Part IA: Mathematics for Natural Sciences A and B
Examples Sheet 0: Basic skills

Model Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

Making friends

1. Speak to your supervision partner, and, if you don't know already, find out: (a) where they are from; (b) their favourite food; (c) what they like to do to relax; (d) what part of the maths course they are most excited about this year.

➼ **Solution:** I'll tell you my answers, for fun:

(a) I'm from a town called Lowestoft, in Suffolk. It's the most easterly place in Britain, and has a rather nice beach!

(b) It's hard to say, but I do love trifle as a dessert.

(c) I like playing piano, reading (mostly science-fiction), cooking and baking, and sometimes running or bouldering.

(d) The part of the first-year maths course that I like the most is the limits and continuity topic in Maths B - it's unusual to see this in a science course, and I think it helps to build an appreciation of what more formal mathematics is about.

So, what was the point of this question? Like all of your courses this year, the maths course is *hard*. It is worth making some friends who are doing the course, so you can discuss the problem sheets together. If you can't do a question, ask someone else in your cohort if they have done it. If someone else in your cohort asks you how to do a problem, try to give them some help if you have time. You should treat the other Natural Scientists (and Chemical Engineers) as team members - try to work together and help each other through the year!

Writing mathematics

2. Read Gareth Wilkes' document 'A Brief Guide to Mathematical Writing', available at: https://www.dpmms.cam.ac.uk/~grw46/Writing_Guide.pdf. [*You can ignore Section 4 for now - but we will study quantifiers in the Maths B course, if you are taking it.*] Hence, make a list of things that you should consider when writing a solution to a mathematics problem.

➼ **Solution:** Some things we should think about are the following:

- **Rewrite your work.** Often, you won't produce a perfect answer to a problem on a first try. Rewriting your work can help organise your ideas so that they are clearer for the reader (I hasten to add, obviously this is something you can't do in an exam setting though!).
- **Write your work as a linear document.** Avoid having lots of arrows on the page, saying 'see this part', or little boxes with extra additional calculations. Your work should flow in a natural way, particularly for very long calculations!
- **State results you are using, and check that they apply.** If you are quoting a result from the course, say what result it is (for example, saying '*using the standard expansion of* $\sinh(x)$ *around* $x = 0$' or saying '*by Stokes' theorem*'). Also, make sure to check the conditions apply (for example, saying '*L'Hôpital's rule applies here because we have an indeterminate form* $0/0$').

- **Diagrams are exceptionally useful.** A lot of parts of the course we will study involve lots of geometry. Having useful diagrams can help you, help the reader, and help the examiner at the end of the year!
- **Use words, where appropriate.** Mathematics should not just be a list of equations one after the other. It can be *extremely* helpful to yourself and the reader to make small remarks telling them *what you are doing*. When you are proving something is true, you can explain why things are true (‘*because **x**, **y** are not linearly independent*’, ‘*because $\sqrt{2}$ is irrational*’, etc) and you can state your assumptions clearly (‘*if A is an $n \times n$ matrix, then A^T is an $n \times n$ matrix*’, etc). In a long calculation, putting notes in the margin can be very useful (‘*integrating by parts*’, ‘*making the substitution $u = x^2$*’, etc).
- **Format your work sensibly.** Try to separate things using paragraphs and indentation. Don’t let everything become one large continuous paragraph! Equations, in particular, should be separated from text by a line break. When trying to prove something, it can also be exceptionally useful to use the format:

  Claim: ...

  Proof: ... . □

  The little box □ means the proof has been completed (you can also write QED if you want to seem more like a 17th century mathematician). This format makes it extremely clear *what* you are trying to prove, what the assumptions are in the proof, and clearly gives space for you to write the proof itself.

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3. Write down your best and most presentable model solution to the following problem:

   Let L be a line passing through the origin with gradient k. Let C be a circle centred on $(2, 0)$ with radius 1. Determine the values of k for which L and C intersect at zero points, one point, or two points: (a) using an algebraic method; (b) using a geometric method.

Compare your solution with your supervision partner, and give each other advice and feedback.

➻ **Solution:** Here is a model solution to this problem.

(a) Algebraic solution. The line L has Cartesian equation $y = kx$, and the circle C has Cartesian equation $(x-2)^2 + y^2 = 1$. To find their intersections algebraically, we solve these equations simultaneously. Substituting $y = kx$ into the equation of the circle, we have:

$$\begin{aligned}(x-2)^2 + k^2x^2 = 1 \quad &\Rightarrow \quad x^2 - 4x + 4 + k^2x^2 = 1 \\ &\Rightarrow \quad (1 + k^2)x^2 - 4x + 3 = 0. \qquad (*)\end{aligned}$$

To determine the number of solutions to this equation, we consider the discriminant of this quadratic, given by:

$$\begin{aligned}16 - 12(1 + k^2) &= 4 - 12k^2 \\ &= 4(1 - 3k^2).\end{aligned}$$

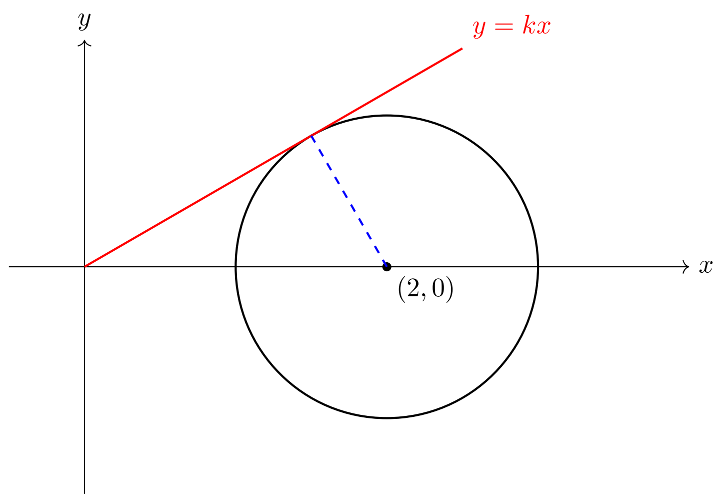
Now, we consider the possible cases:

- Case 1. If the discriminant is negative, there are no solutions to the quadratic equation $(*)$, and hence no intersections. This occurs if $|k| > 1/3$.

- Case 2. If the discriminant is zero, there is exactly one (repeated) solution to the quadratic equation ($*$), and hence one value of x which corresponds to an intersection. Substituting this value into $y = kx$ will give exactly one corresponding value of y, and hence a single intersection. This occurs if $|k| = 1/3$.
- Case 3. If the discriminant is negative, there are two solutions to the quadratic equation ($*$), which each correspond to a single value of y from the equation $y = kx$. This gives exactly two intersections. This occurs if $|k| < 1/3$.

In summary, we have no intersections if $|k| > 1/3$, one intersection if $|k| = 1/3$, and two intersections if $|k| < 1/3$.

(b) Geometric method. To solve this problem geometrically, it will first be useful to produce a diagram. Below, we show the circle and the line $y = kx$ in the case where it is tangent to the circle and $k > 0$ (there is also a case where $k < 0$ and the line is tangent to the circle).



In the case where the line $y = kx$ is tangent to the circle, and $k > 0$, the points $(0, 0)$, $(2, 0)$ and the point of tangency form a right-angled triangle. The length from the origin to the point of tangency is given by Pythagoras' theorem as:

$$\sqrt{2^2 - 1^2} = \sqrt{3}.$$

Hence, using the definition of $\tan(\theta)$ for a right-angled triangle, the angle θ formed by the tangent and the x-axis is given by:

$$\tan(\theta) = \frac{1}{\sqrt{3}}.$$

This implies that the gradient of the line $y = kx$ is $1/\sqrt{3}$. Hence there is precisely one intersection when $|k| = 1/\sqrt{3}$ (the case $k = -1/\sqrt{3}$ follows by the reflectional symmetry in the x-axis). When $|k| < 1/\sqrt{3}$, there are two intersections from the diagram, and when $|k| > 1/\sqrt{3}$ there are no intersections from the diagram.

4. Learn all the letters of the Greek alphabet, and get your supervision partner to test you on them.

Solution: The letters of the Greek alphabet, together with their names, are given below:

- α (uppercase A). This letter is called *alpha*.
- β (uppercase B). This letter is called *beta*.
- γ (uppercase Γ). This letter is called *gamma*.
- δ (uppercase Δ). This letter is called *delta*.
- ϵ (uppercase E). This letter is called *epsilon*.
- ζ (uppercase Z). This letter is called *zeta*.
- η (uppercase H). This letter is called *eta*.
- θ (uppercase Θ). This letter is called *theta*.
- ι (uppercase I). This letter is called *iota*.
- κ (uppercase K). This letter is called *kappa*.
- λ (uppercase Λ). This letter is called *lambda*.
- μ (uppercase M). This letter is called *mu*.
- ν (uppercase N). This letter is called *nu*.
- ξ (uppercase Ξ). This letter is called *xi*.
- o (uppercase O). This letter is called *omicron*.
- π (uppercase Π). This letter is called *pi*.
- ρ (uppercase P). This letter is called *rho*.
- σ (uppercase Σ). This letter is called *sigma*.
- τ (uppercase T). This letter is called *tau*.
- υ (uppercase Υ). This letter is called *upsilon*.
- ϕ (uppercase Φ). This letter is called *phi*.
- χ (uppercase X). This letter is called *chi*.
- ψ (uppercase Ψ). This letter is called *psi*.
- ω (uppercase Ω). This letter is called *omega*.

Basic logic

5. Explain the meaning of the logical symbols $\Rightarrow$, $\Leftarrow$ and $\Leftrightarrow$. [*If you haven't seen them before, look them up online! In general, you should feel free to look up terms you don't understand on an examples sheet.*] Decide which of the following are true:

(a) $x^2 \leq 1 \Rightarrow x \leq 1$;

(b) $x^2 \leq 1 \Leftarrow x \leq 1$;

(c) $x^2 \leq 1 \Leftrightarrow x \leq 1$.

Explain also the meanings of the terms *necessary condition* and *sufficient condition*. Decide which of the following are true:

(d) $|x| = 1$ is sufficient for $x = 1$;

(e) $|x| = 1$ is necessary for $x = 1$;

(f) $|x| = 1$ is necessary and sufficient for $x = 1$.

Solution: Let's start with the definition of the logical symbols:

- The symbol $\Rightarrow$ means 'implies'. If we write $A \Rightarrow B$, then it means that 'if statement A is true, then statement B is true'.
- The symbol $\Leftarrow$ means 'is implied by'. If we write $A \Leftarrow B$, then it means that 'if statement B is true, then statement A is true'.
- The symbol $\Leftrightarrow$ means 'implies and is implied by'. If we write $A \Leftrightarrow B$, then it means that 'statement A is true if and only if statement B is true'. Importantly, the logical implication needs to go both ways!

Now, let's examine the given statements:

(a) The first statement is telling us that if $x^2 \leq 1$, then $x \leq 1$. This is true: the quadratic $x^2 - 1$ is less than or equal to zero in the region $-1 \leq x \leq 1$, which implies $x \leq 1$ as required.

(b) The second statement is telling us that if $x \leq 1$, then $x^2 \leq 1$. This isn't true: if $x = -2$, then $x^2 = 4$, which is not less than or equal to one.

(c) The third statement is telling us that $x \leq 1$ if and only if $x^2 \leq 1$, i.e. these statements are logically equivalent. We have just seen that this isn't the case!

Let A, B be two logical statements. We say that A is *necessary* for B if B implies A, that is, B cannot be true without A being true. In logical symbols, we write $B \to A$. On the other hand, we say that A is *sufficient* for B if A implies B, that is, provided that A is true, it follows that B is also true. In logical symbols, we write $A \to B$.

(d) For the first statement, it is not sufficient that $|x| = 1$ for $x = 1$. That is because given $|x| = 1$, we could have $x = -1$, which does not imply $x = 1$.

(e) For the second statement, it is true that $|x| = 1$ is necessary for $x = 1$. It cannot be the case that $x = 1$ without $|x| = 1$ being true too.

(f) Since $|x| = 1$ is not sufficient for $x = 1$, we see that $|x| = 1$ is not necessary *and* sufficient for $x = 1$.

6. What is the error in the following argument?

> *'Suppose we wish to solve the equation $x-1=2$. We begin by squaring both sides, to obtain $(x-1)^2=4$. Expanding the left hand side we have:*
>
> $$x^2-2x+1=4 \qquad \Leftrightarrow \qquad x^2-2x-3=0 \qquad \Leftrightarrow \qquad (x+1)(x-3)=0.$$
>
> *Since one of these factors must be zero, it follows that there are two solutions to the equation, $x=-1$ or $x=3$.'*

Solution: The problem lies in the very first step, where the implication only goes in one direction! Writing the whole argument in logical symbols, we have:

$$\begin{aligned} x-1=2 \quad &\Rightarrow \quad (x-1)^2=4 \\ &\Leftrightarrow \quad x^2-2x+1=4 \\ &\Leftrightarrow \quad x^2-2x-3=0 \\ &\Leftrightarrow \quad (x+1)(x-3)=0 \\ &\Leftrightarrow \quad x=-1, \text{ or } x=3. \end{aligned}$$

The logical symbol in the first line does *not* reverse. This shows that *if* the equation holds, *then* x must take one of the values in the final line. We have *not* shown that *if* x takes one of the values in the final line, *then* the equation holds.

Importantly, this shows us that when we are solving an equation, if we make a non-reversible step in our argument, we need to check that our solutions satisfy the original equation.

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7. What is meant by *proof by contradiction*? Prove by contradiction that $\sqrt{2}$ and $\sqrt{3}$ are irrational numbers.

Solution: A 'proof by contradiction' is a sequence of logical steps $A \Rightarrow B \Rightarrow C \Rightarrow \ldots \Rightarrow Z$, where Z is *false*. This implies that A must be false too!

Let's use proof by contradiction to show that $\sqrt{2}$ is irrational. We suppose that $\sqrt{2}$ is rational, so that we can write $\sqrt{2}=p/q$, for some integers p, q. We may assume that the fraction p/q is written in its lowest terms, so that p, q are not both even. Then squaring both sides, we obtain:

$$2=p^2/q^2 \qquad \Rightarrow \qquad 2q^2=p^2.$$

Now, the left hand side is even, so p^2 must be even. Thus, p must be even, since the square of any odd number is odd. Thus, we can write $p=2r$ for some integer r. Substituting into the above equation, we have:

$$2q^2=4r^2 \qquad \Rightarrow \qquad q^2=2r^2.$$

But then, by the same argument, q must be even. This contradicts our original assumption that p/q was written in its lowest terms. Hence $\sqrt{2}$ is irrational.

We can perform a similar proof for $\sqrt{3}$. We suppose that $\sqrt{3}$ is rational, so that we can write $\sqrt{3} = p/q$, for some integers p, q. We again assume that the fraction p/q is written in its lowest terms, so that p, q are not both multiples of three. Then squaring both sides, we obtain:

$$3 = p^2/q^2 \qquad \Rightarrow \qquad 3q^2 = p^2.$$

Now, the left hand side is a multiple of three, so p^2 must be a multiple of three. Thus, p must be a multiple of three. Then, we can write $p = 3r$ for some integer r. Substituting into the above equation, we have:

$$3q^2 = 9r^2 \qquad \Rightarrow \qquad q^2 = 3r^2.$$

But then, by the same argument, q must be a multiple of three. This contradicts our original assumption that p/q was written in its lowest terms. Hence $\sqrt{3}$ is irrational.

Sets and functions

8. State what is meant by the sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$. Decide which of the following statements are true:

(a) $\pi \in \mathbb{Q}$, (b) $3 \notin \mathbb{R}$, (c) $\mathbb{Z} \subseteq \mathbb{Q}$, (d) $\mathbb{Q} \supset \mathbb{C}$.

➺ **Solution:** The definitions of the given sets are the following:

- The set $\mathbb{Z}$ is the set of all integers, positive and negative:

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}.$$

- The set $\mathbb{Q}$ is the set of all rational numbers, i.e. numbers which are the ratio of two integers, i.e. numbers which are expressible as fractions.
- The set $\mathbb{R}$ is the set of all real numbers, i.e. it includes both fractions, and numbers with an infinite, non-repeating decimal expansion.
- The set $\mathbb{C}$ is the set of all complex numbers, i.e. numbers of the form $x + iy$, where x, y are real numbers. We will study complex numbers a lot more later in Michaelmas term!

The symbol $\in$ means 'is a member of the set', and the symbol $\notin$ means 'is not a member of the set'. The symbol $\subseteq$ means 'is a subset of, or is equal to, the set'. The symbol $\supset$ means 'is a superset of'. Hence:

(a) $\pi \in \mathbb{Q}$ is false, because π is not rational - it can't be written as a fraction!

(b) $3 \notin \mathbb{R}$ is false, because 3 is a real number.

(c) $\mathbb{Z} \subseteq \mathbb{Q}$ is true, because every integer is also a rational number (we can write the integer p as a fraction $p/1$, if you are worried!).

(d) $\mathbb{Q} \supset \mathbb{C}$ is false, because not every complex number is a ratio of integers. For example, i is not a ratio of two integers!

9. State what is meant by the sets $[a, b]$, (a, b) and $(a, b]$, where a, b are real numbers. Decide which of the following statements are true:

(a) $1 \in [0, 1)$, (b) $3 \notin (3, 4)$, (c) $[2, 3] \subset (2, 5]$, (d) $(-1, 0) \subset [-1, 0]$.

➡ **Solution:** These sets are *intervals* of real numbers. They have the following definitions:

- $[a, b]$ is the set of all real numbers x in the range $a \leq x \leq b$, so it *includes* the endpoints.
- (a, b) is the set of all real numbers x in the range $a < x < b$, so it *excludes* the endpoints.
- $(a, b]$ is the set of all real numbers x in the range $a < x \leq b$, so it excludes the endpoint a, but includes the endpoint b.

Out of the given statements, we have:

(a) $1 \in [0, 1)$ is false, because the curved bracket means that the endpoint of the interval is excluded.

(b) $3 \notin (3, 4)$ is true, because the endpoint 3 is excluded from the interval in this case.

(c) $[2, 3] \subset (2, 5]$ is false, because 2 is in $[2, 3]$ but not in $(2, 5]$.

(d) $(-1, 0) \subset [-1, 0]$ is true, because every real number satisfying $-1 < x < 0$ also satisfies $-1 \leq x \leq 0$.

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10. A function is a mapping from a set to another set. We write $f : A \to B$ to denote the function (or just f when the sets are implied), and we write $f(x)$ for the value of the function at the point $x \in A$. Which of the following define functions, and why?

(a) $f : [0, \infty) \to \mathbb{R}$, given by $f(x) = \sqrt{x}$;

(b) $f : \mathbb{Z} \to \mathbb{Z}$, given by $f(x) = \frac{1}{2}x$;

(c) $f : \mathbb{R} \to \mathbb{R}$, where $f(x)$ is implicitly defined by $2^{f(x)} = x$;

(d) $f : [0, \infty) \to \mathbb{R}$, where $f(x)$ is implicitly defined by $f(x)^2 = x$.

➡ **Solution:**

(a) This defines a function, because for every value of $x \in [0, \infty)$, the function uniquely gives a value $\sqrt{x} \in \mathbb{R}$. Note that we don't hit *every* real number with this function, because the square root is always positive - that doesn't matter though, as long as the result is a member of the set of real numbers.

(b) This does *not* define a function, because if $x = 1$, then $f(1) = 1/2$ is not in the specified range of the function, which is supposed to be the integers. We could fix this by saying the function is a mapping $f : \mathbb{Z} \to \mathbb{Q}$ or $f : \mathbb{Z} \to \mathbb{R}$.

(c) This does not define a function, because for $x = -1$ for example, there is no real number $f(x)$ such that $2^{f(x)} = -1$. This is because exponentiation of a positive real number always gives a positive result. This could be fixed by considering the function to be a mapping $f : (0, \infty) \to \mathbb{R}$; the function that this defines is then just $f(x) = \log_2(x)$.

(d) This does not define a function, because it is multi-valued. For example, take $x = 1$. Then $f(x)^2 = 1$ has two roots, $f(x) = \pm 1$. We don't know which one to pick! If we instead said the function was a mapping $f : [0, \infty) \to [0, \infty)$, we would know to pick the positive root, which then makes the function equivalent to the one in part (a).

Sequences and series

11. Prove the following results using (i) induction; (ii) a direct argument:

(a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, (b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, (c) $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

[*Hint: for the direct arguments, consider summing* $(k+1)^r - k^r$ *for an appropriate integer* r *in each case; note this sum telescopes.*] Is it more useful to prove a result by induction, or by a direct argument? Why?

➼ **Solution:** (a) (i) We first prove the result using induction. The base case, $n = 1$, is obvious since:

$$\sum_{k=1}^{1} k = 1, \qquad \frac{1(1+1)}{2} = 1.$$

Assume the result is true for $n = m$, and consider the case $n = m + 1$. We have:

$$\begin{aligned}
\sum_{k=1}^{m+1} k &= (m+1) + \sum_{k=1}^{m} k \\
&= (m+1) + \frac{m(m+1)}{2} && \text{(induction hypothesis)} \\
&= (m+1)\left(1 + \frac{m}{2}\right) \\
&= \frac{(m+1)(m+2)}{2}.
\end{aligned}$$

Hence, the result is true for $n = m + 1$ if the induction hypothesis holds. So we're done by the principle of mathematical induction.

(a) (ii) To prove the result directly, note the identity:

$$(k+1)^2 - k^2 = 2k + 1.$$

Summing both sides from $k = 1$ to $k = n$, we have:

$$(1+1)^2 - 1^2 + (2+1)^2 - 2^2 + (3+1)^2 - 3^2 + \cdots + (n+1)^2 - n^2 = 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1.$$

Simplifying both sides, noticing that the left hand side forms a telescoping series, we have:

$$(n+1)^2 - 1 = 2\sum_{k=1}^{n} k + n.$$

Rearranging, we have:

$$\sum_{k=1}^{n} k = \frac{(n+1)^2 - n - 1}{2} = \frac{n(n+1)}{2},$$

as required.

(b) (i) We first prove the result using induction. The base case, $n = 1$, is obvious since:

$$\sum_{k=1}^{1} k^2 = 1, \qquad \frac{1(1+1)(2+1)}{6} = 1.$$

Assume the result is true for $n = m$, and consider the case $n = m + 1$. We have:

$$\begin{aligned}
\sum_{k=1}^{m+1} k^2 &= (m+1)^2 + \sum_{k=1}^{m} k^2 \\
&= (m+1)^2 + \frac{m(m+1)(2m+1)}{6} && \text{(induction hypothesis)} \\
&= (m+1)\left(m+1+\frac{m(2m+1)}{6}\right) \\
&= \frac{(m+1)(6m+6+2m^2+m)}{6} \\
&= \frac{(m+1)(2m^2+7m+6)}{6} \\
&= \frac{(m+1)(m+2)(2m+3)}{6}.
\end{aligned}$$

Hence, the result is true for $n = m + 1$ if the induction hypothesis holds. So we're done by the principle of mathematical induction.

(b) (ii) To prove the result directly, note the identity:

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1$$

Summing both sides from $k = 1$ to $k = n$, we have:

$$(1+1)^3 - 1^3 + (2+1)^3 - 2^3 + (3+1)^3 - 3^3 + \cdots + (n+1)^3 - n^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1.$$

Simplifying both sides, noticing that the left hand side forms a telescoping series, we have:

$$(n+1)^3 - 1 = 3\sum_{k=1}^{n} k^2 + \frac{3n(n+1)}{2} + n.$$

Rearranging, we have:

$$\sum_{k=1}^{n} k^2 = \frac{(n+1)^3 - 1 - n - 3n(n+1)/2}{3} = \frac{2n^3+3n^2+n}{6} = \frac{n(n+1)(2n+1)}{6},$$

as required.

(c) (i) We first prove the result using induction. The base case, $n = 1$, is obvious since:

$$\sum_{k=1}^{1} k^3 = 1, \qquad \frac{1^2(1+1)^2}{4} = 1.$$

Assume the result is true for $n = m$, and consider the case $n = m + 1$. We have:

$$\begin{aligned}
\sum_{k=1}^{m+1} k^3 &= (m+1)^3 + \sum_{k=1}^{m} k^3 \\
&= (m+1)^3 + \frac{m^2(m+1)^2}{4} && \text{(induction hypothesis)} \\
&= (m+1)^2 \left(m + 1 + \frac{m^2}{4} \right) \\
&= \frac{(m+1)^2(m^2+4m+4)}{4} \\
&= \frac{(m+1)^2(m+2)^2}{4}.
\end{aligned}$$

Hence, the result is true for $n = m + 1$ if the induction hypothesis holds. So we're done by the principle of mathematical induction.

(c) (ii) To prove the result directly, note the identity:

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$$

Summing both sides from $k = 1$ to $k = n$, we have:

$$(1+1)^4 - 1^4 + (2+1)^4 - 2^4 + (3+1)^4 - 3^4 + \cdots + (n+1)^4 - n^4 = 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1.$$

Simplifying both sides, noticing that the left hand side forms a telescoping series, we have:

$$(n+1)^4 - 1 = 4\sum_{k=1}^{n} k^3 + n(n+1)(2n+1) + 2n(n+1) + n$$

Rearranging, we have:

$$\sum_{k=1}^{n} k^3 = \frac{(n+1)^4 - n(n+1)(2n+1) - 2n(n+1) - n}{4} = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2(n+1)^2}{4},$$

as required.

Often, it is much more useful to prove a result by a direct argument than by induction. This is because induction requires us to know that the result is true in advance! The proof is often simpler through induction, but usually less enlightening about *why* the result is true in the first place.

❊ **Comments:** You might have spotted a very interesting curiosity: the sum of the first n cubes is equal to the sum of the first n natural numbers squared. There is a beautiful visual proof of this fact, which is demonstrated very nicely in this video, if you're interested: https://www.youtube.com/watch?v=NxOcT_VKQR0.

12. What is meant by an *arithmetic sequence* with first term a and common difference d? Prove that the sum of the first n terms of an arithmetic sequence is:

$$S_n = \frac{1}{2}n\left(2a + (n-1)d\right).$$

Hence, find the sum of the series $2, 5, 8, 11, ..., 32$.

➼ **Solution:** The arithmetic sequence with first term a and common difference d is:

$$a, \qquad a+d, \qquad a+2d, \qquad a+3d, \qquad ...$$

The sum of the first n terms is given by:

$$\sum_{k=0}^{n-1}(a+kd) = na + d\sum_{k=0}^{n-1}k = na + \frac{d(n-1)n}{2} = \frac{1}{2}n\left(2a+(n-1)d\right).$$

The given sequence $2, 5, 8, 11, ..., 32$ is an arithmetic sequence with first term $a = 2$ and common difference $d = 3$. The final term is the $(32-2)/3+1 = 11$th term in the series. Hence the sum is:

$$\frac{11}{2}\left(4 + 10\cdot 3\right) = \frac{11\cdot 34}{2} = 11\cdot 17 = 187.$$

---

13. What is meant by a *geometric sequence* with first term a and common ratio r? Prove that the sum of the first n terms of a geometric sequence is:

$$S_n = \frac{a(1-r^n)}{1-r}.$$

What happens if $r = 1$? What is the behaviour of this sum in the limit as $n \to \infty$? Hence, find the sum of the infinite series $2, 2/3, 2/9, 2/27, ...$.

➼ **Solution:** The geometric sequence with first term a and common ratio r is:

$$a, \qquad ar, \qquad ar^2, \qquad ar^3, \qquad ar^4, \qquad ...$$

The sum of the first n terms is:

$$S_n = a + ar + ... + ar^{n-1}.$$

Multiplying by $1-r$, we have:

$$S_n(1-r) = (a + ar + ... + ar^{n-1})(1-r) = a + ar + ... + ar^{n-1} - ar - ar^2 - ... - ar^n = a - ar^n.$$

Hence the sum is indeed given by:

$$S_n = \frac{a(1-r^n)}{1-r},$$

as required. Observe that this formula *only* applies if $r \neq 1$; in the case where $r = 1$, the sum to n terms is just na (it's just $a + a + ... + a$, but n times). Whenever we have a situation where we are dividing by zero in maths in some special case, we should return to the first place where we tried to divided by zero and see what happens if we are in that special case.

If $|r| < 1$, we have $r^n \to 0$ as $n \to \infty$. Hence if $|r| < 1$, we can sum a geometric series to infinity, giving the result:

$$S_\infty = \frac{a}{1-r}.$$

In the case of the series $2, 2/3, 2/9, 2/27, ...$ the first term is $a = 2$ and the common ratio is $r = 1/3$, so the sum to infinity is:

$$S_\infty = \frac{2}{1-1/3} = \frac{2}{2/3} = 3.$$

14. Using the formula for the sum of an infinite geometric series, find a formula for the sum of the infinite series:

$$\sum_{k=1}^{\infty} kr^k,$$

where $|r| < 1$. [*Hint: differentiation!*] Hence determine:

$$\frac{2}{3} + 2\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^3 + 4\left(\frac{2}{3}\right)^4 + \cdots.$$

➾ **Solution:** In the previous question, we proved that:

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}.$$

Multiplying both sides of this formula by r, we have:

$$\sum_{k=1}^{\infty} ar^k = \frac{ar}{1-r}.$$

Differentiating both sides of this formula with respect to r, we have:

$$\sum_{k=1}^{\infty} akr^{k-1} = \frac{d}{dr}\left(\frac{ar}{1-r}\right) = \frac{a(1-r)+ar}{(1-r)^2} = \frac{a}{(1-r)^2}.$$

To finish, we set $a = 1$ and multiplying through by r, giving:

$$\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}.$$

The given series' value is:

$$\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k = \frac{(2/3)}{(1-2/3)^2} = \frac{2/3}{1/9} = 6.$$

Trigonometric functions

15. Define the trigonometric functions $\sin(x), \cos(x), \tan(x)$ in terms of the side lengths of an appropriate right-angled triangle. Define also the reciprocal trigonometric functions $\operatorname{cosec}(x), \sec(x), \cot(x)$. Hence, prove each of the following trigonometric identities:

(a) *The Pythagorean identities:*

$$\sin^2(x) + \cos^2(x) = 1, \qquad \tan^2(x) + 1 = \sec^2(x), \qquad \cot^2(x) + 1 = \operatorname{cosec}^2(x).$$

(b) *The compound angle formulae:*

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \sin(y)\cos(x), \qquad \cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y),$$

$$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}.$$

(c) *The double angle formulae:*

$$\sin(2x) = 2\sin(x)\cos(x), \qquad \cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x).$$

(d) *The power reduction formulae:*

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x)).$$

(e) *The product to sum formulae:*

$$\sin(x)\sin(y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right), \qquad \sin(x)\cos(y) = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right),$$

$$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right).$$

(f) *The sum to product formulae:*

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right), \qquad \cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),$$

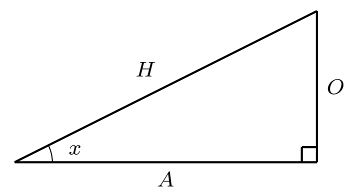
$$\cos(x) - \cos(y) = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

Learn all of these identities off by heart, and get your supervision partner to test you on them.

➭ **Solution:** There are lots of ways to define the trigonometric functions $\sin(x), \cos(x), \tan(x)$. For example, these functions can be defined in terms of complex exponentials, Taylor series, or as solutions of certain differential equations (we shall study all of these options later in the course). Here, however, we will use the most elementary definition, in terms of right-angled triangles.

Given a right-angled triangle with sides O, A, H, and one given angle x, where H is the length of the hypotenuse and O is the length of the sides opposite the angle x, we shall define the trigonometric functions via:

$$\sin(x) := \frac{O}{H}, \qquad \cos(x) := \frac{A}{H}, \qquad \tan(x) := \frac{O}{A} = \frac{O}{H}\cdot\frac{H}{A} = \frac{\sin(x)}{\cos(x)}.$$



The *reciprocal trigonometric functions* are then defined as:

$$\operatorname{cosec}(x) := \frac{1}{\sin(x)}, \qquad \sec(x) := \frac{1}{\cos(x)}, \qquad \cot(x) := \frac{1}{\tan(x)}.$$

Notice that this definition *only* works for x in the range $x \in [0, \pi/2)$, because we need it to be an acute angle. We extend the definition by requiring the standard periodicity properties $\sin(x) = \sin(\pi - x)$ (extending to $[0, \pi]$), $\sin(-x) = -\sin(x)$ (further extending to $[-\pi, \pi]$), $\sin(x) = \sin(x + 2\pi)$ (extending to all real numbers), and $\cos(x) = -\cos(\pi - x)$ (extending to $[0, \pi]$), $\cos(x) = \cos(-x)$ (extending to $[-\pi, \pi]$, $\cos(x) = \cos(x + 2\pi)$ (extending to all real numbers), which allow all of these functions to be defined for all real numbers x.

Notice also the *cofunction identities* which follow immediately from the right-angled triangle definitions:

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x), \qquad \cos\left(\frac{\pi}{2} - x\right) = \sin(x),$$

because the remaining angle in a right-angled triangle, with angle x, is $\pi/2 - x$.

These definitions allow us to prove each of the given identities:

(a) *The Pythagorean identities.* These identities are called the Pythagorean identities because they come from Pythagoras' theorem for the right-angled triangle:

$$O^2 + A^2 = H^2.$$

Dividing by H^2, we have:

$$\left(\frac{O}{H}\right)^2 + \left(\frac{A}{H}\right)^2 = 1 \qquad \Leftrightarrow \qquad \sin^2(x) + \cos^2(x) = 1.$$

Next, we divide this identity by $\cos^2(x)$ to get:

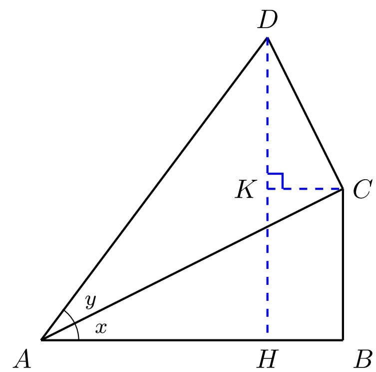
$$\frac{\sin^2(x)}{\cos^2(x)} + 1 = \frac{1}{\cos^2(x)} \qquad \Leftrightarrow \qquad \tan^2(x) + 1 = \sec^2(x).$$

On the other hand, if we had instead divided by $\sin^2(x)$, we would have gotten:

$$1 + \frac{\cos^2(x)}{\sin^2(x)} = \frac{1}{\sin^2(x)} \qquad \Leftrightarrow \qquad 1 + \cot^2(x) + \operatorname{cosec}^2(x),$$

which completes our derivation of the three Pythagorean identities.

(b) *The compound angle formulae.* Consider the following diagram, involving two right-angled triangles ABC and ACD stacked on top of one another. One has an acute angle x at the vertex A, and one has an acute angle y at the vertex A (note that we assume that $x + y$ is acute).



Observe that x and the angle KCA are alternating, so KCA is equal to x. This implies that the triangles DKC and ABC are similar triangles.

Now by the definition of sine, we have (writing AB for the length of the line segment from A to B, etc):

$$\begin{aligned}
\sin(x+y) &= \frac{DH}{AD} \\
&= \frac{DK}{AD} + \frac{KH}{AD} && (\text{since } DH = DK + KH) \\
&= \frac{DK}{AD} + \frac{CB}{AD} && (\text{since } KH = CB) \\
&= \frac{DK}{DC} \cdot \frac{DC}{AD} + \frac{CB}{AC} \cdot \frac{AC}{AD} \\
&= \cos(x)\sin(y) + \sin(x)\cos(y) && (\text{definitions of sine, cosine and using similar triangles})
\end{aligned}$$

This establishes the formula in the case where $x + y, x, y$ are acute; the formula is readily extended to all real values of the angle via the various properties we outlined above.

In particular, this implies we can replace $y \mapsto -y$, which gives:

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y),$$

which gives the full formula $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$. We can also replace $x \mapsto \pi/2 - x$ in the formula giving:

$$\sin\left(\frac{\pi}{2} - x \pm y\right) = \sin\left(\frac{\pi}{2} - x\right)\cos(y) \pm \cos\left(\frac{\pi}{2} - x\right)\sin(y).$$

Using the cofunction identities, this implies:

$$\cos(x \mp y) = \cos(x)\cos(y) \pm \sin(x)\sin(y),$$

as required.

Finally, dividing the formula for $\sin(x \pm y)$ by the formula for $\cos(x \pm y)$, we obtain the tangent compound angle formula:

$$\begin{aligned}\tan(x \pm y) &= \frac{\sin(x \pm y)}{\cos(x \pm y)} \\ &= \frac{\sin(x)\cos(y) \pm \cos(x)\sin(y)}{\cos(x)\cos(y) \mp \sin(x)\sin(y)} \\ &= \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)} \qquad \text{(divide numerator and denominator by } \cos(x)\cos(y)\text{)}\end{aligned}$$

(c) *The double angle formulae.* These follow immediately from the compound angle formulae by taking $x = y$. We have:

$$\sin(2x) = \sin(x + x) = \sin(x)\cos(x) + \cos(x)\sin(x) = 2\sin(x)\cos(x),$$

and similarly:

$$\cos(2x) = \cos(x + x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2(x) - \sin^2(x).$$

To get the alternative forms of the cosine double angle formulae, we use one of the Pythagorean identities in the following forms: $\cos^2(x) = 1 - \sin^2(x)$ and $\sin^2(x) = 1 - \cos^2(x)$. This gives:

$$\cos(2x) = \cos^2(x) - \sin^2(x) = (1 - \sin^2(x)) - \sin^2(x) = 1 - 2\sin^2(x),$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2(x) - (1 - \cos^2(x)) = 2\cos^2(x) - 1.$$

(d) *The power reduction formulae.* These follow immediately from the double angle formulae for $\cos(2x)$, simply by rearrangement:

$$\cos(2x) = 1 - 2\sin^2(x) \qquad \Rightarrow \qquad \sin^2(x) = \frac{1}{2}(1 - \cos(2x)),$$

$$\cos(2x) = 2\cos^2(x) - 1 \qquad \Rightarrow \qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x)).$$

(e) *The product to sum formulae.* We can obtain these formulae by taking sums and differences of the compound angle formuale for $\sin(x + y), \sin(x - y)$ and $\cos(x + y), \cos(x - y)$. We have:

$$\sin(x + y) + \sin(x - y) = \sin(x)\cos(y) + \cos(x)\sin(y) + \sin(x)\cos(y) - \cos(x)\sin(y) = 2\sin(x)\cos(y) \qquad (\dagger)$$

and similarly:

$$\cos(x + y) + \cos(x - y) = \cos(x)\cos(y) - \sin(x)\sin(y) + \cos(x)\cos(y) + \sin(x)\sin(y) = 2\cos(x)\cos(y),$$

$$\cos(x + y) - \cos(x - y) = \cos(x)\cos(y) - \sin(x)\sin(y) - \cos(x)\cos(y) - \sin(x)\sin(y) = -2\sin(x)\sin(y).$$

Rearrangement in each case produces the required identities.

(f) *The sum to product formulae.* These formulae are just the product to sum formulae in reverse. If we take $x + y = A$ and $x - y = B$ in the above, we have $x = (A + B)/2$ and $y = (A - B)/2$. This gives:

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right),$$

when inserted into (†), for example. The other identities follow similarly by substitution into the remaining cosine formulae.

17. Prove the trigonometric inequalities:

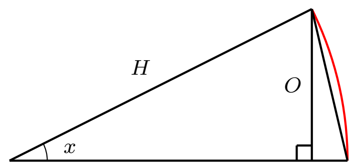
    (a) $|\sin(x)| \leq |x|$, for all real x;

    (b) $\cos(x) \geq 1 - x^2/2$, for all real x.

➾ **Solution:** (a) Since the hypotenuse of a right-angled triangle can never be shorter than its opposite side, we immediately have that $\sin(x) \leq 1$ for acute angles $0 \leq x < \pi/2$. By the extension to negative angles, this gives $-1 \leq \sin(x) \leq 1$ for $-\pi/2 < x < \pi/2$, which then implies by the periodicity properties that $|\sin(x)| \leq 1$ for all x.

For $|x| \geq \pi/2$, since $\pi/2 > 1$, we immediately have $|\sin(x)| \leq |x|$.

For $|x| \leq \pi/2$, we can use the right-angled triangle to prove that $|\sin(x)| \leq |x|$. We consider constructing a circular arc with radius equal to the length of the hypotenuse of the right-angled triangle, as shown below. We also insert an additional chord, as shown in the figure.



From this figure, we see that the area of the triangle contained in the circular sector is:

$$\frac{1}{2}OH = \frac{1}{2}\sin(x)H^2.$$

This is evidently less than the area of the circular sector, which is:

$$\frac{1}{2}xH^2.$$

The inequality $\sin(x) \leq x$ follows for acute x, which implies $-x \leq -\sin(x) = \sin(-x)$ for x in the range $-\pi/2 < x \leq 0$. So we're done!

(b) For the second inequality, we have:

$$\cos(x) = 1 - 2\sin^2\left(\frac{x}{2}\right) \geq 1 - 2\left(\frac{x}{2}\right)^2 = 1 - \frac{x^2}{2},$$

using the result from part (a).

The boring stuff: exams and coursework

18. Look up the format of the first-year maths exams. How much is each exam worth? How long does each exam last? How many questions do you have to answer, and you should you split your time in the exams?

➭ **Solution:** There are two first-year maths exams, which are each worth $6/13 \approx 46\%$ of the your final grade in maths (accounting for a total of $12/13 \approx 92\%$ of your final grade). Each exam lasts 3 hours. Each exam has the following structure:

- **Section A.** The exam begins with 10 short questions, each worth 2 marks, totalling 20 marks. These questions are designed to be answered quickly and credit is always awarded to the correct answer, even if full reasoning is not given. This section is compulsory.

- **Section B.** The second part of the exam contains 10 long questions, which are each worth 20 marks. You must choose **5 questions** out of the **10 questions** to answer. These questions usually award marks for explaining yourself well, in addition to getting the right answers. Out of the 10 questions, 2 questions are only accessible to students who have done the Maths B course, but these will be clearly marked to avoid confusion.

In total, the exam is out of 120 marks, with 20 marks coming from Section A and 5 questions contributing 20 marks each in Section B. This means that you should spend at most 30 minutes on Section A and at most 30 minutes on each Section B question.

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19. Look up the first-year maths coursework. How much is it worth?

➭ **Solution:** The first-year maths coursework is a scientific computing course that is run by the Cavendish Laboratory (**not** by the maths department). It involves 8 practical sessions, each 90 minutes long, 4 of which you attend in Michaelmas and 4 of which you attend in Lent. Each practical involves submission of a small piece of computing work which contributes to your final grade. The total contribution is $1/13 \approx 6\%$ of your grade, so that each piece of work is worth $1/104 \approx 1\%$ of your final grade.

Part IA: Mathematics for Natural Sciences B
Examples Sheet 1: Basics of vector geometry, and the scalar and vector products

Model Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

Basics of vector algebra

1. Let $A = (1, 3, 4)$, $B = (-1, 2, 4)$, and $C = (2, 2, 3)$. Which of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ is the longest?

Solution: The three-dimensional version of Pythagoras' theorem gives the lengths of the vectors as:

$$|\overrightarrow{AB}| = \sqrt{(-1-1)^2 + (2-3)^2 + (4-4)^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$|\overrightarrow{BC}| = \sqrt{(2 - -1)^2 + (2-2)^2 + (3-4)^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$$

$$|\overrightarrow{AC}| = \sqrt{(2-1)^2 + (2-3)^2 + (3-4)^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

Hence the longest vector is $\overrightarrow{BC}$.

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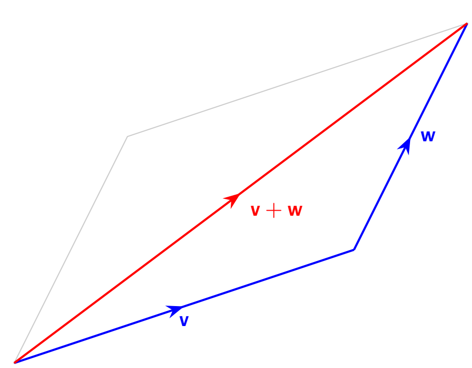
2. (a) State the definition of $\mathbf{v}+\mathbf{w}$, given the vectors $\mathbf{v}$, $\mathbf{w}$. Using this definition, show that $\mathbf{v}+\mathbf{w} = \mathbf{w}+\mathbf{v}$, and $(\mathbf{v}+\mathbf{w})+\mathbf{u} = \mathbf{v} + (\mathbf{w}+\mathbf{u})$ for any vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$.

   (b) Suppose that an aeroplane's engine produces a velocity 125 km h^{-1} due North. If there is a wind travelling at a velocity 80 km h^{-1} at a bearing 60° West of North, use trigonometry to determine how fast the aeroplane travels across the Earth, and the bearing of its direction of travel from North.

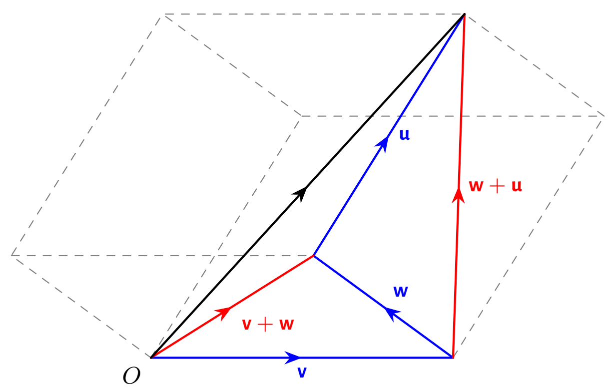
Solution: (a) To produce the vector $\mathbf{v}+\mathbf{w}$ from the vectors $\mathbf{v}$, $\mathbf{w}$, we put the start of the vector $\mathbf{w}$ at the end of the vector $\mathbf{v}$, as shown in the diagram below. The directed line segment from the start of the vector $\mathbf{v}$ to the end of the vector $\mathbf{w}$ in this configuration is then defined to be the vector $\mathbf{v}+\mathbf{w}$.



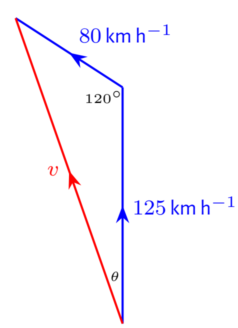
This definition immediately shows that $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$. This is because the if we started with the vector $\mathbf{w}$, then put the start of the vector $\mathbf{v}$ at the end of the vector $\mathbf{w}$, we produce a parallelogram (shown in grey in the above figure). The diagonal of the parallelogram is both equal to $\mathbf{v}+\mathbf{w}$ and $\mathbf{w}+\mathbf{v}$, which shows that it does not matter in which order we add vectors (In fancy language, we say that vector addition is a *commutative operation*).

Next, we are asked to show the property $(\mathbf{v}+\mathbf{w})+\mathbf{u}=\mathbf{v}+(\mathbf{w}+\mathbf{u})$. We shall do this in three dimensions, which is the more general case.

The best proof is just a 'proof by picture'. The vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$ form a parallelepiped, as shown in the diagram below. In red, we show the vectors $\mathbf{v}+\mathbf{w}$ and $\mathbf{w}+\mathbf{u}$, which are diagonals on the faces of the parallelepiped. Finally, the vectors $(\mathbf{v}+\mathbf{w})+\mathbf{u}$ and $\mathbf{v}+(\mathbf{w}+\mathbf{u})$ both correspond to the black diagonal of the parallelepiped shown in the figure below; hence, they are equal.



(b) To find the resultant velocity of the aircraft, we need to add the two given velocities (which are, of course, vector quantities). Here, a good diagram is helpful!



By the cosine rule, we have that the resultant speed of the aircraft is:

$$v=\sqrt{80^2+125^2-2\cdot 80\cdot 125\cos(120^\circ)}\,\text{km h}^{-1}=5\sqrt{1281}\,\text{km h}^{-1}.$$

By the sine rule, we have that the angle θ in the diagram is given by:

$$\frac{\sin(\theta)}{80\,\text{km h}^{-1}}=\frac{\sin(120^\circ)}{v}\qquad\Rightarrow\qquad\theta=\arcsin\left(\frac{80\,\text{km h}^{-1}\sin(120^\circ)}{v}\right)=\arcsin\left(\frac{40\sqrt{3}}{5\sqrt{1281}}\right)=\arcsin\left(\frac{8}{\sqrt{427}}\right).$$

Despite the rather nasty numbers, in general, exact answers are preferred, since calculators are *not* permitted in the first-year mathematics exams.

3. (a) Define a *basis* of vectors.

   (b) Let $\mathbf{v} = (1,2)$, $\mathbf{e}_1 = (1,-1)$ and $\mathbf{e}_2 = (2,3)$. Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbb{R}^2$, and determine the components of $\mathbf{v}$ with respect to this basis.

   (c) Let $\mathbf{w}_1 = (1,2,3)$ with respect to the basis $\{(1,1,0),(1,-1,0),(0,0,1)\}$, and let $\mathbf{w}_2 = (3,2,1)$ with respect to the basis $\{(0,1,2),(2,1,0),(0,1,-2)\}$. Find $\mathbf{w}_1 - 3\mathbf{w}_2$ with respect to the standard basis of $\mathbb{R}^3$.

Solution: (a) A *basis of vectors in n dimensions* is a collection of n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ satisfying two properties:

(i) SPANNING. Any vector $\mathbf{v}$ can be written as a linear combination of the vectors in the basis; that is, for some scalar coefficients $\alpha_1, \ldots, \alpha_n$ we can write:

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n.$$

(ii) LINEAR INDEPENDENCE. If we have:

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

for some coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$, then we must have $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. This property is telling us that no vectors in the basis are redundant - if there *were* some non-zero coefficients that satisfied this equation, we would be able to rearrange this equation to write one of the basis vectors in terms of the others.

It is actually the case that if we have n vectors that span $\mathbb{R}^n$, they must be linearly independent. Similarly, if we have n vectors that are linearly independent in $\mathbb{R}^n$, they must span. The proof is given in the Comments at the end of this question.

(b) First, we are asked to show that $\{(1,-1),(2,3)\}$ is a basis for $\mathbb{R}^2$. We show each of the properties separately (although as commented above, we actually only need to show one of these properties, and the other is guaranteed to follow!):

(i) SPANNING. Let (v,w) be any vector in $\mathbb{R}^2$. We need to find coefficients α, β such that:

$$\begin{pmatrix} v \\ w \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \alpha + 2\beta \\ -\alpha + 3\beta \end{pmatrix}.$$

Hence, we have a system of two linear simultaneous equations for the variables α, β. Adding the first equation to the second, we obtain $5\beta = v + w$. Thus $\beta = \frac{1}{5}(v+w)$. Substituting back into the first equation, we have:

$$v = \alpha + \frac{2}{5}(v+w) \qquad \Rightarrow \qquad \alpha = \frac{1}{5}\left(3v - 2w\right).$$

It follows that we can write:

$$\begin{pmatrix} v \\ w \end{pmatrix} = \frac{1}{5}(3v-2w)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{5}(v+w)\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

for any vector $(v,w) \in \mathbb{R}^2$.

(ii) LINEAR INDEPENDENCE. In the calculation for spanning, we have just shown that if $(0,0) = \alpha(1,-1) + \beta(2,3)$, then the coefficients α, β are given by:

$$\alpha = \frac{1}{5}(3\cdot 0 - 2\cdot 0) = 0, \qquad \beta = \frac{1}{5}(0+0) = 0.$$

Hence we have linear independence.

As mentioned above, we know that it must be the case that if we have already shown the vectors span $\mathbb{R}^2$, they must be linearly independent - this is why the calculations are so similar.

Finally, we are asked the components of $\mathbf{v} = (1,2)$ with respect to this basis. By the above calculation, we have:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{5}(3\cdot 1 - 2\cdot 2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{5}(1+2)\begin{pmatrix} 2 \\ 3 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Hence the components are $-1/5, 3/5$.

(c) We are told that:

$$\mathbf{w}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} + 2\begin{pmatrix}1\\-1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\-1\\3\end{pmatrix},$$

and:

$$\mathbf{w}_2 = 3\begin{pmatrix}0\\1\\2\end{pmatrix} + 2\begin{pmatrix}2\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}4\\6\\8\end{pmatrix}.$$

Hence with respect to the standard basis of $\mathbb{R}^3$, we have:

$$\mathbf{w}_1 - 3\mathbf{w}_2 = \begin{pmatrix}3\\-1\\3\end{pmatrix} - 3\begin{pmatrix}4\\6\\9\end{pmatrix} = \begin{pmatrix}-9\\-19\\-24\end{pmatrix}.$$

❊ **Comments:** We stated in this question that if we have n vectors in $\mathbb{R}^n$ that span, they must be linearly independent. We also stated that if we have n vectors in $\mathbb{R}^n$ that are linearly independent, they must span.

To prove these results, we first prove a smaller result called the *Steinitz exchange lemma*. This is rather technical and mathematical (and *definitely non-examinable!*), but essentially it just tells us: *a set that is linearly independent can never be bigger than a set that spans*, which makes intuitive sense!

The Steinitz exchange lemma (for $\mathbb{R}^n$): Let $U = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ be a set of linear independent vectors in $\mathbb{R}^n$, and let $V = \{\mathbf{v}_1, ..., \mathbf{v}_l\}$ be a set of vectors which spans $\mathbb{R}^n$. Then we can choose vectors $\mathbf{v}_{\alpha_1}, ..., \mathbf{v}_{\alpha_{l-k}}$ in V such that $\{\mathbf{u}_1, ..., \mathbf{u}_k, \mathbf{v}_{\alpha_1}, ..., \mathbf{v}_{\alpha_{l-k}}\}$ spans $\mathbb{R}^n$. Consequently, we must have $k \leq l$ (a set that is linearly independent can never be bigger than a set that spans).

Proof: We use induction on k. When $k = 0$, we trivially have that $\{\mathbf{v}_1, ..., \mathbf{v}_l\}$ already spans by assumption.

Now suppose that result is true for $k = r-1$. Then by the induction hypothesis, we can choose $l-(r-1)$ elements of the set V which when taken together with the set U span. Without loss of generality, we can therefore assume that the set $\{\mathbf{u}_1, ..., \mathbf{u}_{r-1}, \mathbf{v}_r, ..., \mathbf{v}_l\}$ spans.

Since this set spans, there must be coefficients μ_i such that:

$$\mathbf{u}_r = \sum_{j=1}^{r-1} \mu_j \mathbf{u}_j + \sum_{j=r}^{l} \mu_j \mathbf{v}_j.$$

At least one of the coefficients μ_j for $j \geq r$ must be non-zero, else we would contradict linear independence of the set U. By reordering basis elements if necessary, we may assume without loss of generality that $\mu_r \neq 0$. Then we can rearrange the above equation to read:

$$\mathbf{v}_r = \frac{1}{\mu_r}\left(\mathbf{u}_r - \sum_{j=1}^{r-1} \mu_j \mathbf{u}_j - \sum_{j=r+1}^{l} \mu_j \mathbf{v}_j\right).$$

That is, $\mathbf{v}_r$ is in the span of $\{\mathbf{u}_1, ..., \mathbf{u}_r, \mathbf{v}_{r+1}, ..., \mathbf{v}_l\}$. Hence, this set must also span $\mathbb{R}^n$, since by assumption $\{\mathbf{u}_1, ..., \mathbf{u}_{r-1}, \mathbf{v}_r, ..., \mathbf{v}_l\}$ spans $\mathbb{R}^n$. The result follows by induction. □

We immediately have the following consequences:

Corollary: We have the following consequences of the Steinitz exchange lemma:

(i) All bases of $\mathbb{R}^n$ have size n.

(ii) Any linearly independent set of size n in $\mathbb{R}^n$ is a basis (so spans).

(iii) Any set of size n in $\mathbb{R}^n$ that spans is a basis (so is linearly independent).

Proof: The standard Cartesian unit vectors $\mathbf{e}_1 = (1, 0, 0, .., 0)$, $\mathbf{e}_2, ..., \mathbf{e}_n$ can easily be shown to be a basis of $\mathbb{R}^n$. Let $S = \{\mathbf{e}_1, ..., \mathbf{e}_n\}$.

(i) Let T be another basis of $\mathbb{R}^n$. Then T is a linearly independent set so must have size less than n. But S is linearly independent, so must have size less than T. Hence T must have size exactly n.

(ii) Suppose that $T = \{\mathbf{t}_1, ..., \mathbf{t}_n\}$ is a linearly independent set of size n that does not span. Then there exists some $\mathbf{v}$ not in the span of T; we claim that $\{\mathbf{t}_1, ..., \mathbf{t}_n, \mathbf{v}\}$ is linearly independent. Suppose that:

$$\alpha_0\mathbf{v} + \sum_{i=1}^{n} \alpha_i \mathbf{t}_i = \mathbf{0}.$$

Then if $\alpha_0 \neq 0$, we could rearrange to write $\mathbf{v}$ in terms of $\{\mathbf{t}_1, ..., \mathbf{t}_n\}$. But this is a contradiction, because $\mathbf{v}$ is not contained in the span of T. Hence $\alpha_0 = 0$; by linear independence of T, we then have $\alpha_i = 0$ for all $i = 1, ..., n$.

But then $\{\mathbf{t}_1, ..., \mathbf{t}_n, \mathbf{v}\}$ is a linearly independent set of size $n + 1$ which exceeds the size of S, which spans. This contradicts the Steinitz exchange lemma, so T must have spanned in the first place.

(iii) Suppose that $T = \{\mathbf{t}_1, ..., \mathbf{t}_n\}$ is a spanning set of size n that is linearly dependent. Then there exists some coefficients $\alpha_1, ..., \alpha_n$, not all zero, such that:

$$\alpha_1\mathbf{t}_1 + \cdots + \alpha_n\mathbf{t}_n = \mathbf{0}.$$

Without loss of generality, $\alpha_1 \neq 0$, which allows us to rearrange and write $\mathbf{t}_1$ in terms of $\mathbf{t}_2, ..., \mathbf{t}_n$. This shows that $\{\mathbf{t}_2, ..., \mathbf{t}_n\}$ spans. But this contradicts the Steinitz exchange lemma, because a set of size $n-1$ cannot span since S is a set of size n which is linearly independent. Thus T must have been linearly independent in the first place.

Phew! That's lots of technical work, so you are probably only bothered by this if you are interested in very pure maths. Otherwise, remembering the conclusion is fine: *a set of n vectors is a basis for $\mathbb{R}^n$ if it either spans, or is linearly independent*.

The equation of a line

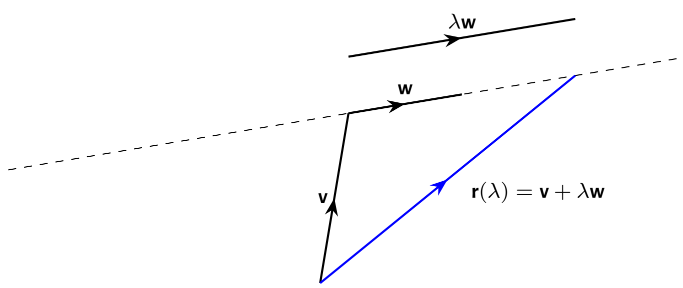
4. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ be 3-vectors, and suppose that $\mathbf{w} \neq \mathbf{0}$.

(a) Explain why the equation $\mathbf{r} = \mathbf{v} + \lambda\mathbf{w}$, as $\lambda \in \mathbb{R}$ varies, represents a line, and summarise its properties. Why is the condition $\mathbf{w} \neq \mathbf{0}$ necessary?

(b) If $\mathbf{v} = (x_0, y_0, z_0)$ and $\mathbf{w} = (a, b, c)$, where $a, b, c \neq 0$, show that the same line may be equivalently described through the system of equations:
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$
What is the corresponding system of equations in the cases where one or more of a, b, c are zero?

(c) Show that the position vectors $(1, 0, 1)$, $(1, 1, 0)$ and $(1, -3, 4)$ lie on a straight line, and find both its vector form, as in (a), and its Cartesian form, as in (b).

Solution: (a) The equation $\mathbf{r} = \mathbf{v} + \lambda\mathbf{w}$ represents the dashed line shown in the figure below. The dashed line is parallel to the vector $\mathbf{w}$, and the point with position vector $\mathbf{v}$ lies on the line. Therefore, to get to any point on the line, we can first follow the position vector $\mathbf{v}$ from the origin onto the line, then follow a scaled version of the vector $\mathbf{w}$ to get to any other point on the line.



In particular, $\mathbf{r} = \mathbf{v} + \lambda\mathbf{w}$ describes a line through the point with position vector $\mathbf{v}$, parallel to $\mathbf{w}$. The condition $\mathbf{w} \neq \mathbf{0}$ is needed, else the equation $\mathbf{r} = \mathbf{v} + \lambda\mathbf{w} = \mathbf{v}$ would describe a single point instead of a line.

(b) We insert $\mathbf{r} = (x, y, z)$, $\mathbf{v} = (x_0, y_0, z_0)$ and $\mathbf{w} = (a, b, c)$ into the vector equation to get:
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_0 + \lambda a \\ y_0 + \lambda b \\ z_0 + \lambda c \end{pmatrix}.$$
This is a system of three equations, $x = x_0 + \lambda a$, $y = y_0 + \lambda b$, $z = z_0 + \lambda c$. Rearranging each equation for λ, and equating them, we have:
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$
as required.

If one of a, b, c is zero, we cannot perform the division in the solution of the equations. For example, if $a = 0$, then we have the equations $x = x_0$, $y = y_0 + \lambda b$, $z = z_0 + \lambda c$. This means that the resulting equation of the line, eliminating λ, is given by:
$$x = x_0, \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$
That is, the x-direction equation is 'decoupled' from the equations in the y-direction and z-direction. Similar conclusions hold when $b = 0$ or $c = 0$.

(c) Observe that $(1,1,0)-(1,0,1) = (0,1,-1)$ and $(1,-3,4)-(1,1,0) = (0,-4,4) = -4(0,1,-1)$. Hence the vectors joining the points $(1,0,1)$ to $(1,1,0)$, and then $(1,1,0)$ to $(1,-3,4)$, are parallel; it follows that all three points lie on a straight line.

In vector form, one possible equation of the line is:

$$\mathbf{r} = (1,1,0) + \lambda(0,1,-1),$$

but other forms are possible. Setting $\mathbf{r} = (x,y,z)$, we have:

$$(x,y,z) = (1,1,0) + \lambda(0,1,-1) \qquad \Rightarrow \qquad \{x=1,\ y-1=\lambda,\ -z=\lambda\}.$$

Therefore, eliminating λ, the Cartesian equation of the line is:

$$\{x=1,\ 1-y=z\}.$$

---

5. Show that the solution of the linear system $x-2y+3z=0, 3x-2y+z=0$ is a line that is equally inclined to the x and z-axes, and makes an angle $\arccos(\sqrt{2/3})$ with the y-axis.

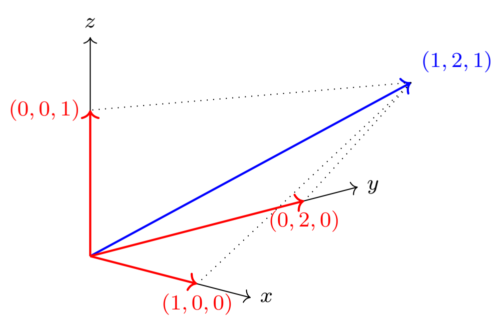
Solution: This is a linear system of equations with two equations, but three free variables. Thus we expect that one variable will be unconstrained. Therefore, we should try to write two of the variables in terms of the third variable.

Subtracting the second equation from the first, we obtain $-2x+2z=0$, which on rearrangement gives $x=z$. Substituting into the first equation, we have $z-2y+3z=0$, which on rearrangement gives $y=2z$. Thus the solution of the system can be written as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 2z \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

where z is a free real parameter. We recognise that this is the equation of a line going through the origin, with direction vector $(1,2,1)$.

To obtain the angles, we do some trigonometry (or, we use the *scalar product* - see later in the sheet).



We see from the figure that the angle the vector makes with both the x-axis and the z-axis is the same, given by:

$$\arccos\left(\frac{1}{\sqrt{1^2+2^2+1^2}}\right) = \arccos\left(\frac{1}{\sqrt{6}}\right).$$

The angle the vector makes with the y-axis is given by:

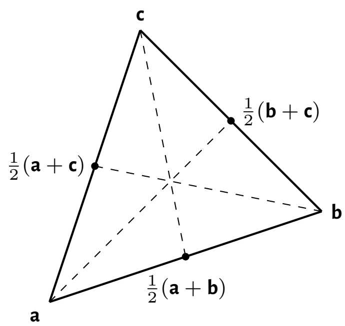
$$\arccos\left(\frac{2}{\sqrt{1^2+2^2+1^2}}\right) = \arccos\left(\frac{2}{\sqrt{6}}\right) = \arccos\left(\sqrt{\frac{2}{3}}\right),$$

as required.

6. (a) A *median* of a triangle is a line joining a vertex to the midpoint of its opposite edge. Prove that the three medians of a triangle are concurrent (the point at which they meet is called the *centroid* of the triangle).

   (b) Similarly, prove that in any tetrahedron, the lines joining the midpoints of opposite edges are concurrent.

➼ **Solution:** (a) Let the vertices of the triangle have position vectors **a**, **b**, **c** respectively, as shown in the figure below.



The midpoints of the edges of the triangle are:

$$\frac{\mathbf{a}+\mathbf{b}}{2}, \qquad \frac{\mathbf{b}+\mathbf{c}}{2}, \qquad \frac{\mathbf{a}+\mathbf{c}}{2}.$$

The vector equations of the lines going the midpoints and the opposite vertices are:

$$\mathbf{r}_1(\lambda) = \mathbf{c} + \lambda\left(\frac{\mathbf{a}+\mathbf{b}}{2} - \mathbf{c}\right), \qquad \mathbf{r}_2(\mu) = \mathbf{a} + \mu\left(\frac{\mathbf{b}+\mathbf{c}}{2} - \mathbf{a}\right), \qquad \mathbf{r}_3(\nu) = \mathbf{b} + \nu\left(\frac{\mathbf{a}+\mathbf{c}}{2} - \mathbf{b}\right).$$

Collecting like terms, these equations can be written as:

$$\mathbf{r}_1(\lambda) = \frac{\lambda}{2}\mathbf{a} + \frac{\lambda}{2}\mathbf{b} + (1-\lambda)\mathbf{c}, \qquad \mathbf{r}_2(\mu) = (1-\mu)\mathbf{a} + \frac{\mu}{2}\mathbf{b} + \frac{\mu}{2}\mathbf{c}, \qquad \mathbf{r}_3(\nu) = \frac{\nu}{2}\mathbf{a} + (1-\nu)\mathbf{b} + \frac{\nu}{2}\mathbf{c}.$$

Now, we would like to set $\mathbf{r}_1(\lambda) = \mathbf{r}_2(\mu) = \mathbf{r}_3(\nu)$, and find values of λ, μ, ν that solve these equations. *However*, in general **a**, **b**, **c** are *not* linearly independent in two dimensions so we cannot compare coefficients!

We can be a bit sneaky here: if we imagine that the triangle is not planar - that is, we imagine that **a**, **b**, **c** are *three-dimensional vectors* - we *can* assume they are linearly independent. Hence, comparing coefficients is completely okay here!

In the first equality, $\mathbf{r}_1(\lambda) = \mathbf{r}_2(\mu)$, we see that $\lambda = \mu$ and $\lambda/2 = 1-\mu$, which gives $\lambda = \mu = 2/3$. This gives the point of intersection of $\mathbf{r}_1, \mathbf{r}_2$ as:

$$\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Symmetrically, we see that $\mathbf{r}_2, \mathbf{r}_3$ and $\mathbf{r}_1, \mathbf{r}_3$ intersect at the same point, so all three lines intersect at the same point.

(b) For the second part, let the vertices of the tetrahedron have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. Then the midpoints of the edges are:

$$\frac{\mathbf{a}+\mathbf{b}}{2}, \qquad \frac{\mathbf{a}+\mathbf{c}}{2}, \qquad \frac{\mathbf{a}+\mathbf{d}}{2}, \qquad \frac{\mathbf{b}+\mathbf{c}}{2}, \qquad \frac{\mathbf{b}+\mathbf{d}}{2}, \qquad \frac{\mathbf{c}+\mathbf{d}}{2}.$$

The lines joining the midpoints of opposite edges are:

$$\mathbf{r}_1(\lambda) = \frac{\mathbf{a}+\mathbf{b}}{2}+\lambda\left(\frac{\mathbf{a}+\mathbf{b}}{2}-\frac{\mathbf{c}+\mathbf{d}}{2}\right), \qquad \mathbf{r}_2(\mu) = \frac{\mathbf{a}+\mathbf{c}}{2}+\mu\left(\frac{\mathbf{a}+\mathbf{c}}{2}-\frac{\mathbf{b}+\mathbf{d}}{2}\right), \qquad \mathbf{r}_3(\nu) = \frac{\mathbf{a}+\mathbf{d}}{2}+\nu\left(\frac{\mathbf{a}+\mathbf{d}}{2}-\frac{\mathbf{b}+\mathbf{c}}{2}\right).$$

Collecting like terms, these equations can be rewritten as:

$$\mathbf{r}_1(\lambda) = \frac{(1+\lambda)}{2}\mathbf{a}+\frac{(1+\lambda)}{2}\mathbf{b}-\frac{\lambda}{2}\mathbf{c}-\frac{\lambda}{2}\mathbf{d}, \qquad \mathbf{r}_2(\mu) = \frac{(1+\mu)}{2}\mathbf{a}-\frac{\mu}{2}\mathbf{b}+\frac{(1+\mu)}{2}\mathbf{c}-\frac{\mu}{2}\mathbf{d},$$

$$\mathbf{r}_3(\nu) = \frac{(1+\nu)}{2}\mathbf{a}-\frac{\nu}{2}\mathbf{b}-\frac{\nu}{2}\mathbf{c}+\frac{(1+\nu)}{2}\mathbf{d}.$$

As in part (a), we can compare coefficients here (even though $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are not linearly independent in three dimensions, we could embed the tetrahedron in a four-dimensional space if we wanted!). The equation $\mathbf{r}_1(\lambda) = \mathbf{r}_2(\mu)$ gives $\mu = \lambda$ and $1+\lambda = -\mu$, which tells us that $\lambda = \mu = -1/2$. This gives the point of intersection between the first two lines:

$$\frac{1}{4}\left(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}\right).$$

The remaining pairs of lines have the same intersection, by symmetry of the calculation, so we're done.

The scalar product

7. Let $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{R}^3$ be 3-vectors.

(a) Give the definition of the *scalar product* $\mathbf{v} \cdot \mathbf{w}$ in terms of lengths and angles. If $\mathbf{v}$ is a unit vector, explain why $\mathbf{v} \cdot \mathbf{w}$ is the signed length of the projection of $\mathbf{w}$ in the direction of $\mathbf{v}$.

(b) Using *only* this definition, prove each of the following properties of the scalar product:

(i) *commutativity*: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$;

(ii) *homogeneity*: $(\lambda \mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{v} \cdot \mathbf{w})$;

(iii) *left-distributivity*: $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$.

[Hint: In (ii), consider the cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ separately.]

Now let $\mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0), \mathbf{e}_3 = (0, 0, 1)$ be the standard basis vectors for $\mathbb{R}^3$, and let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ be 3-vectors.

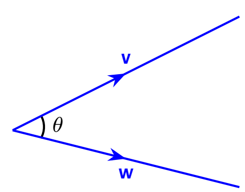
(c) Using *only* the definition of the scalar product in terms of lengths and angles, show that for $i, j = 1, 2, 3$, we have $\mathbf{e}_i \cdot \mathbf{e}_j = 1$ if $i = j$, and $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ if $i \neq j$.

(d) By writing $\mathbf{v}, \mathbf{w}$ as linear combinations of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and applying the properties of the scalar product from the previous question, prove the formula $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$.

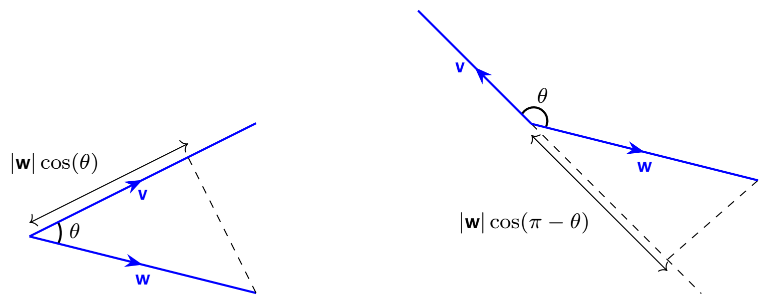
Solution: (a) The scalar product $\mathbf{v} \cdot \mathbf{w}$ is defined by:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos(\theta),$$

where $|\mathbf{v}|$, $|\mathbf{w}|$ are the lengths of the vectors, and θ is the *angle* between the vectors. Importantly, the angle is chosen to be the smaller angle produced when $\mathbf{v}, \mathbf{w}$ are positioned so that they start from the same point (see the diagram below).



Now consider the case where $\mathbf{v} = \hat{\mathbf{v}}$ is a unit vector. In this case, we have $\hat{\mathbf{v}} \cdot \mathbf{w} = |\mathbf{w}| \cos(\theta)$. Compare with the figures below:



If the angle θ is acute, we see that $|\mathbf{w}| \cos(\theta)$ is the length of the projection of $\mathbf{w}$ in the direction of $\mathbf{v}$. Hence $\hat{\mathbf{v}} \cdot \mathbf{w}$ is the length of the projection of $\mathbf{w}$ in the direction of $\mathbf{v}$.

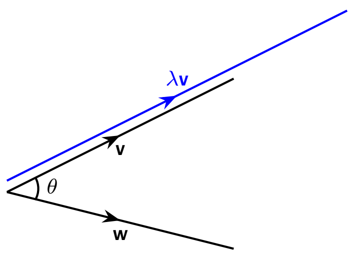
If the angle θ is obtuse, we see that $|\mathbf{w}|\cos(\pi-\theta) = -|\mathbf{w}|\cos(\theta)$ is the length of the projection of **w** in the direction of **v**. Hence $-\hat{\mathbf{v}} \cdot \mathbf{w}$ is the length of the projection of **w** in the direction of **v**.

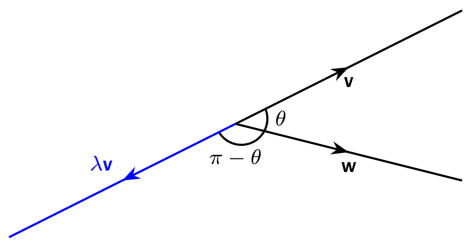
Overall then, we see that $\hat{\mathbf{v}} \cdot \mathbf{w}$ is the *signed length of the projection*; its magnitude is always equal to the length of the projection, but the sign depends on whether the vectors **v**, **w** are pointing in the same direction (the angle between them is acute), or in different directions (the angle between them is obtuse).

(b) The properties can be proved as follows:

(i) Commutativity. This property is obvious. In both the definitions of $\mathbf{v} \cdot \mathbf{w}$ and $\mathbf{w} \cdot \mathbf{v}$, we are required to place the starts of the vectors **v**, **w** in the same location. Both situations are identical, though, so the scalar products must be equal.

(ii) Homogeneity. We split into three cases: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Relevant diagrams in the cases $\lambda > 0$ and $\lambda < 0$ are shown in the figures below.





In the first case, $\lambda > 0$, we are in the situation of the left diagram above. The angle between $\lambda\mathbf{v}$ and **w** is the same as the angle between **v** and **w**. Hence, we have:

$$(\lambda\mathbf{v}) \cdot \mathbf{w} = |\lambda\mathbf{v}||\mathbf{w}|\cos(\theta) = \lambda|\mathbf{v}||\mathbf{w}|\cos(\theta) = \lambda(\mathbf{v} \cdot \mathbf{w}),$$

as required.

In the second case, $\lambda = 0$, the vector $\lambda\mathbf{v}$ has zero length. Hence $(\lambda\mathbf{v}) \cdot \mathbf{w} = 0 = \lambda(\mathbf{v} \cdot \mathbf{w})$ holds trivially.

In the third and final case, $\lambda < 0$, we are in the situation of the right diagram above. The angle between $\lambda\mathbf{v}$ and **w** is now *not* the same as the angle between **v** and **w**; instead, if the angle between **v** and **w** is θ, we see that the angle between $\lambda\mathbf{v}$ and **w** is $\pi - \theta$. Hence we have:

$$\begin{aligned}
(\lambda\mathbf{v}) \cdot \mathbf{w} &= |\lambda\mathbf{v}||\mathbf{w}|\cos(\pi-\theta) \\
&= (-\lambda)|\mathbf{v}||\mathbf{w}|\cos(\pi-\theta) && (\text{since } \lambda < 0, \text{ we have } |\lambda\mathbf{v}| = -\lambda|\mathbf{v}|) \\
&= \lambda|\mathbf{v}||\mathbf{w}|\cos(\theta) && (\text{since } \cos(\pi-\theta) = -\cos(\theta)) \\
&= \lambda(\mathbf{v} \cdot \mathbf{w}).
\end{aligned}$$

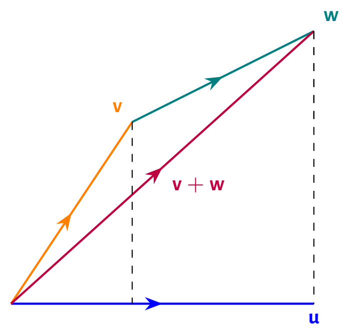
Thus the formula holds for all values of λ, as required.

(iii) Distributivity. The property $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$ is equivalent to:

$$(\mathbf{v} + \mathbf{w}) \cdot \hat{\mathbf{u}} = \mathbf{v} \cdot \hat{\mathbf{u}} + \mathbf{w} \cdot \hat{\mathbf{u}},$$

by the homogeneity property, that we proved in part (b). That is, the property is equivalent to the fact that: *the signed length of the projection of* $\mathbf{v} + \mathbf{w}$ *onto the direction of* $\mathbf{u}$ *is the same as the sum of the signed lengths of the projections of* $\mathbf{v}$ *and* $\mathbf{w}$ *onto the direction of* $\mathbf{u}$.

This property is very easy to see in two dimensions, with a nice, clear diagram! See the one below, for example.



This also holds in 3D, but drawing a diagram is much more fiddly, so we won't do it. One way of thinking about it is imagining that the above diagram is a 3D diagram, just with $\mathbf{v}$, $\mathbf{w}$ pointing at angles into and out of the page; the projections still look like those in the diagram.

(c) Each of the standard basis vectors has length 1, and is inclined at an angle $\pi/2$ to all the other basis vectors. Since $\cos(\pi/2) = 0$ and $\cos(0) = 1$, this implies:

$$\mathbf{e}_i \cdot \mathbf{e}_j = |\mathbf{e}_i||\mathbf{e}_j| \cos(0) = 1 \cdot 1 \cdot 1 = 1,$$

if $i = j$, and:

$$\mathbf{e}_i \cdot \mathbf{e}_j = |\mathbf{e}_i||\mathbf{e}_j| \cos(\pi/2) = 1 \cdot 1 \cdot 0 = 0,$$

if $i \neq j$, as required.

(d) We have $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ and $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$. For efficiency, we break down the computation of $\mathbf{v} \cdot \mathbf{w}$ into two parts:

- Observe that:

$$\begin{aligned}
(v_i\mathbf{e}_i) \cdot \mathbf{w} &= v_i\mathbf{e}_i \cdot \mathbf{w} && \text{(homogeneity)} \\
&= v_i\left(\mathbf{e}_i \cdot (w_1\mathbf{e}_1) + \mathbf{e}_i \cdot (w_2\mathbf{e}_2) + \mathbf{e}_i \cdot (w_3\mathbf{e}_3)\right) && \text{(distributivity, twice)} \\
&= v_iw_1\mathbf{e}_i \cdot \mathbf{e}_1 + v_iw_2\mathbf{e}_i \cdot \mathbf{e}_2 + v_iw_3\mathbf{e}_i \cdot \mathbf{e}_3 && \text{(symmetry and homogeneity)} \\
&= v_iw_i,
\end{aligned}$$

using the fact that $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for $i \neq j$, and 1 for $i = j$.

- Next, we have:

$$
\begin{aligned}
\mathbf{v}\cdot\mathbf{w} &= \mathbf{w}\cdot\mathbf{v} && \text{(symmetry)}\\
&= \mathbf{w}\cdot(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3)\\
&= \mathbf{w}\cdot(v_1\mathbf{e}_1) + \mathbf{w}\cdot(v_2\mathbf{e}_2) + \mathbf{w}\cdot(v_3\mathbf{e}_3) && \text{(distributivity, twice)}\\
&= (v_1\mathbf{e}_1)\cdot\mathbf{w} + (v_2\mathbf{e}_2)\cdot\mathbf{w} + (v_3\mathbf{e}_3)\cdot\mathbf{w} && \text{(symmetry)}\\
&= v_1w_1 + v_2w_2 + v_3w_3,
\end{aligned}
$$

  using the earlier part of the calculation in the final line. Hence, we're done!

---

8. Explain how we can use the two different formulae for the scalar product to determine the angles between vectors. Hence:
   (a) determine the angles AOB and OAB, where the points A, B have coordinates $(0,3,4)$, $(3,2,1)$ respectively;
   (b) find the acute angle at which two diagonals of a cube intersect.

Solution: From the previous question, we now have two formulae for the scalar product:

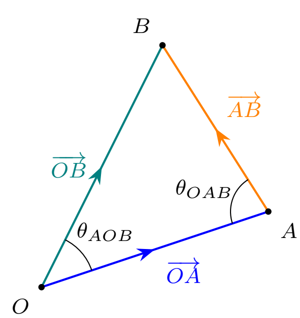
$$\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos(\theta) = v_1w_1 + v_2w_2 + v_3w_3.$$

Rearranging the second equality, we have:

$$\cos(\theta) = \frac{v_1w_1 + v_2w_2 + v_3w_3}{|\mathbf{v}||\mathbf{w}|},$$

which allows us to compute the angle between two vectors easily.

(a) In the first case, we want the angle between the vector $\overrightarrow{OA}$ and $\overrightarrow{OB}$ (remember that the angle we get from this formula is the smaller angle produced when the vectors start at the same point - see the diagram below).



The relevant vectors are:

$$\overrightarrow{OA} = (0,3,4), \qquad \overrightarrow{OB} = (3,2,1).$$

Hence we have:

$$\cos(\theta_{AOB}) = \frac{0\cdot 3 + 3\cdot 2 + 4\cdot 1}{\sqrt{3^2+4^2}\sqrt{3^2+2^2+1^2}} = \frac{10}{5\sqrt{14}} = \sqrt{\frac{2}{7}}.$$

Thus the angle is:

$$\theta_{AOB} = \arccos\sqrt{\frac{2}{7}}.$$

In the second case, we want the angle between the vector $\overrightarrow{AO}$ and $\overrightarrow{AB}$. The relevant vectors are:

$$\overrightarrow{AO} = (0, -3, -4), \qquad \overrightarrow{AB} = (3, 2, 1) - (0, 3, 4) = (3, -1, -3).$$

Hence we have:

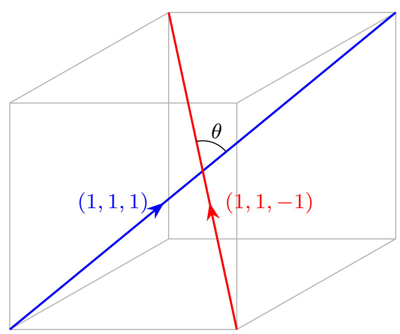
$$\cos(\theta_{OAB}) = \frac{0 \cdot 3 + (-3) \cdot (-1) + (-4) \cdot (-3)}{\sqrt{3^2 + 4^2}\sqrt{3^2 + 1^2 + 3^2}} = \frac{15}{5\sqrt{19}} = \frac{3}{\sqrt{19}}.$$

Thus the angle is:

$$\theta_{OAB} = \arccos\frac{3}{\sqrt{19}}.$$

(b) Without loss of generality, we may assume that we are working with a unit cube, of side length 1, since the angles are unchanged by scaling the cube up or down. Let's put the vertices of the cube at the points:

$$(0, 0, 0), \quad (0, 0, 1), \quad (0, 1, 1), \quad (0, 1, 0), \quad (1, 0, 0), \quad (1, 0, 1), \quad (1, 1, 1), \quad (1, 1, 0).$$



The vector from the origin to the opposite vertex is $(1, 1, 1)$. On the other hand, the vector from the vertex $(1, 0, 0)$ to the opposite vertex at $(0, 1, 1)$ is given by $(0, 1, 1) - (1, 0, 0) = (-1, 1, 1)$. The required angle θ therefore satisfies:

$$\cos(\theta) = \frac{1 \cdot (-1) + 1 \cdot 1 + 1 \cdot 1}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{3}.$$

Hence the angle is:

$$\theta = \arccos\frac{1}{3}.$$

9. Consider the line with vector equation $\mathbf{r} = (-1, 0, 1) + \lambda(3, 2, 1)$, where λ is a real parameter.

(a) Using the scalar product, compute the projection of the vector $(1, 2, 3)$ in the direction $(3, 2, 1)$.

(b) Hence, determine the point on the line which is closest to the point $(0, 2, 4)$, and the shortest distance from the line to the point $(0, 2, 4)$.

(c) Now, generalise your result: find a formula for the point on the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ which is closest to the point with position vector $\mathbf{p}$, and a formula for the the shortest distance from the line to the point.

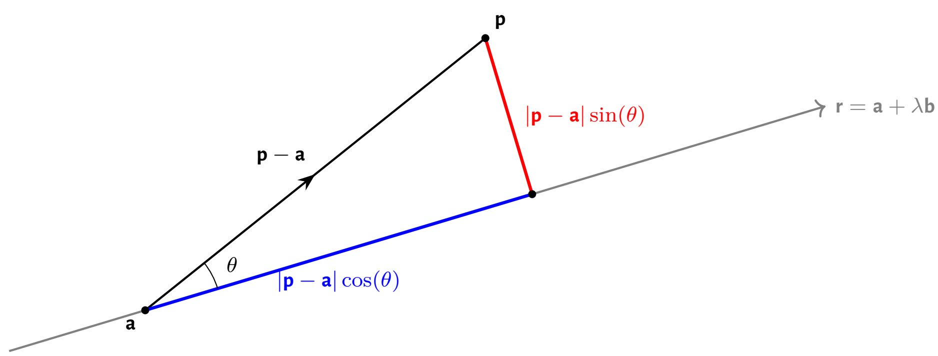
Solution: (a) As we learned in Question 7, the projection is:

$$(1, 2, 3) \cdot \frac{(3, 2, 1)}{\sqrt{3^2 + 2^2 + 1^2}} = \frac{10}{\sqrt{14}}.$$

(b) Our strategy will be the following:

- Find a vector joining the point to another special point on the line.
- Compute the projection of our vector onto the line.
- Move the length of the projection along the line, from the special point, to find the closest point.

This can be visualised with the diagram below, where $\mathbf{p} = (0, 2, 4)$ and $\mathbf{a} = (-1, 0, 1)$. The direction of the line is $\mathbf{b} = (3, 2, 1)$.



A vector joining the line to the point is $(0, 2, 4) - (-1, 0, 1) = (1, 2, 3)$. The projection of the vector onto the line is therefore:

$$(1, 2, 3) \cdot \frac{(3, 2, 1)}{|(3, 2, 1)|} = \frac{10}{\sqrt{14}},$$

from part (a). Therefore, the closest point on the line to the point is:

$$(-1, 0, 1) + \frac{10}{\sqrt{14}}\frac{(3, 2, 1)}{\sqrt{14}} = (-1, 0, 1) + (30/14, 20/14, 10/14) = (16/14, 20/14, 24/14) = \frac{1}{7}(8, 10, 12).$$

The shortest distance between the point and the line is therefore:

$$\left|(0, 2, 4) - \frac{1}{7}(8, 10, 12)\right| = \frac{1}{7}\left|(-8, 4, 16)\right| = \frac{4}{7}\left|(-2, 1, 4)\right| = \frac{4}{7}\sqrt{4 + 1 + 16} = \frac{4\sqrt{21}}{7} = 4\sqrt{\frac{3}{7}}.$$

(c) Now, we do the general argument. Following the diagram above, the vector joining the point to the line is $\mathbf{p} - \mathbf{a}$. The projection of the point onto the direction of the line is $\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})$. Hence the closest point on the line to the point is:

$$\mathbf{a} + (\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a}))\hat{\mathbf{b}}.$$

The shortest distance between the point and the line is therefore:

$$\left|\mathbf{p} - \mathbf{a} - \hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\hat{\mathbf{b}}\right|.$$

Alternatively, the shortest distance can be computed by Pythagoras:

$$\sqrt{|\mathbf{p} - \mathbf{a}|^2 - (|\mathbf{p} - \mathbf{a}|\cos(\theta))^2} = \sqrt{|\mathbf{p} - \mathbf{a}|^2 - (\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a}))^2}.$$

These expressions are the same, because:

$$\begin{aligned}
\left|\mathbf{p} - \mathbf{a} - \hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\hat{\mathbf{b}}\right| &= \sqrt{\left(\mathbf{p} - \mathbf{a} - \hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\hat{\mathbf{b}}\right) \cdot \left(\mathbf{p} - \mathbf{a} - \hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\hat{\mathbf{b}}\right)} \\
&= \sqrt{|\mathbf{p} - \mathbf{a}|^2 - 2\left(\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\right)^2 + \left(\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})|\right)^2} \\
&= \sqrt{|\mathbf{p} - \mathbf{a}|^2 - \left(\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\right)^2}.
\end{aligned}$$

It is not very useful to remember either version of this formula - but it *is* useful to know how to derive it. Knowing the method here is much more important than remembering a formula and being able to substitute things into it!

---

10. Show that if four points A, B, C, D are such that $AD \perp BC$ and $BD \perp AC$, then $CD \perp AB$.

Solution: Let the position vectors of the points be $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. We are given that $AD \perp BC$ and $BD \perp AC$, which in terms of vectors become:

$$(\mathbf{d} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{b}) = 0, \qquad (\mathbf{d} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{a}) = 0.$$

We want to show that:

$$(\mathbf{d} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) = 0.$$

Expanding all of these conditions using the properties of the scalar product, we have:

$$\mathbf{d} \cdot \mathbf{c} - \mathbf{d} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{b} = 0 \ (\dagger 1), \qquad \mathbf{d} \cdot \mathbf{c} - \mathbf{d} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{a} = 0 \ (\dagger 2),$$

and we want to show that:

$$\mathbf{d} \cdot \mathbf{b} - \mathbf{d} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{a} = 0. \tag{$*$}$$

But notice that equation ($*$) is just equation ($\dagger 2$) subtract equation ($\dagger 1$). So we are done!

11. Using the scalar product, prove that for any tetrahedron, the sum of the squares of the lengths of the edges equals four times the sum of the squares of the lengths of the lines joining the mid-points of opposite edges.

➾ **Solution:** Let the vertices of the tetrahedron have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$. Then the sum of the squares of the lengths of the edges of the tetrahedron is:

$$(\mathbf{b}-\mathbf{a})^2+(\mathbf{c}-\mathbf{a})^2+(\mathbf{d}-\mathbf{a})^2+(\mathbf{c}-\mathbf{b})^2+(\mathbf{d}-\mathbf{b})^2+(\mathbf{d}-\mathbf{c})^2. \tag{†}$$

The mid-points of the edges are:

$$\frac{\mathbf{a}+\mathbf{b}}{2},\qquad \frac{\mathbf{a}+\mathbf{c}}{2},\qquad \frac{\mathbf{a}+\mathbf{d}}{2},\qquad \frac{\mathbf{b}+\mathbf{c}}{2},\qquad \frac{\mathbf{b}+\mathbf{d}}{2},\qquad \frac{\mathbf{c}+\mathbf{d}}{2}.$$

The sum of the squares of the lengths joining the opposite mid-points is therefore:

$$\left(\frac{\mathbf{c}+\mathbf{d}}{2}-\frac{\mathbf{a}+\mathbf{b}}{2}\right)^2+\left(\frac{\mathbf{b}+\mathbf{d}}{2}-\frac{\mathbf{a}+\mathbf{c}}{2}\right)^2+\left(\frac{\mathbf{a}+\mathbf{d}}{2}-\frac{\mathbf{b}+\mathbf{c}}{2}\right)^2.$$

We need to make this look like (†). We can do this by organising the terms as follows:

$$\begin{aligned}
\left(\frac{\mathbf{c}+\mathbf{d}}{2}-\frac{\mathbf{a}+\mathbf{b}}{2}\right)^2 &= \frac{1}{4}\left((\mathbf{c}-\mathbf{a})+(\mathbf{d}-\mathbf{b})\right)^2=\frac{1}{4}\left((\mathbf{c}-\mathbf{a})^2+2(\mathbf{c}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{b})+(\mathbf{d}-\mathbf{b})^2\right),\\
\left(\frac{\mathbf{b}+\mathbf{d}}{2}-\frac{\mathbf{a}+\mathbf{c}}{2}\right)^2 &= \frac{1}{4}\left((\mathbf{b}-\mathbf{c})+(\mathbf{d}-\mathbf{a})\right)^2=\frac{1}{4}\left((\mathbf{b}-\mathbf{c})^2+2(\mathbf{b}-\mathbf{c})\cdot(\mathbf{d}-\mathbf{a})+(\mathbf{d}-\mathbf{a})^2\right),\\
\left(\frac{\mathbf{a}+\mathbf{d}}{2}-\frac{\mathbf{b}+\mathbf{c}}{2}\right)^2 &= \frac{1}{4}\left((\mathbf{a}-\mathbf{b})+(\mathbf{d}-\mathbf{c})\right)^2=\frac{1}{4}\left((\mathbf{a}-\mathbf{b})^2+2(\mathbf{a}-\mathbf{b})\cdot(\mathbf{d}-\mathbf{c})+(\mathbf{d}-\mathbf{c})^2\right).
\end{aligned}$$

We see that the sum of the right hand side contains all the terms we need, apart from the cross-terms; we just need to show that these all cancel. We have:

$$\begin{aligned}
&2(\mathbf{c}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{b})+2(\mathbf{b}-\mathbf{c})\cdot(\mathbf{d}-\mathbf{a})+2(\mathbf{a}-\mathbf{b})\cdot(\mathbf{d}-\mathbf{c})\\
&\qquad=2\left(\mathbf{c}\cdot\mathbf{d}-\mathbf{c}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{d}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{d}-\mathbf{b}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{d}+\mathbf{c}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{d}-\mathbf{a}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{d}+\mathbf{b}\cdot\mathbf{c}\right)\\
&\qquad=0,
\end{aligned}$$

and hence the result follows.

12.(a) Using the geometric definition of the scalar product, prove the *Cauchy-Schwarz inequality* $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

(b) From the Cauchy-Schwarz inequality, deduce the *triangle inequality* $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$. What is the geometrical significance of this inequality? Learn this equality off by heart; it will be useful later when we study limits!

(c) From the triangle inequality, deduce the *reverse triangle inequality* $||\mathbf{a}| - |\mathbf{b}|| \leq |\mathbf{a} - \mathbf{b}|$.

➾ **Solution:** (a) Since $|\cos(\theta)| \leq 1$, we have:

$$|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos(\theta)| \leq |\mathbf{a}||\mathbf{b}|,$$

which proves the Cauchy-Schwarz inequality, as required.

(b) We just proved an inequality to do with the scalar product, so to prove the triangle inequality, we should try to relate things to a scalar product. To do so, consider the square of $|\mathbf{a} + \mathbf{b}|$. We have:

$$\begin{aligned}
|\mathbf{a} + \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) && \text{(since } |\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}) \\
&= \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\
&= |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 \\
&\leq |\mathbf{a}|^2 + 2|\mathbf{a} \cdot \mathbf{b}| + |\mathbf{b}|^2 && \text{(since for any } x \text{, we have } x \leq |x|) \\
&\leq |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 && \text{(Cauchy-Schwarz)} \\
&= (|\mathbf{a}| + |\mathbf{b}|)^2.
\end{aligned}$$

To finish, we take the square root of both sides, which gives:

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|,$$

which is the triangle inequality as required.

If we consider the triangle formed by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{a} - \mathbf{b}$, then the triangle inequality tells us that the lengths of these vectors satisfy $|\mathbf{a} - \mathbf{b}| \leq |\mathbf{a}| + |-\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$. In particular, it tells us that the third side of a triangle cannot be longer than the sum of the lengths of the other two sides.

(c) Using the triangle inequality, we have:

$$|\mathbf{a}| = |\mathbf{a} - \mathbf{b} + \mathbf{b}| \leq |\mathbf{a} - \mathbf{b}| + |\mathbf{b}| \qquad \Rightarrow \qquad |\mathbf{a}| - |\mathbf{b}| \leq |\mathbf{a} - \mathbf{b}|.$$

Additionally, we have:

$$|\mathbf{b}| = |\mathbf{b} - \mathbf{a} + \mathbf{a}| \leq |\mathbf{a} - \mathbf{b}| + |\mathbf{a}| \qquad \Rightarrow \qquad |\mathbf{b}| - |\mathbf{a}| \leq |\mathbf{a} - \mathbf{b}|.$$

Putting these together, we see that:

$$||\mathbf{a}| - |\mathbf{b}|| \leq |\mathbf{a} - \mathbf{b}|,$$

as required. This inequality tells us that the length of the third side of a triangle is always at least the difference of the lengths of the other two sides.

The equation of a plane

13. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ be fixed 3-vectors, with $\mathbf{b} \neq \mathbf{0}$.

(a) Explain why the equation $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$ represents a plane, and summarise its properties. Show using properties of the scalar product that an equivalent representation of this plane is $\mathbf{r} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$. What is the geometric significance of the quantity $|\mathbf{a} \cdot \mathbf{b}|/|\mathbf{b}|$ here?

(b) By writing $\mathbf{r} = (x, y, z)$, $\mathbf{b} = (l, m, n)$, and $\mathbf{a} \cdot \mathbf{b} = d$, show that the equation of a plane may equivalently be written in the Cartesian form $lx + my + nz = d$.

(c) Find the equation of the plane containing the point $(3, 2, 1)$ with normal $(1, 2, 3)$ in both the vector form, as in (a), and the Cartesian form, as in (b). What is the shortest distance from the origin to the plane?

Solution: (a) The equation $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$ states that the vector from the point **a** to the point **r** is orthogonal to the vector **b**. Thus this equation represents a plane which is normal to the vector **b**, and passes through the point **a**. Using distributivity of the scalar product, we have $0 = (\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = \mathbf{r} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b}$. Rearranging, we obtain $\mathbf{r} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ as required.

Let θ be the angle between **r** and **b**. Then the equation of the plane may be written as:

$$|\mathbf{r}||\mathbf{b}| \cos(\theta) = \mathbf{a} \cdot \mathbf{b}.$$

Since $\mathbf{a} \cdot \mathbf{b}$ is fixed, $|\mathbf{r}|$ is minimised when the magnitude of $\cos(\theta)$ is maximised (the sign of $\cos(\theta)$ must match the sign of $\mathbf{a} \cdot \mathbf{b}$). Since the maximum value of the magnitude of $\cos(\theta)$ is ± 1, this shows that *the shortest distance from the origin to the plane* is given by:

$$|\mathbf{r}| = \frac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{b}|}.$$

(b) Writing $d = \mathbf{a} \cdot \mathbf{b}$, $\mathbf{r} = (x, y, z)$ and $\mathbf{b} = (l, m, n)$, the equation $\mathbf{r} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ becomes $(x, y, z) \cdot (l, m, n) = d$, which multiplies out to $lx + my + nz = d$. This is the Cartesian form of a plane.

(c) The equation of the plane in vector form is:

$$\left(\mathbf{r} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\right) \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0.$$

Letting $\mathbf{r} = (x, y, z)$, this multiplies out to:

$$x + 2y + 3z - 3 - 4 - 3 = 0 \qquad \Rightarrow \qquad x + 2y + 3z = 10,$$

which is the Cartesian equation of the plane. By part (a), the shortest distance from the origin to the plane is given by:

$$\frac{10}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{10}{\sqrt{14}}.$$

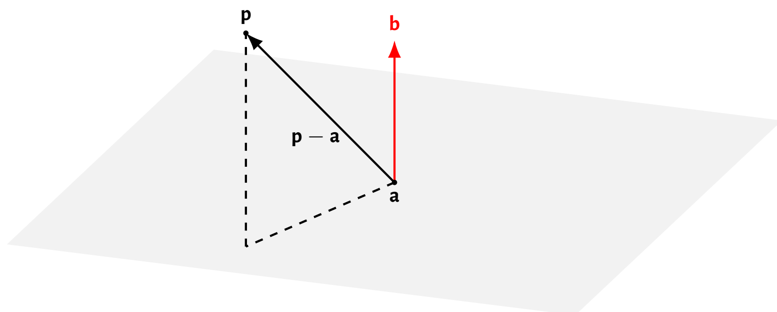
14. Consider the plane with vector equation $(\mathbf{r} - (1, 0, 1)) \cdot (2, -1, 0) = 0$.

(a) Using the scalar product, compute the projection of the vector $(2, 0, 3)$ in the direction $(2, -1, 0)$.

(b) Using the result of part (a), determine the point on the plane which is closest to the point $(3, 0, 4)$, and the shortest distance from the plane to the point $(3, 0, 4)$.

(c) Now, generalise your result: find a formula for the point on the plane $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$ which is the closest to the point with position vector $\mathbf{p}$, and a formula for the shortest distance from the plane to this point.

Solution: When doing this question, we should have in mind the following diagram:



(a) For the first part, we have:

$$(2, 0, 3) \cdot \frac{(2, -1, 0)}{\sqrt{2^2 + 1^2}} = \frac{4}{\sqrt{5}}.$$

(b) Next, we note that a vector joining the point to the plane is $(3, 0, 4) - (1, 0, 1) = (2, 0, 3)$. The length of the projection of this vector in the direction of the normal $(2, -1, 0)$ will be the shortest distance to the plane, which by part (a) is $4/\sqrt{5}$.

To get the closest point, we subtract the component of $(2, 0, 3)$ in the direction normal to the plane, giving us:

$$(2, 0, 3) - \frac{4}{\sqrt{5}} \frac{(2, -1, 0)}{\sqrt{5}} = (2, 0, 3) - (8/5, -4/5, 0) = (2/5, 4/5, 3).$$

This vector is the planar component of the vector joining the plane and the point. Adding this vector to $(1, 0, 1)$, we must get the closest point on the plane to the point:

$$(1, 0, 1) + (2/5, 4/5, 3) = (7/5, 4/5, 3).$$

(c) Now, we do the same argument with abstract vectors. A vector joining the plane to the point is $\mathbf{p} - \mathbf{a}$. The length of the projection of this vector in a direction normal to the plane will be the shortest distance to the plane, which is given by:

$$\left| \hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a}) \right|$$

The modulus is required, because the projection is a *signed* projection. To finish, the component of the vector $\mathbf{p} - \mathbf{a}$ parallel to the plane is then:

$$\mathbf{p} - \mathbf{a} + (\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a}))\hat{\mathbf{b}}.$$

Adding this to the point $\mathbf{a}$, we get the closest point on the plane to the point:

$$\mathbf{p} + \left(\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a}) \right) \hat{\mathbf{b}}.$$

15. Using the results of Question 14, calculate the shortest distances between the plane $5x + 2y - 7z + 9 = 0$ and the points $(1, -1, 3)$ and $(3, 2, 3)$. Are the points on the same side of the plane?

➾ **Solution:** By inspection, a point on the plane is $(-1, -2, 0)$, and the normal to the plane is $(5, 2, -7)$. Therefore vectors joining the points to the plane are $(1, -1, 3) - (-1, -2, 0) = (2, 1, 3)$ and $(3, 2, 3) - (-1, -2, 0) = (4, 4, 3)$. The shortest distances are then given by:

$$\left|(2, 1, 3) \cdot \frac{(5, 2, -7)}{\sqrt{5^2 + 2^2 + 7^2}}\right| = \left|-\frac{9}{\sqrt{78}}\right| = \frac{9}{\sqrt{78}},$$

and:

$$\left|(4, 4, 3) \cdot \frac{(5, 2, -7)}{\sqrt{78}}\right| = \frac{7}{\sqrt{78}}.$$

Importantly, we see that the modulus was relevant in the first case but not in the second. This means that the angle between the normal and the vector joining a point in the plane to the point off the plane is obtuse in the first case, and acute in the second case (think about the signs of the scalar product). Hence, they must be on opposite sides of the plane.

Equations of other 3D surfaces

16. Let k, m be positive constants, with $m < 1$. Describe the following surfaces: (a) $|\mathbf{r}| = k$; (b) $\mathbf{r} \cdot \mathbf{u} = m|\mathbf{r}|$.

➾ **Solution:**

(a) This surface, $|\mathbf{r}| = k$, comprises the set of all vectors whose distance from the origin is equal to k. Hence this is a sphere centred on the origin of radius k.

(b) Write $\mathbf{r} \cdot \mathbf{u} = |\mathbf{r}| \cos(\theta)$ (using the fact that $\mathbf{u}$ is a unit vector). Then the equation can be rewritten as:

$$\cos(\theta) = m.$$

Hence, this surface consists of all vectors which are a constant angle $\arccos(m)$ to the vector $\mathbf{u}$. Thus the surface is a *cone*, with axis along $\mathbf{u}$. The tip of the cone is at the origin, since $\mathbf{r} = \mathbf{0}$ satisfies the equation.

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17. Describe the surface given by the vector equation:

$$|\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}| = 2,$$

where $\mathbf{u} = \frac{1}{\sqrt{2}}(1, 0, 1)$. What is the intersection of this surface and the surface $x + z = 0$?

➾ **Solution:** Note that $\mathbf{u}$ is a unit vector, so $\mathbf{r} \cdot \mathbf{u}$ is the length of the projection of $\mathbf{u}$ in the direction of $\mathbf{u}$. Hence the vector $\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}$ represents the vector $\mathbf{r}$ with its $\mathbf{u}$ component entirely 'removed'. That is, $\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}$ is the *projection of the vector* $\mathbf{r}$ *orthogonal to the* $\mathbf{u}$ *direction*.

The equation of the surface tells us that the component of $\mathbf{r}$ orthogonal to $\mathbf{u}$ is constant, and equal to 2. Thus this equation describes a *cylinder of radius* 2, with axis along the vector $\mathbf{u}$.

Since the direction of $\mathbf{u}$ in this question is $(1, 0, 1)$, this is the same as the direction orthogonal to the plane $x + z = 0$. Hence, the intersection of $x + z = 0$ and the figure $|\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}| = 2$ must be a circle of radius 2, centred on the origin, in the plane orthogonal to the vector $(1, 0, 1)$.

18.(a) Write down a vector equation for the sphere with centre at the point with position vector **a**, and radius $p > 0$.

(b) If there is a second sphere with centre at the point with position vector **b**, and radius $q > 0$, what conditions are required on **a**, **b**, p and q for the two spheres to intersect in a circle?

(c) Show that, if the two spheres do intersect, then the plane in which their intersection occurs is given by the equation $2\mathbf{r} \cdot (\mathbf{b} - \mathbf{a}) = p^2 - q^2 + |\mathbf{b}|^2 - |\mathbf{a}|^2$.

Solution: (a) The vector equation of the sphere is $|\mathbf{r} - \mathbf{a}| = p$, since the distance between any point on the sphere **r** and the sphere's centre **a** must always be equal to the radius p.

(b) If there is a second sphere $|\mathbf{r} - \mathbf{b}| = q$, we need the distance between the centres of the spheres to be less than the sum of the radii of the spheres. That is, we need:

$$|\mathbf{b} - \mathbf{a}| < p + q.$$

We need strict inequality here, because if $|\mathbf{b} - \mathbf{a}| = p + q$, then the spheres just 'touch' at a single point.

(c) To get the plane of intersection, consider squaring the two equations of the spheres and using properties of the scalar product:

$$p^2 = |\mathbf{r} - \mathbf{a}|^2 = (\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{a}) = |\mathbf{r}|^2 - 2\mathbf{r} \cdot \mathbf{a} + |\mathbf{a}|^2$$

$$q^2 = |\mathbf{r} - \mathbf{b}|^2 = (\mathbf{r} - \mathbf{b}) \cdot (\mathbf{r} - \mathbf{b}) = |\mathbf{r}|^2 - 2\mathbf{r} \cdot \mathbf{b} + |\mathbf{b}|^2.$$

Subtracting the second equation from the first, we have:

$$2\mathbf{r} \cdot (\mathbf{b} - \mathbf{a}) + |\mathbf{a}|^2 - |\mathbf{b}|^2 = p^2 - q^2.$$

Rearranging, we have:

$$2\mathbf{r} \cdot (\mathbf{b} - \mathbf{a}) = p^2 - q^2 + |\mathbf{b}|^2 - |\mathbf{a}|^2,$$

as required.

The vector product

19. Let **v**, **w**, **u** $\in \mathbb{R}^3$ be 3-vectors.

(a) Give the geometrical definition of the *vector product* (or *cross product*) $\mathbf{v} \times \mathbf{w}$ in terms of lengths, angles and an appropriate perpendicular vector.

(b) Using *only* this definition, prove each of the following properties of the vector product:

(i) *anti-commutativity*: $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$;

(ii) *homogeneity*: $(\lambda \mathbf{v}) \times \mathbf{w} = \lambda(\mathbf{v} \times \mathbf{w})$;

(iii) *left-distributivity*: $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$.

*[Hint: In (ii), consider the cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ separately. In (iii), start by explaining why $\mathbf{v} \times \hat{\mathbf{u}}$ is the projection of **v** onto the plane through the origin perpendicular to **u**, followed by a rotation by $\frac{1}{2}\pi$.]*

Now let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the standard basis vectors for $\mathbb{R}^3$, and let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ be 3-vectors.

(c) Using *only* the definition of the vector product in terms of lengths, angles, and a perpendicular vector, show that $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ and $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$.

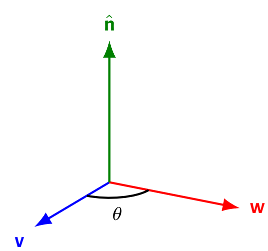
(d) By writing **v**, **w** as linear combinations of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and applying the properties of the vector product from the previous question, prove the standard formula:

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2, \quad v_3 w_1 - v_1 w_3, \quad v_1 w_2 - v_2 w_1).$$

➼ **Solution:** (a) We define the *vector product* of **v**, **w** to be the vector:

$$\mathbf{v} \times \mathbf{w} := |\mathbf{v}||\mathbf{w}| \sin(\theta)\hat{\mathbf{n}},$$

where θ is the angle between the vectors **v**, **w** and $\hat{\mathbf{n}}$ is a unit vector chosen to be orthogonal to the plane of **v**, **w**, in a direction such that the vectors **v**, **w**, $\hat{\mathbf{n}}$ taken in that order have a *right-handed orientation*.

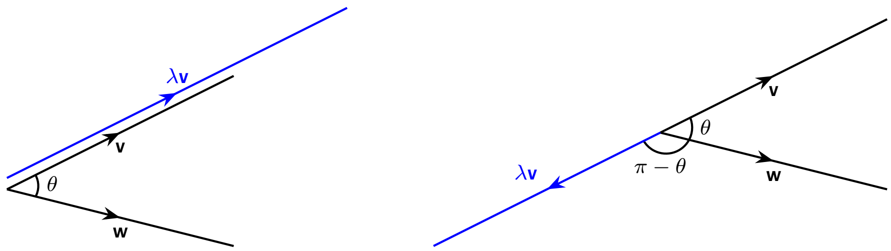


The above figure illustrates the right-handedness of the system. If you put your right index finger in the direction of **v**, and your right middle finger in the direction of **w**, then your thumb points in the direction of $\hat{\mathbf{n}}$, i.e. the direction of the vector product.

(b) We prove each of the properties in turn:

(i) ANTI-COMMUTATIVITY. This property follows from the right-hand rule. The vectors $\mathbf{v} \times \mathbf{w}$, $\mathbf{w} \times \mathbf{v}$ evidently have the same magnitude, because they have the same angle θ between them in both cases. However, in the first case our vector $\hat{\mathbf{n}}$ is pointing in the opposite direction to our vector in the second case, because of the right-hand rule. Hence $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$, as required.

(ii) Homogeneity. We draw some similar diagrams to those that we did for the scalar product. Below, we are showing the plane containing the two vectors **v**, **w**.



In the case where $\lambda > 0$, we have a diagram like the one on the left of the page. The vector product $(\lambda \mathbf{v}) \times \mathbf{w}$ points into the page, by the right-hand rule; let $\hat{\mathbf{n}}$ be a unit vector pointing into the page throughout. Similarly, the vector product $\mathbf{v} \times \mathbf{w}$ points into the page by the right-hand rule. Both $\lambda\mathbf{v}$, **w** and **v**, **w** have the angle θ between them. So we have:

$$(\lambda \mathbf{v}) \times \mathbf{w} = |\lambda \mathbf{v}||\mathbf{w}| \sin(\theta)\hat{\mathbf{n}} = \lambda|\mathbf{v}||\mathbf{w}| \sin(\theta)\hat{\mathbf{n}} = \lambda(\mathbf{v} \times \mathbf{w}),$$

as required.

In the case where $\lambda < 0$, we have a diagram like the one on the right of the page. The vector product $(\lambda \mathbf{v}) \times \mathbf{w}$ now points *out* of the page, by the right-hand rule. However, the vector product $\mathbf{v} \times \mathbf{w}$ points into the page by the right-hand rule. The vectors $\lambda\mathbf{v}$, **w** have an angle $\pi - \theta$ between them, and the vectors **v**, **w** have an angle θ between them. So we have:

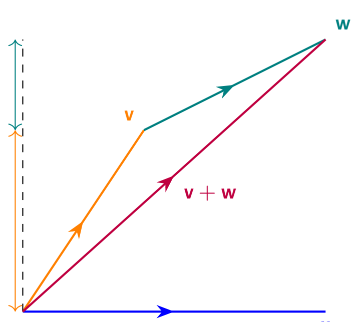
$$(\lambda \mathbf{v}) \times \mathbf{w} = |\lambda \mathbf{v}||\mathbf{w}| \sin(\pi - \theta)(-\hat{\mathbf{n}}) = -|\lambda||\mathbf{v}||\mathbf{w}| \sin(\theta)\hat{\mathbf{n}} = \lambda|\mathbf{v}||\mathbf{w}| \sin(\theta)\hat{\mathbf{n}} = \lambda(\mathbf{v} \times \mathbf{w}),$$

as required. Lastly, in the case $\lambda = 0$, we have that both sides are equal to the zero vector **0**. So we're done!

(iii) Left-distributivity. Here, we follow the hint. If θ is the angle between the vector **v** and the vector **u**, then note that $|\mathbf{v}| \sin(\theta)$ is the length of the projection of **v** orthogonal to **u**. Hence, $|\mathbf{v}||\hat{\mathbf{u}}| \sin(\theta) = |\mathbf{v} \times \hat{\mathbf{u}}|$ is the magnitude of the projection of **v** orthogonal to **u**. Importantly though, $\mathbf{v} \times \hat{\mathbf{u}}$ lies in the plane *perpendicular* to the plane containing both **v**, $\hat{\mathbf{u}}$. So $\mathbf{v} \times \hat{\mathbf{u}}$ is not quite the projection of the vector **v** orthogonal to **u**, but is instead this projection *rotated by* $\pi/2$.

As a result, to prove left-distributivity, it is sufficient by homogeneity to prove $(\mathbf{v} + \mathbf{w}) \times \hat{\mathbf{u}} = \mathbf{v} \times \hat{\mathbf{u}} + \mathbf{w} \times \hat{\mathbf{u}}$; in other words, the projection of $\mathbf{v} + \mathbf{w}$ onto a direction orthogonal to **u** is equal to the sum of the projections of **v**, **w** in a direction orthogonal to **u** (in both cases, a rotation by $\pi/2$ is carried out afterwards - so we can ignore it, because it is equivalent in both cases).

If all the vectors lie in the same plane, this is easily illustrated using a similar diagram to the scalar product one:



Similarly to the scalar product, this also holds if the vectors are not all in the same plane, but drawing a diagram is much more fiddly, so we won't do it. One way of thinking about it is imagining that the above diagram is a 3D diagram, just with **v**, **w** pointing at angles into and out of the page; the projections still look like those in the diagram.

(c) Since $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are all perpendicular to one another, we have:

$$|\mathbf{e}_1 \times \mathbf{e}_2| = |\mathbf{e}_1||\mathbf{e}_2| \sin(\pi/2) = 1,$$

and similarly for the other combinations. By the right-hand rule, the directions of the given vector products are determined, giving the results in the question.

It is also helpful to notice that $\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{0}$, etc, because these vectors have angle $\theta = 0$ between them.

(d) We break down the calculation into two parts. First, we have:

$$\begin{aligned}
(v_i\mathbf{e}_i) \times \mathbf{w} &= (v_i\mathbf{e}_i) \times (w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3) \\
&= v_i\mathbf{e}_i \times (w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3) && \text{(homogeneity)} \\
&= -v_i(w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3) \times \mathbf{e}_i && \text{(anti-commutativity)} \\
&= -v_i\left((w_1\mathbf{e}_1) \times \mathbf{e}_i + (w_2\mathbf{e}_2) \times \mathbf{e}_i + (w_3\mathbf{e}_3) \times \mathbf{e}_i\right) && \text{(left-distributivity, twice)} \\
&= -v_iw_1\mathbf{e}_1 \times \mathbf{e}_i - v_iw_2\mathbf{e}_2 \times \mathbf{e}_i - v_iw_3\mathbf{e}_3 \times \mathbf{e}_i && \text{(homogeneity)}
\end{aligned}$$

Now, depending on the value of i, we get:

$$(v_i\mathbf{e}_i) \times \mathbf{w} = \begin{cases} v_1w_2\mathbf{e}_3 - v_1w_3\mathbf{e}_2, & i = 1, \\ -v_2w_1\mathbf{e}_3 + v_2w_3\mathbf{e}_1, & i = 1, \\ v_3w_1\mathbf{e}_2 - v_3w_2\mathbf{e}_1, & i = 3. \end{cases}$$

To finish, we use left-distributivity twice again:

$$\begin{aligned}
\mathbf{v} \times \mathbf{w} &= (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \times \mathbf{w} \\
&= (v_1\mathbf{e}_1) \times \mathbf{w} + (v_2\mathbf{e}_2) \times \mathbf{w} + (v_3\mathbf{e}_3) \times \mathbf{w} \\
&= v_1w_2\mathbf{e}_3 - v_1w_3\mathbf{e}_2 - v_2w_1\mathbf{e}_3 + v_2w_3\mathbf{e}_1 + v_3w_1\mathbf{e}_2 - v_3w_2\mathbf{e}_1 \\
&= (v_2w_2 - v_3w_2)\mathbf{e}_1 + (v_3w_1 - v_1w_2)\mathbf{e}_2 + (v_1w_2 - v_2w_1)\mathbf{e}_3,
\end{aligned}$$

as required. Phew, that was hard work!

20. Find the angle between the position vectors of the points $(2,1,1)$ and $(3,-1,-5)$, and find the direction cosines of a vector perpendicular to both. Can both the angle and vector be computed using *only* the vector product?

➾ **Solution:** Computing the scalar product, we have $(2,1,1)\cdot(3,-1,-5) = 6-1-5 = 0$, thus the vectors are orthogonal, so are at an angle $\pi/2$. To find the direction cosines of a vector perpendicular to both, we compute their vector product:

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix} = \begin{pmatrix} -4 \\ 13 \\ -5 \end{pmatrix}.$$

The *direction cosines* are the just the cosines of the angles that this vector makes with the x, y, z axes. To find these, we just normalise the vector, which has length $\sqrt{4^2+13^2+5^2} = \sqrt{210}$. This gives the direction cosines:

$$-\frac{4}{\sqrt{210}}, \qquad \frac{13}{\sqrt{210}}, \qquad -\frac{5}{\sqrt{210}}.$$

The formula for the vector product, $\mathbf{v} \times \mathbf{w} = |\mathbf{v}||\mathbf{w}|\sin(\theta)\hat{\mathbf{n}}$ makes it look like the angle can be computed using the vector product - just take the lengths of both sides to give $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}|\sin(\theta)$, which rearranges to:

$$\sin(\theta) = \frac{|\mathbf{v} \times \mathbf{w}|}{|\mathbf{v}||\mathbf{w}|}.$$

In general, the problem is that this equation usually has two solutions in the range $[0,\pi]$, say θ and $\pi-\theta$. We can't tell if vectors are at an acute or an obtuse angle if we just use the vector product normally!

However, in this question since the vectors are inclined at $\pi/2$, there is precisely *one* solution to the equation. So we could have used the vector product in this case! But, it is a bad method in general.

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21. Find all points $\mathbf{r}$ which satisfy $\mathbf{r} \times \mathbf{a} = \mathbf{b}$ where $\mathbf{a} = (1,1,0)$ and $\mathbf{b} = (1,-1,0)$.

➾ **Solution:** Let $\mathbf{r} = (x,y,z)$. Then taking the vector product, our equation becomes:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -z \\ z \\ x-y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Comparing components, we see that $z=-1$ and $x=y$. Thus the set of points that satisfy this equation is $(\lambda,\lambda,-1)$. This is a line going through the point $(0,0,-1)$ parallel to the vector $(1,1,0)$.

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22. Using properties of the vector product, prove the identity $(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}) = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$.

➾ **Solution:** By the left and right distributive properties, we have:

$$(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}) = (\mathbf{b}-\mathbf{a}) \times \mathbf{c} - (\mathbf{b}-\mathbf{a}) \times \mathbf{a} = \mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} + \mathbf{a} \times \mathbf{a}.$$

Now, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ since the angle between a vector and itself is zero. Finally, using antisymmetry of the vector product we have $\mathbf{a} \times \mathbf{c} = -\mathbf{c} \times \mathbf{a}$ and $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$. The result follows.

Part IA: Mathematics for Natural Sciences B
Examples Sheet 2: Further vector geometry, triple products, vector area, and polar coordinate systems

Model Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

More on the equation of a line

1. (a) Explain why the line through the points with positions vectors $\mathbf{a}$, $\mathbf{b}$ is $(\mathbf{r}-\mathbf{a})\times(\mathbf{b}-\mathbf{a})=\mathbf{0}$. Show using properties of the vector product that an equivalent representation of this line is $\mathbf{r}\times(\mathbf{b}-\mathbf{a})=\mathbf{a}\times\mathbf{b}$. What is the geometric significance of the quantity $|\mathbf{a}\times\mathbf{b}|/|\mathbf{b}-\mathbf{a}|$ here?

   (b) Express the line $\mathbf{r}=(1,0,1)+\lambda(3,-1,0)$ in the form $\mathbf{r}\times\mathbf{c}=\mathbf{d}$.

Solution: (a) The direction of the line is $\mathbf{b}-\mathbf{a}$. For any point on the line $\mathbf{r}$, we must have $\mathbf{r}-\mathbf{a}$ parallel to this direction. This happens if and only if $(\mathbf{r}-\mathbf{a})\times(\mathbf{b}-\mathbf{a})=\mathbf{0}$, as required. Hence, this is an alternative way of writing the equation of a line.

Using the distributive property of the vector product, we can expand this to give $\mathbf{r}\times(\mathbf{b}-\mathbf{a})-\mathbf{a}\times(\mathbf{b}-\mathbf{a})=\mathbf{0}$. Rearranging, and using the distributive property again, we have:

$$\begin{aligned}\mathbf{r}\times(\mathbf{b}-\mathbf{a})&=\mathbf{a}\times(\mathbf{b}-\mathbf{a})\\&=\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{a}\\&=\mathbf{a}\times\mathbf{b}&&(\text{since }\mathbf{a}\times\mathbf{a}=\mathbf{0})\end{aligned}$$

We can interpret the quantity $|\mathbf{a}\times\mathbf{b}|/|\mathbf{b}-\mathbf{a}|$ in a similar way to Question 13(a) from Sheet 1. Writing the left hand side of this equation as $|\mathbf{r}||\mathbf{b}-\mathbf{a}|\sin(\theta)\hat{\mathbf{n}}$, then taking the length of both sides, we have:

$$|\mathbf{r}|\sin(\theta)=\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}-\mathbf{a}|}.$$

This tells us that $|\mathbf{r}|$ is minimised when $\sin(\theta)$ is maximised, i.e. when $\sin(\theta)=1$ occurring at $\theta=\pi/2$. This tells us that the shortest distance between the origin and the line is given by:

$$\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}-\mathbf{a}|}.$$

(b) The given line goes through the point $(1,0,1)$ and has direction $(3,-1,0)$. Hence, it has the equation:

$$(\mathbf{r}-(1,0,1))\times(3,-1,0)=\mathbf{0}.$$

Rearranging, we have:

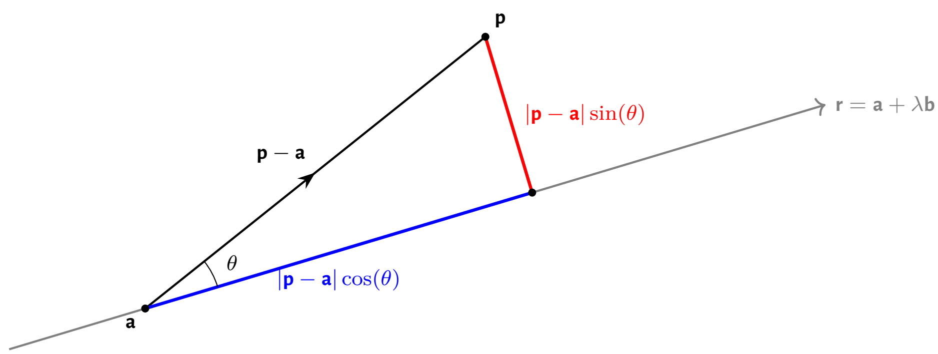
$$\mathbf{r}\times\begin{pmatrix}3\\-1\\0\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}3\\-1\\0\end{pmatrix}=\begin{pmatrix}1\\3\\-1\end{pmatrix}.$$

2. (a) Show that the shortest distance between the point **p** and the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ can be written as $\left|\hat{\mathbf{b}} \times (\mathbf{p} - \mathbf{a})\right|$.

   (b) ($*$) Does this formula agree with the one you derived in terms of the scalar product in Question 10(c) of Sheet 1? [*Hint: Try squaring the formula in part (a), and using properties of the scalar triple product - see later in the sheet!*]

   (c) Find the shortest distance from a vertex of a unit cube to a diagonal excluding that vertex using both the formula in (a), and the formula from Question 10(c) of Sheet 1, and check that your answers agree.

Solution: (a) Consider the diagram below:



The shortest distance between the line and the point is $|\mathbf{p}-\mathbf{a}|\sin(\theta)$, which is the magnitude of the vector product $\hat{\mathbf{b}}\times(\mathbf{p}-\mathbf{a})$, as required.

(b) Yes, this formula agrees with the one we derived in Question 10(c) of Sheet 1. To see this, we square the formula from part (a):

$$\begin{aligned}
\left|\hat{\mathbf{b}} \times (\mathbf{p} - \mathbf{a})\right|^2 &= \left(\hat{\mathbf{b}} \times (\mathbf{p} - \mathbf{a})\right) \cdot \left(\hat{\mathbf{b}} \times (\mathbf{p} - \mathbf{a})\right) \\
&= \hat{\mathbf{b}} \cdot \left((\mathbf{p} - \mathbf{a}) \times (\hat{\mathbf{b}} \times (\mathbf{p} - \mathbf{a}))\right) && \text{(property of scalar triple product)} \\
&= \hat{\mathbf{b}} \cdot \left(\hat{\mathbf{b}}|\mathbf{p} - \mathbf{a}|^2 - (\mathbf{p} - \mathbf{a})\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\right) && \text{(Lagrange's formula)} \\
&= |\mathbf{p} - \mathbf{a}|^2 - (\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a}))^2.
\end{aligned}$$

Taking the square root, we obtain the Pythagorean formula we obtained in terms of the scalar product on Sheet 1.

(c) Let the unit cube have vertices $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ and $(1,1,1)$. A diagonal of the cube is then $\lambda(1,1,1)$ (this is the easiest one to pick!). A separate vertex that does not lie on this diagonal is $(0,0,1)$. Hence using the formula from (a), the shortest distance is the magnitude of:

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\0\end{pmatrix},$$

which is $\sqrt{2/3}$.

Using the formula from Question 10(c) of Sheet 1, the shortest distance is instead given by:

$$\sqrt{1^2 - \left(\frac{(1,1,1)}{\sqrt{3}} \cdot (0,0,1)\right)^2} = \sqrt{1 - \frac{1}{3}} = \sqrt{\frac{2}{3}},$$

in perfect agreement.

More on the equation of a plane

3. (a) Explain why the plane through the points with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is $(\mathbf{r}-\mathbf{a})\cdot((\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}))=0$. Show using properties of the vector product, and the result from Question 22 of Sheet 1, that this may equivalently be written in the more symmetric form $\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.

   (b) Find an equation of the form $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$ for the plane passing through $(1,1,1)$, $(1,2,3)$ and $(0,0,4)$.

Solution: (a) Two vectors contained in the plane are $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$. Taking their cross product, we produce a vector orthogonal to the plane, $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})$. Hence, using the standard equation of a plane from Sheet 1, we have that $(\mathbf{r}-\mathbf{a})\cdot((\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}))=0$ is indeed the equation of the plane.

The result from Question 22 of Sheet 1 gives $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$. Hence the equation can be rewritten as:

$$(\mathbf{r}-\mathbf{a})\cdot(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})=0.$$

Using the distributive property of the vector product, and rearranging, we have:

$$\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})=\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})+\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a}).$$

Finally, the terms $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})$ and $\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})$ vanish, because $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{a}$ are orthogonal to $\mathbf{a}$ by definition.

(b) We follow the procedure outline in part (a). Two vectors contained in the plane are $(1,2,3)-(1,1,1)=(0,1,2)$ and $(0,0,4)-(1,1,1)=(0,0,3)$. Taking their cross product, we have:

$$\begin{pmatrix}0\\1\\2\end{pmatrix}\times\begin{pmatrix}0\\0\\3\end{pmatrix}=\begin{pmatrix}3\\0\\0\end{pmatrix}.$$

Hence the equation of the plane is:

$$\left(\mathbf{r}-\begin{pmatrix}1\\1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}3\\0\\0\end{pmatrix}=0.$$

4. You need to drill a hole in a piece of metal starting at a right angle to a flat surface passing through the points $A = (1, 0, 0)$, $B = (1, 1, 1)$ and $C = (0, 2, 0)$, with the hole emerging at the point $D = (2, 1, 0)$. How long a drill must you use and where (in the plane ABC) must you start drilling?

➭ **Solution:** Two vectors contained in the piece of metal are $\overrightarrow{AB} = (1, 1, 1) - (1, 0, 0) = (0, 1, 1)$ and $\overrightarrow{AC} = (0, 2, 0) - (1, 0, 0) = (-1, 2, 0)$. Hence a vector orthogonal to the plane is:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$

To obtain the length of the drill, we must find the shortest distance from the point D to the plane of metal. A vector from the plane to the point is $\overrightarrow{AD} = (2, 1, 0) - (1, 0, 0) = (1, 1, 0)$. The length of the projection onto the direction normal to the metal is:

$$\left| (1, 1, 0) \cdot \frac{(-2, -1, 1)}{\sqrt{4 + 1 + 1}} \right| = \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}.$$

Hence, we must use a drill that is $\sqrt{3/2}$ units long.

To answer where in the plane must we start drilling, we must add to the point $A = (1, 0, 0)$ the projection of $\overrightarrow{AD}$ parallel to the plane. Subtracting the component orthogonal to the plane gives us the parallel component:

$$\overrightarrow{AD}_{\parallel} = (1, 1, 0) + \sqrt{\frac{3}{2}} \cdot \frac{(-2, -1, 1)}{\sqrt{6}} = (1, 1, 0) + (-1, -1/2, 1/2) = (0, 1/2, 1/2).$$

Adding this to the point A, we have $(1, 0, 0) + (0, 1/2, 1/2) = (1, 1/2, 1/2)$.

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5. Determine whether:

(a) the points $\mathbf{P}_1 = (0, 0, 2)$, $\mathbf{P}_2 = (0, 1, 3)$, $\mathbf{P}_3 = (1, 2, 3)$, $\mathbf{P}_4 = (2, 3, 4)$ are coplanar;

(b) the points $\mathbf{Q}_1 = (-2, 1, 1)$, $\mathbf{Q}_2 = (-1, 2, 2)$, $\mathbf{Q}_3 = (-3, 3, 2)$, $\mathbf{Q}_4 = (-2, 4, 3)$ are coplanar.

➭ **Solution:** In each case, we construct the planes going through the first three points, then check if the fourth point lies in the plane. We have:

(a) Two vectors parallel to the plane containing the first three points are:

$$\mathbf{P}_2 - \mathbf{P}_1 = (0, 1, 3) - (0, 0, 2) = (0, 1, 1), \qquad \mathbf{P}_3 - \mathbf{P}_1 = (1, 2, 3) - (0, 0, 2) = (1, 2, 1).$$

Taking the cross product, a vector orthogonal to the plane is:

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

Hence, the equation of the plane is $(\mathbf{r} - (0, 0, 2)) \cdot (-1, 1, -1) = 0$. Substituting $\mathbf{r} = \mathbf{P}_4 = (2, 3, 4)$, we have $(2, 3, 4) = (0, 0, 2) = (2, 3, 2)$, but $(2, 3, 2) \cdot (-1, 1, -1) = -1 \neq 0$, hence these points are not coplanar.

(b) Two vectors parallel to the plane containing the first three points are:

$$\mathbf{Q}_2 - \mathbf{Q}_1 = (-1, 2, 2) - (-2, 1, 1) = (1, 1, 1), \qquad \mathbf{Q}_3 - \mathbf{Q}_1 = (-3, 3, 2) - (-1, 2, 2) = (-2, 2, 0).$$

Taking the cross product, a vector orthogonal to the plane is:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}.$$

Hence, the equation of the plane is $(\mathbf{r} - (-2, 1, 1)) \cdot (-1, -1, 2) = 0$ (scaling the normal). Substituting $\mathbf{r} = \mathbf{Q}_4 = (-2, 4, 3)$, we have $(-2, 4, 3) - (-2, 1, 1) = (0, 3, 2)$, but $(0, 3, 2) \cdot (-1, -1, 2) = 1 \neq 0$, hence these points are not coplanar.

Shortest distances

6. *Without using a formula*, find the shortest distance between the lines $\mathbf{r}_1 = (1, 0, 1) + \lambda(2, -1, 3)$ and $\mathbf{r}_2 = (0, 1, -2) + \mu(1, 0, 2)$, justifying the steps you take. [*Sometimes, it is better to understand a method, than to quote a formula*.]

Solution: If we wanted to connect the lines in the shortest way possible, we would draw a line that was *orthogonal* to both lines. Thus we would connect them along the direction:

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$

Now, let's pick any vector joining the two lines, say $(0, 1, -2) - (1, 0, 1) = (-1, 1, -3)$. If we project this vector along the orthogonal direction joining both lines, we will get the shortest distance between the two. We have:

$$(-1, 1, -3) \cdot \frac{(-2, -1, 1)}{\sqrt{4 + 1 + 1}} = -\frac{2}{\sqrt{6}}.$$

Taking the modulus, we obtain the shortest length $\sqrt{2/3}$.

7. So far, we have developed formulae for the shortest distance from points to lines, and from points to planes. Now, using the scalar and vector products, establish formulae for the following:

(a) the shortest distance from the line $\mathbf{r}_1 = \mathbf{v}_1 + \lambda \mathbf{w}_1$ to the line $\mathbf{r}_2 = \mathbf{v}_2 + \mu \mathbf{w}_2$; [*Hint: Take care when the lines are parallel!*]

(b) the shortest distance from the line $\mathbf{r} = \mathbf{v} + \lambda \mathbf{w}$ to the plane $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$;

(c) the shortest distance from the plane $(\mathbf{r}_1 - \mathbf{a}_1) \cdot \mathbf{b}_1 = 0$ to the plane $(\mathbf{r}_2 - \mathbf{a}_2) \cdot \mathbf{b}_2 = 0$.

Solution: (a) Following the procedure we outlined in the previous question, we note that a direction orthogonal to both lines is $\mathbf{w}_1 \times \mathbf{w}_2$. An arbitrary vector joining the lines is $\mathbf{v}_2 - \mathbf{v}_1$. Projecting this vector in the direction orthogonal to both lines, we get the shortest distance:

$$\left| \frac{(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{w}_1 \times \mathbf{w}_2)}{|\mathbf{w}_1 \times \mathbf{w}_2|} \right|.$$

This is fine unless $|\mathbf{w}_1 \times \mathbf{w}_2| = 0$, when the lines are parallel. In this case, the lines both have direction $\mathbf{w} \equiv \mathbf{w}_1 = \mathbf{w}_2$. We then need the projection of $\mathbf{v}_2 - \mathbf{v}_1$ orthogonal to $\mathbf{w}$, which is simply given by:

$$|(\mathbf{v}_2 - \mathbf{v}_1) \times \hat{\mathbf{w}}| .$$

(b) If $\mathbf{w}$ is not parallel to the plane, then the line and the plane must intersect, giving the shortest distance zero. This occurs if $\mathbf{w} \cdot \mathbf{b} \neq 0$.

In the case where $\mathbf{w} \cdot \mathbf{b} = 0$, then we need the projection of a vector $\mathbf{v} - \mathbf{a}$ perpendicular to the plane, which is just $|(\mathbf{v} - \mathbf{a}) \cdot \hat{\mathbf{b}}|$ (this agrees with our standard point-to-plane formula, because any point on the line is equally acceptable).

(c) If the planes are not parallel, that is $\mathbf{b}_1, \mathbf{b}_2$ are not parallel, then the planes intersect, giving the shortest distance zero.

In the case where $\mathbf{b}_1, \mathbf{b}_2$ are parallel, then we can just take the projection of $\mathbf{a}_1 - \mathbf{a}_2$ parallel to $\mathbf{b}_1$, say, giving $|(\mathbf{a}_1 - \mathbf{a}_2) \cdot \hat{\mathbf{b}}_1|$ (again, this agrees with our standard point-to-plane formula, because any pair of points on the planes is equally acceptable).

The vector triple product, and vector equations

8. (a) By expanding in terms of the standard basis vectors, prove *Lagrange's formula* for the vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

Think of a way of remembering this formula off by heart - it is very useful!

(b) Hence, construct an example of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

(c) $(*)$ Prove the vector triple product using a geometric argument. [*Hint:* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ *is the projection of* $\mathbf{a}$ *into the plane through the origin perpendicular to* $\mathbf{b} \times \mathbf{c}$*, rotated by* $\frac{1}{2}\pi$*, and scaled by the magnitude of* $\mathbf{b} \times \mathbf{c}$.]

➾ **Solution:** (a) Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$. Then:

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \left(\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right) \\
&= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix} \\
&= \begin{pmatrix} a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3 \\ a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1 \\ a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 \end{pmatrix} \\
&= \begin{pmatrix} b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3) \\ b_2(a_1c_1 + a_2c_2 + a_3c_3) - c_2(a_1b_1 + a_2b_2 + a_3b_3) \\ b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3(a_1b_1 + a_2b_2 + a_3b_3) \end{pmatrix} \\
&= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).
\end{aligned}$$

We can remember this formula using the phrase 'BACk of the CAB', which tells us which order the vectors come in.

(b) An easy example can be constructed using the Cartesian unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. We have:

$$\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_2) = -\mathbf{e}_2,$$

but:

$$(\mathbf{e}_1 \times \mathbf{e}_1) \times \mathbf{e}_2 = \mathbf{0},$$

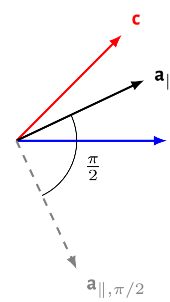
since $\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{0}$.

(c) This part is rather tricky, and certainly non-examinable, hence it is 'starred'! Using the hint, we observe that:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

is the projection of the vector **a** parallel to the plane with normal $\mathbf{b} \times \mathbf{c}$, rotated by $\pi/2$, and scaled by the magnitude of $\mathbf{b} \times \mathbf{c}$.

Alternatively, we can produce the same vector as follows. Let $\mathbf{a}_{||}$ be the projection of **a** into the plane perpendicular to $\mathbf{b} \times \mathbf{c}$. Let θ be the angle between **b**, **c**, and let ϕ be the angle between $\mathbf{a}_{||}$, **b**.



Since $\mathbf{a}_{||,\pi/2}$ is in the plane spanned by **b**, **c**, we can write:

$$\mathbf{a}_{||,\pi/2} = \beta\mathbf{b} + \gamma\mathbf{c}.$$

for some coefficients β, γ. To obtain the coefficients, we can compute:

$$\mathbf{a}_{||,\pi/2} \times \mathbf{c} = \beta\mathbf{b} \times \mathbf{c} \qquad \Leftrightarrow \qquad \beta = \frac{|\mathbf{a}_{||,\pi/2}||\mathbf{c}|\sin(\pi/2 + (\theta - \phi))}{|\mathbf{b}||\mathbf{c}|\sin(\theta)} = \frac{|\mathbf{a}_{||}||\mathbf{c}|\cos(\theta - \phi)}{|\mathbf{b}||\mathbf{c}|\sin(\theta)} = \frac{\mathbf{a}_{||} \cdot \mathbf{c}}{|\mathbf{b} \times \mathbf{c}|}$$

Similarly, we have:

$$\mathbf{a}_{||,\pi/2} \times \mathbf{b} = \gamma\mathbf{c} \times \mathbf{b} \qquad \Leftrightarrow \qquad \gamma = -\frac{|\mathbf{a}_{||,\pi/2}||\mathbf{b}|\sin(\pi/2 - \phi)}{|\mathbf{b}||\mathbf{c}|\sin(\theta)} = -\frac{|\mathbf{a}_{||}||\mathbf{b}|\cos(\phi)}{|\mathbf{b}||\mathbf{c}|\sin(\theta)} = -\frac{\mathbf{a}_{||} \cdot \mathbf{b}}{|\mathbf{b} \times \mathbf{c}|}.$$

Overall then, if we scale $\mathbf{a}_{||,\pi/2}$ by $|\mathbf{b} \times \mathbf{c}|$, we get:

$$|\mathbf{b} \times \mathbf{c}|\mathbf{a}_{||,\pi/2} = (\mathbf{a}_{||} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a}_{||} \cdot \mathbf{b})\mathbf{c}.$$

In the final line, we may replace $\mathbf{a}_{||}$ by **a** because the perpendicular component of **a** is orthogonal to both **b** and **c** by construction. Hence, we're done!

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9. Using the vector triple product, prove the *Jacobi identity*, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$.

Solution: We have:

$$\begin{aligned}\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) + \mathbf{c}(\mathbf{b} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) \\ &= 0,\end{aligned}$$

as required.

10. Two vector operators, $P_{\hat{\mathbf{u}}} : \mathbb{R}^3 \to \mathbb{R}^3$ and $R_{\hat{\mathbf{u}}} : \mathbb{R}^3 \to \mathbb{R}^3$ are defined by $P_{\hat{\mathbf{u}}}(\mathbf{r}) = (\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}$ and $R_{\hat{\mathbf{u}}}(\mathbf{r}) = \hat{\mathbf{u}} \times (\mathbf{r} \times \hat{\mathbf{u}})$ respectively. Interpret these operators geometrically, and hence explain why $P_{\hat{\mathbf{u}}}(\mathbf{r}) + R_{\hat{\mathbf{u}}}(\mathbf{r}) = \mathbf{r}$ for all vectors $\mathbf{r}$. Also explain why $P_{\hat{\mathbf{u}}}^2 = P_{\hat{\mathbf{u}}}$ and $R_{\hat{\mathbf{u}}}^2 = R_{\hat{\mathbf{u}}}$.

Solution: The operator $P_{\hat{\mathbf{u}}}$ gives the projection of a vector in the $\hat{\mathbf{u}}$ direction. Using the vector triple product, we have:

$$R_{\hat{\mathbf{u}}}(\mathbf{r}) = \hat{\mathbf{u}} \times (\mathbf{r} \times \hat{\mathbf{u}}) = \mathbf{r} - (\hat{\mathbf{u}} \cdot \mathbf{r})\hat{\mathbf{u}},$$

hence this operator removes the component of a vector in the $\hat{\mathbf{u}}$ direction. Hence, it gives the projection of a vector perpendicular to the $\hat{\mathbf{u}}$ direction. This immediately implies that:

$$P_{\hat{\mathbf{u}}}(\mathbf{r}) + R_{\hat{\mathbf{u}}}(\mathbf{r}) = \mathbf{r}$$

for all vectors $\mathbf{r}$, as required.

It is also straightforward that $P_{\hat{\mathbf{u}}}^2 = P_{\hat{\mathbf{u}}}$, because it we apply the projection operator twice, after the first application, the resulting vector already points in the $\hat{\mathbf{u}}$ direction, so the second application does nothing. We can check this explicitly using some algebra:

$$P_{\hat{\mathbf{u}}}^2(\mathbf{r}) = P_{\hat{\mathbf{u}}}((\hat{\mathbf{u}} \cdot \mathbf{r})\hat{\mathbf{u}}) = (\hat{\mathbf{u}} \cdot \hat{\mathbf{u}})(\hat{\mathbf{u}} \cdot \mathbf{r})\hat{\mathbf{u}} = (\hat{\mathbf{u}} \cdot \mathbf{r})\hat{\mathbf{u}}.$$

The same is true for $R_{\hat{\mathbf{u}}}^2 = R_{\hat{\mathbf{u}}}$, because after one application of the projection operator, we are already pointing in a direction orthogonal to $\hat{\mathbf{u}}$.

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11. Solve the following vector equations, and give geometric interpretations of their solutions:

(a) $\mathbf{a} \times \mathbf{r} + \lambda \mathbf{r} = \mathbf{c}$, where $\lambda \neq 0$, and $\mathbf{a}, \mathbf{c} \in \mathbb{R}^3$ are arbitrary 3-vectors;

(b) $\mathbf{r} \times \mathbf{a} = \mathbf{b}$, where $\mathbf{a} \in \mathbb{R}^3$ is an arbitrary non-zero 3-vector;

(c) $\mathbf{r} = \mathbf{a} + (\mathbf{b} \cdot \mathbf{r})\mathbf{c}$, where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ are arbitrary 3-vectors;

(d) $2\mathbf{r} + \hat{\mathbf{n}} \times \mathbf{r} + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{r})^2 = \mathbf{a}$, where $\hat{\mathbf{n}}$ is a unit vector, and $\hat{\mathbf{n}} \cdot \mathbf{a} = -1$.

Solution:

(a) Begin by taking the scalar product of the equation with $\mathbf{a}$ (this hopefully gives us information parallel to $\mathbf{a}$). We then have:

$$\lambda \mathbf{r} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{a},$$

giving us the component of $\mathbf{r}$ parallel to $\mathbf{a}$. Now consider taking the vector product of the equation with $\mathbf{a}$ (this hopefully gives us information perpendicular to $\mathbf{a}$). We then have:

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{r}) + \lambda \mathbf{a} \times \mathbf{r} = \mathbf{a} \times \mathbf{c} \qquad \Leftrightarrow \qquad \mathbf{a}(\mathbf{a} \cdot \mathbf{r}) - |\mathbf{a}|^2 \mathbf{r} + \lambda \mathbf{a} \times \mathbf{r} = \mathbf{a} \times \mathbf{c}.$$

To finish, we substitute for $\mathbf{a} \cdot \mathbf{r}$ using the expression we found earlier when computing the scalar product. We also substitute for $\mathbf{a} \times \mathbf{r}$ using the original equation, $\mathbf{a} \times \mathbf{r} = \mathbf{c} - \lambda \mathbf{r}$. This gives:

$$\frac{(\mathbf{a} \cdot \mathbf{c})\mathbf{a}}{\lambda} - |\mathbf{a}|^2 \mathbf{r} + \lambda \mathbf{c} - \lambda^2 \mathbf{r} = \mathbf{a} \times \mathbf{c}.$$

Rearranging for $\mathbf{r}$, we have:

$$\mathbf{r} = \frac{(\mathbf{a} \cdot \mathbf{c})\mathbf{a} - \lambda \mathbf{a} \times \mathbf{c} + \lambda^2 \mathbf{c}}{\lambda(|\mathbf{a}|^2 + \lambda^2)}.$$

This is a single point.

(b) First, let's take the scalar product with **a** to give us information parallel to **a**. We find:

$$0 = \mathbf{a} \cdot \mathbf{b}.$$

Hence, we see the equation has no solutions unless $\mathbf{a} \cdot \mathbf{b}$. Next, we take the vector product with **a** to give us information perpendicular to **a**. We have:

$$\mathbf{a} \times (\mathbf{r} \times \mathbf{a}) = \mathbf{a} \times \mathbf{b} \qquad \Leftrightarrow \qquad |\mathbf{a}|^2\mathbf{r} - \mathbf{a}(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a} \times \mathbf{b}.$$

Rearranging, we have:

$$\mathbf{r} = \frac{\mathbf{a}(\mathbf{a} \cdot \mathbf{r})}{|\mathbf{a}|^2} + \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}|^2}.$$

It appears we cannot get any more information on $\mathbf{a} \cdot \mathbf{r}$, because if we add any vector parallel to **a** to **r**, then this just gets annihilated. So the equation must have many solutions, of the form:

$$\mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}|^2} + \lambda\mathbf{a},$$

for any $\lambda \in \mathbb{R}$. We have so far only proved that solutions of this form are *necessary* (if we assume the equation, this is the form that the solutions must take). We must also prove that they are *sufficient*, by showing that these actually solve the equation in practice. We have:

$$\mathbf{r} \times \mathbf{a} = \frac{(\mathbf{a} \times \mathbf{b}) \times \mathbf{a}}{|\mathbf{a}|^2} = -\frac{\mathbf{a}(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{a}|^2} + \frac{|\mathbf{a}|^2\mathbf{b}}{|\mathbf{a}|^2} = \mathbf{b},$$

provided that $\mathbf{a} \cdot \mathbf{b} = 0$. So we're done!

Summarising: the equation has no solutions if $\mathbf{a} \cdot \mathbf{b} \neq 0$, but it has many solutions of the form:

$$\mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}|^2} + \lambda\mathbf{a}$$

otherwise. This is a line through the point $(\mathbf{a} \times \mathbf{b})/|\mathbf{a}|^2$ parallel to the vector **a**.

(c) Taking the scalar product with **b**, we aim to get information parallel to **b**:

$$\mathbf{r} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + (\mathbf{r} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{b}). \qquad (*)$$

Rearranging, we have:

$$\mathbf{r} \cdot \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{1 - \mathbf{b} \cdot \mathbf{c}},$$

provided that $\mathbf{b} \cdot \mathbf{c} \neq 1$. Hence, if $\mathbf{b} \cdot \mathbf{c}$, we get the solution:

$$\mathbf{r} = \mathbf{a} + \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{c}}{1 - \mathbf{b} \cdot \mathbf{c}},$$

which is just a single point.

On the other hand, if $\mathbf{b} \cdot \mathbf{c} = 1$, we see that equation $(*)$ implies $\mathbf{a} \cdot \mathbf{b} = 0$. Thus there are no solutions unless $\mathbf{a} \cdot \mathbf{b} = 0$ too. We get no further information on $\mathbf{r} \cdot \mathbf{b}$, so we guess that this is a free parameter and the solution is of the form:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{c}.$$

Certainly the solution is *necessarily* of this form, because the equation looks like this in the first place! We also must check it is *sufficient* by substituting back into the original equation. We have:

$$\mathbf{a} + (\mathbf{b} \cdot \mathbf{r})\mathbf{c} = \mathbf{a} + (\mathbf{b} \cdot \mathbf{a} + \lambda\mathbf{b} \cdot \mathbf{c})\mathbf{c} = \mathbf{a} + \lambda\mathbf{c} = \mathbf{r},$$

since $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 1$. Thus this is indeed the general solution in this case.

Summarising: if $\mathbf{b} \cdot \mathbf{c} \neq 1$, the solution is a point $\mathbf{a} + (\mathbf{a} \cdot \mathbf{b})/(1 - \mathbf{b} \cdot \mathbf{c})$; if $\mathbf{b} \cdot \mathbf{c} = 1$ and $\mathbf{a} \cdot \mathbf{b} \neq 0$, there are no solutions; if $\mathbf{b} \cdot \mathbf{c} = 1$ and $\mathbf{a} \cdot \mathbf{b} = 0$, the solution is a line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{c}$.

(d) Take the scalar product of the equation with $\hat{\mathbf{n}}$ to get information parallel to $\hat{\mathbf{n}}$. Then:

$$2\mathbf{r}\cdot\hat{\mathbf{n}} + (\hat{\mathbf{n}}\cdot\mathbf{r})^2 = \mathbf{a}\cdot\hat{\mathbf{n}} = -1 \qquad \Leftrightarrow \qquad (\hat{\mathbf{n}}\cdot\mathbf{r})^2 + 2\hat{\mathbf{n}}\cdot\mathbf{r} + 1 = 0.$$

This is a quadratic equation for $\hat{\mathbf{n}}\cdot\mathbf{r}$; it has a repeated root:

$$\hat{\mathbf{n}}\cdot\mathbf{r} = -1.$$

Substituting this back into the original equation, we have:

$$2\mathbf{r} + \hat{\mathbf{n}}\times\mathbf{r} = \mathbf{a} - \hat{\mathbf{n}}. \qquad (**)$$

Next, we take the vector product of the equation with $\hat{\mathbf{n}}$ to get information perpendicular to $\hat{\mathbf{n}}$. We have:

$$2\hat{\mathbf{n}}\times\mathbf{r} + \hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{r}) = \hat{\mathbf{n}}\times\mathbf{a} \qquad \Leftrightarrow \qquad 2\hat{\mathbf{n}}\times\mathbf{r} + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r}) - \mathbf{r} = \hat{\mathbf{n}}\times\mathbf{a}.$$

Substituting for $\hat{\mathbf{n}}\cdot\mathbf{r} = -1$ and for $\hat{\mathbf{n}}\times\mathbf{r}$ using $(**)$, we have:

$$2\left(\mathbf{a} - \hat{\mathbf{n}} - 2\mathbf{r}\right) - \hat{\mathbf{n}} - \mathbf{r} = \hat{\mathbf{n}}\times\mathbf{a}.$$

Rearranging for $\mathbf{r}$, we have:

$$\mathbf{r} = \frac{2\mathbf{a} - 3\hat{\mathbf{n}} - \hat{\mathbf{n}}\times\mathbf{a}}{5}.$$

This is a single point.

The scalar triple product, and non-orthonormal bases

12. Let **a**, **b**, **c** $\in \mathbb{R}^3$ be 3-vectors.

(a) Give the definition of the *scalar triple product* $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of the 3-vectors **a**, **b**, **c** $\in \mathbb{R}^3$. Hence show that the volume of the parallelepiped defined by the positions vectors **a**, **b**, **c** is $|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|$. Why is the modulus necessary?

(b) Using the relation between the scalar triple product and a parallelepiped, explain why:

(i) the scalar triple product is antisymmetric on odd permutations of its entries, and symmetric on even permutations of its entries;

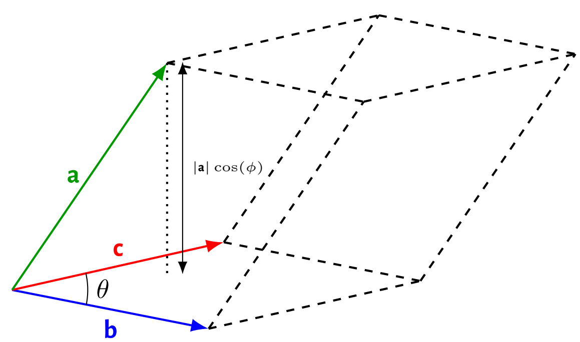
(ii) the condition $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$ implies that **a**, **b**, **c** are not coplanar, and thus form a basis.

(c) Compute the volume of a parallelepiped defined by the three position vectors $\mathbf{a} = (0, \frac{1}{2}, \frac{1}{2})$, $\mathbf{b} = (\frac{1}{2}, 0, \frac{1}{2})$, $\mathbf{c} = (\frac{1}{2}, \frac{1}{2}, 0)$, and comment on whether these vectors form a basis.

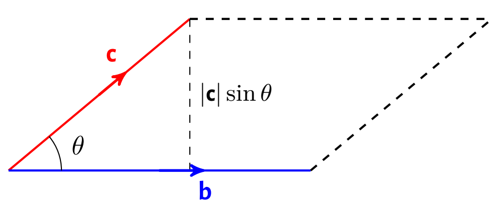
Solution: (a) The scalar triple product is defined by:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] := \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

If the vectors are arranged in a right-handed way, then **a** forms an acute angle ϕ with $\mathbf{b} \times \mathbf{c}$, so $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \equiv [\mathbf{a}, \mathbf{b}, \mathbf{c}] > 0$. Suppose that the vectors **b**, **c** make an angle θ.



The height of the parallelogram formed by **a**, **b**, **c** is given by $|\mathbf{a}| \cos(\phi)$, as shown in the figure above. The area of the base can be computed by considering the figure below:



We see that the height of the parallelogram base is $|\mathbf{c}| \sin(\theta)$, so that its area is $|\mathbf{b}||\mathbf{c}| \sin(\theta)$, which is equal to the magnitude of $\mathbf{b} \times \mathbf{c}$. Therefore, the volume of the parallelepiped is $|\mathbf{a}||\mathbf{b}||\mathbf{c}| \sin(\theta) \cos(\phi)$, which is equal to $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \equiv [\mathbf{a}, \mathbf{b}, \mathbf{c}]$, as required.

In the case where **a**, **b**, **c** is left-handed, we have that **a** makes an *obtuse* angle ϕ with $\mathbf{b} \times \mathbf{c}$. We still have that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is equal in magnitude to the volume, but it now has a relative negative sign. Hence $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is the volume of the parallelepiped in general, explaining the need for the modulus.

(b) Recall from part (a) that the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is the signed volume of the parallelepiped formed by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$; it is positive if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is right-handed and negative if left-handed. Hence, we have:

(i) Swapping the arguments of $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ an even number of times retains the right-handedness of the parallelepiped; hence, the value is unchanged. Swapping the arguments of $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ an odd number of times changes the handedness of the parallelepiped to a left-handed orientation; hence, the value acquires a minus sign. The result follows.

(ii) Since $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is the signed volume of a parallelepiped, if it is non-zero, then the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ do not lie in the same plane (in this case, the volume degenerates to zero). Thus, they form a basis for three-dimensional space, as required.

(c) The volume is:

$$\begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \times \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} -1/4 \\ 1/4 \\ 1/4 \end{pmatrix} = \frac{1}{4} \neq 0,$$

hence these vectors do indeed form a basis.

---

13. Simplify the scalar triple products $(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{b}+\mathbf{c}) \times (\mathbf{c}+\mathbf{a})$ and $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$.

➭ **Solution:** In the first case, we have:

$$(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{b}+\mathbf{c}) \times (\mathbf{c}+\mathbf{a}) = (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = 2\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \equiv 2[\mathbf{a}, \mathbf{b}, \mathbf{c}].$$

In the second case, we have:

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] &= (\mathbf{a} \times \mathbf{b}) \cdot [\mathbf{c}((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}) - \mathbf{a}((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c})] \qquad \text{(Lagrange's formula)} \\ &= (\mathbf{a} \times \mathbf{b}) \cdot [\mathbf{c}[\mathbf{a}, \mathbf{b}, \mathbf{c}]] \\ &= [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2. \end{aligned}$$

14. Let **0**, **a**, **b**, **c** form the vertices of a tetrahedron, with $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$. Write down conditions in terms of the scalar triple product for the vector **r** to lie inside the tetrahedron.

Solution: First, observe that since $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$, we have that the vectors are right-handed.

Now, the vector **r** must lie on the correct side of each of the four planes bounding the tetrahedron. Beginning with the side spanned by **a**, **b**, the normal $\mathbf{a} \times \mathbf{b}$ points in the direction of the tetrahedron. Hence:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) > 0$$

is needed to ensure that **r** is on the correct side of the plane. The same is true for the other pairs (taking care to consider handedness), giving:

$$\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) > 0, \qquad \mathbf{r} \cdot (\mathbf{c} \times \mathbf{a}) > 0.$$

Finally, we consider the plane that goes through points with position vectors **a**, **b**, **c**. We saw in Question 3(a) that this plane can be written in the form:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

The normal $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ is outward-pointing from the tetrahedron, so the side contained inside the tetrahedron satisfies:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{c}) < \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

Overall then, the four conditions are:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) > 0, \qquad \mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) > 0, \qquad \mathbf{r} \cdot (\mathbf{c} \times \mathbf{a}) > 0,$$

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{c}) < \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

15. Let **a**, **b**, **c** $\in \mathbb{R}^3$ be 3-vectors.

(a) If these vectors form an orthonormal basis, derive expressions for the coefficients α, β, γ in the formula $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$. Hence express $(2,3,4)$ in terms of the basis $\{(1,1,0),(1,-1,0),(0,0,1)\}$.

(b) If instead these vectors do *not* form an orthonormal basis, derive expressions for the coefficients α, β, γ in the formula $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$. [*Hint: consider scalar triple products.*] Hence express $(1,1,1)$ in terms of the basis $\{(1,2,1),(0,0,1),(2,-1,1)\}$.

(c) We define the *reciprocal vectors* to **a**, **b**, **c** to be the vectors:

$$\mathbf{A} = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a},\mathbf{b},\mathbf{c}]}, \qquad \mathbf{B} = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a},\mathbf{b},\mathbf{c}]}, \qquad \mathbf{C} = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a},\mathbf{b},\mathbf{c}]}.$$

Show that $\mathbf{A}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{b} = \mathbf{C}\cdot\mathbf{c} = 1$, and $\mathbf{A}\cdot\mathbf{b} = \mathbf{A}\cdot\mathbf{c} = \mathbf{B}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{c} = \mathbf{C}\cdot\mathbf{a} = \mathbf{C}\cdot\mathbf{b} = 0$. Hence, by comparing to part (b), explain how the reciprocal basis can be used to express a general vector **d** in terms of a non-orthonormal basis $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$. Compute the reciprocal basis to the basis $\{(1,2,1),(0,0,1),(2,-1,1)\}$.

Solution: (a) If the vectors are orthonormal, we have $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} = 0$ and $\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = 1$. Hence given:

$$\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c},$$

we can take the scalar product with the vectors **a**, **b**, **c** in turn to find the coefficients. Taking the scalar product with **a**, **b**, **c** in turn, we have:

$$\mathbf{a}\cdot\mathbf{d} = \alpha, \qquad \mathbf{b}\cdot\mathbf{d} = \beta, \qquad \mathbf{c}\cdot\mathbf{d}\gamma.$$

To express $(2,3,4)$ in terms of the basis $\{(1,1,0),(1,-1,0),(0,0,1)\}$, we cannot immediately use the formula above, because this is not an *orthonormal basis*. It is orthogonal however, since $(1,1,0)\cdot(1,-1,0) = 0$, $(1,1,0)\cdot(0,0,1) = 0$ and $(1,-1,0)\cdot(0,0,1) = 0$. To make it orthonormal, we consider the following related basis:

$$\{(1,1,0)/\sqrt{2},(1,-1,0)/\sqrt{2},(0,0,1)\}.$$

Using the formulae for the coefficients from above, we therefore have:

$$\begin{aligned}(2,3,4) &= \left((2,3,4)\cdot\frac{(1,1,0)}{\sqrt{2}}\right)\frac{(1,1,0)}{\sqrt{2}} + \left((2,3,4)\cdot\frac{(1,-1,0)}{\sqrt{2}}\right)\frac{(1,-1,0)}{\sqrt{2}} + ((2,3,4)\cdot(0,0,1))\,(0,0,1)\\ &= \frac{5}{2}(1,1,0) - \frac{1}{2}(1,-1,0) + 4(0,0,1),\end{aligned}$$

which is an expression for $(2,3,4)$ in terms of the desired basis.

(b) Now, **a**, **b**, **c** form a *non*-orthonormal basis. We still would like to find the coefficients α, β, γ in:

$$\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c},$$

but it is no longer so easy. Can we use the same trick, to somehow take the scalar product with something, leaving only one coefficient leftover?

The answer is *yes*, if we dot with something perpendicular to two of the basis vectors. For example, to get the coefficient α, we consider taking the scalar product with $\mathbf{b}\times\mathbf{c}$. Then:

$$\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c}) = \alpha\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \qquad \Leftrightarrow \qquad \alpha = \frac{\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}.$$

Similarly, we have:

$$\beta = \frac{\mathbf{d} \cdot (\mathbf{c} \times \mathbf{a})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}, \qquad \gamma = \frac{\mathbf{d} \cdot (\mathbf{a} \times \mathbf{b})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}.$$

To calculate the coefficients for the given example then, we first compute the cross products:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

The scalar triple product is:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 5.$$

Hence we have:

$$\alpha = \frac{1}{5} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \frac{3}{5},$$

$$\beta = \frac{1}{5} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix} = \frac{1}{5},$$

$$\gamma = \frac{1}{5} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{5}.$$

That is, we have:

$$(1,1,1) = \frac{3}{5}(1,2,1) + \frac{1}{5}(0,0,1) + \frac{1}{5}(2,-1,1).$$

(c) The properties $\mathbf{A} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{b} = \mathbf{C} \cdot \mathbf{c} = 1$ are obvious from the permutation properties of the scalar triple product. Similarly $\mathbf{A} \cdot \mathbf{b} = \mathbf{A} \cdot \mathbf{c} = \mathbf{B} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{c} = \mathbf{C} \cdot \mathbf{a} = \mathbf{C} \cdot \mathbf{b} = 0$ are all obvious from the fact that the scalar triple product vanishes when two of its arguments are equal.

We have shown that if $\mathbf{d} = \alpha \mathbf{a} + \beta \mathbf{b} + \gamma \mathbf{c}$, then the coefficients α, β, γ can be written as:

$$\alpha = \mathbf{d} \cdot \mathbf{A}, \qquad \beta = \mathbf{d} \cdot \mathbf{B}, \qquad \gamma = \mathbf{d} \cdot \mathbf{C}.$$

Hence if we use the reciprocal basis $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, we can happily dot as if we were dealing with an orthonormal basis. An orthonormal basis has the special property that it is its own reciprocal basis.

For the given basis, we already computed in part (b) that:

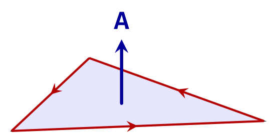
$$\mathbf{A} = \frac{1}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \qquad \mathbf{B} = \frac{1}{5} \begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix}, \qquad \mathbf{C} = \frac{1}{5} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

Vector area

16.(a) Define the *vector area* **A** of a surface composed of k flat faces with areas $A_1, \ldots, A_k$ and unit normals $\hat{\mathbf{n}}_1, \ldots, \hat{\mathbf{n}}_k$. What are the conventions usually used when choosing the unit normal(s)?

(b) In terms of the position vectors **a**, **b** determine the vector areas of: (i) the parallelogram defined by **a**, **b**; (ii) the triangle defined by **a**, **b**. Hence, given points $O = (0,0,0)$, $A = (1,0,0)$, $B = (1,1,1)$, $C = (0,2,0)$, compute the vector area of the triangle OAB (with vertices taken in that order), and the vector area of the surface bounded by the loop $OABC$ comprising the two triangular surfaces OAB and BCO.

➾ **Solution:** The *vector area* of a flat surface A is the vector **A** with magnitude A and direction normal to the surface. The choice of which normal is a matter of convention, but if the orientation of the boundary of **A** is specified, then the normal is usually taken in a right-handed sense. This means that if you fingers curl round the orientation of the boundary, then your thumb points in the direction of the normal.

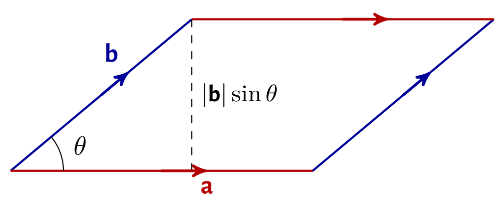


For a surface composed of multiple flat faces with area $A_1, \ldots, A_k$ and unit normals $\hat{\mathbf{n}}_1, \ldots, \hat{\mathbf{n}}_k$, the total vector area is defined to be the sum of the vector areas of the individual faces:

$$\mathbf{A} = \sum_{i=1}^{k} A_i \hat{\mathbf{n}}_i.$$

(b) First, we compute the vector areas of the basic shapes asked for in the question:

(i) The parallelogram formed by **a**, **b** having an angle θ between the vectors takes the form:



Evidently, the area of the parallelogram is $|\mathbf{a}||\mathbf{b}|\sin(\theta) = |\mathbf{a} \times \mathbf{b}|$.

Now, if we choose an orientation of the parallelogram where we follow the vector **a** first, then the vector area will be a unit vector pointing out of the page scaled by the area of the parallelogram. That is, the vector area will just be the *cross product*, $\mathbf{a} \times \mathbf{b}$.

(ii) For a triangle, we just have half the area of the parallelogram above. So assuming the same orientation (following **a** first), the vector area is $\frac{1}{2}\mathbf{a} \times \mathbf{b}$.

Now, consider the explicit coordinates we are given. First, we are asked for the vector area of the triangle OAB, which by the above work is simply:

$$\frac{1}{2}\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\-1\\1\end{pmatrix}.$$

The vector area of the surface comprised of the triangle OAB together with the triangle BCO is the sum of the previous vector area, together with:[1]

$$\frac{1}{2}\overrightarrow{BC} \times \overrightarrow{BO} = \frac{1}{2}\begin{pmatrix}-1\\1\\-1\end{pmatrix} \times \begin{pmatrix}-1\\-1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-2\\0\\2\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

Hence, the total vector area is:

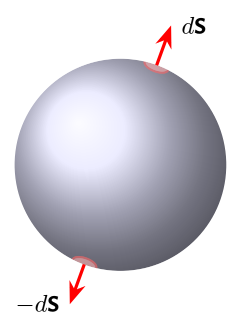
$$\frac{1}{2}\begin{pmatrix}0\\-1\\1\end{pmatrix} + \begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\-1/2\\3/2\end{pmatrix}$$

17.(a) Give a very general explanation of how the idea of vector area could be extended to *curved surfaces*, and hence explain why we expect the vector area of any *closed* surface to be **0**.

(b) Compute the vector area of the square with vertices $(0,0,0)$, $(2,0,0)$, $(2,2,0)$, $(0,2,0)$, taken in that order. Hence, compute the vector area of the pyramid extending this square with the point $(1,1,1)$, excluding its square face.

(c) Compute the vector area of the curved surface of a truncated hollow cone, bounded by a horizontal circle of radius 4 units and a horizontal circle of radius 3 units at some height above the first (note the result is independent of the height!).

Solution: (a) For a closed surface, we could split the surface into lots of small *approximately* flat pieces, calculate each of their vector areas, then add them all up. This amounts to performing an *integral* over the surface. We will study surface integration properly in Lent term.



For a *closed* surface, i.e. one with no boundary, we expect each little flat piece that we split the surface up into to have a neighbour on the opposite side of the surface, but with an equal and opposite vector area. These will exactly cancel giving the vector area of a closed surface as zero. Note, this is not a proof, just a hopeful argument - we will see a proof properly in Lent when we study Stokes' theorem.

(b) The square clearly has area 4, so has vector area $(0,0,4)$, since $(0,0,1)$ is a unit normal to the square.

Let **A** be the vector area of the top part of the square-based pyramid. Then since the vector area of a closed surface is always zero, we must have $\mathbf{A} + (0,0,4) = \mathbf{0}$, so $\mathbf{A} = -(0,0,4)$. This assumes a certain orientation of the upper part of the square-based pyramid; if we had used a different orientation, we would have obtained $(0,0,4)$.

[1] The solution to Question 18(c) has a good diagram for this part!

(c) Let the axis of the cone be along the z-axis. Then if the complete shape has an outward pointing normal everywhere, the smaller top disc has vector area $\pi(3^2) \cdot (0,0,1) = (0,0,9\pi)$. The larger bottom disc has vector area $\pi(4^2) \cdot (0,0,-1) = (0,0,-16\pi)$. Let the vector area of the curved surface be **A**. Then:

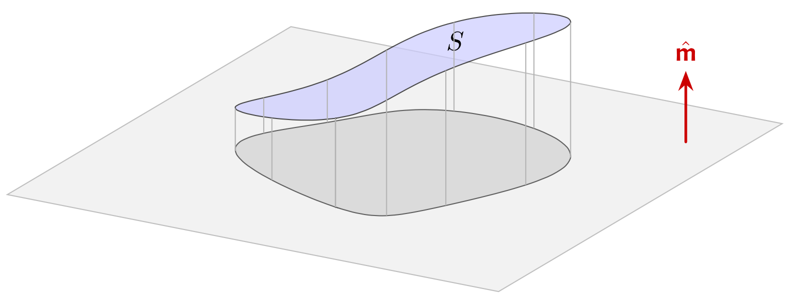
$$\mathbf{A} + (0,0,9\pi) + (0,0,-16\pi) = \mathbf{0} \qquad \Leftrightarrow \qquad \mathbf{A} = (0,0,7\pi).$$

18.(a) Let **S** be the vector area of the surface S. Prove that the area of the projection of the surface S onto the plane with unit normal $\hat{\mathbf{m}}$ is $|\mathbf{S} \cdot \hat{\mathbf{m}}|$. [*Hint: consider joining the surface to its 'shadow' on the plane to create a closed surface.*]

(b) Compute the vector area of the projection of the square with vertices $(0,0,0)$, $(2,0,0)$, $(2,2,0)$, $(0,2,0)$ onto the plane with unit normal $\hat{\mathbf{m}} = (0,-1,1)/\sqrt{2}$.

(c) By projecting areas onto the yz, xz, and xy planes, compute the vector area of the surface comprised of the two triangles OAB, BCO with vertices $O = (0,0,0)$, $A = (1,0,0)$, $B = (1,1,1)$, $C = (0,2,0)$, taken in that order. [*Your answer should match your answer to Question 17(b)!*] What is the area of the surface projected onto: (i) the plane with normal $(0,-1,1)$; (ii) the plane that maximises the projected area?

Solution: (a) This part is a very cute argument, but it is tricky if you have never seen it before! We imagine joining the shadow of S to the plane with normal $\hat{\mathbf{m}}$ via some prism-like surface, as shown in the figure.



Let S_{proj} be the area of the projection of the surface S onto the plane with normal $\hat{\mathbf{m}}$. Let $\mathbf{S}_\perp$ be the *vector* area of the curved surface joining S to the plane, which is necessarily perpendicular to $\hat{\mathbf{m}}$. Then since we now have a closed surface, we must have:

$$\mathbf{S} + \mathbf{S}_\perp - S_{\text{proj}}\hat{\mathbf{m}} = \mathbf{0}.$$

Taking the scalar product with $\hat{\mathbf{m}}$, and using the fact $\mathbf{S}_\perp \cdot \hat{\mathbf{m}} = 0$, we immediately get the result $\mathbf{S} \cdot \hat{\mathbf{m}} = S_{\text{proj}}$, as required.

This argument assumed that **S** pointed *out* of the volume created, so that the appropriate vector area for the shadow was $-S_{\text{proj}}\hat{\mathbf{m}}$. If instead **S** pointed *into* the volume created, the appropriate vector area for the shadow is instead $S_{\text{proj}}\hat{\mathbf{m}}$ (this ensures that the normal is continuous across the surface, if we imagine moving it around). This implies the need for the modulus sign if we are calculating area in general: $|\mathbf{S} \cdot \hat{\mathbf{m}}|$. This will be an important point in part (c) of this question.

(b) The square has vector area $(0,0,4)$. Projecting onto the plane with unit normal $\hat{\mathbf{m}}$, the area of the projection is:

$$(0,0,4) \cdot (0,-1,1)/\sqrt{2} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

Therefore, the *vector area* of the projection is $2\sqrt{2}\hat{\mathbf{m}} = (0,-2,2)$.

(c) The vector area may be expressed as:

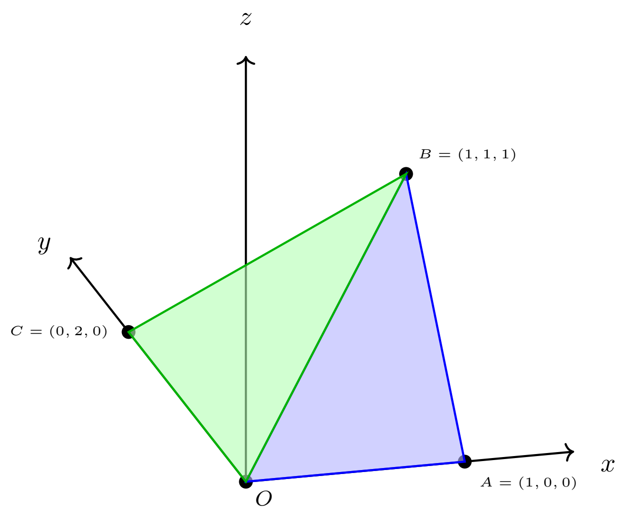
$$\mathbf{S} = S_x\hat{\mathbf{e}}_x + S_y\hat{\mathbf{e}}_y + S_z\hat{\mathbf{e}}_z,$$

where S_x, S_y, S_z are the coefficients in the expansion in an orthonormal basis, hence are given by:

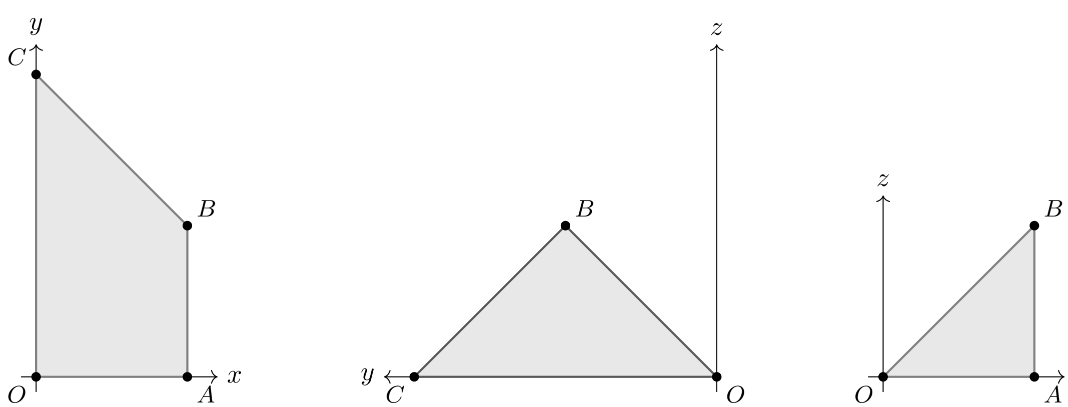
$$S_x = \mathbf{S}\cdot\hat{\mathbf{e}}_x, \qquad S_y = \mathbf{S}\cdot\hat{\mathbf{e}}_y, \qquad S_z = \mathbf{S}\cdot\hat{\mathbf{e}}_z.$$

In particular, the coefficients in the expansion are the (signed) projected areas of the surface onto the planes with normals $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ respectively, i.e. the coordinate planes.

Drawing a figure of the full 3D situation first, we have:



Now, imagine the projections of the surface onto each of the coordinate planes. We have:



Calculating the areas of each of the projections, we immediately see that:

$$|S_z| = \frac{3}{2}, \qquad |S_x| = 1, \qquad |S_y| = \frac{1}{2}.$$

To work out the signs correctly, we do the following. Following the argument presented in (a), we imagine joining up the surface $OABC$ to its shadow in a coordinate plane with normal $\hat{\mathbf{n}}$ (where $\hat{\mathbf{n}} = \hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$). If the vector area points *into* the volume that is created, then for consistency the vector area from the coordinate plane is the projection multiplied by $\hat{\mathbf{n}}$, giving:

$$\mathbf{S} + S_{\text{proj}}\hat{\mathbf{n}} + \mathbf{S}_\perp = \mathbf{0}.$$

This implies that $\mathbf{S} \cdot \hat{\mathbf{n}} = -S_{\text{proj}}$, so we get a negative sign. On the other hand, if the vector area points *out* of the volume that is created, then there is no change of sign.

In the xy plane, the orientation of $OABC$ implies the vector area points out of the volume it would create with the xy-plane. This means that the sign of S_z is positive, $S_z = 3/2$.

On the other hand, in both the yz and the xz planes, the orientation of $OABC$ implies that the vector area points *into* the volume it would create with the corresponding planes; hence $S_x = -1$ and $S_y = -1/2$. Overall we have:

$$\mathbf{S} = (-1, -1/2, 3/2),$$

consistent with our findings earlier on in the sheet.

To finish, we compute the projected area of $OABC$ onto two planes.

(i) Projecting onto the plane with normal $(0, -1, 1)$, we get the area:

$$\left|\mathbf{S} \cdot \frac{(0,-1,1)}{\sqrt{2}}\right| = \frac{1/2 + 3/2}{\sqrt{2}} = \sqrt{2}.$$

(ii) We now seek the plane which would maximise the projected area. If we have a plane with normal $\hat{\mathbf{m}}$, then the projected area is $|\mathbf{S} \cdot \hat{\mathbf{m}}| = |\mathbf{S}||\hat{\mathbf{m}}||\cos(\theta)|$, where θ is the angle between the vectors $\mathbf{S}$, $\hat{\mathbf{m}}$. Hence, the plane that maximises the projected area is the one with unit normal $\hat{\mathbf{S}}$. The projected area in this direction is:

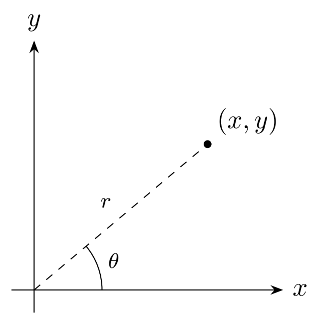
$$\left|\mathbf{S} \cdot \hat{\mathbf{S}}\right| = |\mathbf{S}| = \sqrt{1 + \frac{1}{4} + \frac{9}{4}} = \sqrt{\frac{7}{2}}.$$

Polar coordinate systems

19. Draw (convincing) diagrams defining plane, cylindrical, and spherical polar coordinates. In each case, derive the coordinate transform laws from polars to Cartesians, and from Cartesians to polars. Hence, find the cylindrical polar and spherical polar coordinates of the point $(3, 4, 5)$.

➾ **Solution:** We address each coordinate system in turn.

- **Plane polar coordinates.** This system is defined by the diagram below:



From basic trigonometry, we have that:

$$x = r\cos(\theta), y = r\sin(\theta).$$

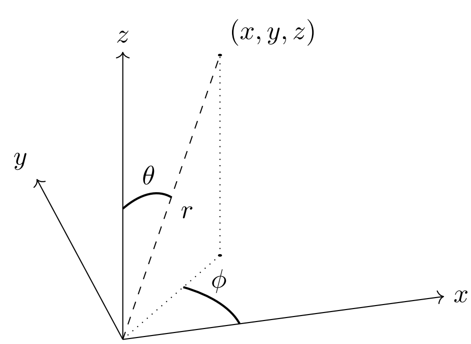
On the other hand, r is the length from the origin to the point (x, y), so:

$$r = \sqrt{x^2 + y^2}.$$

Finally, the angle θ satisfies $\tan(\theta) = x/y$. Depending on conventions, the root of this equation might be chosen to lie in the range $[0, 2\pi)$ or $(-\pi, \pi)$ (or indeed, other ranges).

- **Cylindrical polar coordinates.** These are the same as plane polars, just with an added z-direction. The formula for plane polars still hold when transforming $(r, \theta, z) \leftrightarrow (x, y, z)$.

- **Spherical polar coordinates.** Spherical polar coordinates for three dimensions are defined according to a *radial coordinate* r, which tells us how far from the origin we are, a *polar angle* θ, which tells us the angle between the radius from the origin to the point of interest and the z-axis (called the *polar axis*), and an *azimuthal angle* ϕ, which tells us the angle between the projection of the radius onto the xy-plane and the x-axis.



Some basic trigonometry tells us that:

$$x = r\sin(\theta)\cos(\phi), \qquad y = r\sin(\theta)\sin(\phi), \qquad z = r\cos(\theta).$$

Inverting these formula, we have:

$$r = \sqrt{x^2+y^2+z^2}, \qquad \tan(\phi) = \frac{x}{y}, \qquad \cos(\theta) = \frac{z}{\sqrt{x^2+y^2+z^2}},$$

where the coordinate θ is chosen to lie in the range $[0, \pi/2]$, and the coordinate ϕ is usually chosen to lie either in the range $[0, 2\pi)$ or $[-\pi, \pi)$.

Applying these formulae to the point $(3, 4, 5)$, we have:

- In cylindrical polar coordinates, the coordinates of the point are:

$$(r, \theta, z) = \left(\sqrt{3^2+4^2}, \arctan\left(\frac{4}{3}\right), 5\right) = \left(5, \arctan\left(\frac{4}{3}\right), 5\right).$$

- In spherical polar coordinates, the coordinates of the point are:

$$(r, \theta, \phi) = \left(\sqrt{3^2+4^2+5^2}, \arccos\left(\frac{5}{\sqrt{3^2+4^2+5^2}}\right), \arctan\left(\frac{4}{3}\right)\right) = \left(5\sqrt{2}, \frac{\pi}{4}, \arctan\left(\frac{4}{3}\right)\right).$$

20(a) In 2D Cartesian coordinates, a circle is specified by $(x-1)^2 + y^2 = 1$. Find its equation in polar coordinates.

(b) In 3D Cartesian coordinates, a sphere is specified by $(x-1)^2 + y^2 + z^2 = 1$. Find its equation in spherical polar coordinates.

➡ **Solution:** (a) In plane polar coordinates, we have $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Hence the equation in polar coordinates is:

$$(r\cos(\theta) - 1)^2 + r^2\sin^2(\theta) = 1.$$

Simplifying, we have:

$$1 = r^2\cos^2(\theta) - 2r\cos(\theta) + 1 + r^2\sin^2(\theta) = r^2 - 2r\cos(\theta) + 1,$$

which can be rearranged to $r^2 = 2r\cos(\theta)$. Dividing by r, we have $r = 2\cos(\theta)$. This allows for the case $r = 0$ when $\theta = \pi/2$, so we have not lost anything by dividing through by r.

(b) In spherical polar coordinates, we have $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$ and $z = r\cos(\theta)$. Hence the equation in polar coordinates is:

$$\begin{aligned}1 &= (r\sin(\theta)\cos(\phi) - 1)^2 + r^2\sin^2(\theta)\sin^2(\phi) + r^2\cos^2(\theta)\\ &= r^2\sin^2(\theta)\cos^2(\phi) - 2r\sin(\theta)\cos(\phi) + 1 + r^2\sin^2(\theta)\sin^2(\phi) + r^2\cos^2(\theta)\\ &= r^2 - 2r\sin(\theta)\cos(\phi) + 1.\end{aligned}$$

This can be rearranged to $r^2 = 2r\sin(\theta)\cos(\phi)$, which similarly can be reduced to $r = 2\sin(\theta)\cos(\phi)$.

21. Let $a > 0$ be a constant. Describe the following loci:

(a) (i) $\phi = a$; (ii) $r = \phi$, in plane polar coordinates.

(b) (i) $z = a$; (ii) $r = a$; (iii) $r = a$ and $z = \phi$, in cylindrical polar coordinates.

(c) (i) $\theta = a$; (ii) $\phi = a$; (iii) $r = a$; (iv) $r = \theta = a$, in spherical polar coordinates.

➡ **Solution:** We have:

(a) (i) $\phi = a$ corresponds to a half-line emanating from the origin, at an angle a to the x-axis. (ii) $r = \phi$ corresponds to a spiral, starting out at the origin then as ϕ increases, r increases too - it crosses the y-axis at $\pi/2$, then the negative x-axis at $-\pi$, then the negative y-axis at $-3\pi/2$. Eventually it gets back to the positive x-axis at the point 2π.

(b) (i) $z = a$ is a plane with normal $(0, 0, 1)$ a distance a from the origin. (ii) $r = a$ is a cylinder of radius a, with axis along the z-axis. (iii) The locus $r = a$ says that we on the cylinder of radius a, with axis along the z-axis. The equation $z = \phi$ says the angle and z-coordinate are related; in particular, the higher we are along the z-axis, the greater the angle with the x-axis. Thus this shape is a *helix* (a corkscrew type shape).

(c) (i) $\theta = a$ is a semi-infinite cone, with opening angle a, and axis along the z-axis. Its point is at the origin. (ii) $\phi = a$ is an infinite half-plane, inclined at an angle a to the x-axis, with its edge along the z-axis. (iii) $r = a$ is a sphere of radius a centred on the origin. (iv) $r = \theta = a$ is the intersection of the figures in (i) and (iii); this is a circle, at a height $a\cos(a)$ from the origin and radius $a\sin(a)$.

---

22. Consider a point with position vector $\hat{\mathbf{n}}$ on the unit sphere S.

(a) Explain why $\hat{\mathbf{n}} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, where θ, ϕ are the spherical coordinates of $\hat{\mathbf{n}}$.

(b) Show that the vector area $d\mathbf{S}$ of a small patch near $\hat{\mathbf{n}}$, subtending a small angle $d\theta$ in the θ direction, and a small angle $d\phi$ in the ϕ direction, is given *approximately* by $d\mathbf{S} = \hat{\mathbf{n}}\sin(\theta)d\theta d\phi$.

(c) $(*)$ Hence, by integrating $d\mathbf{S}$ first over ϕ whilst keeping θ constant, then over θ, show that the vector area of the sphere is zero. [*Hint: what are the limits on ϕ, θ?*] You have now performed your first **surface integral**, a topic we shall cover properly in Lent. In fact, it is possible to use surface integration to show that the vector area of *any* closed shape is zero through a theorem called *Stokes' theorem*, which we shall also see in Lent.

(d) $(*)$ Without computing it, what is the value of the surface integral $\displaystyle\int_S \hat{\mathbf{n}} \cdot d\mathbf{S}$?

➡ **Solution:** (a) In general, a position vector in spherical polar coordinates is given by:

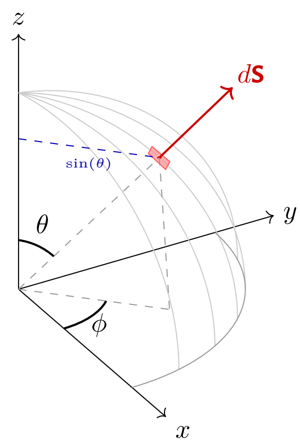
$$\mathbf{r} = (x, y, z) = (r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta)).$$

If a vector lies on the surface of the unit sphere, the only condition that needs to be satisfied is $r = 1$. This gives the general position vector on the surface of the unit sphere to be:

$$\hat{\mathbf{n}} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)),$$

as required.

(b) Consider the following diagram:



We are interested in obtaining the approximate size of the patch; its direction will be $\hat{\mathbf{n}}$, which allows us to form the vector area. Note that the length of the arc in the θ direction defining the vertical extent of the patch is just $d\theta$, corresponding to the fact that we have a unit sphere (recall the standard formula 'radius times angle' gives arc length). On the other hand, the length of the arc in the ϕ direction depends on the height we are above the xy-plane; at an angle θ, the arc is part of a circle of radius $\sin(\theta)$, so the length of the arc is approximately $\sin(\theta)d\phi$. Thus the area of the patch is approximately $\sin(\theta)d\theta d\phi$, and so the vector area is approximately $d\mathbf{S} = \sin(\theta)d\theta d\phi\hat{\mathbf{n}}$, as required.

(c) ($*$) Now for the fun part! We compute the integral:

$$\int\limits_{\text{sphere}} d\mathbf{S} = \int\limits_{\text{sphere}} \hat{\mathbf{n}}\sin(\theta)d\theta d\phi,$$

to give the total vector area of the sphere. We need to perform a double integral over both the ranges $0 \leq \theta\pi$ and $0 \leq \phi \leq 2\pi$. We are told in the question that the way to do this is to keep one variable fixed, and integrate with respect to the other. Then, we integrate with respect to the remaining variable we kept fixed in the first step.

Carrying out this procedure, and treating each part of the vector in the integrand as a separate integral, we have:

$$\int_0^\pi \left[\int_0^{2\pi} \hat{\mathbf{n}} d\phi\right]\sin(\theta)d\theta = \int_0^\pi \left[\int_0^{2\pi} \begin{pmatrix}\sin(\theta)\cos(\phi)\\ \sin(\theta)\sin(\phi)\\ \cos(\theta)\end{pmatrix} d\phi\right]\sin(\theta)d\theta = \int_0^\pi \begin{pmatrix}0\\0\\2\pi\cos(\theta)\end{pmatrix}\sin(\theta)\,d\theta = \int_0^\pi \begin{pmatrix}0\\0\\\pi\sin(2\theta)\end{pmatrix}d\theta = \mathbf{0},$$

so indeed, we find the vector area is zero!

(d) ($*$) In the last part, observe that $\hat{\mathbf{n}} \cdot d\mathbf{S} = \sin(\theta)d\theta d\phi$ is just the area of a small patch on the surface of the sphere, not the vector area. Hence, if we integrate over the entire sphere, we must just get the surface area of the sphere:

$$\int\limits_{\text{sphere}} \hat{\mathbf{n}} \cdot d\mathbf{S} = 4\pi.$$

You can check this for yourself, if you are keen, by performing a similar multiple integral calculation to part (c).

Part IA: Mathematics for Natural Sciences B
Examples Sheet 3: Complex numbers and hyperbolic functions

Model Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

Real and imaginary parts

1. Find the real and imaginary parts of the following numbers (where n is an integer):

(a) i^3, (b) i^{4n}, (c) $\left(\frac{1+i}{\sqrt{2}}\right)^2$, (d) $\left(\frac{1-i}{\sqrt{2}}\right)^2$, (e) $\left(\frac{1+\sqrt{3}i}{2}\right)^3$, (f) $\frac{1+i}{2-5i}$, (g) $\left(\frac{1+i}{1-i}\right)^2$.

Solution:

(a) We have $i^3 = i^2 \cdot i = -i$. Hence the real part is 1 and the imaginary part is -1 (recall that the imaginary part does *not* include i).

(b) We have $i^{4n} = (i^2)^{2n} = (-1)^{2n} = 1$, since an even power of -1 is always 1. Hence the real part is 1 and the imaginary part is 0.

(c) Carrying out the multiplication, we have:

$$\left(\frac{1+i}{\sqrt{2}}\right)^2 = \left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{1+i}{\sqrt{2}}\right) = \frac{1+2i-1}{2} = i.$$

Thus, the real part is 0 and the imaginary part is 1. In particular, we learn that $\pm(1+i)/\sqrt{2}$ are the square roots of i.

(d) Carrying out the multiplication, we have:

$$\left(\frac{1-i}{\sqrt{2}}\right)^2 = \left(\frac{1-i}{\sqrt{2}}\right)\left(\frac{1-i}{\sqrt{2}}\right) = \frac{1-2i-1}{2} = -i.$$

Thus, the real part is 0 and the imaginary part is -1. In particular, we learn that $\pm(1-i)/\sqrt{2}$ are the square roots of $-i$.

(e) Using the binomial expansion, we have:

$$\left(\frac{1+\sqrt{3}i}{2}\right)^3 = \frac{1+3\sqrt{3}i-9-3\sqrt{3}i}{8} = -1.$$

Thus, the real part is -1 and the imaginary part is 0. In particular, we learn that $(1+\sqrt{3}i)/2$ is one of the cube roots of -1.

(f) Realising the denominator, we have:

$$\frac{1+i}{2-5i} = \frac{(1+i)(2+5i)}{4+25} = \frac{2-5+(5+2)i}{29} = -\frac{3}{29} + \frac{7i}{29}.$$

Thus, the real part is $-3/29$ and the imaginary part is $7/29$.

(g) Realising the denominator, then squaring, we have:

$$\left(\frac{1+i}{1-i}\right)^2 = \left(\frac{(1+i)(1+i)}{2}\right)^2 = \left(\frac{2i}{2}\right)^2 = i^2 = -1.$$

Thus, the real part is -1 and the imaginary part is 0.

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2. If $z = x + iy$, find the real and imaginary parts of the following functions in terms of x and y:

(a) z^2, (b) iz, (c) $(1+i)z$, (d) $z^2(z-1)$, (e) $z^*(z^2 - zz^*)$.

Solution: (a) We have:

$$z^2 = (x+iy)^2 = x^2 + 2xyi - y^2 = (x^2 - y^2) + 2xyi,$$

so that the real part is $x^2 - y^2$ and the imaginary part is $2xy$.

(b) We have:

$$iz = i(x+iy) = -y + ix,$$

so that the real part is $-y$ and the imaginary part is x.

(c) We have:

$$(1+i)z = (1+i)(x+iy) = x - y + (x+y)i,$$

so that the real part is $x - y$ and the imaginary part is $x + y$.

(d) We have:

$$z^2(z-1) = (x+iy)^2((x-1)+iy) = (x^2-y^2+2xyi)((x-1)+iy) = (x^2-y^2)(x-1)-2xy^2+\big((x^2-y^2)y + 2xy(x-1)\big)\,i.$$

Hence the real part is:

$$(x^2 - y^2)(x-1) - 2xy^2 = x^3 - x^2 + y^2 - 3xy^2.$$

The imaginary part is:

$$(x^2 - y^2)y + 2xy(x-1) = x^2y - y^3 + 2x^2y - 2xy.$$

(e) Recall that the complex conjugate of $z = x + iy$ is given by $z^* = x - iy$. Hence we have:

$$z^*(z^2 - zz^*) = z^*z(z - z^*) = (x-iy)(x+iy)(x+iy-(x-iy)) = 2iy(x^2+y^2).$$

Thus, the real part is 0 and the imaginary part is $2y(x^2 + y^2)$.

3. Define u and v to be the real and imaginary parts, respectively, of the complex function $w = 1/z$. Show that the contours of constant u and v are circles. Show also that the contours of u and the contours of v intersect at right angles.

➡ **Solution:** Let $z = x + iy$. Then:

$$w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2},$$

by realising the denominator. This implies that:

$$u = \frac{x}{x^2+y^2}, \qquad v = -\frac{y}{x^2+y^2}.$$

The contours of constant u are the surfaces where u is constant. Hence treating u as a constant, and rearranging the equation we obtained for u, we see that such surfaces satisfy:

$$x^2+y^2 = \frac{x}{u} \qquad \Rightarrow \qquad \left(x - \frac{1}{2u}\right)^2 + y^2 = \frac{1}{4u^2},$$

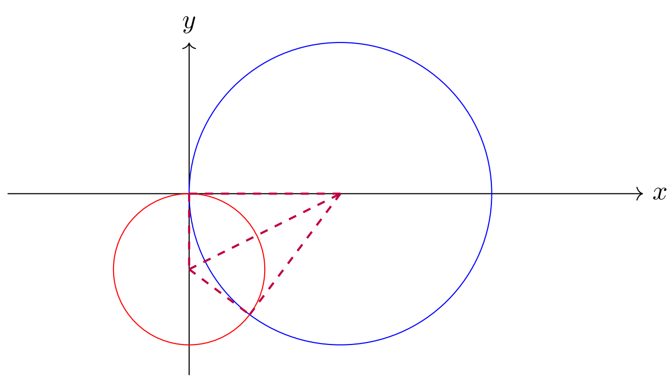
so that they are circles with centre $(1/2u, 0)$ and radius $1/2u$. Similarly, the contours of constant v obey the equation:

$$x^2+y^2 = -\frac{y}{v} \qquad \Rightarrow \qquad x^2 + \left(y + \frac{1}{2v}\right)^2 = \frac{1}{4v^2},$$

so that they are circles with centre $(0, -1/2v)$ and radius $1/2v$.

The only exception is when $u = 0$ or $v = 0$. In these cases, we have either $x = 0$ or $y = 0$, where the contours of constant u, v are just the coordinate axes themselves.

To show that the contours intersect at right angles, we use some geometry. Below, we take $u = 1$ and $v = 2$.



Any pair of these circles intersect at the origin, where the tangents to the circles are the $x = 0$ and $y = 0$ coordinate axes respectively. Thus, they intersect at right angles there.

At their other intersection, we can draw radii to form two similar triangles (shown in dashed purple lines in the above figure), implying that the other intersection also takes place at right angles.

Factoring polynomials and solving equations

4. Factorise the following expressions: (a) z^2+1; (b) z^2-2z+2; (c) z^2+i; (d) $z^2+(1-i)z-i$. *[Hint: you have already computed the two square roots of i in Question 1(c).]*

➾ **Solution:** Throughout this question, we use the identity for the *sum of two squares*, $a^2+b^2=(a+ib)(a-ib)$, which applies for all *complex* numbers a,b.

(a) The first expression is the sum of two squares, so can be factorised simply as $z^2+1=(z+i)(z-i)$.

(b) For the second expression, we complete the square, and then use the fact that the resulting expression is the sum of two squares:

$$z^2-2z+2=(z-1)^2+1=(z-1+i)(z-1-i).$$

(c) We computed the square roots of i in Question 1(c), so we can again use the fact that the expression is the sum of two squares here:

$$z^2+i=\left(z+\frac{1+i}{\sqrt{2}}i\right)\left(z-\frac{1+i}{\sqrt{2}}i\right)=\left(z+\frac{i-1}{\sqrt{2}}\right)\left(z+\frac{1-i}{\sqrt{2}}\right)$$

(d) Again, completing the square, we have:

$$z^2+(1-i)z-i=\left(z+\frac{1-i}{2}\right)^2-\frac{1-2i-1}{4}-i=\left(z+\frac{1-i}{2}\right)^2-\frac{1}{2}i.$$

This is the difference of two squares. Using the square roots of i that we computed in Question 1(c), we can factorise this as:

$$\left(z+\frac{1-i}{2}-\frac{1+i}{2}\right)\left(z+\frac{1-i}{2}+\frac{1+i}{2}\right)=(z-i)(z+1),$$

which is pleasantly simple!

---

5. Given that $z=2+i$ solves the equation $z^3-(4+2i)z^2+(4+5i)z-(1+3i)=0$, find the remaining solutions.

➾ **Solution:** We factorise the left hand side. We have:

$$\begin{aligned}z^3-(4+2i)z^2+(4+5i)z-(1+3i)&=(z-(2+i))\left(z^2-(2+i)z-\frac{1+3i}{2+i}\right)\\&=(z-(2+i))\left(z^2-(2+i)z-(1+i)\right).\end{aligned}$$

We can now apply the quadratic formula to find the roots of the second factor. The roots are:

$$z_\pm=\frac{2+i\pm\sqrt{(2+i)^2-4(1+i)}}{2}=\frac{2+i\pm i}{2},$$

which give the roots $1+i$ and 1. Hence the complete set of solutions to the cubic is $\{2+i,1+i,1\}$.

6. Consider the polynomial equation $a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0$, where the coefficients $a_n, a_{n-1}, ..., a_0$ are real. Show that the solutions to this equation come in complex conjugate pairs. Deduce that if n is odd, there is at least one real solution.

➠ **Solution:** For this question, we will need to use the following important properties of complex conjugation:

Proposition: Let z, w be complex numbers. The following properties hold:

(i) $(z+w)^* = z^* + w^*$;

(ii) $(zw)^* = z^* w^*$.

Proof: In each case, we just write things out in terms of Cartesian components. Let $z = x + iy$ and $w = u + iv$. Then we have:

(i) $(z+w)^* = (x+iy+u+iv)^* = ((x+u)+i(y+v))^* = (x+u)-i(y+v) = (x-iy)+(u-iv) = z^*+w^*$.

(ii) $(zw)^* = ((x+iy)(u+iv))^* = ((xu-yv)+i(xv+yu))^* = (xu-yv)-i(xv+yu)$.

Now let's start the question proper. Let z be a solution of the equation:

$$a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0.$$

Let's take the complex conjugate of the entire equation:

$$\left(a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0\right)^* = 0.$$

Using the properties of complex conjugation, and the fact that the coefficients are real (i.e. $a_k^* = a_k$), we can rewrite the left hand side as:

$$a_n (z^*)^n + a_{n-1} (z^*)^{n-1} + ... + a_1 (z^*) + a_0 = 0.$$

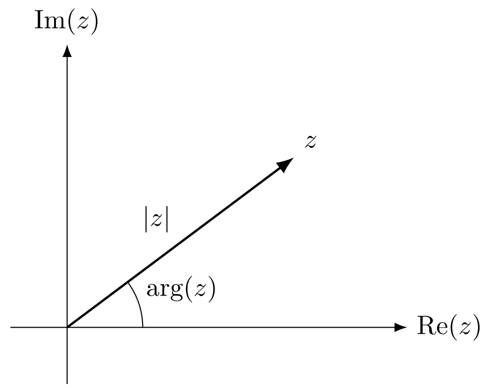
Hence, we see that z^* satisfies the same equation. Thus if z satisfies a polynomial equation with real coefficients, then its complex conjugate also satisfies the same equation.

We have just shown that complex solutions (with a non-zero imaginary part) of real polynomial equations come in *pairs*. Hence, for an odd degree equation which has an odd number of roots, one of them must be purely real.

Geometry of complex numbers

7. Using a diagram, explain the geometric meaning of the *modulus*, $|z|$, and *argument*, $\arg(z)$, of a complex number z. Find the moduli and (principal) arguments of: (a) $1+\sqrt{3}i$; (b) $-1+i$; (c) $-\sqrt{3}-i/\sqrt{3}$.

Solution: Let $z = x + iy$ be a complex number. We can associate this complex number with a point (x, y) in the plane, where the x-axis is the real axis and the y-axis is the imaginary axis. In this context, the plane is called an *Argand diagram*.



The *modulus* of $z = x + iy$ is the length of the vector joining the origin to the point (x, y). By Pythagoras' theorem, we have $|z| = \sqrt{x^2 + y^2}$. The *argument* of $z = x + iy$ is the angle between the positive x-axis (i.e. the positive real axis) and the vector joining the origin to the point (x, y). Of course, there are multiple choices of angle; standard choices include the range $[0, 2\pi)$ or the range $[-\pi, \pi)$. Both are sometimes called the 'principal' choice of argument.

For the given complex numbers, we have:

(a) The modulus is $\sqrt{1+3} = 2$. The argument is:

$$\arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}.$$

(b) The modulus is $\sqrt{(-1)^2 + 1^2} = \sqrt{2}$. The complex number is in the second quadrant, so the argument is:

$$\pi - \arctan\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

(c) The modulus is $\sqrt{3 + 1/3} = \sqrt{11/3}$. The complex number is in the third quadrant, so the argument is:

$$\pi + \arctan\left(\frac{1/\sqrt{3}}{\sqrt{3}}\right) = \pi + \arctan\left(\frac{1}{3}\right),$$

in the range $[0, 2\pi)$, or alternatively:

$$\arctan\left(\frac{1}{3}\right) - \pi,$$

in the range $[-\pi, \pi)$.

8. For $z \in \mathbb{C}$, show that $|z|^2 = zz^*$. Hence prove that $|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2)$, where $a, b \in \mathbb{C}$, and interpret this result geometrically. *[Hint: you don't need to split a, b into real and imaginary parts.]*

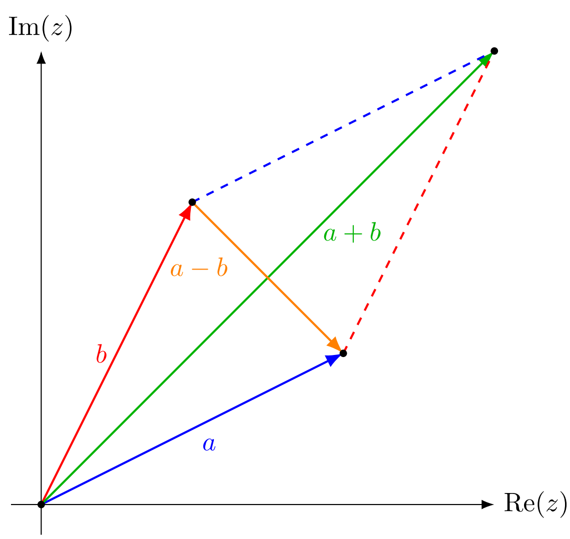
Solution: Let $z = x + iy$. Then $|z|^2 = x^2 + y^2$, whilst $zz^* = (x+iy)(x-iy) = x^2 + y^2$, so $|z|^2 = zz^*$ as required.

To prove the given identity, we use the fact we just showed about moduli. We have:

$$\begin{aligned} |a+b|^2 + |a-b|^2 &= (a+b)(a^*+b^*) + (a-b)(a^*-b^*) \\ &= aa^* + ab^* + ba^* + bb^* + aa^* - ab^* - ba^* + bb^* \\ &= 2aa^* + 2bb^* \\ &= 2\left(|a|^2 + |b|^2\right), \end{aligned}$$

as required.

This result shows that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides of the parallelogram. This is shown in the diagram below, illustrating the parallelogram formed by the complex numbers a, b. The sum of its side lengths squared given by $|a|^2 + |b|^2 + |a|^2 + |b|^2 = 2(|a|^2 + |b|^2)$, and the sum of its diagonal lengths squared is $|a+b|^2 + |a-b|^2$.



9. By writing $z = |z|(\cos(\arg(z)) + i\sin(\arg(z)), w = |w|(\cos(\arg(w)) + i\sin(\arg(w))$, compute the modulus and argument of the product zw. Hence give the geometrical interpretation of multiplying one complex number by another complex number. Give also a geometrical interpretation of division of one complex number by another complex number, z/w.

➡ **Solution:** We have:

$$\begin{aligned}zw &= |z||w| \left(\cos(\arg(z)) + i\sin(\arg(z))\right)\left(\cos(\arg(w)) + i\sin(\arg(w))\right)\\ &= |z||w| \left(\cos(\arg(z))\cos(\arg(w)) - \sin(\arg(z))\sin(\arg(w)) + \left(\cos(\arg(z))\sin(\arg(w)) + \sin(\arg(z))\cos(\arg(w))\right) i\right)\\ &= |z||w| \left(\cos(\arg(z) + \arg(w)) + \sin(\arg(z) + \arg(w)) i\right),\end{aligned}$$

where in the last line we used the compound angle formulae for cosine and sine. It follows that the modulus and argument of the product zw are given by:

$$|zw| = |z||w|, \qquad \arg(zw) = \arg(z) + \arg(w).$$

In particular, we see that multiplying the complex number z by the complex number w results in a *scaled rotation* of z. Indeed, z is scaled by a factor of $|w|$ and rotated by an angle $\arg(w)$ about the origin anticlockwise.

For division, we can view the quotient z/w as the product $z \cdot (1/w)$. The complex number $1/w$ satisfies:

$$\frac{1}{w} = \frac{1}{|w|}\cdot\frac{1}{\cos(\arg(w)) + \sin(\arg(w))i} = \frac{1}{|w|}\frac{\cos(\arg(w)) - \sin(\arg(w))i}{\cos^2(\arg(w)) + \sin^2(\arg(w))} = \frac{1}{|w|}\left(\cos(-\arg(w)) + \sin(-\arg(w))i\right).$$

Hence, we see that $|1/w| = 1/|w|$ and $\arg(1/w) = -\arg(w)$. It follows from the multiplication rule we proved above that:

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}, \qquad \arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w).$$

In particular, division still corresponds to a scaled rotation. However, z is now scaled *down* by a factor of $|w|$ instead of being scaled up, and the rotation is by $\arg(w)$ about the origin *clockwise*.

10. Let $z_1 = 2+i, z_2 = 3+4i$. Find z_1z_2 by: (a) adding arguments and multiplying moduli; (b) using the rules of complex algebra. Verify that your results agree.

Solution: (a) The moduli of these complex numbers are $|z_1| = \sqrt{2^2+1^2} = \sqrt{5}$ and $|z_2| = \sqrt{3^2+4^2} = 5$. The arguments of these complex numbers are $\arg(z_1) = \arctan(1/2)$ and $\arg(z_2) = \arctan(4/3)$. Hence the modulus and argument of the product are given by:

$$|z_1z_2| = 5\sqrt{5}, \qquad \arg(z_1z_2) = \arctan(1/2) + \arctan(4/3).$$

To obtain a Cartesian expression for z_1z_2, we will need to compute the cosine and sine of the argument of z_1z_2, which seems to be quite hard at the moment! To make things easier, we need to *add the arctangents first*. We could do this using the result of Question 11. For practice though, we shall present an alternative method. Let:

$$t = \arctan(1/2) + \arctan(4/3),$$

and consider taking the tangent of both sides and applying the compound angle formula for tangent:

$$\tan(t) = \tan\left(\arctan(1/2) + \arctan(4/3)\right) = \frac{\tan(\arctan(1/2)) + \tan(\arctan(4/3))}{1 - \tan(\arctan(1/2))\tan(\arctan(4/3))} = \frac{1/2+4/3}{1-2/3} = \frac{11}{2}.$$

Hence, we see that:

$$t = \arctan(11/2).$$

This is the angle between the adjacent and hypotenuse of a right-angled triangle with opposite side of length 11 and adjacent side of length 2. The hypotenuse is of length $\sqrt{11^2+2^2} = \sqrt{125} = 5\sqrt{5}$. Hence:

$$5\sqrt{5}\cos(\arctan(11/2)) = 2, \qquad 5\sqrt{5}\sin(\arctan(11/2)) = 11.$$

It follows that:

$$z_1z_2 = 2+11i.$$

(b) We can also compute the product using the rules of complex algebra. We have:

$$z_1z_2 = (2+i)(3+4i) = 6-4+(8+3)i = 2+11i,$$

which agrees perfectly. Much simpler!

11. By considering multiplication of the complex numbers $z = 1 + iA$ and $w = 1 + iB$, derive the arctangent addition formula:

$$\arctan(A) + \arctan(B) = \arctan\left(\frac{A+B}{1-AB}\right).$$

➾ **Solution:** The argument of z is $\arctan(A)$, whilst the argument of w is $\arctan(B)$. Multiplication of complex numbers results in addition of their arguments, so the argument of zw is $\arctan(A) + \arctan(B)$, which is the left hand side of the arctangent addition formula.

On the other hand, we can instead first compute the product of the complex numbers algebraically. We have

$$zw = (1 + iA)(1 + iB) = 1 - AB + (A + B)i.$$

This has argument:

$$\arctan\left(\frac{A+B}{1-AB}\right).$$

Hence, the arctangent addition formula follows.

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12. Give a geometrical interpretation (in terms of *vectors*) of the real and imaginary parts of the quantity $Q = z_1 z_2^*$. Show also that Q is invariant under a rotation of z_1, z_2 about the origin, and confirm that this is consistent with your geometrical interpretation. *[Hint: In Question 9, you showed that multiplying by a complex number u of unit modulus is equivalent to a rotation about the origin.]*

➾ **Solution:** Let $z_1 = x_1 + iy_1$ and let $z_2 = x_2 + iy_2$. Then the given quantity is:

$$Q = (x_1 + iy_1)(x_2 - iy_2) = x_1x_2 + y_1y_2 + (x_2y_1 - x_1y_2)i.$$

The real part is $x_1x_2 + y_1y_2$, which is the scalar product of the vectors $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$. The imaginary part is $x_2y_1 - x_1y_2$, which is the z-component of the cross product of the vectors $(x_2, y_2, 0)$ and $(x_1, y_1, 0)$; this is also the magnitude of this cross product since it points purely in the z-direction.

A rotation about the origin is equivalent to multiplication by a unit modulus complex number, u, satisfying $|u|^2 = uu^* = 1$. Hence under such a rotation, we have $z_1 \mapsto uz_1$ and $z_2 \mapsto uz_2$. This gives:

$$Q = z_1 z_2^* \mapsto (uz_1)(uz_2)^* = uu^* z_1 z_2^* = z_1 z_2^* = Q.$$

Hence Q is invariant under such a rotation. This is consistent with the geometric interpretation we gave earlier in the question, since the scalar product won't change under a rotation (it depends on lengths and angles, which are preserved under a rotation), and the cross produce won't change under a rotation (since the rotation is in the complex plane, and therefore the direction of the cross product will still be out of the complex plane, in the z-direction).

Loci in the complex plane

13. (**Circles**) Describe the sets of points $z \in \mathbb{C}$ satisfying:

(a) $|z| = 4$, (b) $|z-1| = 3$, (c) $|z-i| = 2$, (d) $|z-(1-2i)| = 3$, (e) $|z^*-1| = 1$, (f) $|z^*-i| = 1$.

➻ **Solution:**

(a) This is a circle, centred at 0, radius 4.

(b) This is a circle, centred at 1, radius 3.

(c) This is a circle, centred at i, radius 2.

(d) This is a circle, centred at $1-2i$, radius 3.

(e) Note that $|z^*-1| = |(z-1)^*| = |z-1|$. Hence this is a circle centred at 1, radius 1.

(f) Note that $|z^*-i| = |(z+i)^*| = |z+i|$. Hence this is a circle centred at $-i$, radius 1.

---

14. (**Transformations of circles**) Describe the set of points $z \in \mathbb{C}$ satisfying $|z-2-i| = 6$. Without further calculation, describe the sets of points $u \in \mathbb{C}, v \in \mathbb{C}, w \in \mathbb{C}$ satisfying:

(a) $u = z + 5 - 8i$, (b) $v = iz + 2$, (c) $w = \frac{3}{2}z + \frac{1}{2}z^*$,

where $|z-2-i| = 6$.

➻ **Solution:** The set of points $z \in \mathbb{C}$ satisfying $|z-2-i| = 6$ is a circle centred at $2+i$, radius 6.

(a) If we define $u = z + 5 - 8i$, we have translated the circle by $5-8i$. Hence the locus of u is a circle centred at $7-7i$, radius 6.

(b) If we define $v = iz + 2$, we have rotated the circle by $\pi/2$ clockwise about the origin (this is the multiplication by i), then translated the circle by 2. Since this is a rigid motion, that does not involve bending or squashing the circle, it is sufficient to keep track of where the centre goes. We note:

$$i(2+i) + 2 = 2i - 1 + 2 = 2i + 1,$$

so the locus of v is a circle centred at $1+2i$, radius 6.

(c) This part is more difficult. This is not an obvious transformation from the lectures, so we might consider splitting z into real and imaginary parts. We have:

$$w = \frac{3}{2}(x+iy) + \frac{1}{2}(x-iy) = 2x + iy.$$

Hence, we see that the point $x+iy$ gets mapped to the point $2x+iy$ under the transformation from z to w. Hence, this transformation is a *scaling* in the x-direction (i.e. along the real axis). The result is therefore an *ellipse* with centre $4+i$, major diameter 12 in the x-direction, and minor diameter 6 in the y-direction.

15. (**Circles of Apollonius**) Let $a, b \in \mathbb{C}$. Show that the set of points satisfying $|z-a| = \lambda|z-b|$, where $\lambda \neq 1$, is a circle in the complex plane. *[Hint: start by squaring the equation. You don't need to split z into real and imaginary parts.]* Determine the centre and radius of the circle $|z| = 2|z-2|$.

Solution: We follow the hint, and start by squaring the given equation:

$$\begin{aligned}
|z-a| = \lambda|z-b| \qquad &\Rightarrow \qquad |z-a|^2 = \lambda^2|z-b|^2 \\
&\Rightarrow \qquad (z-a)(z^*-a^*) = \lambda^2(z-b)(z^*-b^*) \\
&\Rightarrow \qquad |z|^2 - a^*z - az^* + |a|^2 = \lambda^2(|z|^2 - b^*z - bz^* + |b|^2) \\
&\Rightarrow \qquad (1-\lambda^2)|z|^2 - (a^* - \lambda^2 b^*)z - (a - \lambda^2 b)z^* = \lambda^2|b|^2 - |a|^2 \\
&\Rightarrow \qquad |z|^2 - \left(\frac{a^* - \lambda^2 b^*}{1-\lambda^2}\right)z - \left(\frac{a-\lambda^2 b}{1-\lambda^2}\right)z^* = \frac{\lambda^2|b|^2 - |a|^2}{1-\lambda^2}.
\end{aligned}$$

We now notice that the terms on the left look like the first three terms in the expansion of:

$$\left|z - \frac{a-\lambda^2 b}{1-\lambda^2}\right|^2,$$

just as we expanding $|z-a|^2$, $|z-b|^2$ in the first couple of lines. Therefore, collecting terms and subtracting the extra fourth term, we are left with:

$$\left|z - \frac{a-\lambda^2 b}{1-\lambda^2}\right|^2 - \left|\frac{a-\lambda^2 b}{1-\lambda^2}\right|^2 = \frac{\lambda^2|b|^2 - |a|^2}{1-\lambda^2}.$$

We can simplify this by moving the second term on the left hand side to the right hand side. We obtain the right hand side:

$$\begin{aligned}
\frac{\lambda^2|b|^2 - |a|^2}{1-\lambda^2} + \left|\frac{a-\lambda^2 b}{1-\lambda^2}\right|^2 &= \frac{\lambda^2|b|^2 - |a|^2 - \lambda^4|b|^2 + \lambda^2|a|^2 + |a|^2 - \lambda^2 ab^* - \lambda^2 a^* b + \lambda^4|b|^2}{(1-\lambda^2)^2} \\
&= \frac{\lambda^2(|b|^2 - ab^* - a^*b + |a|^2)}{(1-\lambda^2)^2} \\
&= \frac{\lambda^2|a-b|^2}{(1-\lambda^2)^2}.
\end{aligned}$$

Hence we see that the original equation can be recast in the form:

$$\left|z - \frac{a-\lambda^2 b}{1-\lambda^2}\right|^2 = \frac{\lambda^2|a-b|^2}{(1-\lambda^2)^2}.$$

Taking the square root, we have:

$$\left|z - \frac{a-\lambda^2 b}{1-\lambda^2}\right| = \frac{\lambda|a-b|}{|1-\lambda^2|}$$

Hence, this is indeed a circle with centre and radius, respectively:

$$\frac{a-\lambda^2 b}{1-\lambda^2}, \qquad \frac{\lambda|a-b|}{|1-\lambda^2|}.$$

For the given example, $|z| = 2|z-2|$, we have $a = 0$, $b = 2$ and $\lambda = 2$. Hence the centre is:

$$\frac{0-8}{1-4} = \frac{8}{3},$$

and the radius is:

$$\frac{2 \cdot 2}{|1-4|} = \frac{4}{3}.$$

---

16. (**Lines and half-lines**) Describe the sets of points $z \in \mathbb{C}$ satisfying:

(a) $|z-2| = |z+i|$, (b) $|z-2| = |z^*+i|$, (c) $\arg(z) = \pi/2$, (d) $\arg(z^*) = \pi/4$.

➭ **Solution:**

(a) This is a line bisecting the line joining the points 2 and $-i$.

(b) This is a line bisecting the line joining the points 2 and i (since $|z^*+i| = |(z-i)^*| = |z-i|$).

(c) This is a half-line, emanating from the origin along the imaginary axis.

(d) Since $\arg(z^*) = -\arg(z)$, this is a half-line, emanating from the origin and inclined at an angle $\pi/4$ below the real axis.

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17. (**Lines and circles**) Let $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$. *Without setting* $z = x + iy$, describe the locus $azz^* + bz + b^*z^* + c = 0$ for different values of a, b, c. How does the locus change under the maps: (a) $z \mapsto \alpha z$ for $\alpha \in \mathbb{C}$; (b) $z \mapsto 1/z$?

➭ **Solution:** We attempt to factorise this expression, like a circle of Apollonius as discussed in Question 15. First, we divide by a, assuming that $a \neq 0$:

$$zz^* + \frac{bz + b^*z^*}{a} + \frac{c}{a} = 0.$$

Completing the square on the first three terms (and using the fact that a is real), we have:

$$\left|z + \frac{b^*}{a}\right|^2 - \left|\frac{b}{a}\right|^2 + \frac{c}{a} = 0 \qquad \Rightarrow \qquad \left|z + \frac{b^*}{a}\right| = \frac{|b|^2 - ca}{a^2}.$$

Hence, if $a \neq 0$, the locus is:

- A circle centred on $-b^*/a$ with radius $\sqrt{|b|^2 - ca}/a$ and $|b|^2 - ca > 0$.
- A point at $-b^*/a$ and $|b|^2 = ca$.
- Empty if $|b|^2 < ca$.

On the other hand, if $a = 0$, the locus is $bz + b^*z^* + c = 0$. The real part of any complex number $w = x + iy$ may be written as $x = \frac{1}{2}(w + w^*)$, hence we recognise this equation as:

$$2\,\mathrm{Re}(bz) + c = 0 \qquad \Leftrightarrow \qquad \mathrm{Re}(bz) = -c/2.$$

The equation $\mathrm{Re}(bz) = -c/2$ tells us that the imaginary part of the expression bz is constant; if we define $w = bz$, then it tells us that in the w-plane, we have a vertical line at $-c/2$.

To understand what things look like in the z-plane, we need to write $z = w/b$. Note that if $b = 0$, then the equation becomes $0 = -c/2$, and we need $c = 0$ too for consistency; then, the original equation just looks like $0 = 0$ which is very uninteresting! In the case $b \neq 0$, $z = w/b$ is a scaled rotation of w by angle $-\arg(b)$ anticlockwise about the origin. Hence the figure in the z-plane looks like a line inclined at angle $\arg(b)$ to the vertical, going through the point $-c/2b$.

(a) The transformation $z \mapsto \alpha z$ is a scaled rotation, enlarging the figure by a factor $|\alpha|$ and rotating it by an angle $\arg(\alpha)$ anticlockwise about the origin.

(b) The transformation $z \mapsto 1/z$ is an *inversion*. To see its effect, we set $w = 1/z$ in the defining equation of the locus:

$$azz^* + bz + b^*z^* + c = 0 \qquad \Leftrightarrow \qquad \frac{a}{ww^*} + \frac{b}{w} + \frac{b^*}{w^*} + c = 0$$
$$\Leftrightarrow \qquad a + bw^* + b^*w + cww^* = 0.$$

In particular, we see that we interchange the roles $a \leftrightarrow c$ and $b \leftrightarrow b^*$ under this transformation. So we have the following cases:

- If $a, c \neq 0$, then this map transforms a circle into another circle. The radius is scaled by a factor a/c and the centre is mapped to $-b/c$.
- If $a \neq 0$ and $c = 0$, then this map transforms a circle into a line. The new line goes through $-a/2b^*$ and is inclined at an angle $\arg(b^*)$ to the vertical.
- If $a = 0$ and $c \neq 0$, then this map transforms a line into a circle. The new circle has centre $-b/a$ and radius $|b|^2/c$.
- If $a = 0$ and $c = 0$, then this map transforms a line into a line. The line is just a line through the origin, and is mapped from having an angle $\arg(b)$ with the vertical to having an angle $\arg(b^*)$ with the vertical.

18. (**More complex figures**) Sketch the sets of points $z \in \mathbb{C}$ satisfying:

(a) $\mathrm{Re}(z^2) = \mathrm{Im}(z^2)$, (b) $\dfrac{\mathrm{Im}(z^2)}{z^2} = -i$, (c) $|z^* + 2i| + |z| = 4$, (d) $|2z - z^* - 3i| = 2$.

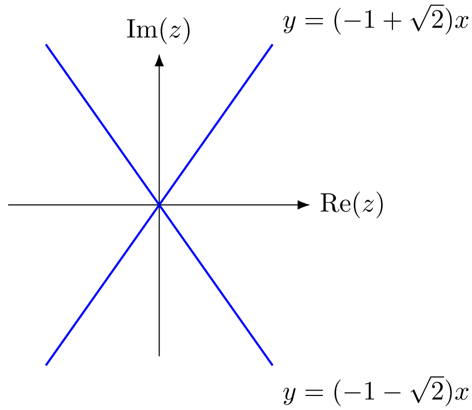
➡ **Solution:** (a) Let $z = x + iy$. Then $z^2 = x^2 - y^2 + 2xyi$, so the locus $\mathrm{Re}(z^2) = \mathrm{Im}(z^2)$ is equivalent to:

$$x^2 - y^2 = 2xy \qquad \Rightarrow \qquad 0 = y^2 + 2xy - x^2.$$

Solving this equation for y, we have:

$$y = \frac{-2x \pm \sqrt{4x^2 + 4x^2}}{2} = -x \pm x\sqrt{2} = (-1 \pm \sqrt{2})x.$$

Thus the locus is a pair of lines passing through the origin.



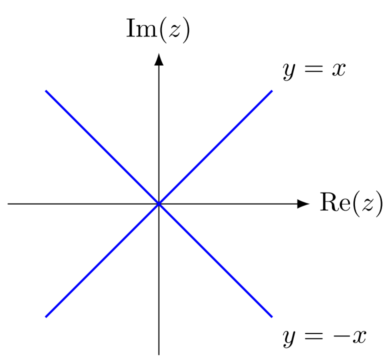
(b) Let $z = x + iy$. Then using part (a), we have $\mathrm{Im}(z^2) = 2xy$. Inserting this into the locus $\mathrm{Im}(z^2)/z^2 = -i$, we have:

$$\frac{2xy}{x^2 - y^2 + 2xyi} = -i.$$

Multiplying up, we have:

$$2xy = i(y^2 - x^2) + 2xy.$$

Cancelling $2xy$ from both sides, we see that $x^2 = y^2$, so that $y = \pm x$. Thus the locus is again a pair of lines passing through the origin.



The locus excludes the origin where the left hand side, $\mathrm{Im}(z^2)/z^2$, is undefined.

(c) Let $z = x + iy$. Then the locus $|z^* + 2i| + |z| = 4$ can be rewritten as:

$$\sqrt{x^2 + (y-2)^2} + \sqrt{x^2 + y^2} = 4.$$

Squaring both sides, we have:

$$x^2 + (y-2)^2 + x^2 + y^2 + 2\sqrt{(x^2+y^2)(x^2+(y-2)^2)} = 16$$

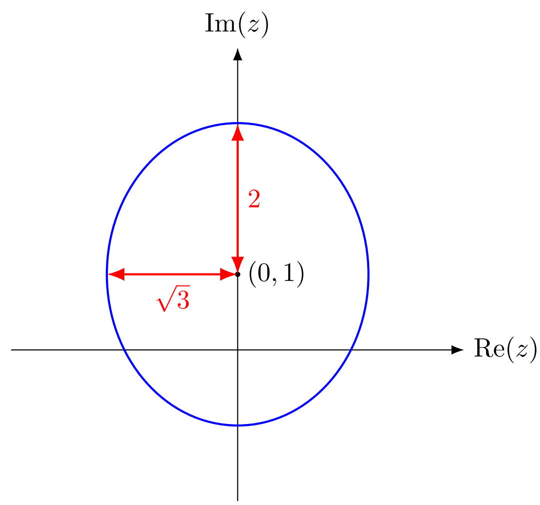
$$\Leftrightarrow \qquad x^2 + y^2 - 2y - 6 = -\sqrt{(x^2+y^2)(x^2+y^2-4y+4)} = -\sqrt{x^4 + 2x^2y^2 - 4x^2y + 4x^2 + y^4 - 4y^3 + 4y^2}$$

Squaring both sides again, we have:

$$x^4 + y^4 + 4y^2 + 36 + 2x^2y^2 - 4x^2y - 12x^2 - 4y^3 - 12y^2 + 24y = x^4 + 2x^2y^2 - 4x^2y + 4x^2 + y^4 - 4y^3 + 4y^2$$

Simplifying, this reduces to:

$$9 = 4x^2 + 3y^2 - 6y \qquad \Leftrightarrow \qquad 1 = \left(\frac{x}{\sqrt{3}}\right)^2 + \left(\frac{y-1}{2}\right)^2.$$



(d) Let $z = x + iy$. Then:

$$2 = |2z - z^* - 3i| = |2(x+iy) - (x-iy) - 3i| = |x + 3i(y-1)| = \sqrt{x^2 + 9(y-1)^2}.$$

Rearranging, we have:

$$1 = \left(\frac{x}{2}\right)^2 + \left(\frac{y-1}{2/3}\right)^2.$$

This is an ellipse, centred on $(0, 1)$, with semi-minor axis $2/3$ and semi-major axis 2. This is the same as the figure above, just scaled in the x, y directions.

Exponential form of a complex number

19. State *Euler's formula* for the complex exponential $e^{i\theta}$. Hence provide a simpler derivation of the modulus-argument multiplication law proved in Question 9.

➾ **Solution:** *Euler's formula* states that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. To rederive the modulus-argument multiplication law, let $z = |z|e^{i\arg(z)}$ and $w = |w|e^{i\arg(w)}$. Then:

$$zw = |z||w|e^{i(\arg(z)+\arg(w))},$$

which shows $|zw| = |z||w|$ and $\arg(zw) + \arg(z) + \arg(w)$.

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20. Find (a) the real and imaginary parts; (b) the modulus and argument, of:

$$\frac{e^{i\omega t}}{R + i\omega L + (i\omega C)^{-1}},$$

where ω, t, R, L, C are real, quoting your answers in terms of $X = \omega L - (\omega C)^{-1}$. (*) If you are taking IA Physics, can you think of what each of ω, t, R, L, C might represent?

➾ **Solution:** (a) To find the real and imaginary parts, we need to realise the denominator. Note that:

$$R + i\omega L + (i\omega C)^{-1} = R + i\left(\omega L - \frac{1}{\omega C}\right) = R + iX.$$

Hence we have:

$$\frac{e^{i\omega t}}{R+iX} = \frac{(R-iX)e^{i\omega t}}{R^2+X^2} = \frac{(R-iX)(\cos(\omega t) + i\sin(\omega t))}{R^2+X^2}.$$

Therefore, the real and imaginary parts, are, respectively:

$$\frac{R\cos(\omega t) + X\sin(\omega t)}{R^2+X^2}, \qquad \frac{R\sin(\omega t) - X\cos(\omega t)}{R^2+X^2}.$$

(b) To find the modulus, we use the property $|z/w| = |z|/|w|$. The numerator has modulus 1, and the denominator has modulus $\sqrt{R^2+X^2}$. Hence the modulus is $1/\sqrt{R^2+X^2}$.

To find the argument, we use the property $\arg(z/w) = \arg(z) - \arg(w)$. The numerator has argument ωt, and the denominator has argument $\arctan(X/R)$. Hence the argument is:

$$\omega t - \arctan\left(\frac{X}{R}\right).$$

This result is useful in alternating current circuits. The quantities here represent resistance (R), inductance (L), capacitance (C), frequency of the current (ω) and time (t).

21. Express each of the following in Cartesian form: (a) $e^{-i\pi/2}$; (b) $e^{-i\pi}$; (c) $e^{i\pi/4}$; (d) e^{1+i}; (e) $e^{2e^{i\pi/4}}$.

➡ **Solution:** We use Euler's formula in each case:

(a) $e^{-i\pi/2} = \cos(-\pi/2) + i\sin(-\pi/2) = -i$.

(b) $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$.

(c) $e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = \dfrac{1+i}{\sqrt{2}}$.

(d) $e^{1+i} = e \cdot e^{i} = e\left(\cos(1) + i\sin(1)\right) = e\cos(1) + ie\sin(1)$. This cannot be further simplified.

(e) $e^{2e^{i\pi/4}} = e^{2(1+i)/\sqrt{2}} = e^{\sqrt{2}+i\sqrt{2}} = e^{\sqrt{2}}e^{i\sqrt{2}} = e^{\sqrt{2}}\left(\cos(\sqrt{2}) + i\sin(\sqrt{2})\right) = e^{\sqrt{2}}\cos(\sqrt{2}) + ie^{\sqrt{2}}\sin(\sqrt{2})$. This cannot be further simplified.

---

22. Let a, b, ω be real constants. Show that $a\cos(\omega x) + b\sin(\omega x) = \mathrm{Re}((a-bi)e^{i\omega x})$, and hence, by writing $a - bi$ in exponential form, deduce that $a\cos(\omega x) + b\sin(\omega x) = \sqrt{a^2+b^2}\cos(\omega x - \arctan(b/a))$.

➡ **Solution:** We have:

$$\mathrm{Re}((a-bi)e^{i\omega x}) = \mathrm{Re}\left((a-bi)(\cos(\omega x) + i\sin(\omega x))\right) = a\cos(\omega x) + b\sin(\omega x),$$

as required. In exponential form, we have $a - bi = \sqrt{a^2+b^2}e^{-i\arctan(b/a)}$. Hence we have:

$$(a-bi)e^{i\omega x} = \sqrt{a^2+b^2}e^{i(\omega x - \arctan(b/a))}.$$

Taking the real part, we see that:

$$a\cos(\omega x) + b\sin(\omega x) = \sqrt{a^2+b^2}\cos\left(\omega x - \arctan(b/a)\right),$$

as required. This result is useful, because it shows that the linear combination of trigonometric functions can always be combined to produce a single trigonometric function, albeit with a shifted phase.

Multi-valued functions: logarithms and powers

23. Explain why the complex logarithm $\log : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ is a *multi-valued function*, and give its possible values. Using the complex logarithm, find all complex numbers satisfying: (a) $e^{2z} = -1$; (b) $e^{z^*} = i + 1$.

➡ **Solution:** The *complex logarithm* of the complex number z, written $\log(z)$, is the solution of the equation:

$$e^{\log(z)} = z.$$

Write $\log(z) = u(z) + iv(z)$, where $u(z), v(z)$ are the real and imaginary parts of the complex logarithm respectively. Then:

$$z = e^{u(z)+iv(z)} = e^{u(z)}e^{iv(z)}.$$

Write $z = |z|e^{i\arg(z)}$. Comparing the modulus, we see that $u(z) = \log|z|$. Comparing the argument, we see that $v(z) = \arg(z) + 2\pi n$, where n is an integer. Hence:

$$\log(z) = \log|z| + i\arg(z) + 2\pi i n,$$

for any integer n. This shows that the complex logarithm is a multi-valued function.

Applying this to the given equations:

(a) Taking the logarithm of $e^{2z} = -1$, we have:

$$2z = \log(-1) = \log|1| + i\arg(-1) + 2\pi i n = i\pi + 2\pi i n.$$

Hence $z = \frac{1}{2}i\pi + \pi i n$ for n an integer.

(b) Taking the logarithm of $e^{z^*} = 1 + i$, we have:

$$z^* = \log(1+i) = \log|1+i| + i\arg(1+i) + 2\pi i n = \log(\sqrt{2}) + \frac{\pi i}{4} + 2\pi i n.$$

Hence $z = \log(\sqrt{2}) - \frac{\pi i}{4} + 2\pi i n$, where n is an integer.

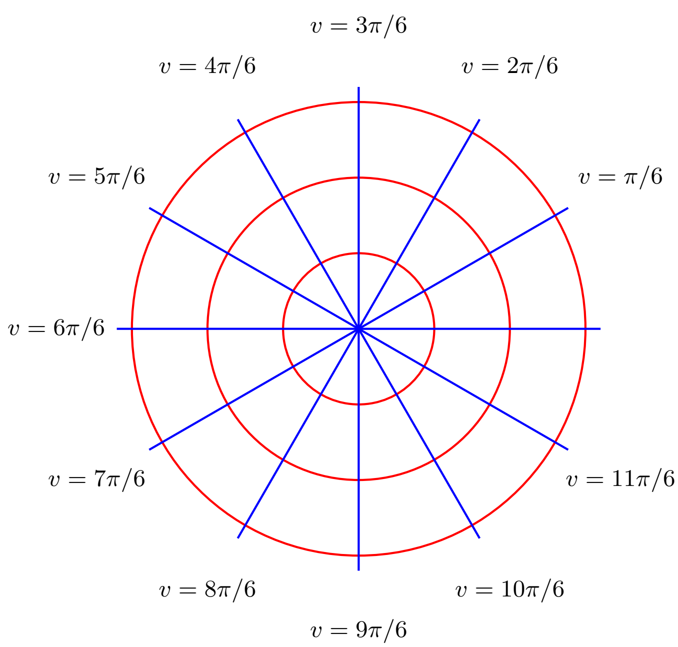
---

24. Let the real and imaginary parts of the complex logarithm $\log(z)$ be u, v respectively. Sketch the contours of constant u, v in the complex plane, and show that they intersect at right angles.

Solution: The complex logarithm, $\log(z) = \log|z| + i\arg(z) + 2\pi i n$, has $u = \log|z|$, $v = \arg(z) + 2\pi n$, for n an integer.

Therefore, for u constant, we have $|z| = e^u$. This is a circle centred on the origin. All radii are allowed, as u varies from $-\infty$ to ∞. For v constant, we have $\arg(z) + 2\pi n = v$, which described a half-line emanating from the origin, at angle $v - 2\pi n$, or equivalently v, from the x-axis.

Below, we display a diagram showing contours of constant v in blue, and contours of constant u in red. Since the contours of constant v correspond to radii of the circles which comprise the contours of constant u, they must intersect at right angles.



25. Explain how the complex logarithm can be used to define complex powers, z^w, and hence describe the multi-valued nature of complex exponentiation. Compute all values of the multi-valued exponentials: (a) i^i; (b) $i^{1/3}$.

➼ **Solution:** If w is a complex number, we define the *complex power* z^w by:

$$z^w := e^{w\log(z)} = e^{w(\log(z)+i\arg(z)+2\pi in)},$$

where n is an integer. This means that :

- If w is an integer, then the $2\pi in$ part of the exponent has no effect - $e^{2\pi inw} = 1$, so we're safe! Therefore, integer powers of complex numbers are single-valued.
- If w is a rational number, then there are some n such that $e^{2\pi inw} = 1$. For example, if $w = 1/2$, we have that $n = 2, 4, ...$ These n will periodically repeat with the period of the denominator of w (when it is written in its lowest terms). Hence, rational powers of complex numbers are *multi-valued*, but can only take *finitely many different values*.
- If w is an irrational number, then the powers are *multi-valued*, but can take *infinitely many* different values.
- If $w = a + bi$ is a complex number, with $b \neq 0$, then we always have a term $(bi)\cdot(2\pi in) = -2\pi bn$ in the exponent. This implies that the powers are *multi-valued*, and again always take *infinitely many* different values.

Examining the exponentials we are given:

(a) $i^i = e^{i\log(i)} = e^{i(\log|i|+i\arg(i)+2\pi in)} = e^{-\pi/2-2\pi n}$, for all integers n. Hence, there are infinitely many possible values of this exponential, but all possible values of i^i are in fact real!

(b) $i^{1/3} = e^{\log(i)/3} = e^{(\log|i|+i\arg(i)+2\pi in)/3} = e^{i\pi/6+2\pi in/3}$. There are only finitely many possible values of this exponential, which vary as we take $n = 0, 1, 2$. The possible values are:

$$\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{i}{2},$$

$$\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{i}{2},$$

$$\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i.$$

---

26. Compute all possible values of $\left(i^i\right)^i$ and $i^{(i^i)}$.

➼ **Solution:** We already computed i^i in the previous question, with $i^i = e^{-\pi/2-2\pi n}$ for all integers n. Taking another power of i, we have:

$$\left(e^{-\pi/2-2\pi n}\right)^i = e^{i\left(\log(e^{-\pi/2-2\pi n})\right)} = e^{-i\pi/2-2\pi in} = e^{-i\pi/2} = -i.$$

In particular, we see that $(i^i)^i = -i$ is single-valued. On the other hand, we have:

$$i^{(i^i)} = e^{e^{-\pi/2-2\pi n}\log(i)} = e^{e^{-\pi/2-2\pi n}(\log|i|+i\arg(i)+2\pi im)} = e^{e^{-\pi/2-2\pi n}(i\pi/2+2\pi im)}.$$

Expressing this in Cartesian form, we see that we have a doubly-multi-valued result,

$$\cos\left(e^{-\pi/2-2\pi n}\left(\frac{\pi}{2}+2\pi m\right)\right) + \sin\left(e^{-\pi/2-2\pi n}\left(\frac{\pi}{2}+2\pi m\right)\right),$$

where n, m are integers. This cannot be further simplified.

27. Find the real and imaginary parts of the function $f(z) = \log(z^{1+i})$. Hence, sketch the locus $\mathrm{Re}(f(z)) = 0$.

➼ **Solution:** Since:

$$f(z) = \log(z^{1+i}) = \log\left(e^{(1+i)(\log|z|+i\arg(z)+2\pi in)}\right) = (1+i)\left(\log|z| + i\arg(z) + 2\pi in\right),$$

for n an integer, we have:

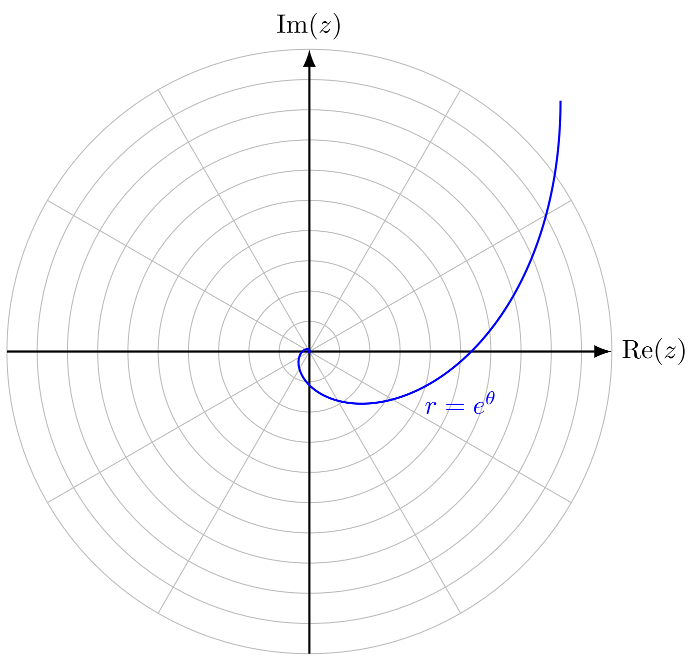
$$f(z) == \log|z| - \arg(z) - 2\pi n + (\log|z| + \arg(z) + 2\pi n)\, i,$$

where n is an integer, which gives the real and imaginary parts.

The locus $\mathrm{Re}(f(z)) = 0$ is given by $\log|z| = \arg(z) + 2\pi n$. Writing this in terms of polar coordinates, we have $|z| = r$ and $\arg(z) = \theta \in [-\pi, \pi)$, say. Then:

$$r = e^{\theta + 2\pi n}.$$

This implies that the complete locus is a *logarithmic spiral*, shown in the figure below.



It grows pretty rapidly! More so than the *Archimedean spiral*, $r = \theta$, that we saw on Examples Sheet 2.

Roots of unity

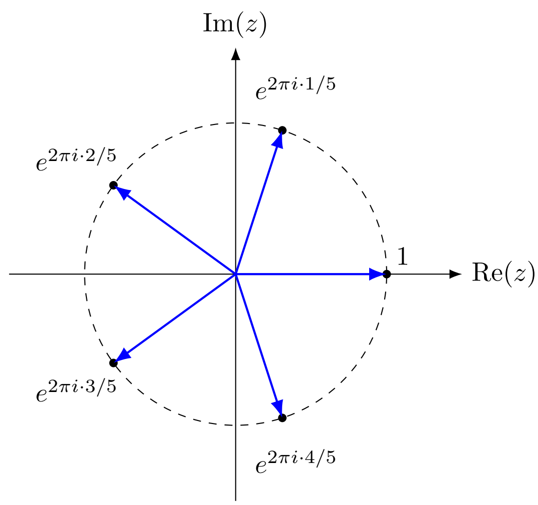
28. Write down the solutions to the equation $z^n = 1$ in terms of complex exponentials, and plot the solutions on an Argand diagram. *[Recall that the solutions are called the* nth roots of unity.*]*

Solution: The solutions are:

$$z = 1^{1/n} = e^{(1/n)\cdot(\log|1|+i\arg(1)+2\pi im)} = e^{2\pi im/n},$$

where m is an integer. On an Argand diagram, these solutions form the vertices of an n-sided regular polygon on the unit circle, with one vertex at the point 1.

For the case $n = 5$, for example, the figure takes the form:



The roots form a regular pentagon in this case.

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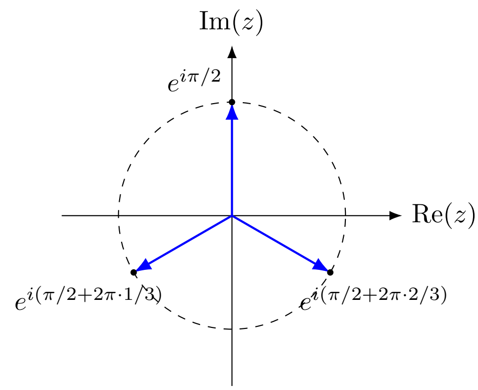
29. Find and plot the solutions to the following equations: (a) $z^3 = -1$; (b) $z^4 = 1$; (c) $z^2 = i$; (d) $z^3 = -i$.

Solution:

(a) The solutions are:

$$z = (-1)^{1/3} = e^{(1/3)\cdot(\log|-1|+i\arg(-1)+2\pi in)} = e^{i(\pi/2+2\pi n/3)}.$$

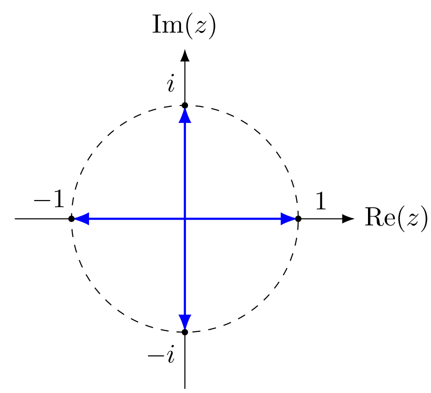
These form a triangle in the complex plane, shown below.



(b) The solutions are:

$$z = 1^{1/4} = e^{(1/4)\cdot(\log|1|+i\arg(1)+2\pi i n)} = e^{\pi i n/2}.$$

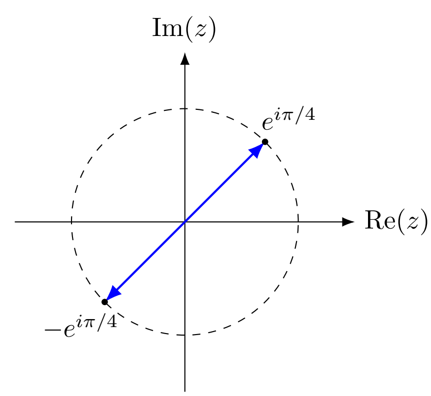
Equivalently, these can be written as $\{1, -1, i, -i\}$. These form a square in the Argand diagram, as shown in the figure below.



(c) The solutions are:

$$z = i^{1/2} = e^{(1/2)\cdot(\log|i|+i\arg(i)+2\pi i n)} = e^{i\pi/4+\pi i n} = \pm e^{i\pi/4}.$$

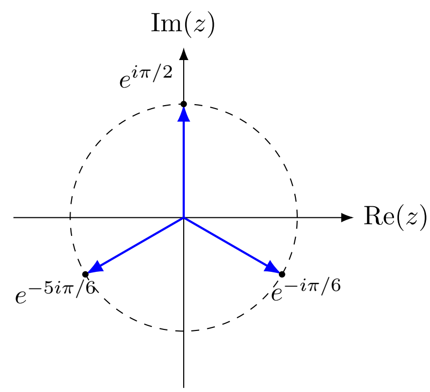
Equivalently, these can be written in Cartesian form as $\pm(1+i)/\sqrt{2}$. These are two points on opposite sides of the origin, as shown in the figure below.



(d) The solutions are:

$$z = (-i)^{1/3} = e^{(1/3)\cdot(\log|-i|+i\arg(-i)+2\pi i n)} = e^{-i\pi/6+2\pi i n/3}.$$

These form a triangle in the complex plane, as shown in the figure below.



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30. If $\omega^n = 1$, determine the possible values of $1+\omega+\omega^2+\cdots+\omega^{n-1}$, and interpret your result geometrically.

➾ **Solution:** This is a geometric progression, so summing the terms we have:

$$1+\omega+\omega^2+\cdots+\omega^{n-1} = \frac{1-\omega^n}{1-\omega} = 0,$$

provided that $\omega \neq 1$. Hence the possible values are:

$$1+\omega+\omega^2+\cdots+\omega^{n-1} = \begin{cases} n, & \text{if } \omega = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Geometrically, in the case $\omega \neq 1$, this corresponds to us following the position vectors of the vertices of the polygon that is formed by the roots of unity (or, a sub-polygon). In the case $\omega \neq 1$, this necessarily ends up taking us to zero.

31. Show that the roots of the equation $z^{2n} - 2bz^n + c = 0$ will, for general complex values of b and c and integral values of n, lie on two circles in the Argand diagram. Give a condition on b and c such that the circles coincide. Find the largest possible value for $|z_1 - z_2|$, if z_1 and z_2 are roots of $z^6 - 2z^3 + 2 = 0$.

➻ **Solution:** Solving the quadratic, we have:

$$z^n = \frac{2b \pm \sqrt{4b^2 - 4c}}{2} = b \pm \sqrt{b^2 - c}.$$

Taking the $1/n$th power, we have:

$$z = \left(b \pm \sqrt{b^2 - c}\right)^{1/n} = \left|b \pm \sqrt{b^2 - c}\right|^{1/n} e^{i\arg(b \pm \sqrt{b^2-c})/n + 2\pi i k/n}.$$

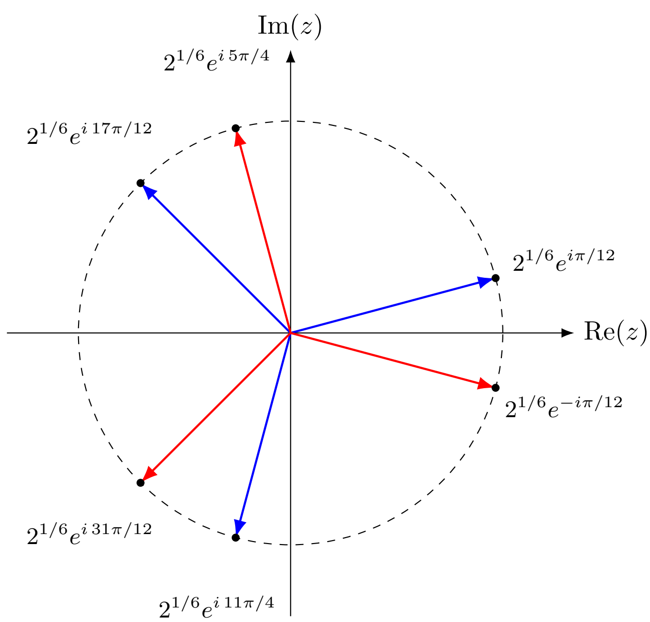
where k is an integer. Thus the solutions lie on two circles, centred on the origin, of radii $|b \pm \sqrt{b^2 - c}|^{1/n}$ respectively. The circles coincide if and only if:

$$|b + \sqrt{b^2 - c}| = |b - \sqrt{b^2 - c}|.$$

In the case where $n = 3$, $b = 1$ and $c = 2$, we have the roots:

$$\begin{aligned} z &= \left|1 \pm \sqrt{1 - 2}\right|^{1/3} e^{i\arg(1 \pm \sqrt{1-2})/3 + 2\pi i k/3} \\ &= \left|1 \pm i\right|^{1/3} e^{i\arg(1\pm i)/3 + 2\pi i k/3} \\ &= 2^{1/6} e^{\pm i\pi/12 + 2\pi i k/3}, \end{aligned}$$

for k an integer. Therefore, we have clusters of pairs of roots which have an angle $\pi/6$ between them, separated into three groups which are rotated by $2\pi/3$.



From the figure, we see that the roots are furthest apart when they are inclined at an angle $2\pi/3 + \pi/6 = 5\pi/6$. By the cosine rule, the distance between the roots is:

$$\sqrt{2^{2/6} + 2^{2/6} - 2 \cdot 2^{2/6}\cos(5\pi/6)} = 2^{1/6}\sqrt{2 + \sqrt{3}}.$$

Trigonometry with complex numbers

32. Prove *De Moivre's formula*, $(\cos(\theta)+i\sin(\theta))^n = \cos(n\theta)+i\sin(n\theta)$. Hence, solve the equation $16\sin^5(\theta) = \sin(5\theta)$ by expressing $\sin(5\theta)$ in terms of $\sin(\theta)$ and its powers.

➡ **Solution:** Using Euler's formula, we have:

$$(\cos(\theta)+i\sin(\theta))^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta)+i\sin(n\theta).$$

To solve the given equation, note that:

$$\sin(5\theta) = \mathrm{Im}\left((\cos(\theta)+i\sin(\theta))^5\right) = \sin^5(\theta) - 10\sin^3(\theta)\cos^2(\theta) + 5\sin(\theta)\cos^4(\theta).$$

Using the identity $\cos^2(\theta) = 1-\sin^2(\theta)$, we can simplify this to read:

$$\begin{aligned}\sin(5\theta) &= \sin^5(\theta) - 10\sin^3(\theta)(1-\sin^2(\theta)) + 5\sin(\theta)(1-\sin^2(\theta))^2 \\ &= \sin^5(\theta) - 10\sin^3(\theta) + 10\sin^5(\theta) + 5\sin(\theta) - 10\sin^3(\theta) + 5\sin^5(\theta) \\ &= 16\sin^5(\theta) - 20\sin^3(\theta) + 5\sin(\theta).\end{aligned}$$

Therefore, the equation $16\sin^5(\theta) = \sin(5\theta)$ is equivalent to the equation:

$$0 = 4\sin^3(\theta) - \sin(\theta) = \sin(\theta)(2\sin(\theta)-1)(2\sin(\theta)+1).$$

Setting each factor to zero, we have:

- $\sin(\theta) = 0$ if and only if $\theta = n\pi$ for n an integer;
- $\sin(\theta) = \frac{1}{2}$ if and only if $\theta = \pi/6 + 2n\pi, 5\pi/6 + 2n\pi$ for n an integer;
- $\sin(\theta) = -\frac{1}{2}$ if and only if $\theta = -\pi/6 + 2n\pi, 7\pi/6 + 2n\pi$ for n an integer.

33. Starting from Euler's formula, show that the trigonometric functions can be written in terms of complex exponentials as:

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Learn these formulae off by heart. Hence, express $\sin^5(\theta)$ in terms of $\sin(\theta), \sin(3\theta)$ and $\sin(5\theta)$.

➼ **Solution:** Euler's formula applied to $e^{i\theta}$ and $e^{-i\theta}$ gives:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$
$$e^{-i\theta} = \cos(\theta) - i\sin(\theta).$$

Adding these formulae, we get:

$$2\cos(\theta) = e^{i\theta} + e^{-i\theta} \qquad \Leftrightarrow \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Subtracting these formulae, we get:

$$2i\sin(\theta) = e^{i\theta} - e^{-i\theta} \qquad \Leftrightarrow \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Hence, we have:

$$\sin^5(\theta) = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^5 = \frac{e^{5i\theta} - 5e^{3i\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - e^{-5i\theta}}{32i}.$$

Collecting like terms, we see that:

$$\sin^5(\theta) = \frac{\sin(5\theta)}{16} - \frac{5\sin(3\theta)}{16} + \frac{5\sin(\theta)}{8}.$$

34. Show that if $x, y \in \mathbb{R}$, the equation $\cos(y) = x$ has the solutions $y = \pm i \log\left(x + i\sqrt{1-x^2}\right) + 2n\pi$ for integer n.

➠ **Solution:** Using the formula for $\cos(y)$ in terms of complex exponentials, the equation $\cos(y) = x$ can be rewritten as:

$$\frac{e^{iy} + e^{-iy}}{2} = x \qquad \Leftrightarrow \qquad e^{2iy} - 2xe^{iy} + 1 = 0.$$

This is a quadratic equation for e^{iy}; solving using the quadratic formula we have:

$$e^{iy} = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1} = x \pm i\sqrt{1 - x^2}.$$

Taking the complex logarithm, we have:

$$iy = \log\left(x \pm i\sqrt{1-x^2}\right) + 2\pi i n,$$

where n is an integer (we assume here that $\log$ takes its principal value, so that a specific argument choice is made). Dividing by i, we have:

$$y = -i\log\left(x \pm i\sqrt{1-x^2}\right) + 2\pi n,$$

where n is an integer. This is close to the final answer. To finish, observe that:

$$x - i\sqrt{1-x^2} = \frac{(x - i\sqrt{1-x^2})(x + i\sqrt{1-x^2})}{x + i\sqrt{1-x^2}} = \frac{x^2 + 1 - x^2}{x + i\sqrt{1-x^2}} = \frac{1}{x + i\sqrt{1-x^2}}.$$

Hence, $\log(x - i\sqrt{1-x^2}) = -\log(x + i\sqrt{1-x^2})$. This implies that the solution of the equation may be written as:

$$y = -i\log\left(x \pm i\sqrt{1-x^2}\right) + 2\pi n = \pm i\log\left(x + i\sqrt{1-x^2}\right) + 2\pi n,$$

where n is an integer, as required.

---

35. Let $\theta \neq 2p\pi$ for $p \in \mathbb{Z}$. Show that $\displaystyle\sum_{n=0}^{N-1}\cos(n\theta) = \frac{\cos\left((N-1)\theta/2\right)\sin\left(N\theta/2\right)}{\sin\left(\theta/2\right)}$. What happens if $\theta = 2p\pi$?

➠ **Solution:** We have:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos(n\theta) &= \mathrm{Re}\left[\sum_{n=0}^{N-1} e^{in\theta}\right] \\
&= \mathrm{Re}\left[\frac{1 - e^{iN\theta}}{1 - e^{i\theta}}\right] && (\text{if } e^{i\theta} \neq 1) \\
&= \mathrm{Re}\left[\frac{e^{iN\theta/2}}{e^{i\theta/2}}\frac{e^{-iN\theta/2} - e^{iN\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}}\right] \\
&= \mathrm{Re}\left[e^{i(N-1)\theta/2} \cdot \frac{-2i\sin(N\theta/2)}{-2i\sin(\theta/2)}\right] \\
&= \frac{\cos((N-1)\theta/2)\sin(N\theta/2)}{\sin(\theta/2)},
\end{aligned}$$

as required. This holds provided that $e^{i\theta} \neq 1$, in which case we cannot sum the geometric series in the second line. This occurs if and only if $\theta = 2p\pi$ for an integer p. In this case, we have the sum:

$$\sum_{n=0}^{N-1}\cos(2p\pi n) = \sum_{n=0}^{N-1} 1 = N.$$

Hyperbolic functions

36 (a) Give the definitions of $\cosh(x)$ and $\sinh(x)$ in terms of exponentials.

(b) Hence, show that $\cos(x) = \cosh(ix)$ and $i\sin(x) = \sinh(ix)$. Deduce *Osborn's rule*: 'a hyperbolic trigonometric identity can be deduced from a circular trigonometric identity[2] by replacing each trigonometric function with its hyperbolic counterpart *except* where sine enters quadratically, where we include an extra factor of -1.'

(c) Using Osborn's rule, write down the formula for $\tanh(x+y)$ in terms of $\tanh(x), \tanh(y)$.

➾ **Solution:** (a) We have:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

(b) Comparing to Question 33, we immediately notice that $\cosh(ix) = \cos(x)$ and $\sinh(ix) = i\sin(x)$, as required. In particular, we see that if we have a trigonometric identity, we can turn it into a hyperbolic identity by replacing cosine with hyperbolic cosine, and replacing sine with hyperbolic cosine multiplied by i - this means that whenever we have a sine squared, then it becomes *negative* hyperbolic sine squared.

(c) We have to be a bit careful here - we just said that terms that are quadratic in sine receive a minus sign when we convert from trigonometric to hyperbolic identities. However, this *also* applies to produces of tangents, since $\tan(x) = \sin(x)/\cos(x)$. Hence the compound angle identity:

$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

gets converted to the hyperbolic identity:

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}.$$

---

37. Find the real and imaginary parts of the following complex numbers:

(a) $\log\left[\sinh\left(\frac{i\pi}{2}\right) + \cosh\left(\frac{9i\pi}{2}\right)\right]$, (b) $\sum_{n=1}^{121}\left[\tanh\left(\frac{in\pi}{4}\right) - \tanh\left(\frac{in\pi}{4} - \frac{i\pi}{4}\right)\right]$.

➾ **Solution:** (a) We have:

$$\sinh\left(\frac{i\pi}{2}\right) = i\sin\left(\frac{\pi}{2}\right) = i, \qquad \cosh\left(\frac{9i\pi}{2}\right) = \cos\left(\frac{9\pi}{2}\right) = 0.$$

Hence we must evaluate:

$$\log(i) = \log|i| + i\arg(i) + 2n\pi i = \frac{i\pi}{2} + 2n\pi i,$$

where n is an integer.

[2] Provided the arguments of all the circular trigonometric functions are homogeneous linear polynomials in the variables of interest.

(b) Here, we spot that this is a telescoping sum:

$$\sum_{n=1}^{121}\left[\tanh\left(\frac{in\pi}{4}\right)-\tanh\left(\frac{in\pi}{4}-\frac{i\pi}{4}\right)\right]$$

$$\begin{aligned}
&= \tanh\left(\frac{i\pi}{4}\right)-\tanh(0)+\tanh\left(\frac{2i\pi}{4}\right)-\tanh\left(\frac{i\pi}{4}\right)+\cdots+\tanh\left(\frac{121i\pi}{4}\right)-\tanh\left(\frac{120i\pi}{4}\right)\\
&= \tanh\left(\frac{121i\pi}{4}\right)\\
&= i\tan\left(\frac{121\pi}{4}\right)\\
&= i\tan\left(30\pi+\frac{\pi}{4}\right)\\
&= i.
\end{aligned}$$

---

38. Find the real and imaginary parts of the function $\tan(z^*)$.

➾ **Solution:** Observe that:

$$\tan(iy)=\frac{\sin(iy)}{\cos(iy)}=\frac{e^{-y}-e^{y}}{i(e^{-y}+e^{y})}=\frac{i\sinh(y)}{\cosh(y)}=i\tanh(y).$$

Hence, we have:

$$\tan(z^*)=\tan(x-iy)=\frac{\tan(x)-\tan(iy)}{1+\tan(x)\tan(iy)}=\frac{\tan(x)-i\tanh(y)}{1+i\tan(x)\tanh(y)}.$$

Realising the denominator, we have:

$$\tan(z^*)=\frac{(\tan(x)-i\tanh(y))(1-i\tan(x)\tanh(y))}{1+\tan^2(x)\tanh^2(y)}=\frac{\tan(x)-\tan(x)\tanh^2(y)-i\tanh(y)(1+\tan^2(x))}{1+\tan^2(x)\tanh^2(y)}.$$

It follows that the real part is:

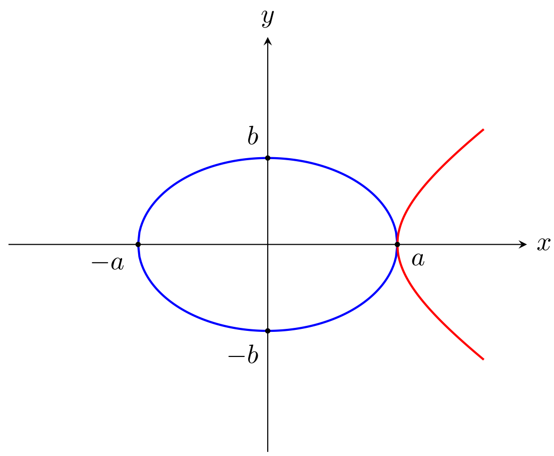
$$\tan(x)\cdot\frac{1-\tanh^2(y)}{1+\tan^2(x)\tanh^2(y)}=\frac{\tan(x)}{\cosh^2(y)+\tan^2(x)\sinh^2(y)}=\frac{\sin(x)\cos(x)}{\cos^2(x)\cosh^2(y)+\cos^2(x)\sinh^2(y)}.$$

The imaginary part is:

$$-\tanh(y)\cdot\frac{1+\tan^2(x)}{1+\tan^2(x)\tanh^2(y)}=\frac{\tanh(y)}{\cos^2(x)+\sin^2(x)\tanh^2(y)}=\frac{\sinh(x)\cosh(x)}{\cos^2(x)\cosh^2(y)+\sin^2(x)\sinh^2(y)}.$$

39. Let $b \geq a > 0$ be fixed, and let θ be a variable parameter. Find the Cartesian equations of the two parametric curves: (a) $(x, y) = (a\cos(\theta), b\sin(\theta))$; (b) $(x, y) = (a\cosh(\theta), b\sinh(\theta))$, and sketch them in the plane. [*This explains why hyperbolic functions are called hyperbolic functions!*]

➾ **Solution:** (a) We have $(x/a)^2 + (y/b)^2 = \cos^2(\theta) + \sin^2(\theta) = 1$. (b) We have $(x/a)^2 - (y/b)^2 = \cosh^2(\theta) - \sinh^2(\theta) = 1$. In the first case (a), we have an ellipse with major semi-axis b and minor semi-axis a. In the second case, we have a hyperbola (although only the right branch, because $x > 0$). Sketches are given below.



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40. Express $\cosh^{-1}(x)$, $\sinh^{-1}(x)$ and $\tanh^{-1}(x)$ as logarithms, justifying any sign choices you make.

➾ **Solution:** Let $y = \cosh^{-1}(x)$. Then:

$$\cosh(y) = x \qquad \Leftrightarrow \qquad \frac{e^y + e^{-y}}{2} = x \qquad \Leftrightarrow \qquad e^{2y} - 2xe^y + 1 = 0.$$

This is a quadratic equation for e^y, which we can solve to give:

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

We need $x > 1$ for this to exist, which is perfectly consistent with taking the inverse function $\cosh^{-1}(x)$, which should only exist on this range. Hence $x + \sqrt{x^2 - 1} > 1$, whilst $x - \sqrt{x^2 - 1} < 1$. The first case would give $y > 0$, and the second case would give $y < 0$. By convention, we choose $\cosh^{-1}(x) > 0$, which gives:

$$y = \log(x + \sqrt{x^2 - 1}).$$

Now, let $y = \sinh^{-1}(x)$. Then:

$$\sinh(y) = x \qquad \Leftrightarrow \qquad \frac{e^y - e^{-y}}{2} = x \qquad \Leftrightarrow \qquad e^{2y} - 2xe^y - 1 = 0.$$

This is a quadratic equation for e^y, which we can solve to give:

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

Note that $x - \sqrt{x^2+1} < 0$, hence this cannot correspond to a real solution of the equation. Thus we have:

$$y = \log(x + \sqrt{x^2+1}).$$

Finally, let $y = \tanh^{-1}(x)$. Then:

$$\tanh(y) = x \qquad \Leftrightarrow \qquad \frac{e^y - e^{-y}}{e^y + e^{-y}} = x \qquad \Leftrightarrow \qquad \frac{e^{2y}-1}{e^{2y}+1} = x.$$

Rearranging, we have:

$$e^{2y} - 1 = xe^{2y} + x \qquad \Leftrightarrow \qquad 1 + x = (1-x)e^{2y} \qquad \Leftrightarrow \qquad y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

---

41. Solve the equation $\cosh(x) = \sinh(x) + 2\text{sech}(x)$, giving the solutions as logarithms.

➡ **Solution:** Dividing by $\cosh(x)$ (which is never zero), we have:

$$1 = \tanh(x) + 2\text{sech}^2(x).$$

Using the identity $1 - \tanh^2(x) = \text{sech}^2(x)$, we can rearrange this to a quadratic equation for $\tanh(x)$:

$$1 = \tanh(x) + 2(1 - \tanh^2(x)) \qquad \Leftrightarrow \qquad 0 = 2\tanh^2(x) - \tanh(x) - 1 = (2\tanh(x)+1)(\tanh(x)-1).$$

Hence we have:

$$\tanh(x) = 1 \qquad \text{or} \qquad \tanh(x) = -\frac{1}{2}.$$

The first case is impossible, so we get the unique solution:

$$x = \tanh^{-1}\left(-\frac{1}{2}\right) = \frac{1}{2}\log\left(\frac{1/2}{3/2}\right) = -\frac{1}{2}\log(3).$$

---

42. Find all solutions to the equations: (a) $\cosh(z) = i$; (b) $\sinh(z) = -2$.

➡ **Solution:**

(a) We have:

$$\frac{e^z + e^{-z}}{2} = i \qquad \Leftrightarrow \qquad e^{2z} - 2ie^z + 1 = 0.$$

Solving this quadratic equation, we have:

$$e^z = \frac{-2i \pm \sqrt{-4-4}}{2} = i\left(-1 \pm \sqrt{2}\right)$$

Hence:

$$\begin{aligned} z = \log\left(i\left(-1\pm\sqrt{2}\right)\right) &= \log\left|i\left(-1\pm\sqrt{2}\right)\right| + i\arg\left(i\left(-1\pm\sqrt{2}\right)\right) + 2n\pi i \\ &= \log\left|\sqrt{2}\pm 1\right| + \frac{i\pi}{2} + 2n\pi i, \end{aligned}$$

for n an integer.

(b) We have:

$$\frac{e^z - e^{-z}}{2} = -2 \qquad \Leftrightarrow \qquad e^{2z} + 4e^z - 1 = 0.$$

Solving this quadratic equation, we have:

$$e^z = \frac{-4 \pm \sqrt{16+4}}{2} = -2 \pm \sqrt{5}.$$

Taking the logarithm, we have:

$$z = \log\left(-2 \pm \sqrt{5}\right) = \log\left|\sqrt{5} \pm 2\right| + i \arg\left(-2 \pm \sqrt{5}\right) + 2n\pi i,$$

which gives two families of solutions:

$$z = \log\left|\sqrt{5} + 2\right| + i\pi + 2n\pi, \qquad z = \log\left|\sqrt{5} - 2\right| + 2n\pi,$$

for n an integer.

Part IA: Mathematics for Natural Sciences B
Examples Sheet 4: Differential calculus, limits and continuity

Model Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

Limit definition of the derivative

1. Let $y \equiv y(x)$ be a function of x. Define the *derivative* dy/dx of y as a limit. Using the limit definition:

   (a) show that differentiation is a linear operation;

   (b) find the derivative of $y(x) = x^n$, for $n = 0, 1, 2, 3, \ldots$.

Hence obtain the derivative of $ax + bx^2 \sin(\theta)$, where a, b, θ are real constants.

Solution: The derivative, as a function of x, is defined by:

$$\frac{dy}{dx}(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h},$$

when this limit exists. The derivative is a new function of x.

(a) Let $y_1(x), y_2(x)$ be two functions of x. Then for any two real numbers a, b, we have:

$$\begin{aligned}\frac{d}{dx}\left(ay_1 + by_2\right) &= \lim_{h \to 0} \frac{ay_1(x+h) + by_2(x+h) - ay_1(x) - by_2(x)}{h} \\ &= a \lim_{h \to 0} \frac{y_1(x+h) - y_1(x)}{h} + b \lim_{h \to 0} \frac{y_2(x+h) - y_2(x)}{h} \\ &= a\frac{dy_1}{dx} + b\frac{dy_2}{dx},\end{aligned}$$

using basic properties of limits. Hence differentiation is a linear operation.

(b) Let $y(x) = x^n$. Using the binomial expansion, $(x+h)^n = x^n + \binom{n}{1}hx^{n-1} + \binom{n}{2}h^2x^{n-2} + \cdots + h^n$, we have:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{x^n + hnx^{n-1} + h^2 g(x,h) - x^n}{h} = nx^{n-1} + \lim_{h \to 0}\left[hg(x,h)\right] = nx^{n-1},$$

where $g(x,h)$ is a function of x, h which is finite when $h \to 0$. Thus the derivative is given by nx^{n-1}.

Using linearity and the power rule, we have:

$$\frac{d}{dx}\left(ax + bx^2\sin(\theta)\right) = a\frac{dx}{dx} + b\sin(\theta)\frac{d(x^2)}{dx} = a + 2bx\sin(\theta).$$

2. (a) Using *only* the limit definition, show that for $a > 0$, the derivative of $y(x) = a^x$ is proportional to a^x.

   (b) One definition of the number e is the value of a for which the proportionality constant in the previous part is 1. Using *only* this definition, show that the derivative of a^x is given by $\log(a)a^x$.

➡ **Solution:** (a) We have:

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = a^x \lim_{h\to 0}\frac{a^h-1}{h},$$

using properties of limits. The remaining limit is independent of x, so is just a constant. Thus we have:

$$\frac{dy}{dx} = ka^x$$

for some proportionality constant k.

(b) We are given that e is the value of a for which the proportionality constant is 1. Therefore, we have been told that:

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

To compute the proportionality constant for general a, we then have:

$$\lim_{h\to 0}\frac{a^h-1}{h} = \log(a)\lim_{h\to 0}\frac{e^{h\log(a)}-1}{h\log(a)} = \log(a)\lim_{h'\to 0}\frac{e^{h'}-1}{h'}.$$

where we have substituted $h' = h\log(a)$ in the limit in the final step. Using the definition of e, we see that the proportionality constant for general a is $\log(a)$, as required.

Rules of differentiation

3. Let $y \equiv y(x)$, $u \equiv u(x)$ and $v \equiv v(x)$ be functions of x. Using the limit definition of the derivative, prove the following rules of differentiation:

(a) $\dfrac{d}{dx}(u(v)) = \dfrac{dv}{dx}\dfrac{du}{dv}$, (b) $\dfrac{d}{dx}(uv) = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$, (c) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{\dfrac{du}{dx}v - u\dfrac{dv}{dx}}{v^2}$, (d) $\dfrac{dy}{dx} = \left(\dfrac{dx}{dy}\right)^{-1}$.

[Recall that these rules are called the chain rule, the product rule, the quotient rule, and the reciprocal rule, respectively. Make sure you know them off by heart!] Rewrite these rules in terms of Lagrange's 'primed' notation for derivatives.

➡ **Solution:** (a) For the chain rule, we have:

$$\begin{aligned}
\frac{d}{dx}(u(v)) &= \lim_{h\to 0}\frac{u(v(x+h))-u(v(x))}{h}\\
&= \lim_{h\to 0}\frac{u(v(x)+v(x+h)-v(x))-u(v(x))}{v(x+h)-v(x)}\cdot\frac{v(x+h)-v(x)}{h}\\
&= \lim_{h\to 0}\frac{u(v(x)+v(x+h)-v(x))-u(v(x))}{v(x+h)-v(x)}\cdot\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}.
\end{aligned}$$

Now consider substituting $h' = v(x+h) - v(x)$ in the first limit. This is okay, because $h' = v(x+h) - v(x) \to 0$ as $h \to 0$, if we assume that v is a continuous function (see later on the sheet!). This leaves us with:

$$\frac{d}{dx}(u(v)) = \lim_{h' \to 0} \frac{u(v(x) + h') - u(v(x))}{h'} \cdot \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} = \frac{du}{dv} \cdot \frac{dv}{dx},$$

where the notation du/dv means differentiate $u(v)$, considered as a function of v (not of x!).

(b) For the product rule, we have:

$$\begin{aligned}
\frac{d}{dx}(uv) &= \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\
&= \lim_{h \to 0} \left[\frac{(u(x+h) - u(x))v(x+h)}{h} + \frac{u(x)(v(x+h) - v(x))}{h} \right] \\
&= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \cdot \lim_{h \to 0} v(x+h) + u(x) \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} \\
&= \frac{du}{dx} v + u \frac{dv}{dx},
\end{aligned}$$

as required.

(c) For the quotient rule, we have:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h \to 0} \frac{u(x+h)/v(x+h) - u(x)/v(x)}{h} \\
&= \lim_{h \to 0} \frac{u(x+h)v(x) - u(x)v(x+h)}{hv(x)v(x+h)} \\
&= \frac{1}{v} \cdot \lim_{h \to 0} \frac{1}{v(x+h)} \cdot \lim_{h \to 0} \frac{u(x+h)v(x) - u(x)v(x+h)}{h} \\
&= \frac{1}{v^2} \cdot \lim_{h \to 0} \left[\frac{(u(x+h) - u(x))v(x)}{h} + \frac{u(x)(v(x) - v(x+h))}{h} \right] \\
&= \frac{\dfrac{du}{dx} v - u \dfrac{dv}{dx}}{v^2}.
\end{aligned}$$

(d) Finally, for the reciprocal rule, we must first figure out what all the derivatives depend on. On the left hand side, we have:

$$\frac{dy}{dx} \equiv \frac{dy}{dx}(x)$$

is a function of x. On the right hand side, we have:

$$\frac{dx}{dy} \equiv \frac{dx}{dy}(y) \equiv \frac{dx}{dy}(y(x)),$$

which is a function of y, but can be considered to be a function of x if we substitute the formula for y in terms of x, $y \equiv y(x)$. We will need to compare both sides as functions of x in order to prove the result!

On the right hand side, we have:

$$\frac{dx}{dy}(y) = \lim_{h \to 0} \frac{x(y+h) - x(y)}{h}.$$

Substitute $h = y(x+h') - y(x)$, so that as $h' \to 0$, we also have $h \to 0$. Then this quotient becomes:

$$\lim_{h' \to 0} \frac{x(y(x) + y(x+h') - y(x)) - x(y(x))}{y(x+h') - y(x)} = \lim_{h \to 0} \frac{h'}{y(x+h') - y(x)} = \left(\frac{dy}{dx}\right)^{-1},$$

where we have used the fact that $x(y(x)) = x$, since the first x denotes x as a function of y, then $y(x)$ denotes y rewritten as a function of x, which must be inverses of one another.

We are also asked to rewrite these rules in Lagrange's 'primed' notation for derivatives. We have:

- The chain rule applies to a function of a function. The derivative of $f(g(x))$ is given by:

$$(f(g(x)))' = g'(x)f'(g(x)).$$

- The product rule applies to products of functions. The derivative of $f(x)g(x)$ is given by:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

- The quotient rule applies to functions divided by functions. The derivative of $f(x)/g(x)$ is given by:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

- The reciprocal rule applies to *inverses* of functions. In Leibniz notation, it states that:

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}.$$

  Importantly, the left hand side is a function of x, but the right hand side is a function of y. In order to write it in function notation, we should not that y itself can be written as a function of x as $y \equiv y(x)$. Then if $y = f^{-1}(x)$, in Lagrange's primed notation, the rule is telling us that:

$$\left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))}.$$

4. Using the rules you derived in the previous question, compute the derivatives of:

(a) $\log(x)$, (b) 3^{x^2}, (c) $\dfrac{e^x}{x^3-1}$, (d) $x^3\log(x^2-7)$, (e) $\sqrt{x^3-e^x\log(x)}$.

➾ **Solution:** We use each of the rules in turn:

(a) Let $y=\log(x)$. Then we have $x=e^y$, and so by the reciprocal rule we have:

$$\frac{dy}{dx}=\left(\frac{dx}{dy}\right)^{-1}=(e^y)^{-1}=\frac{1}{e^y}=\frac{1}{x}.$$

(b) Rewriting this exponential to base e, we have $u=3^{x^2}=e^{x^2\log(3)}$. Then by the chain rule (with $v=x^2\log(3)$), we have:

$$\frac{du}{dx}=2x\log(3)e^{x^2\log(3)}=2\log(3)x\cdot 3^{x^2}.$$

(c) Using the quotient rule, the derivative is given by:

$$\frac{e^x(3x^2)-(x^3-1)e^x}{(x^3-1)^2}=\frac{\left(1+3x^2-x^3\right)e^x}{(x^3-1)^2}.$$

(d) Using the product rule *and* the chain rule for the second factor, the derivative is given by:

$$3x^2\log(x^2-7)+\frac{x^3\cdot 2x}{x^2-7}=3x^2\log(x^2-7)+\frac{2x^4}{x^2-7}.$$

(e) Using the chain rule and the product rule, we have:

$$\frac{d}{dx}\sqrt{x^3-e^x\log(x)}=\frac{1}{2}\frac{3x^2-e^x\log(x)-e^x/x}{\sqrt{x^3-e^x\log(x)}}=\frac{3x^3-e^x(x\log(x)+1)}{2x\sqrt{x^3-e^x\log(x)}}.$$

5. By writing each of the following trigonometric and hyperbolic functions in terms of exponentials, compute their derivatives: (a) $\cos(x)$; (b) $\sin(x)$; (c) $\cosh(x)$; (d) $\sinh(x)$; (e) $\tan(x)$; (f) $\tanh(x)$. Learn these derivatives off by heart.

Solution: We tackle each of the trigonometric and hyperbolic functions in turn.

(a) Using the complex formula for cosine from Sheet 3, we have $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$. Differentiating, we have:

$$\frac{d}{dx}\cos(x) = \frac{1}{2}(ie^{ix} - ie^{-ix}) = -\frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = -\sin(x),$$

using the complex formula for sine from Sheet 3.

(b) Similarly, we have $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$, so differentiating we have:

$$\frac{d}{dx}\sin(x) = \frac{1}{2i}\left(ie^{ix} + ie^{-ix}\right) = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos(x).$$

(c) Recall that $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$, we can differentiate to give:

$$\frac{d}{dx}\cosh(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x).$$

(d) Similarly, we have $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$, so differentiating we have:

$$\frac{d}{dx}\sinh(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x).$$

(e) For $\tan(x)$, we have:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\frac{1}{2i}(e^{ix} - e^{-ix})}{\frac{1}{2}(e^{ix} + e^{-ix})} = -i\left(\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}\right).$$

Differentiating using the quotient rule, we have:

$$\frac{d}{dx}\tan(x) = -i\left(\frac{i(e^{ix} + e^{-ix})^2 - i(e^{ix} - e^{-ix})^2}{(e^{ix} + e^{-ix})^2}\right) = \frac{(e^{ix} + e^{-ix})^2 - (e^{ix} - e^{-ix})^2}{(e^{ix} + e^{-ix})^2}$$

$$= \frac{(2\cos(x))^2 - (2i\sin(x))^2}{(2\cos(x))^2} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \sec^2(x).$$

(f) The calculation for $\tanh(x)$ is extremely similar. We have:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Differentiating using the quotient rule, we have:

$$\frac{d}{dx}\tanh(x) = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{(2\cosh(x))^2 - (2\sinh(x))^2}{(2\cosh(x))^2}$$

$$= \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \operatorname{sech}^2(x).$$

6. Using: (i) the logarithmic formulae for the inverse hyperbolic functions you derived on Sheet 3; (ii) the reciprocal rule, compute the derivatives of: (a) $\cosh^{-1}(x)$; (b) $\sinh^{-1}(x)$; (c) $\tanh^{-1}(x)$. Learn these derivatives off by heart.

➫ **Solution:** We have the following derivatives:

(a) (i) We previously derived $\cosh^{-1}(x) = \log(x + \sqrt{x^2-1})$. Hence using the chain rule, we have the derivative:

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1 + x(x^2-1)^{-1/2}}{x + (x^2-1)^{1/2}} = \frac{1}{\sqrt{x^2-1}} \cdot \frac{1 + x(x^2-1)^{-1/2}}{1 + x(x^2-1)^{-1/2}} = \frac{1}{\sqrt{x^2-1}}.$$

(ii) Let $y(x) = \cosh^{-1}(x)$. Then $\cosh(y) = x$, which gives:

$$\frac{dx}{dy} = \sinh(y).$$

Using the reciprocal rule, and hyperbolic trigonometric identities, we then have:

$$\frac{dy}{dx} = \frac{1}{\sinh(y)} = \frac{1}{\sqrt{\cosh^2(y) - 1}} = \frac{1}{\sqrt{x^2-1}}.$$

(b) (i) We previously derived $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$. Hence using the chain rule, we have the derivative:

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1 + x(x^2+1)^{-1/2}}{x + (x^2+1)^{1/2}} = \frac{1}{\sqrt{x^2+1}} \cdot \frac{1 + x(x^2+1)^{-1/2}}{1 + x(x^2+1)^{-1/2}} = \frac{1}{\sqrt{x^2+1}}.$$

(ii) Let $y(x) = \sinh^{-1}(x)$. Then $\sinh(y) = x$, which gives:

$$\frac{dx}{dy} = \cosh(y).$$

Using the reciprocal rule, and hyperbolic trigonometric identities, we then have:

$$\frac{dy}{dx} = \frac{1}{\cosh(y)} = \frac{1}{\sqrt{\sinh^2(y) + 1}} = \frac{1}{\sqrt{x^2+1}}.$$

(c) (i) We previously derived:

$$\tanh^{-1}(x) = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|.$$

Hence using the chain rule, we have the derivative (assuming x is in the range where $(1+x)/(1-x)$ is positive):

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{2}\frac{d}{dx}\left(\log(1+x) - \log(1-x)\right) = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = \frac{1}{1-x^2}.$$

(ii) Let $y(x) = \tanh^{-1}(x)$. Then $\tanh(y) = x$, which gives:

$$\frac{dx}{dy} = \operatorname{sech}^2(y).$$

Using the reciprocal rule, and hyperbolic trigonometric identities, we then have:

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2(y)} = \frac{1}{\tanh^2(y) - 1} = \frac{1}{x^2-1}.$$

7. If $y \equiv y(x)$ is a function of x, show that $\dfrac{d^3x}{dy^3} = -\left(\dfrac{dy}{dx}\right)^{-4}\dfrac{d^3y}{dx^3} + 3\left(\dfrac{dy}{dx}\right)^{-5}\left(\dfrac{d^2y}{dx^2}\right)^2$. Verify this when $y = e^{2x}$.

Solution: This question is about expressing a higher derivative of x with respect to y in terms of derivatives of y with respect to x. We start with a simple derivative, then differentiate repeatedly using the chain rule and reciprocal rule.

First, note that:

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$$

by the reciprocal rule.

Taking another derivative with respect to y, we have:

$$\frac{d^2x}{dy^2} = \frac{d}{dy}\left[\left(\frac{dy}{dx}\right)^{-1}\right].$$

But the argument of the derivative on the right hand side is a function of x, not a function of y - how can we differentiate it? We must consider it to be a function of y *through the dependence of x on y*, $x \equiv x(y)$:

$$\left(\frac{dy}{dx}\right)^{-1} \equiv \left(\frac{dy}{dx}(x)\right)^{-1} \equiv \left(\frac{dy}{dx}(x(y))\right)^{-1}.$$

This implies that we should use the chain rule to differentiate. We have:

$$\begin{aligned}
\frac{d}{dy}\left[\left(\frac{dy}{dx}\right)^{-1}\right] &= \frac{dx}{dy}\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^{-1}\right] && \text{(chain rule)}\\
&= -\frac{dx}{dy}\left(\frac{dy}{dx}\right)^{-2}\frac{d^2y}{dx^2} && \text{(chain rule again)}\\
&= -\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2} && \text{(reciprocal rule)}
\end{aligned}$$

Now we have got the hang of things, the last derivative is easy. We repeat the same method:

$$\begin{aligned}
\frac{d^3x}{dy^3} &= \frac{d}{dy}\left[-\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2}\right]\\
&= \frac{dx}{dy}\frac{d}{dx}\left[-\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2}\right] && \text{(chain rule)}\\
&= 3\frac{dx}{dy}\left(\frac{dy}{dx}\right)^{-4}\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dx}{dy}\left(\frac{dy}{dx}\right)^{-3}\frac{d^3y}{dx^3} && \text{(product rule and chain rule)}\\
&= 3\left(\frac{dy}{dx}\right)^{-5}\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^{-4}\frac{d^3y}{dx^3} && \text{(reciprocal rule)}
\end{aligned}$$

To finish, we are asked to verify this result when $y = e^{2x}$. In this case, we have that x as a function of y is given by $x(y) \equiv \frac{1}{2}\log(y)$. Hence all the derivatives are:

$$\frac{dy}{dx} = 2e^{2x}, \qquad \frac{d^2y}{dx^2} = 4e^{2x}, \qquad \frac{d^3y}{dx^3} = 8e^{2x},$$

and:

$$\frac{dx}{dy} = \frac{1}{2y}, \qquad \frac{d^2x}{dy^2} = -\frac{1}{2y^2}, \qquad \frac{d^3x}{dy^3} = \frac{1}{y^3}.$$

Verifying the rule we derived above, we check:

$$3\left(\frac{dy}{dx}\right)^{-5}\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^{-4}\frac{d^3y}{dx^3} = \frac{3}{32e^{10x}} \cdot 16e^{4x} - \frac{1}{16e^{8x}}8e^{2x} = \left(\frac{3}{2} - \frac{1}{2}\right)\frac{1}{e^{6x}} = \frac{1}{y^3},$$

as required.

8. What is *implicit differentiation*, and why is it called implicit? Using: (a) implicit differentiation; (b) the reciprocal rule, find dy/dx given $y + e^y \sin(y) = 1/x$, and make sure that your answers agree.

Solution: Sometimes, we define y as a function of x through an equation, such as:

$$y^3 + \sin(y)\cos(x) + x = 0. \tag{$*$}$$

It is impossible to solve this equation *explicitly* to give something of the form $y = y(x)$, but we can still solve the equation in principle (e.g. numerically). Hence we say that this equation defines y as a function of x *implicitly*. In general, an implicit definition of y as a function of x is an equation of the form:

$$f(x, y) = 0$$

where f is a function of two variables.

Implicit differentiation is the differentiation of y when it is implicitly defined (which explains why it is called implicit differentiation). For example, when $y \equiv y(x)$ is defined through the equation ($*$), we can differentiate to get:

$$3y^2\frac{dy}{dx} + \cos(y)\cos(x)\frac{dy}{dx} - \sin(y)\sin(x) + 1 = 0,$$

where we have used the chain rule and product rules to compute the derivatives of each of the terms. Implicit differentiation is just a fancy name for the chain rule applied to implicit equations really!

(a) Given the equation $y + e^y \sin(y) = 1/x$, we can differentiate using implicit differentiation to give:

$$\frac{dy}{dx} + e^y \sin(y)\frac{dy}{dx} + e^y \cos(y)\frac{dy}{dx} = -\frac{1}{x^2}.$$

Rearranging, we obtain:

$$\frac{dy}{dx} = -\frac{1}{x^2(1 + e^y \sin(y) + e^y \cos(y))}.$$

(b) On the other hand, we can take the reciprocal of both sides to give:

$$x = \frac{1}{y + e^y \sin(y)}. \tag{$\dagger$}$$

Taking the derivative with respect to y, we have:

$$\frac{dx}{dy} = -\frac{1 + e^y \sin(y) + e^y \cos(y)}{(y + e^y \sin(y))^2}.$$

Taking the reciprocal, we have:

$$\frac{dy}{dx} = -\frac{(y + e^y \sin(y))^2}{1 + e^y \sin(y) + e^y \cos(y)} = -\frac{1}{x^2(1 + e^y \sin(y) + e^y \cos(y))},$$

where in the last step we substituted using equation (†). This is in agreement with the implicit differentiation method we used in part (a).

Curve-sketching

9. State what it means for a function to be *even* and for a function to be *odd*, and explain the geometric significance of these definitions. Hence, decide whether the following functions are even, odd, both, or neither:

(a) x, (b) $\sin(x)$, (c) e^x, (d) $\sin(\frac{\pi}{2} - x)$, (e) $|x|\cos(x)$, (f) $\sqrt{x}$, (g) 2, (h) 0, (i) $\log\left|\frac{1+x}{1-x}\right|$.

➭ **Solution:** A function is *even* if it satisfies $f(x) = f(-x)$. This means that it is invariant under a reflection in the y-axis. A function is *odd* if it satisfies $f(x) = -f(-x)$. This means that it is invariant under a rotation by π around the origin.

For the given functions:

(a) For $f(x) = x$, we have $f(-x) = -x = -f(x)$, so this function is odd. Alternatively, we could spot that it is invariant under a rotation by π around the origin. The function is *not* even, because it is not invariant under a reflection in the y-axis.

(b) For $f(x) = \sin(x)$, we have $f(-x) = \sin(-x) = -\sin(x) = -f(x)$, so this function is odd. It is not even.

(c) For $f(x) = e^x$, the function is neither reflectionally invariant in the y-axis, nor rotationally invariant under a rotation by π around the origin. Hence this function is neither even nor odd.

(d) For $f(x) = \sin(\pi/2 - x)$, we notice that this function is a transformation of sine. It is a translation by $\pi/2$ in the negative x-direction $\sin(x) \mapsto \sin(x + \pi/2)$, then a reflection in the y-axis, $\sin(x + \pi/2) \mapsto \sin(\pi/2 - x)$. This just gives a cosine graph though, so $f(x) = \cos(x)$. This is evidently even, and not odd.

(e) For $f(x) = |x|\cos(x)$, we note that $f(-x) = |-x|\cos(-x) = |x|\cos(x) = f(x)$, so this function is even. It is not odd.

(f) For $f(x) = \sqrt{x}$, the function is not defined for $x < 0$, so this function is neither even nor odd.

(g) For $f(x) = 2$, we have $f(-x) = 2 = f(x)$, so the function is even. The function is not odd.

(h) For $f(x) = 0$, we have *both* $f(-x) = 0 = f(x)$ and $f(-x) = 0 = -f(x)$. Hence the function is *both* even *and* odd. The zero function is in fact the only example of a function which is both even and odd.

(i) For:

$$f(x) = \log\left|\frac{1+x}{1-x}\right|,$$

we could observe that $f(x) = 2\tanh^{-1}(x)$, which immediately shows the function is odd and not even (consider the graph of $\tanh(x)$, from Sheet 3!). Alternatively, we can calculate directly using laws of logarithms:

$$f(-x) = \log\left|\frac{1-x}{1+x}\right| = \log\left|\frac{1+x}{1-x}\right|^{-1} = -\log\left|\frac{1+x}{1-x}\right| = -f(x).$$

10. Write down a list of things you should consider when sketching the graph of a function. Compare with your supervision partner before the supervision, and exchange ideas!

➻ **Solution:** A possible list is the following:

- Look at points where the function is undefined.
- Consider the behaviour of the function in the limits $x \to \pm\infty$.
- Find the x-intercepts, by setting $y = 0$, and the y-intercepts, by setting $x = 0$.
- Differentiate to find stationary points.
- Think about sums and products of functions.
- Think about regions of positivity and negativity.

11. Sketch the graphs of the following functions, explaining your reasoning in each case:

(a) $(x-3)^3 + 2x$, (b) $\dfrac{x}{1+x^2}$, (c) $\dfrac{x^2+3}{x-1}$, (d) xe^x, (e) $\dfrac{\log(x)}{1+x}$, (f) $\dfrac{1}{1-e^x}$, (g) $e^x\cos(x)$.

Solution: Performing the analysis in each case:

(a) This is just a cubic. We expand the bracket first to get:

$$y = x^3 - 9x^2 + 29x - 27.$$

Hence we see that the y-intercept is -27. Taking the derivative, we have:

$$\frac{dy}{dx} = 3x^2 - 18x + 29,$$

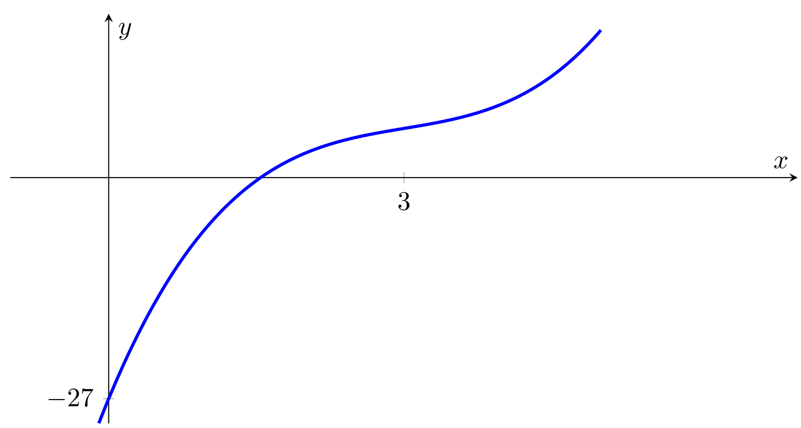
which has discriminant $18^2 - 4\cdot 3\cdot 29 = -24 < 0$. Hence, there are no stationary points. It is a positive cubic, so it simply increases from $-\infty$ to ∞.

If we want to be incredibly precise, we could compute the second derivative. This gives:

$$\frac{d^2y}{dx^2} = 6x - 18,$$

so there is a point of inflection at $x = 3$. This is where the cubic changes from reducing its rate of growth to increasing its rate of growth.

We now have enough information to draw the complete sketch:



(b) This function is defined for all values of x, since the denominator is always positive. It approaches zero as $x \to \infty$, coming from the positive direction since the numerator $x > 0$ in this region. It approaches zero as $x \to -\infty$ coming from the negative direction since the numerator $x < 0$ in this region. The y-intercept is at $x = 0$. The graph is also odd, since:

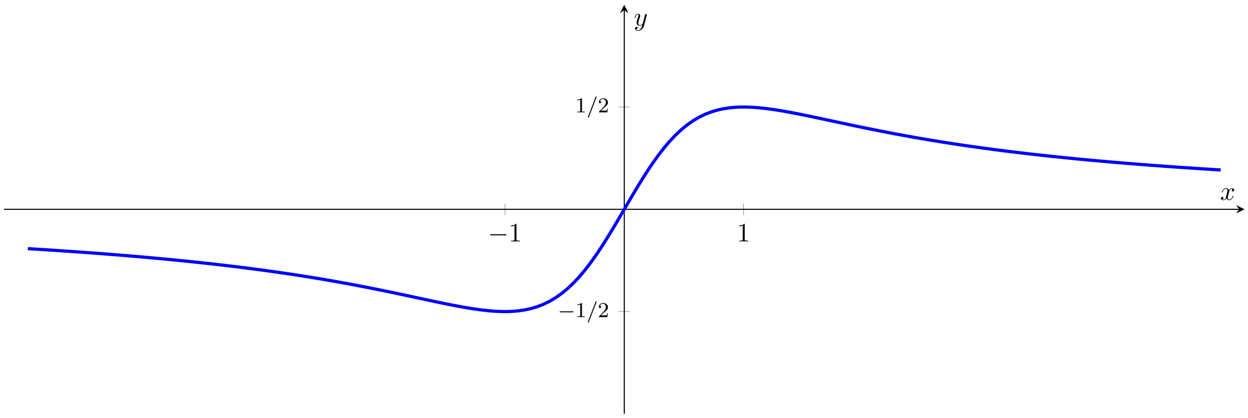
$$\frac{(-x)}{1+(-x)^2} = -\left(\frac{x}{1+x^2}\right).$$

Finally, we check for stationary points. Differentiating $y = x/(1+x^2)$, we have:

$$\frac{dy}{dx} = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}.$$

Hence there are stationary points at $x = \pm 1$. Thinking about the shape of the graph, and the information we have already obtained, the only possibility is that $x = 1$ is a maximum and $x = -1$ is a minimum.

We now have everything we need to sketch the graph:



(c) This graph has a vertical asymptote at $x = 1$. Just to the right of the asymptote, we have $x - 1 > 0$ and $x^2 + 3 > 0$, hence the graph is positive. So the graph approaches $+\infty$ from the right of $x = 1$. Just to the left of the asymptote, we have $x - 1 < 0$ and $x^2 + 3 > 0$, hence the graph is negative. So the graph approaches $-\infty$ from the left of $x = 1$.

Indeed, the sign of the graph is entirely determined by the sign of the denominator, so we see that the graph never crosses the x-axis. It is entirely positive for $x > 1$ and entirely negative for $x < 1$.

As x approaches infinity, the behaviour of the graph is roughly linear. To work out the exact behaviour, we perform polynomial long division:

$$\frac{x^2+3}{x-1} = \frac{(x-1)^2 + 2x + 2}{x-1} = \frac{(x-1)^2 + 2(x-1) + 4}{x-1} = x - 1 + 2 + \frac{4}{x-1} = x + 1 + \frac{4}{x-1}.$$

Therefore, as $x \to \infty$, the graph approaches the *oblique asymptote* $x + 1$. The same applies as $x \to -\infty$.

Finally, we check for stationary points. We have:

$$\frac{d}{dx}\left(\frac{x^2+3}{x-1}\right) = \frac{2x(x-1) - (x^2+3)}{(x-1)^2} = \frac{x^2 - 2x - 3}{(x-1)^2} = \frac{(x-3)(x+1)}{(x-1)^2}.$$

Hence there are stationary points at $x = 3$ and $x = -1$. By the above work, the shape of the graph implies that $x = 3$ is necessarily a minimum and $x = -1$ is necessarily a maximum.

We now have everything we need to sketch the graph. We have:

(†) Leibniz's formula

12. Using mathematical induction, prove *Leibniz's formula* for the nth derivative of a product:

$$\frac{d^n}{dx^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)},$$

where $f^{(k)}$ denotes the kth derivative of f. Hence compute: (a) the third derivative of $\log^2(x)$; (b) the 100th derivative of $x^2 e^x$.

➾ Solution: The formula clearly holds when $n = 0$. Now assume that the formula holds for $n = m$, and consider the case $n = m + 1$. We have:

$$\begin{aligned}
\frac{d^{m+1}}{dx^{m+1}}(fg) &= \frac{d}{dx}\left(\frac{d^m}{dx^m}(fg)\right) \\
&= \frac{d}{dx}\left(\sum_{k=0}^{m} \binom{m}{k} f^{(k)} g^{(m-k)}\right) && \text{(induction hypothesis)} \\
&= \sum_{k=0}^{m} \binom{m}{k} \left(f^{(k+1)} g^{(m-k)} + f^{(k)} g^{(m-k+1)}\right) && \text{(product rule)} \\
&= \sum_{k=0}^{m} \binom{m}{k} f^{(k+1)} g^{(m-k)} + \sum_{k=0}^{m} \binom{m}{k} f^{(k)} g^{(m-k+1)} \\
&= \sum_{k=1}^{m+1} \binom{m}{k-1} f^{(k)} g^{(m-k+1)} + \sum_{k=0}^{m} \binom{m}{k} f^{(k)} g^{(m-k+1)} && \text{(reindexing sum)} \\
&= \binom{m}{0} f^{(0)} g^{(m)} + \sum_{k=1}^{m} \left(\binom{m}{k-1} + \binom{m}{k}\right) f^{(k)} g^{(m-k+1)} + \binom{m}{m} f^{(m+1)} g^{(0)} \\
&= \binom{m+1}{0} f^{(0)} g^{(m)} + \sum_{k=1}^{m} \binom{m+1}{k} f^{(k)} g^{(m-k+1)} + \binom{m+1}{m+1} f^{(m+1)} g^{(0)} \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} f^{(k)} g^{(m-k+1)}.
\end{aligned}$$

Note that in the penultimate line, we used the Pascal's triangle property of binomial coefficients (if you haven't seen this identity before, don't worry - we will prove it again in the probability section of the course), namely:

$$\binom{m}{k-1} + \binom{m}{k} = \binom{m+1}{k}.$$

Hence, by the principle of mathematical induction, we are done.

Computing the required higher derivatives, we have:

(a) The third derivative of $\log^2(x)$ is:

$$\begin{aligned}\frac{d^3}{dx^2}\log^2(x) &= \binom{3}{0}\log(x)\frac{d^3}{dx^3}\log(x) + \binom{3}{1}\frac{d}{dx}\log(x)\frac{d^2}{dx^2}\log(x) \\ &\qquad + \binom{3}{2}\frac{d^2}{dx^2}\log(x)\frac{d}{dx}\log(x) + \binom{3}{3}\left(\frac{d^3}{dx^3}\log(x)\right)\cdot\log(x) \\ &= 2\log(x)\frac{d^3}{dx^3}\log(x) + 6\frac{d}{dx}\log(x)\frac{d^2}{dx^2}\log(x),\end{aligned}$$

collecting like terms in the final line. The derivatives of the logarithms are:

$$\frac{d}{dx}\log(x) = \frac{1}{x}, \qquad \frac{d^2}{dx^2}\log(x) = -\frac{1}{x^2}, \qquad \frac{d^3}{dx^3}\log(x) = \frac{2}{x^3}.$$

Inserting these into the formula, we have:

$$\frac{d^3}{dx^3}\log^2(x) = \frac{4\log(x)}{x^3} - \frac{6}{x^3}.$$

(b) The 100th derivative of x^2e^x is given by:

$$\frac{d^{100}}{dx^{100}}(x^2e^x) = \sum_{k=0}^{100}\binom{100}{k}\frac{d^k}{dx^k}(x^2)\frac{d^{100-k}}{dx^{100-k}}e^x.$$

All of the exponential derivatives are equal to e^x. The only derivatives of x^2 that survive in the sum are the $k=0, k=1, k=2$ derivatives, given by:

$$x^2, \qquad \frac{d}{dx}(x^2) = 2x, \qquad \frac{d^2}{dx^2}(x^2) = 2,$$

with all other higher derivatives vanishing. Hence the sum reduces to:

$$\binom{100}{0}x^2e^x + \binom{100}{1}(2x)e^x + \binom{100}{2}\cdot 2e^x = \left(x^2 + 200x + 9900\right)e^x,$$

evaluating the binomial coefficients to finish.

13. Use Leibniz's formula to prove that the nth derivative of $e^{-x^2/2}$ is a solution of the equation $Z'' + xZ' + (n+1)Z = 0$.

➻ **Solution:** Suppose that:

$$Z = \frac{d^n}{dx^n} e^{-x^2/2}.$$

Then the first derivative is given by:

$$\frac{dZ}{dx} = \frac{d^n}{dx^n}\left(-xe^{-x^2/2}\right) = -\sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k} x \frac{d^{n-k}}{dx^{n-k}} e^{-x^2/2}.$$

The $k = 0$, $k = 1$ terms are the only terms which are non-vanishing, giving:

$$\frac{dZ}{dx} = -x\frac{d^n}{dx^n} e^{-x^2/2} - n\frac{d^{n-1}}{dx^{n-1}} e^{-x^2/2} = -xZ - n\frac{d^{n-1}}{dx^{n-1}} e^{-x^2/2}.$$

Taking an additional derivative, we have:

$$\frac{d^2Z}{dx^2} = -Z - x\frac{dZ}{dx} - nZ.$$

Rearranging, we have:

$$\frac{d^2Z}{dx^2} + x\frac{dZ}{dx} + (n+1)Z = 0,$$

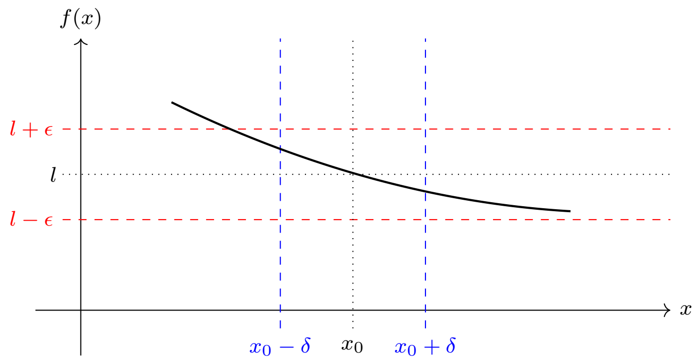
as required.

(†) Formal definition of a limit

14. Suppose that $f : (a, b)\backslash\{x_0\} \to \mathbb{R}$ is a real function defined on a (possibly infinite) open interval excluding a point $x_0 \in \mathbb{R}$. Give the formal mathematical definition of the phrase '$f(x) \to l$ as $x \to x_0$', and explain this definition using a diagram. How should this definition be modified for the cases $l = \pm\infty$?

Solution: We say that $f(x) \to l$ as $x \to x_0$ if for any given accuracy $\epsilon > 0$, there exists some tolerance $\delta > 0$ (which can depend on ϵ) such that if x lies within δ of the point x_0, $0 < |x - x_0| < \delta$, then $f(x)$ lies within ϵ of l, $|f(x) - l| < \epsilon$.

This can be explained with the following diagram:



We want to express the idea that as x approaches x_0, $f(x)$ approaches l. To do this mathematically, we say: suppose that we pick any desired accuracy of $f(x)$ being close enough to l, say ϵ. Then there must exist some tolerance δ such that if x is close enough to x_0, within δ, then $f(x)$ is within the desired accuracy ϵ of l.

The figure above shows that for a particular accuracy $\epsilon > 0$, there does indeed exist a tolerance δ such that within the region $x_0 - \delta < x < x_0 + \delta$ (equivalent to $|x - x_0| < \delta$), we have $l - \epsilon < f(x) < l + \epsilon$ (equivalent to $|f(x) - l| < \epsilon$). Notice that in this figure, we could have chosen δ larger - but this doesn't really matter, as long as there exists *some* tolerance δ. We aren't interested in the optimal tolerance in the definition of a limit!

To define $f(x) \to \infty$ as $x \to x_0$, we want that given any large number K, if x is sufficiently close to x_0, then $f(x)$ exceeds K. More mathematically: given any real number K, there exists some tolerance $\delta > 0$ such that if $0 < |x - x_0| < \delta$, we have $f(x) > K$.

Similarly, we define $f(x) \to -\infty$ as $x \to x_0$ as: given any real number K, there exists some tolerance $\delta > 0$ such that if $0 < |x - x_0| < \delta$, we have $f(x) < K$.

15. Here is a model example of a formal mathematical argument, from first principles, showing that $x^2 \to 1$ as $x \to 1$:

> *'Suppose we are given some arbitrary tolerance $\epsilon > 0$. Choose some closeness $\delta = \min(1, \epsilon/3)$. Then for all x which are δ-close to 1, i.e. $0 < |x-1| < \delta$, we have:*
>
> $$\begin{aligned} |x^2 - 1| &= |((x-1)+1)^2 - 1| && (1) \\ &= |(x-1)^2 + 2(x-1)| && (2) \\ &\leq |x-1|^2 + 2|x-1| && (3) \\ &< \delta^2 + 2\delta && (4) \\ &\leq \delta + 2\delta && (5) \\ &= 3\delta && (6) \\ &\leq \epsilon. && (7) \end{aligned}$$
>
> *Hence if x is δ-close to 1, we have that $|x^2 - 1| < \epsilon$, so that x^2 is ϵ-close to 1. We conclude that, by the definition of a limit, we have $x^2 \to 1$ as $x \to 1$.'*

(a) Which of ϵ, δ are we given, and which of ϵ, δ must we choose?

(b) Why do we express $|x^2 - 1|$ in terms of $x - 1$ in line (1)?

(c) What law from earlier in the course have we used in going from line (2) to line (3)?

(d) What have we used in going from line (4) to line (5)? What about in going from line (6) to line (7)?

(e) Would the proof still have been successful if we had chosen $\delta = \min(1, \epsilon/4)$? In terms of ϵ, what is the largest possible value of δ we could have chosen for the proof to still work?

Solution: (a) We are given $\epsilon > 0$, and we must choose δ (which can depend on δ). We think of finding a limit like a game - some enemy is challenging us to say 'I bet that if I choose a small enough tolerance ϵ, you can't find a region close enough to x_0 where all values of the function are within ϵ of the limit l'. Our aim is to beat this enemy by finding such a δ!

(b) We choose to express $|x^2 - 1|$ in terms of $x - 1$ because we have 'control' over $|x - 1|$ - we can make this smaller by choosing δ smaller.

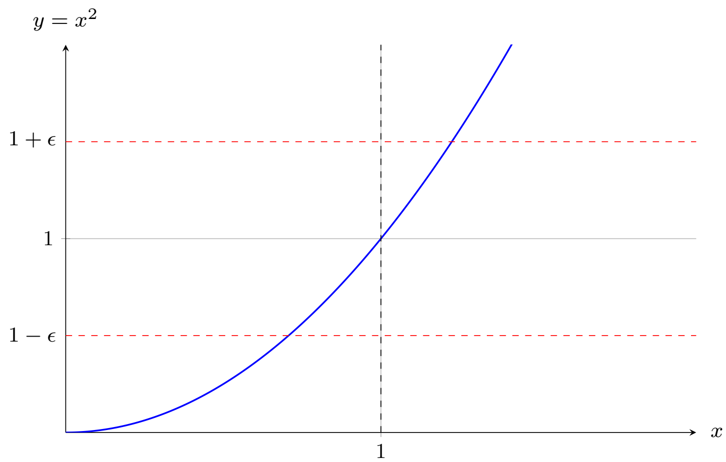
(c) Here, we used the triangle inequality, $|a + b| \leq |a| + |b|$. We proved it earlier for vectors, but of course it works for one-dimensional vectors too (i.e. just numbers!).

(d) To go from line (4) to line (5), we used the fact that $\delta = \min(1, \epsilon/3)$, so that $\delta \leq 1$. This means that $\delta^2 \leq \delta$. To go from line (6) to line (7), we used the fact that $\delta = \min(1, \epsilon/3)$, so that $\delta \leq \epsilon/3$.

(e) The proof would still have worked if $\delta = \min(1, \epsilon/4)$. This is because in going from line (6) to line (7), we could have used the fact that $\delta \leq \epsilon/4$. So we would have had:

$$3\delta \leq \frac{3\epsilon}{4} < \epsilon.$$

To find the largest possible value of δ that would have worked, it's useful to draw a diagram.



We see that if $0 < \epsilon \leq 1$, then the largest possible δ can be found by calculating the intersections of the lines $y = 1+\epsilon$ and $y = 1-\epsilon$ with the graph of $y = x^2$. In particular, we see that we need:

$$\sqrt{1-\epsilon} < x < \sqrt{1+\epsilon}$$

for x^2 to lie between $1-\epsilon$ and $1+\epsilon$. To find the largest δ, we must decide which of the points $x = \sqrt{1-\epsilon}, \sqrt{1+\epsilon}$ is closest to 1; from the diagram, it certainly looks like it is $\sqrt{1+\epsilon}$. The distances we need to compare are $1-\sqrt{1-\epsilon}$ and $\sqrt{1+\epsilon}-1$. Observe that:

$$\begin{aligned}
\sqrt{1+\epsilon}-1 < 1-\sqrt{1-\epsilon} \quad &\Leftrightarrow \quad \sqrt{1+\epsilon}+\sqrt{1-\epsilon} < 2 \\
&\Leftrightarrow \quad 1+\epsilon+2\sqrt{1-\epsilon^2}+1-\epsilon < 4 \qquad \text{(squaring both sides)} \\
&\Leftrightarrow \quad \sqrt{1-\epsilon^2} < 1 \\
&\Leftrightarrow \quad 1-\epsilon^2 < 1 \qquad \text{(squaring both sides)} \\
&\Leftrightarrow \quad 0 < \epsilon^2.
\end{aligned}$$

This is true, so $\sqrt{1+\epsilon}$ is closer to 1 than $\sqrt{1-\epsilon}$. It follows that if $0 < \epsilon \leq 1$, the maximum possible δ that we can choose is $\delta = \sqrt{1+\epsilon}-1$.

On the other hand, if $\epsilon > 1$, there are no intersections between $y = x^2$ and $y = 1-\epsilon$, and instead there are two (symmetric) intersections between $y = x^2$ and $y = 1+\epsilon$. Clearly the closer intersection to $x = 1$ is at $\sqrt{1+\epsilon}$. So it follows that if $\epsilon > 1$, the maximum possible δ that we can choose is $\delta = \sqrt{1+\epsilon}-1$.

Hence, in all cases the maximum possible δ we can choose is $\delta = \sqrt{1+\epsilon}-1$.

16. Using the model example in Question 14 as a template, provide proofs from first principles showing that:

(a) $4x^3 \to 0$ as $x \to 0$, (b) $x^2 \to a^2$ as $x \to a$, for $a \in \mathbb{R}$, (c) $\sin(x) \to 1$ as $x \to \pi/2$,

(d) $x \sin(1/x) \to 0$ as $x \to 0$, (e) $1/x^2 \to \infty$ as $x \to 0$.

➾ **Solution:** (a) Let $\epsilon > 0$. Assuming that $0 < |x| < \delta$, where δ is something we shall choose later, we then have:

$$\begin{aligned} |4x^3| &= 4|x|^3 \\ &< 4\delta^3. \end{aligned}$$

So we see that if we choose $\delta = \min(1, \epsilon/4)$, then we have:

$$4\delta^3 \leq 4\delta \leq \epsilon.$$

Thus $|4x^3| < \epsilon$, and we're done.

(b) Let $\epsilon > 0$. Assuming that $0 < |x - a| < \delta$, where δ is something we shall choose later, we then have:

$$\begin{aligned} |x^2 - a^2| &= |(x-a)^2 + 2a(x-a)| \\ &\leq |x-a|^2 + 2|a||x-a| && \text{(triangle inequality)} \\ &= \delta^2 + 2|a|\delta. \end{aligned}$$

So we see that if we choose $\delta = \min(1, \epsilon/(1 + 2|a|))$, then we have:

$$\delta^2 + 2|a|\delta \leq \delta(1 + 2|a|) \leq \epsilon.$$

Thus $|x^2 - a^2| < \epsilon$, and we're done.

(c) Let $\epsilon > 0$. Assuming that $0 < |x - \pi/2| < \delta$, where δ is something we shall choose later, we then have:

$$\begin{aligned} |\sin(x) - 1| &= |\sin(x) - \sin(\pi/2)| \\ &= \left| 2\sin\left(\frac{x - \pi/2}{2}\right)\cos\left(\frac{x + \pi/2}{2}\right) \right| && \text{(sum to product formula)} \\ &\leq 2\left|\sin\left(\frac{x - \pi/2}{2}\right)\right| && \text{(since the modulus of cosine is less than } 1\text{)} \end{aligned}$$

Now, recall from the zeroth examples sheet, we have $|\sin(u)| \leq |u|$ for all u. Hence we have:

$$|\sin(x) - 1| \leq 2\left|\sin\left(\frac{x - \pi/2}{2}\right)\right| \leq 2\left|\frac{x - \pi/2}{2}\right| = |x - \pi/2| < \delta.$$

Hence we see that if choose $\delta = \epsilon$, we have $|\sin(x) - 1| < \epsilon$, and we're done.

(d) Let $\epsilon > 0$. Assuming that $0 < |x| < \delta$, where δ is something we shall choose later, we then have:

$$\begin{aligned} |x \sin(1/x)| &\leq |x| && \text{(since the modulus of sine is less than 1)} \\ &< \delta. \end{aligned}$$

Hence we see that if we choose $\delta = \epsilon$, we have $|x \sin(1/x)| < \epsilon$, and we're done.

(e) Given a constant K, assume that $0 < |x| < \delta$, where δ is something we shall choose later. We then have:

$$\left|\frac{1}{x^2}\right| = \frac{1}{|x|^2} > \frac{1}{\delta^2}.$$

Hence we see that if we choose $1/\delta^2 > K$, that is, choose $\delta < \sqrt{1/|K|}$, then we're done.

17. Suppose that $f : (a, \infty) \to \mathbb{R}$ is a real function defined on an open interval up to positive infinity. Give the formal mathematical definition of the phrase '$f(x) \to l$ as $x \to \infty$', and explain this definition using a diagram. How should this definition be modified for the cases $l = \pm\infty$? Hence, show directly from the definition that: (a) $1/x \to 0$ as $x \to \infty$; (b) $\sin(x)/x \to 0$ as $x \to \infty$; (c) $x^3 \to -\infty$ as $x \to -\infty$.

Solution: We say that $f(x) \to l$ as $x \to \infty$ if for all $\epsilon > 0$, there exists some M such that if $x > M$, we have $|f(x) - l| < \epsilon$.

If $l = \infty$, then the definition becomes: for all $K \in \mathbb{R}$, there exists some M such that if $x > M$, we have $f(x) > K$. On the other hand if $l = -\infty$, then the definition becomes: for all $K \in \mathbb{R}$, there exists some M such that if $x > M$, we have $f(x) < K$.

Examining the limits in the question, we have:

(a) Given $\epsilon > 0$, let $x > M$ where M is a constant we will choose later. Then:

$$\left|\frac{1}{x}\right| = \frac{1}{|x|} < \frac{1}{M}.$$

since if $x > M$, we have $|x| > M$ (if we then assume M positive, $1/M > 1/|x|$). So we should choose $M = 1/\epsilon$, then we're done.

(b) Given $\epsilon > 0$, let $x > M$ where M is a constant we will choose later. Then:

$$\left|\frac{\sin(x)}{x}\right| \leq \frac{1}{|x|} < \frac{1}{M},$$

since if $x > M$, we have $|x| > M$ (if we then assume M positive, $1/M > 1/|x|$). So we should choose $M = 1/\epsilon$, then we're done.

(c) Given any K, let $x < M$, where M is a constant we will choose later. Then:

$$x^3 < M^3.$$

since cubing something is an increasing function. So if we choose $M = K^{1/3}$, then we're done.

(†) Laws of limits

18. Let $x_0 \in \mathbb{R}$, and let $f, g : \mathbb{R} \to \mathbb{R}$ be real functions. From the formal definition of a limit, prove that:

$$\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x),$$

provided that (i) both the limits on the right hand side exist in $\mathbb{R} \cup \{\infty, -\infty\}$ (the set of real numbers with infinity and negative infinity adjoined), *and* (ii) if one of the limits on the right hand side is ∞, the other is *not* $-\infty$.

➾ **Solution:** We split everything into cases:

- **Both limits are finite.** Let's start with the cases where both limits on the right hand side are finite. Then given $\epsilon > 0$, there must exist some $\delta_1, \delta_2 > 0$ such that if $0 < |x - x_0| < \min(\delta_1, \delta_2)$ we have both:

$$|f(x) - l| < \epsilon/2 \qquad \text{and} \qquad |g(x) - m| < \epsilon/2,$$

  where l, m are the finite values of the limits of $f(x), g(x)$ respectively. It follows that if we take $\delta = \min(\delta_1, \delta_2)$, we have for all x satisfying $0 < |x - x_0| < \delta$:

$$|f(x) + g(x) - l - m| = |f(x) - l + g(x) - m| \leq |f(x) - l| + |g(x) - m| < \epsilon/2 + \epsilon/2 = \epsilon.$$

  Hence $f(x) + g(x)$ tends to $l + m$.

- **One limit is infinite, one is finite.** Let's suppose that one of the limits is infinite, and the other is finite now, $f(x) \to \infty$ and $g(x) \to m$ as $x \to x_0$. Then given $\epsilon > 0$, there must exist some $\delta_1, \delta_2 > 0$ such that if $0 < |x - x_0| < \min(\delta_1, \delta_2)$ we have both:

$$f(x) > \frac{1}{\epsilon} - m + \frac{\epsilon}{2}, \qquad \text{and} \qquad |g(x) - m| < \frac{\epsilon}{2}.$$

  The second inequality implies:

$$m - g(x) < \frac{\epsilon}{2} \qquad \Leftrightarrow \qquad g(x) > m - \frac{\epsilon}{2}.$$

  It follows that given any K, if we take $\epsilon = \min(1/K, 1)$, then for $\delta = \min(\delta_1, \delta_2)$ with δ_1, δ_2 as above, we have for all x satisfying $0 < |x - x_0| < \delta$:

$$f(x) + g(x) > \frac{1}{\epsilon} - m + \frac{\epsilon}{2} + m - \frac{\epsilon}{2} = \frac{1}{\epsilon} \geq K.$$

  Therefore, $f(x) + g(x) \to \infty$, as required. A similar argument applies when $f(x) \to -\infty$ and $g(x) \to m$, for m finite, as $x \to x_0$.

- **Both limits are infinite, but have the same sign.**

- **Both limits are infinite, with opposite sign.** This is the case where the law fails. If $f(x) \to \infty$ but $g(x) \to -\infty$, then given any $K \in \mathbb{R}$, there exist δ_1, δ_2 such that if $0 < |x - x_0| < \min(\delta_1, \delta_2)$ we have both:

$$f(x) > K, \qquad g(x) < K.$$

19. From the formal mathematical definition of a limit, it is possible to prove results about the limits of sums, products, quotients and compositions of functions, similarly to Question 18. State these '*laws of limits*' clearly (making sure to take particular care when the limits are infinite), and use them to evaluate the following:

(a) $\displaystyle\lim_{x\to 0}\frac{x+1}{2-x^2}$, (b) $\displaystyle\lim_{x\to\infty}\sin\left(\frac{x^2+x+1}{3x^2-4}\right)$, (c) $\displaystyle\lim_{x\to 0}\left(\exp\left(\frac{x^4-1}{x^4+1}\right)\right)^{1/x^2}$, (d) $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+7x}-x\right)$.

Solution: Here is a large, comprehensive summary of the laws of limits:

Laws of limits

- ADDITION LAW. We have:
$$\lim_{x\to x_0}(f(x)+g(x))=\lim_{x\to x_0}f(x)+\lim_{x\to x_0}g(x),$$
provided that all limits exist in the set $\mathbb{R}\cup\{\infty,-\infty\}$ (that is, they can be infinite), ***and***:
$$\left\{\lim_{x\to x_0}f(x),\lim_{x\to x_0}g(x)\right\}\neq\{\infty,-\infty\}.$$
Here, the rules $l+\infty=\infty$ (for finite l) and $\infty+\infty=\infty$ are relevant for evaluating the right hand side in the case of infinite limits.

- MULTIPLICATION LAW. We have:
$$\lim_{x\to x_0}(f(x)g(x))=\lim_{x\to x_0}f(x)\cdot\lim_{x\to x_0}g(x),$$
provided that all limits exist in the set $\mathbb{R}\cup\{\infty,-\infty\}$ (that is, they can be infinite), ***and***:
$$\left\{\lim_{x\to x_0}f(x),\lim_{x\to x_0}g(x)\right\}\neq\{0,\infty\},\{0,-\infty\}.$$
Here, the rules $l\cdot\infty=\infty$ (for finite non-zero l) and $\infty\cdot\infty=\infty$ are relevant for evaluating the right hand side in the case of infinite limits.

- DIVISION LAW. We have:
$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{\lim_{x\to x_0}f(x)}{\lim_{x\to x_0}g(x)},$$
provided that all limits exist in the set $\mathbb{R}\cup\{\infty,-\infty\}$ (that is, they can be infinite), ***and***:
$$\left\{\lim_{x\to x_0}f(x),\lim_{x\to x_0}g(x)\right\}\neq\ldots$$
Here, the rules $l/\infty=0$ (for finite l) and $\infty/l=\infty$ (for finite non-zero l) are relevant for evaluate the right hand side in the case of infinite limits.

- POWER LAW.

- CONTINUITY LAW.

(a) Trivially, we have:
$$\lim_{x\to 0}x=0.$$

Since $(x+1)/(2-x^2)$ is a continuous function of x away from $x=\pm\sqrt{2}$, we have by the continuity law that:

$$\lim_{x\to 0}\frac{x+1}{2-x^2}=\frac{0+1}{2-0^2}=\frac{1}{2}.$$

(b) By the continuity law, we have:

$$\sin\left(\lim_{x\to\infty}\frac{x^2+x+1}{3x^2-4}\right)=\sin\left(\lim_{x\to\infty}\frac{1+1/x+1/x^2}{3-4/x^2}\right)=\sin\left(\frac{1+0+0^2}{3-0^2}\right)=\sin\left(\frac{1}{3}\right).$$

(d) This limit involves a bit of a trick. We have:

$$\sqrt{x^2+7x}-x=\frac{(\sqrt{x^2+7x}-x)(\sqrt{x^2+7x}+x)}{\sqrt{x^2+7x}+x}=\frac{x^2+7x-x^2}{\sqrt{x^2+7x}+x}=\frac{7x}{\sqrt{x^2+7x}+x}=\frac{7}{\sqrt{1+7/x}+1}.$$

This is a continuous function of $1/x$, which tends to zero as $x\to\infty$. Hence the limit is:

$$\frac{7}{\sqrt{1+0}+1}=\frac{7}{2}.$$

20. State *L'Hôpital's rule* for evaluating limits of differentiable functions, carefully specifying the conditions under which it is valid. Assuming $\alpha > 0$ throughout, use L'Hôpital's rule - where appropriate - to evaluate the limits of the following functions both (i) as $x \to 0^+$ (a *one-sided limit*), and (ii) as $x \to \infty$:

(a) $x^\alpha \log(x)$, (b) $x^{-\alpha} \log(x)$, (c) $x^\alpha e^{-x}$, (d) $x^{-\alpha} e^x$, (e) $\sin(\alpha x)/x$.

➾ **Solution:** L'Hôpital's rule states that:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

provided that:

(a) both limits exist;

(b)

Examining the limits given in the question, we have:

(a) (i) As $x \to 0^+$, we have $x^\alpha \to 0$, since $\alpha > 0$. Hence:

$$\lim_{x \to 0^+} x^\alpha \log(x) = \lim_{x \to 0^+} \frac{\log(x)}{x^{-\alpha}}$$

is an indeterminate form $(-\infty)/(+\infty)$. Therefore, we can apply L'Hôpital's rule:

$$\lim_{x \to 0^+} \frac{\log(x)}{x^{-\alpha}} = \lim_{x \to 0^+} \frac{1/x}{-\alpha x^{-\alpha-1}} = -\frac{1}{\alpha} \lim_{x \to 0^+} x^\alpha = 0.$$

Therefore, the original limit is also zero.

(ii) As $x \to \infty$, we have $x^\alpha \to \infty$, since $\alpha > 0$. We also have $\log(x) \to \infty$. Hence by the product rule for limits, we have:

$$\lim_{x \to \infty} x^\alpha \log(x) = \infty.$$

(b) (i) As $x \to 0^+$, we have $x^{-\alpha} \to \infty$, since $\alpha > 0$. We also have $\log(x) \to -\infty$ as $x \to 0^+$. Hence by the product rule for limits, we have:

$$\lim_{x \to 0^+} x^{-\alpha} \log(x) = -\infty.$$

(ii) As $x \to \infty$, we have $x^{-\alpha} \to 0$, since $\alpha > 0$. Hence:

$$\lim_{x \to \infty} x^{-\alpha} \log(x) = \lim_{x \to \infty} \frac{\log(x)}{x^\alpha}$$

is an indeterminate form ∞/∞. Therefore, we can apply L'Hôpital's rule:

$$\lim_{x \to \infty} \frac{\log(x)}{x^\alpha} = \lim_{x \to \infty} \frac{1/x}{\alpha x^{\alpha-1}} = \frac{1}{\alpha} \lim_{x \to \infty} x^{-\alpha} = 0.$$

Therefore, the original limit is also zero.

21. Using L'Hôpital's rule, evaluate the following 'power law' limits:

$$\text{(a)} \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x, \qquad \text{(b)} \lim_{x\to\infty}\log^{1/x}(x), \qquad \text{(c)} \lim_{x\to 0^+} x^x, \qquad \text{(d)} \lim_{x\to\infty} x^{1/x}.$$

➻ **Solution:** In all cases, we aim to rewrite the limits in terms of some quotients, where we can apply L'Hôpital's rule. We have:

(a) Using the continuity law, we have:

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = \exp\left(\lim_{x\to\infty} x\log\left(1-\frac{1}{x}\right)\right)$$
$$\exp\left(\lim_{x\to\infty}\frac{\log(1-1/x)}{1/x}\right).$$

The limit in the exponent is now of the form $\log(1)/0 = 0/0$, which is an indeterminate form. Hence we can apply L'Hôpital's rule to the limit in the exponent:

$$\lim_{x\to\infty}\frac{\log(1-1/x)}{1/x} = \lim_{x\to\infty}\frac{(1/x^2)}{(1-1/x)(-1/x^2)} = -1.$$

Hence, the original limit is:

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = e^{-1}.$$

(b) Using the continuity law, we have:

$$\lim_{x\to\infty}\log^{1/x}(x) = \exp\left(\lim_{x\to\infty}\frac{\log(\log(x))}{x}\right).$$

The limit in the exponent is now of the form ∞/∞, which is an indeterminate form. Hence we can apply L'Hôpital's rule to the limit in the exponent:

$$\lim_{x\to\infty}\frac{\log(\log(x))}{x} = \lim_{x\to\infty}\frac{1/x}{\log(x)} = \lim_{x\to\infty}\frac{1}{x\log(x)} = 0.$$

Hence, the original limit is:

$$\lim_{x\to\infty}\log^{1/x}(x) = 1.$$

(c) Using the continuity law, we have:

$$\lim_{x\to 0^+} x^x = \exp\left(\lim_{x\to 0^+} x\log(x)\right).$$

Using the result of the previous question, the limit in the exponent is 0. Hence the original limit is 1.

(d) Using the continuity law, we have:

$$\lim_{x\to\infty} x^{1/x} = \exp\left(\lim_{x\to\infty}\frac{\log(x)}{x}\right).$$

Using the result of the previous question, the limit in the exponent is 0. Hence the original limit is 1.

22. Explain why the following arguments with limits are *wrong*.

(a) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}(1+0)^x = 1$, using $\lim_{x\to\infty}\frac{1}{x} = 0$ in the first step.

(b) $\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \lim_{x\to 0}\frac{1-1}{x^2} = 0$, using $\cos(x)\approx 1$ for small enough x.

Now, evaluate the limits correctly.

➻ **Solution:** (a) It is incorrect to replace only 'part' of a limit when trying to work it out - that is not one of the laws of limits! To evaluate it correctly, we use L'Hôpital's rule:

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x &= \lim_{x\to\infty}\exp\left(x\log\left(1+\frac{1}{x}\right)\right) \\
&= \exp\left(\lim_{x\to\infty}\frac{\log(1+1/x)}{1/x}\right) && \text{(continuity law)} \\
&= \exp\left(\lim_{x\to\infty}\frac{-1/x^2}{(1+1/x)(-1/x^2)}\right) && \text{(L'Hôpital's rule)} \\
&= \exp\left(\lim_{x\to\infty}\frac{1}{1+1/x}\right) \\
&= e^1 = e.
\end{aligned}$$

(b) It is incorrect to make 'approximations' in limits to try to help evaluate them - that is not one of the laws of limits! To evaluate it correctly, we use L'Hôpital's rule:

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\frac{\cos(x)}{1} = 1.$$

23. Using L'Hôpital's rule where appropriate, compute the limit:

$$\lim_{x\to\infty}\left(1+a^x+\left(\frac{a^2}{2}\right)^x\right)^{\frac{1}{x}}.$$

for all values of $a \geq 0$.

Solution: If $a < 1$ and $a^2/2 < 1$, then $1+a^x+(a^2/2)^x \to 1$ as $x \to \infty$. Thus we get the form $1^0 = 1$, and the limit is 1.

Similarly, if $a = 1$, then $a^2/2 = 1/2$, so that $1+1^x+(1/2)^x \to 2$ as $x \to \infty$. Thus we get the form $2^0 = 1$, and the limit is also 1.

If either $a > 1$ or $a^2/2 > 1$, we instead get the indeterminate form ∞^0, and we will need to apply L'Hôpital's rule. Solving the second inequality, we have $a^2/2 > 1 \Leftrightarrow a > \sqrt{2}$ (since $a > 0$), so this case occurs if and only if $a > 1$.

Manipulating the limit to apply L'Hôpital's rule, we have:

$$\begin{aligned}\lim_{x\to\infty}\left(1+a^x+\left(\frac{a^2}{2}\right)^x\right)^{1/x} &= \lim_{x\to\infty}\exp\left(\frac{1}{x}\log\left(1+a^x+\left(\frac{a^2}{2}\right)^x\right)\right)\\ &= \exp\left(\lim_{x\to\infty}\frac{\log\left(1+e^{x\log(a)}+e^{x\log(a^2/2)}\right)}{x}\right) \qquad \text{(continuity law)}\end{aligned}$$

We can now apply L'Hôpital's rule, since $a > 1$, so $e^{x\log(a)}$ approaches infinity as $x \to \infty$, and hence the numerator approaches infinite as $x \to \infty$. Differentiating the numerator and denominator, the required limit becomes:

$$\exp\left(\lim_{x\to\infty}\frac{\log(a)e^{x\log(a)}+\log(a^2/2)e^{x\log(a^2/2)}}{e^{x\log(a)}+e^{x\log(a^2/2)}}\right).$$

Now, depending on the relative sizes of a and $a^2/2$, some of these exponentials will grow faster than others. Here are the cases:

- If $a > a^2/2$, then we have $\log(a) > \log(a^2/2)$ so that $0 > \log(a^2/2) - \log(a)$. Hence we perform the following manipulation:

  $$\lim_{x\to\infty}\frac{\log(a)e^{x\log(a)}+\log(a^2/2)e^{x\log(a^2/2)}}{e^{x\log(a)}+e^{x\log(a^2/2)}} = \lim_{x\to\infty}\frac{\log(a)+\log(a^2/2)e^{x\log(a^2/2)-x\log(a)}}{1+e^{x\log(a^2/2)-x\log(a)}} = \log(a).$$

  So overall, we find the original limit is a. This case occurs when $a > a^2/2 \Leftrightarrow 1 < a < 2$ (since we have already assumed that $a > 1$).

- If $a^2/2 > a$, then we have $\log(a) < \log(a^2/2)$ so that $0 > \log(a) - \log(a^2/2)$. Hence we perform the following manipulation:

  $$\lim_{x\to\infty}\frac{\log(a)e^{x\log(a)}+\log(a^2/2)e^{x\log(a^2/2)}}{e^{x\log(a)}+e^{x\log(a^2/2)}} = \lim_{x\to\infty}\frac{\log(a)e^{x\log(a)-x\log(a^2/2)}+\log(a^2/2)}{e^{x\log(a)-x\log(a^2/2)}+1} = \log(a^2/2).$$

  So overall, we find the original limit is $a^2/2$. This case occurs when $a^2/2 > a \Leftrightarrow a > 2$ (since we have already assumed that $a > 1$).

- If $a^2/2 = a$, i.e. we have $a = 2$, then all the exponentials cancel. The limit is:

$$\lim_{x\to\infty} \frac{\log(2) + \log(4/2)}{1+1} = \log(2)$$

  So overall, we find the original limit is 2.

In conclusion, we have shown that:

$$\lim_{x\to\infty} \left(1 + a^x + \left(\frac{a^2}{2}\right)^x\right)^{\frac{1}{x}} = \begin{cases} 1, & 0 \le a \le 1, \\ a, & 1 < a < 2, \\ 2, & a = 2, \\ a^2/2, & a > 2. \end{cases}$$

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24. Consider the limit:

$$\lim_{x\to\infty} \frac{x}{x + \sin(x)}.$$

Show that this limit is equal to one. Show that if we instead naïvely apply L'Hôpital's rule, we incorrectly conclude that the limit does not exist.

Solution: We have:

$$\lim_{x\to\infty} \frac{x}{x+\sin(x)} = \lim_{x\to\infty} \frac{1}{1 + \sin(x)/x}$$

$$= \frac{1}{1+0} = 1,$$

since $\sin(x)/x \to 0$ as $x \to \infty$, and $1/(1+u)$ is a continuous function.

If we instead apply L'Hôpital's rule, we have (incorrectly):

$$\lim_{x\to\infty} \frac{x}{x+\sin(x)} = \lim_{x\to\infty} \frac{1}{1+\cos(x)},$$

which does not exist, because cosine is infinitely oscillatory.

(†) Miscellaneous limits

[*This section contains a large collection of limits from past papers for you to evaluate. If you feel like you are getting too much of a good thing, feel free to save some of them for us to do together in the supervision*.]

25. Evaluate the following limits, using the most efficient method in each case:

(a) $\lim_{x\to 0^+} x\log(x)$;

(b) $\lim_{x\to a} \dfrac{x^x - a^a}{x-a}$ where $a > 0$;

(c) $\lim_{x\to 0} \dfrac{\tan(x)-\sin(x)}{\sin^3(x)}$;

(d) $\lim_{x\to a} \dfrac{\sin(x)-\sin(a)}{x-a}$;

(e) $\lim_{x\to\infty} \left(\dfrac{x+a}{x-a}\right)^x$;

(f) $\lim_{x\to 0} \dfrac{\cos(x)-\cos(3x)}{x^2}$;

(g) $\lim_{x\to 0} \dfrac{\log(\cos(x))}{\log(\cos(3x))}$;

(h) $\lim_{x\to 0} \dfrac{\sin(3x)}{\sinh(x)}$.

Solution: (a) We use L'Hôpital's rule:

$$\lim_{x\to 0^+} x\log(x) = \lim_{x\to 0^+} \frac{\log(x)}{1/x} = \lim_{x\to 0^+} \frac{-1/x^2}{-1/x^2} = 1.$$

(b) We could use L'Hôpital's rule here, but it is quicker to recognise that this is just the derivative of the function x^x at the point $x = a$. We have:

$$\frac{d}{dx}x^x = \frac{d}{dx}e^{x\log(x)} = (\log(x)+1)\,e^{x\log(x)} = (\log(x)+1)\,x^x.$$

Hence the limit is $(\log(a)+1)a^a$.

(c) We can do this with some trigonometric identities. We have:

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(x)-\sin(x)}{\sin^3(x)} &= \lim_{x\to 0} \frac{\sin(x)/\cos(x)-\sin(x)}{\sin^3(x)} \\
&= \lim_{x\to 0} \frac{1-\cos(x)}{\sin^2(x)\cos(x)} \\
&= \lim_{x\to 0} \frac{1-\cos(x)}{(1-\cos^2(x))\cos(x)} \\
&= \lim_{x\to 0} \frac{1-\cos(x)}{(1-\cos(x))(1+\cos(x))\cos(x)} \\
&= \lim_{x\to 0} \frac{1}{(1+\cos(x))\cos(x)} \\
&= \frac{1}{2}.
\end{aligned}$$

(d) We could use L'Hôpital's rule here, or use some trigonometric identities, but it is quicker to recognise that this is just the derivative of $\sin(x)$ at the point $x = a$. Hence the limit is $\cos(a)$.

(e) Here, the most appropriate method is L'Hôpital's rule, since we have an indeterminate form 1^∞. We have:

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x &= \exp\left(\lim_{x\to\infty} x\log\left(\frac{x+a}{x-a}\right)\right) && \text{(continuity law)}\\ &= \exp\left(\lim_{x\to\infty}\frac{\log((x+a)/(x-a))}{1/x}\right).\end{aligned}$$

Inside the limit, the numerator approaches $\log(1) = 0$, and the denominator approaches 0, so we can apply L'Hôpital's rule. We have:

$$\lim_{x\to\infty}\frac{\log((x+a)/(x-a))}{1/x} = \lim_{x\to\infty}\frac{\left(\dfrac{(x-a)-(x+a)}{(x-a)^2}\right)}{\left(\dfrac{x+a}{x-a}\right)\left(-\dfrac{1}{x^2}\right)} = \lim_{x\to\infty}\frac{2ax^2}{(x+a)(x-a)} = \lim_{x\to\infty}\frac{2a}{(1+a/x)(1-a/x)} = 2a.$$

Hence the original limit is e^{2a}.

(f) We can use L'Hôpital's rule. We have:

$$\begin{aligned}\lim_{x\to 0}\frac{\cos(x)-\cos(3x)}{x^2} &= \lim_{x\to 0}\frac{-\sin(x)+3\sin(3x)}{2x}\\ &= \lim_{x\to 0}\frac{-\cos(x)+9\cos(3x)}{2}\\ &= 4.\end{aligned}$$

(g) We can use L'Hôpital's rule (twice actually!). We have:

$$\begin{aligned}\lim_{x\to 0}\frac{\log(\cos(x))}{\log(\cos(3x))} &= \lim_{x\to 0}\frac{\sin(x)/\cos(x)}{3\sin(3x)/\cos(3x)}\\ &= \frac{1}{3}\lim_{x\to 0}\frac{\cos(3x)}{\cos(x)}\cdot\lim_{x\to 0}\frac{\sin(x)}{\sin(3x)}\\ &= \frac{1}{3}\cdot\frac{1}{1}\cdot\lim_{x\to 0}\frac{\cos(x)}{3\cos(3x)}\\ &= \frac{1}{3}.\end{aligned}$$

(h) We can use L'Hôpital's rule. We have:

$$\lim_{x\to 0}\frac{\sin(3x)}{\sinh(x)} = \lim_{x\to 0}\frac{3\cos(3x)}{\cosh(x)} = 3.$$

(†) Continuity of functions

26. Let $f : (a, b) \to \mathbb{R}$ be a real function, and let $x_0 \in (a, b)$ be a point in its domain.

(a) State the formal ϵ, δ definition of f being *continuous* at x_0. Explain this condition by drawing a diagram.

(b) Using the formal definition of a limit, explain why this condition is equivalent to the statement:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

➻ **Solution:** (a) We say that f is continuous at x_0 if for any given tolerance $\epsilon > 0$, there exists some closeness δ, such that if we are at some point x within δ of x_0, $|x - x_0| < \delta$, we have that $f(x)$ is within ϵ of $f(x_0)$, $|f(x) - f(x_0)| < \epsilon$.

(b) Comparing with the definition of a limit, continuity is just saying that we require:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

27. Using the formal ϵ, δ definition of continuity, show directly that the following functions are continuous everywhere:

(a) x, (b) $|x|$, (c) x^2, (d) $\sin(x)$.

At what points are these functions differentiable? [*Hint: for part (c), look at your answer to Question 16(b).*]

➼ **Solution:**

(a) Let's prove continuity at the point $x = a$, for general a. Given $\epsilon > 0$, we consider x such that $|x - a| < \delta$, for some δ we shall decide at a later time. Then trivially we have:

$$|x - a| < \delta,$$

so that we see that if we choose $\delta = \epsilon$, the condition for continuity at $x = a$ will be satisfied. So we're done.

(b) Similarly, we shall prove continuity at the point $x = a$, for general a. Given $\epsilon > 0$, we consider x such that $|x - a| < \delta$, for some δ we shall decide at a later time. Then we have:

$$||x| - |a|| \leq |x - a| < \delta,$$

using the *reverse triangle inequality*, which we proved on Examples Sheet 1, Question 12(c). Hence we see that if we choose $\delta = \epsilon$, we have $||x| - |a|| < \epsilon$. So we're done.

(c) We already saw that $x^2 \to a^2$ as $x \to a$ in Question 16(b), using an ϵ-δ argument. This is precisely what we need to show for continuity.

(d) Let's prove continuity at the point $x = a$, for general a. Given $\epsilon > 0$, we consider x such that $|x - a| < \delta$, for some δ we shall decide at a later time. Then we have:

$$\begin{aligned} |\sin(x) - \sin(a)| &= 2\left|\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{x+a}{2}\right)\right| \\ &\leq 2\left|\sin\left(\frac{x-a}{2}\right)\right| \\ &\leq 2\left|\frac{x-a}{2}\right| && \text{(since } |\sin(u)| \leq |u| \text{ for all } u\text{)} \\ &\leq |x - a| < \delta. \end{aligned}$$

Hence, if we choose $\delta = \epsilon$, we have $|\sin(x) - \sin(a)| < \epsilon$. Hence $\sin(x)$ is a continuous function.

These functions are all differentiable everywhere, except for $|x|$, which is not differentiable at $x = 0$.

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28. Using the formal ϵ, δ definition of continuity, show directly that the function $f(x) = 0$ for $x \leq 0$, $f(x) = 1$ for $x > 0$, is discontinuous at $x = 0$.

➼ **Solution:** We need to show that there exists some ϵ for which given any $\delta > 0$, there is some point with $|x| < \delta$ and $|f(x) - f(0)| = |f(x)| > \epsilon$.

Choose $\epsilon = 1/2$. Then given any $\delta > 0$, if we choose $x = \delta/2$, we have $|f(\delta/2) - f(0)| = 1 > 1/2$. So the function is not continuous at $x = 0$.

29. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x\sin(1/x)$ for $x \neq 0$, and $f(0) = 0$. Show that f is continuous everywhere, and is differentiable everywhere except at $x = 0$.

➾ **Solution:** $x\sin(1/x)$ is evidently continuous at $x \neq 0$, since $x\sin(1/x) \to a\sin(1/a)$ as $x \to a$, by the product, quotient and continuity law of limits (we have already shown that $\sin(x)$ is continuous). The only problem point is $x = 0$, where $x\sin(1/x) \to 0$ as $x \to 0$ (we also showed this earlier on in the sheet). Therefore the function is continuous everywhere.

It is differentiable everywhere except $x = 0$ by the product and chain rules. It is not differentiable at $x = 0$ because the limit:

$$\lim_{h\to 0} \frac{h\sin(1/h) - 0}{h} = \lim_{h\to 0} \sin(1/h)$$

does not exist.

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30. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} |x|^p \sin(x), & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad g(x) = \begin{cases} |x|^q \sin(\pi\sin(1/x)), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where p, q are real numbers. For which values of p, q are f, g: (a) continuous; (b) differentiable? Justify your answers.

➾ **Solution:** (a) Let's start with $f(x)$. Clearly $f(x)$ is continuous everywhere except $x = 0$. At $x = 0$, for continuity we require:

$$\lim_{x\to 0} |x|^p \sin(x) = 0.$$

This is certainly true if $p \geq 0$. If $p < 0$, we may be able to use L'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sin(x)}{|x|^{-p}} = -\frac{1}{p} \lim_{x\to 0} \frac{\cos(x)}{|x|^{-p-1}}.$$

If $-p - 1 < 0$, i.e. $-1 < p < 0$, this is convergent to 0. Else, if $p = -1$, this is convergent to $-1/p$, which demonstrates a lack of continuity. For $p < -1$, this is divergent to $-\infty$, which demonstrates lack of continuity. Thus it is continuous precisely for $p > -1$.

Similarly, $g(x)$ is clearly continuous everywhere except $x = 0$. At $x = 0$, for continuity we require:

$$\lim_{x\to 0} |x|^q \sin(\pi\sin(1/x)) = 0.$$

This is certainly true if $q > 0$. The difference with $f(x)$ is that the sine term no longer converges to zero on its own, it infinitely oscillates as $x \to 0$. Indeed, suppose we pick $\sin(1/x) = 1/2$, by choosing $1/x = 2n\pi + \pi/6$. This defines a sequence of points:

$$x_n = \frac{1}{2n\pi + \pi/6},$$

which converges to zero. This means that we have a sequence of points that satisfy:

$$|x_n|^q \sin(\pi\sin(1/x_n)) = |x_n|^q \to \infty$$

as $n \to \infty$, for any $q < 0$. If $q = 0$ this converges to 1. In particular, these points are not approaching zero, so we cannot have continuity.

(b) Clearly $f(x)$ is differentiable everywhere except $x = 0$. Differentiability of $f(x)$ at $x = 0$ requires existence of the limit:

$$\lim_{h\to 0} \frac{|h|^p \sin(h) - 0}{h} = \lim_{h\to 0} \frac{|h|^p \sin(h)}{h} = \lim_{h\to 0} |h|^p \cdot \lim_{h\to 0} \frac{\sin(h)}{h} = \lim_{h\to 0} |h|^p \lim_{h\to 0} \frac{\cos(h)}{1} = \lim_{h\to 0} |h|^p.$$

This limit exists if and only if $p \geq 0$, hence the function is differentiable at $x = 0$ if and only if $p \geq 0$.

In the case of $g(x)$, again, clearly $g(x)$ is differentiable everywhere except $x = 0$. Differentiability of $g(x)$ at $x = 0$ requires existence of the limit:

$$\lim_{h\to 0} \frac{|h|^q \sin(\pi \sin(1/h)) - 0}{h} = \lim_{h\to 0} \frac{|h|^q \sin(\pi \sin(1/h))}{h}.$$

This time, we can't separate the limit out and use L'Hôpital's rule. Instead, observe that if $q > 1$, we can write:

$$\lim_{h\to 0} \frac{|h|^q \sin(\pi \sin(1/h))}{h} = \lim_{h\to 0} \operatorname{sign}(h)|h|^{q-1} \sin(\pi \sin(1/h)) = 0,$$

where $\operatorname{sign}(h)$ is the sign of h (positive for $h > 0$, negative for $h < 0$). On the other hand if $q \leq 1$, we can choose a similar sequence to part (a), say:

$$h_n = \frac{1}{n\pi + \pi/6}.$$

We then observe that this sequence satisfies:

$$\frac{|h_n|^q \sin(\pi \sin(1/h_n))}{h_n} = h_n^{q-1} \sin\left(\pi \left((-1)^n \cdot \frac{1}{2}\right)\right) = \frac{(-1)^n}{h_n^{1-q}}.$$

If $q \leq 1$, this is not convergent. Hence the original limit cannot be convergent either. It follows that $g(x)$ is differentiable at $x = 0$ if and only if $q > 1$.

31. Three functions f_0, f_1, f_2 are defined by:

$$f_n(x) = \left(\frac{x - \pi/2}{x}\right)^n \sin(\tan(x))$$

for $n = 0, 1, 2$, at all points except $x = m\pi/2$ for integer m, where the functions are defined to be zero. For each n, determine with justification all points in the range $(-\pi, \pi)$ where the function is: (a) continuous; (b) differentiable.

Solution: (a) When $n = 0$, we have $f_0(x) = \sin(\tan(x))$. This is continuous everywhere that $x \neq m\pi/2$. At $x = m\pi/2$, we have that the function $f_0(x)$ is defined to be zero. However, the limit:

$$\lim_{x \to m\pi/2} \sin(\tan(x))$$

does not exist, because $\tan(x)$ approaches infinity from two different directions here, and the limit of sine does not exist near infinity.

When $n = 1$, we have:

$$f_1(x) = \left(\frac{x - \pi/2}{x}\right) \sin(\tan(x)).$$

This is continuous everywhere that $x \neq m\pi/2$. At all points where $x = m\pi/2$, with $m \neq 0, 1$, we have the same issue as before, because $(x - \pi/2)/x$ is finite at these points. The problem is even worse at $m = 0$, because we have an infinite limit multiplied by a non-existent limit. However, at $m = 1$, we have:

$$\lim_{x \to \pi/2} \frac{x - \pi/2}{x} \sin(\tan(x)) = \frac{2}{\pi} \lim_{x \to \pi/2} (x - \pi/2) \sin(\tan(x)).$$

Since sine is always bounded, and $(x - \pi/2) \to 0$ as $x \to \pi/2$, we have that this tends to zero. Thus the function is continuous at $x = \pi/2$.

When $n = 2$, we have:

$$f_2(x) = \left(\frac{x - \pi/2}{x}\right)^2 \sin(\tan(x)).$$

The same reasoning as the case $n = 1$ gives that this function is continuous everywhere except $x = m\pi/2$ with $m \neq 1$.

(b) Now consider differentiability. The first function $f_0(x)$ is differentiable everywhere except points where $x = m\pi/2$, by the chain rule. But it is not differentiable at points of discontinuity; a function *must* be continuous at a point for it to be differentiable there (see Question 32).

The second function $f_1(x)$ is differentiable everywhere except points where $x = m\pi/2$, as above. We just need to check $x = \pi/2$, where the function is continuous. We require that the limit:

$$\lim_{x \to \pi/2} \frac{((x - \pi/2)/x) \sin(\tan(x)) - 0}{x - \pi/2} = \lim_{x \to \pi/2} \frac{\sin(\tan(x))}{x}$$

exists. But it evidently does not, so the function is indeed not differentiable there.

Finally, the third function $f_2(x)$ is differentiable everywhere except points where $x = m\pi/2$, as above. We just need to check $x = \pi/2$, where the function is continuous. We require that the limit:

$$\lim_{x \to \pi/2} \frac{((x - \pi/2)/x)^2 \sin(\tan(x)) - 0}{x - \pi/2} = \lim_{x \to \pi/2} \frac{(x - \pi/2)}{x^2} \sin(\tan(x)),$$

exists. Evidently it does, so the function is also differentiable at $x = \pi/2$.

32. Show that if a function is differentiable at a point x_0 in its domain, then it must be continuous at x_0. ($*$) Is it true that a continuous function $f : \mathbb{R} \to \mathbb{R}$ must be differentiable at *some* point?

➾ **Solution:** Let $f(x)$ be differentiable at x_0. We then have that:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, and is finite. Further, the limit $\lim_{x \to x_0}(x - x_0) = 0$ exists. Thus the product of these limits exists, and we can use the product rule for limits to give:

$$0 = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} = \lim_{x \to x_0} (f(x) - f(x_0)).$$

This gives:

$$\lim_{x \to x_0} f(x) = f(x_0),$$

so that $f(x)$ is continuous at x_0 as required.

It is *not* true that a function that is everywhere continuous must be differentiable at some point. Great counterexamples include *fractal patterns* (for example, look up the *Weierstrass function*, https://en.wikipedia.org/wiki/Weierstrass_function).

Part IA: Mathematics for Natural Sciences B
Examples Sheet 5: Infinite series and Taylor series

Model Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

(†) Basics of infinite series

1. State clearly what it means for an infinite series to be: (i) *convergent*; (ii) *absolutely convergent*. If a series is absolutely convergent, must it be convergent? Is the converse true?

Solution: Let:

$$\sum_{n=1}^{\infty} a_n$$

be an infinite series. We say that:

(i) the infinite series is *convergent* if the limit:

$$\lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

of partial sums exists (and is finite); otherwise, the series is *divergent*;

(ii) the infinite series is *absolutely convergent* if the series:

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent.

If a series if absolutely convergent, then it must be convergent (the proof is beyond the scope of the course). The converse is not true; there exist series which are convergent but not absolutely convergent (we shall see examples later on).

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2. By evaluating the partial sums, determine whether $\sum\limits_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$ is convergent. Is it absolutely convergent?

Solution: We have:

$$\sum_{n=1}^{N} \left(\sqrt{n+1} - \sqrt{n}\right) = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + \cdots + (\sqrt{N} - \sqrt{N-1}) + (\sqrt{N+1} - \sqrt{N}).$$

We see that this is a *telescoping series*, where most terms cancel out, leaving only:

$$\sum_{n=1}^{N} \left(\sqrt{n+1} - \sqrt{n}\right) = \sqrt{N+1} - 1.$$

Hence, this has an infinite limit as $N \to \infty$, so is not convergent. It cannot be absolutely convergent, because it is not convergent (and anything that is absolutely convergent is necessarily convergent too).

3. By evaluating the partial sums, prove that the geometric series $\sum_{n=0}^{\infty} ar^n$ is absolutely convergent for $-1 < r < 1$.

➼ **Solution:** Let:

$$S_{N+1} = |a| + |a||r| + |a||r|^2 + ... + |a||r|^N$$

be the $(N+1)$th partial sum of the absolute values. Multiplying by $1 - |r|$, we have:

$$\begin{aligned}(1-|r|)S_{N+1} &= (1-|r|)\left(|a| + |a||r| + |a||r|^2 + ... + |a||r|^N\right)\\ &= |a| + |a||r| + |a||r|^2 + ... + |a||r|^N - |a||r| - |a||r|^2 - |a||r|^3 - ... - |a||r|^{N+1}\\ &= |a|\left(1 - |r|^{N+1}\right).\end{aligned}$$

Hence the partial sums are given by:

$$S_{N+1} = \frac{|a|\left(1-|r|^{N+1}\right)}{1-|r|},$$

provided that $|r| \neq 1$. If $|r| < 1$, then as $N \to \infty$ we have $|r|^{N+1} \to 0$. Hence the limit of the partial sums is given by:

$$\frac{|a|}{1-|r|},$$

which is finite. It follows that if $|r| < 1$, the sum is absolutely convergent (and hence convergent).

(†) Tests for convergence

4. (a) Clearly state the *comparison test* for series convergence or divergence.

   (b) Using the comparison test, prove that the harmonic series diverges. Hence, show that the following definition of the Riemann zeta function:

   $$\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p},$$

   with p real, converges if and only if $p > 1$. Is it absolutely convergent when $p > 1$?

➼ **Solution:** (a) Let:

$$\sum_{n=1}^{\infty} a_n, \qquad \sum_{n=1}^{\infty} b_n$$

be two series. The *comparison test* states that if there exists some N such that $0 \leq a_n \leq b_n$ for all $n \geq N$, then:

- if the first series diverges, then the second series diverges;
- if the second series converges, then the first series converges.

(b) The harmonic series is:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots.$$

Grouping the terms, we can rewrite the harmonic series as:

$$\begin{aligned}
1+&\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\left(\frac{1}{8}+\cdots\right)+\cdots\\
&>\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\cdots\right)+\cdots\\
&=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots.
\end{aligned}$$

Hence, we see by comparison with the divergent series $1/2+1/2+1/2+1/2+$..., the harmonic series must be divergent.

Now consider the Riemann zeta function $\zeta(p)$. If $p=1$, the series diverges because it is the harmonic series. Further, if $p<1$, we have $n^p \leq n$ for all positive integers n, hence:

$$\begin{aligned}
\zeta(p)&=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\cdots\\
&>1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots,
\end{aligned}$$

so by comparison we have that the series diverges for $p<1$. Finally, observe that if $p>1$, we have:

$$\begin{aligned}
\zeta(p)&=\frac{1}{1^p}+\frac{1}{2^p}+\left(\frac{1}{3^p}+\frac{1}{4^p}\right)+\left(\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}+\frac{1}{8^p}\right)+\cdots\\
&\leq\frac{1}{1^p}+\frac{1}{2^p}+\left(\frac{1}{4^p}+\frac{1}{4^p}\right)+\left(\frac{1}{8^p}+\frac{1}{8^p}+\frac{1}{8^p}+\frac{1}{8^p}\right)+\cdots\\
&=1+2^{-p}+2\cdot 4^{-p}+4\cdot 8^{-p}+\cdots\\
&=1+2^{-p}+2^{-2p+1}+2^{-3p+2}+\cdots\\
&=1+\frac{2^{-p}}{1-2^{-p+1}},
\end{aligned}$$

since for $p>1$, we have $2^{-p+1}=1/2^{p-1}<1$, so we can bound things above with a convergent geometric series. It follows that the series converges for $p>1$.

It is absolutely convergent for $p>1$, since all of the terms are positive already.

5. (a) Clearly state, *and prove*, the *alternating series test* for series convergence or divergence.

(b) Hence, show that the following definition of the Dirichlet eta function:

$$\eta(p) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p},$$

with p real, converges if and only if $p > 0$. Is it absolutely convergent when $p > 0$?

➾ **Solution:** (a) Let:

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

satisfying the following three key properties:

(i) $a_n \geq 0$;

(ii) $a_{n+1} \leq a_n$;

(iii) $a_n \to 0$ as $n \to \infty$.

Then the *alternating series test* states the series converges.

To prove the test, we split the partial sums into even and odd cases. The even partial sums satisfy:

$$\begin{aligned}
\sum_{n=1}^{2N} (-1)^{n+1} a_n &= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2N-1} - a_{2N} \\
&= a_1 + \underbrace{(a_3 - a_2)}_{<0} + \underbrace{(a_5 - a_4)}_{<0} + \cdots + \underbrace{(a_{2N} - a_{2N-1})}_{<0} \\
&\leq a_1.
\end{aligned}$$

Therefore, the even partial sums are bounded above. Furthermore:

$$\sum_{n=1}^{2N} (-1)^{n+1} a_n = \sum_{n=1}^{2(N-1)} (-1)^{n+1} a_n + \underbrace{(a_{2N-1} - a_{2N})}_{>0}.$$

Hence, the even partial sums are increasing. These two facts combined imply that the even partial sums converge.

The odd partial sums satisfy:

$$\begin{aligned}
\sum_{n=1}^{2N+1} (-1)^{n+1} a_n &= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2N-1} - a_{2N} + a_{2N+1} \\
&= \underbrace{(a_1 - a_2)}_{>0} + \underbrace{(a_3 - a_4)}_{>0} + \cdots + \underbrace{(a_{2N-1} - a_{2N})}_{>0} + a_{2N+1} \\
&\geq a_{2N+1} > 0.
\end{aligned}$$

Therefore, the odd partial sums are bounded below. Furthermore:

$$\sum_{n=1}^{2N+1}(-1)^{n+1}a_n = \sum_{n=1}^{2N-1}(-1)^{n+1}a_n + \underbrace{(-a_{2N}+a_{2N+1})}_{<0}.$$

Hence, the odd partial sums are decreasing. These two facts combined imply that the odd partial sums converge.

To see that these sequences of partial sums converge to the same limit, observe that:

$$\lim_{N\to\infty}\left(\sum_{n=1}^{2N+1}(-1)^{n+1}a_n - \sum_{n=1}^{2N}(-1)^{n+1}a_n\right) = \lim_{N\to\infty} a_{2N+1} = 0,$$

by assumption (iii). Hence the two limits must be equal.

(b) For the Dirichlet eta function:

$$\eta(p) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p},$$

we observe that if $p > 0$, we have $1/n^p \to 0$ as $n \to \infty$, and:

$$\frac{1}{(n+1)^p} < \frac{1}{n^p}$$

for all positive integers n. Hence, by the alternating series test, we have convergence for $p > 0$.

When $p = 0$, the series with terms $(-1)^{n+1}$ does not converge, so the nth term of the series does not tend to zero. Hence the series is not convergent.

When $p < 0$, the series with terms $(-1)^{n+1}/n^p$ oscillates between being large and positive, and large and negative, and hence does not converge. Thus, the nth term of the series does not tend to zero. Hence the series is not convergent.

Finally, observe that if we take the absolute value of all the terms, we get $\zeta(p)$ instead. Hence the series is absolutely convergent if and only if $p > 1$, from Question 4(b).

6. Clearly state the *ratio test* for series convergence or divergence. Use it to show that the series:

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

is absolutely convergent. To what value does it converge? [*Hint: If you're unsure, come back after studying Taylor series.*]

➾ **Solution:** Let:

$$\sum_{n=1}^{\infty} a_n$$

be an infinite series of complex numbers a_n, containing only non-zero terms. Suppose also that the limit:

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists in $\mathbb{R} \cup \{\infty\}$ (that is, we allow infinite limits). Then the *ratio test* states:

- if $r > 1$, the series diverges;
- if $r < 1$, the series absolutely converges, and hence is convergent.

If $r = 1$, the test is inconclusive.

Applying this to the series in the question, we have the nth term $a_n = 2^n/n!$. Hence:

$$r = \lim_{n\to\infty} \left| \frac{2^{n+1}/(n+1)!}{2^n/n!} \right| = \lim_{n\to\infty} \left| \frac{2}{n+1} \right| = 0.$$

This is less than 1, hence the series is absolutely convergent, and hence is convergent.

We spot that the series looks like the Taylor series for e^x about $x = 0$, given by:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Comparing with the series in the question, we see that:

$$\sum_{n=1}^{\infty} \frac{2^n}{n!} = e^2 - 1,$$

since the Taylor series starts at the term $n = 0$, but the series in the question starts at the term $n = 1$, so we need to subtract the $n = 0$ term.

7. Another test that was not lectured (but has come up in exams before!) is the *integral test*.

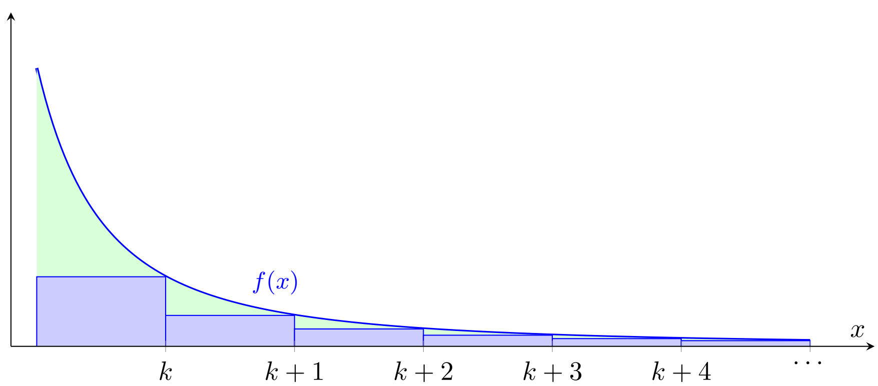
(a) Suppose that $f : [k, \infty) \to \mathbb{R}$ is a continuous, positive, decreasing function, where k is an integer. By drawing a convincing diagram, show that:

$$\int_k^\infty f(x)\,dx \text{ converges} \qquad \Rightarrow \qquad \sum_{n=k}^\infty f(n) \text{ converges.}$$

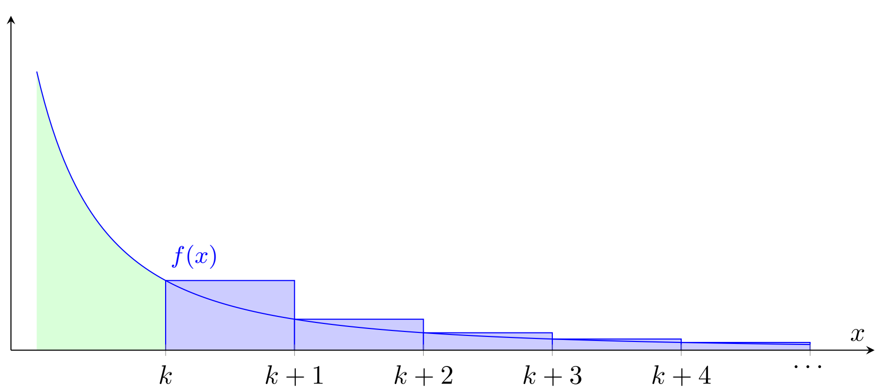
Show also that if the integral diverges, then the series diverges. This test is called the *integral test*.

(b) Using the integral test, reanalyse the convergence of the definition of $\zeta(p)$ given in Question 4.

Solution: (a) Since $f(x)$ is continuous, positive, and decreasing, it takes the form shown in the picture below. Inserting rectangles which are of height $f(k), f(k+1), f(k+2)$, etc, under the curve, we see immediately that if the integral converges (i.e. the area under the curve is finite), then the sum must converge too.



On the other hand, we could equally place the rectangles above the curve, by aligning their left edges with the curve instead, as shown in the figure below.



This shows that if the integral diverges (i.e. there is infinite area under the curve), then the sum must diverge too.

(b) For:

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p},$$

we should consider the integral:

$$\int_1^\infty \frac{dx}{x^p}.$$

- If $p > 1$, we have:

  $$\int_1^\infty \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^\infty = 0 - \frac{1}{1-p} = \frac{1}{p-1}.$$

  This is finite, hence the original series is convergent in this case.

- If $p = 1$, we have:

  $$\int_1^\infty \frac{dx}{x} = [\log(x)]_1^\infty = \infty.$$

  This is infinite, hence the original series is divergent in this case.

- If $p < 1$, we have:

  $$\int_1^\infty \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^\infty = \infty.$$

  This is infinite, hence the original series is divergent in this case.

Hence, the analysis agrees with the analysis we performed earlier using the comparison test.

(†) Miscellaneous series

8. Applying an appropriate test in each case, determine which of the following series are convergent, and which are absolutely convergent:

(a) $\sum_{n=1}^{\infty} \frac{n^2+1}{3n^2+4}$;

(b) $\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$;

(c) $\sum_{n=1}^{\infty} \frac{n^{10}}{n!}$;

(d) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$;

(e) $\sum_{n=1}^{\infty} \frac{5 \cdot 8 \cdot 11 \cdot \ldots \cdot (3n+2)}{1 \cdot 5 \cdot 9 \cdot \ldots \cdot (4n-3)}$;

(f) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$;

(g) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2n+5}{3n+1}\right)^n$;

(h) $\sum_{n=2}^{\infty} \frac{1}{n \log(n)}$;

(i) $\sum_{n=1}^{\infty} \frac{n^4}{3^n}$;

(j) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}$;

(k) $\sum_{n=2}^{\infty} \frac{1}{n^2 \log(n)}$;

(l) $\sum_{n=1}^{\infty} \frac{n^2+2n}{n^3+3n^2+1}$;

(m) $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$;

(n) $\sum_{n=0}^{\infty} \frac{a^{2n+1}}{2n+1}$, where $a > 0$;

(o) $\sum_{n=1}^{\infty} n^p \sin(\omega n)$ where $\omega > 0$, and $p < -1$;

(p) $\sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi)}{n}$;

(q) $\sum_{n=2}^{\infty} \frac{2^n}{n \log(n)}$;

(r) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$;

(s) $\sum_{n=1}^{\infty} \frac{n^3}{\log^n(2)}$;

(t) $\sum_{n=1}^{\infty} \left(\sqrt{n^4+a^2} - n^2\right)$, for $a > 0$;

(u) $\sum_{n=1}^{\infty} \frac{n}{2^n - 1}$;

(v) $\sum_{n=1}^{\infty} \frac{(n!)^3 e^{3n}}{(3n)!}$.

Solution: One important test that we have not mentioned on the sheet so far is the 'nth term test' or the 'divergence test': the series:

$$\sum_{n=1}^{\infty} a_n$$

is divergent if $a_n \not\to 0$ as $n \to \infty$. We will use this in some of the examples.

(a) The divergence test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{n^2+1}{3n^2+4} = \lim_{n\to\infty} \frac{1+1/n^2}{3+4/n^2} = \frac{1}{3} \neq 0.$$

Hence, the series diverges.

(b) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{(n+1)^{10}}{2^{n+1}} \cdot \frac{2^n}{n^{10}} = \frac{1}{2} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{10} = \frac{1}{2} < 1.$$

Hence, the series converges. All terms of the series are positive, so the series converges absolutely.

(c) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{(n+1)^{10}}{(n+1)!} \cdot \frac{n!}{n^{10}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{10} \cdot \lim_{n\to\infty} \frac{1}{n+1} = 1 \cdot 0 = 0 < 1.$$

Hence, the series converges. All terms of the series are positive, so the series converges absolutely.

(d) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \frac{1}{10} \lim_{n\to\infty} (n+1) = \infty.$$

Hence, the series diverges.

(e) The ratio test is appropriate here. Observe that:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{5 \cdot 8 \cdot 11 \cdot \ldots \cdot (3(n+1)+2)}{1 \cdot 5 \cdot 9 \cdot \ldots \cdot (4(n+1)-3)} \cdot \frac{1 \cdot 5 \cdot 9 \cdot \ldots \cdot (4n-3)}{5 \cdot 8 \cdot 11 \cdot \ldots \cdot (3n+2)} \\
&= \lim_{n\to\infty} \frac{3(n+1)+1)}{4(n+1)-3} \\
&= \lim_{n\to\infty} \frac{3n+4}{4n+1} \\
&= \lim_{n\to\infty} \frac{3+4/n}{4+1/n} \\
&= \frac{3}{4}.
\end{aligned}$$

Hence, the series converges. All terms of the series are positive, so the series converges absolutely.

(f) The alternating series test is appropriate here. Observe that: $1/\sqrt{n} > 0$, $1/\sqrt{n} \to 0$ as $n \to \infty$, and:

$$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}.$$

Hence the three conditions of the alternating series test are satisfied, so the series converges.

The series converges absolutely if:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

converges. The comparison test is appropriate here. Note that for all integers $n \geq 1$, we have $\sqrt{n} < n$, so:

$$\frac{1}{\sqrt{n}} > \frac{1}{n}.$$

The harmonic series diverges, so this implies that this series diverges too.

Alternatively, this question can be done by simply spotting that the series is the Dirichlet eta function $\eta(1/2)$, which we already discussed in Question 5(b).

(g) Superficially, it looks like the alternating series test is the right test to use here. Instead though, we can try to use the ratio test to prove absolute convergence, from which we can immediately deduce convergence.

Consider the limit:

$$\lim_{n\to\infty}\left(\frac{2(n+1)+5}{3(n+1)+1}\right)^{n+1}\left(\frac{3n+1}{2n+5}\right)^{n}=\frac{2}{3}\lim_{n\to\infty}\left(\frac{n+1+5/2}{n+1+1/3}\right)^{n+1}\left(\frac{n+1/3}{n+5/2}\right)^{n}.$$

Note that the final limit is the product:

$$\lim_{n\to\infty}\left(\frac{n+1+5/2}{n+1+1/3}\right)^{n+1}\left(\frac{n+1/3}{n+5/2}\right)^{n}=\lim_{n\to\infty}\left(\frac{n+1+5/2}{n+1+1/3}\right)^{n+1}\lim_{n\to\infty}\left(\frac{n+1/3}{n+5/2}\right)^{n}.$$

Making the substitution $m=n+1$ in the first limit on the right hand side, we immediately see that the two limits on the right hand side are reciprocals of one another. Hence the product is 1, and it follows that:

$$\lim_{n\to\infty}\left(\frac{2(n+1)+5}{3(n+1)+1}\right)^{n+1}\left(\frac{3n+1}{2n+5}\right)^{n}=\frac{2}{3}<1.$$

Hence, the series is absolutely convergent. Absolute convergence implies convergence, so the series is also convergent.

(h) The integral test is appropriate here. Observe that (using the reverse chain rule - or the substitution $u=\log(x)$, if you're worried):

$$\int_{2}^{\infty}\frac{dx}{x\log(x)}=[\log(\log(x))]_2^{\infty}=\infty.$$

Hence, the integral is divergent, so the series is also divergent.

(i) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty}\frac{(n+1)^4}{3^{n+1}}\cdot\frac{3^n}{n^4}=\frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^4=\frac{1}{3}<1.$$

Hence, the series converges. All terms of the series are positive, so the series converges absolutely.

(j) We could use the alternating series test. However, it is actually better to look at the series:

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \tag{†}$$

first. Observe that:

$$\frac{1}{(2n-1)^2}\leq\frac{1}{n^2}$$

for all integers $n\geq 1$. Since the series definition of $\zeta(2)$ converges by Question 4(b), we have by the comparison test that the series (†) converges. Hence the series in the question is absolutely convergent, which additionally implies convergence of the original series.

(k) The comparison test is appropriate here. Note that for all integers $n\geq 2$, we have:

$$\frac{1}{n^2\log(n)}\leq\frac{1}{n^2}.$$

Since the series definition of $\zeta(2)$ converges by Question 4(b), we have by the comparison test that the original series converges. All terms of the series are positive, so the series converges absolutely.

(l) The comparison test is appropriate here. Note that for all positive integers $n \geq 1$, we have:

$$\frac{n^2+2n}{n^3+3n^2+1} \geq \frac{n^2+2n}{n^3+3n^2+n} = \frac{n+2}{n^2+3n+1} \geq \frac{n+2}{n^2+3n+2} = \frac{n+2}{(n+2)(n+1)} = \frac{1}{n+1}.$$

The harmonic series diverges, hence the original series diverges too.

(m) The comparison test is appropriate here. Observe that:

$$\frac{1}{1+n^2} \leq \frac{1}{n^2}.$$

Since the series definition of $\zeta(2)$ converges by Question 4(b), we have by the comparison test that the original series converges. All terms of the series are positive, so the series converges absolutely.

(n) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{a^{2n+3}}{2n+3} \cdot \frac{2n+1}{a^{2n+1}} = a^2 \lim_{n\to\infty} \frac{2+1/n}{2+3/n} = a^2.$$

Hence if $a^2 < 1$, i.e. $a < 1$, we have that the series converges. All terms are positive in this case, so the series converges absolutely too. If $a^2 > 1$, i.e. $a > 1$, we have that the series diverges.

The case $a = 1$ cannot be dealt with by the ratio test. Instead, we use the comparison test. Observe that:

$$\frac{1}{2n+1} \geq \frac{1}{2n+2} = \frac{1}{2} \cdot \frac{1}{n+1}.$$

But the harmonic series diverges, hence the series with $a = 1$ must diverge too.

(o) The comparison test is appropriate here. Since $|\sin(\omega n)| \leq 1$, we have:

$$n^p |\sin(\omega n)| \leq n^p.$$

For $p < -1$, by Question 4(b), the series with terms on the right hand side converges. Hence the original series is absolutely convergent for $p < -1$, and hence convergent.[3]

(p) This is a bit of a trick question; note that $\cos((2n-1)\pi) = -1$. Hence the series is just the negative harmonic series, which diverges.

(q) The divergence test is appropriate here (the numerator of the terms grows exponentially, while the denominator of the terms grows only polynomially). We have:

$$\lim_{n\to\infty} \frac{2^n}{n \log(n)} = \lim_{x\to\infty} \frac{e^{x\log(2)}/x}{\log(x)},$$

since the limit of this sequence must equal the limit of the real function on the right hand side. The numerator on the right hand side approaches ∞, because exponential growth is faster than polynomial growth, a fact we proved on the previous Examples Sheet. The denominator on the right hand side also approaches ∞. Hence by L'Hôpital's rule, this limit equals:

$$\lim_{x\to\infty} \frac{\log(2)e^{x\log(2)}/x - e^{x\log(2)}/x^2}{1/x} = \lim_{x\to\infty} \left(\log(2) - \frac{1}{x}\right) e^{x\log(2)} = \infty.$$

Hence the nth term of the series does not converge, so the original series is divergent.

[3] Note that the case $p = -1$ is actually *very* hard to deal with - it requires another test, called Dirichlet's test, which you can look up online. It turns out that the series converges when $p = -1$, but is not absolutely convergent. The series diverges when $p > -1$.

(r) The comparison test is appropriate here. Observe that:

$$\frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2},$$

since $|\sin(n)| \leq 1$. The series with terms on the right hand side converges by Question 4(b), hence the original series is absolutely convergent. Absolute convergence implies convergence, so the original series is also convergent.

(s) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{(n+1)^3}{\log^{n+1}(2)} \cdot \frac{\log^n(2)}{n^3} = \frac{1}{\log(2)} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3 = \frac{1}{\log(2)} < 1.$$

Hence the series is convergent. All terms of the series are positive, so the series converges absolutely.

(t) We can use a nice trick here. Observe that:

$$\sqrt{n^4 + a^2} - n^2 = \frac{(\sqrt{n^4 + a^2} - n^2)(\sqrt{n^4 + a^2} + n^2)}{\sqrt{n^4 + a^2} + n^2} = \frac{n^4 + a^2 - n^4}{\sqrt{n^4 + a^2} + n^2} = \frac{a^2}{\sqrt{n^4 + a^2} + n^2}.$$

Next, note that:

$$\frac{a^2}{\sqrt{n^4 + a^2} + n^2} \leq \frac{a^2}{\sqrt{n^4} + n^2} = \frac{a^2}{2n^2}.$$

But the series with these terms converges by Question 4(b), hence by the comparison test the original series in this question converges. Note that all the terms of the original series are also positive, so the series converges absolutely.

(u) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{n+1}{2^{n+1} - 1} \cdot \frac{2^n - 1}{n} = \lim_{n\to\infty} \frac{1/2 - 1/2^{n+1}}{1 - 1/2^{n+1}} \cdot \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \frac{1}{2} < 1.$$

Hence, the series converges. All terms of the series are positive, so the series converges absolutely.

(v) The ratio test is appropriate here. Observe that:

$$\begin{aligned}
\lim_{n\to\infty} \frac{((n+1)!)^3 e^{3(n+1)}}{(3(n+1))!} \cdot \frac{(3n)!}{(n!)^3 e^{3n}} &= e^3 \lim_{n\to\infty} \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \\
&= e^3 \lim_{n\to\infty} \frac{1 + 1/n}{3 + 3/n} \cdot \frac{1 + 1/n}{3 + 2/n} \cdot \frac{1 + 1/n}{3 + 1/n} \\
&= \frac{e^3}{27}.
\end{aligned}$$

Since $e < 3$, this fraction is less than 1. Hence, the series converges. All terms of the series are positive, so the series converges absolutely.

Taylor series

9. Carefully state *Taylor's theorem*, giving Lagrange's formula for the remainder term. Hence, obtain the first three non-zero terms in the Taylor series of $\log(x)$ about $x = 1$ by direct differentiation. Using this expansion, together with Lagrange's form of the remainder, show that:

$$|\log(3/2) - 5/12| \leq 1/64,$$

and hence give an approximation of $\log(3/2)$ valid to one decimal place.

➻ **Solution:** Taylor's theorem states that for a real function f which is $(n+1)$-times differentiable about the point x_0, we have:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_{n+1},$$

where the remainder R_{n+1} is given by:

$$R_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

for some point ξ between x and x_0.

For $\log(x)$, we have:

$$\frac{d}{dx}\log(x) = \frac{1}{x}, \qquad \frac{d^2}{dx^2}\log(x) = -\frac{1}{x^2}, \qquad \frac{d^3}{dx^3}\log(x) = \frac{2}{x^3}, \qquad \frac{d^4}{dx^4}\log(x) = -\frac{6}{x^4}.$$

Hence the Taylor series about $x = 1$, up to the third non-zero term, is given by:

$$\begin{aligned}\log(x) &= \log(1) + \frac{1}{1}(x-1) - \frac{1}{1^2 \cdot 2!}(x-1)^2 + \frac{2}{1^3 \cdot 3!}(x-1)^3 + R_4 \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + R_4,\end{aligned}$$

where the remainder term is given by:

$$R_4 = -\frac{1}{4\xi^4}(x-1)^4,$$

for some ξ between x and 1.

Put $x = 3/2$ in the above formula. Then:

$$\log\left(\frac{3}{2}\right) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 + R_4 = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} + R_4 = \frac{5}{12} + R_4.$$

It follows that:

$$\left|\log\left(\frac{3}{2}\right) - \frac{5}{12}\right| = |R_4|.$$

But note that for ξ satisfying $1 < \xi < 3/2$, we have $3/2 < 1/\xi < 1$, so:

$$|R_4| = \frac{1}{4\xi^4} \cdot \frac{1}{2^4} \leq \frac{1}{4(1)^4} \cdot \frac{1}{2^4} = \frac{1}{64},$$

as required. To one decimal place, $5/12$ is 0.4, which is within $1/64$ of $\log(3/2)$, so is a correct approximation to $\log(3/2)$ to within one decimal place.

10. Write down the Taylor series about $x = 0$ for the following functions, finding their range of convergence by appropriate tests in each case:

(a) e^x, (b) $\log(1+x)$, (c) $\sin(x)$, (d) $\cos(x)$, (e) $\sinh(x)$, (f) $\cosh(x)$, (g) $(1+x)^a$,

where in the final part $a \in \mathbb{R}$ is any real number (ignore endpoints of the range of convergence in the final case, where convergence is subtle). What happens when a is a non-negative integer? Learn these series off by heart, and get your supervision partner to test you on them.

Solution:

(a) We have:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Consider:

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

Hence, by the ratio test, this series is absolutely convergent for all values of x, and hence convergent for all values of x.

(b) We have:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

Consider:

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} \cdot \frac{n}{|x|^n} = |x| \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = |x|.$$

By the ratio test, we see that the series is absolutely convergent for $|x| < 1$. Hence, the series converges for $|x| < 1$.

On the boundaries, we have $x = -1$ is the negative harmonic series $-1 - 1/2 - 1/3 - 1/4 - \ldots$, which diverges. On the other hand, at $x = 1$, we have the alternating series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

which has $1/n \to 0$ as $n \to \infty$, with $1/n$ monotonically decreasing. Hence the alternating series test implies that the series converges at $x = 1$. Thus the range of convergence is $-1 <\leq x \leq 1$.

(c) We have:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Consider:

$$\lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{|x|^{2n+1}} = |x|^2 \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = 0.$$

Hence, by the ratio test this series is absolutely convergent (and thus convergent) for all values of x.

(d) We have:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Consider:

$$\lim_{n\to\infty} \frac{|x|^{2n+2}}{(2n+2)!} \frac{(2n)!}{|x|^{2n}} = |x|^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

Hence, by the ratio test this series is absolutely convergent (and thus convergent) for all values of x.

(e) We have:

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Similarly to part (c), this is convergent for all values of x.

(f) We have:

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Similarly to part (d), this is convergent for all values of x.

(g) We have:

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots = 1 + \sum_{n=1}^{\infty} \frac{a(a-1)(a-2)...(a-n+1)}{n!}x^n.$$

Consider:

$$\lim_{n\to\infty} \frac{|a(a-1)...(a-n)||x|^{n+1}}{(n+1)!} \frac{n!}{|a(a-1)...(a-n+1)||x|^n} = |x| \lim_{n\to\infty} \frac{|a-n|}{n+1} = |x| \lim_{n\to\infty} \frac{|1-a/n|}{1+1/n} = |x|.$$

Hence this series is absolutely convergent, and thus convergent, if $|x| < 1$. It may or may not converge at the end-points (it depends on the value of a, mostly, and is rather complicated - not something we need to worry about usually).

When a is a non-negative integer, the series terminates after finitely many terms (all terms afterwards are zero). This correspond to the case of the standard binomial expansion of $(1+x)^a$.

11. *Without differentiating*, find the first three terms in the Taylor series of the following functions. [*Note: there are lots of examples from past papers here to practise with, but if you are getting bored, we can do some in the supervision together. The next few questions, 12-17, have more of a problem-solving element.*]

(a) $\dfrac{1}{\sqrt{1+x}}$ about $x=0$;

(b) $\dfrac{1}{(x^2+2)^{3/2}}$ about $x=0$;

(c) $\tan(x)$ about $x=0$;

(d) $\log(\cos(x))$ about $x=0$;

(e) $\arcsin(x)$ about $x=0$;

(f) $\arctan(x)$ about $x=1$;

(g) $(\cosh(x))^{-1/2}$ about $x=0$;

(h) $e^{\sin(x)}$ about $x=\pi/2$;

(i) $x\sinh(x^2)$ about $x=0$;

(j) $\log(1+\log(1+x))$ about $x=0$;

(k) $\sin^6(x)$ about $x=0$;

(l) $\dfrac{\cosh(x)}{\cos(x)}$ about $x=0$;

(m) $\cosh(\log(x))$ about $x=2$;

(n) $\log(2-e^x)$ about $x=0$;

(o) $\dfrac{\sin(x)}{\sinh(x)}$ about $x=0$;

(p) $\sinh(\log(x))$ about $x=1$;

(q) $\sin\left(\dfrac{\pi e^x}{2}\right)$ about $x=0$;

(r) $\dfrac{\sinh(x+1)}{x+2}$ about $x=-1$;

(s) $\dfrac{\log(1+x^3)}{\cosh(x)}$ about $x=0$;

(t) $\dfrac{\cosh(x)}{\sqrt{1+x^2}}$ about $x=0$;

(u) $\dfrac{e^{-x^2}}{\cosh(x)}$ about $x=0$;

(v) $\dfrac{\log(2+x)}{2-x}$ about $x=0$;

(w) $\log(\cosh(x))$ about $x=0$;

(x) $\cosh(\sqrt{x})$ about $x=2$;

(y) $\dfrac{\sin(x)}{(1+x)^2}$ about $x=0$;

(z) $\dfrac{x\sin(x)}{\log(1+x^2)}$ about $x=0$;

(a′) $\cos\left(\sqrt{\dfrac{\pi^2}{16}+x}\right)$ about $x=0$;

(b′) $\log((2+x)^3)$ about $x=0$.

Solution:

(a) Using the binomial expansion, we have:

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^2 + \cdots = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots .$$

(b) Using the binomial expansion, we have:

$$\begin{aligned}\frac{1}{(x^2+2)^{3/2}} = \frac{1}{2^{3/2}}\left(1+\frac{x^2}{2}\right)^{-3/2} &= \frac{1}{2^{3/2}}\left(1 - \frac{3x^2}{4} + \frac{1}{2!}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x^2}{2}\right)^2 + \cdots\right)\\ &= \frac{1}{2^{3/2}}\left(1 - \frac{3x^2}{4} + \frac{15}{32}x^4 + \cdots\right)\\ &= \frac{1}{2^{3/2}} - \frac{3x^2}{4\cdot 2^{3/2}} + \frac{15}{32\cdot 2^{3/2}}x^4 + \cdots .\end{aligned}$$

(c) Observe that:

$$
\begin{aligned}
\tan(x) = \frac{\sin(x)}{\cos(x)} &= \frac{x - x^3/3! + x^5/5! - \cdots}{1 - x^2/2! + x^4/4! - \cdots} \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^{-1} \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(-\frac{x^2}{2!} + \cdots\right)^2 + \cdots\right) \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{5}{24}x^4 + \cdots\right) \\
&= x + \left(\frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(\frac{5}{24} - \frac{1}{2\cdot 6} + \frac{1}{5!}\right)x^5 + \cdots \\
&= x + \frac{1}{3}x^3 + \left(\frac{25}{120} - \frac{10}{120} + \frac{1}{120}\right)x^5 + \cdots \\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots .
\end{aligned}
$$

(d) Observe that:

$$
\begin{aligned}
\log(\cos(x)) &= \log\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \\
&= \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 + \frac{1}{3}\left(-\frac{x^2}{2!} + \cdots\right)^3 \\
&= -\frac{1}{2}x^2 + \left(\frac{1}{4!} - \frac{1}{8}\right)x^4 + \left(-\frac{1}{6!} + \frac{2}{2\cdot 2!\cdot 4!} - \frac{1}{3\cdot 2^3}\right)x^6 + \cdots \\
&= -\frac{1}{2}x^2 - \frac{1}{12}x^4 + \left(-\frac{1}{720} + \frac{1}{48} - \frac{1}{24}\right)x^6 + \cdots \\
&= -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 + \cdots .
\end{aligned}
$$

(e) This one involves a bit of a trick. Since $\arcsin(x)$ differentiates to $1/\sqrt{1-x^2}$ (we can prove this using the reciprocal rule, for example, just as you did on Examples Sheet 4), we have:

$$
\begin{aligned}
\arcsin(x) = \int \frac{dx}{\sqrt{1-x^2}} &= \int \left(1 - x^2\right)^{-1/2}\, dx \\
&= \int \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + \cdots\right) dx \\
&= x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots .
\end{aligned}
$$

There is no constant of integration, because $\arcsin(0) = 0$.

(f) This is similar to the previous part, but we need to be careful because the expansion is around $x = 1$ instead of around $x = 0$. We have:

$$\arctan(x) = \int \frac{dx}{1+x^2}.$$

Rewriting the integrand in terms of $x - 1$, the small quantity we wish to expand around, we have:

$$\frac{1}{1+x^2} = \frac{1}{1+(x-1)^2+2x-1} = \frac{1}{(x-1)^2+2x} = \frac{1}{(x-1)^2+2(x-1)+2} = \frac{1}{2}\left(1+(x-1)+\frac{(x-1)^2}{2}\right)^{-1}.$$

Performing the binomial expansion of the integrand, we have:

$$\begin{aligned}\frac{1}{2}\left(1+(x-1)+\frac{(x-1)^2}{2}\right)^{-1} &= \frac{1}{2}\left(1-\left((x-1)-\frac{(x-1)^2}{2}+\cdots\right)+((x-1)+\cdots)^2+\cdots\right)\\ &= \frac{1}{2}\left(1-(x-1)+\frac{3}{2}(x-1)^2+\cdots\right)\\ &= \frac{1}{2}-\frac{(x-1)}{2}+\frac{3(x-1)^2}{4}+\cdots.\end{aligned}$$

Integrating term by term, we have:

$$\frac{(x-1)}{2}-\frac{(x-1)^2}{4}+\frac{3(x-1)^3}{12}+\cdots,$$

up to a constant of integration. At $x = 1$, the left hand side is $\arctan(1) = \pi/4$, so the constant of integration must be $\pi/4$. This gives:

$$\arctan(x) = \frac{\pi}{4}+\frac{1}{2}(x-1)-\frac{1}{4}(x-1)^2+\cdots.$$

(g) Observe that:

$$\begin{aligned}(\cosh(x))^{-1/2} &= \left(1+\frac{x^2}{2}+\frac{x^4}{4!}+\cdots\right)^{-1/2}\\ &= \left(1-\frac{1}{2}\left(\frac{x^2}{2}+\frac{x^4}{4!}+\cdots\right)+\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x^2}{2}+\cdots\right)^2+\cdots\right)\\ &= 1-\frac{1}{4}x^2+\left(-\frac{1}{2\cdot 4!}+\frac{3}{2!\cdot 2^2\cdot 2^2}\right)x^4+\cdots\\ &= 1-\frac{1}{4}x^2+\frac{7}{96}x^4+\cdots.\end{aligned}$$

(h) Note that $\sin(x) = \sin(x - \pi/2 + \pi/2) = \sin(x - \pi/2)\cos(\pi/2) + \cos(x - \pi/2)\sin(\pi/2) = \cos(x - \pi/2)$. Hence we have:

$$\begin{aligned} e^{\sin(x)} = e^{\cos(x-\pi/2)} &= \exp\left(1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + \cdots\right) \\ &= e \cdot \exp\left(-\frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + \cdots\right) \\ &= e\left(1 + \left(-\frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + \cdots\right) + \frac{1}{2!}\left(-\frac{(x-\pi/2)^2}{2!} + \cdots\right)^2 + \cdots\right) \\ &= e\left(1 - \frac{(x-\pi/2)^2}{2} + \left(\frac{1}{4!} + \frac{1}{8}\right)(x-\pi/2)^4 + \cdots\right) \\ &= e - \frac{e}{2}(x-\pi/2)^2 + \frac{e}{6}(x-\pi/2)^4 + \cdots . \end{aligned}$$

Note that in the second line, we needed to factor out e, because we know the expansion of e^u where u is small, but we do not know the expansion of e^{1+u} where u is small.

(i) We have:

$$x\sinh(x^2) = x\left(x^2 + \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} + \cdots\right) = x^3 + \frac{1}{6}x^7 + \frac{1}{120}x^{11} + \cdots .$$

(j) Observe that:

$$\begin{aligned} \log(1 + \log(1+x)) &= \log\left(1 + x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) - \frac{1}{2}\left(x - \frac{x^2}{2} + \cdots\right)^2 + \frac{1}{3}(x + \cdots)^3 + \cdots \\ &= x + \left(-\frac{1}{2} - \frac{1}{2}\right)x^2 + \left(\frac{1}{3} + \frac{2}{4} + \frac{1}{3}\right)x^3 + \cdots \\ &= x - x^2 + \frac{7}{6}x^3 + \cdots . \end{aligned}$$

(k) Combining the expansion of $\sin(x)$ with the binomial expansion, we have:

$$\begin{aligned} \sin^6(x) &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^6 \\ &= x^6\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)^6 \\ &= x^6\left(1 + 6\left(-\frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) + 15\left(-\frac{x^2}{3!} + \cdots\right)^2 + \cdots\right) \\ &= x^6 - x^8 + \left(\frac{6}{5!} + \frac{15}{(3!)^2}\right)x^{10} + \cdots \\ &= x^6 - x^8 + \frac{7}{15}x^{10} + \cdots . \end{aligned}$$

(l) Observe that:

$$\begin{aligned}
\frac{\cosh(x)}{\cos(x)} &= \frac{1 + x^2/2! + x^4/4! + \cdots}{1 - x^2/2! + x^4/4! + \cdots} \\
&= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^{-1} \\
&= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(-\frac{x^2}{2!} + \cdots\right)^2 + \cdots\right) \\
&= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{5}{24}x^4 + \cdots\right) \\
&= 1 + x^2 + \left(\frac{5}{24} + \frac{1}{4} + \frac{1}{4!}\right)x^4 + \cdots \\
&= 1 + x^2 + \frac{1}{2}x^4 + \cdots .
\end{aligned}$$

(m) In this problem, we want to expand around $x = 2$. Thus we should rewrite everything in terms of the 'small' quantity $x - 2$. We shall call this $u = x - 2$ for convenience. Note that:

$$\begin{aligned}
\cosh(\log(x)) = \cosh(\log(x - 2 + 2)) &= \cosh(\log(u + 2)) \\
&= \cosh(\log(2) + \log(1 + u/2)) \\
&= \cosh\left(\log(2) + \frac{u}{2} - \frac{u^2}{8} + \frac{u^3}{24} + \cdots\right).
\end{aligned}$$

This gets even more messy now. We want to expand something of the form $\cosh(\log(2) + v)$, where v is small. Using hyperbolic compound angle identities, we have:

$$\cosh(\log(2) + v) = \cosh(\log(2))\cosh(v) + \sinh(\log(2))\sinh(v).$$

For simplicity, also note that $\cosh(\log(2)) = \frac{1}{2}(2 + 1/2) = 5/4$ and $\sinh(\log(2)) = \frac{1}{2}(2 - 1/2) = 3/4$. Expanding, we now have:

$$\begin{aligned}
\cosh(\log(x)) &= \frac{5}{4}\cosh\left(\frac{u}{2} - \frac{u^2}{8} + \frac{u^3}{24} + \cdots\right) + \frac{3}{4}\sinh\left(\frac{u}{2} - \frac{u^2}{8} + \frac{u^3}{24} + \cdots\right) \\
&= \frac{5}{4}\left(1 + \frac{1}{2}\left(\frac{u}{2} - \frac{u^2}{8}\cdots\right)^2 + \cdots\right) + \frac{3}{4}\left(\left(\frac{u}{2} - \frac{u^2}{8} + \cdots\right) + \frac{1}{3!}\left(\frac{u}{2} + \cdots\right)^3 + \cdots\right) \\
&= \frac{5}{4} + \frac{3u}{8} + \left(\frac{5}{4}\cdot\frac{1}{8} - \frac{3}{32}\right)u^2 + \cdots \\
&= \frac{5}{4} + \frac{3}{8}(x - 2) + \frac{1}{16}(x - 2)^2 + \cdots .
\end{aligned}$$

(n) Expanding the exponential first, we have:

$$\begin{aligned}
\log(2-e^x) &= \log\left(2-1-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\cdots\right) \\
&= \log\left(1-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\cdots\right) \\
&= \left(-x-\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right)-\frac{1}{2}\left(-x-\frac{x^2}{2!}+\cdots\right)^2+\frac{1}{3}\left(-x+\cdots\right)^3+\cdots \\
&= -x+\left(-\frac{1}{2}-\frac{1}{2}\right)x^2+\left(-\frac{1}{3!}-\frac{2}{2\cdot 2}-\frac{1}{3}\right)x^3+\cdots \\
&= -x-x^2-x^3+\cdots .
\end{aligned}$$

(o) We have:

$$\begin{aligned}
\frac{\sin(x)}{\sinh(x)} &= \frac{x-x^3/3!+x^5/5!+\cdots}{x+x^3/3!+x^5/5!+\cdots} \\
&= \left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)^{-1} \\
&= \frac{1}{x}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)\left(1+\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)^{-1} \\
&= \left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)\left(1-\left(\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)+\left(\frac{x^2}{3!}+\cdots\right)^2+\cdots\right) \\
&= \left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)\left(1-\frac{x^2}{3!}+\left(\frac{1}{3!^2}-\frac{1}{5!}\right)x^4+\cdots\right) \\
&= \left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)\left(1-\frac{x^2}{3!}+\frac{7}{360}x^4+\cdots\right) \\
&= 1-\frac{2}{3!}x^2+\left(\frac{7}{360}+\frac{1}{36}+\frac{1}{120}\right)x^4+\cdots \\
&= 1-\frac{1}{3}x^2+\frac{1}{18}x^4+\cdots .
\end{aligned}$$

(p) In this problem, we want to expand around $x = 1$. Thus we should rewrite everything in terms of the 'small' quantity $x - 1$. We shall call this $u = x - 1$ for convenience. Note that:

$$\begin{aligned}
\sinh(\log(x)) = \sinh(\log(x-1+1)) &= \sinh(\log(1+u)) \\
&= \sinh\left(u - \frac{u^2}{2} + \frac{u^3}{3} + \cdots\right) \\
&= \left(u - \frac{u^2}{2} + \frac{u^3}{3} + \cdots\right) + \frac{1}{3!}\left(u - \frac{u^2}{2} + \cdots\right)^3 + \frac{1}{5!}(u + \cdots)^5 \\
&= u - \frac{u^2}{2} + \left(\frac{1}{3} + \frac{1}{6}\right)u^3 + \cdots \\
&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \cdots .
\end{aligned}$$

(q) Expanding the exponential first, we have:

$$\sin\left(\frac{\pi e^x}{2}\right) = \sin\left(\frac{\pi}{2} + \frac{\pi}{2}x + \frac{\pi}{4}x^2 + \frac{\pi}{12}x^3 + \cdots\right).$$

Now, we need to expand an expression of the form $\sin(\pi/2 + u)$, where u is a small quantity. By the compound angle formula, we have:

$$\sin(\pi/2 + u) = \sin(\pi/2)\cos(u) + \cos(\pi/2)\sin(u) = \cos(u).$$

Hence, we can rewrite the above as:

$$\begin{aligned}
\sin\left(\frac{\pi e^x}{2}\right) &= \cos\left(\frac{\pi}{2}x + \frac{\pi}{4}x^2 + \frac{\pi}{12}x^3 + \cdots\right) \\
&= 1 - \frac{1}{2!}\left(\frac{\pi}{2}x + \frac{\pi}{4}x^2 + \cdots\right)^2 + \frac{1}{4!}\left(\frac{\pi}{2}x + \cdots\right)^4 + \cdots \\
&= 1 - \frac{\pi^2}{8}x^2 - \frac{\pi^2}{8}x^3 + \cdots .
\end{aligned}$$

(r) In this problem, we want to expand around $x = -1$. Thus we should rewrite everything in terms of the 'small' quantity $x + 1$. We shall call this $u = x + 1$ for convenience. Note that:

$$\begin{aligned}
\frac{\sinh(x+1)}{x+2} &= \frac{\sinh(u)}{1+u} \\
&= \sinh(u)(1+u)^{-1} \\
&= \left(u + \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots\right)(1 - u + u^2 - u^3 + \cdots) \\
&= u - u^2 + \frac{7}{6}u^3 + \cdots \\
&= (x+1) - (x+1)^2 + \frac{7}{6}(x+1)^3 + \cdots .
\end{aligned}$$

(s) Observe that:

$$\begin{aligned}
\frac{\log(1+x^3)}{\cosh(x)} &= \frac{x^3 - x^6/2 + x^9/3 + \cdots}{1 + x^2/2! + x^4/4! + \cdots} \\
&= \left(x^3 - \frac{x^6}{2} + \frac{x^9}{3} + \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^{-1} \\
&= \left(x^3 - \frac{x^6}{2} + \frac{x^9}{3} + \cdots\right)\left(1 - \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(\frac{x^2}{2!} + \cdots\right)^2 + \cdots\right) \\
&= \left(x^3 - \frac{x^6}{2} + \frac{x^9}{3} + \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots\right) \\
&= x^3 - \frac{1}{2}x^5 - \frac{1}{2}x^6 + \cdots .
\end{aligned}$$

(t) Observe that:

$$\begin{aligned}
\frac{\cosh(x)}{\sqrt{1+x^2}} = \cosh(x)(1+x^2)^{-1/2} &= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \cdots\right) \\
&= 1 + \left(\frac{3}{8} - \frac{1}{4} + \frac{1}{4!}\right)x^4 + \left(-\frac{5}{16} + \frac{3}{16} - \frac{1}{4!\cdot 2} + \frac{1}{6!}\right)x^6 + \cdots \\
&= 1 + \frac{1}{6}x^4 - \frac{13}{90}x^6 + \cdots .
\end{aligned}$$

(u) Observe that:

$$\begin{aligned}
\frac{e^{-x^2}}{\cosh(x)} &= \frac{1 - x^2 + x^4/2! - x^6/3! + \cdots}{1 + x^2/2! + x^4/4! + \cdots} \\
&= \left(1 - x^2 + \frac{x^4}{2!} + \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^{-1} \\
&= \left(1 - x^2 + \frac{x^4}{2!} + \cdots\right)\left(1 - \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 + \cdots\right) \\
&= \left(1 - x^2 + \frac{x^4}{2!} + \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots\right) \\
&= 1 - \frac{3}{2}x^2 + \left(\frac{5}{24} + \frac{1}{2} + \frac{1}{2}\right)x^4 + \cdots \\
&= 1 - \frac{3}{2}x^2 + \frac{29}{24}x^4 + \cdots .
\end{aligned}$$

(v) Observe that:

$$\frac{\log(2+x)}{2-x} = \frac{\log(2)+\log(1+x/2)}{2(1-x/2)} = \frac{1}{2}\left(\log(2)+\frac{x}{2}-\frac{x^2}{4}+\cdots\right)\left(1-\frac{x}{2}\right)^{-1}$$
$$= \frac{1}{2}\left(\log(2)+\frac{x}{2}-\frac{x^2}{4}+\cdots\right)\left(1+\frac{x}{2}+\frac{x^2}{4}+\cdots\right)$$
$$= \frac{1}{2}\left(\log(2)+\left(\frac{1}{2}+\frac{1}{2}\log(2)\right)x+\left(\frac{1}{4}\log(2)+\frac{1}{4}-\frac{1}{4}\right)x^2+\cdots\right)$$
$$= \frac{1}{2}\log(2)+\frac{(1+\log(2))}{4}x+\frac{\log(2)}{8}x^2+\cdots.$$

(w) We have:

$$\log(\cosh(x)) = \log\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)$$
$$= \left(\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)-\frac{1}{2}\left(\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2+\frac{1}{3}\left(\frac{x^2}{2!}+\cdots\right)^3+\cdots$$
$$= \frac{1}{2}x^2+\left(\frac{1}{24}-\frac{1}{8}\right)x^4+\left(\frac{1}{6!}-\frac{1}{2!4!}+\frac{1}{3\cdot 8}\right)x^6+\cdots$$
$$= \frac{1}{2}x^2-\frac{1}{12}x^4+\frac{1}{45}x^6+\cdots.$$

(x) In this problem, we want to expand around $x=2$. Thus we should rewrite everything in terms of the 'small' quantity $x-2$. We shall call this $u=x-2$ for convenience. Note that:

$$\cosh(\sqrt{x}) = \cosh(\sqrt{2+x-2}) = \cosh(\sqrt{2+u}) = \cosh\left(\sqrt{2}(1+u/2)^{1/2}\right)$$

Expanding using the binomial theorem, we have:

$$\cosh(\sqrt{2}(1+u/2)^{1/2}) = \cosh\left(\sqrt{2}\left(1+\frac{u}{4}-\frac{u^2}{32}+\cdots\right)\right).$$

We don't the expansion of $\cosh(\sqrt{2}+v)$ for small v, so we now use the hyperbolic compound angle identities to give:

$$\cosh(\sqrt{2}+v) = \cosh(\sqrt{2})\cosh(v)+\sinh(\sqrt{2})\sinh(v).$$

Applying this to the above, we have:

$$\cosh\left(\sqrt{2}(1+u/2)^{1/2}\right) = \cosh(\sqrt{2})\cosh\left(\frac{\sqrt{2}u}{4}-\frac{\sqrt{2}u^2}{32}+\cdots\right)+\sinh(\sqrt{2})\sinh\left(\frac{\sqrt{2}u}{4}-\frac{\sqrt{2}u^2}{32}+\cdots\right)$$
$$= \cosh(\sqrt{2})\left(1-\frac{1}{2!}\left(\frac{\sqrt{2}u}{4}\right)^2+\cdots\right)+\sinh(\sqrt{2})\left(\frac{\sqrt{2}u}{4}-\frac{\sqrt{2}u^2}{32}+\cdots\right)$$
$$= \cosh(\sqrt{2})+\frac{\sqrt{2}\sinh(\sqrt{2})}{4}u-\left(\frac{\sqrt{2}\sinh(\sqrt{2})}{32}+\frac{\cosh(\sqrt{2})}{16}\right)u^2+\cdots$$
$$= \cosh(\sqrt{2})+\frac{\sqrt{2}\sinh(\sqrt{2})}{4}(x-2)-\left(\frac{\sqrt{2}\sinh(\sqrt{2})}{32}+\frac{\cosh(\sqrt{2})}{16}\right)(x-2)^2+\cdots.$$

(y) We have:

$$\frac{\sin(x)}{(1+x)^2} = \sin(x)(1+x)^{-2} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\left(1 - 2x + 3x^2 + \cdots\right)$$
$$= x - 2x^2 + \left(3 - \frac{1}{3!}\right)x^3 + \cdots$$
$$= x - 2x^2 + \frac{17}{6}x^3 + \cdots .$$

(z) We have:

$$\frac{x\sin(x)}{\log(1+x^2)} = \frac{x(x - x^3/3! + x^5/5! + \cdots)}{x^2 - x^4/2 + x^6/3 + \cdots}$$
$$= \frac{1 - x^2/3! + x^4/5! + \cdots}{1 - x^2/2 + x^4/3 + \cdots}$$
$$= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{3} + \cdots\right)^{-1}$$
$$= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)\left(1 - \left(-\frac{x^2}{2} + \frac{x^4}{3} + \cdots\right) + \left(-\frac{x^2}{2} + \cdots\right)^2 + \cdots\right)$$
$$= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)\left(1 + \frac{x^2}{2} - \frac{x^4}{12} + \cdots\right)$$
$$= 1 + \frac{1}{3}x^2 + \left(-\frac{1}{12} - \frac{1}{12} + \frac{1}{120}\right)x^4 + \cdots$$
$$= 1 + \frac{1}{3}x^2 - \frac{19}{120}x^4 + \cdots .$$

(a′) Using the binomial expansion first, we have:

$$\cos\left(\sqrt{\frac{\pi^2}{16} + x}\right) = \cos\left(\frac{\pi}{4}\left(1 + \frac{16x}{\pi^2}\right)^{1/2}\right)$$
$$= \cos\left(\frac{\pi}{4}\left(1 + \frac{8x}{\pi^2} - \frac{32x^2}{\pi^4} + \cdots\right)\right)$$
$$= \cos\left(\frac{\pi}{4} + \frac{2x}{\pi} - \frac{8x^2}{\pi^3} + \cdots\right).$$

We don't know the expansion of $\cos(\pi/4 + u)$, where u is small, so we now use the compound angle identities for the trigonometric functions:

$$\cos(\pi/4 + u) = \cos(\pi/4)\cos(u) - \sin(\pi/4)\sin(u) = \frac{1}{\sqrt{2}}\left(\cos(u) - \sin(u)\right).$$

Applying this to the above expansion, we have:

$$\begin{aligned}
\cos\left(\sqrt{\frac{\pi^2}{16}+x}\right) &= \frac{1}{\sqrt{2}}\left(\cos\left(\frac{2x}{\pi}-\frac{8x^2}{\pi^3}+\cdots\right)-\sin\left(\frac{2x}{\pi}-\frac{8x^2}{\pi^3}\right)\right) \\
&= \frac{1}{\sqrt{2}}\left(1-\frac{1}{2!}\left(\frac{2x}{\pi}+\cdots\right)^2-\left(\frac{2x}{\pi}-\frac{8x^2}{\pi^3}+\cdots\right)+\cdots\right) \\
&= \frac{1}{\sqrt{2}}\left(1-\frac{2x}{\pi}+\left(\frac{2}{\pi^3}-\frac{8}{\pi^2}\right)x^2+\cdots\right) \\
&= \frac{1}{\sqrt{2}}-\frac{\sqrt{2}x}{\pi}+\frac{\sqrt{2}(4-\pi)}{\pi^3}x^2+\cdots.
\end{aligned}$$

(b′) Finally, we have:

$$\begin{aligned}
\log((2+x)^3) = 3\log(2+x) &= 3\log(2)+3\log(1+x/2) \\
&= 3\log(2)+3\left(\frac{x}{2}-\frac{x^2}{8}+\cdots\right) \\
&= 3\log(2)+\frac{3x}{2}-\frac{3x^2}{8}+\cdots.
\end{aligned}$$

We finished on an easy one!

12. *Without differentiating*, find the value of the thirty-second derivative of $\cos(x^4)$ at $x = 0$.

➼ **Solution:** Using the standard Taylor series for cosine about $x = 0$, we have:

$$\begin{aligned}\cos(x^4) &= 1 - \frac{(x^4)^2}{2!} + \frac{(x^4)^4}{4!} - \frac{(x^4)^6}{6!} + \frac{(x^4)^8}{8!} - \cdots \\ &= 1 - \frac{x^8}{2!} + \frac{x^{16}}{4!} + \frac{x^{24}}{6!} + \frac{x^{32}}{8!} - \cdots .\end{aligned}$$

But recall that the coefficient of x^{32} in the Taylor expansion of $\cos(x^4)$ about $x = 0$ is given by:

$$\frac{1}{32!}\frac{d^{32}}{dx^{32}}\cos(x^4)\bigg|_{x=0}.$$

Hence the value of the required derivative is:

$$\frac{d^{32}}{dx^{32}}\cos(x^4)\bigg|_{x=0} = \frac{32!}{8!}.$$

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13. Find the first three non-zero terms in a series approximation of $\log(1 + x + 2x^2) - \log(x^2)$ valid for $x \to \infty$.

➼ **Solution:** When $x \to \infty, 1/x \to 0$. Hence we should write everything in terms of $1/x$, and expand assuming that $1/x$ is close to zero.

We have:

$$\begin{aligned}\log(1 + x + 2x^2) - \log(x^2) &= \log(2x^2) + \log\left(1 + \frac{1}{2x} + \frac{1}{2x^2}\right) - \log(x^2) \\ &= \log(2) + \log\left(1 + \frac{1}{2x} + \frac{1}{2x^2}\right) \\ &= \log(2) + \left(\frac{1}{2x} + \frac{1}{2x^2}\right) - \frac{1}{2}\left(\frac{1}{2x} + \frac{1}{2x^2}\right)^2 + \cdots \quad \text{(Taylor series for } \log(1+u) \text{ about } u = 0\text{)} \\ &= \log(2) + \frac{1}{2x} + \frac{1}{2x^2} - \frac{1}{2}\left(\frac{1}{2x}\right)^2 + \cdots \\ &= \log(2) + \frac{1}{2x} + \frac{3}{8x^2} + \cdots .\end{aligned}$$

14. Let $f(x)$ be a function which can be expanded as a Taylor series. Find the first two terms in the Taylor series of the function $\log(1+f(x))$, assuming that $1+f(0)>0$, $f'(0)\neq 0$ and $f''(0)(1+f(0))\neq(f'(0))^2$. Why are these conditions necessary?

➺ **Solution:** The Taylor series of $f(x)$ about $x=0$ is given by:

$$f(x)=f(0)+f'(0)x+\frac{f^{(2)}(0)}{2!}x^2+\cdots .$$

Inserting this into $\log(1+f(x))$, we have:

$$\log\left(1+f(x)\right)=\log\left(1+f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots\right).$$

We know the Taylor series of $\log(1+u)$ about $u=0$. So we need to rewrite our logarithm in this form, where u is a quantity close to 0. To do so, we factor out $1+f(0)$ from the argument of the logarithm, giving:

$$\log\left(1+f(0)\right)+\log\left(1+\frac{f'(0)}{1+f(0)}x+\frac{f''(0)}{2!(1+f(0))}x^2+\cdots\right).$$

We can now expand the second logarithmic term in a Taylor series, since it is of the form $\log(1+u)$ where u is a quantity close to 0. We have:

$$\begin{aligned}
&\log\left(1+f(0)\right)+\left(\frac{f'(0)}{1+f(0)}x+\frac{f''(0)}{2!(1+f(0))}x^2+\cdots\right)-\frac{1}{2}\left(\frac{f'(0)}{1+f(0)}x+\cdots\right)^2+\cdots\\
&=\log(1+f(0))+\frac{f'(0)}{1+f(0)}x+\left(\frac{f''(0)}{2!(1+f(0))}-\frac{1}{2}\frac{(f'(0))^2}{(1+f(0))^2}\right)x^2+\cdots\\
&=\log(1+f(0))+\frac{f'(0)}{1+f(0)}x+\frac{1}{2}\left(\frac{f''(0)(1+f(0))-(f'(0))^2}{(1+f(0))^2}\right)x^2+\cdots .
\end{aligned}$$

We need $1+f(0)>0$ for the logarithm to exist. We need $f'(0)\neq 0$, else the first term in the expansion would vanish (and we would have to calculate to higher order!). Similarly, we need $f''(0)(1+f(0))\neq(f'(0))^2$, else the second term in the expansion would vanish (and we would have to calculate to higher order).

15. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ be the Taylor series of $f(x)$ about $x = 0$, with $a_0 > 0$, $a_1 \neq 0$, $a_1^2 \neq a_2 a_0$ and $a_1^2 \neq 4a_2 a_0$. Find the first three terms in the Taylor series of (a) $1/f(x)$ about $x = 0$; (b) $\sqrt{f(x)}$ about $x = 0$. Explain where you used the assumptions on the a_n in your answer.

➽ **Solution:** (a) We have:

$$\begin{aligned}
\frac{1}{f(x)} &= \frac{1}{a_0 + a_1 x + a_2 x^2 + \cdots} \\
&= \frac{1}{a_0}\left(1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \cdots\right)^{-1} \\
&= \frac{1}{a_0}\left(1 - \left(\frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \cdots\right) + \left(\frac{a_1}{a_0}x + \cdots\right)^2 + \cdots\right) \\
&= \frac{1}{a_0}\left(1 - \frac{a_1}{a_0}x + \left(\frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}\right)x^2 + \cdots\right) \\
&= \frac{1}{a_0} - \frac{a_1}{a_0^2}x + \frac{(a_1^2 - a_2 a_0)}{a_0^3}x^2 + \cdots .
\end{aligned}$$

The coefficients are all non-zero since $a_0 > 0$, $a_1 \neq 0$ and $a_1^2 \neq a_2 a_0$.

(b) We have:

$$\begin{aligned}
\sqrt{f(x)} &= \left(a_0 + a_1 x + a_2 x^2 + \cdots\right)^{1/2} \\
&= a_0^{1/2}\left(1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \cdots\right)^{1/2} \\
&= a_0^{1/2}\left(1 + \frac{1}{2}\left(\frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \cdots\right) + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{a_1}{a_0}x + \cdots\right)^2 + \cdots\right) \\
&= a_0^{1/2}\left(1 + \frac{a_1}{2a_0}x + \left(\frac{a_2}{2a_0} - \frac{a_1^2}{8a_0^2}\right)x^2 + \cdots\right) \\
&= a_0^{1/2} + \frac{a_1}{2a_0^{1/2}}x + \frac{(4a_2 a_0 - a_1^2)}{8a_0^{3/2}}x^2 + \cdots .
\end{aligned}$$

The coefficients are all non-zero since $a_0 > 0$ (indeed, this implies the square root of a_0 exists!), $a_1 \neq 0$ and $a_1^2 \neq 4a_2 a_0$.

16. (†) By considering a Taylor series expansion in each case, evaluate the limits:

$$\text{(a) } \lim_{x\to 0}\frac{\tan(x)-\tanh(x)}{\sinh(x)-x}, \qquad \text{(b) } \lim_{x\to 0}\left(\frac{\operatorname{cosec}(x)}{x^3}-\frac{\sinh(x)}{x^5}\right).$$

➾ **Solution:** (a) For the first limit, we observe that the denominator has the expansion:

$$\sinh(x)-x = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots - x = \frac{x^3}{3!}+\frac{x^5}{5!}+\cdots.$$

Meanwhile, tangent has the expansion:

$$\begin{aligned}\tan(x)=\frac{\sin(x)}{\cos(x)} &= \frac{x-x^3/3!+\cdots}{1-x^2/2!+\cdots}\\ &= \left(x-\frac{x^3}{3!}+\cdots\right)\left(1-\frac{x^2}{2!}+\cdots\right)^{-1}\\ &= \left(x-\frac{x^3}{3!}+\cdots\right)\left(1+\frac{x^2}{2!}+\cdots\right)\\ &= x+\left(\frac{1}{2!}-\frac{1}{3!}\right)x^3+\cdots\\ &= x+\frac{1}{3}x^3+\cdots.\end{aligned}$$

Similarly, hyperbolic tangent has the expansion:

$$\begin{aligned}\tanh(x)=\frac{\sinh(x)}{\cosh(x)} &= \frac{x+x^3/3!+\cdots}{1+x^2/2!+\cdots}\\ &= \left(x+\frac{x^3}{3!}+\cdots\right)\left(1+\frac{x^2}{2!}+\cdots\right)^{-1}\\ &= \left(x+\frac{x^3}{3!}+\cdots\right)\left(1-\frac{x^2}{2!}+\cdots\right)\\ &= x+\left(\frac{1}{3!}-\frac{1}{2!}\right)x^3+\cdots\\ &= x-\frac{1}{3}x^3+\cdots.\end{aligned}$$

Putting this altogether, we have:

$$\frac{\tan(x)-\tanh(x)}{\sinh(x)-x}=\frac{x+x^3/3+\cdots-x+x^3/3+\cdots}{x^3/3!+x^5/5!+\cdots}=\frac{2x^3/3+\cdots}{x^3/6+\cdots}=\frac{2/3+\cdots}{1/6+\cdots}.$$

In the limit as $x\to 0$, the terms we have neglected vanish. Thus the limit is:

$$\frac{2}{3}\cdot 6 = 4.$$

(b) We have:

$$\begin{aligned}
\frac{\operatorname{cosec}(x)}{x^3} - \frac{\sinh(x)}{x^5} &= \frac{1}{x^3 \sin(x)} - \frac{\sinh(x)}{x^5} \\
&= \frac{1}{x^3(x - x^3/3! + x^5/5! + ...)} - \frac{x + x^3/3! + x^5/5! + ...}{x^5} \\
&= \frac{1}{x^4}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)^{-1} - \frac{1}{x^4}\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) \\
&= \frac{1}{x^4}\left(1 - \left(-\frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) + \left(-\frac{x^2}{3!} + \cdots\right)^2 + \cdots\right) - \frac{1}{x^4}\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) \\
&= \frac{1}{x^4}\left(1 + \frac{x^2}{3!} + \left(\frac{1}{3!^2} - \frac{1}{5!}\right)x^4 + \cdots\right) - \frac{1}{x^4}\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) \\
&= \left(\frac{1}{3!^2} - \frac{1}{5!}\right) - \frac{1}{5!} + \cdots \\
&= \frac{1}{90} + \cdots .
\end{aligned}$$

The terms that are remaining tend to zero as $x \to 0$. Hence the required limit is $1/90$.

17.(a) Using the Taylor series for $\log(1+x)$, show that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots=\log(2)$.

(b) $(*)$ Hence, by an appropriate sequence of transformations of the series in part (a), show that:

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots=\frac{3}{2}\log(2).$$

(c) $(*)$ Comment on this result in relation to absolute convergence. [*Look up the Riemann rearrangement theorem afterwards!*]

Solution: (a) The Taylor series for $\log(1+x)$ is:

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots.$$

Letting $x=1$, we have:

$$\log(2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots,$$

as required.

(b) This is just a trick:

$$\begin{array}{ccccccccc}
1 & -\frac{1}{2} & +\frac{1}{3} & -\frac{1}{4} & +\frac{1}{5} & -\frac{1}{6} & +\frac{1}{7} & -\frac{1}{8}+\cdots & =\log(2) \\
 & 1 & & -\frac{1}{2} & & +\frac{1}{3} & & -\frac{1}{4}+\cdots & =\log(2) \\
 & \frac{1}{2} & & -\frac{1}{4} & & +\frac{1}{6} & & -\frac{1}{8}+\cdots & =\frac{1}{2}\log(2)
\end{array}$$

In the second line, we have just aligned the terms in a slightly different way. In the third line, we have multiplied everything in the second line by $1/2$. Now consider adding the first and third lines:

$$\begin{array}{ccccccccc}
1 & & +\frac{1}{3} & -\frac{1}{2} & +\frac{1}{5} & & +\frac{1}{7} & -\frac{1}{4}+\cdots & =\frac{3}{2}\log(2)
\end{array}$$

Hence, the result follows.

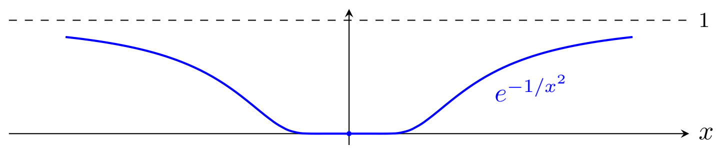
(c) This is allowed to occur here because the series is not *absolutely convergent*. If a series is absolutely convergent, we can rearrange its terms arbitrarily and still obtain the same infinite sum. If a series is not absolutely convergent, but still converges, then the *Riemann rearrangement theorem* says that we can rearrange its terms arbitrarily to obtain *any* value we wish via some rearrangement!

18. ($*$) Sketch the graph of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-1/x^2}$ for $x \neq 0$, and $f(0) = 0$. Show that this function is infinitely differentiable at $x = 0$, and determine its Taylor series at $x = 0$. Comment on the general utility of Taylor series.

Solution: As $x \to \pm\infty$, the graph goes to 1, since $-1/x^2 \to 0$. Further, the graph is everywhere positive, and since $-1/x^2 < 0$, we have $e^{-1/x^2} < 1$, so that the graph is everywhere less than 1 too. Stationary points occur when:

$$\frac{d}{dx} e^{-1/x^2} = \frac{2}{x^3} e^{-1/x^2} = 0,$$

which never happens, hence there are none. Thus the graph takes the 'bowl' shape given below:



Away from zero, we claim that the nth derivative of the function is of the form:

$$p_n\left(\frac{1}{x}\right) e^{-1/x^2},$$

where p_n is some polynomial. To prove this, we can use induction. Certainly the function itself is of this form, so the case $n = 0$ works. Assuming the result is true for $n = k$, we consider the case $n = k + 1$:

$$\frac{d}{dx}\left(p_k\left(\frac{1}{x}\right) e^{-1/x^2}\right) = -\frac{1}{x^2} p_k'\left(\frac{1}{x}\right) e^{-1/x^2} + \frac{2}{x^3} p_k\left(\frac{1}{x}\right) e^{-1/x^2} = \left(\frac{2}{x^3} p_k\left(\frac{1}{x}\right) - \frac{1}{x^2} p_k'\left(\frac{1}{x}\right)\right) e^{-1/x^2},$$

so indeed the $(k+1)$th derivative is also of this form, if the kth derivative is. We see that the polynomials that multiply e^{-1/x^2} satisfy the recurrence relation $2y^3 p_k(y) - y^2 p_k'(y) = p_{k+1}(y)$.

Next, we claim that the nth derivative of the function at $x = 0$ is 0. We shall prove this with induction too. Certainly this is true of the function itself, since $f(0) = 0$, so the case $n = 0$ works. Now assuming the result is true for $n = k$, we note that the $(k+1)$th derivative at $x = 0$ is given by:

$$\lim_{h \to 0}\left[\frac{p_k(1/h)e^{-1/h^2} - 0}{h}\right] = \lim_{h \to 0}\left[\frac{p_k(1/h)/h}{e^{1/h^2}}\right].$$

This limit is zero. To see why, consider making the substitution $u = 1/h$ in the limit, and taking the limit as $u \to \pm\infty$ and as $u \to -\infty$ to give the left and right limits as $h \to 0$. Both of these limits correspond to taking the limit of a polynomial $p_k(u)u$ over an exponential e^{u^2}, which by repeated use of L'Hôpital's rule will result in zero. Hence, the result is proved by induction.

It follows that indeed the function is infinitely differentiable at $x = 0$, as required.

Using the previous results, the Taylor series of this function about $x = 0$ is:

$$0 + 0x + \frac{0x^2}{2!} + \frac{0x^3}{3!} + \cdots \equiv 0.$$

Whoops! The Taylor series is identically equal to zero. This is a *useless* approximation to the function in a neighbourhood of $x = 0$. In general, a Taylor series may converge, but might not converge to the original function (here the Taylor series is convergent everywhere, because it is just a sum of zeroes, but only equals the original function when $x = 0$).

(†) Landau's big-O notation

19. Give the formal definition of Landau's big-O notation, '$f(x) = O(g(x))$ as $x \to x_0$', including the cases where $x_0 = \pm\infty$. Decide which of the following statements are true, justifying your reasoning with careful proofs:

(a) $x = O(x^2)$ as $x \to 0$, (b) $x^2 = O(x)$ as $x \to 0$, (c) $x = O(x^2)$ as $x \to \infty$, (d) $x^2 = O(x)$ as $x \to \infty$.

➾ **Solution:** We say that $f(x) = O(g(x))$ as $x \to x_0$ if there exist $\delta > 0, M > 0$ such that whenever $0 < |x - x_0| < \delta$, we have:

$$|f(x)| \leq M|g(x)|.$$

This definition is saying that there is a neighbourhood of x_0 where the size of $f(x)$ is at most some constant M times the size of $g(x)$.

In the case of $x_0 = \infty$, we say that $f(x) = O(g(x))$ as $x \to \infty$ if there exist $K > 0, M > 0$ such that whenever $x > K$, we have:

$$|f(x)| \leq M|g(x)|.$$

This definition is saying that if we take x sufficiently large, the size of $f(x)$ is at most some constant M times the size of $g(x)$. A similar definition applies when $x_0 = -\infty$.

(a) As x gets close to zero from above, we have that x^2 is smaller. So we expect the statement to be false. To prove it, note that if x is positive, we have $x < Mx^2$ if and only if:

$$0 < x(Mx - 1) \qquad \Rightarrow \qquad x > \frac{1}{M}.$$

In particular, there is no neighbourhood of $x = 0$ on which the inequality $x < Mx^2$ holds. Hence, the statement is indeed false.

(b) This statement is true. Take $\delta = 1$ and $M = 1$. Then if $|x| < 1$, we have:

$$|x^2| = |x|^2 = |x| \cdot |x| < 1 \cdot |x|.$$

Thus $x^2 = O(x)$ as $x \to 0$.

(c) This statement is true. Take $K = 1$ and $M = 1$. Then if $x > 1$, we have:

$$|x| = 1 \cdot |x| < x \cdot |x| = |x^2|,$$

since $x > 0$.

(d) Similarly to part (a), this statement is false. To prove it, note that if x is positive, we have $x^2 < Mx$ if and only if:

$$x(x - M) < 0 \qquad \Rightarrow \qquad x < M.$$

In particular, for any constant M, there is always a point at which the growth of x^2 surpasses the growth of x (at the point M itself). Hence, the statement is indeed false.

20. Give the leading terms in an approximation to each of the following functions in the given limits, indicating the leading behaviour of the remainder in Landau's big-O notation:

(a) $\dfrac{x^3+x}{x+2}$ as $x \to 0$, (b) $\dfrac{\cos(x)-1}{x^3}$ as $x \to 0$, (c) $\dfrac{1+2x+2x^2}{3x+3}$ as $x \to \infty$.

➾ **Solution:** We will now be less formal with our proofs, and use Landau's big O-notation more intuitively.

(a) Using the binomial expansion around $x = 0$, we have:

$$\begin{aligned}
\frac{x^3+x}{x+2} &= \frac{1}{2}(x^3+x)\left(1+\frac{x}{2}\right)^{-1} \\
&= \frac{1}{2}(x^3+x)\left(1-\frac{x}{2}+O(x^2)\right) \\
&= \frac{x}{2}+O(x^2).
\end{aligned}$$

(b) Using the Taylor expansion of cosine around $x = 0$, we have:

$$\begin{aligned}
\frac{\cos(x)-1}{x^3} &= \frac{1-x^2/2!+x^4/4!+O(x^6)-1}{x^3} \\
&= -\frac{1}{2x}+\frac{x}{4!}+O(x^3) \\
&= -\frac{1}{2x}+O(x).
\end{aligned}$$

(c) Here, $1/x$ is the small quantity we should expand in as $x \to \infty$. Using the binomial expansion, we have:

$$\begin{aligned}
\frac{1+2x+2x^2}{3x+3} &= \frac{1}{3x}\left(1+2x+2x^2\right)\left(1+\frac{1}{x}\right)^{-1} \\
&= \frac{1}{3x}\left(1+2x+2x^2\right)\left(1-\frac{1}{x}+\frac{1}{x^2}+O\left(\frac{1}{x^3}\right)\right) \\
&= \frac{1}{3x}\left(2x^2+2x-2x+2-2+1+O\left(\frac{1}{x}\right)\right) \\
&= \frac{1}{3x}\left(2x^2+O(1)\right) \\
&= \frac{2x}{3}+O\left(\frac{1}{x}\right).
\end{aligned}$$

21. Show that:

$$(x^3+x^2+1)^{1/3}-(x^2+x)^{1/2}=-\frac{1}{6}+\frac{1}{72x}+O\left(\frac{1}{x^2}\right) \qquad \text{as } x\to\infty.$$

➭ **Solution:** Since we are taking the limit as $x\to\infty$, we will expand in powers of $1/x$. We have:

$$\begin{aligned}
(x^3+x^2+1)^{1/3}-(x^2+x)^{1/2} &= x\left(1+\frac{1}{x}+\frac{1}{x^3}\right)^{1/3}-x\left(1+\frac{1}{x}\right)^{1/2}\\
&= x\left(1+\frac{1}{3}\left(\frac{1}{x}+\frac{1}{x^3}\right)-\frac{1}{9}\left(\frac{1}{x}+\frac{1}{x^3}\right)^2+O\left(\frac{1}{x^3}\right)\right)-x\left(1+\frac{1}{2x}-\frac{1}{8x^2}+O\left(\frac{1}{x^3}\right)\right)\\
&= \left(\frac{1}{3}-\frac{1}{2}\right)+\left(-\frac{1}{9}+\frac{1}{8}\right)\frac{1}{x}+O\left(\frac{1}{x^2}\right)\\
&= -\frac{1}{6}+\frac{1}{72x}+O\left(\frac{1}{x^2}\right),
\end{aligned}$$

as required.

Newton-Raphson root finding

22. Give an explanation of the Newton-Raphson algorithm for root finding, including an appropriate sketch. Under what general conditions is it guaranteed that Newton-Raphson will converge to the root of interest? Prove that, when it converges to the root of interest, the Newton-Raphson method enjoys *quadratic convergence*.

➾ **Solution:** Suppose that we want to find a specific root of the equation $f(x) = 0$. Let $x = x^*$ be an exact root. We might start with some guess $x = x_0$ which is close to the root of interest, $x = x^*$. Then:

$$0 = f(x^*) = f(x_0) + f'(x_0)(x^* - x_0) + \cdots$$

Assuming that we can neglect higher order terms, this suggests that a more accurate guess for x^* is given by:

$$f(x_0) + f'(x_0)(x^* - x_0) \approx 0 \qquad \Leftrightarrow \qquad x^* \approx x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This suggests the definition of an iterative process:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

which gets closer to x^* on each step. In general, it can be shown that the process will converge if: (i) $f'(x^*) \neq 0$; (ii) we start sufficiently close to the true root of interest.

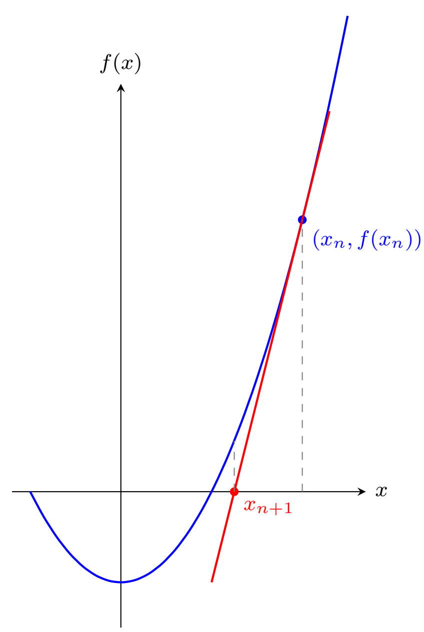
This iterative algorithm has a nice geometric interpretation. Imagine we are at the point x_n, with corresponding function value $f(x_n)$. The tangent to the graph at this point is then:

$$y - f(x_n) = f'(x_n)(x - x_n).$$

This implies that the tangent crosses the x-axis at the point satisfying:

$$-f(x_n) = f'(x_n)(x - x_n) \qquad \Leftrightarrow \qquad x = x_n - \frac{f(x_n)}{f'(x_n)},$$

which geometrically we hope is closer to a root of $f(x) = 0$. This is displayed graphically in the figure below.



Let us now define $\epsilon_{n+1} = x_{n+1} - x^*$ to be the difference between the $(n+1)$th Newton-Raphson iterate, and the true root. Then:

$$\epsilon_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - x^* = \epsilon_n - \frac{f(x_n)}{f'(x_n)}.$$

Performing a Taylor expansion of $f(x_n) = f(x_n - x^* + x^*) = f(\epsilon_n + x^*)$ and $f'(x_n) = f'(x_n - x^* + x^*) = f'(\epsilon_n + x^*)$, assuming that ϵ_n is small, we have:

$$\begin{aligned}
\epsilon_{n+1} &= \epsilon_n - \frac{\epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*) + \cdots}{f'(x^*) + \epsilon_n f''(x^*) + \cdots} && \text{(since } f(x^*) = 0\text{)} \\
&= \epsilon_n - \left(\epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*) + \cdots\right)\left(f'(x^*) + \epsilon_n f''(x^*) + \cdots\right)^{-1} \\
&= \epsilon_n - \frac{1}{f'(x^*)}\left(\epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*) + \cdots\right)\left(1 + \frac{\epsilon_n f''(x^*)}{f'(x^*)} + \cdots\right)^{-1} \\
&= \epsilon_n - \frac{1}{f'(x^*)}\left(\epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*) + \cdots\right)\left(1 - \frac{\epsilon_n f''(x^*)}{f'(x^*)} + \cdots\right) \\
&= \frac{1}{2}\epsilon_n^2 \frac{f''(x^*)}{f'(x^*)} + \cdots .
\end{aligned}$$

Hence, we see that $\epsilon_{n+1} \propto \epsilon_n^2$, so the error in the algorithm converges quadratically fast to zero.

23(a) Find the value of the first iterate of Newton-Raphson iteration for the function $f(x) = x - 2 + \log(x)$ with a starting guess of $x_0 = 1$.

(b) Find the value of the first and second iterates of Newton-Raphson iteration, valid to two decimal places, for the function $f(x) = x^2 - 2$ with a starting guess of $x_0 = 1$.

[*Both parts of this question are based on old (short) tripos questions, so try doing them*without *a calculator!*]

➻ Solution:

(a) Note that $f'(x) = 1 + 1/x$. Hence, we have:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1 - 2 + \log(1)}{1 + 1/1} = \frac{3}{2}.$$

(b) Note that $f'(x) = 2x$. Hence, we have:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^2 - 2}{2} = \frac{3}{2}.$$

Similarly, we have:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{3}{2} - \frac{(3/2)^2 - 2}{3} = \frac{17}{12}.$$

24. [*You may use a calculator for this question, but remember that you* won't *be able to use a calculator in the exam. Newton-Raphson questions will be more theoretical in the exams, like the next question, or involve easy calculations, like the previous question.*]

(a) Sketch the graph of $f(x) = x^3 - 3x^2 + 2$, indicating the coordinates of the turning points and the coordinates of the intersections with the x-axis.

(b) Use Newton-Raphson with an initial guess of $x_0 = 2.5$ to find an estimate of the largest root of the equation $f(x) = 0$, accurate to 5 decimal places. Draw a sketch showing the progress of the algorithm.

(c) To which roots (if any) does the algorithm converge if we instead start at: (i) $x_0 = 1.5$; (ii) $x_0 = 1.9$; (iii) $x_0 = 2$?

Solution: (a) The given function is a positive cubic. The stationary points occur when:

$$0 = f'(x) = 3x^2 - 6x = 3x(x-2) \qquad \Leftrightarrow \qquad x = 0, 2.$$

The point of inflection of the graph occurs when $0 = f''(x) = 6x - 6$, which is $x = 1$.

The graph intersects with the x-axis when $f(x) = 0$. Guessing a solution, we see that $x = 1$ works. This allows us to factorise the equation $f(x) = 0$ as:

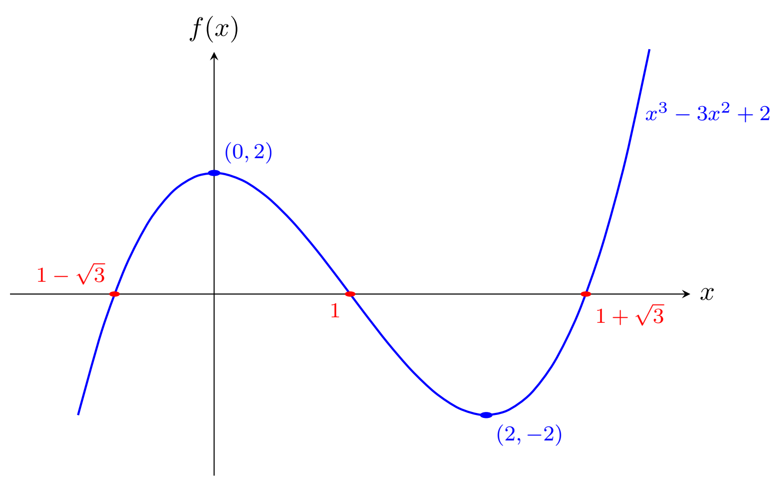
$$f(x) = (x-1)(x^2 - 2x - 2).$$

The quadratic factor can be further reduced by finding its roots using the quadratic formula. We have:

$$x_\pm = \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}.$$

Since $\sqrt{3} > 1$, we have that one of these roots is positive and one is negative. They occur symmetrically around $x = 1$.

We now have enough information to draw a fairly accurate graph:



(b) Let's now apply Newton-Raphson iteration to this function, starting with an initial guess of $x_0 = 2.5$. The iterative algorithm is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n^2 + 2}{3x_n^2 - 6x_n}.$$

Iterating using a calculator, we have:

$$x_1 = 2.8$$

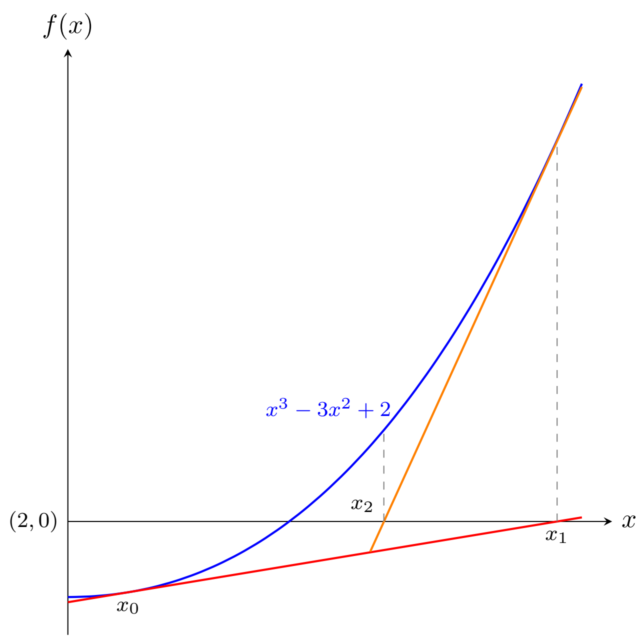
$$x_2 = 2.7357142857...$$

$$x_3 = 2.7320623734...$$

$$x_4 = 2.7320508076...$$

$$x_5 = 2.7320508075...$$

This shows that the correct root to 5 decimal places is 2.73205, which corresponds to the root $1+\sqrt{3}$. A sketch showing the first two steps of the algorithm is given below:



(c) In this part of the question, we experiment with some other starting values. In the case of $x_0 = 1$, the closest root is 1, because we just start at a root. In the case of $x_0 = 1.9$, the closest root is actually $1+\sqrt{3}$, since $1+\sqrt{3}-1.9 \approx 0.83$. In the case of $x_0 = 2$, the closest root is $1+\sqrt{3}$, but this is also a stationary point of the function. We shall see that the behaviour is not what we might expect in (ii)!

(i) Starting at $x_0 = 1$, we already have $f(x_0) = 0$. Hence the second term in the Newton-Raphson iteration formula gives zero, and all of our iterates are the same: $x_0 = 1, x_1 = 1, x_2 = 1, ...$. This is what we would expect for an exact root.

(ii) Applying the algorithm in this case, we have:

$$x_1 = -1.5578947...$$

$$x_2 = -1.0129154...$$

$$x_3 = -0.7816615...$$

$$x_4 = -0.7340488...$$

$$x_5 = -0.7320542...$$

$$x_6 = -0.7320508...$$

This shows that the algorithm is convergent to $1-\sqrt{3}$ in this case. We were not expecting this, because this is actually the furthest root from our starting point $x_0 = 1.9$! This demonstrates the *chaotic* nature of the Newton-Raphson algorithm - we will only get to a particular root if we start sufficiently close. This is explored in more detail in Question 26.

(iii) Starting at $x_0 = 2$, the algorithm fails completely. This is because $f'(x_0) = 0$ here, so the denominator in the Newton-Raphson iteration formula is zero, giving a singular result. We cannot use the algorithm with this starting guess.

25. The real function f is defined by $f(x) = x^2 - 2\epsilon x - 1$, where ϵ is a small positive parameter ($0 < \epsilon \ll 1$). Let x_i be the ith Newton-Raphson iterate, with a starting guess of $x_0 = 1$, and let x_* be the unique positive root satisfying $f(x_*) = 0$. By Taylor expansion, show that $|x_i - x_*| \propto \epsilon^{n_i}$, where: (a) $n_0 = 1$; (b) $n_1 = 2$; (c) $n_2 = 4$.

➾ **Solution:** First, let's find the unique positive root of $f(x) = 0$. Using the quadratic equation, the positive root will be:

$$x_* = \frac{2\epsilon + \sqrt{4\epsilon^2 + 4}}{2} = \epsilon + \sqrt{1 + \epsilon^2}.$$

Expanding for small ϵ, we have:

$$x_* = \epsilon + 1 + \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 + \cdots = 1 + \epsilon + \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 + \cdots .$$

(a) The zeroth iterate is $x_0 = 1$. Hence:

$$x_0 - x_* = 1 - \epsilon - \sqrt{1 + \epsilon^2} = -\epsilon - \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 - \cdots .$$

This is indeed proportional to $\epsilon^{n_0} = \epsilon$ as required.

(b) The first iterate is:

$$x_1 = x_0 - \frac{x_0^2 - 2\epsilon x_0 - 1}{2(x_0 - \epsilon)} = 1 + \frac{\epsilon}{1 - \epsilon} = \frac{1}{1 - \epsilon}.$$

Taylor expanding for small ϵ, we have:

$$x_1 = (1 - \epsilon)^{-1} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \cdots .$$

Consequently, we have:

$$x_1 - x_* = \frac{1}{2}\epsilon^2 + \epsilon^3 + \frac{9}{8}\epsilon^4 + \cdots .$$

This is indeed proportional to $\epsilon^{n_1} = \epsilon^2$ as required.

(c) The second iterate is:

$$x_2 = x_1 - \frac{x_1^2 - 2\epsilon x_1 - 1}{2(x_1 - \epsilon)}$$

Substituting $x_1 = 1/(1 - \epsilon)$, we have:

$$\begin{aligned}
x_2 &= \frac{1}{1 - \epsilon} - \frac{1/(1-\epsilon)^2 - 2\epsilon/(1-\epsilon) - 1}{2(1/(1-\epsilon) - \epsilon)} \\
&= \frac{1}{1 - \epsilon} - \frac{1 - 2\epsilon(1-\epsilon) - (1-\epsilon)^2}{2(1 - \epsilon - \epsilon(1-\epsilon)^2)} \\
&= \frac{1}{1 - \epsilon} - \frac{\epsilon^2}{2(1 - 2\epsilon + 2\epsilon^2 - \epsilon^3)} \\
&= (1 - \epsilon)^{-1} - \frac{1}{2}\epsilon^2 \left(1 - 2\epsilon + 2\epsilon^2 - \epsilon^3\right)^{-1} .
\end{aligned}$$

Now, expanding using a Taylor expansion for small ϵ, we have:

$$\begin{aligned}
x_2 &= 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + ... - \frac{1}{2}\epsilon^2 \left(1 + \left(2\epsilon - 2\epsilon^2 + \epsilon^3\right) + \left(2\epsilon - 2\epsilon^2 + \epsilon^3\right)^2 + ...\right) \\
&= 1 + \epsilon + \frac{1}{2}\epsilon^2 + 0 \cdot \epsilon^3 + 0 \cdot \epsilon^4 + \cdots ,
\end{aligned}$$

where we notice the terms in ϵ^3, ϵ^4 vanish. Hence $x_2 - x_* = \epsilon^4/8 + \cdots$. This implies $|x_2 - x_*| \propto \epsilon^4$, so $n_2 = 4$, as required.

26. ($*$) Consider the cubic equation $x^3 - 2x + 2 = 0$. Perform a numerical investigation (for example, by writing some simple code) to determine the ranges in $\mathbb{R}$ which converge to the various roots, if any. Comment on the sensitivity of Newton-Raphson to the choice of initial guess x_0. [*Afterwards, look up Newton fractals - it is particularly interesting to see the behaviour of Newton-Raphson in the complex plane!*]

Solution: Let $f(x) = x^3 - 2x + 2$. Observe that $f'(x) = 3x^2 - 2$ for this function, so the minimum value occurs at $x = \sqrt{2/3}$. This has corresponding value $f(\sqrt{2/3}) = (2/3)\sqrt{2/3} - 2\sqrt{2/3} + 2 \approx 0.911 > 0$, which implies that the cubic only has one real root.

The Newton-Raphson recurrence relation for the given function is:

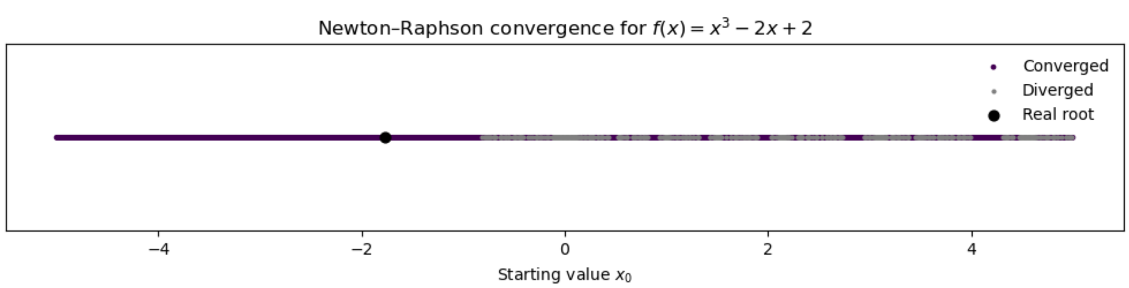
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n + 2}{3x_n^2 - 2}.$$

After finding this recurrence relation, I asked ChatGPT to write some Python code that performs the recurrence on a range of initial values on the real line, to determine which root the algorithm converges to:

```python
import numpy as np
import matplotlib.pyplot as plt

# Define the function and derivative
def f(x):
    return x**3 - 2*x + 2

def df(x):
    return 3*x**2 - 2

# Newton-Raphson iteration that tests true convergence
def newton(x0, max_iter=50, tol=1e-8):
    x = x0
    for _ in range(max_iter):
        dfx = df(x)
        if abs(dfx) < 1e-12:  # avoid division by zero
            return np.nan
        x_new = x - f(x)/dfx
        if abs(f(x_new)) < tol:  # true convergence check
            return x_new
        x = x_new
    return np.nan  # did not converge

# Compute roots (complex, but we only expect one real)
roots = np.roots([1, 0, -2, 2])
print("Roots:", roots)

# Sample starting values on the real line
x_vals = np.linspace(-5, 5, 2000)
final_vals = np.array([newton(x0) for x0 in x_vals])

# Classify convergence
def classify_root(x):
    if np.isnan(x):  # diverged or failed
        return np.nan
    distances = [abs(x - r) for r in roots]
    return np.argmin(distances)
```

```python

colors = np.array([classify_root(x) for x in final_vals])

# Plot setup
plt.figure(figsize=(10, 2))

# Points that converged (coloured)
mask_converged = ~np.isnan(colors)
plt.scatter(x_vals[mask_converged],
            np.zeros_like(x_vals[mask_converged]),
            c=colors[mask_converged],
            cmap='viridis', s=6, label="Converged")

# Points that diverged (gray)
mask_diverged = np.isnan(colors)
plt.scatter(x_vals[mask_diverged],
            np.zeros_like(x_vals[mask_diverged]),
            color='gray', s=4, label="Diverged")

# Axes and labels
plt.title("-NewtonRaphson convergence for $f(x)=x^3 - 2x + 2$")
plt.yticks([])
plt.xlabel("Starting value $x_0$")

# Mark the real root with a small circle
real_root = roots[np.isclose(roots.imag, 0)][0].real
plt.scatter([real_root], [0], color='black', s=40, zorder=5, label="Real root")

plt.legend(frameon=False, loc="upper right")
plt.tight_layout()
plt.show()
```

The resulting plot is the following:



In particular, we see that the purple starting points arrive at the real root, whilst the grey points *diverge*, not converging to any particular root. The behaviour appears a bit random as to whether the algorithm actually converges or not; indeed, we say that the Newton-Raphson method is *chaotic*. The method only converges to a root if we start sufficiently close to it (and how close is close enough can depend strongly on the function we are working with).

This behaviour is even more fascinating in the complex plane; I encourage you to look at the Wikipedia article about *Newton fractals*, which you can find here: https://en.wikipedia.org/wiki/Newton_fractal.

Part IA: Mathematics for Natural Sciences B
Examples Sheet 6: Single-variable integration

Model Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

Riemann sums and the definition of the integral

1. Explain what is meant by a *Riemann sum* for a function $f : [a,b] \to \mathbb{R}$ using a *partition* $P = (x_0, ..., x_n)$ (with $x_0 = a, x_n = b$) and *tagging* $T = (t_1, ..., t_n)$. By choosing appropriate partitions and taggings in each case, use sequences of Riemann sums to evaluate the definite integrals of the following functions on $[0,1]$ from first principles:

(a) x, (b) x^2, (c) x^3, (d) $\sqrt{x}$, (e) $\cos(x)$.

[*Hint: For part (d), consider a non-uniform tagging. For part (e), consider the integral of* $\mathrm{Re}(e^{ix})$ *instead of* $\cos(x)$.]

Solution: If we want to approximate the integral of the function $f(x)$ on the interval $[a,b]$, we can do so by considering a sum of *rectangles*. We split the interval $[a,b]$ into a *partition* $P = (x_0, ..., x_n)$ which satisfies:

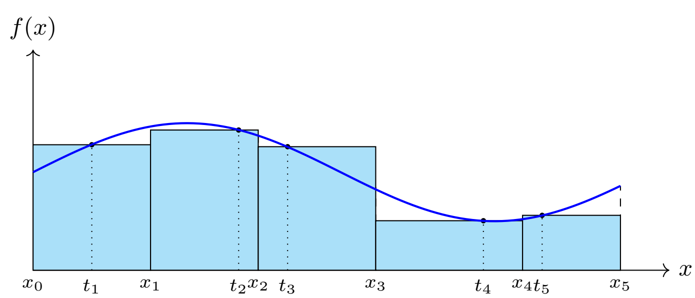
$$a = x_0 < x_1 < ... < x_n = b,$$

where $[x_0, x_1]$ is the base of the first rectangle, $[x_1, x_2]$ is the base of the second rectangle, etc. Notice we start at zero, because then the highest index n matches the number of rectangles we get!

Within each rectangle, we then pick a point called a *tag* for the rectangle. This produces a *tagging* $T = (t_1, ..., t_n)$ which satisfies:

$$x_0 \leq t_1 \leq x_1 \leq t_2 \leq x_2 \leq ... \leq x_{n-1} \leq t_n \leq x_n.$$

At the tags, we evaluate the function to give us the heights of the rectangles in our approximation, $f(t_1), ..., f(t_n)$. This is illustrated in the diagram below.



The *Riemann sum* for $f : [a,b] \to \mathbb{R}$ that results from this partition and tagging is defined to be the area of the rectangles:

$$R = \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}).$$

We say that f is a *Riemann integrable function*, with *integral* I, if for all sequences of partitions P_n and taggings T_n, such that the width of the largest sub-interval in the partition P_n tends to zero as $n \to \infty$, we have that the associated sequences of Riemann sums R_n all converge to I.

Proving that a function is Riemann integrable is very hard, because it requires us to consider all possible partitions and taggings in one go.[4] However, if we already know that a function is Riemann integrable, we can find its integral by just considering *one* sequence of partitions and taggings; that is what we do in this question. In general, any continuous function is integrable, so this is completely fine in this case!

Finding integrals in this way is not straightforward, and is definitely not the best approach - however, it is the most fundamental, and is similar to the 'first principles' limit approach we used when we first introduced differentiation. Integration in practice involves knowing a standard set of integrals, and a set of techniques (namely integration by substitution and integration by parts), which allows us to determine integrals of most common functions.

(a) Use the uniform partition $(0, 1/n, .., n/n)$ with right-handed tagging $(1/n, 2/n, ..., n/n)$. Then we have the sequence of Riemann sums:

$$R_n = \sum_{k=1}^{n} \left(\frac{k}{n}\right) \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{n(n+1)}{2n^2} = \frac{n+1}{2n}$$

Taking the limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1+1/n}{2} = \frac{1}{2}.$$

This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}.$$

(b) Use the uniform partition $(0, 1/n, ..., n/n)$ with right-handed tagging $(1/n, 2/n, ..., n/n)$. Then we have the sequence of Riemann sums:

$$R_n = \sum_{k=1}^{n} \left(\frac{k^2}{n^2}\right) \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{(n+1)(2n+1)}{6n^2}.$$

Taking the limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{(1+1/n)(2+1/n)}{6} = \frac{2}{6} = \frac{1}{3}.$$

This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}.$$

[4] It is possible to do this for very simple functions though. You might like to show that a constant function is integrable from first principles, if you are feeling adventurous!

(c) Use the uniform partition $(0, 1/n, ..., n/n)$ with right-handed tagging $(1/n, 2/n, ..., n/n)$. Then we have the sequence of Riemann sums:

$$R_n = \sum_{k=1}^{n} \left(\frac{k^3}{n^3}\right) \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^4} \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4n^4} = \frac{(n+1)^2}{4n^2}.$$

Taking the limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)^2}{4n^2} = \lim_{n\to\infty} \frac{(1+1/n)^2}{4} = \frac{1}{4}.$$

This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 x^3 \, dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}.$$

(d) This time, things get more exciting, because using a uniform partition won't work. Instead, let's use a *quadratically spaced* partition, to try to clear the square root. We use the partition $(0, 1/n^2, 4/n^2, 9/n^2, ..., n^2/n^2)$ with right-handed tagging $(1/n^2, 4/n^2, ..., n^2/n^2)$. Then we have the sequence of Riemann sums:

$$R_n = \sum_{k=1}^{n} \sqrt{\frac{k^2}{n^2}} \cdot \left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2}\right) = \frac{1}{n^3} \sum_{k=1}^{n} k\left(k^2 - (k^2 - 2k + 1)\right) = \frac{1}{n^3} \sum_{k=1}^{n} \left(2k^2 - k\right).$$

Performing the sum, we have:

$$\frac{1}{n^3} \sum_{k=1}^{n} \left(2k^2 - k\right) = \frac{2}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2n^3} = \frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2}.$$

Taking the limit as $n \to \infty$, the second term vanishes, and the first term gives $2/3$. This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 \sqrt{x} \, dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}.$$

(e) This is an even more exciting problem, because we get to use complex numbers. Observe that:

$$\int_0^1 \cos(x) \, dx = \mathrm{Re}\left[\int_0^1 e^{ix} \, dx\right],$$

so instead of constructing a Riemann sum for $\cos(x)$, we will construct a Riemann sum for e^{ix}, as hinted at in the question. Choose a uniform partition $(0, 1/n, ..., n/n)$ with right-handed tagging $(1/n, 2/n, ..., n/n)$. Then we have the sequence of Riemann sums for the complex integral:

$$R_n = \sum_{k=1}^{n} e^{ik/n} \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n} \sum_{k=1}^{n} e^{ik/n}.$$

This is a geometric progression with first term $e^{i/n}$ and common ratio $e^{i/n}$. Hence the sum is:

$$R_n = \frac{e^{i/n}(1 - e^i)}{n(1 - e^{i/n})}.$$

We now need to take the limit of this expression as $n \to \infty$. The numerator approaches $1 - e^i$, but the denominator is of the form $\infty \cdot 0$, so is an indeterminate form. We could use L'Hôpital's rule to evaluate this:

$$\lim_{n\to\infty} n(1 - e^{i/n}) = \lim_{n\to\infty} \frac{1 - e^{i/n}}{1/n} = \lim_{n\to\infty} \frac{ie^{i/n}/n^2}{-1/n^2} = -i.$$

Alternatively, we can use a Taylor series expansion for $e^{i/n}$ (see the next sheet, if you are unfamiliar!). We have:

$$\lim_{n\to\infty} n(1 - e^{i/n}) = \lim_{n\to\infty} n\left(1 - 1 - \frac{i}{n} - \frac{1}{2!}\left(\frac{i}{n}\right)^2 + \cdots\right) = -i.$$

Hence we have:

$$\lim_{n\to\infty} \frac{e^{i/n}(1 - e^i)}{n(1 - e^{i/n})} = \frac{1 - e^i}{-i} = i - ie^i = i - i\cos(1) + \sin(1).$$

Taking the real part, this leaves $\sin(1)$. This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 \cos(x)\, dx = [\sin(x)]_0^1 = \sin(1).$$

In fact, this proof also shows (by taking imaginary parts), that the integral of $\sin(x)$ over $[0, 1]$ is given by $1 - \cos(1)$. This also agrees with what we would expect from our knowledge of integration:

$$\int_0^1 \sin(x)\, dx = [-\cos(x)]_0^1 = 1 - \cos(1).$$

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2. Using a non-uniform tagging, use a sequence of Riemann sums to evaluate the integral $\displaystyle\int_1^\infty \frac{dx}{x^{1+\alpha}}$, where $\alpha > 0$.

➾ **Solution:** Let us define $r_n = 1 + 1/n$ to be a ratio larger than one. Then, we use a geometric partition $(1, r_n, r_n^2, r_n^3, ...)$ with right-handed tagging $(r_n, r_n^2, r_n^3, ...)$. The corresponding sequence of Riemann sums is:

$$R_n = \sum_{k=0}^{\infty} \frac{1}{(r_n^k)^{1+\alpha}} \left(r_n^{k+1} - r_n^k\right) = (r_n - 1)\sum_{k=0}^{\infty} r_n^{-k\alpha} = \frac{r_n - 1}{1 - r_n^{-\alpha}}.$$

Here, we could take the sum to infinity since $r_n > 1$, so $r_n^{-\alpha} < 1$. We now take the limit as $n \to \infty$, which is equivalent to the limit $r_n \to 1$. To do this, we can use L'Hôpital's rule:

$$\lim_{n\to\infty} R_n = \lim_{r\to 1} \frac{r - 1}{1 - r^{-\alpha}} = \lim_{r\to 1} \frac{1}{\alpha r^{-\alpha}} = \frac{1}{\alpha}.$$

Observe that this agrees with the expected result, since:

$$\int_1^\infty \frac{dx}{x^{1+\alpha}} = \left[-\frac{1}{\alpha x^\alpha}\right]_1^\infty = \frac{1}{\alpha}.$$

3. Show by considering Riemann sums that $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2}=\frac{\pi}{4}$.

➾ **Solution:** Observe that:

$$\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2}=\sum_{k=1}^{n}\sqrt{1-\left(\frac{k}{n}\right)^2}\cdot\left(\frac{k}{n}-\frac{k-1}{n}\right)$$

is a sequence of Riemann sums for the function $\sqrt{1-x^2}$ on the interval $[0,1]$ using the partition $0=0/n<1/n<...<n/n=1$, with the (right handed) tagging $1/n, 2/n, ..., n/n$. Hence:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2}=\int_0^1\sqrt{1-x^2}\,dx.$$

To evaluate this integral, we could use a substitution. Alternatively, we could note it is the area under the graph $y=\sqrt{1-x^2}$ between 0 and 1. Rearranging this equation, we see that $x^2+y^2=1$, which tells us the graph is a *circle*. It immediately follows that the integral is just the area of quarter of a circle, hence:

$$\int_0^1\sqrt{1-x^2}\,dx=\frac{\pi}{4},$$

as required.

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4. $(*)$ If a sequence of Riemann sums for a function $f:[a,b]\to\mathbb{R}$ converges, must the function be integrable?

➾ **Solution:** The answer is *no*; just because one sequence of Riemann sums converges, it does not mean that all sequences of Riemann sums will converge to the same value. Consider, for example, the function $f:[0,1]\to\mathbb{R}$ given by:

$$f(x)=\begin{cases}0, & \text{if } x \text{ is rational,}\\ 1, & \text{if } x \text{ is irrational.}\end{cases}$$

Then using the uniform partition $(0,1/n,2/n,...,n/n)$ with right-handed tagging $(1/n,2/n,...,n/n)$, we have a sequence of Riemann sums:

$$R_n=\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\left(\frac{k}{n}-\frac{k-1}{n}\right)=0,$$

since $f(k/n)=0$, since k/n is rational for each k. This suggests that the integral of $f(x)$ on $[0,1]$ is zero.

However, if we instead use a tagging consisting of *irrational* numbers $(s_1,...,s_n)$, chosen such that $0=0/n<s_1<1/n<s_2<2/n<...<s_n<n/n=1$, then we have a sequence of Riemann sums:

$$R_n'=\sum_{k=1}^{n}f(s_k)\left(\frac{k}{n}-\frac{k-1}{n}\right)=\frac{n}{n}=1.$$

This suggests that the integral of $f(x)$ on $[0,1]$ is one. In particular, we see that the function is *not* integrable, because two different sequences of Riemann sums for $f(x)$ on $[0,1]$ converge to two different values.

Basic integrals

5. *Write down* the indefinite integrals of each of the following functions, where $a \neq 0$, $\alpha \neq -1$, and f is any (differentiable, non-zero) function:

(a) $(ax+b)^\alpha$, (b) e^{ax+b}, (c) $(ax+b)^{-1}$, (d) $\sin(ax+b)$, (e) $\cos(ax+b)$,

(f) $\sec^2(ax+b)$, (g) $\operatorname{cosec}^2(ax+b)$, (h) $\sinh(ax+b)$, (i) $\cosh(ax+b)$, (j) $f'(x)f(x)^\alpha$,

(k) $f'(x)/f(x)$.

Learn these integrals off by heart, and get your supervision partner to test you on them.

➾ **Solution:** These are all standard integrals. We have (where c is an arbitrary constant in each case):

(a) $\displaystyle\int (ax+b)^\alpha\, dx = \frac{(ax+b)^{\alpha+1}}{a(\alpha+1)} + c$, if $a \neq 0$ and $\alpha \neq -1$.

(b) $\displaystyle\int e^{ax+b}\, dx = \frac{1}{a}e^{ax+b} + c$, if $a \neq 0$.

(c) $\displaystyle\int \frac{1}{ax+b}\, dx = \frac{\log(ax+b)}{a} + c$, if $a \neq 0$.

(d) $\displaystyle\int \sin(ax+b)\, dx = -\frac{\cos(ax+b)}{a} + c$, if $a \neq 0$.

(e) $\displaystyle\int \cos(ax+b)\, dx = \frac{\sin(ax+b)}{a} + c$, if $a \neq 0$.

(f) $\displaystyle\int \sec^2(ax+b)\, dx = \frac{\tan(ax+b)}{a} + c$, if $a \neq 0$.

(g) $\displaystyle\int \operatorname{cosec}^2(ax+b)\, dx = -\frac{\cot(ax+b)}{a} + c$, if $a \neq 0$.

(h) $\displaystyle\int \sinh(ax+b)\, dx = \frac{\cosh(ax+b)}{a} + c$, if $a \neq 0$.

(i) $\displaystyle\int \cosh(ax+b)\, dx = \frac{\sinh(ax+b)}{a} + c$, if $a \neq 0$.

(j) $\displaystyle\int f'(x)f(x)^\alpha\, dx = \frac{f(x)^{\alpha+1}}{\alpha+1}$, if $\alpha \neq -1$.

(k) $\displaystyle\int \frac{f'(x)}{f(x)}\, dx = \log(f(x)) + c.$

6. Using the results of the previous question, evaluate the definite integrals:

(a) $\displaystyle\int_0^2 (x-1)^2\,dx$, (b) $\displaystyle\int_0^\pi e^{i\theta}\,d\theta$, (c) $\displaystyle\int_0^\pi \cos(x)\,dx$, (d) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx$, (e) $\displaystyle\int_0^1 \frac{2x+4}{x^2+4x+1}\,dx$.

Solution: We have:

(a) $\displaystyle\int_0^2 (x-1)^2\,dx = \left[\frac{(x-1)^3}{3}\right]_0^2 = \frac{1}{3} - -\frac{1}{3} = \frac{2}{3}.$

(b) $\displaystyle\int_0^\pi e^{i\theta}\,d\theta = \left[\frac{e^{i\theta}}{i}\right]_0^\pi = -\frac{1}{i} - \frac{1}{i} = -\frac{2}{i} = 2i.$

(c) $\displaystyle\int_0^\pi \cos(x)\,dx = [\sin(x)]_0^\pi = \sin(\pi) - \sin(0) = 0$. Alternatively, spot that $\cos(x)$ is rotationally symmetric about $x = \pi/2$, and the positive and negative contributions to the integral from $0 < x < \pi/2$ and $\pi/2 < x < \pi$ therefore exactly cancel out.

(d) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx = [\tan(x)]_{-\pi/4}^{\pi/4} = 1 - -1 = 2.$

(e) $\displaystyle\int_0^1 \frac{2x+4}{x^2+4x+1}\,dx = \left[\log(x^2+4x+1)\right]_0^1 = \log(6) - \log(1) = \log(6).$

7. By writing $\cos(bx)$ as the real part of a complex exponential, determine the indefinite integral of $e^{ax}\cos(bx)$. Similarly, determine the indefinite integrals of $e^{x}(\sin(x)-\cos(x))$ and $e^{x}(\sin(x)+\cos(x))$.

➬ **Solution:** We have:

$$\begin{aligned}\int e^{ax}\cos(bx)\,dx &= \mathrm{Re}\left[\int e^{ax}e^{ibx}\,dx\right]\\ &= \mathrm{Re}\left[\int e^{(a+ib)x}\,dx\right]\\ &= \mathrm{Re}\left[\frac{e^{(a+ib)x}}{a+ib}+c\right]\\ &= \mathrm{Re}\left[\frac{e^{ax}(a-ib)(\cos(bx)+i\sin(bx))}{a^2+b^2}+c\right]\\ &= \frac{e^{ax}(a\cos(bx)+b\sin(bx)}{a^2+b^2}+c,\end{aligned}$$

where c is a real constant of integration.

Similarly, we have:

$$\int e^{ax}\sin(bx)\,dx = \mathrm{Im}\left[\int e^{ax}e^{ibx}\,dx\right] = \frac{e^{ax}(a\sin(bx)-b\cos(bx))}{a^2+b^2}+c,$$

using the penultimate line of the calculation from above, but just taking the imaginary part instead of the real part. It follows that:

$$\int e^{x}(\sin(x)-\cos(x))\,dx = \frac{e^{x}(\sin(x)-\cos(x))}{2}-\frac{e^{x}(\cos(x)+\sin(x))}{2}+c = -e^{x}\cos(x)+c,$$

for a real constant of integration c. Similarly, we have:

$$\int e^{x}(\sin(x)+\cos(x))\,dx = \frac{e^{x}(\sin(x)-\cos(x))}{2}+\frac{e^{x}(\cos(x)+\sin(x))}{2}+c = e^{x}\sin(x)+c,$$

for a real constant of integration c.

Integration by substitution

8. By means of an appropriate substitution in each case, determine the indefinite integrals of the following functions:

(a) $\dfrac{1}{\sqrt{1-x^2}}$, (b) $\dfrac{1}{\sqrt{x^2-1}}$, (c) $\dfrac{1}{\sqrt{1+x^2}}$, (d) $\dfrac{1}{1+x^2}$, (e) $\dfrac{1}{1-x^2}$

Learn these integrals off by heart, and get your supervision partner to test you on them.

➾ **Solution:** Each of these integrals can be performed by making an appropriate trigonometric or hyperbolic substitution.

(a) Use the substitution $x = \sin(\theta)$, so that $dx = \cos(\theta)d\theta$. This is appropriate here, since for the square root to make sense, we need $-1 < x < 1$. This is covered by the range of $\sin(\theta)$, so the substitution is valid. We then have:

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos(\theta)d\theta}{\sqrt{1-\sin^2(\theta)}} = \int d\theta = \theta + c = \arcsin(x) + c.$$

(b) Use the substitution $x = \cosh(\theta)$, so that $dx = \sinh(\theta)d\theta$. This is appropriate here, since for the square root to make sense, we need $x > 1$ or $x < -1$. Since the range is disjoint, we can focus on $x > 1$, which is covered by the range of $\cosh(\theta)$ (we could have chosen to make the substitution $x = -\cosh(\theta)$ if $x < -1$ was the range of interest). We then have:

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sinh(\theta)d\theta}{\sqrt{\cosh^2(\theta)-1}} = \int d\theta = \theta + c = \operatorname{arcosh}(x) + c.$$

(c) Use the substitution $x = \sinh(\theta)$, so that $dx = \cosh(\theta)d\theta$. This is appropriate here, since x can take any value, and $\sinh(\theta)$ has range $\mathbb{R}$. We then have:

$$\int \frac{dx}{\sqrt{x^2+1}} = \int \frac{\cosh(\theta)d\theta}{\sqrt{\sinh^2(\theta)+1}} = \int d\theta = \theta + c = \operatorname{arsinh}(x) + c.$$

(d) Use the substitution $x = \tan(\theta)$, so that $dx = \sec^2(\theta)d\theta$. This is appropriate here, since x can take any value, and $\sinh(\theta)$ has range $\mathbb{R}$. We then have:

$$\int \frac{dx}{1+x^2} = \int \frac{\sec^2(\theta)d\theta}{1+\tan^2(\theta)} = \int d\theta = \theta + c = \arctan(x) + c.$$

(e) Use the substitution $x = \tanh(\theta)$, so that $dx = \operatorname{sech}^2(\theta)d\theta$. This is appropriate here since x takes values in $x < -1$, $-1 < x < 1$ and $x > 1$. These ranges are disjoint, so if we are interested in $-1 < x < 1$, we are safe to make this substitution. We then have:

$$\int \frac{dx}{1-x^2} = \int \frac{\operatorname{sech}^2(\theta)d\theta}{1-\tanh^2(\theta)} = \int d\theta = \theta + c = \operatorname{artanh}(x) + c.$$

9. Using the results of the previous question, determine: (a) $\displaystyle\int \frac{dx}{\sqrt{x^2+x+1}}$; (b) $\displaystyle\int \frac{8-2x}{\sqrt{6x-x^2}}\,dx$.

Solution:

(a) We complete the square in the denominator, $x^2+x+1=(x+1/2)^2+3/4$. Then:

$$\int \frac{dx}{\sqrt{(x+1/2)^2+3/4}} = \frac{2}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1}}.$$

This is now a linear function of one of our standard integrals from the previous question. Hence the integral is just given by:

$$\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2}\operatorname{arsinh}\left(\frac{2x+1}{\sqrt{3}}\right)+c = \operatorname{arsinh}\left(\frac{2x+1}{\sqrt{3}}\right)+c.$$

To get the constants that multiply the inverse hyperbolic function correct, it can be useful to think about what happens when you differentiate the final expression. By the chain rule, a $2/\sqrt{3}$ will pop out, so we need to cancel that when we integrate!

(b) This is more complicated, because even completing the square on the denominator will not give us an integral of the form of the previous question. Instead, we observe that the derivative of $6x-x^2$ is $6-2x$, so the integral is close to being of the form $f'(x)f(x)^\alpha$. If we rewrite the integrand as:

$$\frac{8-2x}{\sqrt{6x-x^2}} = \frac{6-2x}{\sqrt{6x-x^2}}+\frac{2}{\sqrt{6x-x^2}} = (6-2x)\left(6x-x^2\right)^{-1/2}+\frac{2}{\sqrt{6x-x^2}},$$

then the first term is now of the form $f'(x)f(x)^\alpha$ and can be directly integrated, whilst the second term can be transformed to one of our standard forms studied in the previous question.

Completing the square in the denominator of the second term, we have $6x-x^2=9-(x-3)^2$. Hence we have:

$$\begin{aligned}\int \frac{8-2x}{\sqrt{6x-x^2}}\,dx &= \int (6-2x)(6x-x^2)^{-1/2}\,dx+\frac{2}{3}\int\frac{dx}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}}\\ &= 2\sqrt{6x-x^2}+\frac{2}{3}\cdot 3\cdot\arcsin\left(\frac{x-3}{3}\right)+c\\ &= 2\sqrt{6x-x^2}+2\arcsin\left(\frac{x-3}{3}\right)+c.\end{aligned}$$

10. By means of an appropriate substitution in each case, determine the indefinite integrals of the following functions:

(a) $x\sqrt{x+3}$, (b) $\tan(x)\sqrt{\sec(x)}$, (c) $\dfrac{e^x}{\sqrt{1-e^{2x}}}$, (d) $\dfrac{1}{x\sqrt{x^2-1}}$.

Solution:

(a) Consider the substitution $u = x+3$, trying to clear the square root. We have $du = dx$, so that the integral can be rewritten as:

$$\int x\sqrt{x+3}\,dx = \int (u-3)u^{1/2}\,du = \int \left(u^{3/2} - 3u^{1/2}\right)\,du = \frac{2}{5}u^{5/2} - 2u^{3/2} + c.$$

Hence the integral is given by:

$$\frac{2}{5}(x+1)^{5/2} + 2(x+1)^{3/2} + c.$$

(b) Consider the substitution $u = \sec(x)$. This is a good choice, because the derivative of $\sec(x)$ is $\sec(x)\tan(x)$, so we will be able to clear the $\tan(x)$ and leave only $\sec(x)$ terms behind. We have $du = \sec(x)\tan(x)dx$, so:

$$\int \tan(x)\sqrt{\sec(x)}\,dx = \int \frac{du}{\sqrt{u}} = 2u^{1/2} + c = 2\sqrt{\sec(x)} + c.$$

(c) Here, the obvious substitution is $u = e^x$. We have $du = e^x\,dx$, so:

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \arcsin(u) + c = \arcsin(e^x) + c.$$

(d) Consider a trigonometric substitution $x = \sec(\theta)$, since this will clear the square root ($\sqrt{\sec^2(\theta)-1} = \tan(\theta)$), but also give us $dx = \sec(\theta)\tan(\theta)d\theta$, so that the remaining $\sec(\theta)$ on the denominator will cancel. We have:

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec(\theta)\tan(\theta)d\theta}{\sec(\theta)\tan(\theta)} = \int d\theta = \theta + c = \operatorname{arcsec}(x) + c.$$

Another way of writing $\operatorname{arcsec}(x)$ in terms of more standard function is as $\arccos(1/x)$ - to see this, note that:

$$y = \operatorname{arcsec}(x) \qquad \Rightarrow \qquad \sec(y) = x \qquad \Rightarrow \qquad \cos(y) = \frac{1}{x} \qquad \Rightarrow \qquad y = \arccos\left(\frac{1}{x}\right).$$

11. This question shows that any trigonometric integral can be turned into an algebraic integral through the use of the powerful *half-tangent substitution*.

(a) Show that if $t = \tan\left(\frac{1}{2}x\right)$, then $\sin(x) = 2t/(1+t^2)$, $\cos(x) = (1-t^2)/(1+t^2)$ and $dx/dt = 2/(1+t^2)$. Deduce that for any function f, we have:

$$\int f\left(\sin(x), \cos(x)\right)\, dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}.$$

(b) Using the method derived in (a), find the indefinite integrals of the following functions:

(i) $\operatorname{cosec}(x)$, (ii) $\sec(x)$, (iii) $\dfrac{1}{2+\cos(x)}$.

➾ **Solution:** (a) We have:

$$\sin(x) = 2\sin(x/2)\cos(x/2) = \frac{2\sin(x/2)\cos(x/2)}{\cos^2(x/2)+\sin^2(x/2)} = \frac{2\tan(x/2)}{1+\tan^2(x/2)} = \frac{2t}{1+t^2}$$

and:

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2) = \frac{\cos^2(x/2)-\sin^2(x/2)}{\cos^2(x/2)+\sin^2(x/2)} = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-t^2}{1+t^2},$$

as required. We also have:

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{1}{2}x\right) = \frac{1}{2}\left(1+t^2\right) \qquad \Leftrightarrow \qquad \frac{dx}{dt} = \frac{2}{1+t^2}.$$

The required equality then follows immediately from the substitution $t = \tan(x/2)$.

(b) Applying the substitution in each of the given cases, we have:

(i) $$\int \operatorname{cosec}(x)\, dx = \int \frac{dx}{\sin(x)} = \int \frac{2(1+t^2)dt}{2t(1+t^2)} = \int \frac{dt}{t} = \log(t) + c = \log(\tan(x/2)) + c.$$

(ii) $$\int \sec(x)\, dx = \int \frac{dx}{\cos(x)} = \int \frac{2(1+t^2)dt}{(1-t^2)(1+t^2)} = 2\int \frac{dt}{1-t^2} = 2\operatorname{artanh}\left(\tan(x/2)\right) + c.$$

(iii) $$\int \frac{dx}{2+\cos(x)} = \int \frac{2dt}{(2(1+t^2)+(1-t^2))} = \int \frac{2dt}{3+t^2} = \frac{2}{3}\int \frac{dt}{1+\left(\frac{t}{\sqrt{3}}\right)^2} = \frac{2}{\sqrt{3}}\arctan\left(\frac{\tan(x/2)}{\sqrt{3}}\right) + c.$$

Partial fractions and rational functions

12. Explain the general strategy that one should adopt when integrating a rational function. Hence, determine the indefinite integrals of the following rational functions by decomposing into partial fractions:

$$\text{(a)}\ \frac{1}{1-x^2}, \qquad \text{(b)}\ \frac{3x}{2x^2+x-1}, \qquad \text{(c)}\ \frac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}.$$

Compare your answer to (a) with your answer to Question 7(e), where you evaluated the same integral using a substitution. Are your results compatible?

Solution: Consider the rational function $p(x)/q(x)$, where $q(x)$ can be factorised in the form:

$$q(x) = (x-a_1)^{j_1}...(x-a_m)^{j_m}(x^2+b_1x+c_1)^{k_1}...(x^2+b_nx+c_n)^{k_n},$$

where the quadratic factors have no real roots. Then $p(x)/q(x)$ can be decomposed in the form:

$$\frac{p(x)}{q(x)} = r(x) + \sum_{i=1}^{m}\sum_{r=1}^{j_i}\frac{A_{ir}}{(x-a_i)^r} + \sum_{i=1}^{n}\sum_{r=1}^{k_i}\frac{B_{ir}x+C_{ir}}{(x^2+b_ix+c_i)^r},$$

where $r(x)$ is a polynomial, which is called the *partial fraction decomposition* of the rational function. This gives us a general strategy for integrating a rational function:

- First, perform the partial fraction decomposition.
- The polynomial term $r(x)$ can be integrated straightforwardly.
- The terms in the partial fraction decomposition involving the real roots can be integrated straightforwardly via:

$$\int \frac{A_{ir}}{(x-a_i)^r} = \begin{cases} -\dfrac{A_{ir}}{(r-1)(x-a_i)^{r-1}}, & \text{if } r\neq 1, \\ A_{ir}\log(x-a_i), & \text{if } r = 1. \end{cases}$$

- For the terms in the partial fraction decomposition involving a simple quadratic factor, i.e. a quadratic factor with $k_i = 1$, we can write:

$$\frac{B_{i1}x+C_{i1}}{x^2+b_{k_i}x+c_i} = \frac{B_{i1}}{2}\frac{2x+b_{k_i}}{x^2+b_{k_i}x+c_i} + \frac{C_{i1}-B_{i1}b_{k_i}/2}{x^2+b_{k_i}x+c_i}.$$

  The first term is now a *logarithmic derivative*, and can be integrated directly. Meanwhile, the remaining term is a constant multiplied by the reciprocal of a quadratic; by completing the square in the denominator, this can be made into a derivative of an *arctangent* or *hyperbolic arctangent*.
- For terms in the partial fraction decomposition involving non-simple quadratic factors, i.e. quadratic factors with $k_i > 1$, things are more complicated. These functions are all integrable, through an arctangent or hyperbolic arctangent substitution, but it is unlikely that integrals of this kind will be in your standard arsenal.

We shall now apply this technique in the cases of three given rational functions.

(a) We have:

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right).$$

Hence, integrating, we have:

$$\int\frac{dx}{1-x^2} = -\frac{1}{2}\log(1-x) + \frac{1}{2}\log(1+x) + c = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) + c.$$

This is perfectly consistent with Question 7(e), because $\tanh^{-1}(x) = \frac{1}{2}\log((1+x)/(1-x))$.

(b) Factorising, we have $2x^2 + x - 1 = (2x-1)(x+1)$. Hence decomposing into partial fractions, we have:

$$\frac{3x}{2x^2+x-1} = \frac{1}{2x-1} + \frac{1}{x+1}.$$

Hence, integrating, we have:

$$\int \frac{3x}{2x^2+x-1}\,dx = \frac{1}{2}\log(2x-1) + \log(x+1) + c.$$

(c) First, we perform polynomial division:

$$\frac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4} = -1 + \frac{9+6x-x^2-2x^3}{3+2x-2x^2-2x^3-x^4}.$$

We can now decompose the second term into partial fractions. First, we need to factorise the denominator. We spot that $x=1$ is a factor, so:

$$3+2x-2x^2-2x^3-x^4 = (1-x)\left(3+5x+3x^2+x^3\right)$$

We spot that $x=-1$ is a factor of the second bracket, so:

$$(1-x)(x^3+3x^2+5x+3) = (1-x)(1+x)(x^2+2x+3).$$

The final factor has discriminant $4-4\cdot 3 = -8 < 0$, hence there are no more real roots. Thus the partial fractions take the form:

$$\frac{9+6x-x^2-2x^3}{3+2x-2x^2-2x^3-x^4} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{x^2+2x+3}.$$

Multiplying up, we have:

$$9+6x-x^2-2x^3 = A(1+x)(x^2+2x+3) + B(1-x)(x^2+2x+3) + (Cx+D)(1-x^2). \qquad (\dagger)$$

Setting $x=1$ in (†), we have:

$$12A = 12 \qquad \Rightarrow \qquad A = 1.$$

Setting $x=-1$ in (†), we have:

$$4B = 4 \qquad \Rightarrow \qquad B = 1.$$

Setting $x=0$ in (†), we have:

$$9 = 3A + 3B + D \qquad \Rightarrow \qquad D = 9-3-3 = 3.$$

Finally, comparing coefficients of x^3 on both sides of (†), we have:

$$-2 = A - B - C \qquad \Rightarrow \qquad C = A - B + 2 = 2.$$

Hence, the partial fractions for the original rational function are:

$$-1 + \frac{1}{1-x} + \frac{1}{1+x} + \frac{2x+3}{x^2+2x+3}.$$

To integrate, we need to split the final term into a logarithmic derivative, and a constant divided by a quadratic. We have:

$$\frac{2x+3}{x^2+2x+3} = \frac{2x+2}{x^2+2x+3} + \frac{1}{x^2+2x+3} = \frac{2x+2}{x^2+2x+3} + \frac{1}{(x+1)^2+2} = \frac{2x+2}{x^2+2x+3} + \frac{1}{2}\frac{1}{((x+1)/\sqrt{2})^2+1}$$

Therefore, integrating the original rational function, we have:

$$\begin{aligned}\int \frac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}\,dx &= \int \left(-1 + \frac{1}{1-x} + \frac{1}{1+x} + \frac{2x+2}{x^2+2x+3} + \frac{1}{2}\frac{1}{((x+1)/\sqrt{2})^2+1}\right)dx \\ &= -x - \log(1-x) + \log(1+x) + \log(2x^2+2x+3) + \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c.\end{aligned}$$

Integration by parts

13. Using integration by parts, determine the following integrals:

$$\text{(a)} \int\limits_{-\pi/2}^{\pi/2} x\sin(2x)\,dx, \qquad \text{(b)} \int\limits_{0}^{\infty} xe^{-2x}\,dx, \qquad \text{(c)} \int\limits_{0}^{1} x\log\left(\frac{1}{x}\right)\,dx, \qquad \text{(d)} \int\limits_{0}^{\infty} x^3 e^{-x^2}\,dx.$$

Solution:

(a) We have:

$$\begin{aligned}\int\limits_{-\pi/2}^{\pi/2} x\sin(2x)\,dx &= \left[-\frac{1}{2}x\cos(2x)\right]_{-\pi/2}^{\pi/2} + \frac{1}{2}\int\limits_{-\pi/2}^{\pi/2}\cos(2x)\,dx \\ &= \frac{\pi}{2} + \frac{1}{2}\left[\frac{1}{2}\sin(2x)\right]_{-\pi/2}^{\pi/2} \\ &= \frac{\pi}{2}.\end{aligned}$$

(b) We have:

$$\begin{aligned}\int\limits_{0}^{\infty} xe^{-2x}\,dx &= \left[-\frac{1}{2}xe^{-2x}\right]_{0}^{\infty} + \frac{1}{2}\int\limits_{0}^{\infty} e^{-2x}\,dx \\ &= \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_{0}^{\infty} \\ &= \frac{1}{4}.\end{aligned}$$

(c) We have:

$$\begin{aligned}\int\limits_{0}^{1} x\log\left(\frac{1}{x}\right)\,dx = -\int\limits_{0}^{1} x\log(x)\,dx &= -\left[\frac{1}{2}x^2\log(x)\right]_{0}^{1} + \frac{1}{2}\int\limits_{0}^{1} x\,dx \\ &= \frac{1}{2}\left[\frac{1}{2}x^2\right]_{0}^{1} \\ &= \frac{1}{4}.\end{aligned}$$

Here, we used the fact that $x^2\log(x) \to 0$ as $x \to 0$. This is because the polynomial approaches zero faster than the logarithm approaches negative infinity. This is a general phenomena, as we proved using L'Hôpital's rule earlier in the course.

(d) Observe that the derivative of e^{-x^2} is $-2xe^{-x^2}$, so that the integral of xe^{-x^2} is $-\frac{1}{2}e^{-x^2}$. Hence, we have:

$$\begin{aligned}\int_0^\infty x^3 e^{-x^2}\,dx &= \left[-\frac{1}{2}x^2e^{-x^2}\right]_0^\infty + \frac{1}{2}\int_0^\infty (2x)e^{-x^2}\,dx \\ &= \left[-\frac{1}{2}e^{-x^2}\right]_0^\infty \\ &= \frac{1}{2}.\end{aligned}$$

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14. By writing each of the following functions $f(x)$ in the form $1 \cdot f(x)$, and using integration by parts, determine their indefinite integrals:

(a) $\log(x)$, (b) $\log^3(x)$, (c) $\cosh^{-1}(x)$, (d) $\tanh^{-1}(x)$, (e) $\sin(\log(x))$.

Solution:

(a) $\displaystyle\int 1 \cdot \log(x)\,dx = x\log(x) - \int x \cdot (1/x)\,dx = x\log(x) - x + c.$

(b) Here, we use integration by parts multiple times. We have:

$$\begin{aligned}\int 1 \cdot \log^3(x)\,dx &= x\log^3(x) - 3\int \log^2(x)\,dx \\ &= x\log^3(x) - 3x\log^2(x) + 6\int \log(x)\,dx \\ &= x\log^3(x) - 3x\log^2(x) + 6x\log(x) - 6x + c.\end{aligned}$$

(c) First, observe that:

$$\int 1 \cdot \cosh^{-1}(x)\,dx = x\cosh^{-1}(x) - \int \frac{x}{\sqrt{x^2-1}}\,dx.$$

Since the derivative of x^2 is $2x$, the remaining integral is of the form $f'(x)f(x)^\alpha$, hence can be directly integrated. We have:

$$x\cosh^{-1}(x) - \sqrt{x^2-1} + c.$$

(d) $\displaystyle\int 1 \cdot \tanh^{-1}(x)\,dx = x\tanh^{-1}(x) - \int \frac{x}{1-x^2}\,dx = x\tanh^{-1}(x) + \frac{1}{2}\log(x^2-1) + c.$

(e) We begin by performing one integration by parts:

$$\int 1 \cdot \sin(\log(x))\,dx = x\sin(\log(x)) - \int \cos(\log(x))\,dx.$$

We now iterate, performing a second integration by parts:

$$\int \sin(\log(x))\,dx = x\sin(\log(x)) - x\cos(\log(x)) - \int \sin(\log(x))\,dx.$$

But notice that the new integral is the same as the original one - hence, rearranging this equation, we see that:

$$\int \sin(\log(x))\,dx = \frac{x}{2}\left(\sin(\log(x)) - \cos(\log(x))\right) + c.$$

Reduction formulae

15.(a) Show that for $n \geq 1$, we have:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx,$$

Hence, evaluate $\int \sin^6(x)\,dx$.

(b) Using (a), show that the integral $I_n = \int_0^{\pi/2} \sin^n(x)\,dx$ satisfies $I_n = (n-1)I_n/n$. Hence, evaluate I_2 and I_4.

Solution: (a) Integrating by parts, we have:

$$\int \sin^n(x)\,dx = \int \sin(x)\cdot\sin^{n-1}(x)\,dx = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x)\,dx.$$

Expanding $\cos^2(x) = 1 - \sin^2(x)$, we can rearrange this to read:

$$\int \sin^n(x)\,dx = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx.$$

Rearranging, we then have:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx,$$

as required.

This allows us to evaluate the given integral, by applying the recurrence relation repeatedly:

$$\begin{aligned}\int \sin^6(x)\,dx &= -\frac{1}{6}\cos(x)\sin^5(x) + \frac{5}{6}\int \sin^4(x)\,dx \\ &= -\frac{1}{6}\cos(x)\sin^5(x) + \frac{5}{6}\left(-\frac{1}{4}\cos(x)\sin^3(x) + \frac{3}{4}\int \sin(x)\,dx\right) \\ &= -\frac{1}{6}\cos(x)\sin^5(x) - \frac{5}{24}\cos(x)\sin^3(x) - \frac{5}{8}\cos(x) + c.\end{aligned}$$

(b) Simply inserting the limits into our recurrence relation, we have:

$$\int_0^{\pi/2} \sin^n(x)\,dx = \left[-\frac{1}{n}\cos(x)\sin^{n-1}(x)\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}(x)\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}(x)\,dx.$$

This immediately gives $I_n = (n-1)I_{n-2}/n$. To evaluate the given integrals, we note:

$$I_0 = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}.$$

Hence:

$$I_2 = \frac{1}{2}I_0 = \frac{\pi}{4}, \qquad I_4 = \frac{3}{4}I_2 = \frac{3\pi}{8}.$$

16. Establish reduction formulae for each of the following parametric integrals:

$$\text{(a) } I_n = \int_0^\infty x^n e^{-x^2}\,dx, \quad \text{(b) } J_n = \int_0^\pi x^{2n}\cos(x)\,dx, \quad \text{(c) } K_n = \int_0^\infty x^{n-1}e^{-x}\,dx, \quad \text{(d) } L_n = \int_0^\infty \frac{dx}{(1+x^2)^n}.$$

Hence: (i) evaluate I_3, I_5; (ii) evaluate J_3; (iii) establish a general formula for K_n; (iv) evaluate L_4. $(*)$ Using part (c), suggest a reasonable definition of $z!$ where z is a complex number. Will this work for all complex numbers?

Solution:

(a) Since the derivative of e^{-x^2} is $-2xe^{-x^2}$, we can integrate xe^{-x^2} to get $-\frac{1}{2}e^{-x^2}$. Hence:

$$I_n = \int_0^\infty x^n e^{-x^2}\,dx = \left[-\frac{1}{2}x^{n-1}e^{-x^2}\right]_0^\infty + \frac{(n-1)}{2}\int_0^\infty x^{n-2}e^{-x^2}\,dx = \frac{(n-1)}{2}I_{n-2},$$

assuming that $n \geq 2$. To answer (i), we note that this implies:

$$I_3 = I_1 = \int_0^\infty xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^\infty = \frac{1}{2}.$$

We also have:

$$I_5 = 2I_3 = 1.$$

(b) Integrating by parts twice, we have:

$$\begin{aligned} J_n = \int_0^\pi x^{2n}\cos(x)\,dx &= \left[x^{2n}\sin(x)\right]_0^\pi - 2n\int_0^\pi x^{2n-1}\sin(x)\,dx \\ &= 2n\left[x^{2n-1}\cos(x)\right]_0^\pi - 2n(2n-1)\int_0^\pi x^{2n-2}\cos(x)\,dx \\ &= -2n\pi^{2n-1} - 2n(2n-1)J_{n-1}. \end{aligned}$$

To answer (ii), we note that this implies:

$$J_3 = -6\pi^5 - 30J_2 = -6\pi^5 - 30\left(-4\pi^3 - 12J_1\right) = -6\pi^5 + 120\pi^3 + 360\left(-2\pi\right) = -6\pi^5 + 120\pi^3 - 720\pi,$$

since $J_0 = 0$.

(c) Integrating by parts, we have:

$$K_n = \int_0^\infty x^{n-1}e^{-x}\,dx = \left[-x^{n-1}e^{-x}\right]_0^\infty + (n-1)\int_0^\infty x^{n-2}e^{-x}\,dx = (n-1)K_{n-1}.$$

In particular, to answer (iii), we can use this relation iteratively to give:

$$K_n = (n-1)K_{n-1} = (n-1)(n-2)K_{n-2} = (n-1)(n-2)(n-3)K_{n-3} = \cdots = (n-1)!K_1.$$

But we have:

$$K_1 = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty = 1.$$

Hence, $K_n = (n-1)!$ for all positive integers n.

(d) Here, we can only integrate 1 and differentiate the integrand. We have:

$$L_n = \int_0^\infty \frac{dx}{(1+x^2)^n} = \left[\frac{x}{(1+x^2)^n}\right]_0^\infty + 2n\int_0^\infty \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

$$= 2n\int_0^\infty \frac{(1+x^2)-1}{(1+x^2)^{n+1}}\,dx$$

$$= 2n\int_0^\infty \frac{1}{(1+x^2)^n}\,dx - 2n\int_0^\infty \frac{1}{(1+x^2)^{n+1}}\,dx.$$

This can be written as $L_n = 2nL_n - 2nL_{2n+1}$, which can be rearranged to read:

$$L_{n+1} = \frac{2n-1}{2n}L_n,$$

for all $n \geq 1$. To answer (iv), and evaluate L_4, we note:

$$L_4 = \frac{5}{6}L_3 = \frac{5}{6}\cdot\frac{3}{4}L_2 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}L_1.$$

But now we have:

$$L_1 = \int_0^\infty \frac{dx}{1+x^2} = [\arctan(x)]_0^\infty = \frac{\pi}{2}.$$

Hence:

$$L_4 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}.$$

From part (c), we know that:

$$(n-1)! = \int_0^\infty x^{n-1}e^{-x}\,dx.$$

This suggests that a possible definition of $z!$ for z a complex number is:

$$z! = \int_0^\infty x^z e^{-x}\,dx.$$

Importantly, for this definition to make sense, the integral needs to converge. For large values of x, the integrand is exponentially suppressed, so will be convergent. For small values of x, the integrand resembles $x^z e^{-z} \approx x^z$, which gives a contribution to the integral:

$$\int_0^\epsilon x^z\,dx = \left[\frac{x^{z+1}}{z+1}\right]_0^\epsilon.$$

This is finite only if $\mathrm{Re}(z)+1>0$, hence this definition only makes sense if $\mathrm{Re}(z) > -1$.

We can extend the definition by *analytic continuation* however, a tool from complex analysis, which you will learn about in Part IB Mathematics for Natural Sciences.

Miscellaneous integrals

17. Evaluate the following integrals, using the most efficient method in each case:

(a) $\displaystyle\int_4^9 \frac{dx}{\sqrt{x}-1}$

(b) $\displaystyle\int_{\pi/3}^{\pi/4} \frac{1+\tan^2(x)}{(1+\tan(x))^2}\,dx$

(c) $\displaystyle\int \frac{e^{2x}-2e^x}{e^{2x}+1}\,dx$

(d) $\displaystyle\int \frac{dx}{1+3\cos^2(x)}$

(e) $\displaystyle\int_2^3 \frac{2x+1}{x(x+1)}\,dx$

(f) $\displaystyle\int \frac{1}{2\sqrt{x}}e^{\sqrt{x}}\,dx$

(g) $\displaystyle\int x^3 e^{-x^4}\,dx$

(h) $\displaystyle\int \left(\frac{\sin(2x)}{\sin^2(x)+\log(x)} + \frac{1}{x(\sin^2(x)+\log(x))}\right) dx$

(i) $\displaystyle\int x\sqrt{3-2x}\,dx$

(j) $\displaystyle\int \frac{\sin(x)}{\cos^2(x)-5\cos(x)+6}\,dx$

(k) $\displaystyle\int \frac{\log(x)}{x^4}\,dx$

(l) $\displaystyle\int \sqrt{1-x^2}\,dx$

(m) $\displaystyle\int_{\pi/3}^{\pi/2} \tan(x)\cos^4(x)\,dx$

(n) $\displaystyle\int_1^5 x^2\log(x)\,dx$

(o) $\displaystyle\int e^x\sinh(3x)\,dx$

(p) $\displaystyle\int \frac{\arctan(x)}{x^2}\,dx$

(q) $\displaystyle\int_{e^3}^{e^4} \frac{3\log(x)-4}{x\log^2(x)-3x\log(x)+2x}\,dx$

(r) $\displaystyle\int_0^{\pi/6} x\sin(3x)\,dx$

(s) $\displaystyle\int \sin(2x)e^{\sin^2(x)}\,dx$

(t) $\displaystyle\int \frac{dx}{\cos^2(x)(\tan^3(x)-\tan(x))}$

(u) $\displaystyle\int_{-1/\pi}^{1/\pi} \sin^2(3x^3+2x)\log\left[\frac{1-x^5}{1+x^5}\right] dx$

(v) $\displaystyle\int \sin(2x)\cos(x)\,dx$

(w) $\displaystyle\int x\log(x)\,dx$

(x) $\displaystyle\int \frac{dx}{x\log(x)}$

(y) $\displaystyle\int \frac{\sinh^3(x)}{\cosh^2(x)}\,dx$

(z) $\displaystyle\int \frac{1}{\sin^2(3x+1)}\,dx$

Solution:

(a) Let $u=\sqrt{x}$, to clear the square root. Then $u^2=x$, so that $2udu=dx$. Thus we have:

$$\int_4^9 \frac{dx}{\sqrt{x}-1} = \int_2^3 \frac{2udu}{u-1} = \int_2^3 \left(\frac{2(u-1)}{u-1}+\frac{2}{u-1}\right) du = 2+2\left[\log(u-1)\right]_2^3 = 2+2\log(2).$$

(b) Note that $1+\tan^2(x) = \sec^2(x)$, so the numerator is just $\sec^2(x)$ in disguise. This suggests the obvious substitution $u = \tan(x)$, which gives $du = \sec^2(x)dx$ and the limits change from $[\pi/3, \pi/4] \mapsto [\sqrt{3}, 1]$. Hence we have:

$$\int_{\pi/3}^{\pi/4} \frac{1+\tan^2(x)}{(1+\tan(x))^2}\, dx = \int_{\sqrt{3}}^{1} \frac{du}{(1+u)^2} = \left[-\frac{1}{1+u}\right]_{\sqrt{3}}^{1} = \frac{1}{1+\sqrt{3}} - \frac{1}{2}.$$

(c) An obvious substitution is $u = e^x$. We have $du = e^x dx$, so:

$$\begin{aligned}\int \frac{e^{2x} - 2e^x}{e^{2x}+1}\, dx &= \int \frac{u-2}{u^2+1}\, du \\ &= \int \left(\frac{u}{u^2+1} - \frac{2}{u^2+1}\right) du \\ &= \frac{1}{2}\log(u^2+1) - 2\arctan(u) + c \\ &= \frac{1}{2}\log(e^{2x}+1) - 2\arctan(e^x) + c.\end{aligned}$$

(d) This is quite sneaky. Multiply the numerator and denominator by $\sec^2(x)$, to give:

$$\int \frac{\sec^2(x)dx}{\sec^2(x)+3} = \int \frac{\sec^2(x)dx}{\tan^2(x)+4}.$$

Now, the obvious substitution is $u = \tan(x)$. We have $du = \sec^2(x)dx$, which gives:

$$\int \frac{du}{u^2+4} = \frac{1}{4}\int \frac{du}{(u/2)^2+1} = \frac{2}{4}\arctan\left(\frac{u}{2}\right) + c = \frac{1}{2}\arctan\left(\frac{1}{2}\tan(x)\right) + c.$$

(e) Note that:

$$\int_2^3 \frac{2x+1}{x(x+1)}\, dx = \int_2^3 \frac{2x+1}{x^2+x}\, dx = \left[\log(x^2+x)\right]_2^3 = \log(12) - \log(6) = \log(2).$$

(f) Note that the derivative of $\sqrt{x}$ is $1/2\sqrt{x}$. This gives:

$$\int \frac{1}{2\sqrt{x}} e^{\sqrt{x}}\, dx = e^{\sqrt{x}} + c,$$

by thinking about the reverse chain rule.

(g) Note that the derivative of $-x^4$ is $-4x^3$. Hence:

$$\int x^3 e^{-x^4}\, dx = -\frac{1}{4}e^{-x^4} + c,$$

by thinking about the reverse chain rule.

(h) We have:

$$\begin{aligned}\int \left(\frac{\sin(2x)}{\sin^2(x)+\log(x)} + \frac{1}{x(\sin^2(x)+\log(x))}\right) dx &= \int \left(\frac{2\sin(x)\cos(x) + 1/x}{\sin^2(x)+\log(x)}\right) dx \\ &= \log\left(\sin^2(x)+\log(x)\right) + c.\end{aligned}$$

(i) Make the substitution $u = 3 - 2x$, to clear the square root. Then $du = -2dx$, and $x = \frac{3}{2} - \frac{1}{2}u$. Hence:

$$\begin{aligned}\int x\sqrt{3-2x}\,dx &= -\frac{1}{2}\int\left(\frac{3}{2}-\frac{1}{2}u\right)u^{1/2}\,du\\ &= \frac{1}{4}\int\left(u^{3/2}-3u^{1/2}\right)\,du\\ &= \frac{1}{4}\left(\frac{2}{5}u^{5/2}-2u^{3/2}\right)+c\\ &= \frac{1}{10}(3-2x)^{5/2}-\frac{1}{2}(3-2x)^{3/2}+c.\end{aligned}$$

(j) Evidently, the substitution $u = \cos(x)$ will work. We have $du = -\sin(x)dx$, hence:

$$\int \frac{\sin(x)}{\cos^2(x)-5\cos(x)+6}\,dx = -\int\frac{du}{u^2-5u+6} = -\int\frac{du}{(u-5/2)^2-1/4} = 4\int\frac{du}{1-(2u-5)^2}.$$

Now, using a standard inverse hyperbolic integral, we have:

$$4\int\frac{du}{1-(2u-5)^2} = 2\tanh^{-1}(2u-5)+c = 2\tanh^{-1}(2\cos(x)-5)+c.$$

(k) We use integration by parts. We have:

$$\int\frac{\log(x)}{x^4}\,dx = -\frac{\log(x)}{3x^3}+\frac{1}{3}\int\frac{1}{x^4}\,dx = -\frac{\log(x)}{3x^3}-\frac{1}{9}\frac{1}{x^3}+c.$$

(l) Here, a trigonometric substitution is appropriate. Let $x = \cos(\theta)$. Then $dx = -\sin(\theta)d\theta$, which gives:

$$\int\sqrt{1-x^2}\,dx = -\int\sqrt{1-\cos^2(\theta)}\sin(\theta)\,d\theta = -\int\sin^2(\theta)\,d\theta.$$

Using the standard trick for integrating $\sin^2(\theta)$, we have:

$$-\int\sin^2(\theta)\,d\theta = \frac{1}{2}\int\left(\cos(2\theta)-1\right)\,d\theta = \frac{1}{2}\left(\frac{\sin(2\theta)}{2}-\theta\right)+c = \frac{1}{4}\sin(2\arcsin(x))-\frac{1}{2}\arcsin(x)+c.$$

We can slightly simplify this by using $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\sin(\theta)\sqrt{1-\sin^2(\theta)}$. This leaves:

$$\frac{1}{2}x\sqrt{1-x^2}-\frac{1}{2}\arcsin(x)+c.$$

(m) We have:

$$\int_{\pi/3}^{\pi/2}\tan(x)\cos^4(x)\,dx = \int_{\pi/3}^{\pi/2}\sin(x)\cos^3(x)\,dx = \left[-\frac{1}{4}\cos^4(x)\right]_{\pi/3}^{\pi/2} = \frac{1}{4}\cdot\left(\frac{1}{2}\right)^4 = \frac{1}{64}.$$

(n) We have:

$$\int_1^5 x^2\log(x)\,dx = \left[\frac{1}{3}x^3\log(x)\right]_1^5 - \frac{1}{3}\int_1^5 x^2\,dx = \frac{125}{3}\log(5) - \frac{1}{9}(125-1) = \frac{125}{3}\log(5) - \frac{124}{9}.$$

(o) Writing the hyperbolic function in terms of exponentials, we have:

$$\int e^x\sinh(3x)\,dx = \frac{1}{2}\int\left(e^{4x}-e^{-2x}\right)\,dx = \frac{1}{2}\left(\frac{1}{4}e^{4x}-\frac{1}{2}e^{-2x}\right)+c = \frac{1}{8}e^{4x}-\frac{1}{4}e^{-2x}+c.$$

Alternatively, this can be done by parts.

(p) Here, we can integrate by parts:

$$\int\frac{\arctan(x)}{x^2}\,dx = -\frac{\arctan(x)}{x}+\int\frac{1}{x(1+x^2)}\,dx.$$

Using partial fractions, we have:

$$\frac{1}{x(1+x^2)} = \frac{A}{x}+\frac{Bx+C}{1+x^2} \qquad \Rightarrow \qquad 1 = A(1+x^2)+Bx^2+Cx.$$

Here, $A=1, B=-1, C=0$. Thus we have:

$$-\frac{\arctan(x)}{x}+\int\left(\frac{1}{x}-\frac{x}{1+x^2}\right)\,dx = -\frac{\arctan(x)}{x}+\log(x)-\frac{1}{2}\log(1+x^2)+c.$$

(q) Evidently, a sensible substitution is $u=\log(x)$; note that we have $du=dx/x$, and the limits transform as $[e^3,e^4]\mapsto[3,4]$. Thus:

$$\begin{aligned}\int_{e^3}^{e^4}\frac{3\log(x)-4}{x\log^2(x)-3x\log(x)+2x}\,dx &= \int_3^4\frac{3u-4}{u^2-3u+2}\,du\\
&= \frac{3}{2}\int_3^4\frac{2u-3}{u^2-3u+2}\,du+\frac{1}{2}\int_3^4\frac{1}{u^2-3u+2}\,du\\
&= \frac{3}{2}\left[\log\left(u^2-3u+2\right)\right]_3^4+\frac{1}{2}\int_3^4\frac{1}{(u-3/2)^2-1/4}\,du\\
&= \frac{3}{2}\left(\log(6)-\log(2)\right)-2\int_3^4\frac{du}{1-(2u-3)^2}\,du\\
&= \frac{3}{2}\log(3)-\left[\operatorname{artanh}(2u-3)\right]_3^4\\
&= \frac{3}{2}\log(3)-\operatorname{artanh}(5)+\operatorname{artanh}(3).\end{aligned}$$

If we really want to tidy everything up, we could simplify the two inverse hyperbolic functions. We let:

$$t = \operatorname{artanh}(5)-\operatorname{artanh}(3).$$

Now by the hyperbolic compound angle identity, we have:

$$\tanh(t) = \frac{5-3}{1-15} = -\frac{2}{14} = -\frac{1}{7}.$$

This leaves us with:

$$\frac{3}{2}\log(3) + \operatorname{artanh}\left(\frac{1}{7}\right).$$

(r) This can be done easily with integration by parts:

$$\begin{aligned}\int_0^{\pi/6} x\sin(3x)\,dx &= \left[-\frac{1}{3}x\cos(3x)\right]_0^{\pi/6} + \frac{1}{3}\int_0^{\pi/6}\cos(3x)\,dx \\ &= \frac{1}{3}\left[\frac{1}{3}\sin(3x)\right]_0^{\pi/6} \\ &= \frac{1}{9}.\end{aligned}$$

(s) Observe that the derivative of $\sin^2(x)$ is $2\sin(x)\cos(x) = \sin(2x)$. Hence:

$$\int \sin(2x)e^{\sin^2(x)}\,dx = e^{\sin^2(x)} + c.$$

(t) The integral in question can be rewritten in the form:

$$\int \frac{\sec^2(x)}{\tan^3(x) - \tan(x)}\,dx.$$

This suggests the substitution $u = \tan(x)$, which gives $du = \sec^2(x)dx$. Hence we have:

$$\int \frac{du}{u^3 - u} = \int \frac{du}{u(u-1)(u+1)}.$$

We decompose the integrand into partial fractions via:

$$\frac{1}{u(u-1)(u+1)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1}.$$

This implies:

$$1 = A(u-1)(u+1) + Bu(u+1) + Cu(u-1).$$

Taking $u = 0$ gives $A = -1$. Taking $u = 1$ gives $B = 1/2$. Taking $u = -1$ gives $C = -1/2$. Thus we have:

$$\int \left(-\frac{1}{u} + \frac{1}{2(u-1)} - \frac{1}{2(u+1)}\right)\,du = -\log(u) + \frac{1}{2}\log(u-1) - \frac{1}{2}\log(u+1) + c.$$

Thus the final integral is:

$$\frac{1}{2}\log\left(\frac{u-1}{u^2(u+1)}\right) + c = \frac{1}{2}\log\left(\frac{\tan(x)-1}{\tan^2(x)(\tan(x)+1)}\right) + c.$$

(u) This is a trick question - it is an odd function integrated over a symmetric domain, so the integral is just zero.

(v) We have:

$$\int \sin(2x)\cos(x)\,dx = 2\int \sin(x)\cos^2(x)\,dx = -\frac{2}{3}\cos^3(x) + c.$$

(w) Using integration by parts, we have:

$$\int x\log(x)\,dx = \frac{1}{2}x^2\log(x) - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\log(x) - \frac{1}{4}x^2 + c.$$

(x) Here, an obvious substitution is $u = \log(x)$, so that $du = dx/x$. We then have:

$$\int \frac{dx}{x\log(x)} = \int \frac{du}{u} = \log(u) + c = \log(\log(x)) + c.$$

(y) This integral can be rewritten as:

$$\int \frac{\sinh^3(x)}{\cosh^2(x)}\,dx = \int \frac{(\cosh^2(x) - 1)}{\cosh^2(x)}\sinh(x)\,dx,$$

suggesting the substitution $u = \cosh(x)$, which gives $du = \sinh(x)dx$. We then have:

$$\int \frac{u^2 - 1}{u^2}\,du = \int \left(1 - \frac{1}{u^2}\right)\,du = u + \frac{1}{u} + c = \cosh(x) + \operatorname{sech}(x) + c.$$

(z) An easy one to finish! This is a standard integral, from the Basic Integrals section of the start of the sheet. We have:

$$\int \frac{1}{\sin^2(3x+1)}\,dx = \int \operatorname{cosec}^2(3x+1)\,dx = -\frac{1}{3}\cot(3x+1) + c.$$

The fundamental theorem of calculus

18. State both parts of the *fundamental theorem of calculus*. Use the fundamental theorem of calculus to evaluate the following derivatives:

$$\text{(a) } \frac{d}{dx}\int_1^x \frac{\log(t)\sin^2(t)}{t^2+7}\,dt, \quad \text{(b) } \frac{d}{dx}\left[\sum_{n=0}^N \binom{N}{n}\int_n^x \sin(y^2+y^6)\,dy\right], \quad \text{(c) } \frac{d}{dx}\left[\sin(x)\int_x^0 \sin(\cos(t))\,dt\right].$$

Solution: The fundamental theorem of calculus states two things:

- Integration reverses differentiation, in the sense that:

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a).$$

- Differentiation reverses integration, in the sense that:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Applying this result to the given derivatives:

(a) Straightforwardly, this derivative is given by: $\dfrac{\log(x)\sin^2(x)}{x^2+7}$.

(b) Since differentiation is linear, we have:

$$\begin{aligned}\frac{d}{dx}\left[\sum_{n=0}^N \binom{N}{n}\int_n^x \sin(y^2+y^6)\,dy\right] &= \sum_{n=0}^N \binom{N}{n}\frac{d}{dx}\int_n^x \sin(y^2+y^6)\,dy \\ &= \sum_{n=0}^N \binom{N}{n}\sin(x^2+x^6) \\ &= \sin(x^2+x^6)\sum_{n=0}^N \binom{N}{n}.\end{aligned}$$

Here's a fun thing: the sum of the binomial coefficients is always 2^N. To see why, study the identity (which is just the binomial expansion):

$$2^N = (1+1)^N = \sum_{n=0}^N \binom{N}{n}1^n\cdot 1^{N-n} = \sum_{n=0}^N \binom{N}{n}.$$

Hence, the derivative simplifies to $2^N\sin(x^2+x^6)$.

(c) First, we have:

$$\int_x^0 \sin(\cos(t))\,dt = -\int_0^x \sin(\cos(t))\,dt.$$

Hence, using the product rule, we have:

$$\frac{d}{dx}\left[\sin(x)\int_x^0 \sin(\cos(t))\,dt\right] = -\frac{d}{dx}\left[\sin(x)\int_0^x \sin(\cos(t))\,dt\right] = -\cos(x)\int_0^x \sin(\cos(t))\,dt - \sin(x)\sin(\cos(x)).$$

This cannot be further simplified (we cannot perform the integral in terms of elementary functions).

19. Without evaluating the integrals, determine the local extrema of the functions F_1, F_2 defined by:

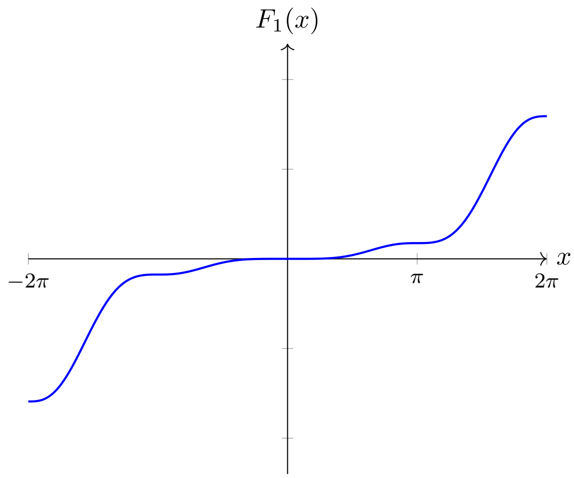
$$\text{(a) } F_1(x) = \int_0^x t^2 \sin^2(t)\, dt, \qquad \text{(b) } F_2(x) = \int_{-\infty}^x e^{-t^2}\, dt.$$

Hence, sketch the graphs of the functions F_1, F_2. [*Note:* $F_2(x) \to \sqrt{\pi}$ *as* $x \to \infty$*; see Question 23!*]

Solution: (a) We have:

$$F_1'(x) = x^2 \sin^2(x).$$

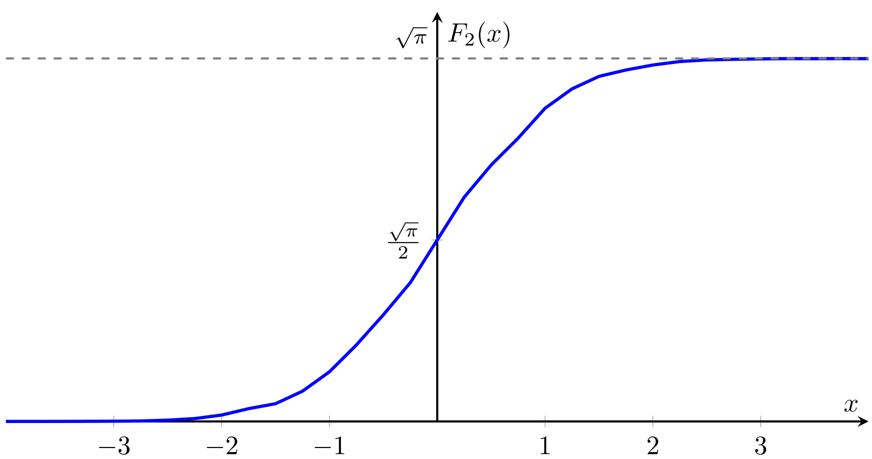
Hence, the stationary points of the function occur at $x = n\pi$, for n an integer. The function is zero at $x = 0$, and is strictly increasing (because as x increases, more area under the function $t^2 \sin^2(t)$ contributes!). Therefore, the function looks like a series of inflection points.



(b) We have:

$$F_2'(x) = e^{-x^2},$$

so this function has no stationary points. The function is again strictly increasing. We are given that $F_2(x) \to \sqrt{\pi}$ as $x \to \infty$, and we note that $F_2(x) \to 0$ as $x \to -\infty$. The function is everywhere positive. Hence the graph looks like:



(†) Leibniz's integral rule

20. Using the multivariable chain rule (we'll study it properly next term!), derive *Leibniz's integral rule*:

$$\frac{d}{dx}\int\limits_{a(x)}^{b(x)} f(x,t)\,dt = f(x,b(x))\frac{db}{dx} - f(x,a(x))\frac{da}{dx} + \int\limits_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt.$$

Give a geometric explanation for the rule in terms of changing areas. Verify that the rule holds in the following cases:

(a) $a(x) = 0$, $b(x) = 1 + x$, and $f(x,t) = t(x-t)$;

(b) $a(x) = \pi x^2$, $b(x) = x$, and $f(x,t) = 2x^2 t + x\sin(t)$.

Solution: Define a multivariable function:

$$F(a,b,x) = \int\limits_a^b f(x,t)\,dt.$$

Consider the case where the arguments a, b depend on x, so that we are considering $F(a(x), b(x), x)$. To take the derivative of this function with respect to x, we apply the chain rule separately to each argument, and then sum the results; this is the *multivariable chain rule*, which we shall study properly in Lent term. We have:

$$\frac{d}{dx}F(a(x),b(x),x) = \frac{\partial F}{\partial a}a'(x) + \frac{\partial F}{\partial b}b'(x) + \frac{\partial F}{\partial x}.$$

Observe that, by the fundamental theorem of calculus:

$$\frac{\partial F}{\partial b} = \frac{\partial}{\partial b}\int\limits_a^b f(x,t)\,dt = f(x,b).$$

Similarly, observe that:

$$\frac{\partial F}{\partial a} = \frac{\partial}{\partial a}\int\limits_a^b f(x,t)\,dt = -\frac{\partial}{\partial a}\int\limits_b^a f(x,t)\,dt = -f(x,a).$$

Finally, observe that the partial derivative of F with respect to its third argument is just:

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\int\limits_a^b f(x,t)\,dt = \int\limits_a^b \frac{\partial f}{\partial x}(x,t)\,dt,$$

since the arguments a, b are kept constant. Putting everything together, this gives:

$$\frac{d}{dx}F(a(x),b(x),x) = f(x,b(x))\frac{db}{dx} - f(x,a(x))\frac{da}{dx} + \int\limits_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)\,dt,$$

as required.

This has a very nice interpretation in terms of changing areas. The integral:

$$\int_{a(x)}^{b(x)} f(x,t)\,dt$$

represents the area under the curve $f(x,t)$, *considered as a function of* t, between $t = a(x)$ and $t = b(x)$. When we ask about the derivative with respect to x, we ask how this area changes as a function of x. There are three ways that this can happen:

- The function itself can change. If x changes by an amount Δx, resulting in a change Δf in f, then the change in area is:
$$\Delta A = \int_a^b \Delta f(x,t)\,dt.$$
Thus the rate of change of area is given by:
$$\int_a^b \frac{\partial f}{\partial x}(x,t)\,dt.$$
- The upper limit can change. If x changes by an amount Δx, resulting in a change Δb in b, then the change in area is:
$$\Delta A = \int_a^{b+\Delta b} f(x,t)\,dt - \int_a^b f(x,t)\,dt = \int_b^{b+\Delta b} f(x,t)\,dt \approx f(x,b)\Delta b.$$
Hence the rate of change of area is given by:
$$\frac{db}{dx} f(x,b(x)).$$
- Finally, the lower limit can change. If x changes by an amount Δx, resulting in a change Δa in a, then the change in area is:
$$\Delta A = \int_{a+\Delta a}^{b} f(x,t)\,dt - \int_a^b f(x,t)\,dt = -\int_a^{a+\Delta a} f(x,t)\,dt \approx -f(x,a)\Delta a.$$
Hence the rate of change of area is given by:
$$-\frac{da}{dx} f(x,a(x)).$$

Putting together all these possible ways of changing the area, we get the Leibniz integral rule.

(a) For $a(x) = 0, b(x) = 1 + x$ and $f(x,t) = t(x-t)$, the integral in question is:

$$\int_0^{1+x} t(x-t)\,dt = \int_0^{1+x} \left(tx - t^2\right)\,dt = \left[\frac{1}{2}xt^2 - \frac{1}{3}t^3\right]_0^{1+x} = \frac{1}{2}x(1+x)^2 - \frac{1}{3}(1+x)^3 = \frac{1}{6}x^3 - \frac{1}{2}x - \frac{1}{3}.$$

Taking the derivative, we see that:

$$\frac{d}{dx}\int_0^{1+x} t(x-t)\,dt = \frac{1}{2}x^2 - \frac{1}{2}.$$

Compare with the result from using the Leibniz integral rule. We have:

$$\frac{d}{dx}\int\limits_0^{1+x} t(x-t)\,dt = (1+x)(x-(1+x))\frac{d}{dx}(1+x) + \int\limits_0^{1+x} t\,dt = -(1+x) + \frac{1}{2}(1+x)^2 = \frac{1}{2}x^2 - \frac{1}{2}.$$

Thus, the rule works.

(b) For $a(x) = \pi x^2, b(x) = x$ and $f(x,t) = 2x^2t + x\sin(t)$, the integral in question is:

$$\begin{aligned}\int\limits_{\pi x^2}^{x} \left(2x^2t + x\sin(t)\right)\,dt &= \left[x^2t^2 - x\cos(t)\right]_{\pi x^2}^{x} \\ &= x^4 - x\cos(x) - \pi^2x^6 + x\cos(\pi x^2).\end{aligned}$$

Taking the derivative, we see that:

$$\frac{d}{dx}\int\limits_{\pi x^2}^{x} \left(2x^2t + x\sin(t)\right)\,dt = 4x^3 - \cos(x) + x\sin(x) - 6\pi^2x^5 + \cos(\pi x^2) - 2\pi x^2\sin(\pi x^2).$$

Compare with the result from using the Leibniz integral rule. We have:

$$\begin{aligned}\frac{d}{dx}\int\limits_{\pi x^2}^{x} \left(2x^2t + x\sin(t)\right)\,dt &= \left(2x^3 + x\sin(x)\right) - \left(2\pi x^4 + x\sin(\pi x^2)\right)(2\pi x) + \int\limits_{\pi x^2}^{x} \left(4xt + \sin(t)\right)\,dt \\ &= 2x^3 + x\sin(x) - 4\pi^2x^5 - 2\pi x^2\sin(\pi x^2) + \left[2xt^2 - \cos(t)\right]_{\pi x^2}^{x} \\ &= 2x^3 + x\sin(x) - 4\pi^2x^5 - 2\pi x^2\sin(\pi x^2) + 2x^3 - \cos(x) - 2\pi^2x^5 + \cos(\pi x^2) \\ &= 4x^3 - \cos(x) + x\sin(x) - 6\pi^2x^5 + \cos(\pi x^2) - 2\pi x^2\sin(\pi x^2).\end{aligned}$$

Thus, the rule works again!

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21. Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{d}{dx}\displaystyle\int\limits_{\sin(1/x)}^{\sqrt{x}} \frac{2t^4+1}{(t-2)(t^2+3)}\,dt.$

➾ **Solution:** We start by evaluating the derivative using the Leibniz integral rule. We have:

$$\frac{d}{dx}\int\limits_{\sin(1/x)}^{\sqrt{x}} \frac{2t^4+1}{(t-2)(t^2+3)}\,dt = \frac{2x^2+1}{(\sqrt{x}-2)(x+3)}\cdot\frac{1}{2\sqrt{x}} - \frac{2\sin^4(1/x)+1}{(\sin(1/x)-2)(\sin^2(1/x)+3)}\left(-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right).$$

We now take the limit as $x\to\infty$. The second term vanishes, because $1/x^2\to 0$ as $x\to\infty$, and $\sin(1/x), \cos(1/x)$ are bounded between -1 and 1.

The first term gives:

$$\lim_{x\to\infty}\frac{2x^2+1}{(\sqrt{x}-2)(x+3)}\cdot\frac{1}{2\sqrt{x}} = \lim_{x\to\infty}\frac{2+1/x^2}{2(1-2/\sqrt{x})(1+3/x)} = 1.$$

Hence, the limit is 1.

22. For all values of x, evaluate the integrals:

$$\text{(a) } f(x) = \int_0^1 \frac{t^x - 1}{\log(t)}\, dt, \qquad \text{(b) } g(x) = \int_0^\infty \frac{\log(1 + x^2t^2)}{1 + t^2}\, dt, \qquad \text{(c) } h(x) = \int_0^1 \frac{\sin(x\log(t))}{\log(t)}\, dt,$$

by considering the derivatives $f'(x), g'(x), h'(x)$. This method is sometimes called *Feynman's trick* for integration.

Solution:

(a) Taking the derivative, by differentiating under the integral sign, we have:

$$f'(x) = \int_0^1 \frac{\partial}{\partial x}\left(\frac{t^x - 1}{\log(t)}\right) dt.$$

To take the derivative, observe that $t^x = e^{x\log(t)}$. Hence we have:

$$f'(x) = \int_0^1 t^x \, dt = \left[\frac{t^{x+1}}{x+1}\right]_0^1 = \frac{1}{x+1},$$

provided that $x > -1$ (else the derivative does not exist). Integrating this expression, we have:

$$f(x) = \log(x+1) + c.$$

To evaluate the value of the constant, observe that $f(0) = 0$, since $t^0 = 1$. Hence $f(x) = \log(x+1)$.

(b) Taking the derivative, by differentiating under the integral sign, we have:

$$g'(x) = \int_0^\infty \frac{2xt^2}{(1+t^2)(1+x^2t^2)}\, dt.$$

Splitting the integrand into partial fractions, we note:

$$\frac{2xt^2}{(1+t^2)(1+x^2t^2)} = \frac{2x}{x^2-1}\left(\frac{1}{1+t^2} - \frac{1}{1+x^2t^2}\right).$$

Hence we can integrate directly to give:

$$g'(x) = \frac{2x}{x^2-1}\left[\arctan(t) - \frac{1}{x}\arctan(xt)\right]_0^\infty = \frac{2x}{x^2-1}\left(1 - \frac{1}{x}\right)\frac{\pi}{2} = \frac{\pi}{x+1}.$$

Integrating this expression, we have:

$$g(x) = \pi\log(x+1) + c.$$

To evaluate the value of the constant, observe that $g(0) = 0$, since $\log(1) = 0$. Hence $g(x) = \pi\log(x+1)$.

(c) Taking the derivative, by differentiating under the integral sign, we have:

$$h'(x) = \int_0^1 \cos(x \log(t))\, dt.$$

Integrating by parts, by writing $1 \cdot \cos(x \log(t))$, we have:

$$\int_0^1 \cos(x \log(t))\, dt = [t \cos(x \log(t))]_0^1 + x \int_0^1 \sin(x \log(t))\, dt = 1 + x \int_0^1 \sin(x \log(t))\, dt.$$

Integrating by parts again, we have:

$$\int_0^1 \sin(x \log(t))\, dt = [t \sin(x \log(t))]_0^1 - x \int_0^1 \cos(x \log(t))\, dt = -x \int_0^1 \cos(x \log(t))\, dt.$$

Hence, we have shown that:

$$h'(x) = 1 + x(-xh'(x)) = 1 - x^2 h'(x).$$

Rearranging, we have:

$$h'(x) = \frac{1}{1 + x^2}.$$

Integrating, we have:

$$h(x) = \arctan(x) + c,$$

for a constant c. Taking $x = 0$, we see that $h(0) = 0$. Hence $c = 0$, and it follows that $h(x) = \arctan(x)$.

23. This question determines the Gaussian integral in a different way to the lectures (you will use a transformation to polar coordinates on the next sheet!). Define:

$$f(x) = \left(\int_0^x e^{-t^2}\, dt \right)^2, \qquad \text{and} \qquad g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{1+t^2}\, dt.$$

Show that $f'(x) + g'(x) = 0$, and hence deduce that $f(x) + g(x) = \pi/4$. Conclude that $\int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}$.

➾ **Solution:** By the chain rule, and the fundamental theorem of calculus, we have:

$$f'(x) = 2e^{-x^2} \int_0^x e^{-t^2}\, dt.$$

On the other hand, by differentiating under the integral sign, we have:

$$g'(x) = -2x \int_0^1 e^{-x^2(t^2+1)}\, dt = -2xe^{-x^2} \int_0^1 e^{-x^2t^2}\, dt.$$

Substituting $u = xt$, so that $du = xdt$, in the final integral in the expression for $g'(x)$, we have:

$$g'(x) = -2e^{-x^2} \int_0^x e^{-u^2}\, du.$$

Hence $f'(x) + g'(x) = 0$, as required.

We immediately deduce that $f(x) + g(x)$ is a constant. To evaluate the value of the constant, we consider $x = 0$. Then $f(0) = 0$, whilst:

$$g(0) = \int_0^1 \frac{dt}{1+t^2} = [\arctan(t)]_0^1 = \frac{\pi}{4}.$$

So we see that indeed $f(x) + g(x) = \pi/4$, as required.

To finish, take the limit of the equation $f(x)+g(x) = \pi/4$ as $x \to \infty$. Inside the integrand of $g(x)$, we have $e^{-x^2(t^2+1)} \to 0$ for all t, so that the contribution from $g(x)$ vanishes. Meanwhile,

$$f(x) \to \left(\int_0^\infty e^{-t^2}\, dt \right)^2.$$

The right hand side of the equation is a constant in x, so is unchanged by taking the limit. Hence we have shown:

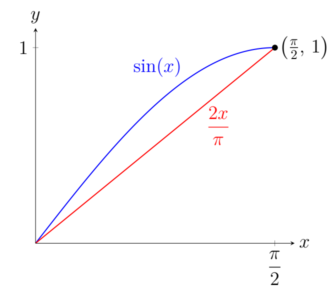
$$\left(\int_0^\infty e^{-t^2}\, dt \right)^2 = \frac{\pi}{4} \qquad \Rightarrow \qquad \int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2},$$

as required.

(†) Integral inequalities

24. Using a sketch, show that $\sin(x) \geq 2x/\pi$ for $0 \leq x \leq \pi/2$. Hence show that $\displaystyle\int_0^{\pi/2} \frac{x^2}{1+\sin^2(x)}\,dx < \frac{\pi^3}{8}\left(1-\frac{\pi}{4}\right)$.

➡ Solution: Plotting the graphs of $\sin(x)$ and $2x/\pi$, we have:



Hence, we see from the figure that $\sin(x) \geq 2x/\pi$ for $x \in [0, \pi/2]$. Applying this to the integral, we have:

$$\int_0^{\pi/2} \frac{x^2}{1+\sin^2(x)}\,dx \leq \int_0^{\pi/2} \frac{x^2}{1+4x^2/\pi^2}\,dx = \frac{\pi^2}{4}\int_0^{\pi/2}\left(1-\frac{1}{1+4x^2/\pi^2}\right)dx = \frac{\pi^2}{4}\left(\frac{\pi}{2}-\left[\frac{\pi}{2}\arctan\left(\frac{2x}{\pi}\right)\right]_0^{\pi/2}\right) = \frac{\pi^3}{8}\left(1-\frac{\pi}{4}\right),$$

as required.

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25. State and prove *Schwarz's inequality* for integrals. Use it to show that $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{x^2+1}}\,dx < \sqrt{\frac{\pi}{4}\arctan\left(\frac{\pi}{2}\right)}$.

➡ Solution: *Schwarz's inequality* states $\displaystyle\left(\int_a^b f(x)g(x)\,dx\right)^2 \leq \int_a^b f(x)^2\,dx \cdot \int_a^b g(x)^2\,dx$. To prove this, let $\lambda \in \mathbb{R}$, and consider:

$$0 \leq \int_a^b (f(x)+\lambda g(x))^2\,dx = \lambda^2\int_a^b g(x)^2\,dx + 2\lambda\int_a^b f(x)g(x)\,dx + \int_a^b f(x)^2.$$

The right hand side is a quadratic in λ. But this quadratic is positive for *all* values of λ, hence it must have discriminant less than or equal to zero. The discriminant is:

$$4\left(\int_a^b f(x)g(x)\right)^2 - 4\int_a^b f(x)^2\,dx \cdot \int_a^b g(x)^2\,dx \leq 0,$$

which on rearrangement produces Schwarz's inequality.

Applying Schwarz's inequality to the given integral, we have:

$$\left(\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{x^2+1}}\,dx\right)^2 \leq \int_0^{\pi/2}\sin^2(x)\,dx\int_0^{\pi/2}\frac{1}{x^2+1}\,dx = \left(\frac{1}{2}\int_0^{\pi/2}(1-\cos(x))\,dx\right)[\arctan(x)]_0^{\pi/2} = \frac{\pi}{4}\arctan\left(\frac{\pi}{2}\right).$$

Taking the square root, we get the desired result.

Part IA: Mathematics for Natural Sciences B
Examples Sheet 8: Probability spaces, conditional probability, and combinatorics

Model Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

Sample spaces and events

1. In an experiment, two fair four-sided dice are rolled. We define:

$$S_1 = \{(i,j) : i \text{ is the result of the first die, } j \text{ is the result of the second die}\},$$
$$S_2 = \{\text{the sum of the results is odd, the sum of the results is even}\},$$
$$S_3 = \{\text{the sum of the results is prime, the first die shows } 1\text{, the first die shows } 2\}.$$

Which of S_1, S_2, S_3 are valid sample spaces for the experiment?

Solution: Recall that a *sample space* for an experiment is a set that includes all possible outcomes of an experiment. In the above cases:

- S_1 covers all possible pairs of dice rolls, hence covers all outcomes of the experiment. Thus it is a valid sample space.
- S_2 also covers all possible outcomes of the experiment, since the sum of the dice must be odd *or* even. Thus it is also a valid sample space.
- S_3 does not cover all possible outcomes of the experiment, since if the first dice shows 2 and the second dice shows 4, then the sum of the results is 6. This is not prime, so this outcome is not included in the proposed sample space. Hence S_3 is not a valid sample space.

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2. Given a (discrete) sample space S, define an *event*. Write down in set notation:

(a) a sample space for the result of a 12-sided die roll, the event corresponding to getting a three, the event corresponding to getting an even result, and the event corresponding to getting a prime result;

(b) a sample space for the result of flipping three coins, the event corresponding to getting all tails, the event corresponding to getting an even number of tails, and the event corresponding to getting more heads than tails.

Solution: An *event* is a subset $E \subseteq S$ of the sample space. For the given experiments, we have:

(a) A sample space for the result of a 12-sided die roll is just $S = \{1, 2, 3, 4, 5, 6, ..., 12\}$ corresponding to all possible numbers shown by the dice. The event corresponding to getting a three is $E_3 = \{3\}$. The event corresponding to getting an even result if $E_{\text{even}} = \{2, 4, 6, 8, 10, 12\}$. The event corresponding to getting a prime result is $E_{\text{prime}} = \{2, 3, 5, 7, 11\}$.

(b) A sample space for the result of flipping three coins is just:

$$S = \{(i, j, k) : i, j, k \in \{\text{heads}, \text{tails}\}\}.$$

The event corresponding to getting all tails is $E_{\text{all tails}} = \{(\text{tails}, \text{tails}, \text{tails})\}$. The even corresponding to getting an even number of tails is:

$$E_{\text{even tails}} = \{(\text{tails}, \text{tails}, \text{heads}), (\text{tails}, \text{heads}, \text{tails}), (\text{heads}, \text{tails}, \text{tails}), (\text{heads}, \text{heads}, \text{heads})\}.$$

Finally, the event corresponding to getting more heads than tails is:

$$E_{\text{heads}>\text{tails}} = \{(\text{heads}, \text{heads}, \text{tails}), (\text{heads}, \text{tails}, \text{heads}), (\text{tails}, \text{heads}, \text{heads})\}.$$

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3. Suppose that S is a sample space and A, B, C are events. By drawing appropriate diagrams, show that:

(a) $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$, (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,

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➻ **Solution:**

4. Suppose that S is a sample space and A is an event. Simplify the expressions:

(a) $A \cap S$, (b) $(A \cap \overline{A}) \cup (A \cup \overline{A}) \cup \overline{A}$.

➡ **Solution:** We have:

(a) $A \cap S$ is the set of all outcomes which are in both the set A and the set S. But all outcomes are in S by definition, hence $A \cap S = A$.

(b) $A \cap \overline{A}$ is the set of all outcomes which are in the set A, and are also outside of the set A. Hence $A \cap \overline{A} = \emptyset$ is the empty set. $A \cup \overline{A}$ is the set of all outcomes which are in the set A or are out of the set A. Hence $A \cup \overline{A} = S$ is the entire sample space. This leaves:

$$\emptyset \cup A \cup \overline{A},$$

which is the set of all outcomes which are either in the empty set (there are none there though!), or in the set A, or out of the set A. This gives the entire sample space as a result, S.

Probability measures

5. Let S be a (discrete) sample space, and let $\mathcal{F}$ be the set of all associated events. What are the three basic *Kolmogorov axioms* that a probability measure $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ must satisfy?

Now suppose that $S = \{\omega_1, \omega_2, \omega_3\}$ is a sample space containing three outcomes.

(a) Write down the set of all possible events associated with this sample space.

(b) Show that if we are given the probabilities $\mathbb{P}(\{\omega_1\}), \mathbb{P}(\{\omega_2\})$, then we may deduce the probabilities of all other events using the basic axioms.

(c) Similarly, show that if we are instead given the probabilities $\mathbb{P}(\{\omega_1, \omega_2\}), \mathbb{P}(\{\omega_1, \omega_3\})$, then we may deduce the probabilities of all other events using the basic axioms.

➭ **Solution:** The *Kolmogorov axioms* state that a probability measure $\mathbb{P} : \mathcal{F} \to \mathbb{R}$, a function on the space of events which tells us the probability of that event, must satisfy the following:

(K1) For any event $E \subseteq S$, we have $\mathbb{P}(E) \geq 0$.

(K2) The entire sample space has probability 1, $\mathbb{P}(S) = 1$.

(K3) If $E_1, E_2, \ldots$ are a series of disjoint events (that is, they all have zero intersection with one another), we have:

$$\mathbb{P}\left(E_1 \cup E_2 \cup \cdots\right) = \mathbb{P}(E_1) + \mathbb{P}(E_2) + \cdots .$$

From these basic axioms, we can deduce other results. For example:

Complement rule: For any event E, we have $\mathbb{P}(\overline{E}) = 1 - \mathbb{P}(E)$.

Proof: The events $E, \overline{E}$ are disjoint. Hence $\mathbb{P}(E \cup \overline{E}) = \mathbb{P}(E) + \mathbb{P}(\overline{E})$. However, $E \cup \overline{E} = S$, the entire sample space, so $\mathbb{P}(E \cup \overline{E}) = \mathbb{P}(S) = 1$. Thus we have:

$$\mathbb{P}(E) + \mathbb{P}(\overline{E}) = 1 \qquad \Leftrightarrow \qquad \mathbb{P}(\overline{E}) = 1 - \mathbb{P}(E),$$

as required. □

We now consider the sample space $S = \{\omega_1, \omega_2, \omega_3\}$ given in the question.

(a) The set of all events associated with this sample space is:

$$\mathcal{F} = \{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_3\}, \{\omega_1, \omega_2, \omega_3\}\}.$$

(b) Given the probabilities $\mathbb{P}(\{\omega_1\}), \mathbb{P}(\{\omega_2\})$, we can deduce that probability of the event $\{\omega_3\}$:

$$1 = \mathbb{P}(S) = \mathbb{P}(\{\omega_1\} \cup \{\omega_2\} \cup \{\omega_3\}) = \mathbb{P}(\{\omega_1\}) + \mathbb{P}(\{\omega_2\}) + \mathbb{P}(\{\omega_3\}),$$

so we have:

$$\mathbb{P}(\{\omega_3\}) = 1 - \mathbb{P}(\{\omega_2\}) + \mathbb{P}(\{\omega_3\}).$$

This allows us to deduce the probabilities of all other events, since we can write $\{\omega_1, \omega_2\} = \{\omega_1\} \cup \{\omega_2\}$, etc.

(c) Given the probabilities

6.

(a) From the axioms for a probability measure, prove that for any two events A, B (not necessarily exclusive), we have $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. Generalise this formula to three events A, B, C.

(b) A card is drawn randomly from a standard pack. Using the generalised formula in part (a), determine the probability that the card either shows a prime number, is a spade, or is red.

Solution: (a) Observe that we can split the event $A \cup B$ into three disjoint sets as:

$$A \cup B = (A \cap \overline{B}) \cup (A \cap B) \cup (B \cap \overline{A})$$

Hence we have:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \cap \overline{B}) + \mathbb{P}(A \cap B) + \mathbb{P}(B \cap \overline{A}). \tag{†}$$

Next, observe that we can split the events A, B into two disjoint sets respectively as:

$$A = (A \cap \overline{B}) \cup (A \cap B), \qquad B = (B \cap \overline{A}) \cup (A \cap B).$$

Hence we have:

$$\mathbb{P}(A) = \mathbb{P}(A \cap \overline{B}) + \mathbb{P}(A \cap B), \qquad \mathbb{P}(B) = \mathbb{P}(B \cap \overline{A}) + \mathbb{P}(A \cap B).$$

Substituting for $\mathbb{P}(A \cap \overline{B})$ and $\mathbb{P}(B \cap \overline{A})$ in equation (†), we get the result:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

The obvious generalisation to three events is:

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C).$$

(b) Let S be the sample space comprising the possible cards drawn from the pack, so that the size of the sample space is 52. Let E_{prime} be the event corresponding to drawing a prime number; the prime cards in each suit are $2, 3, 5, 7$, so we have $\mathbb{P}(E_{\text{prime}}) = 4/13$. Let E_{spade} be the event corresponding to drawing a spade; the spades are one of four suits, hence $\mathbb{P}(E_{\text{spade}}) = 1/4$. Let E_{red} be the event corresponding to drawing a red card; there are equal numbers of red and black cards in the deck, hence $E_{\text{red}} = 1/2$.

Next, there are four prime cards which are also spades, which gives $\mathbb{P}(E_{\text{prime}} \cap E_{\text{spade}}) = 4/52 = 1/13$. There are eight prime cards which are also red, which gives $\mathbb{P}(E_{\text{prime}} \cap E_{\text{red}}) = 8/52 = 2/13$. There are no spades which are red, because spades are a black suit, hence $\mathbb{P}(E_{\text{spade}} \cap E_{\text{red}}) = 0$. Finally, there are no cards which are all three of spades, red and prime, because spades are a black suit. Hence $\mathbb{P}(E_{\text{spade}} \cap E_{\text{red}} \cap E_{\text{prime}}) = 0$.

Putting all this information together, we have the required probability:

$$\mathbb{P}(E_{\text{prime}} \cup E_{\text{spade}} \cup E_{\text{red}}) = \frac{4}{13} + \frac{1}{4} + \frac{1}{2} - \frac{1}{13} - \frac{2}{13} = \frac{43}{52}.$$

Conditional probability

7. Suppose that S is a (discrete) sample space, $\mathcal{F}$ is the set of all events, and $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ is a probability measure.

(a) Define the *conditional probability* $\mathbb{P}(B|A)$ of an event B given an event A.

(b) Show that, for a fixed A, the conditional probability function $\mathbb{P}(\cdot|A) : \mathcal{F} \to \mathbb{R}$ satisfies the three basic axioms for a probability measure.

(c) State the definition for two events A, B being *independent* under a probability measure, and explain why this definition makes sense using the definition of conditional probability.

➪ **Solution:** (a) The conditional probability is defined by:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

This tells us the probability of B, *given that A has already occurred*. This makes sense, because the probability of B should now be shrunk down to the probability of $B \cap A$ (since we know that A has already happened, so any outcomes in the original B that were not contained in A are no longer allowed). We also need to normalise to the size of the new effective sample space, which becomes A itself, hence the division by $\mathbb{P}(A)$.

(b) If we fix A, and consider the probability measure $\mathbb{P}(B|A)$ for events B, we can prove it satisfies each of Kolmogorov's axioms. We have:

(K1) $\mathbb{P}(B|A) = \mathbb{P}(A \cap B)/\mathbb{P}(A) \geq 0$ for all events B, since $\mathbb{P}(A \cap B), \mathbb{P}(A)$ are both non-negative (and we assume that $\mathbb{P}(A) > 0$).

(K2) $\mathbb{P}(S|A) = \mathbb{P}(S \cap A)/\mathbb{P}(A) = \mathbb{P}(A)/\mathbb{P}(A) = 1$, as required.

(K3) Finally, for an events $B_1, B_2, \ldots$ which are all mutually disjoint, we have:

$$\begin{aligned}\mathbb{P}(B_1 \cup B_2 \cdots |A) &= \frac{\mathbb{P}((B_1 \cup B_2 \cdots) \cap A)}{\mathbb{P}(A)} \\ &= \frac{\mathbb{P}((B_1 \cap A) \cup (B_2 \cap A) \cup \cdots)}{\mathbb{P}(A)} \\ &= \frac{\mathbb{P}(B_1 \cap A)}{\mathbb{P}(A)} + \frac{\mathbb{P}(B_2 \cap A)}{\mathbb{P}(A)} + \cdots \\ &= \mathbb{P}(B_1|A) + \mathbb{P}(B_2|A) + \cdots,\end{aligned}$$

using the distributive law for intersection over union, which we proved in Question 3(b).

Hence, conditional probability indeed satisfies all the axioms of a usual probability measure.

(c) A, B are said to be *independent events* if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. This makes sense, because if A, B are independent, we have:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B)\mathbb{P}(A)}{\mathbb{P}(A)} = \mathbb{P}(B).$$

That is, the event A occurring does not affect the probability of the event B occurring.

8. A box of 100 gaskets contains ten gaskets with type-A defects only, five with type-B defects only, and two with both types of defect. Given that a gasket drawn at random has a type-A defect, what is the probability that it also has a type-B defect?

➾ **Solution:** Let S be the sample space, which is $S = \{\text{no defects, type A only, type B only, both defects}\}$. We are given the probabilities:

$$\mathbb{P}(\{\text{type A only}\}) = \frac{10}{100} = \frac{1}{10}, \qquad \mathbb{P}(\{\text{type B only}\}) = \frac{5}{100} = \frac{1}{20}, \qquad \mathbb{P}(\{\text{both defects}\}) = \frac{2}{100} = \frac{1}{50}.$$

All of these events are mutually disjoint. The probability we wish to calculate is:

$$\begin{aligned}\mathbb{P}(\{\text{both defects}\}|\{\text{type A only, both defects}\}) &= \frac{\mathbb{P}(\{\text{both defects}\} \cap \{\text{type A only, both defects}\})}{\mathbb{P}(\{\text{type A only, both defects}\})} \\ &= \frac{\mathbb{P}(\{\text{both defects}\})}{1/10 + 1/50} \\ &= \frac{1/50}{6/50} = \frac{1}{6}.\end{aligned}$$

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9. Your supervisor has two children, who are either boys or girls. Assuming equal probability of either gender, determine: (a) the probability that at least one child is a boy, given that at least one is a girl; (b) the probability that at least one child is a boy, given that the *younger* child is a girl.

➾ **Solution:** Let B represent a boy and G a girl. Let the pair (i, j) represent a pair of children, with the ith child older than the jth child. Then the sample space for this experiment is:

$$S = \{(B, B), \qquad (B, G), \qquad (G, B), \qquad (G, G)\}.$$

All of these outcomes are equally likely, so:

$$\mathbb{P}(\{(B, B)\}) = \mathbb{P}(\{(B, G)\}) = \mathbb{P}(\{(G, B)\}) = \mathbb{P}(\{(G, G)\}) = \frac{1}{4}.$$

Notice all these events are also mutually disjoint. Beginning the question proper, we have:

(a) We wish to calculate the probability:

$$\mathbb{P}(\{(B, G), (B, B), (G, B)\}|\{(B, G), (G, G), (G, B)\}) = \frac{\mathbb{P}(\{(B, G), (G, B)\})}{\mathbb{P}(\{(B, G), (G, G), (G, B)\})} = \frac{2/4}{3/4} = \frac{2}{3}.$$

(b) Now, we instead wish to calculate the probability:

$$\mathbb{P}(\{(B, G), (B, B), (G, B)\}|\{(B, G), (G, G)\}) = \frac{\mathbb{P}(\{(B, G)\})}{\mathbb{P}(\{(B, G), (G, G)\})} = \frac{1/4}{2/4} = \frac{1}{2}.$$

Bayes' theorem

10. State and prove *Bayes' theorem*. Give an interpretation of each of the terms that arise.

➾ **Solution:** Bayes' theorem states:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)}.$$

To prove this, simply use the definition of conditional probability:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{(\mathbb{P}(B \cap A)/\mathbb{P}(B)) \cdot \mathbb{P}(B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)}.$$

To interpret Bayes' theorem, we think about 'updating' our knowledge according to experimental results. On the right hand side, $\mathbb{P}(B)$ represents our 'prior' probability of the event B being observed. If we observe A however, the probability gets updated by: (i) multiplying $\mathbb{P}(B)$ by $\mathbb{P}(A|B)$, the probability of A being observed under the assumption that B occurs (called the *likelihood*); (ii) normalising by the overall probability of A happening (called the *evidence*). Overall, $\mathbb{P}(B|A)$ is the *posterior* probability of B occurring given that A has already occurred.

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11. You randomly choose a biscuit from one of two seemingly identical jars. Jar A has 10 chocolate biscuits and 30 plain; jar B has 20 chocolate and 20 plain biscuits. Unfortunately, you choose a plain biscuit. What is the probability that you chose from jar A?

➾ **Solution:** The sample space is $S = \{(A, \text{plain}), (A, \text{choc}), (B, \text{plain}), (B, \text{choc})\}$, where the first half of the pair represents the jar we have selected from. Let $A = \{(A, \text{plain}), (A, \text{choc})\}$ be the event of choosing from jar A, and similarly let B be the event of choosing from jar B. Let $P = \{(A, \text{plain}), (B, \text{plain})\}$ be the event of choosing a plain biscuit, and similarly let C be the event of choosing a chocolate biscuit. Observe that:

- The probability of selecting from jar A and jar B is equal, so that:

  $$\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{2}$$

- Since A, B are mutually disjoint and $A \cup B = S$, the probability of choosing a plain biscuit may be decomposed as:

  $$\mathbb{P}(P) = \mathbb{P}((P \cap A) \cup (P \cap B)) = \mathbb{P}(P \cap A) + \mathbb{P}(P \cap B) = \mathbb{P}(P|A)\mathbb{P}(A) + \mathbb{P}(P|B)\mathbb{P}(B).$$

  The decomposition of this event in this way is extremely useful, and very often used alongside Bayes' theorem (it is sometimes called the *law of total probability*). The probability of choosing a plain biscuit given we have selected A is $3/4$, and the probability of choosing a plain biscuit given we have selected B is $1/2$. Hence we can simplify this probability to:

  $$\mathbb{P}(P) = \frac{3}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{8}.$$

Now by Bayes' theorem, we have:

$$\mathbb{P}(A|P) = \frac{\mathbb{P}(P|A)\mathbb{P}(A)}{\mathbb{P}(P)} = \frac{(3/4) \cdot (1/2)}{5/8} = \frac{3}{5}.$$

12. (**The base rate fallacy**) Suppose that a disease affects one person in a thousand, and that a medical test for the disease accurately classifies 99% of all cases. What is the probability that, in a random screening exercise, a person who tests positively for the disease actually has the disease?

➡ **Solution:** The sample space is $S = \{(\text{positive, disease}), (\text{positive, no disease}), (\text{negative, disease}), (\text{negative, no disease})\}$, with each outcome representing the outcome of the test and whether the person actually has the disease. Define:

- $D = \{(\text{positive, disease}), (\text{negative, disease})\}$ to be the event of having the disease. We are given that $\mathbb{P}(D) = 1/1000$, and hence $\mathbb{P}(\overline{D}) = 999/1000$.
- $P = \{(\text{positive, disease}), (\text{positive, no disease})\}$ to be the event of testing positive. We are given that the test accurately classifies 99% of all cases, which tells us that if you have the disease, then the probability of testing positive is $99/100$. That is, $\mathbb{P}(P|D) = 99/100$. It also tells us that if you don't have the disease, then the probability of testing negative is $1/100$. That is, $\mathbb{P}(P|\overline{D}) = 1/100$.

Observe also that $D, \overline{D}$ are disjoint, and $D \cup \overline{D} = S$, so we can write:

$$\mathbb{P}(P) = \mathbb{P}((P \cap D) \cup (P \cap \overline{D})) = \mathbb{P}(P \cap D) + \mathbb{P}(P \cap \overline{D}) = \mathbb{P}(P|D)\mathbb{P}(D) + \mathbb{P}(P|\overline{D})\mathbb{P}(\overline{D}).$$

Substituting the known values from above, we have that the overall probability of testing positive is:

$$\mathbb{P}(P) = \frac{99}{100} \cdot \frac{1}{1000} + \frac{1}{100} \cdot \frac{999}{1000} = \frac{1098}{100000}.$$

We wish to calculate $\mathbb{P}(D|P)$, the probability of having the disease if you tested positive. We apply Bayes' theorem:

$$\mathbb{P}(D|P) = \frac{\mathbb{P}(P|D)\mathbb{P}(D)}{\mathbb{P}(P)} = \frac{(99/100) \cdot (1/1000)}{1098/100000} = \frac{99}{1098} = \frac{11}{122}.$$

Combinatorics

13.

(a) How many ways are there to order n distinct objects?

(b) How many ways are there to order r objects from a set of n distinct objects?

(c) How many ways are there to choose a subset of r objects from a set of n distinct objects?

(d) How many ways are there to arrange n identical objects into r groups?

➡ **Solution:** Most of these were covered in the lectures.

(a) There are $n!$ ways of ordering n distinct objects. This is because there are n ways to pick the first object, $n-1$ ways to pick the second, $n-2$ ways to pick the third, etc. This gives a total of $n \cdot (n-1) \cdot (n-2) \cdot \ldots = n!$ ways.

(b) There are

14. Prove the following property of the binomial coefficients:

$$\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r},$$

using: (a) the expression for the binomial coefficients in terms of factorials; (b) the combinatorial interpretation of the binomial coefficients in terms of combinations. Explain how this property of binomial coefficients relates to *Pascal's triangle*.

➾ **Solution:** We perform the proof in both of the required ways:

(a) In terms of factorials, we have:

$$\begin{aligned}
\binom{n}{r+1} + \binom{n}{r} &= \frac{n!}{(r+1)!(n-r-1)!} + \frac{n!}{r!(n-r)!} \\
&= \frac{n!}{r!(n-r-1)!}\left(\frac{1}{r+1} + \frac{1}{n-r}\right) \\
&= \frac{n!}{r!(n-r-1)!}\left(\frac{n+1}{(r+1)(n-r)}\right) \\
&= \frac{(n+1)!}{(r+1)!((n+1)-(r+1))!} \\
&= \binom{n+1}{r+1},
\end{aligned}$$

as required.

(b) We know that $\binom{n+1}{r+1}$ is the number of ways to choose a subset of $r+1$ objects from a set of $n+1$ distinct objects. We can perform this choice by splitting the set in a different way though: imagine we choose one special element from the set of $n+1$ objects. Then, we can *either* choose all of our $r+1$ objects from the remaining n; there are $\binom{n}{r+1}$ ways of doing this. *Or, alternatively,* we can choose our special element and then choose the remaining r objects from the remaining n objects; there are $\binom{n}{r}$ ways of doing this. This establishes the identity:

$$\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$$

in a purely combinatorial (i.e. counting) way!

This relates to Pascal's triangle as follows.

15. In one of the National Lottery games, six balls are drawn at random from 49 balls, numbered from one to 49. You pick six different numbers.

(a) What is the probability that your six numbers match those drawn?

(b) What is the probability that exactly r of the numbers you choose match those drawn?

(c) What is the probability that five numbers of those you choose match those drawn and that your sixth number matches a 'bonus ball' drawn from those remaining after the first six balls are drawn?

16. Suppose that n distinguishable particles are placed randomly into N different states. A particular configuration of this system is such that there are n_s particles in state s, where $1 \leq s \leq N$. If the ordering of particles in any particular state does not matter, show that the number of ways of realising a particular configurations is:

$$\frac{n!}{n_1!n_2!...n_N!}.$$

17. Letters $A, B, C, D, E,$ and F are written in a random order, but without repetition, into places $1, 2, 3, 4, 5$ and 6. Explaining your reasoning in each case, how many distinct orderings:

(a) exist in total?

(b) have F in the sixth place?

(c) have E or F in the sixth place?

(d) have E in the fifth place, and F in the sixth place?

(e) have E in the fifth place, or F in the sixth place?

(f) have E in the fifth place, or F in the sixth place, but *not both*?

Now, the letters A, B, C, D, E and F are instead partitioned into two bins, where order does not matter in a given bin. We say that a partition is of *type* $[a, b]$, if a letters are placed into the first bin, and b letters are placed into the second bin.

(g) How many partitions of type $[4, 2]$ are there?

(h) Assuming that from all the possible orderings given enumerated in part (a), the letters in the first four places are placed into the first bin, and the letters in the final two places are placed into the second bin, how many times do A, B, C, D end up in the first bin overall?

(i) Calculate the product of your answers to the two previous parts, and explain the value you obtain.

(j) Repeat the calculation of parts (g)-(i) for each of the possible types of partitions.

Miscellaneous probability space problems

18. A box contains $N_B \geq 2$ blue balls and $N - N_B \geq 2$ non-blue balls. An experiment consists of three consecutive stages: drawing a ball from a box, returning it or not returning it, then drawing a second ball from the box. The event B_i represents a blue ball being drawn on the ith draw, for $i = 1, 2$. The event R represents returning a ball on the second stage of the experiment. The probability of event R is $\mathbb{P}(R) = r$.

(a) Write down the sample space of the experiment, and find the probabilities of all of the possible outcomes.

(b) Hence, find:

(i) $\mathbb{P}(B_2)$;

(ii) $\mathbb{P}(B_1 \cap B_2)$;

(iii) $\mathbb{P}(R|B_1 \cap B_2)$.

(c) By sketching the graph of $\mathbb{P}(R|\overline{B}_1 \cap B_2)$ as a function of r, show that $\mathbb{P}(R|\overline{B}_1 \cap B_2) \leq r$.

Solution: (a) A valid sample space is:

$$S = \{(c_1, \text{return}, c_2), (c_1, \text{not return}, c_2) : c_1, c_2 = \text{blue, not blue}\}.$$

The required probabilities are:

$$\begin{aligned}
\mathbb{P}(\{(\text{blue, return, blue})\}) &= \frac{N_B}{N} \cdot r \cdot \frac{N_B}{N} = \frac{N_B^2 r}{N^2}, \\
\mathbb{P}(\{(\text{blue, return, not blue})\}) &= \frac{N_B}{N} \cdot r \cdot \frac{N - N_B}{N} = \frac{N_B(N - N_B)r}{N^2}, \\
\mathbb{P}(\{(\text{not blue, return, blue})\}) &= \frac{N - N_B}{N} \cdot r \cdot \frac{N_B}{N} = \frac{N_B(N - N_B)r}{N^2}, \\
\mathbb{P}(\{(\text{not blue, return, not blue})\}) &= \frac{N - N_B}{N} \cdot r \cdot \frac{N - N_B}{N} = \frac{(N - N_B)^2 r}{N^2}, \\
\mathbb{P}(\{(\text{blue, not return, blue})\}) &= \frac{N_B}{N} \cdot (1 - r) \cdot \frac{N_B}{N - 1} = \frac{N_B^2(1 - r)}{N(N - 1)}, \\
\mathbb{P}(\{(\text{blue, not return, not blue})\}) &= \frac{N_B}{N} \cdot (1 - r) \cdot \frac{N - N_B}{N - 1} = \frac{N_B(N - N_B)(1 - r)}{N(N - 1)}, \\
\mathbb{P}(\{(\text{not blue, not return, blue})\}) &= \frac{N - N_B}{N} \cdot (1 - r) \cdot \frac{N_B}{N - 1} = \frac{N_B(N - N_B)(1 - r)}{N(N - 1)}, \\
\mathbb{P}(\{(\text{not blue, not return, not blue})\}) &= \frac{N - N_B}{N} \cdot (1 - r) \cdot \frac{N - N_B}{N - 1} = \frac{(N - N_B)^2(1 - r)}{N(N - 1)}.
\end{aligned}$$

(b) Using the result from part (a), we have:

(i) Since B_2 contains all outcomes which have 'blue' in the third stage of the experiment, we have:

$$\begin{aligned}
\mathbb{P}(B_2) &= \frac{N_B^2 r}{N^2} + \frac{N_B(N - N_B)r}{N^2} + \frac{N_B^2(1 - r)}{N(N - 1)} + \frac{N_B(N - N_B)(1 - r)}{N(N - 1)} \\
&= \frac{N_B r}{N} + \frac{N_B(1 - r)}{N - 1}.
\end{aligned}$$

(ii) The event $B_1 \cap B_2$ corresponds to both the first stage and the third stage of the experiment showing 'blue'. This gives the probability:

$$\mathbb{P}(B_1 \cap B_2) = \frac{N_B^2 r}{N^2} + \frac{N_B^2(1-r)}{N(N-1)}.$$

(iii) The conditional probability $\mathbb{P}(R|B_1 \cap B_2)$ can be written out as:

$$\mathbb{P}(R|B_1 \cap B_2) = \frac{\mathbb{P}(R \cap B_1 \cap B_2)}{\mathbb{P}(B_1 \cap B_2)}.$$

The event $R \cap B_1 \cap B_2$ corresponds to exactly one outcome, when the first stage is 'blue', the second stage is 'return', and the third stage is 'blue'. which has probability $\mathbb{P}(R \cap B_1 \cap B_2) = N_B^2 r/N^2$. Hence the required probability is:

$$\begin{aligned}\mathbb{P}(R|B_1 \cap B_2) &= \frac{N_B^2 r/N^2}{N_B^2 r/N^2 + N_B^2(1-r)/N(N-1)} \\ &= \frac{rN - r}{N - r}.\end{aligned}$$

(c) The conditional probability $\mathbb{P}(R|\overline{B}_1 \cap B_2)$ can be written out as:

$$\mathbb{P}(R|\overline{B}_1 \cap B_2) = \frac{\mathbb{P}(R \cap \overline{B}_1 \cap B_2)}{\mathbb{P}(\overline{B}_1 \cap B_2)},$$

and hence we can compute it similarly to part (b)(iii). We have:

$$\mathbb{P}(\overline{B}_1 \cap B_2) = \frac{N_B(N-N_B)r}{N^2} + \frac{N_B(N-N_B)(1-r)}{N(N-1)}$$

Since $\mathbb{P}(R \cap \overline{B}_1 \cap B_2) = N_B(N-N_B)r/N^2$, we have:

$$\begin{aligned}\mathbb{P}(R|\overline{B}_1 \cap B_2) &= \frac{N_B(N-N_B)r/N^2}{N_B(N-N_B)r/N^2 + N_B(N-N_B)(1-r)/N(N-1)} \\ &= \frac{r/N}{r/N + (1-r)/(N-1)} \\ &= \frac{rN - r}{N - r},\end{aligned}$$

similarly to part (b)(iii).

At $r = 0$, the probability is zero. At $r = 1$, the probability is 1. Computing the derivative of the probability with respect to r, we have:

$$\frac{d}{dr}\left(\frac{rN-r}{N-r}\right) = \frac{(N-1)(N-r) + (rN-r)}{N-r)^2} = \frac{N(N-1)}{(N-r)^2}.$$

This is everywhere positive, so the function is strictly increasing on the interval $[0,1]$. However,

$$\frac{d^2}{dr^2}\left(\frac{rN-r}{N-r}\right) = \frac{2N(N-1)}{(N-r)^3},$$

so that the second derivative is also positive. This implies that the gradient of the function is also strictly increasing on the interval $[0,1]$.

19. A factory produces good bananas with probability p and bad bananas with probability $1-p$. The bananas are placed on a conveyor belt and inspected by n different workers sequentially. Worker k notices a good banana with probability g_k, and removes it from the conveyor belt in this instance. Worker k notices a bad banana with probability b_k, and also removes it from the conveyor belt in this instance. Assume that $0 < p < 1$ and that $g_k > 0, b_k > 0$ for all $k = 1, ..., n$.
An experiment is conducted where a banana, which may be good or bad, is placed on the conveyor belt for inspection.

(a) Write down the sample space for the experiment.

(b) Let G be the event that a good banana is produced, and let X_k be the event that the banana is removed from the conveyor belt by worker k. Find:

(i) $\mathbb{P}(G \cap X_1)$;

(ii) $\mathbb{P}(X_1)$;

(iii) $\mathbb{P}(\overline{G}|X_2)$;

(iv) $\mathbb{P}(G|X_2 \cup X_3 \cup ... \cup X_n)$;

(v) $\mathbb{P}(X_{k+1} \cup X_{k+2} \cup ... \cup X_n)$;

(vi) $\mathbb{P}(G|X_1 \cup X_2 \cup ... \cup X_k)$.

(c) If $b_1 = b_2 = ... = b_n$, and $n = 98$, find the minimal value of p for which $\mathbb{P}(G|X_1) = 0.99$.

20. (**Sampling with replacement**) A bag is filled with N white balls and M black balls. Balls are drawn from the bag sequentially without replacement. Let W_i denote the event that the ith ball drawn is white, and let B_i denote the event that the ith ball drawn is black.

(a) Find $\mathbb{P}(W_1)$, $\mathbb{P}(W_2)$, and $\mathbb{P}(W_3)$. Hence, conjecture and prove a general formula for $\mathbb{P}(W_i)$.

(b) What does the event $W_i \cap W_j$ represent? Find $\mathbb{P}(W_1 \cap W_2)$, $\mathbb{P}(W_1 \cap W_3)$ and $\mathbb{P}(W_2 \cap W_3)$. Hence, conjecture and prove a general formula for $\mathbb{P}(W_i \cap W_j)$.

(c) What does the event $W_i \cup W_j$ represent? Find $\mathbb{P}(W_1 \cup W_2)$, $\mathbb{P}(W_1 \cup W_3)$ and $\mathbb{P}(W_2 \cup W_3)$. Hence, conjecture and prove a general formula for $\mathbb{P}(W_i \cup W_j)$.

(d) Using Bayes' theorem, compute $\mathbb{P}(W_2|W_1)$ and $\mathbb{P}(W_3|W_2)$. Hence, conjecture and prove a general formula for $\mathbb{P}(W_{i+1}|W_i)$.

(e) Show that the probability of obtaining exactly $n \leq N$ white balls in a total of x draws is given by:

$$\frac{\binom{N}{n}\binom{M}{x-n}}{\binom{N+M}{x}}.$$