

## Part IA: Mathematics for Natural Sciences A

### Examples Sheet 11: Linear ordinary differential equations

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#### Basic definitions

1. Consider the general linear  $n$ th-order ordinary differential equation:

$$\alpha_n(x) \frac{d^n y}{dx^n} + \alpha_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + \alpha_1(x) \frac{dy}{dx} + \alpha_0(x)y = f(x).$$

where  $\alpha_n(x) \not\equiv 0$ .

- (a) Give the definitions of the following terms: (i) homogeneous equation; (ii) coefficient functions; (iii) forcing.
  - (b) Define a *complementary function* for this equation. How many arbitrary constants feature in the complementary function for this equation?
  - (c) Define a *particular integral* for this equation. Is a particular integral for this equation unique?
  - (d) Show that if  $y_{CF}$  is the complementary function for this equation, and  $y_{PI}$  is a particular integral, then the sum  $y = y_{CF} + y_{PI}$  solves the equation.
  - (e) Suppose that we now seek a particular solution of this equation satisfying certain boundary conditions. How many boundary conditions are needed to fully specify a particular solution?
2. By direct differentiation, verify that the following ordinary differential equations have the given complementary functions:
- (a)  $y_{CF} = Ax + Be^x$  is the complementary function for  $(x - 1)y'' - xy' + y = 0$ ;
  - (b)  $y_{CF} = A + B \log(x)$  is the complementary function for  $xy'' + y' = \cos(x)e^{x^2}$ ;
  - (c)  $y_{CF} = Ax + B \sin(x)$  is the complementary function for  $(1 - x \cot(x))y'' - xy' + y = x$ ;
  - (d)  $y_{CF} = A + Bx + Ce^x$  is the complementary function for  $y''' - y'' = x$ .
3. By direct differentiation, verify that the following ordinary differential equations have the given particular integrals:
- (a)  $y_{PI} = \cos(x)$  is a particular integral for  $-y'' + y = 2 \cos(x)$ ;
  - (b)  $y_{PI} = x^2$  is a particular integral for  $xy'' + y' = 4x$ ;
  - (c)  $y_{PI} = e^{x^2}$  is a particular integral for  $y''' - 2xy'' - 2y' - y = (4x - 1)e^{x^2}$ ;
  - (d)  $y_{PI} = \sin(x)/x$  is a particular integral for  $xy^{(4)} + 4y^{(3)} + xy^{(2)} + 2y^{(1)} + xy = \sin(x)$ .
4. Verify that the equation:

$$(3 + x)y'' + (2 + x)y' - y = x^2$$

has complementary function  $y_{CF}(x) = Ae^{-x} + B(x + 2)$ . Hence, by finding a particular integral of the form

$$y_{PI}(x) = \alpha x^2 + \beta x + \gamma,$$

determine the full solution to the equation subject to the boundary conditions  $y(0) = 0$  and  $y'(0) = 1$ .

**Constant coefficient equations**

5. Consider the linear second-order ordinary differential equation with *constant coefficients*:

$$\alpha \frac{d^2y}{dx^2} + \beta \frac{dy}{dx} + \gamma y = f(x),$$

where  $\alpha, \beta, \gamma$  are *constants*, with  $\alpha \neq 0$ .

- (a) Show that  $y = e^{\mu x}$  solves the homogeneous equation if and only if  $\mu$  satisfies the *auxiliary equation*:

$$\mu^2 + \beta\mu + \gamma = 0.$$

Deduce the complementary function of the equation in the case that the auxiliary equation does not have repeated roots.

- (b) Show that if the auxiliary equation has a repeated root  $\omega$ , then the complementary function is given by

$$y_{CF} = (Ax + B)e^{\omega x}.$$

- (c) How do these results generalise to  $n$ th order linear differential equations with constant coefficients?

6. Determine the solutions of the following differential equations:

(a)  $y'' + 6y' + 5y = 0;$

(b)  $y'' + 3y' + 4y = 0;$

(c)  $y'' + 4y = x;$

(d)  $y'' - 2y' + 2y = 2x^2;$

(e)  $y'' + y = |x|;$

(f)  $y'' + 3y' + 2y = e^{-x};$

(g)  $y'' - 2y' + 5y = e^x \cos(2x);$

(h)  $y'' + 2y' + y = 2xe^{-x}.$

7. Determine the solutions of the following differential equations subject to the given constraints:

(a)  $y'' - 4y' + 13y = 0$ , subject to  $y(0) = \pi$  and  $y(-\pi/2) = 1$ ;

(b)  $y'' - 4y' + 5y = 125x^2$ , subject to  $y(0) = 1$  and  $y(\frac{\pi}{2}) = \frac{25\pi^2}{4} + 20\pi + 22$ ;

(c)  $y'' + 7y' + 12y = 6$ , subject to  $y(0) = 0$  and  $y(\frac{1}{3}) = \frac{1-e^{-1}}{2}$ ;

(d)  $y'' + 7y' + 12y = 2e^{-3x}$ , subject to  $y(0) = 1$  and  $y'(0) = 0$ .

8. Find the value of  $a$  for which the complementary function of the ODE:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + ay = 0,$$

is given by  $y_{CF} = Axe^{-2x} + Be^{-2x}$ .

9. Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + y = \cos(kx),$$

where  $k$  is a real number.

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10. The differential operator  $\mathcal{L}$  is defined by:

$$\mathcal{L} = \frac{d^2}{dx^2} + \sqrt{3} \frac{d}{dx} + 3.$$

Solve the equation  $\mathcal{L}y = 0$ , and hence solve the equations:

- (a)  $\mathcal{L}y = e^{-\sqrt{3}x}$ ;
- (b)  $\mathcal{L}y = x$ .

Without further calculation, state the general solution of  $\mathcal{L}y = 2x + e^{-\sqrt{3}x}$ . Find also the solution of this equation satisfying the boundary conditions  $y(0) = 0, y(\pi) = \frac{e^{-\sqrt{3}\pi}}{3} - \frac{2}{3\sqrt{3}}$ .

### Harmonic oscillators

11. Consider the constant coefficient linear second-order ordinary differential equation:

$$\frac{d^2y}{dt^2} + 2\gamma \frac{dy}{dt} + \omega_0^2 y = f(t),$$

modelling an oscillating system which depends on time  $t$ . The coefficients  $\gamma, \omega_0$  are positive.

- (a) What is the physical interpretation of the constant  $\gamma$ ? What is the physical interpretation of the function  $f(t)$ ?
- (b) Find the complementary function of this equation. Discuss the different forms the complementary function can take (in particular, defining the terms *underdamping*, *critical damping*, and *overdamping*), and how this relates to the *transient* behaviour of the oscillator.
- (c) In the underdamped case, find the long-term behaviour of the oscillator in the case of resonant forcing:

$$f(t) = e^{-\gamma t} \sin\left(t\sqrt{\omega_0^2 - \gamma^2}\right).$$

### Coupled systems of differential equations

[This section is labelled 'non-examinable' in the lecture notes, but has appeared on Tripos papers - see e.g. 2023 Paper 2 or 2021 Paper 2.]

12. (a) Consider the system of differential equations:

$$\frac{dx}{dt} = ax + by + p, \quad \frac{dy}{dt} = cx + dy + q,$$

for the variables  $x(t), y(t)$ , where  $a, b, c, d, p, q$  are constants. Show that:

$$\frac{d^2x}{dt^2} = (a+d)\frac{dx}{dt} + (bc-ad)x + bq - pd.$$

(b) Hence:

(i) Find the general solution of the system:

$$\frac{dx}{dt} = 4y + 2, \quad \frac{dy}{dt} = x.$$

(ii) Solve the system:

$$\frac{dx}{dt} = 3x - y, \quad \frac{dy}{dt} = x + y,$$

subject to the initial conditions  $x(0) = 0$  and  $y(0) = 1$ .

(iii) Solve the system:

$$\frac{dx}{dt} = -3x + y, \quad \frac{dy}{dt} = -5x + y,$$

subject to the initial conditions  $x(0) = 1, y(0) = 1$ .