

Part IA: Mathematics for Natural Sciences A

Examples Sheet 10: First-order ordinary differential equations

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Separable equations

1. Explain what is meant by a *separable differential equation*, and how we can solve one. Find the general solution of the following separable differential equations:

(a) $\frac{dy}{dx} = x,$

(b) $\frac{dy}{dx} = (2 - y)(1 - y),$

(c) $\frac{dy}{dx} = -\frac{x^3}{(1 + y)^2},$

(d) $\frac{dy}{dx} = \frac{4y}{x(y - 3)},$

(e) $\frac{dy}{dx} = xe^{x-2y},$

(f) $\frac{dy}{dx} = \sin(y + x) - \sin(y - x).$

2. Find the solution of the differential equation:

$$\frac{dy}{dx} = 1 - y^2,$$

subject to the initial condition $y(0) = y_0$. Sketch representative solution curves for various values of y_0 . Could you have drawn the sketches without solving the equation?

3. Determine the half-life of thorium-234 if a sample of mass 5g is reduced to 4g in one week. What amount of thorium is left after twelve weeks?
4. *Newton's law of cooling* states that the rate of heat loss from a body is proportional to the difference between the temperature of the object and its ambient environment. Assuming that Newton's law of cooling applies, calculate the time at which a cup of tea in a 20°C room was made, given that: (i) the tea is measured to have temperature 40°C at 5pm; (ii) the tea is measured to have temperature 30°C at 3pm; (iii) the water was initially at boiling point.
5. Consider the family of curves $C = \{y = ax^2 : a \in \mathbb{R}\}$. Sketch a few representative curves in C . Determine a family of curves $C' = \{y = f(x, b) : b \in \mathbb{R}\}$ such that each curve in C' is orthogonal to all curves in C , and sketch a few representative curves in the family C' .
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Linear equations

6. Write down the general form of a *linear first-order differential equation*. What is meant by an *integrating factor* for such an equation? Solve the following linear first-order differential equations by finding an appropriate integrating factor:

(a) $\frac{dy}{dx} + 2xy = 4x,$

(b) $\frac{dy}{dx} + \frac{y}{(2 - 3x^2)^3} = 1,$

(c) $\frac{dy}{dx} - y \tan(x) = 1,$

(d) $\frac{dy}{dx} + (1 + \log(x))y = x^{-x},$

(e) $\frac{dy}{dx} + \frac{y}{\tan(x)} = \cos^2(x),$

(f) $\frac{1}{x} \frac{dy}{dx} + y - 5e^{-x^2} = 0.$

7. Solve the equation:

$$(x^2 + 1) \frac{dy}{dx} + 3x^3 y = 6xe^{-3x^2/2},$$

subject to the boundary condition $y(0) = 1$.

8. Establish a formula for the general solution of the linear differential equation:

$$\alpha(x) \frac{dy}{dx} + \beta(x)y = \gamma(x),$$

stating any conditions you must assume for your formula to be valid.

Some 'easy to spot' substitutions

[Hint: most 'easy' substitutions come from thinking about implicit differentiation.]

9. Using an appropriate substitution, find the general solution of the equation:

$$y^3 + x + 3y^2 \frac{dy}{dx} = 0.$$

Find also the solution satisfying the boundary condition $y(0) = 1$.

10. Using an appropriate substitution, find the general solution of the equation:

$$(x + y + 1)^2 \frac{dy}{dx} + (x + y + 1)^2 + x^3 = 0.$$

Find also the solution satisfying the boundary condition $y(0) = 0$.

Some standard substitutions: homogeneous, Bernoulli, and affine transformations

11. Define a *homogeneous equation*, and state the substitution which renders them solvable. Hence, solve the equations:

$$(i) \frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right), \quad (ii) (y - x) \frac{dy}{dx} + (2x + 3y) = 0, \quad (iii) \frac{dy}{dx} = \frac{x^3 + y^2}{3xy^2}.$$

12. Define a *Bernoulli equation*, and state the substitution which renders them solvable. Hence, solve the equations:

$$(i) \frac{dy}{dx} - y = xy^5, \quad (ii) \frac{dy}{dx} + y = y^2(\cos(x) - \sin(x)), \quad (iii) xy \frac{dy}{dx} + (x^2 + y^2 + x) = 0.$$

13. (a) Show that equations of the form:

$$\frac{dy}{dx} = f(ax + by + c),$$

with $b \neq 0$ may be reduced to a separable equation by making the substitution $u = ax + by + c$.

- (b) Hence, solve the equations:

$$(i) \frac{dy}{dx} = (4x + y)^3, \quad (ii) \cos(x + y - 1) \frac{dy}{dx} = \sin(x + y - 1), \quad (iii) \frac{dx}{dy} = \operatorname{sech}^2(2x - y + 2).$$