Manifolds: definitions and examples

Abstract

We review the definition of a manifold, and give examples. We start by defining *topological n-manifolds*, which realise our intuition of spaces which look *locally* like open subsets of $\mathbb{R}^n$. Subsequently, we discuss the problems with defining *smooth functions* on such spaces, which leads us to the definition of a *smooth atlas* on a topological n-manifold. We describe equivalence of smooth atlases, which leads us to introduce *smooth structures*, and hence *n-manifolds*. We close by giving basic examples of manifolds, namely $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathrm{Mat}_n(\mathbb{R})$, $\mathrm{Mat}_n(\mathbb{C})$ and S^1.

1. Definitions

In this course, the types of groups which we are typically interested in *Lie groups*, which are groups which can be 'parametrised in terms of some *smooth* coordinates' (the word 'smooth' implies some ability to *calculus* on the group). In this handout, we make the notion of spaces which can be 'parametrised by smooth coordinates' precise; such spaces are called *manifolds*.

The basic idea is that, locally, a manifold looks like an open subset of $\mathbb{R}^n$; in particular, this allows us to port all definitions from real multivariable calculus to the manifold, including notions such as *smoothness*. With this in mind, we might posit the following initial definition:

Definition: A *topological n-manifold X* is a topological space satisfying the following axioms:

(M1) The topology on X is *Hausdorff*: given any two points $p, q \in X$, there exists open neighbourhoods $U \ni p, V \ni q$ such that $U \cap V \neq \emptyset$ (i.e. 'points can be separated by open sets').

(M2) The topology on X is *second-countable*: the topology is generated by a countable basis of open sets.

(M3) The topology on X is *locally n-Euclidean*: for all points $p \in X$, there exists an open neighbourhood $U \ni p$ on which there is some homeomorphism $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$. The pair (U, ϕ) is called a *chart*, and the map ϕ is called a set of *local coordinates* on the domain U. The inverse $\phi^{-1} : \phi(U) \to U$ is called a *parametrisation* of the domain U.

The integer n is called the *dimension* of the topological n-manifold.[1]

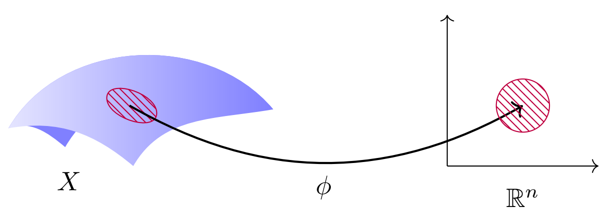

Figure 1: *A topological n-manifold is a topological space which looks locally like an open subset of $\mathbb{R}^n$.*

The first two axioms (M1) and (M2) are purely technical, and ensure the statements of some theorems are not too cluttered. The important axiom realising our intuition was (M3).

[1] It can be shown using methods from algebraic topology that if a topological space is both a topological n-manifold and a topological m-manifold, then $n = m$; therefore, the dimension of a topological n-manifold is well-defined.

Topological n-manifolds are not yet spaces on which we can do calculus. Suppose, for example, that we wish to define smoothness of a real-valued function $f : X \to \mathbb{R}$ on the topological n-manifold X at a point p. Reasonably, we might declare that f is smooth at the point $p \in X$ if there is some chart (U, ϕ) whose domain contains p such that the function $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth as a real, multivariable function (i.e. all its partial derivatives exist to all orders). We think of $f \circ \phi^{-1}$ as the 'local coordinate expression of the function f'.

However, this definition is a little unsatisfying. Suppose that we change coordinates about p, so that we now work with another chart (V, ψ) whose domain contains p. Since our only assumption on ϕ and ψ is that they are *homeomorphisms*, the most we can say about the 'change of coordinates function' $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is that it is a *continuous function*. In particular it does not follow that $f \circ \psi^{-1} = f \circ \phi^{-1} \circ \phi \circ \psi^{-1}$ is necessarily smooth as a real, multivariable function. Changing coordinates can change the smoothness of a function!

To remedy this, we restrict to a collection of charts where *changes of coordinates* are smooth. In particular, we choose some subset of charts $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$, whose domains U_α cover X, such that for any charts (U_α, ϕ_α), (U_β, ϕ_β), the transition function $\phi_\alpha \circ \phi_\beta^{-1}$ is a smooth function in the sense of real, multivariable calculus. This fixes the problem, since if $f \circ \phi_\alpha^{-1}$ is smooth, then $f \circ \phi_\alpha^{-1} \circ \phi_\alpha \circ \phi_\beta^{-1}$ is smooth by the chain rule.

Baking this into a definition, we have the following:

Definition: Let X be a topological n-manifold. A *smooth atlas* on X is a collection of charts $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$, whose domains U_α cover X, such that for any charts (U_α, ϕ_α), (U_β, ϕ_β) the *transition function*:

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is a smooth function in the sense of real, multivariable calculus.

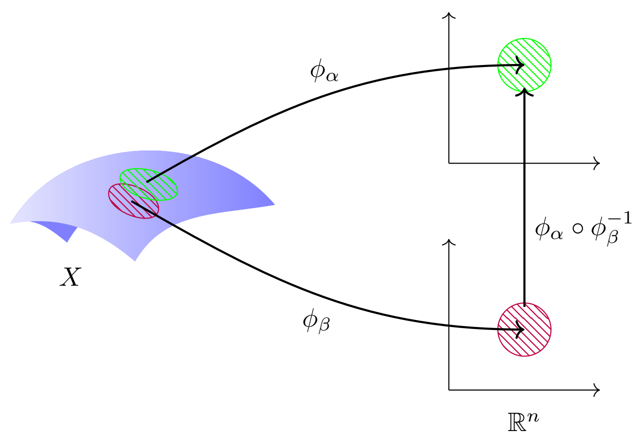

Figure 2: *A smooth atlas on a topological n-manifold is a collection of charts for which all transition functions, i.e. changes of coordinates, are smooth (in the sense of real, multivariable calculus).*

We can now define smooth functions on a topological n-manifold *with respect to a smooth atlas*, avoiding the problem we described above.

Definition: Let X be a topological n-manifold, and let $\mathcal{A}$ be a smooth atlas on X. A function $f : X \to \mathbb{R}$ is *smooth* with respect to $\mathcal{A}$ if for some chart $(U, \phi) \in \mathcal{A}$ we have that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is a smooth function in the sense of real multivariable calculus.

We are now in a good position; by choosing a special subset of charts on a topological n-manifold, we can reasonably begin to talk about smooth functions. However, it is quite disappointing that our choice is arbitrary - would another choice have led to the same set of smooth functions on the topological n-manifold?

To remove the arbitrariness, we introduce a natural notion of *equivalence* of smooth atlases:

Definition: Let X be a topological n-manifold and let $\mathcal{A}, \mathcal{B}$ be smooth atlases on X. We say that $\mathcal{A}, \mathcal{B}$ are *smoothly equivalent* if for all functions $f : X \to \mathbb{R}$, f is smooth with respect to $\mathcal{A}$ if and only if f is smooth with respect to $\mathcal{B}$. That is, $\mathcal{A}, \mathcal{B}$ give rise to the same smooth functions on X.

Naturally, smooth equivalence of smooth atlases is an equivalence relation:

Proposition: Smooth equivalence of smooth atlases is an equivalence relation on the set of smooth atlases. We call an equivalence class of smooth atlases a *smooth structure*.

Proof: Easy exercise. $\square$

Now, instead of saying that a function is smooth with respect to some arbitrary choice of smooth atlas, we can reliably say that a function is smooth with respect to some *smooth structure*. This (finally) leads us to make the definition of a *smooth n-manifold*:

Definition: A *smooth n-manifold* (henceforth abbreviated to n-manifold, or just manifold when n is clear) is a topological n-manifold equipped with a smooth structure. The integer n is called the *dimension* of the manifold.

As an interesting aside, it turns out that topological $1, 2$ and 3-manifolds can be given a unique smooth structure. Topological n-manifolds with $n \geq 4$ can be given multiple, inequivalent smooth structures.

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2. Examples

The most basic example of a manifold is $\mathbb{R}^n$ itself.

Example: The standard topology on $\mathbb{R}^n$ is Hausdorff (since it is induced by a metric) and second-countable (a basis for the topology is provided by balls with rational centres and rational radii). Given any point $p \in \mathbb{R}^n$, we have that $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$ is a chart whose domain contains p. Therefore $\mathbb{R}^n$ is a topological n-manifold.

The chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$ is a *global* chart since it covers all of $\mathbb{R}^n$. In particular, $\{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\}$ is trivially a smooth atlas on $\mathbb{R}^n$, since the only transition function is the trivial transition function. Hence $\mathbb{R}^n$ can be made into an n-manifold, with a representative of its smooth structure given by $\{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\}$.

The above example allows us to immediately write down many related examples:

Example:

- $\mathbb{C}^n$ is a $2n$-manifold. To see this, we simply note $\mathbb{C}^n = \mathbb{R}^{2n}$, by writing each component of a vector in $\mathbb{C}^n$ in terms of its real and imaginary parts.
- The space of $n \times n$ matrices over $\mathbb{R}$, $\mathrm{Mat}_n(\mathbb{R})$, is an n^2-manifold. Again, this is because we can identify $\mathrm{Mat}_n(\mathbb{R}) = \mathbb{R}^{n^2}$, simply by stacking all entries of a matrix in a column vector in some way.
- The space of $n \times n$ matrices over $\mathbb{C}$, $\mathrm{Mat}_n(\mathbb{C})$, is a $2n^2$-manifold. This follows from the chain of equalities $\mathrm{Mat}_n(\mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$, using both of the identifications in the previous two bullet points.

A less trivial example of a manifold is the circle, S^1.

Example: Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane. We can endow S^1 with the subspace topology, inherited from its ambient space $\mathbb{C} = \mathbb{R}^2$; in particular, its topology is therefore Hausdorff and second-countable. We define two charts on S^1:

- $\phi : S^1 \backslash \{1\} \to (0, 2\pi)$ given by $\phi(e^{i\theta}) = \theta$ for $\theta \in (0, 2\pi)$.
- $\psi : S^1 \backslash \{-1\} \to (-\pi, \pi)$ given by $\psi(e^{i\theta}) = \theta$ for $\theta \in (-\pi, \pi)$.

Figure 3: *Two charts covering S^1. On their overlap, their transition functions must be smooth.*

The existence of these charts shows that S^1 is locally 1-Euclidean, hence S^1 is a topological 1-manifold. Furthermore, $\{(S^1\backslash\{1\}, \phi), (S^1\backslash\{-1\}, \psi)\}$ is a smooth atlas on S^1, since:

$$\phi \circ \psi^{-1} : (-\pi, 0) \cup (0, \pi) \to (0, \pi) \cup (0, 2\pi), \qquad \phi \circ \psi^{-1}(\theta) = \begin{cases} \theta & \text{if } \theta \in (0, \pi), \\ \theta + 2\pi & \text{if } \theta \in (-\pi, 0), \end{cases}$$

is smooth, and similarly $\psi \circ \phi^{-1}$ is smooth. It follows that S^1 is a 1-manifold with $\{(S^1\backslash\{1\}, \phi), (S^1\backslash\{-1\}, \psi)\}$ a representative of its smooth structure.