)\int_C \frac{(t^2-1)^l}{(t-z)^{l+2}}\,dt, \qquad w''(z)=(l+2)(l+1)\int_C \frac{(t^2-1)^l}{(t-z)^{l+3}}\,dt.$$

Inserting these into the equation, we find that we must show that the following expression is zero:

$$\int_C \left[\frac{(l+1)(l+2)(z^2-1)}{(t-z)^2}+\frac{2z(l+1)}{t-z}-l(l+1)\right]\frac{(t^2-1)^l}{(t-z)^{l+1}}\,dt.$$

As usual, we have factored out the kernel part, $(t-z)^{-l-1}$, and we now try to shuffle all the z dependence in the large bracket into the kernel using integration by parts.

This is easy in the case of the Laplace kernel, for example, since we can just use:

$$ze^{zt} = \frac{d}{dt}\left(e^{zt}\right).$$

to absorb our z's, and then integrate by parts. In the case of the Euler kernel, things are harder. We instead have:

$$\frac{d}{dt}\left(\frac{1}{(t-z)^{\alpha}}\right) = -\frac{\alpha}{(t-z)^{\alpha+1}} \qquad \Rightarrow \qquad \frac{1}{(t-z)^{\alpha+1}} = -\frac{1}{\alpha}\frac{d}{dt}\left(\frac{1}{(t-z)^{\alpha}}\right). \tag{$*$}$$

Thus we see that to absorb our z's, we should instead rewrite everything so that our z's only appear in factors of $t-z$. Trying to do this with the big bracket we are interested in, we have:

$$\begin{aligned}&\frac{(l+1)(l+2)(z^2-1)}{(t-z)^2} + \frac{2z(l+1)}{t-z} - l(l+1)\\ &= \frac{(l+1)(l+2)((t-z)^2 - 2(t-z)t + t^2 - 1)}{(t-z)^2} - \frac{2(t-z)(l+1)}{t-z} + \frac{2t(l+1)}{t-z} - l(l+1),\end{aligned}$$

where whenever we saw a z, we replaced it with $t-z$ (or $z-t$), and then added or subtracted the extra terms off. We've repeated this until all our z's only appear in factors of $(t-z)$, as we desired. Simplifying this expression, we have

$$\begin{aligned}&(l+1)(l+2) - \frac{2t(l+1)(l+2)}{t-z} + \frac{(t^2-1)(l+1)(l+2)}{(t-z)^2} - 2(l+1) + \frac{2t(l+1)}{t-z} - l(l+1)\\ &= \frac{(t^2-1)(l+1)(l+2)}{(t-z)^2} - \frac{2t(l+1)^2}{t-z}.\end{aligned}$$

We are now in a position where we can use integration by parts to absorb all our z's into the kernel function. Inserting our simplified expression for the large bracket back into the integral, we have:

$$\int\limits_C \left[\frac{(t^2-1)(l+1)(l+2)}{(t-z)^2} - \frac{2t(l+1)^2}{t-z}\right]\frac{(t^2-1)^l}{(t-z)^{l+1}}\,dt.$$

Using the identity ($*$), we can rewrite this as:

$$\int\limits_C \left[-(t^2-1)^{l+1}(l+1)\frac{d}{dt}\left(\frac{1}{(t-z)^{l+2}}\right) - \frac{2t(t^2-1)^l(l+1)^2}{(t-z)^{l+2}}\right]\,dt.$$

Using integration by parts on the first term (recall there is no boundary term since C is closed), we're left with:

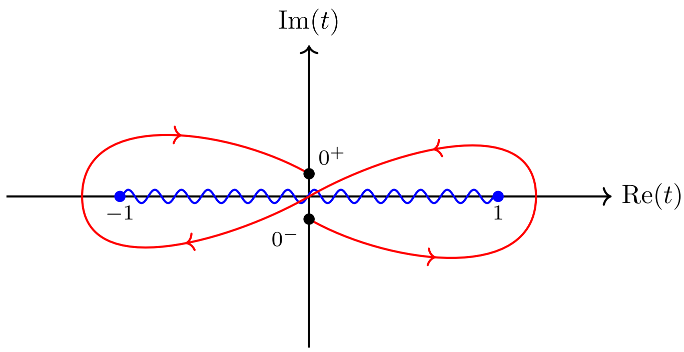
$$\int\limits_C \left[\frac{d}{dt}\left((t^2-1)^{l+1}(l+1)\right) - 2t(t^2-1)^l(l+1)^2\right]\frac{1}{(t-z)^{l+2}}\,dt = 0,$$

where to see the integral is zero, we just carry out the differentiation of the first term. Phew!

(b) Half the battle with this part of the question is drawing the contour! First, we note that the integrand of

$$Q_\nu(z) = \frac{1}{4i\sin(\nu\pi)} \int_{C'} \frac{(t^2-1)^\nu}{(t-z)^{\nu+1}}\, dt$$

has branch points at $t = 1$, $t = -1$ and $t = z$ only. We are told explicitly that our contour does *not* encircle the point z, so we can just ignore the multi-valued behaviour near $t = z$. However, we *must* account for the multi-valuedness around the points $t = 1$ and $t = -1$. We do so by inserting a branch cut along $[-1, 1]$. Then our contour looks like (for example) a figure of eight:



Notice in particular: *the contour crosses the branch cut!* This is rather strange as we haven't had to deal with this during the lectures or the examples sheets. However, there's a fairly nice *intuitive* notion of what should happen when we cross the cut - since we are crossing the cut by encircling the point $t = 1$, there must be a change in the arguments of our branch relative to the point $t = 1$. We don't have to worry about changes to the arguments relative to $t = -1$, as we never fully encircle this point (we don't cross the branch cut a second time).

Before we go ahead then, we should make sure that we are very clear about the branches that we will be using.

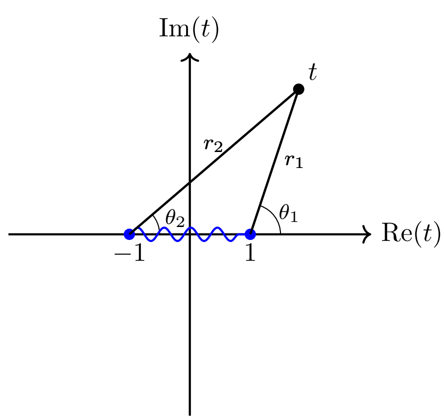
- **Branch 1:** Let's assume that the contour starts at 0^+, which is the origin just *underneath* the branch cut (as in the figure). We will construct the initial branch for $(t^2-1)^\nu$ that we are working on by letting:

$$t - 1 = r_1 e^{i\theta_1}, \qquad t + 1 = r_2 e^{i\theta_2},$$

  which explicitly gives:

$$(t^2-1)^\nu = \left(r_1 r_2 e^{i\theta_1} e^{i\theta_2}\right)^\nu = (r_1 r_2)^\nu e^{i(\theta_1+\theta_2)\nu}.$$

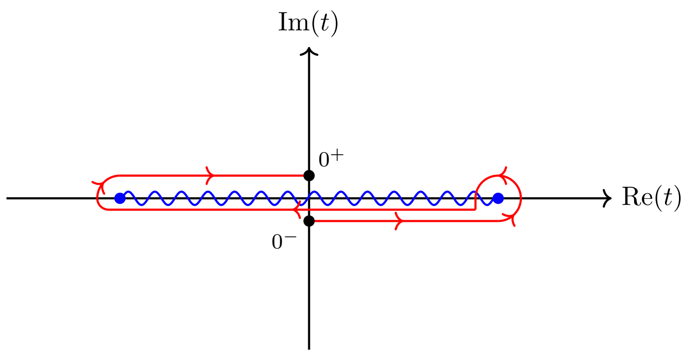
  We then choose the arguments θ_1, θ_2 such that $\theta_1 \in (0, 2\pi)$, $\theta_2 \in (0, 2\pi)$; one can check that this gives a continuous function $(t^2-1)^\nu$ as usual, provided we don't cross the branch cut. The arguments θ_1, θ_2 and moduli r_1, r_2 are represented in the below figure to remind you how this all works:



- **Branch 2:** If we look at the figure-of-eight contour, we see that our contour wraps around the branch point $t = 1$ such that it eventually crosses the branch cut once. When we cross the branch cut, we will move onto a *different branch*. In our setup this corresponds to different ranges of possible θ_1, θ_2.

  As we move through the branch cut, we encircle (in an anticlockwise fashion) the point $t = 1$ exactly, so we must increase the argument relative to the point $t = 1$ by 2π. It follows that **Branch 2** is described by the argument choices $\theta_1 \in (2\pi, 4\pi)$ and $\theta_2 \in (0, 2\pi)$ instead.

With both our branches carefully sorted out, let's now deform the contour to near the branch cut and hence evaluate $Q_\nu(z)$. The deformed contour will look something like this:



Now, let's evaluate the integral on each section of the contour separately:

- Along the first straight section, we are working with **Branch 1**. The relevant choices of arguments are $\theta_1 = \pi$ and $\theta_2 = 2\pi$ (because we are just underneath the branch cut). We notice also that $r_1 = 1 - x$ where x is the distance we have travelled along the straight segment from 0^-; it follows that

$$t = 1 + (1-x)e^{i\pi} = x$$

  for $x \in (0, 1)$ on this section of the contour. Hence the contribution to the integral is:

$$\frac{1}{4i\sin(\pi\nu)} \int_0^1 \frac{(x^2-1)^\nu e^{i(\pi+2\pi)\nu}}{(x-z)^{\nu+1}}\, dx = \frac{e^{3\pi i\nu}}{4i\sin(\pi\nu)} \int_0^1 \frac{(x^2-1)^\nu}{(x-z)^{\nu+1}}\, dx.$$

- The next part of the contour is a small circular piece which remains on **Branch 1**. Here, we have $r_1 = \epsilon, r_2 = 1 + O(\epsilon)$, for some small ϵ. The integrand's measure will also be of order $O(\epsilon)$. In particular this implies that the integrand will be at least of order $O(\epsilon)$ as $\epsilon \to 0$, and hence gives zero contribution.

- We now cross the branch and reach another straight section of the contour. On this section, $\theta_1 = 3\pi$ and $\theta_2 = 2\pi$ since we are now on **Branch 2** where $\theta_1 \in (2\pi, 4\pi)$. We have $r_1 = 1 - x$ as before, and hence $t = x$ along this section of the contour again. It follows that the contribution is:

$$\frac{1}{4i\sin(\pi\nu)} \int_1^{-1} \frac{(x^2-1)^\nu e^{i(3\pi+2\pi)\nu}}{(x-z)^{\nu+1}}\, dx = -\frac{e^{5\pi i\nu}}{4i\sin(\pi\nu)} \int_{-1}^1 \frac{(x^2-1)^\nu}{(x-z)^{\nu+1}}\, dx.$$

- There is now another small semi/quarter circle part of the contour, depending on how you draw it, but again it gives zero contribution as it is of order $O(\epsilon)$ as $\epsilon \to 0$ for some small radius ϵ.

- Finally, there is another straight section of the contour. The arguments here are determined by **Branch 2** and hence $\theta_1 = 3\pi, \theta_2 = 0$. It follows that the contribution to the integral is given by:

$$\frac{1}{4i\sin(\pi\nu)}\int_{-1}^{0}\frac{(x^2-1)^\nu e^{i(3\pi+0)\nu}}{(x-z)^{\nu+1}}\,dx = \frac{e^{3\pi i\nu}}{4i\sin(\pi\nu)}\int_{-1}^{0}\frac{(x^2-1)^\nu}{(x-z)^{\nu+1}}\,dx.$$

Putting everything together, we have found:

$$\begin{aligned}
Q_\nu(z) &= \frac{e^{3\pi i\nu}}{4i\sin(\pi\nu)}\int_{0}^{1}\frac{(x^2-1)^\nu}{(x-z)^{\nu+1}}\,dx - \frac{e^{5\pi i\nu}}{4i\sin(\pi\nu)}\int_{-1}^{1}\frac{(x^2-1)^\nu}{(x-z)^{\nu+1}}\,dx + \frac{e^{3\pi i\nu}}{4i\sin(\pi\nu)}\int_{-1}^{0}\frac{(x^2-1)^\nu}{(x-z)^{\nu+1}}\,dx \\
&= \frac{e^{4\pi i\nu}(e^{-\pi i\nu}-e^{\pi i\nu})}{4i\sin(\pi\nu)}\int_{-1}^{1}\frac{(x^2-1)^\nu}{(x-z)^{\nu+1}}\,dx = -\frac{e^{4\pi i\nu}}{2}\int_{-1}^{1}\frac{(x^2-1)^\nu}{(x-z)^{\nu+1}}\,dx.
\end{aligned}$$

Taking the limit as $\nu \to l$, for l an integer, we have $e^{4\pi i\nu} \to 1$, and hence we're left with:

$$Q_l(z) = -\frac{1}{2}\int_{-1}^{1}\frac{(x^2-1)^l}{(x-z)^{l+1}}\,dx.$$

Finally, we are asked to find $Q_0(z)$ and check that it indeed solves Legendre's equation. We have:

$$Q_0(z) = -\frac{1}{2}\int_{-1}^{1}\frac{1}{x-z}\,dx = -\frac{1}{2}\left[\log(x-z)\right]_{-1}^{1} = -\frac{1}{2}\log\left(\frac{z-1}{z+1}\right) = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right).$$

Differentiating, we have:

$$Q_0'(z) = \frac{1}{2(z+1)} - \frac{1}{2(z-1)} = \frac{1}{1-z^2}, \qquad Q_0''(z) = \frac{2z}{(1-z^2)^2}.$$

Substituting into the left hand side of Legendre's equation with $l = 0$, we have:

$$(z^2-1)Q_0''(z) + 2zQ_0'(z) = -\frac{2z}{1-z^2} + \frac{2z}{1-z^2} = 0,$$

so indeed this solution satisfies Legendre's equation with $l = 0$, as required.

2016, Paper 2, Section II, 12A

The Hurwitz zeta function $\zeta_H(s,q)$ is defined for $\mathrm{Re}(q) > 0$ by

$$\zeta_H(s,q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^s}.$$

State without proof the complex values of s for which this series converges.

Consider the integral

$$I(s,q) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1}e^{qz}}{1-e^z}\, dz,$$

where C is the Hankel contour. Show that $I(s,q)$ provides an analytic continuation of the Hurwitz zeta function for all $s \neq 1$. Include in your account a careful discussion of removable singularities. [*Hint:* $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$.]

Show that $I(s,q)$ has a simple pole at $s = 1$ and find its residue.

➾ **Solution:** The given series for the Hurwitz zeta function, $\zeta_H(s,q)$ converges for $\mathrm{Re}(s) > 1$ (it is of exactly the same form as the Riemann zeta function series). We are now asked to construct the analytic continuation of the Hurwitz zeta function. In analogy with the Riemann zeta function itself, we first construct an integral representation that is valid for $\mathrm{Re}(s) > 1$.

Begin by recalling that the gamma function has the following integral representation for $\mathrm{Re}(s) > 0$:

$$\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}\, dt.$$

Making the substitution $t = (q+n)x$, we see that the measure transforms to $dt = (q+n)dx$ and the limits transform as $[0,\infty) \mapsto [0,\infty)$. We thus see that:

$$\Gamma(s) = (q+n)^s \int_0^{\infty} x^{s-1}e^{-qx}e^{-nx}\, dx.$$

Now divide by $(q+n)^s$ and sum this expression from $n = 0$ to infinity. We see that:

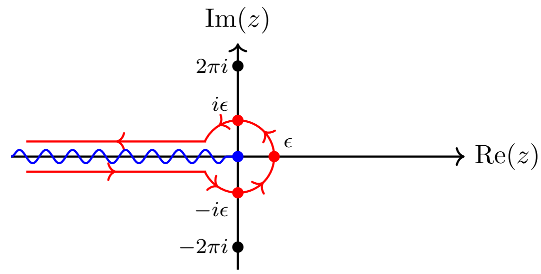
$$\zeta_H(s,q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1}e^{-qx} \left(\sum_{n=0}^{\infty} e^{-nx} \right) dx.$$

where the sum on the left converges only when $\mathrm{Re}(s) > 1$. The sum on the right is a geometric progression with first term 1 and common ratio e^{-x}. Hence performing the sum of the geometric progression, we are left with:

$$\zeta_H(s,q) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}e^{-qx}}{1-e^{-x}}\, dx.$$

This integral formula holds for the Hurwitz zeta function only when $\mathrm{Re}(s) > 1$.

We now show that the given Hankel contour expression provides the analytic continuation of the Hurwitz zeta function. Recall that the Hankel contour C can be deformed to look like:



With this picture of the contour, we see that the straight segments give a contribution to $I(s,q)$:

$$\frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{\epsilon}\frac{e^{-i\pi(s-1)}x^{s-1}e^{-qx}}{1-e^{-x}}e^{-i\pi}dx+\frac{\Gamma(1-s)}{2\pi i}\int_{\epsilon}^{\infty}\frac{e^{i\pi(s-1)}x^{s-1}e^{-qx}}{1-e^{-x}}e^{i\pi}dx$$

$$=\frac{\Gamma(1-s)(e^{i\pi s}-e^{-i\pi s})}{2\pi i}\int_{\epsilon}^{\infty}\frac{x^{s-1}e^{-qx}}{1-e^{-x}}\,dx=\frac{\Gamma(1-s)}{\pi/\sin(\pi s)}\int_{\epsilon}^{\infty}\frac{x^{s-1}e^{-qx}}{1-e^{-x}}\,dx=\frac{1}{\Gamma(s)}\int_{\epsilon}^{\infty}\frac{x^{s-1}e^{-qx}}{1-e^{-x}}\,dx,$$

where in the last step we used the reflection formula for the gamma function. Notice that this recovers the integral formula for the Hurwitz zeta function in the limit as $\epsilon\to 0$, which corresponds to us deforming the Hankel contour close to 0.

The remaining contribution to $I(s,q)$ comes from the circular part of the contour. This gives a contribution:

$$\frac{\Gamma(1-s)}{2\pi i}\int_{-\pi}^{\pi}\frac{\epsilon^{s-1}e^{i\theta(s-1)}e^{q\epsilon e^{i\theta}}}{1-e^{\epsilon e^{i\theta}}}i\epsilon e^{i\theta}\,d\theta=\frac{\Gamma(1-s)}{2\pi i}\int_{-\pi}^{\pi}\frac{\epsilon^{s-1}e^{i\theta(s-1)}e^{q\epsilon e^{i\theta}}}{-e^{i\theta}+O(\epsilon)}i\epsilon e^{i\theta}\,d\theta=O(\epsilon^{s-1}),$$

where we got the second equality by expanding the exponential in the denominator. We see that provided $\mathrm{Re}(s)>1$, when we deform the circular Hankel contour to the origin via $\epsilon\to 0$, the contribution from the circular part of the contour vanishes. We deduce that for $\mathrm{Re}(s)>1$, we have

$$I(s,q)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{x^{s-1}e^{-qx}}{1-e^{-x}}\,dx,$$

and so $I(s,q)=\zeta_H(s,q)$ for $\mathrm{Re}(s)>1$. It follows that $I(s,q)$ is the analytic continuation of the Hurwitz zeta function.

There is one slight subtlety in the argument: $I(s,q)$ appears to be singular at the points $s=2,3,\ldots$ (due to the presence of the $\Gamma(1-s)$ factor) which are included in the range $\mathrm{Re}(s)>1$. As the question suggests, these are in fact *removable singularities*. In this case, the singularities are removable because $\Gamma(1-s)$ has simple poles at all these locations which are cancelled by zeroes from the Hankel contour integral. We can see that the integral has zeroes at the points $s=2,3,\ldots$ by computing its value at these points using the residue theorem:

$$\int_C\frac{z^{s-1}e^{qz}}{1-e^z}\,dz=2\pi i\mathrm{Res}\left(\frac{z^{s-1}e^{qz}}{1-e^z};0\right).$$

The residue can be calculated via series expansion:

$$\frac{z^{s-1}e^{qz}}{1-e^z}=\frac{z^{s-1}e^{qz}}{1-1-z-\frac{1}{2}z^2+O(z^3)}=\frac{z^{s-2}e^{qz}}{-1-\frac{1}{2}z+O(z^2)}.$$

We see that if $s=2,3,4\ldots$, the function is analytic near $z=0$, and hence has residue zero. It follows that indeed the integral has zeroes at these locations that cancel off the simple poles from $\Gamma(1-s)$, leaving a finite result at $s=2,3,\ldots$.

Finally, we are asked to show that $\zeta_H(s,q)$ has a simple pole at $s = 1$ and to find its residue. It is sufficient to attempt to find the residue at $s = 1$ assuming that it is a simple pole; if we find an infinite answer, the pole was in fact of a higher order, and if we get zero as an answer, the singularity at $s = 1$ was in fact removable.

Hence we compute:

$$\lim_{s\to 1}[(s-1)\zeta_H(s,q)] = \lim_{s\to 1}\left[\frac{(s-1)\Gamma(1-s)}{2\pi i}\int\limits_C \frac{z^{s-1}e^{qz}}{1-e^z}\,dz\right].$$

The integral is finite in the limit $s \to 1$ and can as usual be evaluated using the residue theorem. Note the integrand can be expanded near $z = 0$ (the only singularity contained in the contour) as:

$$\frac{e^{qz}}{1-e^z} = \frac{e^{qz}}{1-1-z-\frac{1}{2}z^2+O(z^3)} = -\frac{1}{z}\left(\frac{1+qz}{1+\frac{1}{2}z+O(z^2)}\right) = -\frac{1}{z}+O(1).$$

Hence by the residue theorem we have:

$$\int\limits_C \frac{e^{qz}}{1-e^z}\,dz = -2\pi i.$$

The other factor we need to deal with is the limit of $(s-1)\Gamma(1-s)$ as $s \to 1$. Recall that the residue of the gamma function at $t = 0$ is given by $(-1)^0/0! = 1$, hence we have:

$$\lim_{t\to 0}[t\Gamma(t)] = 1 \qquad \Rightarrow \qquad \lim_{s\to 1}[(1-s)\Gamma(1-s)] = 1 \qquad \Rightarrow \qquad \lim_{s\to 1}[(s-1)\Gamma(1-s)] = -1.$$

Hence putting everything together, we see that:

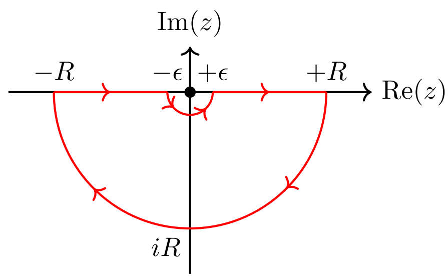
$$\mathrm{Res}\,(\zeta_H(s,q);s) = \frac{(-1)}{2\pi i}\cdot -2\pi i = +1.$$

This implies the Hurwitz zeta function has a simple pole at $s = 1$ with residue $+1$.

2017, Paper 1, Section I, 7E

Calculate the value of the integral $\mathcal{P}\int\limits_{-\infty}^{\infty} \frac{e^{-ix}}{x^n}\,dx$ where $\mathcal{P}$ stands for the principal value and n is a positive integer.

➾ **Solution:** Consider instead the contour integral $\oint\limits_C \frac{e^{-iz}}{z^n}\,dz$, where the contour C is shown below.



Since the contour C encloses no singularities of the integrand, we have by Cauchy's Theorem: $\oint\limits_C \frac{e^{-iz}}{z^n}\,dz = 0.$

Now consider evaluating the integral on each section of the contour separately. We have:

$$0 = \int\limits_{-R}^{-\epsilon} \frac{e^{-ix}}{x^n}\,dx + \int\limits_{-\pi}^{0} \frac{e^{-i\epsilon e^{i\theta}}}{\epsilon^n e^{i\theta n}} i\epsilon e^{i\theta}\,d\theta + \int\limits_{\epsilon}^{R} \frac{e^{-ix}}{x^n}\,dx + \int\limits_{C_R} \frac{e^{-iz}}{z^n}\,dz, \qquad (*)$$

where C_R is large semicircular arc part of the contour, in the lower half plane. Recall Jordan's Lemma states that if $f(z)$ is a function such that $|f(z)| \to 0$ uniformly on C_R as $R \to \infty$, and $\lambda > 0$, the integral of $e^{-i\lambda z}f(z)$ over C_R tends to zero as $R \to \infty$. Taking the limit as $R \to \infty$, $\epsilon \to 0$ in $(*)$ then, and using both Jordan's Lemma (note $n \geq 1$, so it applies) and the definition of the principal value, we find that:

$$\mathcal{P}\int\limits_{-\infty}^{\infty} \frac{e^{-ix}}{x^n}\,dx = -i \lim_{\epsilon \to 0} \int\limits_{-\pi}^{0} \frac{e^{-i\theta(n-1)}e^{-i\epsilon e^{i\theta}}}{\epsilon^{n-1}}\,d\theta.$$

It remains to take the limit of the integral on the right hand side. Expanding the exponential of the order ϵ term, the integral simplifies to:

$$\int\limits_{-\pi}^{0} \frac{e^{-i\theta(n-1)}e^{-i\epsilon e^{i\theta}}}{\epsilon^{n-1}}\,d\theta = \sum_{m=0}^{\infty} \frac{(-i)^m}{m!\,\epsilon^{n-m-1}} \int\limits_{-\pi}^{0} e^{-i\theta(n-m-1)}\,d\theta = \frac{(-i)^{n-1}\pi}{(n-1)!} + \sum_{\substack{m=0 \\ m\neq n-1}}^{\infty} \frac{(-i)^{m-1}\epsilon^{m-(n-1)}}{m!} \cdot \frac{1-(-1)^{n-m-1}}{n-m-1},$$

where we have split the sum up according to the cases $m = n-1$ or $m \neq n-1$. We see that the leftover sum of powers of ϵ *could* contain poles depending on the value of n. If $n = 1$, this doesn't occur - the lowest power of ϵ in the sum is just ϵ^1. If $n = 2$, the lowest power of ϵ in the sum is ϵ^{-1}, so we get a divergence! Similarly for $n \geq 3$, we see that either the $m = 0$ term or $m = 1$ term will arise to an ϵ divergence (one of them can cancel from the factor of $1-(-1)^{n-m-1}$, but the other cannot).

Putting everything together then, we have:

$$\mathcal{P}\int\limits_{-\infty}^{\infty} \frac{e^{-ix}}{x}\,dx = -i\pi, \qquad \mathcal{P}\int\limits_{-\infty}^{\infty} \frac{e^{-ix}}{x^n}\,dx \text{ diverges for } n \geq 2.$$

❄ **Comments from the Examiner:** *A good question. Some students struggled to find the residue, making trivial arithmetic mistakes.*

2017, Paper 2, Section I, 7E

Euler's formula for the gamma function is

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}.$$

Use Euler's formula to show

$$\frac{\Gamma(2z)}{2^{2z}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)} = C$$

where C is a constant. Evaluate C. [*Hint*: You may use $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$.]

Solution: Recall from the Examples Sheet that it's useful to use *Stirling's formula*, which states $\sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}/n! \to 1$. Using this, we define the nth *partial product* for the gamma function via:

$$\Gamma_n(z) = \frac{(1+1/1)^z ... (1+1/n)^z}{z(1+z/1)...(1+z/n)}\cdot\frac{\sqrt{2\pi}e^{-n}n^{n+1/2}}{n!}.$$

which by the Gauss product formula, along with Stirling's formula, obeys $\Gamma_n(z) \to \Gamma(z)$ as $n \to \infty$. Notice that we can simplify this as:

$$\Gamma_n(z) = \frac{(1+1)^z...(n+1)^z}{1^z...n^z\cdot z(z+1)...(z+n)}\cdot\sqrt{2\pi}e^{-n}n^{n+1/2} = \frac{\sqrt{2\pi}e^{-n}n^{n+1/2}(n+1)^z}{z(z+1)...(z+n)}.$$

Now consider the quotient:

$$\frac{\Gamma_{2n}(2z)}{2^{2z}\Gamma_n(z)\Gamma_n(z+1/2)},$$

which converges to $\Gamma(2z)/(2^{2z}\Gamma(z)\Gamma(z+1/2))$ as $n\to\infty$. Writing out each of the terms in this quotient, and simplifying, we have:

$$\frac{(2n)^{2n+1/2}}{\sqrt{2\pi}n^{n+1/2}\cdot n^{n+1/2}}\cdot\frac{(2n+1)^{2z}}{2^{2z}(n+1)^z(n+1)^{z+1/2}}\cdot\frac{z(z+1)...(z+n)(z+\frac{1}{2})(z+\frac{1}{2}+1)...(z+\frac{1}{2}+n)}{(2z)(2z+1)...(2z+2n)}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{n^{1/2}(n+1)^{1/2}}\cdot\left(\frac{n+1/2}{n+1}\right)^{2z}\cdot\frac{2^{2n+1/2}z(z+1)...(z+n)(z+\frac{1}{2})(z+\frac{1}{2}+1)...(z+\frac{1}{2}+n)}{(2z)(2z+1)...(2z+2n)}$$

$$= \frac{1}{4\sqrt{\pi}}\cdot\frac{1}{n^{1/2}(n+1)^{1/2}}\cdot\left(\frac{n+1/2}{n+1}\right)^{2z}\cdot\frac{z(z+1)...(z+n)(2z+1)(2z+3)...(2z+2n+1)}{(2z)(2z+1)...(2z+2n)}$$

$$= \frac{1}{2\sqrt{\pi}}\cdot\frac{(z+n/2+1/2)}{n^{1/2}(n+1)^{1/2}}\cdot\left(\frac{n+1/2}{n+1}\right)^{2z}.$$

Hence we see as $n\to\infty$, we have

$$\frac{\Gamma_{2n}(2z)}{2^{2z}\Gamma_n(z)\Gamma_n(z+1/2)} \to \frac{1}{2\sqrt{\pi}}.$$

This establishes the required identity:

$$\frac{\Gamma(2z)}{2^{2z}\Gamma(z)\Gamma(z+1/2)} = \frac{1}{2\sqrt{\pi}},$$

noting that in particular we have found the value of the constant.

Comments from the Examiner: *This questions was pretty hard. Students either knew the trick (likely mentioned in lectures) or did not do it well. Many skipped the proof and simply found the numeric value of the constant.*

2017, Paper 3, Section I, 7E

Find all the singular points of the differential equation

$$z\frac{d^2y}{dz^2} + (2-z)\frac{dy}{dz} - y = 0$$

and determine whether they are regular or irregular singular points.

By writing $y(z) = f(z)/z$, find two linearly independent solutions to this equation. Comment on the relationship of your solutions to the nature of the singular points of the original differential equation.

➾ **Solution:** First rewrite the differential equation as:

$$y'' + \left(\frac{2}{z} - 1\right) y' - \frac{y}{z} = 0.$$

We see that the coefficient functions are analytic everywhere in the finite complex plane except $z = 0$. Hence all points in the finite complex plane are ordinary except $z = 0$. Noting that

$$z\,(2/z - 1) = 2 - z, \qquad z^2\,(-1/z) = -z$$

are analytic functions at $z = 0$, we see that $z = 0$ is a regular singular point.

For the point at infinity, recall that for an ordinary point, we require the coefficient of y' to be of the form $2/z + O(1/z^2)$ as $z \to \infty$, and for a regular singular point, we require the coefficient of y' to be of the form $O(1/z)$ as $z \to \infty$. Neither condition is satisfied because of the -1 present in the coefficient of y' in this case. Thus the point at infinity is an irregular singular point of this equation.

We now consider making the substitution $y = f(z)/z$ as suggested by the question. We have:

$$y = \frac{f}{z} \qquad \Rightarrow \qquad y' = \frac{f'}{z} - \frac{f}{z^2} \qquad \Rightarrow \qquad y'' = \frac{f''}{z} - \frac{2f'}{z^2} + \frac{2f}{z^3},$$

so substituting into the equation, we have:

$$\frac{f''}{z} - \frac{2f'}{z^2} + \frac{2f}{z^3} + \frac{2f'}{z^2} - \frac{2f}{z^3} - \frac{f'}{z} + \frac{f}{z^2} - \frac{f}{z^2} = 0.$$

Simplifying, this reduces simply to $f'' - f' = 0$. Integrating directly, we get $f' = f + C$ for some constant C. This equation is now separable, with solution:

$$\int \frac{df}{f + C} = \int dz \qquad \Rightarrow \qquad \log(f + C) = z + D \qquad \Rightarrow \qquad f = A + Be^z$$

for constants A and B. Thus the general solution to the original ODE is:

$$y(z) = A/z + Be^z/z.$$

Near $z = 0$, we see we can expand the solution into a Frobenius-style series solution: $(A+B)/z + B + Bz/2 + \cdots$, which is consistent with $z = 0$ being a regular singular point. The solution is *not* a power series (note the $1/z$ term corresponding to an exponent -1 at the RSP), and hence $z = 0$ is not an ordinary point.

Setting $t = 1/z$, we see that the solution near $z = \infty$, i.e. $t = 0$, has an essential singularity due to the $te^{1/t}$ term. In particular, this has no lowest power of $1/t$ in its series expansion, and hence there is no Frobenius-style series solution near $t = 0$, i.e. $z = \infty$ (recall Frobenius series have a lowest power in their expansion, corresponding to an *exponent* of the RSP). This is consistent with $z = \infty$ being irregular singular.

❈ **Comments from the Examiner:** *A relatively easy question, most students did well.*

2017, Paper 4, Section I, 7E

Consider the differential equation

$$z\frac{d^2y}{dz^2} - 2\frac{dy}{dz} + zy = 0. \tag{$*$}$$

Laplace's method finds a solution of this differential equation by writing $y(z)$ in the form

$$y(z) = \int_C e^{zt} f(t)\, dt,$$

where C is a closed contour. Determine $f(t)$. Hence find two linearly independent solutions of ($*$) for z real.

Solution: Inserting $y(z)$ into the equation, we have

$$\int_C \left(zft^2 - 2ft + zf\right) e^{zt}\, dt = 0.$$

Using integration by parts, we can transform this into the equation:

$$\int_C \left(-\frac{d}{dt}\left(f(t^2+1)\right) - 2ft\right) e^{zt}\, dt + \left[f(t^2+1)e^{zt}\right]_C = 0.$$

Hence we see that $f(t)$ must obey the equation:

$$\frac{d}{dt}\left(f(t^2+1)\right) = -2ft \qquad \Rightarrow \qquad \frac{1}{f}\frac{df}{dt} = -\frac{4t}{t^2+1}.$$

This is a separable equation with solution:

$$\log(f) = -2\int \frac{2t}{t^2+1}\, dt = -2\log(t^2+1) + C \qquad \Rightarrow \qquad f(t) = \frac{A}{(t^2+1)^2}.$$

Hence we have a solution of the equation:

$$y(z) = A\int_C \frac{e^{zt}}{(t^2+1)^2}\, dt, \qquad \text{provided the contour } C \text{ obeys } \left[\frac{e^{zt}}{t^2+1}\right]_C = 0.$$

We now wish to explicitly construct two linearly independent solutions. The first thing we check is whether C can be a closed contour; since the integrand contains double poles at $z = i$ and $z = -i$, this is certainly possible. Therefore, let's use C closed and enclosing the pole at $z = i$ only; then by the residue theorem we have:

$$y(z) \propto \operatorname{Res}\left(\frac{e^{zt}}{(t^2+1)^2}; i\right) = \lim_{t\to i}\left[\frac{d}{dt}\left(\frac{e^{zt}}{(t+i)^2}\right)\right] = \frac{ze^{iz}}{(2i)^2} - \frac{2e^{iz}}{(2i)^3} \propto (z+i)e^{iz}.$$

Hence $y(z) = (z+i)e^{iz}$ is a solution of the equation. We could construct another by using a closed contour C enclosing the pole at $z = -i$ only, but in this case we are restricting to the case $z \in \mathbb{R}$; hence we can simply take the real and imaginary parts of the $y(z)$ we have just found, which must individually satisfy the given equation:

$$y_1(z) = \operatorname{Re}(y(z)) = z\cos(z) - \sin(z), \qquad y_2(z) = \operatorname{Im}(y(z)) = \cos(z) + z\sin(z).$$

These are indeed linearly independent functions for $z \in \mathbb{R}$.

Comments from the Examiner: *A good question. Although maybe slightly too hard for Section I. Most students got the right form of f, although a surprisingly large number of them struggled to solve the first order ODE for f correctly. Then about half of the students did not even attempt to take the contour integral, leaving the result in the explicit form. Out of those who took the integral some got the residues wrong. But the main issue was that even when they would get the correct answer they did not try to give the real solution (as they were asked) leaving it in the complex form. I believe only two students did it fully.*

2017, Paper 1, Section II, 13E

The Riemann zeta function is defined by

$$\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s}$$

for $\mathrm{Re}(s) > 1$. Show that

$$\zeta_R(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt.$$

Let $I(s)$ be defined by

$$I(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{t^{s-1}}{e^{-t} - 1}\, dt$$

where C is the Hankel contour. Show that $I(s)$ provides an analytic continuation of $\zeta_R(s)$ for a range of s which should be determined. Hence evaluate $\zeta_R(-1)$.

Solution: The integral representation of the zeta function is bookwork from lectures. We recall that we start from the integral form of the gamma function, which holds for $\mathrm{Re}(s) > 0$:

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t}\, dt.$$

Now make the substitution $t = nx$. The measure changes as $dt = n dx$ and the limits change as $[0, \infty) \mapsto [0, \infty)$. Thus the integral becomes:

$$\Gamma(s) = n^s \int_0^{\infty} x^{s-1} e^{-nx}\, dx \qquad \Rightarrow \qquad \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-nx}\, dx.$$

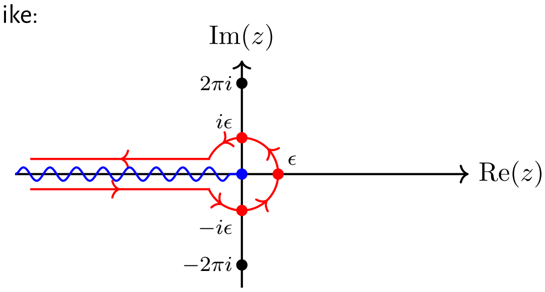
Summing both sides from $n = 1$ to $n = \infty$, we have

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \left(\sum_{n=1}^{\infty} e^{-nx} \right) dx,$$

where we can use the sum representation of the zeta function if we assume $\mathrm{Re}(s) > 1$. The sum inside the integral is a geometric progression with first term e^{-x} and common ratio e^{-x}, so can be summed to give the required result:

$$\zeta_R(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{-x}}{1 - e^{-x}}\, dx = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx.$$

Now we are asked to show that the Hankel contour representation of the zeta function provides its analytic continuation. This is discussed in much more detail in my solution to Examples Sheet 2, Question 7(a). We start by recalling that the Hankel contour can be deformed to look like:



Hence evaluating the given integral $I(s)$, we see that we have the contributions from the straight segments:

$$\frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{\epsilon}\frac{e^{-i\pi(s-1)}x^{s-1}}{e^x-1}\,e^{-i\pi}\,dx+\frac{\Gamma(1-s)}{2\pi i}\int_{\epsilon}^{\infty}\frac{e^{i\pi(s-1)}x^{s-1}}{e^x-1}\,e^{i\pi}\,dx=\frac{\Gamma(1-s)(e^{i\pi s}-e^{-i\pi s})}{2\pi i}\int_{\epsilon}^{\infty}\frac{x^{s-1}}{e^x-1}\,dx$$

$$=\frac{\Gamma(1-s)}{\pi/\sin(\pi s)}\int_{\epsilon}^{\infty}\frac{x^{s-1}}{e^x-1}\,dx=\frac{1}{\Gamma(s)}\int_{\epsilon}^{\infty}\frac{x^{s-1}}{e^x-1}\,dx,$$

where in the last step we used the reflection formula for the gamma function, $\Gamma(s)\Gamma(1-s)=\pi/\sin(\pi s)$. Note we recover the integral representation of the zeta function for $\mathrm{Re}(s)>1$ in the limit as $\epsilon\to 0$. The other contribution to $I(s)$ is given by the circular part of the Hankel contour. We get:

$$\frac{\Gamma(1-s)}{2\pi i}\int_{-\pi}^{\pi}\frac{\epsilon^{s-1}e^{-i\theta(s-1)}}{e^{\epsilon e^{i\theta}}-1}\,i\epsilon e^{i\theta}\,d\theta=\frac{\Gamma(1-s)}{2\pi i}\int_{-\pi}^{\pi}\frac{i\epsilon^s e^{i\theta s}}{1-\epsilon e^{i\theta}+O(\epsilon^2)-1}\,d\theta=O(\epsilon^{s-1}).$$

Hence we see that provided $\mathrm{Re}(s)>1$, we can deform the Hankel contour via $\epsilon\to 0$ and receive zero contribution from the circular arc. Thus putting everything together, we see that if $\mathrm{Re}(s)>1$, we have:

$$I(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x-1}\,dx=\zeta_R(s)$$

In particular, $I(s)$ agrees with $\zeta_R(s)$ in the domain $\mathrm{Re}(s)>1$, and hence it provides the analytic continuation of $\zeta_R(s)$ to the region $\mathrm{Re}(s)\le 1$. Note also that in $I(s)$, the simple poles of $\Gamma(1-s)$ at $s=2,3,4,\ldots$ are precisely cancelled by zeroes from the integral factor in $I(s)$ and hence $I(s)$ is still analytic at these points. We can check that indeed the integral factor has zeroes at $s=2,3,4\ldots$ using the residue theorem:

$$\int_C\frac{t^{s-1}}{e^{-t}-1}\,dt=2\pi i\mathrm{Res}\left(\frac{t^{s-1}}{e^{-t}-1};0\right).$$

The residue can be calculated via series expansion:

$$\frac{t^{s-1}}{e^{-t}-1}=\frac{t^{s-1}}{1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+O(t^4)-1}=-\frac{t^{s-2}}{1-\frac{1}{2}t+\frac{1}{6}t^2+O(t^3)}.$$

We see that if $s=2,3,4...$, the function is analytic near $t=0$, and hence has residue zero, so the integral has a zero at $s=2,3,4...$ exactly cancelling the simple pole from the gamma function.

Finally, we are asked to compute $\zeta_R(-1)$. Using the Hankel contour expression and the residue theorem, we have:

$$\zeta_R(-1)=\frac{\Gamma(1+1)}{2\pi i}\int_C\frac{t^{-2}}{e^{-t}-1}\,dt=\mathrm{Res}\left(\frac{t^{-2}}{e^{-t}-1};0\right).$$

Using the series expansion we wrote down earlier in this question, we can calculate the residue:

$$\frac{t^{-2}}{e^{-t}-1}=-\frac{1}{t^3}\left(1-\frac{1}{2}t+\frac{1}{6}t^2+O(t^3)\right)^{-1}=-\frac{1}{t^3}\left(1+(-1)\left(-\frac{1}{2}t+\frac{1}{6}t^2+\cdots\right)+\frac{(-1)(-2)}{2!}\left(-\frac{1}{2}t+\cdots\right)^2+\cdots\right),$$

where we have expanded the denominator using the binomial theorem. Reading the coefficient of $1/t$ off, we find that $\zeta_R(-1)=(1/6)-(1/4)=-1/12$ (this is of course telling us that $1+2+3+\cdots=-1/12$).

❊ **Comments from the Examiner:** *First part presented no difficulty to most students. In the second part the main issue was the determination of the range of s - very few students had done it correctly. The last evaluation cost many students lost points because of the arithmetic errors, but was quite doable.*

2017, Paper 2, Section II, 12E

The hypergeometric equation is represented by the Papperitz symbol

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix}\right\} \tag{$*$}$$

and has solution $y_0(z) = F(a,b,c;z)$.

Functions $y_1(z)$ and $y_2(z)$ are defined by

$$y_1(z) = F(a,b,a+b+1-c;1-z)$$

and

$$y_2(z) = (1-z)^{c-a-b}F(c-a,c-b,c-a-b+1;1-z),$$

where $c-a-b$ is not an integer. Show that $y_1(z)$ and $y_2(z)$ obey the hypergeometric equation ($*$).

Explain why $y_0(z)$ can be written in the form $y_0(z) = Ay_1(z) + By_2(z)$, where A and B are independent of z, but depend on a, b and c.

Suppose that

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

with $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $|\arg(1-z)| < \pi$. Find expressions for A and B.

Solution: To show that $y_1(z)$, $y_2(z)$ satisfy the hypergeometric equation, we recall the two basic manipulations of the Papperitz symbol:

- For any Möbius transformation M, we have:

$$P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{matrix}\right\} = P\left\{\begin{matrix} M(z_1) & M(z_2) & M(z_3) & \\ \alpha_1 & \beta_1 & \gamma_1 & M(z) \\ \alpha_2 & \beta_2 & \gamma_2 & \end{matrix}\right\}$$

  Thus we notice that under a Möbius transformation of the independent variable in the Papperitz equation, the regular singular points are mapped according to the Möbius transformation, but the exponents are left unchanged.

- We have the *exponent-shifting property* of the Papperitz symbol:

$$\left(\frac{z-z_1}{z-z_2}\right)^k\left(\frac{z-z_2}{z-z_3}\right)^l P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{matrix}\right\} = P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha_1+k & \beta_1+l-k & \gamma_1-l & z \\ \alpha_2+k & \beta_2+l-k & \gamma_2-l & \end{matrix}\right\}.$$

  In our particular case of interest, we will need the limit of this as $z_3 \to \infty$. By introducing a constant z_3^l on the LHS, we do not affect the solutions of the equation with the P-symbol on the left hand side, so we can take the limit as $z_3 \to \infty$ just to get:

$$(z-z_1)^k(z-z_2)^{l-k}P\left\{\begin{matrix} z_1 & z_2 & \infty & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{matrix}\right\} = P\left\{\begin{matrix} z_1 & z_2 & \infty & \\ \alpha_1+k & \beta_1+l-k & \gamma_1-l & z \\ \alpha_2+k & \beta_2+l-k & \gamma_2-l & \end{matrix}\right\}.$$

The given function $y_1(z) = F(a, b, a + b + 1 - c; 1 - z)$ solves the equation with the Papperitz symbol:

$$P \begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & 1-z \\ c-a-b & 1-c & b & \end{Bmatrix}.$$

By applying the Mobius transformation $M(w) = 1-w$, we see that the regular singular points are moved as $0 \mapsto 1, 1 \mapsto 0$, $\infty \mapsto \infty$, and the variable transforms as $M(1-z) = z$. Thus we have:

$$P \begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & 1-z \\ c-a-b & 1-c & b & \end{Bmatrix} = P \begin{Bmatrix} 1 & 0 & \infty & \\ 0 & 0 & a & z \\ c-a-b & 1-c & b & \end{Bmatrix},$$

and it follows that $y_1(z)$ satisfies the hypergeometric equation.

The second given function $y_2(z) = (1-z)^{c-a-b} F(c-a, c-b, c-a-b+1; 1-z)$ solves the equation with the Papperitz symbol:

$$(1-z)^{c-a-b} P \begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & c-a & 1-z \\ a+b-c & 1-c & c-b & \end{Bmatrix}.$$

Using a Möbius transformation, we can move the regular singular points as before:

$$(1-z)^{c-a-b} P \begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & c-a & 1-z \\ a+b-c & 1-c & c-b & \end{Bmatrix} = (1-z)^{c-a-b} P \begin{Bmatrix} 1 & 0 & \infty & \\ 0 & 0 & c-a & z \\ a+b-c & 1-c & c-b & \end{Bmatrix}$$

Now using exponent-shifting, we see we can absorb $(1-z)^{c-a-b}$. We must add $c-a-b$ to the exponents of the regular singular point at $z = 1$ and we must subtract $c-a-b$ from the exponents of the regular singular point at ∞:

$$(1-z)^{c-a-b} P \begin{Bmatrix} 1 & 0 & \infty & \\ 0 & 0 & c-a & z \\ a+b-c & 1-c & c-b & \end{Bmatrix} = P \begin{Bmatrix} 1 & 0 & \infty & \\ c-a-b & 0 & b & z \\ 0 & 1-c & a & \end{Bmatrix}$$

and hence it follows that $y_2(z)$ satisfies the hypergeometric equation, as required (note that the ordering of the columns, and the rows of the exponents, is clearly immaterial in the P-symbol).

The given functions $y_1(z), y_2(z)$ are clearly linearly independent and both satisfy the hypergeometric equation. It follows that any other solution of the hypergeometric equation may be expressed as their linear combination. But clearly $y_0(z) = F(a, b, c; z)$, the hypergeometric function itself, must satisfy the hypergeometric equation and so it follows:

$$y_0(z) = A(a, b, c) y_1(z) + B(a, b, c) y_2(z). \tag{$*$}$$

We can determine $A(a, b, c)$ and $B(a, b, c)$ by evaluating the above equation ($*$) at two locations $z = 0$ and $z = 1$ and solving the resulting simultaneous equations for A and B.

First, note that at $z = 1$, we have:

$$F(a, b, c; 1) = A(a, b, c),$$

since $y_1(1) = 1$ and $y_2(1) = 0$ (by definition the hypergeometric function has the value 1 when evaluated at 0). We see that we need an expression for $F(a, b, c; 1)$; we use the provided integral formula:

$$F(a, b, c; 1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1}\, dt = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} B(b, c-b-a) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)},$$

where we can use the integral representation for the beta function provided $\mathrm{Re}(c-b-a) > 0$ and $\mathrm{Re}(b) > 0$ (which is given). It follows that

$$A(a, b, c) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}.$$

To obtain $B(a, b, c)$, we can use symmetry as per the argument in Examples Sheet 4, Question 8. We evaluate the expression $(*)$ at the point $z = 0$ to obtain:

$$1 = A(a, b, c)F(a, b, a+b+1-c; 1) + B(a, b, c)F(c-a, c-b, c-a-b+1; 1).$$

We now notice that there is a symmetry of the hypergeometric functions here. Making the replacements $a \mapsto c-a$, $b \mapsto c-b$ and $c \mapsto c$, we're left with:

$$1 = A(c-a, c-b, c)F(c-a, c-b, c-a-b+1; 1) + B(c-a, c-b, c)F(a, b, a+b+1-c; 1).$$

Comparing coefficients of hypergeometric functions in these two equations, we find:

$$B(a, b, c) = A(c-a, c-b, c) = \frac{\Gamma(c)\Gamma(c-(c-a)-(c-b))}{\Gamma(c-(c-b))\Gamma(c-(c-a))} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(b)\Gamma(a)}.$$

Hence we have found both the constants A and B, as required.

❊ **Comments from the Examiner:** *A scary looking question. First part was rather easy. In the second part there was a lot of arithmetic issues, although many students still got A correctly. For the B only a few found the compact expression.*

2018, Paper 1, Section I, 7B

The beta and gamma functions are defined by

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \qquad \Gamma(p) = \int_0^\infty e^{-t}t^{p-1}\,dt,$$

where $\mathrm{Re}(p) > 0, \mathrm{Re}(q) > 0$.

(a) By using a suitable substitution, or otherwise, prove that

$$B(z,z) = 2^{1-2z}B\left(z, \frac{1}{2}\right)$$

for $\mathrm{Re}(z) > 0$. Extending B by analytic continuation, for which values of $z \in \mathbb{C}$ does this result hold?

(b) Prove that

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

for $\mathrm{Re}(p) > 0, \mathrm{Re}(q) > 0$.

Solution: (a) We begin by deriving the trigonometric form of the beta function. Let $t = \sin^2(\theta)$ in the given definition of the beta function. Then the measure changes as $dt = 2\sin(\theta)\cos(\theta)d\theta$ and the limits change as $[0,1] \mapsto [0,\pi/2]$. Thus we're left with:

$$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}(\theta)\cos^{2p-1}(\theta)\,d\theta.$$

Let $p = q = z$. Then we have:

$$B(z,z) = 2\int_0^{\pi/2} \sin^{2z-1}(\theta)\cos^{2z-1}(\theta)\,d\theta = \frac{2}{2^{2z-1}}\int_0^{\pi/2} \sin^{2z-1}(2\theta)\,d\theta.$$

We want to return this final integral to beta function form. We're essentially forced to make the substitution $\phi = 2\theta$ then, which causes the measure to changes as $d\phi = 2d\theta$ and the limits to change as $[0,\pi/2] \mapsto [0,\pi]$, so we're left with:

$$B(z,z) = 2^{1-2z}\int_0^{\pi} \sin^{2z-1}(\phi)d\phi = 2^{1-2z}\cdot 2\int_0^{\pi/2} \sin^{2z-1}(\phi)\cos^{2(1/2)-1}(\phi)\,d\phi = 2^{1-2z}B\left(z,\frac{1}{2}\right),$$

where we converted the limits from $[0,\pi]$ to $[0,\pi/2]$ using the fact that sine has reflectional symmetry about the line $\phi = \pi/2$.

We proved this relationship for $\mathrm{Re}(z) > 0$, since this is where the trigonometric form of the beta function is valid. However, it can be extended to all of $z \in \mathbb{C}\backslash\{0,-1,-2,\dots\}$ by analytic continuation. We can see this by comparing with the gamma function expression for the beta function we derive in part (b), for example.

(b) We use the standard proof from lectures. We notice that

$$\Gamma(p)\Gamma(q) = \left(\int_0^\infty e^{-s}s^{p-1}\,ds\right)\left(\int_0^\infty e^{-t}t^{q-1}\,dt\right) = \int_0^\infty\int_0^\infty e^{-s+t}s^{p-1}t^{q-1}dsdt.$$

Making the substitution $s = r\sin^2(\theta)$ and $t = r\cos^2(\theta)$. The Jacobian of this transformation is given by:

$$\frac{\partial(s,t)}{\partial(r,\theta)} = \det\begin{pmatrix} \frac{\partial s}{\partial r} & \frac{\partial s}{\partial \theta} \\ \frac{\partial t}{\partial r} & \frac{\partial t}{\partial \theta}\end{pmatrix} = \det\begin{pmatrix} \sin^2(\theta) & 2r\sin(\theta)\cos(\theta) \\ \cos^2(\theta) & -2r\cos(\theta)\sin(\theta)\end{pmatrix}$$

$$= 2r\cos(\theta)\sin^3(\theta) + 2r\sin(\theta)\cos^3(\theta) = 2r\sin(\theta)\cos(\theta),$$

and the region $[0,\infty)\times[0,\infty)$ in (s,t) space is mapped to the region $[0,\infty)\times[0,\pi/2]$ in (r,θ) space. Hence we're left with:

$$\Gamma(p)\Gamma(q) = 2\int_0^{\pi/2}\int_0^\infty e^{-r}r^{p+q-1}\sin^{2p-1}(\theta)\cos^{2q-1}(\theta)\,drd\theta$$

$$= \left(\int_0^\infty e^{-r}r^{p+q-1}\,dr\right)\left(2\int_0^{\pi/2}\sin^{2p-1}(\theta)\cos^{2q-1}(\theta)\,d\theta\right) = \Gamma(p+q)B(p,q),$$

using the trigonometric formula for the beta function we derived in part (a). Hence we have the required result:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

❊ **Comments from the Examiner:** *This question generally caused little problem. Most students were able to find one of the suitable substitutions and demonstrate the stated results. Some lost points when specifying the values of z for which the first relation holds.*

2018, Paper 2, Section I, 7B

Show that

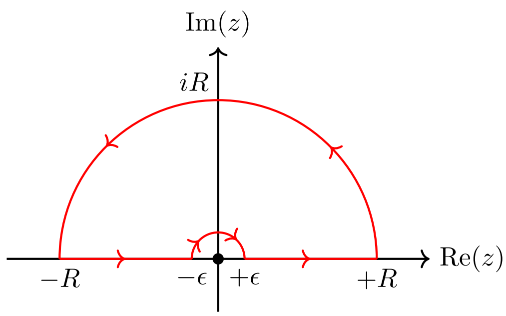
$$\int_{-\infty}^{\infty} \frac{\cos(nx) - \cos(mx)}{x^2} = \pi(m-n),$$

in the sense of Cauchy principal value, where n and m are positive integers. [State clearly any standard results involving contour integrals that you use.]

Solution: Consider instead the contour integral:

$$\oint_C \frac{e^{inz} - e^{imz}}{z^2}\, dz$$

around the contour shown below:



Since the contour encloses no singularities, we have by Cauchy's theorem: $\displaystyle\oint_C \frac{e^{inz} - e^{imz}}{z^2}\, dz = 0.$

Now consider instead evaluating the integral on each section of the contour separately. As $\epsilon \to 0$ and $R \to \infty$, the straight segments from the contour give the desired principal value integral.

For the upper semicircular portion of the contour, recall Jordan's Lemma states that if $f(z)$ is a function such that $|f(z)| \to 0$ uniformly on the upper semicircular part of the contour as $R \to \infty$, and $\lambda > 0$, the integral of $e^{i\lambda z} f(z)$ tends to zero as $R \to \infty$; hence both the e^{inz} and e^{imz} terms in our contour integral give zero contribution from the upper semicircular arc as $R \to \infty$.

The small semicircular arc gives the contribution:

$$\int_{\pi}^{0} \frac{e^{in\epsilon e^{i\theta}} - e^{im\epsilon e^{i\theta}}}{\epsilon^2 e^{2i\theta}} i\epsilon e^{i\theta}\, d\theta = -i\int_{0}^{\pi} \frac{(1 + in\epsilon e^{i\theta} + O(\epsilon^2)) - (1 + im\epsilon e^{i\theta} + O(\epsilon^2)}{\epsilon e^{i\theta}}\, d\theta = \pi(n-m) + O(\epsilon).$$

Hence putting everything together, as $\epsilon \to 0$ and $R \to \infty$ we get the result (after taking real parts):

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{\cos(nx) - \cos(mx)}{x^2}\, dx + \pi(n-m) = 0 \qquad \Rightarrow \qquad \mathcal{P}\int_{-\infty}^{\infty} \frac{\cos(nx) - \cos(mx)}{x^2}\, dx = \pi(m-n),$$

as required.

Comments from the Examiner: *Standard application of the Cauchy principal value method. Having the answer provided to students certainly simplified things a lot. Majority had no issues identifying the appropriate integration contour and taking the integrals.*

2018, Paper 3, Section I, 7B

Using a suitable branch cut, show that

$$\int_{-a}^{a} (a^2 - x^2)^{1/2}\, dx = \frac{a^2\pi}{2},$$

where $a > 0$.

➯ **Solution:** We consider instead the contour integral

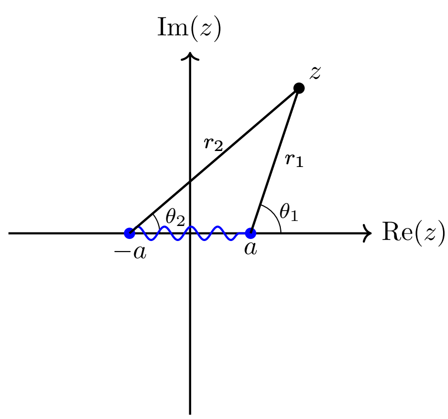
$$\oint_C (a^2 - z^2)^{1/2}\, dz$$

where we will specify the contour shortly.

First, we notice that such a contour integral requires us to pick a branch of the multi-valued integrand first. In order to do so, let us begin by writing:

$$(a^2 - z^2)^{1/2} = \pm i(z^2 - a^2)^{1/2} = \pm i((z-a)(z+a))^{1/2} = \pm i\sqrt{r_1 r_2}e^{i(\theta_1+\theta_2)/2},$$

where $z = a + r_1 e^{i\theta_1}$ and $z = -a + r_2 e^{i\theta_2}$. In particular, we notice that choosing the arguments θ_1, θ_2 such that $0 \leq \theta_1 < 2\pi$ and $0 \leq \theta_2 < 2\pi$ gives a continuous result. This choice is consistent with inserting a branch cut along the real axis in the interval $[-a, a]$ as shown:



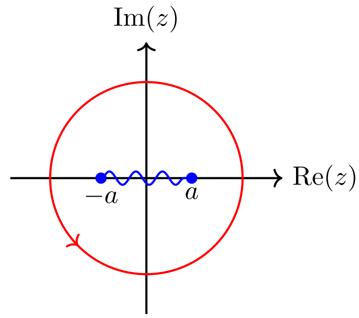
We still need to fix the sign $\pm$ in our choice of branch. We will choose the sign so that $(a^2 - z^2)^{1/2}$ corresponds to the *negative* square root just about the branch cut $[-a, a]$; here we have $\theta_1 = \pi$ and $\theta_2 = 0$, so the correct choice of sign is the $+$ sign. Thus our branch is explicitly:

$$(a^2 - z^2)^{1/2} = +i\sqrt{r_1 r_2}e^{i(\theta_1+\theta_2)/2}, \qquad \text{where } 0 \leq \theta_1 < 2\pi, 0 \leq \theta_2 < 2\pi.$$

Since $r_1 = |z - a|$ and $r_2 = |z + a|$, we can write this as:

$$(a^2 - z^2)^{1/2} = +i\sqrt{|z^2 - a^2|}e^{i(\theta_1+\theta_2)/2}, \qquad \text{where } 0 \leq \theta_1 < 2\pi, 0 \leq \theta_2 < 2\pi.$$

We now specify the contour that we will work with. We use the contour C encircling the branch cut as shown:



There are now two ways of evaluating the contour integral we have constructed. We can either (i) deform the contour C to a very large circle and approximate with an expansion in $1/z$; (ii) deform the contour C onto the branch cut, which should give us (something related to) the real integral we are interested in.

Let's begin by deforming the contour C on the branch cut. Carefully using our definition of the branch cut, we have the contributions from the straight segments:

$$\int_a^{-a} i\sqrt{a^2-x^2}e^{i\pi/2}\,dx + \int_{-a}^{a} i\sqrt{a^2-x^2}e^{3\pi i/2}\,dx = 2\int_{-a}^{a}(a^2-x^2)^{1/2}\,dx.$$

From the semicircular parts of the contour near $z=a$ and $z=-a$, we get zero contribution in the limit as the semicircular radii $\epsilon \to 0$. For example, near $z=-a$ we have the contribution (setting $r_1 = a + O(\epsilon), r_2 = \epsilon, \theta_1 = \pi$):

$$\int_0^{2\pi} i\sqrt{a\epsilon + O(\epsilon^2)}e^{i(\theta+\pi)/2}\,d\theta = O(\sqrt{\epsilon}) \to 0 \qquad \text{as } \epsilon \to 0.$$

We now consider instead evaluating the integral by deforming C to a very large circle and approximating the integrand with a series in $1/z$. Let's begin by rewriting the integrand via:

$$(a^2-z^2)^{1/2} = \left((-z^2)\left(1-\frac{a^2}{z^2}\right)\right)^{1/2} = iz\left(1-\frac{a^2}{z^2}\right)^{1/2}.$$

As usual, we get a possible sign from the branch choice. To get the correct sign of $(-z)^{1/2}$, we imagine setting $a=0$ in $(a^2-z^2)^{1/2}$ - then $r_1 = r_2 = r$ and $\theta_1 = \theta_2 = \theta$, so $(-z^2)^{1/2} = i\sqrt{r_1 r_2}e^{i(\theta_1+\theta_2)/2} = ire^{i\theta} = iz$ with this branch of the square root. Expanding the remaining factor using the binomial theorem, we find:

$$(a^2-z^2)^{1/2} = iz\left(1-\frac{a^2}{z^2}\right)^{1/2} = iz\left(1-\frac{a^2}{2z^2}+\cdots\right) = iz - \frac{ia^2}{2z} + \ldots,$$

and hence

$$\oint_C (a^2-z^2)^{1/2}\,dz = 2\pi i \cdot \left(-\frac{ia^2}{2}\right) = \pi a^2$$

by the residue theorem. Comparing the two ways of evaluating the contour integral, we arrive at the final required result:

$$\int_{-1}^{1}(a^2-x^2)^{1/2}\,dx = \frac{a^2\pi}{2}.$$

❊ **Comments from the Examiner:** *A nice short question testing major concepts. Many students did well on it, with a good share getting the full grade. Things were made a bit simpler for them by giving the final result. Some have struggled with the Laurent expansion. A small number of students did not know how to use the branch cut and took the integral using other means.*

2018, Paper 4, Section I, 7B

State the conditions for a point $z = z_0$ to be a *regular singular point* of a second-order homogeneous ordinary differential equation in the complex plane.

Find all singular points of the Bessel equation

$$z^2 y''(z) + z y'(z) + \left(z^2 - \frac{1}{4}\right) y(z) = 0, \tag{$*$}$$

and determine whether they are regular or irregular.

By writing $y(z) = f(z)/\sqrt{z}$, find two linearly independent solutions of $(*)$. Comment on the relationship of your solutions to the nature of the singular points.

Solution: Let the equation be $w'' + p(z)w' + q(z)w = 0$. A point in the finite complex plane $z = z_0$ is an *ordinary point* if $p(z), q(z)$ are analytic at $z = z_0$. The point is called a *regular singular point* if it is not an ordinary point and $(z - z_0)p(z)$, $(z - z_0)^2 q(z)$ are both analytic at $z = z_0$.

The point $z = \infty$ is an *ordinary point* if as $z \to \infty$ we have: $p(z) = 2/z + O\left(1/z^2\right)$, $q(z) = O\left(1/z^4\right)$. The point $z = \infty$ is a *regular singular point* if it is not an ordinary point as as $z \to \infty$ we have: $p(z) = O\left(1/z\right)$, $q(z) = O\left(1/z^2\right)$.

To determine the regular singular points of $(*)$, we first rewrite it in the standard form:

$$y''(z) + y'(z)/z + \left(1 - 1/(4z^2)\right) y(z) = 0.$$

The coefficient functions are clearly analytic everywhere in the finite complex plane except $z = 0$, hence all points in the finite complex plane except $z = 0$ are ordinary points. Since the coefficient functions are *not* analytic at $z = 0$, we have that $z = 0$ is not ordinary. However:

$$z\left(1/z\right), \qquad z^2\left(1 - 1/(4z^2)\right)$$

are analytic functions at $z = 0$, and hence $z = 0$ is a regular singular point of the equation. We must also consider the point at infinity; since the coefficient of $y(z)$ in this equation does not tend to zero as $z \to \infty$, we see that $z = \infty$ is neither ordinary or a regular singular point. Thus it is an irregular singular point of the equation.

Making the suggested substitution, we have the derivatives:

$$y(z) = \frac{f(z)}{\sqrt{z}} \qquad \Rightarrow \qquad y'(z) = \frac{f'(z)}{\sqrt{z}} - \frac{f(z)}{2z^{3/2}} \qquad \Rightarrow \qquad y''(z) = \frac{f''(z)}{\sqrt{z}} - \frac{f'(z)}{z^{3/2}} + \frac{3f(z)}{4z^{5/2}}.$$

Hence the equation becomes:

$$z^{3/2} f''(z) - z^{1/2} f'(z) + \frac{3f(z)}{4z^{1/2}} + z^{1/2} f'(z) - \frac{f(z)}{2z^{1/2}} + z^{3/2} f(z) - \frac{f(z)}{4z^{1/2}} = 0,$$

which simplifies to just $f''(z) + f(z) = 0$, which has the standard solution $f(z) = Ae^{iz} + Be^{-iz}$. Thus the solution to the original equation was

$$y(z) = Ae^{iz}/\sqrt{z} + Be^{-iz}/\sqrt{z}.$$

We see that both independent solutions have Frobenius series expansions around $z = 0$, consistent with $z = 0$ being a regular singular point. Near $z = \infty$, we set $z = 1/t$ and consider an expansion near $t = 0$; we see that there is an essential singularity in $y(1/t)$ near $t = 0$ and hence the Laurent series of the expansion is unbounded below - it follows that there is *no* Frobenius series expansion near $z = \infty$, consistent with $z = \infty$ being an irregular singular point.

Comments from the Examiner: *Easy question. In the first part the only issues with the singular point classification were at infinity due to errors in making the coordinate change in the equation. With the provided substitution the last part of the problem was easy. Looks like a direct application of the concepts learned in class.*

2018, Paper 1, Section II, 14B

The equation

$$zw'' + 2aw' + zw = 0, \tag{†}$$

where a is a constant with $\mathrm{Re}(a) > 0$, has solutions of the form

$$w(z) = \int_\gamma e^{zt} f(t)\, dt,$$

for suitably chosen contours γ and some suitable function $f(t)$.

(a) Find $f(t)$ and determine the condition on γ, which you should express in terms of z, t and a.

(b) Use the results of part (a) to show that γ can be a finite contour and specify two possible finite contours with the help of a clearly labelled diagram. Hence, find the corresponding solution of the equation (†) in the case $a = 1$.

(c) In the case $a = 1$ and real z, show that γ can be an infinite contour and specify two possible infinite contours with the help of a clearly labelled diagram. [Hint: *Consider separately the cases $z > 0$ and $z < 0$.*] Hence, find a second, linearly independent solution of the equation (†) in this case.

Solution: (a) Inserting $w(z)$ into the equation, we find

$$\int_\gamma \left(zt^2 f + 2atf + zf\right) e^{zt}\, dt = 0.$$

Using integration by parts, we have:

$$\int_\gamma \left(-\frac{d}{dt}\left((t^2+1)f(t)\right) + 2atf\right) e^{zt}\, dt + \left[(t^2+1)f(t)e^{zt}\right]_\gamma = 0.$$

It follows that f satisfies the equation:

$$\frac{d}{dt}\left((t^2+1)f\right) = 2atf \qquad \Rightarrow \qquad \frac{1}{f}\frac{df}{dt} = (a-1)\frac{2t}{t^2+1}.$$

This is a separable equation with the solution:

$$\log(f) = (a-1)\log(t^2+1) + C \qquad \Rightarrow \qquad f(t) = A(t^2+1)^{a-1}.$$

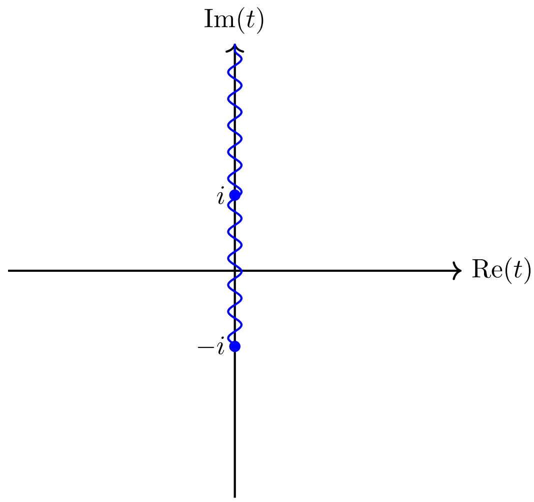
Hence we find the solution:

$$w(z) = \int_\gamma (t^2+1)^{a-1} e^{zt}\, dt \qquad \text{provided that the contour } \gamma \text{ obeys } \left[(t^2+1)^a e^{zt}\right]_\gamma = 0.$$

(b) For the next part of the question, we have to be careful with the wording. We are *first* asked to draw two finite contours that give solutions of the equation, and *then* asked to specialise to $a = 1$. Thus we must start by considering generic a with $\mathrm{Re}(a) > 0$.

In the generic case, a, $a-1$ is non-integral. So in particular, the functions in the solution require a branch cut. The function $(t^2+1)^{a-1}$ has branch points at $t = \pm i$, and possibly a branch point at infinity. Substituting $t = 1/u$, we have $(1/u^2+1)^{a-1} = (1+u^2)^{a-1}/u^{2a-2}$, so generically there is a branch point at $u = 0$, and hence at $t = \infty$.

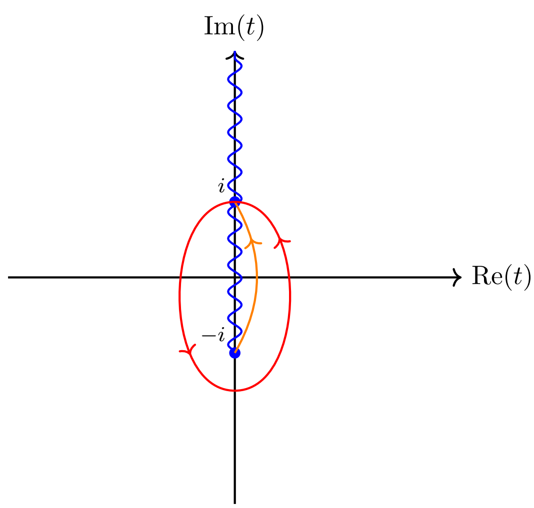
Let's insert a branch cut along imaginary axis from $-i$ to ∞, encompassing the whole positive imaginary axis:



To choose possible contours, we note that $w(z)$ has no pole singularities for $\mathrm{Re}(a) > 0$, so considering the integrand is not too helpful. We still need to satisfy the boundary condition however:

$$\left[(t^2+1)^a e^{zt}\right]_\gamma = 0$$

For $\mathrm{Re}(a) > 0$, we see that a suitable finite contour should start and end at the zeroes $\pm i$ of the boundary function. Due to the presence of the branch cut, we could pick a contour starting at $-i$ and ending at i for example. Another possible contour might start at i, encircle $-i$, and then return to i.



We're not asked to comment on linear independence, but let's do so anyway. It turns out that there is *no* possible choice of two finite contours giving rise to two linearly independent solutions in this case. For example, the two contours we have drawn above are linearly dependent - one is a multiple of the other as it turns out, due to the nature of the branch cut. In this question, there are only a few possible finite contours up to deformation and checking them in turn we find that they are all linearly dependent.

Fortunately, the question doesn't ask us to find two linearly independent contours, just 'draw two possible contours' so we're safe! Also the fact the question asks for *the* solution corresponding to your finite contour suggests strongly that you won't be able to find two just using finite contours.

We are now asked to specialise to the case $a = 1$, where the solution takes the form:

$$w(z) = \int_\gamma e^{zt}\,dt \qquad \text{provided that the contour } \gamma \text{ obeys } \left[(t^2+1)e^{zt}\right]_\gamma = 0.$$

Our loop contour that we drew before deforms to zero now, because the branch cut disappears. Therefore we choose to use our straight line segment contour from $-i$ to i. Parametrising the path via $t = ix$, we find that:

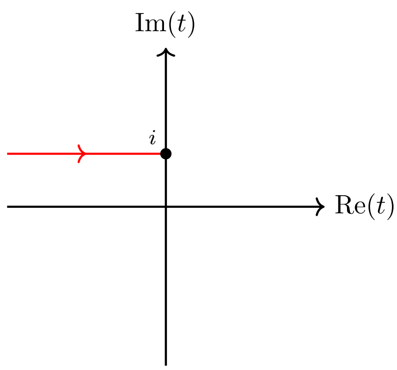
$$w(z) = i\int_{-1}^{1} e^{ixz}\,dx = \frac{i(e^{iz}-e^{-iz})}{iz} = \frac{2i\sin(z)}{z}$$

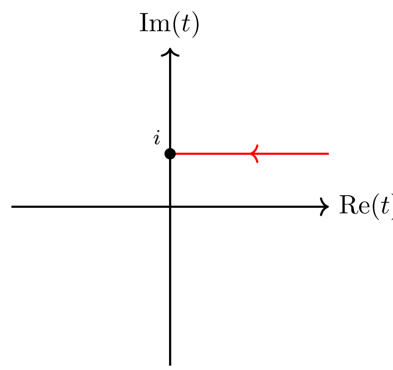
is a solution of the equation where $a = 1$.

(c) We are only asked to specify infinite contours in the special case $a = 1$. Here, we must choose the contours such that:

$$\left[(t^2+1)e^{zt}\right]_\gamma = 0$$

For $z > 0$, $e^{zt} \to 0$ as $t \to \infty$ in the left half plane. For $z < 0$, $e^{zt} \to 0$ as $t \to \infty$ in the right half plane. Hence, using the zeroes of $t^2 + 1$ as well, two possible choices of contour are:





Above: *The contour on the left generates a solution for $z > 0$. The contour on the right generates a solution for $z < 0$.*

Evaluating the integral on both of these contours, we have:

- For $z > 0$, parametrise by $t = x + i$ with $x \in (-\infty, 0]$, so that the solution is:

$$w(z) = e^{iz}\int_{-\infty}^{0} e^{xz}\,dx = \frac{e^{iz}}{z}.$$

- For $z < 0$, parameterise by $t = x + i$ with $x \in [0, \infty)$, so that the solution is:

$$w(z) = e^{iz}\int_{\infty}^{0} e^{xz}\,dz = \frac{e^{iz}}{z}.$$

Hence, for all z, a second linearly independent solution is:

$$w(z) = \frac{e^{iz}}{z}.$$

This solution is clearly linearly independent to the one we found in (a).

2018, Paper 2, Section II, 13B

Consider a multi-valued function $w(z)$.

(a) Explain what is meant by a *branch point* and a *branch cut*.

(b) Consider $z = e^w$.

(i) By writing $z = re^{i\theta}$, where $0 \leq \theta < 2\pi$, and $w = u + iv$, deduce the expression for $w(z)$ in terms of r and θ. Hence, show that w is infinitely valued and state its *principal value*.

(ii) Show that $z = 0$ and $z = \infty$ are the branch points of w. Deduce that the line $\mathrm{Im}(z) = 0, \mathrm{Re}(z) > 0$ is a possible choice of branch cut.

(iii) Use the Cauchy-Riemann conditions to show that w is analytic in the cut plane. Show that $\dfrac{dw}{dz} = \dfrac{1}{z}$.

Solution: (a) Let $z = z_0$ be a point in the complex plane and let $f(z)$ be a multi-valued function. If the function $f(z)$ does not return to its initial value as z winds in a sufficiently small closed circuit around the point $z = z_0$, then the point $z = z_0$ is called a *branch point* of the multi-valued function $f(z)$.

A *branch cut* B for the multi-valued function $f(z)$ is a curve (or collection of curves) removed from the (extended) complex plane such that on the domain $\mathbb{C}\backslash B$ there do not exist any small circuits around the branch points of $f(z)$. In particular, on the domain $\mathbb{C}\backslash B$, the function $f(z)$ can be consistently restricted to a continuous single-valued function.

(b) (i) On the advice of the question, we write $z = re^{i\theta}$ and $w = u + iv$; then $re^{i\theta} = e^{u+iv} = e^u e^{iv}$. Taking the modulus of both sides, we see that $r = e^u$ and hence $u = \log(r)$ (there is no multi-valuedness in this case, because we are taking the real logarithm).

We're left to compare $e^{i\theta} = e^{iv}$, which implies $e^{i(\theta - v)} = 0$ and hence $v = \theta + 2\pi n$ for any integer n. It follows that:

$$w(z) = \log(r) + i\theta + 2n\pi i,$$

where n is any integer. In particular, w is infinitely valued (there are infinitely many choices for n on the right hand side). The *principal value* of $w(z)$ is defined by:

$$w_p(z) = \log(r) + i\theta_p$$

where θ_p is chosen such that $\theta_p \in [0, 2\pi)$, i.e. we have chosen $n = 0$.

(ii) We are now asked to investigate the branch points of the function $w(z)$. There are two things that need to be done here: first, we need to show that $z = 0$ and $z = \infty$ are both branch points, and second we need to show that all other points in the complex plane are *not* branch points.

Near $z = 0$, we might consider the curve $\gamma : [0, 2\pi] \to \mathbb{C}$ specified by $\gamma(t) = \epsilon e^{it}$, i.e. a circle of radius ϵ for ϵ small. Then the multi-valued function $w(z)$ changes around the loop as:

$$w(\gamma(2\pi)) - w(\gamma(0)) = \log(\epsilon) + 2\pi i + 2n\pi i - \log(\epsilon) - 2n\pi i = 2\pi i.$$

So we see that it does not return to its initial value after traversing this small loop. Thus $z = 0$ is a branch point of the multi-valued function $w(z)$.

To evaluate the behaviour near $z = \infty$, we substitute $z = 1/\zeta$ and consider the behaviour near $\zeta = 0$ instead. If $z = re^{i\theta}$, then $1/z = e^{-i\theta}/r$, and hence we have

$$w(\zeta) = \log(1/r) - i\theta + 2n\pi i = -\log(r) - i\theta + 2n\pi i.$$

This new function of ζ is the same as our old function of z, albeit with a minus sign. Thus it must have a branch point at $\zeta = 0$ by the previous argument, and hence $z = \infty$ is a branch point of the multi-valued function $w(z)$.

Finally, let's consider the behaviour near any $z = z_0$, with $z_0 \neq 0$ and $z_0 \neq \infty$. Since on traversing any small loop around the point $z = z_0$ we do not enclose the origin, the argument of all points on the very small loop is always contained completely in a small interval inside $[0, 2\pi]$; in particular, the argument must return to its original value because the choice of different arguments of a given point in the complex plane can only every differ by multiples of 2π, and we never leave our interval $[0, 2\pi]$ during our circuit.

Hence the branch points of $w(z) = \log(r) + i\theta + 2n\pi i$ are precisely $z = 0$ and $z = \infty$. Provided we ban all small loops around $z = 0$ and $z = \infty$, we will be able to restrict to a domain where $w(z)$ can be chosen to be single-valued. Hence a suitable branch cut is a curve that joins $z = 0$ and $z = \infty$, since then we cannot encircle $z = 0$ or $z = \infty$ without crossing the curve; such a curve is provided in the form of $\mathrm{Im}(z) = 0$ and $\mathrm{Re}(z) \geq 0$, as given in the question.

Let's suppose that just above the cut, $w(z)$ takes the form $w(z) = \log(r) + 2N\pi i$ for some integer N (indeed, from the general multi-valued form of $w(z)$, this must be the case). Then since $w(z)$ is now single-valued in the cut plane, we have:

$$w_N(z) = \log(r) + i\theta_p + 2N\pi i = w_p(z) + 2N\pi i.$$

everywhere in the cut plane. In particular, we have infinitely many branches that we can choose from.

(iii) Finally, we must show using the Cauchy-Riemann equations that in the cut-plane, $w(z)$ is an analytic function. By this, the question means that we need to show that each branch $w_N(z)$ in the cut plane is an analytic function; as you know, just having a cut doesn't completely specify the nature of the single-valued function.

We can show that Cauchy-Riemann equations are satisfied by the components of $w_N(z)$ in one of two ways:

- 1. In Cartesian coordinates. If we choose to work in Cartesians, we will have to write down formulae for $\log(r)$ and θ_p in terms of the real and imaginary parts (x, y) of a generic point $z = x + iy$ in the cut plane. The logarithm is easy, and is given by:
$$\log(r) = \log(\sqrt{x^2 + y^2}) = \frac{1}{2}\log(x^2 + y^2).$$
We have to be a bit more careful with θ_p, since the standard $\arctan$ formula doesn't actually work everywhere in the complex plane. Splitting things into cases, we have:
  - Suppose that (x, y) is in the first quadrant and not on the axes. Then $y > 0$ and $x > 0$. The formula $\theta_p = \arctan(y/x)$ works here.
  - Suppose that (x, y) lies on the axis $\mathrm{Re}(z) = 0$ on the boundary between the first and second quadrants. Then $\theta_p = \pi/2$. This corresponds to taking $x \to 0$ from above in $\arctan(y/x)$ with $y > 0$.
  - Suppose that (x, y) lies in the left half plane. Then $x < 0$ and y can vary between being negative and positive. Thinking about the geometry, we see that the corrected formula is $\theta_p = \arctan(y/x) + \pi$ in this region.
  - Suppose that (x, y) lies on the axis $\mathrm{Re}(z) = 0$ on the boundary between the third and fourth quadrants. Then $\theta_p = 3\pi/2$. This corresponds to taking $x \to 0$ from below in the formula $\arctan(y/x) + \pi$ with $y < 0$ (which is true on the boundary between the third and fourth quadrants).
  - Finally, suppose that (x, y) lies in the fourth quadrant. Then $x > 0$ and $y < 0$. The corrected formula is $\theta_p = \arctan(y/x) + 2\pi$.

Hence it follows that the principal value of the argument θ_p can be expressed as $\arctan(y/x)$ everywhere, with some corrective constant depending on the region and how we approach the axes. But when we differentiate, this corrective constant will disappear and hence we can just ignore it after all our hard work!

Let's go ahead and check the Cauchy-Riemann equations are satisfied then. We work with the function:

$$w_N(z) = \frac{1}{2}\log(x^2+y^2) + i\left(\arctan(y/x) + \text{corrective constant} + 2N\pi\right),$$

so $u(x,y) = \frac{1}{2}\log(x^2+y^2)$ and $v(x,y) = \arctan(y/x) + \text{corrective constant} + 2N\pi$. Thus we have the derivatives:

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}, \qquad \frac{\partial v}{\partial x} = -\frac{y}{x^2+y^2}, \qquad \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}.$$

It follows immediately that the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied, as required.

- 2. In polar coordinates. A much cleaner way (that avoids having to check lots of arguments) is to use the Cauchy-Riemann equations in polar coordinates, which are given by:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta_p}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta_p}.$$

These conditions can be derived from the formula $\partial f/\partial \bar{z} = 0$ for example (which is of course equivalent to the Cauchy-Riemann equations, as you know from lectures), but expressing $\partial/\partial\bar{z}$ in terms of polar coordinates using the chain rule, rather than Cartesian coordinates as we normally do. They also only apply away from the origin and with a specific choice of arguments, but since we have cut out the origin and restricted arguments with a branch cut, we're safe.

Checking these instead then, we find:

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \qquad \frac{\partial u}{\partial \theta_p} = 0, \qquad \frac{\partial v}{\partial r} = 0, \qquad \frac{\partial v}{\partial \theta_p} = 1.$$

So it follows that these Cauchy-Riemann equations are trivially satisfied.

Whatever approach we choose to take, we find that the Cauchy-Riemann equations are indeed satisfied. Since in all cases all partial derivatives are continuous, the real and imaginary parts are also differentiable in a real sense. We deduce that the functions $w_N(z)$ are indeed analytic functions in the cut plane.

To finish, recall from lectures the formula that gives the derivative in terms of the partial derivatives of the real and imaginary parts:

$$w_N'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x-iy}{x^2+y^2} = \frac{z}{z\bar{z}} = \frac{1}{z},$$

as required. We have seen that this holds for any choice of branch given this branch cut. The equivalent expression in polar coordinates is simply:

$$w_N'(z) = e^{-i\theta_p}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{e^{-i\theta_p}}{r} = \frac{1}{z}.$$

2019, Paper 1, Section I, 7A

The beta function is defined by

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

where $\mathrm{Re}(p) > 0, \mathrm{Re}(q) > 0$ and Γ is the gamma function.

(a) By using a suitable substitution and properties of beta and gamma functions, show that

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\Gamma(1/4)^2}{\sqrt{32\pi}}.$$

(b) Deduce that

$$K\left(1/\sqrt{2}\right) = \frac{4\Gamma(5/4)^2}{\sqrt{\pi}},$$

where $K(k)$ is the complete elliptic integral, defined as

$$K(k) := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

[Hint: *You might find the change of variable* $x = t(2-t^2)^{-1/2}$ *helpful in part (b).*]

Solution: (a) Let $t = x^4$. Then $dt = 4x^3 dx = 4t^{3/4}\,dx$, so $dx = t^{-3/4}dt/4$. The limits change as $[0,1] \mapsto [0,1]$. Thus the integral transforms to:

$$\frac{1}{4}\int_0^1 t^{-3/4}(1-t)^{-1/2}\,dt = \frac{1}{4}B\left(\frac{1}{4},\frac{1}{2}\right) = \frac{\Gamma(1/4)\Gamma(1/2)}{4\Gamma(3/4)},$$

where in the final step we used the property $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$. To simplify things further, recall the *reflection formula* for the gamma function:

$$\Gamma(z)\Gamma(1-z) = \pi\mathrm{cosec}(\pi z).$$

This formula implies that $\Gamma(1/2)^2 = \pi \Rightarrow \Gamma(1/2) = \sqrt{\pi}$, and $\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2} \Rightarrow \Gamma(3/4) = \pi\sqrt{2}/\Gamma(1/4)$. Substituting into the above, we have the required form:

$$\frac{\Gamma(1/4)\Gamma(1/2)}{4\Gamma(3/4)} = \frac{\Gamma(1/4)^2}{4\sqrt{2\pi}} = \frac{\Gamma(1/4)^2}{\sqrt{32\pi}}.$$

(b) Let's apply the suggested change of variables to the integral in part (a). The limits transform as $x \in [0,1] \mapsto t \in [0,1]$. The measure transforms as $dx = \left((2-t^2)^{-1/2} + t^2(2-t^2)^{-3/2}\right)dt = 2(2-t^2)^{-3/2}dt$. Hence the integral transforms as:

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 2\int_0^1 \frac{dt}{\sqrt{(2-t^2)((2-t^2)^2-t^4)}} = 2\int_0^1 \frac{dt}{\sqrt{(2-t^2)(4-4t^2)}}$$

$$= \frac{1}{\sqrt{2}}\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-t^2/2)}} = \frac{1}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right).$$

Hence, by the result of part (a), we have:

$$K(1/\sqrt{2}) = \frac{\sqrt{2}\Gamma(1/4)^2}{\sqrt{32\pi}} = \frac{\Gamma(1/4)^2}{4\sqrt{\pi}}.$$

Using the functional equation of the gamma function, $\Gamma(z+1) = z\Gamma(z)$, we have $\Gamma(5/4) = \Gamma(1/4)/4$. Thus we have:

$$K(1/\sqrt{2}) = \frac{16\Gamma(5/4)^2}{4\sqrt{\pi}} = \frac{4\Gamma(5/4)^2}{\sqrt{\pi}},$$

as required.

❊ **Comments from the Examiner:** *This question was well-answered. Students typically demonstrated an understanding of the properties of the gamma function, and the ability to manipulate expressions into the required form.*

2019, Paper 2, Section I, 7A

Assume that $|f(z)/z| \to 0$ as $|z| \to \infty$ and that $f(z)$ is analytic in the upper half-plane (including the real axis). Evaluate

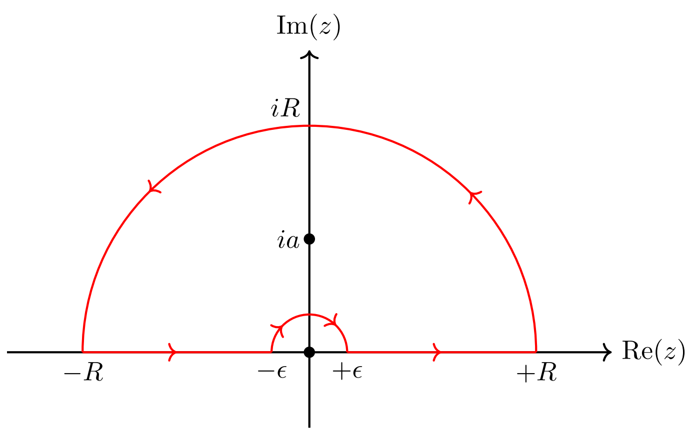
$$\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x(x^2+a^2)}\, dx,$$

where a is a positive real number. [*You must state clearly any standard results involving contour integrals that you use.*]

➻ **Solution:** Consider instead the contour integral

$$\oint_C \frac{f(z)}{z(z^2+a^2)}\, dz,$$

where C is the contour in the diagram below:



There is a single simple pole enclosed by the contour at $z = ia$. Hence we can evaluate the integral using the residue theorem:

$$\oint_C \frac{f(z)}{z(z^2+a^2)}\,dz = 2\pi i \text{Res}\left(\frac{f(z)}{z(z^2+a^2)}; ia\right) = 2\pi i \lim_{z\to ia}\left[\frac{f(z)}{z(z+ia)}\right] = \frac{2\pi i f(ia)}{ia(2ia)} = -\frac{\pi i f(ia)}{a^2}.$$

Now consider evaluating the integral on each section of the contour separately instead. We have:

$$\int_{-R}^{-\epsilon} \frac{f(x)}{x(x^2+a^2)}\,dx + \int_{\epsilon}^{R} \frac{f(x)}{x(x^2+a^2)}\,dx + \int_{\pi}^{0} \frac{f(\epsilon e^{i\theta})}{\epsilon e^{i\theta}(\epsilon^2 e^{2i\theta}+a^2)} i\epsilon e^{i\theta}\,d\theta + \int_{0}^{\pi} \frac{f(Re^{i\theta})}{Re^{i\theta}(R^2 e^{2i\theta}+a^2)}\,iRe^{i\theta}\,d\theta.$$

We are given that $|f(Re^{i\theta})/Re^{i\theta}| \to 0$ as $R\to\infty$. In particular, the last term vanishes in the limit $R\to\infty$. The first two terms combine to give the desired principal value integral in the limit $R\to\infty$ and $\epsilon\to 0$. Thus we have:

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x(x^2+a^2)}\,dx = \lim_{\epsilon\to 0} i\int_0^{\pi} \frac{f(\epsilon e^{i\theta})}{(\epsilon^2 e^{2i\theta}+a^2)}\,d\theta - \frac{\pi i f(ia)}{a^2} = \frac{i\pi}{a^2}\left(f(0)-f(ia)\right).$$

❊ **Comments from the Examiner:** *This question was exceptionally well-answered, with only a few students misapplying the indentation lemma.*

2019, Paper 3, Section I, 7A

The equation

$$zw'' + w = 0$$

has solutions of the form

$$w(z) = \int_\gamma e^{zt} f(t)\,dt,$$

for suitably chosen contours γ and some suitable function $f(t)$.

(a) Find $f(t)$ and determine the required condition on γ, which you should express in terms of z and t.

(b) Use the result of part (a) to specify a possible contour with the help of a clearly labelled diagram.

➬ **Solution:** (a) Inserting the integral representation of $w(z)$ into the given equation, we have:

$$\int_\gamma e^{zt}\left(zt^2 f(t) + f(t)\right)\,dt = 0.$$

Integrating the first term by parts, we have:

$$0 = \int_\gamma e^{zt}\left(-\frac{d}{dt}\left(t^2 f(t)\right) + f(t)\right) + \left[e^{zt}t^2 f(t)\right]_\gamma = \int_\gamma e^{zt}\left(-2tf(t) - t^2 f'(t) + f(t)\right) + \left[e^{zt}t^2 f(t)\right]_\gamma = 0.$$

Thus we choose $f(t)$ such that

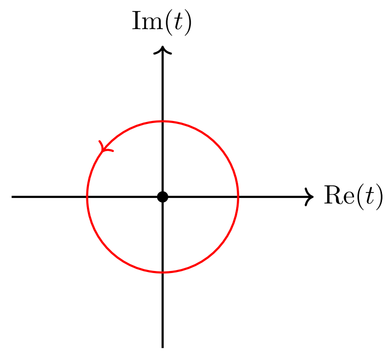
$$\frac{f'}{f} = \frac{1-2t}{t^2} \qquad \Rightarrow \qquad \log(f) = -\frac{1}{t} - 2\log(t) + C \qquad \Rightarrow \qquad f(t) = Ae^{-1/t}/t^2,$$

and we choose γ such that $\left[e^{zt}t^2 f(t)\right]_\gamma = \left[Ae^{zt-1/t}\right]_\gamma = 0$.

(b) We are now asked to pick an appropriate contour γ. As usual, we try a closed contour first. If we pick a closed contour γ encircling the origin, then the condition $[Ae^{zt-1/t}]_\gamma = 0$ is guaranteed, and

$$w(z) \propto \int_\gamma \frac{e^{zt-1/t}}{t^2}\,dt$$

is non-zero because the coefficient of $1/t$ in the expansion of the integrand is non-zero. It follows that a suitable contour is:



❊ **Comments from the Examiner:** *This question proved more challenging than expected, with both a surprisingly large number of calculation errors for the first part, and also sometimes poor understanding of the required properties of the possible contour γ for the second part.*

2019, Paper 4, Section I, 7A

A single-valued function $\mathrm{Arcsin}(z)$ can be defined, for $0 \leq \arg(z) < 2\pi$, by means of an integral as:

$$\mathrm{Arcsin}(z) = \int_0^z \frac{dt}{(1-t^2)^{1/2}}. \tag{†}$$

(a) Choose a suitable branch-cut with the integrand taking a value $+1$ at the origin on the upper side of the cut, i.e. at $t = 0^+$, and describe suitable paths of integration in the two cases $0 \leq \arg(z) \leq \pi$ and $\pi < \arg(z) < 2\pi$.

(b) Construct the multi-valued function $\arcsin(z)$ by analytic continuation.

(c) Express $\arcsin(e^{2\pi i}z)$ in terms of $\mathrm{Arcsin}(z)$ and deduce the periodicity property of $\sin(z)$.

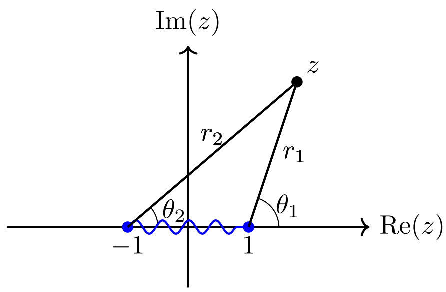
Solution: (a) Let us begin by constructing a branch of the function $(1-t^2)^{-1/2}$. Let us first write:

$$(1-t^2)^{-1/2} = \pm i(t^2-1)^{-1/2} = \pm i((t-1)(t+1))^{-1/2} = \pm i(r_1e^{i\theta_1}r_2e^{i\theta_2})^{-1/2} = \pm\frac{ie^{-i(\theta_1+\theta_2)/2}}{\sqrt{r_1r_2}},$$

where $t-1 = r_1e^{i\theta_1}$, $t+1 = r_2e^{i\theta_2}$, and the sign depends on the branch choice and will be decided in a moment. We will choose to use a branch cut along the real axis from $t=-1$ to $t=+1$, and a choice of arguments:

$$0 \leq \theta_1 < 2\pi, \qquad 0 \leq \theta_2 < 2\pi,$$

as indicated in the diagram below.



This choice of arguments guarantees that the function is continuous everywhere, except across the branch cut. The only place where we might get a discontinuity is across part of the positive real axis, $[1, \infty)$, but we see that the jump in both arguments θ_1 and θ_2 simultaneously actually cancels out:

$$e^{-i(0+0)/2} = e^{-i(2\pi+2\pi)/2}.$$

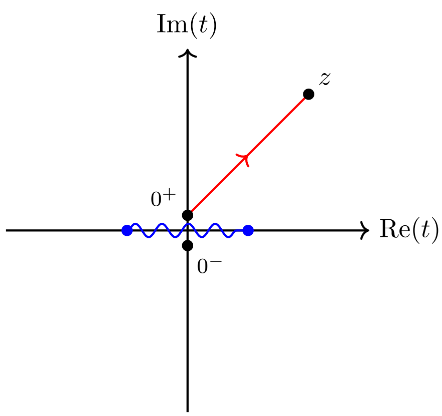
To ensure that the function is equal to $+1$ above the cut at the origin, we should choose the sign appropriately. Above the cut at the origin, we have $\theta_1 = \pi$, $\theta_2 = 0$, $r_1 = r_2 = 1$, hence the function takes the value:

$$\pm \frac{ie^{-i\pi/2}}{\sqrt{1 \cdot 1}} = \pm i(-i) = \pm 1.$$

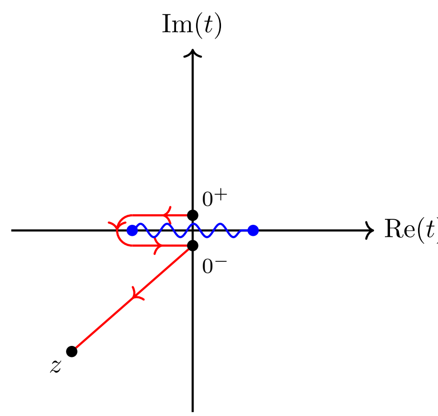
It follows we should choose the $+$ sign. Therefore, an unambiguous definition of the integrand is given by:

$$(1-t^2)^{-1/2} = \frac{ie^{-i(\theta_1+\theta_2)/2}}{\sqrt{r_1 r_2}}, \qquad 0 \leq \theta_1, \theta_2 < 2\pi.$$

We are also asked to describe suitable contours of integration in the defining integral of $\mathrm{Arcsin}(z)$. By Cauchy's theorem, all paths are equally good as each other, but the canonical choices are (i) a straight line from $t = 0^+$ to $t = z$ in the case $0 \leq \arg(z) \leq \pi$, and (ii) a contour that wraps around the branch cut from $t = 0^+$ to $t = 0^-$, and then follows a straight line from $t = 0^-$ to $t = z$, in the case that $\pi < \arg(z) < 2\pi$.



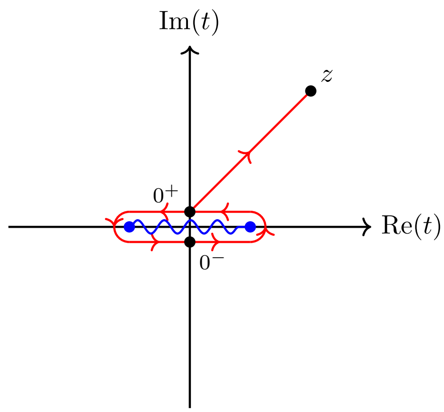
(a) $0 \leq \arg(z) \leq \pi$



(b) $\pi < \arg(z) < 2\pi$

(b) Next, we want to construct the analytic continuation of $\mathrm{Arcsin}(z)$ beyond the region $0 \leq \arg(z) < 2\pi$. Currently, if we increase $\arg(z)$ from $2\pi - \epsilon$ to $2\pi + \epsilon$, there is a jump in the value of $\mathrm{Arcsin}(z)$, since $\arg(z) = 2\pi + \epsilon$ is equivalent to $\arg(z) = \epsilon$, and hence we switch discontinuously from using the contour (ii) above to using the contour (i) above.

Therefore, to construct an *analytic* continuation of $\mathrm{Arcsin}(z)$ beyond $0 \leq \arg(z) < 2\pi$, we must ensure that increasing $\arg(z)$ from $2\pi - \epsilon$ to $2\pi + \epsilon$ is a smooth operation. When $\arg(z) = 2\pi - \epsilon$, we are using the contour (ii) above; hence, we should demand that increasing $\arg(z)$ to $2\pi + \epsilon$ just involves extending the contour in (ii) simply involves passing z through $[1, \infty)$. By deformation of the contour, we can make such a contour look like a loop around the branch cut, then a straight line:



We see that such a contour provides a smooth function of z on the domain $2\pi \leq \arg(z) \leq 3\pi$. To extend further, we can simply demand that we follow one loop around the branch cut, then supplement the contour with the contour (ii) from part (a). We can then extend to the domain $3\pi < \arg(z) < 4\pi$ smoothly.

Iterating this procedure, we can extend to $0 \leq \arg(z)$, in particular by looping anticlockwise around the branch cut as many times as we wish. Similarly, we can extend to $\arg(z) < 0$ by looping clockwise around the branch cut as many times as we wish. The result is a multi-valued analytic function of z, for all possible arguments of z, given by:

$$\arcsin(z) = \mathrm{Arcsin}(e^{-2in\pi} z) + n \int\limits_C \frac{dt}{\sqrt{1-t^2}},$$

where $\arg(z) \in [2n\pi, 2(n+1)\pi)$. The contour C is a single anticlockwise loop around the branch cut. Note the argument of $\mathrm{Arcsin}(z)$ has $\arg(e^{-2in\pi} z) \in [0, 2\pi)$, as is necessary for this single-valued function.

We can evaluate the integral over C explicitly to provide a more concrete representation of $\arcsin(z)$. Using the branch cut specified in part (a), we have (neglecting the small semi-circular contributions, which can be shown to be zero in the limit as the size of each semi-circle tends to zero):

$$\int\limits_C \frac{dt}{\sqrt{1-t^2}} = \int\limits_0^{-1} \frac{dx}{\sqrt{1-x^2}} - \int\limits_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} + \int\limits_{1}^{0} \frac{dx}{\sqrt{1-x^2}} = 2\left(\arcsin(-1) - \arcsin(1)\right) = -2\pi,$$

where $\arcsin : [-1, 1] \to \mathbb{R}$ in the above calculation denotes the standard real-valued function. Hence we have the explicit formula:

$$\arcsin(z) = \mathrm{Arcsin}(e^{-2in\pi} z) - 2n\pi.$$

(c) Finally, we are asked to express $\arcsin(e^{2\pi i}z)$ in terms of $\mathrm{Arcsin}(z)$, and hence deduce the periodicity property of $\sin(z)$. Suppose that z is such that $\arg(z) \in [2n\pi, 2(n+1)\pi)$. Then $\arg(e^{2\pi i}z) \in [2(n+1)\pi, 2(n+2)\pi)$, and so:

$$\begin{aligned}\arcsin(e^{2\pi i}z) &= \mathrm{Arcsin}(e^{-2i(n+1)\pi}e^{2\pi i}z) - 2(n+1)\pi \\ &= \mathrm{Arcsin}(e^{-2in\pi}z) - 2(n+1)\pi.\end{aligned}$$

The question only makes sense in the case $\arg(z) \in [0, 2\pi)$, else $\mathrm{Arcsin}(z)$ isn't defined; this corresponds to the case $n = 0$. However, let's keep things general for now.

Notice that we can rewrite the right hand side using: $\arcsin(z) = \mathrm{Arcsin}(e^{-2in\pi}z) - 2n\pi$. We find that:

$$\arcsin(e^{2\pi i}z) = \arcsin(z) - 2\pi.$$

The inverse of the multi-valued function $\arcsin(z)$ is the function $\sin(z)$. Therefore, setting $z = \sin(w)$ in this equation, then taking the sine of both sides, we have:

$$\arcsin(e^{2\pi i}\sin(w)) = \arcsin(\sin(w)) - 2\pi \qquad \Rightarrow \qquad \sin(w) = \sin(w - 2\pi),$$

which is the periodicity condition as required.

❈ **Comments from the Examiner:** *This apparently straightforward question was very badly answered. In particular, the required construction of a multi-valued function for* $\arcsin$ *was often mangled.*

2019, Paper 1, Section II, 14A

(a) Consider the *Papperitz symbol* (or P-symbol):

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}. \tag{†}$$

Explain in general terms what this P-symbol represents. [You need not write down any differential equations explicitly, but should provide an explanation of the meaning of $a, b, c, \alpha, \beta, \gamma, \alpha', \beta'$ and γ'.]

(b) Prove that the action of $[(z-a)/(z-b)]^\delta$ on (†) results in the *exponential shifting*,

$$P\begin{Bmatrix} a & b & c & \\ \alpha+\delta & \beta-\delta & \gamma & z \\ \alpha'+\delta & \beta'-\delta & \gamma' & \end{Bmatrix} \tag{‡}$$

[Hint: *It may prove useful to start by considering the relationship between two solutions, ω and ω_1, which satisfy the P-equations described by the respective P-symbols (†) and (‡).*]

(c) Explain what is meant be a *Möbius transformation* of a second order differential equation. By using suitable transformations acting on (†), show how to obtain the P-symbol

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix} \tag{$*$}$$

which corresponds to the *hypergeometric equation*.

(d) The hypergeometric function $F(a, b, c; z)$ is defined to be the solution of the differential equation corresponding to ($*$) that is analytic at $z = 0$ with $F(a, b, c; 0) = 1$, which corresponds to the exponent zero. Use exponential shifting to show that the second solution, which corresponds to the exponent $1 - c$, is

$$z^{1-c}F(a-c+1, b-c+1, 2-c; z).$$

Solution: (a) The Papperitz symbol represents the solution space of the Fuchsian equation in the variable z, with exactly three regular singular points at the locations $z = a$, $z = b$, $z = c$, and with corresponding exponents α, α' at $z = a$, exponents β, β' at $z = b$, and exponents γ, γ' at $z = c$.

(b) The proof required here is more of a heuristic argument; this is discussed in the solutions to Examples Sheet 4, Question 3. Suppose that w is in the solution space represented by the Papperitz symbol in (a). Near an ordinary point $z = \bar{z}$ of the equation, we can expand w as:

$$w(z) = C\sum_{n=0}^{\infty} c_n(z-\bar{z})^n + D\sum_{n=0}^{\infty} d_n(z-\bar{z})^n.$$

Multiplying by $(z-a)^\delta/(z-b)^\delta$, we have:

$$\frac{(z-a)^\delta}{(z-b)^\delta}w(z) = \frac{(z-\bar{z}+\bar{z}-a)^\delta}{(z-\bar{z}+\bar{z}-b)^\delta}w(z) = \frac{(\bar{z}-a)^\delta}{(\bar{z}-b)^\delta}\left(1+\delta\left(\frac{z-\bar{z}}{\bar{z}-a}\right)+\cdots\right)\left(1-\delta\left(\frac{z-\bar{z}}{\bar{z}-b}\right)+\cdots\right)w(z).$$

We see that $w(z)$ is multiplied by an analytic function in $z - \bar{z}$; in particular, $w(z)$ is mapped to an analytic function of $z - \bar{z}$ under the multiplication.

Near the regular singular point $z = a$, we can expand w as:

$$w(z) = C(z-a)^{\alpha} \sum_{n=0}^{\infty} c_n (z-a)^n + D(z-a)^{\alpha'} \sum_{n=0}^{\infty} d_n (z-a)^n.$$

Hence multiplying by the given factor, we have near $z = a$:

$$\frac{(z-a)^{\delta}}{(z-b)^{\delta}} w(z) = \frac{(z-a)^{\delta}}{(z-a+a-b)^{\delta}} w(z) = \frac{1}{(a-b)^{\delta}} (z-a)^{\delta} \left(1 - \delta\left(\frac{z-a}{a-b}\right) + \cdots\right) w(z).$$

In particular, we see that $w(z)$ is multiplied by an analytic function of $z - a$, and by the overall factor $(z-a)^{\delta}$. Thus $w(z)$ takes the following form after the multiplication:

$$\frac{(z-a)^{\delta}}{(z-b)^{\delta}} w(z) = C(z-a)^{\alpha+\delta} \sum_{n=0}^{\infty} c_n' (z-a)^n + D(z-a)^{\alpha'+\delta} \sum_{n=0}^{\infty} d_n' (z-a)^n,$$

where c_n' and d_n' are constants. This is the form that a solution of a Fuchsian equation would take near an RSP $z = a$ with exponents $\alpha + \delta$ and $\alpha' + \delta$.

Near the regular singular point $z = b$, we can expand w as:

$$w(z) = C(z-b)^{\beta} \sum_{n=0}^{\infty} c_n (z-b)^n + D(z-b)^{\beta'} \sum_{n=0}^{\infty} d_n (z-b)^n.$$

After multiplication by the given factor, we have near $z = b$:

$$\frac{(z-a)^{\delta}}{(z-b)^{\delta}} w(z) = \frac{(z-b+b-a)^{\delta}}{(z-b)^{\delta}} w(z) = \frac{(b-a)^{\delta}}{(z-b)^{\delta}} \left(1 + \delta\left(\frac{z-b}{b-a}\right) + \cdots\right) w(z),$$

hence similarly we see that the function $w(z)$ multiplied by $(z-a)^{\delta}/(z-b)^{\delta}$ takes the form:

$$\frac{(z-a)^{\delta}}{(z-b)^{\delta}} w(z) = C(z-b)^{\beta-\delta} \sum_{n=0}^{\infty} c_n' (z-b)^n + D(z-b)^{\beta'-\delta} \sum_{n=0}^{\infty} d_n' (z-b)^n.$$

This is the form that a solution of a Fuchsian equation would take near an RSP $z = b$ with exponents $\beta - \delta$ and $\beta' - \delta$.

Finally, we can repeat this exercise for the point $z = c$. This time, the function $(z-a)^{\delta}/(z-b)^{\delta}$ is an analytic function of $z - c$, and hence there is no change to the exponents or positions of regular singular points.

It follows (at least, heuristically) that:

$$\frac{(z-a)^{\delta}}{(z-b)^{\delta}} P \left\{ \begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix} \right\} = P \left\{ \begin{matrix} a & b & c & \\ \alpha+\delta & \beta-\delta & \gamma & z \\ \alpha'+\delta & \beta'-\delta & \gamma' & \end{matrix} \right\},$$

as required.

(c) A *Möbius transformation* is a transformation of the independent variable z of the form:

$$z \mapsto M(z) = \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$ (this condition is required to ensure the map is invertible). If $M : \mathbb{C} \to \mathbb{C}$ is a Möbius transformation, we have the important result for the manipulation of P-symbols:

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} M(a) & M(b) & M(c) & \\ \alpha & \beta & \gamma & M(z) \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}.$$

Using Möbius transformations and the exponent-shifting of part (b), we can transform the P-symbol of part (a) to the P-symbol in this part of the question. Let us first apply the Möbius transformation mapping a, b and c to $0, 1$ and ∞:

$$M(z) = z' = \frac{(b-c)(z-a)}{(b-a)(z-c)}.$$

Then we have:

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ \alpha & \beta & \gamma & z' \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}.$$

We now wish to shift the exponents into hypergeometric form, using the result from (b). Note that:

$$\frac{(z')^{-\alpha}}{(z'-1)^{-\alpha}} P\begin{Bmatrix} 0 & 1 & \infty & \\ \alpha & \beta & \gamma & z' \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & \beta+\alpha & \gamma & z' \\ \alpha'-\alpha & \beta'+\alpha & \gamma' & \end{Bmatrix}.$$

Finally note that we can shift the exponent at infinity by adjusting the result of part (b) slightly. Note that the result of part (b) still holds when we multiply by $b^\delta(z-a)^\delta/(z-b)^\delta$, since multiplying a solution by a constant, we still get a solution of the equation. Taking the limit $b \to \infty$ shows that we can shift the exponent at the regular singular point by multiplying with $(z-a)^\delta$. Hence we have:

$$(z-1)^{-\beta-\alpha} P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & \beta+\alpha & \gamma & z' \\ \alpha'-\alpha & \beta'+\alpha & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha+\beta+\gamma & z' \\ \alpha'-\alpha & \beta'-\beta & \alpha+\beta+\gamma' & \end{Bmatrix}.$$

Identifying $c = 1+\alpha-\alpha'$, $a = \alpha+\beta+\gamma$ and $b = \alpha+\beta+\gamma'$, we see that we have derived the hypergeometric P-symbol as required:

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix}.$$

The only non-trivial step in casting the P-symbol in this form is the evaluation:

$$c - a - b = 1+\alpha-\alpha' - (\alpha+\beta+\gamma) - (\alpha+\beta+\gamma') = (1-\alpha'-\alpha-\beta-\gamma-\gamma') - \beta = \beta' - \beta,$$

which follows from the use of the Fuchsian invariant:

$$\alpha' + \alpha + \beta' + \beta + \gamma' + \gamma = 1.$$

(d) The hypergeometric function $z^{1-c}F(a-c+1, b-c+1, 2-c; z)$ lies in the solution space described by the P-symbol:

$$z^{1-c}P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a-c+1 & z \\ c-1 & c-a-b & b-c+1 & \end{Bmatrix}.$$

Using part (b), and the comments on shifting the exponents of the regular singular point at infinity, we know that we can use the factor z^{1-c} to cause a shift in the exponents at 0 by $+(1-c)$ and a shift in the exponents at ∞ by $-(1-c)$. We have:

$$z^{1-c}P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a-c+1 & z \\ c-1 & c-a-b & b-c+1 & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ 1-c & 0 & a & z \\ 0 & c-a-b & b & \end{Bmatrix}.$$

We see that the result is the hypergeometric equation's solution space, and hence the function

$$z^{1-c}F(a-c+1, b-c+1, 2-c; z)$$

solves the hypergeometric equation as required.

❋ **Comments from the Examiner:** *This question was on the whole well-answered, with the students being able to manipulate P-symbols appropriately. A few found it difficult to construct the appropriate form of the second solution.*

2019, Paper 2, Section II, 13A

The Riemann zeta function is defined as

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{†}$$

for $\mathrm{Re}(z) > 1$, and by analytic continuation to the rest of $\mathbb{C}$ except at singular points. The integral representation of (†) for $\mathrm{Re}(z) > 1$ is given by

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t - 1}\, dt, \tag{‡}$$

where Γ is the gamma function.

(a) The *Hankel representation* is defined as

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i} \int_{-\infty}^{(0+)} \frac{t^{z-1}}{e^{-t} - 1}\, dt. \tag{$*$}$$

Explain briefly why this representation gives an analytic continuation of $\zeta(z)$ as defined in (‡) to all z other than $z = 1$, using a diagram to illustrate what is mean by the upper limit of the integral in ($*$). [You may assume $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$.]

(b) Find

$$\mathrm{Res}\left(\frac{t^{-z}}{e^{-t}-1}; t = 2\pi i n\right),$$

where n is an integer and the poles are simple.

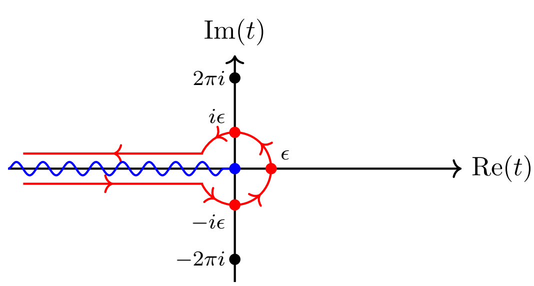
(c) By considering

$$\int_\gamma \frac{t^{-z}}{e^{-t}-1}\, dt,$$

where γ is a suitably modified Hankel contour and using the result of part (b), derive the *reflection formula*:

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\cos\left(\frac{1}{2}\pi z\right)\Gamma(z)\zeta(z).$$

Solution: (a) The given integral ($*$) involves integration around the *Hankel contour*, as indicated in the diagram below.



The standard method for showing that ($*$) provides an analytic continuation of the zeta function is simply to evaluate the integral expression in ($*$). We evaluate the integral on each section of the contour as follows.

On the straight segments of the Hankel contour, we get the contribution:

$$\frac{\Gamma(1-z)}{2\pi i}\int_{\infty}^{\epsilon}\frac{e^{-i\pi(z-1)}x^{z-1}}{e^x-1}e^{-i\pi}\,dx+\frac{\Gamma(1-z)}{2\pi i}\int_{\epsilon}^{\infty}\frac{e^{i\pi(z-1)}x^{z-1}}{e^x-1}e^{i\pi}\,dx=\frac{\Gamma(1-z)(e^{i\pi z}-e^{-i\pi z})}{2\pi i}\int_{\epsilon}^{\infty}\frac{x^{z-1}}{e^x-1}\,dx$$

$$=\frac{\Gamma(1-z)}{\pi/\sin(\pi z)}\int_{\epsilon}^{\infty}\frac{x^{z-1}}{e^x-1}\,dx=\frac{1}{\Gamma(z)}\int_{\epsilon}^{\infty}\frac{x^{z-1}}{e^x-1}\,dx,$$

where in the last step we used the reflection formula for the gamma function, $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$. Note we recover the integral representation of the zeta function for $\mathrm{Re}(z)>1$ in the limit as the Hankel contour is deformed via $\epsilon\to 0$.

The other contribution to the integral (∗) is given by the circular part of the Hankel contour. We get:

$$\frac{\Gamma(1-z)}{2\pi i}\int_{-\pi}^{\pi}\frac{\epsilon^{z-1}e^{-i\theta(z-1)}}{e^{\epsilon e^{i\theta}}-1}i\epsilon e^{i\theta}\,d\theta=\frac{\Gamma(1-z)}{2\pi i}\int_{-\pi}^{\pi}\frac{i\epsilon^z e^{i\theta z}}{1-\epsilon e^{i\theta}+O(\epsilon^2)-1}\,d\theta=O(\epsilon^{z-1}).$$

Hence we see that provided $\mathrm{Re}(z)>1$, we can deform the Hankel contour via $\epsilon\to 0$ and receive zero contribution from the circular arc. Thus putting everything together, we see that if $\mathrm{Re}(z)>1$, we have that the integral expression in (∗) is given by:

$$\frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{x^{z-1}}{e^x-1}\,dx,$$

which agrees with the integral expression for the Riemann zeta function in the region $\mathrm{Re}(z)>1$.

However, the integral (∗) is also defined in the much larger domain $\mathbb{C}\backslash\{0\}$, and hence provides the analytic continuation of the zeta function to this domain.

To see that the integral (∗) is indeed defined in this domain, note that

$$\int_{-\infty}^{(0+)}\frac{t^z}{e^{-t}-1}\,dt \tag{†}$$

has an integrand that is continuous everywhere along the contour, and is exponentially suppressed - hence convergent - at the ends of the contour at $t\to-\infty$. The factor that multiplies the integral in (∗):

$$\frac{\Gamma(1-z)}{2\pi i}$$

is analytic everywhere except for simple poles at $z=1,2,3,...$. However, the poles at $z=2,3,...$ are cancelled by simples zeroes of (†). To check that indeed (†) has zeroes at $z=2,3,4\dots$ using the residue theorem:

$$\int_{-\infty}^{(0+)}\frac{t^{z-1}}{e^{-t}-1}\,dt=2\pi i\mathrm{Res}\left(\frac{t^{z-1}}{e^{-t}-1};0\right).$$

The residue can be calculated via series expansion:

$$\frac{t^{z-1}}{e^{-t}-1}=\frac{t^{z-1}}{1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+O(t^4)-1}=-\frac{t^{z-2}}{1-\frac{1}{2}t+\frac{1}{6}t^2+O(t^3)}.$$

We see that if $z=2,3,4...$, the function is analytic near $t=0$, and hence has residue zero, so the integral has a zero at $z=2,3,4...$ exactly cancelling the simple pole from the gamma function.

In summary, the integral (∗) is an analytic continuation of the zeta function to the domain $\mathbb{C}\backslash\{0\}$, as required.

(b) We are given that the poles in this expression are simple, so we can use the standard formula to find the residue at a simple pole. We find:

$$\mathrm{Res}\left(\frac{t^{-z}}{e^{-t}-1}; t = 2\pi i n\right) = (2\pi i n)^{-z} \lim_{t\to 2\pi i n}\left[\frac{t-2\pi i n}{e^{-t}-1}\right].$$

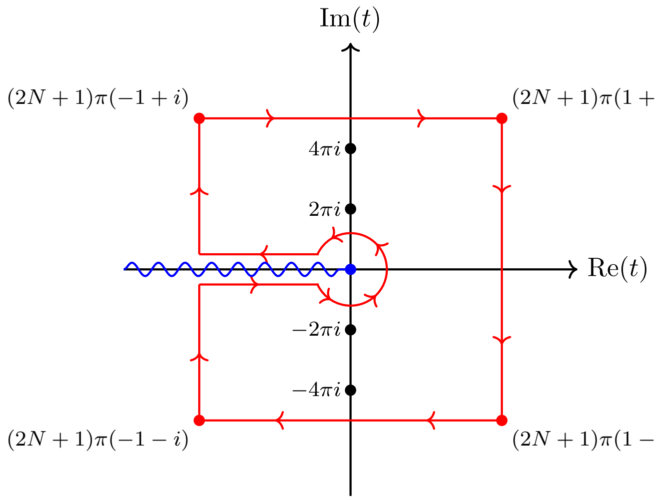
Using l'Hôpital's rule, we have:

$$\lim_{t\to 2\pi i n}\left[\frac{t-2\pi i n}{e^{-t}-1}\right] = -\lim_{t\to 2\pi i n}\left[\frac{1}{e^{-t}}\right] = -e^{2\pi i n} = -1.$$

Hence we have:

$$\mathrm{Res}\left(\frac{t^{-z}}{e^{-t}-1}; t = 2\pi i n\right) = -(2\pi i n)^{-z}.$$

(c) To show the reflection formula, we close the Hankel contour using a large square contour of the following form (for N an integer):



Let this modified Hankel contour be C. By the residue theorem, and the result of part (b), we have (note the plus sign, because the contour is traversed clockwise):

$$\oint_C \frac{t^{-z}}{e^{-t}-1}\,dt = 2\pi i \sum_{n=1}^{N}\left((2\pi i n)^{-z} + (-2\pi i n)^{-z}\right) = 2\pi i (2\pi)^{-z}\left(i^{-z} + (-i)^{-z}\right)\sum_{n=1}^{N} n^{-z}$$

$$= 2\pi i(2\pi)^{-z}\left(e^{-i\pi z/2} + e^{i\pi z/2}\right)\sum_{n=1}^{N} n^{-z} = 2i(2\pi)^{1-z}\cos\left(\frac{\pi z}{2}\right)\sum_{n=1}^{N} n^{-z}.$$

In the limit $N\to\infty$, the right hand side becomes:

$$2i(2\pi)^{1-z}\cos\left(\frac{\pi z}{2}\right)\zeta(z),$$

provided that $\mathrm{Re}(z) > 1$, so we can use the series expression for the zeta function.

We need to decide what happens to the left hand side. The straight parts of the contour that straddle the branch cut, together with the circular arc, reproduce the regular Hankel contour, hence we get the contribution:

$$\int_{-\infty}^{(0+)} \frac{t^{-z}}{e^{-t}-1}\,dt = \frac{2\pi i\zeta(1-z)}{\Gamma(z)},$$

in addition to the contribution from the sides of the large square. Assuming for now that the contributions from the sides of the large square vanish in the limit $N \to 0$, we can put everything together to get the result:

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\cos\left(\frac{\pi z}{2}\right)\Gamma(z)\zeta(z).$$

In the derivation, we used the fact that $\mathrm{Re}(z) > 1$, but we can clearly extend both sides to a maximal domain in the complex plane by analytic continuation (in fact, the domain for which this is true is $\mathbb{C}\backslash\{0\}$).

Hence provided we can show that the contributions from the sides of the large square vanish in the limit $N \to \infty$, we'll be done. Let's look at each side in turn:

- For the side between the points $(2N+1)\pi(1+i)$ and $(2N+1)\pi(1-i)$, the contribution is:

$$\int_{(2N+1)\pi}^{-(2N+1)\pi} \frac{((2N+1)\pi+iy)^{-z}}{e^{-(2N+1)\pi-iy}-1}\,idy.$$

  We can prove this converges to zero as $N \to \infty$ by bounding the modulus of this contribution:

$$\left|\int_{-(2N+1)\pi}^{(2N+1)\pi} \frac{((2N+1)\pi+iy)^{-z}}{e^{-(2N+1)\pi-iy}-1}\,dy\right| \le \int_{-\infty}^{\infty} \frac{((2N+1)^2\pi^2+y^2)^{-z/2}}{|e^{-(2N+1)\pi-iy}-1|}\,dy.$$

  To bound the denominator, note that $e^{-(2N+1)\pi-iy}$ is a point on the circle of radius $e^{-(2N+1)\pi}$. Its distance from the point 1 is minimised when $y = 0$, hence we have:

$$\left|e^{-(2N+1)\pi-iy}-1\right| \ge \left|e^{-(2N+1)\pi}-1\right| \qquad \Rightarrow \qquad \frac{1}{\left|e^{-(2N+1)\pi-iy}-1\right|} \le \frac{1}{\left|e^{-(2N+1)\pi}-1\right|} \to 1$$

  as $N \to \infty$. Thus we're left with:

$$\left|\int_{-(2N+1)\pi}^{(2N+1)\pi} \frac{((2N+1)\pi+iy)^{-z}}{e^{-(2N+1)\pi-iy}-1}\,dy\right| \le \int_{-\infty}^{\infty} \frac{1}{((2N+1)^2\pi^2+y^2)^{z/2}}\,dy.$$

  Provided $\mathrm{Re}(z) > 0$, this converges to zero as $N \to \infty$. It's fine for us only to prove the result for $\mathrm{Re}(z) > 0$, because we can argue that the result must hold in a much larger domain using analytic continuation.

- Similar bounds apply for the remaining three sides. For the left hand side, we can simply use the exponential growth of the denominator to show the contribution goes to zero. For the upper and lower sides of the square, the same argument as above applies.

❋ **Comments from the Examiner:** *This question was solidly answered. Some students did not present a cogent argument for why the Hankel representation is an analytic continuation of $\zeta(z)$ for all $z \neq 1$. Also, not all students constructed and justified the 'suitably modified Hankel contour' appropriately.*