# Part IA: Mathematics for Natural Sciences A
# Examples Sheet 2: The vector product, and triple products of vectors

## Model Solutions

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**The vector product**

1. Find the angle between the position vectors of the points $(2, 1, 1)$ and $(3, -1, -5)$, and find the direction cosines of a vector perpendicular to both. Can both the angle and vector be computed using *only* the vector product?

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➾ **Solution:** Computing the scalar product, we have $(2, 1, 1)\cdot(3, -1, -5) = 6-1-5 = 0$, thus the vectors are orthogonal, so are at an angle $\pi/2$. To find the direction cosines of a vector perpendicular to both, we compute their vector product:

$$\begin{pmatrix}2\\1\\1\end{pmatrix} \times \begin{pmatrix}3\\-1\\-5\end{pmatrix} = \begin{pmatrix}-4\\13\\-5\end{pmatrix}.$$

The *direction cosines* are the just the cosines of the angles that this vector makes with the $x, y, z$ axes. To find these, we just normalise the vector, which has length $\sqrt{4^2+13^2+5^2} = \sqrt{210}$. This gives the direction cosines:

$$-\frac{4}{\sqrt{210}}, \qquad \frac{13}{\sqrt{210}}, \qquad -\frac{5}{\sqrt{210}}.$$

The formula for the vector product, $\mathbf{v} \times \mathbf{w} = |\mathbf{v}||\mathbf{w}|\sin(\theta)\hat{\mathbf{n}}$ makes it look like the angle can be computed using the vector product - just take the lengths of both sides to give $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}|\sin(\theta)$, which rearranges to:

$$\sin(\theta) = \frac{|\mathbf{v} \times \mathbf{w}|}{|\mathbf{v}||\mathbf{w}|}.$$

In general, the problem is that this equation usually has two solutions in the range $[0, \pi]$, say $\theta$ and $\pi - \theta$. We can't tell if vectors are at an acute or an obtuse angle if we just use the vector product normally!

*However*, in this question since the vectors are inclined at $\pi/2$, there is precisely *one* solution to the equation. So we could have used the vector product in this case! But, it is a bad method in general.

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2. Find all points $\mathbf{r}$ which satisfy $\mathbf{r} \times \mathbf{a} = \mathbf{b}$ where $\mathbf{a} = (1, 1, 0)$ and $\mathbf{b} = (1, -1, 0)$.

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➾ **Solution:** Let $\mathbf{r} = (x, y, z)$. Then taking the vector product, our equation becomes:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \times \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-z\\z\\x-y\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

Comparing components, we see that $z = -1$ and $x = y$. Thus the set of points that satisfy this equation is $(\lambda, \lambda, -1)$. This is a line going through the point $(0, 0, -1)$ parallel to the vector $(1, 1, 0)$.

3. Using properties of the vector product, prove the identity $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}) = \mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$.

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**Solution:** By the left and right distributive properties, we have:

$$(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}) = (\mathbf{b}-\mathbf{a})\times\mathbf{c} - (\mathbf{b}-\mathbf{a})\times\mathbf{a} = \mathbf{b}\times\mathbf{c} - \mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{a} + \mathbf{a}\times\mathbf{a}.$$

Now, $\mathbf{a}\times\mathbf{a} = \mathbf{0}$ since the angle between a vector and itself is zero. Finally, using antisymmetry of the vector product we have $\mathbf{a}\times\mathbf{c} = -\mathbf{c}\times\mathbf{a}$ and $\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}$. The result follows.

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**More on the equation of a line**

4. (a) Explain why the line through the points with positions vectors $\mathbf{a}$, $\mathbf{b}$ is $(\mathbf{r}-\mathbf{a})\times(\mathbf{b}-\mathbf{a}) = \mathbf{0}$. Show using properties of the vector product that an equivalent representation of this line is $\mathbf{r}\times(\mathbf{b}-\mathbf{a}) = \mathbf{a}\times\mathbf{b}$. What is the geometric significance of the quantity $|\mathbf{a}\times\mathbf{b}|/|\mathbf{b}-\mathbf{a}|$ here?

   (b) Express the line $\mathbf{r} = (1,0,1)+\lambda(3,-1,0)$ in the form $\mathbf{r}\times\mathbf{c} = \mathbf{d}$.

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**Solution:** (a) The direction of the line is $\mathbf{b}-\mathbf{a}$. For any point on the line $\mathbf{r}$, we must have $\mathbf{r}-\mathbf{a}$ parallel to this direction. This happens if and only if $(\mathbf{r}-\mathbf{a})\times(\mathbf{b}-\mathbf{a}) = \mathbf{0}$, as required. Hence, this is an alternative way of writing the equation of a line.

Using the distributive property of the vector product, we can expand this to give $\mathbf{r}\times(\mathbf{b}-\mathbf{a}) - \mathbf{a}\times(\mathbf{b}-\mathbf{a}) = \mathbf{0}$. Rearranging, and using the distributive property again, we have:

$$\begin{aligned}
\mathbf{r}\times(\mathbf{b}-\mathbf{a}) &= \mathbf{a}\times(\mathbf{b}-\mathbf{a}) \\
&= \mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{a} \\
&= \mathbf{a}\times\mathbf{b} && (\text{since } \mathbf{a}\times\mathbf{a} = \mathbf{0})
\end{aligned}$$

We can interpret the quantity $|\mathbf{a}\times\mathbf{b}|/|\mathbf{b}-\mathbf{a}|$ in a similar way to Question 13(a) from Sheet 1. Writing the left hand side of this equation as $|\mathbf{r}||\mathbf{b}-\mathbf{a}|\sin(\theta)\hat{\mathbf{n}}$, then taking the length of both sides, we have:

$$|\mathbf{r}|\sin(\theta) = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}-\mathbf{a}|}.$$

This tells us that $|\mathbf{r}|$ is minimised when $\sin(\theta)$ is maximised, i.e. when $\sin(\theta) = 1$ occurring at $\theta = \pi/2$. This tells us that the shortest distance between the origin and the line is given by:

$$\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}-\mathbf{a}|}.$$

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(b) The given line goes through the point $(1,0,1)$ and has direction $(3,-1,0)$. Hence, it has the equation:

$$(\mathbf{r}-(1,0,1))\times(3,-1,0) = \mathbf{0}.$$

Rearranging, we have:

$$\mathbf{r}\times\begin{pmatrix}3\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}3\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\3\\-1\end{pmatrix}.$$

5. (a) Show that the shortest distance between the point **p** and the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ can be written as $\left|\hat{\mathbf{b}} \times (\mathbf{p} - \mathbf{a})\right|$.

   (b) Find the shortest distance from a vertex of a unit cube to a diagonal excluding that vertex using both the formula in (a), and the formula from Question 13(c) of Sheet 1, and check that your answers agree.

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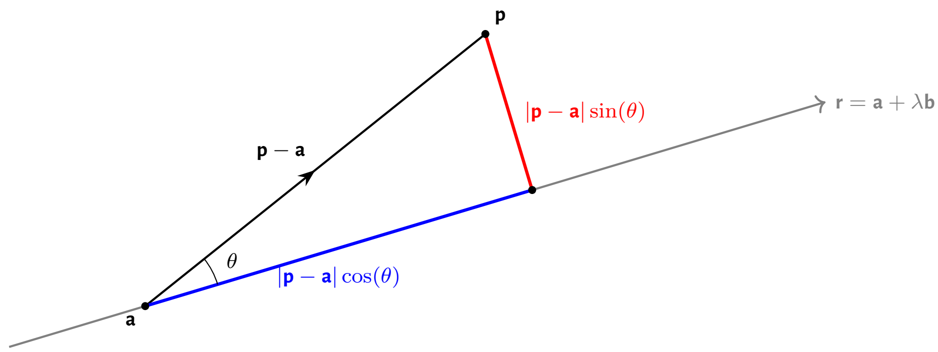
**Solution:** (a) Consider the diagram below:




The shortest distance between the line and the point is $|\mathbf{p}-\mathbf{a}|\sin(\theta)$, which is the magnitude of the vector product $\hat{\mathbf{b}}\times(\mathbf{p}-\mathbf{a})$, as required.

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(b) Let the unit cube have vertices $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ and $(1,1,1)$. A diagonal of the cube is then $\lambda(1,1,1)$ (this is the easiest one to pick!). A separate vertex that does not lie on this diagonal is $(0,0,1)$. Hence using the formula from (a), the shortest distance is the magnitude of:

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\0\end{pmatrix},$$

which is $\sqrt{2/3}$.

Using the formula from Question 13(c) of Sheet 1, the shortest distance is instead given by:

$$\sqrt{1^2 - \left(\frac{(1,1,1)}{\sqrt{3}}\cdot(0,0,1)\right)^2} = \sqrt{1-\frac{1}{3}} = \sqrt{\frac{2}{3}},$$

in perfect agreement.

**More on the equation of a plane**

6. (a) Explain why the plane through the points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is $(\mathbf{r}-\mathbf{a}) \cdot ((\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})) = 0$. Show using properties of the vector product, and the result from Question 3, that this may equivalently be written in the more symmetric form $\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

   (b) Find an equation of the form $(\mathbf{r}-\mathbf{a}) \cdot \mathbf{n} = 0$ for the plane passing through $(1,1,1)$, $(1,2,3)$ and $(0,0,4)$.

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➾ **Solution:** (a) Two vectors contained in the plane are $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$. Taking their cross product, we produce a vector orthogonal to the plane, $(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})$. Hence, using the standard equation of a plane from Sheet 1, we have that $(\mathbf{r}-\mathbf{a}) \cdot ((\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})) = 0$ is indeed the equation of the plane.

The result from Question 3 gives $(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}) = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$. Hence the equation can be rewritten as:

$$(\mathbf{r}-\mathbf{a}) \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = 0.$$

Using the distributive property of the vector product, and rearranging, we have:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \cdot (\mathbf{c} \times \mathbf{a}).$$

Finally, the terms $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ and $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{a})$ vanish, because $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{a}$ are orthogonal to $\mathbf{a}$ by definition.

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(b) We follow the procedure outline in part (a). Two vectors contained in the plane are $(1,2,3)-(1,1,1) = (0,1,2)$ and $(0,0,4)-(1,1,1) = (0,0,3)$. Taking their cross product, we have:

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}.$$

Hence the equation of the plane is:

$$\left(\mathbf{r} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) \cdot \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = 0.$$

7. You need to drill a hole in a piece of metal starting at a right angle to a flat surface passing through the points $A = (1,0,0)$, $B = (1,1,1)$ and $C = (0,2,0)$, with the hole emerging at the point $D = (2,1,0)$. How long a drill must you use and where (in the plane $ABC$) must you start drilling?

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➭ **Solution:** Two vectors contained in the piece of metal are $\overrightarrow{AB} = (1,1,1) - (1,0,0) = (0,1,1)$ and $\overrightarrow{AC} = (0,2,0) - (1,0,0) = (-1,2,0)$. Hence a vector orthogonal to the plane is:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$

To obtain the length of the drill, we must find the shortest distance from the point $D$ to the plane of metal. A vector from the plane to the point is $\overrightarrow{AD} = (2,1,0) - (1,0,0) = (1,1,0)$. The length of the projection onto the direction normal to the metal is:

$$\left| (1,1,0) \cdot \frac{(-2,-1,1)}{\sqrt{4+1+1}} \right| = \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}.$$

Hence, we must use a drill that is $\sqrt{3/2}$ units long.

To answer where in the plane must we start drilling, we must add to the point $A = (1,0,0)$ the projection of $\overrightarrow{AD}$ parallel to the plane. Subtracting the component orthogonal to the plane gives us the parallel component:

$$\overrightarrow{AD}_{\|} = (1,1,0) + \sqrt{\frac{3}{2}} \cdot \frac{(-2,-1,1)}{\sqrt{6}} = (1,1,0) + (-1,-1/2,1/2) = (0,1/2,1/2).$$

Adding this to the point $A$, we have $(1,0,0) + (0,1/2,1/2) = (1,1/2,1/2)$.

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8. Determine whether:

(a) the points $\mathbf{P}_1 = (0,0,2)$, $\mathbf{P}_2 = (0,1,3)$, $\mathbf{P}_3 = (1,2,3)$, $\mathbf{P}_4 = (2,3,4)$ are coplanar;

(b) the points $\mathbf{Q}_1 = (-2,1,1)$, $\mathbf{Q}_2 = (-1,2,2)$, $\mathbf{Q}_3 = (-3,3,2)$, $\mathbf{Q}_4 = (-2,4,3)$ are coplanar.

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➭ **Solution:** In each case, we construct the planes going through the first three points, then check if the fourth point lies in the plane. We have:

(a) Two vectors parallel to the plane containing the first three points are:

$$\mathbf{P}_2 - \mathbf{P}_1 = (0,1,3) - (0,0,2) = (0,1,1), \qquad \mathbf{P}_3 - \mathbf{P}_1 = (1,2,3) - (0,0,2) = (1,2,1).$$

Taking the cross product, a vector orthogonal to the plane is:

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

Hence, the equation of the plane is $(\mathbf{r} - (0,0,2)) \cdot (-1,1,-1) = 0$. Substituting $\mathbf{r} = \mathbf{P}_4 = (2,3,4)$, we have $(2,3,4) = (0,0,2) = (2,3,2)$, but $(2,3,2) \cdot (-1,1,-1) = -1 \neq 0$, hence these points are not coplanar.

(b) Two vectors parallel to the plane containing the first three points are:

$$\mathbf{Q}_2 - \mathbf{Q}_1 = (-1,2,2) - (-2,1,1) = (1,1,1), \qquad \mathbf{Q}_3 - \mathbf{Q}_1 = (-3,3,2) - (-1,2,2) = (-2,2,0).$$

Taking the cross product, a vector orthogonal to the plane is:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}.$$

Hence, the equation of the plane is $(\mathbf{r} - (-2,1,1)) \cdot (-1,-1,2) = 0$ (scaling the normal). Substituting $\mathbf{r} = \mathbf{Q}_4 = (-2,4,3)$, we have $(-2,4,3) - (-2,1,1) = (0,3,2)$, but $(0,3,2) \cdot (-1,-1,2) = 1 \neq 0$, hence these points are not coplanar.

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**Shortest distances**

9. *Without using a formula*, find the shortest distance between the lines $\mathbf{r}_1 = (1,0,1) + \lambda(2,-1,3)$ and $\mathbf{r}_2 = (0,1,-2) + \mu(1,0,2)$, justifying the steps you take. [*Sometimes, it is better to understand a method, than to quote a formula*.]

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**Solution:** If we wanted to connect the lines in the shortest way possible, we would draw a line that was *orthogonal* to both lines. Thus we would connect them along the direction:

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$

Now, let's pick any vector joining the two lines, say $(0,1,-2) - (1,0,1) = (-1,1,-3)$. If we project this vector along the orthogonal direction joining both lines, we will get the shortest distance between the two. We have:

$$(-1,1,-3) \cdot \frac{(-2,-1,1)}{\sqrt{4+1+1}} = -\frac{2}{\sqrt{6}}.$$

Taking the modulus, we obtain the shortest length $\sqrt{2/3}$.

10. So far, we have developed formulae for the shortest distance from points to lines, and from points to planes. Now, using the scalar and vector products, establish formulae for the following:

(a) the shortest distance from the line $\mathbf{r}_1 = \mathbf{v}_1 + \lambda\mathbf{w}_1$ to the line $\mathbf{r}_2 = \mathbf{v}_2 + \mu\mathbf{w}_2$; [*Hint: Take care when the lines are parallel!*]

(b) the shortest distance from the line $\mathbf{r} = \mathbf{v} + \lambda\mathbf{w}$ to the plane $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$;

(c) the shortest distance from the plane $(\mathbf{r}_1 - \mathbf{a}_1) \cdot \mathbf{b}_1 = 0$ to the plane $(\mathbf{r}_2 - \mathbf{a}_2) \cdot \mathbf{b}_2 = 0$.

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**Solution:** (a) Following the procedure we outlined in the previous question, we note that a direction orthogonal to both lines is $\mathbf{w}_1 \times \mathbf{w}_2$. An arbitrary vector joining the lines is $\mathbf{v}_2 - \mathbf{v}_1$. Projecting this vector in the direction orthogonal to both lines, we get the shortest distance:

$$\left| \frac{(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{w}_1 \times \mathbf{w}_2)}{|\mathbf{w}_1 \times \mathbf{w}_2|} \right|.$$

This is fine unless $|\mathbf{w}_1 \times \mathbf{w}_2| = 0$, when the lines are parallel. In this case, the lines both have direction $\mathbf{w} \equiv \mathbf{w}_1 = \mathbf{w}_2$. We then need the projection of $\mathbf{v}_2 - \mathbf{v}_1$ orthogonal to $\mathbf{w}$, which is simply given by:

$$|(\mathbf{v}_2 - \mathbf{v}_1) \times \hat{\mathbf{w}}| \, .$$

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(b) If $\mathbf{w}$ is not parallel to the plane, then the line and the plane must intersect, giving the shortest distance zero. This occurs if $\mathbf{w} \cdot \mathbf{b} \neq 0$.

In the case where $\mathbf{w} \cdot \mathbf{b} = 0$, then we need the projection of a vector $\mathbf{v} - \mathbf{a}$ perpendicular to the plane, which is just $|(\mathbf{v} - \mathbf{a}) \cdot \hat{\mathbf{b}}|$ (this agrees with our standard point-to-plane formula, because any point on the line is equally acceptable).

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(c) If the planes are not parallel, that is $\mathbf{b}_1, \mathbf{b}_2$ are not parallel, then the planes intersect, giving the shortest distance zero.

In the case where $\mathbf{b}_1, \mathbf{b}_2$ are parallel, then we can just take the projection of $\mathbf{a}_1 - \mathbf{a}_2$ parallel to $\mathbf{b}_1$, say, giving $|(\mathbf{a}_1 - \mathbf{a}_2) \cdot \hat{\mathbf{b}}_1|$ (again, this agrees with our standard point-to-plane formula, because any pair of points on the planes is equally acceptable).

**The vector triple product, and vector equations**

11.(a) By expanding in terms of the standard basis vectors, prove *Lagrange's formula* for the vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

Think of a way of remembering this formula off by heart - it is very useful!

(b) Hence, construct an example of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

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**Solution:** (a) Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$. Then:

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \left( \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right) \\
&= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix} \\
&= \begin{pmatrix} a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3 \\ a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1 \\ a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 \end{pmatrix} \\
&= \begin{pmatrix} b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3) \\ b_2(a_1c_1 + a_2c_2 + a_3c_3) - c_2(a_1b_1 + a_2b_2 + a_3b_3) \\ b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3(a_1b_1 + a_2b_2 + a_3b_3) \end{pmatrix} \\
&= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).
\end{aligned}$$

We can remember this formula using the phrase 'BACk of the CAB', which tells us which order the vectors come in.

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(b) An easy example can be constructed using the Cartesian unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. We have:

$$\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_2) = -\mathbf{e}_2,$$

but:

$$(\mathbf{e}_1 \times \mathbf{e}_1) \times \mathbf{e}_2 = \mathbf{0},$$

since $\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{0}$.

12. Prove the *Jacobi identity*, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$.

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➠ **Solution:** We have:

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) + \mathbf{c}(\mathbf{b} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) \\ &= 0, \end{aligned}$$

as required.

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13. Two vector operators, $P_{\hat{\mathbf{u}}} : \mathbb{R}^3 \to \mathbb{R}^3$ and $R_{\hat{\mathbf{u}}} : \mathbb{R}^3 \to \mathbb{R}^3$ are defined by $P_{\hat{\mathbf{u}}}(\mathbf{r}) = (\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}$ and $R_{\hat{\mathbf{u}}}(\mathbf{r}) = \hat{\mathbf{u}} \times (\mathbf{r} \times \hat{\mathbf{u}})$ respectively. Interpret these operators geometrically, and hence explain why $P_{\hat{\mathbf{u}}}(\mathbf{r}) + R_{\hat{\mathbf{u}}}(\mathbf{r}) = \mathbf{r}$ for all vectors $\mathbf{r}$. Also explain why $P_{\hat{\mathbf{u}}}^2 = P_{\hat{\mathbf{u}}}$ and $R_{\hat{\mathbf{u}}}^2 = R_{\hat{\mathbf{u}}}$.

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➠ **Solution:** The operator $P_{\hat{\mathbf{u}}}$ gives the projection of a vector in the $\hat{\mathbf{u}}$ direction. Using the vector triple product, we have:

$$R_{\hat{\mathbf{u}}}(\mathbf{r}) = \hat{\mathbf{u}} \times (\mathbf{r} \times \hat{\mathbf{u}}) = \mathbf{r} - (\hat{\mathbf{u}} \cdot \mathbf{r})\hat{\mathbf{u}},$$

hence this operator removes the component of a vector in the $\hat{\mathbf{u}}$ direction. Hence, it gives the projection of a vector perpendicular to the $\hat{\mathbf{u}}$ direction. This immediately implies that:

$$P_{\hat{\mathbf{u}}}(\mathbf{r}) + R_{\hat{\mathbf{u}}}(\mathbf{r}) = \mathbf{r}$$

for all vectors $\mathbf{r}$, as required.

It is also straightforward that $P_{\hat{\mathbf{u}}}^2 = P_{\hat{\mathbf{u}}}$, because it we apply the projection operator twice, after the first application, the resulting vector already points in the $\hat{\mathbf{u}}$ direction, so the second application does nothing. We can check this explicitly using some algebra:

$$P_{\hat{\mathbf{u}}}^2(\mathbf{r}) = P_{\hat{\mathbf{u}}}((\hat{\mathbf{u}} \cdot \mathbf{r})\hat{\mathbf{u}}) = (\hat{\mathbf{u}} \cdot \hat{\mathbf{u}})(\hat{\mathbf{u}} \cdot \mathbf{r})\hat{\mathbf{u}} = (\hat{\mathbf{u}} \cdot \mathbf{r})\hat{\mathbf{u}}.$$

The same is true for $R_{\hat{\mathbf{u}}}^2 = R_{\hat{\mathbf{u}}}$, because after one application of the projection operator, we are already pointing in a direction orthogonal to $\hat{\mathbf{u}}$.

14. Solve the following vector equations, and give geometric interpretations of their solutions:

(a) $\mathbf{a} \times \mathbf{r} + \lambda \mathbf{r} = \mathbf{c}$, where $\lambda \neq 0$, and $\mathbf{a}, \mathbf{c} \in \mathbb{R}^3$ are arbitrary 3-vectors;

(b) $\mathbf{r} \times \mathbf{a} = \mathbf{b}$, where $\mathbf{a} \in \mathbb{R}^3$ is an arbitrary non-zero 3-vector;

(c) $\mathbf{r} = \mathbf{a} + (\mathbf{b} \cdot \mathbf{r})\mathbf{c}$, where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ are arbitrary 3-vectors;

(d) $2\mathbf{r} + \hat{\mathbf{n}} \times \mathbf{r} + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{r})^2 = \mathbf{a}$, where $\hat{\mathbf{n}}$ is a unit vector, and $\hat{\mathbf{n}} \cdot \mathbf{a} = -1$.

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**Solution:**

(a) Begin by taking the scalar product of the equation with **a** (this hopefully gives us information parallel to **a**). We then have:

$$\lambda \mathbf{r} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{a},$$

giving us the component of **r** parallel to **a**. Now consider taking the vector product of the equation with **a** (this hopefully gives us information perpendicular to **a**). We then have:

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{r}) + \lambda \mathbf{a} \times \mathbf{r} = \mathbf{a} \times \mathbf{c} \qquad \Leftrightarrow \qquad \mathbf{a}(\mathbf{a} \cdot \mathbf{r}) - |\mathbf{a}|^2 \mathbf{r} + \lambda \mathbf{a} \times \mathbf{r} = \mathbf{a} \times \mathbf{c}.$$

To finish, we substitute for $\mathbf{a} \cdot \mathbf{r}$ using the expression we found earlier when computing the scalar product. We also substitute for $\mathbf{a} \times \mathbf{r}$ using the original equation, $\mathbf{a} \times \mathbf{r} = \mathbf{c} - \lambda \mathbf{r}$. This gives:

$$\frac{(\mathbf{a} \cdot \mathbf{c})\mathbf{a}}{\lambda} - |\mathbf{a}|^2 \mathbf{r} + \lambda \mathbf{c} - \lambda^2 \mathbf{r} = \mathbf{a} \times \mathbf{c}.$$

Rearranging for **r**, we have:

$$\mathbf{r} = \frac{(\mathbf{a} \cdot \mathbf{c})\mathbf{a} - \lambda \mathbf{a} \times \mathbf{c} + \lambda^2 \mathbf{c}}{\lambda(|\mathbf{a}|^2 + \lambda^2)}.$$

This is a single point.

(b) First, let's take the scalar product with **a** to give us information parallel to **a**. We find:

$$0 = \mathbf{a} \cdot \mathbf{b}.$$

Hence, we see the equation has no solutions unless $\mathbf{a} \cdot \mathbf{b}$. Next, we take the vector product with **a** to give us information perpendicular to **a**. We have:

$$\mathbf{a} \times (\mathbf{r} \times \mathbf{a}) = \mathbf{a} \times \mathbf{b} \qquad \Leftrightarrow \qquad |\mathbf{a}|^2 \mathbf{r} - \mathbf{a}(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a} \times \mathbf{b}.$$

Rearranging, we have:

$$\mathbf{r} = \frac{\mathbf{a}(\mathbf{a} \cdot \mathbf{r})}{|\mathbf{a}|^2} + \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}|^2}.$$

It appears we cannot get any more information on $\mathbf{a} \cdot \mathbf{r}$, because if we add any vector parallel to **a** to **r**, then this just gets annihilated. So the equation must have many solutions, of the form:

$$\mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}|^2} + \lambda \mathbf{a},$$

for any $\lambda \in \mathbb{R}$. We have so far only proved that solutions of this form are *necessary* (if we assume the equation, this is the form that the solutions must take). We must also prove that they are *sufficient*, by showing that these actually solve the equation in practice. We have:

$$\mathbf{r} \times \mathbf{a} = \frac{(\mathbf{a} \times \mathbf{b}) \times \mathbf{a}}{|\mathbf{a}|^2} = -\frac{\mathbf{a}(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{a}|^2} + \frac{|\mathbf{a}|^2 \mathbf{b}}{|\mathbf{a}|^2} = \mathbf{b},$$

provided that $\mathbf{a} \cdot \mathbf{b} = 0$. So we're done!

Summarising: the equation has no solutions if $\mathbf{a}\cdot\mathbf{b}\neq 0$, but it has many solutions of the form:

$$\mathbf{r}=\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}|^2}+\lambda\mathbf{a}$$

otherwise. This is a line through the point $(\mathbf{a}\times\mathbf{b})/|\mathbf{a}|^2$ parallel to the vector $\mathbf{a}$.

(c) Taking the scalar product with $\mathbf{b}$, we aim to get information parallel to $\mathbf{b}$:

$$\mathbf{r}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{b}+(\mathbf{r}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{b}). \tag{$*$}$$

Rearranging, we have:

$$\mathbf{r}\cdot\mathbf{b}=\frac{\mathbf{a}\cdot\mathbf{b}}{1-\mathbf{b}\cdot\mathbf{c}},$$

provided that $\mathbf{b}\cdot\mathbf{c}\neq 1$. Hence, if $\mathbf{b}\cdot\mathbf{c}$, we get the solution:

$$\mathbf{r}=\mathbf{a}+\frac{(\mathbf{a}\cdot\mathbf{b})\mathbf{c}}{1-\mathbf{b}\cdot\mathbf{c}},$$

which is just a single point.

On the other hand, if $\mathbf{b}\cdot\mathbf{c}=1$, we see that equation $(*)$ implies $\mathbf{a}\cdot\mathbf{b}=0$. Thus there are no solutions unless $\mathbf{a}\cdot\mathbf{b}=0$ too. We get no further information on $\mathbf{r}\cdot\mathbf{b}$, so we guess that this is a free parameter and the solution is of the form:

$$\mathbf{r}=\mathbf{a}+\lambda\mathbf{c}.$$

Certainly the solution is *necessarily* of this form, because the equation looks like this in the first place! We also must check it is *sufficient* by substituting back into the original equation. We have:

$$\mathbf{a}+(\mathbf{b}\cdot\mathbf{r})\mathbf{c}=\mathbf{a}+(\mathbf{b}\cdot\mathbf{a}+\lambda\mathbf{b}\cdot\mathbf{c})\mathbf{c}=\mathbf{a}+\lambda\mathbf{c}=\mathbf{r},$$

since $\mathbf{a}\cdot\mathbf{b}=0$ and $\mathbf{b}\cdot\mathbf{c}=1$. Thus this is indeed the general solution in this case.

Summarising: if $\mathbf{b}\cdot\mathbf{c}\neq 1$, the solution is a point $\mathbf{a}+(\mathbf{a}\cdot\mathbf{b})/(1-\mathbf{b}\cdot\mathbf{c})$; if $\mathbf{b}\cdot\mathbf{c}=1$ and $\mathbf{a}\cdot\mathbf{b}\neq 0$, there are no solutions; if $\mathbf{b}\cdot\mathbf{c}=1$ and $\mathbf{a}\cdot\mathbf{b}=0$, the solution is a line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{c}$.

(d) Take the scalar product of the equation with $\hat{\mathbf{n}}$ to get information parallel to $\hat{\mathbf{n}}$. Then:

$$2\mathbf{r}\cdot\hat{\mathbf{n}}+(\hat{\mathbf{n}}\cdot\mathbf{r})^2=\mathbf{a}\cdot\hat{\mathbf{n}}=-1 \qquad\Leftrightarrow\qquad (\hat{\mathbf{n}}\cdot\mathbf{r})^2+2\hat{\mathbf{n}}\cdot\mathbf{r}+1=0.$$

This is a quadratic equation for $\hat{\mathbf{n}}\cdot\mathbf{r}$; it has a repeated root:

$$\hat{\mathbf{n}}\cdot\mathbf{r}=-1.$$

Substituting this back into the original equation, we have:

$$2\mathbf{r}+\hat{\mathbf{n}}\times\mathbf{r}=\mathbf{a}-\hat{\mathbf{n}}. \tag{$**$}$$

Next, we take the vector product of the equation with $\hat{\mathbf{n}}$ to get information perpendicular to $\hat{\mathbf{n}}$. We have:

$$2\hat{\mathbf{n}}\times\mathbf{r}+\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{r})=\hat{\mathbf{n}}\times\mathbf{a} \qquad\Leftrightarrow\qquad 2\hat{\mathbf{n}}\times\mathbf{r}+\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})-\mathbf{r}=\hat{\mathbf{n}}\times\mathbf{a}.$$

Substituting for $\hat{\mathbf{n}}\cdot\mathbf{r}=-1$ and for $\hat{\mathbf{n}}\times\mathbf{r}$ using $(**)$, we have:

$$2\left(\mathbf{a}-\hat{\mathbf{n}}-2\mathbf{r}\right)-\hat{\mathbf{n}}-\mathbf{r}=\hat{\mathbf{n}}\times\mathbf{a}.$$

Rearranging for $\mathbf{r}$, we have:

$$\mathbf{r}=\frac{2\mathbf{a}-3\hat{\mathbf{n}}-\hat{\mathbf{n}}\times\mathbf{a}}{5}.$$

This is a single point.

### The scalar triple product, and non-orthonormal bases

15. Let **a**, **b**, **c** $\in \mathbb{R}^3$ be $3$-vectors.

(a) Give the definition of the *scalar triple product* $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of the $3$-vectors **a**, **b**, **c** $\in \mathbb{R}^3$. Hence show that the volume of the parallelepiped defined by the positions vectors **a**, **b**, **c** is $|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|$. Why is the modulus necessary?

(b) Using the relation between the scalar triple product and a parallelepiped, explain why:

(i) the scalar triple product is antisymmetric on odd permutations of its entries, and symmetric on even permutations of its entries;

(ii) the condition $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$ implies that **a**, **b**, **c** are not coplanar, and thus form a basis.

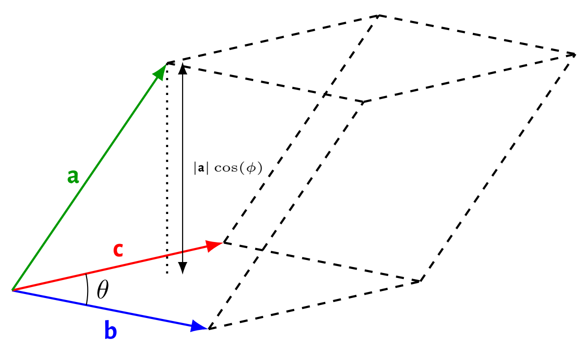
(c) Compute the volume of a parallelepiped defined by the three position vectors $\mathbf{a} = (0, \frac{1}{2}, \frac{1}{2})$, $\mathbf{b} = (\frac{1}{2}, 0, \frac{1}{2})$, $\mathbf{c} = (\frac{1}{2}, \frac{1}{2}, 0)$, and comment on whether these vectors form a basis.

---

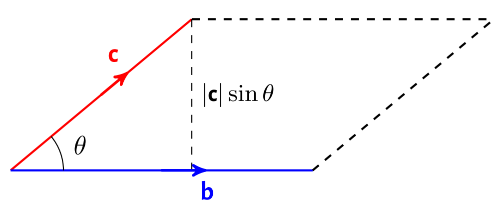
**Solution:** (a) The scalar triple product is defined by:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] := \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

If the vectors are arranged in a right-handed way, then **a** forms an acute angle $\phi$ with $\mathbf{b} \times \mathbf{c}$, so $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \equiv [\mathbf{a}, \mathbf{b}, \mathbf{c}] > 0$. Suppose that the vectors **b**, **c** make an angle $\theta$.




The height of the parallelogram formed by **a**, **b**, **c** is given by $|\mathbf{a}| \cos(\phi)$, as shown in the figure above. The area of the base can be computed by considering the figure below:




We see that the height of the parallelogram base is $|\mathbf{c}| \sin(\theta)$, so that its area is $|\mathbf{b}||\mathbf{c}| \sin(\theta)$, which is equal to the magnitude of $\mathbf{b} \times \mathbf{c}$. Therefore, the volume of the parallelepiped is $|\mathbf{a}||\mathbf{b}||\mathbf{c}| \sin(\theta) \cos(\phi)$, which is equal to $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \equiv [\mathbf{a}, \mathbf{b}, \mathbf{c}]$, as required.

In the case where **a**, **b**, **c** is left-handed, we have that **a** makes an *obtuse* angle $\phi$ with $\mathbf{b} \times \mathbf{c}$. We still have that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is equal in magnitude to the volume, but it now has a relative negative sign. Hence $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is the volume of the parallelepiped in general, explaining the need for the modulus.

(b) Recall from part (a) that the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is the signed volume of the parallelepiped formed by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$; it is positive if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is right-handed and negative if left-handed. Hence, we have:

(i) Swapping the arguments of $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ an even number of times retains the right-handedness of the parallelepiped; hence, the value is unchanged. Swapping the arguments of $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ an odd number of times changes the handedness of the parallelepiped to a left-handed orientation; hence, the value acquires a minus sign. The result follows.

(ii) Since $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is the signed volume of a parallelepiped, if it is non-zero, then the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ do not lie in the same plane (in this case, the volume degenerates to zero). Thus, they form a basis for three-dimensional space, as required.

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(c) The volume is:

$$\begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \times \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} -1/4 \\ 1/4 \\ 1/4 \end{pmatrix} = \frac{1}{4} \neq 0,$$

hence these vectors do indeed form a basis.

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16. Simplify the scalar triple products $(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{b}+\mathbf{c}) \times (\mathbf{c}+\mathbf{a})$ and $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$.

---

**Solution:** In the first case, we have:

$$(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{b}+\mathbf{c}) \times (\mathbf{c}+\mathbf{a}) = (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = 2\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \equiv 2[\mathbf{a}, \mathbf{b}, \mathbf{c}].$$

In the second case, we have:

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] &= (\mathbf{a} \times \mathbf{b}) \cdot [\mathbf{c}((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}) - \mathbf{a}((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c})] && \text{(Lagrange's formula)} \\ &= (\mathbf{a} \times \mathbf{b}) \cdot [\mathbf{c}[\mathbf{a}, \mathbf{b}, \mathbf{c}]] \\ &= [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2. \end{aligned}$$

17. Let $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form the vertices of a tetrahedron, with $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$. Write down conditions in terms of the scalar triple product for the vector $\mathbf{r}$ to lie inside the tetrahedron.

---

➻ **Solution:** First, observe that since $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$, we have that the vectors are right-handed.

Now, the vector $\mathbf{r}$ must lie on the correct side of each of the four planes bounding the tetrahedron. Beginning with the side spanned by $\mathbf{a}$, $\mathbf{b}$, the normal $\mathbf{a} \times \mathbf{b}$ points in the direction of the tetrahedron. Hence:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) > 0$$

is needed to ensure that $\mathbf{r}$ is on the correct side of the plane. The same is true for the other pairs (taking care to consider handedness), giving:

$$\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) > 0, \qquad \mathbf{r} \cdot (\mathbf{c} \times \mathbf{a}) > 0.$$

Finally, we consider the plane that goes through points with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. We saw in Question 3(a) that this plane can be written in the form:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

The normal $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ is outward-pointing from the tetrahedron, so the side contained inside the tetrahedron satisfies:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{c}) < \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

Overall then, the four conditions are:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) > 0, \qquad \mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) > 0, \qquad \mathbf{r} \cdot (\mathbf{c} \times \mathbf{a}) > 0,$$

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{c}) < \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

18. Let **a**, **b**, **c** $\in \mathbb{R}^3$ be $3$-vectors.

(a) If these vectors form an orthonormal basis, derive expressions for the coefficients $\alpha, \beta, \gamma$ in the formula $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$. Hence express $(2,3,4)$ in terms of the basis $\{(1,1,0),(1,-1,0),(0,0,1)\}$.

(b) If instead these vectors do *not* form an orthonormal basis, derive expressions for the coefficients $\alpha, \beta, \gamma$ in the formula $\mathbf{d} = \alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}$. [*Hint: consider scalar triple products.*] Hence express $(1,1,1)$ in terms of the basis $\{(1,2,1),(0,0,1),(2,-1,1)\}$.

(c) We define the *reciprocal vectors* to **a**, **b**, **c** to be the vectors:

$$\mathbf{A} = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a},\mathbf{b},\mathbf{c}]}, \qquad \mathbf{B} = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a},\mathbf{b},\mathbf{c}]}, \qquad \mathbf{C} = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a},\mathbf{b},\mathbf{c}]}.$$

Show that $\mathbf{A}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{b} = \mathbf{C}\cdot\mathbf{c} = 1$, and $\mathbf{A}\cdot\mathbf{b} = \mathbf{A}\cdot\mathbf{c} = \mathbf{B}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{c} = \mathbf{C}\cdot\mathbf{a} = \mathbf{C}\cdot\mathbf{b} = 0$. Hence, by comparing to part (b), explain how the reciprocal basis can be used to express a general vector **d** in terms of a non-orthonormal basis $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$. Compute the reciprocal basis to the basis $\{(1,2,1),(0,0,1),(2,-1,1)\}$.

---

**Solution:** (a) If the vectors are orthonormal, we have $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} = 0$ and $\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = 1$. Hence given:

$$\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c},$$

we can take the scalar product with the vectors **a**, **b**, **c** in turn to find the coefficients. Taking the scalar product with **a**, **b**, **c** in turn, we have:

$$\mathbf{a}\cdot\mathbf{d} = \alpha, \qquad \mathbf{b}\cdot\mathbf{d} = \beta, \qquad \mathbf{c}\cdot\mathbf{d}\gamma.$$

To express $(2,3,4)$ in terms of the basis $\{(1,1,0),(1,-1,0),(0,0,1)\}$, we cannot immediately use the formula above, because this is not an *orthonormal basis*. It is orthogonal however, since $(1,1,0)\cdot(1,-1,0) = 0$, $(1,1,0)\cdot(0,0,1) = 0$ and $(1,-1,0)\cdot(0,0,1) = 0$. To make it orthonormal, we consider the following related basis:

$$\{(1,1,0)/\sqrt{2},(1,-1,0)/\sqrt{2},(0,0,1)\}.$$

Using the formulae for the coefficients from above, we therefore have:

$$\begin{aligned}(2,3,4) &= \left((2,3,4)\cdot\frac{(1,1,0)}{\sqrt{2}}\right)\frac{(1,1,0)}{\sqrt{2}} + \left((2,3,4)\cdot\frac{(1,-1,0)}{\sqrt{2}}\right)\frac{(1,-1,0)}{\sqrt{2}} + ((2,3,4)\cdot(0,0,1))\,(0,0,1)\\ &= \frac{5}{2}(1,1,0) - \frac{1}{2}(1,-1,0) + 4(0,0,1),\end{aligned}$$

which is an expression for $(2,3,4)$ in terms of the desired basis.

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(b) Now, **a**, **b**, **c** form a *non*-orthonormal basis. We still would like to find the coefficients $\alpha, \beta, \gamma$ in:

$$\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c},$$

but it is no longer so easy. Can we use the same trick, to somehow take the scalar product with something, leaving only one coefficient leftover?

The answer is *yes*, if we dot with something perpendicular to two of the basis vectors. For example, to get the coefficient $\alpha$, we consider taking the scalar product with $\mathbf{b}\times\mathbf{c}$. Then:

$$\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c}) = \alpha\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \qquad \Leftrightarrow \qquad \alpha = \frac{\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}.$$

Similarly, we have:

$$\beta = \frac{\mathbf{d} \cdot (\mathbf{c} \times \mathbf{a})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}, \qquad \gamma = \frac{\mathbf{d} \cdot (\mathbf{a} \times \mathbf{b})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}.$$

To calculate the coefficients for the given example then, we first compute the cross products:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

The scalar triple product is:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 5.$$

Hence we have:

$$\alpha = \frac{1}{5} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \frac{3}{5},$$

$$\beta = \frac{1}{5} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix} = \frac{1}{5},$$

$$\gamma = \frac{1}{5} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{5}.$$

That is, we have:

$$(1,1,1) = \frac{3}{5}(1,2,1) + \frac{1}{5}(0,0,1) + \frac{1}{5}(2,-1,1).$$

---

(c) The properties $\mathbf{A} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{b} = \mathbf{C} \cdot \mathbf{c} = 1$ are obvious from the permutation properties of the scalar triple product. Similarly $\mathbf{A} \cdot \mathbf{b} = \mathbf{A} \cdot \mathbf{c} = \mathbf{B} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{c} = \mathbf{C} \cdot \mathbf{a} = \mathbf{C} \cdot \mathbf{b} = 0$ are all obvious from the fact that the scalar triple product vanishes when two of its arguments are equal.

We have shown that if $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$, then the coefficients $\alpha, \beta, \gamma$ can be written as:

$$\alpha = \mathbf{d} \cdot \mathbf{A}, \qquad \beta = \mathbf{d} \cdot \mathbf{B}, \qquad \gamma = \mathbf{d} \cdot \mathbf{C}.$$

Hence if we use the reciprocal basis $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, we can happily dot as if we were dealing with an orthonormal basis. An orthonormal basis has the special property that it is its own reciprocal basis.

For the given basis, we already computed in part (b) that:

$$\mathbf{A} = \frac{1}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \qquad \mathbf{B} = \frac{1}{5} \begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix}, \qquad \mathbf{C} = \frac{1}{5} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$