Part IA: Mathematics for Natural Sciences A
Examples Sheet 10: First-order ordinary differential equations

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Separable equations

1. Explain what is meant by a *separable differential equation*, and how we can solve one. Find the general solution of the following separable differential equations:

   (a) $\dfrac{dy}{dx} = x,$ (b) $\dfrac{dy}{dx} = (2-y)(1-y),$ (c) $\dfrac{dy}{dx} = -\dfrac{x^3}{(1+y)^2},$

   (d) $\dfrac{dy}{dx} = \dfrac{4y}{x(y-3)},$ (e) $\dfrac{dy}{dx} = xe^{x-2y},$ (f) $\dfrac{dy}{dx} = \sin(y+x) - \sin(y-x).$

2. Determine the half-life of thorium-234 if a sample of mass 5g is reduced to 4g in one week. What amount of thorium is left after twelve weeks?

3. *Newton's law of cooling* states that the rate of heat loss from a body is proportional to the difference between the temperature of the object and its ambient environment. Assuming that Newton's law of cooling applies, calculate the time at which a cup of tea in a $20°$C room was made, given that: (i) the tea is measured to have temperature $30°$C at 5pm; (ii) the tea is measured to have temperature $40°$C at 3pm; (iii) the water was initially at boiling point.

4. Consider the family of curves $C = \{y = ax^2 : a \in \mathbb{R}\}$. Sketch a few representative curves in C. Determine a family of curves $C' = \{y = f(x,b) : b \in \mathbb{R}\}$ such that each curve in C' is orthogonal to all curves in C, and sketch a few representative curves in the family C'.

Linear equations

5. Write down the general form of a *linear first-order differential equation*. What is meant by an *integrating factor* for such an equation? Solve the following linear first-order differential equations by finding an appropriate integrating factor:

   (a) $\dfrac{dy}{dx} + 2xy = 4x,$ (b) $\dfrac{dy}{dx} + \dfrac{y}{2-3x} = 1,$ (c) $\dfrac{dy}{dx} - y\tan(x) = 1,$

   (d) $\dfrac{dy}{dx} + (1+\log(x))y = x^{-x},$ (e) $\dfrac{dy}{dx} + \dfrac{y}{\tan(x)} = \cos^2(x),$ (f) $\dfrac{1}{x}\dfrac{dy}{dx} + y - 5e^{-x^2} = 0.$

6. Solve the equation:
$$(x^2+1)\frac{dy}{dx} + 3x^3 y = 6xe^{-3x^2/2},$$
   subject to the boundary condition $y(0) = 1$.

7. Establish a formula for the general solution of the linear differential equation:
$$\alpha(x)\frac{dy}{dx} + \beta(x)y = \gamma(x),$$
   stating any conditions you must assume for your formula to be valid.

Some 'easy to spot' substitutions

[*Hint: most 'easy' substitutions come from thinking about implicit differentiation.*]

8. Using an appropriate substitution, find the general solution of the equation:

$$y^3 + x + 3y^2\frac{dy}{dx} = 0.$$

Find also the solution satisfying the boundary condition $y(0) = 1$.

9. Using an appropriate substitution, find the general solution of the equation:

$$(x+y+1)^2\frac{dy}{dx} + (x+y+1)^2 + x^3 = 0.$$

Find also the solution satisfying the boundary condition $y(0) = 0$.

Some standard substitutions: homogeneous, Bernoulli, and affine transformations

10. Define a *homogeneous equation*, and state the substitution which renders them solvable. Hence, solve the equations:

(a) $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right)$, (b) $(y-x)\dfrac{dy}{dx} + (2x+3y) = 0$, (c) $\dfrac{dy}{dx} = \dfrac{x^3+y^3}{3xy^2}$.

11. Define a *Bernoulli equation*, and state the substitution which renders them solvable. Hence, solve the equations:

(a) $\dfrac{dy}{dx} - y = xy^5$, (b) $\dfrac{dy}{dx} + y = y^2(\cos(x) - \sin(x))$, (c) $xy\dfrac{dy}{dx} + (x^2+y^2+x) = 0$.

12. (a) Show that equations of the form:

$$\frac{dy}{dx} = f(ax+by+c),$$

with $b \neq 0$ may be reduced to a separable equation by making the substitution $u = ax+by+c$.

(b) Hence, solve the equations:

(i) $\dfrac{dy}{dx} = (4x+y)^2$, (ii) $\cos(x+y-1)\dfrac{dy}{dx} = \sin(x+y-1)$, (iii) $\dfrac{dx}{dy} = \dfrac{1}{\cosh^2(2x-y+2)+2}$.