Semi-direct products

Abstract

We motivate internal direct and semi-direct products of groups based on the idea of 'factorising' a group into smaller subgroups. From this construction, we generalise to external direct and semi-direct products.

1: Internal products of groups

Given a complicated group G, we would like to be able to break it down into smaller, more manageable pieces in order to understand its structure. A straightforward way of doing this is to look for a 'factorisation' of the group G into two of its subgroups. Some simple requirements we might ask of such a factorisation are:

- <u>Existence of a factorisation.</u> There exist subgroups H, K of G such that any element $g \in G$ can be written in the form
$$g = hk$$
for some $h \in H$ and some $k \in K$. In other words, the group can be written in the form:
$$G = HK = \{hk : h \in H, k \in K\}.$$
If this condition holds, we say that G is the *product* of the subgroups H and K (in that order!).

- <u>Uniqueness of a factorisation.</u> If the group element $g \in G$ can be written in the form $g = hk$ for $h \in H$ and $k \in K$, then this decomposition is *unique*. Explicitly, if $g = hk$ and $g = h'k'$ for $h, h' \in H$ and $k, k' \in K$, then $h = h'$ and $k = k'$.

- <u>Multiplication rule for factorised elements.</u> Suppose that we want to multiply the elements $g = hk$ and $g' = h'k'$ in the factorised group. The product is given by:
$$g \cdot g' = hk \cdot h'k'$$
Since $g \cdot g' \in G$ by closure of the group multiplication, there must be a way of rewriting the right hand side in the form $h''k''$ for some $h'' \in H$ and some $k'' \in K$.

  Noting that h, k' are already in the right places in the factorisation, we see that all we need do to express $g \cdot g'$ in the correct factorised form is to express kh' in the correct factorised form. Hence we define some functions $\alpha : H \times K \to H$, $\beta : H \times K \to K$ via:
$$kh' = \alpha(h', k)\beta(h', k),$$
for all $k \in K$ and $h' \in H$. These functions α, β completely determine the multiplication in the group; in a sense, they determine how the subgroups H and K 'interact' with one another.

The first two conditions, existence and uniqueness, do not really have any interesting content. The third condition, i.e. the specification of the functions α, β which allow us to perform multiplication in the group, really contains all of the meat of our factorisation.

Typically, if we find a 'factorisation' of a group in the form we have outlined above, the functions α and β are very complicated. In practice, there are two special cases for α and β which prove useful. The first is simply to assume that α and β are the projections onto their respective subgroups, i.e. $\alpha(h, k) = h$ and $\beta(h, k) = k$ (equivalent to saying that the subgroups H and K are completely commuting). Then we have the notion of the *internal direct product* of subgroups:

Definition: Let H and K be subgroups of the group G. Suppose that:

- For all $g \in G$, there exist unique elements $h \in H$ and $k \in K$ such that $g = hk$.
- For all $h \in H$ and $k \in K$, we have $kh = hk$.

Then we say that G is the *internal direct product* of the subgroups H and K. We write $G = H \times K$.

The second, more general, simplification is to assume that β is the projection onto its respective subgroup, i.e. $\beta(h, k) = k$. Then we have the notion of the *internal semi-direct product* of subgroups:

Definition: Let H and K be subgroups of the group G. Suppose that:

- For all $g \in G$, there exist unique elements $h \in H$ and $k \in K$ such that $g = hk$.
- For all $h \in H$ and $k \in K$, we have $kh = \alpha(h, k)k$ for some function $\alpha : H \times K \to H$.

Then we say that G is the *internal semi-direct product* of the subgroups H and K with respect to the map α. We write $G = H \rtimes_\alpha K$.

The reason for the little triangle is that from this definition, it follows $H \trianglelefteq G$. For suppose that $g \in G$ and $h' \in H$; then $g = hk$ for some $h \in H$ and $k \in K$, and hence $gh'g^{-1} = (hk)h'(hk)^{-1} = hkh'k^{-1}h^{-1} = h\alpha(h', k)h^{-1} \in H$.

The map α in an internal semi-direct product satisfies some useful properties:

Theorem: Let $G = H \rtimes_\alpha K$ be an internal semi-direct product. The map α obeys the following properties:

(i) For all $k \in K$, $\alpha(\cdot, k) : H \to H$ is an *automorphism* of groups.

(ii) In light of (i), define a map $\phi : K \to \mathrm{Aut}(H)$ via $\phi_k = \alpha(\cdot, k)$ (writing $\phi(k) = \phi_k$ is conventional here). Then ϕ is a homomorphism of groups, i.e. a structure-preserving map: $\phi_k \phi_{k'} = \phi_{kk'}$.

Proof: Recall that the function $\alpha : H \times K \to H$ is defined by the condition $kh = \alpha(h, k)k$ for all $h \in H$ and all $k \in K$. Rearranging this equation, we get the explicit expression $\alpha(h, k) = khk^{-1}$ for the function α. With this expression, it is easy to verify both (i) and (ii).

For (i), note the following three facts:

- $\alpha(hh', k) = khh'k^{-1} = khk^{-1}kh'k^{-1} = \alpha(h, k)\alpha(h', k)$ so that α is a homomorphism for each k.
- If $\alpha(h, k) = \alpha(h', k)$, then $khk^{-1} = kh'k^{-1}$, from which it follows $h = h'$. Hence α is injective for each k.
- Finally, note that if $h' \in H$, then $\alpha(h', k^{-1}) \in H$, since the codomain of α is H. Then $\alpha(\alpha(h', k^{-1}), k) = \alpha(k^{-1}h'k, k) = h'$. Thus the map $\alpha(\cdot, k)$ is a surjection for each k.

Hence $\alpha(\cdot, k) : H \to H$ is indeed an automorphism for each $k \in K$.

For (ii), note that for all $h \in H$, we have $\phi_{kk'}(h) = \alpha(h, kk') = (kk')h(kk')^{-1} = kk'h(k')^{-1}k = k\phi_{k'}(h)k^{-1} = (\phi_k \circ \phi_{k'})(h)$. Hence $\phi : K \to \mathrm{Aut}(H)$ is a homomorphism of groups. $\square$

This theorem tells us some important information. The map $\alpha : H \times K \to H$, which tells us what happens to a H element when it is exchanged past a K element, should really be thought of as modifying the H element via some automorphism which depends only on the element $k \in K$ which we are commuting past. Hence we arrive at the (more standard) definition:

Definition: Let H and K be subgroups of the group G. Suppose that:

- For all $g \in G$, there exist unique elements $h \in H$ and $k \in K$ such that $g = hk$.
- For all $h \in H$ and $k \in K$, we have $kh = \phi_k(h)k$ for some homomorphism $\phi : K \to \mathrm{Aut}(H)$.

Then we say that G is the *internal semi-direct product* of the subgroups H and K with respect to the map ϕ. We write $G = H \rtimes_\phi K$.

Let's consider a couple of examples of semi-direct products:

Example:

(i) Consider the dihedral group D_n, specified by the group presentation:

$$D_n = \langle r, s \,|\, r^n = e, s^2 = e, rs = sr^{-1} \rangle \,.$$

The elements of D_n can be written explicitly as $D_n = \{e, r, ..., r^{n-1}, s, rs, r^2 s, ..., r^{n-1}s\}$. This shows that we can decompose D_n as the group product of the reflections and rotations:

$$D_n = \{e, r, ..., r^{n-1}\}\{e, s\}.$$

This decomposition is clearly unique. Finally, notice that the group presentation tells us how to commute rotations past reflections (considering the identity e to be a reflection for this purpose):

$$er^m = r^m e, \qquad sr^m = r^{-m}s.$$

This tells us that our semi-direct product function should be $\phi : \{e, s\} \to \mathrm{Aut}(\{e, r, ..., r^{n-1}\})$, given by $\phi_e = \mathrm{id}$ and $\phi_s(r^m) = r^{-m}$. This is indeed a homomorphism as can be readily checked, and ϕ_s is indeed a group automorphism. Hence we have shown:

$$D_n = \{e, r, ..., r^{n-1}\} \rtimes_\phi \{e, s\} \cong \mathbb{Z}_n \rtimes_\phi \mathbb{Z}_2,$$

with $\phi_e(r^m) = r^m$ and $\phi_s(r^m) = r^{-m}$.

(ii) Consider the *Poincaré group*, $ISO(1,3)$, which consists of transformations of Minkowski spacetime of the form:

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\mu,$$

where Λ is a Lorentz transformation and a is a four-translation in spacetime. Let's write this transformation as (Λ, a). Then the group composition law is:

$$(\Lambda, a) \cdot (\Lambda', a') = (\Lambda\Lambda', \Lambda a' + a).$$

The Lorentz transformations form a subgroup of the Poincaré group called the *Lorentz group*, given by $O(1,3) \cong \{(\Lambda, 0) : \Lambda \text{ a Lorentz transformation}\}$. Similarly, the four-translations form a subgroup of the Poincaré group which is isomorphic to $\mathbb{R}^4$ under addition, and is given by $\mathbb{R}^4 \cong \{(I, a) : a \in \mathbb{R}^4\}$.

Notice that the Poincaré group is clearly the product of the translation subgroup and Lorentz group:

$$ISO(1,3) = \{(I, a)\}\{(\Lambda, 0)\},$$

and clearly the decomposition of group elements in this way is unique. When we want to commute a Lorentz transformation past a translation, we have:

$$(\Lambda, 0) \cdot (I, \Lambda^{-1}a) = (\Lambda, a) = (I, a) \cdot (\Lambda, 0).$$

This tells us that our semi-direct product function should be $\phi : O(1,3) \to \mathrm{Aut}(\mathbb{R}^4)$, given by $\phi_\Lambda(a) = \Lambda a$. Clearly, ϕ_Λ is an automorphism of $\mathbb{R}^4$. Also, ϕ is a homomorphism since $\phi_{\Lambda\Lambda'}(a) = (\Lambda\Lambda')a = (\phi_\Lambda \circ \phi_{\Lambda'})(a)$. Hence we have:

$$ISO(1,3) = \{(I, a)\} \rtimes_\phi \{(\Lambda, 0)\} \cong \mathbb{R}^4 \rtimes_\phi O(1,3),$$

with $\phi_\Lambda(a) = \Lambda a$.

2: External products of groups

Now we understand how direct products and semi-direct products work as the *internal* product of groups, we can generalise to the *external product* of groups. Given two unrelated groups H and K, we would like to construct a larger, more complicated group G, such that G has the internal structure of a direct product or semi-direct product of two subgroups, which are isomorphic to H and K.

To this end, let us try to put a group structure on the Cartesian product $H \times K$. The reason for considering this structure is that it already gives us the fact that every group element can be written as the unique decomposition 'hk', which we translate across to the Cartesian product as '(h, k)'. It remains to add the multiplicative structure:

- The internal direct product has the multiplication rule $hk \cdot h'k' = hh' \cdot kk'$, hence we define the multiplication on the Cartesian product as $(h, k) \cdot (h', k') = (hh', kk')$ in analogy.
- The internal semi-direct product has the multiplication rule $hk \cdot h'k' = h\phi_k(h') \cdot kk'$ for some homomorphism $\phi : K \to \mathrm{Aut}(H)$, hence we define the multiplication on the Cartesian product as $(h, k) \cdot (h', k') = (h\phi_k(h'), kk')$ in analogy.

Both of these multiplication rules give a group structure to the Cartesian product as required. Hence we make the definitions:

Definition: Let H and K be groups. We define their *external direct product* to be the Cartesian product $H \times K$ together with the multiplication rule:

$$(h, k) \cdot (h', k') = (hh', kk').$$

We write the external direct product group as $H \times K$.

Definition: Let H and K be groups, and suppose that $\phi : K \to \mathrm{Aut}(H)$ is a homomorphism. We define the *external semi-direct product* of the groups, with respect to the homomorphism ϕ, to be the Cartesian product $H \times K$ together with the multiplication rule:

$$(h, k) \cdot (h', k') = (h\phi_k(h'), kk').$$

We write the external semi-direct product group as $H \rtimes_\phi K$.

As an exercise, one can verify that these definitions indeed lead to group structures on $H \times K$.