Part IA: Mathematics for Natural Sciences A
Examples Sheet 3: Vector area, polar coordinate systems, and complex numbers

Model Solutions

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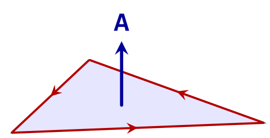
Vector area

1. (a) Define the *vector area* **A** of a surface composed of k flat faces with areas $A_1, \ldots, A_k$ and unit normals $\hat{\mathbf{n}}_1, \ldots, \hat{\mathbf{n}}_k$. What are the conventions usually used when choosing the unit normal(s)?

   (b) In terms of the position vectors **a**, **b** determine the vector areas of: (i) the parallelogram defined by **a**, **b**; (ii) the triangle defined by **a**, **b**. Hence, given points $O = (0,0,0)$, $A = (1,0,0)$, $B = (1,1,1)$, $C = (0,2,0)$, compute the vector area of the triangle OAB (with vertices taken in that order), and the vector area of the surface bounded by the loop $OABC$ comprising the two triangular surfaces OAB and BCO.

Solution: The *vector area* of a flat surface A is the vector **A** with magnitude A and direction normal to the surface. The choice of which normal is a matter of convention, but if the orientation of the boundary of **A** is specified, then the normal is usually taken in a right-handed sense. This means that if you fingers curl round the orientation of the boundary, then your thumb points in the direction of the normal.

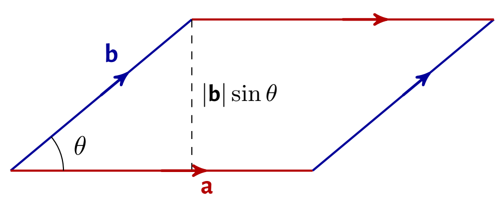


For a surface composed of multiple flat faces with area $A_1, \ldots, A_k$ and unit normals $\hat{\mathbf{n}}_1, \ldots, \hat{\mathbf{n}}_k$, the total vector area is defined to be the sum of the vector areas of the individual faces:

$$\mathbf{A} = \sum_{i=1}^{k} A_i \hat{\mathbf{n}}_i.$$

(b) First, we compute the vector areas of the basic shapes asked for in the question:

(i) The parallelogram formed by **a**, **b** having an angle θ between the vectors takes the form:



Evidently, the area of the parallelogram is $|\mathbf{a}||\mathbf{b}|\sin(\theta) = |\mathbf{a} \times \mathbf{b}|$.

Now, if we choose an orientation of the parallelogram where we follow the vector **a** first, then the vector area will be a unit vector pointing out of the page scaled by the area of the parallelogram. That is, the vector area will just be the *cross product*, $\mathbf{a} \times \mathbf{b}$.

(ii) For a triangle, we just have half the area of the parallelogram above. So assuming the same orientation (following **a** first), the vector area is $\frac{1}{2}\mathbf{a} \times \mathbf{b}$.

Now, consider the explicit coordinates we are given. First, we are asked for the vector area of the triangle OAB, which by the above work is simply:

$$\frac{1}{2}\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\-1\\1\end{pmatrix}.$$

The vector area of the surface comprised of the triangle OAB together with the triangle BCO is the sum of the previous vector area, together with:[1]

$$\frac{1}{2}\overrightarrow{BC} \times \overrightarrow{BO} = \frac{1}{2}\begin{pmatrix}-1\\1\\-1\end{pmatrix} \times \begin{pmatrix}-1\\-1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-2\\0\\2\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

Hence, the total vector area is:

$$\frac{1}{2}\begin{pmatrix}0\\-1\\1\end{pmatrix} + \begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\-1/2\\3/2\end{pmatrix}$$

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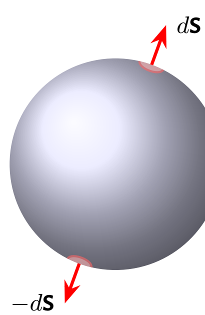
2. (a) Give a very general explanation of how the idea of vector area could be extended to *curved surfaces*, and hence explain why we expect the vector area of any *closed* surface to be **0**.

(b) Compute the vector area of the square with vertices $(0,0,0)$, $(2,0,0)$, $(2,2,0)$, $(0,2,0)$, taken in that order. Hence, compute the vector area of the pyramid extending this square with the point $(1,1,1)$, excluding its square face.

(c) Compute the vector area of the curved surface of a truncated hollow cone, bounded by a horizontal circle of radius 4 units and a horizontal circle of radius 3 units at some height above the first (note the result is independent of the height!).

Solution: (a) For a closed surface, we could split the surface into lots of small *approximately* flat pieces, calculate each of their vector areas, then add them all up. This amounts to performing an *integral* over the surface. We will study surface integration properly in Lent term.



For a *closed* surface, i.e. one with no boundary, we expect each little flat piece that we split the surface up into to have a neighbour on the opposite side of the surface, but with an equal and opposite vector area. These will exactly cancel giving the vector area of a closed surface as zero. Note, this is not a proof, just a hopeful argument - we will see a proof properly in Lent when we study Stokes' theorem.

[1] The solution to Question 3(c) has a good diagram for this part!

(b) The square clearly has area 4, so has vector area $(0,0,4)$, since $(0,0,1)$ is a unit normal to the square.

Let **A** be the vector area of the top part of the square-based pyramid. Then since the vector area of a closed surface is always zero, we must have $\mathbf{A}+(0,0,4)=\mathbf{0}$, so $\mathbf{A}=-(0,0,4)$. This assumes a certain orientation of the upper part of the square-based pyramid; if we had used a different orientation, we would have obtained $(0,0,4)$.

(c) Let the axis of the cone be along the z-axis. Then if the complete shape has an outward pointing normal everywhere, the smaller top disc has vector area $\pi(3^2)\cdot(0,0,1)=(0,0,9\pi)$. The larger bottom disc has vector area $\pi(4^2)\cdot(0,0,-1)=(0,0,-16\pi)$. Let the vector area of the curved surface be **A**. Then:

$$\mathbf{A}+(0,0,9\pi)+(0,0,-16\pi)=\mathbf{0} \qquad \Leftrightarrow \qquad \mathbf{A}=(0,0,7\pi).$$

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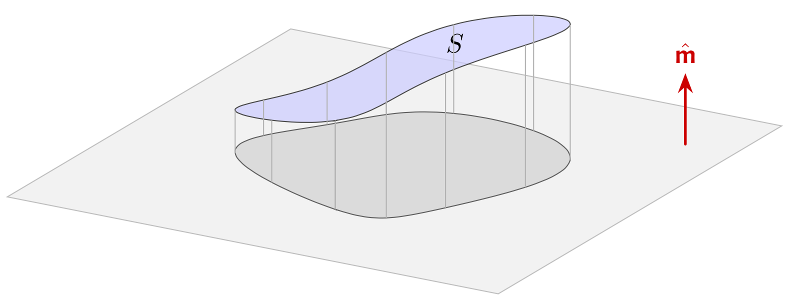
3. (a) Let **S** be the vector area of the surface S. Prove that the area of the projection of the surface S onto the plane with unit normal $\hat{\mathbf{m}}$ is $|\mathbf{S}\cdot\hat{\mathbf{m}}|$. [*Hint: consider joining the surface to its 'shadow' on the plane to create a closed surface.*]

   (b) Compute the vector area of the projection of the square with vertices $(0,0,0)$, $(2,0,0)$, $(2,2,0)$, $(0,2,0)$ onto the plane with unit normal $\hat{\mathbf{m}}=(0,-1,1)/\sqrt{2}$.

   (c) By projecting areas onto the yz, xz, and xy planes, compute the vector area of the surface comprised of the two triangles OAB, BCO with vertices $O=(0,0,0)$, $A=(1,0,0)$, $B=(1,1,1)$, $C=(0,2,0)$, taken in that order. [*Your answer should match your answer to Question 17(b)!*] What is the area of the surface projected onto: (i) the plane with normal $(0,-1,1)$; (ii) the plane that maximises the projected area?

Solution: (a) This part is a very cute argument, but it is tricky if you have never seen it before! We imagine joining the shadow of S to the plane with normal $\hat{\mathbf{m}}$ via some prism-like surface, as shown in the figure.



Let S_{proj} be the area of the projection of the surface S onto the plane with normal $\hat{\mathbf{m}}$. Let $\mathbf{S}_\perp$ be the *vector* area of the curved surface joining S to the plane, which is necessarily perpendicular to $\hat{\mathbf{m}}$. Then since we now have a closed surface, we must have:

$$\mathbf{S}+\mathbf{S}_\perp-S_{\text{proj}}\hat{\mathbf{m}}=\mathbf{0}.$$

Taking the scalar product with $\hat{\mathbf{m}}$, and using the fact $\mathbf{S}_\perp\cdot\hat{\mathbf{m}}=0$, we immediately get the result $\mathbf{S}\cdot\hat{\mathbf{m}}=S_{\text{proj}}$, as required.

This argument assumed that **S** pointed *out* of the volume created, so that the appropriate vector area for the shadow was $-S_{\text{proj}}\hat{\mathbf{m}}$. If instead **S** pointed *into* the volume created, the appropriate vector area for the shadow is instead $S_{\text{proj}}\hat{\mathbf{m}}$ (this ensures that the normal is continuous across the surface, if we imagine moving it around). This implies the need for the modulus sign if we are calculating area in general: $|\mathbf{S}\cdot\hat{\mathbf{m}}|$. This will be an important point in part (c) of this question.

(b) The square has vector area $(0,0,4)$. Projecting onto the plane with unit normal $\hat{\mathbf{m}}$, the area of the projection is:

$$(0,0,4)\cdot(0,-1,1)/\sqrt{2} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

Therefore, the *vector area* of the projection is $2\sqrt{2}\hat{\mathbf{m}} = (0,-2,2)$.

(c) The vector area may be expressed as:

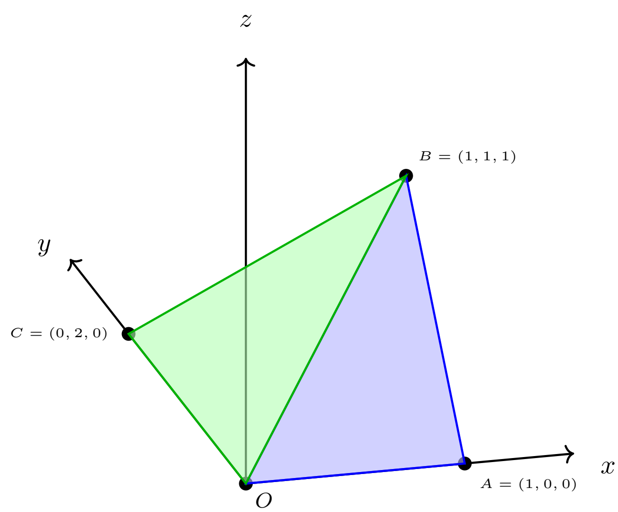
$$\mathbf{S} = S_x\hat{\mathbf{e}}_x + S_y\hat{\mathbf{e}}_y + S_z\hat{\mathbf{e}}_z,$$

where S_x, S_y, S_z are the coefficients in the expansion in an orthonormal basis, hence are given by:

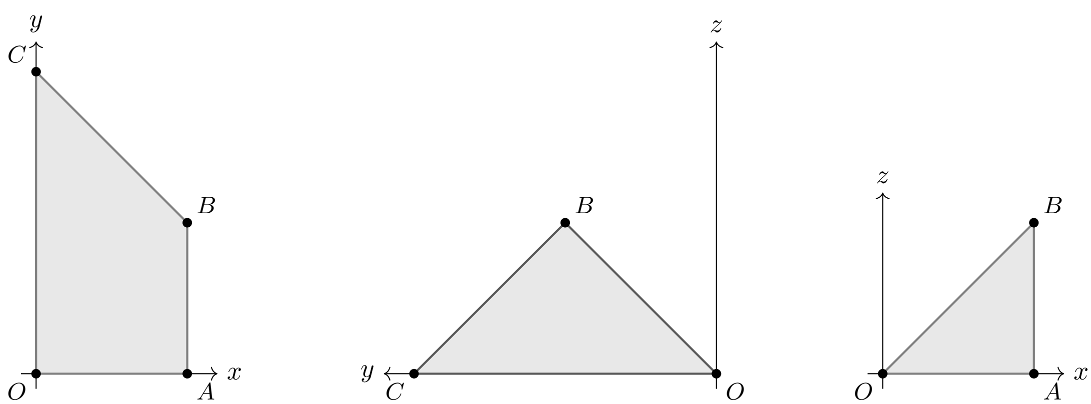
$$S_x = \mathbf{S}\cdot\hat{\mathbf{e}}_x, \qquad S_y = \mathbf{S}\cdot\hat{\mathbf{e}}_y, \qquad S_z = \mathbf{S}\cdot\hat{\mathbf{e}}_z.$$

In particular, the coefficients in the expansion are the (signed) projected areas of the surface onto the planes with normals $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ respectively, i.e. the coordinate planes.

Drawing a figure of the full 3D situation first, we have:



Now, imagine the projections of the surface onto each of the coordinate planes. We have:



Calculating the areas of each of the projections, we immediately see that:

$$|S_z| = \frac{3}{2}, \qquad |S_x| = 1, \qquad |S_y| = \frac{1}{2}.$$

To work out the signs correctly, we do the following. Following the argument presented in (a), we imagine joining up the surface $OABC$ to its shadow in a coordinate plane with normal $\hat{\mathbf{n}}$ (where $\hat{\mathbf{n}} = \hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$). If the vector area points *into* the volume that is created, then for consistency the vector area from the coordinate plane is the projection multiplied by $\hat{\mathbf{n}}$, giving:

$$\mathbf{S} + S_{\text{proj}}\hat{\mathbf{n}} + \mathbf{S}_\perp = \mathbf{0}.$$

This implies that $\mathbf{S} \cdot \hat{\mathbf{n}} = -S_{\text{proj}}$, so we get a negative sign. On the other hand, if the vector area points *out* of the volume that is created, then there is no change of sign.

In the xy plane, the orientation of $OABC$ implies the vector area points out of the volume it would create with the xy-plane. This means that the sign of S_z is positive, $S_z = 3/2$.

On the other hand, in both the yz and the xz planes, the orientation of $OABC$ implies that the vector area points *into* the volume it would create with the corresponding planes; hence $S_x = -1$ and $S_y = -1/2$. Overall we have:

$$\mathbf{S} = (-1, -1/2, 3/2),$$

consistent with our findings earlier on in the sheet.

To finish, we compute the projected area of $OABC$ onto two planes.

(i) Projecting onto the plane with normal $(0, -1, 1)$, we get the area:

$$\left|\mathbf{S} \cdot \frac{(0,-1,1)}{\sqrt{2}}\right| = \frac{1/2 + 3/2}{\sqrt{2}} = \sqrt{2}.$$

(ii) We now seek the plane which would maximise the projected area. If we have a plane with normal $\hat{\mathbf{m}}$, then the projected area is $|\mathbf{S} \cdot \hat{\mathbf{m}}| = |\mathbf{S}||\hat{\mathbf{m}}||\cos(\theta)|$, where θ is the angle between the vectors $\mathbf{S}$, $\hat{\mathbf{m}}$. Hence, the plane that maximises the projected area is the one with unit normal $\hat{\mathbf{S}}$. The projected area in this direction is:

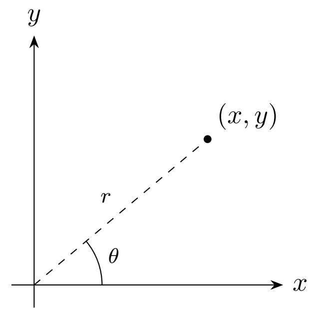
$$\left|\mathbf{S} \cdot \hat{\mathbf{S}}\right| = |\mathbf{S}| = \sqrt{1 + \frac{1}{4} + \frac{9}{4}} = \sqrt{\frac{7}{2}}.$$

Polar coordinate systems

4. Draw (convincing) diagrams defining plane, cylindrical, and spherical polar coordinates. In each case, derive the coordinate transform laws from polars to Cartesians, and from Cartesians to polars. Hence, find the cylindrical polar and spherical polar coordinates of the point $(3, 4, 5)$.

Solution: We address each coordinate system in turn.

- **Plane polar coordinates.** This system is defined by the diagram below:



From basic trigonometry, we have that:

$$x = r\cos(\theta), y = r\sin(\theta).$$

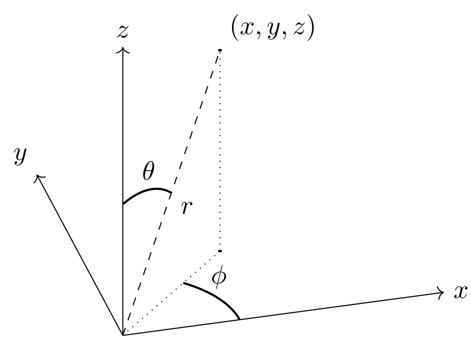
On the other hand, r is the length from the origin to the point (x, y), so:

$$r = \sqrt{x^2 + y^2}.$$

Finally, the angle θ satisfies $\tan(\theta) = x/y$. Depending on conventions, the root of this equation might be chosen to lie in the range $[0, 2\pi)$ or $(-\pi, \pi)$ (or indeed, other ranges).

- **Cylindrical polar coordinates.** These are the same as plane polars, just with an added z-direction. The formula for plane polars still hold when transforming $(r, \theta, z) \leftrightarrow (x, y, z)$.

- **Spherical polar coordinates.** Spherical polar coordinates for three dimensions are defined according to a *radial coordinate* r, which tells us how far from the origin we are, a *polar angle* θ, which tells us the angle between the radius from the origin to the point of interest and the z-axis (called the *polar axis*), and an *azimuthal angle* ϕ, which tells us the angle between the projection of the radius onto the xy-plane and the x-axis.



Some basic trigonometry tells us that:

$$x = r\sin(\theta)\cos(\phi), \qquad y = r\sin(\theta)\sin(\phi), \qquad z = r\cos(\theta).$$

Inverting these formula, we have:

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \tan(\phi) = \frac{x}{y}, \qquad \cos(\theta) = \frac{z}{\sqrt{x^2 + y^2 + z^2}},$$

where the coordinate θ is chosen to lie in the range $[0, \pi/2]$, and the coordinate ϕ is usually chosen to lie either in the range $[0, 2\pi)$ or $[-\pi, \pi)$.

Applying these formulae to the point $(3, 4, 5)$, we have:

- In cylindrical polar coordinates, the coordinates of the point are:

$$(r, \theta, z) = \left(\sqrt{3^2 + 4^2}, \arctan\left(\frac{4}{3}\right), 5\right) = \left(5, \arctan\left(\frac{4}{3}\right), 5\right).$$

- In spherical polar coordinates, the coordinates of the point are:

$$(r, \theta, \phi) = \left(\sqrt{3^2 + 4^2 + 5^2}, \arccos\left(\frac{5}{\sqrt{3^2 + 4^2 + 5^2}}\right), \arctan\left(\frac{4}{3}\right)\right) = \left(5\sqrt{2}, \frac{\pi}{4}, \arctan\left(\frac{4}{3}\right)\right).$$

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5. (a) In 2D Cartesian coordinates, a circle is specified by $(x-1)^2 + y^2 = 1$. Find its equation in polar coordinates.

   (b) In 3D Cartesian coordinates, a sphere is specified by $(x-1)^2 + y^2 + z^2 = 1$. Find its equation in spherical polar coordinates.

➾ **Solution:** (a) In plane polar coordinates, we have $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Hence the equation in polar coordinates is:

$$(r\cos(\theta) - 1)^2 + r^2\sin^2(\theta) = 1.$$

Simplifying, we have:

$$1 = r^2\cos^2(\theta) - 2r\cos(\theta) + 1 + r^2\sin^2(\theta) = r^2 - 2r\cos(\theta) + 1,$$

which can be rearranged to $r^2 = 2r\cos(\theta)$. Dividing by r, we have $r = 2\cos(\theta)$. This allows for the case $r = 0$ when $\theta = \pi/2$, so we have not lost anything by dividing through by r.

(b) In spherical polar coordinates, we have $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$ and $z = r\cos(\theta)$. Hence the equation in polar coordinates is:

$$\begin{aligned}
1 &= (r\sin(\theta)\cos(\phi) - 1)^2 + r^2\sin^2(\theta)\sin^2(\phi) + r^2\cos^2(\theta) \\
&= r^2\sin^2(\theta)\cos^2(\phi) - 2r\sin(\theta)\cos(\phi) + 1 + r^2\sin^2(\theta)\sin^2(\phi) + r^2\cos^2(\theta) \\
&= r^2 - 2r\sin(\theta)\cos(\phi) + 1.
\end{aligned}$$

This can be rearranged to $r^2 = 2r\sin(\theta)\cos(\phi)$, which similarly can be reduced to $r = 2\sin(\theta)\cos(\phi)$.

6. Let $a > 0$ be a constant. Describe the following loci:

(a) (i) $\phi = a$; (ii) $r = \phi$, in plane polar coordinates.

(b) (i) $z = a$; (ii) $r = a$; (iii) $r = a$ and $z = \phi$, in cylindrical polar coordinates.

(c) (i) $\theta = a$; (ii) $\phi = a$; (iii) $r = a$; (iv) $r = \theta = a$, in spherical polar coordinates.

➭ **Solution:** We have:

(a) (i) $\phi = a$ corresponds to a half-line emanating from the origin, at an angle a to the x-axis. (ii) $r = \phi$ corresponds to a spiral, starting out at the origin then as ϕ increases, r increases too - it crosses the y-axis at $\pi/2$, then the negative x-axis at $-\pi$, then the negative y-axis at $-3\pi/2$. Eventually it gets back to the positive x-axis at the point 2π.

(b) (i) $z = a$ is a plane with normal $(0, 0, 1)$ a distance a from the origin. (ii) $r = a$ is a cylinder of radius a, with axis along the z-axis. (iii) The locus $r = a$ says that we on the cylinder of radius a, with axis along the z-axis. The equation $z = \phi$ says the angle and z-coordinate are related; in particular, the higher we are along the z-axis, the greater the angle with the x-axis. Thus this shape is a *helix* (a corkscrew type shape).

(c) (i) $\theta = a$ is a semi-infinite cone, with opening angle a, and axis along the z-axis. Its point is at the origin. (ii) $\phi = a$ is an infinite half-plane, inclined at an angle a to the x-axis, with its edge along the z-axis. (iii) $r = a$ is a sphere of radius a centred on the origin. (iv) $r = \theta = a$ is the intersection of the figures in (i) and (iii); this is a circle, at a height $a\cos(a)$ from the origin and radius $a\sin(a)$.

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7. Consider a point with position vector $\hat{\mathbf{n}}$ on the unit sphere S.

(a) Explain why $\hat{\mathbf{n}} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, where θ, ϕ are the spherical coordinates of $\hat{\mathbf{n}}$.

(b) Show that the vector area $d\mathbf{S}$ of a small patch near $\hat{\mathbf{n}}$, subtending a small angle $d\theta$ in the θ direction, and a small angle $d\phi$ in the ϕ direction, is given *approximately* by $d\mathbf{S} = \hat{\mathbf{n}}\sin(\theta)d\theta d\phi$. Why might this be useful?

➭ **Solution:** (a) In general, a position vector in spherical polar coordinates is given by:

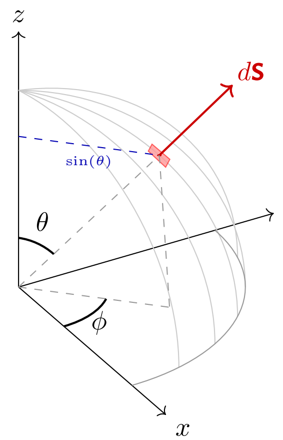
$$\mathbf{r} = (x, y, z) = (r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta)).$$

If a vector lies on the surface of the unit sphere, the only condition that needs to be satisfied is $r = 1$. This gives the general position vector on the surface of the unit sphere to be:

$$\hat{\mathbf{n}} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)),$$

as required.

(b) Consider the following diagram:



We are interested in obtaining the approximate size of the patch; its direction will be $\hat{\mathbf{n}}$, which allows us to form the vector area. Note that the length of the arc in the θ direction defining the vertical extent of the patch is just $d\theta$, corresponding to the fact that we have a unit sphere (recall the standard formula 'radius times angle' gives arc length). On the other hand, the length of the arc in the ϕ direction depends on the height we are above the xy-plane; at an angle θ, the arc is part of a circle of radius $\sin(\theta)$, so the length of the arc is approximately $\sin(\theta)d\phi$. Thus the area of the patch is approximately $\sin(\theta)d\theta d\phi$, and so the vector area is approximately $d\mathbf{S} = \sin(\theta)d\theta d\phi\hat{\mathbf{n}}$, as required.

We could *integrate* this over part of the sphere to give its vector area. We shall do this a lot more in Lent term when we introduce the idea of *surface integration*.

Real and imaginary parts

8. Find the real and imaginary parts of the following numbers (where n is an integer):

(a) i^3, (b) i^{4n}, (c) $\left(\frac{1+i}{\sqrt{2}}\right)^2$, (d) $\left(\frac{1-i}{\sqrt{2}}\right)^2$, (e) $\left(\frac{1+\sqrt{3}i}{2}\right)^3$, (f) $\frac{1+i}{2-5i}$, (g) $\left(\frac{1+i}{1-i}\right)^2$.

Solution:

(a) We have $i^3 = i^2 \cdot i = -i$. Hence the real part is 1 and the imaginary part is -1 (recall that the imaginary part does *not* include i).

(b) We have $i^{4n} = (i^2)^{2n} = (-1)^{2n} = 1$, since an even power of -1 is always 1. Hence the real part is 1 and the imaginary part is 0.

(c) Carrying out the multiplication, we have:

$$\left(\frac{1+i}{\sqrt{2}}\right)^2 = \left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{1+i}{\sqrt{2}}\right) = \frac{1+2i-1}{2} = i.$$

Thus, the real part is 0 and the imaginary part is 1. In particular, we learn that $\pm(1+i)/\sqrt{2}$ are the square roots of i.

(d) Carrying out the multiplication, we have:

$$\left(\frac{1-i}{\sqrt{2}}\right)^2 = \left(\frac{1-i}{\sqrt{2}}\right)\left(\frac{1-i}{\sqrt{2}}\right) = \frac{1-2i-1}{2} = -i.$$

Thus, the real part is 0 and the imaginary part is -1. In particular, we learn that $\pm(1-i)/\sqrt{2}$ are the square roots of $-i$.

(e) Using the binomial expansion, we have:

$$\left(\frac{1+\sqrt{3}i}{2}\right)^3 = \frac{1+3\sqrt{3}i-9-3\sqrt{3}i}{8} = -1.$$

Thus, the real part is -1 and the imaginary part is 0. In particular, we learn that $(1+\sqrt{3}i)/2$ is one of the cube roots of -1.

(f) Realising the denominator, we have:

$$\frac{1+i}{2-5i} = \frac{(1+i)(2+5i)}{4+25} = \frac{2-5+(5+2)i}{29} = -\frac{3}{29} + \frac{7i}{29}.$$

Thus, the real part is $-3/29$ and the imaginary part is $7/29$.

(g) Realising the denominator, then squaring, we have:

$$\left(\frac{1+i}{1-i}\right)^2 = \left(\frac{(1+i)(1+i)}{2}\right)^2 = \left(\frac{2i}{2}\right)^2 = i^2 = -1.$$

Thus, the real part is -1 and the imaginary part is 0.

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9. If $z = x + iy$, find the real and imaginary parts of the following functions in terms of x and y:

(a) z^2, (b) iz, (c) $(1+i)z$, (d) $z^2(z-1)$, (e) $z^*(z^2 - zz^*)$.

➻ **Solution:** (a) We have:

$$z^2 = (x+iy)^2 = x^2 + 2xyi - y^2 = (x^2 - y^2) + 2xyi,$$

so that the real part is $x^2 - y^2$ and the imaginary part is $2xy$.

(b) We have:

$$iz = i(x+iy) = -y + ix,$$

so that the real part is $-y$ and the imaginary part is x.

(c) We have:

$$(1+i)z = (1+i)(x+iy) = x - y + (x+y)i,$$

so that the real part is $x-y$ and the imaginary part is $x+y$.

(d) We have:

$$z^2(z-1) = (x+iy)^2((x-1)+iy) = (x^2-y^2+2xyi)((x-1)+iy) = (x^2-y^2)(x-1)-2xy^2+\big((x^2-y^2)y + 2xy(x-1)\big)\,i.$$

Hence the real part is:

$$(x^2-y^2)(x-1) - 2xy^2 = x^3 - x^2 + y^2 - 3xy^2.$$

The imaginary part is:

$$(x^2-y^2)y + 2xy(x-1) = x^2y - y^3 + 2x^2y - 2xy.$$

(e) Recall that the complex conjugate of $z = x+iy$ is given by $z^* = x - iy$. Hence we have:

$$z^*(z^2 - zz^*) = z^*z(z - z^*) = (x-iy)(x+iy)(x+iy-(x-iy)) = 2iy(x^2+y^2).$$

Thus, the real part is 0 and the imaginary part is $2y(x^2+y^2)$.

10. Define u and v to be the real and imaginary parts, respectively, of the complex function $w = 1/z$. Show that the contours of constant u and v are circles. Show also that the contours of u and the contours of v intersect at right angles.

➾ **Solution:** Let $z = x + iy$. Then:

$$w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2},$$

by realising the denominator. This implies that:

$$u = \frac{x}{x^2+y^2}, \qquad v = -\frac{y}{x^2+y^2}.$$

The contours of constant u are the surfaces where u is constant. Hence treating u as a constant, and rearranging the equation we obtained for u, we see that such surfaces satisfy:

$$x^2 + y^2 = \frac{x}{u} \qquad \Rightarrow \qquad \left(x - \frac{1}{2u}\right)^2 + y^2 = \frac{1}{4u^2},$$

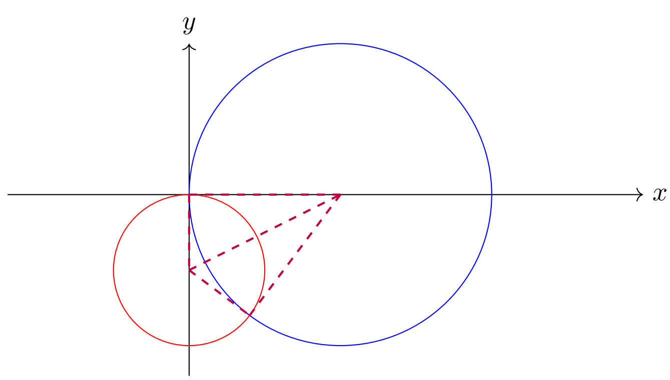
so that they are circles with centre $(1/2u, 0)$ and radius $1/2u$. Similarly, the contours of constant v obey the equation:

$$x^2 + y^2 = -\frac{y}{v} \qquad \Rightarrow \qquad x^2 + \left(y + \frac{1}{2v}\right)^2 = \frac{1}{4v^2},$$

so that they are circles with centre $(0, -1/2v)$ and radius $1/2v$.

The only exception is when $u = 0$ or $v = 0$. In these cases, we have either $x = 0$ or $y = 0$, where the contours of constant u, v are just the coordinate axes themselves.

To show that the contours intersect at right angles, we use some geometry. Below, we take $u = 1$ and $v = 2$.



Any pair of these circles intersect at the origin, where the tangents to the circles are the $x = 0$ and $y = 0$ coordinate axes respectively. Thus, they intersect at right angles there.

At their other intersection, we can draw radii to form two similar triangles (shown in dashed purple lines in the above figure), implying that the other intersection also takes place at right angles.

Factoring polynomials and solving equations

11. Factorise the following expressions: (a) z^2+1; (b) z^2-2z+2; (c) z^2+i; (d) $z^2+(1-i)z-i$. *[Hint: you have already computed the two square roots of i in Question 1(c).]*

➼ **Solution:** Throughout this question, we use the identity for the *sum of two squares*, $a^2+b^2=(a+ib)(a-ib)$, which applies for all *complex* numbers a,b.

(a) The first expression is the sum of two squares, so can be factorised simply as $z^2+1=(z+i)(z-i)$.

(b) For the second expression, we complete the square, and then use the fact that the resulting expression is the sum of two squares:
$$z^2-2z+2=(z-1)^2+1=(z-1+i)(z-1-i).$$

(c) We computed the square roots of i in Question 1(c), so we can again use the fact that the expression is the sum of two squares here:
$$z^2+i=\left(z+\frac{1+i}{\sqrt{2}}i\right)\left(z-\frac{1+i}{\sqrt{2}}i\right)=\left(z+\frac{i-1}{\sqrt{2}}\right)\left(z+\frac{1-i}{\sqrt{2}}\right)$$

(d) Again, completing the square, we have:
$$z^2+(1-i)z-i=\left(z+\frac{1-i}{2}\right)^2-\frac{1-2i-1}{4}-i=\left(z+\frac{1-i}{2}\right)^2-\frac{1}{2}i.$$

This is the difference of two squares. Using the square roots of i that we computed in Question 1(c), we can factorise this as:
$$\left(z+\frac{1-i}{2}-\frac{1+i}{2}\right)\left(z+\frac{1-i}{2}+\frac{1+i}{2}\right)=(z-i)(z+1),$$

which is pleasantly simple!

---

12. Given that $z=2+i$ solves the equation $z^3-(4+2i)z^2+(4+5i)z-(1+3i)=0$, find the remaining solutions.

➼ **Solution:** We factorise the left hand side. We have:
$$\begin{aligned}z^3-(4+2i)z^2+(4+5i)z-(1+3i)&=(z-(2+i))\left(z^2-(2+i)z-\frac{1+3i}{2+i}\right)\\&=(z-(2+i))\left(z^2-(2+i)z-(1+i)\right).\end{aligned}$$

We can now apply the quadratic formula to find the roots of the second factor. The roots are:
$$z_\pm=\frac{2+i\pm\sqrt{(2+i)^2-4(1+i)}}{2}=\frac{2+i\pm i}{2},$$

which give the roots $1+i$ and 1. Hence the complete set of solutions to the cubic is $\{2+i,1+i,1\}$.

13. Consider the polynomial equation $a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0$, where the coefficients $a_n, a_{n-1}, ..., a_0$ are real. Show that the solutions to this equation come in complex conjugate pairs. Deduce that if n is odd, there is at least one real solution.

➼ **Solution:** For this question, we will need to use the following important properties of complex conjugation:

Proposition: Let z, w be complex numbers. The following properties hold:

(i) $(z + w)^* = z^* + w^*$;

(ii) $(zw)^* = z^* w^*$.

Proof: In each case, we just write things out in terms of Cartesian components. Let $z = x + iy$ and $w = u + iv$. Then we have:

(i) $(z+w)^* = (x+iy+u+iv)^* = ((x+u)+i(y+v))^* = (x+u)-i(y+v) = (x-iy)+(u-iv) = z^*+w^*$.

(ii) $(zw)^* = ((x + iy)(u + iv))^* = ((xu - yv) + i(xv + yu))^* = (xu - yv) - i(xv + yu)$.

Now let's start the question proper. Let z be a solution of the equation:

$$a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0.$$

Let's take the complex conjugate of the entire equation:

$$\left(a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0\right)^* = 0.$$

Using the properties of complex conjugation, and the fact that the coefficients are real (i.e. $a_k^* = a_k$), we can rewrite the left hand side as:

$$a_n (z^*)^n + a_{n-1} (z^*)^{n-1} + ... + a_1 (z^*) + a_0 = 0.$$

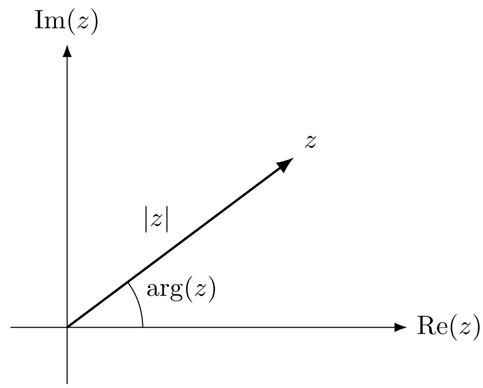
Hence, we see that z^* satisfies the same equation. Thus if z satisfies a polynomial equation with real coefficients, then its complex conjugate also satisfies the same equation.

We have just shown that complex solutions (with a non-zero imaginary part) of real polynomial equations come in *pairs*. Hence, for an odd degree equation which has an odd number of roots, one of them must be purely real.

Geometry of complex numbers

14. Using a diagram, explain the geometric meaning of the *modulus*, $|z|$, and *argument*, $\arg(z)$, of a complex number z. Find the moduli and (principal) arguments of: (a) $1+\sqrt{3}i$; (b) $-1+i$; (c) $-\sqrt{3}-i/\sqrt{3}$.

➾ **Solution:** Let $z = x + iy$ be a complex number. We can associate this complex number with a point (x, y) in the plane, where the x-axis is the real axis and the y-axis is the imaginary axis. In this context, the plane is called an *Argand diagram*.



The *modulus* of $z = x + iy$ is the length of the vector joining the origin to the point (x, y). By Pythagoras' theorem, we have $|z| = \sqrt{x^2 + y^2}$. The *argument* of $z = x + iy$ is the angle between the positive x-axis (i.e. the positive real axis) and the vector joining the origin to the point (x, y). Of course, there are multiple choices of angle; standard choices include the range $[0, 2\pi)$ or the range $[-\pi, \pi)$. Both are sometimes called the 'principal' choice of argument.

For the given complex numbers, we have:

(a) The modulus is $\sqrt{1+3} = 2$. The argument is:

$$\arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}.$$

(b) The modulus is $\sqrt{(-1)^2 + 1^2} = \sqrt{2}$. The complex number is in the second quadrant, so the argument is:

$$\pi - \arctan\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

(c) The modulus is $\sqrt{3 + 1/3} = \sqrt{11/3}$. The complex number is in the third quadrant, so the argument is:

$$\pi + \arctan\left(\frac{1/\sqrt{3}}{\sqrt{3}}\right) = \pi + \arctan\left(\frac{1}{3}\right),$$

in the range $[0, 2\pi)$, or alternatively:

$$\arctan\left(\frac{1}{3}\right) - \pi,$$

in the range $[-\pi, \pi)$.

15. For $z \in \mathbb{C}$, show that $|z|^2 = zz^*$. Hence prove that $|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2)$, where $a, b \in \mathbb{C}$, and interpret this result geometrically. *[Hint: you don't need to split a, b into real and imaginary parts.]*

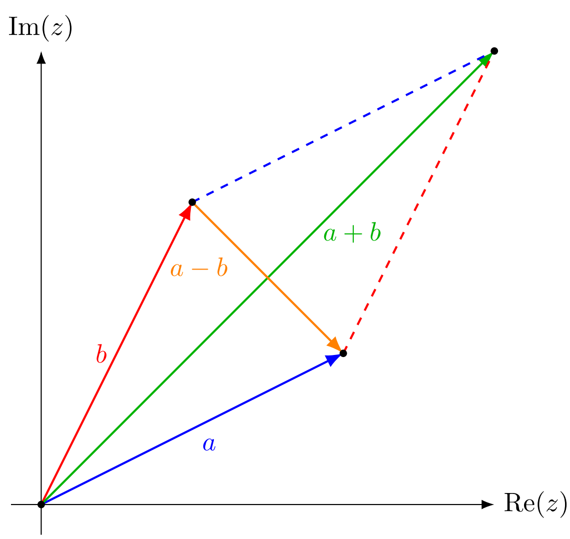
Solution: Let $z = x + iy$. Then $|z|^2 = x^2 + y^2$, whilst $zz^* = (x+iy)(x-iy) = x^2 + y^2$, so $|z|^2 = zz^*$ as required.

To prove the given identity, we use the fact we just showed about moduli. We have:

$$\begin{aligned} |a+b|^2 + |a-b|^2 &= (a+b)(a^* + b^*) + (a-b)(a^* - b^*) \\ &= aa^* + ab^* + ba^* + bb^* + aa^* - ab^* - ba^* + bb^* \\ &= 2aa^* + 2bb^* \\ &= 2\left(|a|^2 + |b|^2\right), \end{aligned}$$

as required.

This result shows that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides of the parallelogram. This is shown in the diagram below, illustrating the parallelogram formed by the complex numbers a, b. The sum of its side lengths squared given by $|a|^2+|b|^2+|a|^2+|b|^2 = 2(|a|^2+|b|^2)$, and the sum of its diagonal lengths squared is $|a+b|^2 + |a-b|^2$.



16. By writing $z = |z|(\cos(\arg(z)) + i\sin(\arg(z)), w = |w|(\cos(\arg(w)) + i\sin(\arg(w))$, compute the modulus and argument of the product zw. Hence give the geometrical interpretation of multiplying one complex number by another complex number. Give also a geometrical interpretation of division of one complex number by another complex number, z/w.

Solution: We have:

$$
\begin{aligned}
zw &= |z||w|\left(\cos(\arg(z)) + i\sin(\arg(z))\right)\left(\cos(\arg(w)) + i\sin(\arg(w))\right)\\
&= |z||w|\left(\cos(\arg(z))\cos(\arg(w)) - \sin(\arg(z))\sin(\arg(w)) + \left(\cos(\arg(z))\sin(\arg(w)) + \sin(\arg(z))\cos(\arg(w))\right)i\right)\\
&= |z||w|\left(\cos(\arg(z) + \arg(w)) + \sin(\arg(z) + \arg(w))i\right),
\end{aligned}
$$

where in the last line we used the compound angle formulae for cosine and sine. It follows that the modulus and argument of the product zw are given by:

$$|zw| = |z||w|, \qquad \arg(zw) = \arg(z) + \arg(w).$$

In particular, we see that multiplying the complex number z by the complex number w results in a *scaled rotation* of z. Indeed, z is scaled by a factor of $|w|$ and rotated by an angle $\arg(w)$ about the origin anticlockwise.

For division, we can view the quotient z/w as the product $z \cdot (1/w)$. The complex number $1/w$ satisfies:

$$\frac{1}{w} = \frac{1}{|w|}\cdot\frac{1}{\cos(\arg(w)) + \sin(\arg(w))i} = \frac{1}{|w|}\frac{\cos(\arg(w)) - \sin(\arg(w))i}{\cos^2(\arg(w)) + \sin^2(\arg(w))} = \frac{1}{|w|}\left(\cos(-\arg(w)) + \sin(-\arg(w))i\right).$$

Hence, we see that $|1/w| = 1/|w|$ and $\arg(1/w) = -\arg(w)$. It follows from the multiplication rule we proved above that:

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}, \qquad \arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w).$$

In particular, division still corresponds to a scaled rotation. However, z is now scaled *down* by a factor of $|w|$ instead of being scaled up, and the rotation is by $\arg(w)$ about the origin *clockwise*.

17. Let $z_1 = 2+i$, $z_2 = 3+4i$. Find z_1z_2 by: (a) adding arguments and multiplying moduli; (b) using the rules of complex algebra. Verify that your results agree.

➾ **Solution:** (a) The moduli of these complex numbers are $|z_1| = \sqrt{2^2+1^2} = \sqrt{5}$ and $|z_2| = \sqrt{3^2+4^2} = 5$. The arguments of these complex numbers are $\arg(z_1) = \arctan(1/2)$ and $\arg(z_2) = \arctan(4/3)$. Hence the modulus and argument of the product are given by:

$$|z_1z_2| = 5\sqrt{5}, \qquad \arg(z_1z_2) = \arctan(1/2) + \arctan(4/3).$$

To obtain a Cartesian expression for z_1z_2, we will need to compute the cosine and sine of the argument of z_1z_2, which seems to be quite hard at the moment! To make things easier, we need to *add the arctangents first*. We could do this using the result of Question 11. For practice though, we shall present an alternative method. Let:

$$t = \arctan(1/2) + \arctan(4/3),$$

and consider taking the tangent of both sides and applying the compound angle formula for tangent:

$$\tan(t) = \tan\left(\arctan(1/2) + \arctan(4/3)\right) = \frac{\tan(\arctan(1/2)) + \tan(\arctan(4/3))}{1 - \tan(\arctan(1/2))\tan(\arctan(4/3))} = \frac{1/2+4/3}{1-2/3} = \frac{11}{2}.$$

Hence, we see that:

$$t = \arctan(11/2).$$

This is the angle between the adjacent and hypotenuse of a right-angled triangle with opposite side of length 11 and adjacent side of length 2. The hypotenuse is of length $\sqrt{11^2+2^2} = \sqrt{125} = 5\sqrt{5}$. Hence:

$$5\sqrt{5}\cos(\arctan(11/2)) = 2, \qquad 5\sqrt{5}\sin(\arctan(11/2)) = 11.$$

It follows that:

$$z_1z_2 = 2+11i.$$

(b) We can also compute the product using the rules of complex algebra. We have:

$$z_1z_2 = (2+i)(3+4i) = 6-4+(8+3)i = 2+11i,$$

which agrees perfectly. Much simpler!

18. By considering multiplication of the complex numbers $z = 1 + iA$ and $w = 1 + iB$, derive the arctangent addition formula:

$$\arctan(A) + \arctan(B) = \arctan\left(\frac{A+B}{1-AB}\right).$$

➾ **Solution:** The argument of z is $\arctan(A)$, whilst the argument of w is $\arctan(B)$. Multiplication of complex numbers results in addition of their arguments, so the argument of zw is $\arctan(A) + \arctan(B)$, which is the left hand side of the arctangent addition formula.

On the other hand, we can instead first compute the product of the complex numbers algebraically. We have

$$zw = (1 + iA)(1 + iB) = 1 - AB + (A + B)i.$$

This has argument:

$$\arctan\left(\frac{A+B}{1-AB}\right).$$

Hence, the arctangent addition formula follows.

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19. Give a geometrical interpretation (in terms of *vectors*) of the real and imaginary parts of the quantity $Q = z_1 z_2^*$. Show also that Q is invariant under a rotation of z_1, z_2 about the origin, and confirm that this is consistent with your geometrical interpretation. *[Hint: In Question 9, you showed that multiplying by a complex number u of unit modulus is equivalent to a rotation about the origin.]*

➾ **Solution:** Let $z_1 = x_1 + iy_1$ and let $z_2 = x_2 + iy_2$. Then the given quantity is:

$$Q = (x_1 + iy_1)(x_2 - iy_2) = x_1x_2 + y_1y_2 + (x_2y_1 - x_1y_2)i.$$

The real part is $x_1x_2 + y_1y_2$, which is the scalar product of the vectors $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$. The imaginary part is $x_2y_1 - x_1y_2$, which is the z-component of the cross product of the vectors $(x_2, y_2, 0)$ and $(x_1, y_1, 0)$; this is also the magnitude of this cross product since it points purely in the z-direction.

A rotation about the origin is equivalent to multiplication by a unit modulus complex number, u, satisfying $|u|^2 = uu^* = 1$. Hence under such a rotation, we have $z_1 \mapsto uz_1$ and $z_2 \mapsto uz_2$. This gives:

$$Q = z_1 z_2^* \mapsto (uz_1)(uz_2)^* = uu^* z_1 z_2^* = z_1 z_2^* = Q.$$

Hence Q is invariant under such a rotation. This is consistent with the geometric interpretation we gave earlier in the question, since the scalar product won't change under a rotation (it depends on lengths and angles, which are preserved under a rotation), and the cross produce won't change under a rotation (since the rotation is in the complex plane, and therefore the direction of the cross product will still be out of the complex plane, in the z-direction).