# Part IA: Mathematics for Natural Sciences B
# Examples Sheet 3: Complex numbers and hyperbolic functions

## Model Solutions

*Please send all comments and corrections to jmm232@cam.ac.uk.*

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### Real and imaginary parts

1. Find the real and imaginary parts of the following numbers (where $n$ is an integer):

(a) $i^3$, (b) $i^{4n}$, (c) $\left(\frac{1+i}{\sqrt{2}}\right)^2$, (d) $\left(\frac{1-i}{\sqrt{2}}\right)^2$, (e) $\left(\frac{1+\sqrt{3}i}{2}\right)^3$, (f) $\frac{1+i}{2-5i}$, (g) $\left(\frac{1+i}{1-i}\right)^2$.

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**Solution:**

(a) We have $i^3 = i^2 \cdot i = -i$. Hence the real part is $1$ and the imaginary part is $-1$ (recall that the imaginary part does *not* include $i$).

(b) We have $i^{4n} = (i^2)^{2n} = (-1)^{2n} = 1$, since an even power of $-1$ is always $1$. Hence the real part is $1$ and the imaginary part is $0$.

(c) Carrying out the multiplication, we have:

$$\left(\frac{1+i}{\sqrt{2}}\right)^2 = \left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{1+i}{\sqrt{2}}\right) = \frac{1+2i-1}{2} = i.$$

Thus, the real part is $0$ and the imaginary part is $1$. In particular, we learn that $\pm(1+i)/\sqrt{2}$ are the square roots of $i$.

(d) Carrying out the multiplication, we have:

$$\left(\frac{1-i}{\sqrt{2}}\right)^2 = \left(\frac{1-i}{\sqrt{2}}\right)\left(\frac{1-i}{\sqrt{2}}\right) = \frac{1-2i-1}{2} = -i.$$

Thus, the real part is $0$ and the imaginary part is $-1$. In particular, we learn that $\pm(1-i)/\sqrt{2}$ are the square roots of $-i$.

(e) Using the binomial expansion, we have:

$$\left(\frac{1+\sqrt{3}i}{2}\right)^3 = \frac{1+3\sqrt{3}i-9-3\sqrt{3}i}{8} = -1.$$

Thus, the real part is $-1$ and the imaginary part is $0$. In particular, we learn that $(1+\sqrt{3}i)/2$ is one of the cube roots of $-1$.

(f) Realising the denominator, we have:

$$\frac{1+i}{2-5i} = \frac{(1+i)(2+5i)}{4+25} = \frac{2-5+(5+2)i}{29} = -\frac{3}{29} + \frac{7i}{29}.$$

Thus, the real part is $-3/29$ and the imaginary part is $7/29$.

(g) Realising the denominator, then squaring, we have:

$$\left(\frac{1+i}{1-i}\right)^2 = \left(\frac{(1+i)(1+i)}{2}\right)^2 = \left(\frac{2i}{2}\right)^2 = i^2 = -1.$$

Thus, the real part is $-1$ and the imaginary part is $0$.

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2. If $z = x + iy$, find the real and imaginary parts of the following functions in terms of $x$ and $y$:

(a) $z^2$, (b) $iz$, (c) $(1+i)z$, (d) $z^2(z-1)$, (e) $z^*(z^2 - zz^*)$.

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➾ **Solution:** (a) We have:

$$z^2 = (x+iy)^2 = x^2 + 2xyi - y^2 = (x^2 - y^2) + 2xyi,$$

so that the real part is $x^2 - y^2$ and the imaginary part is $2xy$.

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(b) We have:

$$iz = i(x+iy) = -y + ix,$$

so that the real part is $-y$ and the imaginary part is $x$.

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(c) We have:

$$(1+i)z = (1+i)(x+iy) = x - y + (x+y)i,$$

so that the real part is $x - y$ and the imaginary part is $x + y$.

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(d) We have:

$$z^2(z-1) = (x+iy)^2((x-1)+iy) = (x^2-y^2+2xyi)((x-1)+iy) = (x^2-y^2)(x-1)-2xy^2+\big((x^2-y^2)y + 2xy(x-1)\big)\, i.$$

Hence the real part is:

$$(x^2 - y^2)(x-1) - 2xy^2 = x^3 - x^2 + y^2 - 3xy^2.$$

The imaginary part is:

$$(x^2 - y^2)y + 2xy(x-1) = x^2y - y^3 + 2x^2y - 2xy.$$

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(e) Recall that the complex conjugate of $z = x + iy$ is given by $z^* = x - iy$. Hence we have:

$$z^*(z^2 - zz^*) = z^*z(z - z^*) = (x-iy)(x+iy)(x+iy-(x-iy)) = 2iy(x^2+y^2).$$

Thus, the real part is $0$ and the imaginary part is $2y(x^2+y^2)$.

3. Define $u$ and $v$ to be the real and imaginary parts, respectively, of the complex function $w = 1/z$. Show that the contours of constant $u$ and $v$ are circles. Show also that the contours of $u$ and the contours of $v$ intersect at right angles.

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**Solution:** Let $z = x + iy$. Then:

$$w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2},$$

by realising the denominator. This implies that:

$$u = \frac{x}{x^2+y^2}, \qquad v = -\frac{y}{x^2+y^2}.$$

The contours of constant $u$ are the surfaces where $u$ is constant. Hence treating $u$ as a constant, and rearranging the equation we obtained for $u$, we see that such surfaces satisfy:

$$x^2 + y^2 = \frac{x}{u} \qquad \Rightarrow \qquad \left(x - \frac{1}{2u}\right)^2 + y^2 = \frac{1}{4u^2},$$

so that they are circles with centre $(1/2u, 0)$ and radius $1/2u$. Similarly, the contours of constant $v$ obey the equation:

$$x^2 + y^2 = -\frac{y}{v} \qquad \Rightarrow \qquad x^2 + \left(y + \frac{1}{2v}\right)^2 = \frac{1}{4v^2},$$

so that they are circles with centre $(0, -1/2v)$ and radius $1/2v$.

The only exception is when $u = 0$ or $v = 0$. In these cases, we have either $x = 0$ or $y = 0$, where the contours of constant $u, v$ are just the coordinate axes themselves.

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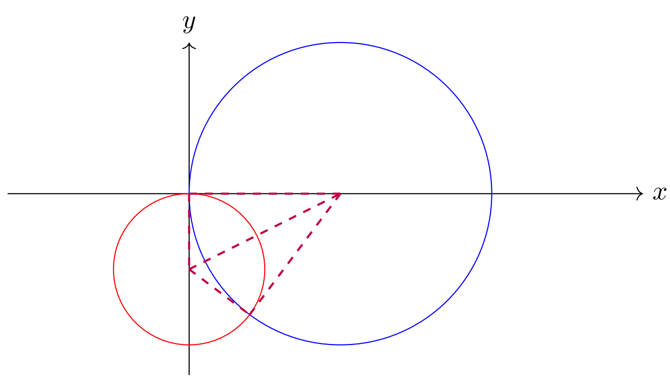
To show that the contours intersect at right angles, we use some geometry. Below, we take $u = 1$ and $v = 2$.




Any pair of these circles intersect at the origin, where the tangents to the circles are the $x = 0$ and $y = 0$ coordinate axes respectively. Thus, they intersect at right angles there.

At their other intersection, we can draw radii to form two similar triangles (shown in dashed purple lines in the above figure), implying that the other intersection also takes place at right angles.

**Factoring polynomials and solving equations**

4. Factorise the following expressions: (a) $z^2+1$; (b) $z^2-2z+2$; (c) $z^2+i$; (d) $z^2+(1-i)z-i$. *[Hint: you have already computed the two square roots of $i$ in Question 1(c).]*

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➾ **Solution:** Throughout this question, we use the identity for the *sum of two squares*, $a^2+b^2=(a+ib)(a-ib)$, which applies for all *complex* numbers $a, b$.

(a) The first expression is the sum of two squares, so can be factorised simply as $z^2+1=(z+i)(z-i)$.

(b) For the second expression, we complete the square, and then use the fact that the resulting expression is the sum of two squares:
$$z^2-2z+2=(z-1)^2+1=(z-1+i)(z-1-i).$$

(c) We computed the square roots of $i$ in Question 1(c), so we can again use the fact that the expression is the sum of two squares here:
$$z^2+i=\left(z+\frac{1+i}{\sqrt{2}}i\right)\left(z-\frac{1+i}{\sqrt{2}}i\right)=\left(z+\frac{i-1}{\sqrt{2}}\right)\left(z+\frac{1-i}{\sqrt{2}}\right)$$

(d) Again, completing the square, we have:
$$z^2+(1-i)z-i=\left(z+\frac{1-i}{2}\right)^2-\frac{1-2i-1}{4}-i=\left(z+\frac{1-i}{2}\right)^2-\frac{1}{2}i.$$

This is the difference of two squares. Using the square roots of $i$ that we computed in Question 1(c), we can factorise this as:
$$\left(z+\frac{1-i}{2}-\frac{1+i}{2}\right)\left(z+\frac{1-i}{2}+\frac{1+i}{2}\right)=(z-i)(z+1),$$

which is pleasantly simple!

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5. Given that $z=2+i$ solves the equation $z^3-(4+2i)z^2+(4+5i)z-(1+3i)=0$, find the remaining solutions.

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➾ **Solution:** We factorise the left hand side. We have:
$$\begin{aligned}z^3-(4+2i)z^2+(4+5i)z-(1+3i)&=(z-(2+i))\left(z^2-(2+i)z-\frac{1+3i}{2+i}\right)\\&=(z-(2+i))\left(z^2-(2+i)z-(1+i)\right).\end{aligned}$$

We can now apply the quadratic formula to find the roots of the second factor. The roots are:
$$z_\pm=\frac{2+i\pm\sqrt{(2+i)^2-4(1+i)}}{2}=\frac{2+i\pm i}{2},$$

which give the roots $1+i$ and $1$. Hence the complete set of solutions to the cubic is $\{2+i,1+i,1\}$.

6. Consider the polynomial equation $a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0$, where the coefficients $a_n, a_{n-1}, ..., a_0$ are real. Show that the solutions to this equation come in complex conjugate pairs. Deduce that if $n$ is odd, there is at least one real solution.

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➾ **Solution:** For this question, we will need to use the following important properties of complex conjugation:

**Proposition:** Let $z, w$ be complex numbers. The following properties hold:

(i) $(z + w)^* = z^* + w^*$;

(ii) $(zw)^* = z^* w^*$.

*Proof:* In each case, we just write things out in terms of Cartesian components. Let $z = x + iy$ and $w = u + iv$. Then we have:

(i) $(z+w)^* = (x+iy+u+iv)^* = ((x+u)+i(y+v))^* = (x+u)-i(y+v) = (x-iy)+(u-iv) = z^*+w^*$.

(ii) $(zw)^* = ((x + iy)(u + iv))^* = ((xu - yv) + i(xv + yu))^* = (xu - yv) - i(xv + yu)$.

Now let's start the question proper. Let $z$ be a solution of the equation:

$$a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0.$$

Let's take the complex conjugate of the entire equation:

$$\left(a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0\right)^* = 0.$$

Using the properties of complex conjugation, and the fact that the coefficients are real (i.e. $a_k^* = a_k$), we can rewrite the left hand side as:

$$a_n (z^*)^n + a_{n-1} (z^*)^{n-1} + ... + a_1 (z^*) + a_0 = 0.$$

Hence, we see that $z^*$ satisfies the same equation. Thus if $z$ satisfies a polynomial equation with real coefficients, then its complex conjugate also satisfies the same equation.

We have just shown that complex solutions (with a non-zero imaginary part) of real polynomial equations come in *pairs*. Hence, for an odd degree equation which has an odd number of roots, one of them must be purely real.

### Geometry of complex numbers

7. Using a diagram, explain the geometric meaning of the *modulus*, $|z|$, and *argument*, $\arg(z)$, of a complex number $z$. Find the moduli and (principal) arguments of: (a) $1 + \sqrt{3}i$; (b) $-1 + i$; (c) $-\sqrt{3} - i/\sqrt{3}$.

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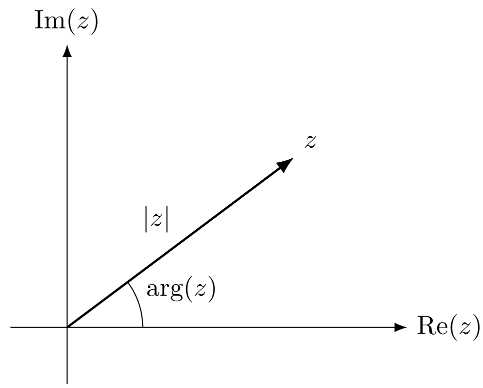
➾ **Solution:** Let $z = x + iy$ be a complex number. We can associate this complex number with a point $(x, y)$ in the plane, where the $x$-axis is the real axis and the $y$-axis is the imaginary axis. In this context, the plane is called an *Argand diagram*.




The *modulus* of $z = x + iy$ is the length of the vector joining the origin to the point $(x, y)$. By Pythagoras' theorem, we have $|z| = \sqrt{x^2 + y^2}$. The *argument* of $z = x + iy$ is the angle between the positive $x$-axis (i.e. the positive real axis) and the vector joining the origin to the point $(x, y)$. Of course, there are multiple choices of angle; standard choices include the range $[0, 2\pi)$ or the range $[-\pi, \pi)$. Both are sometimes called the 'principal' choice of argument.

For the given complex numbers, we have:

(a) The modulus is $\sqrt{1+3} = 2$. The argument is:

$$\arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}.$$

(b) The modulus is $\sqrt{(-1)^2 + 1^2} = \sqrt{2}$. The complex number is in the second quadrant, so the argument is:

$$\pi - \arctan\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

(c) The modulus is $\sqrt{3 + 1/3} = \sqrt{11/3}$. The complex number is in the third quadrant, so the argument is:

$$\pi + \arctan\left(\frac{1/\sqrt{3}}{\sqrt{3}}\right) = \pi + \arctan\left(\frac{1}{3}\right),$$

in the range $[0, 2\pi)$, or alternatively:

$$\arctan\left(\frac{1}{3}\right) - \pi,$$

in the range $[-\pi, \pi)$.

8. For $z \in \mathbb{C}$, show that $|z|^2 = zz^*$. Hence prove that $|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2)$, where $a, b \in \mathbb{C}$, and interpret this result geometrically. *[Hint: you don't need to split $a, b$ into real and imaginary parts.]*

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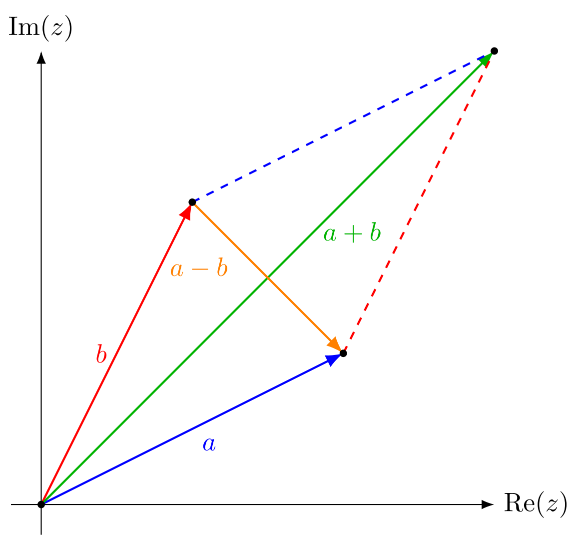
**Solution:** Let $z = x + iy$. Then $|z|^2 = x^2 + y^2$, whilst $zz^* = (x+iy)(x-iy) = x^2 + y^2$, so $|z|^2 = zz^*$ as required.

To prove the given identity, we use the fact we just showed about moduli. We have:

$$\begin{aligned}
|a+b|^2 + |a-b|^2 &= (a+b)(a^* + b^*) + (a-b)(a^* - b^*) \\
&= aa^* + ab^* + ba^* + bb^* + aa^* - ab^* - ba^* + bb^* \\
&= 2aa^* + 2bb^* \\
&= 2\left(|a|^2 + |b|^2\right),
\end{aligned}$$

as required.

This result shows that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides of the parallelogram. This is shown in the diagram below, illustrating the parallelogram formed by the complex numbers $a, b$. The sum of its side lengths squared given by $|a|^2 + |b|^2 + |a|^2 + |b|^2 = 2(|a|^2 + |b|^2)$, and the sum of its diagonal lengths squared is $|a+b|^2 + |a-b|^2$.



9. By writing $z = |z|(\cos(\arg(z)) + i\sin(\arg(z)), w = |w|(\cos(\arg(w)) + i\sin(\arg(w))$, compute the modulus and argument of the product $zw$. Hence give the geometrical interpretation of multiplying one complex number by another complex number. Give also a geometrical interpretation of division of one complex number by another complex number, $z/w$.

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**Solution:** We have:

$$\begin{aligned}
zw &= |z||w|\left(\cos(\arg(z)) + i\sin(\arg(z))\right)\left(\cos(\arg(w)) + i\sin(\arg(w))\right)\\
&= |z||w|\left(\cos(\arg(z))\cos(\arg(w)) - \sin(\arg(z))\sin(\arg(w)) + \left(\cos(\arg(z))\sin(\arg(w)) + \sin(\arg(z))\cos(\arg(w))\right)i\right)\\
&= |z||w|\left(\cos(\arg(z) + \arg(w)) + \sin(\arg(z) + \arg(w))i\right),
\end{aligned}$$

where in the last line we used the compound angle formulae for cosine and sine. It follows that the modulus and argument of the product $zw$ are given by:

$$|zw| = |z||w|, \qquad \arg(zw) = \arg(z) + \arg(w).$$

In particular, we see that multiplying the complex number $z$ by the complex number $w$ results in a *scaled rotation* of $z$. Indeed, $z$ is scaled by a factor of $|w|$ and rotated by an angle $\arg(w)$ about the origin anticlockwise.

For division, we can view the quotient $z/w$ as the product $z \cdot (1/w)$. The complex number $1/w$ satisfies:

$$\frac{1}{w} = \frac{1}{|w|}\cdot\frac{1}{\cos(\arg(w)) + \sin(\arg(w))i} = \frac{1}{|w|}\frac{\cos(\arg(w)) - \sin(\arg(w))i}{\cos^2(\arg(w)) + \sin^2(\arg(w))} = \frac{1}{|w|}\left(\cos(-\arg(w)) + \sin(-\arg(w))i\right).$$

Hence, we see that $|1/w| = 1/|w|$ and $\arg(1/w) = -\arg(w)$. It follows from the multiplication rule we proved above that:

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}, \qquad \arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w).$$

In particular, division still corresponds to a scaled rotation. However, $z$ is now scaled *down* by a factor of $|w|$ instead of being scaled up, and the rotation is by $\arg(w)$ about the origin *clockwise*.

10. Let $z_1 = 2+i, z_2 = 3+4i$. Find $z_1z_2$ by: (a) adding arguments and multiplying moduli; (b) using the rules of complex algebra. Verify that your results agree.

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➾ **Solution:** (a) The moduli of these complex numbers are $|z_1| = \sqrt{2^2+1^2} = \sqrt{5}$ and $|z_2| = \sqrt{3^2+4^2} = 5$. The arguments of these complex numbers are $\arg(z_1) = \arctan(1/2)$ and $\arg(z_2) = \arctan(4/3)$. Hence the modulus and argument of the product are given by:

$$|z_1z_2| = 5\sqrt{5}, \qquad \arg(z_1z_2) = \arctan(1/2) + \arctan(4/3).$$

To obtain a Cartesian expression for $z_1z_2$, we will need to compute the cosine and sine of the argument of $z_1z_2$, which seems to be quite hard at the moment! To make things easier, we need to *add the arctangents first*. We could do this using the result of Question 11. For practice though, we shall present an alternative method. Let:

$$t = \arctan(1/2) + \arctan(4/3),$$

and consider taking the tangent of both sides and applying the compound angle formula for tangent:

$$\tan(t) = \tan\left(\arctan(1/2) + \arctan(4/3)\right) = \frac{\tan(\arctan(1/2)) + \tan(\arctan(4/3))}{1 - \tan(\arctan(1/2))\tan(\arctan(4/3))} = \frac{1/2+4/3}{1-2/3} = \frac{11}{2}.$$

Hence, we see that:

$$t = \arctan(11/2).$$

This is the angle between the adjacent and hypotenuse of a right-angled triangle with opposite side of length $11$ and adjacent side of length $2$. The hypotenuse is of length $\sqrt{11^2+2^2} = \sqrt{125} = 5\sqrt{5}$. Hence:

$$5\sqrt{5}\cos(\arctan(11/2)) = 2, \qquad 5\sqrt{5}\sin(\arctan(11/2)) = 11.$$

It follows that:

$$z_1z_2 = 2+11i.$$

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(b) We can also compute the product using the rules of complex algebra. We have:

$$z_1z_2 = (2+i)(3+4i) = 6-4+(8+3)i = 2+11i,$$

which agrees perfectly. Much simpler!

11. By considering multiplication of the complex numbers $z = 1 + iA$ and $w = 1 + iB$, derive the arctangent addition formula:

$$\arctan(A) + \arctan(B) = \arctan\left(\frac{A+B}{1-AB}\right).$$

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➼ **Solution:** The argument of $z$ is $\arctan(A)$, whilst the argument of $w$ is $\arctan(B)$. Multiplication of complex numbers results in addition of their arguments, so the argument of $zw$ is $\arctan(A) + \arctan(B)$, which is the left hand side of the arctangent addition formula.

On the other hand, we can instead first compute the product of the complex numbers algebraically. We have

$$zw = (1 + iA)(1 + iB) = 1 - AB + (A + B)i.$$

This has argument:

$$\arctan\left(\frac{A+B}{1-AB}\right).$$

Hence, the arctangent addition formula follows.

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12. Give a geometrical interpretation (in terms of *vectors*) of the real and imaginary parts of the quantity $Q = z_1 z_2^*$. Show also that $Q$ is invariant under a rotation of $z_1, z_2$ about the origin, and confirm that this is consistent with your geometrical interpretation. *[Hint: In Question 9, you showed that multiplying by a complex number $u$ of unit modulus is equivalent to a rotation about the origin.]*

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➼ **Solution:** Let $z_1 = x_1 + iy_1$ and let $z_2 = x_2 + iy_2$. Then the given quantity is:

$$Q = (x_1 + iy_1)(x_2 - iy_2) = x_1x_2 + y_1y_2 + (x_2y_1 - x_1y_2)i.$$

The real part is $x_1x_2 + y_1y_2$, which is the scalar product of the vectors $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$. The imaginary part is $x_2y_1 - x_1y_2$, which is the $z$-component of the cross product of the vectors $(x_2, y_2, 0)$ and $(x_1, y_1, 0)$; this is also the magnitude of this cross product since it points purely in the $z$-direction.

A rotation about the origin is equivalent to multiplication by a unit modulus complex number, $u$, satisfying $|u|^2 = uu^* = 1$. Hence under such a rotation, we have $z_1 \mapsto uz_1$ and $z_2 \mapsto uz_2$. This gives:

$$Q = z_1z_2^* \mapsto (uz_1)(uz_2)^* = uu^*z_1z_2^* = z_1z_2^* = Q.$$

Hence $Q$ is invariant under such a rotation. This is consistent with the geometric interpretation we gave earlier in the question, since the scalar product won't change under a rotation (it depends on lengths and angles, which are preserved under a rotation), and the cross produce won't change under a rotation (since the rotation is in the complex plane, and therefore the direction of the cross product will still be out of the complex plane, in the $z$-direction).

### Loci in the complex plane

13. (**Circles**) Describe the sets of points $z \in \mathbb{C}$ satisfying:

(a) $|z| = 4$, (b) $|z-1| = 3$, (c) $|z-i| = 2$, (d) $|z-(1-2i)| = 3$, (e) $|z^*-1| = 1$, (f) $|z^*-i| = 1$.

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**Solution:**

(a) This is a circle, centred at $0$, radius $4$.

(b) This is a circle, centred at $1$, radius $3$.

(c) This is a circle, centred at $i$, radius $2$.

(d) This is a circle, centred at $1-2i$, radius $3$.

(e) Note that $|z^*-1| = |(z-1)^*| = |z-1|$. Hence this is a circle centred at $1$, radius $1$.

(f) Note that $|z^*-i| = |(z+i)^*| = |z+i|$. Hence this is a circle centred at $-i$, radius $1$.

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14. (**Transformations of circles**) Describe the set of points $z \in \mathbb{C}$ satisfying $|z-2-i| = 6$. Without further calculation, describe the sets of points $u \in \mathbb{C}, v \in \mathbb{C}, w \in \mathbb{C}$ satisfying:

(a) $u = z + 5 - 8i$, (b) $v = iz + 2$, (c) $w = \frac{3}{2}z + \frac{1}{2}z^*$,

where $|z-2-i| = 6$.

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**Solution:** The set of points $z \in \mathbb{C}$ satisfying $|z-2-i| = 6$ is a circle centred at $2+i$, radius $6$.

(a) If we define $u = z+5-8i$, we have translated the circle by $5-8i$. Hence the locus of $u$ is a circle centred at $7-7i$, radius $6$.

(b) If we define $v = iz+2$, we have rotated the circle by $\pi/2$ clockwise about the origin (this is the multiplication by $i$), then translated the circle by $2$. Since this is a rigid motion, that does not involve bending or squashing the circle, it is sufficient to keep track of where the centre goes. We note:

$$i(2+i) + 2 = 2i - 1 + 2 = 2i + 1,$$

so the locus of $v$ is a circle centred at $1+2i$, radius $6$.

(c) This part is more difficult. This is not an obvious transformation from the lectures, so we might consider splitting $z$ into real and imaginary parts. We have:

$$w = \frac{3}{2}(x+iy) + \frac{1}{2}(x-iy) = 2x + iy.$$

Hence, we see that the point $x+iy$ gets mapped to the point $2x+iy$ under the transformation from $z$ to $w$. Hence, this transformation is a *scaling* in the $x$-direction (i.e. along the real axis). The result is therefore an *ellipse* with centre $4+i$, major diameter $12$ in the $x$-direction, and minor diameter $6$ in the $y$-direction.

15. (**Circles of Apollonius**) Let $a, b \in \mathbb{C}$. Show that the set of points satisfying $|z-a| = \lambda|z-b|$, where $\lambda \neq 1$, is a circle in the complex plane. *[Hint: start by squaring the equation. You don't need to split $z$ into real and imaginary parts.]* Determine the centre and radius of the circle $|z| = 2|z-2|$.

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➭ **Solution:** We follow the hint, and start by squaring the given equation:

$$\begin{aligned}
|z-a| = \lambda|z-b| \qquad &\Rightarrow \qquad |z-a|^2 = \lambda^2|z-b|^2 \\
&\Rightarrow \qquad (z-a)(z^*-a^*) = \lambda^2(z-b)(z^*-b^*) \\
&\Rightarrow \qquad |z|^2 - a^*z - az^* + |a|^2 = \lambda^2(|z|^2 - b^*z - bz^* + |b|^2) \\
&\Rightarrow \qquad (1-\lambda^2)|z|^2 - (a^* - \lambda^2 b^*)z - (a - \lambda^2 b)z^* = \lambda^2|b|^2 - |a|^2 \\
&\Rightarrow \qquad |z|^2 - \left(\frac{a^* - \lambda^2 b^*}{1-\lambda^2}\right)z - \left(\frac{a-\lambda^2 b}{1-\lambda^2}\right)z^* = \frac{\lambda^2|b|^2 - |a|^2}{1-\lambda^2}.
\end{aligned}$$

We now notice that the terms on the left look like the first three terms in the expansion of:

$$\left|z - \frac{a-\lambda^2 b}{1-\lambda^2}\right|^2,$$

just as we expanding $|z-a|^2$, $|z-b|^2$ in the first couple of lines. Therefore, collecting terms and subtracting the extra fourth term, we are left with:

$$\left|z - \frac{a-\lambda^2 b}{1-\lambda^2}\right|^2 - \left|\frac{a-\lambda^2 b}{1-\lambda^2}\right|^2 = \frac{\lambda^2|b|^2 - |a|^2}{1-\lambda^2}.$$

We can simplify this by moving the second term on the left hand side to the right hand side. We obtain the right hand side:

$$\begin{aligned}
\frac{\lambda^2|b|^2 - |a|^2}{1-\lambda^2} + \left|\frac{a-\lambda^2 b}{1-\lambda^2}\right|^2 &= \frac{\lambda^2|b|^2 - |a|^2 - \lambda^4|b|^2 + \lambda^2|a|^2 + |a|^2 - \lambda^2 ab^* - \lambda^2 a^* b + \lambda^4|b|^2}{(1-\lambda^2)^2} \\
&= \frac{\lambda^2(|b|^2 - ab^* - a^*b + |a|^2)}{(1-\lambda^2)^2} \\
&= \frac{\lambda^2|a-b|^2}{(1-\lambda^2)^2}.
\end{aligned}$$

Hence we see that the original equation can be recast in the form:

$$\left|z - \frac{a-\lambda^2 b}{1-\lambda^2}\right|^2 = \frac{\lambda^2|a-b|^2}{(1-\lambda^2)^2}.$$

Taking the square root, we have:

$$\left|z - \frac{a-\lambda^2 b}{1-\lambda^2}\right| = \frac{\lambda|a-b|}{|1-\lambda^2|}$$

Hence, this is indeed a circle with centre and radius, respectively:

$$\frac{a-\lambda^2 b}{1-\lambda^2}, \qquad \frac{\lambda|a-b|}{|1-\lambda^2|}.$$

For the given example, $|z| = 2|z-2|$, we have $a = 0$, $b = 2$ and $\lambda = 2$. Hence the centre is:

$$\frac{0-8}{1-4} = \frac{8}{3},$$

and the radius is:

$$\frac{2 \cdot 2}{|1-4|} = \frac{4}{3}.$$

---

16. (**Lines and half-lines**) Describe the sets of points $z \in \mathbb{C}$ satisfying:

(a) $|z-2| = |z+i|$, (b) $|z-2| = |z^* + i|$, (c) $\arg(z) = \pi/2$, (d) $\arg(z^*) = \pi/4$.

---

➾ **Solution:**

(a) This is a line bisecting the line joining the points $2$ and $-i$.

(b) This is a line bisecting the line joining the points $2$ and $i$ (since $|z^* + i| = |(z-i)^*| = |z-i|$).

(c) This is a half-line, emanating from the origin along the imaginary axis.

(d) Since $\arg(z^*) = -\arg(z)$, this is a half-line, emanating from the origin and inclined at an angle $\pi/4$ below the real axis.

---

17. (**Lines and circles**) Let $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$. *Without setting* $z = x + iy$, describe the locus $azz^* + bz + b^*z^* + c = 0$ for different values of $a, b, c$. How does the locus change under the maps: (a) $z \mapsto \alpha z$ for $\alpha \in \mathbb{C}$; (b) $z \mapsto 1/z$?

---

➾ **Solution:** We attempt to factorise this expression, like a circle of Apollonius as discussed in Question 15. First, we divide by $a$, assuming that $a \neq 0$:

$$zz^* + \frac{bz + b^*z^*}{a} + \frac{c}{a} = 0.$$

Completing the square on the first three terms (and using the fact that $a$ is real), we have:

$$\left|z + \frac{b^*}{a}\right|^2 - \left|\frac{b}{a}\right|^2 + \frac{c}{a} = 0 \qquad \Rightarrow \qquad \left|z + \frac{b^*}{a}\right| = \frac{|b|^2 - ca}{a^2}.$$

Hence, if $a \neq 0$, the locus is:

- A circle centred on $-b^*/a$ with radius $\sqrt{|b|^2 - ca}/a$ and $|b|^2 - ca > 0$.
- A point at $-b^*/a$ and $|b|^2 = ca$.
- Empty if $|b|^2 < ca$.

On the other hand, if $a = 0$, the locus is $bz + b^*z^* + c = 0$. The real part of any complex number $w = x + iy$ may be written as $x = \frac{1}{2}(w + w^*)$, hence we recognise this equation as:

$$2\,\mathrm{Re}(bz) + c = 0 \qquad \Leftrightarrow \qquad \mathrm{Re}(bz) = -c/2.$$

The equation $\mathrm{Re}(bz) = -c/2$ tells us that the imaginary part of the expression $bz$ is constant; if we define $w = bz$, then it tells us that in the $w$-plane, we have a vertical line at $-c/2$.

To understand what things look like in the $z$-plane, we need to write $z = w/b$. Note that if $b = 0$, then the equation becomes $0 = -c/2$, and we need $c = 0$ too for consistency; then, the original equation just looks like $0 = 0$ which is very uninteresting! In the case $b \neq 0$, $z = w/b$ is a scaled rotation of $w$ by angle $-\arg(b)$ anticlockwise about the origin. Hence the figure in the $z$-plane looks like a line inclined at angle $\arg(b)$ to the vertical, going through the point $-c/2b$.

---

(a) The transformation $z \mapsto \alpha z$ is a scaled rotation, enlarging the figure by a factor $|\alpha|$ and rotating it by an angle $\arg(\alpha)$ anticlockwise about the origin.

(b) The transformation $z \mapsto 1/z$ is an *inversion*. To see its effect, we set $w = 1/z$ in the defining equation of the locus:

$$\begin{aligned} azz^* + bz + b^*z^* + c = 0 \qquad &\Leftrightarrow \qquad \frac{a}{ww^*} + \frac{b}{w} + \frac{b^*}{w^*} + c = 0 \\ &\Leftrightarrow \qquad a + bw^* + b^*w + cww^* = 0. \end{aligned}$$

In particular, we see that we interchange the roles $a \leftrightarrow c$ and $b \leftrightarrow b^*$ under this transformation. So we have the following cases:

- If $a, c \neq 0$, then this map transforms a circle into another circle. The radius is scaled by a factor $a/c$ and the centre is mapped to $-b/c$.
- If $a \neq 0$ and $c = 0$, then this map transforms a circle into a line. The new line goes through $-a/2b^*$ and is inclined at an angle $\arg(b^*)$ to the vertical.
- If $a = 0$ and $c \neq 0$, then this map transforms a line into a circle. The new circle has centre $-b/a$ and radius $|b|^2/c$.
- If $a = 0$ and $c = 0$, then this map transforms a line into a line. The line is just a line through the origin, and is mapped from having an angle $\arg(b)$ with the vertical to having an angle $\arg(b^*)$ with the vertical.

18. (**More complex figures**) Sketch the sets of points $z \in \mathbb{C}$ satisfying:

(a) $\mathrm{Re}(z^2) = \mathrm{Im}(z^2)$, (b) $\dfrac{\mathrm{Im}(z^2)}{z^2} = -i$, (c) $|z^* + 2i| + |z| = 4$, (d) $|2z - z^* - 3i| = 2$.

---

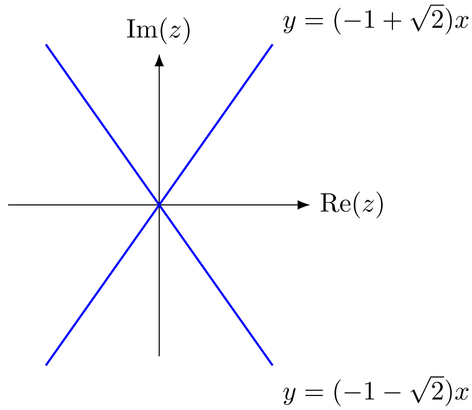
➭ **Solution:** (a) Let $z = x + iy$. Then $z^2 = x^2 - y^2 + 2xyi$, so the locus $\mathrm{Re}(z^2) = \mathrm{Im}(z^2)$ is equivalent to:

$$x^2 - y^2 = 2xy \qquad \Rightarrow \qquad 0 = y^2 + 2xy - x^2.$$

Solving this equation for $y$, we have:

$$y = \frac{-2x \pm \sqrt{4x^2 + 4x^2}}{2} = -x \pm x\sqrt{2} = (-1 \pm \sqrt{2})x.$$

Thus the locus is a pair of lines passing through the origin.

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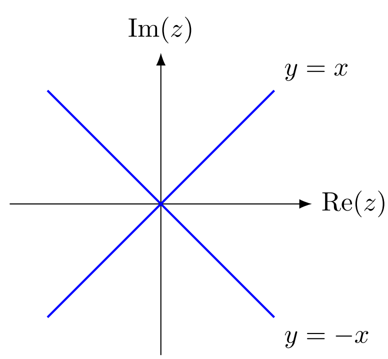
(b) Let $z = x + iy$. Then using part (a), we have $\mathrm{Im}(z^2) = 2xy$. Inserting this into the locus $\mathrm{Im}(z^2)/z^2 = -i$, we have:

$$\frac{2xy}{x^2 - y^2 + 2xyi} = -i.$$

Multiplying up, we have:

$$2xy = i(y^2 - x^2) + 2xy.$$

Cancelling $2xy$ from both sides, we see that $x^2 = y^2$, so that $y = \pm x$. Thus the locus is again a pair of lines passing through the origin.




The locus excludes the origin where the left hand side, $\mathrm{Im}(z^2)/z^2$, is undefined.

(c) Let $z = x + iy$. Then the locus $|z^* + 2i| + |z| = 4$ can be rewritten as:

$$\sqrt{x^2 + (y-2)^2} + \sqrt{x^2 + y^2} = 4.$$

Squaring both sides, we have:

$$x^2 + (y-2)^2 + x^2 + y^2 + 2\sqrt{(x^2+y^2)(x^2+(y-2)^2)} = 16$$

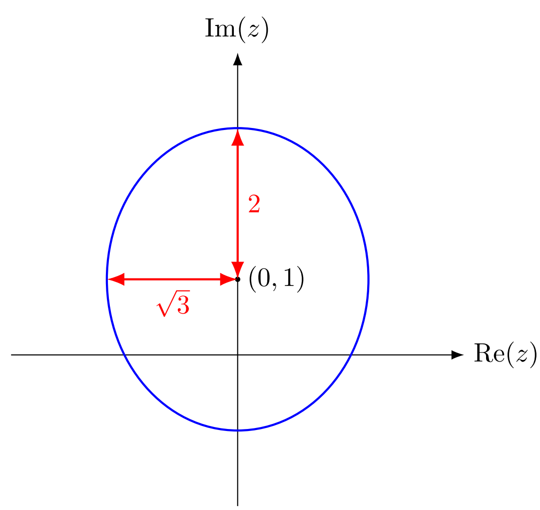
$$\Leftrightarrow \qquad x^2 + y^2 - 2y - 6 = -\sqrt{(x^2+y^2)(x^2+y^2-4y+4)} = -\sqrt{x^4 + 2x^2y^2 - 4x^2y + 4x^2 + y^4 - 4y^3 + 4y^2}$$

Squaring both sides again, we have:

$$x^4 + y^4 + 4y^2 + 36 + 2x^2y^2 - 4x^2y - 12x^2 - 4y^3 - 12y^2 + 24y = x^4 + 2x^2y^2 - 4x^2y + 4x^2 + y^4 - 4y^3 + 4y^2$$

Simplifying, this reduces to:

$$9 = 4x^2 + 3y^2 - 6y \qquad \Leftrightarrow \qquad 1 = \left(\frac{x}{\sqrt{3}}\right)^2 + \left(\frac{y-1}{2}\right)^2.$$

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(d) Let $z = x + iy$. Then:

$$2 = |2z - z^* - 3i| = |2(x+iy) - (x-iy) - 3i| = |x + 3i(y-1)| = \sqrt{x^2 + 9(y-1)^2}.$$

Rearranging, we have:

$$1 = \left(\frac{x}{2}\right)^2 + \left(\frac{y-1}{2/3}\right)^2.$$

This is an ellipse, centred on $(0, 1)$, with semi-minor axis $2/3$ and semi-major axis $2$. This is the same as the figure above, just scaled in the $x, y$ directions.

### Exponential form of a complex number

19. State *Euler's formula* for the complex exponential $e^{i\theta}$. Hence provide a simpler derivation of the modulus-argument multiplication law proved in Question 9.

---

➻ **Solution:** *Euler's formula* states that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. To rederive the modulus-argument multiplication law, let $z = |z|e^{i\arg(z)}$ and $w = |w|e^{i\arg(w)}$. Then:

$$zw = |z||w|e^{i(\arg(z)+\arg(w))},$$

which shows $|zw| = |z||w|$ and $\arg(zw) + \arg(z) + \arg(w)$.

---

20. Find (a) the real and imaginary parts; (b) the modulus and argument, of:

$$\frac{e^{i\omega t}}{R + i\omega L + (i\omega C)^{-1}},$$

where $\omega, t, R, L, C$ are real, quoting your answers in terms of $X = \omega L - (\omega C)^{-1}$. (*) If you are taking IA Physics, can you think of what each of $\omega, t, R, L, C$ might represent?

---

➻ **Solution:** (a) To find the real and imaginary parts, we need to realise the denominator. Note that:

$$R + i\omega L + (i\omega C)^{-1} = R + i\left(\omega L - \frac{1}{\omega C}\right) = R + iX.$$

Hence we have:

$$\frac{e^{i\omega t}}{R+iX} = \frac{(R-iX)e^{i\omega t}}{R^2+X^2} = \frac{(R-iX)(\cos(\omega t) + i\sin(\omega t))}{R^2+X^2}.$$

Therefore, the real and imaginary parts, are, respectively:

$$\frac{R\cos(\omega t) + X\sin(\omega t)}{R^2+X^2}, \qquad \frac{R\sin(\omega t) - X\cos(\omega t)}{R^2+X^2}.$$

---

(b) To find the modulus, we use the property $|z/w| = |z|/|w|$. The numerator has modulus $1$, and the denominator has modulus $\sqrt{R^2+X^2}$. Hence the modulus is $1/\sqrt{R^2+X^2}$.

To find the argument, we use the property $\arg(z/w) = \arg(z) - \arg(w)$. The numerator has argument $\omega t$, and the denominator has argument $\arctan(X/R)$. Hence the argument is:

$$\omega t - \arctan\left(\frac{X}{R}\right).$$

---

This result is useful in alternating current circuits. The quantities here represent resistance ($R$), inductance ($L$), capacitance ($C$), frequency of the current ($\omega$) and time ($t$).

21. Express each of the following in Cartesian form: (a) $e^{-i\pi/2}$; (b) $e^{-i\pi}$; (c) $e^{i\pi/4}$; (d) $e^{1+i}$; (e) $e^{2e^{i\pi/4}}$.

---

➾ **Solution:** We use Euler's formula in each case:

(a) $e^{-i\pi/2} = \cos(-\pi/2) + i\sin(-\pi/2) = -i$.

(b) $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$.

(c) $e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = \dfrac{1+i}{\sqrt{2}}$.

(d) $e^{1+i} = e \cdot e^{i} = e\left(\cos(1) + i\sin(1)\right) = e\cos(1) + ie\sin(1)$. This cannot be further simplified.

(e) $e^{2e^{i\pi/4}} = e^{2(1+i)/\sqrt{2}} = e^{\sqrt{2}+i\sqrt{2}} = e^{\sqrt{2}}e^{i\sqrt{2}} = e^{\sqrt{2}}\left(\cos(\sqrt{2}) + i\sin(\sqrt{2})\right) = e^{\sqrt{2}}\cos(\sqrt{2}) + ie^{\sqrt{2}}\sin(\sqrt{2})$. This cannot be further simplified.

---

22. Let $a, b, \omega$ be real constants. Show that $a\cos(\omega x) + b\sin(\omega x) = \mathrm{Re}((a - bi)e^{i\omega x})$, and hence, by writing $a - bi$ in exponential form, deduce that $a\cos(\omega x) + b\sin(\omega x) = \sqrt{a^2+b^2}\cos(\omega x - \arctan(b/a))$.

---

➾ **Solution:** We have:

$$\mathrm{Re}((a-bi)e^{i\omega x}) = \mathrm{Re}\left((a-bi)(\cos(\omega x) + i\sin(\omega x))\right) = a\cos(\omega x) + b\sin(\omega x),$$

as required. In exponential form, we have $a - bi = \sqrt{a^2+b^2}e^{-i\arctan(b/a)}$. Hence we have:

$$(a-bi)e^{i\omega x} = \sqrt{a^2+b^2}e^{i(\omega x - \arctan(b/a))}.$$

Taking the real part, we see that:

$$a\cos(\omega x) + b\sin(\omega x) = \sqrt{a^2+b^2}\cos\left(\omega x - \arctan(b/a)\right),$$

as required. This result is useful, because it shows that the linear combination of trigonometric functions can always be combined to produce a single trigonometric function, albeit with a shifted phase.

---

### Multi-valued functions: logarithms and powers

23. Explain why the complex logarithm $\log : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ is a *multi-valued function*, and give its possible values. Using the complex logarithm, find all complex numbers satisfying: (a) $e^{2z} = -1$; (b) $e^{z^*} = i + 1$.

---

➾ **Solution:** The *complex logarithm* of the complex number $z$, written $\log(z)$, is the solution of the equation:

$$e^{\log(z)} = z.$$

Write $\log(z) = u(z) + iv(z)$, where $u(z), v(z)$ are the real and imaginary parts of the complex logarithm respectively. Then:

$$z = e^{u(z)+iv(z)} = e^{u(z)}e^{iv(z)}.$$

Write $z = |z|e^{i\arg(z)}$. Comparing the modulus, we see that $u(z) = \log|z|$. Comparing the argument, we see that $v(z) = \arg(z) + 2\pi n$, where $n$ is an integer. Hence:

$$\log(z) = \log|z| + i\arg(z) + 2\pi i n,$$

for any integer $n$. This shows that the complex logarithm is a multi-valued function.

Applying this to the given equations:

(a) Taking the logarithm of $e^{2z} = -1$, we have:

$$2z = \log(-1) = \log|1| + i\arg(-1) + 2\pi i n = i\pi + 2\pi i n.$$

Hence $z = \frac{1}{2}i\pi + \pi i n$ for $n$ an integer.

(b) Taking the logarithm of $e^{z^*} = 1 + i$, we have:

$$z^* = \log(1+i) = \log|1+i| + i\arg(1+i) + 2\pi i n = \log(\sqrt{2}) + \frac{\pi i}{4} + 2\pi i n.$$

Hence $z = \log(\sqrt{2}) - \frac{\pi i}{4} + 2\pi i n$, where $n$ is an integer.

---

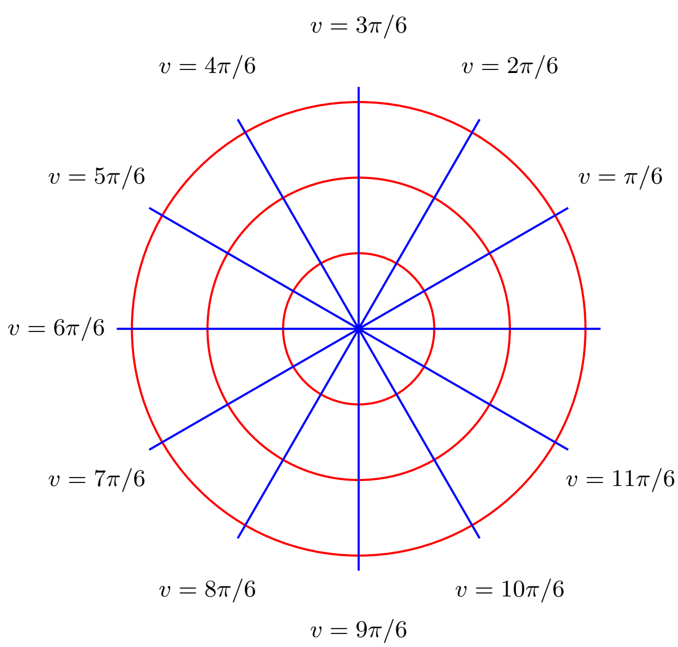
24. Let the real and imaginary parts of the complex logarithm $\log(z)$ be $u, v$ respectively. Sketch the contours of constant $u, v$ in the complex plane, and show that they intersect at right angles.

---

**Solution:** The complex logarithm, $\log(z) = \log|z| + i\arg(z) + 2\pi i n$, has $u = \log|z|$, $v = \arg(z) + 2\pi n$, for $n$ an integer.

Therefore, for $u$ constant, we have $|z| = e^u$. This is a circle centred on the origin. All radii are allowed, as $u$ varies from $-\infty$ to $\infty$. For $v$ constant, we have $\arg(z) + 2\pi n = v$, which described a half-line emanating from the origin, at angle $v - 2\pi n$, or equivalently $v$, from the $x$-axis.

Below, we display a diagram showing contours of constant $v$ in blue, and contours of constant $u$ in red. Since the contours of constant $v$ correspond to radii of the circles which comprise the contours of constant $u$, they must intersect at right angles.



25. Explain how the complex logarithm can be used to define complex powers, $z^w$, and hence describe the multi-valued nature of complex exponentiation. Compute all values of the multi-valued exponentials: (a) $i^i$; (b) $i^{1/3}$.

---

➾ **Solution:** If $w$ is a complex number, we define the *complex power* $z^w$ by:

$$z^w := e^{w \log(z)} = e^{w(\log(z)+i\arg(z)+2\pi i n)},$$

where $n$ is an integer. This means that :

- If $w$ is an integer, then the $2\pi in$ part of the exponent has no effect - $e^{2\pi inw} = 1$, so we're safe! Therefore, integer powers of complex numbers are single-valued.
- If $w$ is a rational number, then there are some $n$ such that $e^{2\pi inw} = 1$. For example, if $w = 1/2$, we have that $n = 2, 4, \ldots$. These $n$ will periodically repeat with the period of the denominator of $w$ (when it is written in its lowest terms). Hence, rational powers of complex numbers are *multi-valued*, but can only take *finitely many different values*.
- If $w$ is an irrational number, then the powers are *multi-valued*, but can take *infinitely many* different values.
- If $w = a + bi$ is a complex number, with $b \neq 0$, then we always have a term $(bi) \cdot (2\pi in) = -2\pi bn$ in the exponent. This implies that the powers are *multi-valued*, and again always take *infinitely many* different values.

Examining the exponentials we are given:

(a) $i^i = e^{i\log(i)} = e^{i(\log|i|+i\arg(i)+2\pi in)} = e^{-\pi/2-2\pi n}$, for all integers $n$. Hence, there are infinitely many possible values of this exponential, but all possible values of $i^i$ are in fact real!

(b) $i^{1/3} = e^{\log(i)/3} = e^{(\log|i|+i\arg(i)+2\pi in)/3} = e^{i\pi/6+2\pi in/3}$. There are only finitely many possible values of this exponential, which vary as we take $n = 0, 1, 2$. The possible values are:

$$\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{i}{2},$$

$$\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{i}{2},$$

$$\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i.$$

---

26. Compute all possible values of $(i^i)^i$ and $i^{(i^i)}$.

---

➾ **Solution:** We already computed $i^i$ in the previous question, with $i^i = e^{-\pi/2-2\pi n}$ for all integers $n$. Taking another power of $i$, we have:

$$\left(e^{-\pi/2-2\pi n}\right)^i = e^{i\left(\log(e^{-\pi/2-2\pi n})\right)} = e^{-i\pi/2-2\pi in} = e^{-i\pi/2} = -i.$$

In particular, we see that $(i^i)^i = -i$ is single-valued. On the other hand, we have:

$$i^{(i^i)} = e^{e^{-\pi/2-2\pi n}\log(i)} = e^{e^{-\pi/2-2\pi n}(\log|i|+i\arg(i)+2\pi im)} = e^{e^{-\pi/2-2\pi n}(i\pi/2+2\pi im)}.$$

Expressing this in Cartesian form, we see that we have a doubly-multi-valued result,

$$\cos\left(e^{-\pi/2-2\pi n}\left(\frac{\pi}{2}+2\pi m\right)\right) + \sin\left(e^{-\pi/2-2\pi n}\left(\frac{\pi}{2}+2\pi m\right)\right),$$

where $n, m$ are integers. This cannot be further simplified.

27. Find the real and imaginary parts of the function $f(z) = \log(z^{1+i})$. Hence, sketch the locus $\mathrm{Re}(f(z)) = 0$.

---

➾ **Solution:** Since:

$$f(z) = \log(z^{1+i}) = \log\left(e^{(1+i)(\log|z|+i\arg(z)+2\pi in)}\right) = (1+i)\left(\log|z| + i\arg(z) + 2\pi in\right),$$

for $n$ an integer, we have:

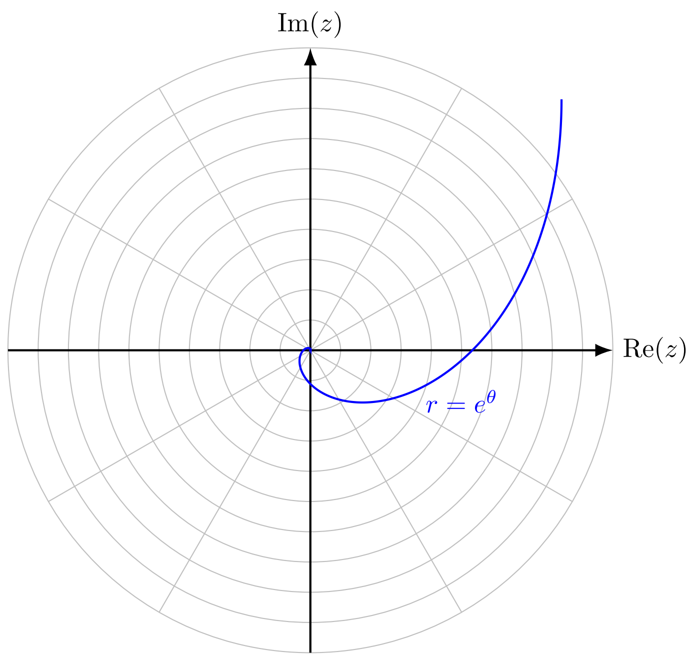
$$f(z) == \log|z| - \arg(z) - 2\pi n + (\log|z| + \arg(z) + 2\pi n)\, i,$$

where $n$ is an integer, which gives the real and imaginary parts.

The locus $\mathrm{Re}(f(z)) = 0$ is given by $\log|z| = \arg(z) + 2\pi n$. Writing this in terms of polar coordinates, we have $|z| = r$ and $\arg(z) = \theta \in [-\pi, \pi)$, say. Then:

$$r = e^{\theta + 2\pi n}.$$

This implies that the complete locus is a *logarithmic spiral*, shown in the figure below.




It grows pretty rapidly! More so than the *Archimedean spiral*, $r = \theta$, that we saw on Examples Sheet 2.

**Roots of unity**

28. Write down the solutions to the equation $z^n = 1$ in terms of complex exponentials, and plot the solutions on an Argand diagram. *[Recall that the solutions are called the* $n$th roots of unity.*]*

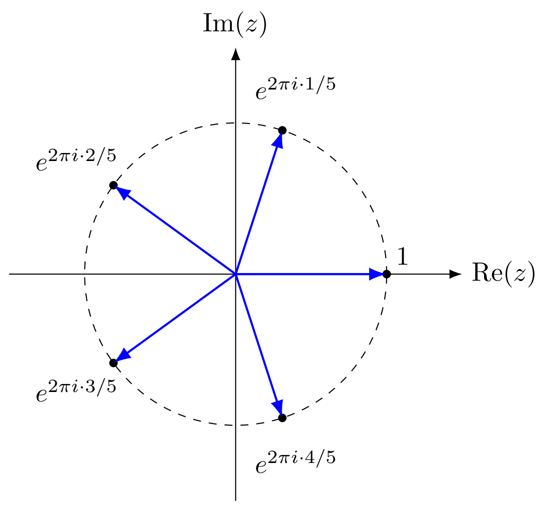
---

**➠ Solution:** The solutions are:

$$z = 1^{1/n} = e^{(1/n)\cdot(\log|1|+i\arg(1)+2\pi im)} = e^{2\pi im/n},$$

where $m$ is an integer. On an Argand diagram, these solutions form the vertices of an $n$-sided regular polygon on the unit circle, with one vertex at the point $1$.

For the case $n = 5$, for example, the figure takes the form:




The roots form a regular pentagon in this case.

---

29. Find and plot the solutions to the following equations: (a) $z^3 = -1$; (b) $z^4 = 1$; (c) $z^2 = i$; (d) $z^3 = -i$.

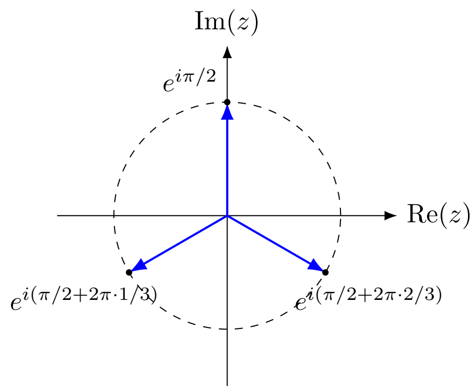
---

**➠ Solution:**

(a) The solutions are:

$$z = (-1)^{1/3} = e^{(1/3)\cdot(\log|-1|+i\arg(-1)+2\pi in)} = e^{i(\pi/2+2\pi n/3)}.$$

These form a triangle in the complex plane, shown below.

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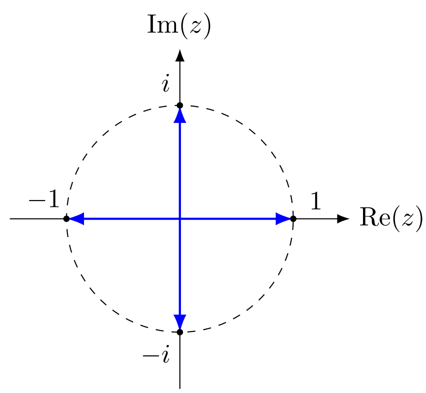
(b) The solutions are:

$$z = 1^{1/4} = e^{(1/4)\cdot(\log|1|+i\arg(1)+2\pi in)} = e^{\pi in/2}.$$

Equivalently, these can be written as $\{1, -1, i, -i\}$. These form a square in the Argand diagram, as shown in the figure below.

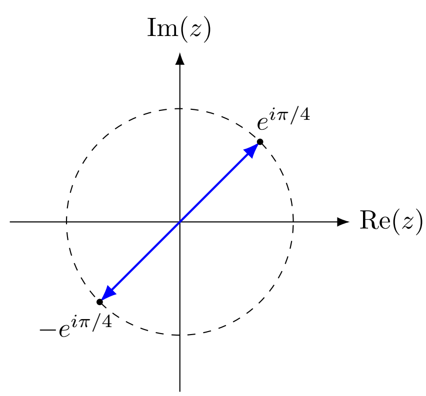



---

(c) The solutions are:

$$z = i^{1/2} = e^{(1/2)\cdot(\log|i|+i\arg(i)+2\pi in)} = e^{i\pi/4+\pi in} = \pm e^{i\pi/4}.$$

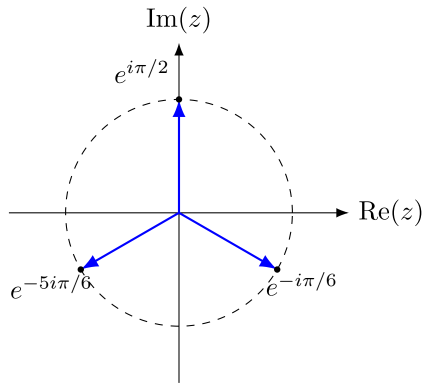
Equivalently, these can be written in Cartesian form as $\pm(1+i)/\sqrt{2}$. These are two points on opposite sides of the origin, as shown in the figure below.



(d) The solutions are:

$$z = (-i)^{1/3} = e^{(1/3)\cdot(\log|-i| + i\arg(-i) + 2\pi i n)} = e^{-i\pi/6 + 2\pi i n/3}.$$

These form a triangle in the complex plane, as shown in the figure below.

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30. If $\omega^n = 1$, determine the possible values of $1 + \omega + \omega^2 + \cdots + \omega^{n-1}$, and interpret your result geometrically.

---

**Solution:** This is a geometric progression, so summing the terms we have:

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = \frac{1 - \omega^n}{1 - \omega} = 0,$$

*provided* that $\omega \neq 1$. Hence the possible values are:

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = \begin{cases} n, & \text{if } \omega = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Geometrically, in the case $\omega \neq 1$, this corresponds to us following the position vectors of the vertices of the polygon that is formed by the roots of unity (or, a sub-polygon). In the case $\omega \neq 1$, this necessarily ends up taking us to zero.

31. Show that the roots of the equation $z^{2n} - 2bz^n + c = 0$ will, for general complex values of $b$ and $c$ and integral values of $n$, lie on two circles in the Argand diagram. Give a condition on $b$ and $c$ such that the circles coincide. Find the largest possible value for $|z_1 - z_2|$, if $z_1$ and $z_2$ are roots of $z^6 - 2z^3 + 2 = 0$.

---

➼ **Solution:** Solving the quadratic, we have:

$$z^n = \frac{2b \pm \sqrt{4b^2 - 4c}}{2} = b \pm \sqrt{b^2 - c}.$$

Taking the $1/n$th power, we have:

$$z = \left(b \pm \sqrt{b^2 - c}\right)^{1/n} = \left|b \pm \sqrt{b^2 - c}\right|^{1/n} e^{i\arg(b \pm \sqrt{b^2-c})/n + 2\pi i k/n}.$$

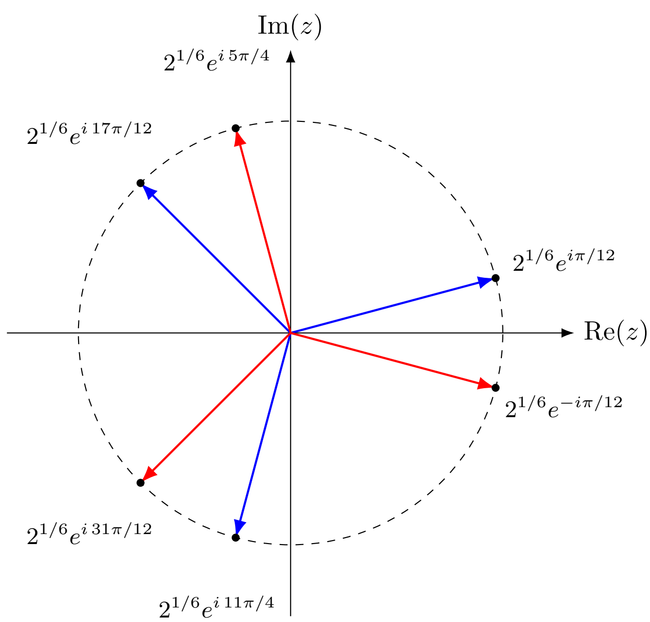
where $k$ is an integer. Thus the solutions lie on two circles, centred on the origin, of radii $|b \pm \sqrt{b^2 - c}|^{1/n}$ respectively. The circles coincide if and only if:

$$|b + \sqrt{b^2 - c}| = |b - \sqrt{b^2 - c}|.$$

In the case where $n = 3$, $b = 1$ and $c = 2$, we have the roots:

$$\begin{aligned} z &= \left|1 \pm \sqrt{1-2}\right|^{1/3} e^{i\arg(1 \pm \sqrt{1-2})/3 + 2\pi i k/3} \\ &= \left|1 \pm i\right|^{1/3} e^{i\arg(1 \pm i)/3 + 2\pi i k/3} \\ &= 2^{1/6} e^{\pm i\pi/12 + 2\pi i k/3}, \end{aligned}$$

for $k$ an integer. Therefore, we have clusters of pairs of roots which have an angle $\pi/6$ between them, separated into three groups which are rotated by $2\pi/3$.




From the figure, we see that the roots are furthest apart when they are inclined at an angle $2\pi/3 + \pi/6 = 5\pi/6$. By the cosine rule, the distance between the roots is:

$$\sqrt{2^{2/6} + 2^{2/6} - 2 \cdot 2^{2/6}\cos(5\pi/6)} = 2^{1/6}\sqrt{2 + \sqrt{3}}.$$

**Trigonometry with complex numbers**

32. Prove *De Moivre's formula*, $(\cos(\theta)+i\sin(\theta))^n = \cos(n\theta)+i\sin(n\theta)$. Hence, solve the equation $16\sin^5(\theta) = \sin(5\theta)$ by expressing $\sin(5\theta)$ in terms of $\sin(\theta)$ and its powers.

---

➾ **Solution:** Using Euler's formula, we have:

$$(\cos(\theta)+i\sin(\theta))^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta)+i\sin(n\theta).$$

To solve the given equation, note that:

$$\sin(5\theta) = \mathrm{Im}\left((\cos(\theta)+i\sin(\theta))^5\right) = \sin^5(\theta) - 10\sin^3(\theta)\cos^2(\theta) + 5\sin(\theta)\cos^4(\theta).$$

Using the identity $\cos^2(\theta) = 1-\sin^2(\theta)$, we can simplify this to read:

$$\begin{aligned}\sin(5\theta) &= \sin^5(\theta) - 10\sin^3(\theta)(1-\sin^2(\theta)) + 5\sin(\theta)(1-\sin^2(\theta))^2\\ &= \sin^5(\theta) - 10\sin^3(\theta) + 10\sin^5(\theta) + 5\sin(\theta) - 10\sin^3(\theta) + 5\sin^5(\theta)\\ &= 16\sin^5(\theta) - 20\sin^3(\theta) + 5\sin(\theta).\end{aligned}$$

Therefore, the equation $16\sin^5(\theta) = \sin(5\theta)$ is equivalent to the equation:

$$0 = 4\sin^3(\theta) - \sin(\theta) = \sin(\theta)(2\sin(\theta)-1)(2\sin(\theta)+1).$$

Setting each factor to zero, we have:

- $\sin(\theta) = 0$ if and only if $\theta = n\pi$ for $n$ an integer;
- $\sin(\theta) = \frac{1}{2}$ if and only if $\theta = \pi/6 + 2n\pi, 5\pi/6 + 2n\pi$ for $n$ an integer;
- $\sin(\theta) = -\frac{1}{2}$ if and only if $\theta = -\pi/6 + 2n\pi, 7\pi/6 + 2n\pi$ for $n$ an integer.

33. Starting from Euler's formula, show that the trigonometric functions can be written in terms of complex exponentials as:

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Learn these formulae off by heart. Hence, express $\sin^5(\theta)$ in terms of $\sin(\theta)$, $\sin(3\theta)$ and $\sin(5\theta)$.

---

➼ **Solution:** Euler's formula applied to $e^{i\theta}$ and $e^{-i\theta}$ gives:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta).$$

Adding these formulae, we get:

$$2\cos(\theta) = e^{i\theta} + e^{-i\theta} \qquad \Leftrightarrow \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Subtracting these formulae, we get:

$$2i\sin(\theta) = e^{i\theta} - e^{-i\theta} \qquad \Leftrightarrow \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Hence, we have:

$$\sin^5(\theta) = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^5 = \frac{e^{5i\theta} - 5e^{3i\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - e^{-5i\theta}}{32i}.$$

Collecting like terms, we see that:

$$\sin^5(\theta) = \frac{\sin(5\theta)}{16} - \frac{5\sin(3\theta)}{16} + \frac{5\sin(\theta)}{8}.$$

34. Show that if $x, y \in \mathbb{R}$, the equation $\cos(y) = x$ has the solutions $y = \pm i \log\left(x + i\sqrt{1-x^2}\right) + 2n\pi$ for integer $n$.

---

➻ **Solution:** Using the formula for $\cos(y)$ in terms of complex exponentials, the equation $\cos(y) = x$ can be rewritten as:

$$\frac{e^{iy} + e^{-iy}}{2} = x \qquad \Leftrightarrow \qquad e^{2iy} - 2xe^{iy} + 1 = 0.$$

This is a quadratic equation for $e^{iy}$; solving using the quadratic formula we have:

$$e^{iy} = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1} = x \pm i\sqrt{1 - x^2}.$$

Taking the complex logarithm, we have:

$$iy = \log\left(x \pm i\sqrt{1-x^2}\right) + 2\pi i n,$$

where $n$ is an integer (we assume here that $\log$ takes its principal value, so that a specific argument choice is made). Dividing by $i$, we have:

$$y = -i\log\left(x \pm i\sqrt{1-x^2}\right) + 2\pi n,$$

where $n$ is an integer. This is close to the final answer. To finish, observe that:

$$x - i\sqrt{1-x^2} = \frac{(x - i\sqrt{1-x^2})(x + i\sqrt{1-x^2})}{x + i\sqrt{1-x^2}} = \frac{x^2 + 1 - x^2}{x + i\sqrt{1-x^2}} = \frac{1}{x + i\sqrt{1-x^2}}.$$

Hence, $\log(x - i\sqrt{1-x^2}) = -\log(x + i\sqrt{1-x^2})$. This implies that the solution of the equation may be written as:

$$y = -i\log\left(x \pm i\sqrt{1-x^2}\right) + 2\pi n = \pm i \log\left(x + i\sqrt{1-x^2}\right) + 2\pi n,$$

where $n$ is an integer, as required.

---

35. Let $\theta \neq 2p\pi$ for $p \in \mathbb{Z}$. Show that $\displaystyle\sum_{n=0}^{N-1} \cos(n\theta) = \frac{\cos\left((N-1)\theta/2\right)\sin\left(N\theta/2\right)}{\sin\left(\theta/2\right)}$. What happens if $\theta = 2p\pi$?

---

➻ **Solution:** We have:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos(n\theta) &= \mathrm{Re}\left[\sum_{n=0}^{N-1} e^{in\theta}\right] \\
&= \mathrm{Re}\left[\frac{1 - e^{iN\theta}}{1 - e^{i\theta}}\right] && (\text{if } e^{i\theta} \neq 1) \\
&= \mathrm{Re}\left[\frac{e^{iN\theta/2}}{e^{i\theta/2}} \frac{e^{-iN\theta/2} - e^{iN\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}}\right] \\
&= \mathrm{Re}\left[e^{i(N-1)\theta/2} \cdot \frac{-2i\sin(N\theta/2)}{-2i\sin(\theta/2)}\right] \\
&= \frac{\cos((N-1)\theta/2)\sin(N\theta/2)}{\sin(\theta/2)},
\end{aligned}$$

as required. This holds provided that $e^{i\theta} \neq 1$, in which case we cannot sum the geometric series in the second line. This occurs if and only if $\theta = 2p\pi$ for an integer $p$. In this case, we have the sum:

$$\sum_{n=0}^{N-1}\cos(2p\pi n) = \sum_{n=0}^{N-1} 1 = N.$$

### Hyperbolic functions

36 (a) Give the definitions of $\cosh(x)$ and $\sinh(x)$ in terms of exponentials.

(b) Hence, show that $\cos(x) = \cosh(ix)$ and $i\sin(x) = \sinh(ix)$. Deduce *Osborn's rule*: 'a hyperbolic trigonometric identity can be deduced from a circular trigonometric identity[1] by replacing each trigonometric function with its hyperbolic counterpart *except* where sine enters quadratically, where we include an extra factor of $-1$.'

(c) Using Osborn's rule, write down the formula for $\tanh(x+y)$ in terms of $\tanh(x), \tanh(y)$.

---

➾ **Solution:** (a) We have:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

(b) Comparing to Question 33, we immediately notice that $\cosh(ix) = \cos(x)$ and $\sinh(ix) = i\sin(x)$, as required. In particular, we see that if we have a trigonometric identity, we can turn it into a hyperbolic identity by replacing cosine with hyperbolic cosine, and replacing sine with hyperbolic cosine multiplied by $i$ - this means that whenever we have a sine squared, then it becomes *negative* hyperbolic sine squared.

(c) We have to be a bit careful here - we just said that terms that are quadratic in sine receive a minus sign when we convert from trigonometric to hyperbolic identities. However, this *also* applies to produces of tangents, since $\tan(x) = \sin(x)/\cos(x)$. Hence the compound angle identity:

$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

gets converted to the hyperbolic identity:

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}.$$

---

37. Find the real and imaginary parts of the following complex numbers:

(a) $\log\left[\sinh\left(\frac{i\pi}{2}\right) + \cosh\left(\frac{9i\pi}{2}\right)\right]$, (b) $\sum_{n=1}^{121}\left[\tanh\left(\frac{in\pi}{4}\right) - \tanh\left(\frac{in\pi}{4} - \frac{i\pi}{4}\right)\right]$.

---

➾ **Solution:** (a) We have:

$$\sinh\left(\frac{i\pi}{2}\right) = i\sin\left(\frac{\pi}{2}\right) = i, \qquad \cosh\left(\frac{9i\pi}{2}\right) = \cos\left(\frac{9\pi}{2}\right) = 0.$$

Hence we must evaluate:

$$\log(i) = \log|i| + i\arg(i) + 2n\pi i = \frac{i\pi}{2} + 2n\pi i,$$

where $n$ is an integer.

[1] Provided the arguments of all the circular trigonometric functions are homogeneous linear polynomials in the variables of interest.

(b) Here, we spot that this is a telescoping sum:

$$\begin{aligned}
&\sum_{n=1}^{121}\left[\tanh\left(\frac{in\pi}{4}\right)-\tanh\left(\frac{in\pi}{4}-\frac{i\pi}{4}\right)\right]\\
&\quad=\tanh\left(\frac{i\pi}{4}\right)-\tanh(0)+\tanh\left(\frac{2i\pi}{4}\right)-\tanh\left(\frac{i\pi}{4}\right)+\cdots+\tanh\left(\frac{121i\pi}{4}\right)-\tanh\left(\frac{120i\pi}{4}\right)\\
&\quad=\tanh\left(\frac{121i\pi}{4}\right)\\
&\quad=i\tan\left(\frac{121\pi}{4}\right)\\
&\quad=i\tan\left(30\pi+\frac{\pi}{4}\right)\\
&\quad=i.
\end{aligned}$$

---

38. Find the real and imaginary parts of the function $\tan(z^*)$.

---

**Solution:** Observe that:

$$\tan(iy)=\frac{\sin(iy)}{\cos(iy)}=\frac{e^{-y}-e^{y}}{i(e^{-y}+e^{y})}=\frac{i\sinh(y)}{\cosh(y)}=i\tanh(y).$$

Hence, we have:

$$\tan(z^*)=\tan(x-iy)=\frac{\tan(x)-\tan(iy)}{1+\tan(x)\tan(iy)}=\frac{\tan(x)-i\tanh(y)}{1+i\tan(x)\tanh(y)}.$$

Realising the denominator, we have:

$$\tan(z^*)=\frac{(\tan(x)-i\tanh(y))(1-i\tan(x)\tanh(y))}{1+\tan^2(x)\tanh^2(y)}=\frac{\tan(x)-\tan(x)\tanh^2(y)-i\tanh(y)(1+\tan^2(x))}{1+\tan^2(x)\tanh^2(y)}.$$

It follows that the real part is:

$$\tan(x)\cdot\frac{1-\tanh^2(y)}{1+\tan^2(x)\tanh^2(y)}=\frac{\tan(x)}{\cosh^2(y)+\tan^2(x)\sinh^2(y)}=\frac{\sin(x)\cos(x)}{\cos^2(x)\cosh^2(y)+\cos^2(x)\sinh^2(y)}.$$

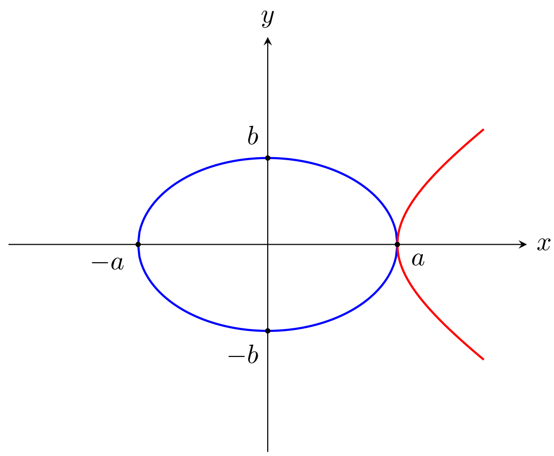
The imaginary part is:

$$-\tanh(y)\cdot\frac{1+\tan^2(x)}{1+\tan^2(x)\tanh^2(y)}=\frac{\tanh(y)}{\cos^2(x)+\sin^2(x)\tanh^2(y)}=\frac{\sinh(x)\cosh(x)}{\cos^2(x)\cosh^2(y)+\sin^2(x)\sinh^2(y)}.$$

39. Let $b \geq a > 0$ be fixed, and let $\theta$ be a variable parameter. Find the Cartesian equations of the two parametric curves: (a) $(x, y) = (a\cos(\theta), b\sin(\theta))$; (b) $(x, y) = (a\cosh(\theta), b\sinh(\theta))$, and sketch them in the plane. [*This explains why hyperbolic functions are called hyperbolic functions!*]

---

➾ **Solution:** (a) We have $(x/a)^2 + (y/b)^2 = \cos^2(\theta) + \sin^2(\theta) = 1$. (b) We have $(x/a)^2 - (y/b)^2 = \cosh^2(\theta) - \sinh^2(\theta) = 1$. In the first case (a), we have an ellipse with major semi-axis $b$ and minor semi-axis $a$. In the second case, we have a hyperbola (although only the right branch, because $x > 0$). Sketches are given below.

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40. Express $\cosh^{-1}(x)$, $\sinh^{-1}(x)$ and $\tanh^{-1}(x)$ as logarithms, justifying any sign choices you make.

---

➾ **Solution:** Let $y = \cosh^{-1}(x)$. Then:

$$\cosh(y) = x \qquad \Leftrightarrow \qquad \frac{e^y + e^{-y}}{2} = x \qquad \Leftrightarrow \qquad e^{2y} - 2xe^y + 1 = 0.$$

This is a quadratic equation for $e^y$, which we can solve to give:

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

We need $x > 1$ for this to exist, which is perfectly consistent with taking the inverse function $\cosh^{-1}(x)$, which should only exist on this range. Hence $x + \sqrt{x^2 - 1} > 1$, whilst $x - \sqrt{x^2 - 1} < 1$. The first case would give $y > 0$, and the second case would give $y < 0$. By convention, we choose $\cosh^{-1}(x) > 0$, which gives:

$$y = \log(x + \sqrt{x^2 - 1}).$$

---

Now, let $y = \sinh^{-1}(x)$. Then:

$$\sinh(y) = x \qquad \Leftrightarrow \qquad \frac{e^y - e^{-y}}{2} = x \qquad \Leftrightarrow \qquad e^{2y} - 2xe^y - 1 = 0.$$

This is a quadratic equation for $e^y$, which we can solve to give:

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

Note that $x - \sqrt{x^2+1} < 0$, hence this cannot correspond to a real solution of the equation. Thus we have:

$$y = \log(x + \sqrt{x^2+1}).$$

---

Finally, let $y = \tanh^{-1}(x)$. Then:

$$\tanh(y) = x \qquad \Leftrightarrow \qquad \frac{e^y - e^{-y}}{e^y + e^{-y}} = x \qquad \Leftrightarrow \qquad \frac{e^{2y}-1}{e^{2y}+1} = x.$$

Rearranging, we have:

$$e^{2y} - 1 = xe^{2y} + x \qquad \Leftrightarrow \qquad 1 + x = (1-x)e^{2y} \qquad \Leftrightarrow \qquad y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

---

41. Solve the equation $\cosh(x) = \sinh(x) + 2\text{sech}(x)$, giving the solutions as logarithms.

---

➾ **Solution:** Dividing by $\cosh(x)$ (which is never zero), we have:

$$1 = \tanh(x) + 2\text{sech}^2(x).$$

Using the identity $1 - \tanh^2(x) = \text{sech}^2(x)$, we can rearrange this to a quadratic equation for $\tanh(x)$:

$$1 = \tanh(x) + 2(1 - \tanh^2(x)) \qquad \Leftrightarrow \qquad 0 = 2\tanh^2(x) - \tanh(x) - 1 = (2\tanh(x)+1)(\tanh(x)-1).$$

Hence we have:

$$\tanh(x) = 1 \qquad \text{or} \qquad \tanh(x) = -\frac{1}{2}.$$

The first case is impossible, so we get the unique solution:

$$x = \tanh^{-1}\left(-\frac{1}{2}\right) = \frac{1}{2}\log\left(\frac{1/2}{3/2}\right) = -\frac{1}{2}\log(3).$$

---

42. Find all solutions to the equations: (a) $\cosh(z) = i$; (b) $\sinh(z) = -2$.

---

➾ **Solution:**

(a) We have:

$$\frac{e^z + e^{-z}}{2} = i \qquad \Leftrightarrow \qquad e^{2z} - 2ie^z + 1 = 0.$$

Solving this quadratic equation, we have:

$$e^z = \frac{-2i \pm \sqrt{-4-4}}{2} = i\left(-1 \pm \sqrt{2}\right)$$

Hence:

$$z = \log\left(i\left(-1 \pm \sqrt{2}\right)\right) = \log\left|i\left(-1 \pm \sqrt{2}\right)\right| + i\arg\left(i\left(-1 \pm \sqrt{2}\right)\right) + 2n\pi i$$

$$= \log\left|\sqrt{2} \pm 1\right| + \frac{i\pi}{2} + 2n\pi i,$$

for $n$ an integer.

(b) We have:

$$\frac{e^z - e^{-z}}{2} = -2 \qquad \Leftrightarrow \qquad e^{2z} + 4e^z - 1 = 0.$$

Solving this quadratic equation, we have:

$$e^z = \frac{-4 \pm \sqrt{16+4}}{2} = -2 \pm \sqrt{5}.$$

Taking the logarithm, we have:

$$z = \log\left(-2 \pm \sqrt{5}\right) = \log\left|\sqrt{5} \pm 2\right| + i \arg\left(-2 \pm \sqrt{5}\right) + 2n\pi i,$$

which gives two families of solutions:

$$z = \log\left|\sqrt{5} + 2\right| + i\pi + 2n\pi, \qquad z = \log\left|\sqrt{5} - 2\right| + 2n\pi,$$

for $n$ an integer.