Part IA: Mathematics for Natural Sciences A
Examples Sheet 2: The vector product, and triple products of vectors

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The vector product

1. Find the angle between the position vectors of the points $(2,1,1)$ and $(3,-1,-5)$, and find the direction cosines of a vector perpendicular to both. Can both the angle and vector be computed using *only* the vector product?

2. Find all points $\mathbf{r}$ which satisfy $\mathbf{r} \times \mathbf{a} = \mathbf{b}$ where $\mathbf{a} = (1,1,0)$ and $\mathbf{b} = (1,-1,0)$.

3. Using properties of the vector product, prove the identity $(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}) = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$.

More on the equation of a line

4. (a) Explain why the line through the points with positions vectors $\mathbf{a}$, $\mathbf{b}$ is $(\mathbf{r}-\mathbf{a}) \times (\mathbf{b}-\mathbf{a}) = \mathbf{0}$. Show using properties of the vector product that an equivalent representation of this line is $\mathbf{r} \times (\mathbf{b}-\mathbf{a}) = \mathbf{a} \times \mathbf{b}$. What is the geometric significance of the quantity $|\mathbf{a} \times \mathbf{b}|/|\mathbf{b}-\mathbf{a}|$ here?

   (b) Express the line $\mathbf{r} = (1,0,1) + \lambda(3,-1,0)$ in the form $\mathbf{r} \times \mathbf{c} = \mathbf{d}$.

5. (a) Show that the shortest distance between the point $\mathbf{p}$ and the line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ can be written as $\left|\hat{\mathbf{b}} \times (\mathbf{p}-\mathbf{a})\right|$.

   (b) Find the shortest distance from a vertex of a unit cube to a diagonal excluding that vertex using both the formula in (a), and the formula from Question 13(c) of Sheet 1, and check that your answers agree.

More on the equation of a plane

6. (a) Explain why the plane through the points with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is $(\mathbf{r}-\mathbf{a}) \cdot ((\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})) = 0$. Show using properties of the vector product, and the result from Question 3, that this may equivalently be written in the more symmetric form $\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

   (b) Find an equation of the form $(\mathbf{r}-\mathbf{a}) \cdot \mathbf{n} = 0$ for the plane passing through $(1,1,1)$, $(1,2,3)$ and $(0,0,4)$.

7. You need to drill a hole in a piece of metal starting at a right angle to a flat surface passing through the points $A = (1,0,0)$, $B = (1,1,1)$ and $C = (0,2,0)$, with the hole emerging at the point $D = (2,1,0)$. How long a drill must you use and where (in the plane ABC) must you start drilling?

8. Determine whether: (a) the points $\mathbf{P}_1 = (0,0,2)$, $\mathbf{P}_2 = (0,1,3)$, $\mathbf{P}_3 = (1,2,3)$, $\mathbf{P}_4 = (2,3,4)$ are coplanar; (b) the points $\mathbf{Q}_1 = (-2,1,1)$, $\mathbf{Q}_2 = (-1,2,2)$, $\mathbf{Q}_3 = (-3,3,2)$, $\mathbf{Q}_4 = (-2,4,3)$ are coplanar.

Shortest distances

9. *Without using a formula*, find the shortest distance between the lines $\mathbf{r}_1 = (1,0,1) + \lambda(2,-1,3)$ and $\mathbf{r}_2 = (0,1,-2) + \mu(1,0,2)$, justifying the steps you take. [*Sometimes, it is better to understand a method, than to quote a formula*.]

10. So far, we have developed formulae for the shortest distance from points to lines, and from points to planes. Now, using the scalar and vector products, establish formulae for the following:

    (a) the shortest distance from the line $\mathbf{r}_1 = \mathbf{v}_1 + \lambda \mathbf{w}_1$ to the line $\mathbf{r}_2 = \mathbf{v}_2 + \mu \mathbf{w}_2$; [*Hint: Take care when the lines are parallel!*]

    (b) the shortest distance from the line $\mathbf{r} = \mathbf{v} + \lambda \mathbf{w}$ to the plane $(\mathbf{r}-\mathbf{a}) \cdot \mathbf{b} = 0$;

    (c) the shortest distance from the plane $(\mathbf{r}_1 - \mathbf{a}_1) \cdot \mathbf{b}_1 = 0$ to the plane $(\mathbf{r}_2 - \mathbf{a}_2) \cdot \mathbf{b}_2 = 0$.

The vector triple product, and vector equations

11. (a) By expanding in terms of components, prove *Lagrange's formula* for the vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$. Think of a way of remembering this formula off by heart - it is very useful!
    (b) Hence, construct an example of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

12. Prove the *Jacobi identity*, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$.

13. Two vector operators, $P_{\hat{\mathbf{u}}} : \mathbb{R}^3 \to \mathbb{R}^3$ and $R_{\hat{\mathbf{u}}} : \mathbb{R}^3 \to \mathbb{R}^3$ are defined by $P_{\hat{\mathbf{u}}}(\mathbf{r}) = (\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}$ and $R_{\hat{\mathbf{u}}}(\mathbf{r}) = \hat{\mathbf{u}} \times (\mathbf{r} \times \hat{\mathbf{u}})$ respectively. Interpret these operators geometrically, and hence explain why $P_{\hat{\mathbf{u}}}(\mathbf{r}) + R_{\hat{\mathbf{u}}}(\mathbf{r}) = \mathbf{r}$ for all vectors $\mathbf{r}$. Also explain why $P_{\hat{\mathbf{u}}}^2 = P_{\hat{\mathbf{u}}}$ and $R_{\hat{\mathbf{u}}}^2 = R_{\hat{\mathbf{u}}}$.

14. Solve the following vector equations, and give geometric interpretations of their solutions:
    (a) $\mathbf{a} \times \mathbf{r} + \lambda \mathbf{r} = \mathbf{c}$, where $\lambda \neq 0$, and $\mathbf{a}, \mathbf{c} \in \mathbb{R}^3$ are arbitrary 3-vectors;
    (b) $\mathbf{r} \times \mathbf{a} = \mathbf{b}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are arbitrary 3-vectors, and $\mathbf{a} \neq \mathbf{0}$;
    (c) $\mathbf{r} = \mathbf{a} + (\mathbf{b} \cdot \mathbf{r})\mathbf{c}$, where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ are arbitrary 3-vectors;
    (d) $2\mathbf{r} + \hat{\mathbf{n}} \times \mathbf{r} + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{r})^2 = \mathbf{a}$, where $\hat{\mathbf{n}}$ is a unit vector, and $\hat{\mathbf{n}} \cdot \mathbf{a} = -1$.

The scalar triple product, and non-orthonormal bases

15. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ be 3-vectors.
    (a) Give the definition of the *scalar triple product* $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of the 3-vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. Hence show that the volume of the parallelepiped defined by the positions vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is $|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|$. Why is the modulus necessary?
    (b) Using the relation between the scalar triple product and a parallelepiped, explain why:
        (i) the scalar triple product is antisymmetric on odd permutations of its entries, and symmetric on even permutations of its entries;
        (ii) the condition $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$ implies that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are not coplanar, and thus form a basis.
    (c) Compute the volume of a parallelepiped defined by the three position vectors $\mathbf{a} = (0, \frac{1}{2}, \frac{1}{2})$, $\mathbf{b} = (\frac{1}{2}, 0, \frac{1}{2})$, $\mathbf{c} = (\frac{1}{2}, \frac{1}{2}, 0)$, and comment on whether these vectors form a basis.

16. Simplify the scalar triple products $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} + \mathbf{c}) \times (\mathbf{c} + \mathbf{a})$ and $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$.

17. Let $\mathbf{0}, \mathbf{a}, \mathbf{b}, \mathbf{c}$ form the vertices of a tetrahedron, with $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$. Write down conditions in terms of the scalar triple product for the vector $\mathbf{r}$ to lie inside the tetrahedron.

18. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ be 3-vectors.
    (a) If these vectors form an orthonormal basis, derive expressions for the coefficients α, β, γ in the formula $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$. Hence express $(2, 3, 4)$ in terms of the basis $\{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$.
    (b) If instead these vectors do *not* form an orthonormal basis, derive expressions for the coefficients α, β, γ in the formula $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$. [*Hint: consider scalar triple products.*] Hence express $(1, 1, 1)$ in terms of the basis $\{(1, 2, 1), (0, 0, 1), (2, -1, 1)\}$.
    (c) We define the *reciprocal vectors* to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to be the vectors:

$$\mathbf{A} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \qquad \mathbf{B} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \qquad \mathbf{C} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}.$$

    Show that $\mathbf{A} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{b} = \mathbf{C} \cdot \mathbf{c} = 1$, and $\mathbf{A} \cdot \mathbf{b} = \mathbf{A} \cdot \mathbf{c} = \mathbf{B} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{c} = \mathbf{C} \cdot \mathbf{a} = \mathbf{C} \cdot \mathbf{b} = 0$. Hence, by comparing to part (b), explain how the reciprocal basis can be used to express a general vector $\mathbf{d}$ in terms of a non-orthonormal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. Compute the reciprocal basis to the basis $\{(1, 2, 1), (0, 0, 1), (2, -1, 1)\}$.