Part IA: Mathematics for Natural Sciences A
Examples Sheet 10: First-order ordinary differential equations

Model Solutions

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Separable equations

1. Explain what is meant by a *separable differential equation*, and how we can solve one. Find the general solution of the following separable differential equations:

(a) $\dfrac{dy}{dx} = x,$ (b) $\dfrac{dy}{dx} = (2-y)(1-y),$ (c) $\dfrac{dy}{dx} = -\dfrac{x^3}{(1+y)^2},$

(d) $\dfrac{dy}{dx} = \dfrac{4y}{x(y-3)},$ (e) $\dfrac{dy}{dx} = xe^{x-2y},$ (f) $\dfrac{dy}{dx} = \sin(y+x) - \sin(y-x).$

Solution: A separable equation is an equation of the form:

$$f(y)\frac{dy}{dx} = g(x).$$

By the fundamental theorem of calculus, and the chain rule, the left hand side can be written as:

$$\frac{d}{dx}\left(\int f(y)\,dy\right) = g(x).$$

Integrating the entire equation, we then have:

$$\int f(y)\,dy = \int g(x)\,dx,$$

which is the solution of the equation, when rearranged for y in terms of x.

(a) Integrating directly, we have:

$$y = \frac{1}{2}x^2 + c,$$

where c is an arbitrary constant.

(b) Rearranging, if $y \neq 1$ and $y \neq 2$, the solution is given by:

$$\int \frac{dy}{(2-y)(1-y)} = \int dx = x + c.$$

To evaluate the integral on the left hand side, we note:

$$\frac{1}{(2-y)(1-y)} = \frac{1}{y^2-3y+2} = \frac{1}{(y-3/2)^2-1/4} = \frac{4}{((y-3/2)/(1/2))^2-1}.$$

The integral is therefore given by:

$$8\mathrm{artanh}\left(\frac{y-3/2}{1/2}\right) = x + c.$$

Rearranging, we see that the solution is:

$$y = \frac{1}{2}\tanh\left(\frac{x+c}{8}\right) + \frac{3}{2}.$$

Notice that the constant solutions $y = 1$ and $y = 2$ that we divided by are in fact solutions themselves though, because they make the right hand side zero, and their derivatives are zero. These two solutions correspond to the asymptotes of the solution we found earlier.

In general, you can always divide by functions of x in differential equations with no issue, because they just introduce 'singular points' in the solutions. For example,

$$x\frac{dy}{dx} = 1 \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{x} \qquad \Rightarrow \qquad y = \ln(x) + c.$$

The singularity in the differential equation at $x = 0$ is reflected in the solution's singularity at $x = 0$. However, if we divide by a function of y, we must keep track of the solution we might have lost. For example,

$$\frac{dy}{dx} = y \qquad \Rightarrow \qquad \int \frac{dy}{y} = \int dx \qquad \Rightarrow \qquad \ln(y) = x + c \qquad \Rightarrow \qquad y = Ae^x,$$

where $A = e^c$ is a positive constant. This family of solutions does *not* include the constant solution $y = 0$ that we divided by!

(c) Separating variables, the equation becomes:

$$\int (1+y)^2\,dy = -\int x^3\,dx$$

Thus:

$$\frac{1}{3}(1+y)^3 = c - \frac{1}{4}x^4,$$

for some arbitrary constant c. Rearranging, we have:

$$y = \left(3c - \frac{3}{4}x^4\right)^{1/3} - 1.$$

(d) Separating variables, if $y \neq 0$ (which is itself a constant solution), the equation becomes:

$$\int \left(1 - \frac{3}{y}\right)\,dy = \int \frac{4}{x}\,dx.$$

Integrating, we have:

$$y - 3\ln(y) = 4\ln(x) + c.$$

This cannot be solved in elementary terms for y, so we just leave it in this form!

(e) Separating variables, by noticing that $e^{x-2y} = e^x e^{-2y}$, we have:

$$\int e^{2y}\,dy = \int xe^x\,dx.$$

Integrating, we have:

$$\frac{1}{2}e^{2y} = xe^x - \int e^x\,dx = xe^x - e^x + c,$$

where c is an arbitrary constant. Rearranging for y, we have:

$$y = \frac{1}{2}\ln\left(2xe^x - 2e^x + 2c\right).$$

(f) This equation is not *obviously* separable, but using some trigonometry it becomes separable. Note that:

$$\sin(y+x) - \sin(y-x) = \sin(y)\cos(x) + \sin(x)\cos(y) - \sin(y)\cos(x) + \sin(x)\cos(y) = 2\sin(x)\cos(y).$$

Hence separating variables, we have (provided that $y \neq (2n+1)\pi/2$ for n an integer, which are themselves constant solutions):

$$\int \frac{dy}{\cos(y)} = 2\int \sin(x)\,dx.$$

Recalling the integral of $\sec(y)$ is $2\mathrm{artanh}(\tan(y/2))$ from the 'half-tangent substitution' question on the integration sheet, we have that:

$$2\mathrm{artanh}(\tan(y/2)) = -2\cos(x) + c.$$

Rearranging, we have for a new arbitrary constant $C = c/2$:

$$y = 2\arctan\left(\tanh(C - \cos(x))\right).$$

2. Determine the half-life of thorium-234 if a sample of mass 5g is reduced to 4g in one week. What amount of thorium is left after twelve weeks?

➻ **Solution:** A radioactive sample decays according to the amount of material left, via the equation:

$$\frac{dm}{dt} = -\alpha m.$$

The solution to this equation is given by separation of variables as:

$$\int \frac{dm}{m} = -\alpha \int dt,$$

which on integration gives:

$$\ln(m) = -\alpha t + c \qquad \Rightarrow \qquad m(t) = Ae^{-\alpha t},$$

for an arbitrary constant A.

To fix the constants, we use the data supplied in the question. At time $t = 0$, there is 5g of thorium-234, so $A = 234$. After $t = 1$ week, there is 4g. Hence:

$$4 = 5e^{-\alpha} \qquad \Rightarrow \qquad \alpha = -\ln\left(\frac{4}{5}\right) \text{ per week.}$$

The half-life is the amount of time it takes for the material's mass to decrease by half. Let the half-life be t_0. Then the half-life must obey:

$$m(t+t_0) = \frac{1}{2}m(t)$$

for all time. Solving this equation, we have:

$$e^{-\alpha(t+t_0)} = \frac{1}{2}e^{-\alpha t} \qquad \Rightarrow \qquad e^{\ln(4/5)t_0} = \frac{1}{2} \qquad \Rightarrow \qquad \left(\frac{4}{5}\right)^{t_0} = \frac{1}{2}.$$

This implies the half-life is $t_0 = \log_{4/5}(1/2)$.

The amount of thorium-234 left at time $t = 12$ weeks is given by:

$$m(12) = 5e^{\ln(4/5)\cdot 12} = 5 \cdot \left(\frac{4}{5}\right)^{12} = \frac{4^{12}}{5^{11}} \text{ g.}$$

3. *Newton's law of cooling* states that the rate of heat loss from a body is proportional to the difference between the temperature of the object and its ambient environment. Assuming that Newton's law of cooling applies, calculate the time at which a cup of tea in a $20°$C room was made, given that: (i) the tea is measured to have temperature $30°$C at 5pm; (ii) the tea is measured to have temperature $40°$C at 3pm; (iii) the water was initially at boiling point.

➬ **Solution:** Let T be the temperature of the body, and let T_0 be the ambient environmental temperature. Then Newton's law of cooling states:

$$\frac{dT}{dt} = -\alpha(T - T_0),$$

for some proportionality constant α. Separating variables, we have:

$$\int \frac{dT}{T - T_0} = -\alpha \int dt \qquad \Rightarrow \qquad \ln(T - T_0) = -\alpha t + c.$$

Rearranging, we have:

$$T(t) = T_0 + Ae^{-\alpha t},$$

for some arbitrary constant A.

Now, we apply the boundary data. The room temperature is $T_0 = 20$. Let us start measuring time, $t = 0$, at 3pm. Then when $t = 0$, we have $T = 40$. Measuring time in units of one hour, we have when $t = 2$, $T = 30$. Thus we have:

$$40 = 20 + A, \qquad 30 = 20 + Ae^{-2\alpha}.$$

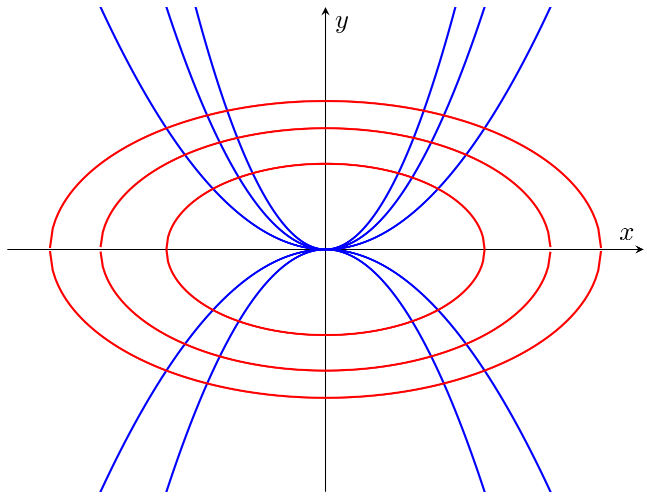
The first equation gives $A = 20$, the second equation gives $\frac{1}{2} = e^{-2\alpha}$, hence $\alpha = \frac{1}{2}\ln(2)$. In particular, we have $T = 100$ when:

$$100 = 20 + 20e^{-\frac{1}{2}\ln(2)t} \qquad \Rightarrow \qquad \ln(4) = -\frac{1}{2}\ln(2)t.$$

Rearranging, we find that $t = -2\ln(4)/\ln(2) = -4\ln(2)/\ln(2) = -4$. Hence the tea was made 4 hours before 5pm, at 1pm.

4. Consider the family of curves $C = \{y = ax^2 : a \in \mathbb{R}\}$. Sketch a few representative curves in C. Determine a family of curves $C' = \{y = f(x,b) : b \in \mathbb{R}\}$ such that each curve in C' is orthogonal to all curves in C, and sketch a few representative curves in the family C'.

➡ **Solution:** The curves are all parabolas of varying gradient, going through the point $(0,0)$. Some are oriented as 'upside-down' parabolas, as shown in blue on the sketch.



Let $y(x)$ be a curve in the family C'. Then at the point $(x, y(x))$, we require that the gradient dy/dx is the negative reciprocal of the gradient of the curve $y' = a(x')^2$ in the family C going through that same point. The curve $(y') = a(x')^2$ that goes through the point $(x, y(x))$ must have $a = y/x^2$, hence has gradient:

$$\frac{dy'}{dx'} = 2a(x') = \frac{2yx}{x^2} = \frac{2y}{x}.$$

Therefore, the curves in C' satisfy the differential equation:

$$\frac{dy}{dx} = -\frac{x}{2y}.$$

Separating variables, we have:

$$\int y\,dy = -\frac{1}{2}\int x\,dx \qquad \Rightarrow \qquad \frac{1}{2}y^2 = -\frac{1}{4}x^2 + b.$$

Rearranging, we see that $y^2 + (x/\sqrt{2})^2 = 2b$, so that the curves in C' are all ellipses of varying radii. Their centres are at $(0,0)$, and their axes lengths are in the ratio $1 : \sqrt{2}$, as shown in red on the sketch.

Linear equations

5. Write down the general form of a *linear first-order differential equation*. What is meant by an *integrating factor* for such an equation? Solve the following linear first-order differential equations by finding an appropriate integrating factor:

(a) $\dfrac{dy}{dx} + 2xy = 4x$, (b) $\dfrac{dy}{dx} + \dfrac{y}{2-3x^2} = 1$, (c) $\dfrac{dy}{dx} - y\tan(x) = 1$,

(d) $\dfrac{dy}{dx} + (1+\log(x))y = x^{-x}$, (e) $\dfrac{dy}{dx} + \dfrac{y}{\tan(x)} = \cos^2(x)$, (f) $\dfrac{1}{x}\dfrac{dy}{dx} + y - 5e^{-x^2} = 0$.

➭ **Solution:** The general first-order *linear* differential equation is:

$$\frac{dy}{dx} + \alpha(x)y = \beta(x), \qquad (*)$$

where $\alpha(x), \beta(x)$ are arbitrary functions. An integrating factor is a function $I(x)$ such that if we multiply the equation by $I(x)$, the left hand side becomes the derivative of a product:

$$I(x)\frac{dy}{dx} + \alpha(x)I(x)y = \beta(x)I(x).$$

For the left hand side to be the derivative of a product, we require that:

$$I'(x) = \alpha(x)I(x) \qquad \Rightarrow \qquad \int \frac{dI}{I} = \int \alpha(x)\,dx \qquad \Rightarrow \qquad I(x) = \exp\left(\int \alpha(x)\,dx\right).$$

Thus the integrating factor can be obtained generally by exponentiating the integral of the coefficient of y in the form $(*)$. By writing the left hand side of $(*)$ as the derivative of a product, we can then integrate directly.

(a) The integrating factor is:

$$\exp\left(\int 2x\,dx\right) = e^{x^2}.$$

Hence the equation becomes:

$$\frac{d}{dx}\left(e^{x^2}y\right) = 4xe^{x^2}.$$

Integrating directly, we have:

$$e^{x^2}y = 2e^{x^2} + c.$$

Hence the general solution is $y = 2 + ce^{-x^2}$.

(b) The integrating factor is:

$$\exp\left(\int \frac{dx}{2-3x}\right) = \exp\left(-\frac{1}{3}\ln(2-3x)\right) = \frac{1}{(2-3x)^{1/3}}.$$

Hence the equation becomes:

$$\frac{d}{dx}\left(\frac{y}{(2-3x)^{1/3}}\right) = \frac{1}{(2-3x)^{1/3}}.$$

Integrating both sides directly, we have:

$$\frac{y}{(2-3x)^{1/3}} = c - \frac{1}{3}(2-3x)^{2/3}.$$

Hence the general solution is:

$$y = c(2-3x)^{-1/3} - \frac{1}{3}(2-3x)^{1/3}.$$

(c) The integrating factor is:

$$\exp\left(-\int \tan(x)\,dx\right) = \exp\left(-\int \frac{\sin(x)}{\cos(x)}\,dx\right) = \exp\left(\ln(\cos(x))\right) = \cos(x).$$

Thus the equation becomes:

$$\frac{d}{dx}\left(\cos(x)y\right) = \cos(x).$$

Integrating directly, we have:

$$\cos(x)y = \sin(x) + c.$$

Hence the solution is:

$$y = \tan(x) + c\sec(x).$$

(d) The integrating factor is:

$$\exp\left(\int (1 + \log(x))\,dx\right) = \exp\left(x + x\log(x) - x\right) = x^x.$$

Hence the equation becomes:

$$\frac{d}{dx}\left(x^x y\right) = 1.$$

Integrating both sides directly, we have:

$$y = x^{1-x} + cx^{-x}.$$

(e) The integrating factor is:

$$\exp\left(\int \cot(x)\,dx\right) = \exp\left(\int \frac{\cos(x)}{\sin(x)}\,dx\right) = \exp\left(\ln(\sin(x))\right) = \sin(x).$$

Therefore, the equation becomes:

$$\frac{d}{dx}\left(\sin(x)y\right) = \sin(x)\cos^2(x).$$

Integrating both sides directly, we have:

$$\sin(x)y = \int \sin(x)\cos^2(x)\,dx = -\frac{1}{3}\cos^3(x) + c.$$

Thus we have:

$$y = c\mathrm{cosec}(x) - \frac{1}{3}\cos^2(x)\cot(x).$$

(f) First, putting the equation in standard form, we have:

$$\frac{dy}{dx} + xy = 5xe^{-x^2}.$$

The integrating factor is evidently $e^{x^2/2}$. Thus the equation becomes:

$$\frac{d}{dx}\left(e^{x^2/2}y\right) = 5xe^{-x^2/2}.$$

Integrating both sides directly, we have:

$$e^{x^2/2}y = c - 5e^{-x^2/2}.$$

Hence the solution is:

$$y = ce^{-x^2/2} - 5e^{-x^2}.$$

6. Solve the equation:

$$(x^2+1)\frac{dy}{dx}+3x^3y=6xe^{-3x^2/2},$$

subject to the boundary condition $y(0)=1$.

➾ **Solution:** Dividing by x^2+1, we have:

$$\frac{dy}{dx}+\frac{3x^3y}{x^2+1}=\frac{6xe^{-3x^2/2}}{x^2+1}.$$

The integrating factor is therefore:

$$\exp\left(\int\frac{3x^3}{x^2+1}\,dx\right).$$

To perform the integral, note that:

$$\int\frac{3x^3}{x^2+1}\,dx=\int\left(\frac{3x(x^2+1)-3x}{x^2+1}\right)\,dx=\frac{3}{2}x^2-\frac{3}{2}\ln(x^2+1).$$

It follows that the integrating factor is:

$$\frac{e^{3x^2/2}}{(x^2+1)^{3/2}}.$$

The equation becomes:

$$\frac{d}{dx}\left(\frac{e^{3x^2/2}y}{(x^2+1)^{3/2}}\right)=\frac{6x}{(x^2+1)^{5/2}}.$$

Integrating both sides directly, we have:

$$\frac{e^{3x^2/2}y}{(x^2+1)^{3/2}}=c-\frac{4}{(x^2+1)^{3/2}}.$$

Hence the general solution is:

$$y(x)=ce^{-3x^2/2}(x^2+1)^{3/2}-4e^{-3x^2/2}.$$

The solution that satisfies the given initial data $y(0)=1$ has:

$$1=c-4,$$

so $c=5$. Thus the specific solution satisfying the boundary data is:

$$y(x)=5e^{-3x^2/2}(x^2+1)^{3/2}-4e^{-3x^2/2}.$$

7. Establish a formula for the general solution of the linear differential equation:

$$\alpha(x)\frac{dy}{dx} + \beta(x)y = \gamma(x),$$

stating any conditions you must assume for your formula to be valid.

➡ **Solution:** Assuming that $\alpha(x) \neq 0$ (by which we mean that α is not *identically zero* on some region - we can have isolated singularities), we have:

$$\frac{dy}{dx} + \frac{\beta(x)}{\alpha(x)}y = \frac{\gamma(x)}{\alpha(x)}.$$

Applying the integrating factor, we then have:

$$\frac{d}{dx}\left(y\exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx\right)\right) = \frac{\gamma(x)}{\alpha(x)}\exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx\right)$$

Integrating both sides with respect to x, we have:

$$y = \exp\left(-\int \frac{\beta(x)}{\alpha(x)}\,dx\right)\int \frac{\gamma(x)}{\alpha(x)}\exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx\right)\,dx + c\exp\left(-\int \frac{\beta(x)}{\alpha(x)}\,dx\right),$$

where c is an arbitrary constant.

It is useful to think about why all the indefinite integrals don't give any extra constants. The constant of integration c arises from the direct integration of both sides of the integral above. The only other possibility is that the integrating factor integral carries a constant of integration, I_0, which would shift the solution to:

$$y = \exp\left(-\int \frac{\beta(x)}{\alpha(x)}\,dx - I_0\right)\int \frac{\gamma(x)}{\alpha(x)}\exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx + I_0\right)\,dx + c\exp\left(-\int \frac{\beta(x)}{\alpha(x)}\,dx - I_0\right).$$

Observe that the constants of integration I_0 cancel in the first term, and they simply shift c in the second term. Thus they have no overall effect. Importantly though, in our formula, any time we see an integral of $\beta(x)/\alpha(x)$ we must use the *same* constant of integration to have this independence of constants.

Some 'easy to spot' substitutions

8. Using an appropriate substitution, find the general solution of the equation:

$$y^3 + x + 3y^2 \frac{dy}{dx} = 0.$$

Find also the solution satisfying the boundary condition $y(0) = 1$.

➡ **Solution:** Here, implicit differentiation suggests that:

$$3y^2 \frac{dy}{dx} = \frac{d}{dx}(y^3).$$

So a sensible substitution to make is just $u = y^3$, giving:

$$\frac{du}{dx} + u = -x.$$

The integrating factor is obviously e^x, giving:

$$ue^x = -\int xe^x \, dx = -xe^x + \int e^x \, dx = e^x(1-x) + c.$$

Hence we have $y^3 = 1 - x + ce^{-x}$, giving:

$$y = (1 - x + ce^{-x})^{1/3}.$$

The solution satisfying $y(0) = 1$ has $1 = (1+c)^{1/3}$, giving $c = 0$. Thus $y = (1-x)^{1/3}$.

9. Using an appropriate substitution, find the general solution of the equation:

$$(x+y+1)^2\frac{dy}{dx}+(x+y+1)^2+x^3=0.$$

Find also the solution satisfying the boundary condition $y(0)=0$.

➾ **Solution:** If we set $u=x+y+1$, then:

$$\frac{du}{dx}=1+\frac{dy}{dx}.$$

This gives:

$$u^2\left(\frac{du}{dx}-1\right)+u^2+x^3=0.$$

Rearranging, we have:

$$u^2\frac{du}{dx}=-x^3.$$

Separating variables, we have:

$$\frac{1}{3}u^3=c-\frac{1}{4}x^4.$$

Thus we have:

$$y=\left(3c-\frac{3}{4}x^4\right)^{1/3}-x-1.$$

Imposing the data $y(0)=0$, we have:

$$0=(3c)^{1/3}-1,$$

so that $3c=1$. Hence:

$$y=\left(1-\frac{3}{4}x^4\right)^{1/3}-x-1.$$

Some standard substitutions: homogeneous, Bernoulli, and affine transformations

10. Define a *homogeneous equation*, and state the substitution which renders them solvable. Hence, solve the equations:

(a) $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right)$, (b) $(y-x)\dfrac{dy}{dx} + (2x+3y) = 0$, (c) $\dfrac{dy}{dx} = \dfrac{x^3+y^3}{3xy^2}$.

➾ **Solution:** A *homogeneous equation* is an equation of the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

These equations can be solved by the substitution $u = y/x$. Rearranging, we see $y = ux$, which gives:

$$\frac{dy}{dx} = \frac{du}{dx}x + u.$$

Inserting into the original equation, we have:

$$\frac{du}{dx}x + u = f(u) \qquad \Rightarrow \qquad \int \frac{du}{f(u)-u} = \int \frac{dx}{x} = \ln(x) + c.$$

Thus, the solution can be obtained by performing the integral on the left hand side.

(a) Here, we have $f(u) = u + \tan(u)$. Hence we calculate:

$$\ln(x) + c = \int \cot(u)\,du = \int \frac{\cos(u)\,du}{\sin(u)} = \ln(\sin(u)).$$

Rearranging, and inserting $u = y/x$, we have $y = x\arcsin(Ax)$ for some constant A.

(b) Rearranging the given equation, we have:

$$\frac{dy}{dx} = \frac{2x+3y}{x-y} = \frac{2+3y/x}{1-y/x}.$$

Hence, $f(u) = (2+3u)/(1-u)$. Note that:

$$f(u) - u = \frac{2+3u}{1-u} - \frac{u-u^2}{1-u} = \frac{2+2u+u^2}{1-u},$$

so we need to compute:

$$\begin{aligned}\int \frac{1-u}{u^2+2u+2}\,du &= -\int \frac{u+1}{u^2+2u+2}\,du + \int \frac{2}{u^2+2u+2}\,du \\ &= -\frac{1}{2}\ln(u^2+2u+2) + \int \frac{2}{(u+1)^2+1}\,du \\ &= -\frac{1}{2}\ln(u^2+2u+2) + 2\arctan(u+1).\end{aligned}$$

Inserting this into the general solution, and recalling that $u = y/x$, we have:

$$-\frac{1}{2}\ln(y^2/x^2 + 2y/x + 2) + 2\arctan(y/x+1) = \ln(x) + c.$$

There is pretty much no hope of rearranging this implicit equation to give y as a function of x, sadly.

(c) Dividing the numerator and denominator of the right hand side by x^3, we have:

$$\frac{dy}{dx} = \frac{1+(y/x)^3}{3(y/x)^2},$$

so we see that $f(u) = (1+u^3)/3u^2$. Thus:

$$f(u) - u = \frac{1}{3u^2} - \frac{2u}{3} = \frac{1-2u^3}{3u^2}.$$

Hence, we need to calculate:

$$\int \frac{du}{f(u)-u} = \int \frac{3u^2}{1-2u^3}\,du = -\frac{1}{2}\ln(1-2u^3).$$

The solution is therefore:

$$-\frac{1}{2}\ln(1-2u^3) = \ln(x) + c \qquad \Rightarrow \qquad 1-2u^3 = \frac{A}{x^2},$$

for some constant A. Substituting $u = y/x$, we have:

$$y = \frac{x}{2^{1/3}}\left(1 - \frac{A}{x^2}\right)^{1/3}.$$

11. Define a *Bernoulli equation*, and state the substitution which renders them solvable. Hence, solve the equations:

$$\text{(i) } \frac{dy}{dx} - y = xy^5, \quad \text{(ii) } \frac{dy}{dx} + y = y^2(\cos(x) - \sin(x)), \quad \text{(iii) } xy\frac{dy}{dx} + (x^2 + y^2 + x) = 0.$$

➻ **Solution:** A *Bernoulli equation* is an equation of the form:

$$\frac{dy}{dx} = P(x)y + Q(x)y^\alpha,$$

for some $\alpha \neq 1$ (not necessarily an integer!). We make the substitution $u = y^{1-\alpha}$. Rearranging for y, we have:

$$y = u^{1/(1-\alpha)} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{1-\alpha}u^{1/(1-\alpha)-1}\frac{du}{dx}.$$

Hence, we have:

$$\frac{1}{1-\alpha}u^{1/(1-\alpha)-1}\frac{du}{dx} = P(x)u^{1/(1-\alpha)} + Q(x)u^{1/(1-\alpha)-1} \qquad \Rightarrow \qquad \frac{1}{1-\alpha}\frac{du}{dx} = P(x)u + Q(x),$$

which is a first-order linear equation, which we know how to solve.

(a) In this case, $P(x) = 1, Q(x) = x, \alpha = 5$, so on substituting $u = y^{1-5} = y^{-4}$, we obtain:

$$-\frac{1}{4}\frac{du}{dx} = u + x \qquad \Rightarrow \qquad \frac{du}{dx} + 4u = -4x.$$

The integrating factor is obviously e^{4x}, which gives:

$$e^{4x}u = -4\int xe^{4x}\,dx = -xe^{4x} + \int e^{4x} = -xe^{4x} + \frac{1}{4}e^{4x} + c.$$

Therefore:

$$y(x) = \left(\frac{1}{4} - x + ce^{-4x}\right)^{-1/4}$$

(b) Here, $P(x) = -1, Q(x) = \cos(x) - \sin(x), \alpha = 2$, so on substituting $u = y^{1-2} = y^{-1}$, we obtain:

$$-\frac{du}{dx} = -u + \cos(x) - \sin(x) \qquad \Rightarrow \qquad \frac{du}{dx} - u = \sin(x) - \cos(x).$$

The integrating factor is evidently e^{-x}, which gives:

$$e^{-x}u = \int e^{-x}(\sin(x) - \cos(x))\,dx = e^{-x}\cos(x) + c \qquad \Rightarrow \qquad y = \frac{1}{\cos(x) + ce^x}.$$

(c) Dividing everywhere by xy (observe that $y = 0$ is not a constant solution), we can convert this equation to Bernoulli form:

$$\frac{dy}{dx} + (x+1)y^{-1} + \frac{y}{x} = 0,$$

with $P(x) = -1/x, Q(x) = -(x+1)$ and $\alpha = -1$. Hence on substituting $u = y^{1-(-1)} = y^2$, we obtain:

$$\frac{1}{2}\frac{du}{dx} = -\frac{u}{x} - x - 1.$$

Rearranging, we have:

$$\frac{du}{dx} + \frac{2u}{x} = -2x - 2.$$

The integrating factor is $\exp\left(2\ln(x)\right) = x^2$, which gives:

$$x^2\frac{du}{dx} + 2xu = -2x^3 - 2x^2.$$

Integrating directly, we have:

$$x^2 u = -\frac{x^4}{2} - \frac{2x^3}{3} + c.$$

Thus we have:

$$y = \sqrt{\frac{c}{x^2} - \frac{x^2}{2} - \frac{2x}{3}}.$$

12.

(a) Show that equations of the form:

$$\frac{dy}{dx} = f(ax+by+c),$$

with $b \neq 0$ may be reduced to a separable equation by making the substitution $u = ax+by+c$.

(b) Hence, solve the equations:

$$\text{(i) } \frac{dy}{dx} = (4x+y)^2, \quad \text{(ii) } \cos(x+y-1)\frac{dy}{dx} = \sin(x+y-1), \quad \text{(iii) } \frac{dx}{dy} = \frac{1}{\cosh^2(2x-y+2)+2}.$$

Solution: (a) We have:

$$\frac{du}{dx} = a + b\frac{dy}{dx},$$

so substituting we get:

$$\frac{1}{b}\frac{du}{dx} - \frac{a}{b} = f(u) \qquad \Rightarrow \qquad \int \frac{du}{bf(u)+a} = \int dx = x + k,$$

where k is an arbitrary constant, using separation of variables in the final step.

(b) Addressing each of the given equations in turn, we have:

(i) Here, $a=4, b=1, c=0$, with $f(u)=u^2$. Thus the substitution is $u = 4x+y$, and the solution is:

$$x + k = \int \frac{du}{u^2+4} = \frac{1}{4}\int \frac{du}{(u/2)^2+1} = \frac{1}{2}\arctan\left(\frac{u}{2}\right).$$

Rearranging, we have:

$$y = 2\tan(2(x+k)) - 4x.$$

(ii) Here, $a=1, b=1, = -1$, with $f(u) = \tan(u)$. Thus the substitution is $u = x+y-1$, and the solution is:

$$x + k = \int \frac{du}{\tan(u)+1} = \int \frac{\cos(u)\,du}{\sin(u)+\cos(u)}.$$

This is a rather fun integral to do, because it can be done with a trick. Observe that:

$$\int \frac{\cos(u)\,du}{\sin(u)+\cos(u)} + \int \frac{\sin(u)\,du}{\sin(u)+\cos(u)} = \int du = u,$$

and also:

$$\int \frac{\cos(u)\,du}{\sin(u)+\cos(u)} - \int \frac{\sin(u)\,du}{\sin(u)+\cos(u)} = \int \frac{\cos(u)-\sin(u)}{\sin(u)+\cos(u)}\,du = \ln(\sin(u)+\cos(u)).$$

Hence, adding these equations, we see that:

$$\int \frac{\cos(u)\,du}{\sin(u)+\cos(u)} = \frac{1}{2}\left(u + \ln(\sin(u)+\cos(u))\right).$$

We therefore have:

$$x + k = \frac{1}{2}\left(x+y-1+\ln(\sin(x+y-1)+\cos(x+y-1))\right).$$

It is fairly hopeless to expect to rearrange this for y.

(iii) The last equation has:

$$\frac{dy}{dx} = \cosh^2(2x - y - 2) + 2.$$

Hence $a = 2, b = -1, c = -2$, $f(u) = \cosh^2(u) + 2$, and the substitution is $u = 2x - y - 2$. Thus:

$$x + k = -\int \frac{du}{\cosh^2(u)} = -\tanh(u) = -\tanh(2x - y - 2).$$

Rearranging, we have:

$$y = 2x + \operatorname{artanh}(x + k) - 2,$$

for a constant k.