Part IA: Mathematics for Natural Sciences B
Examples Sheet 1: Basics of vector geometry, and the scalar and vector products

Model Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

Basics of vector algebra

1. Let $A = (1,3,4)$, $B = (-1,2,4)$, and $C = (2,2,3)$. Which of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ is the longest?

Solution: The three-dimensional version of Pythagoras' theorem gives the lengths of the vectors as:

$$|\overrightarrow{AB}| = \sqrt{(-1-1)^2 + (2-3)^2 + (4-4)^2} = \sqrt{2^2+1^2} = \sqrt{5}$$

$$|\overrightarrow{BC}| = \sqrt{(2--1)^2 + (2-2)^2 + (3-4)^2} = \sqrt{3^2+1^2} = \sqrt{10}$$

$$|\overrightarrow{AC}| = \sqrt{(2-1)^2 + (2-3)^2 + (3-4)^2} = \sqrt{1^2+1^2+1^2} = \sqrt{3}.$$

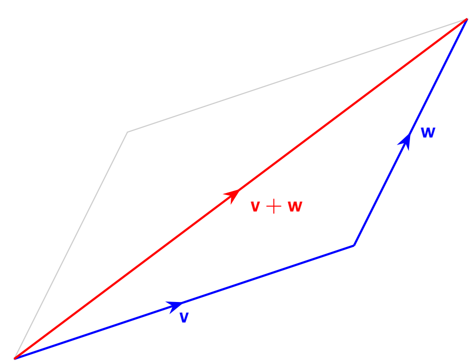
Hence the longest vector is $\overrightarrow{BC}$.

---

2. (a) State the definition of $\mathbf{v}+\mathbf{w}$, given the vectors $\mathbf{v}$, $\mathbf{w}$. Using this definition, show that $\mathbf{v}+\mathbf{w} = \mathbf{w}+\mathbf{v}$, and $(\mathbf{v}+\mathbf{w})+\mathbf{u} = \mathbf{v}+(\mathbf{w}+\mathbf{u})$ for any vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$.

   (b) Suppose that an aeroplane's engine produces a velocity 125 km h^{-1} due North. If there is a wind travelling at a velocity 80 km h^{-1} at a bearing 60° West of North, use trigonometry to determine how fast the aeroplane travels across the Earth, and the bearing of its direction of travel from North.

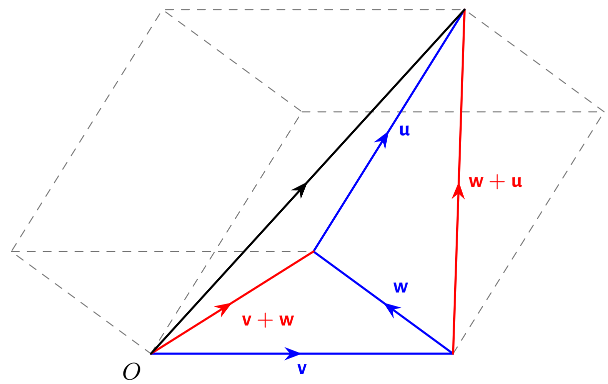
Solution: (a) To produce the vector $\mathbf{v}+\mathbf{w}$ from the vectors $\mathbf{v}$, $\mathbf{w}$, we put the start of the vector $\mathbf{w}$ at the end of the vector $\mathbf{v}$, as shown in the diagram below. The directed line segment from the start of the vector $\mathbf{v}$ to the end of the vector $\mathbf{w}$ in this configuration is then defined to be the vector $\mathbf{v}+\mathbf{w}$.



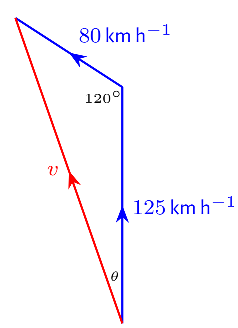
This definition immediately shows that $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$. This is because the if we started with the vector $\mathbf{w}$, then put the start of the vector $\mathbf{v}$ at the end of the vector $\mathbf{w}$, we produce a parallelogram (shown in grey in the above figure). The diagonal of the parallelogram is both equal to $\mathbf{v}+\mathbf{w}$ and $\mathbf{w}+\mathbf{v}$, which shows that it does not matter in which order we add vectors (In fancy language, we say that vector addition is a *commutative operation*).

Next, we are asked to show the property $(\mathbf{v}+\mathbf{w})+\mathbf{u}=\mathbf{v}+(\mathbf{w}+\mathbf{u})$. We shall do this in three dimensions, which is the more general case.

The best proof is just a 'proof by picture'. The vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$ form a parallelepiped, as shown in the diagram below. In red, we show the vectors $\mathbf{v}+\mathbf{w}$ and $\mathbf{w}+\mathbf{u}$, which are diagonals on the faces of the parallelepiped. Finally, the vectors $(\mathbf{v}+\mathbf{w})+\mathbf{u}$ and $\mathbf{v}+(\mathbf{w}+\mathbf{u})$ both correspond to the black diagonal of the parallelepiped shown in the figure below; hence, they are equal.



(b) To find the resultant velocity of the aircraft, we need to add the two given velocities (which are, of course, vector quantities). Here, a good diagram is helpful!



By the cosine rule, we have that the resultant speed of the aircraft is:

$$v=\sqrt{80^2+125^2-2\cdot 80\cdot 125\cos(120^\circ)}\,\mathrm{km\,h^{-1}}=5\sqrt{1281}\,\mathrm{km\,h^{-1}}.$$

By the sine rule, we have that the angle θ in the diagram is given by:

$$\frac{\sin(\theta)}{80\,\mathrm{km\,h^{-1}}}=\frac{\sin(120^\circ)}{v}\qquad\Rightarrow\qquad\theta=\arcsin\left(\frac{80\,\mathrm{km\,h^{-1}}\sin(120^\circ)}{v}\right)=\arcsin\left(\frac{40\sqrt{3}}{5\sqrt{1281}}\right)=\arcsin\left(\frac{8}{\sqrt{427}}\right).$$

Despite the rather nasty numbers, in general, exact answers are preferred, since calculators are *not* permitted in the first-year mathematics exams.

3. (a) Define a *basis* of vectors.

(b) Let **v** $= (1, 2)$, $\mathbf{e}_1 = (1, -1)$ and $\mathbf{e}_2 = (2, 3)$. Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbb{R}^2$, and determine the components of **v** with respect to this basis.

(c) Let $\mathbf{w}_1 = (1, 2, 3)$ with respect to the basis $\{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$, and let $\mathbf{w}_2 = (3, 2, 1)$ with respect to the basis $\{(0, 1, 2), (2, 1, 0), (0, 1, -2)\}$. Find $\mathbf{w}_1 - 3\mathbf{w}_2$ with respect to the standard basis of $\mathbb{R}^3$.

Solution: (a) A *basis of vectors in n dimensions* is a collection of n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ satisfying two properties:

(i) SPANNING. Any vector **v** can be written as a linear combination of the vectors in the basis; that is, for some scalar coefficients $\alpha_1, \ldots, \alpha_n$ we can write:

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n.$$

(ii) LINEAR INDEPENDENCE. If we have:

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

for some coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$, then we must have $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. This property is telling us that no vectors in the basis are redundant - if there *were* some non-zero coefficients that satisfied this equation, we would be able to rearrange this equation to write one of the basis vectors in terms of the others.

It is actually the case that if we have n vectors that span $\mathbb{R}^n$, they must be linearly independent. Similarly, if we have n vectors that are linearly independent in $\mathbb{R}^n$, they must span. The proof is given in the Comments at the end of this question.

(b) First, we are asked to show that $\{(1, -1), (2, 3)\}$ is a basis for $\mathbb{R}^2$. We show each of the properties separately (although as commented above, we actually only need to show one of these properties, and the other is guaranteed to follow!):

(i) SPANNING. Let (v, w) be any vector in $\mathbb{R}^2$. We need to find coefficients α, β such that:

$$\begin{pmatrix} v \\ w \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \alpha + 2\beta \\ -\alpha + 3\beta \end{pmatrix}.$$

Hence, we have a system of two linear simultaneous equations for the variables α, β. Adding the first equation to the second, we obtain $5\beta = v + w$. Thus $\beta = \frac{1}{5}(v + w)$. Substituting back into the first equation, we have:

$$v = \alpha + \frac{2}{5}(v + w) \qquad \Rightarrow \qquad \alpha = \frac{1}{5}(3v - 2w).$$

It follows that we can write:

$$\begin{pmatrix} v \\ w \end{pmatrix} = \frac{1}{5}(3v - 2w) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{5}(v + w) \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

for any vector $(v, w) \in \mathbb{R}^2$.

(ii) LINEAR INDEPENDENCE. In the calculation for spanning, we have just shown that if $(0, 0) = \alpha(1, -1) + \beta(2, 3)$, then the coefficients α, β are given by:

$$\alpha = \frac{1}{5}(3 \cdot 0 - 2 \cdot 0) = 0, \qquad \beta = \frac{1}{5}(0 + 0) = 0.$$

Hence we have linear independence.

As mentioned above, we know that it must be the case that if we have already shown the vectors span $\mathbb{R}^2$, they must be linearly independent - this is why the calculations are so similar.

Finally, we are asked the components of **v** $= (1, 2)$ with respect to this basis. By the above calculation, we have:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{5}(3 \cdot 1 - 2 \cdot 2) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{5}(1 + 2) \begin{pmatrix} 2 \\ 3 \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{3}{5} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Hence the components are $-1/5, 3/5$.

(c) We are told that:

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix},$$

and:

$$\mathbf{w}_2 = 3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}.$$

Hence with respect to the standard basis of $\mathbb{R}^3$, we have:

$$\mathbf{w}_1 - 3\mathbf{w}_2 = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 4 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} -9 \\ -19 \\ -24 \end{pmatrix}.$$

❊ **Comments:** We stated in this question that if we have n vectors in $\mathbb{R}^n$ that span, they must be linearly independent. We also stated that if we have n vectors in $\mathbb{R}^n$ that are linearly independent, they must span.

To prove these results, we first prove a smaller result called the *Steinitz exchange lemma*. This is rather technical and mathematical (and *definitely non-examinable!*), but essentially it just tells us: *a set that is linearly independent can never be bigger than a set that spans*, which makes intuitive sense!

The Steinitz exchange lemma (for $\mathbb{R}^n$): Let $U = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ be a set of linear independent vectors in $\mathbb{R}^n$, and let $V = \{\mathbf{v}_1, ..., \mathbf{v}_l\}$ be a set of vectors which spans $\mathbb{R}^n$. Then we can choose vectors $\mathbf{v}_{\alpha_1}, ..., \mathbf{v}_{\alpha_{l-k}}$ in V such that $\{\mathbf{u}_1, ..., \mathbf{u}_k, \mathbf{v}_{\alpha_1}, ..., \mathbf{v}_{\alpha_{l-k}}\}$ spans $\mathbb{R}^n$. Consequently, we must have $k \leq l$ (a set that is linearly independent can never be bigger than a set that spans).

Proof: We use induction on k. When $k = 0$, we trivially have that $\{\mathbf{v}_1, ..., \mathbf{v}_l\}$ already spans by assumption.

Now suppose that result is true for $k = r - 1$. Then by the induction hypothesis, we can choose $l - (r-1)$ elements of the set V which when taken together with the set U span. Without loss of generality, we can therefore assume that the set $\{\mathbf{u}_1, ..., \mathbf{u}_{r-1}, \mathbf{v}_r, ..., \mathbf{v}_l\}$ spans.

Since this set spans, there must be coefficients μ_i such that:

$$\mathbf{u}_r = \sum_{j=1}^{r-1} \mu_j \mathbf{u}_j + \sum_{j=r}^{l} \mu_j \mathbf{v}_j.$$

At least one of the coefficients μ_j for $j \geq r$ must be non-zero, else we would contradict linear independence of the set U. By reordering basis elements if necessary, we may assume without loss of generality that $\mu_r \neq 0$. Then we can rearrange the above equation to read:

$$\mathbf{v}_r = \frac{1}{\mu_r}\left(\mathbf{u}_r - \sum_{j=1}^{r-1} \mu_j \mathbf{u}_j - \sum_{j=r+1}^{l} \mu_j \mathbf{v}_j\right).$$

That is, $\mathbf{v}_r$ is in the span of $\{\mathbf{u}_1, ..., \mathbf{u}_r, \mathbf{v}_{r+1}, ..., \mathbf{v}_l\}$. Hence, this set must also span $\mathbb{R}^n$, since by assumption $\{\mathbf{u}_1, ..., \mathbf{u}_{r-1}, \mathbf{v}_r, ..., \mathbf{v}_l\}$ spans $\mathbb{R}^n$. The result follows by induction. $\square$

We immediately have the following consequences:

Corollary: We have the following consequences of the Steinitz exchange lemma:

(i) All bases of $\mathbb{R}^n$ have size n.

(ii) Any linearly independent set of size n in $\mathbb{R}^n$ is a basis (so spans).

(iii) Any set of size n in $\mathbb{R}^n$ that spans is a basis (so is linearly independent).

Proof: The standard Cartesian unit vectors $\mathbf{e}_1 = (1, 0, 0, .., 0)$, $\mathbf{e}_2, ..., \mathbf{e}_n$ can easily be shown to be a basis of $\mathbb{R}^n$. Let $S = \{\mathbf{e}_1, ..., \mathbf{e}_n\}$.

(i) Let T be another basis of $\mathbb{R}^n$. Then T is a linearly independent set so must have size less than n. But S is linearly independent, so must have size less than T. Hence T must have size exactly n.

(ii) Suppose that $T = \{\mathbf{t}_1, ..., \mathbf{t}_n\}$ is a linearly independent set of size n that does not span. Then there exists some $\mathbf{v}$ not in the span of T; we claim that $\{\mathbf{t}_1, ..., \mathbf{t}_n, \mathbf{v}\}$ is linearly independent. Suppose that:

$$\alpha_0 \mathbf{v} + \sum_{i=1}^{n} \alpha_i \mathbf{t}_i = \mathbf{0}.$$

Then if $\alpha_0 \neq 0$, we could rearrange to write $\mathbf{v}$ in terms of $\{\mathbf{t}_1, ..., \mathbf{t}_n\}$. But this is a contradiction, because $\mathbf{v}$ is not contained in the span of T. Hence $\alpha_0 = 0$; by linear independence of T, we then have $\alpha_i = 0$ for all $i = 1, ..., n$.

But then $\{\mathbf{t}_1, ..., \mathbf{t}_n, \mathbf{v}\}$ is a linearly independent set of size $n + 1$ which exceeds the size of S, which spans. This contradicts the Steinitz exchange lemma, so T must have spanned in the first place.

(iii) Suppose that $T = \{\mathbf{t}_1, ..., \mathbf{t}_n\}$ is a spanning set of size n that is linearly dependent. Then there exists some coefficients $\alpha_1, ..., \alpha_n$, not all zero, such that:

$$\alpha_1 \mathbf{t}_1 + \cdots + \alpha_n \mathbf{t}_n = \mathbf{0}.$$

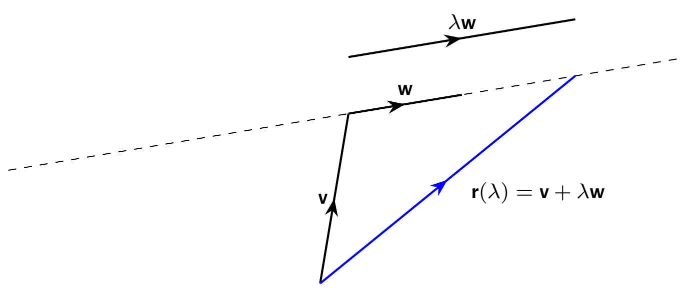
Without loss of generality, $\alpha_1 \neq 0$, which allows us to rearrange and write $\mathbf{t}_1$ in terms of $\mathbf{t}_2, ..., \mathbf{t}_n$. This shows that $\{\mathbf{t}_2, ..., \mathbf{t}_n\}$ spans. But this contradicts the Steinitz exchange lemma, because a set of size $n - 1$ cannot span since S is a set of size n which is linearly independent. Thus T must have been linearly independent in the first place.

Phew! That's lots of technical work, so you are probably only bothered by this if you are interested in very pure maths. Otherwise, remembering the conclusion is fine: *a set of n vectors is a basis for $\mathbb{R}^n$ if it either spans, or is linearly independent*.

The equation of a line

4. Let **v**, **w** $\in \mathbb{R}^3$ be 3-vectors, and suppose that **w** $\neq$ **0**.
   - (a) Explain why the equation **r** $=$ **v** $+ \lambda$**w**, as $\lambda \in \mathbb{R}$ varies, represents a line, and summarise its properties. Why is the condition **w** $\neq$ **0** necessary?
   - (b) If **v** $= (x_0, y_0, z_0)$ and **w** $= (a, b, c)$, where $a, b, c \neq 0$, show that the same line may be equivalently described through the system of equations:
   $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$
   What is the corresponding system of equations in the cases where one or more of a, b, c are zero?
   - (c) Show that the position vectors $(1, 0, 1)$, $(1, 1, 0)$ and $(1, -3, 4)$ lie on a straight line, and find both its vector form, as in (a), and its Cartesian form, as in (b).

Solution: (a) The equation **r** $=$ **v** $+ \lambda$**w** represents the dashed line shown in the figure below. The dashed line is parallel to the vector **w**, and the point with position vector **v** lies on the line. Therefore, to get to any point on the line, we can first follow the position vector **v** from the origin onto the line, then follow a scaled version of the vector **w** to get to any other point on the line.



In particular, **r** $=$ **v** $+ \lambda$**w** describes a line through the point with position vector **v**, parallel to **w**. The condition **w** $\neq$ **0** is needed, else the equation **r** $=$ **v** $+ \lambda$**w** $=$ **v** would describe a single point instead of a line.

(b) We insert **r** $= (x, y, z)$, **v** $= (x_0, y_0, z_0)$ and **w** $= (a, b, c)$ into the vector equation to get:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_0 + \lambda a \\ y_0 + \lambda b \\ z_0 + \lambda c \end{pmatrix}.$$

This is a system of three equations, $x = x_0 + \lambda a$, $y = y_0 + \lambda b$, $z = z_0 + \lambda c$. Rearranging each equation for λ, and equating them, we have:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

as required.

If one of a, b, c is zero, we cannot perform the division in the solution of the equations. For example, if $a = 0$, then we have the equations $x = x_0$, $y = y_0 + \lambda b$, $z = z_0 + \lambda c$. This means that the resulting equation of the line, eliminating λ, is given by:

$$x = x_0, \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

That is, the x-direction equation is 'decoupled' from the equations in the y-direction and z-direction. Similar conclusions hold when $b = 0$ or $c = 0$.

(c) Observe that $(1,1,0)-(1,0,1) = (0,1,-1)$ and $(1,-3,4)-(1,1,0) = (0,-4,4) = -4(0,1,-1)$. Hence the vectors joining the points $(1,0,1)$ to $(1,1,0)$, and then $(1,1,0)$ to $(1,-3,4)$, are parallel; it follows that all three points lie on a straight line.

In vector form, one possible equation of the line is:

$$\mathbf{r} = (1,1,0) + \lambda(0,1,-1),$$

but other forms are possible. Setting $\mathbf{r} = (x,y,z)$, we have:

$$(x,y,z) = (1,1,0) + \lambda(0,1,-1) \qquad \Rightarrow \qquad \{x = 1,\ y-1 = \lambda,\ -z = \lambda\}.$$

Therefore, eliminating λ, the Cartesian equation of the line is:

$$\{x = 1,\ 1-y = z\}.$$

---

5. Show that the solution of the linear system $x-2y+3z = 0, 3x-2y+z = 0$ is a line that is equally inclined to the x and z-axes, and makes an angle $\arccos(\sqrt{2/3})$ with the y-axis.

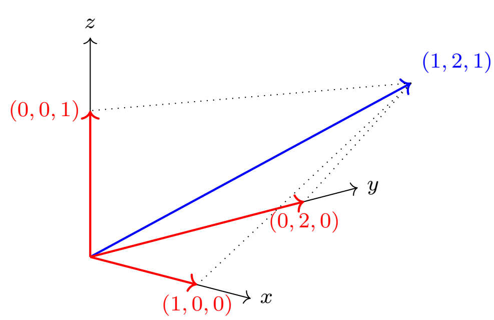
Solution: This is a linear system of equations with two equations, but three free variables. Thus we expect that one variable will be unconstrained. Therefore, we should try to write two of the variables in terms of the third variable.

Subtracting the second equation from the first, we obtain $-2x+2z = 0$, which on rearrangement gives $x = z$. Substituting into the first equation, we have $z - 2y + 3z = 0$, which on rearrangement gives $y = 2z$. Thus the solution of the system can be written as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 2z \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

where z is a free real parameter. We recognise that this is the equation of a line going through the origin, with direction vector $(1,2,1)$.

To obtain the angles, we do some trigonometry (or, we use the *scalar product* - see later in the sheet).



We see from the figure that the angle the vector makes with both the x-axis and the z-axis is the same, given by:

$$\arccos\left(\frac{1}{\sqrt{1^2+2^2+1^2}}\right) = \arccos\left(\frac{1}{\sqrt{6}}\right).$$

The angle the vector makes with the y-axis is given by:

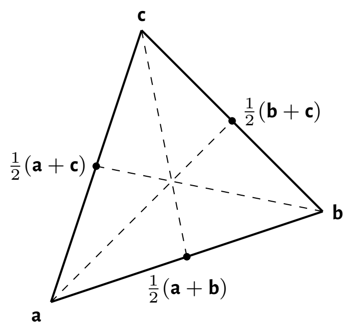
$$\arccos\left(\frac{2}{\sqrt{1^2+2^2+1^2}}\right) = \arccos\left(\frac{2}{\sqrt{6}}\right) = \arccos\left(\sqrt{\frac{2}{3}}\right),$$

as required.

6. (a) A *median* of a triangle is a line joining a vertex to the midpoint of its opposite edge. Prove that the three medians of a triangle are concurrent (the point at which they meet is called the *centroid* of the triangle).

   (b) Similarly, prove that in any tetrahedron, the lines joining the midpoints of opposite edges are concurrent.

➾ **Solution:** (a) Let the vertices of the triangle have position vectors **a**, **b**, **c** respectively, as shown in the figure below.



The midpoints of the edges of the triangle are:

$$\frac{\mathbf{a}+\mathbf{b}}{2}, \qquad \frac{\mathbf{b}+\mathbf{c}}{2}, \qquad \frac{\mathbf{a}+\mathbf{c}}{2}.$$

The vector equations of the lines going the midpoints and the opposite vertices are:

$$\mathbf{r}_1(\lambda) = \mathbf{c} + \lambda\left(\frac{\mathbf{a}+\mathbf{b}}{2} - \mathbf{c}\right), \qquad \mathbf{r}_2(\mu) = \mathbf{a} + \mu\left(\frac{\mathbf{b}+\mathbf{c}}{2} - \mathbf{a}\right), \qquad \mathbf{r}_3(\nu) = \mathbf{b} + \nu\left(\frac{\mathbf{a}+\mathbf{c}}{2} - \mathbf{b}\right).$$

Collecting like terms, these equations can be written as:

$$\mathbf{r}_1(\lambda) = \frac{\lambda}{2}\mathbf{a} + \frac{\lambda}{2}\mathbf{b} + (1-\lambda)\mathbf{c}, \qquad \mathbf{r}_2(\mu) = (1-\mu)\mathbf{a} + \frac{\mu}{2}\mathbf{b} + \frac{\mu}{2}\mathbf{c}, \qquad \mathbf{r}_3(\nu) = \frac{\nu}{2}\mathbf{a} + (1-\nu)\mathbf{b} + \frac{\nu}{2}\mathbf{c}.$$

Now, we would like to set $\mathbf{r}_1(\lambda) = \mathbf{r}_2(\mu) = \mathbf{r}_3(\nu)$, and find values of λ, μ, ν that solve these equations. *However*, in general **a**, **b**, **c** are *not* linearly independent in two dimensions so we cannot compare coefficients!

We can be a bit sneaky here: if we imagine that the triangle is not planar - that is, we imagine that **a**, **b**, **c** are *three-dimensional vectors* - we *can* assume they are linearly independent. Hence, comparing coefficients is completely okay here!

In the first equality, $\mathbf{r}_1(\lambda) = \mathbf{r}_2(\mu)$, we see that $\lambda = \mu$ and $\lambda/2 = 1 - \mu$, which gives $\lambda = \mu = 2/3$. This gives the point of intersection of $\mathbf{r}_1, \mathbf{r}_2$ as:

$$\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Symmetrically, we see that $\mathbf{r}_2, \mathbf{r}_3$ and $\mathbf{r}_1, \mathbf{r}_3$ intersect at the same point, so all three lines intersect at the same point.

(b) For the second part, let the vertices of the tetrahedron have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. Then the midpoints of the edges are:

$$\frac{\mathbf{a}+\mathbf{b}}{2}, \qquad \frac{\mathbf{a}+\mathbf{c}}{2}, \qquad \frac{\mathbf{a}+\mathbf{d}}{2}, \qquad \frac{\mathbf{b}+\mathbf{c}}{2}, \qquad \frac{\mathbf{b}+\mathbf{d}}{2}, \qquad \frac{\mathbf{c}+\mathbf{d}}{2}.$$

The lines joining the midpoints of opposite edges are:

$$\mathbf{r}_1(\lambda) = \frac{\mathbf{a}+\mathbf{b}}{2} + \lambda\left(\frac{\mathbf{a}+\mathbf{b}}{2} - \frac{\mathbf{c}+\mathbf{d}}{2}\right), \qquad \mathbf{r}_2(\mu) = \frac{\mathbf{a}+\mathbf{c}}{2} + \mu\left(\frac{\mathbf{a}+\mathbf{c}}{2} - \frac{\mathbf{b}+\mathbf{d}}{2}\right), \qquad \mathbf{r}_3(\nu) = \frac{\mathbf{a}+\mathbf{d}}{2} + \nu\left(\frac{\mathbf{a}+\mathbf{d}}{2} - \frac{\mathbf{b}+\mathbf{c}}{2}\right).$$

Collecting like terms, these equations can be rewritten as:

$$\mathbf{r}_1(\lambda) = \frac{(1+\lambda)}{2}\mathbf{a} + \frac{(1+\lambda)}{2}\mathbf{b} - \frac{\lambda}{2}\mathbf{c} - \frac{\lambda}{2}\mathbf{d}, \qquad \mathbf{r}_2(\mu) = \frac{(1+\mu)}{2}\mathbf{a} - \frac{\mu}{2}\mathbf{b} + \frac{(1+\mu)}{2}\mathbf{c} - \frac{\mu}{2}\mathbf{d},$$

$$\mathbf{r}_3(\nu) = \frac{(1+\nu)}{2}\mathbf{a} - \frac{\nu}{2}\mathbf{b} - \frac{\nu}{2}\mathbf{c} + \frac{(1+\nu)}{2}\mathbf{d}.$$

As in part (a), we can compare coefficients here (even though $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are not linearly independent in three dimensions, we could embed the tetrahedron in a four-dimensional space if we wanted!). The equation $\mathbf{r}_1(\lambda) = \mathbf{r}_2(\mu)$ gives $\mu = \lambda$ and $1 + \lambda = -\mu$, which tells us that $\lambda = \mu = -1/2$. This gives the point of intersection between the first two lines:

$$\frac{1}{4}\left(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}\right).$$

The remaining pairs of lines have the same intersection, by symmetry of the calculation, so we're done.

The scalar product

7. Let **v**, **w**, **u** $\in \mathbb{R}^3$ be 3-vectors.

(a) Give the definition of the *scalar product* $\mathbf{v} \cdot \mathbf{w}$ in terms of lengths and angles. If **v** is a unit vector, explain why $\mathbf{v} \cdot \mathbf{w}$ is the signed length of the projection of **w** in the direction of **v**.

(b) Using *only* this definition, prove each of the following properties of the scalar product:

(i) *commutativity*: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$;

(ii) *homogeneity*: $(\lambda \mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{v} \cdot \mathbf{w})$;

(iii) *left-distributivity*: $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$.

[Hint: In (ii), consider the cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ separately.]

Now let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the standard basis vectors for $\mathbb{R}^3$, and let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ be 3-vectors.

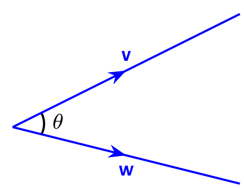
(c) Using *only* the definition of the scalar product in terms of lengths and angles, show that for $i, j = 1, 2, 3$, we have $\mathbf{e}_i \cdot \mathbf{e}_j = 1$ if $i = j$, and $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ if $i \neq j$.

(d) By writing **v**, **w** as linear combinations of the standard basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and applying the properties of the scalar product from the previous question, prove the formula $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$.

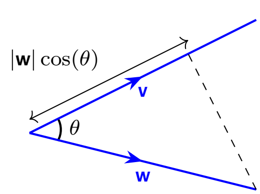
Solution: (a) The scalar product $\mathbf{v} \cdot \mathbf{w}$ is defined by:

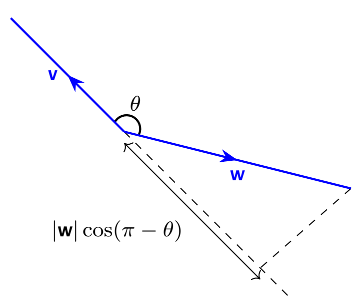
$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos(\theta),$$

where $|\mathbf{v}|$, $|\mathbf{w}|$ are the lengths of the vectors, and θ is the *angle* between the vectors. Importantly, the angle is chosen to be the smaller angle produced when **v**, **w** are positioned so that they start from the same point (see the diagram below).



Now consider the case where $\mathbf{v} = \hat{\mathbf{v}}$ is a unit vector. In this case, we have $\hat{\mathbf{v}} \cdot \mathbf{w} = |\mathbf{w}| \cos(\theta)$. Compare with the figures below:





If the angle θ is acute, we see that $|\mathbf{w}| \cos(\theta)$ is the length of the projection of **w** in the direction of **v**. Hence $\hat{\mathbf{v}} \cdot \mathbf{w}$ is the length of the projection of **w** in the direction of **v**.

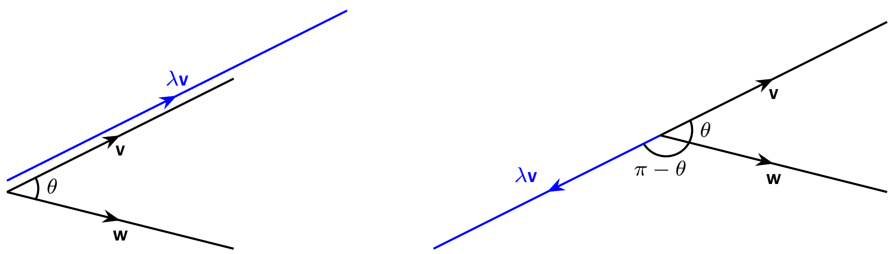
If the angle θ is obtuse, we see that $|\mathbf{w}|\cos(\pi - \theta) = -|\mathbf{w}|\cos(\theta)$ is the length of the projection of **w** in the direction of **v**. Hence $-\hat{\mathbf{v}} \cdot \mathbf{w}$ is the length of the projection of **w** in the direction of **v**.

Overall then, we see that $\hat{\mathbf{v}} \cdot \mathbf{w}$ is the *signed length of the projection*; its magnitude is always equal to the length of the projection, but the sign depends on whether the vectors **v**, **w** are pointing in the same direction (the angle between them is acute), or in different directions (the angle between them is obtuse).

(b) The properties can be proved as follows:

(i) Commutativity. This property is obvious. In both the definitions of $\mathbf{v} \cdot \mathbf{w}$ and $\mathbf{w} \cdot \mathbf{v}$, we are required to place the starts of the vectors **v**, **w** in the same location. Both situations are identical, though, so the scalar products must be equal.

(ii) Homogeneity. We split into three cases: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Relevant diagrams in the cases $\lambda > 0$ and $\lambda < 0$ are shown in the figures below.



In the first case, $\lambda > 0$, we are in the situation of the left diagram above. The angle between $\lambda\mathbf{v}$ and **w** is the same as the angle between **v** and **w**. Hence, we have:

$$(\lambda\mathbf{v}) \cdot \mathbf{w} = |\lambda\mathbf{v}||\mathbf{w}|\cos(\theta) = \lambda|\mathbf{v}||\mathbf{w}|\cos(\theta) = \lambda(\mathbf{v} \cdot \mathbf{w}),$$

as required.

In the second case, $\lambda = 0$, the vector $\lambda\mathbf{v}$ has zero length. Hence $(\lambda\mathbf{v}) \cdot \mathbf{w} = 0 = \lambda(\mathbf{v} \cdot \mathbf{w})$ holds trivially.

In the third and final case, $\lambda < 0$, we are in the situation of the right diagram above. The angle between $\lambda\mathbf{v}$ and **w** is now *not* the same as the angle between **v** and **w**; instead, if the angle between **v** and **w** is θ, we see that the angle between $\lambda\mathbf{v}$ and **w** is $\pi - \theta$. Hence we have:

$$\begin{aligned}(\lambda\mathbf{v}) \cdot \mathbf{w} &= |\lambda\mathbf{v}||\mathbf{w}|\cos(\pi - \theta) \\ &= (-\lambda)|\mathbf{v}||\mathbf{w}|\cos(\pi - \theta) && (\text{since } \lambda < 0, \text{ we have } |\lambda\mathbf{v}| = -\lambda|\mathbf{v}|) \\ &= \lambda|\mathbf{v}||\mathbf{w}|\cos(\theta) && (\text{since } \cos(\pi - \theta) = -\cos(\theta)) \\ &= \lambda(\mathbf{v} \cdot \mathbf{w}).\end{aligned}$$

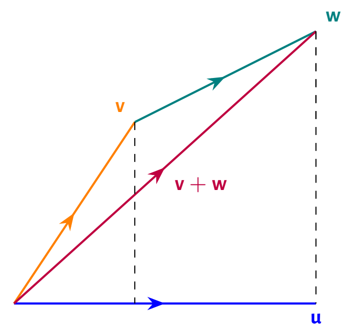
Thus the formula holds for all values of λ, as required.

(iii) Distributivity. The property $(\mathbf{v}+\mathbf{w})\cdot\mathbf{u}=\mathbf{v}\cdot\mathbf{u}+\mathbf{w}\cdot\mathbf{u}$ is equivalent to:

$$(\mathbf{v}+\mathbf{w})\cdot\hat{\mathbf{u}}=\mathbf{v}\cdot\hat{\mathbf{u}}+\mathbf{w}\cdot\hat{\mathbf{u}},$$

by the homogeneity property, that we proved in part (b). That is, the property is equivalent to the fact that: *the signed length of the projection of $\mathbf{v}+\mathbf{w}$ onto the direction of $\mathbf{u}$ is the same as the sum of the signed lengths of the projections of $\mathbf{v}$ and $\mathbf{w}$ onto the direction of $\mathbf{u}$.*

This property is very easy to see in two dimensions, with a nice, clear diagram! See the one below, for example.



This also holds in 3D, but drawing a diagram is much more fiddly, so we won't do it. One way of thinking about it is imagining that the above diagram is a 3D diagram, just with $\mathbf{v}$, $\mathbf{w}$ pointing at angles into and out of the page; the projections still look like those in the diagram.

(c) Each of the standard basis vectors has length 1, and is inclined at an angle $\pi/2$ to all the other basis vectors. Since $\cos(\pi/2)=0$ and $\cos(0)=1$, this implies:

$$\mathbf{e}_i\cdot\mathbf{e}_j=|\mathbf{e}_i||\mathbf{e}_j|\cos(0)=1\cdot1\cdot1=1,$$

if $i=j$, and:

$$\mathbf{e}_i\cdot\mathbf{e}_j=|\mathbf{e}_i||\mathbf{e}_j|\cos(\pi/2)=1\cdot1\cdot0=0,$$

if $i\neq j$, as required.

(d) We have $\mathbf{v}=v_1\mathbf{e}_1+v_2\mathbf{e}_2+v_3\mathbf{e}_3$ and $\mathbf{w}=w_1\mathbf{e}_1+w_2\mathbf{e}_2+w_3\mathbf{e}_3$. For efficiency, we break down the computation of $\mathbf{v}\cdot\mathbf{w}$ into two parts:

- Observe that:

$$\begin{aligned}(v_i\mathbf{e}_i)\cdot\mathbf{w}&=v_i\mathbf{e}_i\cdot\mathbf{w} && \text{(homogeneity)}\\ &=v_i\left(\mathbf{e}_i\cdot(w_1\mathbf{e}_1)+\mathbf{e}_i\cdot(w_2\mathbf{e}_2)+\mathbf{e}_i\cdot(w_3\mathbf{e}_3)\right) && \text{(distributivity, twice)}\\ &=v_iw_1\mathbf{e}_i\cdot\mathbf{e}_1+v_iw_2\mathbf{e}_i\cdot\mathbf{e}_2+v_iw_3\mathbf{e}_i\cdot\mathbf{e}_3 && \text{(symmetry and homogeneity)}\\ &=v_iw_i,\end{aligned}$$

using the fact that $\mathbf{e}_i\cdot\mathbf{e}_j=0$ for $i\neq j$, and 1 for $i=j$.

- Next, we have:

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{w} &= \mathbf{w}\cdot\mathbf{v} && \text{(symmetry)}\\
&= \mathbf{w}\cdot(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3)\\
&= \mathbf{w}\cdot(v_1\mathbf{e}_1) + \mathbf{w}\cdot(v_2\mathbf{e}_2) + \mathbf{w}\cdot(v_3\mathbf{e}_3) && \text{(distributivity, twice)}\\
&= (v_1\mathbf{e}_1)\cdot\mathbf{w} + (v_2\mathbf{e}_2)\cdot\mathbf{w} + (v_3\mathbf{e}_3)\cdot\mathbf{w} && \text{(symmetry)}\\
&= v_1w_1 + v_2w_2 + v_3w_3,
\end{aligned}$$

using the earlier part of the calculation in the final line. Hence, we're done!

---

8. Explain how we can use the two different formulae for the scalar product to determine the angles between vectors. Hence:
   (a) determine the angles AOB and OAB, where the points A, B have coordinates $(0,3,4)$, $(3,2,1)$ respectively;
   (b) find the acute angle at which two diagonals of a cube intersect.

Solution: From the previous question, we now have two formulae for the scalar product:

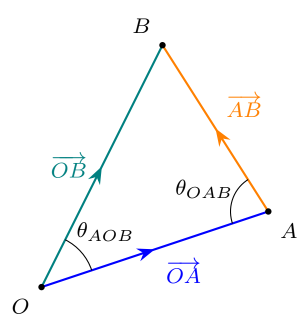
$$\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos(\theta) = v_1w_1 + v_2w_2 + v_3w_3.$$

Rearranging the second equality, we have:

$$\cos(\theta) = \frac{v_1w_1 + v_2w_2 + v_3w_3}{|\mathbf{v}||\mathbf{w}|},$$

which allows us to compute the angle between two vectors easily.

(a) In the first case, we want the angle between the vector $\overrightarrow{OA}$ and $\overrightarrow{OB}$ (remember that the angle we get from this formula is the smaller angle produced when the vectors start at the same point - see the diagram below).



The relevant vectors are:

$$\overrightarrow{OA} = (0,3,4), \qquad \overrightarrow{OB} = (3,2,1).$$

Hence we have:

$$\cos(\theta_{AOB}) = \frac{0\cdot 3 + 3\cdot 2 + 4\cdot 1}{\sqrt{3^2+4^2}\sqrt{3^2+2^2+1^2}} = \frac{10}{5\sqrt{14}} = \sqrt{\frac{2}{7}}.$$

Thus the angle is:

$$\theta_{AOB} = \arccos\sqrt{\frac{2}{7}}.$$

In the second case, we want the angle between the vector $\overrightarrow{AO}$ and $\overrightarrow{AB}$. The relevant vectors are:

$$\overrightarrow{AO} = (0,-3,-4), \qquad \overrightarrow{AB} = (3,2,1) - (0,3,4) = (3,-1,-3).$$

Hence we have:

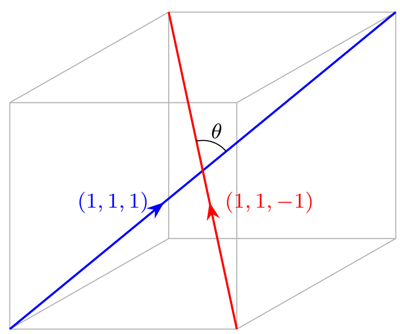
$$\cos(\theta_{OAB}) = \frac{0\cdot 3 + (-3)\cdot(-1) + (-4)\cdot(-3)}{\sqrt{3^2+4^2}\sqrt{3^2+1^2+3^2}} = \frac{15}{5\sqrt{19}} = \frac{3}{\sqrt{19}}.$$

Thus the angle is:

$$\theta_{OAB} = \arccos\frac{3}{\sqrt{19}}.$$

(b) Without loss of generality, we may assume that we are working with a unit cube, of side length 1, since the angles are unchanged by scaling the cube up or down. Let's put the vertices of the cube at the points:

$$(0,0,0), \quad (0,0,1), \quad (0,1,1), \quad (0,1,0), \quad (1,0,0), \quad (1,0,1), \quad (1,1,1), \quad (1,1,0).$$



The vector from the origin to the opposite vertex is $(1,1,1)$. On the other hand, the vector from the vertex $(1,0,0)$ to the opposite vertex at $(0,1,1)$ is given by $(0,1,1)-(1,0,0) = (-1,1,1)$. The required angle θ therefore satisfies:

$$\cos(\theta) = \frac{1\cdot(-1) + 1\cdot 1 + 1\cdot 1}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+1^2+1^2}} = \frac{1}{3}.$$

Hence the angle is:

$$\theta = \arccos\frac{1}{3}.$$

9. Consider the line with vector equation $\mathbf{r} = (-1, 0, 1) + \lambda(3, 2, 1)$, where λ is a real parameter.

(a) Using the scalar product, compute the projection of the vector $(1, 2, 3)$ in the direction $(3, 2, 1)$.

(b) Hence, determine the point on the line which is closest to the point $(0, 2, 4)$, and the shortest distance from the line to the point $(0, 2, 4)$.

(c) Now, generalise your result: find a formula for the point on the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ which is closest to the point with position vector $\mathbf{p}$, and a formula for the the shortest distance from the line to the point.

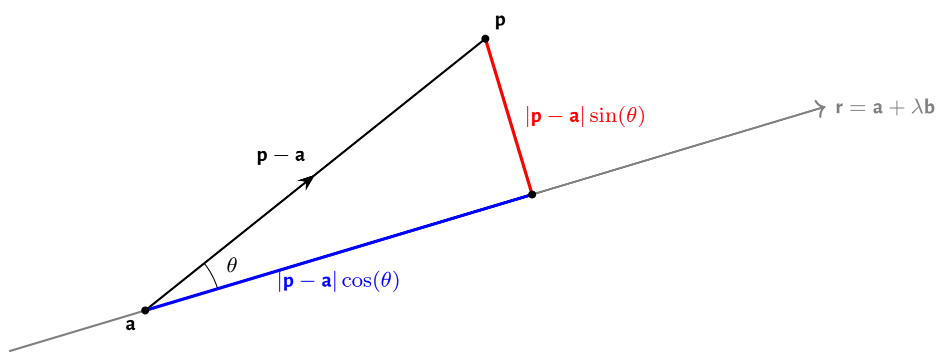
Solution: (a) As we learned in Question 7, the projection is:

$$(1, 2, 3) \cdot \frac{(3, 2, 1)}{\sqrt{3^2 + 2^2 + 1^2}} = \frac{10}{\sqrt{14}}.$$

(b) Our strategy will be the following:

- Find a vector joining the point to another special point on the line.
- Compute the projection of our vector onto the line.
- Move the length of the projection along the line, from the special point, to find the closest point.

This can be visualised with the diagram below, where $\mathbf{p} = (0, 2, 4)$ and $\mathbf{a} = (-1, 0, 1)$. The direction of the line is $\mathbf{b} = (3, 2, 1)$.



A vector joining the line to the point is $(0, 2, 4) - (-1, 0, 1) = (1, 2, 3)$. The projection of the vector onto the line is therefore:

$$(1, 2, 3) \cdot \frac{(3, 2, 1)}{|(3, 2, 1)|} = \frac{10}{\sqrt{14}},$$

from part (a). Therefore, the closest point on the line to the point is:

$$(-1, 0, 1) + \frac{10}{\sqrt{14}} \frac{(3, 2, 1)}{\sqrt{14}} = (-1, 0, 1) + (30/14, 20/14, 10/14) = (16/14, 20/14, 24/14) = \frac{1}{7}(8, 10, 12).$$

The shortest distance between the point and the line is therefore:

$$\left|(0, 2, 4) - \frac{1}{7}(8, 10, 12)\right| = \frac{1}{7}\,|(-8, 4, 16)| = \frac{4}{7}\,|(-2, 1, 4)| = \frac{4}{7}\sqrt{4 + 1 + 16} = \frac{4\sqrt{21}}{7} = 4\sqrt{\frac{3}{7}}.$$

(c) Now, we do the general argument. Following the diagram above, the vector joining the point to the line is $\mathbf{p} - \mathbf{a}$. The projection of the point onto the direction of the line is $\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})$. Hence the closest point on the line to the point is:

$$\mathbf{a} + (\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a}))\hat{\mathbf{b}}.$$

The shortest distance between the point and the line is therefore:

$$\left|\mathbf{p} - \mathbf{a} - \hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\hat{\mathbf{b}}\right|.$$

Alternatively, the shortest distance can be computed by Pythagoras:

$$\sqrt{|\mathbf{p} - \mathbf{a}|^2 - (|\mathbf{p} - \mathbf{a}|\cos(\theta))^2} = \sqrt{|\mathbf{p} - \mathbf{a}|^2 - (\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a}))^2}.$$

These expressions are the same, because:

$$\begin{aligned}
\left|\mathbf{p} - \mathbf{a} - \hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\hat{\mathbf{b}}\right| &= \sqrt{\left(\mathbf{p} - \mathbf{a} - \hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\hat{\mathbf{b}}\right) \cdot \left(\mathbf{p} - \mathbf{a} - \hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\hat{\mathbf{b}}\right)} \\
&= \sqrt{|\mathbf{p} - \mathbf{a}|^2 - 2\left(\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\right)^2 + \left(\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})|\right)^2} \\
&= \sqrt{|\mathbf{p} - \mathbf{a}|^2 - \left(\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\right)^2}.
\end{aligned}$$

It is not very useful to remember either version of this formula - but it *is* useful to know how to derive it. Knowing the method here is much more important than remembering a formula and being able to substitute things into it!

---

10. Show that if four points A, B, C, D are such that $AD \perp BC$ and $BD \perp AC$, then $CD \perp AB$.

Solution: Let the position vectors of the points be $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. We are given that $AD \perp BC$ and $BD \perp AC$, which in terms of vectors become:

$$(\mathbf{d} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{b}) = 0, \qquad (\mathbf{d} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{a}) = 0.$$

We want to show that:

$$(\mathbf{d} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) = 0.$$

Expanding all of these conditions using the properties of the scalar product, we have:

$$\mathbf{d} \cdot \mathbf{c} - \mathbf{d} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{b} = 0 \ (\dagger 1), \qquad \mathbf{d} \cdot \mathbf{c} - \mathbf{d} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{a} = 0 \ (\dagger 2),$$

and we want to show that:

$$\mathbf{d} \cdot \mathbf{b} - \mathbf{d} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{a} = 0. \tag{$*$}$$

But notice that equation ($*$) is just equation ($\dagger 2$) subtract equation ($\dagger 1$). So we are done!

11. Using the scalar product, prove that for any tetrahedron, the sum of the squares of the lengths of the edges equals four times the sum of the squares of the lengths of the lines joining the mid-points of opposite edges.

➾ **Solution:** Let the vertices of the tetrahedron have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$. Then the sum of the squares of the lengths of the edges of the tetrahedron is:

$$(\mathbf{b}-\mathbf{a})^2+(\mathbf{c}-\mathbf{a})^2+(\mathbf{d}-\mathbf{a})^2+(\mathbf{c}-\mathbf{b})^2+(\mathbf{d}-\mathbf{b})^2+(\mathbf{d}-\mathbf{c})^2. \tag{†}$$

The mid-points of the edges are:

$$\frac{\mathbf{a}+\mathbf{b}}{2},\qquad \frac{\mathbf{a}+\mathbf{c}}{2},\qquad \frac{\mathbf{a}+\mathbf{d}}{2},\qquad \frac{\mathbf{b}+\mathbf{c}}{2},\qquad \frac{\mathbf{b}+\mathbf{d}}{2},\qquad \frac{\mathbf{c}+\mathbf{d}}{2}.$$

The sum of the squares of the lengths joining the opposite mid-points is therefore:

$$\left(\frac{\mathbf{c}+\mathbf{d}}{2}-\frac{\mathbf{a}+\mathbf{b}}{2}\right)^2+\left(\frac{\mathbf{b}+\mathbf{d}}{2}-\frac{\mathbf{a}+\mathbf{c}}{2}\right)^2+\left(\frac{\mathbf{a}+\mathbf{d}}{2}-\frac{\mathbf{b}+\mathbf{c}}{2}\right)^2.$$

We need to make this look like (†). We can do this by organising the terms as follows:

$$\left(\frac{\mathbf{c}+\mathbf{d}}{2}-\frac{\mathbf{a}+\mathbf{b}}{2}\right)^2=\frac{1}{4}\left((\mathbf{c}-\mathbf{a})+(\mathbf{d}-\mathbf{b})\right)^2=\frac{1}{4}\left((\mathbf{c}-\mathbf{a})^2+2(\mathbf{c}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{b})+(\mathbf{d}-\mathbf{b})^2\right),$$

$$\left(\frac{\mathbf{b}+\mathbf{d}}{2}-\frac{\mathbf{a}+\mathbf{c}}{2}\right)^2=\frac{1}{4}\left((\mathbf{b}-\mathbf{c})+(\mathbf{d}-\mathbf{a})\right)^2=\frac{1}{4}\left((\mathbf{b}-\mathbf{c})^2+2(\mathbf{b}-\mathbf{c})\cdot(\mathbf{d}-\mathbf{a})+(\mathbf{d}-\mathbf{a})^2\right),$$

$$\left(\frac{\mathbf{a}+\mathbf{d}}{2}-\frac{\mathbf{b}+\mathbf{c}}{2}\right)^2=\frac{1}{4}\left((\mathbf{a}-\mathbf{b})+(\mathbf{d}-\mathbf{c})\right)^2=\frac{1}{4}\left((\mathbf{a}-\mathbf{b})^2+2(\mathbf{a}-\mathbf{b})\cdot(\mathbf{d}-\mathbf{c})+(\mathbf{d}-\mathbf{c})^2\right).$$

We see that the sum of the right hand side contains all the terms we need, apart from the cross-terms; we just need to show that these all cancel. We have:

$$\begin{aligned}
&2(\mathbf{c}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{b})+2(\mathbf{b}-\mathbf{c})\cdot(\mathbf{d}-\mathbf{a})+2(\mathbf{a}-\mathbf{b})\cdot(\mathbf{d}-\mathbf{c})\\
&\qquad=2\left(\mathbf{c}\cdot\mathbf{d}-\mathbf{c}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{d}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{d}-\mathbf{b}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{d}+\mathbf{c}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{d}-\mathbf{a}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{d}+\mathbf{b}\cdot\mathbf{c}\right)\\
&\qquad=0,
\end{aligned}$$

and hence the result follows.

12.(a) Using the geometric definition of the scalar product, prove the *Cauchy-Schwarz inequality* $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

(b) From the Cauchy-Schwarz inequality, deduce the *triangle inequality* $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$. What is the geometrical significance of this inequality? Learn this equality off by heart; it will be useful later when we study limits!

(c) From the triangle inequality, deduce the *reverse triangle inequality* $||\mathbf{a}| - |\mathbf{b}|| \leq |\mathbf{a} - \mathbf{b}|$.

➻ **Solution:** (a) Since $|\cos(\theta)| \leq 1$, we have:

$$|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos(\theta)| \leq |\mathbf{a}||\mathbf{b}|,$$

which proves the Cauchy-Schwarz inequality, as required.

(b) We just proved an inequality to do with the scalar product, so to prove the triangle inequality, we should try to relate things to a scalar product. To do so, consider the square of $|\mathbf{a} + \mathbf{b}|$. We have:

$$\begin{aligned}
|\mathbf{a} + \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) && (\text{since } |\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}) \\
&= \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\
&= |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 \\
&\leq |\mathbf{a}|^2 + 2|\mathbf{a} \cdot \mathbf{b}| + |\mathbf{b}|^2 && (\text{since for any } x, \text{ we have } x \leq |x|) \\
&\leq |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 && (\text{Cauchy-Schwarz}) \\
&= (|\mathbf{a}| + |\mathbf{b}|)^2.
\end{aligned}$$

To finish, we take the square root of both sides, which gives:

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|,$$

which is the triangle inequality as required.

If we consider the triangle formed by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{a} - \mathbf{b}$, then the triangle inequality tells us that the lengths of these vectors satisfy $|\mathbf{a} - \mathbf{b}| \leq |\mathbf{a}| + |-\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$. In particular, it tells us that the third side of a triangle cannot be longer than the sum of the lengths of the other two sides.

(c) Using the triangle inequality, we have:

$$|\mathbf{a}| = |\mathbf{a} - \mathbf{b} + \mathbf{b}| \leq |\mathbf{a} - \mathbf{b}| + |\mathbf{b}| \qquad \Rightarrow \qquad |\mathbf{a}| - |\mathbf{b}| \leq |\mathbf{a} - \mathbf{b}|.$$

Additionally, we have:

$$|\mathbf{b}| = |\mathbf{b} - \mathbf{a} + \mathbf{a}| \leq |\mathbf{a} - \mathbf{b}| + |\mathbf{a}| \qquad \Rightarrow \qquad |\mathbf{b}| - |\mathbf{a}| \leq |\mathbf{a} - \mathbf{b}|.$$

Putting these together, we see that:

$$||\mathbf{a}| - |\mathbf{b}|| \leq |\mathbf{a} - \mathbf{b}|,$$

as required. This inequality tells us that the length of the third side of a triangle is always at least the difference of the lengths of the other two sides.

The equation of a plane

13. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ be fixed 3-vectors, with $\mathbf{b} \neq \mathbf{0}$.

(a) Explain why the equation $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$ represents a plane, and summarise its properties. Show using properties of the scalar product that an equivalent representation of this plane is $\mathbf{r} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$. What is the geometric significance of the quantity $|\mathbf{a} \cdot \mathbf{b}|/|\mathbf{b}|$ here?

(b) By writing $\mathbf{r} = (x, y, z)$, $\mathbf{b} = (l, m, n)$, and $\mathbf{a} \cdot \mathbf{b} = d$, show that the equation of a plane may equivalently be written in the Cartesian form $lx + my + nz = d$.

(c) Find the equation of the plane containing the point $(3, 2, 1)$ with normal $(1, 2, 3)$ in both the vector form, as in (a), and the Cartesian form, as in (b). What is the shortest distance from the origin to the plane?

Solution: (a) The equation $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$ states that the vector from the point **a** to the point **r** is orthogonal to the vector **b**. Thus this equation represents a plane which is normal to the vector **b**, and passes through the point **a**. Using distributivity of the scalar product, we have $0 = (\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = \mathbf{r} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b}$. Rearranging, we obtain $\mathbf{r} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ as required.

Let θ be the angle between **r** and **b**. Then the equation of the plane may be written as:

$$|\mathbf{r}||\mathbf{b}| \cos(\theta) = \mathbf{a} \cdot \mathbf{b}.$$

Since $\mathbf{a} \cdot \mathbf{b}$ is fixed, $|\mathbf{r}|$ is minimised when the magnitude of $\cos(\theta)$ is maximised (the sign of $\cos(\theta)$ must match the sign of $\mathbf{a} \cdot \mathbf{b}$). Since the maximum value of the magnitude of $\cos(\theta)$ is ± 1, this shows that *the shortest distance from the origin to the plane* is given by:

$$|\mathbf{r}| = \frac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{b}|}.$$

(b) Writing $d = \mathbf{a} \cdot \mathbf{b}$, $\mathbf{r} = (x, y, z)$ and $\mathbf{b} = (l, m, n)$, the equation $\mathbf{r} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ becomes $(x, y, z) \cdot (l, m, n) = d$, which multiplies out to $lx + my + nz = d$. This is the Cartesian form of a plane.

(c) The equation of the plane in vector form is:

$$\left(\mathbf{r} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0.$$

Letting $\mathbf{r} = (x, y, z)$, this multiplies out to:

$$x + 2y + 3z - 3 - 4 - 3 = 0 \qquad \Rightarrow \qquad x + 2y + 3z = 10,$$

which is the Cartesian equation of the plane. By part (a), the shortest distance from the origin to the plane is given by:

$$\frac{10}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{10}{\sqrt{14}}.$$

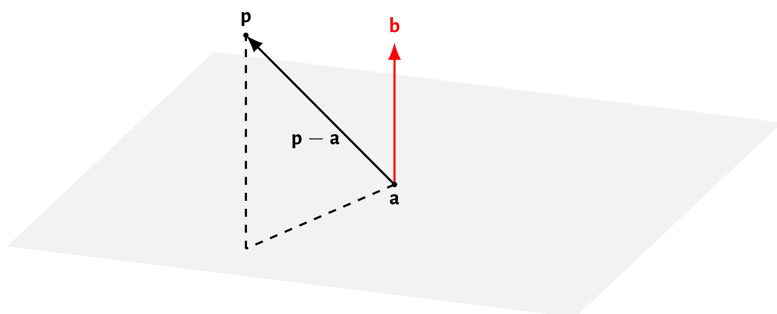
14. Consider the plane with vector equation $(\mathbf{r} - (1, 0, 1)) \cdot (2, -1, 0) = 0$.

(a) Using the scalar product, compute the projection of the vector $(2, 0, 3)$ in the direction $(2, -1, 0)$.

(b) Using the result of part (a), determine the point on the plane which is closest to the point $(3, 0, 4)$, and the shortest distance from the plane to the point $(3, 0, 4)$.

(c) Now, generalise your result: find a formula for the point on the plane $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$ which is the closest to the point with position vector $\mathbf{p}$, and a formula for the shortest distance from the plane to this point.

➾ **Solution:** When doing this question, we should have in mind the following diagram:



(a) For the first part, we have:

$$(2, 0, 3) \cdot \frac{(2, -1, 0)}{\sqrt{2^2 + 1^2}} = \frac{4}{\sqrt{5}}.$$

(b) Next, we note that a vector joining the point to the plane is $(3, 0, 4) - (1, 0, 1) = (2, 0, 3)$. The length of the projection of this vector in the direction of the normal $(2, -1, 0)$ will be the shortest distance to the plane, which by part (a) is $4/\sqrt{5}$.

To get the closest point, we subtract the component of $(2, 0, 3)$ in the direction normal to the plane, giving us:

$$(2, 0, 3) - \frac{4}{\sqrt{5}} \frac{(2, -1, 0)}{\sqrt{5}} = (2, 0, 3) - (8/5, -4/5, 0) = (2/5, 4/5, 3).$$

This vector is the planar component of the vector joining the plane and the point. Adding this vector to $(1, 0, 1)$, we must get the closest point on the plane to the point:

$$(1, 0, 1) + (2/5, 4/5, 3) = (7/5, 4/5, 3).$$

(c) Now, we do the same argument with abstract vectors. A vector joining the plane to the point is $\mathbf{p} - \mathbf{a}$. The length of the projection of this vector in a direction normal to the plane will be the shortest distance to the plane, which is given by:

$$\left| \hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a}) \right|$$

The modulus is required, because the projection is a *signed* projection. To finish, the component of the vector $\mathbf{p} - \mathbf{a}$ parallel to the plane is then:

$$\mathbf{p} - \mathbf{a} + (\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a}))\hat{\mathbf{b}}.$$

Adding this to the point $\mathbf{a}$, we get the closest point on the plane to the point:

$$\mathbf{p} + \left(\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a}) \right) \hat{\mathbf{b}}.$$

15. Using the results of Question 14, calculate the shortest distances between the plane $5x+2y-7z+9=0$ and the points $(1,-1,3)$ and $(3,2,3)$. Are the points on the same side of the plane?

➾ Solution: By inspection, a point on the plane is $(-1,-2,0)$, and the normal to the plane is $(5,2,-7)$. Therefore vectors joining the points to the plane are $(1,-1,3)-(-1,-2,0)=(2,1,3)$ and $(3,2,3)-(-1,-2,0)=(4,4,3)$. The shortest distances are then given by:

$$\left|(2,1,3)\cdot\frac{(5,2,-7)}{\sqrt{5^2+2^2+7^2}}\right| = \left|-\frac{9}{\sqrt{78}}\right| = \frac{9}{\sqrt{78}},$$

and:

$$\left|(4,4,3)\cdot\frac{(5,2,-7)}{\sqrt{78}}\right| = \frac{7}{\sqrt{78}}.$$

Importantly, we see that the modulus was relevant in the first case but not in the second. This means that the angle between the normal and the vector joining a point in the plane to the point off the plane is obtuse in the first case, and acute in the second case (think about the signs of the scalar product). Hence, they must be on opposite sides of the plane.

Equations of other 3D surfaces

16. Let k, m be positive constants, with $m<1$. Describe the following surfaces: (a) $|\mathbf{r}|=k$; (b) $\mathbf{r}\cdot\mathbf{u}=m|\mathbf{r}|$.

➾ Solution:

(a) This surface, $|\mathbf{r}|=k$, comprises the set of all vectors whose distance from the origin is equal to k. Hence this is a sphere centred on the origin of radius k.

(b) Write $\mathbf{r}\cdot\mathbf{u}=|\mathbf{r}|\cos(\theta)$ (using the fact that $\mathbf{u}$ is a unit vector). Then the equation can be rewritten as:

$$\cos(\theta)=m.$$

Hence, this surface consists of all vectors which are a constant angle $\arccos(m)$ to the vector $\mathbf{u}$. Thus the surface is a *cone*, with axis along $\mathbf{u}$. The tip of the cone is at the origin, since $\mathbf{r}=\mathbf{0}$ satisfies the equation.

---

17. Describe the surface given by the vector equation:

$$|\mathbf{r}-(\mathbf{r}\cdot\mathbf{u})\mathbf{u}|=2,$$

where $\mathbf{u}=\frac{1}{\sqrt{2}}(1,0,1)$. What is the intersection of this surface and the surface $x+z=0$?

➾ Solution: Note that $\mathbf{u}$ is a unit vector, so $\mathbf{r}\cdot\mathbf{u}$ is the length of the projection of $\mathbf{u}$ in the direction of $\mathbf{u}$. Hence the vector $\mathbf{r}-(\mathbf{r}\cdot\mathbf{u})\mathbf{u}$ represents the vector $\mathbf{r}$ with its $\mathbf{u}$ component entirely 'removed'. That is, $\mathbf{r}-(\mathbf{r}\cdot\mathbf{u})\mathbf{u}$ is the *projection of the vector* $\mathbf{r}$ *orthogonal to the* $\mathbf{u}$ *direction*.

The equation of the surface tells us that the component of $\mathbf{r}$ orthogonal to $\mathbf{u}$ is constant, and equal to 2. Thus this equation describes a *cylinder of radius* 2, with axis along the vector $\mathbf{u}$.

Since the direction of $\mathbf{u}$ in this question is $(1,0,1)$, this is the same as the direction orthogonal to the plane $x+z=0$. Hence, the intersection of $x+z=0$ and the figure $|\mathbf{r}-(\mathbf{r}\cdot\mathbf{u})\mathbf{u}|=2$ must be a circle of radius 2, centred on the origin, in the plane orthogonal to the vector $(1,0,1)$.

18.(a) Write down a vector equation for the sphere with centre at the point with position vector **a**, and radius $p > 0$.

(b) If there is a second sphere with centre at the point with position vector **b**, and radius $q > 0$, what conditions are required on **a**, **b**, p and q for the two spheres to intersect in a circle?

(c) Show that, if the two spheres do intersect, then the plane in which their intersection occurs is given by the equation $2\mathbf{r} \cdot (\mathbf{b} - \mathbf{a}) = p^2 - q^2 + |\mathbf{b}|^2 - |\mathbf{a}|^2$.

Solution: (a) The vector equation of the sphere is $|\mathbf{r} - \mathbf{a}| = p$, since the distance between any point on the sphere **r** and the sphere's centre **a** must always be equal to the radius p.

(b) If there is a second sphere $|\mathbf{r} - \mathbf{b}| = q$, we need the distance between the centres of the spheres to be less than the sum of the radii of the spheres. That is, we need:

$$|\mathbf{b} - \mathbf{a}| < p + q.$$

We need strict inequality here, because if $|\mathbf{b} - \mathbf{a}| = p + q$, then the spheres just 'touch' at a single point.

(c) To get the plane of intersection, consider squaring the two equations of the spheres and using properties of the scalar product:

$$p^2 = |\mathbf{r} - \mathbf{a}|^2 = (\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{a}) = |\mathbf{r}|^2 - 2\mathbf{r} \cdot \mathbf{a} + |\mathbf{a}|^2$$

$$q^2 = |\mathbf{r} - \mathbf{b}|^2 = (\mathbf{r} - \mathbf{b}) \cdot (\mathbf{r} - \mathbf{b}) = |\mathbf{r}|^2 - 2\mathbf{r} \cdot \mathbf{b} + |\mathbf{b}|^2.$$

Subtracting the second equation from the first, we have:

$$2\mathbf{r} \cdot (\mathbf{b} - \mathbf{a}) + |\mathbf{a}|^2 - |\mathbf{b}|^2 = p^2 - q^2.$$

Rearranging, we have:

$$2\mathbf{r} \cdot (\mathbf{b} - \mathbf{a}) = p^2 - q^2 + |\mathbf{b}|^2 - |\mathbf{a}|^2,$$

as required.

The vector product

19. Let $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{R}^3$ be 3-vectors.

(a) Give the geometrical definition of the *vector product* (or *cross product*) $\mathbf{v} \times \mathbf{w}$ in terms of lengths, angles and an appropriate perpendicular vector.

(b) Using *only* this definition, prove each of the following properties of the vector product:

(i) *anti-commutativity*: $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$;

(ii) *homogeneity*: $(\lambda \mathbf{v}) \times \mathbf{w} = \lambda(\mathbf{v} \times \mathbf{w})$;

(iii) *left-distributivity*: $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$.

[Hint: In (ii), consider the cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ separately. In (iii), start by explaining why $\mathbf{v} \times \hat{\mathbf{u}}$ is the projection of $\mathbf{v}$ onto the plane through the origin perpendicular to $\mathbf{u}$, followed by a rotation by $\frac{1}{2}\pi$.]

Now let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the standard basis vectors for $\mathbb{R}^3$, and let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ be 3-vectors.

(c) Using *only* the definition of the vector product in terms of lengths, angles, and a perpendicular vector, show that $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ and $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$.

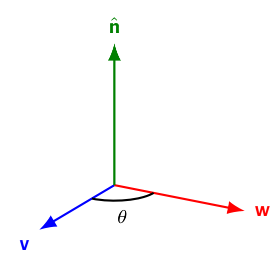
(d) By writing $\mathbf{v}, \mathbf{w}$ as linear combinations of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and applying the properties of the vector product from the previous question, prove the standard formula:

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2, \quad v_3 w_1 - v_1 w_3, \quad v_1 w_2 - v_2 w_1).$$

Solution: (a) We define the *vector product* of $\mathbf{v}, \mathbf{w}$ to be the vector:

$$\mathbf{v} \times \mathbf{w} := |\mathbf{v}||\mathbf{w}| \sin(\theta) \hat{\mathbf{n}},$$

where θ is the angle between the vectors $\mathbf{v}, \mathbf{w}$ and $\hat{\mathbf{n}}$ is a unit vector chosen to be orthogonal to the plane of $\mathbf{v}, \mathbf{w}$, in a direction such that the vectors $\mathbf{v}, \mathbf{w}, \hat{\mathbf{n}}$ taken in that order have a *right-handed orientation*.

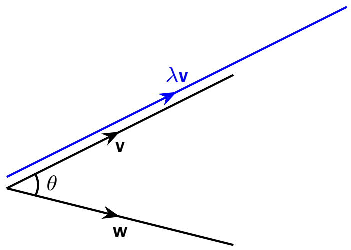


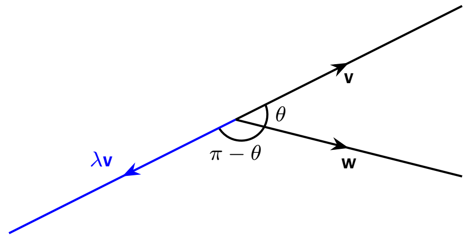
The above figure illustrates the right-handedness of the system. If you put your right index finger in the direction of $\mathbf{v}$, and your right middle finger in the direction of $\mathbf{w}$, then your thumb points in the direction of $\hat{\mathbf{n}}$, i.e. the direction of the vector product.

(b) We prove each of the properties in turn:

(i) ANTI-COMMUTATIVITY. This property follows from the right-hand rule. The vectors $\mathbf{v} \times \mathbf{w}$, $\mathbf{w} \times \mathbf{v}$ evidently have the same magnitude, because they have the same angle θ between them in both cases. However, in the first case our vector $\hat{\mathbf{n}}$ is pointing in the opposite direction to our vector in the second case, because of the right-hand rule. Hence $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$, as required.

(ii) HOMOGENEITY. We draw some similar diagrams to those that we did for the scalar product. Below, we are showing the plane containing the two vectors **v**, **w**.





In the case where $\lambda > 0$, we have a diagram like the one on the left of the page. The vector product $(\lambda \mathbf{v}) \times \mathbf{w}$ points into the page, by the right-hand rule; let $\hat{\mathbf{n}}$ be a unit vector pointing into the page throughout. Similarly, the vector product $\mathbf{v} \times \mathbf{w}$ points into the page by the right-hand rule. Both λ**v**, **w** and **v**, **w** have the angle θ between them. So we have:

$$(\lambda \mathbf{v}) \times \mathbf{w} = |\lambda \mathbf{v}||\mathbf{w}| \sin(\theta)\hat{\mathbf{n}} = \lambda |\mathbf{v}||\mathbf{w}| \sin(\theta)\hat{\mathbf{n}} = \lambda(\mathbf{v} \times \mathbf{w}),$$

as required.

In the case where $\lambda < 0$, we have a diagram like the one on the right of the page. The vector product $(\lambda \mathbf{v}) \times \mathbf{w}$ now points *out* of the page, by the right-hand rule. However, the vector product $\mathbf{v} \times \mathbf{w}$ points into the page by the right-hand rule. The vectors λ**v**, **w** have an angle $\pi - \theta$ between them, and the vectors **v**, **w** have an angle θ between them. So we have:

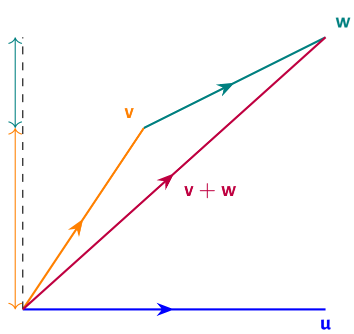
$$(\lambda \mathbf{v}) \times \mathbf{w} = |\lambda \mathbf{v}||\mathbf{w}| \sin(\pi - \theta)(-\hat{\mathbf{n}}) = -|\lambda||\mathbf{v}||\mathbf{w}| \sin(\theta)\hat{\mathbf{n}} = \lambda |\mathbf{v}||\mathbf{w}| \sin(\theta)\hat{\mathbf{n}} = \lambda(\mathbf{v} \times \mathbf{w}),$$

as required. Lastly, in the case $\lambda = 0$, we have that both sides are equal to the zero vector **0**. So we're done!

(iii) LEFT-DISTRIBUTIVITY. Here, we follow the hint. If θ is the angle between the vector **v** and the vector **u**, then note that $|\mathbf{v}| \sin(\theta)$ is the length of the projection of **v** orthogonal to **u**. Hence, $|\mathbf{v}||\hat{\mathbf{u}}| \sin(\theta) = |\mathbf{v} \times \hat{\mathbf{u}}|$ is the magnitude of the projection of **v** orthogonal to **u**. Importantly though, $\mathbf{v} \times \hat{\mathbf{u}}$ lies in the plane *perpendicular* to the plane containing both **v**, $\hat{\mathbf{u}}$. So $\mathbf{v} \times \hat{\mathbf{u}}$ is not quite the projection of the vector **v** orthogonal to **u**, but is instead this projection *rotated by* $\pi/2$.

As a result, to prove left-distributivity, it is sufficient by homogeneity to prove $(\mathbf{v} + \mathbf{w}) \times \hat{\mathbf{u}} = \mathbf{v} \times \hat{\mathbf{u}} + \mathbf{w} \times \hat{\mathbf{u}}$; in other words, the projection of $\mathbf{v} + \mathbf{w}$ onto a direction orthogonal to **u** is equal to the sum of the projections of **v**, **w** in a direction orthogonal to **u** (in both cases, a rotation by $\pi/2$ is carried out afterwards - so we can ignore it, because it is equivalent in both cases).

If all the vectors lie in the same plane, this is easily illustrated using a similar diagram to the scalar product one:



Similarly to the scalar product, this also holds if the vectors are not all in the same plane, but drawing a diagram is much more fiddly, so we won't do it. One way of thinking about it is imagining that the above diagram is a 3D diagram, just with **v**, **w** pointing at angles into and out of the page; the projections still look like those in the diagram.

(c) Since $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are all perpendicular to one another, we have:

$$|\mathbf{e}_1 \times \mathbf{e}_2| = |\mathbf{e}_1||\mathbf{e}_2|\sin(\pi/2) = 1,$$

and similarly for the other combinations. By the right-hand rule, the directions of the given vector products are determined, giving the results in the question.

It is also helpful to notice that $\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{0}$, etc, because these vectors have angle $\theta = 0$ between them.

(d) We break down the calculation into two parts. First, we have:

$$\begin{aligned}
(v_i\mathbf{e}_i) \times \mathbf{w} &= (v_i\mathbf{e}_i) \times (w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3) \\
&= v_i\mathbf{e}_i \times (w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3) && \text{(homogeneity)} \\
&= -v_i(w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3) \times \mathbf{e}_i && \text{(anti-commutativity)} \\
&= -v_i\left((w_1\mathbf{e}_1) \times \mathbf{e}_i + (w_2\mathbf{e}_2) \times \mathbf{e}_i + (w_3\mathbf{e}_3) \times \mathbf{e}_i\right) && \text{(left-distributivity, twice)} \\
&= -v_iw_1\mathbf{e}_1 \times \mathbf{e}_i - v_iw_2\mathbf{e}_2 \times \mathbf{e}_i - v_iw_3\mathbf{e}_3 \times \mathbf{e}_i && \text{(homogeneity)}
\end{aligned}$$

Now, depending on the value of i, we get:

$$(v_i\mathbf{e}_i) \times \mathbf{w} = \begin{cases} v_1w_2\mathbf{e}_3 - v_1w_3\mathbf{e}_2, & i = 1, \\ -v_2w_1\mathbf{e}_3 + v_2w_3\mathbf{e}_1, & i = 1, \\ v_3w_1\mathbf{e}_2 - v_3w_2\mathbf{e}_1, & i = 3. \end{cases}$$

To finish, we use left-distributivity twice again:

$$\begin{aligned}
\mathbf{v} \times \mathbf{w} &= (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \times \mathbf{w} \\
&= (v_1\mathbf{e}_1) \times \mathbf{w} + (v_2\mathbf{e}_2) \times \mathbf{w} + (v_3\mathbf{e}_3) \times \mathbf{w} \\
&= v_1w_2\mathbf{e}_3 - v_1w_3\mathbf{e}_2 - v_2w_1\mathbf{e}_3 + v_2w_3\mathbf{e}_1 + v_3w_1\mathbf{e}_2 - v_3w_2\mathbf{e}_1 \\
&= (v_2w_2 - v_3w_2)\mathbf{e}_1 + (v_3w_1 - v_1w_2)\mathbf{e}_2 + (v_1w_2 - v_2w_1)\mathbf{e}_3,
\end{aligned}$$

as required. Phew, that was hard work!

20. Find the angle between the position vectors of the points $(2,1,1)$ and $(3,-1,-5)$, and find the direction cosines of a vector perpendicular to both. Can both the angle and vector be computed using *only* the vector product?

➾ **Solution:** Computing the scalar product, we have $(2,1,1)\cdot(3,-1,-5) = 6-1-5 = 0$, thus the vectors are orthogonal, so are at an angle $\pi/2$. To find the direction cosines of a vector perpendicular to both, we compute their vector product:

$$\begin{pmatrix}2\\1\\1\end{pmatrix} \times \begin{pmatrix}3\\-1\\-5\end{pmatrix} = \begin{pmatrix}-4\\13\\-5\end{pmatrix}.$$

The *direction cosines* are the just the cosines of the angles that this vector makes with the x, y, z axes. To find these, we just normalise the vector, which has length $\sqrt{4^2+13^2+5^2} = \sqrt{210}$. This gives the direction cosines:

$$-\frac{4}{\sqrt{210}}, \qquad \frac{13}{\sqrt{210}}, \qquad -\frac{5}{\sqrt{210}}.$$

The formula for the vector product, $\mathbf{v}\times\mathbf{w} = |\mathbf{v}||\mathbf{w}|\sin(\theta)\hat{\mathbf{n}}$ makes it look like the angle can be computed using the vector product - just take the lengths of both sides to give $|\mathbf{v}\times\mathbf{w}| = |\mathbf{v}||\mathbf{w}|\sin(\theta)$, which rearranges to:

$$\sin(\theta) = \frac{|\mathbf{v}\times\mathbf{w}|}{|\mathbf{v}||\mathbf{w}|}.$$

In general, the problem is that this equation usually has two solutions in the range $[0,\pi]$, say θ and $\pi-\theta$. We can't tell if vectors are at an acute or an obtuse angle if we just use the vector product normally!

However, in this question since the vectors are inclined at $\pi/2$, there is precisely *one* solution to the equation. So we could have used the vector product in this case! But, it is a bad method in general.

---

21. Find all points $\mathbf{r}$ which satisfy $\mathbf{r}\times\mathbf{a} = \mathbf{b}$ where $\mathbf{a} = (1,1,0)$ and $\mathbf{b} = (1,-1,0)$.

➾ **Solution:** Let $\mathbf{r} = (x,y,z)$. Then taking the vector product, our equation becomes:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \times \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-z\\z\\x-y\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

Comparing components, we see that $z=-1$ and $x=y$. Thus the set of points that satisfy this equation is $(\lambda,\lambda,-1)$. This is a line going through the point $(0,0,-1)$ parallel to the vector $(1,1,0)$.

---

22. Using properties of the vector product, prove the identity $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}) = \mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$.

➾ **Solution:** By the left and right distributive properties, we have:

$$(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}) = (\mathbf{b}-\mathbf{a})\times\mathbf{c} - (\mathbf{b}-\mathbf{a})\times\mathbf{a} = \mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}+\mathbf{a}\times\mathbf{a}.$$

Now, $\mathbf{a}\times\mathbf{a} = \mathbf{0}$ since the angle between a vector and itself is zero. Finally, using antisymmetry of the vector product we have $\mathbf{a}\times\mathbf{c} = -\mathbf{c}\times\mathbf{a}$ and $\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}$. The result follows.