Part IA: Mathematics for Natural Sciences B
Examples Sheet 5: Formal real analysis

Please send all comments and corrections to jmm232@cam.ac.uk.

Questions marked with a (†) require you to carefully review your lecture notes to make sure you understand basic definitions and properties. Questions marked with a (∗) are difficult and should not be attempted at the expense of the other questions.

Limits of functions

1. (†) Suppose that $f : (a,b)\backslash\{x_0\} \to \mathbb{R}$ is a real function defined on a (possibly infinite) open interval excluding a point $x_0 \in \mathbb{R}$. Give the formal mathematical definition of the phrase '$f(x) \to l$ as $x \to x_0$', and explain this definition using a diagram. Hence, show directly from the definition that $x\sin(1/x) \to 0$ as $x \to 0$.

2. (†) Suppose that $f : (a,\infty) \to \mathbb{R}$ is a real function defined on an open interval up to positive infinity. Give the formal mathematical definition of the phrase '$f(x) \to l$ as $x \to \infty$', and explain this definition using a diagram. Hence, show directly from the definition that $1/x \to 0$ as $x \to \infty$.

3. (†) From the formal mathematical definition of a limit, it is possible to prove results about the limits of sums, products, quotients and compositions of functions.[1] State these '*laws of limits*' clearly, and use them to evaluate the following:

$$\text{(a)} \lim_{x\to 0}\frac{x+1}{2-x^2}, \qquad \text{(b)} \lim_{x\to\infty}\sin\left(\frac{x^2+x+1}{3x^2-4}\right), \qquad \text{(c)} \lim_{x\to\infty}\frac{1}{x}\exp\left(\frac{x^4-1}{x^4+1}\right).$$

4. (†) State *L'Hôpital's Rule* for evaluating limits of differentiable functions, carefully specifying the conditions under which it is valid. Assuming $\alpha > 0$ throughout, use L'Hôpital's Rule - where appropriate - to evaluate the limits of the following functions both (i) as $x \to 0^+$ (a *one-sided limit*), and (ii) as $x \to \infty$:

$$\text{(a) } x^\alpha\log(x), \qquad \text{(b) } x^{-\alpha}\log(x), \qquad \text{(c) } x^\alpha e^{-x}, \qquad \text{(d) } x^{-\alpha}e^x, \qquad \text{(e) } \sin(\alpha x)/x.$$

5. Using L'Hôpital's rule, evaluate the limits: (a) $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$; (b) $\displaystyle\lim_{x\to\infty}\log^{1/x}(x)$; (c) $\displaystyle\lim_{x\to\infty}x^x$.

6. Consider the limit:
$$\lim_{x\to\infty}\frac{x}{x+\sin(x)}.$$
Show that the limit is equal to one. Show that if we instead apply L'Hôpital's Rule without checking the conditions, we incorrectly conclude that the limit does not exist.

Continuity of functions

7. (†) Let $f : (a,b) \to \mathbb{R}$ be a real function, and let $x_0 \in (a,b)$ be a point in its domain.
   (a) State the formal definition of f being *continuous* at x_0. Explain this condition by drawing a diagram.
   (b) Using the formal definition of a limit, explain why this condition is equivalent to the statement:
   $$\lim_{x\to x_0} f(x) = f(x_0).$$

8. Using the formal ϵ, δ definition of continuity, show directly that the following functions are continuous everywhere in $\mathbb{R}$: (a) x; (b) $|x|$; (c) x^2. At what points are these functions differentiable?

[1] The proofs are a bit too complicated for this course though; see Part IA Analysis in the mathematical tripos if you would like further details on how these results are derived.

9. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x\sin(1/x)$ for $x \neq 0$, and $f(0) = 0$. Show that f is continuous everywhere, and is differentiable everywhere except at $x = 0$.

10. Show that if a function is differentiable at a point x_0 in its domain, then it must be continuous at that point.

11. ($*$) Is it true that a continuous function $f : \mathbb{R} \to \mathbb{R}$ must be differentiable at *some* point?

Limits of sequences

12. (†) Suppose that $a_1, a_2, \ldots$ is an infinite sequence of real numbers. Give the formal mathematical definition of the phrase '$a_n \to l$ as $n \to \infty$'. Hence, show directly from the definition that:

$$\text{(a)} \lim_{n\to\infty} \frac{1}{n} = 0, \qquad \text{(b)} \lim_{n\to\infty} r^n = 0, \text{ where } 0 < r < 1, \qquad \text{(c)} \lim_{n\to\infty} \frac{n}{2^n} = 0.$$

[*Hint: in (c), it might be useful to prove that* $2^{n+1} > n(n-1)$.]

13. (†) From the formal mathematical definition of the limit of an infinite sequence, it is possible to prove results about the limits of sums, products and quotients of infinite sequences, and continuous functions of infinite sequences.[2] State these '*laws of limits*' clearly, and use them to evaluate the following limits:

$$\text{(a)} \lim_{n\to\infty} \frac{5^{n+2} - 7^{n+2}}{5^n - 7^n}, \qquad \text{(b)} \lim_{n\to\infty} \frac{n + (-1)^n}{n - (-1)^n}, \qquad \text{(c)} \lim_{n\to\infty} \frac{n + (-2)^n}{n - (-2)^n}, \qquad \text{(d)} \lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right).$$

[*Hint: in (d), consider rewriting the general term in the form* $1/f(n)$ *for an appropriate function* f.]

Infinite series

14. (†) State clearly what it means for an infinite series to be: (i) *convergent*; (ii) *absolutely convergent*. If a series is absolutely convergent, must it be convergent? Is the converse true?

15. (†) By directly evaluating the partial sums, determine whether the series:

$$\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$$

is convergent. Is it absolutely convergent?

16. (†) Prove that the geometric series:

$$\sum_{n=0}^{\infty} ar^n$$

is absolutely convergent for $-1 < r < 1$.

17. (†) Clearly state the following tests for convergence of series: (i) the divergence test; (ii) the comparison test; (iii) the ratio test; (iv) the alternating series test. Applying an appropriate test in each case, determine which of the following series are convergent, and which are absolutely convergent:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2+1}{3n^2+4}, \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{n^{10}}{2^n}, \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{n^{10}}{n!}, \qquad \text{(d)} \sum_{n=1}^{\infty} \frac{n!}{10^n}, \qquad \text{(e)} \sum_{n=1}^{\infty} \frac{5 \cdot 8 \cdot 11 \cdot \ldots \cdot (3n+2)}{1 \cdot 5 \cdot 9 \cdot \ldots \cdot (4n-3)},$$

$$\text{(f)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}, \qquad \text{(g)} \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2n+5}{3n+1}\right)^n, \qquad \text{(h)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}.$$

[2] And once again, the proof is beyond the scope of the course.

Past paper questions

Only attempt these questions before the supervision if you have lots of spare time. Sometimes, we might do them together in the supervision; otherwise, you can use these questions as revision material in the holidays.

Paper 1, Question 19, 2023 (20 marks)

(a) State without proof the ratio test for series convergence.

(b) By using any test for series convergence, determine whether the following series converge or diverge:

(i) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{1+k^2}$, (ii) $\displaystyle\sum_{k=0}^{\infty} \frac{a^{2k+1}}{2k+1}$, where $a > 0$.

(c) State L'Hôpital's rule for evaluation of limits of functions.

(d) By using any method, find the following limits for real variable x and real parameter a:

(i) $\displaystyle\lim_{x\to 0^+} x\log(x)$, (ii) $\displaystyle\lim_{x\to a} \frac{x^x - a^a}{x-a}$, where $a > 0$, (iii) $\displaystyle\lim_{x\to\infty} \left(1 + a^x + \left(\frac{a^2}{2}\right)^x\right)^{1/x}$, where $a \geq 0$.

Paper 1, Question 19, 2018 (20 marks)

(a) (i) A real function $f(x)$ is defined on an interval containing the interior point x_0. Explain what is meant by the statement that the function $f(x)$ is (1) continuous at $x = x_0$; (2) differentiable at $x = x_0$.

(ii) For $m = 0, 1, 2$, the function $f_m(x)$ is defined as:

$$f_m(x) = \begin{cases} x^m \sin(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

For each $m = 0, 1, 2$, determine whether the function $f_m(x)$ is (1) continuous at $x = 0$, (2) differentiable at $x = 0$. Justify your answers.

(b) Using the comparison or the ratio test, or otherwise, determine whether the following series converge or diverge. You may use without proof standard results relating to the series:

$$\sum_{n=1}^{\infty} n^{-p},$$

where $p > 0$.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}$, (ii) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n^4 + a^2} - n^2\right)$, where $a > 0$.