Part IB: Electromagnetism
Examples Sheet 1 Solutions

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1. Suppose that the current density is $\mathbf{J}(t, \mathbf{x}) = C\mathbf{x}e^{-at|\mathbf{x}|^2}$, where C and a are constants. Show that the charge conservation equation can be satisfied by writing the charge density in the form

$$\rho = (f(\mathbf{x}) + tg(\mathbf{x}))e^{-at|\mathbf{x}|^2},$$

where f and g are to be determined.

Solution: In lectures, we saw that the charge density and the current density are related by the *charge conservation equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

We can compute ρ using this equation, since we know $\mathbf{J}$. First we need to find $\nabla \cdot \mathbf{J}$. It's best to do this with some suffix notation; writing the ith component of the vector $\mathbf{x}$ as x_i, we have:

$$\begin{aligned}
\nabla \cdot \mathbf{J} &= \nabla \cdot (C\mathbf{x}e^{-at|\mathbf{x}|^2}) && \text{(definition of } \mathbf{J}) \\
&= C\frac{\partial}{\partial x_i}(x_i e^{-atx_jx_j}) && \text{(suffix notation gives } |\mathbf{x}|^2 = x_jx_j \text{ and } \nabla \cdot \mathbf{V} = \partial V_i/\partial x_i) \\
&= C\frac{\partial x_i}{\partial x_i}e^{-atx_jx_j} + Cx_i\frac{\partial}{\partial x_i}\left(e^{-atx_jx_j}\right) && \text{(product rule)} \\
&= 3Ce^{-at|\mathbf{x}|^2} + Cx_ie^{-atx_jx_j}\frac{\partial}{\partial x_i}(-atx_jx_j) && (\partial x_i/\partial x_i = 3 \text{ and chain rule}) \\
&= 3Ce^{-at|\mathbf{x}|^2} - Catx_ie^{-atx_jx_j}\left(\delta_{ij}x_j + x_j\delta_{ij}\right) && \text{(product rule on second term)} \\
&= 3Ce^{-at|\mathbf{x}|^2} - 2Cat|\mathbf{x}|^2e^{-at|\mathbf{x}|^2} \\
&= C(3 - 2at|\mathbf{x}|^2)e^{-at|\mathbf{x}|^2}.
\end{aligned}$$

Substituting this into the charge conservation equation, we find that we must solve the equation:

$$\frac{\partial \rho}{\partial t} = C(2at|\mathbf{x}|^2 - 3)e^{-at|\mathbf{x}|^2}. \qquad (*)$$

The question demands that we use the ansatz $\rho = (f(\mathbf{x}) + tg(\mathbf{x}))e^{-at|\mathbf{x}|^2}$, where $f = f(\mathbf{x})$ and $g = g(\mathbf{x})$ are both functions of position (and are, in particular, time-independent). Computing the time-derivative of this form of ρ, we have:

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}\left((f + tg)e^{-at|\mathbf{x}|^2}\right) = -fa|\mathbf{x}|^2e^{-at|\mathbf{x}|^2} + ge^{-at|\mathbf{x}|^2} - tga|\mathbf{x}|^2e^{-at|\mathbf{x}|^2} = \left(g(\mathbf{x}) - f(\mathbf{x})a|\mathbf{x}|^2 - g(\mathbf{x})a|\mathbf{x}|^2t\right)e^{-at|\mathbf{x}|^2}.$$

Comparing the t dependence with $(*)$ we see that we should set:

$$g(\mathbf{x}) = -2C, \qquad \text{and} \qquad f(\mathbf{x}) = \frac{C}{a|\mathbf{x}|^2}.$$

Thus we discover that the charge density is given by:

$$\rho(t, \mathbf{x}) = C\left(\frac{1}{a|\mathbf{x}|^2} - 2t\right)e^{-at|\mathbf{x}|^2}.$$

2. In a fluid environment, charge undergoes *diffusion*. This is described empirically by Fick's law,

$$\mathbf{J} = -D\nabla\rho.$$

where D is called the diffusion coefficient. Show that ρ obeys the heat equation. Show that this is solved by a spreading Gaussian of the form,

$$\rho(t,\mathbf{x}) = \frac{\rho_0 a^3}{(4D(t-t_0)+a^2)^{3/2}}\exp\left(-\frac{|\mathbf{x}|^2}{4D(t-t_0)+a^2}\right),$$

where a, t_0 and ρ_0 are constants.

Solution: Fick's law, given in the question, states that $\mathbf{J} = -D\nabla\rho$. Substituting into the continuity equation we find:

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{J} = -\nabla\cdot(-D\nabla\rho) = D\nabla^2\rho.$$

This is the heat equation, as required. We must now show that this is solved by a spreading Gaussian of the form given in the question:

$$\rho(t,\mathbf{x}) = \frac{\rho_0 a^3}{(4D(t-t_0)+a^2)^{3/2}}\exp\left(-\frac{|\mathbf{x}|^2}{4D(t-t_0)+a^2}\right).$$

First, we evaluate the time-derivative of this form of ρ:

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\rho_0 a^3}{(4D(t-t_0)+a^2)^{3/2}}\right)\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right) + \frac{\rho_0 a^3}{(4D(t-t_0)+a^2)^{3/2}}\frac{\partial}{\partial t}\left(\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right)\right)$$

$$= \frac{\rho_0 a^3\left(-\frac{3}{2}\right)\cdot 4D}{(4D(t-t_0)+a^2)^{\frac{5}{2}}}\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right) + \frac{\rho_0 a^3}{(4D(t-t_0)+a^2)^{\frac{3}{2}}}\cdot\frac{4Dr^2}{(4D(t-t_0)+a^2)^2}\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right)$$

$$= -\frac{6D\rho_0 a^3}{(4D(t-t_0)+a^2)^{5/2}}\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right) + \frac{4Dr^2\rho_0 a^3}{(4D(t-t_0)+a^2)^{7/2}}\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right),$$

where we have used the chain rule extensively.

Now we'll find $D\nabla^2\rho$. The given form of $\rho(t,\mathbf{x})$ only depends on $\mathbf{x}$ through $r^2 = |\mathbf{x}|^2$, so it's best to use the Laplacian in spherical polar coordinates. Recall from Part IA Vector Calculus that the Laplacian in sphericals is:

$$\nabla^2\rho = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\rho}{\partial r}\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\rho}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2\rho}{\partial\phi^2}.$$

We can simplify the right hand side because ρ depends only on $r = |\mathbf{x}|$ (at least in the spatial variables), and hence all derivatives are zero except the radial derivatives. Thus we're left with:

$$D\nabla^2\rho = \frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\rho}{\partial r}\right) = \frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{\rho_0 a^3}{(4D(t-t_0)+a^2)^{3/2}}\cdot\frac{-2r}{4D(t-t_0)+a^2}\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right)\right)$$

$$= \frac{D}{r^2}\frac{\partial}{\partial r}\left(\frac{-2r^3\rho_0 a^3}{(4D(t-t_0)+a^2)^{5/2}}\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right)\right)$$

$$= -\frac{6D\rho_0 a^3}{(4D(t-t_0)+a^2)^{\frac{5}{2}}}\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right) + \frac{4Dr^2\rho_0 a^3}{(4D(t-t_0)+a^2)^{\frac{7}{2}}}\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right).$$

Comparing the formula we've found for $\partial\rho/\partial t$ and $D\nabla^2\rho$, we see that the expressions are exactly the same. Thus the given spreading Gaussian indeed satisfies the heat equation. We interpret this as charge diffusing through the medium over time.

3. Show that the motion of a particle of charge q and mass m in a constant magnetic field $(0, 0, B)$ consists of motion with constant velocity in the z-direction combined with circular motion with angular frequency qB/m in the xy plane. Thus the trajectory of a charged particle in a constant magnetic field is a helix. How is the geometry of the helix related to the initial velocity of the particle?

➥ **Solution:** The force on a charged particle is given by the *Lorentz force law*:

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}),$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively, and $\mathbf{x}(t)$ is the time-dependent position of the particle. Using Newton's second law $\mathbf{F} = m\ddot{\mathbf{x}}$ we can transform this equation into an ordinary differential equation for $\mathbf{x}$:

$$m\ddot{\mathbf{x}} = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}). \tag{$*$}$$

We are given that the magnetic field is constant and given by $\mathbf{B} = B\hat{\mathbf{e}}_z$. We assume that the electric field is zero, i.e. $\mathbf{E} = \mathbf{0}$. Note that the charged particle itself will generate its own magnetic and electric fields, but in this question we only consider the response of the particle to an applied magnetic and electric field (you can think about this as neglecting the effect of the electric and magnetic field produced by the particle, e.g. by supposing that the particle has a very small charge).

Substituting into the ODE $(*)$, we have:

$$\ddot{\mathbf{x}} = \frac{qB}{m}\dot{\mathbf{x}} \times \hat{\mathbf{e}}_z.$$

Notice that we can immediately integrate with respect to time, since both sides of the equation are total time derivatives. Therefore, we have:

$$\dot{\mathbf{x}} = \frac{qB}{m}\mathbf{x} \times \hat{\mathbf{e}}_z + \mathbf{c}$$

where $\mathbf{c}$ is a constant vector of integration. Without loss of generality, assume the particle starts at the origin, $\mathbf{x}(0) = \mathbf{0}$ with initial velocity $\dot{\mathbf{x}}(0) = \mathbf{v}$. Then the constant $\mathbf{c}$ is simply $\mathbf{c} = \mathbf{v}$. Thus our ODE $(*)$ has been reduced to the simpler equation:

$$\dot{\mathbf{x}} = \frac{qB}{m}\mathbf{x} \times \hat{\mathbf{e}}_z + \mathbf{v}.$$

Let's work in coordinates where initially $\mathbf{v}$ has components only in the x-direction and the z-direction, i.e. $\mathbf{v} = v_x\mathbf{e}_x + v_z\mathbf{e}_z$. Then evaluating the cross product, and writing out the ODE in full column vector notation, we have:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \frac{qB}{m}\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} + \begin{pmatrix} v_x \\ 0 \\ v_z \end{pmatrix}.$$

Hence we must solve the coupled system of ODEs:

$$\begin{aligned} \dot{x} &= \frac{qB}{m}y + v_x \\ \dot{y} &= -\frac{qB}{m}x \\ \dot{z} &= v_z. \end{aligned}$$

The third equation trivially has the solution $z = v_z t$ (recall we are working in coordinates where the particle starts at the origin at time $t = 0$). It remains to decouple the first two equations.

Differentiating the first equation with respect to time, we have:

$$\ddot{x} = \frac{qB}{m}\dot{y}.$$

We can now substitute in for $\dot{y}$ using the second equation from our system. We find that an equation for x is:

$$\ddot{x} = \frac{qB}{m}\dot{y} = \frac{qB}{m}\left(-\frac{qB}{m}x\right) = -\frac{q^2B^2}{m^2}x \qquad \Rightarrow \qquad \ddot{x} + \frac{q^2B^2}{m^2}x = 0.$$

We notice that this is the equation of a simple harmonic oscillator with frequency qB/m, which has the standard solution:

$$x(t) = C_1 \cos\left(\frac{qB}{m}t\right) + C_2 \sin\left(\frac{qB}{m}t\right),$$

where C_1 and C_2 are constants. Using the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_x$, we find that $C_1 = 0$ and $C_2 = mv_x/qB$. Hence we have:

$$x(t) = \frac{mv_x}{qB}\sin\left(\frac{qB}{m}t\right).$$

Now let's determine y. The second equation in our coupled system gives y as:

$$\dot{y} = -\frac{qB}{m}x = -\frac{qB}{m}\left(\frac{mv_x}{qB}\sin\left(\frac{qB}{m}t\right)\right) = -v_x \sin\left(\frac{qB}{m}t\right).$$

after substituting in for $x(t)$ using the from we found already. We can integrate this equation directly to get:

$$y(t) = \frac{mv_x}{qB}\cos\left(\frac{qB}{m}t\right) - \frac{mv_x}{qB},$$

where we get the constant of integration since $y(0) = 0$ (the particle starts at the origin). Hence the complete solution to the problem is:

$$x(t) = \frac{mv_x}{qB}\sin\left(\frac{qB}{m}t\right), \qquad y(t) = \frac{mv_x}{qB}\cos\left(\frac{qB}{m}t\right) - \frac{mv_x}{qB}, \qquad z(t) = v_z t.$$

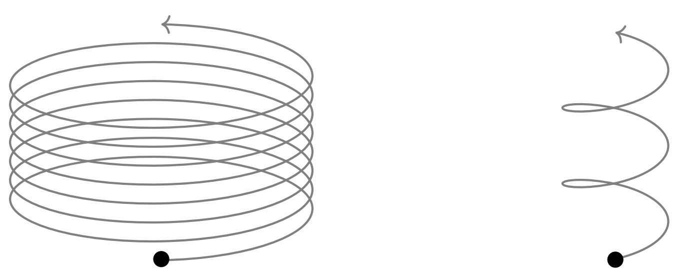
This is indeed the parametric form of a helix. We see that there is constant motion in the z direction of velocity v_z coupled with circular motion in the xy plane of frequency qB/m, as required.

We are finally asked to comment on the relevance of the initial velocity **v** $= (v_x, 0, v_z)$ to the geometry of the helix. By considering $x^2 + (y + mv_x/qB)^2$, we see that

$$x^2 + \left(y + \frac{mv_x}{qB}\right)^2 = \frac{m^2v_x^2}{q^2B^2},$$

and hence v_x is related to the radius of the little circles that make up the helix. That is, the initial velocity perpendicular to the magnetic field determines how large the circles are forming the helix (note that we can always translate the helix via $y \mapsto y - \frac{mv_x}{qB}$ so that the circles are centred on the origin - there is no real dependence of the starting position on v_x).

Looking at $z(t) = v_z t$, we see that the initial velocity parallel to the magnetic field, v_z, determines how fast the particle moves in the z-direction. Thus v_z determines how coiled up the helix is - for small v_z compared to v_x, the helix is highly 'compressed' and there are a lot of turns in a very small z region. For large v_z compared to v_x, the helix is highly 'stretched'.



(a) $v_x \gg v_z$

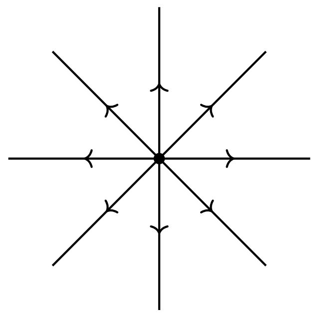
(b) $v_x \ll v_z$

4. Roughly sketch the electric field lines (including arrows to denote sense) and equipotentials (surfaces of constant Φ) for the following system of point charges: (i) a single charge $+q$; (ii) two charges $+q$ separated by a distance $2a$; (iii) two charges $\pm q$ separated by a distance $2a$.

➾ **Solution:** (i) For this part of the question, it's best to recall *Coulomb's law*. For a point charge $+q$, without loss of generality placed at the origin, the electric field is given by:

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0} \cdot \frac{\hat{\mathbf{x}}}{|\mathbf{x}|^2}.$$

In particular, we see that for $q > 0$, the electric field points outwards radially from $\mathbf{0}$. Thus the electric field lines (which recall are the curves *tangent* to the electric field at each point) for a positive point charge look like:



The electric potential Φ is such that $\mathbf{E} = -\nabla\Phi$. We could find Φ explicitly in this case, but it's much easier in the general case to use the result:

Theorem: Let $f(\mathbf{x})$ be a scalar function of multiple variables. At any point $\mathbf{x}_0$, $\nabla f(\mathbf{x}_0)$ is orthogonal to the surface of constant f given by $f(\mathbf{x}) = f(\mathbf{x}_0)$.

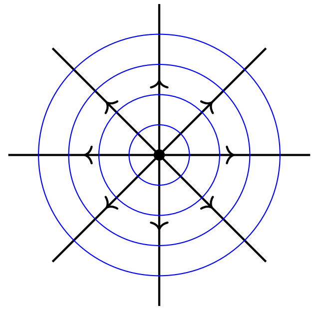
Proof: Let $\mathbf{r}(t)$ be any curve in the surface $f(\mathbf{x}) = f(\mathbf{x}_0)$ such that $\mathbf{r}(t_0) = \mathbf{x}_0$. In particular, since $\mathbf{r}(t)$ lies in the surface $f(\mathbf{x}) = f(\mathbf{x}_0)$ for all t, we must have $f(\mathbf{r}(t)) = f(\mathbf{x}_0)$ for all t. Hence it follows that:

$$0 = \frac{d}{dt}\left(f(\mathbf{r}(t))\right)\bigg|_{t=t_0} = \frac{\partial f}{\partial r_i}\bigg|_{\mathbf{x}=\mathbf{x}_0} \cdot \frac{dr_i}{dt}\bigg|_{t=t_0} = \mathbf{r}'(t_0) \cdot \nabla f(\mathbf{x}_0)$$

It follows that $\nabla f(\mathbf{x}_0)$ is perpendicular to any tangent vector to the surface $f(\mathbf{x}) = f(\mathbf{x}_0)$ at $\mathbf{x}_0$; hence it is perpendicular to the surface at this point. This is what we wanted to show. □

Applied to our particular case, we see that any location $\mathbf{x}_0$, we have that the surface of constant potential $\Phi(\mathbf{x}) = \Phi(\mathbf{x}_0)$ going through that point must be orthogonal to $\nabla\Phi(\mathbf{x}_0) = -\mathbf{E}(\mathbf{x}_0)$, and hence orthogonal also to $\mathbf{E}(\mathbf{x}_0)$.

Filling in the diagram with equipotential surfaces thus amounts to drawing curves orthogonal to those we have already drawn. Imagining enough electric field lines emanating from the point, it becomes clear that the equipotentials in this situation are simply *circles*; they are plotted below.



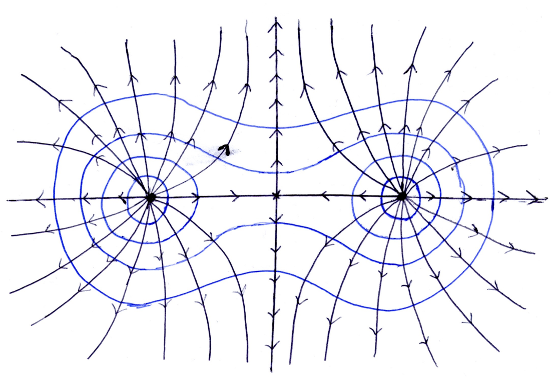
(ii) In the case of two point charges $+q, +q$, separated by a distance $2a$, things are more complicated. It's useful to note some general principles of field line plotting first:

General principles of electric field line plotting:

- Away from charges, we have $\nabla \cdot \mathbf{E} = 0$. In particular, this implies that the electric field is *differentiable* and hence *continuous*.
- Very close to a particular charge, we can ignore the effects of all other charges since the electric field due to the other charges decreases away from them via an inverse square law. Thus the electric field close to any of our point charges will look similar to that of a single, isolated point charge.
- The electric field lines are curves $\mathbf{x}(\lambda)$ that are tangential to the electric field; that is, they obey the defining differential equation:
$$\frac{d\mathbf{x}}{d\lambda} = \mathbf{E}(\mathbf{x}(\lambda)),$$
where λ is some parameter. In particular, we notice that the tangent vector is always smooth and continuous except for when we are at the location of a point charge.

  However, it *is* possible for the tangent to vanish too: $d\mathbf{x}/d\lambda = \mathbf{0}$. This occurs at locations where $\mathbf{E} = \mathbf{0}$ (these points are called *neutral* or *null* points); it is unclear what the tangent vector will be doing there, and hence we require further analysis at points where $\mathbf{E} = \mathbf{0}$.
- Finally, note that we can often exploit symmetry principles in these problems. For example, we might be able to reflect part of a diagram to get another part of the diagram.

Using these principles, it's possible to sketch the field lines and equipotentials as follows:



Let's discuss how we would get to this sketch using the plotting principles:

- First, we might notice that there is a reflectional symmetry in the line through the two $+q$ charges, and a reflectional symmetry in the perpendicular bisector to this line through the origin. Thus it's sufficient only to draw one quadrant, and extend to the rest by symmetry.

- On the lines of symmetry, it must be the case that the electric field is parallel to the line of symmetry. For, if it wasn't, then under the appropriate reflection we would get a charge distribution that was invariant, but an electric field that wasn't invariant.

- Near each of the charges, the electric field lines look like that of a point charge $+q$.

- Recall that we must be careful around points of zero electric field. In this case, the points of zero electric field satisfy:

$$\frac{q(\mathbf{x}-\mathbf{x}_1)}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}_1|^3}+\frac{q(\mathbf{x}-\mathbf{x}_2)}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}_2|^3}=\mathbf{0}, \tag{$*$}$$

  where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the positions of the two charges $+q$, $+q$. Moving one of the terms to the right hand side, and taking the modulus, we find that this equation implies:

$$\frac{1}{|\mathbf{x}-\mathbf{x}_1|^2}=\frac{1}{|\mathbf{x}-\mathbf{x}_1|^2} \qquad \Rightarrow \qquad |\mathbf{x}-\mathbf{x}_1|=|\mathbf{x}-\mathbf{x}_2|.$$

  Thus any point of zero electric field must be equidistant from both $+q$ charges. Substituting this into ($*$) we find:

$$\frac{\mathbf{x}-\mathbf{x}_1}{|\mathbf{x}-\mathbf{x}_1|^3}+\frac{\mathbf{x}-\mathbf{x}_2}{|\mathbf{x}-\mathbf{x}_1|^3}=\mathbf{0} \qquad \Rightarrow \qquad \mathbf{x}-\mathbf{x}_1+\mathbf{x}-\mathbf{x}_2=\mathbf{0},$$

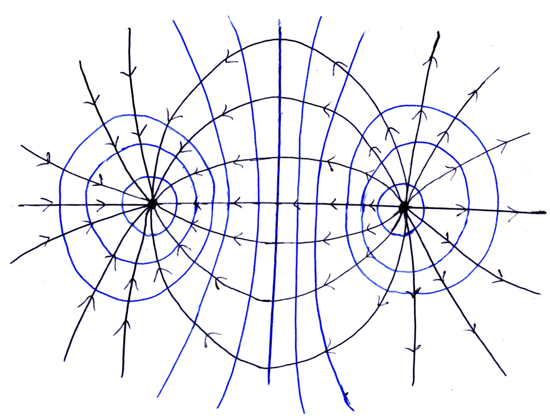
  and hence the only point of zero electric field is:

$$\mathbf{x}=\frac{\mathbf{x}_1+\mathbf{x}_2}{2}.$$

  This is the midpoint between the two $+q$ charges. In particular, looking at the sketch, we see that the electric field lines look like they're crossing here; they are allowed to do this because the electric field goes to zero in the middle, so the tangent to the electric field has zero length.

- This summarises all the salient points of the diagram. To finish, we simply fill in more electric field lines using continuity of the electric field lines. We then add the equipotentials using orthogonality.

(iii) The diagram for two charges $+q$, $-q$ follows by similar logic to the second case (ii):



This time, however, there are no points of zero electric field, which one can prove directly from the equation $\mathbf{E}_1+\mathbf{E}_2=\mathbf{0}$. This means the electric field lines look continuous everywhere except at the point charges. Also note that the $-q$ charge has field lines going *into* it because it has the opposite charge this time.

5. A charge density is given by $\rho = \rho_0 e^{-k|z|}$ with ρ_0 and k positive constants. This is invariant under translations in the xy-plane, rotations about the z-axis, and reflections in the xy-plane, so we assume the electric field has the same symmetries. Show that this implies $\mathbf{E} = (0, 0, E(z))$ with $E(z) = -E(-z)$. Use Gauss' law to show that, for $z > 0$,

$$E(z) = \frac{\rho_0}{\epsilon_0 k}\left(1 - e^{-kz}\right).$$

➾ **Solution:** Let the electric field be $\mathbf{E}(\mathbf{x}) = \mathbf{E}(x, y, z)$. We now apply each of the symmetries in turn to simplify the electric field.

- 1. Translational invariance. Assuming translational invariance in the xy-plane, it must be the case that for all $\Delta x, \Delta y$, we have

$$\mathbf{E}(x + \Delta x, y + \Delta y, z) = \mathbf{E}(x, y, z).$$

  In particular, since this holds for all values of Δx and Δy, it follows that $\mathbf{E}$ cannot depend on x, y. Thus $\mathbf{E}(\mathbf{x}) \equiv \mathbf{E}(z)$.

- 2. Rotational invariance. Now let's apply invariance under rotations about the z-axis. Write the electric field out in terms of its components as: $\mathbf{E}(z) = (E_x(z), E_y(z), E_z(z))$. A general rotation around the z-axis has the matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

  Hence invariance under rotations in the z-axis implies that for all θ, we have

$$\begin{pmatrix} E_x(z) \\ E_y(z) \\ E_z(z) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x(z) \\ E_y(z) \\ E_z(z) \end{pmatrix} = \begin{pmatrix} \cos(\theta)E_x(z) - \sin(\theta)E_y(z) \\ \sin(\theta)E_x(z) + \cos(\theta)E_y(z) \\ E_z(z) \end{pmatrix}.$$

  Since this must hold for all possible values of θ, it follows immediately that $E_x(z) = 0 = E_y(z)$. Thus we see that the electric field is of the form $\mathbf{E}(\mathbf{x}) \equiv (0, 0, E_z(z))$. Since this is the only non-trivial component of $\mathbf{E}$, we can drop the subscript z and instead write $\mathbf{E}(\mathbf{x}) \equiv (0, 0, E(z))$.

- 3. Reflectional invariance. Finally, let's apply invariance under reflections in the xy-plane. Notice that the electric field at a point with z-coordinate z, $(0, 0, E(z))$ is flipped and moved to a point with z-coordinate $-z$. In particular, we see that under the reflection the electric field transforms as $(0, 0, E(z)) \mapsto (0, 0, -E(-z))$.

  Since the electric field is invariant under this reflection, it must be the case that $E(z) = -E(-z)$. Thus we have established all the required properties of the electric field in this case.

We are now ready to use Gauss' law to derive the form of the electric field. In this case, it's simple to use Gauss' law in differential form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}.$$

Using all the symmetry principles, we found that $\mathbf{E} = (0, 0, E(z))$; substituting this form for $\mathbf{E}$, and the form of the charge density ρ, we have

$$\frac{\partial E}{\partial z} = \frac{\rho_0 e^{-k|z|}}{\epsilon_0}.$$

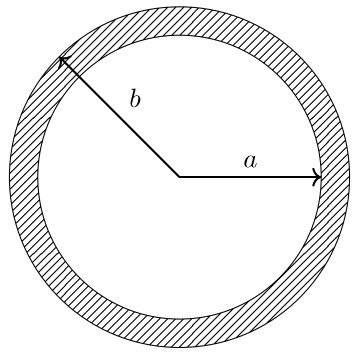
Since we have the relation $E(z) = -E(-z)$, we are only interested in the solution of this equation for $z \geq 0$. It's then trivial to integrate to get:

$$E(z) = \frac{\rho_0}{\epsilon_0 k}\left(C - e^{-kz}\right) \qquad \text{when } z \geq 0,$$

where C is some constant. Now using $E(0) = -E(-0)$, we see that $E(0) = 0$. It follows that $C = 1$, and we get the required electric field.

6. (a) Use Gauss' law to obtain the electric field due to a uniform charge density ρ occupying the region $a < r < b$, with r the radial distance from the origin. (b) Show that in the limit $b \to a$, $\rho \to \infty$ with $(b-a)\rho = \sigma$ fixed, the electric field suffers the expected discontinuity due to surface charge.

Solution: (a) A slice through the spherical shell, going through its centre, is shown in the figure below.



Recall that Gauss' law, in integral form, states that for a closed surface S containing a total charge Q, we have:

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\epsilon_0}.$$

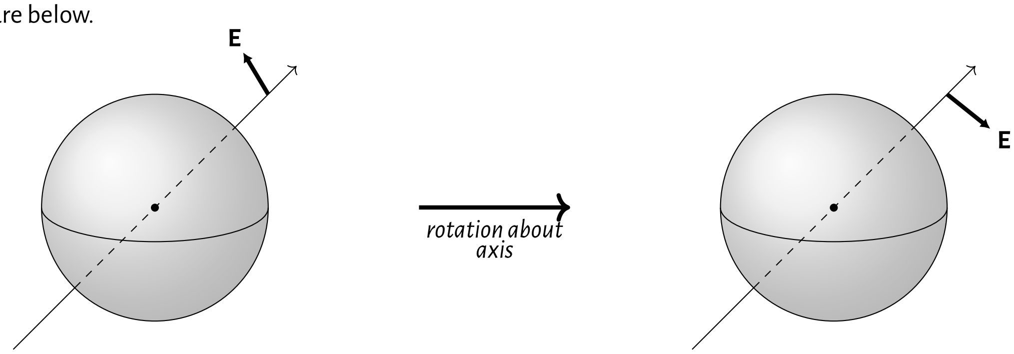
We can use this to determine the electric field for the spherical shell charge distribution as follows. First, let's restrict the functional form of the electric field $\mathbf{E}(\mathbf{x})$ using symmetry arguments:

- 1. Invariance of field under arbitrary rotations of the sphere. Let's work in spherical coordinates (r, θ, ϕ). Note that the charge density $\rho(r)$ is invariant under rotations of the spherical angular coordinates $\theta \mapsto \theta + \Delta\theta$, $\phi \mapsto \phi + \Delta\phi$, and hence the electric field is also invariant under such transformations:

$$\mathbf{E}(r, \theta, \phi) = \mathbf{E}(r, \theta + \Delta\theta, \phi + \Delta\phi).$$

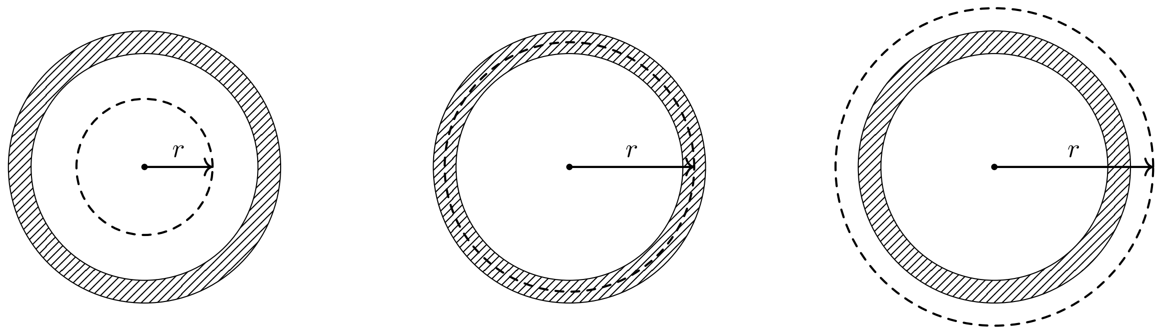
It follows that the electric field is a function of the radius only: $\mathbf{E}(\mathbf{x}) \equiv \mathbf{E}(r)$.

- 2. Invariance of field under rotation through an axis. Now, consider the electric field at any point away from the origin $\mathbf{0}$. Draw an axis through the centre of the sphere and the point at which we are considering the electric field, as in the figure below.



Notice that the charge distribution is invariant under rotations about this axis, hence the electric field must also be invariant under rotations about this axis. But we see from the figure that this can only be true if the electric field is in the *same direction as the axis of rotation*. It follows that at this point, $\mathbf{E} \equiv E_r \hat{\mathbf{r}}$. By rotational invariance of the coordinates θ, ϕ, this is true at all points, and hence we have $\mathbf{E}(\mathbf{x}) \equiv E(r)\hat{\mathbf{r}}$ for some scalar function $E(r)$.

Hence we have shown that $\mathbf{E}(\mathbf{x}) \equiv E(r)\hat{\mathbf{r}}$ for this problem. We are now in a position where we can apply Gauss' law. Let's insert a Gaussian surface S_r as a sphere of radius r centred at the origin $\mathbf{0}$.



(a) $r < a$ (b) $a < r < b$ (c) $r > b$

We will get different results from Gauss' law depending on the radius of the Gaussian surface S_r; in particular, we will get different results in each of the cases $r < a$, $a < r < b$ and $r > b$. In all three cases however, we will need the integral:

$$\int\limits_{S_r} \mathbf{E} \cdot \mathbf{dS} = \int\limits_{S_r} E(r)\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} r^2 d\Omega = 4\pi r^2 E(r),$$

where we have used the fact that the surface element on a sphere is $\mathbf{dS} = r^2 \hat{\mathbf{r}} d\Omega$, where $d\Omega$ is the integral over solid angles: $d\Omega = \sin(\theta) d\theta d\phi$. We have also used the fact that the total solid angle of a sphere is 4π.

We are now ready to use Gauss' law in each of the three cases:

- In the case $r < a$, there is no charge enclosed by the Gaussian surface. Hence Gauss' law gives:

$$\int\limits_{S_r} \mathbf{E} \cdot \mathbf{dS} = 0 \qquad \Rightarrow \qquad 4\pi r^2 E(r) = 0 \qquad \Rightarrow \qquad E(r) = 0.$$

  Hence the electric field vanishes in the region $r < a$.

- In the case $a < r < b$, we enclose some charge in our Gaussian surface. Since there is a uniform charge density ρ in the spherical shell, the amount of charge we enclose is:

$$\rho \cdot (\text{volume of region between } S_r \text{ and } S_a) = \frac{4\pi\rho}{3}(r^3 - a^3).$$

  It follows by Gauss' law that

$$\int\limits_{S_r} \mathbf{E} \cdot \mathbf{dS} = \frac{4\pi\rho}{3\epsilon_0}(r^3 - a^3) \qquad \Rightarrow \qquad 4\pi r^2 E(r) = \frac{4\pi\rho}{3\epsilon_0}(r^3 - a^3) \qquad \Rightarrow \qquad E(r) = \frac{\rho}{3\epsilon_0}\left(r - \frac{a^3}{r^2}\right).$$

- Finally, in the case $r > b$, we enclose all charge on the shell, which by the above work must be

$$\frac{4\pi\rho}{3}(b^3 - a^3).$$

  Hence we see immediately by Gauss' law that the electric field in the region $r > b$ is given by

$$E(r) = \frac{\rho}{3\epsilon_0}\left(\frac{b^3 - a^3}{r^2}\right).$$

Hence, the electric field is given by

$$\mathbf{E}(r,\theta,\phi) = \begin{cases} \mathbf{0} & \text{for } r < a \\ \dfrac{\rho}{3\epsilon_0}\left(r - \dfrac{a^3}{r^2}\right)\hat{\mathbf{r}} & \text{for } a < r < b \\ \dfrac{\rho}{3\epsilon_0}\left(\dfrac{b^3 - a^3}{r^2}\right)\hat{\mathbf{r}} & \text{for } r > b. \end{cases}$$

(b) We are now asked to consider the limit $b \to a, \rho \to \infty$, with $\rho(b-a) = \sigma$ fixed. In the limit $b \to a$, the middle section of the shell disappears, whilst the potential outside the shell obeys:

$$\frac{\rho}{3\epsilon_0 r^2}(b^3 - a^3) = \frac{\rho(b-a)}{3\epsilon_0 r^2}(b^2 + ab + a^2) \to \frac{\sigma a^2}{\epsilon_0 r^2}.$$

Thus the electric field for this problem becomes:

$$E(r) = \begin{cases} 0 & \text{for } 0 \leq r < a \\ \dfrac{\sigma a^2}{\epsilon_0 r^2} & \text{for } a < r. \end{cases}$$

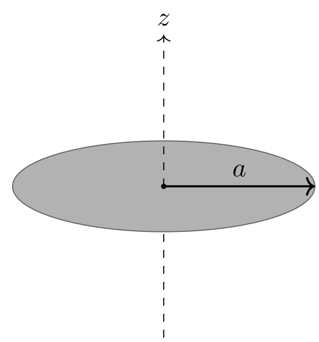
In particular, we see that at $r = a$, the electric field suffers a discontinuity:

$$\lim_{r \to a^+} E(r) - \lim_{r \to a^-} E(r) = \frac{\sigma}{\epsilon_0},$$

which agrees with the matching condition you saw in lectures.

7. A circular disk of radius a has uniform surface charge density σ. Compute the potential at a point on the axis of symmetry at distance z from the centre. Compute the electric field at this point. Find the discontinuity in the normal electric field at the centre of the disk. Show that, far along the axis of symmetry, the electric field looks approximately like that of a charged point particle.

➾ **Solution:** The charged disk is pictured below. We want to find the electric potential Φ along the axis of symmetry through the disk's centre.



Working in cylindrical polar coordinates, we can write the charge density $\rho(r, \theta, z)$ as

$$\rho(r,\theta,z) = \begin{cases} \sigma\delta(z) & \text{for } r < a \\ 0 & \text{otherwise,} \end{cases}$$

since we are given that the disk has uniform surface charge density σ.

Recall from lectures that for an arbitrary bounded charge density the electric potential (assuming $\Phi \to 0$ at infinity) is given by the Green's function solution as:

$$\Phi(\mathbf{x}) = \int\limits_{\mathbb{R}^3} \frac{\rho(\mathbf{x}')}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}'.$$

The cylindrical volume element is $d^3\mathbf{x}' = r'dr'd\theta'dz'$, and the distance between two points in cylindrical coordinates is given by:

$$|\mathbf{x}-\mathbf{x}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} = \sqrt{(r\cos(\theta)-r'\cos(\theta'))^2+(r\sin(\theta)-r'\sin(\theta'))^2+(z-z')^2}.$$

Hence in cylindrical coordinates, our integral becomes:

$$\Phi(r,\theta,z) = \sigma\int\limits_0^a dr' \int\limits_0^{2\pi} d\theta'\, \frac{r'}{4\pi\epsilon_0\sqrt{(r\cos(\theta)-r'\cos(\theta'))^2+(r\sin(\theta)-r'\sin(\theta'))^2+z^2}}.$$

Note we have absorbed the dz' integral and set $z' = 0$ using the delta function $\delta(z')$ in the charge density $\rho(r', \theta', z')$.

This integral is very hard to do - in fact, we can't do it in terms of elementary functions unless r, θ and z take special values. Fortunately, in this question we are only asked to consider the values of Φ along the z-axis, which is described in cylindrical polars by $r = 0$ (and θ arbitrary). Thus the potential along the z-axis is given by:

$$\Phi(z) = \sigma\int\limits_0^a dr' \int\limits_0^{2\pi} d\theta'\, \frac{r'}{4\pi\epsilon_0\sqrt{(r')^2+z^2}} = \frac{\sigma}{2\epsilon_0}\int\limits_0^a dr'\, \frac{r'}{\sqrt{(r')^2+z^2}},$$

where we have performed the trivial integral over θ'.

We can do the r' integral by inspection; notice that the derivative of $\sqrt{(r')^2+z^2}$ is, by the chain rule,

$$\frac{d}{dr'}\left(\sqrt{(r')^2+z^2}\right)=\frac{2r'\cdot\frac{1}{2}}{\sqrt{(r')^2+z^2}}=\frac{r'}{\sqrt{(r')^2+z^2}}.$$

Hence we have

$$\int_0^a dr'\,\frac{r'}{\sqrt{(r')^2+z^2}}=\sqrt{a^2+z^2}-\sqrt{0^2+z^2}=\sqrt{a^2+z^2}-|z|.$$

It follows that the potential along the z-axis is:

$$\Phi(z)=\frac{\sigma}{2\epsilon_0}\left(\sqrt{a^2+z^2}-|z|\right).$$

We can use this to find the electric field by recalling $\mathbf{E}=-\nabla\Phi$. Since we have found that the potential depends only on z along the z axis, we have $\mathbf{E}=-(\partial\Phi/\partial z)\hat{\mathbf{z}}$. Thus the electric field along the z-axis is just given by

$$\mathbf{E}=\begin{cases}\dfrac{\sigma}{2\epsilon_0}\left(1-\dfrac{z}{\sqrt{a^2+z^2}}\right)\hat{\mathbf{z}} & \text{for } z>0\\ \dfrac{\sigma}{2\epsilon_0}\left(-1-\dfrac{z}{\sqrt{a^2+z^2}}\right)\hat{\mathbf{z}} & \text{for } z<0.\end{cases}$$

We see that a discontinuity arises precisely because of the modulus sign in the potential Φ. To calculate this discontinuity, we evaluate the limits:

$$\lim_{z\to 0^+}\mathbf{E}=\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}},\qquad \lim_{z\to 0^-}\mathbf{E}=-\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}.$$

Hence the discontinuity in the normal electric field at the centre of the disk is $(\sigma/\epsilon_0)\hat{\mathbf{z}}$ which exactly agrees with the result you saw in lectures.

Finally, we are asked to consider the limit $|z|\to\infty$. Since the electric field is different for $z>0$ and $z<0$, we need to consider these limits separately.

- Case 1: $z>0$. Let's first consider $z>0$; in this region we can write the electric field along the axis of symmetry as

$$\mathbf{E}=\frac{\sigma}{2\epsilon_0}\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)=\frac{\sigma}{2\epsilon_0}\left(1-\frac{\sqrt{z^2}}{\sqrt{a^2+z^2}}\right)=\frac{\sigma}{2\epsilon_0}\left(1-\frac{1}{\sqrt{1+(a/z)^2}}\right),$$

  For $z\gg a$, we can expand the denominator using the binomial theorem:

$$\mathbf{E}=\frac{\sigma}{2\epsilon_0}\left(1-1+\frac{1}{2}\left(\frac{a}{z}\right)^2+\cdots\right)\sim\frac{\sigma a^2}{4\epsilon_0 z^2}.$$

- Case 2: $z<0$. In the case $z<0$, we perform the same trick, but we instead write $z=-\sqrt{z^2}$:

$$\mathbf{E}=\frac{\sigma}{2\epsilon_0}\left(-1-\frac{z}{\sqrt{a^2+z^2}}\right)=\frac{\sigma}{2\epsilon_0}\left(-1+\frac{\sqrt{z^2}}{\sqrt{a^2+z^2}}\right)=\frac{\sigma}{2\epsilon_0}\left(-1+\frac{1}{\sqrt{1+(a/z)^2}}\right),$$

  Expanding using the binomial theorem, we once again find that:

$$\mathbf{E}\sim\frac{\sigma a^2}{4\epsilon_0 z^2}.$$

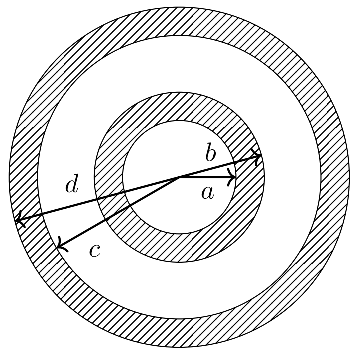
Hence, as $|z|\to\infty$ we find

$$\mathbf{E}\sim\frac{\sigma a^2}{4\epsilon_0 z^2}$$

in all cases. That is, the electric field along the axis is that of a point charge, as required.

8. A capacitor consists of two conductors occupying the regions $a < r < b$ and $c < r < d$ in spherical polar coordinates, where $b < c$. Calculate its capacitance.

➻ **Solution:** A slice through the centre of the two conductors is shown in the figure.



First, we will assume that both conductors carry equal and opposite charge; that is, we assume that the inner conductor has total charge Q and the outer conductor has total charge $-Q$. We make this assumption so that there is no overall charge on the two-conductor system; as we saw in lectures, this would also induce a *self-capacitance* between the system and the ground at infinity - in this question we just want the *mutual capacitance* between the two conductors.

We note that by the same symmetry arguments we used in Question 6, the electric field of the system must take the form:

$$\mathbf{E} = E(r)\hat{\mathbf{r}},$$

where r is a radial coordinate and $\hat{\mathbf{r}}$ is an outward pointing radial unit vector. We now analyse the problem one r region at a time:

- Inside the regions $a < r < b$ and $c < r < d$, we must have $\mathbf{E} = \mathbf{0}$, since these regions are conductors.
- For $r < a$, we have by Gauss' law applied to a Gaussian sphere S_r of radius r inserted into the system:

$$0 = \int\limits_{S_r} \mathbf{E} \cdot \mathbf{dS} = 4\pi r^2 E(r) \qquad \Rightarrow \qquad E(r) = 0.$$

  since S_r encloses no charge. That is, we have $\mathbf{E} = \mathbf{0}$ in $r < a$ also.
- For $r > d$; inserting a Gaussian sphere of radius r into the system and applying Gauss' law, we see that $\mathbf{E} = \mathbf{0}$ similarly, as S_r again encloses zero total charge for $r > d$.
- Finally, for $b < r < c$, if we insert a Gaussian sphere S_r of radius r into the system, we have by Gauss' law:

$$\frac{Q}{\epsilon_0} = \int\limits_{S_r} \mathbf{E} \cdot \mathbf{dS} = 4\pi r^2 E(r) \qquad \Rightarrow \qquad \mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}.$$

  since S_r encloses a total charge Q.

We conclude that the electric field of the system is given by:

$$\mathbf{E} = \begin{cases} \mathbf{0} & \text{for } r < b, \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} & \text{for } b < r < c \\ \mathbf{0} & \text{for } r > c. \end{cases}$$

It follows that a suitable electric potential of the system is given by:

$$\Phi = \begin{cases} A & \text{for } r < b, \\ \dfrac{Q}{4\pi\epsilon_0 r} + B & \text{for } b < r < c \\ C & \text{for } r > c, \end{cases}$$

where A, B and C are constants. Imposing the condition $\Phi \to 0$ as $r \to \infty$, and imposing continuity of Φ at the boundaries, we find that:

$$C = 0, \qquad B = -\frac{Q}{4\pi\epsilon_0 c}, \qquad A = \frac{Q}{4\pi\epsilon_0 b} - \frac{Q}{4\pi\epsilon_0 c}.$$

Thus a suitable electric potential is given by:

$$\Phi = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 b} - \dfrac{Q}{4\pi\epsilon_0 c} & \text{for } r < b, \\ \dfrac{Q}{4\pi\epsilon_0 r} - \dfrac{Q}{4\pi\epsilon_0 c} & \text{for } b < r < c \\ 0 & \text{for } r > c. \end{cases}$$

We are now in a position where we can compute the capacitance. Recall that in lectures we defined the capacitance of two plates carrying equal and opposite charge Q to be:

$$C = \frac{Q}{\Delta\Phi},$$

where $\Delta\Phi$ is the potential difference between the plates. In our case, the two plates are at $r = b, r = c$, and hence the potential difference is:

$$\Delta\Phi = |\Phi(b) - \Phi(c)| = \left(\frac{Q}{4\pi\epsilon_0 b} - \frac{Q}{4\pi\epsilon_0 c}\right) - \left(\frac{Q}{4\pi\epsilon_0 c} - \frac{Q}{4\pi\epsilon_0 c}\right) = \frac{Q(c-b)}{4\pi\epsilon_0 bc}.$$

It follows that the mutual capacitance between the two plates is:

$$C = \frac{4\pi\epsilon_0 bc}{c-b}.$$

9.

(a) This question proves a uniqueness theorem for Poisson's equation. Let V be a connected region with boundary S. Consider two solutions Φ_1 and Φ_2 for Poisson's equation (with the same RHS) in V. Define $\Psi = \Phi_1 - \Phi_2$. Show that

$$\int_V |\nabla\Psi|^2 dV = \int_S \Psi\nabla\Psi \cdot \mathbf{dS}.$$

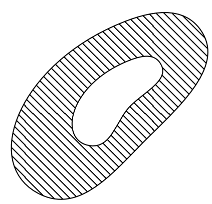
Now assume that Φ_1 and Φ_2 are constant on S. Use Gauss' law to deduce that Ψ is constant throughout V (so the two solutions give the same electric field).

(b) Consider a conductor with a cavity V inside it. Assume that the charge density vanishes in V. Use the result of (a) to prove that $\mathbf{E} = \mathbf{0}$ in V. Thus the conductor *screens* the cavity from the effects of any electric field outside the conductor.

(c) Consider a conductor occupying a finite region V', possible containing cavities. Let S' be a surface lying infinitesimally outside V'. Show that the electric field outside S' depends only on the total charge enclosed by S', i.e. any surface charge on the conductor plus charge inside any cavities. [*Hint: apply the method of (a) to the infinite region outside the conductor.*]

(d) Consider a generalisation of Question 8 where we now allow the two conductors to have arbitrary shape, but still with spherical topology. Explain why (b) and (c) imply that $\mathbf{E}$ vanishes inside the inner conductor and outside the outer conductor. What does this imply about the surface charges?

Solution: (a) This question is quite subtle and we have to be careful not to omit any important details. First of all, we notice that the surface S of V could be made up of many different parts, e.g. consider a conductor with a cavity inside it:



We see that in this case, the boundary of the conductor volume V is the surface $S = S_1 \cup S_2$, formed of the surface of the cavity S_1 and the external surface of the conductor S_2. Throughout we also assume that the volume V is finite, perhaps by putting it in some very large ball, but at the end one can always take the size of this bounding ball to infinity.

In general, we write the surface of the volume V as $S = S_1 \cup S_2 \cup ... \cup S_n$ where the S_i are the boundary surfaces of the volume V. Then, by the divergence theorem, we have

$$\sum_i \int_{S_i} \Psi\nabla\Psi \cdot \mathbf{dS}_i = \int_S \Psi\nabla\Psi \cdot \mathbf{dS} = \int_V \nabla \cdot (\Psi\nabla\Psi) dV,$$

where the outward-pointing normals are included in the area elements in each case. Notice we can simplify $\nabla \cdot (\Psi\nabla\Psi)$ via:

$$\begin{aligned}
\nabla \cdot (\Psi\nabla\Psi) &= \frac{\partial}{\partial x_i}\left(\Psi\frac{\partial\Psi}{\partial x_i}\right) && \text{(in suffix notation)}\\
&= \frac{\partial\Psi}{\partial x_i}\frac{\partial\Psi}{\partial x_i} + \Psi\frac{\partial^2\Psi}{\partial x_i^2} && \text{(by the product rule)}\\
&= (\nabla\Psi)\cdot(\nabla\Psi) + \Psi\nabla^2\Psi && \text{(returning to vector notation)}\\
&= |\nabla\Psi|^2 + \Psi\nabla^2(\Phi_1 - \Phi_2) && \text{(definition of } \Psi\text{)}\\
&= |\nabla\Psi|^2.
\end{aligned}$$

In the last step, we used the fact that both Φ_1 and Φ_2 solve Poisson's equation in the volume V:

$$\nabla^2\Phi_1 = -\rho/\epsilon_0, \qquad \nabla^2\Phi_2 = -\rho/\epsilon_0 \qquad \Rightarrow \qquad \nabla^2(\Phi_1 - \Phi_2) = 0.$$

Hence we have shown that:

$$\sum_i \int_{S_i} \Psi\nabla\Psi \cdot \mathbf{dS}_i = \int_S \Psi\nabla\Psi \cdot \mathbf{dS} = \int_V |\nabla\Psi|^2 dV, \tag{$*$}$$

which is the required result.

We are now told to assume that Φ_1, Φ_2 are constant on S. Since S is a possibly disconnected surface, it could be the case that Φ_1, Φ_2 take certain constant values on some parts other surface, and different constant values on other parts of the surface. That is, it is possible for

$$\Phi_{1,2}\big|_{S_1}, \qquad \Phi_{1,2}\big|_{S_2}, \qquad \dots \qquad \Phi_{1,2}\big|_{S_n}$$

all to take different values, all of which are themselves just constants.

Each surface S_i will also enclose a definite, fixed amount of charge, say Q_i, independent of whether we use the potential Φ_1, Φ_2 (this includes any large bounding surface which contains all other surfaces). Letting $\mathbf{E}_1 = -\nabla\Phi_1$ and $\mathbf{E}_2 = -\nabla\Phi_2$ be the electric fields induced by the potentials Φ_1, Φ_2 respectively *inside* the region bounded by the surface, we hence have by Gauss' law:

$$0 = \frac{Q_i}{\epsilon_0} - \frac{Q_i}{\epsilon_0} = (\pm)\int_{S_i} (\mathbf{E}_1 - \mathbf{E}_2) \cdot \mathbf{dS} = (\pm)\int_{S_i} (\nabla\Phi_2 - \nabla\Phi_1) \cdot \mathbf{dS},$$

where the $\pm$ arises because small, inner surfaces inside the volume V require integrating over a normal which points *into* V in order to integrate inside the region bounded by the surface, whilst on the other hand integrating over a large bounding surface does not require this sign change. Since the result is zero anyway, this subtlety is fairly unimportant.

We use this in the result ($*$) as follows:

$$\begin{aligned}
\int_V |\nabla\Psi|^2 dV &= \sum_i \int_{S_i} \Psi\nabla\Psi \cdot \mathbf{dS}_i \\
&= \sum_i \int_{S_i} (\Phi_1 - \Phi_2)\nabla(\Phi_1 - \Phi_2) \cdot \mathbf{dS}_i && \text{(since } \Psi = \Phi_1 - \Phi_2) \\
&= \sum_i (\Phi_1 - \Phi_2)\big|_{S_i} \int_{S_i} \nabla(\Phi_1 - \Phi_2) \cdot \mathbf{dS}_i && \text{(since } \Phi_1, \Phi_2 \text{ are constant on } S_i) \\
&= 0 && \text{(by Gauss' law argument above)}.
\end{aligned}$$

Hence we see that if Φ_1, Φ_2 are constant on the surface S, we must have

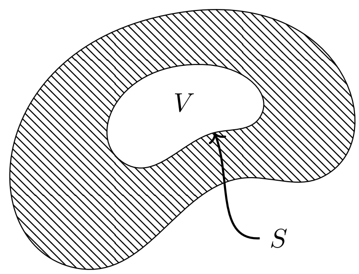
$$\int_V |\nabla\Psi|^2 = 0$$

which implies $\nabla\Psi = 0$ throughout V (since $|\nabla\Psi|^2 \geq 0$). It follows that $\nabla\Phi_1 = \nabla\Phi_2$, i.e. the two potentials induce the same electric field: $\mathbf{E}_1 = \mathbf{E}_2$.

This result is sometimes referred to as the *second uniqueness theorem* for electrostatics (see for example David Griffith's book *Introduction to Electrodynamics*). Stated slightly more clearly, we have:

Second uniqueness theorem of electrostatics: Let V be a connected region bounded by conducting surfaces. Then the electric field throughout V is uniquely determined by specifying: (i) the charge density throughout V; (ii) the *total* charges bounded by each of the conducting surfaces. This includes a 'conductor at infinity', if required.

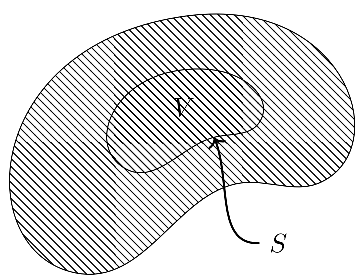
(b) The rest of this question consists of applying the uniqueness theorem we proved in (a) to specific examples. In the first application, we are asked to consider an arbitrary conductor with an empty cavity inside it, say:



Let the volume inside the cavity be V, and let the surface of the cavity be S (technically we can take a surface on the inside of the cavity infinitesimally close to the boundary of the cavity, if we are concerned that we are dealing with a charged conductor - the question does not say). We are asked to prove that the electric field is zero inside V.

We apply the second uniqueness theorem of electrostatics. We are given that the charge density vanishes inside V, and the boundary of V is a conductor. Let's assume a fixed charge Q on the conductor.

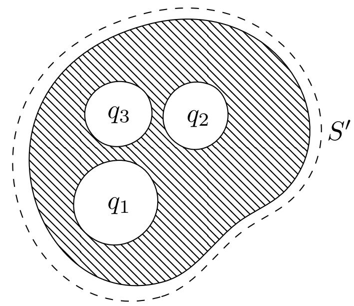
Now consider a different problem. Imagine we fill in the volume V in with the conducting material:



The charge density throughout this volume V is still vanishing, because charges always reside on the surface of conductors. Furthermore the boundary of V is a conductor, and the total charge on the conductor which bounds V is again Q (though all the charge now resides on the outside of the conductor).

But we know the solution to the second problem, since the electric field must vanish inside a conductor. Thus $\mathbf{E} = \mathbf{0}$ for this second problem. By the uniqueness theorem, we also have $\mathbf{E} = \mathbf{0}$ in the first problem too.

(c) In our second example application, we consider a conductor with many cavities, possible containing charges, and possible having an overall surface charge:

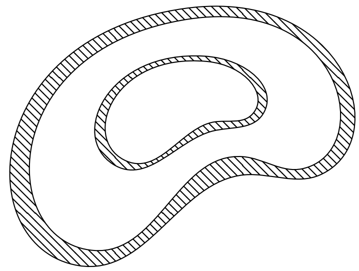


As suggested in the question, we have drawn a surface S' around the outer surface of the conductor, lying infinitesimally close to the surface of the conductor. The phrase 'lying infinitesimally close but outside the conductor' means that the surface S' can be considered to be an equipotential surface (just as the conductor is an equipotential), but can also be considered to contain the surface charge on the outer surface of the conductor if there is any. We are asked to show that the electric field outside of the conductor depends only on the total charge enclosed by the surface S' (and technically the geometry of S').

Again, we consider applying the second uniqueness theorem. In the above scenario, we take the volume V to be the region outside S' (i.e. an infinite region - enclose it in some very large ball if you are concerned with this). In this region, we have vanishing charge density, and the region is also bounded by conductors (with a 'conductor at infinity', if you like). Further, we have that S' contains a total charge $\sigma + q_1 + ... + q_n$, where σ is the total surface charge on the conductor, and q_1, q_2, ... , q_n are the total charges in the cavities; similarly, the 'conductor at infinity' contains this same total charge.

Now we can immediately apply the second uniqueness theorem. If we were to change the internal geometry of the conductor inside S', but keep the total charge contained by S' the same, then the uniqueness theorem tells us we will get exactly the same electric field in the region V. This is precisely the result required by the question.

(d) The final part of this question asks us to apply the results of part (b) and part (c) to a deformed version of the capacitor in Question 8. The phrase 'having arbitrary shape, but still with spherical topology' means that we should just smoothly squish the spherical shells in Question 8, to get something that looks like this:



As in Question 8, we assume that the charge on the inner conductor is Q and the charge on the outer conductor is $-Q$, i.e. they have equal and opposite charges.

By part (b) of this question, we know that since there is no charge in the inner cavity of the internal conductor, the electric field must vanish in this cavity, $\mathbf{E} = \mathbf{0}$, as required

By part (c) of this question, we know that the electric field outside the outer conductor depends only on the geometry of its outer surface and the total charge contained within it, which in this case is $Q - Q = 0$. But this means that the electric field outside the outer conductor is the same as the electric field outside a region which contains no charge and has $\mathbf{E} = \mathbf{0}$ everywhere within. We see that $\mathbf{E} = \mathbf{0}$ outside of the region is a solution of this problem, and hence is the unique solution to the problem by the uniqueness theorem. It follows that $\mathbf{E} = \mathbf{0}$ outside the outer conductor, as required.

Finally, we are asked what this implies for the surface charges in this problem. We note:

- There can only be surface charges on the surfaces of the conductors: the outer surface of the outer conductor, the inner surface of the outer conductor, the outer surface of the inner conductor, and the inner surface of the outer conductor.
- Inside the outer conductor, we have $\mathbf{E} = \mathbf{0}$. Furthermore, outside the outer surface of the outer conductor, we also have $\mathbf{E} = \mathbf{0}$ as we showed above. It follows there can be no surface charge on the outer surface of the outer conductor; for, if there was, there would be some discontinuity in the electric field across this outer surface of the outer conductor, due to non-zero surface charge.
- Similarly, we have $\mathbf{E} = \mathbf{0}$ inside the inner conductor, and we also have $\mathbf{E} = \mathbf{0}$ inside the inner conductor's cavity. By the same argument then, we see that there can be no surface charge on the inner surface of the inner conductor.

It follows that all surface charges reside on the inner surface of the outer conductor, or the outer surface of the inner conductor. In particular, there is a surface charge Q on the outer surface of the inner conductor, and a surface charge $-Q$ on the inner surface of the outer conductor.

10. ($*$) A spherical conducting shell has radius R. A charge q is placed inside the shell at a point $\mathbf{x}_0 = (0,0,d)$ from the centre, with $d < R$. Show that the potential inside the shell can be determined (up to a constant) by placing an appropriate image charge outside the shell at $\mathbf{x}_0' = (0,0,R^2/d)$. Show that the induced surface charge on the shell is

$$\sigma = -\frac{q}{4\pi}\frac{R^2-d^2}{R(R^2-2dR\cos(\theta)+d^2)^{3/2}},$$

where θ is the angle between the point on the shell and the z-axis.

➾ **Solution:** The conducing spherical shell must have constant potential on it; without loss of generality, we can choose to take $\Phi = 0$ on the surface of the sphere. The electric potential created by the charge at $\mathbf{x}_0 = (0,0,d)$ is given by:

$$\Phi_1(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{x}-\mathbf{x}_0|}.$$

Writing the Cartesian components of $\mathbf{x}$ in spherical polar coordinates, we have

$$\mathbf{x} = \begin{pmatrix} r\sin(\theta)\cos(\phi) \\ r\sin(\theta)\sin(\phi) \\ r\cos(\theta) \end{pmatrix} \qquad \Rightarrow \qquad \mathbf{x}-\mathbf{x}_0 = \begin{pmatrix} r\sin(\theta)\cos(\phi) \\ r\sin(\theta)\sin(\phi) \\ r\cos(\theta)-d \end{pmatrix}.$$

It follows that:

$$|\mathbf{x}-\mathbf{x}_0|^2 = r^2\sin^2(\theta)\cos^2(\phi) + r^2\sin^2(\theta)\sin^2(\phi) + (r\cos(\theta)-d)^2 = r^2+d^2-2rd\cos(\theta).$$

Therefore, we can write the electric potential created by the charge as:

$$\Phi_1(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\cdot\frac{1}{\sqrt{r^2+d^2-2rd\cos(\theta)}}.$$

If we place an image charge q' at $\mathbf{x}_0' = (0,0,R^2/d)$ (which for now we allow to have undetermined charge q', which we will vary to ensure the appropriate boundary conditions are satisfied), we can similarly write down the electric potential produced by this charge:

$$\Phi_2(\mathbf{x}) = \frac{q'}{4\pi\epsilon_0}\cdot\frac{1}{\sqrt{r^2+R^4/d^2-2r(R^2/d)\cos(\theta)}}.$$

Thus we see that the total electric potential is the superposition:

$$\Phi(\mathbf{x}) = \Phi_1(\mathbf{x})+\Phi_2(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\cdot\frac{1}{\sqrt{r^2+d^2-2rd\cos(\theta)}} + \frac{q'}{4\pi\epsilon_0}\cdot\frac{1}{\sqrt{r^2+R^4/d^2-2r(R^2/d)\cos(\theta)}}.$$

The location of the surface of the conductor is $r = R$, hence we set $r = R$ and $\Phi = 0$ and then solve for q'. We have:

$$0 = \frac{q}{\sqrt{R^2+d^2-2Rd\cos(\theta)}} + \frac{q'}{\sqrt{R^2+R^4/d^2-2R(R^2/d)\cos(\theta)}}$$

Factoring out R/d from the denominator of the second term, we have:

$$0 = \frac{q}{\sqrt{R^2+d^2-2Rd\cos(\theta)}} + \frac{q'd}{R}\cdot\frac{1}{\sqrt{d^2+R^2-2Rd\cos(\theta)}}.$$

Hence we see that $\Phi = 0$ on $r = R$ if we pick $q' = -qR/d$; the resulting total potential is:

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{\sqrt{r^2 + d^2 - 2rd\cos(\theta)}} - \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{\sqrt{d^2r^2/R^2 + R^2 - 2rd\cos(\theta)}}.$$

In particular, this potential (i) satisfies Poisson's equation inside the shell, and (ii) satisfies the boundary condition $\Phi = 0$ on $r = R$. Thus it is the unique solution for the potential in the region $r \leq R$ (up to a constant).

Notice that adding a constant V to this potential, we can enforce the condition $\Phi = V$ on the surface $r = R$ for any V; this does not affect the electric field because the constant gets differentiated away. Hence we did not lose any generality by imposing $\Phi = 0$ on the surface $r = R$.

Before we compute the induced surface charge on the shell, it is useful to find the potential outside of the shell. Since we want the *induced* surface charge, i.e. the part due to the presence of the charged point particle inside the shell, we can assume that there is no other charge on the shell. Then by the result of part (c) from Question 9, we know that electric field outside the shell depends only on the total charge contained within the conductor and its cavities, and its outer surface's geometry.

But the total charge contained in the conductor and in this case 0. It follows that we can take $\mathbf{E} = \mathbf{0}$ outside the spherical shell, and hence we may choose to take $\Phi = 0$ everywhere outside the shell (in particular, this enforces continuity with our choice that $\Phi = 0$ on the surface of the shell from earlier).

We can now go ahead and calculate the surface charge. We know from the general theory that the discontinuity in the electric field at $r = R$ is given by:

$$\lim_{r \to R^+} \mathbf{E}(\mathbf{x}) - \lim_{r \to R^-} \mathbf{E}(\mathbf{x}) = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}},$$

where σ is the surface charge on the conductor at $r = R$, and $\hat{\mathbf{n}}$ is the normal to the surface $r = R$. But we know $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ in this case, hence we can simply compute:

$$\sigma = \epsilon_0 \left(\lim_{r \to R^+} \hat{\mathbf{r}} \cdot \mathbf{E}(\mathbf{x}) - \lim_{r \to R^-} \hat{\mathbf{r}} \cdot \mathbf{E}(\mathbf{x}) \right) = -\epsilon_0 \lim_{r \to R^-} \hat{\mathbf{r}} \cdot \mathbf{E}(\mathbf{x}),$$

where the second equality follows because $\mathbf{E} = \mathbf{0}$ outside the shell. We thus need to find:

$$\hat{\mathbf{r}} \cdot \mathbf{E}(\mathbf{x}) = -\hat{\mathbf{r}} \cdot \nabla\Phi = -\hat{\mathbf{r}} \cdot \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi} \right) \Phi = -\frac{\partial \Phi}{\partial r},$$

which in the region $r \leq R$ is given by:

$$-\frac{\partial \Phi}{\partial r} = -\frac{q}{4\pi\epsilon_0}\left(\frac{d\cos(\theta) - r}{(r^2 + d^2 - 2rd\cos(\theta))^{3/2}} \right) + \frac{q}{4\pi\epsilon_0}\left(\frac{d\cos(\theta) - d^2r/R^2}{(r^2 + d^2 - 2rd\cos(\theta))^{3/2}} \right).$$

In the limit as $r \to R^-$, we find:

$$-\frac{\partial \Phi}{\partial r}\bigg|_{r=R} = -\frac{q}{4\pi\epsilon_0} \cdot \frac{d^2 - R^2}{R(R^2 + d^2 - 2Rd\cos(\theta))^{3/2}},$$

from which it follows that the induced surface charge is:

$$\sigma = -\frac{q}{4\pi} \cdot \frac{R^2 - d^2}{R(R^2 + d^2 - 2Rd\cos(\theta))^{3/2}}$$

as required.

11.

(a) Show that, far from a charge distribution $\rho(\mathbf{x})$ localised in a region V, the potential takes the form

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2}\frac{\mathbb{Q}_{ij}x_ix_j}{r^5} + \cdots\right),$$

where Q is the total charge, $\mathbf{p}$ is the dipole moment, $\mathbb{Q}_{ij}$ is the quadrupole moment tensor and $r = |\mathbf{x}|$.

(b) Compute Q, $\mathbf{p}$ and $\mathbb{Q}_{ij}$ for: (i) two charges, $+q$ and $-q$, at points $(0,0,0)$ and $(d,0,0)$ respectively; (ii) two charges $+q$ and two charges $-q$ placed on the corners of a square, with sides of length d, such that every charge has an opposite charge to each of its neighbours; (iii) four charges $+q$ and four charges $-q$ placed on the corners of a cube, with sides of length d, such that every charge has an opposite charge from each of its neighbours.

Solution: (a) The first part was done in lectures; we'll review the derivation here. Recall that the Green's function solution for the potential is given by:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int\limits_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}',$$

where V is the volume occupied by the charge density ρ. Since V is localised, we can assume it is contained in some sphere of radius d; 'far away from the charge distribution' then means we are interested in the potential in the region $|\mathbf{x}| \gg d$.

In this region, we can expand the factor of $|\mathbf{x}-\mathbf{x}'|^{-1}$ in the Green's function solution. First, notice that we can write this factor as:

$$|\mathbf{x}-\mathbf{x}'|^{-1} = \left((\mathbf{x}-\mathbf{x}')^2\right)^{-1/2} = \left(|\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{x}' + |\mathbf{x}'|^2\right)^{-1/2} = \frac{1}{|\mathbf{x}|}\left(1 - 2\hat{\mathbf{x}}\cdot\frac{\mathbf{x}'}{|\mathbf{x}|} + \frac{|\mathbf{x}'|^2}{|\mathbf{x}|^2}\right)^{-1/2}.$$

The integration variable $\mathbf{x}'$ in the Green's function solution is such that $|\mathbf{x}| \gg d > |\mathbf{x}'|$, and hence we can expand the expression above using the binomial theorem:

$$\left(1 - 2\hat{\mathbf{x}}\cdot\frac{\mathbf{x}'}{|\mathbf{x}|} + \frac{|\mathbf{x}'|^2}{|\mathbf{x}|^2}\right)^{-1/2} = 1 + \left(-\frac{1}{2}\right)\left(-2\hat{\mathbf{x}}\cdot\frac{\mathbf{x}'}{|\mathbf{x}|} + \frac{|\mathbf{x}'|^2}{|\mathbf{x}|^2}\right) + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-2\hat{\mathbf{x}}\cdot\frac{\mathbf{x}'}{|\mathbf{x}|} + \frac{|\mathbf{x}'|^2}{|\mathbf{x}|^2}\right)^2 + O\left(\frac{|\mathbf{x}'|^3}{|\mathbf{x}|^3}\right)$$

$$= 1 + \frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{|\mathbf{x}|} - \frac{1}{2}\frac{|\mathbf{x}'|^2}{|\mathbf{x}|^2} + \frac{3}{2}\left(\hat{\mathbf{x}}\cdot\frac{\mathbf{x}'}{|\mathbf{x}|}\right)^2 + O\left(\frac{|\mathbf{x}'|^3}{|\mathbf{x}|^3}\right)$$

$$= 1 + \frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{|\mathbf{x}|} + \frac{3(\hat{\mathbf{x}}\cdot\mathbf{x}')^2 - |\mathbf{x}'|^2}{2|\mathbf{x}|^2} + O\left(\frac{|\mathbf{x}'|^3}{|\mathbf{x}|^3}\right).$$

Substituting this back into the Green's function solution, we find:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int\limits_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}|}\left(1 + \frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{|\mathbf{x}|} + \frac{3(\hat{\mathbf{x}}\cdot\mathbf{x}')^2 - |\mathbf{x}'|^2}{2|\mathbf{x}|^2} + O\left(\frac{|\mathbf{x}'|^3}{|\mathbf{x}|^3}\right)\right) d^3\mathbf{x}'$$

$$= \frac{1}{4\pi\epsilon_0}\int\limits_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}|}\left(1 + \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^2} + \frac{3(\mathbf{x}\cdot\mathbf{x}')^2 - |\mathbf{x}|^2|\mathbf{x}'|^2}{2|\mathbf{x}|^4} + O\left(\frac{|\mathbf{x}'|^3}{|\mathbf{x}|^3}\right)\right) d^3\mathbf{x}'$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2}\frac{\mathbb{Q}_{ij}x_ix_j}{r^5} + \cdots\right),$$

where in the last line, we use the definitions of total charge, dipole moment and quadrupole moment:

$$Q = \int\limits_V \rho(\mathbf{x}')\, d^3\mathbf{x}', \qquad \mathbf{p} = \int\limits_V \rho(\mathbf{x}')\mathbf{x}'\, d^3\mathbf{x}', \qquad \mathbb{Q}_{ij} = \int\limits_V \rho(\mathbf{x}')(3x_i'x_j' - \delta_{ij}|\mathbf{x}'|^2)\, d^3\mathbf{x}'.$$

(b) For each of the parts (i), (ii) and (iii) of this question, we are dealing with point charges, so let's derive formulas for the charge, dipole moment and quadrupole moment for a charge density consisting only of point charges.

In general, the charge density looks like:

$$\rho(\mathbf{x}') = \sum_{i=1}^{N} q_r \delta(\mathbf{x}' - \mathbf{x}_r).$$

where there are N point particles of charge q_i respectively. The total charge is given by:

$$Q = \int\limits_V \rho(\mathbf{x}')\, d^3\mathbf{x}' = \sum_{r=1}^{N} q_r.$$

The dipole moment is given by:

$$\mathbf{p} = \int\limits_V \mathbf{x}'\rho(\mathbf{x}')\, d^3\mathbf{x}' = \sum_{r=1}^{N} q_r\mathbf{x}_r.$$

Finally, the quadrupole moment is given by:

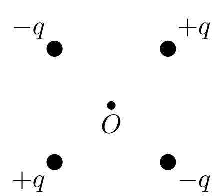
$$\mathbb{Q}_{ij} = \int\limits_V (3x_i'x_j' - \delta_{ij}|\mathbf{x}'|^2)\rho(\mathbf{x}')\, d^3\mathbf{x}' = \sum_{r=1}^{N} q_r \left(3\mathbf{x}_r\mathbf{x}_r^T - |\mathbf{x}_r|^2\mathbb{I}\right),$$

where $\mathbf{x}_i\mathbf{x}_i^T$ is the outer product of vectors (forming a matrix), and $\mathbb{I}$ is the identity matrix. In the first case then, we have a charge q at $\mathbf{0}$ and a charge $-q$ at $(d, 0, 0)$, which gives:

$$Q = q - q = 0, \qquad \mathbf{p} = q\mathbf{0} - q(d, 0, 0)^T = (-qd, 0, 0),$$

$$\mathbb{Q}_{ij} = -q\left(3\begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} d & 0 & 0 \end{pmatrix} - \begin{pmatrix} d^2 & 0 & 0 \\ 0 & d^2 & 0 \\ 0 & 0 & d^2 \end{pmatrix}\right) = \begin{pmatrix} -2qd^2 & 0 & 0 \\ 0 & qd^2 & 0 \\ 0 & 0 & qd^2 \end{pmatrix}.$$

(ii) Let's arrange the charges around the origin like so:



That is, let's put the charges $+q$ at the coordinates $\pm(d/2, d/2, 0)$ and $-q$ at the coordinates $\pm(d/2, -d/2, 0)$. Then, by our formulas from above, we have:

$$Q = q + q - q - q = 0, \qquad \mathbf{p} = q\begin{pmatrix} d/2 \\ d/2 \\ 0 \end{pmatrix} + q\begin{pmatrix} -d/2 \\ -d/2 \\ 0 \end{pmatrix} - q\begin{pmatrix} d/2 \\ -d/2 \\ 0 \end{pmatrix} - q\begin{pmatrix} -d/2 \\ d/2 \\ 0 \end{pmatrix} = \mathbf{0},$$

For the quadrupole moment we note that the outer product of each of our vectors is given by:

$$\begin{pmatrix} A \\ B \\ 0 \end{pmatrix} \begin{pmatrix} C & D & 0 \end{pmatrix} = \begin{pmatrix} AC & AD & 0 \\ BC & BD & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the length of any one of our vectors is just

$$\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2} = \frac{d}{2}.$$

Hence choosing $A = \pm d/2, B = \pm d/2, C = \pm d/2, D = \pm d/2$ as appropriate in our outer product formula, we have

$$\begin{aligned}
\mathbb{Q}_{ij} &= q\left(3\begin{pmatrix} d^2/4 & d^2/4 & 0 \\ d^2/4 & d^2/4 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{d^2}{4}\mathbb{I}\right) + q\left(3\begin{pmatrix} d^2/4 & d^2/4 & 0 \\ d^2/4 & d^2/4 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{d^2}{4}\mathbb{I}\right) \\
&\quad - q\left(3\begin{pmatrix} d^2/4 & -d^2/4 & 0 \\ -d^2/4 & d^2/4 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{d^2}{4}\mathbb{I}\right) - q\left(3\begin{pmatrix} d^2/4 & -d^2/4 & 0 \\ -d^2/4 & d^2/4 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{d^2}{4}\mathbb{I}\right) \\
&= \begin{pmatrix} 0 & 3qd^2 & 0 \\ 3qd^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

(iii) This part is similar to (ii), but now we put the $+q$ charges at the coordinates

$$(d/2, d/2, d/2), \quad (d/2, -d/2, -d/2), \quad (-d/2, d/2, -d/2) \quad \text{and} \quad (-d/2, -d/2, d/2).$$

We put the $-q$ charges at the coordinates

$$(d/2, d/2, -d/2), \quad (d/2, -d/2, d/2), \quad (-d/2, d/2, d/2) \quad \text{and} \quad (-d/2, -d/2, -d/2).$$

The total charge is clearly $Q = 0$, and the dipole moment is easily computed as before:

$$\mathbf{p} = q\left(\begin{pmatrix} d/2 \\ d/2 \\ d/2 \end{pmatrix} + \begin{pmatrix} -d/2 \\ d/2 \\ -d/2 \end{pmatrix} + \begin{pmatrix} -d/2 \\ -d/2 \\ d/2 \end{pmatrix} + \begin{pmatrix} d/2 \\ -d/2 \\ -d/2 \end{pmatrix}\right) - q\left(\begin{pmatrix} -d/2 \\ -d/2 \\ -d/2 \end{pmatrix} + \begin{pmatrix} -d/2 \\ d/2 \\ d/2 \end{pmatrix} + \begin{pmatrix} d/2 \\ -d/2 \\ d/2 \end{pmatrix} + \begin{pmatrix} d/2 \\ d/2 \\ -d/2 \end{pmatrix}\right) = \mathbf{0}.$$

Finally, we must compute the more complicated quadrupole moment. To make things simpler, first notice that all the vectors we are considering have the same length, and there are four with positive $+q$ charge and four with negative $-q$ charge. This means the term:

$$\sum_{i=1}^{8} q_i |\mathbf{x}_i|^2 \mathbb{I} = 0$$

in the quadrupole moment cancels out completely.

The outer products for the quadrupole moment all look something like:

$$\frac{d^2}{4}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

with an appropriate sprinkling of signs in the matrix.

Considering the signs, we see that the outer products for the coordinates of the $+q$ charges are:

$$\frac{d^2}{4}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}, \qquad \frac{d^2}{4}\begin{pmatrix}1 & -1 & -1\\ -1 & 1 & 1\\ -1 & 1 & 1\end{pmatrix}, \qquad \frac{d^2}{4}\begin{pmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{pmatrix}, \qquad \frac{d^2}{4}\begin{pmatrix}1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 1\end{pmatrix},$$

which sum to:

$$d^2\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = d^2\mathbb{I}.$$

Since the coordinates of the $-q$ charges are exactly negative the coordinates of the $+q$ charges, they give the same outer products (since $\mathbf{x}_r\mathbf{x}_r^T = (-\mathbf{x}_r)(-\mathbf{x}_r)^T$), and hence they also sum to $d^2\mathbb{I}$.

It follows that the quadrupole moment of this charge distribution is thus:

$$\mathbb{Q}_{ij} = q(3d^2\mathbb{I}) - q(3d^2\mathbb{I}) = 0.$$

❊ **Comments:** In this question, we have constructed distributions of charges which have zero total charge, zero dipole moment and then zero quadrupole moment successively. It is possible to do so in this case because each of the charge distributions we consider exhibits more and more symmetry.

These situations can arise in physical systems, for example in chemistry, where carbon dioxide molecules have zero dipole moment. This has important implications for modelling, because it means that we can't always treat everything like a point charge/point dipole/point quadrupole...!

12. Show that the force and torque on a point electric dipole at position $\mathbf{x}$ in an electrostatic field are $\mathbf{F} = \mathbf{p} \cdot \nabla \mathbf{E}$ and $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} + \mathbf{x} \times \mathbf{F}$. Deduce that the potential energy of the dipole is $-\mathbf{p} \cdot \mathbf{E}$. Hence show that the electrostatic energy of a pair of dipoles is:

$$E = \frac{1}{4\pi\epsilon_0}\left(\frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} - \frac{3\mathbf{p}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2)\mathbf{p}_2 \cdot (\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^5}\right).$$

Solution: The key fact that we need to remember for this question is that a point dipole $\mathbf{p}$ situated at the point $\mathbf{x}$ has charge distribution

$$\rho(\mathbf{x}') = -\mathbf{p} \cdot \nabla \delta^3(\mathbf{x}' - \mathbf{x}).$$

You saw this in lectures, but to remind you we'll prove it again here:

Theorem: The charge distribution of a point dipole $\mathbf{p}$ at $\mathbf{x}$ is given by $\rho(\mathbf{x}') = -\mathbf{p} \cdot \nabla \delta^3(\mathbf{x}' - \mathbf{x})$.

Proof: A point dipole is the limit of a physical dipole, with charges $+q$ and $-q$ say, as their separation vector $\mathbf{a}$ tends to $\mathbf{0}$ and their dipole moment $\mathbf{p} = q\mathbf{a}$ remains fixed. Without loss of generality, put the centre of the dipole at zero and translate at the end. Then the charge distribution of two point charges separated by a vector $\mathbf{a}$ and centre at the origin is given by:

$$\rho(\mathbf{x}') = q\delta^3\left(\mathbf{x}' - \frac{\mathbf{a}}{2}\right) - q\delta^3\left(\mathbf{x}' + \frac{\mathbf{a}}{2}\right).$$

Taylor expanding the delta functions for small $\mathbf{a}$, we have

$$\rho(\mathbf{x}') = -\frac{q\mathbf{a}}{2} \cdot \nabla \delta^3\left(\mathbf{x}'\right) - \frac{q\mathbf{a}}{2} \cdot \nabla \delta^3\left(\mathbf{x}'\right) + O(q\mathbf{a}^2) = -q\mathbf{a} \cdot \nabla + O(q\mathbf{a}^2)\delta^3(\mathbf{x}').$$

In the limit $\mathbf{a} \to \mathbf{0}$ with $q\mathbf{a} = \mathbf{p}$ fixed, we recover

$$\rho(\mathbf{x}') = -\mathbf{p} \cdot \nabla \delta^3(\mathbf{x}').$$

Translating the initial position of the dipole to $\mathbf{x}$ rather than zero, we obtain the result. $\square$

We can now compute the force and the torque on the dipole, which recall are given by:

$$\mathbf{F} = \int\limits_V \rho(\mathbf{x}')\mathbf{E}(\mathbf{x}')\, d^3\mathbf{x}', \qquad \boldsymbol{\tau} = \int\limits_V \rho(\mathbf{x}')\mathbf{x}' \times \mathbf{E}(\mathbf{x}')\, d^3\mathbf{x}',$$

where V is a volume containing our charge distribution. Note these formulae only hold when the magnetic field is zero: $\mathbf{B} = \mathbf{0}$.

Substituting in our expression for the charge distribution $\rho(\mathbf{x}') = -\mathbf{p} \cdot \nabla \delta^3(\mathbf{x}' - \mathbf{x})$, we find that the force is given by:

$$\begin{aligned}
\mathbf{F} &= -\int\limits_V \mathbf{p} \cdot \left(\nabla \delta^3(\mathbf{x}' - \mathbf{x})\right) \mathbf{E}(\mathbf{x}')\, d^3\mathbf{x}' \\
&= -\int\limits_V p_i \frac{\partial}{\partial x_i'}\left(\delta^3(\mathbf{x}' - \mathbf{x})\right) \mathbf{E}(\mathbf{x}')\, d^3\mathbf{x}' && \text{(using suffix notation)} \\
&= -\int\limits_V p_i \frac{\partial}{\partial x_i'}\left(\delta^3(\mathbf{x}' - \mathbf{x})\mathbf{E}(\mathbf{x}')\right)\, d^3\mathbf{x}' + \int\limits_V p_i \delta^3(\mathbf{x}' - \mathbf{x}) \frac{\partial}{\partial x_i'}\mathbf{E}(\mathbf{x}')\, d^3\mathbf{x}' \\
&= -\left(\int\limits_{\partial V} p_i \delta^3(\mathbf{x}' - \mathbf{x})\mathbf{E}(\mathbf{x}')n_i\, d^3\mathbf{x}'\right) + p_i \frac{\partial}{\partial x_i}\mathbf{E}(\mathbf{x}) && \text{(divergence theorem on first integral)}
\end{aligned}$$

To finish, note that we can take the surface of V to be away from $\mathbf{x}$, and hence we have that $\delta^3(\mathbf{x}' - \mathbf{x}) = 0$ for all $\mathbf{x}'$ on ∂V. It follows that we have

$$\mathbf{F} = \mathbf{p} \cdot \nabla \mathbf{E}(\mathbf{x})$$

as required.

We can find the torque similarly:

$$\begin{aligned}
\boldsymbol{\tau} &= -\int_V \mathbf{p} \cdot \left(\nabla \delta^3(\mathbf{x}' - \mathbf{x})\right) \mathbf{x}' \times \mathbf{E}(\mathbf{x}')\, d^3\mathbf{x}' \\
&= -\int_V p_i \frac{\partial}{\partial x_i'} \left(\delta^3(\mathbf{x}' - \mathbf{x})\right) \mathbf{x}' \times \mathbf{E}(\mathbf{x}')\, d^3\mathbf{x}' && \text{(using suffix notation)} \\
&= -\int_V p_i \frac{\partial}{\partial x_i'} \left(\delta^3(\mathbf{x}' - \mathbf{x})\mathbf{x}' \times \mathbf{E}(\mathbf{x}')\right)\, d^3\mathbf{x}' + \int_V p_i \delta^3(\mathbf{x}' - \mathbf{x}) \frac{\partial}{\partial x_i'} \left(\mathbf{x}' \times \mathbf{E}(\mathbf{x}')\right)\, d^3\mathbf{x}' \\
&= -\left(\int_{\partial V} p_i \delta^3(\mathbf{x}' - \mathbf{x})\mathbf{x}' \times \mathbf{E}(\mathbf{x}') n_i\, d^3\mathbf{x}'\right) + p_i \frac{\partial}{\partial x_i} \left(\mathbf{x} \times \mathbf{E}(\mathbf{x})\right) && \text{(divergence theorem on first integral)}
\end{aligned}$$

Thus we see that:

$$\boldsymbol{\tau} = \mathbf{p} \cdot \nabla (\mathbf{x} \times \mathbf{E}(\mathbf{x})) = p_i \frac{\partial}{\partial x_i} \left(\epsilon_{jkl} x_j E_k\right) = \epsilon_{jkl} p_j E_k + p_i \epsilon_{jkl} x_j \frac{\partial E_k}{\partial x_i} = \mathbf{p} \times \mathbf{E} + \mathbf{x} \times (\mathbf{p} \cdot \nabla \mathbf{E}(\mathbf{x})) = \mathbf{p} \times \mathbf{E} + \mathbf{x} \times \mathbf{F},$$

as required.

Finally, we are asked to find the electrostatic energy due to a pair of dipoles. Imagine keeping a point dipole $\mathbf{p}_1$ stationary at the point $\mathbf{x}_1$, and then bringing in another point dipole $\mathbf{p}_2$ from infinity to the point $\mathbf{x}_2$. The amount of energy stored in the system is then the work done against the field by the second dipole $\mathbf{p}_2$, given by:

$$W = -\int_{\infty}^{\mathbf{x}_2} \mathbf{F} \cdot \mathbf{dx},$$

where $\mathbf{F}$ is the force on the dipole $\mathbf{p}_2$. But we calculated this force earlier - it is given by $\mathbf{F} = \mathbf{p}_2 \cdot \nabla \mathbf{E}(\mathbf{x})$, where $\mathbf{E}(\mathbf{x})$ is the electric field the dipole is moving through.

In this case, the electric field $\mathbf{E}(\mathbf{x})$ is generated by the initial stationary dipole $\mathbf{p}_1$ at position $\mathbf{x}_1$, which as we saw in lectures is given by:

$$\mathbf{E}(\mathbf{x}) = \frac{3((\hat{\mathbf{x}} - \hat{\mathbf{x}}_1) \cdot \mathbf{p}_1)(\hat{\mathbf{x}} - \hat{\mathbf{x}}_1) - \mathbf{p}_1}{4\pi\epsilon_0 |\mathbf{x} - \mathbf{x}_1|^3}.$$

It follows that the energy of a pair of dipoles is:

$$W = -\int_{\infty}^{\mathbf{x}_2} \mathbf{F} \cdot \mathbf{dx} = \mathbf{p}_2 \cdot \int_{\infty}^{\mathbf{x}_2} \nabla \mathbf{E}(\mathbf{x}) \cdot \mathbf{dx} = -\mathbf{p}_2 \cdot \mathbf{E}(\mathbf{x}_2) = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2 - 3\mathbf{p}_1 \cdot (\hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_1)\mathbf{p}_2 \cdot (\hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_1)}{4\pi\epsilon_0 |\mathbf{x}_2 - \mathbf{x}_1|^3},$$

which can be written as:

$$W = \frac{1}{4\pi\epsilon_0} \left(\frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{x}_2 - \mathbf{x}_1|^3} - \frac{3\mathbf{p}_1 \cdot (\mathbf{x}_2 - \mathbf{x}_1)\mathbf{p}_2 \cdot (\mathbf{x}_2 - \mathbf{x}_1)}{|\mathbf{x}_2 - \mathbf{x}_1|^5} \right),$$

as required.

Part IB: Electromagnetism
Examples Sheet 2 Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

1. A constant magnetic field points along the z-axis: $\mathbf{B} = B\hat{\mathbf{e}}_z$. Verify that each of the following vector potentials satisfies $\mathbf{B} = \nabla \times \mathbf{A}$: (a) $\mathbf{A} = xB\hat{\mathbf{e}}_y$; (b) $\mathbf{A} = \frac{1}{2}B(x\hat{\mathbf{e}}_y - y\hat{\mathbf{e}}_x)$; (c) in cylindrical polar coordinates, $\mathbf{A} = \frac{1}{2}rB\hat{\mathbf{e}}_\theta$; (d) in spherical polar coordinates, $\mathbf{A} = \frac{1}{2}r\sin(\theta)B\hat{\mathbf{e}}_\phi$.

➻ **Solution:** For this question, it's useful to recall the general expression for the curl of a vector field in orthogonal curvilinear coordinates (you saw this expression in Part IA Vector Calculus):

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix},$$

where $1, 2, 3$ denote the orthogonal components in some orthogonal curvilinear coordinates. The factors h_1, h_2, h_3 are called *Lamé coefficients*, and are given by:

- $h_x = 1$, $h_y = 1$ and $h_z = 1$ in Cartesian coordinates;
- $h_r = 1$, $h_\theta = r$ and $h_z = 1$ in cylindrical coordinates;
- $h_r = 1$, $h_\theta = r$ and $h_\phi = r\sin(\theta)$ in spherical coordinates.

(a) For $\mathbf{A} = xB\hat{\mathbf{e}}_y$, we have:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \partial_x & \partial_y & \partial_z \\ 0 & xB & 0 \end{vmatrix} = \hat{\mathbf{e}}_x \begin{vmatrix} \partial_y & \partial_z \\ xB & 0 \end{vmatrix} - \hat{\mathbf{e}}_y \begin{vmatrix} \partial_x & \partial_z \\ 0 & 0 \end{vmatrix} + \hat{\mathbf{e}}_z \begin{vmatrix} \partial_x & \partial_y \\ 0 & xB \end{vmatrix} = B\hat{\mathbf{e}}_z.$$

(b) For $\mathbf{A} = \frac{1}{2}B(x\hat{\mathbf{e}}_y - y\hat{\mathbf{e}}_x)$, we have:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \partial_x & \partial_y & \partial_z \\ -\frac{1}{2}yB & \frac{1}{2}xB & 0 \end{vmatrix} = \hat{\mathbf{e}}_x \begin{vmatrix} \partial_y & \partial_z \\ \frac{1}{2}xB & 0 \end{vmatrix} - \hat{\mathbf{e}}_y \begin{vmatrix} \partial_x & \partial_z \\ -\frac{1}{2}yB & 0 \end{vmatrix} + \hat{\mathbf{e}}_z \begin{vmatrix} \partial_x & \partial_y \\ -\frac{1}{2}yB & \frac{1}{2}xB \end{vmatrix} = B\hat{\mathbf{e}}_z.$$

(c) For $\mathbf{A} = \frac{1}{2}rB\hat{\mathbf{e}}_\theta$ in cylindrical coordinates, we have:

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \partial_r & \partial_\theta & \partial_z \\ 0 & \frac{1}{2}r^2 B & 0 \end{vmatrix} = \frac{\hat{\mathbf{e}}_r}{r} \begin{vmatrix} \partial_\theta & \partial_z \\ \frac{1}{2}r^2 B & 0 \end{vmatrix} - \hat{\mathbf{e}}_\theta \begin{vmatrix} \partial_r & \partial_z \\ 0 & 0 \end{vmatrix} + \frac{\hat{\mathbf{e}}_z}{r} \begin{vmatrix} \partial_r & \partial_\theta \\ 0 & \frac{1}{2}r^2 B \end{vmatrix} = B\hat{\mathbf{e}}_z.$$

(d) For $\mathbf{A} = \frac{1}{2}r\sin(\theta)B\hat{\mathbf{e}}_\phi$ in spherical coordinates, we have:

$$\begin{aligned} \nabla \times \mathbf{A} &= \frac{1}{r^2\sin(\theta)}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & r\sin(\theta)\hat{\mathbf{e}}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & \frac{1}{2}Br^2\sin^2(\theta) \end{vmatrix} \\ &= \frac{\hat{\mathbf{e}}_r}{r^2\sin(\theta)} \begin{vmatrix} \partial_\theta & \partial_\phi \\ 0 & \frac{1}{2}Br^2\sin^2(\theta) \end{vmatrix} - \frac{\hat{\mathbf{e}}_\theta}{r\sin(\theta)} \begin{vmatrix} \partial_r & \partial_\phi \\ 0 & \frac{1}{2}Br^2\sin^2(\theta) \end{vmatrix} + \frac{\hat{\mathbf{e}}_\phi}{r} \begin{vmatrix} \partial_r & \partial_\theta \\ 0 & 0 \end{vmatrix} \\ &= B\left(\cos(\theta)\hat{\mathbf{e}}_r - \sin(\theta)\hat{\mathbf{e}}_\theta\right) \end{aligned}$$

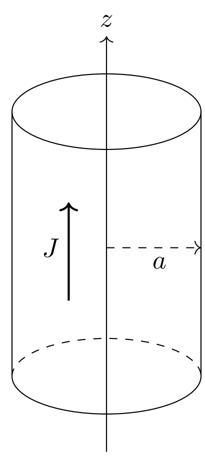
To finish, note that $\cos(\theta)\hat{\mathbf{e}}_r - \sin(\theta)\hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_z$, and hence $\nabla \times \mathbf{A} = B\hat{\mathbf{e}}_z$.

2. (a) A cylindrical conductor of radius a, with axis along the z-axis, carries a uniform current density $\mathbf{J} = J\hat{\mathbf{e}}_z$. Show that the magnetic field within the conductor is given, in cylindrical polar coordinates, by $\mathbf{B} = \frac{1}{2}\mu_0 J r \hat{\mathbf{e}}_\theta$.

(b) A steady current I flows in the z-direction uniformly in the region between the cylinders $x^2 + y^2 = a^2$ and $(x+d)^2 + y^2 = b^2$, where $0 < d < b - a$. Use the superposition principle to show that the associated magnetic field $\mathbf{B}$ throughout the region $x^2 + y^2 < a^2$ is given by

$$\mathbf{B} = \frac{\mu_0 I d}{2\pi(b^2 - a^2)}\hat{\mathbf{e}}_y.$$

➻ **Solution:** (a) The cylindrical conductor is pictured below. We work in cylindrical coordinates throughout for convenience.

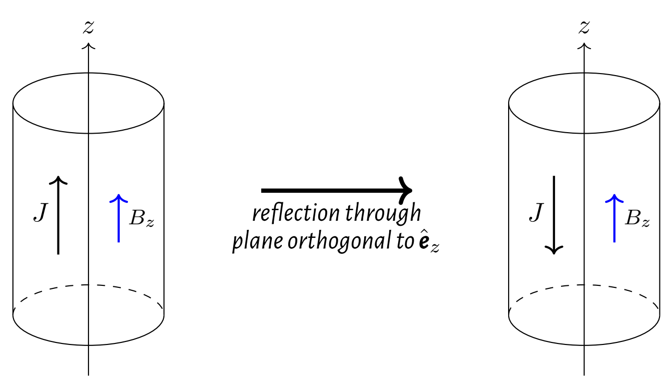


To begin, let's try to reduce the possible functional forms of the magnetic field $\mathbf{B}(r, \theta, z)$ using symmetry principles. We first note that the system is invariant under a rotation of the coordinates $\theta \mapsto \theta + \Delta\theta$ or a translation of the coordinates $z \mapsto z + \Delta z$. Thus we have:

$$\mathbf{B}(r, \theta, z) = \mathbf{B}(r, \theta + \Delta\theta, z + \Delta z),$$

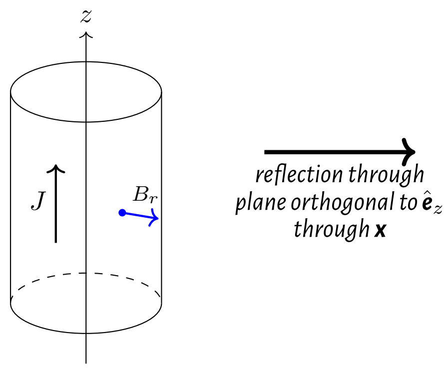
for all $\Delta\theta$, Δz. It follows that $\mathbf{B}(r, \theta, z) \equiv \mathbf{B}(r)$, i.e. $\mathbf{B}$ is a function of radius only.

To further restrict the possible forms of $\mathbf{B}$, suppose that $\mathbf{B}$ has a component parallel to the z axis, $B_z\hat{\mathbf{e}}_z$. Under a reflection in a plane orthogonal to the z-axis, the current flow changes direction $\mathbf{J} \mapsto -\mathbf{J}$, but since $\mathbf{B}$ is a *pseudovector* the component $B_z\hat{\mathbf{e}}_z$ is invariant under this reflection; it follows that we need $B_z = 0$.



Under a reflection through a plane orthogonal to $\hat{\mathbf{e}}_z$, any component of $\mathbf{B}$ in the z-direction, say B_z, is invariant because $\mathbf{B}$ is a pseudovector; however, the current $\mathbf{J}$ does change direction. Hence, uniqueness of the solution implies we need $B_z = 0$.

Similarly, we can show that **B** has no component in the $\hat{\mathbf{e}}_r$ direction. Suppose that at some point **x**, we have that $\mathbf{B}(\mathbf{x})$ has a radial component $B_r\hat{\mathbf{r}}$. Consider reflecting in a plane orthogonal to $\hat{\mathbf{e}}_z$ through the point **x**:

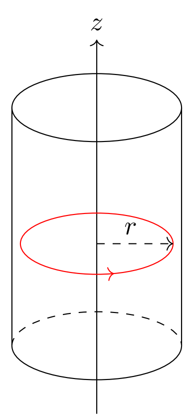


The current changes direction as before. A normal vector would not be reflected in this case, because the vector lies in the plane of reflection; however *pseudovectors* transform in the opposite ways under reflections. Hence $B_r\hat{\mathbf{e}}_r$ *is reflected*. It follows by uniqueness of solution to the problem that we need $B_r = 0$.

We thus conclude by the above argument that **B** must have the functional form: $\mathbf{B} = B(r)\hat{\mathbf{e}}_\theta$. We are now in a position to apply *Ampère's law*. Recall Ampère's law states

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I,$$

where C is a curve forming the boundary of some surface A, and I is the current flowing through that surface. Choose C to be a circle centred on the conductor's axis and of radius $r < a$, traversed anticlockwise.



The left hand side of Ampère's law thus becomes:

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \int_0^{2\pi} B(r) r d\theta = 2\pi r B(r).$$

Choose the surface that C bounds, A, to be the circular area enclosed by C, in a plane orthogonal to $\hat{\mathbf{e}}_z$. Then the amount of current I flowing through the surface A is given by:

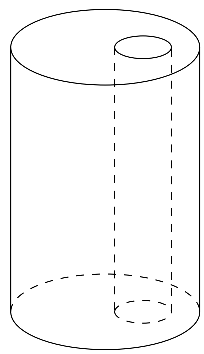
$$\mu_0 I = \mu_0 \int_A \mathbf{J} \cdot d\mathbf{A} = \mu_0 \pi r^2 J.$$

Putting everything together, we find that $B(r) = \frac{1}{2}\mu_0 J r$, and thus

$$\mathbf{B} = \frac{1}{2}\mu_0 J r \hat{\mathbf{e}}_\theta,$$

as required.

(b) The setup for this problem is that a cylinder of radius b, with axis a distance d from the z axis, is now placed around our cylinder in part (a):



We are given that a current I flows in the z-direction in the gap between the big cylinder and the small cylinder. It follows that the current density, i.e. the 'current per unit area', for the system is given by:

$$\mathbf{J} = \left(\frac{I}{\text{cross-sectional area of big cylinder} - \text{cross-sectional area of small cylinder}}\right)\hat{\mathbf{e}}_z = \left(\frac{I}{\pi(b^2 - a^2)}\right)\hat{\mathbf{e}}_z,$$

since no current flows in the small cylinder, so we must discount that area in our calculation.

We now use the superposition principle to compute the magnetic field everywhere. We know from part (a) that if we ignore the big cylinder and suppose that we only have the small cylinder, filled with conducting material and carrying the same current density, that the magnetic field generated is:

$$\mathbf{B}_{\text{small}} = \frac{1}{2}\mu_0 J r \hat{\mathbf{e}}_\theta = \frac{I\mu_0 r}{2\pi(b^2 - a^2)}\hat{\mathbf{e}}_\theta = \frac{I\mu_0}{2\pi(b^2 - a^2)}\left(r\cos(\theta)\hat{\mathbf{e}}_y - r\sin(\theta)\hat{\mathbf{e}}_x\right) = \frac{I\mu_0}{2\pi(b^2 - a^2)}\left(x\hat{\mathbf{e}}_y - y\hat{\mathbf{e}}_x\right)$$

Now suppose that we include the bigger cylinder and fill in the gap with conducting material, and also fill the smaller cylinder with conducting material. Then, translating the solution above using $x \mapsto x + d$, we get the magnetic field in this scenario:

$$\mathbf{B}_{\text{big}} = \frac{I\mu_0}{2\pi(b^2 - a^2)}\left((x + d)\hat{\mathbf{e}}_y - y\hat{\mathbf{e}}_x\right).$$

Now, the equations of electromagnetism are linear so we are free to consider the superposition of these two scenarios. We want to subtract the first scenario from the second, so that no current flows in the smaller cylinder. Thus we get the required magnetic field:

$$\mathbf{B} = \mathbf{B}_{\text{big}} - \mathbf{B}_{\text{small}} = \frac{\mu_0 I d}{2\pi(b^2 - a^2)}\hat{\mathbf{e}}_y.$$

3. Use the Biot-Savart law to determine the magnetic field:

   (a) around an infinite, straight wire carrying current I;

   (b) at the centre of a square loop of wire, with sides of length a, carrying current I;

   (c) at the point $(0,0,z)$ above a loop of wire of radius a, lying in the (x,y) plane, with centre at the origin, carrying current I.

Solution: (a) Since this question is all about the Biot-Savart law, we should really write it down. We recall from lectures that the Biot-Savart law states that for a current density $\mathbf{J}(\mathbf{x})$, the magnetic field is given by:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int\limits_V \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3\mathbf{x}'.$$

For the first scenario of an infinite wire, let's suppose the wire lies in the z-direction. Then the current density is given by:

$$\mathbf{J}(\mathbf{x}') = I\delta(x')\delta(y')\hat{\mathbf{e}}_z$$

Before we substitute directly into the Biot-Savart law, it's useful to consider the form that $\mathbf{x}-\mathbf{x}'$ will take. In cylindrical polar coordinates, we have $\mathbf{x} = r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z$. We also note that because of the delta functions in the current, we only get a contribution from the integral in the Biot-Savart law if the integration variable $\mathbf{x}'$ is such that $\mathbf{x}' = z'\hat{\mathbf{e}}_z$. Therefore the generic form for $\mathbf{x}-\mathbf{x}'$ in this case is:

$$\mathbf{x}-\mathbf{x}' = r\hat{\mathbf{e}}_r + (z-z')\hat{\mathbf{e}}_z, \qquad \Rightarrow \qquad |\mathbf{x}-\mathbf{x}'|^2 = r^2 + (z-z')^2.$$

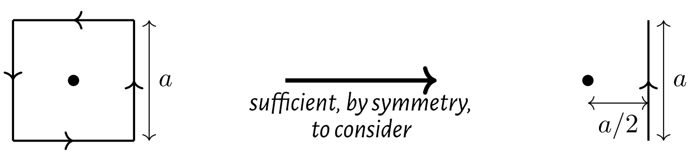
We now substitute into the Biot-Savart law:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\int\limits_{-\infty}^{\infty} \frac{\hat{\mathbf{e}}_z \times (r\hat{\mathbf{e}}_r + (z-z')\hat{\mathbf{e}}_z)}{(r^2+(z-z')^2)^{3/2}}\, dz' = \frac{\mu_0 I r}{4\pi}\hat{\mathbf{e}}_\theta \int\limits_{-\infty}^{\infty} \frac{1}{(r^2+(z-z')^2)^{3/2}}\, dz'.$$

It remains to evaluate the integral. A good substitution to make is $z' = r\tan(u)$. Then the measure transforms as $dz' = r\sec^2(u)du$, and the limits transform as $(-\infty,\infty)\mapsto(-\pi/2,\pi/2)$. It follows that the magnetic field is given by:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I r}{4\pi}\hat{\mathbf{e}}_\theta \int\limits_{-\pi/2}^{\pi/2} \frac{r\sec^2(u)}{(r^2+r^2\tan^2(u))^{3/2}}\, du = \frac{\mu_0 I}{4\pi r}\hat{\mathbf{e}}_\theta \int\limits_{-\pi/2}^{\pi/2} \cos(u)\, du = \frac{\mu_0 I}{2\pi r}\hat{\mathbf{e}}_\theta.$$

(b) By symmetry, the contribution to the magnetic field at the centre of the square will be the same from each side of the square. Hence we can simply work out the magnetic field due to a single side of the square:



We can describe the current density due to one side of the square as:

$$\mathbf{J}(\mathbf{x}') = \begin{cases} I\delta(x'-a/2)\delta(z')\hat{\mathbf{e}}_y & \text{for } |y| < a/2, \\ \mathbf{0} & \text{otherwise} \end{cases}$$

if we put the origin at the centre of the square, and put the wire in the xy-plane aligned parallel to the y-axis.

Again, it's useful to simplify $\mathbf{x}-\mathbf{x}'$ first. In this case, we only want to know the magnetic field at the origin: $\mathbf{x}=\mathbf{0}$. The delta functions also mean we only get a contribution from the integral if the integration variable is such that $\mathbf{x}' = \frac{1}{2}a\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y$. Hence, by the Biot-Savart law, we have the magnetic field due to one side of the square:

$$\mathbf{B}(\mathbf{0}) = \frac{\mu_0 I}{4\pi}\int\limits_{-a/2}^{a/2} \frac{\hat{\mathbf{e}}_y \times (-\frac{1}{2}a\hat{\mathbf{e}}_x - y\hat{\mathbf{e}}_y)}{\left((y')^2+\frac{1}{4}a^2\right)^{3/2}}\,dy' = \frac{\mu_0 a I}{8\pi}\hat{\mathbf{e}}_z \int\limits_{-a/2}^{a/2} \frac{1}{\left((y')^2+\frac{1}{4}a^2\right)^{3/2}}\,dy'.$$

Make the substitution $y' = \frac{1}{2}a\tan(u)$ to evaluate the integral, just as before. The measure changes as $dy' = \frac{1}{2}a\sec^2(u)\,du$, and the limits change as $[-a/2, a/2] \mapsto [-\pi/4, \pi/4]$. Thus we find:

$$\mathbf{B}(\mathbf{0}) = \frac{\mu_0 I}{2\pi a}\hat{\mathbf{e}}_z \int\limits_{-\pi/4}^{\pi/4} \cos(u)\,du = \frac{\sqrt{2}\mu_0 I}{2\pi a}\hat{\mathbf{e}}_z.$$

Hence the total electric field, due to all four sides of the square, at the centre of the square is:

$$\mathbf{B}_{\text{tot}}(\mathbf{0}) = 4\cdot\frac{\sqrt{2}\mu_0 I}{2\pi a}\hat{\mathbf{e}}_z = \frac{2\sqrt{2}\mu_0 I}{\pi a}\hat{\mathbf{e}}_z.$$

(c) The current density for the current loop is given by:

$$\mathbf{J}(\mathbf{x}') = I\delta(r'-a)\delta(z')\hat{\mathbf{e}}_\theta$$

in cylindrical polar coordinates. Before substituting directly into the Biot-Savart law, we note that we want the value of the magnetic field at the point $(0,0,z) = z\hat{\mathbf{e}}_z$, and because of the delta functions we will only get a contribution from the integral in the Biot-Savart Law if the integration variable $\mathbf{x}'$ is given by $\mathbf{x}' = a\hat{\mathbf{e}}_r$. Notice also that the squared length of $\mathbf{x}-\mathbf{x}'$ in this case is given by:

$$|\mathbf{x}-\mathbf{x}'|^2 = |\mathbf{x}|^2 + |\mathbf{x}'|^2 - 2\mathbf{x}\cdot\mathbf{x}' = z^2 + a^2 - 2az\hat{\mathbf{e}}_z\cdot\hat{\mathbf{e}}_r = z^2 + a^2.$$

Therefore, in this case the Biot-Savart law gives us:

$$\mathbf{B}(0,0,z) = \frac{\mu_0 I}{4\pi}\int\limits_0^{2\pi} \frac{\hat{\mathbf{e}}_\theta \times (z\hat{\mathbf{e}}_z - a\hat{\mathbf{e}}_r)}{(z^2+a^2)^{3/2}}\,a d\theta = \frac{a^2\mu_0 I}{2(z^2+a^2)^{3/2}}\hat{\mathbf{e}}_z,$$

since the integral over $\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_r$ vanishes as we go around the circle, by symmetry.

4. (a) A steady current I_1 flows around a closed loop C_1. Use the Biot-Savart law to show that the force exerted on this loop by the magnetic field produced by a second loop C_2 carrying current I_2 is

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} d\mathbf{x}_1 \times \left(d\mathbf{x}_2 \times \frac{(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \right).$$

(b) Show how to write this in a form which satisfies Newton's third law $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

➼ **Solution:** (a) We can describe the current loop C_2 by the current density:

$$\mathbf{J}_2(\mathbf{x}') = I\delta(x_1')\delta(x_2')\hat{\mathbf{e}}_3,$$

where x_1', x_2', x_3' are local, orthogonal, curvilinear coordinates at the point $\mathbf{x}'$ such that $\hat{\mathbf{e}}_3$ is tangential to the curve C_2 at the point $\mathbf{x}'$. Substituting into the Biot-Savart law, we find:

$$\mathbf{B}_2(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{I_2 \delta(x_1')\delta(x_2')\hat{\mathbf{e}}_3 \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} dx_1' dx_2' dx_3' = \frac{\mu_0 I_2}{4\pi} \oint_{C_2} \frac{dx_3' \hat{\mathbf{e}}_3 \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} = \frac{\mu_0 I_2}{4\pi} \oint_{C_2} \frac{\mathbf{dx}' \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}.$$

The force exerted by C_2 on a current loop C_1 is given by:

$$\mathbf{F}_{12} = \int_V \mathbf{J}_1(\mathbf{x}) \times \mathbf{B}_2(\mathbf{x})\, d^3\mathbf{x} = I_1 \oint_{C_1} \mathbf{dx} \times \mathbf{B}_2(\mathbf{x}),$$

by substituting the form of $\mathbf{J}_1(\mathbf{x})$ in terms of delta functions. Inserting the formula for $\mathbf{B}_2(\mathbf{x})$ from above, and relabelling the integration variables, we get:

$$\mathbf{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \mathbf{dx}_1 \times \left(\mathbf{dx}_2 \times \frac{(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \right),$$

as required.

(b) We are now asked to exhibit the antisymmetry of the integral expression for $\mathbf{F}_{12}$ we found above; this is not manifest in its current representation. Since we have a vector triple product in the integrand, however, we have a good idea for where to start - expand the triple product!

Recall, from Part IA Vectors and Matrices, *Lagrange's formula* for the vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Applying this to our integrand, we have:

$$\mathbf{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_2} \left(\oint_{C_1} \frac{(\mathbf{x}_1 - \mathbf{x}_2) \cdot \mathbf{dx}_1}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \right) \mathbf{dx}_2 - \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3} (\mathbf{dx}_1 \cdot \mathbf{dx}_2).$$

Now notice that the first integral vanishes. This is because the integrand is a total derivative:

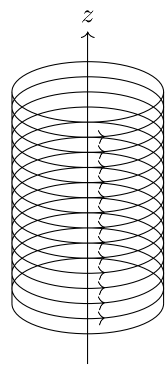
$$\frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} = -\nabla_{\mathbf{x}_1} \left(\frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} \right).$$

Hence the integrand of the first integral is a conservative vector field, and thus its integral around any loop is zero. The remaining term manifestly obeys Newton's third law:

$$\mathbf{F}_{12} = -\frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3} (\mathbf{dx}_1 \cdot \mathbf{dx}_2) = -\mathbf{F}_{21}.$$

5. A surface current experiences a Lorentz force from the *average* magnetic field on either side of the surface. A wire carrying current I winds N times per unit length to form an infinite cylindrical solenoid. Show that the force per unit area on the cylinder is $(\mu_0 I^2 N^2/2)\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the outward unit normal.

➾ **Solution:** The setup is pictured below. We work in cylindrical coordinates throughout for convenience.



In a small section of length dz along the cylinder's axis, there are Ndz turns since there are N turns per unit length. It follows that in a small section of length dz, a current $INdz$ passes through. Hence we can write the current density as:

$$\mathbf{J}(\mathbf{x}) = IN\delta(r-a)\hat{\mathbf{e}}_\theta$$

where a is the radius of the cylinder. This is usually written in the form: $\mathbf{J}(\mathbf{x}) = \mathbf{K}\delta(r-a)$, where $\mathbf{K}$ is called the *surface current*. In our case, we have $\mathbf{K} = IN\hat{\mathbf{e}}_\theta$.

We are given that the Lorentz force on the solenoid is induced by the *average* of the magnetic field on both sides of the surface current. Thus in order to find the force, we should first find the magnetic field. Note first that we can reduce the functional form of the magnetic field by using symmetries:

- By rotational and translational symmetry of the coordinates, we must have $\mathbf{B} = \mathbf{B}(r)$.
- Suppose that $\mathbf{B}$ has a component in the radial direction. Then reflecting in the plane parallel to this component, and orthogonal to the axis of the solenoid, we see that the current does not change direction but the component in the radial direction does (because $\mathbf{B}$ is a pseudovector). Hence $\mathbf{B}$ has no radial component.
- Similarly, suppose that $\mathbf{B}$ has a component in the θ direction. Then again reflecting in the plane parallel to this component, and orthogonal to the axis of the solenoid, we see that the current does not change direction but the component in the θ direction does. Hence $\mathbf{B}$ has no θ component.

We conclude that the $\mathbf{B}$ field for this problem has the functional form $\mathbf{B} = B(r)\hat{\mathbf{e}}_z$. Now recall Ampère's law in differential form:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Away from the surface, we therefore have $\nabla \times \mathbf{B} = \mathbf{0}$. It follows, using the expression for the curl in cylindrical coordinates, that

$$\mathbf{0} = \nabla \times \mathbf{B} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \partial_r & \partial_\theta & \partial_z \\ 0 & 0 & B(r) \end{vmatrix} = -\frac{dB}{dr}\hat{\mathbf{e}}_\theta.$$

We conclude that $B =$ constant away from the surface. Imposing the boundary condition $\mathbf{B} \to \mathbf{0}$ as $\mathbf{x} \to \infty$ (the magnetic field at infinity shouldn't be affected by the solenoid), we see that $B = 0$ for $r > a$.

To finish, we need to find the magnetic field inside the solenoid. We could used Ampère's law in integral form to do this, but a quicker way is just to recall that at a surface current the magnetic field suffers the discontinuity:

$$\mathbf{B}_+ - \mathbf{B}_- = \mu_0 \mathbf{K} \times \hat{\mathbf{e}}_r = \mu_0 IN \hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_r = -\mu_0 IN \hat{\mathbf{e}}_z.$$

It follows that in our case, the magnetic field inside the solenoid must be given by:

$$\mathbf{B} = \mu_0 IN \hat{\mathbf{e}}_z.$$

Thus the magnetic field in this problem is given by:

$$\mathbf{B} = \begin{cases} \mu_0 IN \hat{\mathbf{e}}_z & \text{for } r < a; \\ \mathbf{0} & \text{for } r > a. \end{cases}$$

We are now in a position to calculate the force per unit area on the cylinder. Recall that the Lorentz force law gives the force per unit volume as:

$$\mathbf{f} = \mathbf{J} \times \mathbf{B}.$$

Integrating over radial distance, the delta function in **J** absorbs the integral leaving us the force per unit area:

$$\mathbf{f}_A = \mathbf{K} \times \mathbf{B}.$$

We are told in the question that to determine the force on the surface current, we must insert the average of the magnetic field on both sides of the surface. We thus find:

$$\mathbf{f}_A = \mathbf{K} \times \mathbf{B} = IN \hat{\mathbf{e}}_\theta \times \left(\frac{1}{2} \left(\mathbf{0} + \mu_0 IN \hat{\mathbf{e}}_z \right) \right) = \frac{1}{2} \mu_0 I^2 N^2 \hat{\mathbf{e}}_r,$$

as required.

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6. (a) An atom has magnetic dipole moment **m**. It is arranged experimentally that **m** is anti-parallel to the local magnetic field. Show that the potential energy V_{dipole} of the atom is minimised when $|\mathbf{B}|$ is minimised.

(b) Two infinite straight wires lie at $x = \pm a$ and $y = 0$ and carry a current I in the z-direction. There is also an external magnetic field $\mathbf{B} = B_0 \hat{\mathbf{e}}_z$. By expanding in x and y, show that the potential energy of the atom has a local minimum at $x = y = 0$ and determine the frequency of small oscillations about this minimum when $B_0 \gg (\mu_0 I)/(\pi a^2)$. (The result will depend on the mass M of the atom. This is an example of *magnetic trapping*.)

Solution: (a) We saw in lectures that the energy of a magnetic dipole **m** located at the position **x** in the presence of a magnetic field **B** is given by:

$$V_{\text{dipole}} = -\mathbf{m} \cdot \mathbf{B}(\mathbf{x}).$$

In this question, we are additionally told that the magnetic dipole is arranged experimentally to be anti-parallel to the *local* magnetic field; hence we have $\mathbf{m} = -|\mathbf{m}|\mathbf{B}(\mathbf{x})/|\mathbf{B}(\mathbf{x})|$. It follows that we can write the energy of the dipole as:

$$V_{\text{dipole}}(\mathbf{x}) = \frac{|\mathbf{m}|}{|\mathbf{B}(\mathbf{x})|} \mathbf{B}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}) = |\mathbf{m}||\mathbf{B}|.$$

Hence to minimise the potential energy, we must minimise $|\mathbf{B}|$, as required.

(b) We first need to find the magnetic field of the experiment. By the linearity of the equations of electromagnetism, it is sufficient to consider the magnetic field due to each of the individual wires, and the external field, separately, then combine them by superposition at the end.

We already found in Question 3(a) that the magnetic field due to an infinite straight wire carrying a current I positioned along the z-axis is given by:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{2\pi r}\hat{\mathbf{e}}_\theta.$$

Writing $r^2 = x^2 + y^2$ and $\hat{\mathbf{e}}_\theta = \cos(\theta)\hat{\mathbf{e}}_y - \sin(\theta)\hat{\mathbf{e}}_x = (x\hat{\mathbf{e}}_y - y\hat{\mathbf{e}}_x)/r$ in terms of Cartesian coordinates, this can be written instead as:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{2\pi(x^2+y^2)}\left(x\hat{\mathbf{e}}_y - y\hat{\mathbf{e}}_x\right).$$

Translating the wires away from the z-axis so that they lie along $x = \pm a$, $y = 0$, we see that the two magnetic fields that are relevant in this question are:

$$\mathbf{B}_1(\mathbf{x}) = \frac{\mu_0 I}{2\pi((x-a)^2+y^2)}\left((x-a)\hat{\mathbf{e}}_y - y\hat{\mathbf{e}}_x\right), \qquad \mathbf{B}_2(\mathbf{x}) = \frac{\mu_0 I}{2\pi((x+a)^2+y^2)}\left((x+a)\hat{\mathbf{e}}_y - y\hat{\mathbf{e}}_x\right).$$

Thus superposing these magnetic fields, and the external field, the total magnetic field for the system is given by:

$$\mathbf{B}_{\text{tot}}(\mathbf{x}) = \frac{\mu_0 I}{2\pi((x-a)^2+y^2)}\left((x-a)\hat{\mathbf{e}}_y - y\hat{\mathbf{e}}_x\right) + \frac{\mu_0 I}{2\pi((x+a)^2+y^2)}\left((x+a)\hat{\mathbf{e}}_y - y\hat{\mathbf{e}}_x\right) + B_0\hat{\mathbf{e}}_z.$$

We are now asked to show that $x = y = 0$ is a local minimum of the potential energy of the atom. Using part (a), we must show that $|\mathbf{B}_{\text{tot}}|$ is minimised at $x = y = 0$. In this case, we have:

$$|\mathbf{B}_{\text{tot}}|^2 = B_0^2 + \frac{\mu_0^2 I^2}{4\pi^2}\left(\left(\frac{y}{((x-a)^2+y^2)} + \frac{y}{((x+a)^2+y^2)}\right)^2 + \left(\frac{x-a}{((x-a)^2+y^2)} + \frac{x+a}{((x+a)^2+y^2)}\right)^2\right).$$

We expand in small x, y piece by piece:

- Note that the square roots can be expanded using the binomial theorem:

$$\begin{aligned}\left((x-a)^2+y^2\right)^{-1} &= \left(a^2 - 2xa + x^2 + y^2\right)^{-1} = \frac{1}{a^2}\left(1 - \frac{2x}{a} + \frac{x^2}{a^2} + \frac{y^2}{a^2}\right)^{-1} \\ &= \frac{1}{a^2}\left(1 - \left(-\frac{2x}{a} + \frac{x^2}{a^2} + \frac{y^2}{a^2}\right) + \frac{1}{2}(-1)(-2)\left(\frac{2x}{a}\right)^2 + O(3)\right) \\ &= \frac{1}{a^2} + \frac{2x}{a^3} + \frac{3x^2}{a^4} - \frac{y^2}{a^4} + O(3).\end{aligned}$$

  where we have written $O(3)$ to mean terms cubic in x, y.

- We can expand $((x+a)^2+y^2)^{-1}$ similarly to get:

$$\begin{aligned}\left((x+a)^2+y^2\right)^{-1} &= \left(a^2 + 2xa + x^2 + y^2\right)^{-1} = \frac{1}{a^2}\left(1 + \frac{2x}{a} + \frac{x^2}{a^2} + \frac{y^2}{a^2}\right)^{-1} \\ &= \frac{1}{a^2} - \frac{2x}{a^3} + \frac{3x^2}{a^4} - \frac{y^2}{a^4} + O(3).\end{aligned}$$

- We can now start to expand the large brackets:

$$\begin{aligned} &\frac{y}{(x-a)^2+y^2}+\frac{y}{(x+a)^2+y^2} \\ &= y\left(\frac{1}{a^2}+\frac{2x}{a^3}+\frac{3x^2}{a^4}-\frac{y^2}{a^4}+O(3)\right)+y\left(\frac{1}{a^2}-\frac{2x}{a^3}+\frac{3x^2}{a^4}-\frac{y^2}{a^4}+O(3)\right) \\ &= \frac{2y}{a^2}+\frac{6x^2y}{a^4}-\frac{2y^3}{a^4}+O(4). \end{aligned}$$

Similarly, we have:

$$\begin{aligned} &\frac{x-a}{(x-a)^2+y^2}+\frac{x+a}{(x+a)^2+y^2} \\ &= (x-a)\left(\frac{1}{a^2}+\frac{2x}{a^3}+\frac{3x^2}{a^4}-\frac{y^2}{a^4}+O(3)\right)+(x+a)\left(\frac{1}{a^2}-\frac{2x}{a^3}+\frac{3x^2}{a^4}-\frac{y^2}{a^4}+O(3)\right) \\ &= -\frac{2x}{a^2}+O(3). \end{aligned}$$

Putting everything together, we have:

$$|\mathbf{B}_{\text{tot}}|^2 = B_0^2+\frac{\mu_0^2 I^2}{4\pi^2}\left(\left(\frac{2y}{a^2}+\frac{6x^2y}{a^4}-\frac{2y^3}{a^4}+O(4)\right)^2+\left(-\frac{2x}{a^2}+O(3)\right)^2\right) = B_0^2+\frac{\mu_0^2I^2}{\pi^2}\left(\frac{y^2}{a^4}+\frac{x^2}{a^4}+O(4)\right).$$

It follows that if we are at $x=y=0$, and move away, then $|\mathbf{B}_{\text{tot}}|^2$ will increase. Thus we see that indeed $x=y=0$ is a minimum of $|\mathbf{B}_{\text{tot}}|^2$, and hence a minimum of $V_{\text{dipole}}=|\mathbf{m}||\mathbf{B}_{\text{tot}}|$.

Finally, we are asked to determine the frequency of small oscillations of the atom about the minimum. We could go back to the Lorentz force law to work things out, but actually it's much easier to use the fact that the force on the atom will be the derivative of its potential:

$$\mathbf{F} = -\nabla V_{\text{dipole}} = -|\mathbf{m}|\nabla\left(|\mathbf{B}_{\text{tot}}|\right).$$

From above, we have:

$$|\mathbf{B}_{\text{tot}}| = B_0\sqrt{1+\frac{\mu_0^2I^2}{\pi^2B_0^2}\left(\frac{y^2}{a^4}+\frac{x^2}{a^4}+O(4)\right)}.$$

We are given that $B_0 \gg \mu_0 I/\pi a^2$, which tells us that we can neglect all the $O(4)$ terms. Expanding the remaining square root using the binomial theorem:

$$|\mathbf{B}_{\text{tot}}| = B_0\left(1+\frac{\mu_0^2I^2(x^2+y^2)}{2B_0^2\pi^2a^4}+\cdots\right).$$

Now Newton's second law gives:

$$M\ddot{\mathbf{x}} = \mathbf{F} = -B_0|\mathbf{m}|\nabla\left(1+\frac{\mu_0^2I^2(x^2+y^2)}{2B_0^2\pi^2a^4}\right) = -\frac{|\mathbf{m}|\mu_0^2I^2x}{B_0\pi^2a^4}\hat{\mathbf{e}}_x-\frac{|\mathbf{m}|\mu_0^2I^2y}{B_0\pi^2a^4}\hat{\mathbf{e}}_y,$$

where M is the mass of the atom. Comparing components, we see that only the x, y equations are non-trivial; they are both simple-harmonic oscillators with the same frequency. We see that the frequency of oscillation is:

$$\omega = \sqrt{\frac{|\mathbf{m}|\mu_0^2I^2}{MB_0\pi^2a^4}} = \frac{\mu_0 I}{\pi a^2}\sqrt{\frac{|\mathbf{m}|}{MB_0}}.$$

7. (a) A current creates time-dependent electric and magnetic fields, which in cylindrical polar coordinates are given by $\mathbf{E} = e^{-t}\hat{\mathbf{e}}_\theta$ and $\mathbf{B} = r^{-1}e^{-t}\hat{\mathbf{e}}_z$. Verify that these satisfy Maxwell's equations for vanishing charge density and determine the current density.

(b) Let $S(t)$ be a disc in the plane $z = 0$ with radius $R(t) = 1 + t$ and centred on the origin. Let $C(t)$ be the circular boundary of $S(t)$. Evaluate the magnetic flux through $S(t)$ and the emf around $C(t)$. Verify that your results satisfy Faraday's law of induction.

Solution: (a) Recall Maxwell's equations are given by:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \mu_0\left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right).$$

We want to check that the given $\mathbf{E}$, $\mathbf{B}$ satisfy these equations for $\rho = 0$, and some $\mathbf{J}$. Let's check each of the equations in turn:

- For the equations involving divergences of $\mathbf{E}$, $\mathbf{B}$, it is useful to recall that divergence in cylindrical coordinates is given by:
$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}.$$
Thus the divergence of the given $\mathbf{E}$ field is:
$$\nabla \cdot \mathbf{E} = \frac{1}{r}\frac{\partial(e^{-t})}{\partial \theta} = 0,$$
and the divergence of the given $\mathbf{B}$ field is:
$$\nabla \cdot \mathbf{B} = \frac{\partial}{\partial z}\left(r^{-1}e^{-t}\right) = 0.$$
So indeed Maxwell's equations are satisfied when $\rho = 0$.

- For the equations involving the curl of $\mathbf{E}$, $\mathbf{B}$, we can use the expression for the curl in cylindrical coordinates we wrote down in Question 1 of this sheet:
$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \partial_r & \partial_\theta & \partial_z \\ A_r & rA_\theta & A_z \end{vmatrix}.$$
In our case, we have for the given electric field:
$$\nabla \times \mathbf{E} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \partial_r & \partial_\theta & \partial_z \\ 0 & re^{-t} & 0 \end{vmatrix} = \frac{1}{r}\partial_r(re^{-t})\hat{\mathbf{e}}_z = \frac{e^{-t}}{r}\hat{\mathbf{e}}_z = -\frac{\partial \mathbf{B}}{\partial t}.$$
For the given magnetic field, we have:
$$\nabla \times \mathbf{B} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \partial_r & \partial_\theta & \partial_z \\ 0 & 0 & r^{-1}e^{-t} \end{vmatrix} = -\partial_r(r^{-1}e^{-t})\hat{\mathbf{e}}_\theta = \frac{e^{-t}}{r^2}\hat{\mathbf{e}}_\theta.$$
Comparing this with
$$\mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} = \mu_0\mathbf{J} - \mu_0\epsilon_0 e^{-t}\hat{\mathbf{e}}_\theta,$$
we see that for Maxwell's equations to be satisfied, we require the current density to be
$$\mathbf{J} = \left(\epsilon_0 + \frac{1}{\mu_0 r^2}\right)e^{-t}\hat{\mathbf{e}}_\theta.$$

(b) A point on the surface of the circle $C(t) = \{x^2 + y^2 = (1+t)^2, z = 0\}$ movies with velocity $\mathbf{v} = \hat{\mathbf{e}}_r$ since all points on the circle are moving away from the origin radially, with displacement linear in t. It follows that on $C(t)$, we have:

$$\mathbf{v} \times \mathbf{B} = \hat{\mathbf{e}}_r \times (r^{-1}e^{-t})\hat{\mathbf{e}}_z = -r^{-1}e^{-t}\hat{\mathbf{e}}_\theta.$$

The electromotive force around $C(t)$ is therefore given by:

$$\mathcal{E} = \oint\limits_{C(t)} (\mathbf{E}(\mathbf{x}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{x}, t)) \cdot \mathbf{dx} = \int\limits_0^{2\pi} \left(e^{-t} - \frac{e^{-t}}{1+t} \right) (1+t) d\theta = 2\pi t e^{-t}.$$

The magnetic flux through the surface $S(t)$ is given by:

$$\mathcal{F} = \int\limits_{S(t)} \mathbf{B} \cdot \mathbf{dS} = \int\limits_0^{2\pi} \int\limits_0^{1+t} \frac{e^{-t}}{r} r dr d\theta = 2\pi e^{-t}(1+t).$$

Recall that *Faraday's law of induction* relates the electromotive force and the magnetic flux via:

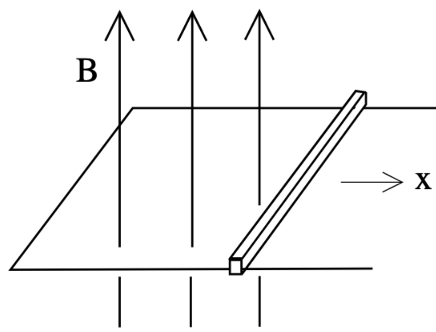
$$\mathcal{E} = -\frac{d\mathcal{F}}{dt}.$$

In our case, we have:

$$-\frac{d\mathcal{F}}{dt} = -\left(-2\pi e^{-t}(1+t) + 2\pi e^{-t}\right) = 2\pi t e^{-t} = \mathcal{E},$$

so Faraday's law of induction is indeed satisfied.

8. A horizontal, rectangular circuit, shown in the figure, has a sliding bar of mass m and length L which moves, without friction, in the x-direction. The sides of the rectangle have lengths L and $x(t)$. The bar has resistance R and the other edges of the circuit have negligible resistance. A uniform vertical magnetic field $\mathbf{B} = B(t)\hat{\mathbf{e}}_z$ is applied for time $t > 0$.



Obtain an expression for the current $I(t)$ that flows around the circuit. Also obtain an expression for the Lorentz force **F** on the bar in terms of $B(t)$ and $I(t)$. Hence show that $x(t)$ must satisfy a differential equation:

$$\frac{d^2x}{dt^2} = -\frac{BL^2}{MR}\frac{d}{dt}(Bx).$$

Solve this equation in the case of constant $B(t)$. Sketch the solution $x(t)$ for $\dot{x}(0) > 0$.

[In this question, you may assume that the effect of the magnetic field due to any current flow is negligible compared to the background **B**.]

Solution: The current in this problem can be obtained by computing the electromotive force $\mathcal{E}$, and then applying *Ohm's law*

$$|\mathcal{E}(t)| = I(t)R.$$

It is tempting to try to calculate the electromotive force directly:

$$\mathcal{E} = \oint_C \left(\mathbf{E}(t,\mathbf{x}) + \mathbf{v} \times \mathbf{B}(t,\mathbf{x})\right) \cdot \mathbf{dx},$$

and whilst this will work when we ignore the induced electric field, it is fraught with danger, since this induced electric field is sometimes important.

Instead, we use *Faraday's law of induction*:

$$\mathcal{E} = -\frac{d\mathcal{F}}{dt}.$$

The magnetic flux through the loop is clearly $B(t)Lx$ in this case (the area of the loop multiplied by the magnitude of the magnetic field). Hence we find the electromotive force is given by:

$$\mathcal{E} = -\frac{d}{dt}\left(B(t)Lx\right),$$

and hence the current is given by (throughout our convention is to always take current positive, and remember its sense when required):

$$I(t) = \left|\frac{L}{R}\frac{d}{dt}\left(B(t)x\right)\right|.$$

Next, we are asked to determine the force **F** on the bar in terms of $B(t)$ and $I(t)$. Recall that we can express **F** in terms of $B(t)$ and $I(t)$ using the following result from lectures:

Theorem: The force on a section of wire C carrying current I is given by

$$\mathbf{F} = I \int_C \mathbf{dx} \times \mathbf{B}(t, \mathbf{x})$$

Proof: The Lorentz force law tells us that the force per unit volume on the wire is given by $\mathbf{f} = \mathbf{J} \times \mathbf{B}$ where $\mathbf{J}$ is the current density. For our wire, the current density is given by:

$$\mathbf{J}(\mathbf{x}) = I\delta(x_1)\delta(x_2)\hat{\mathbf{e}}_3,$$

where x_1, x_2, x_3 are a local system of orthogonal, curvilinear coordinates with $\hat{\mathbf{e}}_3$ tangent to the wire C at the point $\mathbf{x}$, and hence the directions $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ orthogonal to C at the point $\mathbf{x}$. It follows that the total force on the wire is given by:

$$\mathbf{F} = \int_V \mathbf{f}\, d^3\mathbf{x} = \int_V \mathbf{J} \times \mathbf{B}\, d^3\mathbf{x} = I \int_V \delta(x_1)\delta(x_2)\hat{\mathbf{e}}_3 \times \mathbf{B}\, dx_1 dx_2 dx_3 = I \int_C \mathbf{dx} \times \mathbf{B}(\mathbf{x}),$$

as required. □

In our case then, the force on the bar is given by:

$$\mathbf{F} = -I(t) \left(\int_0^L B(t)\, dx \right) \hat{\mathbf{e}}_x = -I(t)B(t)L\hat{\mathbf{e}}_x,$$

where $\hat{\mathbf{e}}_x$ is a unit normal orthogonal to the bar and the magnetic field, and parallel to the sides of the wire of length x. The vector $\hat{\mathbf{e}}_x$ also points *away* from the fixed end of the wire. Note we had to introduce a minus sign because the modulus in Ohm's law $|\mathcal{E}(t)| = IR$ made a difference when we computed the current $I(t)$ above; that is, we are integrating around the wire in the opposite direction to the current.

It follows by Newton's second law, and the expression we found earlier for the current, that we have the relationship:

$$M\frac{d^2x}{dt^2} = -I(t)B(t)L = -\frac{BL^2}{R}\frac{d}{dt}\left(B(t)x\right) \qquad \Rightarrow \qquad \frac{d^2x}{dt^2} = -\frac{BL^2}{MR}\frac{d}{dt}\left(B(t)x\right),$$

as required.

Finally, we are asked to solve this equation for $B(t)$ constant. In that case, we have:

$$\ddot{x} + \frac{B^2L^2}{MR}\dot{x} = 0.$$

Solving this ODE, we start by integrating directly with respect to t:

$$\dot{x} + \frac{B^2L^2}{MR}x = C,$$

where C is a constant. Considering a homogeneous and particular solution to this equation, we arrive at the final expression for x:

$$x(t) = A\exp\left(-\frac{B^2L^2}{MR}t\right) + K,$$

where A and K are constants.

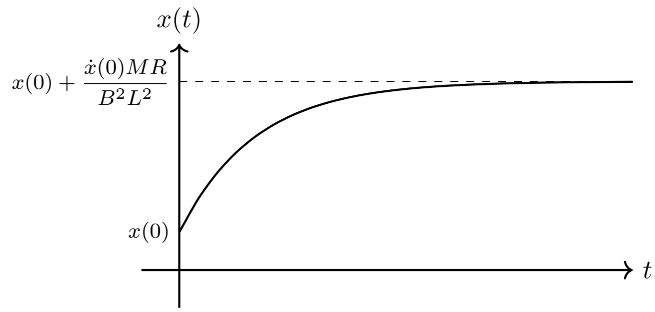
We can express A and K in terms of the initial data $x(0), \dot{x}(0)$:

$$x(0) = A + K, \quad \dot{x}(0) = -\frac{AB^2L^2}{MR}.$$

Thus the solution can be written as:

$$x(t) = \frac{\dot{x}(0)MR}{B^2L^2}\left(1 - \exp\left(-\frac{B^2L^2}{MR}t\right)\right) + x(0).$$

For $\dot{x}(0) > 0$, the solution looks like:



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9. A vector potential is given by $\mathbf{A} = \frac{1}{2}Brz\hat{\mathbf{e}}_\theta$ in cylindrical polar coordinates, where B is a constant.

   (a) Compute the magnetic field $\mathbf{B}$.

   (b) A thin conducting wire of resistance R is formed into a circular loop of radius a. The loop lies in the plane $z = z(t)$ with its centre on the z-axis. Find the induced current in the loop.

   (c) Compute the force exerted on the loop by the magnetic field. To overcome this, an equal and opposite force is applied to the loop. Show that the work done per unit time by this force is equal to the rate of dissipation of energy due to the resistance in the loop.

[In this question, you may assume that the effect of the magnetic field due to any current flow is negligible compared to the background $\mathbf{B}$.]

Solution: (a) We use the formula we wrote down for $\nabla \times \mathbf{A}$ in Question 1:

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \partial_r & \partial_\theta & \partial_z \\ 0 & \frac{1}{2}Br^2z & 0 \end{vmatrix} = -\frac{\hat{\mathbf{e}}_r}{r}\partial_z\left(\frac{1}{2}Br^2z\right) + \frac{\hat{\mathbf{e}}_z}{r}\partial_r\left(\frac{1}{2}Br^2z\right) = Bz\hat{\mathbf{e}}_z - \frac{1}{2}Br\hat{\mathbf{e}}_r.$$

(b) We can calculate the induced current using Ohm's law $I(t) = |\mathcal{E}|/R$, where $\mathcal{E}$ is the electromotive force in the loop. The electromotive force can be obtained using Faraday's law of induction:

$$\mathcal{E}(t) = -\frac{d\mathcal{F}}{dt}.$$

The flux of the magnetic field through the circle can be calculated as follows. The magnetic field is composed of a radial part, parallel to the current loop, and a part in the z-direction, orthogonal to the current loop. Only the part orthogonal to the current loop can contribute to a flux *through* the loop. The flux is therefore given by:

$$\mathcal{F} = (Bz(t)) \cdot \pi a^2.$$

It follows that the electromotive force in this case is given by:

$$\mathcal{E}(t) = -B\dot{z}\pi a^2,$$

and hence the induced current in the loop is:

$$I(t) = \frac{|\mathcal{E}(t)|}{R} = \frac{B\dot{z}\pi a^2}{R}.$$

(c) The force exerted on the loop is given by:

$$\mathbf{F} = -I(t)\oint\limits_C \mathbf{dx} \times \mathbf{B}(t,\mathbf{x}) = -I(t)\cdot\int\limits_0^{2\pi} \hat{\mathbf{e}}_\theta \times \left(Bz\hat{\mathbf{e}}_z - \frac{1}{2}Ba\hat{\mathbf{e}}_r\right)\, ad\theta = -\pi I(t)Ba^2\hat{\mathbf{e}}_z,$$

where the integral of $\hat{\mathbf{e}}_r$ over a circle vanishes. Note the need for a minus sign; this is because our current is flowing clockwise whilst the direction of integration is anticlockwise (we could tell this would be required because the modulus made a difference when we computed $I(t) = |\mathcal{E}(t)|/R$). Therefore, the equal and opposite force need to overcome this force is given by:

$$\mathbf{F}' = \pi I(t)Ba^2\hat{\mathbf{e}}_z.$$

The work done by this force on the loop per unit time is the *power*, given by:

$$\mathbf{F}'\cdot\mathbf{v} = \pi I(t)Ba^2\hat{\mathbf{e}}_z\cdot(\dot{z}\hat{\mathbf{e}}_z) = \pi a^2\dot{z}BI(t) = RI(t)^2,$$

where we used the fact that the velocity of the loop is $\mathbf{v} = \dot{z}\hat{\mathbf{e}}_z$.

Finally, we must compare this to the rate of energy dissipation in the loop due to resistance. Recall from lectures that the formula for this rate is given by:

$$\frac{dW}{dt} = RI(t)^2,$$

and hence the formulas match up, as required. In particular, we note that energy is conserved in the system.

❊ **Comments:** The formula for the rate of work on charged matter $RI(t)^2$ (i.e. power) was derived in lectures assuming an induced electric field resulting from a changing magnetic field, and a fixed wire. Here, we have no induced electric field (we assume it is negligible) since the magnetic field is constant, and the wire *is* moving; however, the formula for power still holds (indeed, in principle nothing should change as the two situations are related to each other by a change of reference frame). We can prove it directly from the formula for Lorentz force per unit volume:

$$\mathbf{f}(\mathbf{x},t) = \mathbf{J}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x}) = I(t)\oint\limits_{C(t)} d\mathbf{x}'\times\mathbf{B}(\mathbf{x})\delta^3(\mathbf{x}'-\mathbf{x}).$$

Dotting with the velocity $\mathbf{v}(\mathbf{x},t)$ of the loop $C(t)$ at the point $\mathbf{x}$ and at time t, the power per unit volume is given by:

$$\mathbf{f}(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t) = I(t)\oint\limits_{C(t)} \mathbf{v}(\mathbf{x},t)\cdot(d\mathbf{x}'\times\mathbf{B}(\mathbf{x}))\delta^3(\mathbf{x}'-\mathbf{x}).$$

Integrating over a volume containing $C(t)$, and using the cyclicity of the scalar triple product, we obtain the total power:

$$P = I(t)\oint\limits_{C(t)} d\mathbf{x}'\cdot(\mathbf{B}(\mathbf{x}')\times\mathbf{v}(\mathbf{x}',t)) = -I(t)\mathcal{E}(t) = -I(t)^2R,$$

applying Ohm's law in the final step. The formula follows.

10. A steady current I flows along a cylindrical conductor of constant circular cross-section and uniform conductivity σ. Show, using the relevant equations for **E** and **J**, that the current is distributed uniformly across the cross-section of the cylinder, and calculate the electric and magnetic fields just outside the surface of the cylinder.

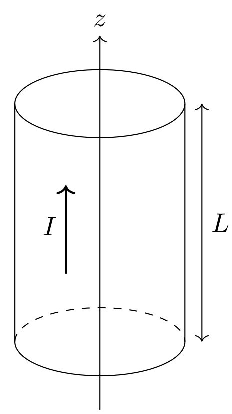
Verify that the integral of the Poynting vector over unit length of the surface is equal to the rate per unit length of dissipation of electrical energy as heat.

➡ **Solution:** The most important equation for the first part of the question is *Ohm's law*, written in the form:

$$\sigma\mathbf{E} = \mathbf{J}.$$

In particular, we note that to show the current is distributed uniformly across the cross-section, it's sufficient to show that the electric field is distributed uniformly across the cross-section. Therefore, let's begin by finding the electric field in the conductor.

To determine the electric field, we first determine an electric potential Φ for the system, and then use $\mathbf{E} = -\nabla\Phi$. It turns out it's a little easier to consider a cylinder of finite length L first, and then take the limit as $L \to \infty$:



Note the following features of a potential Φ for this system:

- Within the conductor, the potential obeys Laplace's equation $\nabla^2\Phi = 0$.
- At the flat ends of the conductor, we can take the potential to be constant, since we will eventually take these ends to infinity. Let Φ_1 be the value of the constant at the start of the conductor, say at $z = 0$, and let Φ_2 be the value of the constant at the end of the conductor, say at $z = L$. Note that $\Phi_1 \neq \Phi_2$ typically, since it is the potential *difference* that drives the current in a wire.
- On the curved surface of the conductor, we can't have charges leaking out, so we must have:

  $$\mathbf{J} \cdot \hat{\mathbf{n}} = 0,$$

  where $\hat{\mathbf{n}}$ is the normal to the curved surface. Using Ohm's law, this can be rewritten as:

  $$0 = \mathbf{J} \cdot \hat{\mathbf{n}} = \sigma\mathbf{E} \cdot \hat{\mathbf{n}} = -\sigma\nabla\Phi \cdot \hat{\mathbf{n}} = -\sigma\frac{\partial\Phi}{\partial\mathbf{n}}.$$

The above setup implies that we have a boundary value problem for Φ. We can guess a solution to this problem:

$$\Phi = \Phi_1 + \frac{\Phi_2}{L}z.$$

But now by the uniqueness theorem for Laplace's equation, it follows that this is the *unique* solution to this boundary value problem. Thus the electric field must be given by:

$$\mathbf{E} = -\nabla\Phi = \frac{\Phi_2}{L}\hat{\mathbf{e}}_z.$$

In particular, $\mathbf{E}$ is distributed uniformly across the cross-section of the cylinder. It follows from Ohm's law that $\mathbf{J} = \sigma\mathbf{E}$ is also distributed uniformly across the cross-section of the cylinder.

To finish, we just need to take $L \to \infty$, while the current carried by the cylinder I is kept constant . We note that since the current density is given by:

$$\mathbf{J} = \sigma\mathbf{E} = \frac{\Phi_2\sigma}{L}\hat{\mathbf{e}}_z,$$

we must have:

$$\int\limits_{\substack{\text{cross-}\\\text{section}}} \mathbf{J}\cdot\mathbf{dS} = \frac{\Phi_2\sigma}{L}\cdot\pi a^2 = I.$$

Thus it follows that the current density, and the electric field inside the conductor, can be written as:

$$\mathbf{J} = \frac{I}{\pi a^2}\hat{\mathbf{e}}_z, \qquad \mathbf{E} = \frac{I}{\pi a^2\sigma}\hat{\mathbf{e}}_z.$$

Taking the limit as $L \to \infty$ with I constant, we see that nothing happens to $\mathbf{J}$ and $\mathbf{E}$; so we're done.

We are now asked to determine the electric and magnetic fields just outside the surface of the conductor. Since there is no surface charge on this conductor, we have that $\mathbf{E}$ is continuous across the surface of the conductor. It follows that:

$$\mathbf{E} = \frac{I}{\pi a^2\sigma}\hat{\mathbf{e}}_z$$

just outside the surface of the conductor.

We can use Ampère's law to determine the magnetic field just outside the surface of the conductor. First, note that the functional form of $\mathbf{B}$ in this problem is such that $\mathbf{B} = B(r)\hat{\mathbf{e}}_\theta$. We can see this using the exact same symmetry arguments we used in Question 2(a). Now consider integrating around a circle just outside of the cylinder. Then by Ampére's law, we have:

$$\mu_0 I = \oint\limits_C \mathbf{B}\cdot\mathbf{dx} = \int\limits_0^{2\pi} B(a)\,ad\theta = 2\pi a B(a) \qquad \Rightarrow \qquad \mathbf{B} = \frac{\mu_0 I}{2\pi a}\hat{\mathbf{e}}_\theta,$$

just outside the cylinder.

For the final part of the problem, we are asked to calculate the integral of the Poynting vector on the surface per unit length, and the energy dissipation of the system, and compare the two.

First recall that the *Poynting vector* is defined by:

$$\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}.$$

Hence, near the surface of the conductor, the Poynting vector for this problem is given by:

$$\mathbf{S} = \frac{1}{\mu_0}\cdot\frac{I}{\pi a^2\sigma}\cdot\frac{\mu_0 I}{2\pi a}\hat{\mathbf{e}}_z\times\hat{\mathbf{e}}_\theta = -\frac{I^2}{2\pi^2\sigma a^3}\hat{\mathbf{e}}_r.$$

The integral of the Poynting vector around a unit length of the surface is given by integrating the vector around a circle on the conductor's surface:

$$\oint_C \mathbf{S} \cdot (a\hat{\mathbf{e}}_r d\theta) = -\frac{I^2}{2\pi^2 \sigma a^3} \int_0^{2\pi} a d\theta = -\frac{I^2}{\pi \sigma a^2}.$$

We are asked to compare this to the rate of electrical energy dissipation. Recall:

Theorem: The rate at which electrical energy dissipates from a system V is given by:

$$\int_V \mathbf{E} \cdot \mathbf{J}\, dV.$$

Proof: Recall that the 'rate of work done' is the same as power, specified by the formula $\mathbf{F} \cdot \dot{\mathbf{x}}$, where $\mathbf{F}$ is force and $\dot{\mathbf{x}}$ is velocity. For a collection of N point particles with velocities $\dot{\mathbf{x}}_i$, all acted on by the Lorentz force, the rate of work done by the Lorentz force is thus:

$$\sum_{i=1}^N \mathbf{F}_i \cdot \dot{\mathbf{x}}_i = \sum_{i=1}^N q_i \dot{\mathbf{x}}_i \cdot (\mathbf{E}(t, \mathbf{x}_i) + \dot{\mathbf{x}}_i \times \mathbf{B}(t, \mathbf{x}_i)) = \sum_{i=1}^N q_i \dot{\mathbf{x}}_i \cdot \mathbf{E}(t, \mathbf{x}_i).$$

Generalising to the continuum, we get the result:

$$\int_V \mathbf{E} \cdot \mathbf{J}\, dV,$$

using the fact that $\mathbf{J}$ is the sum over all the particle's velocities weighted by their charge. To finish, simply note that energy dissipation is the amount of work done *against* the Lorentz force, so we just put a minus sign in front. $\square$

To calculate the rate of dissipation per unit length in our case then, we must compute:

$$-\int_S \mathbf{E} \cdot \mathbf{J}\, dS,$$

where S is a circular cross-section through the conductor. Inserting our formulas for $\mathbf{E}$ and $\mathbf{J}$, we find:

$$-\int_S \mathbf{E} \cdot \mathbf{J}\, dS = -\frac{I^2}{\pi^2 a^4 \sigma} \cdot \pi a^2 = -\frac{I^2}{\pi a^2 \sigma},$$

which is exactly what we found above for the integral of the Poynting vector around a unit length of the surface.

❊ **Comments:** What was the point of this question? Recall from lectures the formula:

$$\frac{d}{dt} \underbrace{\left(\int_V w\, dV \right)}_{\text{energy of EM field in } V} = -\int_S \mathbf{S} \cdot \mathbf{n}\, dA - \int_V \mathbf{E} \cdot \mathbf{J}\, dV.$$

Here, w is the energy density of the electromagnetic field, and $\mathbf{S}$ is the Poynting vector. V is some volume and S is its surface. The three terms in this equation are interpreted as (i) the change in energy of the system (on the LHS); (ii) the energy *flux* out of the system, determined by the Poynting vector; (iii) the energy lost due to the electromagnetic field working on charged matter, causing the charges to collide, transfer energy and dissipate energy as heat. Since we found that the two terms on the right cancel in this problem, we have shown energy is conserved!

Part IB: Electromagnetism
Examples Sheet 3 Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

1. A monochromatic plane wave propagates in empty space $z < 0$ with fields

$$\mathbf{E}_{\text{inc}} = \hat{\mathbf{e}}_x \operatorname{Re}\left(\alpha e^{i(kz-\omega t)}\right), \qquad \mathbf{B}_{\text{inc}} = \frac{1}{c}\hat{\mathbf{e}}_y \operatorname{Re}\left(\alpha e^{i(kz-\omega t)}\right).$$

A perfect conductor fills the region $z \geq 0$. Show that if the reflected fields are given by

$$\mathbf{E}_{\text{ref}} = -\hat{\mathbf{e}}_x \operatorname{Re}\left(\alpha e^{i(-kz-\omega t)}\right), \qquad \mathbf{B}_{\text{ref}} = \frac{1}{c}\hat{\mathbf{e}}_y \operatorname{Re}\left(\alpha e^{i(-kz-\omega t)}\right)$$

then the total fields $\mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{ref}}$ and $\mathbf{B} = \mathbf{B}_{\text{inc}} + \mathbf{B}_{\text{ref}}$ satisfy the Maxwell equations and the relevant boundary conditions at $z = 0$.

What surface current flows in the plane $z = 0$? Compute the Poynting vector in the region $z < 0$ and determine its value averaged over a period $T = 2\pi/\omega$.

Recall from Question 5 of Examples Sheet 2 that a surface current experiences a Lorentz force from the average magnetic field on either side of the surface. Use this to show that the time-averaged force per unit area on the conductor is $\langle f \rangle = \epsilon_0 |\alpha|^2$.

Solution: Recall that Maxwell's equations in the region $z < 0$, which is a vacuum (i.e. with vanishing charge density and current density, $\rho = 0, \mathbf{J} = \mathbf{0}$) are:

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

Substituting $\mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{ref}}$, $\mathbf{B} = \mathbf{B}_{\text{inc}} + \mathbf{B}_{\text{ref}}$ into each of Maxwell's equations in turn, we have:

- $\nabla \cdot \mathbf{E} = \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z} = \dfrac{\partial}{\partial x}\left(\operatorname{Re}\left(\alpha e^{i(kz-\omega t)}\right) - \operatorname{Re}\left(\alpha e^{i(-kz-\omega t)}\right)\right) = 0.$
- $\nabla \cdot \mathbf{B} = \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} + \dfrac{\partial B_z}{\partial z} = \dfrac{\partial}{\partial y}\left(\dfrac{1}{c}\operatorname{Re}\left(\alpha e^{i(kz-\omega t)}\right) + \dfrac{1}{c}\operatorname{Re}\left(\alpha e^{i(-kz-\omega t)}\right)\right) = 0.$
- For Faraday's Law, we have both:

$$-\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{c}\hat{\mathbf{e}}_y \operatorname{Re}\left(i\omega\alpha e^{i(kz-\omega t)}\right) + \frac{1}{c}\hat{\mathbf{e}}_y \operatorname{Re}\left(i\omega\alpha e^{i(-kz-\omega t)}\right),$$

  and

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{vmatrix} = \hat{\mathbf{e}}_y \frac{\partial}{\partial z}\left(\operatorname{Re}\left(\alpha e^{i(kz-\omega t)}\right) - \operatorname{Re}\left(\alpha e^{i(-kz-\omega t)}\right)\right)$$

$$= \hat{\mathbf{e}}_y \operatorname{Re}\left(\alpha i k e^{i(kz-\omega t)}\right) + \hat{\mathbf{e}}_y \operatorname{Re}\left(\alpha i k e^{i(-kz-\omega t)}\right).$$

  These two expressions agree assuming the dispersion relation $\omega = kc$.

- Finally, for the Ampère-Maxwell law, we have both:

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2}\hat{\mathbf{e}}_x \operatorname{Re}\left(-i\omega\alpha e^{i(kz-\omega t)}\right) - \frac{1}{c^2}\hat{\mathbf{e}}_x \operatorname{Re}\left(-i\omega\alpha e^{i(-kz-\omega t)}\right)$$

and

$$\nabla \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & B_z \end{vmatrix} = -\hat{\mathbf{e}}_x \frac{\partial}{\partial z}\left(\frac{1}{c}\operatorname{Re}\left(\alpha e^{i(kz-\omega t)}\right) + \frac{1}{c}\operatorname{Re}\left(\alpha e^{i(-kz-\omega t)}\right)\right)$$

$$= -\frac{1}{c}\hat{\mathbf{e}}_x \left(\operatorname{Re}\left(\alpha i k e^{i(kz-\omega t)}\right) + \operatorname{Re}\left(-\alpha i k e^{i(-kz-\omega t)}\right)\right),$$

hence again we find that these expressions agree assuming the dispersion relation $\omega = kc$.

Hence we have verified that Maxwell's equations are satisfied in the region $z < 0$. In the region $z > 0$, we assume that there is a perfect conductor, and hence $\mathbf{E} = \mathbf{B} = \mathbf{0}$ in the region $z > 0$ (note these trivially satisfy Maxwell's equations in this region).

Now recall from lectures that the boundary conditions near the surface of a perfect conductor are:

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}, \qquad \mathbf{B} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}},$$

where $\mathbf{E}$, $\mathbf{B}$ are the electric and magnetic fields just outside of the conductor, $\hat{\mathbf{n}}$ is a unit outward-pointing normal, σ is the charge density on the surface of the conductor and $\mathbf{K}$ is the current density on the surface of the conductor.

For our given electric and magnetic field, we have just outside the conductor (i.e. setting $z = 0$):

$$\mathbf{E} = \hat{\mathbf{e}}_x \operatorname{Re}\left(\alpha e^{-i\omega t}\right) - \hat{\mathbf{e}}_x \operatorname{Re}\left(\alpha e^{-i\omega t}\right) = \mathbf{0},$$

and

$$\mathbf{B} = \frac{2}{c}\hat{\mathbf{e}}_y \operatorname{Re}\left(\alpha e^{-i\omega t}\right).$$

Hence we see the matching conditions are obeyed provided $\sigma = 0$, and $\mathbf{K}$ is a specific vector that we must find. Note that $\hat{\mathbf{n}} = -\hat{\mathbf{e}}_z$ is the outward-pointing normal in this case, so to get the directions right we need $\mathbf{K} = K\hat{\mathbf{e}}_x$ for some K. Then

$$\mathbf{K} \times \hat{\mathbf{n}} = -K\hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_z = K\hat{\mathbf{e}}_y$$

shows that

$$K = \frac{2}{c\mu_0}\operatorname{Re}\left(\alpha e^{-i\omega t}\right).$$

Thus we see that the boundary conditions are obeyed provided the conductor has zero charge density on its surface, $\sigma = 0$, and current density:

$$\mathbf{K} = \frac{2}{c\mu_0}\operatorname{Re}\left(\alpha e^{-i\omega t}\right)\hat{\mathbf{e}}_x.$$

Next we are asked to determine the Poynting vector in the region $z < 0$ and determine its valued averaged over a period $T = 2\pi/\omega$, which is the period of the monochromatic light we are considering. Recall from lectures, and the end of Examples Sheet 2, that the Poynting vector is given by:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = \frac{1}{c\mu_0}\hat{\mathbf{e}}_z \left(\operatorname{Re}\left(\alpha e^{i(kz-\omega t)}\right) - \operatorname{Re}\left(\alpha e^{i(-kz-\omega t)}\right)\right)\left(\operatorname{Re}\left(\alpha e^{i(kz-\omega t)}\right) + \operatorname{Re}\left(\alpha e^{i(-kz-\omega t)}\right)\right)$$

$$= \frac{4}{c\mu_0}\hat{\mathbf{e}}_z \operatorname{Re}\left(i\alpha e^{-i\omega t}\sin(kz)\right)\operatorname{Re}\left(\alpha e^{-i\omega t}\cos(kz)\right).$$

Let's let $\alpha = |\alpha|e^{i\theta}$ so that we can actually take the real parts in the Poynting vector; this will make it much clearer what we have to integrate when we take the time average. We have:

$$\mathbf{S} = \frac{4}{c\mu_0}\hat{\mathbf{e}}_z \operatorname{Re}\left(i|\alpha|e^{i(\theta-\omega t)}\sin(kz)\right)\operatorname{Re}\left(|\alpha|e^{i(\theta-\omega t)}\cos(kz)\right).$$

Actually taking the real parts then, we have

$$\mathbf{S} = \frac{4|\alpha|^2}{c\mu_0}\hat{\mathbf{e}}_z \sin(kz)\cos(kz)\sin(\omega t-\theta)\cos(\omega t-\theta) = \frac{|\alpha|^2}{c\mu_0}\hat{\mathbf{e}}_z\sin(2kz)\sin(2(\omega t-\theta)).$$

It remains to find the time average over a period. This is given by:

$$\frac{\omega}{2\pi}\int\limits_{-\pi/\omega}^{\pi/\omega}\mathbf{S}\,dt = \frac{\omega|\alpha|^2}{2c\pi\mu_0}\sin(2kz)\int\limits_{-\pi/\omega}^{\pi/\omega}\sin(2(\omega t-\theta))\,dt.$$

Drawing a sketch of the integrand on the given interval shows that the integral is zero. Therefore, the time-average of the Poynting vector over a period of the monochromatic wave is simply $\langle\mathbf{S}\rangle = \mathbf{0}$. The physical interpretation of this result is that on average, there is no energy flux out of the system.

Finally, we are asked to find the time-averaged force per unit area on the conductor. Following the advice of the question, we recall from Examples Sheet 2, Question 5, that the force per unit area on the conductor is given by:

$$\mathbf{f}_A = \mathbf{K}\times\left(\frac{1}{2}\mathbf{B}_{z=0^+}+\frac{1}{2}\mathbf{B}_{z=0^-}\right) = \frac{1}{2}\mathbf{K}\times\mathbf{B}_{z=0^-}$$

in this case. Inserting the formula for $\mathbf{B}_{z=0^-}$ and the formula we found for $\mathbf{K}$, the force per unit area on the conductor is given by:

$$\begin{aligned}\mathbf{f}_A &= \frac{1}{c^2\mu_0}\operatorname{Re}\left(\alpha e^{-i\omega t}\right)\left(\operatorname{Re}\left(\alpha e^{i(-\omega t)}\right)+\operatorname{Re}\left(\alpha e^{i(-\omega t)}\right)\right)\hat{\mathbf{e}}_x\times\hat{\mathbf{e}}_y\\ &= \frac{2}{c^2\mu_0}\operatorname{Re}\left(\alpha e^{-i\omega t}\right)^2\hat{\mathbf{e}}_z = \frac{2|\alpha|^2}{c^2\mu_0}\cos^2(\omega t-\theta)\hat{\mathbf{e}}_z.\end{aligned}$$

Taking the time average, we find:

$$\frac{\omega}{2\pi}\int\limits_{-\pi/\omega}^{\pi/\omega}\mathbf{f}_A\,dt = \frac{\omega|\alpha|^2}{c^2\pi\mu_0}\hat{\mathbf{e}}_z\int\limits_{-\pi/\omega}^{\pi/\omega}\cos^2(\omega t-\theta)\,dt.$$

Note that by the periodicity of cosine, we can simply drop the θ from the integrand. Therefore, we just perform the integral:

$$\int\limits_{-\pi/\omega}^{\pi/\omega}\cos^2(\omega t)\,dt = \frac{1}{2}\int\limits_{-\pi/\omega}^{\pi/\omega}\left(1+\cos(2\omega t)\right)\,dt = \frac{\pi}{\omega}.$$

Putting everything together, we find the time average force per unit area:

$$\langle\mathbf{f}_A\rangle = \frac{|\alpha|^2}{c^2\mu_0}\hat{\mathbf{e}}_z = \epsilon_0|\alpha|^2\hat{\mathbf{e}}_z,$$

as required (using $\epsilon_0 = 1/c^2\mu_0$).

2. Perfectly conducting plates are positioned at $y = 0$ and $y = a$. Show that a monochromatic plane wave can propagate between the plates in the y direction only if the frequency is given by $\omega = n\pi c/a, n \in \mathbb{Z}$.

➼ **Solution:** Any monochromatic wave travelling in the y direction that we insert between the two plates will simply bounce between them, constantly being reflected. Thus we need to consider solutions of the type we used in Question 1. Monochromatic waves between the two plates will (without loss of generality, relabelling coordinates as $z \mapsto y, y \mapsto x, x \mapsto z$) take the form:

$$\mathbf{E} = \hat{\mathbf{e}}_z \operatorname{Re}\left(\alpha e^{i(ky-\omega t)}\right) - \hat{\mathbf{e}}_z \operatorname{Re}\left(\alpha e^{i(-ky-\omega t)}\right), \qquad \mathbf{B} = \frac{1}{c}\hat{\mathbf{e}}_x \operatorname{Re}\left(\alpha e^{i(ky-\omega t)}\right) + \frac{1}{c}\hat{\mathbf{e}}_x \operatorname{Re}\left(\alpha e^{i(-ky-\omega t)}\right),$$

where we have combined the reflected and incident parts defined in Question 1. We must impose the boundary condition $\mathbf{E} = \mathbf{0}$ (assuming neither of the plates have any surface charge) at both $y = 0$ and $y = a$, which translates to the two conditions:

$$\hat{\mathbf{e}}_z \operatorname{Re}\left(\alpha e^{-i\omega t}\right) - \hat{\mathbf{e}}_z \operatorname{Re}\left(\alpha e^{-i\omega t}\right) = \mathbf{0}, \qquad \hat{\mathbf{e}}_z \operatorname{Re}\left(\alpha e^{i(ka-\omega t)}\right) - \hat{\mathbf{e}}_z \operatorname{Re}\left(\alpha e^{i(-ka-\omega t)}\right) = \mathbf{0}.$$

The first condition is trivially satisfied. The remaining condition is then:

$$\operatorname{Re}\left(2i\alpha e^{-i\omega t}\sin(ka)\right) = 0.$$

Since this must hold for all t, we see that for $\alpha \neq 0$, we need $\sin(ka) = 0$. This implies $k = n\pi/a$ for some integer n, which by the dispersion relation $\omega = c|\mathbf{k}|$ implies that $\omega = n\pi c/a$ for some integer n as required.

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3. Perfectly conducting plates are positioned at $y = 0$ and $y = a$. Show that a monochromatic wave may propagate between the plates in the direction z if the field components are

$$E_x = \omega A \sin\left(\frac{n\pi y}{a}\right)\sin(kz - \omega t)$$

and

$$B_y = kA\sin\left(\frac{n\pi y}{a}\right)\sin(kz-\omega t), \qquad B_z = \frac{n\pi A}{a}\cos\left(\frac{n\pi y}{a}\right)\cos(kz-\omega t)$$

with A a constant and $n \in \mathbb{Z}$. Show that the wavelength λ is given by $1/\lambda^2 = 1/\lambda_\infty^2 - n^2/4a^2$, where λ_∞ is the wavelength of waves of the same frequency in the absence of conducting plates.

➼ **Solution:** We just need to verify that the given components satisfy Maxwell's equations, and the correct boundary conditions at the plates.

First, let's do the boundary conditions:

- Near $y = 0$ and $y = a$, clearly $E_x = 0$ because of the factor of sine. It follows that the electric field satisfies all relevant boundary conditions.
- Near $y = 0$ and $y = a$, the $\mathbf{B}$ field satisfies:

$$\mathbf{B}\Big|_{y=0} = \left(0, 0, \frac{n\pi A}{a}\cos(kz-\omega t)\right), \qquad \mathbf{B}\Big|_{y=a} = \left(0, 0, \frac{n\pi A}{a}(-1)^n\cos(kz-\omega t)\right).$$

  So at both plates, the only non-zero component is the $\hat{\mathbf{e}}_z$ component. Hence $\mathbf{B} = \mathbf{K} \times \hat{\mathbf{e}}_y$ is possible at both plates if a surface current flows such that $\mathbf{K}$ is parallel to $\hat{\mathbf{e}}_x$.

Now let's show that the components satisfy Maxwell's (unsourced) equations. We have:

- $\nabla \cdot \mathbf{E} = \dfrac{\partial E_x}{\partial x} = 0.$
- $\nabla \cdot \mathbf{B} = \dfrac{\partial B_y}{\partial y} + \dfrac{\partial B_z}{\partial z} = \dfrac{n\pi k A}{a}\cos\left(\dfrac{n\pi y}{a}\right)\sin(kz-\omega t) - \dfrac{n\pi k A}{a}\cos\left(\dfrac{n\pi y}{a}\right)\cos(kz-\omega t) = 0.$
- For Faraday's law, we have both:

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \partial_x & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = k\omega A \sin\left(\frac{n\pi y}{a}\right)\cos(kz-\omega t)\hat{\mathbf{e}}_y - \frac{n\pi\omega A}{a}\cos\left(\frac{n\pi y}{a}\right)\sin(kz-\omega t)\hat{\mathbf{e}}_z,$$

  and

$$-\frac{\partial \mathbf{B}}{\partial t} = k\omega A \sin\left(\frac{n\pi y}{a}\right)\cos(kz-\omega t)\hat{\mathbf{e}}_y - \frac{n\pi\omega A}{a}\cos\left(\frac{n\pi y}{a}\right)\sin(kz-\omega t)\hat{\mathbf{e}}_z,$$

  hence it is identically satisfied.

- Finally, for the Ampère-Maxwell law, we have

$$\nabla \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \partial_x & \partial_y & \partial_z \\ 0 & B_y & B_z \end{vmatrix} = \left[-\frac{n^2\pi^2}{a^2}A\sin\left(\frac{n\pi y}{a}\right)\cos(kz-\omega t) - k^2 A \sin\left(\frac{n\pi y}{a}\right)\cos(kz-\omega t)\right]\hat{\mathbf{e}}_x$$

$$= -\left(\frac{n^2\pi^2}{a^2} + k^2\right) A \sin\left(\frac{n\pi y}{a}\right)\cos(kz-\omega t)\hat{\mathbf{e}}_z,$$

  and

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = -\frac{\omega^2}{c^2}A\sin\left(\frac{n\pi y}{a}\right)\cos(kz-\omega t)\hat{\mathbf{e}}_x.$$

  Hence we see that for the Ampère-Maxwell law to be satisfied, we require the waves to obey the dispersion relation:

$$\frac{\omega^2}{c^2} = \frac{n^2\pi^2}{a^2} + k^2.$$

At first glance, this looks very strange! Surely we should have the dispersion relation $\omega = ck$, which we found in lectures? The difference here is that the k is not the same k! For example, putting the electric field in this question in the standard form from lectures we have:

$$E_x = \omega A \sin\left(\frac{n\pi y}{a}\right)\sin(kz-\omega t) = \frac{1}{2}\omega A\left[\cos\left(kz - \frac{n\pi y}{a} - \omega t\right) - \cos\left(\frac{n\pi y}{a} + kz - \omega t\right)\right]$$

$$= \frac{1}{2}\omega A\left[\mathrm{Re}\left(e^{i[(0,-n\pi/a,k)\cdot\mathbf{x}-\omega t]}\right) - \mathrm{Re}\left(e^{i[(0,n\pi/a,k)\cdot\mathbf{x}-\omega t]}\right)\right].$$

So actually, we see that in this question we are dealing with the composition of two monochromatic waves of a well-defined frequency ω, and two distinct directions of propagation:

$$\mathbf{k}_1 = \left(0, -\frac{n\pi}{a}, k\right), \qquad \mathbf{k}_2 = \left(0, \frac{n\pi}{a}, k\right).$$

Of course, each of these monochromatic waves needs to obey the standard dispersion relation from lectures $\omega = c|\mathbf{k}_1|$ and $\omega = c|\mathbf{k}_2|$, and hence we get the condition:

$$\frac{\omega}{c} = \sqrt{\frac{n^2\pi^2}{a^2} + k^2} \qquad \Rightarrow \qquad \frac{\omega^2}{c^2} = \frac{n^2\pi^2}{a^2} + k^2.$$

Actually, this provides a much quicker method of doing this question in general - just write the given wave in the standard form, and then say 'by linearity, done!'. Then derive the standard dispersion relation.

Finally, we are asked to comment on the wavelengths of the given electromagnetic waves. The given waves are of the form:

$$E_x = \omega A \sin\left(\frac{n\pi y}{a}\right)\sin(kz - \omega t),$$

so in their direction of propagation, i.e. the z-direction, the wavenumber of the waves is k, and hence the wavelength obeys:

$$k = \frac{2\pi}{\lambda}.$$

Notice that here λ is related to k, *not* the individual component wavevectors $\mathbf{k}_1$, $\mathbf{k}_2$ - this is because we are considering the waves that result from superposing waves of such wavevectors, the result of which is a wave that have some 'effective' wavevector $(0, 0, k)$.

Without the plates, waves of the same frequency ω can propagate completely freely and independently of one another, with generic wavevectors $\mathbf{k}$. Therefore, we don't have to restrict to special combinations that conspire to give special effective wavevectors. Instead, for these individual free waves, we get the standard relationship:

$$\frac{\omega^2}{c^2} = |\mathbf{k}|^2 = \frac{(2\pi)^2}{\lambda_\infty^2},$$

where λ_∞ is the wavelength of such waves.

Hence substituting all our findings into the relationship

$$\frac{\omega^2}{c^2} = \frac{n^2\pi^2}{a^2} + k^2,$$

we have:

$$\frac{(2\pi)^2}{\lambda_\infty^2} = \frac{n^2\pi^2}{a^2} + \frac{(2\pi)^2}{\lambda^2} \qquad \Rightarrow \qquad \frac{1}{\lambda^2} = \frac{1}{\lambda_\infty^2} - \frac{n^2}{4a^2},$$

which is the required relationship.

4. Consider a plane polarised electromagnetic wave described by the vector and scalar potentials $\mathbf{A}(t,\mathbf{x}) = \mathrm{Re}\left(\mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)$ and $\Phi(t,\mathbf{x}) = \mathrm{Re}\left(\Phi_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)$ with constant $\mathbf{A}_0$ and Φ_0. Use Maxwell's equations to find a relationship between $\mathbf{A}_0$ and Φ_0.

Find a gauge transformation such that the new vector potential is 'transversely polarised', i.e. $\mathbf{A}_0 \cdot \mathbf{k} = 0$. What is the scalar potential Φ in this gauge?

Solution: Recall that the electric and magnetic fields can be expressed in terms of scalar and vector potentials via the formulae:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

For this question, it is useful to recall that a *gauge transformation* of the vector and scalar potentials is a transformation of the form:

$$\mathbf{A} \mapsto \mathbf{A}' = \mathbf{A} + \nabla\chi, \qquad \Phi \mapsto \Phi' = \Phi - \frac{\partial \chi}{\partial t}, \tag{$*$}$$

where χ is some scalar function. Under such a transformation, the $\mathbf{E}$, $\mathbf{B}$ fields remain invariant, but our calculations can be simplified by choosing χ appropriately.

Let's substitute the original fields $\mathbf{A}$, Φ into Maxwell's equations and see what happens. It turns out that the relevant Maxwell equation we consider here is:

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \Rightarrow \qquad \nabla \times (\nabla \times \mathbf{A}) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}\right).$$

On the left hand side we have:

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla\times\left(\nabla\times\mathrm{Re}\left(\mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)\right) = -\mathrm{Re}\left(\mathbf{k}\times(\mathbf{k}\times\mathbf{A}_0)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right) = -\mathrm{Re}\left(\left((\mathbf{k}\cdot\mathbf{A}_0)\,\mathbf{k} - |\mathbf{k}|^2\mathbf{A}_0\right) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right),$$

where in the last step we have used Lagrange's formula $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$. On the right hand side we have:

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}\right) = \frac{1}{c^2}\mathrm{Re}\left(\frac{\partial}{\partial t}\left(-i\mathbf{k}\Phi_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + i\omega\mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)\right) = \frac{1}{c^2}\mathrm{Re}\left(-\omega\mathbf{k}\Phi_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \omega^2\mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)$$

Comparing these two expressions, we have:

$$\mathrm{Re}\left(\left((\mathbf{k}\cdot\mathbf{A}_0)\,\mathbf{k} - |\mathbf{k}|^2\mathbf{A}_0\right) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right) = \frac{1}{c^2}\mathrm{Re}\left(\left(\omega\mathbf{k}\Phi_0 - \omega^2\mathbf{A}_0\right) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right).$$

Since this equation must hold for all t and $\mathbf{x}$, we have the conditions:

$$c^2\left(\mathbf{k}\cdot\mathbf{A}_0\right)\mathbf{k} - c^2|\mathbf{k}|^2\mathbf{A}_0 = \omega\mathbf{k}\Phi_0 - \omega^2\mathbf{A}_0,$$

and the dispersion relation $\omega = c|\mathbf{k}|$ then implies $c^2(\mathbf{k}\cdot\mathbf{A}_0)\mathbf{k} = \omega\Phi_0\mathbf{k}$. Comparing components gives:

$$c^2\mathbf{k}\cdot\mathbf{A}_0 = \omega\Phi_0 \qquad \Rightarrow \qquad c^2\mathbf{k}\cdot\left(\mathbf{A}_0 - \frac{\Phi_0}{\omega}\mathbf{k}\right) = 0$$

Thus to satisfy the condition of transverse polarisation, we must transform to a new gauge $\mathbf{A}'$, Φ' where $\Phi_0' = 0$ and $\mathbf{A}_0' = \mathbf{A}_0 - \Phi_0\mathbf{k}/\omega$; then we will have $\mathbf{k}\cdot\mathbf{A}_0' = 0$ as required. Consider a gauge transformation of the form ($*$) with χ given by:

$$\chi = \mathrm{Re}\left(\chi_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right), \qquad \text{so that} \qquad \mathbf{A}' = \mathrm{Re}\left((\mathbf{A}_0 + i\mathbf{k}\chi_0)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right), \quad \Phi' = \mathrm{Re}\left((\Phi_0 + i\omega\chi_0)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right).$$

Hence we see that choosing such a gauge transformation with $\chi_0 = i\Phi_0/\omega$, we get the condition $\mathbf{k}\cdot\mathbf{A}_0' = 0$ as required. In the new gauge, $\Phi \equiv 0$ identically.

5. (a) A tensor of type $(0, 2)$ has components $T_{\mu\nu}$. View these components as a 4×4 matrix. Show that if this matrix is invertible in one inertial frame, then it is invertible in any inertial frame, and that the components of the inverse matrix $(T^{-1})^{\mu\nu}$ define a tensor of type $(2, 0)$.

(b) Show that the object with components $\epsilon_{\mu\nu\rho\sigma}$ with respect to any inertial frame is an isotropic pseudo-tensor of type $(0, 4)$.

➼ **Solution:** (a) Before we proceed, it is useful to recall the defining property of a Lorentz transformation Λ:

Definition: A Lorentz transformation Λ is a transformation obeying:

$$\eta = \Lambda^T \eta \Lambda.$$

where η is the Minkowski metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$. In index notation, this equation reads:

$$\eta_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha} \eta_{\mu\nu} \Lambda^{\nu}{}_{\beta}.$$

In particular, we have the following properties:

Theorem: Lorentz transformations obey the following properties:

(i) the composition of two Lorentz transformations is a Lorentz transformation;

(ii) Lorentz transformations are all invertible and the inverse of a Lorentz transformation is a Lorentz transformation;

(iii) the identity is a Lorentz transformation.

Notice that these three conditions (together with associativity of matrix multiplication) imply that the set of all Lorentz transformations forms a group called the *Lorentz group*.

Proof: (i) Let Λ, Λ' be Lorentz transformations. Then

$$(\Lambda'\Lambda)^T \eta (\Lambda'\Lambda) = \Lambda^T (\Lambda')^T \eta \Lambda' \Lambda = \Lambda^T \eta \Lambda = \eta,$$

since each Λ', Λ is a Lorentz transformation individually. It follows $\Lambda'\Lambda$ is a Lorentz transformation.

(ii) Just take determinants of both sides of the defining equation. We have

$$-1 = \det(\eta) = \det(\Lambda^T) \det(\eta) \det(\Lambda) = -\det(\Lambda)^2,$$

using the fact that $\det(\Lambda^T) = \det(\Lambda)$. It follows that $\det(\Lambda) = \pm 1$ and hence all Lorentz transformations are invertible. It remains to show that the inverse of a Lorentz transformation is a Lorentz transformation. To do so, just multiply the defining equation on the left by $(\Lambda^{-1})^T$ and on the right by Λ^{-1} (we now know that all elements of the equation have an inverse):

$$\eta = \Lambda^T \eta \Lambda \qquad \Rightarrow \qquad (\Lambda^{-1})^T \eta \Lambda^{-1} = \eta.$$

(iii) Clear from setting $\Lambda = I$ in the defining equation. $\square$

Now let's get on with the question. Suppose that in the inertial frame S, the components of our tensor **T** (we will denote the tensor itself with boldface type, its components with indices, and the matrices of the components without boldface type) are $T_{\mu\nu}$. Then by the definition of a tensor, in any other inertial frame S', there exists some Lorentz transformation Λ such that the components of the tensor in S' are given by:

$$T'_{\alpha\beta} = (\Lambda^{-1})^{\mu}{}_{\alpha}(\Lambda^{-1})^{\nu}{}_{\beta}T_{\mu\nu}.$$

Let's write T' for the matrix with components $T'_{\alpha\beta}$, T for the matrix with components $T_{\mu\nu}$, and Λ^{-1} for the matrix with components $(\Lambda^{-1})^{\nu}{}_{\beta}$, where ν is the row index and β is the column index. Then we can rewrite the above index equation as a matrix equation:

$$T' = (\Lambda^{-1})^T T\Lambda^{-1}. \tag{$*$}$$

Take the determinant of both sides. Then $\det(T') = \det(\Lambda^{-1})^2\det(T)$. Recall that the inverse of a Lorentz transformation is a Lorentz transformation, and all Lorentz transformations are invertible, so $\det(\Lambda^{-1}) \neq 0$. Recall also that $\det(T) \neq 0$ since T is invertible by assumption. Hence $\det(T') \neq 0$, and it follows that T' is invertible. Hence the components of the tensor **T** form an invertible 4×4 matrix in any inertial frame, as required.

Finally, we are asked to show that the components $(T^{-1})^{\mu\nu}$ define a tensor $\mathbf{T}^{-1}$ of type $(2, 0)$. Let's be very clear what this means before we start. We would like to define a tensor $\mathbf{T}^{-1}$ such that its components in any given inertial frame S are $(T^{-1})^{\mu\nu}$, by which we mean the components of the inverse matrix of **T** in the inertial frame S. This rather wordy definition basically means that in some other inertial frame S', the tensor components $(T^{-1})'^{\mu\nu}$ are the same as the components of the matrix $(T')^{-1}$.

Now let's get started. Taking the inverse of the equation ($*$) gives us:

$$(T')^{-1} = \Lambda T^{-1}\Lambda^T.$$

Adding indices using the above discussion, this equation shows us that

$$(T^{-1})'^{\alpha\beta} = \Lambda^{\alpha}{}_{\mu}(T^{-1})^{\mu\nu}\Lambda^{\beta}{}_{\nu}.$$

This is precisely the condition for $\mathbf{T}'$ to transform as a tensor of type $(2, 0)$.

(b) Recall that the object $\boldsymbol{\epsilon}$ transforms as an *isotropic* pseudo-tensor of type $(0, 4)$ if its components in two inertial frames S and S' are related by the equation:

$$\epsilon_{\alpha\beta\gamma\delta} = \epsilon'_{\alpha\beta\gamma\delta} = \det(\Lambda)(\Lambda^{-1})^{\mu}{}_{\alpha}(\Lambda^{-1})^{\nu}{}_{\beta}(\Lambda^{-1})^{\rho}{}_{\gamma}(\Lambda^{-1})^{\sigma}{}_{\delta}\epsilon_{\mu\nu\rho\sigma},$$

where $\det(\Lambda)$ is the determinant of the relevant Lorentz transformation. In particular, note that the components in S' must be the same as the components in S. Recall also that $\det(\Lambda) = \pm 1$, so the factor of $\det(\Lambda)$ allows for at most a sign difference under specific Lorentz transformations, hence the 'pseudo'.

Hence the aim of this part of the question is simply to prove the identity:

$$\det(\Lambda^{-1})\epsilon_{\alpha\beta\gamma\delta} = (\Lambda^{-1})^{\mu}{}_{\alpha}(\Lambda^{-1})^{\nu}{}_{\beta}(\Lambda^{-1})^{\rho}{}_{\gamma}(\Lambda^{-1})^{\sigma}{}_{\delta}\epsilon_{\mu\nu\rho\sigma}$$

(note that we have moved the determinant onto the left hand side using $\det(\Lambda)\det(\Lambda^{-1}) = 1$). This is actually almost the definition of the determinant of a matrix; we'll remind you of this fact below.

Leibniz's formula: The determinant $\det(A)$ of a 4×4 matrix A with components $A^\mu{}_\alpha$ (where μ refers to the rows and α the columns) obeys the equation:

$$\det(A)\epsilon_{\alpha\beta\gamma\delta} = A^\mu{}_\alpha A^\nu{}_\beta A^\rho{}_\gamma A^\sigma{}_\delta \epsilon_{\mu\nu\rho\sigma}.$$

Proof: The proof is essentially trivial provided we use a sensible definition of the determinant. Let's follow Part IB Linear Algebra, and define the determinant of a general $n \times n$ matrix via:

$$\det(A) := \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} A^{\sigma(i)}{}_i,$$

where S_n is the group of permutations on n symbols, and $\epsilon(\sigma)$ is the sign of the permutation ($+1$ for odd permutations, -1 for even permutations). Then from the definition of the Levi-Civita symbol, we have for a 4×4 matrix:

$$\det(A) = A^\mu{}_1 A^\nu{}_2 A^\rho{}_3 A^\sigma{}_4 \epsilon_{\mu\nu\rho\sigma}.$$

It doesn't take much convincing to see that the formula:

$$\det(A)\epsilon_{\alpha\beta\gamma\delta} = A^\mu{}_\alpha A^\nu{}_\beta A^\rho{}_\gamma A^\sigma{}_\delta \epsilon_{\mu\nu\rho\sigma},$$

follows as required (think about what happens when $\alpha, \beta, \gamma, \delta$ take on specific values, and how we would have to rearrange μ, ν, σ, δ to get $1, 2, 3, 4$ back in the right order on the right hand side). $\square$

The result of this part of the question then follows immediately from this Theorem.

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6. A particle of rest mass m and charge q moves in a constant uniform electric field $\mathbf{E} = (E, 0, 0)$. It starts from the origin with initial 3-momentum $\mathbf{p} = (0, p_0, 0)$. Show that the particle traces out a path in the (x, y) plane given by

$$x = \frac{\mathcal{E}_0}{qE}\left(\cosh\left(\frac{qEy}{p_0 c}\right) - 1\right)$$

where $\mathcal{E}_0 = \sqrt{p_0^2 c^2 + m^2 c^4}$ is the initial kinematic energy of the particle.

➼ **Solution:** Recall from lectures that the relativistic form of the Lorentz force law is:

$$\frac{dP_\mu}{d\tau} = qF_{\mu\nu}U^\nu,$$

where $P = mU$ is the particle's four-momentum, U is the particle's four-velocity, and τ is the particle's proper time. The tensor $F^{\mu\nu}$ is called the *electromagnetic tensor* or the *Maxwell tensor*.

We assign the particle coordinates $X^\mu = (ct, \mathbf{x})$ in some observer's frame S; then the relationship between these coordinates and the proper time is given by the usual formula:

$$\frac{dt}{d\tau} = \gamma = \left(1 - \frac{|\dot{\mathbf{x}}|^2}{c^2}\right)^{-1/2},$$

where $\dot{\mathbf{x}}$ is the velocity of the particle as viewed in the observer's frame S. It follows that the four-velocity and four-momentum in this frame S are given by:

$$U^\mu = \frac{dX^\mu}{d\tau} = \begin{pmatrix} \gamma c \\ \gamma\dot{\mathbf{x}} \end{pmatrix}, \qquad P^\mu = \begin{pmatrix} m\gamma c \\ m\gamma\dot{\mathbf{x}} \end{pmatrix}.$$

Finally, we assume that the given electric field is the one that is applied to the particle in the observer's frame S. Then the electromagnetic tensor takes the form given in lectures:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E/c & 0 & 0 \\ E/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Lowering an index on P^μ (which has the effect of changing the sign of the time component), we can write down the Lorentz force law in the given frame:

$$\gamma \frac{d}{dt} \begin{pmatrix} -m\gamma c \\ m\gamma\dot{x} \\ m\gamma\dot{y} \\ m\gamma\dot{z} \end{pmatrix} = q \begin{pmatrix} 0 & -E/c & 0 & 0 \\ E/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma c \\ \gamma\dot{x} \\ \gamma\dot{y} \\ \gamma\dot{z} \end{pmatrix}.$$

This translates into a system of four equations:

$$mc^2 \frac{d\gamma}{dt} = qE\dot{x}$$

$$\frac{d}{dt}\left(m\gamma\dot{x}\right) = qE$$

$$\frac{d}{dt}\left(m\gamma\dot{y}\right) = 0$$

$$\frac{d}{dt}\left(m\gamma\dot{z}\right) = 0,$$

together with the initial conditions $x(0) = y(0) = z(0) = 0$ (the particle starts at the origin), $\gamma\dot{x}(0) = \gamma\dot{z}(0) = 0$, $\gamma\dot{y}(0) = p_0/m$ (the particle's initial three momentum is $\mathbf{p} = m\gamma\dot{\mathbf{x}} = (0, p_0, 0)$, we assume in the observer's frame S - note we need the γ factor since it implicitly depends on $\dot{\mathbf{x}}$).

It will also be vaguely useful to know that value of γ at $t = 0$. Since γmc^2 is the relativistic energy, we could simply write down $\gamma(0) = \mathcal{E}_0/mc^2$ at time $t = 0$, where $\mathcal{E}_0$ is the initial energy. Alternatively, we could express γ in terms of momentum and then deduce its value at $t = 0$:

$$\mathbf{p} = m\gamma\dot{\mathbf{x}} \qquad \Rightarrow \qquad 1 + \frac{|\mathbf{p}|^2}{m^2c^2} = 1 + \frac{\gamma^2|\dot{\mathbf{x}}|^2}{c^2} = 1 + \frac{|\dot{\mathbf{x}}|^2/c^2}{1 - |\dot{\mathbf{x}}|^2/c^2} = \gamma^2 \qquad \Rightarrow \qquad \gamma = \sqrt{1 + \frac{|\mathbf{p}|^2}{m^2c^4}}.$$

This again shows that at time $t = 0$, $\gamma = \sqrt{m^2c^4 + p_0^2c^2}/mc^2 = \mathcal{E}_0/mc^2$. Knowing the initial value of γ also allows us to write down the initial values $\dot{x}(0) = 0$, $\dot{y}(0) = p_0c^2/\mathcal{E}_0$, $\dot{z}(0) = 0$.

Let's now begin solving these equations. Notice that we can solve the last equation immediately. Integrating, we get:

$$m\gamma\dot{z} = \text{constant},$$

and the constant is zero by the initial condition $\dot{z}(0) = 0$. Note that $m \neq 0$, and $\gamma \neq 0$, so we have

$$\frac{dz}{dt} = 0$$

for all t. Integrating again, we find $z =$ constant, and once again the constant is zero by the initial condition $z(0) = 0$. Thus $z(t) = 0$ for all t, and the motion takes place entirely in the xy-plane.

The other three equations are far more entangled because of the factor of γ, which implicitly depends on $\dot{x}$, $\dot{y}$. One way of dealing with this is to integrate the first equation directly to find a relationship between $\dot{x}$ and $\dot{y}$:

$$\gamma mc^2 = qEx + \text{constant}.$$

At time $t = 0$, we know $x(0) = 0$ and $\gamma(0) = \mathcal{E}_0/mc^2$, and hence the constant is given by $\mathcal{E}_0$. Thus we have:

$$\gamma = \frac{qEx}{mc^2} + \frac{\mathcal{E}_0}{mc^2},$$

and we can proceed to eliminate γ from the other equations, leaving only equations for x and y. The remaining two equations become:

$$\frac{d}{dt}\left(\dot{x}\left(qEx + \mathcal{E}_0\right)\right) = qc^2E$$

$$\frac{d}{dt}\left(\dot{y}\left(qEx + \mathcal{E}_0\right)\right) = 0.$$

The second equation can be integrated directly to find $\dot{y}$ in terms of x. We have, evaluating the constant of integration using $\dot{y}(0) = p_0c^2/\mathcal{E}_0$ and $x(0) = 0$:

$$\frac{dy}{dt} = \frac{p_0c^2}{qEx + \mathcal{E}_0} \tag{$*$}$$

The first equation now contains only x as a dependent variable, so we can solve that separately. Integrating directly, and evaluating the integration constant using the initial conditions, we have

$$\frac{dx}{dt} = \frac{qc^2Et}{qEx + \mathcal{E}_0}. \tag{$**$}$$

This equation is separable, with the solution (immediately evaluating the constant of integration, which is zero):

$$\frac{1}{2}qc^2Et^2 = \frac{1}{2}qEx^2 + \mathcal{E}_0x \qquad \Rightarrow \qquad t = \frac{1}{c}\sqrt{x^2 + \frac{2\mathcal{E}_0x}{qE}} = \frac{1}{c}\sqrt{\left(x + \frac{\mathcal{E}_0}{qE}\right)^2 - \frac{\mathcal{E}_0^2}{q^2E^2}}.$$

Now divide ($*$) by ($**$) and substitute for t using the formula we have just derived:

$$\frac{dy}{dx} = \frac{p_0c}{qE}\left(\left(x + \frac{\mathcal{E}_0}{qE}\right)^2 - \frac{\mathcal{E}_0^2}{q^2E^2}\right)^{-1/2}.$$

We are almost there, we now have an equation that directly relates x and y. This equation is separable, and by a standard hyperbolic cosine substitution, we can find the integral:

$$y = \frac{p_0c}{qE}\operatorname{arcosh}\left(\frac{x + \mathcal{E}_0/qE}{\mathcal{E}_0/qE}\right).$$

Rearranging, we arrive at the required formula:

$$x = \frac{\mathcal{E}_0}{qE}\left(\cosh\left(\frac{qEy}{p_0c}\right) - 1\right).$$

7. For constant electric and magnetic fields, **E** and **B**, show that if $\mathbf{E} \cdot \mathbf{B} = 0$ and $\mathbf{E}^2 - c^2\mathbf{B}^2 \neq 0$, then there exist inertial frames where either **E** or **B** are zero, but not both. [*Hint: show that you can choose axes so that only E_y and B_z are non-zero and then consider a Lorentz transformation in the x-direction.*]

Solution: First of all, it's clear that we can rotate our axes so that the y-direction points in the direction of **E**. We are given $\mathbf{E} \cdot \mathbf{B} = 0$, and hence **B** lies in the plane orthogonal to **E**. But note that we then have the freedom to perform rotations of our coordinates about the new y-axis without affecting the coordinates of **E**; that is, we rotate the xy-plane. Rotate the xy-plane such that the z-axis is aligned with **B**. Then we have

$$\mathbf{E} = (0, E_y, 0), \qquad \mathbf{B} = (0, 0, B_z)$$

in these coordinates.

We are now asked to construct inertial frames where either **E** or **B** is zero; we must also show that no inertial frame gives $\mathbf{E} = \mathbf{B} = \mathbf{0}$. First notice that the rotations between inertial frames don't help us because they preserve the lengths of **E**, **B**. Similar considerations apply to time translations and space translations. Thus the only transformation that could help us set **E** or **B** to zero (or possibly both) is a Lorentz boost.

Furthermore, a Lorentz boost in the direction $\hat{\mathbf{n}}$ preserves the field components in the $\hat{\mathbf{n}}$ (we see this in the equations below). So the only chance we have of eliminating even one of the fields **E**, **B** is through a boost in the x-direction. The transformed electric and magnetic fields were given in lectures to be:

$$\begin{aligned} E_x' &= E_x & B_x' &= B_x \\ E_y' &= \gamma(E_y - vB_z) & B_y' &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\ E_z' &= \gamma(E_z + vB_y) & B_z' &= \gamma\left(B_z - \frac{v}{c^2}E_y\right). \end{aligned}$$

These equations can be derived by considering the transformation of the electromagnetic tensor under such a boost, i.e. $F'_{\mu\nu} = (\Lambda^{-1})^\rho{}_\mu(\Lambda^{-1})^\sigma{}_\nu F_{\rho\sigma}$.

In our particular case, we have $\mathbf{E} = (0, E_y, 0)$ and $\mathbf{B} = (0, 0, B_z)$ in our initial frame, so the transformed fields actually take the simpler forms:

$$\begin{aligned} E_x' &= 0 & B_x' &= 0 \\ E_y' &= \gamma(E_y - vB_z) & B_y' &= 0 \\ E_z' &= 0 & B_z' &= \gamma\left(B_z - \frac{v}{c^2}E_y\right). \end{aligned}$$

If $B_z = 0$ in the initial inertial frame, then we're done because $\mathbf{B} = \mathbf{0}$, and $\mathbf{E}^2 - c^2\mathbf{B}^2 \neq 0$ implies $\mathbf{E} \neq 0$ if $\mathbf{B} = \mathbf{0}$. Similarly if $E_y = 0$ in the initial inertial frame, we're done. So assume without loss of generality that $B_z \neq 0$ and $E_y \neq 0$.

Now notice that the $\mathbf{E}'$ field is zero if and only if $v = E_y/B_z$. Then the $\mathbf{B}'$ field's only non-zero component is given by:

$$B_z' = \frac{\gamma}{B_z}\left(B_z^2 - \frac{E_y^2}{c^2}\right) = \frac{\gamma}{B_z}\left(|\mathbf{B}|^2 - \frac{|\mathbf{E}|^2}{c^2}\right) \neq 0.$$

Hence we can set $\mathbf{E}' = \mathbf{0}$, but we cannot set $\mathbf{B}' = \mathbf{0}$ simultaneously. Similarly $\mathbf{B}'$ is zero if and only if $v = c^2 B_z/E_y$; then the $\mathbf{E}'$ field's only non-zero component is given by:

$$E_y' = \frac{\gamma}{E_y}\left(E_y^2 - c^2B_z^2\right) = \frac{\gamma}{E_y}\left(|\mathbf{E}|^2 - c^2|\mathbf{B}|^2\right) \neq 0.$$

Hence we can set $\mathbf{B}' = \mathbf{0}$, but we cannot set $\mathbf{E}' = \mathbf{0}$ simultaneously. So we're done!

8. An electromagnetic wave is reflected by a perfect conductor at $x=0$. The electric field is $\mathbf{E}(t,\mathbf{x}) = \hat{\mathbf{e}}_y\left[f(t_-)-f(t_+)\right]$ where f is an arbitrary function and $ct_\pm = ct \pm x$. Show that this satisfies the relevant boundary condition at the conductor. Find the corresponding magnetic field $\mathbf{B}$.

Show that under a Lorentz transformation to an inertial frame moving with speed v in the x-direction the electric field is transformed to

$$\mathbf{E}'(t',\mathbf{x}') = \hat{\mathbf{e}}_y\left[\rho f(\rho t'_-) - \frac{1}{\rho} f\left(\frac{t'_+}{\rho}\right)\right] \qquad \text{where} \qquad \rho = \sqrt{\frac{c-v}{c+v}}.$$

Hence for an incident wave $\mathbf{E}(t,\mathbf{x}) = \hat{\mathbf{e}}_y F(t_-)$, find the wave that is reflected after it hits a perfectly conducting mirror moving with speed v in the x-direction.

Solution: Recall that just outside the surface of a perfect conductor, the electric field must look like

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is a vector pointing outwards from the conductor, and σ is the surface charge on the conductor. For our conductor, we assume $\sigma = 0$, so we need $\mathbf{E} = \mathbf{0}$ at $x = 0$. Now for the given electric field, we have:

$$\mathbf{E}(t,x,y,z) = \hat{\mathbf{e}}_y\left[f\left(t - x/c\right) - f\left(t + x/c\right)\right] \qquad \Rightarrow \qquad \mathbf{E}(t,0,y,z) = \mathbf{0},$$

as required.

To construct the magnetic field, we should recall some general facts about electromagnetic waves from lectures. Recall that an electromagnetic wave corresponds to the following ansatz in the Maxwell equations (these could be considered the Fourier modes of the arbitrary functions f in this question):

$$\mathbf{E} = \mathrm{Re}\left(\mathbf{E}_0 e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\right), \qquad \mathbf{B} = \mathrm{Re}\left(\mathbf{B}_0 e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\right).$$

Substituting into the Maxwell equations, we find the following non-trivial relations between the parameters:

$$\mathbf{E}_0 \cdot \mathbf{k} = 0, \qquad \mathbf{B}_0 \cdot \mathbf{k} = 0, \qquad \omega \mathbf{B}_0 = \mathbf{k} \times \mathbf{E}_0, \qquad \omega = c|\mathbf{k}|,$$

where the last relation is called the *dispersion relation*. The first two relations show us that the electric and magnetic fields are both perpendicular to the direction of motion of the wave $\mathbf{k}$. The third relation shows us that furthermore, $\mathbf{B}_0$ and $\mathbf{E}_0$ are orthogonal in the plane perpendicular to $\mathbf{k}$.

In particular, these relations imply we can construct the $\mathbf{B}$ field from the $\mathbf{E}$ field for an electromagnetic wave. We have:

$$\mathbf{B} = \mathrm{Re}\left(\mathbf{B}_0 e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\right) = \mathrm{Re}\left(\frac{|\mathbf{k}|\hat{\mathbf{k}} \times \mathbf{E}_0}{\omega} e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\right) = \frac{1}{c}\mathrm{Re}\left(\hat{\mathbf{k}} \times \mathbf{E}_0 e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\right).$$

Now when we deal with problems of reflection, we have to account for *two* electromagnetic waves - an *incident* electromagnetic wave, and a *reflected* electromagnetic wave. The reflected wave will have the same parameters as the incident wave, except with the direction of motion changed to $-\mathbf{k}$.

Splitting our initial $\mathbf{E}$ given in this problem into its incident and reflected parts, we have:

$$\mathbf{E}_{\text{inc}} = \hat{\mathbf{e}}_y f(t_-), \qquad \mathbf{E}_{\text{ref}} = -\hat{\mathbf{e}}_y f(t_+).$$

Both of these constitute electromagnetic waves in the sense we described above, with $\mathbf{E}_0 = \hat{\mathbf{e}}_y$ and $\hat{\mathbf{k}} = \hat{\mathbf{e}}_x, -\hat{\mathbf{e}}_x$ for the incident and reflected waves respectively. Therefore we can construct the $\mathbf{B}_{\text{inc}}$, $\mathbf{B}_{\text{ref}}$ fields using the relation we wrote down above:

$$\mathbf{B}_{\text{inc}} = \frac{1}{c}\hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_y f(t_-) = \frac{1}{c}\hat{\mathbf{e}}_z f(t_-), \qquad \mathbf{B}_{\text{ref}} = \frac{1}{c}\hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_y f(t_+) = \frac{1}{c}\hat{\mathbf{e}}_z f(t_+).$$

Hence the required field is:

$$\mathbf{B} = \mathbf{B}_{\text{inc}} + \mathbf{B}_{\text{ref}} = \frac{1}{c}\hat{\mathbf{e}}_z\left[f(t_-) + f(t_+)\right].$$

Under a Lorentz boost with speed v in the x-direction, the new components of the electromagnetic field are given by (as we saw in lectures, and reminded you in Question 7):

$$E'_x = E_x \qquad B'_x = B_x$$
$$E'_y = \gamma(E_y - vB_z) \qquad B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$
$$E'_z = \gamma(E_z + vB_y) \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right).$$

So in our particular case, we have:

$$E'_x = 0 \qquad B'_x = 0$$
$$E'_y = \gamma\left(f(t_-) - f(t_+) - \frac{v}{c}f(t_-) - \frac{v}{c}f(t_+)\right) \qquad B'_y = 0$$
$$E'_z = 0 \qquad B'_z = \gamma\left(\frac{1}{c}f(t_-) + \frac{1}{c}f(t_+) - \frac{v}{c^2}f(t_-) + \frac{v}{c^2}f(t_+)\right).$$

Simplifying the transformed electric field, we have

$$\mathbf{E}' = \hat{\mathbf{e}}_y\gamma\left(\left(1 - \frac{v}{c}\right)f(t_-) - \left(1 + \frac{v}{c}\right)f(t_+)\right).$$

Now using the explicit form of γ, we have

$$\gamma\left(1 - \frac{v}{c}\right) = \frac{1 - v/c}{\sqrt{1 - v^2/c^2}} = \frac{1 - v/c}{\sqrt{(1 - v/c)(1 + v/c)}} = \sqrt{\frac{1 - v/c}{1 + v/c}} = \sqrt{\frac{c - v}{c + v}} = \rho.$$

Similarly,

$$\gamma\left(1 + \frac{v}{c}\right) = \frac{1}{\rho}.$$

Also note that we can transform the coordinates t_-, t_+ to the new frame's coordinates. Using the standard formulae from Part IA Dynamics and Relativity, we have:

$$x' = \gamma(x - vt), \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad \Rightarrow \qquad x = \gamma(x' + vt'), \qquad t = \gamma\left(t' + \frac{v}{c^2}x'\right).$$

So it follows that:

$$t_- = t - \frac{x}{c} = \gamma\left(t' + \frac{v}{c^2}x'\right) - \frac{\gamma}{c}(x' + vt') = \rho t' - \rho\frac{x'}{c} = \rho t'_-,$$

and similarly $t_+ = t'_+/\rho$. Hence it follows that the transformed electric field may be written in the final form:

$$\mathbf{E}' = \hat{\mathbf{e}}_y\left[\rho f(\rho t'_-) - \frac{1}{\rho}f\left(\frac{t'_+}{\rho}\right)\right], \qquad (*)$$

as required.

For the last part, we are asked to find the wave that is reflected from a moving mirror. Clearly we must boost to the frame of the mirror, and imagine it is instead stationary. Our incident electric field $\mathbf{E}_{\text{inc}} = \hat{\mathbf{e}}_y F(t_-)$ is boosted to $\mathbf{E}'_{\text{inc}} = \hat{\mathbf{e}}_y \rho F(\rho t'_-)$ in the frame of the moving mirror, by the above arguments.

We are now in the frame where the mirror is stationary. So we apply the boundary condition $\mathbf{E}' = \mathbf{0}$ at the surface of the mirror, $x' = 0$. This tells us that the reflected field must take the form $\mathbf{E}'_{\text{ref}} = -\hat{\mathbf{e}}_y\rho F(\rho t'_+)$. Transforming back to the original frame, and using the results from earlier, we see that $\mathbf{E}_{\text{ref}} = -\hat{\mathbf{e}}_y\rho^2 F(\rho^2 t_+)$ (we do the inverse operations to the reflected part of the wave, so multiply by ρ where we divided by it in the formula $(*)$).

9. (a) A scalar field Φ obeys the wave equation $\partial^\mu\partial_\mu\Phi = 0$. Its *energy-momentum tensor* is $T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\partial_\rho\Phi\partial_\sigma\Phi$. Show that $T_{\mu\nu}$ is *conserved*: $\partial_\nu T^{\mu\nu} = 0$.

(b) The energy-momentum tensor of the Maxwell field is $T_{\mu\nu} = \mu_0^{-1}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$. Explain how T_{00} and T_{0i} are related to the energy density and Poynting vector of the electromagnetic field. Show that Maxwell's equations imply that $\partial_\nu T_{\mu\nu} = -F^\mu{}_\nu j^\nu$ and that the time component of this equation is the energy conservation equation for Maxwell's theory.

➾ **Solution:** (a) We simply take the derivative ∂_ν:

$$\begin{aligned}\partial_\nu T^{\mu\nu} &= \partial_\nu\left(\partial^\mu\Phi\partial^\nu\Phi - \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\partial_\rho\Phi\partial_\sigma\Phi\right)\\ &= \left(\partial_\nu\partial^\mu\Phi\right)\partial^\nu\Phi + \partial^\mu\Phi\left(\partial_\nu\partial^\nu\Phi\right) - \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\left(\partial_\nu\partial_\rho\Phi\right)\partial_\sigma\Phi - \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\partial_\rho\Phi\left(\partial_\nu\partial_\sigma\Phi\right).\end{aligned}$$

Note that the second term is immediately zero, by the wave equation $\partial_\nu\partial^\nu\Phi = 0$ (recall that the summation indices are arbitrary, and we can raise and lower indices at whim).

For the final two terms, let's use the metric to raise and lower all possible indices. For example, we can use $\eta^{\mu\nu}\partial_\nu = \partial^\mu$; this leaves us with:

$$\partial_\nu T^{\mu\nu} = \left(\partial_\nu\partial^\mu\Phi\right)\partial^\nu\Phi - \frac{1}{2}(\partial^\mu\partial_\rho\Phi)\partial^\rho\Phi - \frac{1}{2}\partial^\sigma\Phi\left(\partial^\mu\partial_\sigma\Phi\right).$$

In this form, we see that the final two terms cancel the first term. We see this because (i) partial derivatives commute; (ii) the indices themselves (except for μ) are arbitrary. Hence $\partial_\nu T^{\mu\nu} = 0$ as required.

(b) Let's first recall the form of the electromagnetic tensor given in lectures:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

When we raise and lower the indices on the electromagnetic tensor, the only rule we have to remember is that: *if we raise or lower a 0, we pick up a minus sign from the metric, otherwise we pick up no sign from raising or lowering a spatial index $i = 1, 2, 3$.* It is also sometimes useful to remember that the spatial part of the electromagnetic tensor can be written as $F_{ij} = \epsilon_{ijk}B_k$.

Let's apply this rules to the energy-momentum tensor:

$$T_{\mu\nu} = \mu_0^{-1}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right).$$

For the T_{00} component, we have:

$$T_{00} = \mu_0^{-1}\left(F_{0\rho}F_0{}^\rho - \frac{1}{4}\eta_{00}F_{\rho\sigma}F^{\rho\sigma}\right).$$

Let's evaluate each of the terms separately. We have:

$$\begin{aligned}F_{0\rho}F_0{}^\rho &= F_{00}F_0{}^0 + F_{01}F_0{}^1 + F_{02}F_0{}^2 + F_{03}F_0{}^3\\ &= -F_{00}^2 + F_{01}^2 + F_{02}^2 + F_{03}^2\\ &= \frac{|\mathbf{E}|^2}{c^2}\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
-\frac{1}{4}\eta_{00}F_{\rho\sigma}F^{\rho\sigma} &= \frac{1}{4}\left(F_{00}F^{00} + F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}\right) \\
&= \frac{1}{4}\left(F_{00}^2 - F_{0i}^2 - F_{i0}^2 + 2\sum_{i<j}F_{ij}^2\right) \\
&= \frac{1}{4}\left(-\frac{2|\mathbf{E}|^2}{c^2} + 2|\mathbf{B}|^2\right).
\end{aligned}
$$

Putting everything together, we have

$$
T_{00} = \frac{1}{2\mu_0}\left(\frac{|\mathbf{E}|^2}{c^2} + |\mathbf{B}|^2\right) = \frac{\epsilon_0|\mathbf{E}|^2}{2} + \frac{|\mathbf{B}|^2}{2\mu_0},
$$

where we have used the fact that $\epsilon_0 = 1/c^2\mu_0$. This is the energy density of the electromagnetic field, as we saw earlier in the lecture course.

For T_{0i}, we have:

$$
T_{0i} = \mu_0^{-1}\left(F_{0\rho}F_i{}^{\rho} - \frac{1}{4}\eta_{0i}F_{\rho\sigma}F^{\rho\sigma}\right).
$$

We immediately see that the second term drops out, because $\eta_{0i} = 0$ for a spatial index $i = 1, 2, 3$. The only remaining term is:

$$
\begin{aligned}
\mu_0^{-1}F_{0\rho}F_i{}^{\rho} &= \mu_0^{-1}\left(F_{00}F_0{}^0 + F_{0j}F_i{}^j\right) \\
&= \mu_0^{-1}F_{0j}F_{ij} \\
&= -\frac{1}{c\mu_0}E_j\epsilon_{ijk}B_k \\
&= -\frac{1}{c}\cdot\left[\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right]_i.
\end{aligned}
$$

Hence T_{0i} is related to the Poynting vector as $T_{0i} = -S_i/c$.

Finally, we are asked to show that $T_{\mu\nu}$ is conserved using Maxwell's equations. Recall from lectures that Maxwell's equations in relativistic form may be written as:

$$
\partial_\nu F^{\mu\nu} = \mu_0 j^\mu, \qquad \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0.
$$

Now consider $\partial_\nu T^{\mu\nu}$. We have:

$$
\partial_\nu T^{\mu\nu} = \mu_0^{-1}\left(\left(\partial_\nu F^{\mu\rho}\right)F^\nu{}_\rho + F^{\mu\rho}\left(\partial_\nu F^\nu{}_\rho\right) - \frac{1}{4}\eta^{\mu\nu}\left(\partial_\nu F_{\rho\sigma}\right)F^{\rho\sigma} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}\left(\partial_\nu F^{\rho\sigma}\right)\right)
$$

The second term can be evaluated immediately using the first Maxwell equation, $\partial_\nu F^{\mu\nu} = \mu_0 j^\mu$ implies $\partial_\nu F^\nu{}_\rho = -\mu_0 j_\rho$ by appropriate manipulation of the indices and the antisymmetry of the electromagnetic tensor.

The final two terms can be combined, since they are actually just the same. Furthermore we can raise the index on the derivative via $\eta^{\mu\nu}\partial_\nu = \partial^\mu$.

Hence we can simplify everything to (also moving some indices around):

$$\partial_\nu T^{\mu\nu} = -j_\rho F^{\mu\rho} + \mu_0^{-1}\left(F_{\nu\rho}\partial^\nu F^{\mu\rho} - \frac{1}{2}F_{\rho\sigma}\partial^\mu F^{\rho\sigma}\right).$$

There's a trick to finish. Notice that the last term carries a factor of $1/2$, but the penultimate term does not. To introduce a factor of $1/2$, and hence apply the second Maxwell equation (the *Bianchi identity*), we split up the penultimate term:

$$F_{\nu\rho}\partial^\nu F^{\mu\rho} - \frac{1}{2}F_{\rho\sigma}\partial^\mu F^{\rho\sigma} = \frac{1}{2}F_{\nu\rho}\partial^\nu F^{\mu\rho} + \frac{1}{2}F_{\nu\rho}\partial^\nu F^{\mu\rho} - \frac{1}{2}F_{\rho\sigma}\partial^\mu F^{\rho\sigma}.$$

Now we want to try and make the indices in all the $\partial^\sim F^{\sim\sim}$ factors cyclic somehow. To this end, we relabel $\sigma \mapsto \nu$ in the final term which makes things look a bit nicer.

We also swap the labels $\nu \leftrightarrow \rho$ on the first term. This leaves us with:

$$\frac{1}{2}F_{\rho\nu}\partial^\rho F^{\mu\nu} + \frac{1}{2}F_{\nu\rho}\partial^\nu F^{\mu\rho} - \frac{1}{2}F_{\rho\nu}\partial^\mu F^{\rho\nu}$$

Now using antisymmetry of the electromagnetic tensor, we can factor out $-\frac{1}{2}F_{\rho\nu}$, leaving us with:

$$-\frac{1}{2}F_{\rho\nu}\left(\partial^\rho F^{\nu\mu} + \partial^\nu F^{\mu\rho} + \partial^\mu F^{\rho\nu}\right) = 0,$$

which is zero by the second Maxwell equation (the Bianchi identity). Hence we're left with:

$$\partial_\nu T^{\mu\nu} = -j_\rho F^{\mu\rho}.$$

In particular, we see that in the absence of charges and current, $j^\mu = 0$, the energy-momentum tensor is conserved as required.

Finally, we examine the $\mu = 0$ component of this equation. Rearranging indices, we have:

$$\partial^\nu T_{\mu\nu} = -j^\rho F_{\mu\rho}.$$

When $\mu = 0$, the left hand side becomes:

$$\partial^0 T_{00} + \partial^i T_{0i} = \frac{\partial}{\partial(ct)}\left(\frac{1}{2}\epsilon_0|\mathbf{E}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2\right) - \frac{1}{c}\nabla\cdot\mathbf{S}.$$

On the other hand, the right hand side becomes:

$$-j^\rho F_{\mu\rho} = -j^i F_{0i} - \frac{1}{c}\mathbf{J}\cdot\mathbf{E}.$$

Overall then, we have the conservation equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0|\mathbf{E}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2\right) = -\nabla\cdot\mathbf{S} - \mathbf{J}\cdot\mathbf{E},$$

which indeed the statement of energy conservation for the electromagnetic field (sometimes called *Poynting's theorem*).

10. ($*$) For a general 4-velocity, written as $U^\mu = \gamma(c, \mathbf{v})$, show that

$$F^{\mu\nu}U_\nu = \gamma \begin{pmatrix} \mathbf{E}\cdot\mathbf{v}/c \\ \mathbf{E}+\mathbf{v}\times\mathbf{B} \end{pmatrix}.$$

In the rest-frame of a conducting medium, Ohm's law states that $\mathbf{J} = \sigma\mathbf{E}$ where σ is the conductivity and $\mathbf{J}$ is the 3-current. Assuming that σ is a Lorentz scalar, show that Ohm's law can be written covariantly as

$$j^\mu + \frac{1}{c^2}\left(j^\nu U_\nu\right)U^\mu = \sigma F^{\mu\nu}U_\nu.$$

where j^μ is the charge-current density and U^μ is the (uniform) 4-velocity of the medium. If the medium moves with 3-velocity $\mathbf{v}$ in some inertial frame, show that the current in that frame is

$$\mathbf{J} = \rho\mathbf{v} + \sigma\gamma\left(\mathbf{E}+\mathbf{v}\times\mathbf{B} - \frac{1}{c^2}(\mathbf{v}\cdot\mathbf{E})\mathbf{v}\right)$$

where ρ is the charge density. Simplify this formula, given that the charge density vanishes in the rest-frame of the medium.

Solution: We can evaluate the expression $F^{\mu\nu}U_\nu$ by considering the cases $\mu = 0$ and $\mu = 1,2,3$ separately. For $\mu = 0$, we have:

$$F^{0\nu}U_\nu = F^{00}U_0 = F^{0i}U_i = -F_{0i}U^i = \frac{\gamma}{c}E_i v_i = \gamma\mathbf{E}\cdot\mathbf{v}/c.$$

For $\mu = i = 1,2,3$, we have

$$F^{i\nu}U_\nu = F^{i0}U_0 + F^{ij}U_j = F_{i0}U^0 + F_{ij}U^j = \gamma\left(E_i + \epsilon_{ijk}v_j B_k\right) = \gamma\left(E_i + [\mathbf{v}\times\mathbf{B}]_i\right).$$

Putting everything together, we have

$$F^{\mu\nu}U_\nu = \gamma \begin{pmatrix} \mathbf{E}\cdot\mathbf{v}/c \\ \mathbf{E}+\mathbf{v}\times\mathbf{B} \end{pmatrix},$$

as required.

We are asked to show the general equation:

$$j^\mu + \frac{1}{c^2}\left(j^\nu U_\nu\right)U^\mu = \sigma F^{\mu\nu}U_\nu,$$

given $\mathbf{J} = \sigma\mathbf{E}$ in the rest frame of the conducting medium.

This task illustrates an important general principle. Notice that this equation is entirely composed of objects that transform in a special way under Lorentz transformations: j^μ is a Lorentz vector, U^μ is a Lorentz vector, σ is a Lorentz scalar and $F^{\mu\nu}$ is a tensor. In particular, because the index structure of the equation is correct, under a Lorentz transformation all the terms of this equation transform in the same way, and hence if the equation holds in one frame, then it holds in *all frames*. So it is sufficient to demonstrate that the equation holds in one frame.

Let's demonstrate the validity of this equation in the rest frame of the medium. We have $j^\mu = (\rho c, \mathbf{J})$ by definition, $U^\mu = (c, \mathbf{0})$ since we are in the rest frame so $\mathbf{v} = \mathbf{0}$, and thus

$$j^\nu U_\nu = j^0 U_0 + j^i U_i = -j^0 U^0 + j^i U^i = -\rho c^2.$$

Substituting $\mathbf{v} = \mathbf{0}$ into the previous calculation, we have:

$$F^{\mu\nu}U_\nu = \begin{pmatrix} 0 \\ \mathbf{E} \end{pmatrix}.$$

Putting everything together, we see that the equation

$$j^\mu + \frac{1}{c^2}\left(j^\nu U_\nu\right) U^\mu = \sigma F^{\mu\nu} U_\nu,$$

reduces to

$$\begin{pmatrix} \rho c \\ \mathbf{J} \end{pmatrix} + \frac{1}{c^2}\left(-\rho c^2\right)\begin{pmatrix} c \\ \mathbf{0} \end{pmatrix} = \sigma \begin{pmatrix} 0 \\ \mathbf{E} \end{pmatrix}.$$

Simplifying, we have:

$$\begin{pmatrix} 0 \\ \mathbf{J} \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma \mathbf{E} \end{pmatrix}.$$

This is true in the rest frame, as it's Ohm's law! Therefore:

$$j^\mu + \frac{1}{c^2}\left(j^\nu U_\nu\right) U^\mu = \sigma F^{\mu\nu} U_\nu,$$

holds in all frames, and we deduce that it is indeed the relativistic generalisation of Ohm's law.

We are now asked to evaluate the relativistic form of Ohm's law when the medium is travelling with 3-velocity $\mathbf{v}$. We simply insert $j^\mu = (\rho c, \mathbf{J})$, $U^\mu = \gamma(c, \mathbf{v})$ into the general form of Ohm's law. Note that

$$j^\nu U_\nu = j^0 U_0 + j^i U_i = -j^0 U^0 + j^i U^i = -\rho\gamma c^2 + \gamma \mathbf{J}\cdot\mathbf{v}.$$

Inserting into Ohm's law and using the result for $F^{\mu\nu}U_\nu$ we proved earlier, we have

$$\begin{pmatrix} \rho c \\ \mathbf{J} \end{pmatrix} + \frac{\gamma}{c^2}\left(-\rho\gamma c^2 + \gamma \mathbf{J}\cdot\mathbf{v}\right)\begin{pmatrix} c \\ \mathbf{v} \end{pmatrix} = \sigma\gamma \begin{pmatrix} \mathbf{E}\cdot\mathbf{v}/c \\ \mathbf{E} + \mathbf{v}\times\mathbf{B} \end{pmatrix}.$$

Reading off both equations, we have

$$\rho c - \rho\gamma^2 c + \frac{\gamma^2}{c}\mathbf{J}\cdot\mathbf{v} = \frac{1}{c}\sigma\gamma\mathbf{E}\cdot\mathbf{v}$$

$$\mathbf{J} - \rho\gamma^2\mathbf{v} + \frac{\gamma^2}{c^2}\left(\mathbf{J}\cdot\mathbf{v}\right)\mathbf{v} = \sigma\gamma\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right).$$

The first equation tells us:

$$\rho - \frac{\sigma\gamma}{c^2}\mathbf{E}\cdot\mathbf{v} = \rho\gamma^2 - \frac{\gamma^2}{c^2}\mathbf{J}\cdot\mathbf{v},$$

and hence substituting into the second equation to eliminate the γ^2 terms we have:

$$\mathbf{J} - \left(\rho - \frac{\sigma\gamma}{c^2}\mathbf{E}\cdot\mathbf{v}\right)\mathbf{v} = \sigma\gamma\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right).$$

Rearranging we find:

$$\mathbf{J} = \rho\mathbf{v} + \sigma\gamma\left(\mathbf{E} + \mathbf{v}\times\mathbf{B} - \frac{1}{c^2}(\mathbf{v}\cdot\mathbf{E})\mathbf{v}\right),$$

as required.

Finally, we are given that the charge density vanishes in the rest-frame. In this case, note that in the rest frame we have: $j^\nu U_\nu = -j^0 U^0 + j^i U^i = -\rho c^2 + \mathbf{J}\cdot\mathbf{0} = 0$ (since $\rho = 0$), and hence $j^\nu U_\nu = 0$ in all frames. It follows that Ohm's law in relativistic form reduces to $j^\mu = \sigma F^{\mu\nu}U_\nu$, and hence we get the much simpler result:

$$\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right).$$

Part IB: Electromagnetism
Past Paper Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

2016, Paper 2, Section I, 6D

(a) Derive the integral form of Ampère's law from the differential form of Maxwell's equations with a time-independent magnetic field, $\rho = 0$ and $\mathbf{E} = \mathbf{0}$.

(b) Consider two perfectly-conducting concentric thin cylindrical shells of infinite length with axes along the z-axis and radii a and b ($a < b$). Current I flows in the positive z-direction in each shell. Use Ampère's law to calculate the magnetic field in the three regions (i) $r < a$; (ii) $a < r < b$ and (iii) $r > b$, where $r = \sqrt{x^2 + y^2}$.

(c) If current I now flows in the positive z-direction in the inner shell and in the negative z-direction in the outer shell, calculate the magnetic field in the same three regions.

Solution: (a) The differential form of Ampère's law is $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ in magnetostatics. To derive the integral form, consider an area A with boundary C. By Stokes' Theorem, we have:

$$\oint_C \mathbf{B} \cdot \mathbf{dx} = \int_A \nabla \times \mathbf{B} \cdot \mathbf{dA} = \mu_0 \int_A \mathbf{J} \cdot \mathbf{dA} = \mu_0 I,$$

where I is the current passing through the surface A. We therefore have Ampère's law:

$$\oint_C \mathbf{B} \cdot \mathbf{dx} = \mu_0 I.$$

(b) First, let's note that instead of working with *two* concentric cylindrical shells, we can work with *one* concentric cylindrical shell and then use the principle of superposition to work out the $\mathbf{B}$ field in general (this will also make part (c) considerably easier!). Let our cylindrical shell have radius a and suppose it has current I passing through it in the z-direction.

To use Ampère's law here, we must first restrict the functional form of the $\mathbf{B}$ field. Let's work in cylindrical coordinates so that $\mathbf{B} \equiv \mathbf{B}(r, \theta, z)$. Notice that the current distribution in this case is invariant under the coordinate transformations $z \mapsto z + \Delta z, \theta \mapsto \theta + \Delta\theta$ (note the second is simply a rotation of the coordinates). Thus the $\mathbf{B}$ field is invariant under these coordinate transformations too:

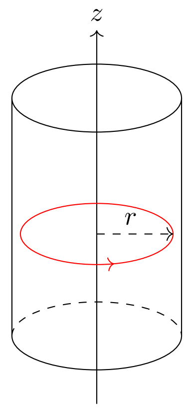
$$\mathbf{B}(r, \theta, z) = \mathbf{B}(r, \theta + \Delta\theta, z + \Delta z),$$

from which it follows that $\mathbf{B} \equiv \mathbf{B}(r)$, i.e. $\mathbf{B}$ is a function of r only.

Now let's consider active transformations of the system. Suppose that $\mathbf{B}$ has a component in the $\hat{\mathbf{e}}_r$ direction at the point $\mathbf{x}$. The under a reflection through a plane orthogonal to the z-axis through the point $\mathbf{x}$, the current changes direction, *and* the $\mathbf{B}$ component changes direction, since it is a *pseudovector*. Comparing the two scenarios, we see that in one the $\mathbf{B}$ field points inwards but in the other the $\mathbf{B}$ field points outwards. Nature cannot be two ways, and we conclude that $\mathbf{B}$ has no component in the $\hat{\mathbf{e}}_r$ direction.

Similarly, suppose that $\mathbf{B}$ has a component in the $\hat{\mathbf{e}}_z$ direction at the point $\mathbf{x}$. Again, considering a reflection in the plane orthogonal to the z-axis and through the point $\mathbf{x}$, the $\mathbf{B}$ component is invariant (since it is a pseudovector) but the current flows in the opposite direction. We conclude that the $\mathbf{B}$ field has no component in the $\hat{\mathbf{e}}_z$ direction.

Thus the most general form of the $\mathbf{B}$ field in this case is: $\mathbf{B} \equiv B(r)\hat{\mathbf{e}}_\theta$.



To work out the function $B(r)$, we use Ampère's law. Begin by considering a circle of radius C_r centre on the z-axis with $r < a$. Then Ampère's law gives:

$$0 = \oint_C B(r)\hat{\mathbf{e}}_\theta \cdot \mathbf{dx} = \int_0^{2\pi} B(r) r \, d\theta = 2\pi r B(r) \qquad \Rightarrow \qquad B(r) = 0.$$

If instead $r > a$, we have by Ampère's law:

$$\mu_0 I = \oint_C B(r)\hat{\mathbf{e}}_\theta \cdot \mathbf{dx} = \int_0^{2\pi} B(r) r \, d\theta = 2\pi r B(r) \qquad \Rightarrow \qquad B(r) = \frac{\mu_0 I}{2\pi r}.$$

We conclude that the **B** field for one shell of radius a is given by:

$$\mathbf{B}_a = \begin{cases} \mathbf{0} & \text{if } r < a, \\ \dfrac{\mu_0 I}{2\pi r}\hat{\mathbf{e}}_\theta & \text{if } r > a. \end{cases}$$

It then follows by the linearity of Maxwell's equations that we can take the superposition of $\mathbf{B}_a$, $\mathbf{B}_b$ (for $a < b$) to solve the problem we initially wanted. We have:

$$\mathbf{B} = \mathbf{B}_a + \mathbf{B}_b = \begin{cases} \mathbf{0} & \text{if } r < a, \\ \dfrac{\mu_0 I}{2\pi r}\hat{\mathbf{e}}_\theta & \text{if } a < r < b, \\ \dfrac{\mu_0 I}{\pi r}\hat{\mathbf{e}}_\theta & \text{if } r > b. \end{cases}$$

(c) In this case, all we need to do is flip the sign of the current in $\mathbf{B}_b$ above. This is equivalent to instead considering the superposition $\mathbf{B} = \mathbf{B}_a - \mathbf{B}_b$, given by:

$$\mathbf{B} = \mathbf{B}_a - \mathbf{B}_b = \begin{cases} \mathbf{0} & \text{if } r < a, \\ \dfrac{\mu_0 I}{2\pi r}\hat{\mathbf{e}}_\theta & \text{if } a < r < b, \\ \mathbf{0} & \text{if } r > b. \end{cases}$$

2016, Paper 4, Section I, 7D

(a) Starting from Maxwell's equations, show that in a vacuum,

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = \mathbf{0} \qquad \text{and} \qquad \nabla \cdot \mathbf{E} = 0 \qquad \text{where} \qquad c = \sqrt{\frac{1}{\epsilon_0 \mu_0}}.$$

(b) Suppose that $\mathbf{E} = \frac{1}{\sqrt{2}} E_0 (1,1,0)\cos(kz - \omega t)$ where E_0, k and ω are real constants.

(i) What are the wavevector and the polarisation? How is ω related to k?

(ii) Find the magnetic field $\mathbf{B}$.

(iii) Compute and interpret the time-averaged value of the Poynting vector $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$.

Solution: (a) Maxwell's equations in a vacuum are given by:

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

Taking the time derivative of the fourth equation (the Ampère-Maxwell law), we arrive at:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla \times \frac{\partial \mathbf{B}}{\partial t}.$$

Substituting for the time-derivative of $\mathbf{B}$ using the third equation (Faraday's law) we have:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\nabla \times (\nabla \times \mathbf{E}).$$

We now use an identity from vector calculus to simplify the right hand side:

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}.$$

The first term is zero, by Gauss' law $\nabla \cdot \mathbf{E}$ in a vacuum. This leaves us with

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = \mathbf{0},$$

as required. This shows us that each component of the electric field, in a vacuum, satisfies the wave equation.

(b) (i) The *wavevector* is the direction of propagation of the waves. We can find this by looking at the argument of the cosine, since this is the part that tells us what spatial variables change as time t changes. We see that the wavevector in this case is given by $\mathbf{k} = (0,0,k)$.

The *polarisation* refers to the direction in which the electric field propagates as the wave propagates. This direction is constant and given by $(1,1,0)/\sqrt{2}$ in this case. We say that the wave is *linearly polarised*.

The frequency ω and wavenumber k of the waves are related by the *dispersion relation* $\omega = ck$. One can derive this simply by substituting the given form for $\mathbf{E}$ into the wave equation. We have, considering the E_x component for simplicity (considering things component-wise is fine for the vector Laplacian in Cartesian coordinates),

$$\frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = \frac{1}{c^2}\left(-\frac{E_0}{\sqrt{2}}\omega^2 \cos(kz - \omega t)\right),$$

and $\nabla^2 E_x = -\frac{1}{\sqrt{2}} E_0 k^2 \cos(kz - \omega t)$. Comparing the two, we have $\omega^2 = k^2 c^2 \Rightarrow \omega = ck$ as required.

(ii) We are now asked to find the magnetic field **B**. There are two ways of doing this. One way is simply quoting the answer from lectures; we recall that **B** will be a wave propagating such that the wavevector, electric field and magnetic fields together form an orthonormal system such that

$$\mathbf{B}_0 \cdot \mathbf{k} = \mathbf{E}_0 \cdot \mathbf{k} = 0, \qquad \omega \mathbf{B}_0 = \mathbf{k} \times \mathbf{E}_0,$$

where $\mathbf{B}_0$ is the polarisation vector for the **B** field. In particular, we can simply write down:

$$\mathbf{B} = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega} \cos(kz - \omega t) = \frac{kE_0}{\omega\sqrt{2}} \left(\hat{\mathbf{e}}_z \times (\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y)\right) \cos(kz - \omega t) = \frac{E_0}{c\sqrt{2}}(-1, 1, 0) \cos(kz - \omega t).$$

Alternatively, we can carefully solve for the **B** field using Maxwell's equations. The third Maxwell equation, Faraday's law, tells us that

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = \frac{1}{\sqrt{2}} E_0 (k\hat{\mathbf{e}}_z) \times (\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y) \sin(kz - \omega t) = \frac{kE_0}{\sqrt{2}}(-1, 1, 0) \sin(kz - \omega t).$$

Integrating, we have

$$\mathbf{B} = \frac{E_0}{c\sqrt{2}}(-1, 1, 0) \cos(kz - \omega t) + \mathbf{f}(\mathbf{x}),$$

where $\mathbf{f}(\mathbf{x})$ is an arbitrary time-independent vector function. Note that we've used the dispersion relation $k/\omega = 1/c$ to simplify things here.

It is not possible in general to fix $\mathbf{f}(\mathbf{x})$ using Maxwell's equations alone (though we can deduce for example that $\mathbf{f}(\mathbf{x})$ must be a harmonic function); we need some boundary data. Since we want **B** to be a wave propagating with wavevector $\mathbf{k} = k\hat{\mathbf{e}}_z = (0, 0, k)$, we must have that **B** is a function of $kz - \omega t$. In particular, this implies that $\mathbf{f}(\mathbf{x})$ must be a constant, and assuming that there is no constant magnetic background field, we can set this constant to zero without loss of generality.

(iii) Finally, we are asked to compute the time-averaged Poynting vector. We are given the expression for the Poynting vector as:

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$

Inserting the **E** and **B** fields, we have

$$\mathbf{S} = \frac{E_0^2}{2c\mu_0} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cos^2(kz - \omega t) = \frac{E_0^2}{\mu_0 c} \hat{\mathbf{e}}_z \cos^2(kz - \omega t).$$

Taking the time-average over a period $T = 2\pi/\omega$, we have

$$\langle \mathbf{S} \rangle = \frac{\omega}{2\pi} \frac{E_0^2}{\mu_0 c} \hat{\mathbf{e}}_z \cdot \int_0^{2\pi/\omega} \cos^2(kz - \omega t)\, dt = \frac{\omega E_0^2}{4\pi \mu_0 c} \hat{\mathbf{e}}_z \int_0^{2\pi/\omega} \left(1 + \cos(2(kz - \omega t))\right)\, dt = \frac{E_0^2}{2\mu_0 c} \hat{\mathbf{e}}_z.$$

The Poynting vector represents the direction of energy flux of the electromagnetic field per unit time. Hence we see that energy is transported by the electromagnetic wave in the $\hat{\mathbf{e}}_z$ direction in this case.

2016, Paper 1, Section II, 16D

(a) From the differential form of Maxwell's equations with $\mathbf{J} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$ and a time-independent electric field, derive the integral form of Gauss' law.

(b) Derive an expression for the electric field $\mathbf{E}$ around an infinitely long line charge lying along the z-axis with charge per unit length μ. Find the electrostatic potential ϕ up to an arbitrary constant.

(c) Now consider the line charge with an ideal earthed conductor filling the region $x > d$. State the boundary conditions satisfied by ϕ and $\mathbf{E}$ on the surface of the conductor.

(d) Show that the same boundary conditions at $x = d$ are satisfied if the conductor is replaced by a second line charge at $x = 2d, y = 0$ with charge per unit length $-\mu$.

(e) Hence or otherwise, returning to the setup in (c), calculate the force per unit length acting on the line charge.

(f) What is the charge per unit area $\sigma(y, z)$ on the surface of the conductor?

Solution: (a) The relevant Maxwell equation here is Gauss' law, which states $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. Integrating over a volume V, we have by the divergence theorem:

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \int_V dV\, \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \int dV\, \rho = \frac{Q}{\epsilon_0},$$

where Q is the total charged contained within the volume V, and S is the volume V's surface. Thus we see that we have derive the integral form of Gauss' law,

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\epsilon_0}.$$

(b) First we use symmetry principles to restrict the functional form of the electric field $\mathbf{E}(\mathbf{x})$ due to the line charge. Let's work in cylindrical coordinates (r, θ, z), so that in principle $\mathbf{E}$ could depend on all of these coordinates, $\mathbf{E} = \mathbf{E}(r, \theta, z)$.

Begin by noticing that the line charge distribution is invariant under the coordinate transformations $z \mapsto z + \Delta z$ and $\theta \mapsto \theta + \Delta\theta$ (the second obviously represents a rotation *around* the line charge). It follows that the electric field must also be invariant under these changes of coordinates, so we have:

$$\mathbf{E}(r, \theta, z) = \mathbf{E}(r, \theta + \Delta\theta, z + \Delta z) \qquad \Rightarrow \qquad \mathbf{E} \equiv \mathbf{E}(r).$$

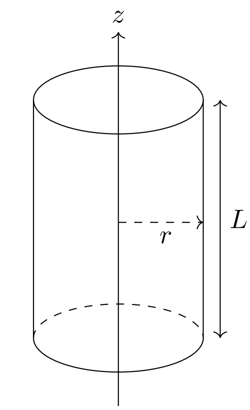
Thus $\mathbf{E}$ is a function only of the radial distance away from the line charge.

Now let's think about active transformations, where we move the system itself rather than just the coordinates. Suppose that $\mathbf{E}$ has some component in the $\hat{\mathbf{e}}_z$ direction at the point $\mathbf{x}$. Then, under a reflection in a plane orthogonal to the wire passing through the $\mathbf{x}$, the charge distribution remains the same but the electric field component is flipped upside down. Nature can't be two ways, and hence the electric field cannot have a $\hat{\mathbf{e}}_z$ component.

Similarly, suppose that $\mathbf{E}$ has some component in the $\hat{\mathbf{e}}_\theta$ direction at the point $\mathbf{x}$. Under a reflection in a plane that contains both the point $\mathbf{x}$ and the wire, the charge distribution is invariant but again the electric field component is flipped. So $\mathbf{E}$ can have no $\hat{\mathbf{e}}_\theta$ component either.

Thus, we have shown that the most general functional form for such a charge distribution is: $\mathbf{E} = E(r)\hat{\mathbf{e}}_r$.

Let's go ahead and find the electric field. Choose V as a cylindrical volume with axis aligned with the wire along the z-axis, radius r and length L. The surface S of V will be our Gaussian surface in this problem.



The amount of charge contained in this volume is $Q = \mu L$, since the line charge carries a charge μ per unit length. The electric flux out of this surface is given by:

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \int_S E(r)\hat{\mathbf{e}}_r \cdot \mathbf{dS} = \int_0^L dz \int_0^{2\theta} d\theta \, rE(r) = 2\pi LrE(r).$$

Thus by Gauss' law, we have:

$$2\pi LrE(r) = \frac{\mu L}{\epsilon_0} \qquad \Rightarrow \qquad \mathbf{E} = \frac{\mu}{2\pi\epsilon_0 r}\hat{\mathbf{e}}_r.$$

An electric potential for this electric field is a function ϕ such that $\mathbf{E} = -\nabla\phi$. Recall that the gradient in cylindrical polars is given by:

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{\partial\phi}{\partial z}\hat{\mathbf{e}}_z,$$

and hence (by inspection) a suitable potential for this problem is simply:

$$\phi = -\frac{\mu\log(r)}{2\pi\epsilon_0} + \phi_0,$$

where ϕ_0 is an arbitrary constant.

(c) We know from lectures that across the boundary of a conductor S, the electric field suffers the discontinuity:

$$\mathbf{E}_+ - \mathbf{E}_- = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}},$$

where $\mathbf{E}_+$ is the electric field just outside of the conductor, $\mathbf{E}_-$ is the electric field just inside of the conductor, and $\hat{\mathbf{n}}$ is the unit normal pointing *out* of the conductor. σ denotes the surface charge on the conductor.

In our case, we are working with an ideal conductor filling the region $x > d$. Inside a conductor, we know the electric field vanishes: $\mathbf{E} = \mathbf{0}$. So our boundary condition reduces to:

$$\mathbf{E}_+ = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{e}}_x,$$

since the outward pointing normal from the region $x \geq d$ is $\hat{\mathbf{e}}_x$.

We are also asked to state the boundary conditions on ϕ. Since $\mathbf{E}_+ - \mathbf{E}_- = \sigma\hat{\mathbf{n}}/\epsilon_0$, we have

$$(\nabla\phi)_- - (\nabla\phi)_+ = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}.$$

Since this is a discontinuity in a derivative, we must have that ϕ is continuous, since if ϕ were discontinuous the derivatives would be worse than discontinuous - they would be delta functions! It follows that ϕ is continuous across the surface $x = d$.

Furthermore, the region $x > d$ is *earthed*, which tells us we should the potential to be 0 in this region. Thus we have that $\phi = 0$ across the surface of the conductor $x = d$, and ϕ is continuous across the surface of the conductor.

(d) We are now asked to show that the same boundary conditions are satisfied at $x = d$ if we insert an additional line charge at $x = 2d, y = 0$ in part (c). In order to verify this, we need to find the electric field and electric potential at $x = d$ in the new setup. We use the principle of superposition for electric fields and electric potentials.

The electric field due to the line charge at $x = 0, y = 0$ (i.e. the z-axis) is simply:

$$\mathbf{E}_1 = \frac{\mu}{2\pi\epsilon_0 r}\hat{\mathbf{e}}_r$$

as we found in part (b). Let's write this in Cartesian coordinates so that we can find the electric field due to the line charge at the new location $x = 2d, y = 0$ by translating this electric field.

Note that

$$\hat{\mathbf{e}}_r = \cos(\theta)\hat{\mathbf{e}}_x + \sin(\theta)\hat{\mathbf{e}}_y = \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_x + \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_y.$$

Hence the electric field in Cartesians is:

$$\mathbf{E}_1 = \frac{\mu}{2\pi\epsilon_0\sqrt{x^2+y^2}}\left(\frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_x + \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_y\right) = \frac{\mu}{2\pi\epsilon_0}\left(\frac{x}{x^2+y^2}\hat{\mathbf{e}}_x + \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_y\right).$$

If we translate our coordinates so that the line charge is at $x = 2d$, $y = 0$, and we map the charge per unit length via $\mu \mapsto -\mu$, we will get the electric field due to the other line charge that we are now considering. We have:

$$\mathbf{E}_2 = -\frac{\mu}{2\pi\epsilon_0}\left(\frac{x-2d}{(x-2d)^2+y^2}\hat{\mathbf{e}}_x + \frac{y}{\sqrt{(x-2d)^2+y^2}}\hat{\mathbf{e}}_y\right)$$

Now by the fact that Maxwell's equations are linear, we can superpose these solutions to obtain the solution to the full problem with two line charges:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{\mu}{2\pi\epsilon_0}\left(\left(\frac{x}{x^2+y^2} - \frac{(x-2d)}{(x-2d)^2+y^2}\right)\hat{\mathbf{e}}_x + \left(\frac{y}{x^2+y^2} - \frac{y}{(x-2d)^2+y^2}\right)\hat{\mathbf{e}}_y\right).$$

We can now verify that the boundary condition is satisfied at $x = d$. As $x \to d$ from below, we see that

$$\mathbf{E} \to \frac{\mu}{2\pi\epsilon_0}\left(\frac{d}{d^2+y^2} + \frac{d}{d^2+y^2}\right)\hat{\mathbf{e}}_x = \frac{\mu d}{\pi\epsilon_0(d^2+y^2)}\hat{\mathbf{e}}_x,$$

so indeed just outside the region $x > d$, we have

$$\mathbf{E} = -\frac{\hat{\mathbf{e}}_x}{\epsilon_0}\left(-\frac{\mu d}{\pi(d^2+y^2)}\right),$$

for some constant $\sigma = -\mu d/(\pi(d^2+y^2))$.

We still need to verify that ϕ is constant on $x = d, y = 0$, and that it is continuous across the surface $x = d$. Recall the potential for a single line charge along the z-axis was:

$$\phi_1 = -\frac{\mu \log(r)}{2\pi\epsilon_0 r} + \phi_0 = -\frac{\mu \log(x^2 + y^2)}{2\pi\epsilon_0 \sqrt{x^2 + y^2}} + \phi_0.$$

Again by the principle of superposition, we have that the electric potential for the new problem is:

$$\phi = -\frac{\mu}{2\pi\epsilon_0}\left(\frac{\log(x^2 + y^2)}{\sqrt{x^2 + y^2}} - \frac{\log((x - 2d)^2 + y^2)}{\sqrt{(x - 2d)^2 + y^2}}\right) + \text{arbitrary constant.}$$

We see that on the surface $x = d, y = 0$, ϕ is indeed constant, and across the surface $x = d$, ϕ is indeed continuous.

(e) We now return to the problem in (c) and determine the force on the line charge per unit length. The appropriate electric field in the region $x < d$ is given by:

$$\mathbf{E} = \frac{\mu}{2\pi\epsilon_0}\left(\left(\frac{x}{x^2 + y^2} - \frac{(x - 2d)}{(x - 2d)^2 + y^2}\right)\hat{\mathbf{e}}_x + \left(\frac{y}{x^2 + y^2} - \frac{y}{(x - 2d)^2 + y^2}\right)\hat{\mathbf{e}}_y\right),$$

and the appropriate electric field in the region $x > d$ is given by $\mathbf{E} = \mathbf{0}$. We have seen in (d) that this satisfies all the necessary boundary condition at $x = d$ (this essentially completes our argument by the *method of images*).

The force on a volume V is given by:

$$\mathbf{F} = \int_V dV\, \rho\mathbf{E}$$

in electrostatics, where ρ is the charge density; this is given by $\rho = \mu\delta(x)\delta(y)$ for the line charge. Hence for a fixed unit length of the line charge, the force per unit length acting on it is given by:

$$\mathbf{f} = \mu \int_0^1 dz\, \mathbf{E}(0, 0, z) = \frac{\mu^2}{2\pi\epsilon_0}\left(\frac{2d}{4d^2}\right)\hat{\mathbf{e}}_x = \frac{\mu^2}{4\pi d\epsilon_0}\hat{\mathbf{e}}_x.$$

(f) We already compute the surface charge induced on the conductor at the end of part (d). It was given by:

$$\sigma(y, z) = -\frac{\mu d}{\pi(d^2 + y^2)}.$$

2016, Paper 2, Section II, 16D

(a) State the covariant form of Maxwell's equations and define all the quantities that appear in these expressions.

(b) Show that **E** · **B** is a Lorentz scalar (invariant under Lorentz transformations) and find another Lorentz scalar involving **E** and **B**.

(c) In some inertial frame S, the electric and magnetic fields are respectively **E** $= (0, E_y, E_z)$ and **B** $= (0, B_y, B_z)$. Find the electric and magnetic fields **E**$' = (0, E'_y, E'_z)$ and **B**$' = (0, B'_y, B'_z)$, in another inertial frame S' that is related to S by the Lorentz transformation,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where v is the velocity of S' in S and $\gamma = (1 - v^2/c^2)^{-1/2}$.

(d) Suppose that **E** $= E_0(0, 1, 0)$ and **B** $= \frac{E_0}{c}(0, \cos(\theta), \sin(\theta))$ where $0 \leq \theta \leq \pi/2$, and E_0 is a real constant. An observer is moving in S with velocity v parallel to the x-axis. What must v be for the electric and magnetic fields to appear to the observer to be parallel? Comment on the case $\theta = \pi/2$.

Solution: (a) In covariant form, Maxwell's equations are given by:

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu, \qquad \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0.$$

The first equation encompasses the *sourced* Maxwell equations, namely Gauss' law and the Ampère Maxwell law (these have sources appearing on the right hand side, in the form of charge density and current density). The second equation is known as the *Bianchi identity* and encompasses the *unsourced* Maxwell equations, namely Faraday's law and Gauss' law for magnetism (i.e. the law that states that magnetic monopoles do not exist).

The *four-current* j^μ is defined in some inertial frame to be $j^\mu = (\rho c, \mathbf{J})$, where ρ is the charge density and **J** is the current density. The *electromagnetic tensor* $F_{\mu\nu}$ is defined in some inertial frame to be:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

Finally, the *four-derivative* is $\partial_\mu = \partial/\partial x^\mu$.

(b) Recall that the construct Lorentz scalars, all we have to do is to ensure all our indices are contracted in some expression. Before we go ahead, it is useful to notice that in the expression for the electromagnetic tensor above, we have:

$$F_{ij} = \epsilon_{ijk} B_k.$$

Now, to construct **E** · **B** as a Lorentz scalar, consider:

$$\begin{aligned} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} &= \epsilon_{0ijk} F^{0i} F^{jk} + \epsilon_{i0jk} F^{i0} F^{jk} + \epsilon_{jk0i} F^{jk} F^{0i} + \epsilon_{jki0} F^{jk} F^{i0} \\ &= -4\epsilon_{ijk} F_{0i} F_{jk} \\ &= 4\epsilon_{ijk} E_i \epsilon_{jkl} B_l / c \end{aligned}$$

Contracting the epsilon symbols, we have $\epsilon_{ijk}\epsilon_{jkl} = \epsilon_{jki}\epsilon_{jkl} = \delta_{kk}\delta_{il} - \delta_{kl}\delta_{ik} = 2\delta_{il}$. Therefore we're left with:

$$\frac{8E_iB_l\delta_{il}}{c} = \frac{8\mathbf{E}\cdot\mathbf{B}}{c}.$$

Hence $\mathbf{E}\cdot\mathbf{B}$ is a Lorentz scalar, since $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ is a Lorentz scalar (it has the correct index structure).[1]

We can construct another Lorentz scalar by considering $F_{\mu\nu}F^{\mu\nu}$. We have:

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= F_{00}F^{00} + F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} \\ &= F_{00}F_{00} - F_{0i}F_{0i} - F_{i0}F_{i0} + F_{ij}F_{ij} \\ &= -\frac{E_iE_i}{c^2} - \frac{E_iE_i}{c^2} + \epsilon_{ijk}B_k\epsilon_{ijl}B_l \\ &= -\frac{2|\mathbf{E}|^2}{c^2} + \left(\delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj}\right)B_kB_l \\ &= -\frac{2|\mathbf{E}|^2}{c^2} + 2|\mathbf{B}|^2. \end{aligned}$$

Hence we deduce that

$$|\mathbf{E}|^2 - c^2|\mathbf{B}|^2$$

is also a Lorentz scalar, since it is proportional to $F_{\mu\nu}F^{\mu\nu}$ which has the correct index structure.

(c) Since the electromagnetic tensor is a tensor, it transforms under this Lorentz boost as: $F'_{\mu\nu} = (\Lambda^{-1})^\alpha{}_\mu(\Lambda^{-1})^\beta{}_\nu F_{\alpha\beta}$. We can rewrite this as the matrix equation:

$$F' = (\Lambda^{-1})^T F\Lambda^{-1},$$

and hence we can compute the electromagnetic tensor in the boosted frame (here F denotes the matrix of the electromagnetic tensor with lower indices). First note that to obtain the inverse of the given Lorentz transformation Λ, just set $v\mapsto -v$, since this is 'boosting in the opposite direction'. It follows that the above matrix equation can be written:

$$F' = \begin{pmatrix} \gamma & \gamma v/c & 0 & 0 \\ \gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}\begin{pmatrix} \gamma & \gamma v/c & 0 & 0 \\ \gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Performing the matrix multiplications, we're left with:

$$\begin{aligned} F' &= \begin{pmatrix} \gamma & \gamma v/c & 0 & 0 \\ \gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\gamma vE_x/c^2 & -\gamma E_x/c & -E_y/c & -E_z/c \\ \gamma E_x/c & \gamma vE_x/c^2 & B_z & -B_y \\ \gamma E_y/c - \gamma vB_z/c & \gamma vE_y/c^2 - \gamma B_z & 0 & B_x \\ \gamma E_z/c + \gamma vB_y/c & \gamma vE_z/c^2 + \gamma B_y & -B_x & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\gamma^2E_x/c + \gamma^2v^2E_x/c & -\gamma(E_y - vB_z)/c & -\gamma(E_z + vB_y)/c \\ -\gamma^2v^2E_x/c^3 + \gamma^2E_x/c & 0 & \gamma(B_z - vE_y/c^2) & -\gamma(B_y + vE_z/c^2) \\ \gamma(E_y - vB_z)/c & -\gamma(B_z - vE_y/c^2) & 0 & B_x \\ \gamma(E_z + vB_y)/c & \gamma(B_y + vE_z/c^2) & -B_x & 0 \end{pmatrix} \end{aligned}$$

[1] Actually, this question is wrong. The quantity here is a Lorentz *pseudoscalar* because of the presence of the epsilon symbol. See Examples Sheet 3, Question 6(b).

Some simplification is possible here, in particular we have:

$$-\frac{\gamma^2 E_x}{c} + \frac{\gamma^2 v^2 E_x}{c} = -\frac{\gamma^2}{c}\left(1 - \frac{v^2}{c^2}\right) E_x = -E_x/c.$$

Thus the final matrix form for the transformed electromagnetic tensor is:

$$F' = \begin{pmatrix} 0 & -E_x/c & -\gamma(E_y - vB_z)/c & -\gamma(E_z + vB_y)/c \\ E_x/c & 0 & \gamma(B_z - vE_y/c^2) & -\gamma(B_y + vE_z/c^2) \\ \gamma(E_y - vB_z)/c & -\gamma(B_z - vE_y/c^2) & 0 & B_x \\ \gamma(E_z + vB_y)/c & \gamma(B_y + vE_z/c^2) & -B_x & 0 \end{pmatrix}$$

In particular, comparing the entries of F' written in terms of the transformed **E**′, **B**′ fields, we see that we have derived the transformation law:

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma(E_y - vB_z) \qquad B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

$$E'_z = \gamma(E_z + vB_y) \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right).$$

In our case, we are given that $E_x = 0$, $B_x = 0$, and hence the formulae simplify (slightly) to:

$$E'_x = 0 \qquad B'_x = 0$$

$$E'_y = \gamma(E_y - vB_z) \qquad B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

$$E'_z = \gamma(E_z + vB_y) \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right).$$

(d) We boost the given electric and magnetic fields to the observer's frame using the transformation law we derived in the previous part. We have:

$$E'_x = 0 \qquad B'_x = 0$$

$$E'_y = E_0\gamma\left(1 - \frac{v}{c}\sin(\theta)\right) \qquad B'_y = E_0\gamma \cdot \frac{1}{c}\cos(\theta)$$

$$E'_z = E_0\gamma \cdot \frac{v}{c}\cos(\theta) \qquad B'_z = E_0\gamma \cdot \frac{1}{c}\left(\sin(\theta) - \frac{v}{c}\right).$$

Recall that two vectors **v**, **w** are parallel if and only if there exists a constant $\lambda \neq 0$ such that $\mathbf{v} = \lambda\mathbf{w}$. Therefore, we need the ratios of the remaining components of the **E**, **B** fields to be equal if they are parallel:

$$\frac{1 - v\sin(\theta)/c}{v\cos(\theta)/c} = \frac{\cos(\theta)/c}{\sin(\theta)/c - v/c^2} \qquad \Rightarrow \qquad \frac{c - v\sin(\theta)}{v\cos(\theta)} = \frac{c\cos(\theta)}{c\sin(\theta) - v}.$$

There could be problems taking this ratio if $\cos(\theta) = 0$ or $\sin(\theta) = v/c$. Let's check these cases carefully:

- In the case $\cos(\theta) = 0$, it follows that $\theta = \pi/2$ (since $0 \leq \theta \leq \pi/2$), and hence $\sin(\theta) = 1$. The initial electric and magnetic fields are $\mathbf{E} = E_0(0,1,0)$, $\mathbf{B} = \frac{1}{c}E_0(0,0,1)$. Therefore in the initial inertial frame, we have $\mathbf{E}\cdot\mathbf{B} = 0$. Since this is a Lorentz invariant, there is *no* inertial frame in which the electric and magnetic fields are parallel - they are always orthogonal. Thus we can discard $\cos(\theta) = 0$.
- Since we are *solving* for v, we can always choose that $v \neq c\sin(\theta)$. This is fine because this choice is unhelpful; it sets $B'_z = 0$ but $E'_z \neq 0$ in this case (unless $v = c$, which is not allowed).

Now going ahead and solving the given equation, we have:

$$c^2 \sin(\theta) - cv - cv\sin^2(\theta) + v^2\sin(\theta) = vc\cos^2(\theta) \qquad \Rightarrow \qquad \sin(\theta)v^2 - 2vc + c^2\sin(\theta) = 0.$$

Solving this quadratic equation, we find

$$v = \frac{c \pm c\sqrt{1-\sin^2(\theta)}}{\sin(\theta)} = \frac{c \pm c\cos(\theta)}{\sin(\theta)}.$$

Since $\sin(\theta) = \sqrt{1-\cos^2(\theta)} = \sqrt{(1-\cos(\theta))(1+\cos(\theta))}$ on this range, we see that

$$v = c\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}, \qquad \text{or} \qquad v = c\sqrt{\frac{1+\cos(\theta)}{1-\cos(\theta)}}.$$

Note that in our case $0 \leq \theta \leq \pi/2$, so $0 < \cos(\theta) \leq 1$ (we discarded the case $\cos(\theta) = 0$ above). Hence we see the second solution is not useful, as it gives $v \geq c$, which is not possible; this also implies that the first solution is valid. Therefore the solution is:

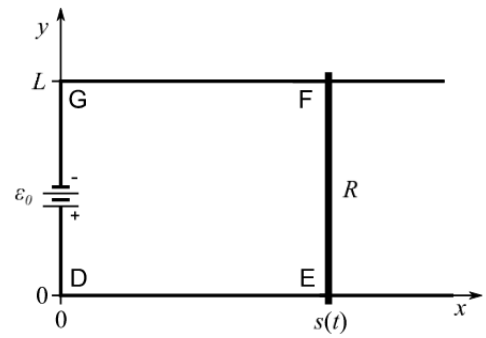
$$v = c\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}.$$

2016, Paper 3, Section II, 17D

(a) State Faraday's law of induction for a moving circuit in a time-dependent magnetic field and define all the terms that appear.

(b) Consider a rectangular circuit DEFG in the $z = 0$ plane as shown in the diagram below. There are two rails parallel to the x-axis for $x > 0$ starting at D at $(x, y) = (0, 0)$ and G at $(0, L)$. A battery provides an electromotive force $\mathcal{E}_0$ between D and G driving current in a positive sense around DEFG. The circuit is completed with a bar resistor of resistance R, length L and mass m that slides without friction on the rails; it connects E at $(s(t), 0)$ and F at $(s(t), L)$. The rest of the circuit has no resistance. The circuit is in a constant uniform magnetic field B_0 parallel to the z-axis.

[*In parts (i)-(iv) you can neglect any magnetic field due to current flows.*]



(i) Calculate the current in the bar and indicate its direction on a diagram of the circuit.

(ii) Find the force acting on the bar.

(iii) If the initial velocity and position of the bar are respectively $\dot{s}(0) = v_0 > 0$ and $s(0) = s_0 > 0$, calculate $\dot{s}(t)$ and $s(t)$ for $t > 0$.

(iv) If $\mathcal{E}_0 = 0$, find the total energy dissipated in the circuit after $t = 0$ and verify that total energy is conserved.

(v) Describe qualitatively the effect of the magnetic field caused by the induced current flowing in the circuit when $\mathcal{E}_0 = 0$.

Solution: (a) Let $C(t)$ be a time-dependent circuit, and let $S(t)$ be a surface with $C(t)$ as its boundary. We recall from lectures that *Faraday's law* states:

$$\mathcal{E} = -\frac{d\mathcal{F}}{dt},$$

where $\mathcal{E}$ is the *electromotive force*, given by:

$$\mathcal{E} = \oint_{C(t)} (\mathbf{E}(t, \mathbf{x}) + \mathbf{v} \times \mathbf{B}(t, \mathbf{x})) \cdot \mathbf{dx},$$

where $\mathbf{v}$ is the velocity of points on the time-dependent curve $C(t)$, and $\mathcal{F}$ is the *magnetic flux*, given by:

$$\mathcal{F} = \int_{S(t)} \mathbf{B} \cdot \mathbf{dx}.$$

(b) (i) This question is complicated by the presence of the battery. The battery induces an electromotive force $\mathcal{E}_0$ in the circuit, which in turn produces a current by *Ohm's law*:

$$|\mathcal{E}_0| = I_0 R \qquad \Rightarrow \qquad I_0 = \frac{\mathcal{E}_0}{R}.$$

Looking at the diagram, we see that the battery has its 'negative' terminal at the top and its 'positive' terminal at the bottom. Electrons flow from positive to negative in a circuit diagram, and hence the current travels in a *anticlockwise* sense in the given figure. This current produces a force on the bar by the Lorentz force law, given by

$$\mathbf{F} = I_0 \int_{\text{bar}} \mathbf{dx} \times \mathbf{B} = \frac{\mathcal{E}_0 B}{R} \int_0^L dx\, \hat{\mathbf{e}}_y \times \hat{\mathbf{e}}_z = \frac{\mathcal{E}_0 BL}{R} \hat{\mathbf{e}}_x.$$

It follows that the bar must initially move towards the *right*.

The movement of the bar will *induce* an electromotive force by Faraday's law too. This is given by:

$$\mathcal{E}(t) = -\frac{d\mathcal{F}}{dt} = -\frac{d}{dt}\left(B_0 L s(t)\right) = -B_0 L \dot{s}(t).$$

Since $\dot{s}(t) > 0$, i.e. the bar is moving towards the right, the electromotive force is induced in the *opposite direction*. That is, the electromotive force is induced in a *clockwise* sense around the given figure (following DGFE).

The current produced by this induced electromotive force is therefore given by Ohm's law to be:

$$I(t) = \frac{|\mathcal{E}(t)|}{R} = \frac{B_0 L \dot{s}(t)}{R},$$

but acts in an clockwise sense around the circuit. It follows that the *overall current* in the circuit, combining the effects of the battery and the motion of the bar, is given by:

$$I_{\text{tot}}(t) = \frac{\mathcal{E}_0}{R} - \frac{B_0 L \dot{s}(t)}{R}.$$

(ii) The force acting on the bar is given by the Lorentz force law to be:

$$\mathbf{F} = I_{\text{tot}}(t) \int_0^L dx\, \hat{\mathbf{e}}_y \times B_0 \hat{\mathbf{e}}_z = \left(\frac{\mathcal{E}_0}{R} - \frac{B_0 L \dot{s}(t)}{R}\right) B_0 L \hat{\mathbf{e}}_x.$$

as expected, the battery creates a force that pushes the bar outwards, but the induced electromotive force in the bar from its motion in the magnetic field creates a force that pushes the bar inwards.

(iii) Newton's second law now tells us that

$$m\ddot{s}(t) = \frac{B_0 L}{R}\left(\mathcal{E}_0 - B_0 L \dot{s}(t)\right).$$

Rearranging, we see that we have a first order differential equation for $\dot{s}(t)$ to solve:

$$\frac{d\dot{s}(t)}{dt} + \frac{B_0^2 L^2}{mR}\dot{s}(t) = \frac{B_0 L \mathcal{E}_0}{mR}.$$

We can solve this equation using an integrating factor:

$$\frac{d}{dt}\left(\dot{s}(t)\exp\left(\frac{B_0^2L^2}{mR}t\right)\right) = \frac{B_0L\mathcal{E}_0}{mR}\exp\left(\frac{B_0^2L^2}{mR}t\right).$$

Integrating directly, we have:

$$\dot{s}(t)\exp\left(\frac{B_0^2L^2}{mR}t\right) = \frac{\mathcal{E}_0}{B_0L}\exp\left(\frac{B_0^2L^2}{mR}t\right) - \frac{\mathcal{E}_0}{B_0L} + v_0,$$

using the initial condition $\dot{s}(0) = v_0$. Rearranging this becomes:

$$\dot{s}(t) = v_0\exp\left(-\frac{B_0^2L^2}{mR}t\right) + \frac{\mathcal{E}_0}{B_0L}\left(1 - \exp\left(-\frac{B_0^2L^2}{mR}t\right)\right).$$

Integrating directly again, we obtain the position of the bar:

$$s(t) = s_0 + \frac{mv_0R}{B_0^2L^2}\left(1 - \exp\left(-\frac{B_0^2L^2}{mR}t\right)\right) + \frac{\mathcal{E}_0}{B_0L}\left(t + \frac{mR}{B_0^2L^2}\left(\exp\left(-\frac{B_0^2L^2}{mR}t\right) - 1\right)\right),$$

using the initial condition $s(0) = s_0$.

(iv) Recall from lectures that the rate of energy dissipation (i.e. the energy dissipation per unit time) from the system is given by $RI_{\text{tot}}(t)^2$, and hence in this case (with $\mathcal{E}_0 = 0$) is given by:

$$R\left(-\frac{B_0L\dot{s}(t)}{R}\right)^2 = \frac{B_0^2L^2\dot{s}^2}{R} = \frac{B_0^2L^2v_0^2}{R}\exp\left(-\frac{2B_0^2L^2}{mR}t\right).$$

Integrating from $t = 0$ to $t = \infty$ gives the total energy dissipated by the system:

$$\frac{B_0^2L^2v_0^2}{R}\int_0^\infty \exp\left(-\frac{2B_0^2L^2}{mR}t\right)dt = \frac{B_0^2L^2v_0^2}{R}\cdot\frac{mR}{2B_0^2L^2} = \frac{1}{2}mv_0^2.$$

Since the initial energy of the circuit is simply the kinetic energy of the bar, $\frac{1}{2}mv_0^2$, this verifies that energy is conserved.

We could also check that energy is conserved by comparing to the work done on the bar. The work done per unit time is given by $\mathbf{F}\cdot\mathbf{v}$, which in this instance is given by:

$$\frac{dW}{dt} = \left(-\frac{B_0L\dot{s}}{R}\right)B_0L\cdot\dot{s} = -\frac{B_0^2L^2}{R}\dot{s}^2.$$

The work done by the force is negative the work done *against* the force, i.e. the energy that needs to be put in to make the motion happen; thus work done on the system equals energy dissipated, and energy is conserved.

(v) Finally we are asked to discuss qualitatively the effect of the magnetic field caused by the induced current flowing in the circuit when $\mathcal{E} = 0$.

Recall that initially, the rod begins moving to the right. As a consequence, more magnetic field is being allowed through the circuit, and hence $\dot{\mathcal{F}} > 0$. It follows by Faraday's law that a current will be induced in the circuit, travelling clockwise around the loop, since $\mathcal{E}(t) = -\dot{\mathcal{F}} < 0$. This current itself induces a magnetic field, which by the right hand rule will point down through the circuit, in the opposite direction to the background, constant magnetic field.

Hence, whilst the amount of background magnetic field passing through the circuit is increasing as the circuit increases in size, the induced magnetic field from the current moving in the bar points in the opposite direction to oppose the change. Thus we avoid having an unstable, runaway effect - this phenomenon is known as *Lenz's law*.

2017, Paper 2, Section I, 6D

State Gauss' Law in the context of electrostatics.

A spherically symmetric capacitor consists of two conductors in the form of concentric spherical shells of radii a and b, with $b > a$. The inner sphere carries a charge Q and the outer sphere carries a charge $-Q$. Determine the electric field **E** and the electrostatic potential ϕ in the regions $r < a$, $a < r < b$ and $r > b$. Show that the capacitance is

$$C = \frac{4\pi\epsilon_0 ab}{b-a}$$

and calculate the electrostatic energy of the system in terms of Q and C.

Solution: The differential form of Gauss' law states that $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, where ρ is charge density and **E** is the electric field. The equivalent integral form of Gauss' law states that for a volume V with boundary S, we have

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\epsilon_0},$$

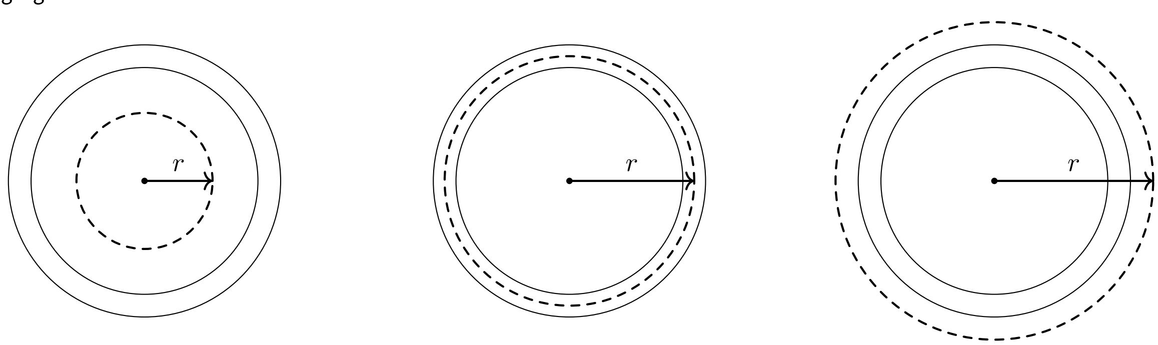
where Q is the total charge contained in the volume V.

The rest of this question is broadly the same as an Examples Sheet question, namely Examples Sheet 1, Question 8. However, things are slightly simplified because our spherical shells have no thickness in this question.

We begin by reducing the functional form of the electric field. In general the electric field will be a function $\mathbf{E} \equiv \mathbf{E}(r, \theta, \phi)$ in spherical coordinates. The charge distribution in this case is invariant under the coordinate transformations $\theta \mapsto \theta + \Delta\theta$, $\phi \mapsto \phi + \Delta\phi$, and it follows that the electric field is too. Thus $\mathbf{E} \equiv \mathbf{E}(r)$.

Considering active transformations of the conductors, we note that if the electric field has either an $\hat{\mathbf{e}}_\theta$ or $\hat{\mathbf{e}}_\phi$ component at the point $\mathbf{x}$, then after a rotation about an axis passing through the centre of the sphere and the point $\mathbf{x}$, these components will change. Hence we get two electric fields for the same problem. Nature cannot be two ways, and it follows that $\mathbf{E} \equiv E(r)\hat{\mathbf{e}}_r$.

We can now begin to apply Gauss' law. Consider inserting the Gaussian surfaces represented by the dashed lines in the following figure:



(a) $r < a$ (b) $a < r < b$ (c) $r > b$

Let S_r denote the Gaussian surface, a sphere of radius r, in each case. The electric flux out of each of these surface is given by:

$$\int_{S_r} \mathbf{E} \cdot \mathbf{dS} = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, E(r) r^2 \sin(\theta) = 4\pi r^2 E(r).$$

The charge contained by these surfaces is 0 if $r < a$, Q if $a < r < b$ and 0 if $r > b$. Thus by Gauss' law, the electric field is given by:

$$\mathbf{E} = \begin{cases} \mathbf{0} & \text{if } r < a, \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{e}}_r & \text{if } a < r < b, \\ \mathbf{0} & \text{if } r > b. \end{cases}$$

The electrostatic potential ϕ is a function such that $\mathbf{E} = -\nabla\phi$. Recalling that the gradient in spherical polars is given by:

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin(\theta)}\frac{\partial\phi}{\partial\phi}\hat{\mathbf{e}}_\phi.$$

So in this case an appropriate electrostatic potential is given by:

$$\phi = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 a} & \text{if } r < a, \\ \dfrac{Q}{4\pi\epsilon_0 r} & \text{if } a < r < b, \\ \dfrac{Q}{4\pi\epsilon_0 b} & \text{if } r > b. \end{cases}$$

Recall that the *capacitance* between the two plates is defined by:

$$C = \frac{Q}{|\Delta\phi|},$$

where is the magnitude of the charge that each plate carries, and $|\Delta\phi|$ is the potential difference between the two plates. In this case, the potential difference is given by:

$$\Delta\phi = \phi(a) - \phi(b) = \frac{Q}{4\pi\epsilon_0 a} - \frac{Q}{4\pi\epsilon_0 b} = \frac{Q(b-a)}{4\pi\epsilon_0 ab}.$$

Hence the capacitance is:

$$C = \frac{4\pi\epsilon_0 ab}{b-a},$$

as required.

There are a couple of ways to compute the electrostatic energy of the capacitor in question. One way is simply to recall the general formula:

$$E = \int dV \, \frac{\epsilon_0}{2}|\mathbf{E}|^2 = \frac{\epsilon_0}{2} \cdot \frac{Q^2}{16\pi^2\epsilon_0^2} \cdot 4\pi \int_a^b dr \, \frac{r^2}{r^4} = \frac{Q^2}{2} \cdot \frac{1}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_a^b = \frac{Q^2}{2} \cdot \frac{(b-a)}{4\pi\epsilon_0 ab} = \frac{Q^2}{2C}.$$

We can also use a simpler argument based on the derivation of this formula. Consider adding a small amount of charge δQ to the inner capacitor, and subtracting a small amount δQ of charge from the other. The work needed to achieve this is given by the formula

$$\delta E = \delta Q\phi(a) - \delta Q\phi(b) = \delta Q \cdot \Delta\phi = Q\delta Q/C.$$

It follows that the energy of the system increases by an amount $\delta E = Q\delta Q/C$. Taking $\delta E, \delta Q$ infinitesimal, we see that

$$\frac{dE}{dQ} = \frac{Q}{C} \qquad \Rightarrow \qquad E = \frac{Q^2}{2C},$$

where the integration constant is found by noting that there is zero energy when the charge on the plates is zero.

2017, Paper 4, Section I, 7C

A thin wire, in the form of a closed curve C, carries a constant current I. Using either the Biot-Savart law or the magnetic vector potential, show that the magnetic field far from the loop is of the approximate form

$$\mathbf{B}(\mathbf{r}) \approx \frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}-\mathbf{m}|\mathbf{r}|^2}{|\mathbf{r}|^5}\right],$$

where **m** is the magnetic dipole moment of the loop. Derive an expression for **m** in terms of I and the vector area spanned by the curve C.

➾ **Solution:** This question is discussed in detail in David Tong's Electromagnetism notes, section 3.4. We recall from lectures that the expression for the vector potential due to a general current distribution $\mathbf{J}(\mathbf{r})$ is given by:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV.$$

Far from the current distribution, we may assume that the integration variable $\mathbf{r}'$, which is confined to the volume V which the current distribution occupes, is such that $|\mathbf{r}'| \ll |\mathbf{r}|$. Hence we can expand the denominator here using the binomial expansion:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \left(|\mathbf{r}|^2 - 2\mathbf{r}\cdot\mathbf{r}' + |\mathbf{r}'|^2\right)^{-1/2} = \frac{1}{|\mathbf{r}|}\left(1-\frac{2\mathbf{r}\cdot\mathbf{r}'}{|\mathbf{r}|^2}+\frac{|\mathbf{r}'|^2}{|\mathbf{r}|^2}\right)^{-1/2} = \frac{1}{|\mathbf{r}|}\left(1+\frac{\mathbf{r}\cdot\mathbf{r}'}{|\mathbf{r}|^2}+O\left(\frac{|\mathbf{r}'|^2}{|\mathbf{r}|^2}\right)\right).$$

It follows that we can write the vector potential as:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}|}+\frac{\mathbf{J}(\mathbf{r}')(\mathbf{r}\cdot\mathbf{r}')}{|\mathbf{r}|^3}+\dots\right)dV.$$

The first term is zero; to see this, we use the identity $J_i = \partial_j'(J_j r_i')$, which follows since $\partial_j J_j = \nabla\cdot\mathbf{J} = 0$, which vanishes by the continuity equation in the absence of charges. We then have that the first term looks like:

$$\int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r}|}\,dV = \frac{1}{|\mathbf{r}|}\int_V J_i(\mathbf{r}')\,dV = \frac{1}{|\mathbf{r}|}\int_V \partial_j'(J_j r_i')\,dV = \frac{1}{|\mathbf{r}|}\int_S J_j r_i' n_j\,dS,$$

by the divergence theorem. We can take the volume V to infinity, since the current distribution is only non-zero in small localised region. This shows us that the surface integral on the right hand side cancels. So we're left with:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi|\mathbf{r}|^3}\int_V \mathbf{J}(\mathbf{r}')(\mathbf{r}\cdot\mathbf{r}')\,dV.$$

To deal with the remaining term, we need a similar trick. We notice that:

$$\partial_j'(J_j r_i' r_k') = J_i r_k' + J_k r_i',$$

so in a volume integral we can replace, using the divergence theorem, $J_i r_k' = -J_k r_i'$. Thus for the remaining volume integral:

$$\int_V J_i r_j r_j'\,dV = \int_V \frac{r_j}{2}(J_i r_j' + J_i r_j')\,dV = \int_V \frac{r_j}{2}\left(J_i r_j' - J_j r_i'\right)\,dV,$$

which in dyadic notation is:

$$\frac{1}{2}\int_V \left(\mathbf{J}(\mathbf{r}')(\mathbf{r}\cdot\mathbf{r}') - \mathbf{r}'(\mathbf{J}(\mathbf{r}')\cdot\mathbf{r})\right)\,dV = \frac{1}{2}\mathbf{r}\times\int_V \left(\mathbf{J}(\mathbf{r}')\times\mathbf{r}'\right)\,dV,$$

using Lagrange's formula for the vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

Hence we can write the vector potential as:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi|\mathbf{r}|^3} \cdot \left(\frac{1}{2}\mathbf{r} \times \int\limits_V \left(\mathbf{J}(\mathbf{r}') \times \mathbf{r}'\right)\, dV \right).$$

The definition of the magnetic dipole moment of the distribution is:

$$\mathbf{m} = \frac{1}{2}\int\limits_V \left(\mathbf{r}' \times \mathbf{J}(\mathbf{r}')\right)\, dV.$$

Hence the final expression for $\mathbf{A}$ is:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 \mathbf{m} \times \mathbf{r}}{4\pi|\mathbf{r}|^3}$$

Taking the curl, we obtain the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$:

$$B_i(\mathbf{r}) = [\nabla \times \mathbf{A}]_i = \frac{\mu_0}{4\pi}\epsilon_{ijk}\frac{\partial}{\partial r_j}\left(\frac{\epsilon_{kab} m_a r_b}{|\mathbf{r}|^3}\right) = \frac{\mu_0}{4\pi}\left(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}\right) m_a \frac{\partial}{\partial r_j}\left(\frac{r_b}{|\mathbf{r}|^3}\right)$$

$$= \frac{\mu_0}{4\pi}\left(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}\right) m_a \left(\frac{\delta_{jb}}{|\mathbf{r}|^3} - \frac{3r_b r_j}{|\mathbf{r}|^5}\right) = \frac{\mu_0}{4\pi}\left(m_i\left(\frac{3}{|\mathbf{r}|^3} - \frac{3|\mathbf{r}|^2}{|\mathbf{r}|^5}\right) - \frac{m_i}{|\mathbf{r}|^3} + \frac{3r_i \mathbf{m}\cdot\mathbf{r}}{|\mathbf{r}|^5}\right)$$

$$= \frac{\mu_0}{4\pi}\left(\frac{3r_i(\mathbf{m}\cdot\mathbf{r}) - m_i|\mathbf{r}|^2}{|\mathbf{r}|^5}\right).$$

Hence we have the required magnetic field:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\mathbf{r}(\mathbf{m}\cdot\mathbf{r}) - \mathbf{m}|\mathbf{r}|^2}{|\mathbf{r}|^5}\right).$$

We are finally asked to determine $\mathbf{m}$ for a current loop C carrying current I in terms of the vector area $\boldsymbol{\mathcal{S}}$ of any surface with perimeter C. By definition, we have:

$$\mathbf{m} = \frac{1}{2}\int\limits_V \left(\mathbf{r}' \times \mathbf{J}(\mathbf{r}')\right)\, dV = \frac{I}{2}\oint\limits_C \mathbf{r}' \times d\mathbf{r}',$$

since for a current loop, we have the local expression $\mathbf{J}(\mathbf{r}') = I\delta(x_1)\delta(x_2)\hat{\mathbf{e}}_{\mathbf{r}'}$, where x_1, x_2 are variables orthogonal to the loop direction and $\hat{\mathbf{e}}_{\mathbf{r}'}$ is a unit vector in the direction of the loop.

To evaluate the integral, consider contracting with an arbitrary constant vector $\mathbf{v}$:

$$\mathbf{v} \cdot \oint\limits_C \mathbf{r}' \times d\mathbf{r}' = \oint\limits_C \mathbf{v} \cdot (\mathbf{r}' \times d\mathbf{r}') = \oint\limits_C (\mathbf{v} \times \mathbf{r}') \cdot d\mathbf{r}'$$

where the last equality is a standard property of the scalar triple product. Using Stokes' theorem, we can write this as:

$$\mathbf{v} \cdot \oint\limits_C \mathbf{r}' \times d\mathbf{r}' = \int\limits_S \nabla' \times (\mathbf{v} \times \mathbf{r}') \cdot d\mathbf{S}.$$

The curl of the vector product can be evaluated using some index notation:

$$[\nabla' \times (\mathbf{v} \times \mathbf{r}')]_i = \epsilon_{ijk}\epsilon_{kab}\frac{\partial}{\partial r_j'}\left(v_a r_b'\right) = \left(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}\right) v_a \delta_{bj} = 3v_i - v_i = 2v_i.$$

Hence we have

$$\mathbf{v} \cdot \oint\limits_C \mathbf{r}' \times d\mathbf{r}' = 2\int\limits_S \mathbf{v} \cdot d\mathbf{S} = 2\mathbf{v} \cdot \boldsymbol{\mathcal{S}},$$

where $\boldsymbol{\mathcal{S}}$ is the vector area of the surface $\boldsymbol{\mathcal{S}}$. Putting everything together, we see that $\mathbf{m} = I\boldsymbol{\mathcal{S}}$.

2017, Paper 1, Section II, 16C

Write down Maxwell's equations for the electric field $\mathbf{E}(\mathbf{x}, t)$ and the magnetic field $\mathbf{B}(\mathbf{x}, t)$ in a vacuum. Deduce that both $\mathbf{E}$ and $\mathbf{B}$ satisfy a wave equation, and relate the wave speed c to the physical constants ϵ_0 and μ_0.

Verify that there exist plane-wave solutions of the form

$$\mathbf{E}(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{e}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right], \qquad \mathbf{B}(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{b}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right],$$

where $\mathbf{e}$ and $\mathbf{b}$ are constant complex vectors, $\mathbf{k}$ is a constant real vector and ω is a real constant. Derive the dispersion relation that relates the angular frequency ω of the wave to the wavevector $\mathbf{k}$, and give the algebraic relations between the vectors $\mathbf{e}$, $\mathbf{b}$ and $\mathbf{k}$ implied by Maxwell's equations.

Let $\mathbf{n}$ be a constant real unit vector. Suppose that a perfect conductor occupies the region $\mathbf{n} \cdot \mathbf{x} < 0$ with a plane boundary $\mathbf{n} \cdot \mathbf{x} = 0$. In the vacuum region $\mathbf{n} \cdot \mathbf{x} > 0$, a plane electromagnetic wave of the above form, with $\mathbf{k} \cdot \mathbf{n} < 0$, is incident on the plane boundary. Write down the boundary conditions on $\mathbf{E}$ and $\mathbf{B}$ at the surface of the conductor. Show that Maxwell's equations and the boundary conditions are satisfied if the solution in the vacuum region is the sum of the incident wave given above and a reflected wave of the form

$$\mathbf{E}'(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{e}'e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega t)}\right], \qquad \mathbf{B}'(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{b}'e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega t)}\right],$$

where

$$\mathbf{e}' = -\mathbf{e} + 2(\mathbf{n}\cdot\mathbf{e})\mathbf{n}, \qquad \mathbf{b}' = \mathbf{b} - 2(\mathbf{n}\cdot\mathbf{b})\mathbf{n}, \qquad \mathbf{k}' = \mathbf{k} - 2(\mathbf{n}\cdot\mathbf{k})\mathbf{n}.$$

Solution: Maxwell's equations in the vacuum are given by:

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

Taking the time derivative of the third Maxwell equation (*Faraday's law*), we have:

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\nabla \times \frac{\partial \mathbf{E}}{\partial t} = -c^2 \nabla \times (\nabla \times \mathbf{B}),$$

substituting for the time derivative of $\mathbf{E}$ using the fourth Maxwell equation (the *Maxwell-Ampère law*). Now we use the vector calculus identity for the curl of the curl: $\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B} + \nabla(\nabla \cdot \mathbf{B})$. Using the second Maxwell equation then (the law says that magnetic monopoles do not exist, or *Gauss' law*), we have:

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \nabla^2 \mathbf{B} \qquad \Rightarrow \qquad \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = \nabla^2 \mathbf{B}.$$

Similarly, taking the time derivative of the fourth Maxwell equation, we have:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\nabla \times \mathbf{E}) = \nabla^2 \mathbf{E},$$

using the third Maxwell equation, the vector calculus identity for the curl of the curl, and the first Maxwell equation (*Gauss' law*). So we have that $\mathbf{E}$, $\mathbf{B}$ both satisfy wave equations with speed c:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E}, \qquad \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = \nabla^2 \mathbf{B}.$$

The relationship between the speed c and the constants ϵ_0, μ_0 is $c^2 = 1/\mu_0\epsilon_0$.

We now verify that the given plane-wave solutions satisfy Maxwell's equations. We have:

- Gauss' law. The first Maxwell equation gives:

$$0 = \nabla \cdot \mathbf{E} = \nabla \cdot \mathrm{Re}\left[\mathbf{e}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] = \mathrm{Re}\left[i\mathbf{k}\cdot\mathbf{e}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right].$$

  Since this must hold for all $\mathbf{x}$, we deduce that $\mathbf{k}\cdot\mathbf{e} = 0$, and then the equation is satisfied.

- Gauss' law for magnetism. The second Maxwell equation gives:

$$0 = \nabla \cdot \mathbf{B} = \nabla \cdot \mathrm{Re}\left[\mathbf{b}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] = \mathrm{Re}\left[i\mathbf{k}\cdot\mathbf{b}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right].$$

  Again, this must hold for all $\mathbf{x}$, and hence we deduce that $\mathbf{k}\cdot\mathbf{b} = 0$, and the equation is satisfied.

- Faraday's law. The left hand side of the third Maxwell equation gives:

$$\nabla \times \mathbf{E} = \nabla \times \mathrm{Re}\left[\mathbf{e}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] = \mathrm{Re}\left[i\mathbf{k}\times\mathbf{e}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right].$$

  The right hand side of the third Maxwell equation gives:

$$-\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\mathrm{Re}\left[\mathbf{b}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] = \mathrm{Re}\left[i\omega\mathbf{b}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right].$$

  Comparing the two, and noting this equation must hold for all $\mathbf{x}$, we deduce that $\omega\mathbf{b} = \mathbf{k}\times\mathbf{e}$ and the equation is satisfied.

- Maxwell-Ampère law. Finally, the left hand side of the fourth Maxwell equation gives:

$$\nabla \times \mathbf{B} = \nabla \times \mathrm{Re}\left[\mathbf{b}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] = \mathrm{Re}\left[i\mathbf{k}\times\mathbf{b}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right].$$

  The right hand side of the fourth Maxwell equation gives:

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mathrm{Re}\left[-\frac{i\omega}{c^2}\mathbf{e}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right].$$

  Comparing again, we deduce that if $\mathbf{k}\times\mathbf{b} = -\omega\mathbf{e}/c^2$, then the equation is satisfied.

Hence the given electromagnetic wave satisfies Maxwell's equations provided we have the relationships:

$$\mathbf{k}\cdot\mathbf{e} = 0, \qquad \mathbf{k}\cdot\mathbf{b} = 0, \qquad \omega\mathbf{b} = \mathbf{k}\times\mathbf{e}, \qquad \mathbf{k}\times\mathbf{b} = -\frac{\omega}{c^2}\mathbf{e}.$$

We notice that the fourth condition is implied by the other three, since taking the vector product of the third equation with $\mathbf{k}$ we have:

$$\omega\mathbf{k}\times\mathbf{b} = \mathbf{k}\times(\mathbf{k}\times\mathbf{e}) = (\mathbf{k}\cdot\mathbf{e})\mathbf{k} - |\mathbf{k}|^2\mathbf{e} = -|\mathbf{k}|^2\mathbf{e}.$$

So provided we have the *dispersion relation* $\omega = c|\mathbf{k}|$, the equations are consistent and the first three equations imply the fourth. Thus we see that our electromagnetic waves satisfy the Maxwell equations provided:

$$\mathbf{k}\cdot\mathbf{e} = 0, \qquad \mathbf{k}\cdot\mathbf{b} = 0, \qquad \omega\mathbf{b} = \mathbf{k}\times\mathbf{e}, \qquad \omega = c|\mathbf{k}|.$$

We now consider the system with a boundary at $\mathbf{n} \cdot \mathbf{x} = 0$. We know from lectures that the relevant boundary conditions just outside the conductor are:

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\mathbf{n}, \qquad \mathbf{B} = \mu_0 \mathbf{K} \times \mathbf{n},$$

where σ is the surface charge on the conductor and $\mathbf{K}$ is the surface current on the conductor.

Let's verify that the given solution satisfies Maxwell's equations and the boundary conditions:

- 1. Maxwell's equations. By linearity of the Maxwell equations, it is sufficient to check that the incident wave $(\mathbf{E}, \mathbf{B})$ and the reflected wave $(\mathbf{E}', \mathbf{B}')$ both individually satisfy Maxwell's equations. The incident electromagnetic wave $(\mathbf{E}, \mathbf{B})$ satisfies Maxwell's equations provided the conditions:

  $$\mathbf{k} \cdot \mathbf{e} = 0, \qquad \mathbf{k} \cdot \mathbf{b} = 0, \qquad \omega \mathbf{b} = \mathbf{k} \times \mathbf{e}, \qquad \omega = c|\mathbf{k}|$$

  hold, as we derived earlier in the question. Therefore the reflected electromagnetic wave $(\mathbf{E}', \mathbf{B}')$ needs to obey the same conditions in terms of the primed variables in order to satisfy Maxwell's equations. Checking each condition in turn, we first have:

  $$\mathbf{k}' \cdot \mathbf{e}' = (\mathbf{k} - 2(\mathbf{n} \cdot \mathbf{k})\mathbf{n}) \cdot (-\mathbf{e} + 2(\mathbf{n} \cdot \mathbf{e})\mathbf{n}) = -\mathbf{k} \cdot \mathbf{e} + 2(\mathbf{n} \cdot \mathbf{e})(\mathbf{n} \cdot \mathbf{k}) + 2(\mathbf{n} \cdot \mathbf{k})(\mathbf{n} \cdot \mathbf{e}) - 4(\mathbf{n} \cdot \mathbf{k})(\mathbf{n} \cdot \mathbf{e})|\mathbf{n}|^2 = 0,$$

  using the fact that the vector $\mathbf{n}$ is a unit vector in the final step. For the second condition, we have:

  $$\mathbf{k}' \cdot \mathbf{b}' = (\mathbf{k} - 2(\mathbf{n} \cdot \mathbf{k})\mathbf{n}) \cdot (\mathbf{b} - 2(\mathbf{n} \cdot \mathbf{b})\mathbf{n}) = \mathbf{k} \cdot \mathbf{b} - 2(\mathbf{n} \cdot \mathbf{b})(\mathbf{n} \cdot \mathbf{k}) - 2(\mathbf{n} \cdot \mathbf{k})(\mathbf{n} \cdot \mathbf{b}) + 4(\mathbf{n} \cdot \mathbf{k})(\mathbf{n} \cdot \mathbf{b})|\mathbf{n}|^2 = 0.$$

  For the third condition, we have:

  $$\mathbf{k}' \times \mathbf{e}' = (\mathbf{k} - 2(\mathbf{n} \cdot \mathbf{k})\mathbf{n}) \times (-\mathbf{e} + 2(\mathbf{n} \cdot \mathbf{e})\mathbf{n}) = -\mathbf{k} \times \mathbf{e} + 2(\mathbf{n} \cdot \mathbf{e})\mathbf{k} \times \mathbf{n} + 2(\mathbf{n} \cdot \mathbf{k})\mathbf{n} \times \mathbf{e}.$$

  It's a little bit more complicated to put everything together here. Note that $-\mathbf{k} \times \mathbf{e} = -\omega \mathbf{b}$ from the conditions on the incident wave, but the rest of the terms require some more simplification. We have:

  $$2(\mathbf{n} \cdot \mathbf{e})\mathbf{k} \times \mathbf{n} + 2(\mathbf{n} \cdot \mathbf{k})\mathbf{n} \times \mathbf{e} = 2\mathbf{n} \times ((\mathbf{n} \cdot \mathbf{k})\mathbf{e} - (\mathbf{n} \cdot \mathbf{e})\mathbf{k}).$$

  Now by Lagrange's formula for the vector triple product, we can write $(\mathbf{n} \cdot \mathbf{k})\mathbf{e} - (\mathbf{n} \cdot \mathbf{e})\mathbf{k} = \mathbf{n} \times (\mathbf{e} \times \mathbf{k}) = -\mathbf{n} \times (\mathbf{k} \times \mathbf{e}) = -\omega \mathbf{n} \times \mathbf{b}$. Thus we're left with:

  $$2(\mathbf{n} \cdot \mathbf{e})\mathbf{k} \times \mathbf{n} + 2(\mathbf{n} \cdot \mathbf{k})\mathbf{n} \times \mathbf{e} = -2\omega \mathbf{n} \times (\mathbf{n} \times \mathbf{b}) = -2\omega(\mathbf{n} \cdot \mathbf{b})\mathbf{n} + 2\omega \mathbf{b},$$

  where in the last step we expanded the vector triple product using Lagrange's formula. Putting everything together:

  $$\mathbf{k}' \times \mathbf{e}' = -\omega \mathbf{b} + 2\omega \mathbf{b} - 2\omega(\mathbf{n} \cdot \mathbf{b})\mathbf{n} = \omega(\mathbf{b} - 2(\mathbf{n} \cdot \mathbf{b})\mathbf{n}) = \omega \mathbf{b}'.$$

  Hence the third condition is satisfied - phew! Finally, the dispersion relation is also satisfied since:

  $$c^2|\mathbf{k}'|^2 = c^2(\mathbf{k} - 2(\mathbf{n} \cdot \mathbf{k})\mathbf{n}) \cdot (\mathbf{k} - 2(\mathbf{n} \cdot \mathbf{k})\mathbf{n}) = c^2|\mathbf{k}|^2 - 4(\mathbf{n} \cdot \mathbf{k})^2 + 4(\mathbf{n} \cdot \mathbf{k})^2 = c^2|\mathbf{k}|^2 = \omega^2.$$

  It follows that $(\mathbf{E}', \mathbf{B}')$ satisfies Maxwell's equations too. So by linearity, $(\mathbf{E} + \mathbf{E}', \mathbf{B} + \mathbf{B}')$ satisfies Maxwell's equations.

- 2. Boundary conditions. We now need to check the boundary conditions at the conducting plane $\mathbf{n}\cdot\mathbf{x}=0$. The first useful thing to note is that if $\mathbf{x}$ is in the plane, then $\mathbf{x}\cdot\mathbf{n}=0$, which implies:

$$\mathbf{k}'\cdot\mathbf{x}=(\mathbf{k}-2(\mathbf{k}\cdot\mathbf{n})\mathbf{n})\cdot\mathbf{x}=\mathbf{k}\cdot\mathbf{x}.$$

Now let's consider the combinations $\mathbf{E}+\mathbf{E}'$, $\mathbf{B}+\mathbf{B}'$ and check they obey the correct boundary conditions at $\mathbf{n}\cdot\mathbf{x}=0$. For the electric field, we have:

$$\mathbf{E}+\mathbf{E}'=\mathrm{Re}\left[\mathbf{e}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}+\mathbf{e}'e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega t)}\right]=\mathrm{Re}\left[(\mathbf{e}+\mathbf{e}')e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right],$$

using the fact that $\mathbf{k}\cdot\mathbf{x}=\mathbf{k}'\cdot\mathbf{x}$ for $\mathbf{x}$ on the plane. Substituting for $\mathbf{e}'$, we have:

$$\mathbf{E}+\mathbf{E}'=\mathrm{Re}\left[2(\mathbf{n}\cdot\mathbf{e})\mathbf{n}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right].$$

Hence the boundary condition on the electric field is satisfied provided that there is an induced surface charge on the plate given by:

$$\mathbf{E}+\mathbf{E}'=\frac{\sigma}{\epsilon}\mathbf{n}\qquad\Rightarrow\qquad\sigma=2\epsilon_0\,\mathrm{Re}\left[(\mathbf{n}\cdot\mathbf{e})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right].$$

For the magnetic field, we have:

$$\mathbf{B}+\mathbf{B}'=\mathrm{Re}\left[\mathbf{b}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}+\mathbf{b}'e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega t)}\right]=\mathrm{Re}\left[(\mathbf{b}+\mathbf{b}')e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right].$$

Substituting for $\mathbf{b}'$, we have:

$$\mathbf{B}+\mathbf{B}'=\mathrm{Re}\left[(2\mathbf{b}-2(\mathbf{n}\cdot\mathbf{b})\mathbf{n})\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right].$$

Notice that $2\mathbf{b}-2(\mathbf{n}\cdot\mathbf{b})\mathbf{n}=2(\mathbf{n}\cdot\mathbf{n})\mathbf{b}-2(\mathbf{n}\cdot\mathbf{b})\mathbf{n}=2\mathbf{n}\times(\mathbf{b}\times\mathbf{n})=2(\mathbf{n}\times\mathbf{b})\times\mathbf{n}$, by Lagrange's formula for the vector triple product. Hence the boundary condition on the magnetic field is satisfied provided that there is an induced surface current on the plate given by:

$$\mathbf{B}+\mathbf{B}'=\mu_0\mathbf{K}\times\mathbf{n}\qquad\Rightarrow\qquad\mathbf{K}=2\mu_0\,\mathrm{Re}\left[\mathbf{n}\times\mathbf{b}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right].$$

2017, Paper 2, Section II, 18C

In special relativity, the electromagnetic fields can be derived from a 4-vector potential $A^\mu = (\phi/c, \mathbf{A})$. Using the Minkowski metric tensor $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$, state how the electromagnetic tensor $F_{\mu\nu}$ is related to the 4-potential and write out explicitly the components of both $F_{\mu\nu}$ and $F^{\mu\nu}$ in terms of those of **E** and **B**.

If $x'^\mu = \Lambda^\mu{}_\nu$ is a Lorentz transformation of the spacetime coordinates from one inertial frame $\mathcal{S}$ to another inertial frame $\mathcal{S}'$, state how $F'^{\mu\nu}$ is related to $F^{\mu\nu}$.

Write down the Lorentz transformation matrix for a boost in standard configuration such that frame $\mathcal{S}'$ moves relative to frame $\mathcal{S}$ with speed v in the x-direction. Deduce the transformation laws:

$$
\begin{aligned}
E'_x &= E_x & B'_x &= B_x \\
E'_y &= \gamma(E_y - vB_z) & B'_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\
E'_z &= \gamma(E_z + vB_y) & B'_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right).
\end{aligned}
$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

In frame $\mathcal{S}$, an infinitely long wire of negligible thickness lies along the x-axis. The wire carries n positive charges $+q$ per unit length, which travel at speed u in the $+x$ direction, and n negative charges $-q$ per unit length, which travel at speed u in the $-x$ direction. There are no other sources of the electromagnetic field. Write down the electric and magnetic fields in $\mathcal{S}$ in terms of Cartesian coordinates. Calculate the electric field in frame $\mathcal{S}'$, which is related to $\mathcal{S}$ by a boost by speed v as described above. Give an explanation of the physical origin of your expression.

Solution: The electromagnetic tensor can be written in terms of the four-potential via:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \eta_{\rho\nu}\partial_\mu A^\rho - \eta_{\rho\mu}\partial_\nu A^\rho.$$

Note this definition makes it completely manifest that the electromagnetic tensor is antisymmetric: $F_{\mu\nu} = -F_{\nu\mu}$. Recall that we can write the electric and magnetic fields in terms of the components of the four-potential as:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

Hence the components are given by:

$$
\begin{aligned}
F_{00} &= \partial_0 A_0 - \partial_0 A_0 = 0, \\
F_{0i} &= \partial_0 A_i - \partial_i A_0 = \frac{1}{c}\left(\frac{\partial A_i}{\partial t} + \partial_i \phi\right) = -\frac{E_i}{c}, \\
F_{i0} &= -F_{0i} = \frac{E_i}{c} \\
F_{ij} &= \partial_i A_j - \partial_j A_i = (\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{ib})\,\partial_a A_b = \epsilon_{ijk}\epsilon_{kab}\partial_a A_b = \epsilon_{ijk}[\nabla \times \mathbf{A}]_k = \epsilon_{ijk}B_k.
\end{aligned}
$$

Hence we have:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

To obtain the electromagnetic tensor with upstairs indices, note that

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\sigma}F_{\alpha\sigma},$$

which in matrix notation reads:

$$F_{\text{up}} = \eta F_{\text{down}}\eta.$$

where $\eta = \mathrm{diag}\{-1,1,1,1\}$ is the Minkowski metric. Hence we have:

$$F^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

We now consider Lorentz transformations of the electromagnetic tensor. Since $F^{\mu\nu}$ has two *upstairs* indices, under a Lorentz transformation Λ it transforms as:

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}.$$

As a matrix equation, this can be written as:

$$F' = \Lambda F \Lambda^T.$$

We will use this immediately in the next part of the question. A Lorentz boost in the x-direction with velocity v has the matrix:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We therefore see that the electromagnetic tensor transforms under a Lorentz boost of velocity v in the x-direction as:

$$F' = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Performing the matrix multiplications, we're left with:

$$F' = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\gamma v E_x/c^2 & \gamma E_x/c & E_y/c & E_z/c \\ -\gamma E_x/c & \gamma v E_x/c^2 & B_z & -B_y \\ -\gamma E_y/c + \gamma v B_z/c & \gamma v E_y/c^2 - \gamma B_z & 0 & B_x \\ -\gamma E_z/c - \gamma v B_y/c & \gamma v E_z/c^2 + \gamma B_y & -B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \gamma^2 E_x/c - \gamma^2 v^2 E_x/c & \gamma(E_y - vB_z)/c & \gamma(E_z + vB_y)/c \\ \gamma^2 v^2 E_x/c^3 - \gamma^2 E_x/c & 0 & \gamma(B_z - vE_y/c^2) & -\gamma(B_y + vE_z/c^2) \\ -\gamma(E_y - vB_z)/c & -\gamma(B_z - vE_y/c^2) & 0 & B_x \\ -\gamma(E_z + vB_y)/c & \gamma(B_y + vE_z/c^2) & -B_x & 0 \end{pmatrix}$$

Some simplification is possible here, in particular we have:

$$\frac{\gamma^2 E_x}{c} - \frac{\gamma^2 v^2 E_x}{c} = \frac{\gamma^2}{c}\left(1 - \frac{v^2}{c^2}\right)E_x = E_x/c.$$

Thus the final matrix form for the transformed electromagnetic tensor is:

$$F' = \begin{pmatrix} 0 & E_x/c & \gamma(E_y - vB_z)/c & \gamma(E_z + vB_y)/c \\ -E_x/c & 0 & \gamma(B_z - vE_y/c^2) & -\gamma(B_y + vE_z/c^2) \\ -\gamma(E_y - vB_z)/c & -\gamma(B_z - vE_y/c^2) & 0 & B_x \\ -\gamma(E_z + vB_y)/c & \gamma(B_y + vE_z/c^2) & -B_x & 0 \end{pmatrix}$$

In particular, comparing the entries of F' written in terms of the transformed $\mathbf{E}'$, $\mathbf{B}'$ fields, we see that we have derived the transformation law:

$$E_x' = E_x \qquad B_x' = B_x$$

$$E_y' = \gamma(E_y - vB_z) \qquad B_y' = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

$$E_z' = \gamma(E_z + vB_y) \qquad B_z' = \gamma\left(B_z - \frac{v}{c^2}E_y\right),$$

as required.

We are now asked to consider a fixed frame $\mathcal{S}$ containing a wire. In a length L of the wire, there is a total charge $Lnq - Lnq = 0$ along each section of the wire; it follows that the charge density is $\rho = 0$ in this frame. The current is not vanishing however; the amount of charge passing a fixed point of the wire per unit time is given by $I = nqu + n(-q)(-u) = 2nqu$.

We now determine the **E**, **B** fields. Since the charge density is vanishing $\rho = 0$, and the current is constant we are in the regime of magnetostatics. Hence $\mathbf{E} = \mathbf{0}$ and **B** is determined by the Maxwell equations:

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

The symmetry of the problem suggests we can use Ampère's law in integral form to solve it. We notice first that the symmetries imply **B** has the functional form $\mathbf{B} \equiv B(r)\hat{\mathbf{e}}_\theta$ in cylindrical coordinates about that $\hat{\mathbf{e}}_x$ axis (for detailed examples of this symmetry analysis see the Examples Sheets). Then Ampère's law in integral form for an Ampèrian loop C_r of radius r, centred on the x-axis gives the result:

$$\mu_0 I = \oint_C \mathbf{B} \times \mathbf{dx} = \int_0^{2\pi} rB(r)\, d\theta = 2\pi r B(r) \qquad \Rightarrow \qquad \mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\mathbf{e}}_\theta.$$

Thus the magnetic field in this case is given by:

$$\mathbf{B} = \frac{\mu_0 nqu}{\pi r}\hat{\mathbf{e}}_\theta.$$

It will be useful when we consider Lorentz boosting to the frame $\mathcal{S}'$ to instead write this in Cartesian components. Recall that $r = \sqrt{y^2 + z^2}$ in these coordinates, and

$$\hat{\mathbf{e}}_\theta = -\sin(\theta)\hat{\mathbf{e}}_y + \cos(\theta)\hat{\mathbf{e}}_z = -\frac{z}{\sqrt{y^2+z^2}}\hat{\mathbf{e}}_y + \frac{y}{\sqrt{y^2+z^2}}\hat{\mathbf{e}}_z.$$

Therefore we can write the magnetic field as:

$$\mathbf{B} = \frac{\mu_0 nqu}{\pi(y^2+z^2)}\left(-z\hat{\mathbf{e}}_y + y\hat{\mathbf{e}}_z\right).$$

We now consider the wire viewed from the frame $\mathcal{S}'$. The transformed $\mathbf{E}'$ field is given by the above transformation law as:

$$E'_x = 0$$

$$E'_y = -v\gamma B_z = -\frac{\mu_0 \gamma v n q u y}{\pi(y^2+z^2)}$$

$$E'_z = +v\gamma B_y = -\frac{\mu_0 \gamma v n q u z}{\pi(y^2+z^2)}.$$

It follows that we can express the transformed electric field as:

$$\mathbf{E}' = -\frac{\mu_0 \gamma v n q u}{\pi r}\left(\frac{y}{r}\hat{\mathbf{e}}_y + \frac{z}{r}\hat{\mathbf{e}}_z\right) = -\frac{\mu_0 \gamma v n q u}{\pi r}\hat{\mathbf{e}}_r.$$

We see that in the boosted frame, the electric field is non-vanishing! This can be explained physically as follows. Suppose that there are n_0^+ particles of charge $+q$ in the wire per unit length when we are in the rest frame of the charge $+q$ particles. Boosting to the frame where they move with velocity u then, by length contraction there are $n = \gamma(u)n_0^+$ particles of charge $+q$ per unit length. Similarly there are $n = \gamma(-u)n_0^-$ particles of charge $-q$ per unit length in the frame where the particles of charge $-q$ move with velocity $-u$ along the wire.

Now imagine boosting from the frame where the particles of charge $+q$ in the wire are at rest to the frame where the wire moves with velocity v. This is equivalent, by the formula for the relativistic addition of velocities, to a boost by

$$u_+ = \frac{u+v}{1-uv/c^2}.$$

Hence the number of particles of charge $+q$ per unit length in the frame $\mathcal{S}'$ is given by length contraction to be:

$$n^+ = \gamma(u_+)n_0^+ = \gamma(u_+)\gamma(u)^{-1}n.$$

Similarly, we the number of particles of charge $-q$ per unit length in the frame $\mathcal{S}'$ is given by:

$$n^- = \gamma(u_-)n_0^- = \gamma(u_-)\gamma(u)^{-1}n \qquad \text{where} \qquad u_- = \frac{-u+v}{1+uv/c^2}.$$

It follows that the total charge per unit length in the frame $\mathcal{S}'$ is:

$$\eta = q(n^+ - n^-) = q\gamma(u)^{-1}n\left(\gamma(u_+) - \gamma(u_-)\right).$$

With some algebra, we can simplify this via:

$$\gamma(u_+) = \frac{1}{\sqrt{1-(u+v)^2/c^2(1-\frac{uv}{c^2})^2}} = \left(1-\frac{uv}{c^2}\right)\frac{1}{\sqrt{\left(1-\frac{uv}{c^2}\right)^2 - \frac{(u+v)^2}{c^2}}}$$

$$= \left(1-\frac{uv}{c^2}\right)\frac{1}{\sqrt{1+\frac{u^2v^2}{c^4}-\frac{u^2}{c^2}-\frac{v^2}{c^2}}} = \left(1-\frac{uv}{c^2}\right)\frac{1}{\sqrt{(1-\frac{u^2}{c^2})(1-\frac{v^2}{c^2})}} = \left(1-\frac{uv}{c^2}\right)\gamma(u)\gamma(v),$$

and similarly $\gamma(u_-) = \left(1+\frac{uv}{c^2}\right)\gamma(u)\gamma(v)$. Hence we deduce that

$$\eta = q(n^+ - n^-) = q\gamma(u)^{-1}n\left(\left(1-\frac{uv}{c^2}\right)\gamma(u)\gamma(v) - \left(1+\frac{uv}{c^2}\right)\gamma(u)\gamma(v)\right) = -\frac{2uvq}{c^2}\gamma n.$$

It follows that the charge per unit length viewed in this frame is non-vanishing. In particular, particles with charge $+q$ now appear to be moving in the x-direction more slowly, and particles with charge $-q$ now appear to be moving in the negative x-direction more quickly, and the overall effect is for a negative charge to appear on the wire (since there appear to be more negative charges per unit length now).

In fact, we recall from lectures that $\eta/2\pi\epsilon_0 r$ is the $\hat{\mathbf{e}}_r$ component (the only non-vanishing component) of the electric field around a line charge with charge per unit length η. In our case, this result exactly matches the formula we found by using boosts of the fields earlier, rather than boosts of particles in the wire (to see they match, recall $c^2\epsilon = 1/\mu_0$).

2017, Paper 3, Section II, 17C

(i) Two point charges, of opposite sign and unequal magnitude, are placed at two different locations. Show that the combined electrostatic potential vanishes on a sphere that encloses only the charge of smaller magnitude.

(ii) A grounded, conducting sphere of radius a is centred at the origin. A point charge q is located outside the sphere at position vector $\mathbf{p}$. Formulate the differential equation and boundary conditions for the electrostatic potential outside the sphere. Using the result of part (i) or otherwise, show that the electric field outside the sphere is identical to that generated (in the absence of any conductors) by the point charge q and an image charge q' located inside the sphere at position vector $\mathbf{p}'$, provided that $\mathbf{p}'$ and q' are chosen correctly.

Calculate the magnitude and direction of the force experience by the charge q.

➾ **Solution:** (i) Recall that *Coulomb's law* states that the electric potential due to a point charge q_a at the location $\mathbf{a}$ is given by:

$$\phi_a(\mathbf{x}) = \frac{q_a}{4\pi\epsilon_0|\mathbf{x}-\mathbf{a}|},$$

where we have chosen to impose the boundary condition $\phi \to 0$ as $|\mathbf{x}| \to \infty$ to fix the arbitrary constant in this potential.

Consider adding another point charge $-q_b$ (where q_a, q_b have the same sign) at another location $\mathbf{b}$. By the principle of superposition (which follows from the linearity of Maxwell's equations), we see that the total electric potential due to the two point charges is given by:

$$\phi(\mathbf{x}) = \frac{q_a}{4\pi\epsilon_0|\mathbf{x}-\mathbf{a}|} - \frac{q_b}{4\pi\epsilon_0|\mathbf{x}-\mathbf{b}|}.$$

We would like to know where this potential is zero. Simply setting $\phi(\mathbf{x}) = 0$, we have:

$$\frac{q_a}{|\mathbf{x}-\mathbf{a}|} = \frac{q_b}{|\mathbf{x}-\mathbf{b}|} \qquad \Rightarrow \qquad q_a|\mathbf{x}-\mathbf{b}| = q_b|\mathbf{x}-\mathbf{a}|$$

To see that this is a sphere, one could quote some general results from geometry (e.g. something to do with circles of Apollonius), but actually for the rest of this question it will be more useful to do a short calculation to show this is a sphere directly. Squaring both sides of the equation, we can expand to give:

$$q_a^2(|\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{b} + |\mathbf{b}|^2) = q_b^2(|\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{a} + |\mathbf{a}|^2).$$

Collecting like terms, we have:

$$(q_a^2 - q_b^2)|\mathbf{x}|^2 - 2\mathbf{x}\cdot\left(q_a^2\mathbf{b} - q_b^2\mathbf{a}\right) + q_a^2|\mathbf{b}|^2 - q_b^2|\mathbf{a}|^2 = 0.$$

Let's assume without loss of generality that q_a has the greater magnitude. Dividing through by $q_a^2 - q_b^2$, and completing the square, we're left with the expression:

$$\left|\mathbf{x} - \left(\frac{q_a^2\mathbf{b} - q_b^2\mathbf{a}}{q_a^2 - q_b^2}\right)\right|^2 - \left|\frac{q_a^2\mathbf{b} - q_b^2\mathbf{a}}{q_a^2 - q_b^2}\right|^2 + \frac{q_a^2|\mathbf{b}|^2 - q_b^2|\mathbf{a}|^2}{q_a^2 - q_b^2} = 0.$$

Hence we have:

$$\left|\mathbf{x} - \left(\frac{q_a^2\mathbf{b} - q_b^2\mathbf{a}}{q_a^2 - q_b^2}\right)\right| = \sqrt{\left|\frac{q_a^2\mathbf{b} - q_b^2\mathbf{a}}{q_a^2 - q_b^2}\right|^2 - \frac{(q_a^2|\mathbf{b}|^2 - q_b^2|\mathbf{a}|^2)}{q_a^2 - q_b^2}}.$$

After performing a little bit of algebra to simplify the right hand side, we see that everything reduces to:

$$\left|\mathbf{x} - \left(\frac{q_a^2\mathbf{b} - q_b^2\mathbf{a}}{q_a^2 - q_b^2}\right)\right| = \frac{q_a q_b|\mathbf{a}-\mathbf{b}|}{q_a^2 - q_b^2}.$$

This is now evidently a sphere. We need to check that **b** is on the interior of the sphere and **a** is on the exterior of the sphere. Substituting **a** into the left hand side, we have:

$$\left|\mathbf{a} - \left(\frac{q_a^2\mathbf{b} - q_b^2\mathbf{a}}{q_a^2 - q_b^2}\right)\right| = \left|\frac{(q_a^2 - q_b^2)\mathbf{a} - q_a^2\mathbf{b} + q_b^2\mathbf{a}}{q_a^2 - q_b^2}\right| = \frac{q_a^2|\mathbf{a} - \mathbf{b}|}{q_a^2 - q_b^2} \geq \frac{q_a q_b|\mathbf{a} - \mathbf{b}|}{q_a^2 - q_b^2},$$

so indeed **a** lies outside the sphere. Similarly

$$\left|\mathbf{b} - \left(\frac{q_a^2\mathbf{b} - q_b^2\mathbf{a}}{q_a^2 - q_b^2}\right)\right| = \frac{q_b^2|\mathbf{a} - \mathbf{b}|}{q_a^2 - q_b^2} \leq \frac{q_b q_a|\mathbf{a} - \mathbf{b}|}{q_a^2 - q_b^2},$$

so **b** lies inside the sphere, as required.

(ii) We now have a sphere centred at the origin **0** of radius a. In the region outside of the sphere, there is a single point charge at the point **p**, so Gauss' law gives

$$\nabla \cdot \mathbf{E} = \frac{q\delta^3(\mathbf{x} - \mathbf{p})}{\epsilon_0}$$

Recalling that $\mathbf{E} = -\nabla\phi$ in terms of the electrostatic potential, we see that the appropriate differential equation for ϕ outside the sphere is *Poisson's equation*:

$$\nabla^2\phi = -\frac{q\delta^3(\mathbf{x} - \mathbf{p})}{\epsilon_0}.$$

We are given that the sphere is grounded, so we also require $\phi = 0$ on $r = a$.

We are now asked to solve the problem using the result of part (i). The idea is to choose an image charge q' at some point $\mathbf{p}'$ such that the sphere of zero potential we found in part (i) is the sphere $r = a$.

First, we require that the sphere is centred on the origin, so using the equation we found for the sphere in part (i), we see that we need:

$$\mathbf{0} = \frac{q^2\mathbf{p}' - {q'}^2\mathbf{p}}{q^2 - {q'}^2} \qquad \Rightarrow \qquad q^2\mathbf{p}' = {q'}^2\mathbf{p}. \tag{$*$}$$

We need the radius of the sphere to be a, so again using the result we found in part (i), we see that we need:

$$-\frac{qq'|\mathbf{p} - \mathbf{p}'|}{q^2 - {q'}^2} = a \qquad \Rightarrow \qquad -qq'|\mathbf{p} - \mathbf{p}'| = a(q^2 - {q'}^2). \tag{$\dagger$}$$

Note that we need the minus sign here, because in our above derivation we assume that we were working with charges q_a and $-q_b$; these are now translated to $q_a = q$ and $-q_b = q'$. It remains to solve the simultaneous vector equations ($*$) and ($\dagger$). First, rearranging ($*$) to give $\mathbf{p}'$ in terms of **p**, q and q', we have $\mathbf{p}' = {q'}^2\mathbf{p}/q^2$. Substituting into ($\dagger$), we find:

$$-qq'\left|\mathbf{p} - \frac{{q'}^2\mathbf{p}}{q^2}\right| = a(q^2 - {q'}^2).$$

We can solve this equation for q'. Since we want $|q'| < |q|$ (the charge on the inside of the sphere should have smaller magnitude), taking the modulus has no effect on the left hand side, and we are left with:

$$q' = -\frac{aq}{|\mathbf{p}|}$$

Substituting back into $(*)$, we see that we should take $\mathbf{p}'$ to be:

$$\mathbf{p}' = \frac{{q'}^2\mathbf{p}}{q^2} = \frac{a^2\mathbf{p}}{|\mathbf{p}|^2}.$$

With this choice of q, $\mathbf{p}'$, the electrostatic potential becomes:

$$\phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0|\mathbf{x}-\mathbf{p}|} + \frac{q'}{4\pi\epsilon_0|\mathbf{x}-\mathbf{p}'|} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{x}-\mathbf{p}|} - \frac{a}{|\mathbf{p}|\cdot|\mathbf{x}-a^2\mathbf{p}/|\mathbf{p}|^2|}\right).$$

Now we move back to the problem of the sphere. We see that:

- The potential $\phi(\mathbf{x})$, by construction, vanishes on the sphere of radius $r = a$.
- The potential satisfies Poisson's equation in the region $r \geq a$. This follows since

  $$\nabla^2\phi = -\frac{q\delta^3(\mathbf{x}-\mathbf{p})}{\epsilon_0} + \frac{qa\delta^3(\mathbf{x}-a^2\mathbf{p}/|\mathbf{p}|^2)}{|\mathbf{p}|\epsilon_0}.$$

  We derive this by recalling from Part IB Methods that

  $$G(\mathbf{x},\mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}$$

  is the Green's function for the Laplacian in three dimensions. Notice that in the region $r > a$, the second delta function does not contribute, since $a^2\mathbf{p}/|\mathbf{p}|^2$ is inside the sphere of radius $r = a$ by construction. Hence

  $$\nabla^2\phi = -\frac{q\delta^3(\mathbf{x}-\mathbf{p})}{\epsilon_0}.$$

Thus the potential obeys the correct equation in the region $r > a$ and the correct boundary condition. Thus, by the uniqueness theorem for Poisson's equation, this must be the potential for the conducting sphere problem too.

Finally, we are asked to determine the force on the particle of charge q. By the method of images we outlined above, the charge q feels a force on it from the conductor identical to a force that it feels due to the presence of the appropriate image charge q'. Thus, by the Lorentz force law, the charge q experiences a force:

$$\mathbf{F} = q\mathbf{E}_{\text{from } q'} = -q\nabla\phi_{\text{from } q'} = -q\nabla\left(\frac{q'}{4\pi\epsilon_0|\mathbf{x}-\mathbf{p}'|}\right)\Bigg|_{\mathbf{x}=\mathbf{p}} = \frac{qq'(\mathbf{p}-\mathbf{p}')}{4\pi\epsilon_0|\mathbf{p}-\mathbf{p}'|^3}.$$

Hence the force is:

$$\mathbf{F} = -\frac{aq^2}{4\pi\epsilon_0|\mathbf{p}|}\frac{(\mathbf{p}-a^2\mathbf{p}/|\mathbf{p}|^2)}{|\mathbf{p}-a^2\mathbf{p}/|\mathbf{p}|^2|^3}.$$

Simplifying, we can reduce this to the expression:

$$\mathbf{F} = -\frac{aq^2}{4\pi\epsilon_0(|\mathbf{p}|^2-a^2)^2}\cdot\mathbf{p}.$$

In particular, the particle experiences an attractive force directed at the centre of the sphere.

2018, Paper 2, Section I, 6C

Derive the Biot-Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dV,$$

from Maxwell's equations, where the time-independent current $\mathbf{j}(\mathbf{r})$ vanishes outside V.

[*You may assume that the vector potential can be chosen to be divergence-free.*]

Solution: The relevant Maxwell equations for magnetostatics are:

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j}.$$

The first equation tells us that the **B** field is divergence-free, and hence allows us to write $\mathbf{B} = \nabla \times \mathbf{A}$ for some *vector potential* **A**. Inserting this into the second equation implies:

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j}.$$

Using an identity for the curl of the curl of a vector field, we can write the left hand side in the form:

$$-\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mu_0 \mathbf{j},$$

where $\nabla^2 \mathbf{A}$ is the vector Laplacian. We are given in the question that there exists some gauge in which $\nabla \cdot \mathbf{A} = 0$ (namely, *Coulomb gauge*). Therefore, we can choose **A** such that this condition holds, and this equation reduces to the vector form of Poisson's equation:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j}.$$

In Cartesian components, this equation reads $\nabla^2 A_i = -\mu_0 j_i$. Using the Green's function solution of Poisson's equation for each component equation then, we have the solution for the vector potential given by:

$$A_i(\mathbf{r}) = \int_V \frac{\mu_0 j_i(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}' \qquad \Rightarrow \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}'.$$

All that's left to do is to take the curl of this expression to obtain the magnetic field. The curl in this case acts on the **r** variable in the integrand, and ignores the integration variable $\mathbf{r}'$. So we just need to work out:

$$\nabla_{\mathbf{r}} \times \left(\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right).$$

One way of doing this is to use index notation. The ith component of the above expression is given by:

$$\epsilon_{ijk} \frac{\partial}{\partial r_j} \left(\frac{j_k}{|\mathbf{r} - \mathbf{r}'|} \right) = \epsilon_{ijk} j_k \left(\frac{-(r_j - r_j')}{|\mathbf{r} - \mathbf{r}'|^3} \right) = \left[\frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right]_i.$$

Hence we have the Biot-Savart law as required:

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV.$$

Comments from the Examiner: *Mostly done well, though candidates often simply asserted $\mathbf{B} = \nabla \times \mathbf{A}$ rather than deducing this as a consequence of the Maxwell equation $\nabla \cdot \mathbf{B} = 0$. There was some minor confusion with signs, both the identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ and (often compensating!) the fact that $\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r})$.*

2018, Paper 4, Section I, 7C

Show that Maxwell's equations imply the conservation of charge.

A conducting medium has $\mathbf{J} = \sigma\mathbf{E}$ where σ is a constant. Show that any charge density decays exponentially in time, at a rate to be determined.

Solution: First, we recall Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \mu_0\left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right).$$

Recall that the divergence of a curl is zero, since in index notation:

$$\nabla \cdot (\nabla \times \mathbf{F}) = \epsilon_{ijk}\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}F_k\right) = 0,$$

by antisymmetry of the epsilon symbol and symmetry of the partial derivatives. Hence taking the divergence of the fourth Maxwell equation (the *Ampère-Maxwell law*) we have

$$0 = \nabla \cdot (\nabla \times \mathbf{B}) = \mu_0\left(\nabla \cdot \mathbf{J} + \epsilon_0 \frac{\partial \nabla \cdot \mathbf{E}}{\partial t}\right) = \mu_0\left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}\right),$$

where in the last step we used the first Maxwell equation, i.e. *Gauss' law*. It follows that charge is conserved:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

We are now given that in a conducting medium, we have *Ohm's law*, $\mathbf{J} = \sigma\mathbf{E}$. Substituting this into the charge conservation law, we have:

$$0 = \frac{\partial \rho}{\partial t} + \sigma \nabla \cdot \mathbf{E} = \frac{\partial \rho}{\partial t} + \frac{\sigma\rho}{\epsilon_0},$$

where in the last step, we used the first Maxwell equation, i.e. Gauss' law, to substitute for $\nabla \cdot \mathbf{E}$. We can solve this separable equation as follows:

$$\frac{\partial \rho}{\partial t} = -\frac{\sigma\rho}{\epsilon_0} \qquad \Rightarrow \qquad \int \frac{d\rho}{\rho} = -\frac{\sigma}{\epsilon_0}\int dt \qquad \Rightarrow \qquad \log(\rho(\mathbf{x}, t)) = -\frac{\sigma t}{\epsilon_0} + f(\mathbf{x}),$$

where $f(\mathbf{x})$ is an arbitrary function, independent of time. Exponentiating, we have

$$\rho(\mathbf{x}, t) = \rho(\mathbf{x}, 0)\exp\left(-\frac{\sigma}{\epsilon_0}t\right).$$

Hence the rate of decay of charge density is σ/ϵ_0 per unit time.

Comments from the Examiner: *This question was done well by almost all candidates who attempted it. There were occasional attempts to justify the conservation equation using something other than Maxwell's equations, but essentially all the candidates who obtained the correct conservation equation then successfully completed the final part of the question.*

2018, Paper 1, Section II, 16C

Starting from the Lorentz force law acting on a current distribution $\mathbf{J}$ obeying $\nabla \cdot \mathbf{J} = 0$, show that the energy of a magnetic dipole **m** in the presence of a time-independent magnetic field **B** is

$$U = -\mathbf{m} \cdot \mathbf{B}.$$

State clearly any approximations you make.

[*You may use without proof the fact that*

$$\int (\mathbf{a} \cdot \mathbf{r})\mathbf{J}(\mathbf{r})\, dV = -\frac{1}{2}\mathbf{a} \times \int (\mathbf{r} \times \mathbf{J}(\mathbf{r}))\, dV$$

for any constant vector **a**, *and the identity*

$$(\mathbf{b} \times \nabla) \times \mathbf{c} = \nabla(\mathbf{b} \cdot \mathbf{c}) - \mathbf{b}(\nabla \cdot \mathbf{c}),$$

*which holds when **b** is constant.*]

A beam of slowly moving, randomly oriented magnetic dipoles enter a region where the magnetic field is

$$\mathbf{B} = \hat{\mathbf{z}}B_0 + (y\hat{\mathbf{x}} + x\hat{\mathbf{y}})B_1,$$

with B_0 and B_1 constants. By considering their energy, briefly describe what happens to those dipoles that are parallel to, and those that are anti-parallel to the direction of **B**.

Solution: The first part of this question is bookwork; the derivation can be found in detail in David Tong's Electromagnetism notes in section 3.4.2, or alternatively in Harvey Reall's Electromagnetism notes in section 3.6, page 45. We begin by recalling that the force on a current distribution is given by the Lorentz force law as:

$$\mathbf{F} = \int_V \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\, dV.$$

We assume that (i) the current is localised in a small region around $\mathbf{r} = \mathbf{0}$ (without loss of generality) and (ii) the magnetic field varies slowly in this region. With these assumptions, it is valid to approximate the magnetic field in this region via the Taylor series expansion:

$$\mathbf{B}(\mathbf{r}) \approx \mathbf{B}(\mathbf{0}) + (\mathbf{r} \cdot \nabla)\mathbf{B}(\mathbf{0}) + \cdots .$$

It follows that the force on the current distribution $\mathbf{J}$ is given in this approximation by:

$$\mathbf{F} = -\mathbf{B}(\mathbf{0}) \times \int_V \mathbf{J}(\mathbf{r})\, dV + \int_V \mathbf{J}(\mathbf{r}) \times (\mathbf{r} \cdot \nabla)\mathbf{B}(\mathbf{0})\, dV + \cdots .$$

The first term vanishes. To see this, let's note that the ith component of the current density $\mathbf{J}(\mathbf{r})$ obeys:

$$J_i = \frac{\partial J_j}{\partial r_j} r_i + J_i = \frac{\partial}{\partial r_j}\left(J_j r_i\right).$$

The first equality follows since the divergence of the current is zero, $\nabla \cdot \mathbf{J} = 0$. Thus we can write the ith component of the integral in the first term of $\mathbf{F}$ above as:

$$\int_V J_i(\mathbf{r})\, dV = \int_V \frac{\partial}{\partial r_j}\left(J_j r_i\right)\, dV = \int_S J_j r_i n_i\, dS,$$

where we used the divergence theorem in the final step. If we keep the current density localised, but integrate over a very large volume V instead, then $J_j \to 0$ on the surface S of the volume, and thus the integral vanishes.

We're left with:

$$\mathbf{F} = \int_V \mathbf{J}(\mathbf{r}) \times (\mathbf{r} \cdot \nabla)\mathbf{B}(\mathbf{0})\, dV + \cdots\,,$$

which requires further manipulation. We note that we can rewrite the integrand using some index notation; the ith component of the integrand is given by:

$$\epsilon_{ijk} J_j r_l \left[\frac{\partial B_k(\mathbf{r}')}{\partial r'_l}\right]_{\mathbf{r}'=\mathbf{0}}.$$

We would like to rewrite this expression as the curl of something. To do so, recall that **B** here is interpreted as an external magnetic field, and hence is unsourced (we don't want the magnetic field produced by the current distribution **J**, we want the response of **J** to the magnetic field **B**). Thus we have: $\nabla \times \mathbf{B} = \mathbf{0}$. In particular the components of $\nabla \times \mathbf{B}$ are all vanishing, and so:

$$\frac{\partial B_k}{\partial r'_l} - \frac{\partial B_l}{\partial r'_k} = 0.$$

Hence we can rewrite:

$$\epsilon_{ijk} J_j r_l \left[\frac{\partial B_k(\mathbf{r}')}{\partial r'_l}\right]_{\mathbf{r}'=\mathbf{0}} = \epsilon_{ijk} J_j r_l \left[\frac{\partial B_l(\mathbf{r}')}{\partial r'_k}\right]_{\mathbf{r}'=\mathbf{0}} = \epsilon_{ijk} \frac{\partial}{\partial r'_k} \left(J_j r_l B_l(\mathbf{r}')\right)\bigg|_{\mathbf{r}'=\mathbf{0}} = -\epsilon_{ikj} \frac{\partial}{\partial r'_k} \left(J_j r_l B_l(\mathbf{r}')\right)\bigg|_{\mathbf{r}'=\mathbf{0}}$$

Returning to dyadic notation, this can be written as:

$$-\nabla_{\mathbf{r}'} \times [\mathbf{J}(\mathbf{r})(\mathbf{r} \cdot \mathbf{B}(\mathbf{r}'))]\bigg|_{\mathbf{r}'=\mathbf{0}}$$

Thus we can rewrite the force as:

$$\mathbf{F} = -\nabla_{\mathbf{r}'} \times \int_V \mathbf{J}(\mathbf{r})(\mathbf{r} \cdot \mathbf{B}(\mathbf{r}'))\, dV\bigg|_{\mathbf{r}'=\mathbf{0}}.$$

Now using the identity given in the question, we can rewrite the remaining integral as:

$$\int_V [\mathbf{J}(\mathbf{r})(\mathbf{r} \cdot \mathbf{B}(\mathbf{r}'))]\, dV = -\frac{1}{2}\mathbf{B}(\mathbf{r}') \times \int_V [\mathbf{r} \times \mathbf{J}(\mathbf{r})]\, dV.$$

Recall that the definition of the magnetic dipole moment is:

$$\mathbf{m} = \frac{1}{2}\int_V [\mathbf{r} \times \mathbf{J}(\mathbf{r})]\, dV.$$

Thus we are left with the force:

$$\mathbf{F} = \nabla_{\mathbf{r}'} \times (\mathbf{B}(\mathbf{r}') \times \mathbf{m})\bigg|_{\mathbf{r}'=\mathbf{0}}.$$

To finish the derivation, we need to take this curl. Let's simplify notation by writing $\mathbf{F} = \nabla \times (\mathbf{B} \times \mathbf{m})$ where the derivatives are understood, along with the evaluation at $\mathbf{r}' = \mathbf{0}$ at the end. We would like to recast this in the form of the identity in the question:

$$(\mathbf{b} \times \nabla) \times \mathbf{c} = \nabla(\mathbf{b} \cdot \mathbf{c}) - \mathbf{b}(\nabla \cdot \mathbf{c}).$$

We notice that in index notation:

$$[\nabla \times (\mathbf{B} \times \mathbf{m})]_i = \epsilon_{ijk}\epsilon_{kab} m_b \frac{\partial B_a}{\partial r'_j}.$$

Also in index notation, we have:

$$[(\mathbf{m} \times \nabla) \times \mathbf{B}]_i = \epsilon_{ijk}\epsilon_{jab} m_a \frac{\partial B_k}{\partial r'_b} = \epsilon_{ika}\epsilon_{kbj} m_b \frac{\partial B_a}{\partial r'_j},$$

after relabelling indices ($a \to b, b \to j, k \to a$ and $j \to k$) to match those of the previous expression.

Comparing the two, we have:

$$\epsilon_{ijk}\epsilon_{kab}m_b\frac{\partial B_a}{\partial r'_j} - \epsilon_{ika}\epsilon_{abj}m_b\frac{\partial B_a}{\partial r'_j} = \left(\epsilon_{ijk}\epsilon_{kab} - \epsilon_{ika}\epsilon_{abj}\right)m_b\frac{\partial B_a}{\partial r'_j}.$$

Simplifying the epsilon symbols, we have:

$$\epsilon_{ijk}\epsilon_{kab} - \epsilon_{ika}\epsilon_{kbj} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja} - \delta_{ij}\delta_{ab} + \delta_{ib}\delta_{ja} = \delta_{ia}\delta_{jb} - \delta_{ij}\delta_{ab} = \epsilon_{ibk}\epsilon_{ajk}.$$

Thus we can rewrite:

$$\epsilon_{ijk}\epsilon_{kab}m_b\frac{\partial B_a}{\partial r'_j} - \epsilon_{ika}\epsilon_{abj}m_b\frac{\partial B_a}{\partial r'_j} = \epsilon_{ibk}\epsilon_{ajk}m_b\frac{\partial B_a}{\partial r'_j}.$$

The right hand side is the ith component of $-\mathbf{m}\times(\nabla\times\mathbf{B})$, and so in particular we have the identity:

$$\nabla\times(\mathbf{B}\times\mathbf{m}) = (\mathbf{m}\times\nabla)\times\mathbf{B} - \mathbf{m}\times(\nabla\times\mathbf{B}).$$

Since $\nabla\times\mathbf{B}=\mathbf{0}$, we can use the result in the question to give us:

$$\mathbf{F} = (\mathbf{m}\times\nabla)\times\mathbf{B} = \nabla(\mathbf{m}\cdot\mathbf{B}) - \mathbf{m}(\nabla\cdot\mathbf{B}).$$

Gauss' law for magnetism tells us that $\nabla\cdot\mathbf{B}=0$, and we conclude that:

$$\mathbf{F} = \nabla_{\mathbf{r}'}(\mathbf{m}\cdot\mathbf{B})\bigg|_{\mathbf{r}'=\mathbf{0}}.$$

Translating the current distribution to be localised at any position we like, we have the more general force: $\mathbf{F}=\nabla(\mathbf{m}\cdot\mathbf{B}(\mathbf{x}))$. The potential energy associated with this force is:

$$U = -\mathbf{m}\cdot\mathbf{B}(\mathbf{x}),$$

as required.

We are now given that the magnetic field is $\mathbf{B}(\mathbf{x}) = (yB_1, xB_1, B_0)$ in a certain region. The energy of a dipole entering this region is, as we derived above, $-\mathbf{m}\cdot\mathbf{B}(\mathbf{x})$ when the dipole is at the position $\mathbf{x}$ in this magnetic field. Let's denote a magnetic dipole at the position $\mathbf{x}$ by $\mathbf{m}(\mathbf{x})$. We want to consider the cases (i) the dipole is parallel to the magnetic field, $\mathbf{m}(\mathbf{x}) = |\mathbf{m}|\hat{\mathbf{B}}(\mathbf{x})$; (ii) the dipole is anti-parallel to the magnetic field, $\mathbf{m}(\mathbf{x}) = -|\mathbf{m}|\hat{\mathbf{B}}(\mathbf{x})$.

In general, we will find the energy of a parallel-aligned/anti-parallel aligned dipole at position $\mathbf{x}$ is:

$$U = -\mathbf{m}(\mathbf{x})\cdot\mathbf{B}(\mathbf{x}) = \mp|\mathbf{m}||\mathbf{B}(\mathbf{x})| = \mp|\mathbf{m}|\sqrt{y^2B_1^2 + x^2B_1^2 + B_0^2}.$$

In particular, for a dipole parallel to the magnetic field, we are in the $-$ case. We see that it is energetically favourable for the dipole to move away from $x=0$, $y=0$, but movement in the z direction does nothing. Furthermore, it is equally energetically favourable to move in the x or y direction, and so we expect the dipole to start moving radially (in a cylindrical sense, with axis on the z-axis) outwards.

In the case that a dipole is anti-parallel to the magnetic field, we are in the $+$ case. It is energetically favourable for the dipole to move towards $x=0, y=0$. So the dipole will start moving towards the z-axis radially.

In general, there will also be a torque on the dipole which we don't consider here. This can cause the dipole to change its alignment. In particular, the most energetically favourable configuration is alignment with the magnetic field, since $U = -\mathbf{m}\cdot\mathbf{B} = -|\mathbf{m}||\mathbf{B}|\cos(\theta)$ is minimised when $\theta = 0$. Therefore we expect the anti-aligned dipole to also rotate and hence become aligned with the magnetic field.

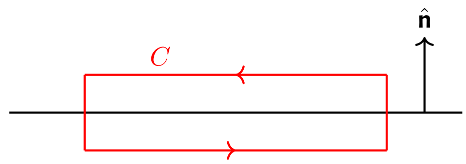
❊ **Comments from the Examiner:** *A disappointing take-up for this question, probably because candidates were put off by the rather long (bookwork) derivation called for at the beginning. There was one perfect solution, whereas all other attempts were mostly just partial.*

2018, Paper 2, Section II, 18C

A plane with unit normal **n** supports a charge density and a current density that are each time-independent. Show that the tangential components of the electric field and the normal component of the magnetic field are continuous across the plane.

Albert moves with constant velocity $\mathbf{v} = v\hat{\mathbf{n}}$ relative to the plane. Find the boundary conditions at the plane on the normal component of the magnetic field and the tangential components of the electric field as seen in Albert's frame.

Solution: The first part of this question follows a standard argument we saw in lectures. Let C be a small rectangular loop of length L and height h arranged so that half the loop lies above the plane and half the loop lies below the plane:



Recall that Faraday's law states:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Let's integrate this over the surface S bounded by the loop C (we can choose to orient S such that it is simply the rectangle with boundary C):

$$\int_S (\nabla \times \mathbf{E}) \cdot \mathbf{dS} = \int_S -\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}.$$

Using Stokes' theorem, the left hand side is simplified; we have:

$$\oint_C \mathbf{E} \cdot \mathbf{dx} = \int_S -\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}.$$

Now consider taking the limit $h \to 0$, i.e. taking the height of the rectangle to zero. The left hand side degenerates into a line integral:

$$\oint_C \mathbf{E} \cdot \mathbf{dx} \to \int_{C_0} (\mathbf{E}_+ - \mathbf{E}_-) \cdot \mathbf{dx},$$

where $\mathbf{E}_+$ is the electric field just above the plane, and $\mathbf{E}_-$ is the electric field just below the plane, and C_0 is the path that the top segment of the rectangle C collapses to. The right hand side collapses into an area integral over an area 0, which must itself be zero provided that $\partial\mathbf{B}/\partial t$ remains bounded as we take the limit (which is true even if **B** is say, discontinuous). We hence see that:

$$\int_{C_0} (\mathbf{E}_+ - \mathbf{E}_-) \cdot \mathbf{dx} = 0.$$

Since this holds for all small line segments C_0 on the plane, we deduce that

$$(\mathbf{E}_+ - \mathbf{E}_-) \cdot \mathbf{t} = 0$$

for any tangent vector **t** in the plane. Equivalently, we can write:

$$(\mathbf{E}_+ - \mathbf{E}_-) \times \mathbf{n} = \mathbf{0},$$

where **n** is the normal to the plane.

Next, we must deal with the normal component of the magnetic field across the plane. This time, we position a small cylinder S of height h and radius r such that it is sliced in half by the plane. Let V be the volume of the cylinder. Then Gauss' law for magnetism $\nabla \cdot \mathbf{B} = 0$ implies:

$$\int_V \nabla \cdot \mathbf{B} \, dV = 0.$$

By the divergence theorem, this can instead be written as:

$$0 = \int_S \mathbf{B} \cdot \mathbf{dS} = \int_{\text{cylinder top}} \mathbf{B} \cdot \mathbf{dS} + \int_{\text{cylinder side}} \mathbf{B} \cdot \mathbf{dS} + \int_{\text{cylinder bottom}} \mathbf{B} \cdot \mathbf{dS}.$$

Letting the height of the cylinder h tend to zero, $h \to 0$, the contribution from the sides of the cylinder vanish, because the area of the sides of the cylinder tends to zero (thus is true because **B** is assumed to be bounded).

Hence we're left with:

$$0 = \int_{D_r} (\mathbf{B}_+ - \mathbf{B}_-) \cdot \mathbf{dS},$$

where D_r is a disk of radius r in the plane, and $\mathbf{B}_+$, $\mathbf{B}_-$ are the limits of the magnetic field as it approaches the plane from above it and below it respectively. Since this must hold for all r, we have

$$(\mathbf{B}_+ - \mathbf{B}_-) \cdot \mathbf{n} = 0,$$

where **n** is the normal to the plane.

For the second part of the question, we must consider what happens to these boundary conditions under an appropriate Lorentz boost. Without loss of generality, position the plane to be $x = 0$, so that its normal vector is $\hat{\mathbf{e}}_x$. In the plane's rest frame, the boundary conditions are:

$$(\mathbf{E}_+ - \mathbf{E}_-) \times \hat{\mathbf{e}}_x = 0, \qquad (\mathbf{B}_+ - \mathbf{B}_-) \cdot \hat{\mathbf{e}}_x = 0,$$

for $\mathbf{E}_+$, $\mathbf{E}_-$ just above and below the plane $x = 0$ respectively; the same definition applies to $\mathbf{B}_+$, $\mathbf{B}_-$.

Albert's frame constitutes a boost by velocity v in the $\hat{\mathbf{e}}_x$ direction. From Albert's perspective, the plane now appears to be moving, hence the first thing we must realise is that the boundary conditions will now be applied in a different place. Recall from Part IA Dynamics and Relativity that under a boost in the x direction by velocity v, the coordinates of our inertial frame change as:

$$x' = \gamma (x - vt) \qquad y' = y$$
$$ct' = \gamma \left(ct - \frac{vx}{c} \right) \qquad z' = z.$$

Inverting these relationships, we have

$$x = \gamma (x' + vt') \qquad y = y'$$
$$ct = \gamma \left(ct' + \frac{vx'}{c} \right) \qquad z = z'.$$

In particular, we see that the plane $x = 0$ is transformed to:

$$x' + vt' = 0 \qquad \Rightarrow \qquad x' = -vt'.$$

This is the plane where Albert thinks the boundary conditions should be applied.

We also recall that there is a standard formula from lectures that relates the components $\mathbf{E}'$, $\mathbf{B}'$ of the electric and magnetic fields to the components of the electric and magnetic fields in the initial frame:

$$
\begin{aligned}
E_x' &= E_x & B_x' &= B_x \\
E_y' &= \gamma(E_y - vB_z) & B_y' &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\
E_z' &= \gamma(E_z + vB_y) & B_z' &= \gamma\left(B_z - \frac{v}{c^2}E_y\right),
\end{aligned}
$$

Inverting this relationship, we have:

$$
\begin{aligned}
E_x &= E_x' & B_x &= B_x' \\
E_y &= \gamma(E_y' + vB_z') & B_y &= \gamma\left(B_y' - \frac{v}{c^2}E_z'\right) \\
E_z &= \gamma(E_z' - vB_y') & B_z &= \gamma\left(B_z' + \frac{v}{c^2}E_y'\right),
\end{aligned}
$$

Therefore substituting into our boundary conditions, we have:

$$(\mathbf{B}_+ - \mathbf{B}_-)\cdot\hat{\mathbf{e}}_x = (B_+)_x - (B_-)_x = 0 \qquad \Rightarrow \qquad (B_+')_x - (B_-')_x = 0.$$

So the boundary condition on the magnetic field is unaltered from Albert's perspective: $(\mathbf{B}_+' - \mathbf{B}_-')\cdot\hat{\mathbf{e}}_x = 0$.

The electric field is a little bit more complicated. We have:

$$(\mathbf{E}_+ - \mathbf{E}_-)\times\hat{\mathbf{e}}_x = \mathbf{0}.$$

Writing this out in full we have:

$$((E_+)_z - (E_-)_z)\hat{\mathbf{e}}_y - ((E_+)_y - (E_-)_y)\hat{\mathbf{e}}_z = \mathbf{0}.$$

In terms of quantities in Albert's frame, we have:

$$\left(\gamma[(E_+')_z - v(B_+')_y] - \gamma[(E_-')_z - v(B_-')_y]\right)\hat{\mathbf{e}}_y - \left(\gamma[(E_+')_y + v(B_+')_z] - \gamma[(E_-')_y + v(B_-')_z]\right)\hat{\mathbf{e}}_z = \mathbf{0}.$$

Collecting everything together, we have:

$$(\mathbf{E}_+' - \mathbf{E}_-')\times\hat{\mathbf{e}}_x - v(\mathbf{B}_+' - \mathbf{B}_-') + v[(\mathbf{B}_+' - \mathbf{B}_-')\cdot\hat{\mathbf{e}}_x]\hat{\mathbf{e}}_x = \mathbf{0}.$$

We can get rid of the last term here because of the boundary condition $(\mathbf{B}_+' - \mathbf{B}_-')\cdot\hat{\mathbf{e}}_x = 0$. Therefore, we're left with:

$$(\mathbf{E}_+' - \mathbf{E}_-')\times\hat{\mathbf{e}}_x = v(\mathbf{B}_+' - \mathbf{B}_-').$$

We can express everything here in coordinate-free form by using spatial rotations in Albert's frame. This leaves us with the conditions:

$$(\mathbf{B}_+' - \mathbf{B}_-')\cdot\mathbf{n} = 0, \qquad (\mathbf{E}_+' - \mathbf{E}_-')\times\mathbf{n} = v(\mathbf{B}_+' - \mathbf{B}_-'),$$

imposed on the plane $\mathbf{x}'\cdot\mathbf{n} = -vt'$.

❊ **Comments from the Examiner:** *I was disappointed that even the bookwork part of this question caused problems for many candidates, who leapt to use Gauss' Law to find a discontinuity in the normal component of **E** and Ampère's Law to find a discontinuity in the tangential components of **B**, whereas the question asks about the tangential components of **E** and the normal components of **B**. Those who avoided this mistake typically went on to do the Lorentz transformation part well, though the final result was not always expressed in terms of quantities in Albert's frame.*

2018, Paper 3, Section II, 17C

Use Maxwell's equations to show that

$$\frac{d}{dt}\int_{\Omega}\left(\frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E}+\frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}\right)dV+\int_{\Omega}\mathbf{J}\cdot\mathbf{E}\,dV=-\frac{1}{\mu_0}\int_{\partial\Omega}(\mathbf{E}\times\mathbf{B})\cdot\mathbf{n}\,dS,$$

where $\Omega\subset\mathbb{R}^3$ is a bounded region, $\partial\Omega$ its boundary and **n** its outward-pointing normal. Give an interpretation for each of the terms in this equation.

A certain capacitor consists of two conducting, circular discs, each of large area A, separated by a small distance along their common axis. Initially, the plates carry charges q_0 and $-q_0$. At time $t=0$, the plates are connected by a resistive wire, causing the charge on the plates to decay slowly as $q(t)=q_0e^{-\lambda t}$ for some constant λ. Construct the Poynting vector and show that energy flows radially out of the capacitor as it discharges.

Solution: We begin by recalling Maxwell's equations:

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_0},\qquad\nabla\cdot\mathbf{B}=0,\qquad\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\qquad\nabla\times\mathbf{B}=\mu_0\left(\mathbf{J}+\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right).$$

It will be useful to make the time-derivatives the subject of the third and fourth equations. We have:

$$\frac{\partial\mathbf{B}}{\partial t}=-\nabla\times\mathbf{E},\qquad\frac{\partial\mathbf{E}}{\partial t}=\frac{1}{\epsilon_0\mu_0}\nabla\times\mathbf{B}-\frac{1}{\epsilon_0}\mathbf{J}.$$

We can use these equations directly to derive the result. We have:

$$\begin{aligned}\frac{d}{dt}\int_{\Omega}\left(\frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E}+\frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}\right)dV&=\int_{\Omega}\left(\epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}+\frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)dV\\&=\int_{\Omega}\left(\frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B})-\mathbf{E}\cdot\mathbf{J}-\frac{1}{\mu_0}\mathbf{B}\cdot(\nabla\times\mathbf{E})\right)dV\\&=-\int_{\Omega}\mathbf{J}\cdot\mathbf{E}\,dV+\frac{1}{\mu_0}\int_{\Omega}\left[\mathbf{E}\cdot(\nabla\times\mathbf{B})-\mathbf{B}\cdot(\nabla\times\mathbf{E})\right]dV.\end{aligned}$$

To finish, we need to simplify the integrand of the final integral. Using index notation, we can write the integrand as:

$$\epsilon_{ijk}E_i\frac{\partial B_k}{\partial x_j}-\epsilon_{ijk}B_i\frac{\partial E_k}{\partial x_j}.$$

If we relabel $i\leftrightarrow k$ in the final term, and then permute $\epsilon_{kji}=-\epsilon_{ijk}$, we can rewrite this expression in the form:

$$\epsilon_{ijk}\left(E_i\frac{\partial B_k}{\partial x_j}+B_k\frac{\partial E_i}{\partial x_j}\right)=\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(E_iB_k\right)=-\nabla\cdot\left(\mathbf{E}\times\mathbf{B}\right).$$

Hence using the divergence theorem, we have the final result:

$$\frac{d}{dt}\int_{\Omega}\left(\frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E}+\frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}\right)dV+\int_{\Omega}\mathbf{J}\cdot\mathbf{E}\,dV=-\frac{1}{\mu_0}\int_{\partial\Omega}(\mathbf{E}\times\mathbf{B})\cdot\mathbf{n}\,dS$$

as required.

We must now interpret each of the terms arising in this equation. The integral on the left hand side is the *energy* of the electromagnetic field; we are taking the time derivative here, and hence the first term on the left hand side represents the *rate of change of energy* of the electromagnetic field.

The second term on the left hand side represents the rate of work done by the electric field on the system. In particular, the integral expression:

$$-\int_{\Omega} \mathbf{J}\cdot\mathbf{E}\,dV$$

should be interpreted as the *rate of energy dissipation* from the system.

Finally, the term on the right hand side is the rate at which energy leaves the volume Ω through the surface $\partial\Omega$. The vector

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$$

is called the *Poynting vector* and tells us the magnitude and direction of energy flow at any point in the system; integrating it over the boundary $\partial\Omega$ gives us the total energy leaving the system per unit time.

We now consider the capacitor problem described in the question. As described above, to construct the Poynting vector, we must first construct the electric and magnetic fields. The main issue is: what assumptions are appropriate in this scenario? It turns out that the most sensible ones to make are:

- We assume that the plates are very large compared to their separation. Then we can instead treat the electric field between the plates as the electric field between *two infinite planes*.
- Since the plates are very large, and the wire joining the plates is infinitesimally thin, we assume that only the charges very close the wire are affected by immediately affected by its presence. Therefore, we assume that the charges on the plates are in general *stationary*, to a good approximation. In particular, we neglect all currents in the plates themselves.
- The mechanism whereby charge is drained from the capacitor is that charges move from one plate to cancel the opposite charge on the other plate; they do so by travelling through the wire. Hence there will be a current flowing in the wire. However, since we are told the discharge is 'slow', and the wire is 'resistive', we will assume that the current flowing in the wire is negligible.

We can now construct the electric and magnetic fields. Recall that we are treating each plate as an 'infinite plane' for this purpose. Let's consider each plane separately and determine the electric field, and then add the results at the end using the linearity of Maxwells equations.

For an infinite plane with surface charge σ with unit normal $\hat{\mathbf{e}}_z$, we can see by the standard symmetry argument that the electric field can only take the functional form $\mathbf{E}_1 = E(z)\hat{\mathbf{e}}_z$ where $E(z) = -E(-z)$ (for a detailed discussion of this exact argument, refer to Examples Sheet 1, Question 5 for example). Now consider integrating over a Gaussian cylinder of radius r with its axis perpendicular to the plane, and such that the plane intersects it. Suppose that its top is at z and its bottom is at $-z$, where $z > 0$. Notice that:

$$\frac{\text{charge in cylinder}}{\epsilon_0} = \frac{2\pi r\sigma}{\epsilon_0}$$

and

$$\int_{\text{cylinder}} \mathbf{E}_1\cdot\mathbf{dS} = 2\pi r(E(z) - E(-z)) = 4\pi r E(z).$$

Hence by Gauss' law, we have $E(z) = \sigma/2\epsilon_0$ for $z > 0$, and then by the parity of $E(z)$ we have $E(z) = -\sigma/2\epsilon_0$ for $z < 0$. Thus the electric field from the plane is:

$$\mathbf{E}_1 = \frac{\sigma}{2\epsilon_0}\mathrm{sign}(z)\hat{\mathbf{e}}_z.$$

In particular, our surface charge density for the plate with charge $q(t)$ is $\sigma = q(t)/\epsilon_0$. For the plate with charge $-q(t)$, the surface charge density is $\sigma = -q(t)/\epsilon_0$. If we therefore imagine superposing the solutions, the solutions will cancel in the regions outside the capacitor, and give an electric field

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{q(t)}{A\epsilon_0}\hat{\mathbf{e}}_z$$

inside the gap between the plates, supposing that the axis of the capacitor is along the z-axis.

We can now construct the magnetic field, which we only need to do in the gap between the plates in order to get the Poynting vector (it is necessarily zero everywhere else since $\mathbf{E} = \mathbf{0}$ everywhere else). We do so using the Ampère-Maxwell law, which states:

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).$$

In this case, we assume there is no current density, because we assume that the current in the wire is negligible. Hence substituting the electric field we found earlier, this expression reduces to:

$$\nabla \times \mathbf{B} = \frac{\mu_0 \dot{q}(t)}{A}\hat{\mathbf{e}}_z = -\frac{\lambda \mu_0 q_0 e^{-\lambda t}}{A}\hat{\mathbf{e}}_z. \tag{$*$}$$

Furthermore, by standard symmetry arguments, we may assume that $\mathbf{B}$ has the functional form $\mathbf{B} = B(r)\hat{\mathbf{e}}_\theta$, in cylindrical coordinates (r, θ, ϕ) about the axis of the capacitor. Therefore there is enough symmetry in the problem to use the integral form of the above equation to solve things. Integrating $(*)$ over the surface S created by a small circular loop C centred on the axis of the capacitors, and with radius r, the left hand side becomes:

$$\int_S (\nabla \times \mathbf{B}) \cdot \mathbf{dS} = \oint_C \mathbf{B} \cdot \mathbf{dr} = 2\pi r B(r),$$

by Stokes' theorem. The right hand side becomes:

$$-\frac{\lambda \mu_0 q_0 e^{-\lambda t}}{A} \int_S \hat{\mathbf{e}}_z \cdot \mathbf{dS} = -\frac{\lambda \pi r^2 \mu_0 q_0 e^{-\lambda t}}{A}.$$

Hence we have constructed the magnetic field:

$$\mathbf{B} = -\frac{\lambda r \mu_0 q_0 e^{-\lambda t}}{2A}\hat{\mathbf{e}}_\theta.$$

It follows that the Poynting vector for this problem is given by:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = -\frac{1}{\mu_0} \cdot \frac{q(t)}{A\epsilon_0} \cdot \frac{\lambda r \mu_0 q_0 e^{-\lambda t}}{2A}\hat{\mathbf{e}}_r = -\frac{\lambda r q_0^2 e^{-2\lambda t}}{2A^2 \epsilon_0}\hat{\mathbf{e}}_r.$$

We see that indeed the direction of energy flow out of the capacitors is *radially inwards*, towards the wire.

❊ **Comments from the Examiner:** *The first part of the question - deriving energy conservation from Maxwell's equations - was usually done well, though several candidates either gave incorrect interpretations of the various terms in this equation, often not specifying that* $\frac{1}{2}\int \left(\epsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0}\mathbf{B} \cdot \mathbf{B} \right) d^3\mathbf{x}$ *was the energy ofthe electromagnetic field. (Several candidates omitted this part of the question altogether.) The second part of the question was poorly done. While most candidates realised they needed to use Ampère's Law to find the magnetic field, few realised that the field in the region between the capacitor's plates was due to the changing electric field (as the plates discharge) rather than due to any current.*