Part IA: Mathematics for Natural Sciences B
Examples Sheet 4: Differential calculus, limits and continuity

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Questions marked with a ($*$) are difficult and should not be attempted at the expense of the other questions. A section marked with a (†) contains content that is unique to the Mathematics B course.

Limit definition of the derivative

1. Let $y \equiv y(x)$ be a function of x. Define the *derivative* dy/dx of y as a limit. Using the limit definition:
   (a) show that differentiation is a linear operation;
   (b) find the derivative of $y(x) = x^n$, for $n = 0, 1, 2, 3, \ldots$.

   Hence obtain the derivative of $ax + bx^2 \sin(\theta)$, where a, b, θ are real constants.

2. (a) Using *only* the limit definition, show that for $a > 0$, the derivative of $y(x) = a^x$ is proportional to a^x.
   (b) One definition of the number e is the value of a for which the proportionality constant in the previous part is 1. Using *only* this definition, show that the derivative of a^x is given by $\log(a)a^x$.

Rules of differentiation

3. Let $y \equiv y(x)$, $u \equiv u(x)$ and $v \equiv v(x)$ be functions of x. Using the limit definition of the derivative, prove the following rules of differentiation:

   (a) $\dfrac{d}{dx}(u(v)) = \dfrac{dv}{dx}\dfrac{du}{dv}$, (b) $\dfrac{d}{dx}(uv) = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$, (c) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{\dfrac{du}{dx}v - u\dfrac{dv}{dx}}{v^2}$, (d) $\dfrac{dy}{dx} = \left(\dfrac{dx}{dy}\right)^{-1}$.

   [Recall that these rules are called the chain rule, the product rule, the quotient rule, and the reciprocal rule, respectively. Make sure you know them off by heart!] Rewrite these rules in terms of Lagrange's 'primed' notation for derivatives.

4. Using the rules you derived in the previous question, compute the derivatives of:

   (a) $\log(x)$, (b) 3^{x^2}, (c) $\dfrac{e^x}{x^3 - 1}$, (d) $x^3 \log(x^2 - 7)$, (e) $\sqrt{x^3 - e^x \log(x)}$.

5. By writing each of the following trigonometric and hyperbolic functions in terms of exponentials, compute their derivatives: (a) $\cos(x)$; (b) $\sin(x)$; (c) $\cosh(x)$; (d) $\sinh(x)$; (e) $\tan(x)$; (f) $\tanh(x)$. Learn these derivatives off by heart.

6. Using: (i) the logarithmic formulae for the inverse hyperbolic functions you derived on Sheet 3; (ii) the reciprocal rule, compute the derivatives of: (a) $\cosh^{-1}(x)$; (b) $\sinh^{-1}(x)$; (c) $\tanh^{-1}(x)$. Learnt these derivatives off by heart.

7. If $y \equiv y(x)$ is a function of x, show that $\dfrac{d^3x}{dy^3} = -\left(\dfrac{dy}{dx}\right)^{-4}\dfrac{d^3y}{dx^3} + 3\left(\dfrac{dy}{dx}\right)^{-5}\left(\dfrac{d^2y}{dx^2}\right)^2$. Verify this when $y = e^{2x}$.

8. What is *implicit differentiation*, and why is it called implicit? Using: (a) implicit differentiation; (b) the reciprocal rule, find dy/dx given $y + e^y \sin(y) = 1/x$, and make sure that your answers agree.

Curve-sketching

9. State what it means for a function to be *even* and for a function to be *odd*, and explain the geometric significance of these definitions. Hence, decide whether the following functions are even, odd, both, or neither:

   (a) x, (b) $\sin(x)$, (c) e^x, (d) $\sin(\frac{\pi}{2} - x)$, (e) $|x|\cos(x)$, (f) $\sqrt{x}$, (g) 2, (h) 0, (i) $\log\left|\dfrac{1+x}{1-x}\right|$.

10. Write down a list of things you should consider when sketching the graph of a function. Compare with your supervision partner before the supervision, and exchange ideas!

11. Sketch the graphs of the following functions, explaining your reasoning in each case:

(a) $(x-3)^3+2x$, (b) $\dfrac{x}{1+x^2}$, (c) $\dfrac{x^2+3}{x-1}$, (d) xe^x, (e) $\dfrac{\log(x)}{1+x}$, (f) $\dfrac{1}{1-e^x}$, (g) $e^x\cos(x)$.

(†) Leibniz's formula

12. Using mathematical induction, prove *Leibniz's formula* for the nth derivative of a product:

$$\frac{d^n}{dx^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)},$$

where $f^{(k)}$ denotes the kth derivative of f. Hence compute: (a) the third derivative of $\log^2(x)$; (b) the 100th derivative of x^2e^x.

13. Use Leibniz's formula to prove that the nth derivative of $e^{-x^2/2}$ is a solution of the equation $Z''+xZ'+(n+1)Z=0$.

(†) Formal definition of a limit

14. Suppose that $f:(a,b)\backslash\{x_0\}\to\mathbb{R}$ is a real function defined on a (possibly infinite) open interval excluding a point $x_0\in\mathbb{R}$. Give the formal mathematical definition of the phrase '$f(x)\to l$ as $x\to x_0$', and explain this definition using a diagram. How should this definition be modified for the cases $l=\pm\infty$?

15. Here is a model example of a formal mathematical argument, from first principles, showing that $x^2\to 1$ as $x\to 1$:

'Suppose we are given some arbitrary tolerance $\epsilon>0$. Choose some closeness $\delta=\min(1,\epsilon/3)$. Then for all x which are δ-close to 1, i.e. $0<|x-1|<\delta$, we have:

$$\begin{aligned}
|x^2-1| &= |((x-1)+1)^2-1| && (1)\\
&= |(x-1)^2+2(x-1)| && (2)\\
&\le |x-1|^2+2|x-1| && (3)\\
&< \delta^2+2\delta && (4)\\
&\le \delta+2\delta && (5)\\
&= 3\delta && (6)\\
&\le \epsilon. && (7)
\end{aligned}$$

Hence if x is δ-close to 1, we have that $|x^2-1|<\epsilon$, so that x^2 is ϵ-close to 1. We conclude that, by the definition of a limit, we have $x^2\to 1$ as $x\to 1$.'

(a) Which of ϵ,δ are we given, and which of ϵ,δ must we choose?
(b) Why do we express $|x^2-1|$ in terms of $x-1$ in line (1)?
(c) What law from earlier in the course have we used in going from line (2) to line (3)?
(d) What have we used in going from line (4) to line (5)? What about in going from line (6) to line (7)?
(e) Would the proof still have been successful if we had chosen $\delta=\min(1,\epsilon/4)$? In terms of ϵ, what is the largest possible value of δ we could have chosen for the proof to still work?

16. Using the model example in Question 15 as a template, provide proofs from first principles showing that:

    (a) $4x^3 \to 0$ as $x \to 0$, (b) $x^2 \to a^2$ as $x \to a$, for $a \in \mathbb{R}$, (c) $\sin(x) \to 1$ as $x \to \pi/2$,

    (d) $x\sin(1/x) \to 0$ as $x \to 0$, (e) $1/x^2 \to \infty$ as $x \to 0$.

17. Suppose that $f : (a, \infty) \to \mathbb{R}$ is a real function defined on an open interval up to positive infinity. Give the formal mathematical definition of the phrase '$f(x) \to l$ as $x \to \infty$', and explain this definition using a diagram. How should this definition be modified for the cases $l = \pm\infty$? Hence, show directly from the definition that: (a) $1/x \to 0$ as $x \to \infty$; (b) $\sin(x)/x \to 0$ as $x \to \infty$; (c) $x^3 \to -\infty$ as $x \to -\infty$.

(†) Laws of limits

18. ($*$) Let $x_0 \in \mathbb{R}$, and let $f, g : \mathbb{R} \to \mathbb{R}$ be real functions. From the formal definition of a limit, prove that:

$$\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x),$$

    provided that (i) both the limits on the right hand side exist in $\mathbb{R} \cup \{\infty, -\infty\}$ (the set of real numbers with infinity and negative infinity adjoined), *and* (ii) if one of the limits on the right hand side is ∞, the other is *not* $-\infty$.

19. From the formal mathematical definition of a limit, it is possible to prove results about the limits of sums, products, quotients and compositions of functions, similarly to Question 18. State these '*laws of limits*' clearly (making sure to take particular care when the limits are infinite), and use them to evaluate the following:

$$\text{(a)} \lim_{x \to 0} \frac{x+1}{2-x^2}, \quad \text{(b)} \lim_{x \to \infty} \sin\left(\frac{x^2+x+1}{3x^2-4}\right), \quad \text{(c)} \lim_{x \to 0} \left(\exp\left(\frac{x^4-1}{x^4+1}\right)\right)^{1/x^2}, \quad \text{(d)} \lim_{x \to \infty} \left(\sqrt{x^2+7x} - x\right).$$

20. State *L'Hôpital's rule* for evaluating limits of differentiable functions, carefully specifying the conditions under which it is valid. Assuming $\alpha > 0$ throughout, use L'Hôpital's rule - where appropriate - to evaluate the limits of the following functions both (i) as $x \to 0^+$ (a *one-sided limit*), and (ii) as $x \to \infty$:

    (a) $x^\alpha \log(x)$, (b) $x^{-\alpha}\log(x)$, (c) $x^\alpha e^{-x}$, (d) $x^{-\alpha}e^x$, (e) $\sin(\alpha x)/x$.

21. Using L'Hôpital's rule, evaluate the following 'power law' limits:

$$\text{(a)} \lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x, \quad \text{(b)} \lim_{x \to \infty} \log^{1/x}(x), \quad \text{(c)} \lim_{x \to 0^+} x^x, \quad \text{(d)} \lim_{x \to \infty} x^{1/x}.$$

22. Explain why the following arguments with limits are *wrong*.

    (a) $\displaystyle\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to \infty} (1+0)^x = 1$, using $\displaystyle\lim_{x \to \infty} \frac{1}{x} = 0$ in the first step.

    (b) $\displaystyle\lim_{x \to 0} \frac{1-\cos(x)}{x^2} = \lim_{x \to 0} \frac{1-1}{x^2} = 0$, using $\cos(x) \approx 1$ for small enough x.

    Now, evaluate the limits correctly.

23. Using L'Hôpital's rule where appropriate, compute the limit:

$$\lim_{x \to \infty} \left(1 + a^x + \left(\frac{a^2}{2}\right)^x\right)^{\frac{1}{x}}.$$

    for all values of $a \geq 0$.

24. Consider the limit:

$$\lim_{x \to \infty} \frac{x}{x + \sin(x)}.$$

    Show that this limit is equal to one. Show that if we instead naïvely apply L'Hôpital's rule, we incorrectly conclude that the limit does not exist.

(†) Miscellaneous limits

[*This section contains a large collection of limits from past papers for you to evaluate. If you feel like you are getting too much of a good thing, feel free to save some of them for us to do together in the supervision.*]

25. Evaluate the following limits, using the most efficient method in each case:

    (a) $\lim_{x \to 0^+} x \log(x)$;

    (b) $\lim_{x \to a} \frac{x^x - a^a}{x - a}$ where $a > 0$;

    (c) $\lim_{x \to 0} \frac{\tan(x) - \sin(x)}{\sin^3(x)}$;

    (d) $\lim_{x \to a} \frac{\sin(x) - \sin(a)}{x - a}$;

    (e) $\lim_{x \to \infty} \left(\frac{x + a}{x - a} \right)^x$;

    (f) $\lim_{x \to 0} \frac{\cos(x) - \cos(3x)}{x^2}$;

    (g) $\lim_{x \to 0} \frac{\log(\cos(x))}{\log(\cos(3x))}$;

    (h) $\lim_{x \to 0} \frac{\sin(3x)}{\sinh(x)}$.

(†) Continuity of functions

26. Let $f : (a, b) \to \mathbb{R}$ be a real function, and let $x_0 \in (a, b)$ be a point in its domain.
    (a) State the formal ϵ, δ definition of f being *continuous* at x_0. Explain this condition by drawing a diagram.
    (b) Using the formal definition of a limit, explain why this condition is equivalent to the statement:
    $$\lim_{x \to x_0} f(x) = f(x_0).$$

27. Using the formal ϵ, δ definition of continuity, show directly that the following functions are continuous everywhere:

    (a) x, (b) $|x|$, (c) x^2, (d) $\sin(x)$.

    At what points are these functions differentiable? [*Hint: for part (c), look at your answer to Question 16(b).*]

28. Using the formal ϵ, δ definition of continuity, show directly that the function $f(x) = 0$ for $x \leq 0$, $f(x) = 1$ for $x > 0$, is discontinuous at $x = 0$.

29. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x \sin(1/x)$ for $x \neq 0$, and $f(0) = 0$. Show that f is continuous everywhere, and is differentiable everywhere except at $x = 0$.

30. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by:
    $$f(x) = \begin{cases} |x|^p \sin(x), & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad g(x) = \begin{cases} |x|^q \sin(\pi \sin(1/x)), & x \neq 0, \\ 0, & x = 0, \end{cases}$$
    where p, q are real numbers. For which values of $p.q$ are f, g: (a) continuous; (b) differentiable? Justify your answers.

31. Three functions f_0, f_1, f_2 are defined by:
    $$f_n(x) = \left(\frac{x - \pi/2}{x} \right)^n \sin(\tan(x))$$
    for $n = 0, 1, 2$, at all points except $x = m\pi/2$ for integer m, where the functions are defined to be zero. For each n, determine with justification all points in the range $(-\pi, \pi)$ where the function is: (a) continuous; (b) differentiable.

32. Show that if a function is differentiable at a point x_0 in its domain, then it must be continuous at x_0. ($*$) Is it true that a continuous function $f : \mathbb{R} \to \mathbb{R}$ must be differentiable at *some* point?