# Part IA: Mathematics for Natural Sciences B
# Examples Sheet 4: Differential calculus, limits and continuity

## Model Solutions

*Please send all comments and corrections to jmm232@cam.ac.uk.*

---

**Limit definition of the derivative**

1. Let $y \equiv y(x)$ be a function of $x$. Define the *derivative* $dy/dx$ of $y$ as a limit. Using the limit definition:

   (a) show that differentiation is a linear operation;

   (b) find the derivative of $y(x) = x^n$, for $n = 0, 1, 2, 3, \ldots$.

Hence obtain the derivative of $ax + bx^2 \sin(\theta)$, where $a, b, \theta$ are real constants.

---

**Solution:** The derivative, as a function of $x$, is defined by:

$$\frac{dy}{dx}(x) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h},$$

when this limit exists. The derivative is a new function of $x$.

---

(a) Let $y_1(x), y_2(x)$ be two functions of $x$. Then for any two real numbers $a, b$, we have:

$$\begin{aligned}\frac{d}{dx}\left(ay_1 + by_2\right) &= \lim_{h\to 0} \frac{ay_1(x+h) + by_2(x+h) - ay_1(x) - by_2(x)}{h} \\ &= a\lim_{h\to 0} \frac{y_1(x+h) - y_1(x)}{h} + b\lim_{h\to 0} \frac{y_2(x+h) - y_2(x)}{h} \\ &= a\frac{dy_1}{dx} + b\frac{dy_2}{dx},\end{aligned}$$

using basic properties of limits. Hence differentiation is a linear operation.

---

(b) Let $y(x) = x^n$. Using the binomial expansion, $(x+h)^n = x^n + \binom{n}{1}hx^{n-1} + \binom{n}{2}h^2x^{n-2} + \cdots + h^n$, we have:

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{x^n + hnx^{n-1} + h^2 g(x,h) - x^n}{h} = nx^{n-1} + \lim_{h\to 0}\left[hg(x,h)\right] = nx^{n-1},$$

where $g(x,h)$ is a function of $x, h$ which is finite when $h \to 0$. Thus the derivative is given by $nx^{n-1}$.

---

Using linearity and the power rule, we have:

$$\frac{d}{dx}\left(ax + bx^2\sin(\theta)\right) = a\frac{dx}{dx} + b\sin(\theta)\frac{d(x^2)}{dx} = a + 2bx\sin(\theta).$$

2. (a) Using *only* the limit definition, show that for $a > 0$, the derivative of $y(x) = a^x$ is proportional to $a^x$.

   (b) One definition of the number $e$ is the value of $a$ for which the proportionality constant in the previous part is $1$. Using *only* this definition, show that the derivative of $a^x$ is given by $\log(a)a^x$.

---

➾ **Solution:** (a) We have:

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h},$$

using properties of limits. The remaining limit is independent of $x$, so is just a constant. Thus we have:

$$\frac{dy}{dx} = ka^x$$

for some proportionality constant $k$.

---

(b) We are given that $e$ is the value of $a$ for which the proportionality constant is $1$. Therefore, we have been told that:

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

To compute the proportionality constant for general $a$, we then have:

$$\lim_{h\to 0} \frac{a^h - 1}{h} = \log(a) \lim_{h\to 0} \frac{e^{h\log(a)} - 1}{h\log(a)} = \log(a) \lim_{h'\to 0} \frac{e^{h'} - 1}{h'}.$$

where we have substituted $h' = h\log(a)$ in the limit in the final step. Using the definition of $e$, we see that the proportionality constant for general $a$ is $\log(a)$, as required.

---

**Rules of differentiation**

3. Let $y \equiv y(x)$, $u \equiv u(x)$ and $v \equiv v(x)$ be functions of $x$. Using the limit definition of the derivative, prove the following rules of differentiation:

(a) $\dfrac{d}{dx}(u(v)) = \dfrac{dv}{dx}\dfrac{du}{dv}$, (b) $\dfrac{d}{dx}(uv) = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$, (c) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{\dfrac{du}{dx}v - u\dfrac{dv}{dx}}{v^2}$, (d) $\dfrac{dy}{dx} = \left(\dfrac{dx}{dy}\right)^{-1}$.

*[Recall that these rules are called the chain rule, the product rule, the quotient rule, and the reciprocal rule, respectively. Make sure you know them off by heart!]* Rewrite these rules in terms of Lagrange's 'primed' notation for derivatives.

---

➾ **Solution:** (a) For the chain rule, we have:

$$\begin{aligned}
\frac{d}{dx}(u(v)) &= \lim_{h\to 0} \frac{u(v(x+h)) - u(v(x))}{h} \\
&= \lim_{h\to 0} \frac{u(v(x) + v(x+h) - v(x)) - u(v(x))}{v(x+h) - v(x)} \cdot \frac{v(x+h) - v(x)}{h} \\
&= \lim_{h\to 0} \frac{u(v(x) + v(x+h) - v(x)) - u(v(x))}{v(x+h) - v(x)} \cdot \lim_{h\to 0} \frac{v(x+h) - v(x)}{h}.
\end{aligned}$$

Now consider substituting $h' = v(x+h) - v(x)$ in the first limit. This is okay, because $h' = v(x+h) - v(x) \to 0$ as $h \to 0$, if we assume that $v$ is a continuous function (see later on the sheet!). This leaves us with:

$$\frac{d}{dx}(u(v)) = \lim_{h' \to 0} \frac{u(v(x) + h') - u(v(x))}{h'} \cdot \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} = \frac{du}{dv} \cdot \frac{dv}{dx},$$

where the notation $du/dv$ means differentiate $u(v)$, considered as a function of $v$ (not of $x$!).

---

(b) For the product rule, we have:

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\ &= \lim_{h \to 0} \left[\frac{(u(x+h) - u(x))v(x+h)}{h} + \frac{u(x)(v(x+h) - v(x))}{h}\right] \\ &= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \cdot \lim_{h \to 0} v(x+h) + u(x) \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} \\ &= \frac{du}{dx} v + u \frac{dv}{dx},\end{aligned}$$

as required.

---

(c) For the quotient rule, we have:

$$\begin{aligned}\frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h \to 0} \frac{u(x+h)/v(x+h) - u(x)/v(x)}{h} \\ &= \lim_{h \to 0} \frac{u(x+h)v(x) - u(x)v(x+h)}{hv(x)v(x+h)} \\ &= \frac{1}{v} \cdot \lim_{h \to 0} \frac{1}{v(x+h)} \cdot \lim_{h \to 0} \frac{u(x+h)v(x) - u(x)v(x+h)}{h} \\ &= \frac{1}{v^2} \cdot \lim_{h \to 0} \left[\frac{(u(x+h) - u(x))v(x)}{h} + \frac{u(x)(v(x) - v(x+h))}{h}\right] \\ &= \frac{\frac{du}{dx} v - u \frac{dv}{dx}}{v^2}.\end{aligned}$$

---

(d) Finally, for the reciprocal rule, we must first figure out what all the derivatives depend on. On the left hand side, we have:

$$\frac{dy}{dx} \equiv \frac{dy}{dx}(x)$$

is a function of $x$. On the right hand side, we have:

$$\frac{dx}{dy} \equiv \frac{dx}{dy}(y) \equiv \frac{dx}{dy}(y(x)),$$

which is a function of $y$, but can be considered to be a function of $x$ if we substitute the formula for $y$ in terms of $x$, $y \equiv y(x)$. We will need to compare both sides as functions of $x$ in order to prove the result!

On the right hand side, we have:

$$\frac{dx}{dy}(y) = \lim_{h\to 0} \frac{x(y+h) - x(y)}{h}.$$

Substitute $h = y(x+h') - y(x)$, so that as $h' \to 0$, we also have $h \to 0$. Then this quotient becomes:

$$\lim_{h'\to 0} \frac{x(y(x) + y(x+h') - y(x)) - x(y(x))}{y(x+h') - y(x)} = \lim_{h\to 0} \frac{h'}{y(x+h') - y(x)} = \left(\frac{dy}{dx}\right)^{-1},$$

where we have used the fact that $x(y(x)) = x$, since the first $x$ denotes $x$ as a function of $y$, then $y(x)$ denotes $y$ rewritten as a function of $x$, which must be inverses of one another.

---

We are also asked to rewrite these rules in Lagrange's 'primed' notation for derivatives. We have:

- The chain rule applies to a function of a function. The derivative of $f(g(x))$ is given by:

$$(f(g(x)))' = g'(x)f'(g(x)).$$

- The product rule applies to products of functions. The derivative of $f(x)g(x)$ is given by:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

- The quotient rule applies to functions divided by functions. The derivative of $f(x)/g(x)$ is given by:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

- The reciprocal rule applies to *inverses* of functions. In Leibniz notation, it states that:

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}.$$

  Importantly, the left hand side is a function of $x$, but the right hand side is a function of $y$. In order to write it in function notation, we should not that $y$ itself can be written as a function of $x$ as $y \equiv y(x)$. Then if $y = f^{-1}(x)$, in Lagrange's primed notation, the rule is telling us that:

$$\left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))}.$$

4. Using the rules you derived in the previous question, compute the derivatives of:

(a) $\log(x)$, (b) $3^{x^2}$, (c) $\dfrac{e^x}{x^3-1}$, (d) $x^3\log(x^2-7)$, (e) $\sqrt{x^3-e^x\log(x)}$.

---

**Solution:** We use each of the rules in turn:

(a) Let $y = \log(x)$. Then we have $x = e^y$, and so by the reciprocal rule we have:

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = (e^y)^{-1} = \frac{1}{e^y} = \frac{1}{x}.$$

(b) Rewriting this exponential to base $e$, we have $u = 3^{x^2} = e^{x^2\log(3)}$. Then by the chain rule (with $v = x^2\log(3)$), we have:

$$\frac{du}{dx} = 2x\log(3)e^{x^2\log(3)} = 2\log(3)x\cdot 3^{x^2}.$$

(c) Using the quotient rule, the derivative is given by:

$$\frac{e^x(3x^2)-(x^3-1)e^x}{(x^3-1)^2} = \frac{\left(1+3x^2-x^3\right)e^x}{(x^3-1)^2}.$$

(d) Using the product rule *and* the chain rule for the second factor, the derivative is given by:

$$3x^2\log(x^2-7) + \frac{x^3\cdot 2x}{x^2-7} = 3x^2\log(x^2-7) + \frac{2x^4}{x^2-7}.$$

(e) Using the chain rule and the product rule, we have:

$$\frac{d}{dx}\sqrt{x^3-e^x\log(x)} = \frac{1}{2}\frac{3x^2-e^x\log(x)-e^x/x}{\sqrt{x^3-e^x\log(x)}} = \frac{3x^3-e^x(x\log(x)+1)}{2x\sqrt{x^3-e^x\log(x)}}.$$

5. By writing each of the following trigonometric and hyperbolic functions in terms of exponentials, compute their derivatives: (a) $\cos(x)$; (b) $\sin(x)$; (c) $\cosh(x)$; (d) $\sinh(x)$; (e) $\tan(x)$; (f) $\tanh(x)$. Learn these derivatives off by heart.

---

**Solution:** We tackle each of the trigonometric and hyperbolic functions in turn.

(a) Using the complex formula for cosine from Sheet 3, we have $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$. Differentiating, we have:

$$\frac{d}{dx}\cos(x) = \frac{1}{2}(ie^{ix} - ie^{-ix}) = -\frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = -\sin(x),$$

using the complex formula for sine from Sheet 3.

(b) Similarly, we have $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$, so differentiating we have:

$$\frac{d}{dx}\sin(x) = \frac{1}{2i}\left(ie^{ix} + ie^{-ix}\right) = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos(x).$$

(c) Recall that $\cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$, we can differentiate to give:

$$\frac{d}{dx}\cosh(x) = \frac{1}{2}(e^{x} - e^{-x}) = \sinh(x).$$

(d) Similarly, we have $\sinh(x) = \frac{1}{2}(e^{x} - e^{-x})$, so differentiating we have:

$$\frac{d}{dx}\sinh(x) = \frac{1}{2}(e^{x} + e^{-x}) = \cosh(x).$$

(e) For $\tan(x)$, we have:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\frac{1}{2i}(e^{ix} - e^{-ix})}{\frac{1}{2}(e^{ix} + e^{-ix})} = -i\left(\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}\right).$$

Differentiating using the quotient rule, we have:

$$\frac{d}{dx}\tan(x) = -i\left(\frac{i(e^{ix} + e^{-ix})^2 - i(e^{ix} - e^{-ix})^2}{(e^{ix} + e^{-ix})^2}\right) = \frac{(e^{ix} + e^{-ix})^2 - (e^{ix} - e^{-ix})^2}{(e^{ix} + e^{-ix})^2}$$

$$= \frac{(2\cos(x))^2 - (2i\sin(x))^2}{(2\cos(x))^2} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \sec^2(x).$$

(f) The calculation for $\tanh(x)$ is extremely similar. We have:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\frac{1}{2}(e^{x} - e^{-x})}{\frac{1}{2}(e^{x} + e^{-x})} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}.$$

Differentiating using the quotient rule, we have:

$$\frac{d}{dx}\tanh(x) = \frac{(e^{x} + e^{-x})^2 - (e^{x} - e^{-x})^2}{(e^{x} + e^{-x})^2} = \frac{(2\cosh(x))^2 - (2\sinh(x))^2}{(2\cosh(x))^2}$$

$$= \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \operatorname{sech}^2(x).$$

6. Using: (i) the logarithmic formulae for the inverse hyperbolic functions you derived on Sheet 3; (ii) the reciprocal rule, compute the derivatives of: (a) $\cosh^{-1}(x)$; (b) $\sinh^{-1}(x)$; (c) $\tanh^{-1}(x)$. Learn these derivatives off by heart.

---

**Solution:** We have the following derivatives:

(a) (i) We previously derived $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$. Hence using the chain rule, we have the derivative:

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1 + x(x^2-1)^{-1/2}}{x + (x^2-1)^{1/2}} = \frac{1}{\sqrt{x^2-1}} \cdot \frac{1 + x(x^2-1)^{-1/2}}{1 + x(x^2-1)^{-1/2}} = \frac{1}{\sqrt{x^2-1}}.$$

(ii) Let $y(x) = \cosh^{-1}(x)$. Then $\cosh(y) = x$, which gives:

$$\frac{dx}{dy} = \sinh(y).$$

Using the reciprocal rule, and hyperbolic trigonometric identities, we then have:

$$\frac{dy}{dx} = \frac{1}{\sinh(y)} = \frac{1}{\sqrt{\cosh^2(y) - 1}} = \frac{1}{\sqrt{x^2-1}}.$$

(b) (i) We previously derived $\sinh^{-1}(x) = \log(x + \sqrt{x^2 + 1})$. Hence using the chain rule, we have the derivative:

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1 + x(x^2+1)^{-1/2}}{x + (x^2+1)^{1/2}} = \frac{1}{\sqrt{x^2+1}} \cdot \frac{1 + x(x^2+1)^{-1/2}}{1 + x(x^2+1)^{-1/2}} = \frac{1}{\sqrt{x^2+1}}.$$

(ii) Let $y(x) = \sinh^{-1}(x)$. Then $\sinh(y) = x$, which gives:

$$\frac{dx}{dy} = \cosh(y).$$

Using the reciprocal rule, and hyperbolic trigonometric identities, we then have:

$$\frac{dy}{dx} = \frac{1}{\cosh(y)} = \frac{1}{\sqrt{\sinh^2(y) + 1}} = \frac{1}{\sqrt{x^2+1}}.$$

(c) (i) We previously derived:

$$\tanh^{-1}(x) = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|.$$

Hence using the chain rule, we have the derivative (assuming $x$ is in the range where $(1+x)/(1-x)$ is positive):

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{2}\frac{d}{dx}\left(\log(1+x) - \log(1-x)\right) = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = \frac{1}{1-x^2}.$$

(ii) Let $y(x) = \tanh^{-1}(x)$. Then $\tanh(y) = x$, which gives:

$$\frac{dx}{dy} = \operatorname{sech}^2(y).$$

Using the reciprocal rule, and hyperbolic trigonometric identities, we then have:

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2(y)} = \frac{1}{\tanh^2(y) - 1} = \frac{1}{x^2-1}.$$

7. If $y \equiv y(x)$ is a function of $x$, show that $\dfrac{d^3x}{dy^3} = -\left(\dfrac{dy}{dx}\right)^{-4}\dfrac{d^3y}{dx^3} + 3\left(\dfrac{dy}{dx}\right)^{-5}\left(\dfrac{d^2y}{dx^2}\right)^2$. Verify this when $y = e^{2x}$.

---

**Solution:** This question is about expressing a higher derivative of $x$ with respect to $y$ in terms of derivatives of $y$ with respect to $x$. We start with a simple derivative, then differentiate repeatedly using the chain rule and reciprocal rule.

First, note that:

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$$

by the reciprocal rule.

Taking another derivative with respect to $y$, we have:

$$\frac{d^2x}{dy^2} = \frac{d}{dy}\left[\left(\frac{dy}{dx}\right)^{-1}\right].$$

But the argument of the derivative on the right hand side is a function of $x$, not a function of $y$ - how can we differentiate it? We must consider it to be a function of $y$ *through the dependence of $x$ on $y$*, $x \equiv x(y)$:

$$\left(\frac{dy}{dx}\right)^{-1} \equiv \left(\frac{dy}{dx}(x)\right)^{-1} \equiv \left(\frac{dy}{dx}(x(y))\right)^{-1}.$$

This implies that we should use the chain rule to differentiate. We have:

$$\begin{aligned}
\frac{d}{dy}\left[\left(\frac{dy}{dx}\right)^{-1}\right] &= \frac{dx}{dy}\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^{-1}\right] && \text{(chain rule)} \\
&= -\frac{dx}{dy}\left(\frac{dy}{dx}\right)^{-2}\frac{d^2y}{dx^2} && \text{(chain rule again)} \\
&= -\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2} && \text{(reciprocal rule)}
\end{aligned}$$

Now we have got the hang of things, the last derivative is easy. We repeat the same method:

$$\begin{aligned}
\frac{d^3x}{dy^3} &= \frac{d}{dy}\left[-\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2}\right] \\
&= \frac{dx}{dy}\frac{d}{dx}\left[-\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2}\right] && \text{(chain rule)} \\
&= 3\frac{dx}{dy}\left(\frac{dy}{dx}\right)^{-4}\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dx}{dy}\left(\frac{dy}{dx}\right)^{-3}\frac{d^3y}{dx^3} && \text{(product rule and chain rule)} \\
&= 3\left(\frac{dy}{dx}\right)^{-5}\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^{-4}\frac{d^3y}{dx^3} && \text{(reciprocal rule)}
\end{aligned}$$

To finish, we are asked to verify this result when $y = e^{2x}$. In this case, we have that $x$ as a function of $y$ is given by $x(y) \equiv \frac{1}{2}\log(y)$. Hence all the derivatives are:

$$\frac{dy}{dx} = 2e^{2x}, \qquad \frac{d^2y}{dx^2} = 4e^{2x}, \qquad \frac{d^3y}{dx^3} = 8e^{2x},$$

and:

$$\frac{dx}{dy} = \frac{1}{2y}, \qquad \frac{d^2x}{dy^2} = -\frac{1}{2y^2}, \qquad \frac{d^3x}{dy^3} = \frac{1}{y^3}.$$

Verifying the rule we derived above, we check:

$$3\left(\frac{dy}{dx}\right)^{-5}\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^{-4}\frac{d^3y}{dx^3} = \frac{3}{32e^{10x}} \cdot 16e^{4x} - \frac{1}{16e^{8x}}8e^{2x} = \left(\frac{3}{2} - \frac{1}{2}\right)\frac{1}{e^{6x}} = \frac{1}{y^3},$$

as required.

8. What is *implicit differentiation*, and why is it called implicit? Using: (a) implicit differentiation; (b) the reciprocal rule, find $dy/dx$ given $y + e^y \sin(y) = 1/x$, and make sure that your answers agree.

---

➼ **Solution:** Sometimes, we define $y$ as a function of $x$ through an equation, such as:

$$y^3 + \sin(y)\cos(x) + x = 0. \tag{$*$}$$

It is impossible to solve this equation *explicitly* to give something of the form $y = y(x)$, but we can still solve the equation in principle (e.g. numerically). Hence we say that this equation defines $y$ as a function of $x$ *implicitly*. In general, an implicit definition of $y$ as a function of $x$ is an equation of the form:

$$f(x, y) = 0$$

where $f$ is a function of two variables.

Implicit differentiation is the differentiation of $y$ when it is implicitly defined (which explains why it is called implicit differentiation). For example, when $y \equiv y(x)$ is defined through the equation ($*$), we can differentiate to get:

$$3y^2\frac{dy}{dx} + \cos(y)\cos(x)\frac{dy}{dx} - \sin(y)\sin(x) + 1 = 0,$$

where we have used the chain rule and product rules to compute the derivatives of each of the terms. Implicit differentiation is just a fancy name for the chain rule applied to implicit equations really!

---

(a) Given the equation $y + e^y \sin(y) = 1/x$, we can differentiate using implicit differentiation to give:

$$\frac{dy}{dx} + e^y \sin(y)\frac{dy}{dx} + e^y \cos(y)\frac{dy}{dx} = -\frac{1}{x^2}.$$

Rearranging, we obtain:

$$\frac{dy}{dx} = -\frac{1}{x^2(1 + e^y \sin(y) + e^y \cos(y))}.$$

(b) On the other hand, we can take the reciprocal of both sides to give:

$$x = \frac{1}{y + e^y \sin(y)}. \tag{$\dagger$}$$

Taking the derivative with respect to $y$, we have:

$$\frac{dx}{dy} = -\frac{1 + e^y \sin(y) + e^y \cos(y)}{(y + e^y \sin(y))^2}.$$

Taking the reciprocal, we have:

$$\frac{dy}{dx} = -\frac{(y + e^y \sin(y))^2}{1 + e^y \sin(y) + e^y \cos(y)} = -\frac{1}{x^2(1 + e^y \sin(y) + e^y \cos(y))},$$

where in the last step we substituted using equation (†). This is in agreement with the implicit differentiation method we used in part (a).

### Curve-sketching

9. State what it means for a function to be *even* and for a function to be *odd*, and explain the geometric significance of these definitions. Hence, decide whether the following functions are even, odd, both, or neither:

(a) $x$, (b) $\sin(x)$, (c) $e^x$, (d) $\sin(\frac{\pi}{2} - x)$, (e) $|x|\cos(x)$, (f) $\sqrt{x}$, (g) $2$, (h) $0$, (i) $\log\left|\dfrac{1+x}{1-x}\right|$.

---

**Solution:** A function is *even* if it satisfies $f(x) = f(-x)$. This means that it is invariant under a reflection in the $y$-axis. A function is *odd* if it satisfies $f(x) = -f(-x)$. This means that it is invariant under a rotation by $\pi$ around the origin.

For the given functions:

(a) For $f(x) = x$, we have $f(-x) = -x = -f(x)$, so this function is odd. Alternatively, we could spot that it is invariant under a rotation by $\pi$ around the origin. The function is *not* even, because it is not invariant under a reflection in the $y$-axis.

(b) For $f(x) = \sin(x)$, we have $f(-x) = \sin(-x) = -\sin(x) = -f(x)$, so this function is odd. It is not even.

(c) For $f(x) = e^x$, the function is neither reflectionally invariant in the $y$-axis, nor rotationally invariant under a rotation by $\pi$ around the origin. Hence this function is neither even nor odd.

(d) For $f(x) = \sin(\pi/2 - x)$, we notice that this function is a transformation of sine. It is a translation by $\pi/2$ in the negative $x$-direction $\sin(x) \mapsto \sin(x + \pi/2)$, then a reflection in the $y$-axis, $\sin(x + \pi/2) \mapsto \sin(\pi/2 - x)$. This just gives a cosine graph though, so $f(x) = \cos(x)$. This is evidently even, and not odd.

(e) For $f(x) = |x|\cos(x)$, we note that $f(-x) = |-x|\cos(-x) = |x|\cos(x) = f(x)$, so this function is even. It is not odd.

(f) For $f(x) = \sqrt{x}$, the function is not defined for $x < 0$, so this function is neither even nor odd.

(g) For $f(x) = 2$, we have $f(-x) = 2 = f(x)$, so the function is even. The function is not odd.

(h) For $f(x) = 0$, we have *both* $f(-x) = 0 = f(x)$ and $f(-x) = 0 = -f(x)$. Hence the function is *both* even *and* odd. The zero function is in fact the only example of a function which is both even and odd.

(i) For:

$$f(x) = \log\left|\frac{1+x}{1-x}\right|,$$

we could observe that $f(x) = 2\tanh^{-1}(x)$, which immediately shows the function is odd and not even (consider the graph of $\tanh(x)$, from Sheet 3!). Alternatively, we can calculate directly using laws of logarithms:

$$f(-x) = \log\left|\frac{1-x}{1+x}\right| = \log\left|\frac{1+x}{1-x}\right|^{-1} = -\log\left|\frac{1+x}{1-x}\right| = -f(x).$$

10. Write down a list of things you should consider when sketching the graph of a function. Compare with your supervision partner before the supervision, and exchange ideas!

---

➻ **Solution:** A possible list is the following:

- **$y$-intercepts.** Find the places where the function crosses the $y$-axis by setting $x = 0$.
- **$x$-intercepts.** Set $y = 0$, and solve the resulting equation to find the places where the function crosses the $x$-axis.
- **Vertical asymptotes.** These occur at points where the function is undefined. Consider the behaviour of the function just to the right of the point and just to the left of the point to work out whether it tends to positive or negative infinity at such a point.
- **Domain.** More generally than vertical asymptotes, you can think about whole regions (instead of just points) where the function is undefined. For example, the function $\log(x)$ does not exist for values of $x < 0$, so you can hatch out that whole region of the $xy$-plane, and not consider the graph there! Similarly for something like $\sqrt{1-x^2}$, where we require $|x| < 1$.
- **Horizontal and oblique asymptotes.** Consider the behaviour of the function in the limits $x \to \pm\infty$. If the function approaches a constant, then it has a *horizontal asymptote*. If the function is not constant in the limit, it might approach a linear function; consider performing polynomial division for a rational function to see whether this is the case - the function has an *oblique asymptote* in this case. For example:

  $$\frac{x^2}{1+x} = \frac{x(1+x)-(1+x)+1}{1+x} = x - 1 + \frac{1}{1+x}$$

  approaches $x - 1$ as $x \to \infty$, so has an oblique asymptote $y = x - 1$.
- **Stationary points.** Differentiate the function to find $dy/dx$. Stationary points of the function occur when $dy/dx = 0$. Often, the nature of a stationary point can be deduce simply by thinking about continuity of the function - for example, if the function has a stationary value of $1$ and approaches $0$ in the limits $x \to \pm\infty$, and is otherwise continuous (i.e. no vertical asymptotes), then that stationary value must be a *maximum value*.

  If you are struggling to tell whether a point is a maximum or minimum just using a continuity argument, you can also check by differentiating again to find $d^2y/dx^2$. If $d^2y/dx^2 > 0$ at a stationary point, then the point is a minimum. If $d^2y/dx^2 < 0$ at a stationary point, then the point is a maximum. On the other hand, these are both *sufficient conditions*, but not *necessary conditions* - for example, the function $y = x^4$ has a minimum at $x = 0$, but $d^2y/dx^2 = 12x^2$ is equal to zero at $x = 0$ (i.e. not positive!).
- **Inflection points.** These are points where $d^2y/dx^2 = 0$. These tell us where the derivative of the function changes sign, telling us whether the rate of increase of the function is itself increasing or decreasing (and similarly for the rate of decrease of the function). Usually these give us little information.
- **Regions of positivity and negativity.** We can hatch out regions of the $x, y$-plane where the function cannot exist through considering positive and negativity. For example, for the graph of $xy = 1$, we must have that $x, y$ have the same sign. Hence the graph cannot exist in the regions $x < 0, y > 0$ and $x > 0, y < 0$, so we can block them out before completing our sketch - this can be very useful for determining whether we have maxima or minima in a stationary point analysis!
- **Parity.** Whether a function is odd or even. Even functions are reflectionally symmetric in the $y$-axis, whilst odd functions are rotationally symmetric on a 180° rotation about the origin.
- **Periodicity.** Is the function periodic? The graphs of sine and cosine repeat every $2\pi$, for example - does the function you are considering have a similar periodic symmetry?

- **Sums and products of functions.** Some functions can be split into sums or products of simpler functions, for which the graphs are easier. The hyperbolic functions are such an example; e.g. $\cosh(x) = (e^x + e^{-x})/2$ is the average of the two graphs $e^x$, $e^{-x}$. An easy way to sketch $\cosh(x)$ is therefore to sketch the graphs of $e^x$, $e^{-x}$ as dashed lines, then try to draw a solid line representing the average of those two graphs.

  Similarly, the graph of something like $e^{-x}\cos(x)$ can be considered to be the graph of $e^{-x}$ modulated by a factor of $\cos(x)$, implying oscillations between $e^{-x}$ and $-e^{-x}$. This is an example given in Question 11.

---

11. Sketch the graphs of the following functions, explaining your reasoning in each case:

(a) $(x-3)^3 + 2x$, (b) $\dfrac{x}{1+x^2}$, (c) $\dfrac{x^2+3}{x-1}$, (d) $xe^x$, (e) $\dfrac{\log(x)}{1+x}$, (f) $\dfrac{1}{1-e^x}$, (g) $e^x\cos(x)$.

---

**Solution:** Performing the analysis in each case:

(a) This is just a cubic. We expand the bracket first to get:

$$y = x^3 - 9x^2 + 29x - 27.$$

Hence we see that the $y$-intercept is $-27$. Taking the derivative, we have:

$$\frac{dy}{dx} = 3x^2 - 18x + 29,$$

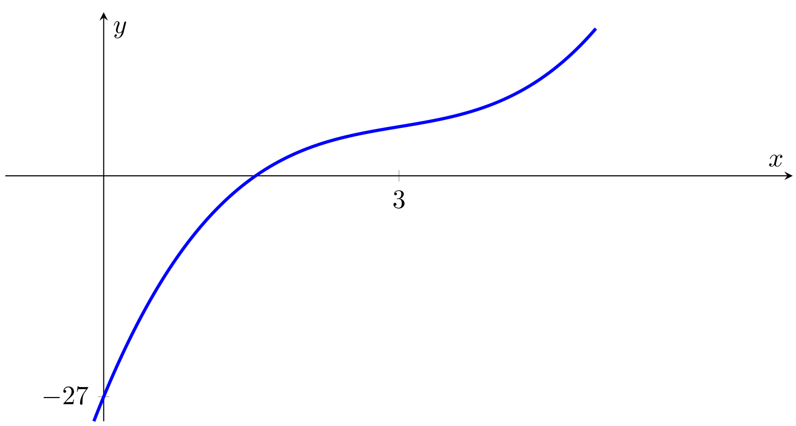
which has discriminant $18^2 - 4 \cdot 3 \cdot 29 = -24 < 0$. Hence, there are no stationary points. It is a positive cubic, so it simply increases from $-\infty$ to $\infty$.

If we want to be incredibly precise, we could compute the second derivative. This gives:

$$\frac{d^2y}{dx^2} = 6x - 18,$$

so there is a point of inflection at $x = 3$. This is where the cubic changes from reducing its rate of growth to increasing its rate of growth.

We now have enough information to draw the complete sketch:



(b) This function is defined for all values of $x$, since the denominator is always positive. It approaches zero as $x \to \infty$, coming from the positive direction since the numerator $x > 0$ in this region. It approaches zero as $x \to -\infty$ coming from the negative direction since the numerator $x < 0$ in this region. The $y$-intercept is at $x = 0$. The graph is also odd, since:

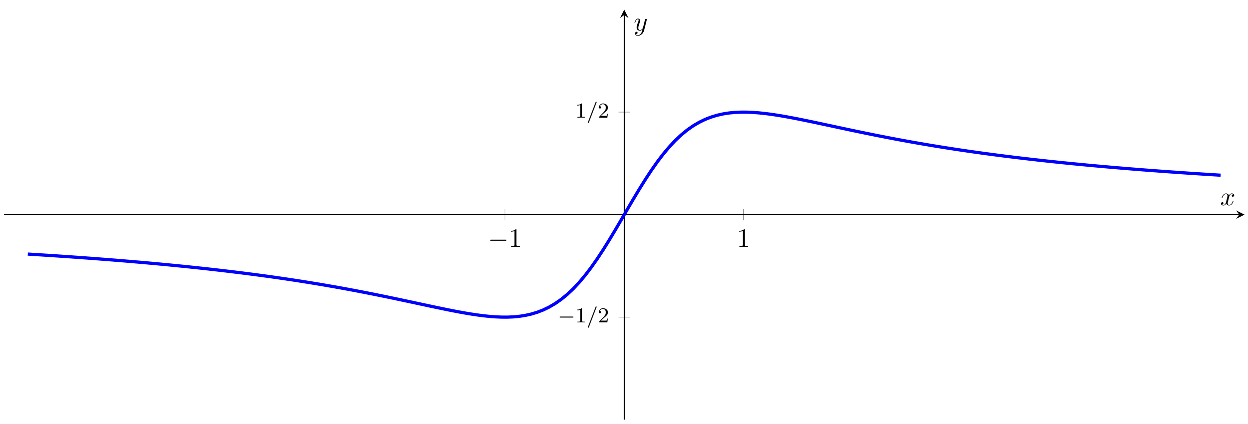
$$\frac{(-x)}{1+(-x)^2} = -\left(\frac{x}{1+x^2}\right).$$

Finally, we check for stationary points. Differentiating $y = x/(1+x^2)$, we have:

$$\frac{dy}{dx} = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}.$$

Hence there are stationary points at $x = \pm 1$. Thinking about the shape of the graph, and the information we have already obtained, the only possibility is that $x = 1$ is a maximum and $x = -1$ is a minimum.

We now have everything we need to sketch the graph:




(c) This graph has a vertical asymptote at $x = 1$. Just to the right of the asymptote, we have $x - 1 > 0$ and $x^2 + 3 > 0$, hence the graph is positive. So the graph approaches $+\infty$ from the right of $x = 1$. Just to the left of the asymptote, we have $x - 1 < 0$ and $x^2 + 3 > 0$, hence the graph is negative. So the graph approaches $-\infty$ from the left of $x = 1$.

Indeed, the sign of the graph is entirely determined by the sign of the denominator, so we see that the graph never crosses the $x$-axis. It is entirely positive for $x > 1$ and entirely negative for $x < 1$.

As $x$ approaches infinity, the behaviour of the graph is roughly linear. To work out the exact behaviour, we perform polynomial long division:

$$\frac{x^2+3}{x-1} = \frac{(x-1)^2+2x+2}{x-1} = \frac{(x-1)^2+2(x-1)+4}{x-1} = x-1+2+\frac{4}{x-1} = x+1+\frac{4}{x-1}.$$

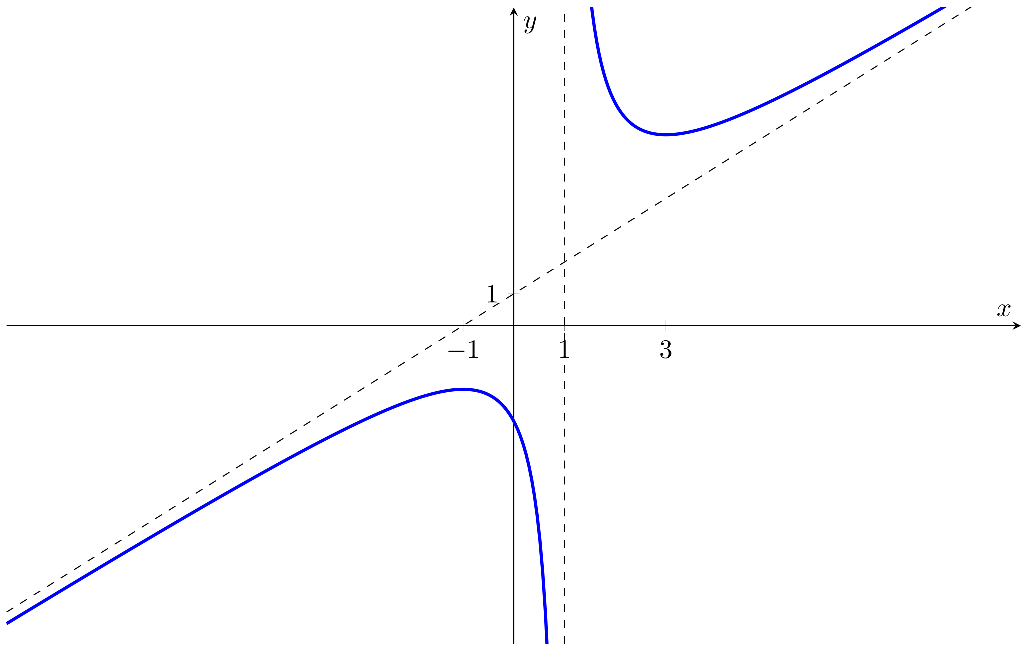
Therefore, as $x \to \infty$, the graph approaches the *oblique asymptote* $x + 1$. The same applies as $x \to -\infty$.

Finally, we check for stationary points. We have:

$$\frac{d}{dx}\left(\frac{x^2+3}{x-1}\right) = \frac{2x(x-1)-(x^2+3)}{(x-1)^2} = \frac{x^2-2x-3}{(x-1)^2} = \frac{(x-3)(x+1)}{(x-1)^2}.$$

Hence there are stationary points at $x = 3$ and $x = -1$. By the above work, the shape of the graph implies that $x = 3$ is necessarily a minimum and $x = -1$ is necessarily a maximum.

We now have everything we need to sketch the graph. We have:




(d) For this function, first note that the $y$-intercept is $0 \cdot e^0 = 0$, and the $x$-intercepts occur when $xe^x = 0$, i.e. only when $x = 0$. Thus the graph crosses the axes exactly once when $(x, y) = (0, 0)$.

Also, observe that $xe^x \to \infty$ as $x \to \infty$, since both $x, e^x \to \infty$ independently. Further as $x \to -\infty$, we have that $e^x \to 0$ exponentially fast, which beats the polynomial growth of $x$ (this can be proved with L'Hôpital's rule, as we shall see later on the sheet). Hence the function tends to zero as $x \to -\infty$.

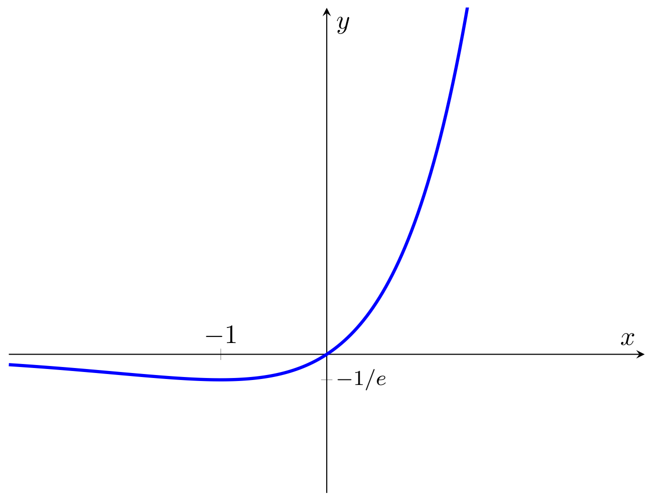
The signs of the functions are helpful here. We have $e^x > 0$, hence the sign of the graph $xe^x$ is the same as the sign of $x$. This means the graph is entirely negative for $x < 0$ and entirely positive for $x > 0$.

Finally, we check for stationary points. We have:

$$\frac{d}{dx}(xe^x) = e^x + xe^x = (x+1)e^x,$$

which is zero if and only if $x = -1$. Thus there is a unique stationary point at $(-1, -1/e)$. Considering all the data we have found on the shape of the function thus far, it follows by continuity of the function that this stationary point must be a minimum.

We now have all the information we need to sketch the graph:




(e) This function only exists for $x > 0$, because of the factor of $\log(x)$. The function would have a singularity at $x = -1$, but this is not in the range of $\log(x)$, so we don't need to consider that.

Next, we note that as $x \to \infty$, we have $\log(x)/(1 + x) \to 0$ since logarithmic growth is slower than polynomial growth (again, this can be proved with L'Hôpital's rule, which we shall see later on in the sheet).

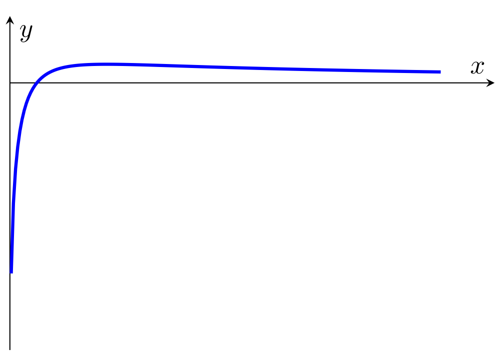
We also observe that $x + 1$ is positive everywhere in the range $x > 0$, and $\log(x)$ is negative for $0 < x < 1$ and positive for $1 < x$. This implies that the function itself is negative in the region $0 < x < 1$ and positive for $1 < x$. There is therefore exactly one $x$-intercept, at $x = 1$.

Finally, we check for stationary points. At stationary points, we have:

$$\frac{d}{dx}\left(\frac{\log(x)}{1+x}\right) = \frac{(1+x)/x - \log(x)}{(1+x)^2} = 0 \qquad \Leftrightarrow \qquad \frac{1}{x} + 1 = \log(x).$$

There is exactly one intersection of the graphs of $\log(x)$ and $1/x + 1$, so there is exactly one solution of this equation. This implies there is a single stationary point (although we don't know its coordinates in this case!). The stationary point must be a maximum, by continuity of the function.

We now have enough information to sketch the complete graph:



(f) This graph is singular when $e^x = 1$, which occurs if and only if $x = 0$. Therefore, there is a vertical asymptote at $x = 0$.

For $x > 0$, we have $e^x > 1$, hence $1 - e^x < 0$, and the graph is negative. For $x < 0$, we have $e^x < 1$, hence $1 - e^x > 0$, and the graph is positive.

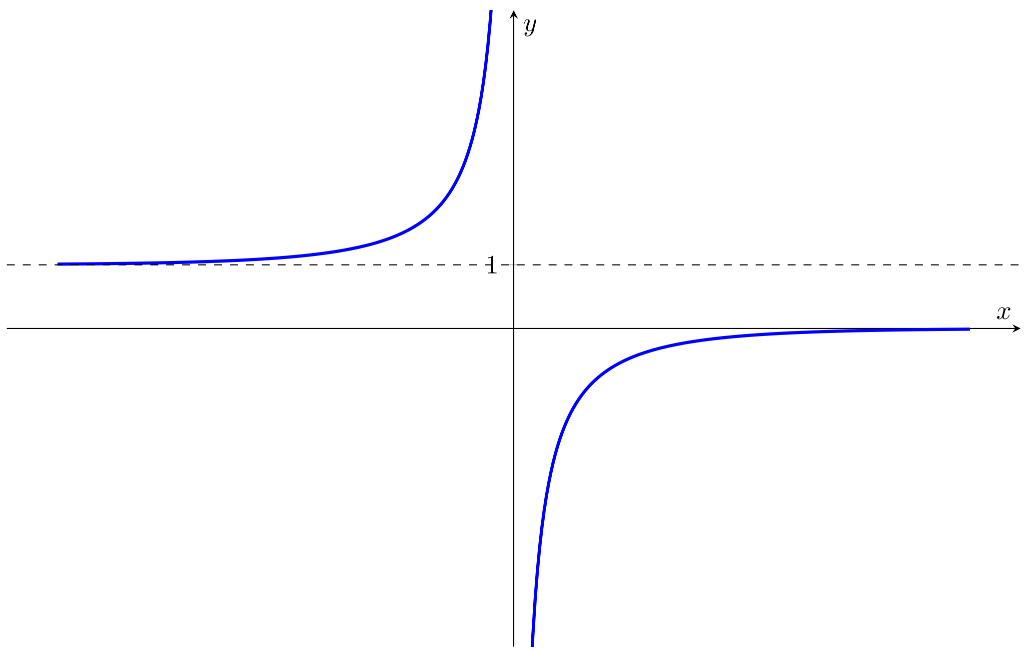
As $x \to \infty$, we have $e^x \to \infty$, so $1/(1 - e^x) \to 0$, which gives a horizontal asymptote at $y = 0$ as $x$ approaches positive infinity. On the other hand, as $x \to -\infty$, we have $e^x \to 0$, so $1/(1 - e^x) \to 1$, which gives a horizontal asymptote at $y = 1$ as $x$ approaches negative infinity.

Finally, we check for stationary points by differentiating. We have:

$$\frac{d}{dx}\left(\frac{1}{1 - e^x}\right) = \frac{e^x}{(1 - e^x)^2}.$$

This is never zero, so there are no stationary points.

We now have enough information to sketch the complete graph:



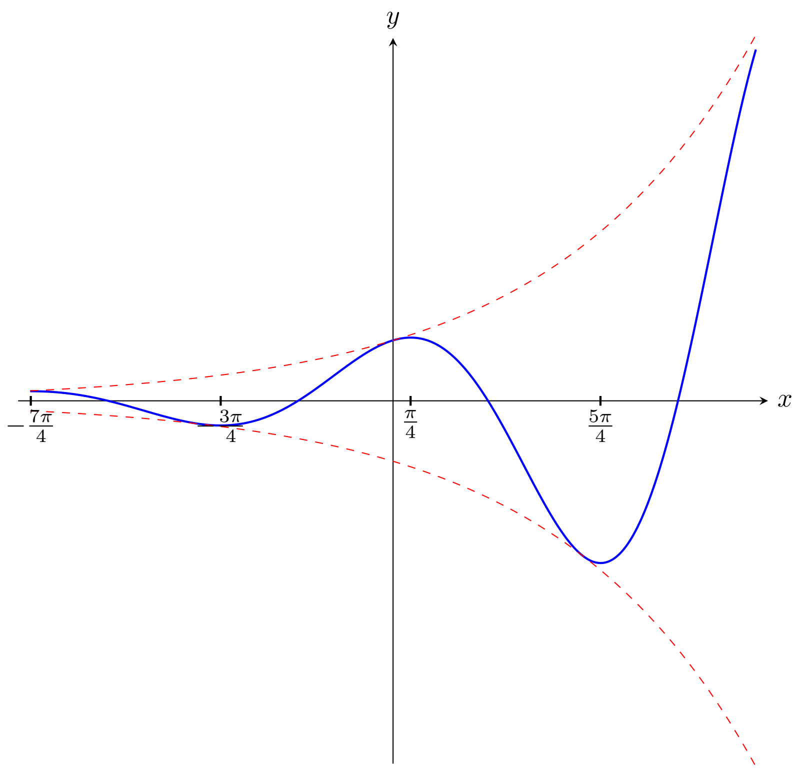
(g) The graph of $e^x \cos(x)$ is the product of the graphs of $e^x$ with $\cos(x)$. In particular, it will look like an $e^x$ curve modulated by a factor of $\cos(x)$. This means that we end up with oscillations between $e^x$ and $-e^x$.

To be a bit more precise, we can calculate the locations of the stationary points. At stationary points we have:

$$\frac{d}{dx}\left(e^x \cos(x)\right) = e^x \cos(x) - e^x \sin(x) = 0,$$

which occurs if and only if $\cos(x) = \sin(x)$, i.e. $\tan(x) = 1$. The full solution to this equation is $x = \pi/4 + n\pi$ an integer. This is not quite what we might expect - maxima and minima of $\cos(x)$ occur at $n\pi/2$, for $n$ an integer. The 'offset' is due to the fact that as we pass a peak or trough of cosine, the exponential is increasing much faster than the cosine is changing its own behaviour (either decreasing or increasing). This results in a small 'offset' which moves the peak away slightly.

The stationary points are essentially the only important information we need to sketch the graph. We include dashed graphs of $e^x$ and $-e^x$ to represent the 'envelope' which the graph of $e^x \cos(x)$ is contained within.



**(†) Leibniz's formula**

12. Using mathematical induction, prove *Leibniz's formula* for the $n$th derivative of a product:

$$\frac{d^n}{dx^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)},$$

where $f^{(k)}$ denotes the $k$th derivative of $f$. Hence compute: (a) the third derivative of $\log^2(x)$; (b) the 100th derivative of $x^2e^x$.

---

➠ **Solution:** The formula clearly holds when $n = 0$. Now assume that the formula holds for $n = m$, and consider the case $n = m + 1$. We have:

$$\begin{aligned}
\frac{d^{m+1}}{dx^{m+1}}(fg) &= \frac{d}{dx}\left(\frac{d^m}{dx^m}(fg)\right) \\
&= \frac{d}{dx}\left(\sum_{k=0}^{m} \binom{m}{k} f^{(k)} g^{(m-k)}\right) && \text{(induction hypothesis)} \\
&= \sum_{k=0}^{m} \binom{m}{k} \left(f^{(k+1)} g^{(m-k)} + f^{(k)} g^{(m-k+1)}\right) && \text{(product rule)} \\
&= \sum_{k=0}^{m} \binom{m}{k} f^{(k+1)} g^{(m-k)} + \sum_{k=0}^{m} \binom{m}{k} f^{(k)} g^{(m-k+1)} \\
&= \sum_{k=1}^{m+1} \binom{m}{k-1} f^{(k)} g^{(m-k+1)} + \sum_{k=0}^{m} \binom{m}{k} f^{(k)} g^{(m-k+1)} && \text{(reindexing sum)} \\
&= \binom{m}{0} f^{(0)} g^{(m)} + \sum_{k=1}^{m} \left(\binom{m}{k-1} + \binom{m}{k}\right) f^{(k)} g^{(m-k+1)} + \binom{m}{m} f^{(m+1)} g^{(0)} \\
&= \binom{m+1}{0} f^{(0)} g^{(m)} + \sum_{k=1}^{m} \binom{m+1}{k} f^{(k)} g^{(m-k+1)} + \binom{m+1}{m+1} f^{(m+1)} g^{(0)} \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} f^{(k)} g^{(m-k+1)}.
\end{aligned}$$

Note that in the penultimate line, we used the Pascal's triangle property of binomial coefficients (if you haven't seen this identity before, don't worry - we will prove it again in the probability section of the course), namely:

$$\binom{m}{k-1} + \binom{m}{k} = \binom{m+1}{k}.$$

Hence, by the principle of mathematical induction, we are done.

Computing the required higher derivatives, we have:

(a) The third derivative of $\log^2(x)$ is:

$$\begin{aligned}\frac{d^3}{dx^2}\log^2(x) &= \binom{3}{0}\log(x)\frac{d^3}{dx^3}\log(x) + \binom{3}{1}\frac{d}{dx}\log(x)\frac{d^2}{dx^2}\log(x)\\ &\qquad + \binom{3}{2}\frac{d^2}{dx^2}\log(x)\frac{d}{dx}\log(x) + \binom{3}{3}\left(\frac{d^3}{dx^3}\log(x)\right)\cdot\log(x)\\ &= 2\log(x)\frac{d^3}{dx^3}\log(x) + 6\frac{d}{dx}\log(x)\frac{d^2}{dx^2}\log(x),\end{aligned}$$

collecting like terms in the final line. The derivatives of the logarithms are:

$$\frac{d}{dx}\log(x) = \frac{1}{x}, \qquad \frac{d^2}{dx^2}\log(x) = -\frac{1}{x^2}, \qquad \frac{d^3}{dx^3}\log(x) = \frac{2}{x^3}.$$

Inserting these into the formula, we have:

$$\frac{d^3}{dx^3}\log^2(x) = \frac{4\log(x)}{x^3} - \frac{6}{x^3}.$$

(b) The 100th derivative of $x^2e^x$ is given by:

$$\frac{d^{100}}{dx^{100}}(x^2e^x) = \sum_{k=0}^{100}\binom{100}{k}\frac{d^k}{dx^k}(x^2)\frac{d^{100-k}}{dx^{100-k}}e^x.$$

All of the exponential derivatives are equal to $e^x$. The only derivatives of $x^2$ that survive in the sum are the $k=0, k=1, k=2$ derivatives, given by:

$$x^2, \qquad \frac{d}{dx}(x^2) = 2x, \qquad \frac{d^2}{dx^2}(x^2) = 2,$$

with all other higher derivatives vanishing. Hence the sum reduces to:

$$\binom{100}{0}x^2e^x + \binom{100}{1}(2x)e^x + \binom{100}{2}\cdot 2e^x = \left(x^2 + 200x + 9900\right)e^x,$$

evaluating the binomial coefficients to finish.

13. Use Leibniz's formula to prove that the $n$th derivative of $e^{-x^2/2}$ is a solution of the equation $Z'' + xZ' + (n+1)Z = 0$.

---

➪ **Solution:** Suppose that:

$$Z = \frac{d^n}{dx^n} e^{-x^2/2}.$$

Then the first derivative is given by:

$$\frac{dZ}{dx} = \frac{d^n}{dx^n}\left(-xe^{-x^2/2}\right) = -\sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k} x \frac{d^{n-k}}{dx^{n-k}} e^{-x^2/2}.$$

The $k = 0$, $k = 1$ terms are the only terms which are non-vanishing, giving:

$$\frac{dZ}{dx} = -x\frac{d^n}{dx^n}e^{-x^2/2} - n\frac{d^{n-1}}{dx^{n-1}}e^{-x^2/2} = -xZ - n\frac{d^{n-1}}{dx^{n-1}}e^{-x^2/2}.$$

Taking an additional derivative, we have:

$$\frac{d^2Z}{dx^2} = -Z - x\frac{dZ}{dx} - nZ.$$

Rearranging, we have:

$$\frac{d^2Z}{dx^2} + x\frac{dZ}{dx} + (n+1)Z = 0,$$

as required.

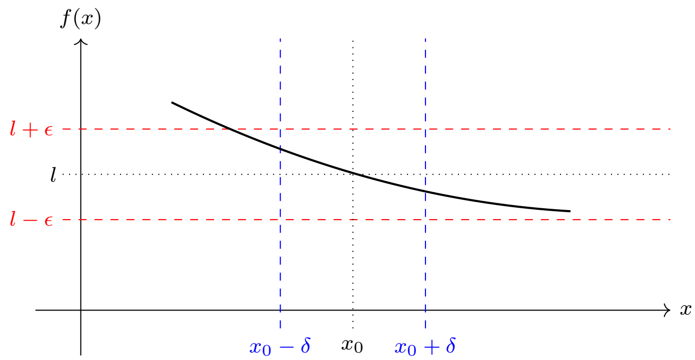
**(†) Formal definition of a limit**

14. Suppose that $f : (a,b)\backslash\{x_0\} \to \mathbb{R}$ is a real function defined on a (possibly infinite) open interval excluding a point $x_0 \in \mathbb{R}$. Give the formal mathematical definition of the phrase '$f(x) \to l$ as $x \to x_0$', and explain this definition using a diagram. How should this definition be modified for the cases $l = \pm\infty$?

---

**Solution:** We say that $f(x) \to l$ as $x \to x_0$ if for any given accuracy $\epsilon > 0$, there exists some tolerance $\delta > 0$ (which can depend on $\epsilon$) such that if $x$ lies within $\delta$ of the point $x_0$, $0 < |x - x_0| < \delta$, then $f(x)$ lies within $\epsilon$ of $l$, $|f(x) - l| < \epsilon$.

This can be explained with the following diagram:




We want to express the idea that as $x$ approaches $x_0$, $f(x)$ approaches $l$. To do this mathematically, we say: suppose that we pick any desired accuracy of $f(x)$ being close enough to $l$, say $\epsilon$. Then there must exist some tolerance $\delta$ such that if $x$ is close enough to $x_0$, within $\delta$, then $f(x)$ is within the desired accuracy $\epsilon$ of $l$.

The figure above shows that for a particular accuracy $\epsilon > 0$, there does indeed exist a tolerance $\delta$ such that within the region $x_0 - \delta < x < x_0 + \delta$ (equivalent to $|x - x_0| < \delta$), we have $l - \epsilon < f(x) < l + \epsilon$ (equivalent to $|f(x) - l| < \epsilon$). Notice that in this figure, we could have chosen $\delta$ larger - but this doesn't really matter, as long as there exists *some* tolerance $\delta$. We aren't interested in the optimal tolerance in the definition of a limit!

---

To define $f(x) \to \infty$ as $x \to x_0$, we want that given any large number $K$, if $x$ is sufficiently close to $x_0$, then $f(x)$ exceeds $K$. More mathematically: given any real number $K$, there exists some tolerance $\delta > 0$ such that if $0 < |x - x_0| < \delta$, we have $f(x) > K$.

Similarly, we define $f(x) \to -\infty$ as $x \to x_0$ as: given any real number $K$, there exists some tolerance $\delta > 0$ such that if $0 < |x - x_0| < \delta$, we have $f(x) < K$.

15. Here is a model example of a formal mathematical argument, from first principles, showing that $x^2 \to 1$ as $x \to 1$:

> *'Suppose we are given some arbitrary tolerance $\epsilon > 0$. Choose some closeness $\delta = \min(1, \epsilon/3)$. Then for all $x$ which are $\delta$-close to $1$, i.e. $0 < |x-1| < \delta$, we have:*
>
> $$\begin{aligned} |x^2 - 1| &= |((x-1)+1)^2 - 1| && (1)\\ &= |(x-1)^2 + 2(x-1)| && (2)\\ &\le |x-1|^2 + 2|x-1| && (3)\\ &< \delta^2 + 2\delta && (4)\\ &\le \delta + 2\delta && (5)\\ &= 3\delta && (6)\\ &\le \epsilon. && (7) \end{aligned}$$
>
> *Hence if $x$ is $\delta$-close to $1$, we have that $|x^2 - 1| < \epsilon$, so that $x^2$ is $\epsilon$-close to $1$. We conclude that, by the definition of a limit, we have $x^2 \to 1$ as $x \to 1$.'*

(a) Which of $\epsilon, \delta$ are we given, and which of $\epsilon, \delta$ must we choose?

(b) Why do we express $|x^2 - 1|$ in terms of $x - 1$ in line (1)?

(c) What law from earlier in the course have we used in going from line (2) to line (3)?

(d) What have we used in going from line (4) to line (5)? What about in going from line (6) to line (7)?

(e) Would the proof still have been successful if we had chosen $\delta = \min(1, \epsilon/4)$? In terms of $\epsilon$, what is the largest possible value of $\delta$ we could have chosen for the proof to still work?

---

**Solution:** (a) We are given $\epsilon > 0$, and we must choose $\delta$ (which can depend on $\delta$). We think of finding a limit like a game - some enemy is challenging us to say 'I bet that if I choose a small enough tolerance $\epsilon$, you can't find a region close enough to $x_0$ where all values of the function are within $\epsilon$ of the limit $l$'. Our aim is to beat this enemy by finding such a $\delta$!

(b) We choose to express $|x^2 - 1|$ in terms of $x - 1$ because we have 'control' over $|x - 1|$ - we can make this smaller by choosing $\delta$ smaller.

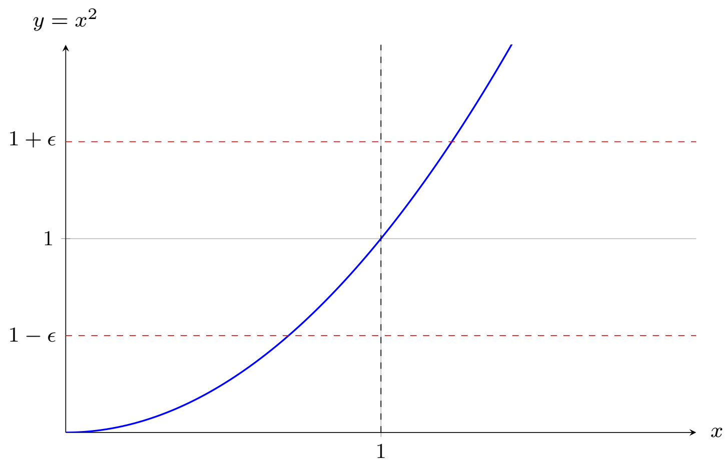
(c) Here, we used the triangle inequality, $|a + b| \le |a| + |b|$. We proved it earlier for vectors, but of course it works for one-dimensional vectors too (i.e. just numbers!).

(d) To go from line (4) to line (5), we used the fact that $\delta = \min(1, \epsilon/3)$, so that $\delta \le 1$. This means that $\delta^2 \le \delta$. To go from line (6) to line (7), we used the fact that $\delta = \min(1, \epsilon/3)$, so that $\delta \le \epsilon/3$.

(e) The proof would still have worked if $\delta = \min(1, \epsilon/4)$. This is because in going from line (6) to line (7), we could have used the fact that $\delta \le \epsilon/4$. So we would have had:

$$3\delta \le \frac{3\epsilon}{4} < \epsilon.$$

To find the largest possible value of $\delta$ that would have worked, it's useful to draw a diagram.




We see that if $0 < \epsilon \leq 1$, then the largest possible $\delta$ can be found by calculating the intersections of the lines $y = 1 + \epsilon$ and $y = 1 - \epsilon$ with the graph of $y = x^2$. In particular, we see that we need:

$$\sqrt{1-\epsilon} < x < \sqrt{1+\epsilon}$$

for $x^2$ to lie between $1 - \epsilon$ and $1 + \epsilon$. To find the largest $\delta$, we must decide which of the points $x = \sqrt{1-\epsilon}, \sqrt{1+\epsilon}$ is closest to $1$; from the diagram, it certainly looks like it is $\sqrt{1+\epsilon}$. The distances we need to compare are $1 - \sqrt{1-\epsilon}$ and $\sqrt{1+\epsilon} - 1$. Observe that:

$$\begin{aligned}
\sqrt{1+\epsilon} - 1 < 1 - \sqrt{1-\epsilon} \quad &\Leftrightarrow \quad \sqrt{1+\epsilon} + \sqrt{1-\epsilon} < 2 \\
&\Leftrightarrow \quad 1 + \epsilon + 2\sqrt{1-\epsilon^2} + 1 - \epsilon < 4 \qquad \text{(squaring both sides)} \\
&\Leftrightarrow \quad \sqrt{1-\epsilon^2} < 1 \\
&\Leftrightarrow \quad 1 - \epsilon^2 < 1 \qquad \text{(squaring both sides)} \\
&\Leftrightarrow \quad 0 < \epsilon^2.
\end{aligned}$$

This is true, so $\sqrt{1+\epsilon}$ is closer to $1$ than $\sqrt{1-\epsilon}$. It follows that if $0 < \epsilon \leq 1$, the maximum possible $\delta$ that we can choose is $\delta = \sqrt{1+\epsilon} - 1$.

On the other hand, if $\epsilon > 1$, there are no intersections between $y = x^2$ and $y = 1 - \epsilon$, and instead there are two (symmetric) intersections between $y = x^2$ and $y = 1 + \epsilon$. Clearly the closer intersection to $x = 1$ is at $\sqrt{1+\epsilon}$. So it follows that if $\epsilon > 1$, the maximum possible $\delta$ that we can choose is $\delta = \sqrt{1+\epsilon} - 1$.

Hence, in all cases the maximum possible $\delta$ we can choose is $\delta = \sqrt{1+\epsilon} - 1$.

16. Using the model example in Question 15 as a template, provide proofs from first principles showing that:

(a) $4x^3 \to 0$ as $x \to 0$, (b) $x^2 \to a^2$ as $x \to a$, for $a \in \mathbb{R}$, (c) $\sin(x) \to 1$ as $x \to \pi/2$,

(d) $x\sin(1/x) \to 0$ as $x \to 0$, (e) $1/x^2 \to \infty$ as $x \to 0$.

---

➾ **Solution:** (a) Let $\epsilon > 0$. Assuming that $0 < |x| < \delta$, where $\delta$ is something we shall choose later, we then have:

$$\begin{aligned} |4x^3| &= 4|x|^3 \\ &< 4\delta^3. \end{aligned}$$

So we see that if we choose $\delta = \min(1, \epsilon/4)$, then we have:

$$4\delta^3 \leq 4\delta \leq \epsilon.$$

Thus $|4x^3| < \epsilon$, and we're done.

---

(b) Let $\epsilon > 0$. Assuming that $0 < |x-a| < \delta$, where $\delta$ is something we shall choose later, we then have:

$$\begin{aligned} |x^2 - a^2| &= |(x-a)^2 + 2a(x-a)| \\ &\leq |x-a|^2 + 2|a||x-a| && \text{(triangle inequality)} \\ &= \delta^2 + 2|a|\delta. \end{aligned}$$

So we see that if we choose $\delta = \min(1, \epsilon/(1+2|a|))$, then we have:

$$\delta^2 + 2|a|\delta \leq \delta(1+2|a|) \leq \epsilon.$$

Thus $|x^2 - a^2| < \epsilon$, and we're done.

---

(c) Let $\epsilon > 0$. Assuming that $0 < |x - \pi/2| < \delta$, where $\delta$ is something we shall choose later, we then have:

$$\begin{aligned} |\sin(x) - 1| &= |\sin(x) - \sin(\pi/2)| \\ &= \left|2\sin\left(\frac{x-\pi/2}{2}\right)\cos\left(\frac{x+\pi/2}{2}\right)\right| && \text{(sum to product formula)} \\ &\leq 2\left|\sin\left(\frac{x-\pi/2}{2}\right)\right| && \text{(since the modulus of cosine is less than } 1) \end{aligned}$$

Now, recall from the zeroth examples sheet, we have $|\sin(u)| \leq |u|$ for all $u$. Hence we have:

$$|\sin(x) - 1| \leq 2\left|\sin\left(\frac{x-\pi/2}{2}\right)\right| \leq 2\left|\frac{x-\pi/2}{2}\right| = |x - \pi/2| < \delta.$$

Hence we see that if choose $\delta = \epsilon$, we have $|\sin(x) - 1| < \epsilon$, and we're done.

(d) Let $\epsilon > 0$. Assuming that $0 < |x| < \delta$, where $\delta$ is something we shall choose later, we then have:

$$\begin{aligned} |x \sin(1/x)| &\leq |x| && \text{(since the modulus of sine is less than } 1) \\ &< \delta. \end{aligned}$$

Hence we see that if we choose $\delta = \epsilon$, we have $|x \sin(1/x)| < \epsilon$, and we're done.

---

(e) Given a constant $K$, assume that $0 < |x| < \delta$, where $\delta$ is something we shall choose later. We then have:

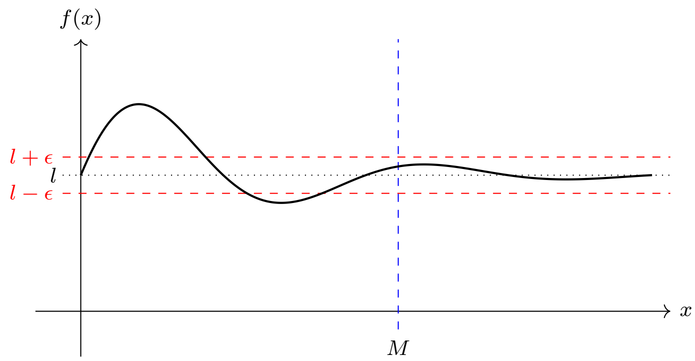
$$\left| \frac{1}{x^2} \right| = \frac{1}{|x|^2} > \frac{1}{\delta^2}.$$

Hence we see that if we choose $1/\delta^2 > K$, that is, choose $\delta < \sqrt{1/|K|}$, then we're done.

17. Suppose that $f : (a, \infty) \to \mathbb{R}$ is a real function defined on an open interval up to positive infinity. Give the formal mathematical definition of the phrase '$f(x) \to l$ as $x \to \infty$', and explain this definition using a diagram. How should this definition be modified for the cases $l = \pm\infty$? Hence, show directly from the definition that: (a) $1/x \to 0$ as $x \to \infty$; (b) $\sin(x)/x \to 0$ as $x \to \infty$; (c) $x^3 \to -\infty$ as $x \to -\infty$.

---

➭ **Solution:** We say that $f(x) \to l$ as $x \to \infty$ if for all $\epsilon > 0$, there exists some $M$ such that if $x > M$, we have $|f(x) - l| < \epsilon$. This can be visualised with the following diagram:




The idea is that for each small tolerance $\epsilon > 0$, if we choose $x$ larger than some $M$, then the function $f(x)$ is constrained to lie in the range $l - \epsilon < f(x) < l + \epsilon$ - equivalently, $|f(x) - l| < \epsilon$.

If $l = \infty$, then the definition becomes: for all $K \in \mathbb{R}$, there exists some $M$ such that if $x > M$, we have $f(x) > K$. On the other hand if $l = -\infty$, then the definition becomes: for all $K \in \mathbb{R}$, there exists some $M$ such that if $x > M$, we have $f(x) < K$.

Examining the limits in the question, we have:

(a) Given $\epsilon > 0$, let $x > M$ where $M$ is a constant we will choose later. Then:

$$\left|\frac{1}{x}\right| = \frac{1}{|x|} < \frac{1}{M}.$$

since if $x > M$, we have $|x| > M$ (if we then assume $M$ positive, $1/M > 1/|x|$). So we should choose $M = 1/\epsilon$, then we're done.

(b) Given $\epsilon > 0$, let $x > M$ where $M$ is a constant we will choose later. Then:

$$\left|\frac{\sin(x)}{x}\right| \leq \frac{1}{|x|} < \frac{1}{M},$$

since if $x > M$, we have $|x| > M$ (if we then assume $M$ positive, $1/M > 1/|x|$). So we should choose $M = 1/\epsilon$, then we're done.

(c) Given any $K$, let $x < M$, where $M$ is a constant we will choose later. Then:

$$x^3 < M^3.$$

since cubing something is an increasing function. So if we choose $M = K^{1/3}$, then we're done.

**(†) Laws of limits**

18. Let $x_0 \in \mathbb{R}$, and let $f, g : \mathbb{R} \to \mathbb{R}$ be real functions. From the formal definition of a limit, prove that:

$$\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x),$$

provided that (i) both the limits on the right hand side exist in $\mathbb{R} \cup \{\infty, -\infty\}$ (the set of real numbers with infinity and negative infinity adjoined), *and* (ii) if one of the limits on the right hand side is $\infty$, the other is *not* $-\infty$.

---

**Solution:** We split everything into cases:

- **Both limits are finite.** Let's start with the cases where both limits on the right hand side are finite. Then given $\epsilon > 0$, there must exist some $\delta_1, \delta_2 > 0$ such that if $0 < |x - x_0| < \min(\delta_1, \delta_2)$ we have both:

  $$|f(x) - l| < \epsilon/2 \qquad \text{and} \qquad |g(x) - m| < \epsilon/2,$$

  where $l, m$ are the finite values of the limits of $f(x), g(x)$ respectively. It follows that if we take $\delta = \min(\delta_1, \delta_2)$, we have for all $x$ satisfying $0 < |x - x_0| < \delta$:

  $$|f(x) + g(x) - l - m| = |f(x) - l + g(x) - m| \leq |f(x) - l| + |g(x) - m| < \epsilon/2 + \epsilon/2 = \epsilon.$$

  Hence $f(x) + g(x)$ tends to $l + m$.

- **One limit is infinite, one is finite.** Let's suppose that one of the limits is infinite, and the other is finite now, $f(x) \to \infty$ and $g(x) \to m$ as $x \to x_0$. Then given any $K$, there must exist some $\delta_1, \delta_2 > 0$ such that if $0 < |x - x_0| < \min(\delta_1, \delta_2)$ we have both:

  $$f(x) > K - m + 1, \qquad \text{and} \qquad |g(x) - m| < 1.$$

  The second inequality implies:

  $$m - g(x) < 1 \qquad \Leftrightarrow \qquad g(x) > m - 1.$$

  Hence, we have:

  $$f(x) + g(x) > K - m + 1 + m - 1 = K.$$

  Therefore, $f(x) + g(x) \to \infty$, as required. A similar argument applies when $f(x) \to -\infty$ and $g(x) \to m$, for $m$ finite, as $x \to x_0$.

- **Both limits are infinite, but have the same sign.** Now, we have $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to x_0$. Then given any $K$, there must exist some $\delta_1, \delta_2$ such that for all $x$ satisfying $0 < |x - x_0| < \min(\delta_1, \delta_2)$ we have:

  $$f(x) > \frac{K}{2}, \qquad g(x) > \frac{K}{2}.$$

  Hence, we have:

  $$f(x) + g(x) > \frac{K}{2} + \frac{K}{2} = K,$$

  which implies that $f(x) + g(x) \to \infty$ as $x \to x_0$.

- **Both limits are infinite, with opposite sign.** This is the case where the law fails. If $f(x) \to \infty$ but $g(x) \to -\infty$, then given any $K \in \mathbb{R}$, there exist $\delta_1, \delta_2$ such that if $0 < |x - x_0| < \min(\delta_1, \delta_2)$ we have both:

  $$f(x) > K, \qquad g(x) < K.$$

  This means that we can only ever get a lower bound on $f(x)$ and an upper bound on $g(x)$. It is impossible to place either an upper bound or lower bound on $f(x) + g(x)$ with only this information.

19. From the formal mathematical definition of a limit, it is possible to prove results about the limits of sums, products, quotients and compositions of functions, similarly to Question 18. State these '*laws of limits*' clearly (making sure to take particular care when the limits are infinite), and use them to evaluate the following:

$$\text{(a)} \lim_{x\to 0}\frac{x+1}{2-x^2}, \quad \text{(b)} \lim_{x\to\infty}\sin\left(\frac{x^2+x+1}{3x^2-4}\right), \quad \text{(c)} \lim_{x\to 0}\left(\exp\left(\frac{x^4-1}{x^4+1}\right)\right)^{1/x^2}, \quad \text{(d)} \lim_{x\to\infty}\left(\sqrt{x^2+7x}-x\right).$$

---

➾ **Solution:** Here is a large, comprehensive summary of the laws of limits:

## Laws of limits

- Addition law. We have:

$$\lim_{x\to x_0}(f(x)+g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x),$$

provided that all limits exist in the set $\mathbb{R}\cup\{\infty,-\infty\}$ (that is, they can be infinite), *and*:

$$\left\{\lim_{x\to x_0} f(x), \lim_{x\to x_0} g(x)\right\} \neq \{\infty,-\infty\}.$$

Here, the intuitive rules $l+\infty=\infty, l-\infty=-\infty$ (for finite $l$) and $\infty+\infty=\infty, -\infty-\infty=-\infty$ are relevant for evaluating the right hand side in the case of infinite limits.

- Multiplication law. We have:

$$\lim_{x\to x_0}(f(x)g(x)) = \lim_{x\to x_0} f(x)\cdot\lim_{x\to x_0} g(x),$$

provided that all limits exist in the set $\mathbb{R}\cup\{\infty,-\infty\}$ (that is, they can be infinite), *and*:

$$\left\{\lim_{x\to x_0} f(x), \lim_{x\to x_0} g(x)\right\} \neq \{0,\infty\}, \{0,-\infty\}.$$

Here, the intuitive rules $l\cdot\infty=\mathrm{sign}(l)\infty, l\cdot(-\infty)=-\mathrm{sign}(l)\infty$ (for finite non-zero $l$, where $\mathrm{sign}(l)$ is the sign of $l$) and $\infty\cdot\infty=\infty, \infty\cdot(-\infty)=-\infty, (-\infty)\cdot(-\infty)=\infty$ are relevant for evaluating the right hand side in the case of infinite limits.

- Division law. We have:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)},$$

provided that all limits exist in the set $\mathbb{R}\cup\{\infty,-\infty\}$ (that is, they can be infinite), *and*:

  - $\lim_{x\to x_0} g(x) \neq 0$;
  - $\lim_{x\to x_0} f(x), \lim_{x\to x_0} g(x)$ are not both infinite.

Here, the intuitive rules $l/\infty=0$ (for finite $l$, including zero) and $\infty/l=\mathrm{sign}(l)\infty$ (for finite non-zero $l$, where $\mathrm{sign}(l)$ is the sign of $l$) are relevant for evaluate the right hand side in the case of infinite limits.

- <u>Power law.</u> We have:

$$\lim_{x\to x_0} f(x)^{g(x)} = \left(\lim_{x\to x_0} f(x)\right)^{\left(\lim\limits_{x\to x_0} g(x)\right)},$$

provided that all limits exist in the set $\mathbb{R}\cup\{\infty,-\infty\}$ (that is, they can be infinite), *and*:

  - $\lim\limits_{x\to x_0} f(x) > 0$ (including the possibility that this limit is infinite). The intuitive rules:

$$l^{\infty} = \begin{cases} 0, & 0<l<1, \\ \infty, & l>1, \end{cases} \qquad l^{-\infty} = \begin{cases} \infty, & 0<l<1, \\ 0, & l>1, \end{cases}$$
$$\infty^{l} = \begin{cases} 0, & l<0, \\ \infty, & l>0, \end{cases}$$

    are relevant for evaluation of the power in the case of infinite limits of $f(x), g(x)$, and include the cases $l=\pm\infty$ as appropriate.

  - $\lim\limits_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) > 0$. The intuitive rule $0^{\infty} = 0$ is relevant in the evaluation of the power in the case of an infinite limit of $g(x)$.

- <u>Continuity law.</u> We have:

$$\lim_{x\to x_0} f(g(x)) = f\left(\lim_{x\to x_0} g(x)\right)$$

provided that $f$ is a *continuous function* at the point $\lim\limits_{x\to x_0} g(x)$. This rule is very useful, appearing in a large number of different contexts!

From the above laws, we see that there are a family of infinite limit combinations that we cannot evaluate using the limit laws; those that can result in finite limits are called *indeterminate forms* (something like $1/0$, for example, is not indeterminate - it is just undefined). The full collection of indeterminate forms are the following:

**Indeterminate forms.**

- $\infty-\infty$;
- $0\cdot(\pm\infty)$;
- $\pm\dfrac{\infty}{\infty}, \quad \dfrac{0}{0}$;
- $0^0, \quad \infty^0, \quad 1^{\pm\infty}$.

We shall see in the later questions that limits involving some indeterminate forms can be tackled using L'Hôpital's rule.

Turning to the limits presented in this question, we have:

(a) Trivially, we have:

$$\lim_{x \to 0} x = 0.$$

Since $(x+1)/(2-x^2)$ is a continuous function of $x$ away from $x = \pm\sqrt{2}$, we have by the continuity law that:

$$\lim_{x \to 0} \frac{x+1}{2-x^2} = \frac{0+1}{2-0^2} = \frac{1}{2}.$$

(b) By the continuity law, we have:

$$\sin\left(\lim_{x \to \infty} \frac{x^2+x+1}{3x^2-4}\right) = \sin\left(\lim_{x \to \infty} \frac{1+1/x+1/x^2}{3-4/x^2}\right) = \sin\left(\frac{1+0+0^2}{3-0^2}\right) = \sin\left(\frac{1}{3}\right).$$

(c) First, observe that:

$$\lim_{x \to 0} \frac{1}{x^2} = \infty, \qquad \lim_{x \to 0} \exp\left(\frac{x^4-1}{x^4+1}\right) = e^{-1},$$

where the second limit was evaluated using the continuity law (and the fact that trivially $x \to 0$ as $x \to 0$). Hence, by the power law we have:

$$\lim_{x \to 0} \left(\exp\left(\frac{x^4-1}{x^4+1}\right)\right)^{1/x^2} = \infty.$$

(d) This limit involves a bit of a trick. We have:

$$\sqrt{x^2+7x} - x = \frac{(\sqrt{x^2+7x}-x)(\sqrt{x^2+7x}+x)}{\sqrt{x^2+7x}+x} = \frac{x^2+7x-x^2}{\sqrt{x^2+7x}+x} = \frac{7x}{\sqrt{x^2+7x}+x} = \frac{7}{\sqrt{1+7/x}+1}.$$

This is a continuous function of $1/x$, which tends to zero as $x \to \infty$. Hence the limit is:

$$\frac{7}{\sqrt{1+0}+1} = \frac{7}{2}.$$

20. State *L'Hôpital's rule* for evaluating limits of differentiable functions, carefully specifying the conditions under which it is valid. Assuming $\alpha > 0$ throughout, use L'Hôpital's rule - where appropriate - to evaluate the limits of the following functions both (i) as $x \to 0^+$ (a *one-sided limit*), and (ii) as $x \to \infty$:

(a) $x^\alpha \log(x)$, (b) $x^{-\alpha} \log(x)$, (c) $x^\alpha e^{-x}$, (d) $x^{-\alpha} e^x$, (e) $\sin(\alpha x)/x$.

---

**Solution:**

**Theorem:** *L'Hôpital's rule* states that:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

provided that:

(a) both limits exist;

(b) the limit on the left hand side is one of the indeterminate forms $0/0$ or $\pm\infty/\infty$.

Examining the limits given in the question, we have:

(a) (i) As $x \to 0^+$, we have $x^\alpha \to 0$, since $\alpha > 0$. Hence:

$$\lim_{x \to 0^+} x^\alpha \log(x) = \lim_{x \to 0^+} \frac{\log(x)}{x^{-\alpha}}$$

is an indeterminate form $(-\infty)/(+\infty)$. Therefore, we can apply L'Hôpital's rule:

$$\lim_{x \to 0^+} \frac{\log(x)}{x^{-\alpha}} = \lim_{x \to 0^+} \frac{1/x}{-\alpha x^{-\alpha-1}} = -\frac{1}{\alpha} \lim_{x \to 0^+} x^\alpha = 0.$$

Therefore, the original limit is also zero.

(ii) As $x \to \infty$, we have $x^\alpha \to \infty$, since $\alpha > 0$. We also have $\log(x) \to \infty$. Hence by the product rule for limits, we have:

$$\lim_{x \to \infty} x^\alpha \log(x) = \infty.$$

(b) (i) As $x \to 0^+$, we have $x^{-\alpha} \to \infty$, since $\alpha > 0$. We also have $\log(x) \to -\infty$ as $x \to 0^+$. Hence by the product rule for limits, we have:

$$\lim_{x \to 0^+} x^{-\alpha} \log(x) = -\infty.$$

(ii) As $x \to \infty$, we have $x^{-\alpha} \to 0$, since $\alpha > 0$. Hence:

$$\lim_{x \to \infty} x^{-\alpha} \log(x) = \lim_{x \to \infty} \frac{\log(x)}{x^\alpha}$$

is an indeterminate form $\infty/\infty$. Therefore, we can apply L'Hôpital's rule:

$$\lim_{x \to \infty} \frac{\log(x)}{x^\alpha} = \lim_{x \to \infty} \frac{1/x}{\alpha x^{\alpha-1}} = \frac{1}{\alpha} \lim_{x \to \infty} x^{-\alpha} = 0.$$

Therefore, the original limit is also zero.

(c) (i) As $x \to 0^+$, we have $x^\alpha \to 0$, since $\alpha > 0$. We also have $e^{-x} \to 1$ as $x \to 0^+$. Hence by the product rule for limits, we have:

$$\lim_{x\to 0^+} x^\alpha e^{-x} = 0.$$

(ii) As $x \to \infty$, we have $x^\alpha \to \infty$, since $\alpha > 0$. We also have $e^{-x} \to 0$ as $x \to \infty$. Thus, the limit of $x^\alpha e^{-x}$ is an indeterminate form $\infty \cdot 0$ as $x \to \infty$. Rewriting it in the form $x^\alpha / e^x$, we can apply L'Hôpital's rule repeatedly to give:

$$\lim_{x\to\infty} \frac{x^\alpha}{e^x} = \lim_{x\to\infty} \frac{\alpha x^{\alpha-1}}{e^x} = \cdots = \lim_{x\to\infty} \frac{\alpha(\alpha-1)...(\alpha-(m-1))x^{\alpha-m}}{e^x},$$

where $m$ is the smallest integer such that $\alpha - m < 0$. We then have the numerator $x^{\alpha-m}$ tending to zero as $x \to \infty$, and the denominator $e^x$ tending to infinity as $x \to \infty$. This gives, by the quotient rule for limits,

$$\lim_{x\to\infty} \frac{x^\alpha}{e^x} = 0.$$

Thus we have proved that 'exponential growth dominates polynomial growth'.

(d) (i) As $x \to 0^+$, we have $x^{-\alpha} \to \infty$, since $\alpha > 0$. We also have $e^x \to 1$ as $x \to 0^+$. Hence by the product rule for limits, we have:

$$\lim_{x\to 0^+} x^{-\alpha} e^x = \infty.$$

(ii) As $x \to \infty$, we have $x^{-\alpha} \to 0$, since $\alpha > 0$. We also have $e^x \to \infty$, so the limit of $x^{-\alpha}e^x$ is an indeterminate form $0 \cdot \infty$. Rewriting this in the form $e^x/x^\alpha$, we have an indeterminate form $\infty/\infty$, which we can apply L'Hôpital's rule repeatedly to give:

$$\lim_{x\to\infty} \frac{e^x}{x^\alpha} = \lim_{x\to\infty} \frac{e^x}{\alpha x^{\alpha-1}} = ... = \lim_{x\to\infty} \frac{e^x}{\alpha(\alpha_1)...(\alpha-(m-1))x^{\alpha-m}},$$

where $m$ is the smallest integer such that $\alpha - m < 0$. At this point we have $e^x \to \infty$ and $x^{m-\alpha} \to \infty$, which implies (by the product rule for limits) that:

$$\lim_{x\to\infty} \frac{e^x}{x^\alpha} = \infty.$$

(e) (i) As $x \to 0^+$, both $\sin(\alpha x)$ and $x$ tend to zero, which results in an indeterminate form $0/0$. Applying L'Hôpital's rule, we have:

$$\lim_{x\to 0^+} \frac{\sin(\alpha x)}{x} = \lim_{x\to 0^+} \frac{\alpha\cos(\alpha x)}{1} = \alpha,$$

since the limit of $\cos(\alpha x)$ as $x \to 0^+$ is $\cos(0) = 1$ by the continuity law (since cosine is a continuous function).

(ii) As $x \to \infty$, we have that $\sin(\alpha x)$ oscillates infinitely in between $-1$ and $1$. On the other hand, $1/x$ tends to zero. Thus we have a bounded oscillation multiplied by a function that approaches zero; the result is a zero limit overall:

$$\lim_{x\to\infty} \frac{\sin(\alpha x)}{x} = 0.$$

21. Using L'Hôpital's rule, evaluate the following 'power law' limits:

(a) $\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^x$, (b) $\lim\limits_{x\to\infty}\log^{1/x}(x)$, (c) $\lim\limits_{x\to 0^+}x^x$, (d) $\lim\limits_{x\to\infty}x^{1/x}$.

---

**Solution:** In all cases, we aim to rewrite the limits in terms of some quotients, where we can apply L'Hôpital's rule. We have:

(a) Using the continuity law, we have:

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = \exp\left(\lim_{x\to\infty} x\log\left(1-\frac{1}{x}\right)\right)$$
$$\exp\left(\lim_{x\to\infty}\frac{\log(1-1/x)}{1/x}\right).$$

The limit in the exponent is now of the form $\log(1)/0 = 0/0$, which is an indeterminate form. Hence we can apply L'Hôpital's rule to the limit in the exponent:

$$\lim_{x\to\infty}\frac{\log(1-1/x)}{1/x} = \lim_{x\to\infty}\frac{(1/x^2)}{(1-1/x)(-1/x^2)} = -1.$$

Hence, the original limit is:

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = e^{-1}.$$

(b) Using the continuity law, we have:

$$\lim_{x\to\infty}\log^{1/x}(x) = \exp\left(\lim_{x\to\infty}\frac{\log(\log(x))}{x}\right).$$

The limit in the exponent is now of the form $\infty/\infty$, which is an indeterminate form. Hence we can apply L'Hôpital's rule to the limit in the exponent:

$$\lim_{x\to\infty}\frac{\log(\log(x))}{x} = \lim_{x\to\infty}\frac{1/x}{\log(x)} = \lim_{x\to\infty}\frac{1}{x\log(x)} = 0.$$

Hence, the original limit is:

$$\lim_{x\to\infty}\log^{1/x}(x) = 1.$$

(c) Using the continuity law, we have:

$$\lim_{x\to 0^+}x^x = \exp\left(\lim_{x\to 0^+}x\log(x)\right).$$

Using the result of the previous question, the limit in the exponent is $0$. Hence the original limit is $1$.

(d) Using the continuity law, we have:

$$\lim_{x\to\infty}x^{1/x} = \exp\left(\lim_{x\to\infty}\frac{\log(x)}{x}\right).$$

Using the result of the previous question, the limit in the exponent is $0$. Hence the original limit is $1$.

22. Explain why the following arguments with limits are *wrong*.

(a) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}(1+0)^x = 1$, using $\lim_{x\to\infty}\frac{1}{x} = 0$ in the first step.

(b) $\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \lim_{x\to 0}\frac{1-1}{x^2} = 0$, using $\cos(x)\approx 1$ for small enough $x$.

Now, evaluate the limits correctly.

---

➾ **Solution:** (a) It is incorrect to replace only 'part' of a limit when trying to work it out - that is not one of the laws of limits! To evaluate it correctly, we use L'Hôpital's rule:

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x &= \lim_{x\to\infty}\exp\left(x\log\left(1+\frac{1}{x}\right)\right) \\
&= \exp\left(\lim_{x\to\infty}\frac{\log(1+1/x)}{1/x}\right) && \text{(continuity law)} \\
&= \exp\left(\lim_{x\to\infty}\frac{-1/x^2}{(1+1/x)(-1/x^2)}\right) && \text{(L'Hôpital's rule)} \\
&= \exp\left(\lim_{x\to\infty}\frac{1}{1+1/x}\right) \\
&= e^1 = e.
\end{aligned}$$

---

(b) It is incorrect to make 'approximations' in limits to try to help evaluate them - that is not one of the laws of limits! To evaluate it correctly, we use L'Hôpital's rule:

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\frac{\cos(x)}{1} = 1.$$

23. Using L'Hôpital's rule where appropriate, compute the limit:

$$\lim_{x\to\infty}\left(1+a^x+\left(\frac{a^2}{2}\right)^x\right)^{\frac{1}{x}}.$$

for all values of $a \geq 0$.

---

**Solution:** If $a < 1$ and $a^2/2 < 1$, then $1+a^x+(a^2/2)^x \to 1$ as $x \to \infty$. Thus we get the form $1^0 = 1$, and the limit is $1$.

Similarly, if $a = 1$, then $a^2/2 = 1/2$, so that $1 + 1^x + (1/2)^x \to 2$ as $x \to \infty$. Thus we get the form $2^0 = 1$, and the limit is also $1$.

If either $a > 1$ or $a^2/2 > 1$, we instead get the indeterminate form $\infty^0$, and we will need to apply L'Hôpital's rule. Solving the second inequality, we have $a^2/2 > 1 \Leftrightarrow a > \sqrt{2}$ (since $a > 0$), so this case occurs if and only if $a > 1$.

Manipulating the limit to apply L'Hôpital's rule, we have:

$$\begin{aligned}\lim_{x\to\infty}\left(1+a^x+\left(\frac{a^2}{2}\right)^x\right)^{1/x} &= \lim_{x\to\infty}\exp\left(\frac{1}{x}\log\left(1+a^x+\left(\frac{a^2}{2}\right)^x\right)\right)\\ &= \exp\left(\lim_{x\to\infty}\frac{\log\left(1+e^{x\log(a)}+e^{x\log(a^2/2)}\right)}{x}\right) \qquad \text{(continuity law)}\end{aligned}$$

We can now apply L'Hôpital's rule, since $a > 1$, so $e^{x\log(a)}$ approaches infinity as $x \to \infty$, and hence the numerator approaches infinite as $x \to \infty$. Differentiating the numerator and denominator, the required limit becomes:

$$\exp\left(\lim_{x\to\infty}\frac{\log(a)e^{x\log(a)}+\log(a^2/2)e^{x\log(a^2/2)}}{e^{x\log(a)}+e^{x\log(a^2/2)}}\right).$$

Now, depending on the relative sizes of $a$ and $a^2/2$, some of these exponentials will grow faster than others. Here are the cases:

- If $a > a^2/2$, then we have $\log(a) > \log(a^2/2)$ so that $0 > \log(a^2/2) - \log(a)$. Hence we perform the following manipulation:

  $$\lim_{x\to\infty}\frac{\log(a)e^{x\log(a)}+\log(a^2/2)e^{x\log(a^2/2)}}{e^{x\log(a)}+e^{x\log(a^2/2)}} = \lim_{x\to\infty}\frac{\log(a)+\log(a^2/2)e^{x\log(a^2/2)-x\log(a)}}{1+e^{x\log(a^2/2)-x\log(a)}} = \log(a).$$

  So overall, we find the original limit is $a$. This case occurs when $a > a^2/2 \Leftrightarrow 1 < a < 2$ (since we have already assumed that $a > 1$).

- If $a^2/2 > a$, then we have $\log(a) < \log(a^2/2)$ so that $0 > \log(a) - \log(a^2/2)$. Hence we perform the following manipulation:

  $$\lim_{x\to\infty}\frac{\log(a)e^{x\log(a)}+\log(a^2/2)e^{x\log(a^2/2)}}{e^{x\log(a)}+e^{x\log(a^2/2)}} = \lim_{x\to\infty}\frac{\log(a)e^{x\log(a)-x\log(a^2/2)}+\log(a^2/2)}{e^{x\log(a)-x\log(a^2/2)}+1} = \log(a^2/2).$$

  So overall, we find the original limit is $a^2/2$. This case occurs when $a^2/2 > a \Leftrightarrow a > 2$ (since we have already assumed that $a > 1$).

- If $a^2/2 = a$, i.e. we have $a = 2$, then all the exponentials cancel. The limit is:

$$\lim_{x\to\infty} \frac{\log(2) + \log(4/2)}{1+1} = \log(2)$$

  So overall, we find the original limit is $2$.

In conclusion, we have shown that:

$$\lim_{x\to\infty} \left(1 + a^x + \left(\frac{a^2}{2}\right)^x\right)^{\frac{1}{x}} = \begin{cases} 1, & 0 \le a \le 1, \\ a, & 1 < a < 2, \\ 2, & a = 2, \\ a^2/2, & a > 2. \end{cases}$$

---

24. Consider the limit:

$$\lim_{x\to\infty} \frac{x}{x + \sin(x)}.$$

Show that this limit is equal to one. Show that if we instead naïvely apply L'Hôpital's rule, we incorrectly conclude that the limit does not exist.

---

**Solution:** We have:

$$\lim_{x\to\infty} \frac{x}{x+\sin(x)} = \lim_{x\to\infty} \frac{1}{1 + \sin(x)/x}$$

$$= \frac{1}{1+0} = 1,$$

since $\sin(x)/x \to 0$ as $x \to \infty$, and $1/(1+u)$ is a continuous function.

If we instead apply L'Hôpital's rule, we have (incorrectly):

$$\lim_{x\to\infty} \frac{x}{x+\sin(x)} = \lim_{x\to\infty} \frac{1}{1+\cos(x)},$$

which does not exist, because cosine is infinitely oscillatory.

**(†) Miscellaneous limits**

[*This section contains a large collection of limits from past papers for you to evaluate. If you feel like you are getting too much of a good thing, feel free to save some of them for us to do together in the supervision.*]

25. Evaluate the following limits, using the most efficient method in each case:

(a) $\lim\limits_{x\to 0^+} x\log(x)$;

(b) $\lim\limits_{x\to a} \dfrac{x^x-a^a}{x-a}$ where $a>0$;

(c) $\lim\limits_{x\to 0} \dfrac{\tan(x)-\sin(x)}{\sin^3(x)}$;

(d) $\lim\limits_{x\to a} \dfrac{\sin(x)-\sin(a)}{x-a}$;

(e) $\lim\limits_{x\to\infty} \left(\dfrac{x+a}{x-a}\right)^x$;

(f) $\lim\limits_{x\to 0} \dfrac{\cos(x)-\cos(3x)}{x^2}$;

(g) $\lim\limits_{x\to 0} \dfrac{\log(\cos(x))}{\log(\cos(3x))}$;

(h) $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{\sinh(x)}$.

---

**Solution:** (a) We use L'Hôpital's rule:

$$\lim_{x\to 0^+} x\log(x) = \lim_{x\to 0^+} \frac{\log(x)}{1/x} = \lim_{x\to 0^+} \frac{-1/x^2}{-1/x^2} = 1.$$

(b) We could use L'Hôpital's rule here, but it is quicker to recognise that this is just the derivative of the function $x^x$ at the point $x=a$. We have:

$$\frac{d}{dx}x^x = \frac{d}{dx}e^{x\log(x)} = (\log(x)+1)\,e^{x\log(x)} = (\log(x)+1)\,x^x.$$

Hence the limit is $(\log(a)+1)a^a$.

(c) We can do this with some trigonometric identities. We have:

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{\sin^3(x)} &= \lim_{x\to 0}\frac{\sin(x)/\cos(x)-\sin(x)}{\sin^3(x)}\\
&= \lim_{x\to 0}\frac{1-\cos(x)}{\sin^2(x)\cos(x)}\\
&= \lim_{x\to 0}\frac{1-\cos(x)}{(1-\cos^2(x))\cos(x)}\\
&= \lim_{x\to 0}\frac{1-\cos(x)}{(1-\cos(x))(1+\cos(x))\cos(x)}\\
&= \lim_{x\to 0}\frac{1}{(1+\cos(x))\cos(x)}\\
&= \frac{1}{2}.
\end{aligned}$$

(d) We could use L'Hôpital's rule here, or use some trigonometric identities, but it is quicker to recognise that this is just the derivative of $\sin(x)$ at the point $x=a$. Hence the limit is $\cos(a)$.

(e) Here, the most appropriate method is L'Hôpital's rule, since we have an indeterminate form $1^\infty$. We have:

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x &= \exp\left(\lim_{x\to\infty} x\log\left(\frac{x+a}{x-a}\right)\right) && \text{(continuity law)}\\ &= \exp\left(\lim_{x\to\infty}\frac{\log((x+a)/(x-a))}{1/x}\right).\end{aligned}$$

Inside the limit, the numerator approaches $\log(1)=0$, and the denominator approaches $0$, so we can apply L'Hôpital's rule. We have:

$$\lim_{x\to\infty}\frac{\log((x+a)/(x-a))}{1/x} = \lim_{x\to\infty}\frac{\left(\dfrac{(x-a)-(x+a)}{(x-a)^2}\right)}{\left(\dfrac{x+a}{x-a}\right)\left(-\dfrac{1}{x^2}\right)} = \lim_{x\to\infty}\frac{2ax^2}{(x+a)(x-a)} = \lim_{x\to\infty}\frac{2a}{(1+a/x)(1-a/x)} = 2a.$$

Hence the original limit is $e^{2a}$.

(f) We can use L'Hôpital's rule. We have:

$$\begin{aligned}\lim_{x\to 0}\frac{\cos(x)-\cos(3x)}{x^2} &= \lim_{x\to 0}\frac{-\sin(x)+3\sin(3x)}{2x}\\ &= \lim_{x\to 0}\frac{-\cos(x)+9\cos(3x)}{2}\\ &= 4.\end{aligned}$$

(g) We can use L'Hôpital's rule (twice actually!). We have:

$$\begin{aligned}\lim_{x\to 0}\frac{\log(\cos(x))}{\log(\cos(3x))} &= \lim_{x\to 0}\frac{\sin(x)/\cos(x)}{3\sin(3x)/\cos(3x)}\\ &= \frac{1}{3}\lim_{x\to 0}\frac{\cos(3x)}{\cos(x)}\cdot\lim_{x\to 0}\frac{\sin(x)}{\sin(3x)}\\ &= \frac{1}{3}\cdot\frac{1}{1}\cdot\lim_{x\to 0}\frac{\cos(x)}{3\cos(3x)}\\ &= \frac{1}{3}.\end{aligned}$$

(h) We can use L'Hôpital's rule. We have:

$$\lim_{x\to 0}\frac{\sin(3x)}{\sinh(x)} = \lim_{x\to 0}\frac{3\cos(3x)}{\cosh(x)} = 3.$$

**(†) Continuity of functions**

26. Let $f:(a,b)\to\mathbb{R}$ be a real function, and let $x_0\in(a,b)$ be a point in its domain.

(a) State the formal $\epsilon,\delta$ definition of $f$ being *continuous* at $x_0$. Explain this condition by drawing a diagram.

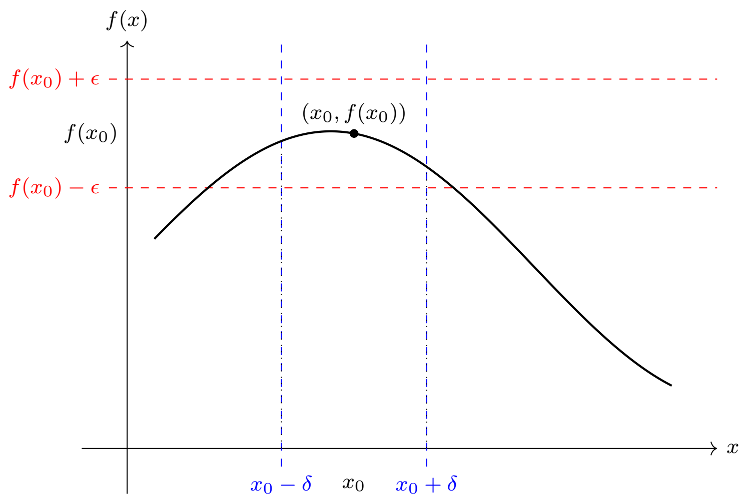
(b) Using the formal definition of a limit, explain why this condition is equivalent to the statement:

$$\lim_{x\to x_0} f(x) = f(x_0).$$

---

**Solution:** (a) We say that $f$ is continuous at $x_0$ if for any given tolerance $\epsilon>0$, there exists some closeness $\delta$, such that if we are at some point $x$ within $\delta$ of $x_0$, $|x-x_0|<\delta$, we have that $f(x)$ is within $\epsilon$ of $f(x_0)$, $|f(x)-f(x_0)|<\epsilon$.

A graphical representation of this definition is the following:




Notice that this is *extremely* similar to our limit definition diagrams!

(b) Comparing with the definition of a limit, continuity is just saying that we require:

$$\lim_{x\to x_0} f(x) = f(x_0).$$

27. Using the formal $\epsilon, \delta$ definition of continuity, show directly that the following functions are continuous everywhere:

(a) $x$, (b) $|x|$, (c) $x^2$, (d) $\sin(x)$.

At what points are these functions differentiable? [*Hint: for part (c), look at your answer to Question 16(b).*]

---

**➾ Solution:**

(a) Let's prove continuity at the point $x = a$, for general $a$. Given $\epsilon > 0$, we consider $x$ such that $|x - a| < \delta$, for some $\delta$ we shall decide at a later time. Then trivially we have:

$$|x - a| < \delta,$$

so that we see that if we choose $\delta = \epsilon$, the condition for continuity at $x = a$ will be satisfied. So we're done.

(b) Similarly, we shall prove continuity at the point $x = a$, for general $a$. Given $\epsilon > 0$, we consider $x$ such that $|x-a| < \delta$, for some $\delta$ we shall decide at a later time. Then we have:

$$||x| - |a|| \leq |x - a| < \delta,$$

using the *reverse triangle inequality*, which we proved on Examples Sheet 1, Question 12(c). Hence we see that if we choose $\delta = \epsilon$, we have $||x| - |a|| < \epsilon$. So we're done.

(c) We already saw that $x^2 \to a^2$ as $x \to a$ in Question 16(b), using an $\epsilon$-$\delta$ argument. This is precisely what we need to show for continuity.

(d) Let's prove continuity at the point $x = a$, for general $a$. Given $\epsilon > 0$, we consider $x$ such that $|x - a| < \delta$, for some $\delta$ we shall decide at a later time. Then we have:

$$\begin{aligned} |\sin(x) - \sin(a)| &= 2\left|\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{x+a}{2}\right)\right| \\ &\leq 2\left|\sin\left(\frac{x-a}{2}\right)\right| \\ &\leq 2\left|\frac{x-a}{2}\right| && \text{(since } |\sin(u)| \leq |u| \text{ for all } u) \\ &\leq |x - a| < \delta. \end{aligned}$$

Hence, if we choose $\delta = \epsilon$, we have $|\sin(x) - \sin(a)| < \epsilon$. Hence $\sin(x)$ is a continuous function.

These functions are all differentiable everywhere, except for $|x|$, which is not differentiable at $x = 0$.

---

28. Using the formal $\epsilon, \delta$ definition of continuity, show directly that the function $f(x) = 0$ for $x \leq 0$, $f(x) = 1$ for $x > 0$, is discontinuous at $x = 0$.

---

**➾ Solution:** We need to show that there exists some $\epsilon$ for which given any $\delta > 0$, there is some point with $|x| < \delta$ and $|f(x) - f(0)| = |f(x)| > \epsilon$.

Choose $\epsilon = 1/2$. Then given any $\delta > 0$, if we choose $x = \delta/2$, we have $|f(\delta/2) - f(0)| = 1 > 1/2$. So the function is not continuous at $x = 0$.

29. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x\sin(1/x)$ for $x \neq 0$, and $f(0) = 0$. Show that $f$ is continuous everywhere, and is differentiable everywhere except at $x = 0$.

---

➾ **Solution:** $x\sin(1/x)$ is evidently continuous at $x \neq 0$, since $x\sin(1/x) \to a\sin(1/a)$ as $x \to a$, by the product, quotient and continuity law of limits (we have already shown that $\sin(x)$ is continuous). The only problem point is $x = 0$, where $x\sin(1/x) \to 0$ as $x \to 0$ (we also showed this earlier on in the sheet). Therefore the function is continuous everywhere.

It is differentiable everywhere except $x = 0$ by the product and chain rules. It is not differentiable at $x = 0$ because the limit:

$$\lim_{h\to 0} \frac{h\sin(1/h) - 0}{h} = \lim_{h\to 0} \sin(1/h)$$

does not exist.

---

30. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} |x|^p \sin(x), & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad g(x) = \begin{cases} |x|^q \sin(\pi\sin(1/x)), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where $p, q$ are real numbers. For which values of $p, q$ are $f, g$: (a) continuous; (b) differentiable? Justify your answers.

---

➾ **Solution:** (a) Let's start with $f(x)$. Clearly $f(x)$ is continuous everywhere except $x = 0$. At $x = 0$, for continuity we require:

$$\lim_{x\to 0} |x|^p \sin(x) = 0.$$

This is certainly true if $p \geq 0$. If $p < 0$, we may be able to use L'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sin(x)}{|x|^{-p}} = -\frac{1}{p} \lim_{x\to 0} \frac{\cos(x)}{|x|^{-p-1}}.$$

If $-p - 1 < 0$, i.e. $-1 < p < 0$, this is convergent to $0$. Else, if $p = -1$, this is convergent to $-1/p$, which demonstrates a lack of continuity. For $p < -1$, this is divergent to $-\infty$, which demonstrates lack of continuity. Thus it is continuous precisely for $p > -1$.

Similarly, $g(x)$ is clearly continuous everywhere except $x = 0$. At $x = 0$, for continuity we require:

$$\lim_{x\to 0} |x|^q \sin(\pi\sin(1/x)) = 0.$$

This is certainly true if $q > 0$. The difference with $f(x)$ is that the sine term no longer converges to zero on its own, it infinitely oscillates as $x \to 0$. Indeed, suppose we pick $\sin(1/x) = 1/2$, by choosing $1/x = 2n\pi + \pi/6$. This defines a sequence of points:

$$x_n = \frac{1}{2n\pi + \pi/6},$$

which converges to zero. This means that we have a sequence of points that satisfy:

$$|x_n|^q \sin(\pi\sin(1/x_n)) = |x_n|^q \to \infty$$

as $n \to \infty$, for any $q < 0$. If $q = 0$ this converges to $1$. In particular, these points are not approaching zero, so we cannot have continuity.

(b) Clearly $f(x)$ is differentiable everywhere except $x = 0$. Differentiability of $f(x)$ at $x = 0$ requires existence of the limit:

$$\lim_{h\to 0}\frac{|h|^p\sin(h)-0}{h} = \lim_{h\to 0}\frac{|h|^p\sin(h)}{h} = \lim_{h\to 0}|h|^p\cdot\lim_{h\to 0}\frac{\sin(h)}{h} = \lim_{h\to 0}|h|^p\lim_{h\to 0}\frac{\cos(h)}{1} = \lim_{h\to 0}|h|^p.$$

This limit exists if and only if $p \geq 0$, hence the function is differentiable at $x = 0$ if and only if $p \geq 0$.

In the case of $g(x)$, again, clearly $g(x)$ is differentiable everywhere except $x = 0$. Differentiability of $g(x)$ at $x = 0$ requires existence of the limit:

$$\lim_{h\to 0}\frac{|h|^q\sin(\pi\sin(1/h))-0}{h} = \lim_{h\to 0}\frac{|h|^q\sin(\pi\sin(1/h))}{h}.$$

This time, we can't separate the limit out and use L'Hôpital's rule. Instead, observe that if $q > 1$, we can write:

$$\lim_{h\to 0}\frac{|h|^q\sin(\pi\sin(1/h))}{h} = \lim_{h\to 0}\operatorname{sign}(h)|h|^{q-1}\sin(\pi\sin(1/h)) = 0,$$

where $\operatorname{sign}(h)$ is the sign of $h$ (positive for $h > 0$, negative for $h < 0$). On the other hand if $q \leq 1$, we can choose a similar sequence to part (a), say:

$$h_n = \frac{1}{n\pi + \pi/6}.$$

We then observe that this sequence satisfies:

$$\frac{|h_n|^q\sin(\pi\sin(1/h_n))}{h_n} = h_n^{q-1}\sin\left(\pi\left((-1)^n\cdot\frac{1}{2}\right)\right) = \frac{(-1)^n}{h_n^{1-q}}.$$

If $q \leq 1$, this is not convergent. Hence the original limit cannot be convergent either. It follows that $g(x)$ is differentiable at $x = 0$ if and only if $q > 1$.

31. Three functions $f_0, f_1, f_2$ are defined by:

$$f_n(x) = \left(\frac{x - \pi/2}{x}\right)^n \sin(\tan(x))$$

for $n = 0, 1, 2$, at all points except $x = m\pi/2$ for integer $m$, where the functions are defined to be zero. For each $n$, determine with justification all points in the range $(-\pi, \pi)$ where the function is: (a) continuous; (b) differentiable.

---

➻ **Solution:** (a) When $n = 0$, we have $f_0(x) = \sin(\tan(x))$. This is continuous everywhere that $x \neq m\pi/2$. At $x = m\pi/2$, we have that the function $f_0(x)$ is defined to be zero. However, the limit:

$$\lim_{x \to m\pi/2} \sin(\tan(x))$$

does not exist, because $\tan(x)$ approaches infinity from two different directions here, and the limit of sine does not exist near infinity.

When $n = 1$, we have:

$$f_1(x) = \left(\frac{x - \pi/2}{x}\right) \sin(\tan(x)).$$

This is continuous everywhere that $x \neq m\pi/2$. At all points where $x = m\pi/2$, with $m \neq 0, 1$, we have the same issue as before, because $(x - \pi/2)/x$ is finite at these points. The problem is even worse at $m = 0$, because we have an infinite limit multiplied by a non-existent limit. However, at $m = 1$, we have:

$$\lim_{x \to \pi/2} \frac{x - \pi/2}{x} \sin(\tan(x)) = \frac{2}{\pi} \lim_{x \to \pi/2} (x - \pi/2) \sin(\tan(x)).$$

Since sine is always bounded, and $(x - \pi/2) \to 0$ as $x \to \pi/2$, we have that this tends to zero. Thus the function is continuous at $x = \pi/2$.

When $n = 2$, we have:

$$f_2(x) = \left(\frac{x - \pi/2}{x}\right)^2 \sin(\tan(x)).$$

The same reasoning as the case $n = 1$ gives that this function is continuous everywhere except $x = m\pi/2$ with $m \neq 1$.

---

(b) Now consider differentiability. The first function $f_0(x)$ is differentiable everywhere except points where $x = m\pi/2$, by the chain rule. But it is not differentiable at points of discontinuity; a function *must* be continuous at a point for it to be differentiable there (see Question 32).

The second function $f_1(x)$ is differentiable everywhere except points where $x = m\pi/2$, as above. We just need to check $x = \pi/2$, where the function is continuous. We require that the limit:

$$\lim_{x \to \pi/2} \frac{((x - \pi/2)/x) \sin(\tan(x)) - 0}{x - \pi/2} = \lim_{x \to \pi/2} \frac{\sin(\tan(x))}{x}$$

exists. But it evidently does not, so the function is indeed not differentiable there.

Finally, the third function $f_2(x)$ is differentiable everywhere except points where $x = m\pi/2$, as above. We just need to check $x = \pi/2$, where the function is continuous. We require that the limit:

$$\lim_{x \to \pi/2} \frac{((x - \pi/2)/x)^2 \sin(\tan(x)) - 0}{x - \pi/2} = \lim_{x \to \pi/2} \frac{(x - \pi/2)}{x^2} \sin(\tan(x)),$$

exists. Evidently it does, so the function is also differentiable at $x = \pi/2$.

32. Show that if a function is differentiable at a point $x_0$ in its domain, then it must be continuous at $x_0$. (∗) Is it true that a continuous function $f : \mathbb{R} \to \mathbb{R}$ must be differentiable at *some* point?

---

**Solution:** Let $f(x)$ be differentiable at $x_0$. We then have that:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, and is finite. Further, the limit $\lim_{x \to x_0}(x - x_0) = 0$ exists. Thus the product of these limits exists, and we can use the product rule for limits to give:

$$0 = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} = \lim_{x \to x_0} (f(x) - f(x_0)).$$

This gives:

$$\lim_{x \to x_0} f(x) = f(x_0),$$

so that $f(x)$ is continuous at $x_0$ as required.

---

It is *not* true that a function that is everywhere continuous must be differentiable at some point. Great counterexamples include *fractal patterns* (for example, look up the *Weierstrass function*, https://en.wikipedia.org/wiki/Weierstrass_function).