Part IA: Mathematics for Natural Sciences
2014 Paper 1 (Unofficial) Mark Scheme

Section A

1. (a) Differentiate:

$$\frac{1}{x^2+4}$$

with respect to x. [1]

Solution: We use the chain rule:

$$\frac{d}{dx}\frac{1}{x^2+4} = -\frac{2x}{(x^2+4)^2}.$$

[*1 mark for correct answer.*]

(b) Differentiate:

$$e^{\sin(x)}$$

with respect to x. [1]

Solution: We again use the the chain rule:

$$\frac{d}{dx}e^{\sin(x)} = \cos(x)e^{\sin(x)}.$$

[*1 mark for correct answer.*]

2. (a) Differentiate a^{-x} with respect to x, where a is a constant which satisfies $a > 0$ and $a \neq 1$. [1]

Solution: Write $a^{-x} = e^{-x\log(a)}$ (this is okay because $a > 0$). We can now differentiate straightforwardly to give:

$$\log(a)e^{-x\log(a)} = \log(a)a^{-x}.$$

This answer still works when $a = 1$, because $a^{-x} = 1$ in that case, and $\log(a) = 0$. [*1 mark for correct answer.*]

(b) Evaluate the indefinite integral: [1]

$$\int \frac{\ln(\ln(x))}{x}\,dx.$$

Solution: Consider the substitution $u = \ln(x)$. Then $du = dx/x$, hence the integral becomes:

$$\int \ln(u)\,du = \int 1\cdot\ln(u)\,du = u\ln(u) - \int u\cdot\frac{1}{u}\,du = u\ln(u) - u + c,$$

using integration by parts to determine the integral of $\ln(u)$. Inserting $u = \ln(x)$, we see the integral is:

$$\ln(x)\ln(\ln(x)) - \ln(x) + c.$$

[*1 mark for correct answer.*]

3. (a) Evaluate the definite integral: [1]

$$\int_{0}^{\frac{\pi}{4}} \tan(x)\,dx.$$

Solution: We have:

$$\int_{0}^{\frac{\pi}{4}} \tan(x)\,dx = \int_{0}^{\frac{\pi}{4}} \frac{\sin(x)}{\cos(x)}\,dx = -\left[\ln(\cos(x))\right]_0^{\pi/4} = -\left(\ln(1/\sqrt{2})\right) = \ln(\sqrt{2}).$$

[*1 mark for correct answer.*]

(b) Evaluate the definite integral: [1]

$$\int_{-2}^{-1} \frac{dx}{x}.$$

Solution: We have:

$$\int_{-2}^{-1} \frac{dx}{x} = \left[\ln|x|\right]_{-2}^{-1} = -\ln(2).$$

Crucially, the modulus is needed here! [*1 mark for correct answer.*]

4. (a) Find the general solution of the differential equation:

$$\frac{dy}{dx} = \cos^2(y)\sin(x)$$

for $-\pi/2 < y < \pi/2$. [1]

Solution: This equation is separable. We have:

$$\int \sec^2(y)\,dy = \int \sin(x)\,dx \qquad \Leftrightarrow \qquad \tan(y) = \sin(x) + c.$$

Rearranging, we have:

$$y = \arctan(\sin(x) + c),$$

which is okay because we are given that $-\pi/2 < y < \pi/2$, so we don't need to use any periodicity to get any other solutions. [*1 mark for correct answer, in simplified form.*]

(b) Find the solution of the differential equation:

$$\frac{dy}{dx} = 3y$$

such that $y = 3$ when $x = 0$. [1]

Solution: There are various ways of solving this (e.g. using an integrating factor, separating variables, or using the auxiliary equation), however we can just spot the general answer $y = Ae^{3x}$. Requiring $y = 3$ when $x = 0$, we get $y = 3e^{3x}$. [*1 mark for correct answer.*]

5. (a) Find all pairs of coordinates where the curve defined by:

$$7x^2 - y^2 = 7$$

meets the straight line $y = x + 1$. [1]

Solution: Substituting $y = x + 1$ into the quadratic, we have:

$$7x^2 - (x+1)^2 = 7 \qquad \Rightarrow \qquad 6x^2 - 2x - 8 = 0.$$

Simplifying, this becomes:

$$0 = 3x^2 - x - 4 = (3x - 4)(x + 1),$$

hence the solutions are $(x, y) = (\frac{4}{3}, \frac{7}{3}), (-1, 0)$. [*1 mark for both intersections correct.*]

(b) Sketch the curve defined by the equation $(x - 1)^2 + 2y^2 = 3$. [1]

Solution: This curve is an ellipse centred on $(x, y) = (1, 0)$. It extends a radial distance $\sqrt{3}$ either side of $x = 1$, and a radial distance of $\sqrt{3/2}$ either side of $y = 0$. [*1 mark for convincing sketch, clearly labelled with centre and lengths of minor and major axes.*]

6. (a) A sphere has radius 10 m. Draw on its surface a circular patch which has area 10 m^2. What fraction of the surface of the sphere does this cover? [1]

Solution: Consider spherical coordinates (r, θ, ϕ) on the surface of the sphere, with $r = 10$ m. A circle on the surface of the sphere centred on the z-axis, subtending an angle θ_0 from the z-axis, has area:

$$A(\theta) = \int_0^{2\pi} \int_0^{\theta_0} r^2 \sin(\theta)\, d\theta d\phi = 2\pi r^2 [-\cos(\theta)]_0^{\theta_0} = 2\pi r^2 (1 - \cos(\theta_0)).$$

In particular, if we need a circular patch of area 10 m^2, we require r m $= 2\pi r^2 (1 - \cos(\theta_0))$, which implies $\cos(\theta_0) = 1 - 1/20\pi$. This is very close to 1, so the angle θ_0 is very small. This allows us to provide a sketch. The fraction of the area is:

$$\frac{10}{4\pi(10)^2} = \frac{1}{400\pi}.$$

[*1 mark for sketch of very small circular area on surface of sphere, AND fraction of area correct.*]

(b) A square pyramid has all of its sides of length 1 m. What is its volume? [1]

Solution: The volume of a pyramid is a third the area of its base, multiplied by its height. The area of the base is 1 m^2 in this case. The height is given by h, satisfying:

$$1^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + h^2,$$

using three-dimensional Pythagoras. Hence $h = \frac{1}{2}$, and it follows the volume is $\frac{1}{6}$ m^3. [*1 mark for correct answer.*]

7. (a) Sketch the graph of: [1]

$$y = \frac{1}{1+\tan(x)}.$$

Solution: The graph is periodic with period π. It has asymptotes at $\tan(x) = -1$, i.e. at $x = 3\pi/4 + n\pi$ where n is an integer. The graph passes zero when $x = \pi/2 + n\pi$ (since there, $\tan(x)$ is singular). In between, the graph looks like an inverted tangent graph. [*1 mark for correct sketch.*]

(b) Sketch the graph of $y = e^{-x^3}$. [1]

Solution: Observe that as $x \to \infty$, $y \to 0$. Additionally, as $x \to -\infty$, $y \to +\infty$. The graph is everywhere positive and passes through the origin at $x = 0$. Finally, note that:

$$\frac{d}{dx}e^{-x^3} = -3x^2 e^{-x^3},$$

so the derivative is everywhere negative; there is a point of inflection at $x = 0$. The graph looks like a decaying exponential graph, just flatter at the origin $x = 0$. [*1 mark for correct sketch.*]

8. (a) Find the values of x at the stationary points of the function: [1]

$$y = x^3 - 2x^2 - 7x + 6.$$

Solution: The stationary points satisfy:

$$0 = \frac{dy}{dx} = 3x^2 - 4x - 7 = (3x-7)(x+1),$$

hence occur at $x = 7/3$ and $x = -1$. [*1 mark for correct answer.*]

(b) The function $y = x^3 - 3x + 7$ has a stationary point at $x = 1$. Is this point a maximum, minimum or a point of inflection? [1]

Solution: Taking two derivatives, we have:

$$\frac{d^2y}{dx^2} = 6x.$$

Hence at $x = 1$, there is a minimum (second derivative is positive). [*1 mark for correct answer.*]

9. (a) What is the area bounded by the curve $y = x^2 - 3x + 2$, the positive half of the x-axis, the positive half of the y-axis, and the line $x = 1/2$? [1]

(b) Sketch the curve(s) defined by the relation $y^2 = x^3$. [1]

Solution: (a) Observe that $0 = x^2 - 3x + 2 = (x-2)(x-1)$, so that there are turning points of the function at $x = 1$ and $x = 2$. Therefore, for $0 < x < 1/2$ the function lies entirely above the x-axis. Therefore, this question is just asking us to compute the integral:

$$\int_0^{1/2} (x^2 - 3x + 2)\, dx = \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_0^{1/2} = \frac{1}{24} - \frac{3}{8} + 1 = \frac{16}{24} = \frac{2}{3}.$$

[1 *mark for correct answer.*]

(b) This is the square root of the graph of $y = x^3$. Hence the graph does not exist for $x < 0$, and for $x > 0$, the graph looks like it has a cusp at $x = 0$, and is mirror-symmetric about the line $y = 0$. [1 *mark for correct graph.*]

10. (a) The straight line L is defined by the equation $y = 2x + 3$. Find the equation of the line L' that is perpendicular to L and intersects L where L crosses the x-axis. [1]

Solution: A perpendicular line is $y = -\frac{1}{2}x + c$. The place where L crosses the x-axis is $(x, y) = (-3/2, 0)$. Hence for the perpendicular to intersect there, we require $0 = 3/4 + c$, which gives:

$$y = -\frac{1}{2}x - \frac{3}{4},$$

as the required equation of the line. [1 *mark for correct answer.*]

(b) Express:

$$\frac{13(x+1)}{(x-4)(x+9)}$$

as a sum of partial fractions. [1]

Solution: We have:

$$\frac{13(x+1)}{(x-4)(x+9)} = \frac{A}{x-4} + \frac{B}{x+9} \qquad \Rightarrow \qquad 13(x+1) = A(x+9) + B(x-4).$$

Set $x = 4$, which gives $13 \cdot 5 = 13A$, which implies $A = 5$. Set $x = -9$, then $13 \cdot (-8) = -13B$, hence $B = 8$. Thus the partial fractions are:

$$\frac{13(x+1)}{(x-4)(x+9)} = \frac{5}{x-4} + \frac{8}{x+9}.$$

[1 *mark for correct answer.*]

Section B

11. (a) Calculate $\mathrm{Det}(A)$ and $\mathrm{Tr}(A)$ where:

$$A = \begin{pmatrix} 1 & -4 & 7 \\ -4 & 4 & -4 \\ 7 & -4 & 1 \end{pmatrix}.$$

From the value of $\mathrm{Det}(A)$ make a deduction about the eigenvalues of A. [4]

Solution: The trace is $\mathrm{Tr}(A) = 1+4+1 = 6$. [*1 mark.*] The determinant can be found more simply by considering equivalent row operations on the matrix. Adding the second row to the first, and adding the second row to the third, we have:

$$\mathrm{Det}(A) = \mathrm{Det}\begin{pmatrix} 1 & -4 & 7 \\ -4 & 4 & -4 \\ 7 & -4 & 1 \end{pmatrix} = \mathrm{Det}\begin{pmatrix} -3 & 0 & 3 \\ -4 & 4 & -4 \\ 3 & 0 & -3 \end{pmatrix}.$$

But note that now the first and last rows are scalar multiples of one another, so the determinant vanishes, $\mathrm{Det}(A) = 0$. [*1 mark for partial progress to calculation of determinant, 1 mark for correct determinant.*] The determinant is the product of the eigenvalues, so at least one eigenvalue must be 0 (the other two must sum to 6 because the trace is the sum of the eigenvalues). [*1 mark for saying at least one eigenvalue is zero.*]

(b) Calculate the eigenvalues and the corresponding normalised eigenvectors of A. Verify that the eigenvectors are mutually orthogonal. [10]

Solution: The eigenvalues satisfy the following equation:

$$\begin{aligned}
0 &= \mathrm{Det}\begin{pmatrix} 1-\lambda & -4 & 7 \\ -4 & 4-\lambda & -4 \\ 7 & -4 & 1-\lambda \end{pmatrix} \\
&= (1-\lambda)((4-\lambda)(1-\lambda)-16) + 4(-4(1-\lambda)+28) + 7(16-7(4-\lambda)) \\
&= (1-\lambda)(-12-5\lambda+\lambda^2) + 4(24+4\lambda) + 7(7\lambda-12) \\
&= -12 + 7\lambda + 6\lambda^2 - \lambda^3 + 96 + 16\lambda - 84 + 49\lambda \\
&= -\lambda^3 + 6\lambda^2 + 72\lambda \\
&= -\lambda(\lambda^2 - 6\lambda - 72) \\
&= -\lambda(\lambda-12)(\lambda+6).
\end{aligned}$$

Hence the eigenvalues are $\lambda = 0$, $\lambda = -6$ and $\lambda = 12$, consistent with the results of the first part. [*1 mark for correct determinant, 1 mark for correct reduction to cubic, 1 mark for factoring cubic correctly and finding the roots.*] For the eigenvectors, observe that for $\lambda = 0$, we require:

$$\begin{pmatrix} 1 & -4 & 7 \\ -4 & 4 & -4 \\ 7 & -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,$$

which is satisfied by any scalar multiple of:

$$\begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ -3 \end{pmatrix},$$

so a normalised eigenvector is $(1,2,1)/\sqrt{6}$. [*1 mark for some correct working, 1 mark for correct eigenvector.*]

For $\lambda = -6$, we require:

$$\begin{pmatrix} 7 & -4 & 7 \\ -4 & 10 & -4 \\ 7 & -4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,$$

which is satisfied by any scalar multiple of:

$$\begin{pmatrix} 7 \\ -4 \\ 7 \end{pmatrix} \times \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} -27 \\ 0 \\ 27 \end{pmatrix},$$

so a normalised eigenvector is $(1,0,-1)/\sqrt{2}$. [*1 mark for some correct working, 1 mark for correct eigenvector.*]

For $\lambda = 12$, we require:

$$\begin{pmatrix} -11 & -4 & 7 \\ -4 & -8 & -4 \\ 7 & -4 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

which is satisfied by any scalar multiple of:

$$\begin{pmatrix} -11 \\ -4 \\ 7 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -18 \\ 18 \\ -18 \end{pmatrix},$$

so a normalised eigenvector is $(1,-1,1)/\sqrt{3}$. [*1 mark for some correct working, 1 mark for correct eigenvector.*]

Finally, we check that they are mutually orthogonal. We have $(1,2,1)\cdot(1,0,-1) = 1-1 = 0$, $(1,2,1)\cdot(1,-1,1) = 1-2+1 = 0$ and $(1,0,-1)\cdot(1,-1,1) = 1-1 = 0$. [*1 mark for verifying mutually orthogonal.*]

(c) By expressing an arbitrary vector $\mathbf{r}$ in terms of the eigenvectors or otherwise, show that a non-zero vector $\mathbf{e}$ exists such that $A\mathbf{r}\cdot\mathbf{e} = 0$ for all $\mathbf{r}$. [4]

Solution: Let the eigenvectors be $\mathbf{e}_0, \mathbf{e}_{-6}, \mathbf{e}_{12}$ respectively. Then if $\mathbf{r} = \alpha\mathbf{e}_0 + \beta\mathbf{e}_{-6} + \gamma\mathbf{e}_{12}$ [*1 mark for expressing **r** in this form.*], we have:

$$A\mathbf{r} = -6\beta\mathbf{e}_{-6} + 12\gamma\mathbf{e}_{12}.$$

[*2 marks for correct action of A.*] Since the eigenvectors are mutually orthogonal, observe that as a result we have $A\mathbf{r}\cdot\mathbf{e}_0 = 0$. Hence there exists a non-zero vector $\mathbf{e} = \mathbf{e}_0$ such that $A\mathbf{r}\cdot\mathbf{e} = 0$ for all $\mathbf{r}$. [*1 mark for correctly identifying $\mathbf{e} = \mathbf{e}_0$.*]

(d) Describe in words the action of A on an arbitrary non-zero vector. [2]

Solution: A projects vectors into the plane with normal $\mathbf{e}_0 = (1, 2, 1)$. It scales the vector by a factor of 12 in the $\mathbf{e}_{12}$ direction, and by a factor of -6 in the $\mathbf{e}_{-6}$ direction. [*1 mark for saying projection into plane, 1 mark for saying scaling behaviour in remaining directions.*]

12. (a) Express the cube roots of $i-1$ in terms of their modulus and argument. [4]

Solution: We begin by writing $i-1$ in modulus argument form. Its modulus is $\sqrt{2}$ and its argument is $3\pi/4$ (think of an Argand diagram), hence:

$$i-1=\sqrt{2}e^{3i\pi/4}.$$

[*1 mark for correct modulus, 1 mark for correct argument.*] It follows that the cube roots are:

$$\sqrt{2}e^{i\pi/4}, \qquad \sqrt{2}e^{i\pi/4+2\pi/3}=\sqrt{2}e^{11\pi i/12}, \qquad \sqrt{2}e^{i\pi/4+4\pi/3}=\sqrt{2}e^{19\pi i/12}=\sqrt{2}e^{-5i\pi/12}.$$

[*1 mark for some correct cube roots, and an additional 1 mark for all correct cube roots.*]

(b) Find all solutions to the equation $\tanh(z)=-i$. [5]

Solution: Writing $\tanh(z)$ in terms of exponentials, we have:

$$\frac{e^z-e^{-z}}{e^z+e^{-z}}=-i.$$

[*1 mark for expressing $\tanh(z)$ in terms of exponentials.*] Multiplying up, then multiplying through by e^z, we have:

$$e^{2z}-1=-ie^{2z}-i.$$

Rearranging, we have:

$$e^{2z}=\frac{1-i}{1+i}=\frac{(1-i)^2}{2}=-i.$$

[*1 mark for rearranging to quadratic equation*.] Hence:

$$2z=\ln(-i)=-\frac{i\pi}{2}+2ni\pi,$$

[*1 mark for correctly taking logarithm using argument of $-i$, additional 1 mark for remembering many solutions.*] from which it follows that:

$$z=-\frac{i\pi}{4}+in\pi.$$

[*1 mark for correct final answer (including many solutions)*.]

(c) Given that $z = 2 + i$ solves the equation:

$$z^3 - (4+2i)z^2 + (4+5i)z - (1+3i) = 0,$$

find the remaining solutions. [6]

Solution: We need to factorise the left hand side. Observe that:

$$\frac{1+3i}{2+i} = \frac{(1+3i)(2-i)}{5} = \frac{5+5i}{5} = 1+i.$$

Hence, we have:

$$z^3 - (4+2i)z^2 + (4+5i)z - (1+3i) = (z-(2+i))(z^2-(2+i)z+(1+i)).$$

[*Up to 3 marks for correctly factorising using given factor; award some marks for working if appropriate.*] Using the quadratic formula on the second factor, we have:

$$z = \frac{2+i \pm \sqrt{(2+i)^2 - 4(1+i)}}{2} = \frac{2+i \pm \sqrt{-1}}{2} = \frac{2+i \pm i}{2} = 1+i, \text{ or } 1.$$

Thus the solutions are:

$$2+i, \qquad 1, \qquad 1+i.$$

[*Up to 3 marks for solution of quadratic; 1 mark for using quadratic formula, 1 mark for correctly simplifying, 1 mark for complete correct statement of roots.*]

(d) Use complex numbers to show that $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$. [5]

Solution: We have, by De Moivre's theorem:

$$\begin{aligned}
\cos(4\theta) &= \mathrm{Re}\left(\cos(4\theta) + i\sin(4\theta)\right) \\
&= \mathrm{Re}\left((\cos(\theta) + i\sin(\theta))^4\right) \\
&= \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta) \\
&= \cos^4(\theta) - 6\cos^2(\theta)(1-\cos^2(\theta)) + (1-\cos^2(\theta))^2 \\
&= \cos^4(\theta) - 6\cos^2(\theta) + 6\cos^4(\theta) + \left(1 - 2\cos^2(\theta) + \cos^4(\theta)\right) \\
&= 8\cos^4(\theta) - 8\cos^2(\theta) + 1,
\end{aligned}$$

as required. [*1 mark for writing as real part, 1 mark for using De Moivre's theorem, 1 mark for correct binomial expansion, 1 mark for some simplification using trigonometric identities, 1 mark for correct final answer.*]

13. (a) Solve the following differential equations for $y(x)$ subject to the listed boundary conditions, making your answer explicit for y.

(i) $\dfrac{dy}{dx} + 3y = 8$, with $y(0) = 4$. [3]

(ii) $\dfrac{dy}{dx} - y\cos(x) = \dfrac{1}{2}\sin(2x)$, with $y(0) = 0$. [7]

Solution: (a) This is a linear differential equation with constant coefficients, so can be solved using the auxiliary equation approach. The auxiliary equation is $\lambda + 3 = 0$ with roots $\lambda = -3$, hence the complementary function is $y_c = Ae^{-3x}$. A particular integral is clearly $y_p = 8/3$, hence the complete general solution is:

$$y = Ae^{-3x} + \frac{8}{3}.$$

Imposing $y(0) = 4$, we have $A + 8/3 = 4 = 12/3$, hence $A = 4/3$. Thus we have:

$$y = \frac{4}{3}\left(e^{-3x} + 2\right).$$

(b) This equation does not have constant coefficients, but is still linear. It can be solved instead using an integrating factor; an appropriate one is clearly $e^{-\sin(x)}$. We get:

$$\frac{d}{dx}\left(ye^{-\sin(x)}\right) = \frac{1}{2}e^{-\sin(x)}\sin(2x) = \sin(x)\cos(x)e^{-\sin(x)}.$$

Integrating both sides, we have:

$$ye^{-\sin(x)} = \int \sin(x)\cos(x)e^{-\sin(x)}\,dx.$$

To do the integral, consider making the substitution $u = \sin(x)$, so that $du = \cos(x)\,dx$. Then we get:

$$\int ue^{-u}\,du = -ue^{-u} + \int e^{-u}\,du = -ue^{-u} - e^{-u} + c.$$

Hence the solution of the differential equation is:

$$y = ce^{\sin(x)} - \sin(x) - 1.$$

Imposing the initial data, we require $y(0) = 0 = c - 1$, so that $c = 1$. Thus the particular solution is:

$$y = e^{\sin(x)} - \sin(x) - 1.$$

(b) The function $y(x)$ satisfies the differential equation:

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 2e^{-3x}.$$

Solve the equation for $y(x)$ subject to $y(0) = 1$ and $\left.\frac{dy}{dx}\right|_{x=0} = 0.$ [10]

Solution: The auxiliary equation is $0 = \lambda^2 + 7\lambda + 12 = (\lambda+4)(\lambda+3)$, hence the complementary function is:

$$y_c = Ae^{-4x} + Be^{-3x}.$$

Since e^{-3x} is already contained in the complementary function, we posit $y_p = \alpha x e^{-3x}$ as a potential particular integral. Then:

$$y_p' = \alpha e^{-3x} - 3\alpha x e^{-3x}, \qquad y_p'' = -6\alpha e^{-3x} + 9\alpha x e^{-3x}.$$

Hence we require:

$$-6\alpha e^{-3x} + 9\alpha x e^{-3x} + 7\alpha e^{-3x} - 21\alpha x e^{-3x} + 12\alpha x e^{-3x} = 2e^{-3x}.$$

Comparing coefficients, we see $\alpha = 2$. Thus the complete solution is:

$$y = Ae^{-4x} + Be^{-3x} + 2xe^{-3x}.$$

It remains to impose the initial data. We have $y(0) = 1$, so $A + B = 1$. We also have $y'(0) = 0$, so we require:

$$-4A - 3B + 2 = 0 \qquad \Leftrightarrow \qquad 4A + 3B = 2.$$

Solving these equations simultaneously, we have:

$$A + 3(A+B) = 2 \qquad \Rightarrow \qquad A + 3 = 2 \qquad \Rightarrow \qquad A = -1, B = 2.$$

Thus the particular solution is:

$$y = 2e^{-3x} - e^{-4x} + 2xe^{-3x}.$$

14. (a) An arbitrary point along a straight line in a three-dimensional space can be written as $\mathbf{r}_1 = \mathbf{a} + \lambda\hat{\mathbf{b}}$, where λ is a scale parameter and $\hat{\mathbf{b}}$ is a unit vector. Obtain a formula for the minimum distance between $\mathbf{r}_1$ and $\mathbf{r}_2 = \mathbf{c} + \mu\hat{\mathbf{d}}$, where $\hat{\mathbf{d}}$ is a unit vector, assuming that the two lines are not parallel. [5]

Solution: A vector orthogonal to both lines is $\hat{\mathbf{b}} \times \hat{\mathbf{d}}$ [1 *mark for explaining this.*]. A vector that joins both lines is $\mathbf{c} - \mathbf{a}$ [1 *mark for explaining this.*]. Projecting this vector onto the orthogonal direction between the lines gives the shortest distance [1 *mark for recognising this.*]; we have:

$$\frac{|(\mathbf{c} - \mathbf{a}) \cdot \hat{\mathbf{b}} \times \hat{\mathbf{d}}|}{|\hat{\mathbf{b}} \times \hat{\mathbf{d}}|}.$$

[1 *mark for correct numerator, 1 mark for normalising correctly.*] Note that $\hat{\mathbf{b}}$, $\hat{\mathbf{d}}$ unit vectors does *not* imply that $\hat{\mathbf{b}} \times \hat{\mathbf{d}}$ is a unit vector!

(b) Find all vectors $\mathbf{x}$ that obey the equation $\mathbf{x} \cdot \mathbf{p} = k$, where $\mathbf{p}$ is a fixed non-zero vector in a three-dimensional space and k is a fixed scalar. [7]

[*Hint: Your answer should contain an arbitrary non-zero vector* $\boldsymbol{q}$ *which can be taken to be non-collinear with* $\boldsymbol{p}$, *i.e.* $\boldsymbol{p} \times \boldsymbol{q} \neq \boldsymbol{0}$. *You should treat the cases* $\boldsymbol{p} \cdot \boldsymbol{q} \neq 0$ *and* $\boldsymbol{p} \cdot \boldsymbol{q} = 0$ *separately.*]

[*Notation:* $\boldsymbol{u} \times \boldsymbol{v}$ *is equivalent to* $\boldsymbol{u} \wedge \boldsymbol{v}$.]

Solution: This is a rather poorly phrased question, so don't worry about it too much if it seemed hard.

Suppose we are given an arbitrary non-zero vector $\mathbf{q}$, which we assume is non-collinear to $\mathbf{p}$. Then $\{\mathbf{p}, \mathbf{q}, \mathbf{p} \times \mathbf{q}\}$ form a (possibly non-orthogonal basis) for $\mathbb{R}^3$, and hence we may express $\mathbf{x}$ in the form:

$$\mathbf{x} = a\mathbf{p} + b\mathbf{q} + c\mathbf{p} \times \mathbf{q}.$$

Imposing the condition $\mathbf{x} \cdot \mathbf{p} = k$, we have:

$$k = a\mathbf{p} \cdot \mathbf{p} + b\mathbf{p} \cdot \mathbf{q}.$$

Rearranging to obtain a, we have:

$$a = \frac{k - b\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}|^2}.$$

Hence we can express $\mathbf{x}$ as:

$$\mathbf{x} = \frac{k - b\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}|^2}\mathbf{p} + b\mathbf{q} + c\mathbf{p} \times \mathbf{q}.$$

There is no need to distinguish the cases $\mathbf{p} \cdot \mathbf{q} = 0$ and $\mathbf{p} \cdot \mathbf{q} \neq 0$ as the question claims.

(c) A particle moves along a path on which the position coordinate in terms of a parameter t is given by:

$$x = \frac{\cos(t)}{\sqrt{1+t^2}}, \qquad y = \frac{\sin(t)}{\sqrt{1+t^2}}, \qquad z = \frac{t}{\sqrt{1+t^2}}.$$

Express the equation for a point on the path in spherical polar coordinates $r(t), \theta(t), \phi(t)$. [8]

Solution: We have:

$$r(t) = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{\cos^2(t)+\sin^2(t)+t^2}{1+t^2}} = \sqrt{\frac{1+t^2}{1+t^2}} = 1.$$

[*1 mark for correct formula for r. 1 mark for correct substitution. 1 mark for correct simplified form.*] Similarly, we have:

$$\theta(t) = \arccos\left(\frac{z}{r}\right) = \arccos\left(\frac{t}{\sqrt{1+t^2}}\right).$$

[*1 mark for correct formula for θ. 1 mark for correct substitution. 1 mark for correct simplified form.*] Finally, we have:

$$\phi(t) = \arctan\left(\frac{y}{x}\right) = \arctan\left(\tan(t)\right) = t.$$

[*1 mark for correct formula for ϕ. 1 mark for correct substitution and simplification.*]

15. (a) The area of integration, D, is defined in plane polar coordinates (r, ϕ) by the inequality $r_2 \le r \le r_1$ where $r_1 = 1 + \cos(\phi)$ and $r_2 = 3/2$.

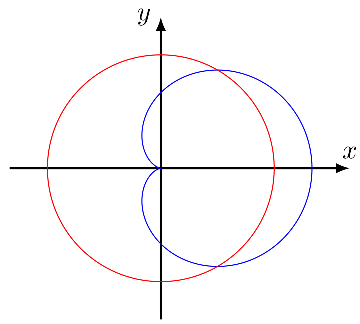
(i) Sketch the area of integration. [4]

(ii) Calculate the value of the area D. [6]

(iii) Evaluate the following integral over this area: [5]

$$\iint_D \frac{x + y + xy}{x^2 + y^2}\, dxdy.$$

Solution: (i) The region of integration is between a circle and a *cardioid*, as shown in the figure below.



This is because the cardioid ranges between 0 and 2, so there are intersections between the circle $r = 3/2$ and the cardioid $r = 1 + \cos(\phi)$. There is an intersection when $3/2 = 1 + \cos(\phi)$, which gives $\cos(\phi) = 1/2$, and hence $\phi = \pm\pi/3$. [*1 mark for circle drawn correctly. 2 marks for cardioid drawn correctly (1 mark for partially). 1 mark for intersection angle computed.*]

(ii) The area of the region D is:

$$\iint_D dA = \int_{-\pi/3}^{\pi/3} \int_{3/2}^{1+\cos(\phi)} r dr d\phi$$

$$= \frac{1}{2} \int_{-\pi/3}^{\pi/3} \left[r^2\right]_{3/2}^{1+\cos(\phi)} d\phi$$

$$= \frac{1}{2} \int_{-\pi/3}^{\pi/3} \left((1 + \cos(\phi))^2 - \frac{9}{4}\right) d\phi$$

$$= \frac{1}{2} \int_{-\pi/3}^{\pi/3} \left(\cos^2(\phi) + 2\cos(\phi) - \frac{5}{4}\right) d\phi$$

$$= \frac{1}{2} \int\limits_{-\pi/3}^{\pi/3} \left(\frac{1}{2} \cos(2\phi) + 2\cos(\phi) - \frac{3}{4} \right) \, d\phi$$

$$= \frac{1}{2} \left[\frac{1}{4} \sin(2\phi) + 2\sin(\phi) - \frac{3}{4}\phi \right]_{-\pi/3}^{\pi/3}$$

$$= \frac{1}{2} \left(\frac{1}{4}\sqrt{3} + 2\sqrt{3} - \frac{\pi}{2} \right)$$

$$= \frac{9\sqrt{3}}{8} - \frac{\pi}{4}$$

[*1 mark for correct polar measure $r dr d\phi$. 1 mark for integral written correctly with correct limits. 1 mark for correct evaluation of radial integral. 1 mark for simplification. 1 mark for correct angular integral. 1 mark for final answer.*]

(iii) The integrand can be expressed in terms of polar coordinates as:

$$\frac{x+y+xy}{x^2+y^2} dxdy = \frac{r\cos(\phi) + r\sin(\phi) + r^2\cos(\phi)\sin(\phi)}{r^2} r dr d\phi$$
$$= (\cos(\phi) + \sin(\phi) + r\sin(\phi)\cos(\phi)) \, dr d\phi.$$

Hence we have:

$$\iint\limits_D \frac{x+y+xy}{x^2+y^2} dxdy = \int\limits_{-\pi/3}^{\pi/3} \int\limits_{3/2}^{1+\cos(\phi)} (\cos(\phi) + \sin(\phi) + r\sin(\phi)\cos(\phi)) \, dr d\phi$$

$$= \int\limits_{-\pi/3}^{\pi/3} \left[r(\cos(\phi) + \sin(\phi)) + \frac{1}{2} r^2 \cos(\phi)\sin(\phi) \right]_{3/2}^{1+\cos(\phi)} d\phi$$

$$= \int\limits_{-\pi/3}^{\pi/3} \Bigg((1+\cos(\phi))(\cos(\phi) + \sin(\phi)) + \frac{1}{2}(1 + 2\cos(\phi) + \cos^2(\phi))\cos(\phi)\sin(\phi)$$

$$- \frac{3}{2}(\cos(\phi) + \sin(\phi)) - \frac{9}{8}\cos(\phi)\sin(\phi) \Bigg) \, d\phi$$

Note that terms involving a single sine will cancel because they are odd functions integrated over an even range. Further, $\cos(\phi)\sin(\phi) = \frac{1}{2}\sin(2\phi)$, so these terms will cancel too. Hence this simplifies to:

$$= \int_{-\pi/3}^{\pi/3} \left(\cos(\phi) + \cos^2(\phi) - \frac{3}{2}\cos(\phi) \right) d\phi$$

$$= \frac{1}{2} \int_{-\pi/3}^{\pi/3} \left(1 + \cos(2\phi) - \cos(\phi)\right)\, d\phi$$

$$= \frac{1}{2} \left[\phi + \frac{1}{2}\sin(2\phi) - \sin(\phi) \right]_{-\pi/3}^{\pi/3}$$

$$= \frac{1}{2} \left[\frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \sqrt{3} \right]$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

[*1 mark for integrand written correctly in polar coordinates. 1 mark for integral written correctly with correct limits. 1 mark for radial integral done correctly. 1 mark for correct simplification of angular integrand. 1 mark for correct angular integral and final answer.*]

(b) Evaluate the triple integral:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\exp\left[-\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{a^2}\right]\,dxdydz,$$

where $a>0, x_0, y_0$ and z_0 are real constants. [5]

Solution: Observe that we can separate the integral into three separate factors:

$$\left(\int_{-\infty}^{\infty} x\exp\left(-\frac{(x-x_0)^2}{a^2}\right)\,dx\right)\cdot\left(\int_{-\infty}^{\infty} \exp\left(-\frac{(y-y_0)^2}{a^2}\right)\,dy\right)\cdot\left(\int_{-\infty}^{\infty} \exp\left(-\frac{(z-z_0)^2}{a^2}\right)\,dz\right).$$

The second two factors are Gaussian integrals. In the first factor, let $u=(y-y_0)/a$. Then $du=dy/a$, and hence:

$$\int_{-\infty}^{\infty} \exp\left(-\frac{(y-y_0)^2}{a^2}\right)\,dy = a\int_{-\infty}^{\infty} e^{-u^2}\,du = a\sqrt{\pi}.$$

This is independent of y_0, hence the final factor is also $a\sqrt{\pi}$. The first factor is:

$$\int_{-\infty}^{\infty} x\exp\left(-\frac{(x-x_0)^2}{a^2}\right)\,dx = \int_{-\infty}^{\infty} (x-x_0)\exp\left(-\frac{(x-x_0)^2}{a^2}\right)\,dx + x_0\int_{-\infty}^{\infty} \exp\left(-\frac{(x-x_0)^2}{a^2}\right)\,dx.$$

Note that the first term has an integrand which is rotationally symmetric about $x=x_0$, so it is like an 'odd' function about the line $x=x_0$. Therefore, it is zero. The remaining term is $x_0a\sqrt{\pi}$. Thus the final integral is:

$$x_0a^3\pi^{3/2}.$$

[*1 mark for noticing separation of integrals. 1 mark for correct substitution in Gaussian integral and 1 mark for correct evaluation of Gaussian integral. 1 mark for correct evaluation of non-Gaussian integral through whatever method. 1 mark for final combination into correct answer.*]

16. (a) A function f of two variables x and y is defined as:

$$f(x, y) = 2x^3 + 6xy^2 - 3y^3 - 150x.$$

Determine the positions of the stationary points of f and their characters (maximum, minimum or saddle point).

[8]

Solution: Differentiating, we have:

$$\nabla f = \left(6x^2 + 6y^2 - 150, 12xy - 9y^2\right).$$

In particular, we have a stationary point wherever $\nabla f = (0, 0)$. The second equation implies that $3y(4x-3y) = 0$, hence either $y = 0$ or $y = \frac{4}{3}x$. Inserting into the first equation, we have two cases:

- If $y = 0$, then $6x^2 = 150$, and hence $x = \pm 5$. Thus we have two stationary points $(x, y) = (\pm 5, 0)$.
- If $y = \frac{4}{3}x$, then $6x^2 + 32x^2/3 = 150$, which implies $50x^2/3 = 150$. Thus $x^2 = 9$, and hence $x = \pm 3$. It follows that $(x, y) = (\pm 3, \pm 4)$ are stationary points.

[*1 mark for correctly evaluating gradient, 1 mark for solving equations for some of stationary points, 1 mark for getting all of stationary points.*]

To do the classification, we notice there is no significant symmetry or boundedness properties of $f(x, y)$ that can help. Thus we will need to do the second derivative tests. We first compute:

$$f_{xx} = 12x, \qquad f_{xy} = 12y, \qquad f_{yy} = 12x - 18y,$$

hence at each of the stationary points:

- At $(x, y) = (\pm 5, 0)$, we have $f_{xx} = \pm 60$, $f_{xy} = 0$, $f_{yy} = \pm 60$. Thus $f_{xx}f_{yy} - f_{xy}^2 = 60^2 > 0$. Further, for $(x, y) = (5, 0)$ we have $f_{xx} + f_{yy} > 0$ and for $(x, y) = (-5, 0)$ we have $f_{xx} + f_{yy} < 0$. It follows that $(5, 0)$ is a minimum and $(-5, 0)$ is a maximum.
- At $(x, y) = (\pm 3, \pm 4)$, we have $f_{xx} = \pm 36$, $f_{xy} = \pm 48$, $f_{yy} = \pm(36-72) = \mp 36$. Hence $f_{xx}+f_{yy} = 0$ and $f_{xx}f_{yy} - f_{xy}^2 = -36^2 - 48^2 < 0$. It follows that both of these points are saddles.

[*1 mark for correctly evaluating second derivatives, 1 mark for applying correct test for $(\pm 5, 0)$, 1 mark for correct conclusion in terms of maximum/minimum, 1 mark for applying correct test for $(\pm 3, \pm 4)$, 1 mark for correct conclusion of saddles.*]

(b) A function g of two variables x and y is defined as:

$$g(x, y) = x^4 + y^4 - 36xy.$$

Sketch the contours of g in the x-y plane, indicating on the sketch the positions and characters of all the stationary points. [12]

Solution: Let's begin by finding the stationary points and their character, since we need to do that already. We have:

$$\nabla g = (4x^3 - 36y, 4y^3 - 36x).$$

Equating these to zero, we have $x^3 = 9y$ and $y^3 = 9x$. Substituting the first equation into the second, we have $(x^3/9)^3 = 9x$, and hence $x^9 = 9^4 x$. Thus either $x = 0$ or $x^8 = 9^4 = 3^8$, and hence $x = \pm 3$. It follows that the stationary points are:

$$(0, 0), \qquad (\pm 3, \pm 3).$$

[*1 mark for correctly evaluating gradient of g. 1 mark for partial progress towards positions of stationary points. 1 mark for finding all stationary points correctly.*]

We could do the second derivative test to work out the character of the stationary points (it is not very hard here). However, we shall use some other properties to work this out. Near $(0, 0)$, the function is dominated by the $-36xy$ part because the $x^4 + y^4$ part is very small. Hence the constant contours look like contours of $-36xy = C$. In particular, they look like $y \propto 1/x$ graphs. Further when $C = 0$ we get $xy = 0$ so the contours look like straight lines passing through the origin. This type of contour implies that $(0, 0)$ is a saddle. [*1 mark for correct argument at origin, second derivative test or otherwise. 1 mark for correctly concluding saddle.*]

For very large values of x, y, the function looks like $x^4 + y^4$, which dominates over the $-36xy$ part. Therefore the contours look like $x^4 + y^4 = C$. This is *almost* a circle, but a bit more squished; for example if we fix $C = 1$, then the circle $x^2 + y^2 = 1$ and the shape $x^4 + y^4 = 1$ have on their locus the points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(1/2^{1/4}, 1/2^{1/4})$ respectively. Since $1/\sqrt{2} < 1/2^{1/4}$, the shape $x^4 + y^4 = 1$ is like a 'stretched circle' instead. [*1 mark for identifying that contours are closed at large distances, and 1 mark for any argument about their 'stretched circle' shape.*]

There is also symmetry in the line $y = x$ since swapping $x \leftrightarrow y$ keeps the function the same. [*1 mark for spotting this.*]

We can now just about construct the plot. The plot is shown in the figure below. [*1 mark for partially correct plot, 1 mark for fully correct plot.*]

To finish, we just need to classify $(\pm 3, \pm 3)$. Considering the information we have above, the only possibility is that $(3, 3)$ is a minimum and $(-3, -3)$ is also a minimum. [*1 mark for identifying $(3, 3)$ minimum, 1 mark for identifying $(3, 3)$ minimum.*]

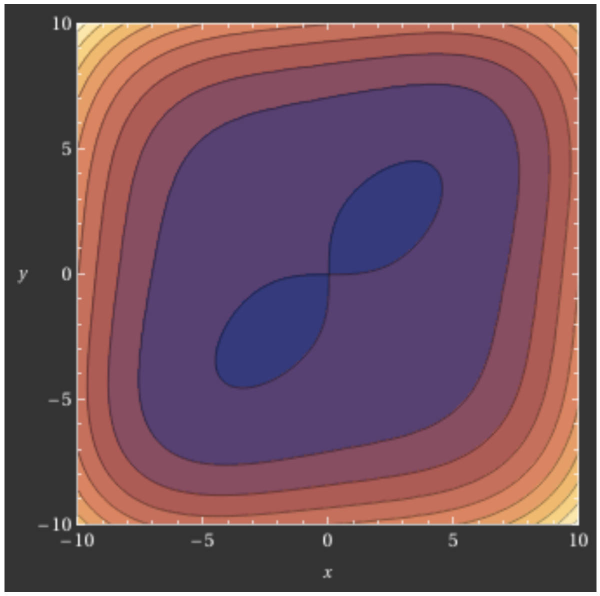


Figure 1: *The contour plot, courtesy of Wolfram Alpha.*

17. (a) Suppose $f(x)$ is a periodic function with period 2π. Write down its Fourier series and give expressions for the coefficients that appear in it. [3]

Solution: The Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

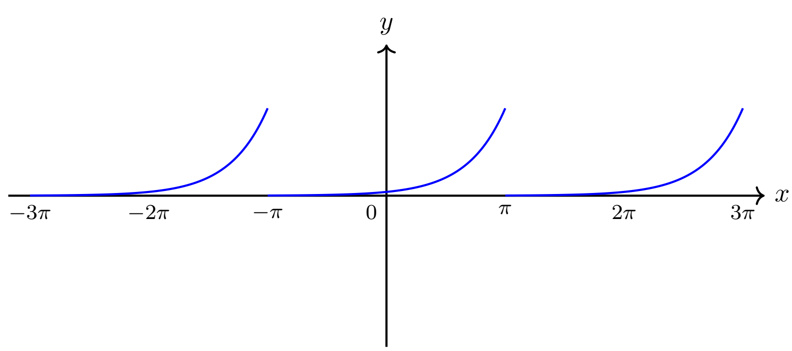
[1 *mark*] The coefficients are given by:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

[*1 mark for some coefficients correctly given, 1 additional mark for all coefficients correctly given.*]

(b) The function $g(x) = e^x$ is defined on the interval $-\pi \le x < \pi$. Sketch the periodic continuation of $g(x)$ with period 2π, between $x = -3\pi$ and $x = 3\pi$. If we were to calculate the Fourier series of this periodic continuation of $g(x)$, what value would it take at the point $x = \pi$? [4]

Solution: The periodic continuation is pictured below.



[*1 mark for correct sketch of e^x. 1 mark for correct periodic extension (repeated three times).*]

At a point of discontinuity, the Fourier series of a function converges to the *average* value across the discontinuity. Thus it would take the value $\frac{1}{2}(e^{\pi} + e^{-\pi})$. [*1 mark for knowing convergence to average. 1 mark for correct value.*]

(c) Consider the function $h(x) = x(\pi - x)$ defined on the interval $0 \leq x < \pi$. By considering the appropriate periodic continuation of $h(x)$ over the real line, show that the half-range sine series for $h(x)$ is: [10]

$$h(x) = \frac{8}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}\sin\left[(2n+1)x\right].$$

Hence demonstrate that: [3]

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

Solution: We extend with an odd extension, so that $h(x) = -h(-x)$. [*1 mark for saying odd extension.*] Then we obtain a sine series. In particular, all the cosine coefficients vanish by symmetry [*1 mark for saying this*] and $a_0 = 0$ by symmetry too [*1 mark for saying this*]. The remaining coefficients are given by:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\sin(nx)\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x(\pi - x)\sin(nx)\,dx \\
&= \frac{2}{\pi}\left(\left[-\frac{x(\pi-x)}{n}\cos(nx)\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}(\pi - 2x)\cos(nx)\,dx\right) \\
&= \frac{2}{n\pi}\left(\left[\frac{(\pi-2x)}{n}\sin(nx)\right]_0^{\pi} + \frac{2}{n}\int_0^{\pi}\sin(nx)\,dx\right) \\
&= \frac{4}{n^2\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi} \\
&= \frac{4}{n^3\pi}(1-(-1)^n).
\end{aligned}$$

[*Up to 4 marks for correct evaluation of coefficients using integration by parts, allocated according to how successful integration is.*] It follows that the even terms vanish, and the odd terms have coefficients $8/n^3\pi$. Thus the Fourier series is:

$$h(x) = \sum_{p=0}^{\infty}\frac{8}{(2p+1)^3\pi}\sin((2p+1)x),$$

as required. [*1 mark for saying odd terms important, and using $n = 2p+1$. 1 mark for saying even terms vanish. 1 mark for convincing derivation of final answer.*] Evaluating this series at the point $x = \pi/2$ [*1 mark*], we have $\sin((2p+1)\pi/2) = (-1)^p$ [*1 mark*]. Thus:

$$\sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)^3} = \frac{\pi}{8}h(\pi/2) = \frac{\pi}{8}\cdot\frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^3}{32},$$

as required. [*1 mark for convincing rearrangement to given answer.*]

18. (a) Suppose X is a discrete random variable that takes the integer values $0, 1, 2, ..., N$. Its normalised probability distribution is denoted by $P(X)$.

Write down expressions for the mean μ, the variance σ^2, and the probability $P(X < Y)$, where Y is a fixed positive integer less than N. [3]

Solution: We have:

$$\mu = \sum_{n=0}^{N} nP(X=n), \qquad \sigma^2 = \sum_{n=0}^{N} n^2 P(X=n) - \mu^2, \qquad P(X<Y) = \sum_{n=0}^{Y-1} P(X=n).$$

[*1 mark for mean, 1 mark for variance (or equivalent correct form), 1 mark for probability.*]

(b) A pond contains K trout and $N-K$ carp. Bruce goes fishing at the pond and in one day catches M fish. Note that Bruce never returns a fish to the pond once it is caught.

Calculate the number of ways M fish (of any species) can be caught from a pond of N fish. Suppose Bruce catches X trout out of his haul of M fish. Show that the number of ways of catching X trout out of the haul of M fish is:

$$\binom{K}{X}\binom{N-K}{M-X}.$$

Assuming that trout and carp are equally likely to be caught, show that the probability that Bruce catches X trout in one day is: [5]

$$P(X) = \binom{K}{X}\binom{N-K}{M-X} \Big/ \binom{N}{M}.$$

Solution: The number of ways of catching M fish from a pond of N fish is:

$$\frac{N(N-1)...(N-M+1)}{M!} = \frac{N!}{M!(N-M)!} = \binom{N}{M},$$

since there are N different ways of catching the first fish, there are $N-1$ different ways of catching the second fish, etc. Since we don't care about the order of the M fish caught, we divide by $M!$. The result is the binomial coefficient. [*1 mark for convincing explanation, and 1 mark for reduction to binomial coefficient.*]

Now if Bruce catches X trout out of his M fish, we require X of the fish to be chosen from the K trout in the pond, and $M-X$ of the fish to be chosen from the remaining $N-K$ carp in the pond. Thus the total number of ways is:

$$\binom{K}{X}\binom{N-K}{M-X}.$$

[*1 mark for convincing explanation (note answer is already given!).*] If any of the fish are equally likely to be caught, then all the ways of catching the fish are equally likely. Hence the probability of any one haul is $1/\binom{N}{M}$ [*1 mark for words to this effect.*], which gives the required probability of getting X trout in the haul:

$$P(X) = \binom{K}{X}\binom{N-K}{M-X} \Big/ \binom{N}{M}.$$

[*1 mark for combining previous answers (again, note answer is already given!).*]

(c) Suppose the pond contains 2 trout and 8 carp, and Bruce catches 2 fish in total. What is the probability that of these two fish (i) none are trout, (ii) one is a trout, and (iii) both are trout? Verify that the three probabilities sum to 1. Consequently, determine the mean and variance of the probability distribution $P(X)$ for this case. (You may leave you answers in reduced fractional form.) [10]

[*Recall that* $\binom{N}{n} = N!/(n!(N-n)!)$.]

Solution: (i) The probability of no trout is:

$$P(0) = \binom{2}{0}\binom{8}{2}\Big/\binom{10}{2} = \frac{(8 \cdot 7)/2}{(9 \cdot 10)/2} = \frac{56}{90} = \frac{28}{45}.$$

[*1 mark for some correct evaluation of binomial coefficients, 1 mark for complete reduction to simplified form.*]

(ii) The probability of one trout is:

$$P(1) = \binom{2}{1}\binom{8}{1}\Big/\binom{10}{2} = \frac{2 \cdot 8}{(9 \cdot 10)/2} = \frac{16}{45}.$$

[*1 mark for some correct evaluation of binomial coefficients, 1 mark for complete reduction to simplified form.*]

(iii) The probability of two trouts is:

$$P(2) = \binom{2}{2}\binom{8}{0}\Big/\binom{10}{2} = \frac{1}{45}.$$

[*1 mark for some correct evaluation of binomial coefficients, 1 mark for complete reduction to simplified form.*]

Observe that:

$$\frac{28}{45} + \frac{16}{45} + \frac{1}{45} = \frac{45}{45} = 1,$$

as expected. [*1 mark for verification.*] The mean is:

$$\mu = \frac{16}{45} + \frac{2}{45} = \frac{18}{45} = \frac{2}{5}.$$

[*1 mark for complete mean calculation.*] Meanwhile, the variance is:

$$\sigma^2 = \frac{16}{45} + \frac{4}{45} - \frac{4}{25} = \frac{20}{45} - \frac{4}{25} = \frac{4}{9} - \frac{4}{25} = \frac{100-36}{225} = \frac{64}{225}.$$

[*1 mark for partial computation of variance, 1 mark for complete variance calculation and simplified.*]

(d) The next day Bruce goes fishing at a large lake that contains only trout and carp but in enormous quantities, i.e. both K and $N-K$ are much larger than M. In this limit $P(X)$ approaches the binomial distribution. Give a qualitative explanation for why this is so. [2]

Solution: A variable X is binomially distributed if it is the number of successful trials in a process where there are (i) only two possible outcomes (in this case, either get a trout or do not get a trout), (ii) the total number of trials is fixed (in this case, Bruce obtains M fish), (iii) the probability of success at each trial is fixed (this is the condition which changes here - since there are so many fish in the pond, the probability of getting a trout is approximately constant, and unchanged by taking a trout out of the pond in the first place). Thus we indeed expect X to be binomially distributed in the large fish limit. [*1 mark for verifying some conditions of binomial, 1 mark for fully convincing explanation.*]

19. (a) A differentiable function $f(x)$ is expanded using the Maclaurin series. Derive an expression which determines the interval for x within which the series is absolutely convergent. [3]

Use your result to determine the interval for x within which the Maclaurin series for $f(x) = \ln(2+x)$ converges absolutely. Determine whether or not the series converges at the end points of the interval. [4]

Solution: The Maclaurin series is given by:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

[*1 mark*] This is absolutely convergent by the ratio test if:

$$1 > \lim_{n\to\infty} \left| \frac{f^{(n+1)}(0) x^{n+1} n!}{f^{(n)}(0) x^n (n+1)!} \right| = |x| \lim_{n\to\infty} \left| \frac{f^{(n+1)}(0)}{f^{(n)}(0)} \frac{1}{n+1} \right|,$$

[*1 mark for applying ratio test.*] which can be rearranged to give:

$$|x| < \lim_{n\to\infty} \left| \frac{f^{(n)}(0)(n+1)}{f^{(n+1)}(0)} \right|.$$

[*1 mark for correct rearrangement to give radius of convergence of series.*] The series diverges if $|x|$ is bigger than the right hand side, hence this is the 'radius of convergence' of the series.

For $f(x) = \ln(2+x)$, to use the above result, we will need the nth derivative. We have $f^{(0)}(0) = \ln(2)$. For $n \geq 1$, we have:

$$f^{(1)}(x) = \frac{1}{2+x}, \qquad f^{(2)}(x) = -\frac{1}{(2+x)^2}, \qquad f^{(3)}(x) = \frac{2}{(2+x)^3}, \qquad \cdots$$

$$f^{(n)}(x) = \frac{(n-1)!(-1)^{n+1}}{(2+x)^n}.$$

Hence, we must consider:

$$\lim_{n\to\infty} \left| \frac{(n+1)\cdot(n-1)!\cdot 2^{n+1}}{2^n \cdot n!} \right| = 2 \lim_{n\to\infty} \left| \frac{n+1}{n} \right| = 2.$$

Thus the radius of convergence is 2, i.e. $|x| < 2$. [*1 mark for correctly differentiating* $\ln(x+2)$ *to give general term. 1 mark for correctly using criterion to deduce radius of convergence* 2.]

We are also asked to determine whether the series converges at the end points of the interval. We have:

$$f(x) = \ln(2) + \sum_{n=1}^{\infty} \frac{(n-1)!(-1)^{n+1}}{n!\cdot 2^n} x^n = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n n} x^n.$$

At $x = 2$, we have:

$$f(2) = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

This is convergent by the alternating series test, since $1/n > 0$, $1/n \to 0$ as $n \to \infty$, and $1/n$ is monotonically decreasing. [*1 mark for saying convergent at* $x = 2$.] On the other hand, when $x = -2$, we have:

$$f(-2) = \ln(2) + \sum_{n=1}^{\infty} \frac{-1}{n},$$

which shows that the Taylor series is proportional to the harmonic series, which diverges. Hence the series is not convergent at this point. [*1 mark for saying divergent at* $x = -2$.]

(b) Find the first three terms in the Maclaurin series for: [3]

$$f(x) = e^{-x}(1+x)^{-1/2}.$$

Solution: We just compute the product of the series for e^{-x} and $(1+x)^{-1/2}$. We have:

$$\begin{aligned} f(x) &= \left(1 - x + \frac{x^2}{2} + \cdots\right)\left(1 - \frac{x}{2} + \frac{3x^2}{8} + \cdots\right) \\ &= 1 - \frac{3x}{2} + \left(\frac{1}{2} + \frac{1}{2} + \frac{3}{8}\right)x^2 + \cdots \\ &= 1 - \frac{3x}{2} + \frac{11x^2}{8} + \cdots. \end{aligned}$$

[*1 mark for correct exponential series, 1 mark for correct binomial series, 1 mark for correct multiplication AND reduction to simplified form.*]

(c) Establish the convergence or divergence of the series $\sum_{n=1}^{\infty} u_n$ whose nth terms, u_n, are:

(i) $2^n/(n\ln(n))$, [2]

(ii) $\sin(n)/n^2$. [2]

Solution: (i) Using the ratio test, we consider:

$$\lim_{n\to\infty} \frac{2^{n+1}n\ln(n)}{2^n(n+1)\ln(n+1)} = 2\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\cdot\left(\frac{\ln(n)}{\ln(n+1)}\right) = 2\lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)}.$$

To take the limit here, we can use L'Hôpital's rule. We have:

$$2\lim_{n\to\infty}\frac{1/n}{1/(n+1)} = 2.$$

This is greater than 1, so the series diverges. [*1 mark for using the ratio test convincingly, 1 mark for conclusion of divergent.*]

(ii) Note that:

$$0 \leq \left|\frac{\sin(n)}{n^2}\right| \leq \frac{1}{n^2},$$

hence the absolute version of the series is strictly less than $1/n^2$. The sum of $1/n^2$ converges, since it is a standard p-series. As a result, the series converges absolutely by the comparison test. Absolute convergence implies convergence, so the original series also converges. [*1 mark for correct use of comparison test - MUST use positive absolute series here. 1 mark for correct conclusion that series converges.*]

(d) Sum the series: [6]

$$S(x) = \frac{x^4}{3(0!} + \frac{x^5}{4(1!)} + \frac{x^6}{5(2!)} + \cdots.$$

Solution: Observe that:

$$\begin{aligned}\frac{d}{dx}\left(\frac{S(x)}{x}\right) &= \frac{d}{dx}\left(\frac{x^3}{3(0!)} + \frac{x^4}{4(1!)} + \frac{x^5}{5(2!)} + \cdots\right)\\ &= \frac{x^2}{0!} + \frac{x^3}{1!} + \frac{x^4}{2!} + \cdots\\ &= x^2\left(\frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)\\ &= x^2e^x.\end{aligned}$$

[*1 mark for dividing by x. 1 mark for taking the derivative. 1 mark for spotting exponential series.*] Integrating both sides, we have:

$$\frac{S(x)}{x} = \int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx = x^2e^x - 2xe^x + 2\int e^x\,dx = (x^2-2x+2)e^x + c.$$

[*1 mark for correct integration by parts.*] Hence:

$$S(x) = (x^2-2x+2)xe^x + cx.$$

Notice that:

$$\lim_{x\to 0}\frac{S(x)}{x} = 0.$$

[*1 mark for noticing this boundary condition.*] Hence, we require:

$$2 + c = 0,$$

and thus:

$$S(x) = (x^2-2x+2)xe^x - 2x.$$

[*1 mark for fully correct answer.*]

20. The point (a, b) is a stationary point of the function $f(x, y)$ subject to the constraint $g(x, y) = 0$. Using the method of Lagrange multipliers, show that: [4]

$$\begin{vmatrix} \dfrac{\partial f}{\partial x}(a,b) & \dfrac{\partial g}{\partial x}(a,b) \\ \dfrac{\partial f}{\partial y}(a,b) & \dfrac{\partial g}{\partial y}(a,b) \end{vmatrix} = 0.$$

Solution: Consider the Lagrangian $L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$. Taking the gradient of the Lagrangian, we have:

$$\nabla L = (\nabla f + \lambda \nabla g, g),$$

hence stationary points occur when $g = 0$ and when:

$$\nabla f = -\lambda \nabla g.$$

This implies that at the point (a, b), we have that the vectors ∇f and ∇g are parallel. Thus the area of the parallelogram formed by them is zero, so the determinant of the matrix whose columns are ∇f and ∇g must be zero. The result in the question follows. [*1 mark for Lagrangian correctly written down. 1 mark for correctly deducing conditions for stationary points. 1 mark for noticing ∇f, ∇g parallel. 1 mark for determinant argument.*]

(a) By considering the function $f(x, y) = x^2 + y^2$, use the method of Lagrange multipliers to find the maximum distance from the origin to the curve $x^2 + y^2 + xy - 4 = 0$. [4]

Solution: The Lagrangian is:

$$L(x, y, \lambda) = x^2 + y^2 + \lambda(x^2 + y^2 + xy - 4).$$

We have just seen that stationary points of this function satisfy:

$$\begin{vmatrix} 2x & 2x + y \\ 2y & 2y + x \end{vmatrix} = 0, \qquad x^2 + y^2 + xy = 4.$$

Multiplying out the determinant, we have:

$$0 = 2x(2y + x) - 2y(2x + y) = 4xy + 2x^2 - 4xy - 2y^2 = 2x^2 - 2y^2,$$

so that either $x = y$ or $x = -y$. In the first case, the constraint implies:

$$3y^2 = 4 \qquad \Rightarrow \qquad y = \pm 4/3.$$

In the second case, the constraint implies:

$$y^2 = 4 \qquad \Rightarrow \qquad y = \pm 2.$$

Thus the stationary points are $\pm(4/3, 4/3)$ and $\pm(2, -2)$. The point at maximum distance from the origin to the curve is therefore one of the following, which we can find by substitution:

- $\pm(4/3, 4/3)$ gives squared distance $16/9 + 16/9 = 32/9$.
- $\pm(2, -2)$ gives squared distance $4 + 4 = 8$, which is strictly greater than $32/9$.

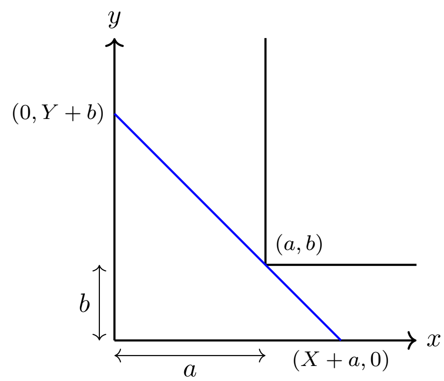
Thus the furthest distance is $\sqrt{8} = 2\sqrt{2}$. [*1 mark for correct simultaneous equations from first part. 1 mark for obtaining one set of solutions. 1 mark for obtaining other set of solutions. 1 mark for substituting into distances and deciding that $2\sqrt{2}$ is the greatest distance.*]

(b) In a school, two horizontal corridors, $0 \leq x \leq a, y \geq 0$ and $x \geq 0, 0 \leq y \leq b$ meet at right angles. The caretaker wishes to know the maximum possible length, L, of a ladder that may be carried horizontally around the corner. Regarding the ladder as a stick, use the method of Lagrange multipliers to calculate L by first placing the ends of the ladder at the points $(a+X, 0)$ and $(0, b+Y)$ and imposing the condition that the corner (a, b) be on the ladder. Then show that at the constrained stationary point, the value of X satisfies the equation:

$$(X^3 - ab^2)(X + a) = 0,$$

and hence show that $L = (a^{2/3} + b^{2/3})^{3/2}$. [12]

Solution: In the ladder's critical position, it is just about to turn around the corner:



Therefore, if we place the end points of the ladder at $(a + X, 0)$ and $(0, b + Y)$, we must now impose the condition that the corner (a, b) is on the ladder. [*1 mark for a good diagram, 1 mark for a convincing explanation of why we need the ladder to be on the corner at the critical point of rotation.*] Hence, we want to maximise the length $(a + X)^2 + (b + Y)^2$ subject to the condition that the ladder passes through the corner (a, b) (any smaller distance will be able to rotate round after that).

The condition that the line through $(a+X, 0)$ and $(0, b+Y)$ passes through the point (a, b) is equivalent to:

$$b = \frac{b+Y}{a+X}(a + X - a) \qquad \Leftrightarrow \qquad b(a+X) = (b+Y)X \qquad \Leftrightarrow \qquad ab = XY.$$

[*1 mark for correctly using line through* $(a + X, 0)$, $(0, b + Y)$, *and 1 mark for imposing that* (a, b) *is on the line. 1 mark for simplifying to form* $ab = XY$.] The Lagrangian is therefore:

$$\mathcal{L} = (X + a)^2 + (Y + b)^2 + \lambda(XY - ab).$$

[*1 mark for correct Lagrangian.*] Taking the gradient, we have:

$$\nabla\mathcal{L} = (2(X + a) + \lambda Y, 2(Y + b) + \lambda X, XY - ab).$$

[*1 mark for correctly taking gradient.*] At stationary points, we have:

$$2(X + a) = -\lambda Y, \qquad 2(Y + b) = -\lambda X, \qquad XY = ab.$$

The third equation implies that $Y = ab/X$. Dividing the first equation by the second, we have:

$$\frac{X+a}{Y+b} = \frac{Y}{X} \qquad \Leftrightarrow \qquad \frac{X+a}{ab/X + b} = \frac{ab}{X^2}.$$

[*Up to 2 marks for eliminating Y, λ to get an equation for X.*] Multiplying up, we have:

$$(X+a)X^2 = ab(ab/X+b) = \frac{ab^2}{X}(X+a) \qquad \Leftrightarrow \qquad (X+a)(X^3-ab^2)=0,$$

as required. [*1 mark for simplifying to required equation.*] We require that $X > 0$, hence $X \neq -a$. Thus $X = (ab^2)^{1/3}$, and it follows that $Y = ab/X = (a^3b^3)^{1/3}/(ab^2)^{1/3} = (a^2b)^{1/3}$. Thus the maximum length of the ladder is:

$$\begin{aligned}
L &= \left(\left((ab^2)^{1/3}+a\right)^2 + \left((a^2b)^{1/3}+b\right)^2\right)^{1/2} \\
&= \left(a^{2/3}(b^{2/3}+a^{2/3})^2 + b^{2/3}(a^{2/3}+b^{2/3})^2\right)^{1/2} \\
&= (a^{2/3}+b^{2/3}) \cdot (a^{2/3}+b^{2/3})^{1/2} \\
&= (a^{2/3}+b^{2/3})^{3/2}.
\end{aligned}$$

[*1 mark for obtaining correct X, 1 mark for simplifying to final form (note the answer is already given!).*]

Part IA: Mathematics for Natural Sciences
2014 Paper 2 (Unofficial) Mark Scheme

Section A

1. Two vectors are given by $\mathbf{u} = (a, 1, -3)$ and $\mathbf{v} = (2, 2, b)$, where a and b are constants. If $\mathbf{u} \times \mathbf{v}$ lies in the x-y plane and $\mathbf{u} \cdot \mathbf{v} = 0$, what values must a and b take? [2]

[*Notation: $\mathbf{u} \times \mathbf{v}$ is equivalent to $\mathbf{u} \wedge \mathbf{v}$.*]

Solution: We have:

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} a \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ b \end{pmatrix} = \begin{pmatrix} b+6 \\ -ab-6 \\ 2a-2 \end{pmatrix}.$$

Hence we have $a = 1$. On the other hand, we have:

$$\mathbf{u} \cdot \mathbf{v} = 2a + 2 - 3b = 0,$$

hence $4 - 3b = 0$, and thus $b = 4/3$. [*1 mark for correct a, 1 mark for correct b.*]

2. Give the real and imaginary parts of $\cosh(\alpha + i\beta)$, where α and β are real numbers. [2]

Solution: We expand $\cosh(\alpha + i\beta)$ using the standard hyperbolic compound angle formula. We have:

$$\cosh(\alpha + i\beta) = \cosh(\alpha)\cosh(i\beta) + \sinh(\alpha)\sinh(i\beta) = \cosh(\alpha)\cos(\beta) + i\sinh(\alpha)\sin(\beta),$$

using the fact that $\cosh(ix) = \cos(x)$ and $\sinh(ix) = i\sin(x)$. Thus the real part is $\cosh(\alpha)\cos(\beta)$ and the imaginary part is $\sinh(\alpha)\sin(\beta)$. [*1 mark for correct identity, 1 mark for correct statement of real and imaginary parts using the $\cosh(ix) = \cos(x)$ and $\sinh(ix) = i\sin(x)$ formulae.*]

3. Find the first two non-zero terms in the Taylor expansion of:

$$f(x) = e^x \ln(1+x)$$

around $x = 0$. [2]

Solution: Expanding both e^x and $\ln(1+x)$ simultaneously, we have:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots, \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots,$$

so that:

$$f(x) = x + x^2 - \frac{x^2}{2} + \cdots = x + \frac{1}{2}x^2 + \cdots.$$

[*1 mark for correct first term, 1 mark for correct second term.*]

4. Write down the complementary function and a particular integral of the ordinary differential equation: [2]

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 3x.$$

Solution: The auxiliary equation is $0 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$, and hence the complementary function is $y(x) = Ae^{3x} + Be^{x}$. [*1 mark*.]

A particular integral can be found by guessing $y = \alpha x + \beta$. Differentiating, we have $y' = \alpha$ and $y'' = 0$, hence:

$$-4\alpha + 3\alpha x + 3\beta = 3x.$$

So we see $\alpha = 1, -4 + 3\beta = 0$, which implies $\beta = 4/3$. Thus $y = x + 4/3$ is a particular integral. [*1 mark*.]

5. Consider the vector field:

$$\mathbf{F} = (e^{-x}\cos(z))\mathbf{i} + (e^{-y}\sin(z))\mathbf{j} + \mathbf{k}.$$

Calculate both the divergence and the curl of $\mathbf{F}$. [2]

Solution: The divergence is:

$$\nabla \cdot \mathbf{F} = -e^{-x}\cos(z) - e^{-y}\sin(z).$$

[*1 mark*.] The curl is:

$$\nabla \times \mathbf{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} e^{-x}\cos(z) \\ e^{-y}\sin(z) \\ 1 \end{pmatrix} = \begin{pmatrix} -e^{-y}\cos(z) \\ -e^{-x}\sin(z) \\ 0 \end{pmatrix}.$$

[*1 mark*.]

6. Show that $u = f(x - ct)$ is a solution to the partial differential equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

where c is a constant and f is an unspecified differentiable function of one variable. [1]

If $u = \cos(x)$ at $t = 0$, what is u at time $t > 0$? [1]

Solution: We have, by the chain rule,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(x - ct) \cdot f'(x - ct) = -cf'(x - ct), \qquad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x - ct) \cdot f'(x - ct) = f'(x - ct).$$

Hence we have:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -cf'(x - ct) + cf'(x - ct) = 0,$$

as required. [*1 mark*.] If $u = \cos(x)$ at $t = 0$, we have $f(x) = \cos(x)$. Thus $u = f(x - ct) = \cos(x - ct)$ for $t > 0$. [*1 mark*.]

7. What are the two eigenvalues of:

$$\begin{pmatrix} \cosh(\phi) & \sinh(\phi) \\ \sinh(\phi) & \cosh(\phi) \end{pmatrix},$$

where ϕ is a real constant? [2]

Solution: The determinant is $\cosh^2(\phi) - \sinh^2(\phi) = 1$, and the trace is $2\cosh(\phi)$. Hence the eigenvalues satisfy $\lambda_1 + \lambda_2 = 2\cosh(\phi) = e^{\phi} + e^{-\phi}$ and $\lambda_1 \lambda_2 = 1$. Evidently the eigenvalues are $\lambda_1 = e^{\phi}, \lambda_2 = e^{-\phi}$. [2 *marks for correct eigenvalues, obtained via any method.*]

8. A surface is described by the equation $z = \cos(x)\sin(y)$, where $0 < x < 2\pi$ and $0 < y < 2\pi$. Find the coordinates (x, y) of any two of its stationary points. [2]

Solution: We have:

$$\nabla z = (-\sin(x)\sin(y), \cos(x)\cos(y)) .$$

Thus we see that $(\pi, \pi/2)$ and $(\pi, 3\pi/2)$ are stationary points of the function. [2 *marks for any correct stationary points (there are several).*]

9. Consider the two-dimensional line integral:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F} = (x^2 + y^2)\mathbf{i}$ and $\mathcal{C}$ is a path in the x-y plane. Calculate the line integral when:

(a) $\mathcal{C}$ is the straight line going from $(0, -1)$ to $(0, 1)$, [1]

(b) $\mathcal{C}$ is the straight line going from $(1, 0)$ to $(-1, 0)$. [1]

Solution: (a) In the first case, notice that the path is in the y-direction, and hence $d\mathbf{r}$ is proportional to $\mathbf{j}$. Thus $\mathbf{F} \cdot d\mathbf{r} = 0$, and the line integral is zero. [1 *mark.*]

(b) In the second case, use the parametrisation:

$$\mathbf{r}(t) = (t - 1, 0),$$

where $0 \leq t \leq 2$. Then the line integral becomes:

$$\int_0^2 ((t-1)^2 + 0^2, 0) \cdot (1, 0)\, dt = \int_0^2 (t-1)^2\, dt = \left[\frac{(t-1)^3}{3}\right]_0^2 = \frac{1}{3} - -\frac{1}{3} = \frac{2}{3}.$$

[1 *mark.*]

10. A continuous random variable X takes values 1 and greater. Its normalised probability distribution is $f(x) = \alpha x^{-\alpha-1}$, where $\alpha > 1$. Evaluate the following probabilities:

(a) $P(X \geq 1)$. [1]

(b) $P(2 \leq X \leq 3)$. [1]

Solution: For both cases, we need to normalise the distribution first. Note that:

$$\int_1^\infty \alpha x^{-\alpha-1}\,dx = \left[\frac{\alpha x^{-\alpha}}{-\alpha}\right]_1^\infty = 1,$$

so the distribution is already normalised. Now, in the first case, note that $P(X \geq 1) = 1$ by the range of the distribution. [*1 mark.*]

For the second case, note that:

$$\int_2^3 \alpha x^{-\alpha-1}\,dx = \left[-x^{-\alpha}\right]_2^3 = -3^{-\alpha} + 2^{-\alpha} = \frac{1}{2^\alpha} - \frac{1}{3^\alpha}.$$

[*1 mark.*]

Section B

11. (a) Addenbrooke's Hospital is conducting a clinical drug trial. The study randomly draws patients from a large population and randomly places them in either a 'control group' or a 'treatment group', each of the same size. The probability that an adverse event occurs to someone in the control group is p_0, and to someone in the treatment group is p_1.

Bruce is taking part in the clinical trial and suffers an adverse event. What is the probability that he is in the control group? [6]

Solution: This is just the *medical testing paradox* which we saw on the examples sheet as an application of Bayes' theorem. Recall Bayes' theorem allows us to write:

$$\begin{aligned}
P(\text{control}|\text{adverse}) &= \frac{P(\text{adverse}|\text{control})P(\text{control})}{P(\text{adverse})} \\
&= \frac{P(\text{adverse}|\text{control})P(\text{control})}{P(\text{adverse}|\text{control})P(\text{control}) + P(\text{adverse}|\text{treatment})P(\text{treatment})} \\
&= \frac{\frac{1}{2}p_0}{\frac{1}{2}p_0 + \frac{1}{2}p_1} \\
&= \frac{p_0}{p_0 + p_1}.
\end{aligned}$$

[*1 mark for any attempt to use Bayes' theorem. 1 mark for correct use of Bayes' theorem. 1 mark for correctly rewriting denominator. 1 mark for noticing probability $1/2$ of control and probability $1/2$ of treatment. 1 mark for correctly substituting. 1 mark for simplifying.*]

(b) The Rayleigh distribution of a positive continuous random variable X is defined by:

$$f(x) = Ax \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

where σ is a positive parameter and A is a normalisation constant.

Find the value of A in terms of σ. [2]

Sketch the distribution $f(x)$. [2]

What is the mean of the distribution? [2]

Solution: To find the value of A, we just need to normalise the distribution. We have:

$$1 = \int_0^\infty Ax \exp\left(-\frac{x^2}{2\sigma^2}\right)\, dx = \left[-A\sigma^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\right]_0^\infty = A\sigma^2.$$

Hence $A = 1/\sigma^2$. [*1 mark for correct setup of normalisation, 1 mark for correct condition.*]

The distribution starts at $x = 0$ and tends to 0 as $x \to \infty$. The distribution does not exist for $x < 0$. The distribution has a turning point, which is necessarily a maximum, when:

$$\frac{d}{dx}\left(Ax \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) = \left(A - \frac{A}{\sigma^2}x^2\right)\exp\left(-\frac{x^2}{2\sigma^2}\right) = 0,$$

which requires $x = \pm\sigma$. Thus there is a maximum when $x = \sigma$. [*1 mark for mostly correct graph, 1 mark for fully correct graph with stationary point marked.*]

The mean is given by:

$$\int_0^\infty Ax^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\, dx = \left[Ax \cdot \left(-\sigma^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\right]_0^\infty + A\sigma^2 \int_0^\infty \exp\left(-\frac{x^2}{2\sigma^2}\right)\, dx$$

$$= A\sigma^2 \cdot \frac{\sqrt{2\pi}\sigma}{2} = \sigma\sqrt{\frac{\pi}{2}},$$

using the Gaussian integral in the final step. [*1 mark for using integration by parts AND Gaussian integral, 1 mark for correct evaluation overall and fully correct answer.*]

(c) Sheila and Bruce are playing a match of table tennis. The first player to win 11 games wins the match. The probability that Sheila wins a game is θ.

What is the probability that Sheila wins m games and loses n games, where both m and n are integers less than 11? [4]

Hence show that the probability that Sheila wins the match is: [4]

$$\sum_{n=0}^{10} \binom{10+n}{10} \theta^{11}(1-\theta)^n.$$

[*Recall the definition* $\binom{N}{n} = N!/(n!(N-n)!)$.]

Solution: We must assume in this question that Sheila either wins or loses a match (i.e. there are no draws, which is actually inconsistent with the rules of real-world table tennis, but never mind). If Sheila wins m games and loses n games, a total of $m+n$ games are played. Thus the outcomes follow a binomial distribution $\mathrm{Bin}(m+n, \theta)$, since $m+n$ games are played, each with a constant probability of Sheila winning, and each with only two outcomes for Sheila. Hence the probability of winning m games and losing n games, given that $m+n$ games in total are played, is:

$$P(\text{win } m, \text{ lose } n | m+n \text{ are played}) = \binom{m+n}{m} \theta^m (1-\theta)^n.$$

[*1 mark for stating we can model with a binomial distribution; 1 mark for correctly explaining why we can model as a binomial distribution (i.e. stating conditions); 1 mark for partially correct formula; 1 mark for fully correct formula.*]

For Sheila to win the game, if a total of N games are played, she must win 10 games of all of the first $N-1$ in any order, *AND* she must win the final Nth game. Thus the probability is:

$$\begin{aligned} P(\text{win final game}) \cdot \left(\sum_{n=0}^{10} P(\text{win } 10, \text{ lose } n | 10+n \text{ are played}) \right) &= \theta \sum_{n=0}^{10} \binom{10+n}{10} \theta^{10}(1-\theta)^n \\ &= \sum_{n=0}^{10} \binom{10+n}{10} \theta^{11}(1-\theta)^n, \end{aligned}$$

as required. [*1 mark for correct argument about winning final game, and other 10 games won in any order. 1 mark for correct unsimplified expression for probability. 1 mark for using previous parts somehow. 1 mark for correct final answer.*]

12. (a) (i) What is an orthogonal matrix? [2]

(ii) Calculate $\mathbf{M}^{-1}$ where:

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -3 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Is $\mathbf{M}$ orthogonal? [6]

(iii) Given that $\mathbf{A}$ and $\mathbf{B}$ are orthogonal matrices, show that $\mathbf{AB}$ is also orthogonal. [2]

Solution: (i) An orthogonal matrix is a matrix satisfying $\mathbf{M}^T\mathbf{M} = I$, where $\mathbf{M}^T$ is the transpose of the matrix (obtained by swapping rows and columns). [*1 mark for saying anything about transpose. 1 mark for correctly stating condition - could also be stated as saying transpose is equal to inverse. There are many other conditions for orthogonality too, but this is probably what is expected.*]

(ii) To calculate $\mathbf{M}^{-1}$, we begin by computing its determinant. We have:

$$\mathrm{Det}(\mathbf{M}) = \mathrm{Det}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -3 & 0 \\ 1 & 1 & 1 \end{pmatrix} = -\mathrm{Det}\begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix} + \mathrm{Det}\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} = -(1+3)+(-3-1) = -8,$$

expanding on the last column. [2 *marks for correctly computing determinant (1 mark for partial progress).*] The inverse is then given by multiplying the determinant by the transpose of the cofactor matrix:

$$\mathbf{M}^{-1} = -\frac{1}{8}\begin{pmatrix} -3 & -1 & 4 \\ -2 & 2 & 0 \\ -3 & -1 & -4 \end{pmatrix}^T = \frac{1}{8}\begin{pmatrix} 3 & 2 & 3 \\ 1 & -2 & 1 \\ -4 & 0 & 4 \end{pmatrix}.$$

[*1 mark for remembering how to compute cofactor matrix. 1 mark for correct calculation of cofactor matrix. 1 mark for multiplication by determinant and for taking transpose.*]

Since $\mathbf{M}^{-1} \neq \mathbf{M}^T$, it follows that $\mathbf{M}$ is not an orthogonal matrix. [*1 mark.*]

(iii) Observe that:

$$(\mathbf{AB})^T\mathbf{AB} = \mathbf{B}^T\mathbf{A}^T\mathbf{AB} = \mathbf{B}^T I\mathbf{B} = \mathbf{B}^T\mathbf{B} = I,$$

using the fact that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$. [*1 mark for correctly stating/using this identity. 1 mark for correct overall proof.*]

(b) (i) Express the following set of simultaneous equations, in which a is a real constant, in matrix form $\mathbf{Ax} = \mathbf{y}$: [2]

$$x + y + z = 3$$

$$2x + 2y + (a-1)z = 4$$

$$ax + y + 3z = 5.$$

(ii) Use matrix methods to solve for the special case $a = 2$. [3]

(iii) Use matrix methods to find values of a for which the equations have no solutions, or multiple solutions. [3]

(iv) Solve for the case which gives rise to multiple solutions. [2]

Solution: (i) As a matrix equation, this can be expressed as:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & a-1 \\ a & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}.$$

[*1 mark for correct* ***A****, 1 mark for correct overall equation.*]

(ii) When $a = 2$, we now wish to solve the system. We shall use Gaussian elimination, and work in the case where a is generic (because the final two parts also require a more general than 2!). Subtracting twice the first equation from the second, and a times the first equation from the final equation, we have:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & a-3 \\ 0 & 1-a & 3-a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5-3a \end{pmatrix}.$$

Swapping the second and third equations, they are now in upper triangular form:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1-a & 3-a \\ 0 & 0 & a-3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5-3a \\ -2 \end{pmatrix}.$$

The final equation implies that $(a-3)z = -2$, which yields:

$$z = -\frac{2}{a-3}, \qquad \text{if } a \neq 3.$$

Substituting into the second equation, $(1-a)y + (3-a)z = 5 - 3a$, we have:

$$y = \frac{5 - 3a - (3-a)z}{1-a} = \frac{3-3a}{1-a} = 3, \qquad \text{if } a \neq 1.$$

Substituting into the first equation, $x + y + z = 3$, we have:

$$x = 3 - y - z = 3 - 3 + \frac{2}{a-3} = \frac{2}{a-3}.$$

Hence, in the case where $a = 2$, we have $(x, y, z) = (-2, 3, 2)$. [*1 mark for any attempt to use Gaussian elimination when $a = 2$, 1 mark for partial progress towards solution, 1 mark for correct solution.*]

(iii) and (iv). From the argument above, we saw that special attention must be paid to $a = 3$ and $a = 1$. [*1 mark for identifying these values via Gaussian elimination.*] If $a = 3$, then the final equation becomes $0 = -2$, which is inconsistent. Hence if $a = 3$ there are no solutions. [*1 mark for identifying that this is the case where there are no solutions.*] If $a = 1$, then we have $z = 1$ and then the second equation gives $2z = 2$, which is automatically satisfied for all values of y. [*1 mark for correctly identifying that this is the case that gives multiple solutions.*] The third equation gives $x = -1$. Thus there are multiple solutions in this case, with solutions $(-1, y, 1)$ for all values of y. [*1 mark for correct x, z values and 1 mark for correct variable y value.*]

13. (a) Evaluate the following indefinite integrals:

(i) $\displaystyle\int \left(\cosh^2(x) + \cosh(x) - \sinh^2(x) + \sinh(x)\right)\, dx,$ [2]

(ii) $\displaystyle\int \frac{x-1}{3x^2+2x+2}\, dx.$ [8]

Solution: (i) Observe that $\cosh^2(x) - \sinh^2(x) = 1$ [*1 mark*] and $\cosh(x) + \sinh(x) = e^x$ [*1 mark*]. Hence the integral is:

$$\int (1 + e^x)\, dx = x + e^x + c.$$

[*1 mark for correct integral. Might also be expressed as* $x + \cosh(x) + \sinh(x) + c$, *if unsimplified, but that's fine.*]

(ii) This integral is significantly more involved. Observe that the denominator has discriminant $2^2 - 4(3)(2) = 4 - 24 = -20 < 0$, so cannot be factored in real terms, thus partial fractions are unlikely to be helpful here. Instead, we do the standard trick from the examples sheet: work out the derivative of the denominator, and express the fraction as a sum of a logarithmic derivative term and a constant divided by a quadratic. Observe that $\frac{d}{dx}(3x^2 + 2x + 2) = 6x + 2$, hence we should write:

$$\frac{x-1}{3x^2+2x+2} = \frac{6x+2}{6(3x^2+2x+2)} - \frac{8}{6(3x^2+2x+2)}.$$

This implies that:

$$\begin{aligned}\int \frac{x-1}{3x^2+2x+2}\, dx &= \frac{1}{6}\int \frac{6x+2}{3x^2+2x+2}\, dx - \frac{4}{3}\int \frac{1}{3x^2+2x+2}\, dx \\ &= \frac{1}{6}\ln(3x^2+2x+2) - \frac{4}{3}\int \frac{1}{3x^2+2x+2}\, dx.\end{aligned}$$

[*1 mark for correct evaluation of derivative of denominator. 1 mark for attempt to decompose quotient into a logarithmic term and a constant divided by a quadratic. 1 mark for completely correct decomposition. 1 mark for logarithmic integral.*]

It remains to evaluate the integral of a constant divided by a quadratic. We know that the result will be trigonometric or hyperbolic. We complete the square in the denominator, giving:

$$3x^2 + 2x + 2 = 3(x^2 + 2x/3 + 2/3) = 3((x + 1/3)^2 + 5/9) = 3(x + 1/3)^2 + 5/3.$$

[*1 mark for correctly completing the square.*] Hence the remaining integral can be re-expressed as:

$$\begin{aligned}\int \frac{dx}{3x^2+2x+2} &= \int \frac{dx}{3(x+1/3)^2 + 5/3} = \frac{3}{5}\int \frac{dx}{9(x+1/3)^2/5 + 1} \\ &= \frac{3}{5}\sqrt{\frac{5}{9}}\arctan\left(\sqrt{\frac{9}{5}}\left(x + \frac{1}{3}\right)\right) + c = \frac{1}{\sqrt{5}}\arctan\left(\frac{3x+1}{\sqrt{5}}\right) + c.\end{aligned}$$

[*2 marks for fiddling around with the arctangent integral, does not need to be simplified; 1 mark for partial progress here - make sure you know your arctangent integrals!*] Thus the complete integral is:

$$\frac{1}{6}\ln(3x^2+2x+2) - \frac{4}{3\sqrt{5}}\arctan\left(\frac{3x+1}{\sqrt{5}}\right) + c.$$

[*1 mark for correct answer, fully simplified.*]

(b) Evaluate the definite integral: [2]

$$\int_{-1/\pi}^{1/\pi} \sin^2(3x^3+2x)\ln\left[\frac{1-x^5}{1+x^5}\right]\,dx.$$

Solution: Observe that this question is not worth very many marks, so it can't be a hard integral! Observe that the range is symmetric about 0, and the integrand is an *odd* function, since:

$$\begin{aligned}\sin^2(3(-x)^3+2(-x))\ln\left[\frac{1-(-x)^5}{1+(-x)^5}\right] &= \sin^2(3x^3+2x)\ln\left[\frac{1+x^5}{1-x^5}\right]\\ &= -\sin^2(3x^2+2x)\ln\left[\frac{1-x^5}{1+x^5}\right].\end{aligned}$$

[*1 mark for convincing demonstration that integrand is odd*.] Hence, the value of the integral is zero. [*1 mark*.]

(c) State the fundamental theorem of calculus and thus find:

$$\frac{d}{dx}\left[\int_a^x f(y)\,dy\right],$$

where a is a real constant. [2]

Solution: The fundamental theorem of calculus states that if $f(x)$ has an antiderivative $F(x)$, then:

$$\int_a^b f(x)\,dx = F(b)-F(a).$$

[*1 mark for statement of fundamental theorem of calculus.*] Hence assuming that $F(x)$ is an antiderivative of $f(x)$ in this problem, we have:

$$\frac{d}{dx}\int_a^x f(y)\,dy = \frac{d}{dx}\left(F(x)-F(a)\right) = F'(x) = f(x).$$

[*1 mark for correct evaluation of integral*.]

(d) Find: [5]

$$\frac{d}{dx}\left[\sum_{n=0}^{N}\binom{N}{n}\int_{n}^{x}\sin(y^2+y^6)\,dy\right].$$

[*Recall the definition* $\binom{N}{n} = N!/(n!(N-n)!)$.]

Solution: Differentiation is linear, [1 *mark for using this fact.*] so we can exchange the order of differentiation and summation. We have:

$$\sum_{n=0}^{N}\binom{N}{n}\frac{d}{dx}\int_{n}^{x}\sin(y^2+y^6)\,dy = \sum_{n=0}^{N}\binom{N}{n}\sin(x^2+x^6),$$

using the previous part of the question. [1 *mark for correctly using previous part of question.*] To finish, observe that $\sin(x^2+x^6)$ is constant so can be removed from the sum. [1 *mark.*] Finally, we have:

$$\sin(x^2+x^6)\sum_{n=0}^{N}\binom{N}{n} = 2^N\sin(x^2+x^6),$$

since the sum of the binomial coefficients is 2^N. [2 *marks for knowing this fact.*] To prove this, simply observe that:

$$2^N = (1+1)^N = \sum_{n=0}^{N}\binom{N}{n}1^n 1^{N-n} = \sum_{n=0}^{N}\binom{N}{n},$$

using the binomial expansion.

14. (a) Show that the differential $(4x + xy^2)dx + (y + x^2 y)dy$ is exact and find a function $f(x, y)$ such that:

$$df = (4x + xy^2)dx + (y + x^2 y)dy.$$

Use your result to solve $df = 0$ for $y(x)$. [4]

Find the particular solution for $y(x)$ given that $y(1) = 2$. [2]

Solution: The condition for the differential $Pdx + Qdy$ being exact is $\partial P/\partial y = \partial Q/\partial x$. Here, we have:

$$\frac{\partial}{\partial y}(4x + xy^2) = 2xy, \qquad \frac{\partial}{\partial x}(y + x^2 y) = 2xy,$$

so indeed the differential is exact. [*1 mark*] We can now find a function f such that df is equal to the differential by setting:

$$\frac{\partial f}{\partial x} = 4x + xy^2, \qquad \frac{\partial f}{\partial y} = y + x^2 y.$$

Integrating the first equation, we have:

$$f(x, y) = 2x^2 + \frac{1}{2}x^2 y^2 + c(y).$$

Differentiating this with respect to y, we have:

$$\frac{\partial f}{\partial y} = x^2 y + c'(y),$$

which reveals that $c'(y) = y$, and hence $c(y) = \frac{1}{2}y^2 + k$. Thus:

$$f(x, y) = 2x^2 + \frac{1}{2}x^2 y^2 + \frac{1}{2}y^2 + k$$

works for any k. [*1 mark for correct setup, 1 mark for correct f (does not need arbitrary constant k at the end).*] Therefore, the solution to $df = 0$ is given by:

$$2x^2 + \frac{1}{2}x^2 y^2 + \frac{1}{2}y^2 = C \qquad \Rightarrow \qquad y = \pm\sqrt{\frac{2(C - 2x^2)}{x^2 + 1}},$$

where C is arbitrary. [*1 mark for correct answer; MUST be expressed with y as the subject, and must have $\pm$ signs.*]

The particular solution with $y(1) = 2$ obeys:

$$2 = \pm\sqrt{\frac{2(C - 2)}{2}} = \pm\sqrt{C - 2}.$$

Hence $C = 6$, and we should choose the plus sign. The particular solution is therefore:

$$y(x) = 2\sqrt{\frac{3 - x^2}{1 + x^2}}.$$

[*1 mark for correct derivation of C. 1 mark for correct final answer, fully simplified.*]

(b) It is given that:

$$M(x,y)dx + N(x,y)dy = 0$$

has an integrating factor μ which depends only on x. By writing μ in the form:

$$\mu = \exp\left(\int f(x)\,dx\right),$$

find an expression for $f(x)$. [6]

If instead the integrating factor is $\psi(y)$, a function of y only, what is the expression for ψ? [2]

Solution: Since μ is an integrating factor, we have:

$$\mu M dx + \mu N dy = 0$$

is exact. [*1 mark for stating this.*] Hence:

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \qquad \Rightarrow \qquad \mu\frac{\partial M}{\partial y} = \mu' N + \mu\frac{\partial N}{\partial x}.$$

[*1 mark for correctly this equation.*] Rearranging this equation, we have:

$$\mu' N = \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right),$$

which is a separable ordinary differential equation for μ. [*1 mark for noticing separable.*] Separating variables, we have:

$$\int \frac{d\mu}{\mu} = \int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\,dx.$$

[*1 mark for correct solution in terms of integrals.*] Hence:

$$\ln(\mu) = \int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\,dx \qquad \Rightarrow \qquad \mu(x) = \exp\left(\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\,dx\right).$$

[*1 mark for correctly evaluating integral, 1 mark for exponentiating to obtain final formula.*]

We simply exchange the roles of x, y and M, N for the function $\psi(y)$. We have:

$$\psi(y) = \exp\left(\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dy\right).$$

[*1 mark for spotting exchange of roles of either x, y or M, N; 1 mark for correct final formula.*]

(c) The equation:

$$(3xy^2 + 2y)dx + (2x^2y + x)dy = 0$$

has an integrating factor $\mu(x)$ which depends only on x. Find $\mu(x)$ and hence solve the equation for $y(x)$ explicitly. [6]

Solution: Using the result from part (b), we have:

$$\begin{aligned}
\mu(x) &= \exp\left(\int \frac{1}{2x^2y + x}\,(6xy + 2 - 4xy - 1)\ dx\right) \\
&= \exp\left(\int \frac{2xy + 1}{2x^2y + x}\,dx\right) \\
&= \exp\left(\int \frac{1}{x}\,dx\right) \\
&= \exp\left(\ln(x)\right) \\
&= x,
\end{aligned}$$

so that an appropriate integrating factor is simply x. [*2 marks for obtaining $\mu(x)$, with amark awarded for progress in evaluating the integral expression from part (b).*] We therefore seek a function $f(x, y)$ such that:

$$\frac{\partial f}{\partial x} = 3x^2y^2 + 2xy, \qquad \frac{\partial f}{\partial y} = 2x^3y + x^2.$$

This time, by inspection we have $f(x, y) = x^3y^2 + x^2y + c$, where c is constant. [*1 mark for correct equations for f, and 1 mark for correctly obtaining f.*] Hence the solution of the equation is:

$$x^3y^2 + x^2y + C = 0,$$

where C is constant. [*1 mark for correct implicit solution.*] Using the quadratic formula, the expression for y explicitly is therefore:

$$y(x) = \frac{-x^2 \pm \sqrt{x^4 - 4Cx^3}}{2x^3} = \frac{-x \pm \sqrt{x^2 - 4Cx}}{2x^2}.$$

[*1 mark for correct formula for $y(x)$, simplified.*]

15. (a) A differentiable function of a real variable x is given by $f(x)$. Suppose one is interested in the value of this function when its argument is close to the point $x = a$. State Taylor's theorem explaining how to calculate the value of this function at the point $x = a + \delta$, where δ is sufficiently small. [4]

> *Solution:* Let f be a function which is at least k-times differentiable at the point a. Then:
>
> $$f(a+\delta) = \sum_{n=0}^{k} \frac{f^{(k)}(a)}{k!}\delta^k + R_k.$$
>
> [*1 mark for correct $f^{(k)}(a)$ behaviour of kth term. 1 mark for other parts of kth term.*] The *remainder* is given by:
>
> $$R_k = \frac{f^{(k+1)}(\xi)}{(k+1)!}\delta^{k+1},$$
>
> for some ξ in the interval between a and $a + \delta$. [*1 mark for correct statement of form of remainder. 1 mark for saying ξ is in the interval between a and $a + \delta$.*]

(b) Write the Maclaurin series for $f(x) = (1 + x)^{1/3}$ in the form:

$$f(x) = \sum_{n=0}^{\infty} u_n x^n$$

and give the general expression for u_n. [6]

> *Solution:* The Maclaurin series is:
>
> $$\begin{aligned}(1+x)^{1/3} &= 1 + \frac{1}{3}x + \frac{(1/3)(-2/3)}{2!}x^2 + \cdots + \frac{(1/3)(-2/3)...((4-3k)/3)}{k!}x^k + \cdots \\ &= 1 + \sum_{k=1}^{\infty} \frac{(1/3)(-2/3)...((4-3k)/3)}{k!}x^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{(1)(-2)(-5)...(4-3k)}{k!3^k}x^k.\end{aligned}$$
>
> [*1 mark for factorials in denominators. 1 mark for recalling that you start at $1/3$ in the first term. 1 mark for knowing the general term looks like $(1/3)(1/3-1)(1/3-2)...(1/3-k)$, etc. 1 mark for correctly writing out general term. 1 mark for correctly simplifying general term.*] Hence the general term is:
>
> $$u_0 = 1, \qquad u_k = \frac{(1)(-2)(-5)...(4-3k)}{k!3^k}, \text{ for } k \geq 1.$$
>
> [*1 mark for distinguishing u_0, u_k for $k \geq 1$.*]

(c) Find the first three non-zero terms in the Maclaurin series for: [6]

$$f(x) = \cos\left(\sqrt{\pi^2/16 + x}\right).$$

Solution: The Maclaurin series here is rather annoying, because expanding $\sqrt{\pi^2/16+x}$ about $x=0$ has first term $\pi/4$, so we need to expand $\cos(y)$, say, about $y=\pi/4$. Note that:

$$\cos(y) = \cos(y-\pi/4+\pi/4) = \cos(y-\pi/4)\cos(\pi/4) - \sin(y-\pi/4)\sin(\pi/4)$$
$$= \frac{1}{\sqrt{2}}\left(1 - (y-\pi/4) - \frac{(y-\pi/4)^2}{2} + \frac{(y-\pi/4)^3}{3!} + \cdots\right).$$

[*1 mark for noticing need to expand cosine around $\pi/4$. 1 mark for correct cosine and sine expansions.*] Now substituting $y=\sqrt{\pi^2/16+x}$, we note that:

$$y = \left(\frac{\pi^2}{16}+x\right)^{1/2} = \frac{\pi}{4}\left(1+\frac{16x}{\pi^2}\right)^{1/2} = \frac{\pi}{4}\left(1+\frac{8x}{\pi^2} - \frac{32x^2}{\pi^4}+\cdots\right).$$

[*1 mark for partially correct binomial expansion, 1 mark for fully correct binomial expansion.*] Hence:

$$y - \frac{\pi}{4} = \frac{2x}{\pi} - \frac{8x^2}{\pi^3} + \cdots.$$

Substituting into our expansion of cosine, we have:

$$\cos\left(\sqrt{\frac{\pi^2}{16}+x}\right) = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}x}{\pi} + \frac{4\sqrt{2}x^2}{\pi^3} - \frac{1}{2\sqrt{2}}\left(\frac{4x^2}{\pi^2}\right) + \cdots$$
$$= \frac{1}{\sqrt{2}} - \frac{\sqrt{2}x}{\pi} + \left(\frac{4\sqrt{2}}{\pi^3} - \frac{\sqrt{2}}{\pi^2}\right)x^2 + \cdots.$$

[*1 mark for substituting binomial expansion into cosine expansion around $\pi/4$. 1 mark for simplifying coefficients and correct final answer.*]

(d) Find the first three non-zero terms in the approximation valid for large x of $\ln(1+x+x^2)$. [4]

Solution: We have:

$$\ln(1+x+x^2) = \ln\left(x^2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)\right)$$
$$= \ln(x^2) + \ln\left(1+\frac{1}{x}+\frac{1}{x^2}\right)$$
$$= 2\ln(x) + \left(\frac{1}{x}+\frac{1}{x^2}\right) - \frac{1}{2}\left(\frac{1}{x}+\frac{1}{x^2}\right)^2 + \cdots$$
$$= 2\ln(x) + \frac{1}{x} + \frac{1}{2x^2} + O\left(\frac{1}{x^3}\right).$$

[*1 mark for correct manipulation of argument of $\ln(x)$ to express it in terms of powers of $1/x$. 1 mark for correct separation of logarithms. 1 mark for correct Taylor expansion of logarithm. 1 mark for simplifying coefficients correctly.*]

16. (a) The vector field $\mathbf{v}$ is given in Cartesian coordinates by:

$$\mathbf{v} = (2x(y+z), x^2 - y^2, y^2 - z^2).$$

(i) Calculate $\nabla \cdot \mathbf{v}$. [2]

(ii) A cube with sides of length unity has corners at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Without using the divergence theorem, calculate:

$$\int \mathbf{v} \cdot d\mathbf{S}$$

over the surface of the cube, where $d\mathbf{S}$ is the outward-pointing differential element of vector surface area. [6]

Solution: (i) We have $\nabla \cdot \mathbf{v} = 2(y+z) - 2y - 2z = 0$. [*1 mark for partially correct, 1 mark for fully correct.*]

(ii) Even though we are only given four corners, the other corners must be at $(0,1,1)$ and $(1,1,1)$ by the geometry of the cube. On the sides of the cube orthogonal to $(0,0,1)$, we have the contributions to the integral (one with unit normal $(0,0,1)$ and one with units normal $(0,0,-1)$):

$$\int_0^1 \int_0^1 (y^2 - 1^2)\, dxdy - \int_0^1 \int_0^1 (y^2 - 0^2)\, dxdy = -1.$$

[2 *marks for correct integrals orthogonal to z-direction.*] On the sides of the cube orthogonal to $(0,1,0)$, we have the contributions to the integral (one with unit normal $(0,1,0)$ and one with unit normal $(0,-1,0)$):

$$\int_0^1 \int_0^1 (x^2 - 1^2)\, dxdz - \int_0^1 \int_0^1 (x^2 - 0^2)\, dxdz = -1.$$

[2 *marks for correct integrals orthogonal to y-direction.*] On the sides of the cube orthogonal to $(1,0,0)$, we have the contributions to the integral (one with unit normal $(1,0,0)$ and one with unit normal $(-1,0,0)$):

$$\int_0^1 \int_0^1 2(1)(y+z)\, dydz - \int_0^1 \int_0^1 2(0)(y+z)\, dydz = 2\int_0^1 \left(\frac{1}{2} + z\right)\, dz = 2\left(\frac{1}{2} + \frac{1}{2}\right) = 2.$$

[2 *marks for correct integrals orthogonal to x-direction.*] Overall, we get $-1 - 1 + 2 = 0$ as the value of the surface integral. [*LOSE one mark if the combination is done incorrectly.*]

(b) By calculating $\nabla \times \mathbf{F}$ in each case, determine which of the following vector fields, given in Cartesian coordinates, are conservative:

$$\mathbf{F}_1 = (3x^2y^2z, 2x^3yz, x^3y^2),$$

$$\mathbf{F}_2 = (3x^2yz^2, 2x^3yz, x^3z^2).$$

For each case, if $\mathbf{F}$ is conservative, find a scalar potential ϕ such that $\mathbf{F} = \nabla\phi$. [6]

[*Notation:* $\boldsymbol{u} \times \boldsymbol{v}$ *is equivalent to* $\boldsymbol{u} \wedge \boldsymbol{v}$.]

For each case, if $\mathbf{F}$ is conservative, explicitly evaluate $\int\limits_A^B \mathbf{F}\cdot d\mathbf{r}$ between the points $A = (0,0,0)$ and $B = (1,1,2)$ along:

(i) the straight line from A to B;

(ii) the curve $x = t, y = t, z = 2t^3$ from $t = 0$ to $t = 1$,

and comment on the relation of your results to the values of the associated scalar field ϕ evaluated at A and B. [6]

Solution: We have:

$$\nabla \times \mathbf{F}_1 = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} 3x^2y^2z \\ 2x^3yz \\ x^3y^2 \end{pmatrix} = \begin{pmatrix} 2yx^3 - 2yx^3 \\ -3x^2y^2 + 3x^2y^2 \\ 6x^2yz - 6x^2yz \end{pmatrix} = \mathbf{0}.$$

Hence $\mathbf{F}_1$ is conservative. [*1 mark for curl computed correctly. 1 mark for stating conservative.*] A scalar potential that works here would be some $\phi(x,y,z)$ satisfying:

$$\frac{\partial\phi}{\partial x} = 3x^2y^2z, \qquad \frac{\partial\phi}{\partial y} = 2x^3yz, \qquad \frac{\partial\phi}{\partial z} = x^3y^2.$$

We see by inspection that $\phi(x,y,z) = x^3y^2z + c$ works for any constant c. [*1 mark for any working, 1 mark for correctly finding a potential*.] We also have:

$$\nabla \times \mathbf{F}_2 = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} 3x^2y^2z \\ 2x^3yz \\ x^3z^2 \end{pmatrix} = \begin{pmatrix} -2x^3y \\ 3x^2y^2 - 3x^2z^2 \\ 0 \end{pmatrix}.$$

Hence $\mathbf{F}_2$ is not conservative, because this does not vanish. [*1 mark for curl computed correctly, 1 mark for stating not conservative.*]

The only case we need to evaluate the integrals for is $\mathbf{F}_1$, since this is conservative. Since we are asked to 'explicitly' evaluate, we probably shouldn't use the gradient theorem. For each case, we have:

(i) The straight line can be parametrised as $\mathbf{r}(t) = (t, t, 2t)$ [*1 mark for parametrisation*]. Hence the integral becomes:

$$\int\limits_0^1 (6t^5, 4t^5, t^5)\cdot(1,1,2)\,dt = 12\int\limits_0^1 t^5\,dt = \frac{12}{6} = 2.$$

[*1 mark for setting up integral correctly, 1 mark for computing integral correctly.*]

(c) The parametrisation is already given in this case. The integral becomes:

$$\int_0^1 (6t^7, 4t^7, t^5) \cdot (1, 1, 6t^2)\, dt = 16 \int_0^1 t^7\, dt = \frac{16}{8} = 2.$$

[*1 mark for setting up integral correctly, 1 mark for computing integral correctly.*]

The answers are the same because the vector field $\mathbf{F}_1$ is conservative, therefore the line integral between A and B of $\mathbf{F}_1$ is independent of the path taken. Indeed, by the gradient theorem, we have:

$$\int_A^B \mathbf{F}_1 \cdot d\mathbf{r} = \int_A^B \nabla\phi \cdot d\mathbf{r} = \phi(B) - \phi(A) = \phi(1, 1, 2) - \phi(0, 0, 0) = 2.$$

This is consistent with the value we found above. [*1 mark for relating to gradient theorem, and the value of ϕ at B.*]

17. (a) Calculate the total differential of the function $p(n, V, T)$, where: [5]

$$p = \frac{nRT}{V} - \frac{n^2 a}{V^2}.$$

Solution: Using the Leibniz rule, we have:

$$dp = \frac{RT}{V}dn + \frac{nR}{V}dT - \frac{nRT}{V^2}dV - \frac{2na}{V^2}dn + \frac{2n^2 a}{V^3}dV.$$

Simplifying, we have:

$$dp = \left(\frac{RT}{V} - \frac{2na}{V^2}\right)dn + \left(\frac{2n^2 a}{V^3} - \frac{nRT}{V^2}\right)dV + \frac{nR}{V}dT.$$

[*1 mark for attempting to use Leibniz rule. 1 mark for fully correct derivative of first term. 1 mark for fully correct derivative of second term. 1 mark for differentiating, but keeping with R and a constant and not treating them as variables. 1 mark for fully simplified final answer.*]

(b) Calculate the gradient of $g(r) = g(\sqrt{x^2 + y^2 + z^2})$, where g is an unspecified differentiable function of one variable. [5]

Solution: We did exactly this on the examples sheet. Note that we have:

$$\frac{\partial}{\partial x}g(r) = \frac{\partial r}{\partial x}g'(r) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}g'(r) = \frac{x}{r}g'(r),$$

using the chain rule. Hence by symmetry we have:

$$\nabla g = \frac{g'(r)}{r}(x, y, z) = g'(r)\hat{\mathbf{x}},$$

where $\mathbf{x} = (x, y, z)$. [*1 mark for partial differentiation (i.e. knowing what the gradient is). 1 mark for using the chain rule correctly. 1 mark for correct evaluation of $\partial r/\partial x$. 1 mark for all $\partial r/\partial x, \partial r/\partial y, \partial r/\partial z$ computed correctly (e.g. by symmetry). 1 mark for final answer (may not be simplified).*]

(c) Let $f(x, y, z) = x^2 + yz$.

(i) Determine the directional derivative of f at the point $(1, 1, -3)$ in the direction $\hat{\mathbf{u}} = \frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. [5]

(ii) Determine the unit vector pointing in the direction in which f increases most rapidly at $(1, 1, -3)$. [5]

Solution: First, we compute the gradient of f, given by:

$$\nabla f = (2x, z, y).$$

[*Up to 2 marks for computing gradient correctly.*] Then:

(i) The directional derivative at the point $(1, 1, -3)$ in the given direction (which note is normalised since $\sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$ is:

$$\hat{\mathbf{u}} \cdot \nabla f(1, 1, -3) = \frac{1}{3}(2, 1, 2) \cdot (2, -3, 1) = \frac{1}{3}(3) = 1.$$

[*1 mark for knowing formula for directional derivative involves dotting with $\hat{\mathbf{u}}$. 1 mark for evaluating ∇f at the correct point. 1 mark for evaluating dot product correctly.*]

(ii) The direction in which f is increasing most rapidly is ∇f. [*1 mark for knowing this fact.*] Evaluating ∇f at the given point, we have $\nabla f(1, 1, -3) = (2, -3, 1)$ [*1 mark for this, again.*], which when we normalise gives:

$$\frac{\nabla f}{|\nabla f|} = \frac{(2, -3, 1)}{\sqrt{2^2 + 3^2 + 1^2}} = \frac{(2, -3, 1)}{\sqrt{14}}.$$

[*1 mark for attempting to normalise. 1 mark for computing normalisation correctly. 1 mark for final answer.*]

18. A function of a real variable x is given by:

$$y = e^{\sin^2(x)}.$$

(a) Find all the stationary points of this function and determine whether each of them is a maximum, minimum or point of inflection. [6]

Solution: Differentiating, we have:

$$\frac{dy}{dx} = 2\sin(x)\cos(x)e^{\sin^2(x)} = \sin(2x)e^{\sin^2(x)}.$$

Therefore the stationary points occur precisely at $\sin(2x) = 0$, i.e. $x = n\pi/2$ for each integer n. [*1 mark for correct derivative, 1 mark for correctly identifying all stationary points.*]

It remains to classify the stationary points. Since $e^{\sin^2(x)} > 0$ everywhere, we note that the gradient near the stationary points has the same sign as $\sin(2x)$ does near each of the stationary points. In particular, thinking about the graph of $\sin(2x)$, we have that if $x = n\pi/2$ is such that n is even, then $\sin(2x)$ is increasing at the point $n\pi/2$. In particular this implies $dy/dx < 0$ left of the stationary point and $dy/dx > 0$ right of the stationary point. Thus $x = n\pi/2$ with n even is a minimum.

On the other hand, near $x = n\pi/2$ with n odd, we have that $\sin(2x)$ is decreasing at the point $n\pi/2$. In particular this implies $dy/dx > 0$ left of the stationary point and $dy/dx < 0$ right of the stationary point. Thus $x = n\pi/2$ with n odd is a maximum.

[*Up to 2 marks for getting minima with justification, up to 2 marks for getting maximum with justification.*]

(b) Sketch y as a function of x. [4]

Solution: We know that y is oscillatory with period π (since $\sin^2(x)$ is oscillatory with period π). At $x = 0$, it goes through the point 1, and then increases in value, before attaining a maximum at $x = \pi/2$, taking the value e there. Then it decreases in value, attaining a minimum at $x = \pi$, taking the value 1 there. The gradient is larger as we approach the maxima, so they are slightly more 'peaked' than the minima are. [*Up to 4 marks for a convincing sketch. Award marks for partial progress.*]

Suppose now that $0 < x < \pi$ and:

$$z = \sqrt{\ln(y)} \tag{†}$$

(a) Calculate $\dfrac{dz}{dy}$ and $\dfrac{d^2z}{dy^2}$. [3]

Solution: We have:

$$\frac{dz}{dy} = \frac{1}{2\sqrt{\ln(y)}} \cdot \frac{1}{y}.$$

[*1 mark for use of chain rule to obtain correct answer.*] Taking a second derivative, we have:

$$\frac{d^2z}{dy^2} = -\frac{1}{4(\ln(y))^{3/2}} \cdot \frac{1}{y^2} - \frac{1}{2\sqrt{\ln(y)}} \cdot \frac{1}{y^2}.$$

[*Up to 2 marks for correct second derivative, one for each term.*]

(b) Use the chain rule to calculate $\dfrac{dz}{dx}$ as a function of x. [2]

Solution: We have:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \frac{1}{2\sqrt{\ln(y)}}\frac{1}{y} \cdot 2\cos(x)\sin(x)e^{\sin^2(x)} = \cos(x).$$

[*1 mark for correct use of chain rule, 1 mark for simplifying.*]

(c) Use the chain rule to calculate $\dfrac{d^2z}{dx^2}$ as a function of x. [3]

Solution: Recall that $z = \sqrt{\ln(y(x))}$, and we have already computed:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}.$$

Taking a further x-derivative, we have:

$$\begin{aligned}\frac{d^2z}{dx^2} &= \frac{d}{dx}\left(\frac{dz}{dy}\right)\frac{dy}{dx} + \frac{dz}{dy}\frac{d^2y}{dx^2}\\ &= \frac{dy}{dx}\frac{d}{dy}\left(\frac{dz}{dy}\right)\frac{dy}{dx} + \frac{dz}{dy}\frac{d^2y}{dx^2}\\ &= \frac{d^2z}{dy^2}\left(\frac{dy}{dx}\right)^2 + \frac{dz}{dy}\frac{d^2y}{dx^2}.\end{aligned}$$

[*1 mark for correct form of chain rule used here.*] Observe that $dy/dx = 2\sin(x)\cos(x)y$. Hence $(dy/dx)^2 = 4\sin^2(x)\cos^2(x)y^2$, so that the first term simplifies to:

$$\begin{aligned}\frac{d^2z}{dy^2}\left(\frac{dy}{dx}\right)^2 &= -\frac{4\sin^2(x)\cos^2(x)}{4(\ln(y))^{3/2}} - \frac{4\sin^2(x)\cos^2(x)}{2\sqrt{\ln(y)}}\\ &= -\frac{\cos^2(x)}{\sin(x)} - 2\sin(x)\cos^2(x)\end{aligned}$$

using the fact that $\ln^{1/2}(y) = \sin(x)$ (since $0 < x < \pi$).

For the second term, note that:

$$\begin{aligned}\frac{d^2y}{dx^2} &= 2\cos^2(x)e^{\sin^2(x)} - 2\sin^2(x)e^{\sin^2(x)} + 4\sin^2(x)\cos^2(x)e^{\sin^2(x)}\\ &= 2(\cos^2(x) + 2\sin^2(x)\cos^2(x) - \sin^2(x))y.\end{aligned}$$

Hence the second term simplifies to:

$$\frac{dz}{dy}\frac{d^2y}{dx^2} = \frac{2(\cos^2(x) + 2\sin^2(x)\cos^2(x) - \sin^2(x))}{2\sqrt{\ln(y)}} = \frac{\cos^2(x)}{\sin(x)} + 2\sin(x)\cos^2(x) - \sin(x).$$

Putting everything together, we obtain:

$$\frac{d^2z}{dx^2} = -\frac{\cos^2(x)}{\sin(x)} - 2\sin(x)\cos^2(x) + \frac{\cos^2(x)}{\sin(x)} + 2\sin(x)\cos^2(x) - \sin(x) = -\sin(x).$$

[*1 mark for evaluation of each term; 1 mark for combining to give final answer.*]

(d) Substitute for y as a function of x in the equation for z in (†) and calculate $\frac{d^2z}{dx^2}$ directly. [2]

Solution: We have $z = \sqrt{\ln(y)} = \sin(x)$, hence $\frac{dz}{dx} = \cos(x)$ and $\frac{d^2z}{dx^2} = -\sin(x)$. [*1 mark for writing* $z = \sin(x)$, *and 1 mark for convincingly taking derivatives.*]

19. A string of uniform mass per unit length ρ is stretched along the x-axis under tension T and undergoes small transverse oscillations in the x-y plane with displacement $y(x,t)$. Derive the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

satisfied by $y(x,t)$, where c is a constant which you should determine in terms of ρ and T. [4]

Solution: Consider applying Newton's second law to the second of string between x_A and x_B, with $x_B - x_A = \delta x$. The tension T is constant throughout the string, and the oscillations are transverse, so resolving in the x-direction we have:

$$T\cos(\theta_A) \approx T\cos(\theta_B),$$

where θ_A is the angle the string makes with the horizontal at x_A, and θ_B is the angle the string makes with the horizontal at x_B, since δx is very small. [*1 mark for resolving in the x-direction.*] Note that this is self-consistent, because:

$$\cos(\theta_A) = \frac{1}{\sqrt{1+\tan^2(\theta_A)}} = \frac{1}{\sqrt{1+(\partial y/\partial x)^2}} \approx 1,$$

by the binomial theorem, assuming $|\partial y/\partial x| \ll 1$ (since each part of the string moves transversely, this must be very small).

On the other hand, resolving the y-direction and using Newton's second law, we have:

$$\rho\delta x \frac{\partial^2 y}{\partial t^2} = T\sin(\theta_B) - T\sin(\theta_A).$$

[*1 mark for resolving in the y-direction.*] Dividing through by $\delta x T \approx \delta x T\cos(\theta_B) \approx \delta x T\cos(\theta_A)$, we have:

$$\frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\tan(\theta_B) - \tan(\theta_A)}{\delta x}.$$

[*1 mark for getting equation in terms of tangents.*] Noting:

$$\tan(\theta_B) - \tan(\theta_A) = \left.\frac{\partial y}{\partial x}\right|_{x=x_B} - \left.\frac{\partial y}{\partial x}\right|_{x=x_A},$$

we have in the limit where $\delta x \to 0$ (using $c^2 = T/\rho$):

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2},$$

as required. [*1 mark for correctly taking limit and identifying wavespeed $c = \sqrt{T/\rho}$.*]

A string of length L is fixed at $x = 0, y = 0$, and at $x = L$ it is attached to a small ring of mass M which is constrained to move without friction on a straight wire parallel to the y-axis. The string undergoes small transverse oscillations in the x-y plane with displacement $y(x, t)$. Show that the equation of motion for the ring is: [4]

$$M \frac{\partial^2 y}{\partial t^2}\bigg|_{x=L} = -T \frac{\partial y}{\partial x}\bigg|_{x=L}.$$

Solution: Consider a small length of string δx on the left hand side of $x = L$. Suppose that the string makes an angle θ_- with the horizontal to the left. Then as before, we have the small angle approximation $\sin(\theta_-) \approx \tan(\theta_-)$ [1 *mark for recognising this approximation*]. We also have by Newton's second law applied in the vertical direction we have:

$$(M + \rho\delta x)\frac{\partial^2 y}{\partial t^2}\bigg|_{x=L} = -T\sin(\theta_-),$$

where the mass is the sum of the mass M and the string's mass $\rho\delta x$ [1 *mark for force resolution in y-direction, using Newton's second law.*]. Using the small angle approximation, this becomes:

$$(M + \rho\delta x)\frac{\partial^2 y}{\partial t^2}\bigg|_{x=L} = -T\tan(\theta_-) \approx -T\frac{\partial y}{\partial x}\bigg|_{x=L}.$$

[1 *mark for getting in terms of tangents.*] Taking the limit as $\delta x \to 0$, this equation approaches:

$$M \frac{\partial^2 y}{\partial t^2}\bigg|_{x=L} = -T \frac{\partial y}{\partial x}\bigg|_{x=L},$$

as required. [1 *mark for correctly taking limit and establishing equation.*]

Find the separable solutions to the equation of motion for the string, and show that the allowed frequencies of vibration $\omega_n > 0$, $n = 0, 1, 2, ...$ satisfy: [4]

$$\cot\left(\frac{\omega_n L}{c}\right) = \frac{M\omega_n}{\rho c}.$$

Solution: We assume separable solutions of the form $y(x, t) = X(x)T(t)$. [1 *mark for this.*] Substituting into the wave equation from the first part of the question, this gives:

$$X\ddot{T} = c^2 X'' T \qquad \Rightarrow \qquad \frac{\ddot{T}}{T} = \frac{c^2 X''}{X} = -\lambda,$$

for some constant λ. [1 *mark for correct separation of variables.*] This implies:

$$T(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t), \qquad X(x) = C\cos(\sqrt{\lambda}x/c) + D\sin(\sqrt{\lambda}x/c)$$

in the case that $\lambda \neq 0$. In the case where $\lambda = 0$, we obtain:

$$T(t) = A + Bt, \qquad X(x) = C + Dx.$$

[1 *mark for obtaining correct solutions in case $\lambda \neq 0$. 1 mark for obtaining correct solutions in case $\lambda = 0$.*]

We now impose the boundary data. We are told that the string is fixed at $x = 0, y = 0$, so we have $y(0, t)$ for all time. Thus $X(0) = 0$. Additionally, we derived in the previous part that:

$$M \frac{\partial^2 y}{\partial t^2}\bigg|_{x=L} + T \frac{\partial y}{\partial x}\bigg|_{x=L} = 0,$$

for all time. Hence we require:

$$MX(L)\ddot{T}(t) + T \cdot T(t)X'(L) = 0$$

for all time. But $\ddot{T}(t) = -\lambda T(t)$, which implies the boundary condition:

$$TX'(L) = \lambda M X(L).$$

[1 *mark for obtaining both boundary conditions correctly.*] In particular, we see that all of our boundary data relates to $X(x)$. Imposing $X(0) = 0$, we see that $C = 0$, and hence:

$$X(x) = \begin{cases} D\sin(\sqrt{\lambda}x/c), & \lambda \neq 0 \\ Dx, & \lambda = 0. \end{cases}$$

Imposing $TX'(L) = \lambda M X(L)$, we observe that if $\lambda = 0$, we would have $X'(L) = 0$ and $X'(x) = D$, implying $X \equiv 0$ identically. Thus $\lambda \neq 0$. In the remaining case, we have:

$$\frac{DT\sqrt{\lambda}}{c}\cos(\sqrt{\lambda}L/c) = \lambda M D\sin(\sqrt{\lambda}x/c),$$

so noting $D \neq 0$, we see that the frequencies of oscillation $\omega_n = \sqrt{\lambda}$ must satisfy:

$$\cot\left(\frac{\omega_n L}{c}\right) = \frac{Mc\omega_n}{T} = \frac{M\omega_n}{\rho c},$$

using the fact that $c/T = c^2/Tc = T/\rho T c = 1/\rho c$. [1 *mark for imposing boundary conditions correctly and deducing equation obeyed by frequencies.*]

It follows that the general solution may be written in the form:

$$y(x,t) = \sum_{n=0}^{\infty} \left(A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\right)\sin\left(\frac{\omega_n x}{c}\right),$$

where ω_n are the solutions to the transcendental equation:

$$\cot\left(\frac{\omega_n L}{c}\right) = \frac{M\omega_n}{\rho c}.$$

Consider now the case $M = 0$, $L = \pi$. Find the displacement, $y(x,t)$, of the string for $t \geq 0$ given the initial conditions: [8]

$$y(x,0) = 0, \qquad \left.\frac{\partial y}{\partial t}\right|_{t=0} = \sin\left(\frac{1}{2}x\right) + \sin\left(\frac{5}{2}x\right).$$

Solution: In this case, the frequencies satisfy:

$$\cot\left(\frac{\omega_n \pi}{c}\right) = 0.$$

so they are the solutions of $\cos(\omega_n\pi/c) = 0$. Thus the frequencies are $\omega_n = (2n+1)c/2$ for n an integer. We may assume the frequencies are positive, so we can take $n \geq 0$. [*Up to 3 marks for obtaining the correct frequencies.*]

Therefore the general solution takes the form:

$$y(x,t) = \sum_{n=0}^{\infty} \left(A_n \cos\left(\frac{(2n+1)ct}{2}\right) + B_n \sin\left(\frac{(2n+1)ct}{2}\right)\right) \sin\left(\frac{(2n+1)x}{2}\right).$$

[1 *mark for stating general solution.*] Imposing the initial data $y(x,0) = 0$, we have:

$$0 = \sum_{n=0}^{\infty} A_n \sin\left(\frac{(2n+1)x}{2}\right).$$

Since all these functions are independent, this implies that $A_n = 0$ identically. [1 *mark for getting all these coefficients are zero from the boundary condition* $y(x,0) = 0$.]

On the other hand, imposing the initial data for the velocity of the string, we have:

$$\sin\left(\frac{1}{2}x\right) + \sin\left(\frac{5}{2}x\right) = \sum_{n=0}^{\infty} \frac{B_n(2n+1)c}{2} \sin\left(\frac{(2n+1)x}{2}\right).$$

Again, by independence, this implies that the only non-zero B_n are $B_0 = 2/c$ and $B_2 = 2/5c$. [2 *marks for imposing this boundary condition, and obtaining* B_0, B_2 *as only non-zero coefficients.*] Hence the complete solution is:

$$y(x,t) = \frac{2}{c}\sin\left(\frac{1}{2}x\right)\sin\left(\frac{ct}{2}\right) + \frac{2}{5c}\sin\left(\frac{5}{2}x\right)\sin\left(\frac{5ct}{2}\right).$$

[1 *mark for fully correct final answer, stated clearly.*]

20. (a) Prove the Schwarz inequality: [5]

$$\int f^2\,dx \int g^2\,dx \geq \left(\int fg\,dx\right)^2.$$

Solution: Observe that:

$$0 \leq \int (f+\lambda g)^2\,dx = \int f^2\,dx + 2\lambda \int fg\,dx + \lambda^2 \int g^2\,dx,$$

for all values of λ. In particular, the discriminant of this quadratic in λ must be non-positive. This gives:

$$4\left(\int fg\,dx\right)^2 \leq 4\int f^2\,dx \int g^2\,dx,$$

which on dividing by 4 gives the result. [*1 mark for first inequality. 1 mark for correctly expanding. 1 mark for noticing discriminant must be non-positive. 1 mark for correct calculation of discriminant and deduction of inequality.*]

(b) Determine the derivative with respect to x of: [6]

$$\int_{3x}^{x^2} \sin(xt)\,dt.$$

Solution: We use the Leibniz integral formula:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = f(x,b(x))\frac{db}{dx} - f(x,a(x))\frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)\,dt.$$

[*1 mark for knowing this formula.*] We have:

$$\frac{d}{dx}\int_{3x}^{x^2} \sin(xt)\,dt = 2x\sin(x^3) - 3\sin(3x^2) + \int_{3x}^{x^2} t\cos(xt)\,dt.$$

[*2 marks for correctly applying the formula here; 1 mark for the first two terms, and 1 mark for the third term.*] Using integration by parts on the final term, we have:

$$\begin{aligned}\int_{3x}^{x^2} t\cos(xt)\,dt &= \left[\frac{t\sin(xt)}{x}\right]_{3x}^{x^2} - \frac{1}{x}\int_{3x}^{x^2} \sin(xt)\,dt \\ &= x\sin(x^3) - 3\sin(3x^2) + \frac{1}{x}\left[\frac{\cos(xt)}{x}\right]_{3x}^{x^2} \\ &= x\sin(x^3) - 3\sin(3x^3) + \frac{\cos(x^3)}{x^2} - \frac{\cos(3x^2)}{x^2}.\end{aligned}$$

[*Up to 2 marks for correct integration by parts.*] Putting everything together, we have the final result:

$$3x\sin(x^3) - 6\sin(3x^2) + \frac{\cos(x^3)}{x^2} - \frac{\cos(3x^2)}{x^2}.$$

[*1 mark for correctly simplified final answer.*]

(c) Let:

$$I(a) = \int_0^1 \frac{\sin(a \ln(x))}{\ln(x)} \, dx.$$

Obtain an expression for the derivative of $I(a)$ with respect to a, $I'(a)$. Evaluate this expression and use it to obtain an expression for $I(a)$ and hence determine the value of $I(1)$. [9]

Solution: We have:

$$I'(a) = \int_0^1 \frac{\partial}{\partial a} \frac{\sin(a \ln(x))}{\ln(x)} \, dx = \int_0^1 \cos(a \ln(x)) \, dx$$

[*Up to 2 marks for computing derivative of integral correctly.*] This integral can be done easily using complex methods. We have:

$$\int_0^1 \cos(a \ln(x)) \, dx = \mathrm{Re} \left[\int_0^1 e^{ia \ln(x)} \, dx \right] = \mathrm{Re} \left[\int_0^1 x^{ia} \, dx \right] = \mathrm{Re} \left[\frac{1}{ia+1} \right] = \frac{1}{a^2+1}.$$

[*Up to 4 marks for correct evaluation of integral, through any method.*] Hence we have:

$$I'(a) = \frac{1}{a^2+1} \qquad \Rightarrow I(a) = \arctan(a),$$

where the constant of integration vanishes since $I(0) = 0$. [*Up to 2 marks for integration here, and evaluation of constant.*] Therefore $I(1) = \pi/4$. [*1 mark for correct final answer.*]