Part IA: Mathematics for Natural Sciences B
Examples Sheet 6: Taylor series, numerical methods and Riemann sums

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Questions marked with a (†) require you to carefully review your lecture notes to make sure you understand basic definitions and properties. Questions marked with a ($*$) are difficult and should not be attempted at the expense of the other questions.

Taylor series

1. (†) State Taylor's theorem, giving Lagrange's formula for the remainder term. Hence obtain the first four non-zero terms in the Taylor expansion of $\sin(x)$ about $x = \pi/6$ by direct differentiation. Using this expansion, find an approximate value for $\sin(31^\circ)$, indicating the relative precision of your answer.

2. (†) Write down the Taylor series about $x = 0$ for the following functions, stating their range of convergence:

   (a) e^x, (b) $\log(1+x)$, (c) $\sin(x)$, (d) $\cos(x)$, (e) $\sinh(x)$, (f) $\cosh(x)$, (g) $(1+x)^a$,

   where in the final part $a \in \mathbb{R}$ is any real number. Learn these series off by heart, and get your supervision partner to test you on them.

3. *Without differentiating*, find the value of the thirty-second derivative of $\cos(x^4)$ at $x = 0$.

4. Let a be a constant. Assuming standard results, use the quickest method you can think of in each case to find the first three non-zero terms in the Taylor series expansion about $x = 0$ of:

   (a) $\dfrac{1}{\sqrt{1+x}}$, (b) $\dfrac{1}{(x^2+a)^{3/2}}$, (c) $\tan(x)$, (d) $\log(\cos(x))$, (e) $\arcsin(x)$.

5. This question combines infinite series, Taylor series, and complex numbers, to produce rapidly convergent approximation formulae for π, called *Machin formulae*.

   (a) Consider a convergent alternating series:
   $$\sum_{n=0}^{\infty} (-1)^n a_n,$$
   satisfying the standard conditions $a_n > 0$, $a_n \geq a_{n+1}$, and $a_n \to 0$ as $n \to \infty$. Prove that the absolute error of the $(k+1)$th partial sum (i.e. the sum up to $n = k$) of this series is bounded above by a_{k+1}.

   (b) Show that for $|x| \leq 1$, we have $\arctan(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$.

   (c) Using the result $\arctan(1) = \pi/4$, and the bound derived in (a), *approximately* how many terms of the series in (b) are needed to calculate π to 10 decimal places?

   (d) By considering the product $(2+i)(3+i)$, show that $\pi/4 = \arctan(1/2) + \arctan(1/3)$, and deduce another series for π. Approximately how many terms of this series are needed to calculate π to 10 decimal places?

   (e) By considering a suitable product of complex numbers, show that $\pi/4 = 4\arctan(1/5) - \arctan(1/239)$, and deduce yet another series for π. Approximately how many terms of this series are needed to calculate π to 10 decimal places?[1]

6. ($*$) Sketch the graph of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-1/x^2}$ for $x \neq 0$, and $f(0) = 0$. Show that this function is infinitely differentiable at $x = 0$, and determine its Taylor series at $x = 0$. Comment on the general utility of Taylor series.

[1] John Machin used this approach to compute π to 100 decimal places in 1706.

Approximation

7. (†) Give the definition of Landau's big O notation, '$f(x) = O(g(x))$ as $x \to x_0$'. Decide which of the following statements are true:

   (a) $x = O(x^2)$ as $x \to 0$, (b) $x^2 = O(x)$ as $x \to 0$, (c) $x = O(x^2)$ as $x \to \infty$, (d) $x^2 = O(x)$ as $x \to \infty$.

8. Give the leading terms in an approximation to each of the following functions in the given limits, indicating the leading behaviour of the remainder in Landau's big O notation:

   (a) $\dfrac{x^3+x}{x+2}$ as $x \to 0$, (b) $\dfrac{\cos(x)-1}{x^2}$ as $x \to 0$, (c) $\dfrac{1+2x+2x^2}{3x+3}$ as $x \to \infty$.

9. Show that:
$$(x^3+x^2+1)^{1/3} - (x^2+x)^{1/2} = -\frac{1}{6} + \frac{1}{72x} + O\left(\frac{1}{x^2}\right) \qquad \text{as } x \to \infty.$$

Newton-Raphson root finding

10. (†) Give an explanation of the Newton-Raphson algorithm for root finding, including an appropriate sketch. Under what conditions is it guaranteed that Newton-Raphson will converge to the root of interest? Prove that, when it converges to the root of interest, the Newton-Raphson method enjoys *quadratic convergence*.

11. (a) Sketch the graph of $f(x) = x^3 - 3x^2 + 2$, indicating the coordinates of the turning points and the coordinates of the intersections with the x-axis.
    (b) Use Newton-Raphson with an initial guess of $x_0 = 2.5$ to find an estimate of the largest root of the equation $f(x) = 0$, accurate to 5 decimal places. Draw a sketch showing the progress of the algorithm.
    (c) To which roots (if any) does the algorithm converge if we instead start at: (i) $x_0 = 1.5$; (ii) $x_0 = 1.9$; (iii) $x_0 = 2$?

Riemann sums and the definition of the integral

12. (†) Let $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$, so that $P = (x_0, \ldots, x_n)$ is a *partition* of the interval $[a, b]$. For $k = 1, \ldots, n$, let $t_k \in [x_{k-1}, x_k]$, so that $T = (t_1, \ldots, t_n)$ is a *tagging* of the partition of the interval. Define the *Riemann sum* $R(f, P, T)$ of $f : [a, b] \to \mathbb{R}$ with respect to the partition P and tagging T. Draw a sketch to explain this definition.

    Carefully state the ϵ, δ definition of the *definite (Riemann) integral* of $f : [a, b] \to \mathbb{R}$. Hence, prove from first principles that the definite integral of a constant c function on $[a, b]$ is $c(b - a)$.

13. (∗) The *fineness* of a partition $P = (x_0, \ldots, x_n)$ of an interval $[a, b]$ is defined to be:
$$|P| = \max_{k=1,\ldots,n} (x_k - x_{k-1}).$$
    Show that if $f : [a, b] \to \mathbb{R}$ has a definite integral, then for any sequence of partitions P_n whose fineness tends to zero, $|P_n| \to 0$, the corresponding sequence of Riemann sums evaluated on these partitions, with arbitrary associated taggings T_n, satisfies:
$$R(f, P_n, T_n) \to \int_a^b f(x)\, dx.$$
    This justifies the 'definition' given in the lectures.

14. Choosing appropriate partitions and taggings in each case, use sequences of Riemann sums to evaluate the definite integrals of the following functions on $[0, 1]$ from first principles: (a) x; (b) x^2; (c) $\sqrt{x}$.

15. Assuming standard integrals, determine the value of the limit: $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt{n^2-k^2}}{n^2}$.

Past paper questions

Only attempt these questions before the supervision if you have lots of spare time. Sometimes, we might do them together in the supervision; otherwise, you can use these questions as revision material in the holidays.

Paper 2, Question 8, 2021 (2 marks)
Without differentiating, find the nth term in the Taylor expansion about $x = 0$ of $f(x) = 1/(1 + x^2)$.

Paper 1, Question 15, 2023 (20 marks)
Find, by any method, the first three non-zero terms in the Taylor series expansions about $x = 0$ of the following functions. [You may quote standard power series without proof.]

(a) $x\sinh(x^2)$, (b) $\log(1 + \log(1 + x))$, (c) $\sin^6(x)$.

Paper 1, Question 15, 2021 (20 marks)

(a) State Taylor's theorem by giving the series expansion about $x = a$ of a function $f(x)$ that is n times differentiable, showing the first n terms, together with an expression for the remainder term R_n.

(b) Show that the Taylor series expansion of $f(x) = \cosh(x)/\cos(x)$ about the point $x = 0$ as far as the term in x^4 is $f(x) \approx 1 + x^2 + x^4/2$.

(c) Find the Taylor series expansion of $g(x) = \cosh(\log(x))$ about the point $x = 2$ as far as the term in $(x - 2)^3$.

(d) Show that the Taylor series expansion of $h(x) = \log(2 - e^x)$ about the point $x = 0$ as far as the term in x^3 is $h(x) \approx -x - x^2 - x^3$ and state the range of x for which the infinite series converges.

Paper 2, Question 6, 2023 (2 marks)
Consider using the Newton-Raphson method to solve the equation $f(x) = 0$ for real variable x.

(a) Let x_0 be an initial guess for the solution. Write down the formula for x_1, the next approximation to the solution, in terms of x_0, $f(x_0)$ and $f'(x_0)$.

(b) Find the value of x_1 for the particular case of $f(x) = x - 2 + \log(x)$ and $x_0 = 1$.

Paper 1, Question 19(b), 2019 (14 marks)
The real function $f(x)$ is defined as $f(x) = x^2 - 2\epsilon x - 1$, where the parameter ϵ is small (i.e. $|\epsilon| \ll 1$). Suppose x_i is the ith Newton-Raphson iterate (with $x_0 = 1$ for the positive root x_* of $f(x)$ (i.e. $f(x_*) = 0$). By considering the leading order term in the Taylor series expansion, show that $|x_i - x_*| \propto \epsilon^{n_i}$, where: (i) $n_0 = 1$; (ii) $n_1 = 2$; (iii) $n_2 > 3$.

Paper 1, Question 20(a), 2019 (6 marks)
The interval $a \le x \le b$ of the x-axis, with $0 < a < b$, is divided into n equal sub-intervals of width $\Delta x = (b - a)/n$. For each sub-interval k, $1 \le k \le n$, a rectangle of width Δx and height $y_k = (a + k\Delta x)^2$ is constructed. Find the sum of the areas of the rectangles as a function of n and show that, as $n \to \infty$, it tends to the area under the parabola $y = x^2$ between $x = a$ and $x = b$. [*Hint: The sum of k^2 from $k = 1$ to $k = n$ is $\frac{1}{6}n(n + 1)(2n + 1)$.*]

Paper 1, Question 20(a), 2021 (9 marks)
Evaluate the following integral by finding the limiting value of Riemann sum:

$$J = \int_{\pi/3}^{2\pi/3} \sin(x)\, dx = \operatorname{Im} \int_{\pi/3}^{2\pi/3} e^{ix}\, dx.$$