Part IA: Mathematics for Natural Sciences A
Examples Sheet 7: Taylor series, and Newton-Raphson iteration

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Taylor series

1. Carefully state *Taylor's theorem*, giving Lagrange's formula for the remainder term. Hence, obtain the first three non-zero terms in the Taylor series of $\log(x)$ about $x = 1$ by direct differentiation. Using this expansion, together with Lagrange's form of the remainder, show that:

$$|\log(3/2) - 5/12| \leq 1/64,$$

and hence give an approximation of $\log(3/2)$ valid to one decimal place.

2. Write down the Taylor series about $x = 0$ for the following functions, stating their range of convergence in each case:

(a) e^x, (b) $\log(1+x)$, (c) $\sin(x)$, (d) $\cos(x)$, (e) $\sinh(x)$, (f) $\cosh(x)$, (g) $(1+x)^a$.

What happens when a is a non-negative integer? Learn these series off by heart, and get your supervision partner to test you on them.

3. *Without differentiating*, find the first three terms in the Taylor series of the following functions. [*Note: there are lots of examples from past papers here to practise with, but if you are getting bored, we can do some in the supervision together. The next few questions, 4-7, have more of a problem-solving element.*]

(a) $\dfrac{1}{\sqrt{1+x}}$ about $x = 0$;

(b) $\dfrac{1}{(x^2+2)^{3/2}}$ about $x = 0$;

(c) $\tan(x)$ about $x = 0$;

(d) $\log(\cos(x))$ about $x = 0$;

(e) $\arcsin(x)$ about $x = 0$;

(f) $\arctan(x)$ about $x = 1$;

(g) $(\cosh(x))^{-1/2}$ about $x = 0$;

(h) $e^{\sin(x)}$ about $x = \pi/2$;

(i) $x\sinh(x^2)$ about $x = 0$;

(j) $\log(1 + \log(1+x))$ about $x = 0$;

(k) $\sin^6(x)$ about $x = 0$;

(l) $\dfrac{\cosh(x)}{\cos(x)}$ about $x = 0$;

(m) $\cosh(\log(x))$ about $x = 2$;

(n) $\log(2 - e^x)$ about $x = 0$;

(o) $\dfrac{\sin(x)}{\sinh(x)}$ about $x = 0$;

(p) $\sinh(\log(x))$ about $x = 1$;

(q) $\sin\left(\dfrac{\pi e^x}{2}\right)$ about $x = 0$;

(r) $\dfrac{\sinh(x+1)}{x+2}$ about $x = -1$;

(s) $\dfrac{\log(1+x^3)}{\cosh(x)}$ about $x = 0$;

(t) $\dfrac{\cosh(x)}{\sqrt{1+x^2}}$ about $x = 0$;

(u) $\dfrac{e^{-x^2}}{\cosh(x)}$ about $x = 0$;

(v) $\dfrac{\log(2+x)}{2-x}$ about $x = 0$;

(w) $\log(\cosh(x))$ about $x = 0$;

(x) $\cosh(\sqrt{x})$ about $x = 2$;

(y) $\dfrac{\sin(x)}{(1+x)^2}$ about $x = 0$;

(z) $\dfrac{x\sin(x)}{\log(1+x^2)}$ about $x = 0$;

(a′) $\cos\left(\sqrt{\dfrac{\pi^2}{16}+x}\right)$ about $x = 0$;

(b′) $\log((2+x)^3)$ about $x = 0$.

4. *Without differentiating*, find the value of the thirty-second derivative of $\cos(x^4)$ at $x = 0$.

5. Find the first three non-zero terms in a series approximation of $\log(1 + x + 2x^2) - \log(x^2)$ valid for $x \to \infty$.

6. Let $f(x)$ be a function which can be expanded as a Taylor series about $x = 0$. Find the first two terms in the Taylor series of the function $\log(1 + f(x))$ about $x = 0$, assuming that $1 + f(0) > 0$, $f'(0) \neq 0$ and $f''(0)(1 + f(0)) \neq (f'(0))^2$. Why are these conditions necessary?

7. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ be the Taylor series of $f(x)$ about $x = 0$, with $a_0 > 0$, $a_1 \neq 0$, $a_1^2 \neq a_2a_0$ and $a_1^2 \neq 4a_2a_0$. Find the first three terms in the Taylor series of (a) $1/f(x)$ about $x = 0$; (b) $\sqrt{f(x)}$ about $x = 0$. Explain where you used the assumptions on the a_n in your answer.

Newton-Raphson root finding

8. Give an explanation of the Newton-Raphson algorithm for root finding, including an appropriate sketch. Under what general conditions is it guaranteed that Newton-Raphson will converge to the root of interest? Prove that, when it converges to the root of interest, the Newton-Raphson method enjoys *quadratic convergence*.

9. (a) Find the value of the first iterate of Newton-Raphson iteration for the function $f(x) = x - 2 + \log(x)$ with a starting guess of $x_0 = 1$.

   (b) Find the value of the first and second iterates of Newton-Raphson iteration, valid to two decimal places, for the function $f(x) = x^2 - 2$ with a starting guess of $x_0 = 1$.

   [*Both parts of this question are based on old (short) tripos questions, so try doing them*without *a calculator!*]

10. [*You may use a calculator for this question, but remember that you* won't *be able to use a calculator in the exam. Newton-Raphson questions will be more theoretical in the exams, like the next question, or involve easy calculations, like the previous question.*]

    (a) Sketch the graph of $f(x) = x^3 - 3x^2 + 2$, indicating the coordinates of the turning points and the coordinates of the intersections with the x-axis.

    (b) Use Newton-Raphson with an initial guess of $x_0 = 2.5$ to find an estimate of the largest root of the equation $f(x) = 0$, accurate to 5 decimal places. Draw a sketch showing the progress of the algorithm.

    (c) To which roots (if any) does the algorithm converge if we instead start at: (i) $x_0 = 1.5$; (ii) $x_0 = 1.9$; (iii) $x_0 = 2$?

11. The real function f is defined by $f(x) = x^2 - 2\epsilon x - 1$, where ϵ is a small positive parameter ($0 < \epsilon \ll 1$). Let x_i be the ith Newton-Raphson iterate, with a starting guess of $x_0 = 1$, and let x_* be the unique positive root satisfying $f(x_*) = 0$. By Taylor expansion, show that $|x_i - x_*| \propto \epsilon^{n_i}$, where: (a) $n_0 = 1$; (b) $n_1 = 2$; (c) $n_2 = 4$.