Part IA: Mathematics for Natural Sciences B
Examples Sheet 5: Infinite series and Taylor series

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Questions marked with a ($*$) are difficult and should not be attempted at the expense of the other questions. A section marked with a (†) contains content that is unique to the Mathematics B course.

(†) Basics of infinite series

1. State clearly what it means for an infinite series to be: (i) *convergent*; (ii) *absolutely convergent*. If a series is absolutely convergent, must it be convergent? Is the converse true?

2. By evaluating the partial sums, determine whether $\sum\limits_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ is convergent. Is it absolutely convergent?

3. By evaluating the partial sums, prove that the geometric series $\sum\limits_{n=0}^{\infty} ar^n$ is absolutely convergent for $-1 < r < 1$.

(†) Tests for convergence

4. (a) Clearly state the *comparison test* for series convergence or divergence.

   (b) Using the comparison test, prove that the harmonic series diverges. Hence, show that the following definition of the Riemann zeta function:
   $$\zeta(p) := \sum_{n=1}^{\infty}\frac{1}{n^p},$$
   with p real, converges if and only if $p > 1$. Is it absolutely convergent when $p > 1$?

5. (a) Clearly state, *and prove*, the *alternating series test* for series convergence or divergence.

   (b) Hence, show that the following definition of the Dirichlet eta function:
   $$\eta(p) := \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p},$$
   with p real, converges if and only if $p > 0$. Is it absolutely convergent when $p > 0$?

6. Clearly state the *ratio test* for series convergence or divergence. Use it to show that the series:
   $$\sum_{n=1}^{\infty}\frac{2^n}{n!}$$
   is absolutely convergent. To what value does it converge? [*Hint: If you're unsure, come back after studying Taylor series.*]

7. Another test that was not lectured (but has come up in exams before!) is the *integral test*.

   (a) Suppose that $f : [k,\infty) \to \mathbb{R}$ is a continuous, positive, decreasing function, where k is an integer. By drawing a convincing diagram, show that:
   $$\int_k^{\infty} f(x)\,dx \text{ converges} \qquad \Rightarrow \qquad \sum_{n=k}^{\infty} f(n) \text{ converges.}$$
   Show also that if the integral diverges, then the series diverges. This test is called the *integral test*.

   (b) Using the integral test, reanalyse the convergence of the definition of $\zeta(p)$ given in Question 4.

(†) Miscellaneous series

[*This section contains a large collection of series from past papers for you to test convergence and absolute convergence. If you feel like you are getting too much of a good thing, feel free to save some of them for us to do together in the supervision.*]

8. Applying an appropriate test in each case, determine which of the following series are convergent, and which are absolutely convergent:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{3n^2+4}$;

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$;

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{n!}$;

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$;

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{5\cdot 8\cdot 11\cdot \ldots \cdot (3n+2)}{1\cdot 5\cdot 9\cdot \ldots \cdot (4n-3)}$;

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$;

(g) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{2n+5}{3n+1}\right)^n$;

(h) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\log(n)}$;

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{3^n}$;

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}$;

(k) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2\log(n)}$;

(l) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2n}{n^3+3n^2+1}$;

(m) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{1+n^2}$;

(n) $\displaystyle\sum_{n=0}^{\infty} \frac{a^{2n+1}}{2n+1}$, where $a>0$;

(o) $\displaystyle\sum_{n=1}^{\infty} n^p \sin(\omega n)$ where $\omega > 0$, and $p < -1$;

(p) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi)}{n}$;

(q) $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{n\log(n)}$;

(r) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$;

(s) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{\log^n(2)}$;

(t) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n^4+a^2}-n^2\right)$, for $a>0$;

(u) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n-1}$;

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3 e^{3n}}{(3n)!}$.

Taylor series

9. Carefully state *Taylor's theorem*, giving Lagrange's formula for the remainder term. Hence, obtain the first three non-zero terms in the Taylor series of $\log(x)$ about $x=1$ by direct differentiation. Using this expansion, together with Lagrange's form of the remainder, show that:

$$|\log(3/2) - 5/12| \leq 1/64,$$

and hence give an approximation of $\log(3/2)$ valid to one decimal place.

10. Write down the Taylor series about $x=0$ for the following functions, finding their range of convergence by appropriate tests in each case:

(a) e^x, (b) $\log(1+x)$, (c) $\sin(x)$, (d) $\cos(x)$, (e) $\sinh(x)$, (f) $\cosh(x)$, (g) $(1+x)^a$,

where in the final part $a\in\mathbb{R}$ is any real number (ignore endpoints of the range of convergence in the final case, where convergence is subtle). What happens when a is a non-negative integer? Learn these series off by heart, and get your supervision partner to test you on them.

11. *Without differentiating*, find the first three terms in the Taylor series of the following functions. [*Note: there are lots of examples from past papers here to practise with, but if you are getting bored, we can do some in the supervision together. The next few questions, 12-18, have more of a problem-solving element.*]

(a) $\dfrac{1}{\sqrt{1+x}}$ about $x=0$;

(b) $\dfrac{1}{(x^2+2)^{3/2}}$ about $x=0$;

(c) $\tan(x)$ about $x=0$;

(d) $\log(\cos(x))$ about $x=0$;

(e) $\arcsin(x)$ about $x=0$;

(f) $\arctan(x)$ about $x=1$;

(g) $(\cosh(x))^{-1/2}$ about $x=0$;

(h) $e^{\sin(x)}$ about $x=\pi/2$;

(i) $x\sinh(x^2)$ about $x=0$;

(j) $\log(1+\log(1+x))$ about $x=0$;

(k) $\sin^6(x)$ about $x=0$;

(l) $\dfrac{\cosh(x)}{\cos(x)}$ about $x=0$;

(m) $\cosh(\log(x))$ about $x=2$;

(n) $\log(2-e^x)$ about $x=0$;

(o) $\dfrac{\sin(x)}{\sinh(x)}$ about $x=0$;

(p) $\sinh(\log(x))$ about $x=1$;

(q) $\sin\left(\dfrac{\pi e^x}{2}\right)$ about $x=0$;

(r) $\dfrac{\sinh(x+1)}{x+2}$ about $x=-1$;

(s) $\dfrac{\log(1+x^3)}{\cosh(x)}$ about $x=0$;

(t) $\dfrac{\cosh(x)}{\sqrt{1+x^2}}$ about $x=0$;

(u) $\dfrac{e^{-x^2}}{\cosh(x)}$ about $x=0$;

(v) $\dfrac{\log(2+x)}{2-x}$ about $x=0$;

(w) $\log(\cosh(x))$ about $x=0$;

(x) $\cosh(\sqrt{x})$ about $x=2$;

(y) $\dfrac{\sin(x)}{(1+x)^2}$ about $x=0$;

(z) $\dfrac{x\sin(x)}{\log(1+x^2)}$ about $x=0$;

(a') $\cos\left(\sqrt{\dfrac{\pi^2}{16}+x}\right)$ about $x=0$;

(b') $\log((2+x)^3)$ about $x=0$.

12. *Without differentiating*, find the value of the thirty-second derivative of $\cos(x^4)$ at $x=0$.

13. Find the first three non-zero terms in a series approximation of $\log(1+x+2x^2)-\log(x^2)$ valid for $x\to\infty$.

14. Let $f(x)$ be a function which can be expanded as a Taylor series about $x=0$. Find the first two terms in the Taylor series of the function $\log(1+f(x))$ about $x=0$, assuming that $1+f(0)>0$, $f'(0)\neq 0$ and $f''(0)(1+f(0))\neq (f'(0))^2$. Why are these conditions necessary?

15. Let $f(x)=a_0+a_1x+a_2x^2+\cdots$ be the Taylor series of $f(x)$ about $x=0$, with $a_0>0$, $a_1\neq 0$, $a_1^2\neq a_2a_0$ and $a_1^2\neq 4a_2a_0$. Find the first three terms in the Taylor series of (a) $1/f(x)$ about $x=0$; (b) $\sqrt{f(x)}$ about $x=0$. Explain where you used the assumptions on the a_n in your answer.

16. (†) By considering a Taylor series expansion in each case, evaluate the limits:

(a) $\displaystyle\lim_{x\to 0}\frac{\tan(x)-\tanh(x)}{\sinh(x)-x}$, (b) $\displaystyle\lim_{x\to 0}\left(\frac{\operatorname{cosec}(x)}{x^3}-\frac{\sinh(x)}{x^5}\right)$.

17. (a) Using the Taylor series for $\log(1+x)$, show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots = \log(2)$.
    (b) ($*$) Hence, by an appropriate sequence of transformations of the series in part (a), show that:
    $$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2}\log(2).$$
    (c) ($*$) Comment on this result in relation to absolute convergence. [*Look up the Riemann rearrangement theorem!*]
18. ($*$) Sketch the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-1/x^2}$ for $x \neq 0$, and $f(0) = 0$. Show that this function is infinitely differentiable at $x = 0$, and find its Taylor series at $x = 0$. Comment on the general utility of Taylor series.

(†) Landau's big-O notation

19. Give the formal definition of Landau's big-O notation, '$f(x) = O(g(x))$ as $x \to x_0$', including the cases where $x_0 = \pm\infty$. Decide which of the following statements are true, justifying your reasoning with careful proofs:

    (a) $x = O(x^2)$ as $x \to 0$, (b) $x^2 = O(x)$ as $x \to 0$, (c) $x = O(x^2)$ as $x \to \infty$, (d) $x^2 = O(x)$ as $x \to \infty$.
20. Give the leading terms in an approximation to each of the following functions in the given limits, indicating the leading behaviour of the remainder in Landau's big-O notation:

    (a) $\dfrac{x^3 + x}{x + 2}$ as $x \to 0$, (b) $\dfrac{\cos(x) - 1}{x^3}$ as $x \to 0$, (c) $\dfrac{1 + 2x + 2x^2}{3x + 3}$ as $x \to \infty$.
21. Show that $(x^3 + x^2 + 1)^{1/3} - (x^2 + x)^{1/2} = -\frac{1}{6} + \frac{1}{72x} + O\left(\frac{1}{x^2}\right)$ as $x \to \infty$.

Newton-Raphson root finding

22. Give an explanation of the Newton-Raphson algorithm for root finding, including an appropriate sketch. Under what general conditions is it guaranteed that Newton-Raphson will converge to the root of interest? Prove that, when it converges to the root of interest, the Newton-Raphson method enjoys *quadratic convergence*.
23. (a) Find the value of the first iterate of Newton-Raphson iteration for the function $f(x) = x - 2 + \log(x)$ with a starting guess of $x_0 = 1$.
    (b) Find the value of the first and second iterates of Newton-Raphson iteration, valid to two decimal places, for the function $f(x) = x^2 - 2$ with a starting guess of $x_0 = 1$.

    [*Both parts of this question are based on old (short) tripos questions, so try doing them*without *a calculator!*]
24. [*You may use a calculator for this question, but remember that you* won't *be able to use a calculator in the exam. Newton-Raphson questions will be more theoretical in the exams, like the next question, or involve easy calculations, like the previous question*.]
    (a) Sketch the graph of $f(x) = x^3 - 3x^2 + 2$, indicating the coordinates of the turning points and the coordinates of the intersections with the x-axis.
    (b) Use Newton-Raphson with an initial guess of $x_0 = 2.5$ to find an estimate of the largest root of the equation $f(x) = 0$, accurate to 5 decimal places. Draw a sketch showing the progress of the algorithm.
    (c) To which roots (if any) does the algorithm converge if we instead start at: (i) $x_0 = 1.5$; (ii) $x_0 = 1.9$; (iii) $x_0 = 2$?
25. The real function f is defined by $f(x) = x^2 - 2\epsilon x - 1$, where ϵ is a small positive parameter ($0 < \epsilon \ll 1$). Let x_i be the ith Newton-Raphson iterate, with a starting guess of $x_0 = 1$, and let x_* be the unique positive root satisfying $f(x_*) = 0$. By Taylor expansion, show that $|x_i - x_*| \propto \epsilon^{n_i}$, where: (a) $n_0 = 1$; (b) $n_1 = 2$; (c) $n_2 = 4$.
26. ($*$) Consider the cubic equation $x^3 - 2x + 2 = 0$. Perform a numerical investigation (for example, by writing some simple code) to determine the ranges in $\mathbb{R}$ which converge to the various roots, if any. Comment on the sensitivity of Newton-Raphson to the choice of initial guess x_0. [*Afterwards, look up Newton fractals - it is particularly interesting to see the behaviour of Newton-Raphson in the complex plane!*]