Part IA: Mathematics for Natural Sciences A
Examples Sheet 5: Differential calculus, Riemann sums, and basic integrals

Model Solutions

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Limit definition of the derivative

1. Let $y \equiv y(x)$ be a function of x. Define the *derivative* dy/dx of y as a limit. Using the limit definition:

   (a) show that differentiation is a linear operation;

   (b) find the derivative of $y(x) = x^n$, for $n = 0, 1, 2, 3, \ldots$.

Hence obtain the derivative of $ax + bx^2 \sin(\theta)$, where a, b, θ are real constants.

Solution: The derivative, as a function of x, is defined by:

$$\frac{dy}{dx}(x) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h},$$

when this limit exists. The derivative is a new function of x.

(a) Let $y_1(x), y_2(x)$ be two functions of x. Then for any two real numbers a, b, we have:

$$\begin{aligned}\frac{d}{dx}\left(ay_1 + by_2\right) &= \lim_{h\to 0} \frac{ay_1(x+h) + by_2(x+h) - ay_1(x) - by_2(x)}{h} \\ &= a\lim_{h\to 0}\frac{y_1(x+h) - y_1(x)}{h} + b\lim_{h\to 0}\frac{y_2(x+h) - y_2(x)}{h} \\ &= a\frac{dy_1}{dx} + b\frac{dy_2}{dx},\end{aligned}$$

using basic properties of limits. Hence differentiation is a linear operation.

(b) Let $y(x) = x^n$. Using the binomial expansion, $(x+h)^n = x^n + \binom{n}{1}hx^{n-1} + \binom{n}{2}h^2x^{n-2} + \cdots + h^n$, we have:

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{x^n + hnx^{n-1} + h^2g(x,h) - x^n}{h} = nx^{n-1} + \lim_{h\to 0}\left[hg(x,h)\right] = nx^{n-1},$$

where $g(x,h)$ is a function of x, h which is finite when $h \to 0$. Thus the derivative is given by nx^{n-1}.

Using linearity and the power rule, we have:

$$\frac{d}{dx}\left(ax + bx^2\sin(\theta)\right) = a\frac{dx}{dx} + b\sin(\theta)\frac{d(x^2)}{dx} = a + 2bx\sin(\theta).$$

2. (a) Using *only* the limit definition, show that for $a > 0$, the derivative of $y(x) = a^x$ is proportional to a^x.

(b) One definition of the number e is the value of a for which the proportionality constant in the previous part is 1. Using *only* this definition, show that the derivative of a^x is given by $\log(a)a^x$.

Solution: (a) We have:

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h},$$

using properties of limits. The remaining limit is independent of x, so is just a constant. Thus we have:

$$\frac{dy}{dx} = ka^x$$

for some proportionality constant k.

(b) We are given that e is the value of a for which the proportionality constant is 1. Therefore, we have been told that:

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

To compute the proportionality constant for general a, we then have:

$$\lim_{h\to 0} \frac{a^h - 1}{h} = \log(a) \lim_{h\to 0} \frac{e^{h\log(a)} - 1}{h\log(a)} = \log(a) \lim_{h'\to 0} \frac{e^{h'} - 1}{h'}.$$

where we have substituted $h' = h\log(a)$ in the limit in the final step. Using the definition of e, we see that the proportionality constant for general a is $\log(a)$, as required.

Rules of differentiation

3. State the chain rule, the product rule, the quotient rule, and the reciprocal rule, respectively. Make sure you know them off by heart!

Solution: The rules are:

- **The chain rule:** $\dfrac{d}{dx}(u(v)) = \dfrac{dv}{dx}\dfrac{du}{dv}$
- **The product rule:** $\dfrac{d}{dx}(uv) = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$
- **The quotient rule:** $\dfrac{\dfrac{du}{dx}v - u\dfrac{dv}{dx}}{v^2}$
- **The reciprocal rule:** $\dfrac{dy}{dx} = \left(\dfrac{dx}{dy}\right)^{-1}$

4. Using the rules you stated in the previous question, compute the derivatives of:

(a) $\log(x)$, (b) 3^{x^2}, (c) $\dfrac{e^x}{x^3-1}$, (d) $x^3\log(x^2-7)$, (e) $\sqrt{x^3 - e^x\log(x)}$.

Solution: We use each of the rules in turn:

(a) Let $y = \log(x)$. Then we have $x = e^y$, and so by the reciprocal rule we have:

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = (e^y)^{-1} = \frac{1}{e^y} = \frac{1}{x}.$$

(b) Rewriting this exponential to base e, we have $u = 3^{x^2} = e^{x^2\log(3)}$. Then by the chain rule (with $v = x^2\log(3)$), we have:

$$\frac{du}{dx} = 2x\log(3)e^{x^2\log(3)} = 2\log(3)x\cdot 3^{x^2}.$$

(c) Using the quotient rule, the derivative is given by:

$$\frac{e^x(3x^2)-(x^3-1)e^x}{(x^3-1)^2} = \frac{\left(1+3x^2-x^3\right)e^x}{(x^3-1)^2}.$$

(d) Using the product rule *and* the chain rule for the second factor, the derivative is given by:

$$3x^2\log(x^2-7) + \frac{x^3\cdot 2x}{x^2-7} = 3x^2\log(x^2-7) + \frac{2x^4}{x^2-7}.$$

(e) Using the chain rule and the product rule, we have:

$$\frac{d}{dx}\sqrt{x^3-e^x\log(x)} = \frac{1}{2}\frac{3x^2-e^x\log(x)-e^x/x}{\sqrt{x^3-e^x\log(x)}} = \frac{3x^3-e^x(x\log(x)+1)}{2x\sqrt{x^3-e^x\log(x)}}.$$

5. By writing each of the following trigonometric and hyperbolic functions in terms of exponentials, compute their derivatives: (a) $\cos(x)$; (b) $\sin(x)$; (c) $\cosh(x)$; (d) $\sinh(x)$; (e) $\tan(x)$; (f) $\tanh(x)$. Learn these derivatives off by heart.

➾ **Solution:** We tackle each of the trigonometric and hyperbolic functions in turn.

(a) Using the complex formula for cosine from Sheet 4, we have $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$. Differentiating, we have:

$$\frac{d}{dx}\cos(x) = \frac{1}{2}(ie^{ix} - ie^{-ix}) = -\frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = -\sin(x),$$

using the complex formula for sine from Sheet 3.

(b) Similarly, we have $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$, so differentiating we have:

$$\frac{d}{dx}\sin(x) = \frac{1}{2i}\left(ie^{ix} + ie^{-ix}\right) = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos(x).$$

(c) Recall that $\cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$, we can differentiate to give:

$$\frac{d}{dx}\cosh(x) = \frac{1}{2}(e^{x} - e^{-x}) = \sinh(x).$$

(d) Similarly, we have $\sinh(x) = \frac{1}{2}(e^{x} - e^{-x})$, so differentiating we have:

$$\frac{d}{dx}\sinh(x) = \frac{1}{2}(e^{x} + e^{-x}) = \cosh(x).$$

(e) For $\tan(x)$, we have:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\frac{1}{2i}(e^{ix} - e^{-ix})}{\frac{1}{2}(e^{ix} + e^{-ix})} = -i\left(\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}\right).$$

Differentiating using the quotient rule, we have:

$$\frac{d}{dx}\tan(x) = -i\left(\frac{i(e^{ix} + e^{-ix})^2 - i(e^{ix} - e^{-ix})^2}{(e^{ix} + e^{-ix})^2}\right) = \frac{(e^{ix} + e^{-ix})^2 - (e^{ix} - e^{-ix})^2}{(e^{ix} + e^{-ix})^2}$$

$$= \frac{(2\cos(x))^2 - (2i\sin(x))^2}{(2\cos(x))^2} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \sec^2(x).$$

(f) The calculation for $\tanh(x)$ is extremely similar. We have:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\frac{1}{2}(e^{x} - e^{-x})}{\frac{1}{2}(e^{x} + e^{-x})} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}.$$

Differentiating using the quotient rule, we have:

$$\frac{d}{dx}\tanh(x) = \frac{(e^{x} + e^{-x})^2 - (e^{x} - e^{-x})^2}{(e^{x} + e^{-x})^2} = \frac{(2\cosh(x))^2 - (2\sinh(x))^2}{(2\cosh(x))^2}$$

$$= \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \mathrm{sech}^2(x).$$

6. Using: (i) the logarithmic formulae for the inverse hyperbolic functions you derived on Sheet 4; (ii) the reciprocal rule, compute the derivatives of: (a) $\cosh^{-1}(x)$; (b) $\sinh^{-1}(x)$; (c) $\tanh^{-1}(x)$. Learn these derivatives off by heart.

Solution: We have the following derivatives:

(a) (i) We previously derived $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$. Hence using the chain rule, we have the derivative:

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1 + x(x^2-1)^{-1/2}}{x + (x^2-1)^{1/2}} = \frac{1}{\sqrt{x^2-1}} \cdot \frac{1 + x(x^2-1)^{-1/2}}{1 + x(x^2-1)^{-1/2}} = \frac{1}{\sqrt{x^2-1}}.$$

(ii) Let $y(x) = \cosh^{-1}(x)$. Then $\cosh(y) = x$, which gives:

$$\frac{dx}{dy} = \sinh(y).$$

Using the reciprocal rule, and hyperbolic trigonometric identities, we then have:

$$\frac{dy}{dx} = \frac{1}{\sinh(y)} = \frac{1}{\sqrt{\cosh^2(y) - 1}} = \frac{1}{\sqrt{x^2-1}}.$$

(b) (i) We previously derived $\sinh^{-1}(x) = \log(x + \sqrt{x^2 + 1})$. Hence using the chain rule, we have the derivative:

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1 + x(x^2+1)^{-1/2}}{x + (x^2+1)^{1/2}} = \frac{1}{\sqrt{x^2+1}} \cdot \frac{1 + x(x^2+1)^{-1/2}}{1 + x(x^2+1)^{-1/2}} = \frac{1}{\sqrt{x^2+1}}.$$

(ii) Let $y(x) = \sinh^{-1}(x)$. Then $\sinh(y) = x$, which gives:

$$\frac{dx}{dy} = \cosh(y).$$

Using the reciprocal rule, and hyperbolic trigonometric identities, we then have:

$$\frac{dy}{dx} = \frac{1}{\cosh(y)} = \frac{1}{\sqrt{\sinh^2(y) + 1}} = \frac{1}{\sqrt{x^2+1}}.$$

(c) (i) We previously derived:

$$\tanh^{-1}(x) = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|.$$

Hence using the chain rule, we have the derivative (assuming x is in the range where $(1+x)/(1-x)$ is positive):

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{2}\frac{d}{dx}\left(\log(1+x) - \log(1-x)\right) = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = \frac{1}{1-x^2}.$$

(ii) Let $y(x) = \tanh^{-1}(x)$. Then $\tanh(y) = x$, which gives:

$$\frac{dx}{dy} = \operatorname{sech}^2(y).$$

Using the reciprocal rule, and hyperbolic trigonometric identities, we then have:

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2(y)} = \frac{1}{\tanh^2(y) - 1} = \frac{1}{x^2-1}.$$

7. If $y \equiv y(x)$ is a function of x, show that $\dfrac{d^3x}{dy^3} = -\left(\dfrac{dy}{dx}\right)^{-4}\dfrac{d^3y}{dx^3} + 3\left(\dfrac{dy}{dx}\right)^{-5}\left(\dfrac{d^2y}{dx^2}\right)^2$. Verify this when $y = e^{2x}$.

Solution: This question is about expressing a higher derivative of x with respect to y in terms of derivatives of y with respect to x. We start with a simple derivative, then differentiate repeatedly using the chain rule and reciprocal rule.

First, note that:

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$$

by the reciprocal rule.

Taking another derivative with respect to y, we have:

$$\frac{d^2x}{dy^2} = \frac{d}{dy}\left[\left(\frac{dy}{dx}\right)^{-1}\right].$$

But the argument of the derivative on the right hand side is a function of x, not a function of y - how can we differentiate it? We must consider it to be a function of y *through the dependence of x on y*, $x \equiv x(y)$:

$$\left(\frac{dy}{dx}\right)^{-1} \equiv \left(\frac{dy}{dx}(x)\right)^{-1} \equiv \left(\frac{dy}{dx}(x(y))\right)^{-1}.$$

This implies that we should use the chain rule to differentiate. We have:

$$\begin{aligned}
\frac{d}{dy}\left[\left(\frac{dy}{dx}\right)^{-1}\right] &= \frac{dx}{dy}\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^{-1}\right] && \text{(chain rule)}\\
&= -\frac{dx}{dy}\left(\frac{dy}{dx}\right)^{-2}\frac{d^2y}{dx^2} && \text{(chain rule again)}\\
&= -\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2} && \text{(reciprocal rule)}
\end{aligned}$$

Now we have got the hang of things, the last derivative is easy. We repeat the same method:

$$\begin{aligned}
\frac{d^3x}{dy^3} &= \frac{d}{dy}\left[-\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2}\right]\\
&= \frac{dx}{dy}\frac{d}{dx}\left[-\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2}\right] && \text{(chain rule)}\\
&= 3\frac{dx}{dy}\left(\frac{dy}{dx}\right)^{-4}\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dx}{dy}\left(\frac{dy}{dx}\right)^{-3}\frac{d^3y}{dx^3} && \text{(product rule and chain rule)}\\
&= 3\left(\frac{dy}{dx}\right)^{-5}\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^{-4}\frac{d^3y}{dx^3} && \text{(reciprocal rule)}
\end{aligned}$$

To finish, we are asked to verify this result when $y = e^{2x}$. In this case, we have that x as a function of y is given by $x(y) \equiv \frac{1}{2}\log(y)$. Hence all the derivatives are:

$$\frac{dy}{dx} = 2e^{2x}, \qquad \frac{d^2y}{dx^2} = 4e^{2x}, \qquad \frac{d^3y}{dx^3} = 8e^{2x},$$

and:

$$\frac{dx}{dy} = \frac{1}{2y}, \qquad \frac{d^2x}{dy^2} = -\frac{1}{2y^2}, \qquad \frac{d^3x}{dy^3} = \frac{1}{y^3}.$$

Verifying the rule we derived above, we check:

$$3\left(\frac{dy}{dx}\right)^{-5}\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^{-4}\frac{d^3y}{dx^3} = \frac{3}{32e^{10x}} \cdot 16e^{4x} - \frac{1}{16e^{8x}}8e^{2x} = \left(\frac{3}{2} - \frac{1}{2}\right)\frac{1}{e^{6x}} = \frac{1}{y^3},$$

as required.

8. What is *implicit differentiation*, and why is it called implicit? Using: (a) implicit differentiation; (b) the reciprocal rule, find dy/dx given $y + e^y \sin(y) = 1/x$, and make sure that your answers agree.

➽ **Solution:** Sometimes, we define y as a function of x through an equation, such as:

$$y^3 + \sin(y)\cos(x) + x = 0. \tag{$*$}$$

It is impossible to solve this equation *explicitly* to give something of the form $y = y(x)$, but we can still solve the equation in principle (e.g. numerically). Hence we say that this equation defines y as a function of x *implicitly*. In general, an implicit definition of y as a function of x is an equation of the form:

$$f(x, y) = 0$$

where f is a function of two variables.

Implicit differentiation is the differentiation of y when it is implicitly defined (which explains why it is called implicit differentiation). For example, when $y \equiv y(x)$ is defined through the equation ($*$), we can differentiate to get:

$$3y^2\frac{dy}{dx} + \cos(y)\cos(x)\frac{dy}{dx} - \sin(y)\sin(x) + 1 = 0,$$

where we have used the chain rule and product rules to compute the derivatives of each of the terms. Implicit differentiation is just a fancy name for the chain rule applied to implicit equations really!

(a) Given the equation $y + e^y \sin(y) = 1/x$, we can differentiate using implicit differentiation to give:

$$\frac{dy}{dx} + e^y \sin(y)\frac{dy}{dx} + e^y\cos(y)\frac{dy}{dx} = -\frac{1}{x^2}.$$

Rearranging, we obtain:

$$\frac{dy}{dx} = -\frac{1}{x^2(1 + e^y\sin(y) + e^y\cos(y))}.$$

(b) On the other hand, we can take the reciprocal of both sides to give:

$$x = \frac{1}{y + e^y\sin(y)}. \tag{$\dagger$}$$

Taking the derivative with respect to y, we have:

$$\frac{dx}{dy} = -\frac{1 + e^y\sin(y) + e^y\cos(y)}{(y + e^y\sin(y))^2}.$$

Taking the reciprocal, we have:

$$\frac{dy}{dx} = -\frac{(y + e^y\sin(y))^2}{1 + e^y\sin(y) + e^y\cos(y)} = -\frac{1}{x^2(1 + e^y\sin(y) + e^y\cos(y))},$$

where in the last step we substituted using equation (†). This is in agreement with the implicit differentiation method we used in part (a).

Curve-sketching

9. State what it means for a function to be *even* and for a function to be *odd*, and explain the geometric significance of these definitions. Hence, decide whether the following functions are even, odd, both, or neither:

(a) x, (b) $\sin(x)$, (c) e^x, (d) $\sin(\frac{\pi}{2} - x)$, (e) $|x|\cos(x)$, (f) $\sqrt{x}$, (g) 2, (h) 0, (i) $\log\left|\dfrac{1+x}{1-x}\right|$.

➻ **Solution:** A function is *even* if it satisfies $f(x) = f(-x)$. This means that it is invariant under a reflection in the y-axis. A function is *odd* if it satisfies $f(x) = -f(-x)$. This means that it is invariant under a rotation by π around the origin.

For the given functions:

(a) For $f(x) = x$, we have $f(-x) = -x = -f(x)$, so this function is odd. Alternatively, we could spot that it is invariant under a rotation by π around the origin. The function is *not* even, because it is not invariant under a reflection in the y-axis.

(b) For $f(x) = \sin(x)$, we have $f(-x) = \sin(-x) = -\sin(x) = -f(x)$, so this function is odd. It is not even.

(c) For $f(x) = e^x$, the function is neither reflectionally invariant in the y-axis, nor rotationally invariant under a rotation by π around the origin. Hence this function is neither even nor odd.

(d) For $f(x) = \sin(\pi/2 - x)$, we notice that this function is a transformation of sine. It is a translation by $\pi/2$ in the negative x-direction $\sin(x) \mapsto \sin(x + \pi/2)$, then a reflection in the y-axis, $\sin(x + \pi/2) \mapsto \sin(\pi/2 - x)$. This just gives a cosine graph though, so $f(x) = \cos(x)$. This is evidently even, and not odd.

(e) For $f(x) = |x|\cos(x)$, we note that $f(-x) = |-x|\cos(-x) = |x|\cos(x) = f(x)$, so this function is even. It is not odd.

(f) For $f(x) = \sqrt{x}$, the function is not defined for $x < 0$, so this function is neither even nor odd.

(g) For $f(x) = 2$, we have $f(-x) = 2 = f(x)$, so the function is even. The function is not odd.

(h) For $f(x) = 0$, we have *both* $f(-x) = 0 = f(x)$ and $f(-x) = 0 = -f(x)$. Hence the function is *both* even *and* odd. The zero function is in fact the only example of a function which is both even and odd.

(i) For:

$$f(x) = \log\left|\frac{1+x}{1-x}\right|,$$

we could observe that $f(x) = 2\tanh^{-1}(x)$, which immediately shows the function is odd and not even (consider the graph of $\tanh(x)$, from Sheet 3!). Alternatively, we can calculate directly using laws of logarithms:

$$f(-x) = \log\left|\frac{1-x}{1+x}\right| = \log\left|\frac{1+x}{1-x}\right|^{-1} = -\log\left|\frac{1+x}{1-x}\right| = -f(x).$$

10. Write down a list of things you should consider when sketching the graph of a function. Compare with your supervision partner before the supervision, and exchange ideas!

➭ **Solution:** A possible list is the following:

- **y-intercepts.** Find the places where the function crosses the y-axis by setting $x = 0$.
- **x-intercepts.** Set $y = 0$, and solve the resulting equation to find the places where the function crosses the x-axis.
- **Vertical asymptotes.** These occur at points where the function is undefined. Consider the behaviour of the function just to the right of the point and just to the left of the point to work out whether it tends to positive or negative infinity at such a point.
- **Domain.** More generally than vertical asymptotes, you can think about whole regions (instead of just points) where the function is undefined. For example, the function $\log(x)$ does not exist for values of $x < 0$, so you can hatch out that whole region of the xy-plane, and not consider the graph there! Similarly for something like $\sqrt{1-x^2}$, where we require $|x| < 1$.
- **Horizontal and oblique asymptotes.** Consider the behaviour of the function in the limits $x \to \pm\infty$. If the function approaches a constant, then it has a *horizontal asymptote*. If the function is not constant in the limit, it might approach a linear function; consider performing polynomial division for a rational function to see whether this is the case - the function has an *oblique asymptote* in this case. For example:

  $$\frac{x^2}{1+x} = \frac{x(1+x)-(1+x)+1}{1+x} = x - 1 + \frac{1}{1+x}$$

  approaches $x - 1$ as $x \to \infty$, so has an oblique asymptote $y = x - 1$.
- **Stationary points.** Differentiate the function to find dy/dx. Stationary points of the function occur when $dy/dx = 0$. Often, the nature of a stationary point can be deduce simply by thinking about continuity of the function - for example, if the function has a stationary value of 1 and approaches 0 in the limits $x \to \pm\infty$, and is otherwise continuous (i.e. no vertical asymptotes), then that stationary value must be a *maximum value*.

  If you are struggling to tell whether a point is a maximum or minimum just using a continuity argument, you can also check by differentiating again to find d^2y/dx^2. If $d^2y/dx^2 > 0$ at a stationary point, then the point is a minimum. If $d^2y/dx^2 < 0$ at a stationary point, then the point is a maximum. On the other hand, these are both *sufficient conditions*, but not *necessary conditions* - for example, the function $y = x^4$ has a minimum at $x = 0$, but $d^2y/dx^2 = 12x^2$ is equal to zero at $x = 0$ (i.e. not positive!).
- **Inflection points.** These are points where $d^2y/dx^2 = 0$. These tell us where the derivative of the function changes sign, telling us whether the rate of increase of the function is itself increasing or decreasing (and similarly for the rate of decrease of the function). Usually these give us little information.
- **Regions of positivity and negativity.** We can hatch out regions of the x, y-plane where the function cannot exist through considering positive and negativity. For example, for the graph of $xy = 1$, we must have that x, y have the same sign. Hence the graph cannot exist in the regions $x < 0, y > 0$ and $x > 0, y < 0$, so we can block them out before completing our sketch - this can be very useful for determining whether we have maxima or minima in a stationary point analysis!
- **Parity.** Whether a function is odd or even. Even functions are reflectionally symmetric in the y-axis, whilst odd functions are rotationally symmetric on a 180° rotation about the origin.
- **Periodicity.** Is the function periodic? The graphs of sine and cosine repeat every 2π, for example - does the function you are considering have a similar periodic symmetry?

- **Sums and products of functions.** Some functions can be split into sums or products of simpler functions, for which the graphs are easier. The hyperbolic functions are such an example; e.g. $\cosh(x) = (e^x + e^{-x})/2$ is the average of the two graphs e^x, e^{-x}. An easy way to sketch $\cosh(x)$ is therefore to sketch the graphs of e^x, e^{-x} as dashed lines, then try to draw a solid line representing the average of those two graphs.

  Similarly, the graph of something like $e^{-x}\cos(x)$ can be considered to be the graph of e^{-x} modulated by a factor of $\cos(x)$, implying oscillations between e^{-x} and $-e^{-x}$. This is an example given in Question 11.

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11. Sketch the graphs of the following functions, explaining your reasoning in each case:

(a) $(x-3)^3 + 2x$, (b) $\dfrac{x}{1+x^2}$, (c) $\dfrac{x^2+3}{x-1}$, (d) xe^x, (e) $\dfrac{\log(x)}{1+x}$, (f) $\dfrac{1}{1-e^x}$, (g) $e^x\cos(x)$.

Solution: Performing the analysis in each case:

(a) This is just a cubic. We expand the bracket first to get:

$$y = x^3 - 9x^2 + 29x - 27.$$

Hence we see that the y-intercept is -27. Taking the derivative, we have:

$$\frac{dy}{dx} = 3x^2 - 18x + 29,$$

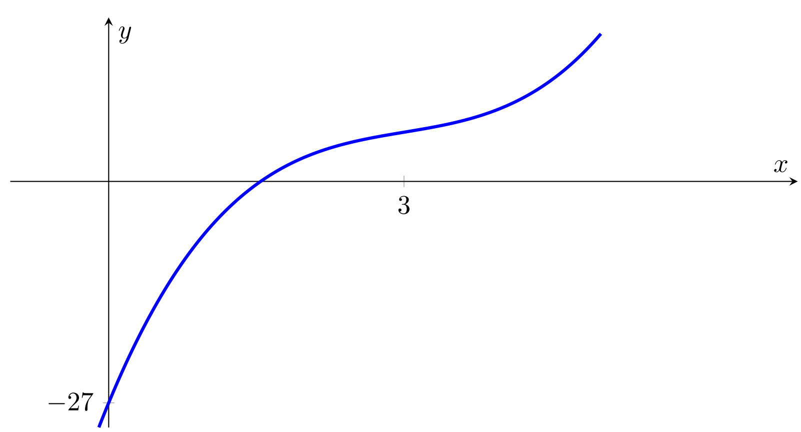
which has discriminant $18^2 - 4 \cdot 3 \cdot 29 = -24 < 0$. Hence, there are no stationary points. It is a positive cubic, so it simply increases from $-\infty$ to ∞.

If we want to be incredibly precise, we could compute the second derivative. This gives:

$$\frac{d^2y}{dx^2} = 6x - 18,$$

so there is a point of inflection at $x = 3$. This is where the cubic changes from reducing its rate of growth to increasing its rate of growth.

We now have enough information to draw the complete sketch:



(b) This function is defined for all values of x, since the denominator is always positive. It approaches zero as $x \to \infty$, coming from the positive direction since the numerator $x > 0$ in this region. It approaches zero as $x \to -\infty$ coming from the negative direction since the numerator $x < 0$ in this region. The y-intercept is at $x = 0$. The graph is also odd, since:

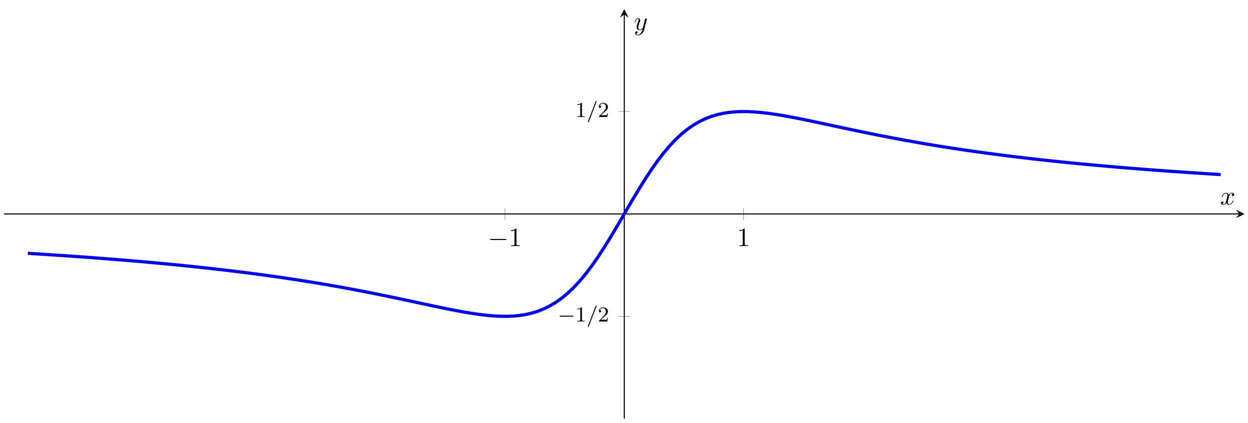
$$\frac{(-x)}{1+(-x)^2} = -\left(\frac{x}{1+x^2}\right).$$

Finally, we check for stationary points. Differentiating $y = x/(1+x^2)$, we have:

$$\frac{dy}{dx} = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}.$$

Hence there are stationary points at $x = \pm 1$. Thinking about the shape of the graph, and the information we have already obtained, the only possibility is that $x = 1$ is a maximum and $x = -1$ is a minimum.

We now have everything we need to sketch the graph:



(c) This graph has a vertical asymptote at $x = 1$. Just to the right of the asymptote, we have $x - 1 > 0$ and $x^2 + 3 > 0$, hence the graph is positive. So the graph approaches $+\infty$ from the right of $x = 1$. Just to the left of the asymptote, we have $x - 1 < 0$ and $x^2 + 3 > 0$, hence the graph is negative. So the graph approaches $-\infty$ from the left of $x = 1$.

Indeed, the sign of the graph is entirely determined by the sign of the denominator, so we see that the graph never crosses the x-axis. It is entirely positive for $x > 1$ and entirely negative for $x < 1$.

As x approaches infinity, the behaviour of the graph is roughly linear. To work out the exact behaviour, we perform polynomial long division:

$$\frac{x^2+3}{x-1} = \frac{(x-1)^2+2x+2}{x-1} = \frac{(x-1)^2+2(x-1)+4}{x-1} = x-1+2+\frac{4}{x-1} = x+1+\frac{4}{x-1}.$$

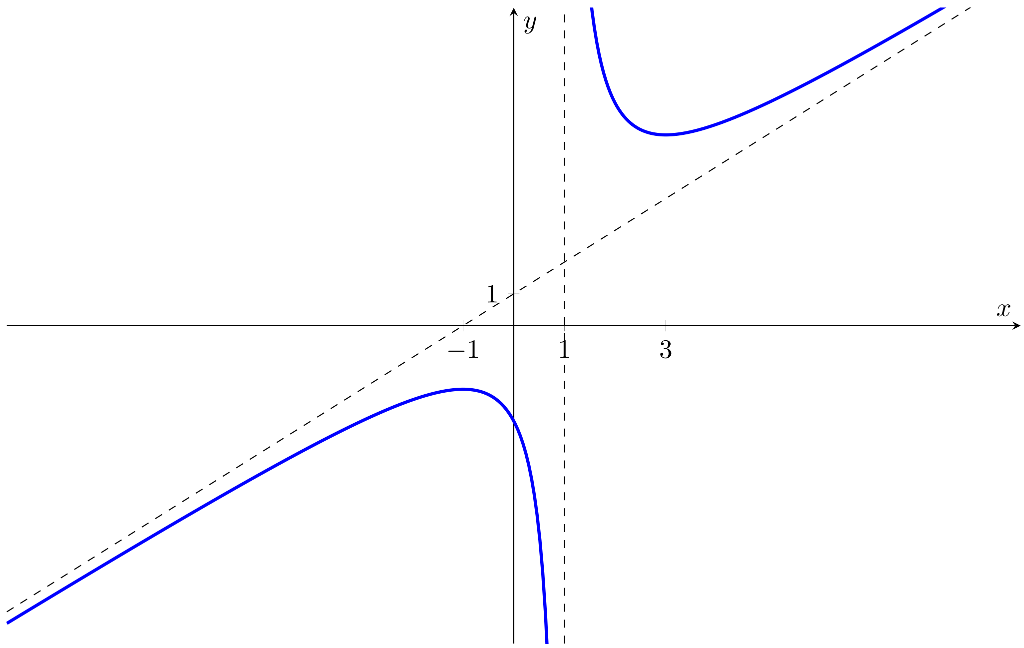
Therefore, as $x \to \infty$, the graph approaches the *oblique asymptote* $x + 1$. The same applies as $x \to -\infty$.

Finally, we check for stationary points. We have:

$$\frac{d}{dx}\left(\frac{x^2+3}{x-1}\right) = \frac{2x(x-1)-(x^2+3)}{(x-1)^2} = \frac{x^2-2x-3}{(x-1)^2} = \frac{(x-3)(x+1)}{(x-1)^2}.$$

Hence there are stationary points at $x = 3$ and $x = -1$. By the above work, the shape of the graph implies that $x = 3$ is necessarily a minimum and $x = -1$ is necessarily a maximum.

We now have everything we need to sketch the graph. We have:



(d) For this function, first note that the y-intercept is $0 \cdot e^0 = 0$, and the x-intercepts occur when $xe^x = 0$, i.e. only when $x = 0$. Thus the graph crosses the axes exactly once when $(x, y) = (0, 0)$.

Also, observe that $xe^x \to \infty$ as $x \to \infty$, since both $x, e^x \to \infty$ independently. Further as $x \to -\infty$, we have that $e^x \to 0$ exponentially fast, which beats the polynomial growth of x (this can be proved with L'Hôpital's rule, as we shall see later on the sheet). Hence the function tends to zero as $x \to -\infty$.

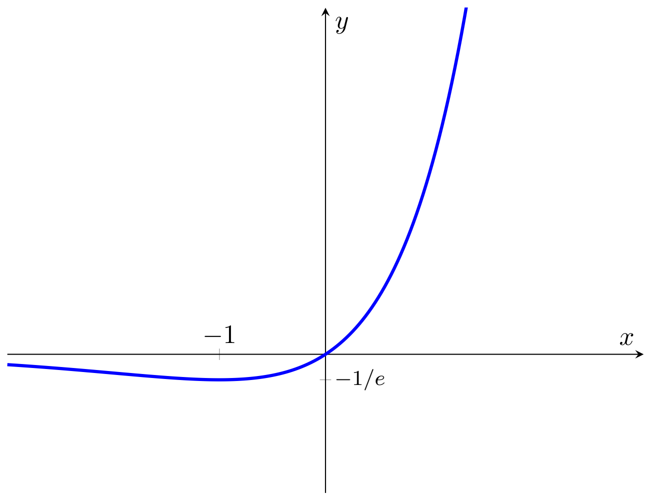
The signs of the functions are helpful here. We have $e^x > 0$, hence the sign of the graph xe^x is the same as the sign of x. This means the graph is entirely negative for $x < 0$ and entirely positive for $x > 0$.

Finally, we check for stationary points. We have:

$$\frac{d}{dx}(xe^x) = e^x + xe^x = (x+1)e^x,$$

which is zero if and only if $x = -1$. Thus there is a unique stationary point at $(-1, -1/e)$. Considering all the data we have found on the shape of the function thus far, it follows by continuity of the function that this stationary point must be a minimum.

We now have all the information we need to sketch the graph:



(e) This function only exists for $x > 0$, because of the factor of $\log(x)$. The function would have a singularity at $x = -1$, but this is not in the range of $\log(x)$, so we don't need to consider that.

Next, we note that as $x \to \infty$, we have $\log(x)/(1+x) \to 0$ since logarithmic growth is slower than polynomial growth (again, this can be proved with L'Hôpital's rule, which we shall see later on in the sheet).

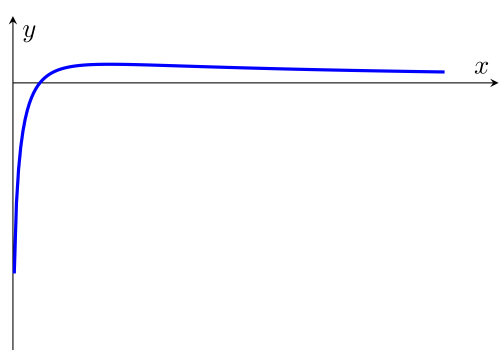
We also observe that $x+1$ is positive everywhere in the range $x > 0$, and $\log(x)$ is negative for $0 < x < 1$ and positive for $1 < x$. This implies that the function itself is negative in the region $0 < x < 1$ and positive for $1 < x$. There is therefore exactly one x-intercept, at $x = 1$.

Finally, we check for stationary points. At stationary points, we have:

$$\frac{d}{dx}\left(\frac{\log(x)}{1+x}\right) = \frac{(1+x)/x - \log(x)}{(1+x)^2} = 0 \qquad \Leftrightarrow \qquad \frac{1}{x} + 1 = \log(x).$$

There is exactly one intersection of the graphs of $\log(x)$ and $1/x+1$, so there is exactly one solution of this equation. This implies there is a single stationary point (although we don't know its coordinates in this case!). The stationary point must be a maximum, by continuity of the function.

We now have enough information to sketch the complete graph:



(f) This graph is singular when $e^x = 1$, which occurs if and only if $x = 0$. Therefore, there is a vertical asymptote at $x = 0$.

For $x > 0$, we have $e^x > 1$, hence $1 - e^x < 0$, and the graph is negative. For $x < 0$, we have $e^x < 1$, hence $1 - e^x > 0$, and the graph is positive.

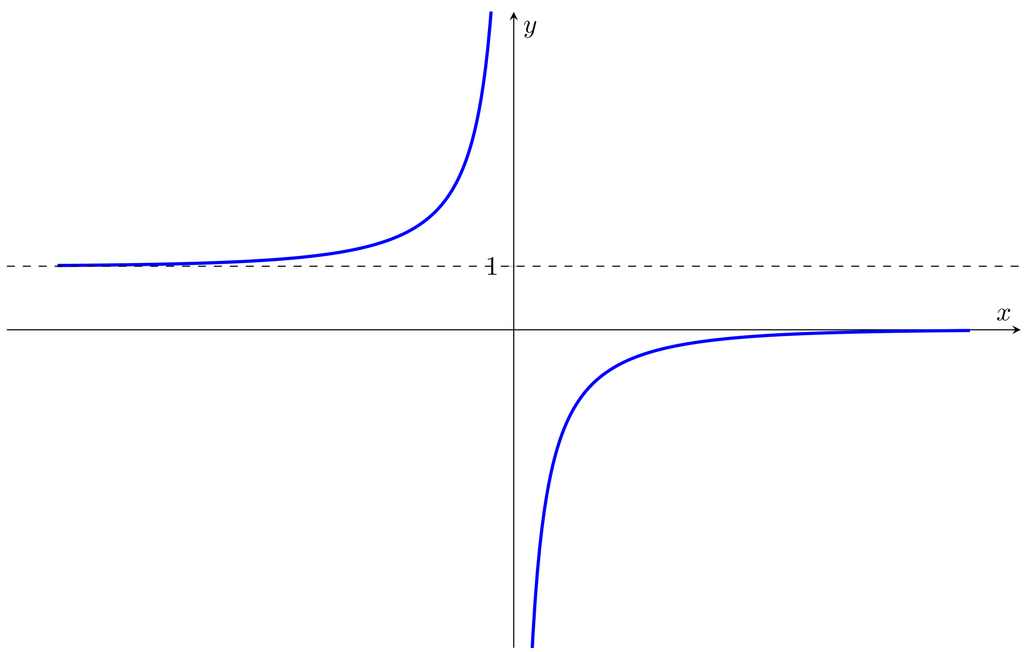
As $x \to \infty$, we have $e^x \to \infty$, so $1/(1 - e^x) \to 0$, which gives a horizontal asymptote at $y = 0$ as x approaches positive infinity. On the other hand, as $x \to -\infty$, we have $e^x \to 0$, so $1/(1 - e^x) \to 1$, which gives a horizontal asymptote at $y = 1$ as x approaches negative infinity.

Finally, we check for stationary points by differentiating. We have:

$$\frac{d}{dx}\left(\frac{1}{1 - e^x}\right) = \frac{e^x}{(1 - e^x)^2}.$$

This is never zero, so there are no stationary points.

We now have enough information to sketch the complete graph:



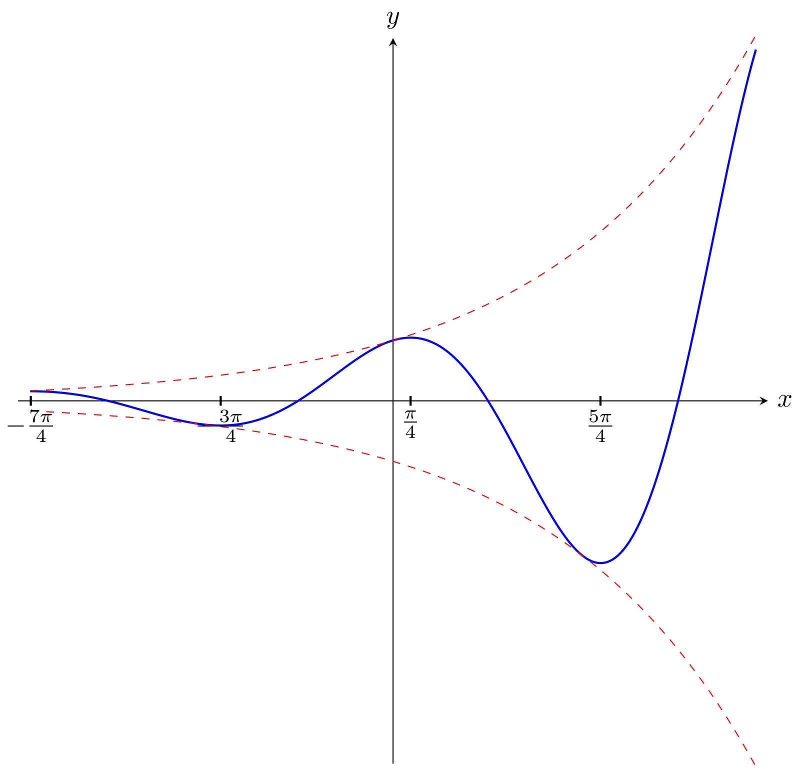
(g) The graph of $e^x \cos(x)$ is the product of the graphs of e^x with $\cos(x)$. In particular, it will look like an e^x curve modulated by a factor of $\cos(x)$. This means that we end up with oscillations between e^x and $-e^x$.

To be a bit more precise, we can calculate the locations of the stationary points. At stationary points we have:

$$\frac{d}{dx}\left(e^x \cos(x)\right) = e^x \cos(x) - e^x \sin(x) = 0,$$

which occurs if and only if $\cos(x) = \sin(x)$, i.e. $\tan(x) = 1$. The full solution to this equation is $x = \pi/4 + n\pi$ an integer. This is not quite what we might expect - maxima and minima of $\cos(x)$ occur at $n\pi/2$, for n an integer. The 'offset' is due to the fact that as we pass a peak or trough of cosine, the exponential is increasing much faster than the cosine is changing its own behaviour (either decreasing or increasing). This results in a small 'offset' which moves the peak away slightly.

The stationary points are essentially the only important information we need to sketch the graph. We include dashed graphs of e^x and $-e^x$ to represent the 'envelope' which the graph of $e^x \cos(x)$ is contained within.



Riemann sums and the definition of the integral

12. Explain what is meant by a *Riemann sum* for a function $f : [a,b] \to \mathbb{R}$ using a *partition* $P = (x_0, ..., x_n)$ (with $x_0 = a, x_n = b$) and *tagging* $T = (t_1, ..., t_n)$. By choosing appropriate partitions and taggings in each case, use sequences of Riemann sums to evaluate the definite integrals of the following functions on $[0,1]$ from first principles:

(a) x, (b) x^2, (c) x^3, (d) $\sqrt{x}$, (e) $\cos(x)$.

[*Hint: For part (d), consider a non-uniform tagging. For part (e), consider the integral of* $\mathrm{Re}(e^{ix})$ *instead of* $\cos(x)$.]

Solution: If we want to approximate the integral of the function $f(x)$ on the interval $[a,b]$, we can do so by considering a sum of *rectangles*. We split the interval $[a,b]$ into a *partition* $P = (x_0, ..., x_n)$ which satisfies:

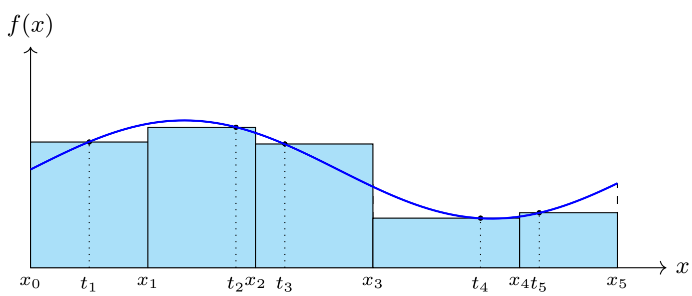
$$a = x_0 < x_1 < ... < x_n = b,$$

where $[x_0, x_1]$ is the base of the first rectangle, $[x_1, x_2]$ is the base of the second rectangle, etc. Notice we start at zero, because then the highest index n matches the number of rectangles we get!

Within each rectangle, we then pick a point called a *tag* for the rectangle. This produces a *tagging* $T = (t_1, ..., t_n)$ which satisfies:

$$x_0 \le t_1 \le x_1 \le t_2 \le x_2 \le ... \le x_{n-1} \le t_n \le x_n.$$

At the tags, we evaluate the function to give us the heights of the rectangles in our approximation, $f(t_1), ..., f(t_n)$. This is illustrated in the diagram below.



The *Riemann sum* for $f : [a,b] \to \mathbb{R}$ that results from this partition and tagging is defined to be the area of the rectangles:

$$R = \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}).$$

We say that f is a *Riemann integrable function*, with *integral* I, if for all sequences of partitions P_n and taggings T_n, such that the width of the largest sub-interval in the partition P_n tends to zero as $n \to \infty$, we have that the associated sequences of Riemann sums R_n all converge to I.

Proving that a function is Riemann integrable is very hard, because it requires us to consider all possible partitions and taggings in one go.[1] However, if we already know that a function is Riemann integrable, we can find its integral by just considering *one* sequence of partitions and taggings; that is what we do in this question. In general, any continuous function is integrable, so this is completely fine in this case!

[1] It is possible to do this for very simple functions though. You might like to show that a constant function is integrable from first principles, if you are feeling adventurous!

Finding integrals in this way is not straightforward, and is definitely not the best approach - however, it is the most fundamental, and is similar to the 'first principles' limit approach we used when we first introduced differentiation. Integration in practice involves knowing a standard set of integrals, and a set of techniques (namely integration by substitution and integration by parts), which allows us to determine integrals of most common functions.

(a) Use the uniform partition $(0, 1/n, .., n/n)$ with right-handed tagging $(1/n, 2/n, ..., n/n)$. Then we have the sequence of Riemann sums:

$$R_n = \sum_{k=1}^{n} \left(\frac{k}{n}\right) \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{n(n+1)}{2n^2} = \frac{n+1}{2n}$$

Taking the limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1+1/n}{2} = \frac{1}{2}.$$

This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}.$$

(b) Use the uniform partition $(0, 1/n, ..., n/n)$ with right-handed tagging $(1/n, 2/n, ..., n/n)$. Then we have the sequence of Riemann sums:

$$R_n = \sum_{k=1}^{n} \left(\frac{k^2}{n^2}\right) \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{(n+1)(2n+1)}{6n^2}.$$

Taking the limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{(1+1/n)(2+1/n)}{6} = \frac{2}{6} = \frac{1}{3}.$$

This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}.$$

(c) Use the uniform partition $(0, 1/n, ..., n/n)$ with right-handed tagging $(1/n, 2/n, ..., n/n)$. Then we have the sequence of Riemann sums:

$$R_n = \sum_{k=1}^{n} \left(\frac{k^3}{n^3}\right) \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^4}\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4n^4} = \frac{(n+1)^2}{4n^2}.$$

Taking the limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)^2}{4n^2} = \lim_{n\to\infty} \frac{(1+1/n)^2}{4} = \frac{1}{4}.$$

This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 x^3\, dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}.$$

(d) This time, things get more exciting, because using a uniform partition won't work. Instead, let's use a *quadratically spaced* partition, to try to clear the square root. We use the partition $(0, 1/n^2, 4/n^2, 9/n^2, ..., n^2/n^2)$ with right-handed tagging $(1/n^2, 4/n^2, ..., n^2/n^2)$. Then we have the sequence of Riemann sums:

$$R_n = \sum_{k=1}^{n} \sqrt{\frac{k^2}{n^2}} \cdot \left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2}\right) = \frac{1}{n^3}\sum_{k=1}^{n} k\left(k^2 - (k^2 - 2k + 1)\right) = \frac{1}{n^3}\sum_{k=1}^{n}\left(2k^2 - k\right).$$

Performing the sum, we have:

$$\frac{1}{n^3}\sum_{k=1}^{n}\left(2k^2 - k\right) = \frac{2}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2n^3} = \frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2}.$$

Taking the limit as $n \to \infty$, the second term vanishes, and the first term gives $2/3$. This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 \sqrt{x}\, dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}.$$

(e) This is an even more exciting problem, because we get to use complex numbers. Observe that:

$$\int_0^1 \cos(x)\, dx = \mathrm{Re}\left[\int_0^1 e^{ix}\, dx\right],$$

so instead of constructing a Riemann sum for $\cos(x)$, we will construct a Riemann sum for e^{ix}, as hinted at in the question. Choose a uniform partition $(0, 1/n, ..., n/n)$ with right-handed tagging $(1/n, 2/n, ..., n/n)$. Then we have the sequence of Riemann sums for the complex integral:

$$R_n = \sum_{k=1}^{n} e^{ik/n}\left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} e^{ik/n}.$$

This is a geometric progression with first term $e^{i/n}$ and common ratio $e^{i/n}$. Hence the sum is:

$$R_n = \frac{e^{i/n}(1 - e^i)}{n(1 - e^{i/n})}.$$

We now need to take the limit of this expression as $n \to \infty$. The numerator approaches $1 - e^i$, but the denominator is of the form $\infty \cdot 0$, so is an indeterminate form. Here, we can use a Taylor series expansion for $e^{i/n}$ (see Sheet 7, if you are unfamiliar!). We have:

$$\lim_{n\to\infty} n(1 - e^{i/n}) = \lim_{n\to\infty} n\left(1 - 1 - \frac{i}{n} - \frac{1}{2!}\left(\frac{i}{n}\right)^2 + \cdots\right) = -i.$$

Hence we have:

$$\lim_{n\to\infty} \frac{e^{i/n}(1-e^i)}{n(1-e^{i/n})} = \frac{1-e^i}{-i} = i - ie^i = i - i\cos(1) + \sin(1).$$

Taking the real part, this leaves $\sin(1)$. This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 \cos(x)\, dx = [\sin(x)]_0^1 = \sin(1).$$

In fact, this proof also shows (by taking imaginary parts), that the integral of $\sin(x)$ over $[0, 1]$ is given by $1 - \cos(1)$. This also agrees with what we would expect from our knowledge of integration:

$$\int_0^1 \sin(x)\, dx = [-\cos(x)]_0^1 = 1 - \cos(1).$$

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13. Using a non-uniform tagging, use a sequence of Riemann sums to evaluate the integral $\displaystyle\int_1^\infty \frac{dx}{x^{1+\alpha}}$, where $\alpha > 0$. [*Hint: To take the required limit, it might be useful to use the binomial expansion* $(1+\epsilon)^{-\alpha} = 1 - \alpha\epsilon + \ldots$.]

➾ **Solution:** Let us define $r_n = 1 + 1/n$ to be a ratio larger than one. Then, we use a geometric partition $(1, r_n, r_n^2, r_n^3, \ldots)$ with right-handed tagging $(r_n, r_n^2, r_n^3, \ldots)$. The corresponding sequence of Riemann sums is:

$$R_n = \sum_{k=0}^{\infty} \frac{1}{(r_n^k)^{1+\alpha}}\left(r_n^{k+1} - r_n^k\right) = (r_n - 1)\sum_{k=0}^{\infty} r_n^{-k\alpha} = \frac{r_n - 1}{1 - r_n^{-\alpha}}.$$

Here, we could take the sum to infinity since $r_n > 1$, so $r_n^{-\alpha} < 1$. We now take the limit as $n \to \infty$, which is equivalent to the limit $r_n \to 1$. To do this, we use the hint in the question, and first write $r_n = 1 + \epsilon_n$, and consider the limit as $\epsilon_n \to 0$.

$$\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{r\to 1} \frac{r-1}{1-r^{-\alpha}} \\
&= \lim_{\epsilon\to 0} \frac{(1+\epsilon)-1}{1-(1+\epsilon)^{-\alpha}} \\
&= \lim_{\epsilon\to 0} \frac{\epsilon}{\alpha\epsilon + \cdots} && \text{(binomial expansion hint; the dotted terms are} \propto \epsilon^2\text{)} \\
&= \frac{1}{\alpha}.
\end{aligned}$$

Observe that this agrees with the expected result, since:

$$\int_1^\infty \frac{dx}{x^{1+\alpha}} = \left[-\frac{1}{\alpha x^\alpha}\right]_1^\infty = \frac{1}{\alpha}.$$

14. Show by considering Riemann sums that $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2}=\frac{\pi}{4}$.

➼ **Solution:** Observe that:

$$\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2}=\sum_{k=1}^{n}\sqrt{1-\left(\frac{k}{n}\right)^2}\cdot\left(\frac{k}{n}-\frac{k-1}{n}\right)$$

is a sequence of Riemann sums for the function $\sqrt{1-x^2}$ on the interval $[0,1]$ using the partition $0=0/n<1/n<\ldots<n/n=1$, with the (right handed) tagging $1/n, 2/n, \ldots, n/n$. Hence:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2}=\int_0^1\sqrt{1-x^2}\,dx.$$

To evaluate this integral, we could use a substitution. Alternatively, we could note it is the area under the graph $y=\sqrt{1-x^2}$ between 0 and 1. Rearranging this equation, we see that $x^2+y^2=1$, which tells us the graph is a *circle*. It immediately follows that the integral is just the area of quarter of a circle, hence:

$$\int_0^1\sqrt{1-x^2}\,dx=\frac{\pi}{4},$$

as required.

Basic integrals

15. *Write down* the indefinite integrals of each of the following functions, where $a \neq 0, \alpha \neq -1$, and f is any (differentiable, non-zero) function:

(a) $(ax+b)^\alpha$, (b) e^{ax+b}, (c) $(ax+b)^{-1}$, (d) $\sin(ax+b)$, (e) $\cos(ax+b)$,

(f) $\sec^2(ax+b)$, (g) $\operatorname{cosec}^2(ax+b)$, (h) $\sinh(ax+b)$, (i) $\cosh(ax+b)$, (j) $f'(x)f(x)^\alpha$,

(k) $f'(x)/f(x)$.

Learn these integrals off by heart, and get your supervision partner to test you on them.

Solution: These are all standard integrals. We have (where c is an arbitrary constant in each case):

(a) $\displaystyle\int (ax+b)^\alpha \, dx = \frac{(ax+b)^{\alpha+1}}{a(\alpha+1)} + c$, if $a \neq 0$ and $\alpha \neq -1$.

(b) $\displaystyle\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c$, if $a \neq 0$.

(c) $\displaystyle\int \frac{1}{ax+b} \, dx = \frac{\log(ax+b)}{a} + c$, if $a \neq 0$.

(d) $\displaystyle\int \sin(ax+b) \, dx = -\frac{\cos(ax+b)}{a} + c$, if $a \neq 0$.

(e) $\displaystyle\int \cos(ax+b) \, dx = \frac{\sin(ax+b)}{a} + c$, if $a \neq 0$.

(f) $\displaystyle\int \sec^2(ax+b) \, dx = \frac{\tan(ax+b)}{a} + c$, if $a \neq 0$.

(g) $\displaystyle\int \operatorname{cosec}^2(ax+b) \, dx = -\frac{\cot(ax+b)}{a} + c$, if $a \neq 0$.

(h) $\displaystyle\int \sinh(ax+b) \, dx = \frac{\cosh(ax+b)}{a} + c$, if $a \neq 0$.

(i) $\displaystyle\int \cosh(ax+b) \, dx = \frac{\sinh(ax+b)}{a} + c$, if $a \neq 0$.

(j) $\displaystyle\int f'(x)f(x)^\alpha \, dx = \frac{f(x)^{\alpha+1}}{\alpha+1}$, if $\alpha \neq -1$.

(k) $\displaystyle\int \frac{f'(x)}{f(x)} \, dx = \log(f(x)) + c.$

16. Using the results of the previous question, evaluate the definite integrals:

(a) $\int_0^2 (x-1)^2\,dx$, (b) $\int_0^\pi e^{i\theta}\,d\theta$, (c) $\int_0^\pi \cos(x)\,dx$, (d) $\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx$, (e) $\int_0^1 \frac{2x+4}{x^2+4x+1}\,dx$.

Solution: We have:

(a) $\int_0^2 (x-1)^2\,dx = \left[\frac{(x-1)^3}{3}\right]_0^2 = \frac{1}{3} - -\frac{1}{3} = \frac{2}{3}$.

(b) $\int_0^\pi e^{i\theta}\,d\theta = \left[\frac{e^{i\theta}}{i}\right]_0^\pi = -\frac{1}{i} - \frac{1}{i} = -\frac{2}{i} = 2i$.

(c) $\int_0^\pi \cos(x)\,dx = [\sin(x)]_0^\pi = \sin(\pi) - \sin(0) = 0$. Alternatively, spot that $\cos(x)$ is rotationally symmetric about $x = \pi/2$, and the positive and negative contributions to the integral from $0 < x < \pi/2$ and $\pi/2 < x < \pi$ therefore exactly cancel out.

(d) $\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx = [\tan(x)]_{-\pi/4}^{\pi/4} = 1 - -1 = 2$.

(e) $\int_0^1 \frac{2x+4}{x^2+4x+1}\,dx = \left[\log(x^2+4x+1)\right]_0^1 = \log(6) - \log(1) = \log(6)$.

17. By writing $\cos(bx)$ as the real part of a complex exponential, determine the indefinite integral of $e^{ax}\cos(bx)$. Similarly, determine the indefinite integrals of $e^x(\sin(x) - \cos(x))$ and $e^x(\sin(x) + \cos(x))$.

➻ **Solution:** We have:

$$\begin{aligned}
\int e^{ax}\cos(bx)\,dx &= \mathrm{Re}\left[\int e^{ax}e^{ibx}\,dx\right] \\
&= \mathrm{Re}\left[\int e^{(a+ib)x}\,dx\right] \\
&= \mathrm{Re}\left[\frac{e^{(a+ib)x}}{a+ib} + c\right] \\
&= \mathrm{Re}\left[\frac{e^{ax}(a-ib)(\cos(bx) + i\sin(bx))}{a^2+b^2} + c\right] \\
&= \frac{e^{ax}(a\cos(bx) + b\sin(bx))}{a^2+b^2} + c,
\end{aligned}$$

where c is a real constant of integration.

Similarly, we have:

$$\int e^{ax}\sin(bx)\,dx = \mathrm{Im}\left[\int e^{ax}e^{ibx}\,dx\right] = \frac{e^{ax}(a\sin(bx) - b\cos(bx))}{a^2+b^2} + c,$$

using the penultimate line of the calculation from above, but just taking the imaginary part instead of the real part. It follows that:

$$\int e^x(\sin(x) - \cos(x))\,dx = \frac{e^x(\sin(x) - \cos(x))}{2} - \frac{e^x(\cos(x) + \sin(x))}{2} + c = -e^x\cos(x) + c,$$

for a real constant of integration c. Similarly, we have:

$$\int e^x(\sin(x) + \cos(x))\,dx = \frac{e^x(\sin(x) - \cos(x))}{2} + \frac{e^x(\cos(x) + \sin(x))}{2} + c = e^x\sin(x) + c,$$

for a real constant of integration c.