Part IA: Mathematics for Natural Sciences B
Examples Sheet 11: Linear ordinary differential equations

Model Solutions

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Basic definitions

1. Consider the general linear nth-order ordinary differential equation:

$$\alpha_n(x)\frac{d^n y}{dx^n} + \alpha_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + \alpha_1(x)\frac{dy}{dx} + \alpha_0(x)y = f(x).$$

where $\alpha_n(x) \not\equiv 0$.

(a) Give the definitions of the following terms: (i) homogeneous equation; (ii) coefficient functions; (iii) forcing.

(b) Define a *complementary function* for this equation. How many arbitrary constants feature in the complementary function for this equation?

(c) Define a *particular integral* for this equation. Is a particular integral for this equation unique?

(d) Show that if y_{CF} is the complementary function for this equation, and y_{PI} is a particular integral, then the sum $y = y_{\mathsf{CF}} + y_{\mathsf{PI}}$ solves the equation.

(e) Suppose that we now seek a particular solution of this equation satisfying certain boundary conditions. How many boundary conditions are needed to fully specify a particular solution?

Solution:

(a) The *homogeneous equation* is the same differential equation, but with $f(x) = 0$ on the right hand side. The *coefficient functions* are the functions $\alpha_i(x)$ for $i = 0, ..., n$. The *forcing* is the term on the right hand side, $f(x)$.

(b) A complementary function y_{CF} is a solution of the homogeneous equation. The general complementary function contains two arbitrary constants of integration, corresponding to the fact we are dealing with a second-order ODE.

(c) A particular integral y_{PI} is a solution of the inhomogeneous equation (i.e. including the forcing term). It is not unique, because we can add on parts of the complementary function and still solve the inhomogeneous equation.

(d) If y_{CF} is the general complementary function, and y_{PI} is a particular integral, the sum satisfies:

$$\begin{aligned}
&\alpha_n(x)\frac{d^n}{dx^n}(y_{\mathsf{CF}} + y_{\mathsf{PI}}) + \alpha_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}(y_{\mathsf{CF}} + y_{\mathsf{PI}}) + \cdots + \alpha_1(x)\frac{d}{dx}(y_{\mathsf{CF}} + y_{\mathsf{PI}}) + \alpha_0(x)(y_{\mathsf{CF}} + y_{\mathsf{PI}}) \\
&= \left(\alpha_n(x)\frac{d^n y_{\mathsf{CF}}}{dx^n} + \alpha_{n-1}(x)\frac{d^{n-1}y_{\mathsf{CF}}}{dx^{n-1}} + \cdots + \alpha_1(x)\frac{dy_{\mathsf{CF}}}{dx} + y_{\mathsf{CF}}\right) \\
&\quad + \left(\alpha_n(x)\frac{d^n y_{\mathsf{PI}}}{dx^n} + \alpha_{n-1}(x)\frac{d^{n-1}y_{\mathsf{PI}}}{dx^{n-1}} + \cdots + \alpha_1(x)\frac{dy_{\mathsf{PI}}}{dx} + y_{\mathsf{PI}}\right) \\
&= f(x),
\end{aligned}$$

since the first term goes to zero as y_{CF} is the general complementary function, and the second term gives $f(x)$ since y_{PI} is a solution of the inhomogeneous equation.

(e) We need n boundary conditions to fix n arbitrary constants.

2. By direct differentiation, verify that the following ordinary differential equations have the given complementary functions:

(a) $y_{\text{CF}} = Ax + Be^x$ is the complementary function for $(x-1)y'' - xy' + y = 0$;

(b) $y_{\text{CF}} = A + B\log(x)$ is the complementary function for $xy'' + y' = \cos(x)e^{x^2}$;

(c) $y_{\text{CF}} = Ax + B\sin(x)$ is the complementary function for $(1 - x\cot(x))y'' - xy' + y = x$;

(d) $y_{\text{CF}} = A + Bx + Ce^x$ is the complementary function for $y''' - y'' = x$.

Solution:

(a) Computing the derivatives, we have $y'_{\text{CF}} = A + Be^x$, $y''_{\text{CF}} = Be^x$. Substituting, we have:

$$(x-1)Be^x - Ax - Bxe^x + Ax + Be^x = 0,$$

as required.

(b) Computing the derivatives, we have $y'_{\text{CF}} = B/x$, $y''_{\text{CF}} = -B/x^2$. Substituting into the homogeneous version of the equation, we have:

$$x\left(-\frac{B}{x^2}\right) + \frac{B}{x} = 0,$$

as required.

(c) Computing the derivatives, we have $y'_{\text{CF}} = A + B\cos(x)$, $y''_{\text{CF}} = -B\sin(x)$. Substituting into the homogeneous version of the equation, we have:

$$(1 - x\cot(x))(-B\sin(x)) - x(A + B\cos(x)) + Ax + B\sin(x) = 0,$$

as required.

(d) Computing the derivatives, we have $y'_{\text{CF}} = B + Ce^x$, $y''_{\text{CF}} = Ce^x$, $y'''_{\text{CF}} = Ce^x$. Substituting into the homogeneous version of the equation, we have:

$$Ce^x - Ce^x = 0,$$

as required.

3. By direct differentiation, verify that the following ordinary differential equations have the given particular integrals:

   (a) $y_{\mathsf{PI}} = \cos(x)$ is a particular integral for $-y'' + y = 2\cos(x)$;

   (b) $y_{\mathsf{PI}} = x^2$ is a particular integral for $xy'' + y' = 4x$;

   (c) $y_{\mathsf{PI}} = e^{x^2}$ is a particular integral for $y''' - 2xy'' - 2y' - y = (4x-1)e^{x^2}$;

   (d) $y_{\mathsf{PI}} = \sin(x)/x$ is a particular integral for $xy^{(4)} + 4y^{(3)} + xy^{(2)} + 2y^{(1)} + xy = \sin(x)$.

Solution:

(a) We have $y'_{\mathsf{PI}} = -\sin(x), y''_{\mathsf{PI}} = -\cos(x)$, hence $-(-\cos(x)) + \cos(x) + 2\cos(x)$, showing that y_{PI} is indeed a particular integral.

(b) We have $y'_{\mathsf{PI}} = 2x, y''_{\mathsf{PI}} = 2$, hence $2x + 2x = 4x$, showing that y_{PI} is indeed a particular integral.

(c) We have $y'_{\mathsf{PI}} = 2xe^{x^2}, y''_{\mathsf{PI}} = (2+4x^2)e^{x^2}, y'''_{\mathsf{PI}} = (12x+8x^3)e^{x^2}$. Substituting into the equation, we have:

$$(12x+8x^3)e^{x^2} - 2x(2+4x^2)e^{x^2} - 2(2xe^{x^2}) - e^{x^2} = (4x-1)e^{x^2},$$

showing that y_{PI} is indeed a particular integral.

(d) To help, observe that:

$$\frac{d}{dx}\left(\frac{\sin(x)}{x^n}\right) = \frac{\cos(x)}{x^n} - \frac{n\sin(x)}{x^{n+1}}, \qquad \frac{d}{dx}\left(\frac{\cos(x)}{x^n}\right) = -\frac{\sin(x)}{x^n} - \frac{n\cos(x)}{x^{n+1}}.$$

Then, we have:

$$\begin{aligned}
y^{(1)}_{\mathsf{PI}} &= \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \\
y^{(2)}_{\mathsf{PI}} &= -\frac{\sin(x)}{x} - \frac{2\cos(x)}{x^2} + \frac{2\sin(x)}{x^3} \\
y^{(3)}_{\mathsf{PI}} &= -\frac{\cos(x)}{x} + \frac{3\sin(x)}{x^2} + \frac{6\cos(x)}{x^3} - \frac{6\sin(x)}{x^4} \\
y^{(4)}_{\mathsf{PI}} &= \frac{\sin(x)}{x} + \frac{4\cos(x)}{x^2} - \frac{12\sin(x)}{x^3} - \frac{24\cos(x)}{x^4} + \frac{24\sin(x)}{x^5}.
\end{aligned}$$

Summing with the required coefficients, we obtain the result.

4. Verify that the equation:

$$(3+x)y'' + (2+x)y' - y = x^2 + 6x + 6$$

has complementary function $y_{\text{CF}}(x) = Ae^{-x} + B(x+2)$. Hence, by finding a particular integral of the form

$$y_{\text{PI}}(x) = \alpha x^2 + \beta x + \gamma,$$

determine the full solution to the equation subject to the boundary conditions $y(0) = 0$ and $y'(0) = 1$.

➾ **Solution:** Differentiating, we have $y'_{\text{CF}} = -Ae^{-x} + B, y''_{\text{CF}} = Ae^{-x}$. Substituting into the equation, we have:

$$(3+x)Ae^{-x} + (2+x)(-Ae^{-x} + B) - (Ae^{-x} + B(x+2)) = 0,$$

so indeed this is the complementary function. Trialling $y_{\text{PI}} = \alpha x^2 + \beta x + \gamma$ as a particular integral, we have:

$$(3+x)(2\alpha) + (2+x)(2\alpha x + \beta) - (\alpha x^2 + \beta x + \gamma) = x^2 + 6x + 6.$$

Comparing coefficients, we see that $\alpha = 1, 6\alpha = 6$, and $6\alpha + 2\beta - \gamma = 6$. So provided we choose $2\beta - \gamma = 0$, we will have found a particular integral. This corresponds to the freedom we have to add on any multiple of $x+2$ to the particular integral, because $x+2$ is in the complementary function:

$$y_{\text{PI}} = x^2 + \beta(x+2),$$

for arbitrary β. Taking $\beta = 0$, we have the complete solution:

$$y(x) = Ae^{-x} + B(x+2) + x^2.$$

The solution satisfying $y(0) = 0$ and $y'(0) = 1$ must obey $A + 2B = 0$ and $-A + B = 1$. Hence $A = -2/3, B = 1/3$. Thus the particular solution is:

$$y(x) = x^2 + \frac{1}{3}x + \frac{2}{3} - \frac{2}{3}e^{-x}.$$

Constant coefficient equations

5. Consider the linear second-order ordinary differential equation with *constant coefficients*:

$$\alpha\frac{d^2y}{dx^2}+\beta\frac{dy}{dx}+\gamma y=f(x),$$

where α,β,γ are *constants*, with $\alpha\neq 0$.

(a) Show that the equation may be rewritten in the 'factorised' form:

$$\alpha\left(\frac{d}{dx}-\omega_1\right)\left(\frac{d}{dx}-\omega_2\right)y=f(x),$$

where ω_1,ω_2 are the roots of the *auxiliary equation* $\alpha\mu^2+\beta\mu+\gamma=0$.

(b) Deduce that the complementary function of this equation is:

$$y_{\mathsf{CF}}(x)=\begin{cases}Ae^{\omega_1 x}+Be^{\omega_2 x}, & \text{if } \omega_1\neq\omega_2,\\ (A+Bx)e^{\omega x}, & \text{if } \omega_1=\omega_2=\omega.\end{cases}$$

How does this result generalise to an nth order differential equation of this form?

(c) $(*)$ Deduce also that we may construct an analytic particular integral, given by:

$$y_{\mathsf{PI}}(x)=\frac{1}{\alpha}e^{\omega_2 x}\int_{x_0}^{x}\left(e^{(\omega_1-\omega_2)\eta}\int_{\eta_0}^{\eta}e^{-\omega_1\xi}f(\xi)d\xi\right)d\eta,$$

where x_0,η_0 are arbitrary constants. By setting $\eta_0=x_0$ and changing the order of integration in the double integral, deduce the simpler form:

$$y_{\mathsf{PI}}(x)=\begin{cases}\dfrac{1}{\alpha(\omega_1-\omega_2)}\displaystyle\int_{x_0}^{x}\left(e^{\omega_1(x-\xi)}-e^{\omega_2(x-\xi)}\right)f(\xi)\,d\xi, & \text{if } \omega_1\neq\omega_2,\\ \dfrac{1}{\alpha}\displaystyle\int_{x_0}^{x}(x-\xi)e^{\omega(x-\xi)}f(\xi)\,d\xi, & \text{if } \omega_1=\omega_2=\omega.\end{cases}$$

[In practice, it is often just easier to guess a particular integral rather than use this formula, though!]

➼ Solution:

(a) Observe that:

$$\left(\frac{d}{dx}-\omega_1\right)\left(\frac{d}{dx}-\omega_2\right)y=\frac{d^2y}{dx^2}-(\omega_1+\omega_2)\frac{dy}{dx}+\omega_1\omega_2 y.$$

Now, if ω_1,ω_2 are the roots of $\alpha\mu^2+\beta\mu+\gamma=0$, then we have $\omega_1+\omega_2=-\beta/\alpha$ and $\omega_1\omega_2=\gamma/\alpha$. Hence the factorisation of the operator follows.

(b) To obtain the complementary function, set $f(x)=0$. Now let $z=(d/dx-\omega_2)y$. Then we first must solve:

$$\alpha\left(\frac{d}{dx}-\omega_1\right)z=0\qquad\Leftrightarrow\qquad\frac{dz}{dx}=\omega_1 z,$$

which has the solution $z(x) = ce^{\omega_1 x}$, where c is a constant. We now solve the equation $z = (d/dx - \omega_2)y$ for y, that is,

$$\frac{dy}{dx} - \omega_2 y = ce^{\omega_1 x}.$$

Multiplying by the integrating factor $e^{-\omega_2 x}$, we have:

$$\frac{d}{dx}(e^{-\omega_2 x}y) = ce^{(\omega_1 - \omega_2)x}.$$

Integrating both sides directly, we have:

$$e^{-\omega_2 x}y = \begin{cases} \dfrac{c}{\omega_1 - \omega_2}e^{(\omega_1-\omega_2)x} + d, & \text{if } \omega_1 \neq \omega_2, \\ cx + d, & \text{if } \omega_1 = \omega_2. \end{cases}$$

Multiplying both sides by $e^{\omega_2 x}$, and relabelling constants, we obtain:

$$y_{\text{CF}} = \begin{cases} Ae^{\omega_1 x} + Be^{\omega_2 x}, & \text{if } \omega_1 \neq \omega_2, \\ (A + Bx)e^{\omega x}, & \text{if } \omega = \omega_1 = \omega_2, \end{cases}$$

as required. For an nth order equation, whenever we have a m-fold repeated root μ, we get a term

$$(A_0 + A_1 x + ... + A_{m-1}x^{m-1})e^{\mu x};$$

otherwise, if the root is not repeated we get terms $e^{\mu x}$ as usual.

(c) We simply extend part (b) to the case where $f(x) \neq 0$. In this case, our z instead satisfies:

$$\frac{dz}{dx} - \omega_1 z = \frac{f(x)}{\alpha}.$$

Multiplying by the integrating factor $e^{-\omega_1 x}$, we have:

$$\frac{d}{dx}\left(e^{-\omega_1 x}z\right) = \frac{e^{-\omega_1 x}f(x)}{\alpha}.$$

Integrating both sides, we see that a particular integral is:

$$z(\eta) = \frac{1}{\alpha}e^{\omega_1 \eta}\int\limits_{\eta_0}^{\eta} e^{-\omega_1 \xi}f(\xi)\,d\xi,$$

where we have pre-emptively written everything in terms of the variable η instead (we have ignored the usual constant of integration on the right hand side - in fact, this has been absorbed into the lower limit of the integral, η_0). Inserting into the equation for y, we have:

$$\frac{dy}{dx} - \omega_2 y = z(x),$$

which by the same solution method, gives a particular integral:

$$y(x) = e^{\omega_2 x}\int\limits_{x_0}^{x} e^{-\omega_2 \eta}z(\eta)\,d\eta = \frac{1}{\alpha}e^{\omega_2 x}\int\limits_{x_0}^{x}\left(e^{(\omega_1-\omega_2)\eta}\int\limits_{\eta_0}^{\eta} e^{-\omega_1 \xi}f(\xi)\,d\xi\right)\,d\eta,$$

as required.

The constants η_0, x_0 are completely arbitrary, and define different choices of particular integral. Hence, we are free to take $\eta_0 = x_0$. The regions of integration are then $x_0 \leq \eta \leq x, x_0 \leq \xi \leq \eta$. These regions can be rewritten as $x_0 \leq \xi \leq x$ and $\xi \leq \eta \leq x$ (which can also be obtained from a good diagram), yielding the exchanged order of integration:

$$y(x) = \frac{1}{\alpha} e^{\omega_2 x} \int_{x_0}^{x} e^{-\omega_1 \xi} f(\xi) \left(\int_{\xi}^{x} e^{(\omega_1 - \omega_2)\eta} \, d\eta \right) \, d\xi.$$

Performing the integral, we have:

$$\int_{\xi}^{x} e^{(\omega_1 - \omega_2)\eta} \, d\eta = \begin{cases} \dfrac{1}{\omega_1 - \omega_2} \left(e^{(\omega_1 - \omega_2)x} - e^{(\omega_1 - \omega_2)\xi} \right), & \text{if } \omega_1 \neq \omega_2, \\ x - \xi, & \text{if } \omega = \omega_1 = \omega_2. \end{cases}$$

Inserting into the formula we obtained for y, we have the final particular integral:

$$y_{\text{PI}}(x) = \begin{cases} \dfrac{1}{\alpha(\omega_1 - \omega_2)} \displaystyle\int_{x_0}^{x} \left(e^{\omega_1 (x - \xi)} - e^{\omega_2 (x - \xi)} \right) f(\xi) \, d\xi, & \text{if } \omega_1 \neq \omega_2, \\ \dfrac{1}{\alpha} \displaystyle\int_{x_0}^{x} (x - \xi) e^{\omega (x - \xi)} f(\xi) \, d\xi, & \text{if } \omega_1 = \omega_2 = \omega. \end{cases}$$

as required.

6. Determine the solutions of the following differential equations:

(a) $y'' + 6y' + 5y = 0$;

(b) $y'' + 3y' + 4y = 0$;

(c) $y'' + 4y = x$;

(d) $y'' - 2y' + 2y = 2x^2$;

(e) $y'' + y = |x|$;

(f) $y'' + 3y' + 2y = e^{-x}$;

(g) $y'' - 2y' + 5y = e^x \cos(2x)$;

(h) $y'' + 2y' + y = 2xe^{-x}$.

Solution:

(a) The auxiliary equation is $\mu^2 + 6\mu + 5 = 0$, with roots $-5, -1$. Hence the solution is $y = Ae^{-5x} + Be^{-x}$.

(b) The auxiliary equation is $\mu^2 + 3\mu + 4 = 0$, which can be solved by the quadratic equation to give:

$$\mu = \frac{-3 \pm i\sqrt{7}}{2}.$$

Hence the solution is:

$$y = Ae^{-(3+i\sqrt{7})x/2} + Be^{-(3-i\sqrt{7})x/2}.$$

This can be simplified as:

$$\begin{aligned} y &= Ae^{-3x/2}e^{-i\sqrt{7}x/2} + Be^{-3x/2}e^{i\sqrt{7}x/2} \\ &= Ae^{-3x/2}\left(\cos(\sqrt{7}x/2) - i\sin(\sqrt{7}x/2)\right) + Be^{-3x/2}\left(\cos(\sqrt{7}x/2) + i\sin(\sqrt{7}x/2)\right) \\ &= A'e^{-3x/2}\cos(\sqrt{7}x/2) + B'e^{-3x/2}\sin(\sqrt{7}x/2), \end{aligned}$$

where we have defined $A' = A + B$ and $B' = -iA + iB$ as new constants. This is the more standard way of writing the solution.

(c) The auxiliary equation is $\mu^2 + 4 = 0$, with roots $\pm 2i$. Hence the complementary function is $y_{\mathsf{CF}} = Ae^{2ix} + Be^{-2ix} = A'\cos(2x) + B'\sin(2x)$.

We now seek a particular integral, which we guess is of the form $y_{\mathsf{PI}} = \alpha x + \beta$. We then have:

$$4(\alpha x + \beta) = x,$$

giving us $\alpha = 1/4, \beta = 0$. Thus the complete solution is:

$$y = A\cos(2x) + B\sin(2x) + \frac{1}{4}x.$$

(d) The auxiliary equation is $\mu^2 - 2\mu + 2 = 0$, which has roots $\mu = 2 \pm i$. Hence the complementary function is $y_{\mathsf{CF}} = e^{2x}\left(A\cos(x) + B\sin(x)\right)$.

We now seek a particular integral, which we guess is of the form $y_{\mathsf{PI}} = \alpha x^2 + \beta x + \gamma$. We then have:

$$2\alpha - 2(2\alpha x + \beta) + 2(\alpha x^2 + \beta x + \gamma) = 2x^2.$$

Comparing coefficients, we see that $\alpha = 1, \beta = 2, \gamma = 1$. Thus the complete solution is:

$$y = e^{2x}\left(A\cos(x) + B\sin(x)\right) + x^2 + 2x + 1.$$

(e) The auxiliary equation is $\mu^2 + 1 = 0$, which has roots $\mu = \pm i$. Hence the complementary function is $y_{\text{CF}} = A\cos(x) + B\sin(x)$.

We now seek a particular integral. Since the equation involves a modulus sign, we consider the regions $x > 0$ and $x < 0$ separately.

- In the region $x > 0$, guess $y_{\text{PI}} = \alpha x + \beta$. Then:
$$\alpha x + \beta = x,$$
giving $\alpha = 1, \beta = 0$.
- In the region $x < 0$, guess $y_{\text{PI}} = \alpha x + \beta$. Then:
$$\alpha x + \beta = -x,$$
giving $\alpha = -1, \beta = 0$.

So the particular integral is x when $x > 0$ and $-x$ when $x < 0$. This gives $y_{\text{PI}} = |x|$. Hence the complete solution is:
$$y = A\cos(x) + B\sin(x) + |x|.$$

(f) The auxiliary equation is $\mu^2 + 3\mu + 2 = 0$, which has roots $\mu = 2, 1$. Thus the complementary function is $y_{\text{CF}} = Ae^{-x} + Be^{-2x}$. Since e^{-x} is already contained in the complementary function, as a particular integral we trial $y_{\text{PI}} = \alpha x e^{-x}$. This gives:
$$\alpha(-2e^{-x} + xe^{-x}) + 3\alpha(e^{-x} - xe^{-x}) + 2\alpha x e^{-x} = e^{-x}.$$
Comparing coefficients, we see that $3\alpha - 2\alpha = 1$, so that $\alpha = 1$. Thus we have complete solution:
$$y = Ae^{-x} + Be^{-2x} + xe^{-x}.$$

(g) The auxiliary equation is $\mu^2 - 2\mu + 5 = 0$, which has roots $\mu = 1 \pm 2i$. Thus the complementary function is $y_{\text{CF}} = e^x(A\cos(2x) + B\sin(2x))$. Since the forcing is contained in the complementary function, for the particular integral we guess $y_{\text{PI}} = xe^x(\alpha\cos(2x) + \beta\sin(2x))$. Then:
$$y_{\text{PI}}' = e^x(\alpha\cos(2x) + \beta\sin(2x)) + xe^x\left((\alpha + 2\beta)\cos(2x) + (\beta - 2\alpha)\sin(2x)\right)$$
$$y_{\text{PI}}'' = e^x((2\alpha + 4\beta)\cos(2x) + (2\beta - 4\alpha)\sin(2x)) + xe^x\left((-3\alpha + 4\beta)\cos(2x) + (-4\alpha - 3\beta)\sin(2x)\right).$$
Combining these according to the equation, we see that:
$$4e^x\left(\beta\cos(2x) - \alpha\sin(2x)\right) = e^x\cos(2x).$$
Comparing coefficients, we see that $4\beta = 1$ and $-4\alpha = 0$, giving $\beta = 1/4$. Thus the general solution is:
$$y = e^x(A\cos(2x) + B\sin(2x)) + \frac{1}{4}xe^x\sin(2x).$$

(h) The auxiliary equation is $\mu^2 + 2\mu + 1 = (\mu + 1)^2 = 0$, which has a repeated root $\mu = -1$. This gives the complementary function $y_{\text{CF}} = (A + Bx)e^{-x}$.

Since xe^{-x} is already contained in the complementary function, we now consider x^2 times the complementary function as a particular integral. This gives $y_{\text{PI}} = (\alpha x^2 + \beta x^3)e^{-x}$, which on differentiation yields:
$$y_{\text{PI}}' = (2\alpha x + 3\beta x^2)e^{-x} - (\alpha x^2 + \beta x^3)e^{-x}$$
$$y_{\text{PI}}'' = (2\alpha + 6\beta x)e^{-x} - (4\alpha x + 6\beta x^2)e^{-x} + (\alpha x^2 + \beta x^3)e^{-x}.$$
Combining these according to the equation, we see that:
$$\alpha = 0, \qquad \beta = \frac{1}{3}.$$
Hence the complete solution is:
$$y_{\text{CF}} = (A + Bx)e^{-x} + \frac{1}{3}x^3e^{-x}.$$

7. Determine the solutions of the following differential equations subject to the given constraints:

(a) $y'' - 4y' + 13y = 0$, subject to $y(0) = \pi$ and $y(-\pi/2) = 1$;

(b) $y'' - 4y' + 5y = 125x^2$, subject to $y(0) = 1$ and $y\left(\frac{\pi}{2}\right) = \frac{25\pi^2}{4} + 20\pi + 22$;

(c) $y'' + 7y' + 12y = 6$, subject to $y(0) = 0$ and $y\left(\frac{1}{3}\right) = \frac{1-e^{-1}}{2}$;

(d) $y'' + 7y' + 12y = 2e^{-3x}$, subject to $y(0) = 1$ and $y'(0) = 0$.

Solution:

(a) The auxiliary equation is $\mu^2 - 4\mu + 13 = 0$, with roots $\mu = 2 \pm 3i$, which gives the complementary function $y_{\mathsf{CF}} = e^{2x}(A\cos(3x) + B\sin(3x))$. Imposing the boundary data, we have $A = \pi$ and:

$$1 = -Be^{-\pi} \qquad \Rightarrow \qquad B = -e^{\pi}.$$

Hence the particular solution is $y(x) = e^{2x}(\pi\cos(3x) - e^{\pi}\sin(3x))$.

(b) The auxiliary equation is $\mu^2 - 4\mu + 5 = 0$, with roots $\mu = 2 \pm i$, hence the complementary function is $y_{\mathsf{CF}} = e^{2x}(A\cos(x) + B\sin(x))$. Trialling $y_{\mathsf{PI}} = \alpha x^2 + \beta x + \gamma$ as a particular integral, we have:

$$2\alpha - 4(2\alpha x + \beta) + 5(\alpha x^2 + \beta x + \gamma) = 125x^2.$$

Comparing coefficients, we see that $\alpha = 25$, $\beta = 40$ and $\gamma = 22$. Hence we have the complete solution:

$$y(x) = e^{2x}\left(A\cos(x) + B\sin(x)\right) + 25x^2 + 40x + 22.$$

Imposing $y(0) = 1$, we have:

$$1 = A + 22 \qquad \Rightarrow \qquad A = -21.$$

Imposing $y(\pi/2) = 25\pi^2/4 + 20\pi + 22$, we have:

$$\frac{25\pi^2}{4} + 20\pi + 22 = e^{\pi}B + \frac{25\pi^2}{4} + 20\pi + 22 \qquad \Rightarrow \qquad B = 0.$$

Hence the particular solution is:

$$y(x) = 25x^2 + 40x + 22 - 21e^{2x}\cos(x).$$

(c) The auxiliary equation is $\mu^2 + 7\mu + 12 = 0$, which has roots $\mu = -3, -4$. Hence the complementary function is $y_{\mathsf{CF}} = Ae^{-3x} + Be^{-4x}$. The particular integral is obviously $y_{\mathsf{PI}} = 1/2$. This gives the complete solution:

$$y(x) = Ae^{-3x} + Be^{-4x} + \frac{1}{2}.$$

Imposing the boundary conditions, we have $y(0) = 0$ implying $0 = A + B + 1/2$. We also have $y(1/3) = (1 - e^{-1})/2$ implying:

$$\frac{1 - e^{-1}}{2} = Ae^{-1} + Be^{-4/3} + \frac{1}{2}.$$

Comparing, we see that $B = 0$, $A = -1/2$. This gives the solution:

$$y(x) = \frac{1}{2}\left(1 - e^{-3x}\right).$$

(d) This part has the same complementary function as the previous part, $y_{\mathsf{CF}} = Ae^{-3x} + Be^{-4x}$, but the particular integral should now be $y_{\mathsf{PI}} = \alpha xe^{-3x}$. Inserting this into the equation, we have:

$$-6\alpha e^{-3x} + 9\alpha xe^{-3x} + 7\alpha e^{-3x} - 21\alpha xe^{-3x} + 12\alpha xe^{-3x} = 2e^{-3x}$$

we see that $\alpha = 2$, giving $\alpha = 2$, hence the complete solution is:

$$y(x) = Ae^{-3x} + Be^{-4x} + 2xe^{-3x}.$$

Imposing the boundary conditions, we have $1 = A + B$ and $0 = -3A - 4B + 2$. These simultaneous equations have solution $A = 2, B = -1$. Hence the solution is:

$$y(x) = 2e^{-3x} - e^{-4x} + 2xe^{-3x}.$$

---

8. Find the value of a for which the complementary function of the ODE:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + ay = 0,$$

is given by $y_{\mathsf{CF}} = Axe^{-2x} + Be^{-2x}$.

➪ **Solution:** The auxiliary equation is $\mu^2 + 4\mu + a = 0$, which has roots:

$$\mu = \frac{-4 \pm \sqrt{16 - 4a}}{2} = -2 \pm \sqrt{4 - a}.$$

There is a repeated root when $a = 4$, which gives rise to the solution $y_{\mathsf{CF}} = (Ax + B)e^{-2x}$.

9. Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + y = \cos(kx),$$

where k is a real number.

➾ **Solution:** The auxiliary equation is $\mu^2 + 1 = 0$, with roots $\mu = \pm i$. This gives complementary function $y_{\text{CF}} = A\cos(x) + B\sin(x)$.

In the case where $k \neq \pm 1$, a particular integral is obviously $\cos(kx)/(1 - k^2)$. Hence the complete solution is:

$$y = A\cos(x) + B\sin(x) + \frac{\cos(kx)}{1 - k^2}.$$

When $k = \pm 1$, we need a different particular integral. We guess $y_{\text{PI}} = \alpha x\cos(x) + \beta x\sin(x)$. Then:

$$y'_{\text{PI}} = \alpha\cos(x) + \beta\sin(x) - \alpha x\sin(x) + \beta x\cos(x),$$

$$y''_{\text{PI}} = -2\alpha\sin(x) + 2\beta\cos(x) - \alpha x\cos(x) - \beta x\sin(x).$$

Comparing coefficients, we see that $\alpha = 0$ and $\beta = 1/2$. Thus the complete solution is:

$$y = A\cos(x) + B\sin(x) + \frac{1}{2}x\sin(x).$$

For those who are interested, this can be considered the limit as $k \to 1$ of the previous case. Rewriting the constants in the form:

$$y = A'\cos(x) + B\sin(x) + \frac{\cos(kx) - \cos(x)}{1 - k^2}.$$

Taking the limit as $k \to 1$, we have:

$$\lim_{k\to 1}\frac{\cos(kx) - \cos(x)}{1 - k^2} = \lim_{k\to 1}\frac{-x\sin(kx)}{-2k} = \frac{1}{2}x\sin(x),$$

using L'Hôpital's rule in the final step. Nice that we can do this, huh!

10. The differential operator $\mathcal{L}$ is defined by:

$$\mathcal{L} = \frac{d^2}{dx^2} + \sqrt{3}\frac{d}{dx} + 3.$$

Solve the equation $\mathcal{L}y = 0$, and hence solve the equations:

(a) $\mathcal{L}y = e^{-\sqrt{3}x}$;

(b) $\mathcal{L}y = x$.

Without further calculation, state the general solution of $\mathcal{L}y = 2x + e^{-\sqrt{3}x}$. Find also the solution of this equation satisfying the boundary conditions:

$$y(0) = 0, \qquad y(\pi) = \frac{e^{-\sqrt{3}\pi}}{3} - \frac{2}{3\sqrt{3}}.$$

Solution: The solution to the equation $\mathcal{L}y = 0$ is just the complementary function of the equation $y'' + \sqrt{3}y' + 3y = 0$. The auxiliary equation is $\mu^2 + \sqrt{3}\mu + 3 = 0$, which has roots $\mu = (-\sqrt{3} \pm 3i)/2$, hence the solution to $\mathcal{L}y = 0$ is:

$$y(x) = e^{-\sqrt{3}x/2}\left(A\cos(3x/2) + B\sin(3x/2)\right).$$

In the given cases, we have:

(a) A particular integral for $\mathcal{L}y = e^{-\sqrt{3}x}$ is $y_{\text{PI}} = \alpha e^{-\sqrt{3}x}$. Substituting into the equation, we have:

$$3\alpha e^{-\sqrt{3}x} - 3\alpha e^{-\sqrt{3}x} + 3\alpha e^{-\sqrt{3}x} = e^{-\sqrt{3}x},$$

so that $\alpha = 1/3$. Thus the general solution is:

$$y(x) = e^{-\sqrt{3}x/2}\left(A\cos(3x/2) + B\sin(3x/2)\right) + \frac{1}{3}e^{-\sqrt{3}x}.$$

(b) A particular integral for $\mathcal{L}y = x$ is $y_{\text{PI}} = \alpha x + \beta$, which gives:

$$\sqrt{3}\alpha + 3\alpha x + 3\beta = x.$$

This implies $\alpha = 1/3$, and $\sqrt{3}/3 + 3\beta = 0$, so that $\beta = -1/3\sqrt{3}$. Thus the general solution is:

$$y(x) = e^{-\sqrt{3}x/2}\left(A\cos(3x/2) + B\sin(3x/2)\right) + \frac{1}{3}x - \frac{1}{3\sqrt{3}}.$$

The general solution of the equation $\mathcal{L}y = 2x + e^{-\sqrt{3}x}$ is now given by linearity as:

$$y(x) = e^{-\sqrt{3}x/2}\left(A\cos(3x/2) + B\sin(3x/2)\right) + \frac{1}{3}e^{-\sqrt{3}x} + \frac{2}{3}x - \frac{2}{3\sqrt{3}}.$$

Imposing the boundary condition $y(0) = 0$, we have:

$$0 = A + \frac{1}{3} - \frac{2}{3\sqrt{3}} \qquad \Rightarrow \qquad A = \frac{2}{3\sqrt{3}} - \frac{1}{3}.$$

Imposing the boundary condition at $x = \pi$, we have:

$$\frac{e^{-\sqrt{3}\pi}}{3} - \frac{2}{3\sqrt{3}} = -Be^{-\sqrt{3}\pi/2} + \frac{1}{3}e^{-\sqrt{3}\pi} + \frac{2}{3}\pi - \frac{2}{3\sqrt{3}} \qquad \Rightarrow \qquad B = -\frac{2}{3}\pi e^{\sqrt{3}\pi/2}.$$

The particular solution is therefore:

$$y(x) = e^{-\sqrt{3}x/2}\left(\left(\frac{2}{3\sqrt{3}} - \frac{1}{3}\right)\cos(3x/2) - \frac{2}{3}\pi e^{\sqrt{3}\pi/2}\sin(3x/2)\right) + \frac{1}{3}e^{-\sqrt{3}x} + \frac{2}{3}x - \frac{2}{3\sqrt{3}}$$

Harmonic oscillators

11. Consider the constant coefficient linear second-order ordinary differential equation:

$$\frac{d^2y}{dt^2} + 2\gamma\frac{dy}{dt} + \omega_0^2 y = f(t),$$

modelling an oscillating system which depends on time t. The coefficients γ, ω_0 are positive.

(a) What is the physical interpretation of the constant γ? What is the physical interpretation of the function $f(t)$?

(b) Find the complementary function of this equation. Discuss the different forms the complementary function can take (in particular, defining the terms *underdamping*, *critical damping*, and *overdamping*), and how this relates to the *transient* behaviour of the oscillator.

(c) In the underdamped case, find the long-term behaviour of the oscillator in the case of resonant forcing:

$$f(t) = e^{-\gamma t}\sin\left(t\sqrt{\omega_0^2-\gamma^2}\right).$$

➾ Solution:

(a) γ is the strength of the damping of the oscillator. The function $f(t)$ is the external driving force exerted on the oscillator per unit mass.

(b) The auxiliary equation is:

$$\mu^2 + 2\gamma\mu + \omega_0^2 = 0 \qquad \Rightarrow \qquad \mu = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2-\omega_0^2}.$$

- If $\gamma^2 > \omega_0^2$, then the complementary function takes the form:

$$y_{\mathsf{CF}} = Ae^{(-\gamma+\sqrt{\gamma^2-\omega_0^2})t} + Be^{(-\gamma-\sqrt{\gamma^2-\omega_0^2})t}.$$

  This is the case of *overdamping*. The displacement y tends to zero exponentially without oscillation. In the general case, there are two modes of decay, with the slower decay happening at a rate $-\gamma + \sqrt{\gamma^2-\omega_0^2}$ per second.

- If $\gamma^2 = \omega_0^2$, then the complementary function takes the form:

$$y_{\mathsf{CF}} = (A+Bt)e^{-\gamma t}.$$

  This is the case of *critical damping*. The displacement y tends to zero exponentially without oscillation. There is only one mode of decay with rate γ, so that for general initial conditions, the decay is faster than the case of overdamping.

- If $\gamma^2 < \omega_0^2$, then the complementary function takes the form:

$$y_{\mathsf{CF}} = e^{-\gamma t}\left(A\cos\left(t\sqrt{\omega_0^2-\gamma^2}\right) + B\sin\left(t\sqrt{\omega_0^2-\gamma^2}\right)\right).$$

  This is the case of *underdamping*. The displacement y tends to zero exponentially with oscillation.

(c) In this case, the complementary function contains the forcing, so we should trial a solution of the form

$$y_{\mathsf{PI}} = te^{-\gamma t}\left(\alpha\cos\left(t\sqrt{\omega_0^2-\gamma^2}\right) + \beta\sin\left(t\sqrt{\omega_0^2-\gamma^2}\right)\right)$$

Differentiating, we have:

$$
\begin{aligned}
y'_{\mathsf{PI}} = e^{-\gamma t}\Big[&(1-\gamma t)\left(\alpha\cos\left(t\sqrt{\omega_0^2-\gamma^2}\right)+\beta\sin\left(t\sqrt{\omega_0^2-\gamma^2}\right)\right) \\
&+t\sqrt{\omega_0^2-\gamma^2}\left(-\alpha\sin\left(t\sqrt{\omega_0^2-\gamma^2}\right)+\beta\cos\left(t\sqrt{\omega_0^2-\gamma^2}\right)\right)\Big].
\end{aligned}
$$

and:

$$
\begin{aligned}
y''_{\mathsf{PI}} = e^{-\gamma t}\Big\{&\left[-2\gamma+t\left(2\gamma^2-\omega_0^2\right)\right]\left(\alpha\cos\left(t\sqrt{\omega_0^2-\gamma^2}\right)+\beta\sin\left(t\sqrt{\omega_0^2-\gamma^2}\right)\right) \\
&+2(1-\gamma t)\sqrt{\omega_0^2-\gamma^2}\left(-\alpha\sin\left(t\sqrt{\omega_0^2-\gamma^2}\right)+\beta\cos\left(t\sqrt{\omega_0^2-\gamma^2}\right)\right)\Big\}.
\end{aligned}
$$

Putting everything together, we have:

$$
2\sqrt{\omega_0^2-\gamma^2}\,e^{-\gamma t}\left(-\alpha\sin\left(t\sqrt{\omega_0^2-\gamma^2}\right)+\beta\cos\left(t\sqrt{\omega_0^2-\gamma^2}\right)\right)=e^{-\gamma t}\sin\left(t\sqrt{\omega_0^2-\gamma^2}\right).
$$

Comparing coefficients, we see that $\beta = 0$ and:

$$
\alpha = -\frac{1}{2\sqrt{\omega_0^2-\gamma^2}}.
$$

Hence the dominant long-term behaviour of the oscillator in this case is:

$$
y_{\mathsf{PI}} = -\frac{1}{2\sqrt{\omega_0^2-\gamma^2}}te^{-\gamma t}\cos\left(t\sqrt{\omega_0^2-\gamma^2}\right),
$$

which is an exponentially decaying curve, but one that is enhanced by the linear polynomial term. Note also that it is a cosine response to a sine forcing - there is a $\pi/2$ phase shift between the forcing and response of the oscillator.

Coupled systems of differential equations

12.

(a) Consider the system of differential equations:

$$\frac{dx}{dt} = ax + by + p, \qquad \frac{dy}{dt} = cx + dy + q,$$

for the variables $x(t), y(t)$, where a, b, c, d, p, q are constants. Show that:

$$\frac{d^2x}{dt^2} = (a+d)\frac{dx}{dt} + (bc - ad)x + bq - pd.$$

(b) Hence:

(i) Find the general solution of the system:

$$\frac{dx}{dt} = 4y + 2, \qquad \frac{dy}{dt} = x.$$

(ii) Solve the system:

$$\frac{dx}{dt} = 3x - y, \qquad \frac{dy}{dt} = x + y,$$

subject to the initial conditions $x(0) = 0$ and $y(0) = 1$.

(iii) Solve the system:

$$\frac{dx}{dt} = -3x + y, \qquad \frac{dy}{dt} = -5x + y,$$

subject to the initial conditions $x(0) = 1, y(0) = 1$.

Solution:

(a) Differentiating the first equation, we have:

$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b\frac{dy}{dt}.$$

Multiplying the second equation through by b, and inserting it into this equation, we have:

$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + bcx + bdy + bq.$$

Finally, note from the original first equation that $by = -dx/dt - ax - p$. So substituting for by, we have:

$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + bcx - d\frac{dx}{dt} - adx - dp + bq.$$

Grouping terms, we have:

$$\frac{d^2x}{dt^2} = (a+d)\frac{dx}{dt} + (bc - ad)x + bq - pd,$$

as required.

(b) (i) Here, we have $a = 0, b = 4, p = 2$ and $c = 1, d = 0, q = 0$. Thus the second-order equation we derived for x becomes:

$$\frac{d^2x}{dt^2} = 4x.$$

This has auxiliary equation $\mu^2 - 4 = 0$ with roots $\mu = \pm 2$, hence the general solution of this equation for x is:

$$x(t) = Ae^{-2t} + Be^{2t}.$$

We must now insert this back into the original system for two reasons: (i) to check that we actually do solve the system for all A, B and that they are not fixed by some other conditions; (ii) to find the expression for y. Inserting back into the first original equation, we can derive an expression for y:

$$-2Ae^{-2t} + 2Be^{2t} = 4y + 2 \qquad \Rightarrow \qquad y(t) = -\frac{1}{2}Ae^{-2t} + \frac{1}{2}Be^{2t} - \frac{1}{2}.$$

It remains to check the consistency with the original second equation. Inserting, we have:

$$Ae^{-2t} + Be^{2t} = Ae^{-2t} + Be^{2t}.$$

So, all is well. Thus the general solution is:

$$x(t) = Ae^{-2t} + Be^{2t}, \qquad y(t) = -\frac{1}{2}Ae^{-2t} + \frac{1}{2}Be^{2t} - \frac{1}{2},$$

for arbitrary constants A, B.

(ii) We employ a similar method. In this case, we have $a = 3, b = -1, p = 0$ and $c = 1, d = 1, q = 0$. Thus the second order equation for x becomes:

$$\frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 4x.$$

The auxiliary equation is $\mu^2 - 4\mu + 4 = 0$, which has a repeated roots $\mu = 2$. Thus the solution for $x(t)$ is $x(t) = (A + Bt)e^{2t}$. Imposing the boundary data $x(0) = 0$, we see that $x(t) = Bte^{2t}$.

Inserting into the first original equation, we can determine $y(t)$. We have:

$$2Bte^{2t} + Be^{2t} = 3Bte^{2t} - y \qquad \Rightarrow \qquad y(t) = Bte^{2t} - Be^{2t}.$$

Imposing $y(0) = 1$, we see that $B = -1$, so that $y(t) = (1 - t)e^{2t}$. Finally, checking the consistency with the second original equation, we see that:

$$-e^{2t} + 2(1 - t)e^{2t} = -te^{2t} + (1 - t)e^{2t},$$

as anticipated. Hence the solution is:

$$x(t) = -te^{2t}, \qquad y(t) = (1 - t)e^{2t}.$$

(iii) For our final system, we have $a = -3, b = 1, p = 0$ and $c = -5, d = 1, q = 0$. Thus the second order equation for x is:

$$\frac{d^2x}{dt^2} = -2\frac{dx}{dt} - 2x.$$

The auxiliary equation is $\mu^2 + 2\mu + 2 = 0$, which has roots $\mu = -1 \pm i$. Hence, we have:

$$x(t) = e^{-t}\left(A\cos(t) + B\sin(t)\right).$$

Imposing the boundary condition $x(0) = 1$, we see that $A = 1$. Differentiating $x(t)$ to obtain $y(t)$ from the first original equation, we have:

$$-e^{-t}\left(\cos(t) + B\sin(t)\right) + e^{-t}\left(-\sin(t) + B\cos(t)\right) = -3e^{-t}\left(\cos(t) + B\sin(t)\right) + y(t).$$

Rearranging, we see that:

$$y(t) = e^{-t}\left((B+2)\cos(t) + (2B-1)\sin(t)\right).$$

Imposing $y(0) = 1$, we see that $B = -1$. Thus the complete solution is:

$$x(t) = e^{-t}\left(\cos(t) - \sin(t)\right),$$
$$y(t) = e^{-t}\left(\cos(t) - 3\sin(t)\right).$$

($*$) Equidimensional equations

13. Consider a linear second-order ordinary differential equation with *non-constant coefficients*:

$$\alpha x^2 \frac{d^2y}{dx^2} + \beta x \frac{dy}{dx} + \gamma y = f(x),$$

where α, β, γ are *constants*, with $\alpha \neq 0$. This type of equation is called an *equidimensional equation*. If you are doing Part IA Physics, suggest a reason for this name.

(a) Show that the equation may be written in the form:

$$\alpha \left(x\frac{d}{dx} - \omega_1 \right) \left(x\frac{d}{dx} - \omega_2 \right) y = f(x),$$

where ω_1, ω_2 are the roots of the *auxiliary equation* $\alpha\mu(\mu-1) + \beta\mu + \gamma = 0$.

(b) Deduce that the complementary function of this equation is:

$$y_{\mathsf{CF}}(x) = \begin{cases} Ax^{\omega_1} + Bx^{\omega_2}, & \text{if } \omega_1 \neq \omega_2, \\ (A + B\log(x))x^{\omega}, & \text{if } \omega_1 = \omega_2 = \omega. \end{cases}$$

How does this result generalise to an nth order differential equation of this form?

Solution:

(a) Expanding the differential operator on the left hand side, we have:

$$\alpha \left(x\frac{d}{dx} - \omega_1 \right) \left(x\frac{dy}{dx} - \omega_2 y \right) = \alpha x^2 \frac{d^2y}{dx^2} - x(\omega_1 + \omega_2 - 1) + \alpha\omega_1\omega_2 y.$$

Comparing coefficients, we have $\alpha(\omega_1 + \omega_2 - 1) = -\beta$, which can be rewritten as $\omega_1 + \omega_2 = -\beta/\alpha + 1$ and $\alpha\omega_1\omega_2 = \gamma$. This shows that ω_1, ω_2 are the roots of the quadratic equation:

$$\mu^2 + \left(\frac{\beta}{\alpha} - 1\right)\mu + \frac{\gamma}{\alpha} = 0.$$

Rearranging, we have:

$$\alpha\mu(\mu - 1) + \beta\mu + \gamma = 0,$$

as required.

(b) Setting $f(x) = 0$, we first solve the equation:

$$\left(x\frac{d}{dx} - \omega_1 \right) z = 0 \qquad \Rightarrow \qquad x\frac{dz}{dx} = \omega_1 z.$$

Separating variables and integrating, we have:

$$\log(z) = \omega_1 \log(x) + c \qquad \Rightarrow \qquad z(x) = Ax^{\omega_1}.$$

To obtain $y(x)$, we now solve the equation:

$$\left(x\frac{d}{dx} - \omega_2 \right) y = Ax^{\omega_1} \qquad \Rightarrow \qquad \frac{dy}{dx} - \frac{\omega_2}{x} y = Ax^{\omega_1 - 1}.$$

The integrating factor is evidently $e^{-\omega_2 \log(x)} = x^{-\omega_2}$. Hence we have:

$$\frac{d}{dx}\left(x^{-\omega_2}y\right) = Ax^{\omega_1-\omega_2-1}.$$

Integrating both sides directly, we have:

$$x^{-\omega_2}y = \begin{cases} \frac{A}{\omega_1-\omega_2}x^{\omega_1-\omega_2} + B, & \text{if } \omega_1 \neq \omega_2, \\ A\log(x) + B, & \text{if } \omega = \omega_1 = \omega_2. \end{cases}$$

Multiplying across by x^{ω_2} and renaming constants, we obtain precisely the result stated in the question.

For an nth order equation, we get a polynomial of degree $m-1$ in $\log(x)$ multiplied by x^{ω} if ω is an m-fold repeated root.

---

14. Using the results of Question 13, determine the solutions of the following differential equations:
  (a) $x^2y'' - 2xy' + y = 0$, subject to the initial data $y(1) = 1, y'(1) = 0$;
  (b) $x^2y'' - xy' + y = x^2$, subject to the initial data $y(1) = 2, y'(1) = 3$;
  (c) $x^2y'' - xy' + y = x\log(x)$, subject to the initial data $y(1) = 0, y'(1) = 1$.

Solution:

(a) The auxiliary equation is $\mu(\mu-1) - 2\mu + 1 = \mu^2 - 3\mu + 1 = 0$. This has roots $\mu = \frac{1}{2}(3 \pm \sqrt{5})$. There is no particular integral, so the general solution is:

$$y(x) = Ax^{(3+\sqrt{5})/2} + Bx^{(3-\sqrt{5})/2}.$$

Requiring $y(1) = 1$ and $y'(1) = 0$, we see that:

$$A + B = 1, \qquad A\left(\frac{3+\sqrt{5}}{2}\right) + B\left(\frac{3-\sqrt{5}}{2}\right) = 0.$$

These equations have solution:

$$A = \frac{5-3\sqrt{5}}{10}, \qquad B = \frac{5+3\sqrt{5}}{10}.$$

Overall, we have the solution:

$$y(x) = \left(\frac{5-3\sqrt{5}}{10}\right)x^{(3+\sqrt{5})/2} + \left(\frac{5+3\sqrt{5}}{10}\right)x^{(3-\sqrt{5})/2}.$$

(b) The auxiliary equation is $\mu(\mu-1) - \mu + 1 = \mu^2 - 2\mu + 1 = 0$, with repeated root $\mu = 1$. Hence the complementary function is $y_{\text{CF}} = Ax\log(x) + Bx$. Guessing $y_{\text{PI}} = \alpha x^2 + \beta x + \gamma$ for the particular integral, we have:

$$2\alpha - (2\alpha x + \beta) + \alpha x^2 + \beta x + \gamma = x^2.$$

Comparing coefficients, we see that $\alpha = 1, \beta = 2$, and $\gamma = 0$. Thus the complete general solution is:

$$y(x) = Ax\log(x) + Bx + x^2 + 2x.$$

Imposing the boundary conditions, we have $y(1) = 2$, which gives $B + 3 = 2$, hence $B = -1$. We also have:

$$y'(1) = A + B + 2 + 2 = 3 \qquad \Rightarrow \qquad A = 0.$$

Hence the solution is $y(x) = x^2 + x$.

(c) In this part, the complementary function remains the same as the previous part, $y_{\mathsf{CF}} = Ax\log(x) + Bx$. Our particular integral guess instead becomes $y_{\mathsf{PI}} = \alpha x\log^3(x) + \beta x\log^2(x)$ (since multiplication by $\log(x)$ of y_{CF} still contains terms in the complementary function). Differentiating, we have:

$$y'_{\mathsf{PI}} = \alpha\left(\log^3(x) + 3\log^2(x)\right) + \beta(\log^2(x) + 2\log(x)),$$

$$y''_{\mathsf{PI}} = \frac{1}{x}\left[\alpha\left(3\log^2(x) + 6\log(x)\right) + 2\beta(\log(x) + 1)\right].$$

Combining these derivatives according to the equation $x^2 y''_{\mathsf{PI}} - xy'_{\mathsf{PI}} + y_{\mathsf{PI}} = x\log(x)$, we see that the $\log^3(x)$ and $\log^2(x)$ terms cancel, leaving us with the simultaneous equations:

$$6\alpha = 1, \qquad 2\beta = 0.$$

Thus the particular integral is $y_{\mathsf{PI}} = \frac{1}{6}x\log^3(x)$. As a result, we have the general solution:

$$y(x) = Ax\log(x) + Bx + \frac{1}{6}x\log^3(x).$$

Imposing the data $y(1) = 0$, we see that $B = 0$. Imposing the data $y'(1) = 1$, we have $A = 1$, hence the solution is:

$$y(x) = x\log(x) + \frac{1}{6}x\log^3(x).$$