Part IA: Mathematics for Natural Sciences A
Examples Sheet 6: Methods of integration

Model Solutions

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Integration by substitution

1. By means of an appropriate substitution in each case, determine the indefinite integrals of the following functions:

$$\text{(a) } \frac{1}{\sqrt{1-x^2}}, \qquad \text{(b) } \frac{1}{\sqrt{x^2-1}}, \qquad \text{(c) } \frac{1}{\sqrt{1+x^2}}, \qquad \text{(d) } \frac{1}{1+x^2}, \qquad \text{(e) } \frac{1}{1-x^2}$$

Learn these integrals off by heart, and get your supervision partner to test you on them.

Solution: Each of these integrals can be performed by making an appropriate trigonometric or hyperbolic substitution.

(a) Use the substitution $x = \sin(\theta)$, so that $dx = \cos(\theta)d\theta$. This is appropriate here, since for the square root to make sense, we need $-1 < x < 1$. This is covered by the range of $\sin(\theta)$, so the substitution is valid. We then have:

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos(\theta)d\theta}{\sqrt{1-\sin^2(\theta)}} = \int d\theta = \theta + c = \arcsin(x) + c.$$

(b) Use the substitution $x = \cosh(\theta)$, so that $dx = \sinh(\theta)d\theta$. This is appropriate here, since for the square root to make sense, we need $x > 1$ or $x < -1$. Since the range is disjoint, we can focus on $x > 1$, which is covered by the range of $\cosh(\theta)$ (we could have chosen to make the substitution $x = -\cosh(\theta)$ if $x < -1$ was the range of interest). We then have:

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sinh(\theta)d\theta}{\sqrt{\cosh^2(\theta)-1}} = \int d\theta = \theta + c = \operatorname{arcosh}(x) + c.$$

(c) Use the substitution $x = \sinh(\theta)$, so that $dx = \cosh(\theta)d\theta$. This is appropriate here, since x can take any value, and $\sinh(\theta)$ has range $\mathbb{R}$. We then have:

$$\int \frac{dx}{\sqrt{x^2+1}} = \int \frac{\cosh(\theta)d\theta}{\sqrt{\sinh^2(\theta)+1}} = \int d\theta = \theta + c = \operatorname{arsinh}(x) + c.$$

(d) Use the substitution $x = \tan(\theta)$, so that $dx = \sec^2(\theta)d\theta$. This is appropriate here, since x can take any value, and $\sinh(\theta)$ has range $\mathbb{R}$. We then have:

$$\int \frac{dx}{1+x^2} = \int \frac{\sec^2(\theta)d\theta}{1+\tan^2(\theta)} = \int d\theta = \theta + c = \arctan(x) + c.$$

(e) Use the substitution $x = \tanh(\theta)$, so that $dx = \operatorname{sech}^2(\theta)d\theta$. This is appropriate here since x takes values in $x < -1$, $-1 < x < 1$ and $x > 1$. These ranges are disjoint, so if we are interested in $-1 < x < 1$, we are safe to make this substitution. We then have:

$$\int \frac{dx}{1-x^2} = \int \frac{\operatorname{sech}^2(\theta)d\theta}{1-\tanh^2(\theta)} = \int d\theta = \theta + c = \operatorname{artanh}(x) + c.$$

2. Using the results of the previous question, determine: (a) $\displaystyle\int \frac{dx}{\sqrt{x^2+x+1}}$; (b) $\displaystyle\int \frac{8-2x}{\sqrt{6x-x^2}}\,dx$.

Solution:

(a) We complete the square in the denominator, $x^2+x+1=(x+1/2)^2+3/4$. Then:

$$\int \frac{dx}{\sqrt{(x+1/2)^2+3/4}} = \frac{2}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1}}.$$

This is now a linear function of one of our standard integrals from the previous question. Hence the integral is just given by:

$$\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2}\operatorname{arsinh}\left(\frac{2x+1}{\sqrt{3}}\right)+c = \operatorname{arsinh}\left(\frac{2x+1}{\sqrt{3}}\right)+c.$$

To get the constants that multiply the inverse hyperbolic function correct, it can be useful to think about what happens when you differentiate the final expression. By the chain rule, a $2/\sqrt{3}$ will pop out, so we need to cancel that when we integrate!

(b) This is more complicated, because even completing the square on the denominator will not give us an integral of the form of the previous question. Instead, we observe that the derivative of $6x-x^2$ is $6-2x$, so the integral is close to being of the form $f'(x)f(x)^\alpha$. If we rewrite the integrand as:

$$\frac{8-2x}{\sqrt{6x-x^2}} = \frac{6-2x}{\sqrt{6x-x^2}}+\frac{2}{\sqrt{6x-x^2}} = (6-2x)\left(6x-x^2\right)^{-1/2}+\frac{2}{\sqrt{6x-x^2}},$$

then the first term is now of the form $f'(x)f(x)^\alpha$ and can be directly integrated, whilst the second term can be transformed to one of our standard forms studied in the previous question.

Completing the square in the denominator of the second term, we have $6x-x^2=9-(x-3)^2$. Hence we have:

$$\begin{aligned}\int \frac{8-2x}{\sqrt{6x-x^2}}\,dx &= \int (6-2x)(6x-x^2)^{-1/2}\,dx+\frac{2}{3}\int \frac{dx}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}}\\ &= 2\sqrt{6x-x^2}+\frac{2}{3}\cdot 3\cdot\arcsin\left(\frac{x-3}{3}\right)+c\\ &= 2\sqrt{6x-x^2}+2\arcsin\left(\frac{x-3}{3}\right)+c.\end{aligned}$$

3. By means of an appropriate substitution in each case, determine the indefinite integrals of the following functions:

$$\text{(a) } x\sqrt{x+3}, \qquad \text{(b) } \tan(x)\sqrt{\sec(x)}, \qquad \text{(c) } \frac{e^x}{\sqrt{1-e^{2x}}}, \qquad \text{(d) } \frac{1}{x\sqrt{x^2-1}}.$$

➾ **Solution:**

(a) Consider the substitution $u = x+3$, trying to clear the square root. We have $du = dx$, so that the integral can be rewritten as:

$$\int x\sqrt{x+3}\,dx = \int (u-3)u^{1/2}\,du = \int \left(u^{3/2} - 3u^{1/2}\right)\,du = \frac{2}{5}u^{5/2} - 2u^{3/2} + c.$$

Hence the integral is given by:

$$\frac{2}{5}(x+1)^{5/2} + 2(x+1)^{3/2} + c.$$

(b) Consider the substitution $u = \sec(x)$. This is a good choice, because the derivative of $\sec(x)$ is $\sec(x)\tan(x)$, so we will be able to clear the $\tan(x)$ and leave only $\sec(x)$ terms behind. We have $du = \sec(x)\tan(x)dx$, so:

$$\int \tan(x)\sqrt{\sec(x)}\,dx = \int \frac{du}{\sqrt{u}} = 2u^{1/2} + c = 2\sqrt{\sec(x)} + c.$$

(c) Here, the obvious substitution is $u = e^x$. We have $du = e^x\,dx$, so:

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \arcsin(u) + c = \arcsin(e^x) + c.$$

(d) Consider a trigonometric substitution $x = \sec(\theta)$, since this will clear the square root ($\sqrt{\sec^2(\theta)-1} = \tan(\theta)$), but also give us $dx = \sec(\theta)\tan(\theta)d\theta$, so that the remaining $\sec(\theta)$ on the denominator will cancel. We have:

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec(\theta)\tan(\theta)d\theta}{\sec(\theta)\tan(\theta)} = \int d\theta = \theta + c = \mathrm{arcsec}(x) + c.$$

Another way of writing $\mathrm{arcsec}(x)$ in terms of more standard function is as $\arccos(1/x)$ - to see this, note that:

$$y = \mathrm{arcsec}(x) \qquad \Rightarrow \qquad \sec(y) = x \qquad \Rightarrow \qquad \cos(y) = \frac{1}{x} \qquad \Rightarrow \qquad y = \arccos\left(\frac{1}{x}\right).$$

4. This question shows that any trigonometric integral can be turned into an algebraic integral through the use of the powerful *half-tangent substitution*.

(a) Show that if $t = \tan\left(\frac{1}{2}x\right)$, then $\sin(x) = 2t/(1+t^2)$, $\cos(x) = (1-t^2)/(1+t^2)$ and $dx/dt = 2/(1+t^2)$. Deduce that for any function f, we have:

$$\int f\left(\sin(x), \cos(x)\right)\, dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}.$$

(b) Using the method derived in (a), find the indefinite integrals of the following functions:

(i) $\operatorname{cosec}(x)$, (ii) $\sec(x)$, (iii) $\dfrac{1}{2+\cos(x)}$.

➾ **Solution:** (a) We have:

$$\sin(x) = 2\sin(x/2)\cos(x/2) = \frac{2\sin(x/2)\cos(x/2)}{\cos^2(x/2)+\sin^2(x/2)} = \frac{2\tan(x/2)}{1+\tan^2(x/2)} = \frac{2t}{1+t^2}$$

and:

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2) = \frac{\cos^2(x/2)-\sin^2(x/2)}{\cos^2(x/2)+\sin^2(x/2)} = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-t^2}{1+t^2},$$

as required. We also have:

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{1}{2}x\right) = \frac{1}{2}\left(1+t^2\right) \qquad \Leftrightarrow \qquad \frac{dx}{dt} = \frac{2}{1+t^2}.$$

The required equality then follows immediately from the substitution $t = \tan(x/2)$.

(b) Applying the substitution in each of the given cases, we have:

(i) $$\int \operatorname{cosec}(x)\, dx = \int \frac{dx}{\sin(x)} = \int \frac{2(1+t^2)dt}{2t(1+t^2)} = \int \frac{dt}{t} = \log(t) + c = \log(\tan(x/2)) + c.$$

(ii) $$\int \sec(x)\, dx = \int \frac{dx}{\cos(x)} = \int \frac{2(1+t^2)dt}{(1-t^2)(1+t^2)} = 2\int \frac{dt}{1-t^2} = 2\operatorname{artanh}\left(\tan(x/2)\right) + c.$$

(iii) $$\int \frac{dx}{2+\cos(x)} = \int \frac{2dt}{(2(1+t^2)+(1-t^2))} = \int \frac{2dt}{3+t^2} = \frac{2}{3}\int \frac{dt}{1+\left(\frac{t}{\sqrt{3}}\right)^2} = \frac{2}{\sqrt{3}}\arctan\left(\frac{\tan(x/2)}{\sqrt{3}}\right) + c.$$

Partial fractions and rational functions

5. Explain the general strategy that one should adopt when integrating a rational function. Hence, determine the indefinite integrals of the following rational functions by decomposing into partial fractions:

$$\text{(a) } \frac{1}{1-x^2}, \qquad \text{(b) } \frac{3x}{2x^2+x-1}, \qquad \text{(c) } \frac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}.$$

Compare your answer to (a) with your answer to Question 7(e), where you evaluated the same integral using a substitution. Are your results compatible?

➾ **Solution:** Consider the rational function $p(x)/q(x)$, where $q(x)$ can be factorised in the form:

$$q(x) = (x-a_1)^{j_1}...(x-a_m)^{j_m}(x^2+b_1x+c_1)^{k_1}...(x^2+b_nx+c_n)^{k_n},$$

where the quadratic factors have no real roots. Then $p(x)/q(x)$ can be decomposed in the form:

$$\frac{p(x)}{q(x)} = r(x) + \sum_{i=1}^{m}\sum_{r=1}^{j_i}\frac{A_{ir}}{(x-a_i)^r} + \sum_{i=1}^{n}\sum_{r=1}^{k_i}\frac{B_{ir}x+C_{ir}}{(x^2+b_ix+c_i)^r},$$

where $r(x)$ is a polynomial, which is called the *partial fraction decomposition* of the rational function. This gives us a general strategy for integrating a rational function:

- First, perform the partial fraction decomposition.
- The polynomial term $r(x)$ can be integrated straightforwardly.
- The terms in the partial fraction decomposition involving the real roots can be integrated straightforwardly via:

$$\int \frac{A_{ir}}{(x-a_i)^r} = \begin{cases} -\dfrac{A_{ir}}{(r-1)(x-a_i)^{r-1}}, & \text{if } r \neq 1, \\ A_{ir}\log(x-a_i), & \text{if } r = 1. \end{cases}$$

- For the terms in the partial fraction decomposition involving a simple quadratic factor, i.e. a quadratic factor with $k_i = 1$, we can write:

$$\frac{B_{i1}x+C_{i1}}{x^2+b_{k_i}x+c_i} = \frac{B_{i1}}{2}\frac{2x+b_{k_i}}{x^2+b_{k_i}x+c_i} + \frac{C_{i1}-B_{i1}b_{k_i}/2}{x^2+b_{k_i}x+c_i}.$$

  The first term is now a *logarithmic derivative*, and can be integrated directly. Meanwhile, the remaining term is a constant multiplied by the reciprocal of a quadratic; by completing the square in the denominator, this can be made into a derivative of an *arctangent* or *hyperbolic arctangent*.

- For terms in the partial fraction decomposition involving non-simple quadratic factors, i.e. quadratic factors with $k_i > 1$, things are more complicated. These functions are all integrable, through an arctangent or hyperbolic arctangent substitution, but it is unlikely that integrals of this kind will be in your standard arsenal.

We shall now apply this technique in the cases of three given rational functions.

(a) We have:

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right).$$

Hence, integrating, we have:

$$\int \frac{dx}{1-x^2} = -\frac{1}{2}\log(1-x) + \frac{1}{2}\log(1+x) + c = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) + c.$$

This is perfectly consistent with Question 7(e), because $\tanh^{-1}(x) = \frac{1}{2}\log((1+x)/(1-x))$.

(b) Factorising, we have $2x^2 + x - 1 = (2x-1)(x+1)$. Hence decomposing into partial fractions, we have:

$$\frac{3x}{2x^2+x-1} = \frac{1}{2x-1} + \frac{1}{x+1}.$$

Hence, integrating, we have:

$$\int \frac{3x}{2x^2+x-1}\,dx = \frac{1}{2}\log(2x-1) + \log(x+1) + c.$$

(c) First, we perform polynomial division:

$$\frac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4} = -1 + \frac{9+6x-x^2-2x^3}{3+2x-2x^2-2x^3-x^4}.$$

We can now decompose the second term into partial fractions. First, we need to factorise the denominator. We spot that $x=1$ is a factor, so:

$$3+2x-2x^2-2x^3-x^4 = (1-x)\left(3+5x+3x^2+x^3\right)$$

We spot that $x=-1$ is a factor of the second bracket, so:

$$(1-x)(x^3+3x^2+5x+3) = (1-x)(1+x)(x^2+2x+3).$$

The final factor has discriminant $4 - 4\cdot 3 = -8 < 0$, hence there are no more real roots. Thus the partial fractions take the form:

$$\frac{9+6x-x^2-2x^3}{3+2x-2x^2-2x^3-x^4} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{x^2+2x+3}.$$

Multiplying up, we have:

$$9+6x-x^2-2x^3 = A(1+x)(x^2+2x+3) + B(1-x)(x^2+2x+3) + (Cx+D)(1-x^2). \qquad (\dagger)$$

Setting $x=1$ in (†), we have:

$$12A = 12 \qquad \Rightarrow \qquad A = 1.$$

Setting $x=-1$ in (†), we have:

$$4B = 4 \qquad \Rightarrow \qquad B = 1.$$

Setting $x=0$ in (†), we have:

$$9 = 3A + 3B + D \qquad \Rightarrow \qquad D = 9-3-3 = 3.$$

Finally, comparing coefficients of x^3 on both sides of (†), we have:

$$-2 = A - B - C \qquad \Rightarrow \qquad C = A - B + 2 = 2.$$

Hence, the partial fractions for the original rational function are:

$$-1 + \frac{1}{1-x} + \frac{1}{1+x} + \frac{2x+3}{x^2+2x+3}.$$

To integrate, we need to split the final term into a logarithmic derivative, and a constant divided by a quadratic. We have:

$$\frac{2x+3}{x^2+2x+3} = \frac{2x+2}{x^2+2x+3} + \frac{1}{x^2+2x+3} = \frac{2x+2}{x^2+2x+3} + \frac{1}{(x+1)^2+2} = \frac{2x+2}{x^2+2x+3} + \frac{1}{2}\frac{1}{((x+1)/\sqrt{2})^2+1}$$

Therefore, integrating the original rational function, we have:

$$\int \frac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}\,dx = \int \left(-1 + \frac{1}{1-x} + \frac{1}{1+x} + \frac{2x+2}{x^2+2x+3} + \frac{1}{2}\frac{1}{((x+1)/\sqrt{2})^2+1}\right)dx$$

$$= -x - \log(1-x) + \log(1+x) + \log(2x^2+2x+3) + \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c.$$

Integration by parts

6. Using integration by parts, determine the following integrals:

(a) $\displaystyle\int_{-\pi/2}^{\pi/2} x\sin(2x)\,dx,$ (b) $\displaystyle\int_0^{\infty} xe^{-2x}\,dx,$ (c) $\displaystyle\int_0^1 x\log\left(\frac{1}{x}\right)\,dx,$ (d) $\displaystyle\int_0^{\infty} x^3e^{-x^2}\,dx.$

Solution:

(a) We have:

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} x\sin(2x)\,dx &= \left[-\frac{1}{2}x\cos(2x)\right]_{-\pi/2}^{\pi/2} + \frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos(2x)\,dx \\ &= \frac{\pi}{2} + \frac{1}{2}\left[\frac{1}{2}\sin(2x)\right]_{-\pi/2}^{\pi/2} \\ &= \frac{\pi}{2}.\end{aligned}$$

(b) We have:

$$\begin{aligned}\int_0^{\infty} xe^{-2x}\,dx &= \left[-\frac{1}{2}xe^{-2x}\right]_0^{\infty} + \frac{1}{2}\int_0^{\infty} e^{-2x}\,dx \\ &= \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_0^{\infty} \\ &= \frac{1}{4}.\end{aligned}$$

(c) We have:

$$\begin{aligned}\int_0^1 x\log\left(\frac{1}{x}\right)\,dx = -\int_0^1 x\log(x)\,dx &= -\left[\frac{1}{2}x^2\log(x)\right]_0^1 + \frac{1}{2}\int_0^1 x\,dx \\ &= \frac{1}{2}\left[\frac{1}{2}x^2\right]_0^1 \\ &= \frac{1}{4}.\end{aligned}$$

Here, we used the fact that $x^2\log(x) \to 0$ as $x \to 0$. This is because the polynomial approaches zero faster than the logarithm approaches negative infinity. This is a general phenomena, as we proved using L'Hôpital's rule earlier in the course.

(d) Observe that the derivative of e^{-x^2} is $-2xe^{-x^2}$, so that the integral of xe^{-x^2} is $-\frac{1}{2}e^{-x^2}$. Hence, we have:

$$\begin{aligned}\int_0^\infty x^3 e^{-x^2}\,dx &= \left[-\frac{1}{2}x^2 e^{-x^2}\right]_0^\infty + \frac{1}{2}\int_0^\infty (2x)e^{-x^2}\,dx \\ &= \left[-\frac{1}{2}e^{-x^2}\right]_0^\infty \\ &= \frac{1}{2}.\end{aligned}$$

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7. By writing each of the following functions $f(x)$ in the form $1 \cdot f(x)$, and using integration by parts, determine their indefinite integrals:

(a) $\log(x)$, (b) $\log^3(x)$, (c) $\cosh^{-1}(x)$, (d) $\tanh^{-1}(x)$, (e) $\sin(\log(x))$.

Solution:

(a) $\displaystyle\int 1 \cdot \log(x)\,dx = x\log(x) - \int x \cdot (1/x)\,dx = x\log(x) - x + c.$

(b) Here, we use integration by parts multiple times. We have:

$$\begin{aligned}\int 1 \cdot \log^3(x)\,dx &= x\log^3(x) - 3\int \log^2(x)\,dx \\ &= x\log^3(x) - 3x\log^2(x) + 6\int \log(x)\,dx \\ &= x\log^3(x) - 3x\log^2(x) + 6x\log(x) - 6x + c.\end{aligned}$$

(c) First, observe that:

$$\int 1 \cdot \cosh^{-1}(x)\,dx = x\cosh^{-1}(x) - \int \frac{x}{\sqrt{x^2-1}}\,dx.$$

Since the derivative of x^2 is $2x$, the remaining integral is of the form $f'(x)f(x)^\alpha$, hence can be directly integrated. We have:

$$x\cosh^{-1}(x) - \sqrt{x^2-1} + c.$$

(d) $\displaystyle\int 1 \cdot \tanh^{-1}(x)\,dx = x\tanh^{-1}(x) - \int \frac{x}{1-x^2}\,dx = x\tanh^{-1}(x) + \frac{1}{2}\log(x^2-1) + c.$

(e) We begin by performing one integration by parts:

$$\int 1 \cdot \sin(\log(x))\,dx = x\sin(\log(x)) - \int \cos(\log(x))\,dx.$$

We now iterate, performing a second integration by parts:

$$\int \sin(\log(x))\,dx = x\sin(\log(x)) - x\cos(\log(x)) - \int \sin(\log(x))\,dx.$$

But notice that the new integral is the same as the original one - hence, rearranging this equation, we see that:

$$\int \sin(\log(x))\,dx = \frac{x}{2}\left(\sin(\log(x)) - \cos(\log(x))\right) + c.$$

Reduction formulae

8. (a) Show that for $n \geq 1$, we have:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx,$$

Hence, evaluate $\int \sin^6(x)\,dx$.

(b) Using (a), show that the integral $I_n = \int\limits_0^{\pi/2} \sin^n(x)\,dx$ satisfies $I_n = (n-1)I_n/n$. Hence, evaluate I_2 and I_4.

➾ **Solution:** (a) Integrating by parts, we have:

$$\int \sin^n(x)\,dx = \int \sin(x)\cdot\sin^{n-1}(x)\,dx = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x)\,dx.$$

Expanding $\cos^2(x) = 1 - \sin^2(x)$, we can rearrange this to read:

$$\int \sin^n(x)\,dx = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx.$$

Rearranging, we then have:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx,$$

as required.

This allows us to evaluate the given integral, by applying the recurrence relation repeatedly:

$$\begin{aligned}\int \sin^6(x)\,dx &= -\frac{1}{6}\cos(x)\sin^5(x) + \frac{5}{6}\int \sin^4(x)\,dx \\ &= -\frac{1}{6}\cos(x)\sin^5(x) + \frac{5}{6}\left(-\frac{1}{4}\cos(x)\sin^3(x) + \frac{3}{4}\int \sin(x)\,dx\right) \\ &= -\frac{1}{6}\cos(x)\sin^5(x) - \frac{5}{24}\cos(x)\sin^3(x) - \frac{5}{8}\cos(x) + c.\end{aligned}$$

(b) Simply inserting the limits into our recurrence relation, we have:

$$\int\limits_0^{\pi/2} \sin^n(x)\,dx = \left[-\frac{1}{n}\cos(x)\sin^{n-1}(x)\right]_0^{\pi/2} + \frac{n-1}{n}\int\limits_0^{\pi/2} \sin^{n-2}(x)\,dx = \frac{n-1}{n}\int\limits_0^{\pi/2} \sin^{n-2}(x)\,dx.$$

This immediately gives $I_n = (n-1)I_{n-2}/n$. To evaluate the given integrals, we note:

$$I_0 = \int\limits_0^{\pi/2} 1\,dx = \frac{\pi}{2}.$$

Hence:

$$I_2 = \frac{1}{2}I_0 = \frac{\pi}{4}, \qquad I_4 = \frac{3}{4}I_2 = \frac{3\pi}{8}.$$

9. Establish reduction formulae for each of the following parametric integrals:

$$\text{(a) } I_n = \int_0^\infty x^n e^{-x^2}\,dx, \quad \text{(b) } J_n = \int_0^\pi x^{2n}\cos(x)\,dx, \quad \text{(c) } K_n = \int_0^\infty x^{n-1}e^{-x}\,dx, \quad \text{(d) } L_n = \int_0^\infty \frac{dx}{(1+x^2)^n}.$$

Hence: (i) evaluate I_3, I_5; (ii) evaluate J_3; (iii) establish a general formula for K_n; (iv) evaluate L_4. $(*)$ Using part (c), suggest a reasonable definition of $z!$ where z is a complex number. Will this work for all complex numbers?

Solution:

(a) Since the derivative of e^{-x^2} is $-2xe^{-x^2}$, we can integrate xe^{-x^2} to get $-\frac{1}{2}e^{-x^2}$. Hence:

$$I_n = \int_0^\infty x^n e^{-x^2}\,dx = \left[-\frac{1}{2}x^{n-1}e^{-x^2}\right]_0^\infty + \frac{(n-1)}{2}\int_0^\infty x^{n-2}e^{-x^2}\,dx = \frac{(n-1)}{2}I_{n-2},$$

assuming that $n \geq 2$. To answer (i), we note that this implies:

$$I_3 = I_1 = \int_0^\infty xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^\infty = \frac{1}{2}.$$

We also have:

$$I_5 = 2I_3 = 1.$$

(b) Integrating by parts twice, we have:

$$J_n = \int_0^\pi x^{2n}\cos(x)\,dx = \left[x^{2n}\sin(x)\right]_0^\pi - 2n\int_0^\pi x^{2n-1}\sin(x)\,dx$$

$$= 2n\left[x^{2n-1}\cos(x)\right]_0^\pi - 2n(2n-1)\int_0^\pi x^{2n-2}\cos(x)\,dx$$

$$= -2n\pi^{2n-1} - 2n(2n-1)J_{n-1}.$$

To answer (ii), we note that this implies:

$$J_3 = -6\pi^5 - 30J_2 = -6\pi^5 - 30\left(-4\pi^3 - 12J_1\right) = -6\pi^5 + 120\pi^3 + 360\left(-2\pi\right) = -6\pi^5 + 120\pi^3 - 720\pi,$$

since $J_0 = 0$.

(c) Integrating by parts, we have:

$$K_n = \int_0^\infty x^{n-1}e^{-x}\,dx = \left[-x^{n-1}e^{-x}\right]_0^\infty + (n-1)\int_0^\infty x^{n-2}e^{-x}\,dx = (n-1)K_{n-1}.$$

In particular, to answer (iii), we can use this relation iteratively to give:

$$K_n = (n-1)K_{n-1} = (n-1)(n-2)K_{n-2} = (n-1)(n-2)(n-3)K_{n-3} = \cdots = (n-1)!K_1.$$

But we have:

$$K_1 = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty = 1.$$

Hence, $K_n = (n-1)!$ for all positive integers n.

(d) Here, we can only integrate 1 and differentiate the integrand. We have:

$$L_n = \int_0^\infty \frac{dx}{(1+x^2)^n} = \left[\frac{x}{(1+x^2)^n}\right]_0^\infty + 2n\int_0^\infty \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

$$= 2n\int_0^\infty \frac{(1+x^2)-1}{(1+x^2)^{n+1}}\,dx$$

$$= 2n\int_0^\infty \frac{1}{(1+x^2)^n}\,dx - 2n\int_0^\infty \frac{1}{(1+x^2)^{n+1}}\,dx.$$

This can be written as $L_n = 2nL_n - 2nL_{2n+1}$, which can be rearranged to read:

$$L_{n+1} = \frac{2n-1}{2n}L_n,$$

for all $n \geq 1$. To answer (iv), and evaluate L_4, we note:

$$L_4 = \frac{5}{6}L_3 = \frac{5}{6}\cdot\frac{3}{4}L_2 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}L_1.$$

But now we have:

$$L_1 = \int_0^\infty \frac{dx}{1+x^2} = [\arctan(x)]_0^\infty = \frac{\pi}{2}.$$

Hence:

$$L_4 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}.$$

From part (c), we know that:

$$(n-1)! = \int_0^\infty x^{n-1}e^{-x}\,dx.$$

This suggests that a possible definition of $z!$ for z a complex number is:

$$z! = \int_0^\infty x^z e^{-x}\,dx.$$

Importantly, for this definition to make sense, the integral needs to converge. For large values of x, the integrand is exponentially suppressed, so will be convergent. For small values of x, the integrand resembles $x^z e^{-z} \approx x^z$, which gives a contribution to the integral:

$$\int_0^\epsilon x^z\,dx = \left[\frac{x^{z+1}}{z+1}\right]_0^\epsilon.$$

This is finite only if $\mathrm{Re}(z)+1>0$, hence this definition only makes sense if $\mathrm{Re}(z) > -1$.

We can extend the definition by *analytic continuation* however, a tool from complex analysis, which you will learn about in Part IB Mathematics for Natural Sciences.

Miscellaneous integrals

10. Evaluate the following integrals, using the most efficient method in each case:

(a) $\displaystyle\int_4^9 \frac{dx}{\sqrt{x}-1}$

(b) $\displaystyle\int_{\pi/3}^{\pi/4} \frac{1+\tan^2(x)}{(1+\tan(x))^2}\,dx$

(c) $\displaystyle\int \frac{e^{2x}-2e^x}{e^{2x}+1}\,dx$

(d) $\displaystyle\int \frac{dx}{1+3\cos^2(x)}$

(e) $\displaystyle\int_2^3 \frac{2x+1}{x(x+1)}\,dx$

(f) $\displaystyle\int \frac{1}{2\sqrt{x}}e^{\sqrt{x}}\,dx$

(g) $\displaystyle\int x^3 e^{-x^4}\,dx$

(h) $\displaystyle\int \left(\frac{\sin(2x)}{\sin^2(x)+\log(x)}+\frac{1}{x(\sin^2(x)+\log(x))}\right)dx$

(i) $\displaystyle\int x\sqrt{3-2x}\,dx$

(j) $\displaystyle\int \frac{\sin(x)}{\cos^2(x)-5\cos(x)+6}\,dx$

(k) $\displaystyle\int \frac{\log(x)}{x^4}\,dx$

(l) $\displaystyle\int \sqrt{1-x^2}\,dx$

(m) $\displaystyle\int_{\pi/3}^{\pi/2} \tan(x)\cos^4(x)\,dx$

(n) $\displaystyle\int_1^5 x^2\log(x)\,dx$

(o) $\displaystyle\int e^x\sinh(3x)\,dx$

(p) $\displaystyle\int \frac{\arctan(x)}{x^2}\,dx$

(q) $\displaystyle\int_{e^3}^{e^4} \frac{3\log(x)-4}{x\log^2(x)-3x\log(x)+2x}\,dx$

(r) $\displaystyle\int_0^{\pi/6} x\sin(3x)\,dx$

(s) $\displaystyle\int \sin(2x)e^{\sin^2(x)}\,dx$

(t) $\displaystyle\int \frac{dx}{\cos^2(x)(\tan^3(x)-\tan(x))}$

(u) $\displaystyle\int_{-1/\pi}^{1/\pi} \sin^2(3x^3+2x)\log\left[\frac{1-x^5}{1+x^5}\right]dx$

(v) $\displaystyle\int \sin(2x)\cos(x)\,dx$

(w) $\displaystyle\int x\log(x)\,dx$

(x) $\displaystyle\int \frac{dx}{x\log(x)}$

(y) $\displaystyle\int \frac{\sinh^3(x)}{\cosh^2(x)}\,dx$

(z) $\displaystyle\int \frac{1}{\sin^2(3x+1)}\,dx$

Solution:

(a) Let $u=\sqrt{x}$, to clear the square root. Then $u^2=x$, so that $2udu=dx$. Thus we have:

$$\int_4^9 \frac{dx}{\sqrt{x}-1}=\int_2^3 \frac{2udu}{u-1}=\int_2^3\left(\frac{2(u-1)}{u-1}+\frac{2}{u-1}\right)du=2+2\left[\log(u-1)\right]_2^3=2+2\log(2).$$

(b) Note that $1+\tan^2(x) = \sec^2(x)$, so the numerator is just $\sec^2(x)$ in disguise. This suggests the obvious substitution $u = \tan(x)$, which gives $du = \sec^2(x)dx$ and the limits change from $[\pi/3, \pi/4] \mapsto [\sqrt{3}, 1]$. Hence we have:

$$\int_{\pi/3}^{\pi/4} \frac{1+\tan^2(x)}{(1+\tan(x))^2}\,dx = \int_{\sqrt{3}}^{1} \frac{du}{(1+u)^2} = \left[-\frac{1}{1+u}\right]_{\sqrt{3}}^{1} = \frac{1}{1+\sqrt{3}} - \frac{1}{2}.$$

(c) An obvious substitution is $u = e^x$. We have $du = e^x dx$, so:

$$\begin{aligned}\int \frac{e^{2x}-2e^x}{e^{2x}+1}\,dx &= \int \frac{u-2}{u^2+1}\,du \\ &= \int \left(\frac{u}{u^2+1} - \frac{2}{u^2+1}\right) du \\ &= \frac{1}{2}\log(u^2+1) - 2\arctan(u) + c \\ &= \frac{1}{2}\log(e^{2x}+1) - 2\arctan(e^x) + c.\end{aligned}$$

(d) This is quite sneaky. Multiply the numerator and denominator by $\sec^2(x)$, to give:

$$\int \frac{\sec^2(x)dx}{\sec^2(x)+3} = \int \frac{\sec^2(x)dx}{\tan^2(x)+4}.$$

Now, the obvious substitution is $u = \tan(x)$. We have $du = \sec^2(x)dx$, which gives:

$$\int \frac{du}{u^2+4} = \frac{1}{4}\int \frac{du}{(u/2)^2+1} = \frac{2}{4}\arctan\left(\frac{u}{2}\right) + c = \frac{1}{2}\arctan\left(\frac{1}{2}\tan(x)\right) + c.$$

(e) Note that:

$$\int_2^3 \frac{2x+1}{x(x+1)}\,dx = \int_2^3 \frac{2x+1}{x^2+x}\,dx = \left[\log(x^2+x)\right]_2^3 = \log(12) - \log(6) = \log(2).$$

(f) Note that the derivative of $\sqrt{x}$ is $1/2\sqrt{x}$. This gives:

$$\int \frac{1}{2\sqrt{x}} e^{\sqrt{x}}\,dx = e^{\sqrt{x}} + c,$$

by thinking about the reverse chain rule.

(g) Note that the derivative of $-x^4$ is $-4x^3$. Hence:

$$\int x^3 e^{-x^4}\,dx = -\frac{1}{4}e^{-x^4} + c,$$

by thinking about the reverse chain rule.

(h) We have:

$$\begin{aligned}\int \left(\frac{\sin(2x)}{\sin^2(x)+\log(x)} + \frac{1}{x(\sin^2(x)+\log(x))}\right) dx &= \int \left(\frac{2\sin(x)\cos(x)+1/x}{\sin^2(x)+\log(x)}\right) dx \\ &= \log\left(\sin^2(x)+\log(x)\right) + c.\end{aligned}$$

(i) Make the substitution $u = 3 - 2x$, to clear the square root. Then $du = -2dx$, and $x = \frac{3}{2} - \frac{1}{2}u$. Hence:

$$\begin{aligned}\int x\sqrt{3-2x}\,dx &= -\frac{1}{2}\int\left(\frac{3}{2}-\frac{1}{2}u\right)u^{1/2}\,du \\ &= \frac{1}{4}\int\left(u^{3/2}-3u^{1/2}\right)\,du \\ &= \frac{1}{4}\left(\frac{2}{5}u^{5/2}-2u^{3/2}\right)+c \\ &= \frac{1}{10}(3-2x)^{5/2}-\frac{1}{2}(3-2x)^{3/2}+c.\end{aligned}$$

(j) Evidently, the substitution $u = \cos(x)$ will work. We have $du = -\sin(x)dx$, hence:

$$\int \frac{\sin(x)}{\cos^2(x)-5\cos(x)+6}\,dx = -\int\frac{du}{u^2-5u+6} = -\int\frac{du}{(u-5/2)^2-1/4} = 4\int\frac{du}{1-(2u-5)^2}.$$

Now, using a standard inverse hyperbolic integral, we have:

$$4\int\frac{du}{1-(2u-5)^2} = 2\tanh^{-1}(2u-5)+c = 2\tanh^{-1}(2\cos(x)-5)+c.$$

(k) We use integration by parts. We have:

$$\int\frac{\log(x)}{x^4}\,dx = -\frac{\log(x)}{3x^3}+\frac{1}{3}\int\frac{1}{x^4}\,dx = -\frac{\log(x)}{3x^3}-\frac{1}{9}\frac{1}{x^3}+c.$$

(l) Here, a trigonometric substitution is appropriate. Let $x = \cos(\theta)$. Then $dx = -\sin(\theta)d\theta$, which gives:

$$\int\sqrt{1-x^2}\,dx = -\int\sqrt{1-\cos^2(\theta)}\sin(\theta)\,d\theta = -\int\sin^2(\theta)\,d\theta.$$

Using the standard trick for integrating $\sin^2(\theta)$, we have:

$$-\int\sin^2(\theta)\,d\theta = \frac{1}{2}\int\left(\cos(2\theta)-1\right)\,d\theta = \frac{1}{2}\left(\frac{\sin(2\theta)}{2}-\theta\right)+c = \frac{1}{4}\sin(2\arcsin(x))-\frac{1}{2}\arcsin(x)+c.$$

We can slightly simplify this by using $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\sin(\theta)\sqrt{1-\sin^2(\theta)}$. This leaves:

$$\frac{1}{2}x\sqrt{1-x^2}-\frac{1}{2}\arcsin(x)+c.$$

(m) We have:

$$\int_{\pi/3}^{\pi/2}\tan(x)\cos^4(x)\,dx = \int_{\pi/3}^{\pi/2}\sin(x)\cos^3(x)\,dx = \left[-\frac{1}{4}\cos^4(x)\right]_{\pi/3}^{\pi/2} = \frac{1}{4}\cdot\left(\frac{1}{2}\right)^4 = \frac{1}{64}.$$

(n) We have:

$$\int_1^5 x^2 \log(x)\,dx = \left[\frac{1}{3}x^3\log(x)\right]_1^5 - \frac{1}{3}\int_1^5 x^2\,dx = \frac{125}{3}\log(5) - \frac{1}{9}(125-1) = \frac{125}{3}\log(5) - \frac{124}{9}.$$

(o) Writing the hyperbolic function in terms of exponentials, we have:

$$\int e^x \sinh(3x)\,dx = \frac{1}{2}\int \left(e^{4x} - e^{-2x}\right)\,dx = \frac{1}{2}\left(\frac{1}{4}e^{4x} - \frac{1}{2}e^{-2x}\right) + c = \frac{1}{8}e^{4x} - \frac{1}{4}e^{-2x} + c.$$

Alternatively, this can be done by parts.

(p) Here, we can integrate by parts:

$$\int \frac{\arctan(x)}{x^2}\,dx = -\frac{\arctan(x)}{x} + \int \frac{1}{x(1+x^2)}\,dx.$$

Using partial fractions, we have:

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} \qquad \Rightarrow \qquad 1 = A(1+x^2) + Bx^2 + Cx.$$

Here, $A = 1, B = -1, C = 0$. Thus we have:

$$-\frac{\arctan(x)}{x} + \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right)\,dx = -\frac{\arctan(x)}{x} + \log(x) - \frac{1}{2}\log(1+x^2) + c.$$

(q) Evidently, a sensible substitution is $u = \log(x)$; note that we have $du = dx/x$, and the limits transform as $[e^3, e^4] \mapsto [3, 4]$. Thus:

$$\begin{aligned}
\int_{e^3}^{e^4} \frac{3\log(x) - 4}{x\log^2(x) - 3x\log(x) + 2x}\,dx &= \int_3^4 \frac{3u-4}{u^2-3u+2}\,du \\
&= \frac{3}{2}\int_3^4 \frac{2u-3}{u^2-3u+2}\,du + \frac{1}{2}\int_3^4 \frac{1}{u^2-3u+2}\,du \\
&= \frac{3}{2}\left[\log\left(u^2-3u+2\right)\right]_3^4 + \frac{1}{2}\int_3^4 \frac{1}{(u-3/2)^2 - 1/4}\,du \\
&= \frac{3}{2}\left(\log(6) - \log(2)\right) - 2\int_3^4 \frac{du}{1-(2u-3)^2}\,du \\
&= \frac{3}{2}\log(3) - \left[\operatorname{artanh}(2u-3)\right]_3^4 \\
&= \frac{3}{2}\log(3) - \operatorname{artanh}(5) + \operatorname{artanh}(3).
\end{aligned}$$

If we really want to tidy everything up, we could simplify the two inverse hyperbolic functions. We let:

$$t = \operatorname{artanh}(5) - \operatorname{artanh}(3).$$

Now by the hyperbolic compound angle identity, we have:

$$\tanh(t) = \frac{5-3}{1-15} = -\frac{2}{14} = -\frac{1}{7}.$$

This leaves us with:

$$\frac{3}{2}\log(3) + \operatorname{artanh}\left(\frac{1}{7}\right).$$

(r) This can be done easily with integration by parts:

$$\begin{aligned}
\int_0^{\pi/6} x\sin(3x)\,dx &= \left[-\frac{1}{3}x\cos(3x)\right]_0^{\pi/6} + \frac{1}{3}\int_0^{\pi/6}\cos(3x)\,dx \\
&= \frac{1}{3}\left[\frac{1}{3}\sin(3x)\right]_0^{\pi/6} \\
&= \frac{1}{9}.
\end{aligned}$$

(s) Observe that the derivative of $\sin^2(x)$ is $2\sin(x)\cos(x) = \sin(2x)$. Hence:

$$\int \sin(2x)e^{\sin^2(x)}\,dx = e^{\sin^2(x)} + c.$$

(t) The integral in question can be rewritten in the form:

$$\int \frac{\sec^2(x)}{\tan^3(x) - \tan(x)}\,dx.$$

This suggests the substitution $u = \tan(x)$, which gives $du = \sec^2(x)dx$. Hence we have:

$$\int \frac{du}{u^3 - u} = \int \frac{du}{u(u-1)(u+1)}.$$

We decompose the integrand into partial fractions via:

$$\frac{1}{u(u-1)(u+1)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1}.$$

This implies:

$$1 = A(u-1)(u+1) + Bu(u+1) + Cu(u-1).$$

Taking $u = 0$ gives $A = -1$. Taking $u = 1$ gives $B = 1/2$. Taking $u = -1$ gives $C = -1/2$. Thus we have:

$$\int \left(-\frac{1}{u} + \frac{1}{2(u-1)} - \frac{1}{2(u+1)}\right)\,du = -\log(u) + \frac{1}{2}\log(u-1) - \frac{1}{2}\log(u+1) + c.$$

Thus the final integral is:

$$\frac{1}{2}\log\left(\frac{u-1}{u^2(u+1)}\right) + c = \frac{1}{2}\log\left(\frac{\tan(x)-1}{\tan^2(x)(\tan(x)+1)}\right) + c.$$

(u) This is a trick question - it is an odd function integrated over a symmetric domain, so the integral is just zero.

(v) We have:

$$\int \sin(2x)\cos(x)\,dx = 2\int \sin(x)\cos^2(x)\,dx = -\frac{2}{3}\cos^3(x) + c.$$

(w) Using integration by parts, we have:

$$\int x\log(x)\,dx = \frac{1}{2}x^2\log(x) - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\log(x) - \frac{1}{4}x^2 + c.$$

(x) Here, an obvious substitution is $u = \log(x)$, so that $du = dx/x$. We then have:

$$\int \frac{dx}{x\log(x)} = \int \frac{du}{u} = \log(u) + c = \log(\log(x)) + c.$$

(y) This integral can be rewritten as:

$$\int \frac{\sinh^3(x)}{\cosh^2(x)}\,dx = \int \frac{(\cosh^2(x) - 1)}{\cosh^2(x)}\sinh(x)\,dx,$$

suggesting the substitution $u = \cosh(x)$, which gives $du = \sinh(x)dx$. We then have:

$$\int \frac{u^2 - 1}{u^2}\,du = \int \left(1 - \frac{1}{u^2}\right)\,du = u + \frac{1}{u} + c = \cosh(x) + \operatorname{sech}(x) + c.$$

(z) An easy one to finish! This is a standard integral, from the Basic Integrals section of the start of the sheet. We have:

$$\int \frac{1}{\sin^2(3x+1)}\,dx = \int \operatorname{cosec}^2(3x+1)\,dx = -\frac{1}{3}\cot(3x+1) + c.$$

The fundamental theorem of calculus

11. State both parts of the *fundamental theorem of calculus*. Use the fundamental theorem of calculus to evaluate the following derivatives:

$$\text{(a) } \frac{d}{dx}\int_1^x \frac{\log(t)\sin^2(t)}{t^2+7}\,dt, \quad \text{(b) } \frac{d}{dx}\left[\sum_{n=0}^{N}\binom{N}{n}\int_n^x \sin(y^2+y^6)\,dy\right], \quad \text{(c) } \frac{d}{dx}\left[\sin(x)\int_x^0 \sin(\cos(t))\,dt\right].$$

Solution: The fundamental theorem of calculus states two things:

- Integration reverses differentiation, in the sense that:

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a).$$

- Differentiation reverses integration, in the sense that:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Applying this result to the given derivatives:

(a) Straightforwardly, this derivative is given by: $\dfrac{\log(x)\sin^2(x)}{x^2+7}$.

(b) Since differentiation is linear, we have:

$$\begin{aligned}\frac{d}{dx}\left[\sum_{n=0}^{N}\binom{N}{n}\int_n^x \sin(y^2+y^6)\,dy\right] &= \sum_{n=0}^{N}\binom{N}{n}\frac{d}{dx}\int_n^x \sin(y^2+y^6)\,dy\\ &= \sum_{n=0}^{N}\binom{N}{n}\sin(x^2+x^6)\\ &= \sin(x^2+x^6)\sum_{n=0}^{N}\binom{N}{n}.\end{aligned}$$

Here's a fun thing: the sum of the binomial coefficients is always 2^N. To see why, study the identity (which is just the binomial expansion):

$$2^N = (1+1)^N = \sum_{n=0}^{N}\binom{N}{n}1^n\cdot 1^{N-n} = \sum_{n=0}^{N}\binom{N}{n}.$$

Hence, the derivative simplifies to $2^N\sin(x^2+x^6)$.

(c) First, we have:

$$\int_x^0 \sin(\cos(t))\,dt = -\int_0^x \sin(\cos(t))\,dt.$$

Hence, using the product rule, we have:

$$\frac{d}{dx}\left[\sin(x)\int_x^0 \sin(\cos(t))\,dt\right] = -\frac{d}{dx}\left[\sin(x)\int_0^x \sin(\cos(t))\,dt\right] = -\cos(x)\int_0^x \sin(\cos(t))\,dt - \sin(x)\sin(\cos(x)).$$

This cannot be further simplified (we cannot perform the integral in terms of elementary functions).

12. Without evaluating the integrals, determine the local extrema of the functions F_1, F_2 defined by:

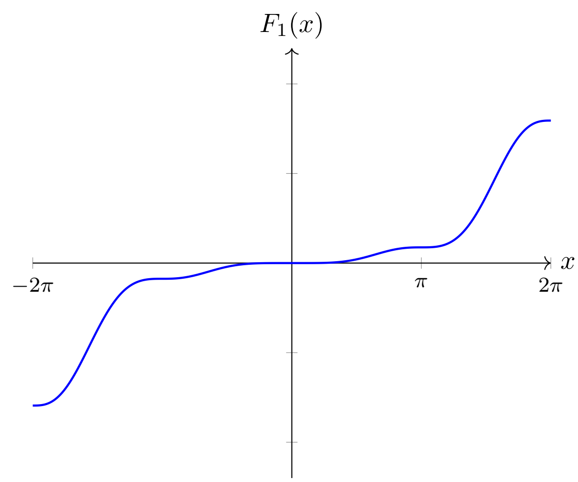
$$\text{(a) } F_1(x) = \int_0^x t^2 \sin^2(t)\,dt, \qquad \text{(b) } F_2(x) = \int_{-\infty}^x e^{-t^2}\,dt.$$

Hence, sketch the graphs of the functions F_1, F_2. [*Note: $F_2(x) \to \sqrt{\pi}$ as $x \to \infty$;* *see Question 23!*]

➾ **Solution:** (a) We have:

$$F_1'(x) = x^2 \sin^2(x).$$

Hence, the stationary points of the function occur at $x = n\pi$, for n an integer. The function is zero at $x = 0$, and is strictly increasing (because as x increases, more area under the function $t^2 \sin^2(t)$ contributes!). Therefore, the function looks like a series of inflection points.



(b) We have:

$$F_2'(x) = e^{-x^2},$$

so this function has no stationary points. The function is again strictly increasing. We are given that $F_2(x) \to \sqrt{\pi}$ as $x \to \infty$, and we note that $F_2(x) \to 0$ as $x \to -\infty$. The function is everywhere positive. Hence the graph looks like: