

Part IA: Mathematics for Natural Sciences A

Examples Sheet 15: Multivariable integration and Gaussian integrals

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Area integrals in Cartesian coordinates

1. Draw a convincing diagram showing the volume that is represented by the *area integral*:

$$\iint_D f(x, y) dA,$$

where D is a region of the xy -plane, f is a function of two variables, and dA is a small area element in the xy -plane. What is the formula for dA in Cartesian coordinates?

2. Evaluate the area integral:

$$\iint_D 2xy^2 dA$$

over the triangle D bounded by the lines $y = 1$, $y = \frac{1}{2}x$ and $x = 0$ by (a) integrating with respect to y first, then with respect to x ; (b) integrating with respect to x first, then with respect to y .

3. Evaluate the following integrals, including a suitable sketch of the regions in each case:

$$(a) \iint_{D_1} xe^{xy} dA,$$

$$(b) \iint_{D_2} \sin(x + y) dA,$$

$$(c) \iint_{D_3} x^2 y dA,$$

where:

- $D_1 = [0, \pi] \times [0, \pi]$,
- D_2 is the triangle bounded by the lines $y = 0$, $x = \frac{1}{2}\pi$, $y = x$,
- D_3 is the region contained in the first quadrant of the plane, bounded by the curves $y = 2$, $y = x^2$ and $y = 1/x$.

Area integrals in polar coordinates

4. With the aid of a diagram, derive the formula for the area element dA in plane polar coordinates. Hence, evaluate the following integrals, including a suitable sketch of the regions in each case:

$$(a) \iint_{D_1} x^2(1 - x^2 - y^2) dA,$$

$$(b) \iint_{D_2} \sqrt{x^2 + y^2} dA,$$

where D_1 is the unit disk, and D_2 is the region bounded by the curve $r = 2a(1 + \cos(\theta))$ in plane polar coordinates, with a constant.

5. Evaluate the double integral:

$$\int_0^3 \int_{-\sqrt{9-x^2}}^0 e^{x^2+y^2} dy dx.$$

6. A heap of building materials, of volume V , is stockpiled in a circle of radius R . Within the heap, $0 \leq r \leq R$, its height profile is given by $Re^{-r/R}$, where r is radial distance from the centre of the heap. Determine R in terms of V .

7. (a) Sketch the surfaces $x^2 + y^2 = 16$ and $z = 2x^2 + 2y^2$ in 3D. Using an appropriate double integral, determine the volume of the region bounded by these surfaces and the plane $z = 0$.
- (b) Repeat part (a) with the second surface replaced by the surface $z^2 = 2x^2 + 2y^2$.

Volume integrals in Cartesian coordinates

8. Evaluate the volume integral:

$$\iiint_V xyz^2 \, dx dy dz$$

where V is:

- (a) the prism bounded by the planes $x = 0$, $y = 0$, $y = 1$, $z = 0$ and $x + z = 1$;
- (b) the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.
9. (a) Calculate the volume of the region defined by the constraints $|x| \leq L$, $|y| \leq b(2 - z/H)$ and $H(x/L)^2 \leq z \leq H$, where L, H, b are constants.
- (b) Calculate the volume of the intersection of this region with an infinite cylinder of diameter D , centred on the line $x = 0$, $z = \frac{1}{2}D$. [Hint: you don't need to do any integration here!]

Volume integrals in cylindrical and spherical polar coordinates

10. With the aid of diagrams, derive the formulae for the volume elements dV in cylindrical polar coordinates and in spherical polar coordinates. Hence, using appropriate coordinates in each case, determine using multiple integration:
- (a) the volume of a cylinder with radius a and height h ;
- (b) the volume of a sphere with radius a ;
- (c) the volume of a cone with height a , with sloped edge inclined at angle $\pi/4$ to the vertical;
- (d) the volume of a cone with sloped edge of length a , with sloped edge inclined at angle $\pi/4$ to the vertical, whose base is replaced by a sector of a sphere of radius a .
11. Find the volume bounded by the plane $z = 0$ and the surface:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{z}{d} + 1,$$

where a, d are positive constants.

12. (a) Find the volume of the cylindrically symmetric object bounded by the planes $z = 0$, $z = H$ and the surface of revolution $r = Re^{-z/H}$, where R, H are constants.
- (b) Now suppose that the cross-section of the shape is altered so that the cylindrical symmetry is broken. Specifically, suppose that a sector of each cross-sectional disk is removed such that at $z = H$, half the disk is removed, but at $z = 0$, none of the disk is removed, and in-between the amount of area removed interpolates linearly between these two extremes. Calculate the volume of the resulting object.
13. (a) By considering scaled coordinates $x' = x/a$, $y' = y/b$ and $z' = z/c$, and by arguing how the volume element dV changes in these coordinates, show that the volume bounded by the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is given by $4\pi abc/3$. Verify that this reduces to the expected result for a sphere.

- (b) Without detailed calculation, determine:

$$\iiint_V (x^4 y^2 z + xy^2 z^4) e^{-(x/a)^2 - (y/b)^2 - (z/c)^2} \, dx dy dz,$$

where V is the interior of the ellipsoid in part (a).

Mass and centre of mass

14. Find the mass of a spherical body of radius R whose mass density is:

$$\rho(r) = Ar^{-2} \left(1 - e^{-r/r_0}\right),$$

where r is radial distance from the centre of the object, and A, r_0 are positive constants.

15. The centre of mass of an object occupying a volume V is given by:

$$\bar{\mathbf{x}} = \frac{1}{M} \iiint_V \rho(\mathbf{x}) \mathbf{x} dV,$$

where M is the mass of the object, and $\rho(\mathbf{x})$ is its mass density at the point \mathbf{x} .

- (a) Consider a wedge cut from a uniform sphere of radius a , defined by the spherical coordinates $0 \leq r \leq a$, $0 \leq \theta \leq \pi$, $-\beta/2 \leq \phi \leq \beta/2$. Show that the centre of mass of the wedge is a distance $af(\beta)$ from its straight edge, where:

$$f(\beta) = \frac{3\pi}{8\beta} \sin(\beta/2).$$

Sketch the graph of $f(\beta)$ for $0 < \beta < 2\pi$, and comment on the physical relevance of your graph.

- (b) Compute the vector area of the curved surface of the wedge.

Gaussian integrals

16. Prove from first principles that $\int_{-\infty}^{\infty} e^{-x^2/a^2} dx = a\sqrt{\pi}$. Deduce the value of $\iiint_{\mathbb{R}^3} e^{-(x^2+y^2+z^2)/a^2} dV$.

17. Evaluate:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2+b/x^2)} dx,$$

by making the substitution $y = \frac{1}{2}(\sqrt{ax} - \sqrt{b/x})$. Hence or otherwise, evaluate:

$$\int_{-\infty}^{\infty} \frac{1}{x^2} e^{-\frac{1}{2}(ax^2+b/x^2)} dx.$$

18. By considering the double integral:

$$I = \iint_D x e^{-xy} dA$$

over the region $D = \{(x, y) : x \geq 0, x/2 \leq y \leq x\}$, evaluate the integral:

$$\int_0^{\infty} \left(e^{-x^2} - e^{-2x^2} \right) \frac{dx}{x^2}.$$

19. Prove by induction that for non-negative integers n and positive a :

$$\int_0^{\infty} x^{2n} e^{-ax^2} dx = \frac{(2n-1)(2n-3)\dots 1}{2(2a)^n} \sqrt{\frac{\pi}{a}}.$$

Verify this result by repeatedly differentiating $\int_0^{\infty} e^{-ax^2} dx$ with respect to a .