

Part IA: Mathematics for Natural Sciences B

Examples Sheet 15: Scalar and vector fields, conservative vector fields, and line integrals

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Questions marked with a (*) are difficult and should not be attempted at the expense of the other questions.

Gradient, directional derivatives, and tangent planes

1. Define the *gradient* ∇f of a scalar function. Calculate the gradient of each of the following three-dimensional functions:

$$(a) xyz, \quad (b) e^{-1/(x+y+z)}, \quad (c) e^{-\alpha^2 x^2 - \beta^2 y^2 - \gamma^2 z^2},$$

where α, β, γ are constants.

2. Let $f(r)$ be a function only of the spherical radial coordinate $r = \sqrt{x^2 + y^2 + z^2}$. Show that $\nabla f(r) = f'(r)\hat{\mathbf{x}}$, where $\mathbf{x} = (x, y, z)$ is the general position vector. Hence evaluate:

$$\nabla(r), \quad \nabla\left(\frac{1}{r}\right), \quad \nabla\left(\frac{e^{-r}}{r}\right).$$

(*) If you are studying Part IA Physics, what is the physical relevance of the second example? What is the physical relevance of the third example?

3. Let \mathbf{a} be a fixed vector in \mathbb{R}^n , and let $\mathbf{x} = (x_1, \dots, x_n)$ be the general position vector in n dimensions. Calculate the gradient of each of the following three-dimensional functions:

$$(a) \mathbf{a} \cdot \mathbf{x}, \quad (b) \mathbf{x} \cdot \mathbf{x}, \quad (c) |\mathbf{x}|^{-1}.$$

4. (a) Using the multivariable Taylor expansion, show that for a function $f(\mathbf{x}) = f(x_1, x_2, \dots, x_n)$ of n variables, we have:

$$f(\mathbf{x}) = f(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0) \cdot \nabla f + \dots,$$

where \mathbf{x}_0 is a fixed point, and the gradient ∇f is evaluated at \mathbf{x}_0 .

- (b) Hence, give the geometrical meaning of the vector $\hat{\mathbf{v}} \cdot \nabla f$ for $\hat{\mathbf{v}}$ a unit vector.
(c) Using part (b), explain why ∇f points in the direction of greatest increase of a scalar function.
5. (a) Find the rate of change of the function $f(x, y, z) = \ln(x^2 + y^2) + z$ at the point $(3, -4, 5)$ in the directions:
(i) $(3, -4, 0)$; (ii) $(1, 2, 0)$.
(b) Find the directional derivative of the function $f(x, y, z) = \frac{1}{2}[(x-1)^2 + (y-1)^2 + (z-1)^2]$ at the origin $(0, 0, 0)$ in the direction $(1, 1, 0)$.
6. Find the equations of the tangent planes to the following surfaces at the specified points:
 - $z = 3x^2y \sin(\pi x/2)$ at the point $x = y = 1$;
 - $xz + z^2 - xy^2 = 5$ at the point $(1, 1, 2)$.

If you placed a ball on the surface in part (a) at the point $x = 1, y = 1/2$, which way would it roll?

Vector fields and vector derivatives

7. (a) Sketch the three-dimensional vector fields $\mathbf{G}_1(\mathbf{x}) = k(\mathbf{x} - \mathbf{x}_0)$ and $\mathbf{G}_2(\mathbf{x}) = \mathbf{a} \times (\mathbf{x} - \mathbf{x}_0)$ where k is a real constant, and \mathbf{a} is a three-dimensional real vector.
(b) Define the *divergence* $\nabla \cdot \mathbf{F}$ and *curl* $\nabla \times \mathbf{F}$ of a vector field \mathbf{F} . Using the multivariable Taylor expansion, show that to first order, we may write:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{x}_0) + \frac{1}{3}(\nabla \cdot \mathbf{F})(\mathbf{x} - \mathbf{x}_0) + (\nabla \times \mathbf{F}) \times (\mathbf{x} - \mathbf{x}_0) + \dots,$$

where $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ are evaluated at \mathbf{x}_0 .

- (c) Considering both parts (a) and (b), give a geometrical interpretation of $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.
(d) Without performing any calculations, state the divergence and curl of the vector field $\mathbf{F}(\mathbf{x}) = 3\mathbf{x} + \mathbf{x} \times \hat{\mathbf{e}}_x$, where $\hat{\mathbf{e}}_x$ is a unit vector in the x -direction.
8. Find the divergence and curl of each of the following vector fields, where \mathbf{a}, \mathbf{b} are constant vectors:

(a) \mathbf{x} , (b) $\mathbf{a}(\mathbf{x} \cdot \mathbf{b})$, (c) $\mathbf{a} \times \mathbf{x}$, (d) $\mathbf{x}/|\mathbf{x}|^3$, (e) $(\mathbf{a} \cdot \mathbf{b})\mathbf{x} + \mathbf{a}$, (f) $\mathbf{a} \times (\mathbf{b} \times \mathbf{x})$.

(*) Do your expressions for divergence and curl in part (d) make sense at the origin, $\mathbf{0}$?

9. For each of the following pairs of vector fields, calculate $\nabla \cdot (\mathbf{F} \times \mathbf{G})$ and $\nabla \times (\mathbf{F} \times \mathbf{G})$:

- (a) $\mathbf{F} = (x, y, z)$, $\mathbf{G} = (y, -x, 0)$;
(b) $\mathbf{F} = (\sin(x), \sin(y), \sin(z))$, $\mathbf{G} = (\cos(x), \cos(y), \cos(z))$.

Verify that the identities:

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} + \mathbf{F} \cdot (\nabla \times \mathbf{G}),$$

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

hold in each case, where $(\mathbf{G} \cdot \nabla)\mathbf{F} = G_x \partial \mathbf{F} / \partial x + G_y \partial \mathbf{F} / \partial y + G_z \partial \mathbf{F} / \partial z$. [We will learn how to efficiently prove these identities using 'suffix notation' in Part IB Mathematics.]

Second-order derivatives

10. Define the *Laplacian* $\nabla^2 \phi$ of a scalar field ϕ . Evaluate the Laplacian of the scalar field $\phi = \frac{1}{2}(x^2 + y^2 + z^2)$.
11. Show that for any scalar field ϕ and any vector field \mathbf{F} , we have $\nabla \times (\nabla \phi) = \mathbf{0}$ and $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

Line integrals

12. Let $f(\mathbf{x})$ be a scalar field and let $\mathbf{F}(\mathbf{x})$ be a vector field. Suppose that $\gamma(t)$ is a curve. Give a summary of the steps required to evaluate the following types of line integrals:

$$\int_{\gamma} f(\mathbf{x}) |\mathbf{dx}|, \quad \int_{\gamma} \mathbf{F} \cdot d\mathbf{x}, \quad \int_{\gamma} \mathbf{F} \times d\mathbf{x}.$$

Which of the integrals evaluate to scalars, and which evaluate to vectors? (*) If you are doing Part IA Physics, suggest a use of each type of line integral.

13. By directly parametrising the curves, evaluate the following line integrals over the specified curves:

$$(a) \int_{C_1} (x^2 + y^2) ds, \quad (b) \int_{C_2} (x^2 + y^2) ds, \quad (c) \int_{C_3} xy ds;$$

where $ds = |\mathbf{dx}|$ is the infinitesimal arclength, and:

- C_1 is a straight line from $(0, 0)$ to $(1, 1)$ in two dimensions;
- C_2 is the unit circle centred on the origin in two dimensions;
- C_3 is a helix, $x(t) = \cos(6t)$, $y(t) = \sin(6t)$, $z(t) = 8t$ in three dimension, with $0 \leq t \leq \pi/12$.

Give a sketch of the paths of integration in each case.

14. By directly parametrising the curves, evaluate the following line integrals over the specified curves:

$$(a) \int_{C_1} \begin{pmatrix} y \\ -x \\ -1 \end{pmatrix} \cdot d\mathbf{x}, \quad (b) \int_{C_2} \begin{pmatrix} xy \\ x^2 + y \\ 0 \end{pmatrix} \cdot d\mathbf{x}, \quad (c) \int_{C_3} \begin{pmatrix} xy \\ x^2 + y \\ 0 \end{pmatrix} \cdot d\mathbf{x};$$

where:

- C_1 is the helix $x(t) = \cos(t)$, $y(t) = \sin(t)$, $z(t) = t$, with $0 \leq t \leq 2\pi$.
- C_2 is the planar curve $x(t) = t$, $y(t) = t^2$, $z(t) = 0$, with $0 \leq t \leq 1$;
- C_3 is the planar curve $x(t) = 2t - t^2$, $y(t) = t^4 - 4t^3 + 4t^2$, $z(t) = 0$, with $0 \leq t \leq 1$;

Give a sketch of the paths of integration in each case.

15. By directly parametrising the curve, evaluate each of the line integrals:

$$\int_C d\mathbf{x}, \quad \int_C \mathbf{x} \times d\mathbf{x}, \quad \int_C \mathbf{x} \times (\mathbf{x} \times d\mathbf{x}),$$

where C is the unit circle centred on the origin in the x - y plane, traversed anticlockwise about the z -axis. How does the result of the second integral relate to the vector area of the unit disk centred on the origin in the x - y plane?

Conservative vector fields

16. State what it means for a vector field \mathbf{F} to be (a) *conservative*; (b) *irrotational*. How are these conditions related?

17. Show that the following vector fields are conservative by finding potential functions in each case:

$$(a) \mathbf{F}_1 = (yz, xz, xy), \quad (b) \mathbf{F}_2 = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

Now, check directly that each of these vector fields is irrotational. (*) If you are doing Part IA Physics, what is the physical relevance of the second example?

18. Two vector fields are given by $\mathbf{F}(\mathbf{r}) = \mathbf{a} \times \mathbf{r}$ and $\mathbf{G} = f(|\mathbf{r}|)\mathbf{r}$, where \mathbf{a} is a constant vector, $\mathbf{r} = (x, y, z)$ is the general position vector, and f is an arbitrary scalar function. Which, if any, of these functions can be written as the gradient of a scalar function? For those that can, find an appropriate scalar function.

19. Show that \mathbf{F} is conservative if and only if $\mathbf{F} \cdot d\mathbf{x}$ is an exact differential, where $d\mathbf{x} = (dx, dy, dz)$ is an infinitesimal displacement vector.

Line integrals of conservative vector fields

20. State and prove the *gradient theorem* for line integrals. Using the gradient theorem, show that if \mathbf{F} is conservative, and C_1, C_2 are paths of integration with the same start and end points, then:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{x} = \int_{C_2} \mathbf{F} \cdot d\mathbf{x}.$$

Deduce also that the line integral of a conservative field around a closed loop is always zero.

21. (a) Without using the gradient theorem, evaluate the integral of the vector field:

$$\mathbf{F}(\mathbf{r}) = e^{-r^2} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $\mathbf{r} = (x, y, z)$ and $r^2 = x^2 + y^2 + z^2$, along the curve:

$$\mathbf{r}(t) = \begin{pmatrix} \cos(2\pi t^n) \\ \sin(2\pi t^n) \\ 1 \end{pmatrix},$$

for $0 \leq t \leq 1$, and where n is a fixed positive integer.

- (b) Now, show that \mathbf{F} is conservative. Hence, use the gradient theorem to verify your answer in part (a) is correct.

22. Consider the following line integrals:

$$\int_{\gamma_i} \mathbf{F} \cdot d\mathbf{x}, \quad \int_{\gamma_i} \mathbf{G} \cdot d\mathbf{x},$$

where:

$$\mathbf{F} = \begin{pmatrix} 4x - 2y - 2z \\ -2x + 2y + az \\ -2x + ay + 2z \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} y \cos(xy) - by + (b+c)z \\ x \cos(xy) + bx - bz \\ (c-b)x + by \end{pmatrix},$$

where a, b and c are real constants, and:

- γ_1 is the set of straight lines connecting the points $(0, 0, 0)$ to $(1, 1, 0)$, followed by $(1, 1, 0)$ to $(1, 1, 1)$;
- γ_2 is the curve from $(0, 0, 0)$ to $(1, 1, 1)$ on which the position vector is parametrised by $\mathbf{x}(t) = (t, t, t^2)$ for $0 \leq t \leq 1$.

For each value of a, b, c :

- (a) Evaluate each of the line integrals by directly parametrising the paths of integration.
- (b) Decide whether \mathbf{F}, \mathbf{G} are conservative vector fields, and relate this result to the calculation you performed in part (a).