Part IA: Mathematics for Natural Sciences
2015 Paper 1 (Unofficial) Mark Scheme

Section A

1. (a) Factorise the expression $x^3 - x^2 - x + 1$, and [1]

Solution: Spot that $x = 1$ is a solution, so we have:

$$x^3 - x^2 - x + 1 = (x-1)(x^2-1) = (x-1)(x-1)(x+1) = (x-1)^2(x+1).$$

[*1 mark.*]

(b) express its reciprocal as a sum of partial fractions. [1]

Solution: We have:

$$\frac{1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}.$$

Multiplying up, we have:

$$1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2.$$

Set $x = 1$ to see $B = \frac{1}{2}$. Set $x = -1$ to see $C = \frac{1}{4}$. Finally, set $x = 0$ to see $1 = -A + B + C$ which yields $A = B + C - 1 = -\frac{1}{4}$. Thus the partial fractions are:

$$\frac{1}{(x-1)^2(x+1)} = -\frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{1}{4(x+1)}.$$

[*1 mark for correct answer.*]

2. The line $y = mx$, where m is a positive constant, has only a single point of contact with the curve $y = e^x$. Calculate:

(a) the value of m, and [1]

(b) the point of contact. [1]

Solution: Considering the geometry, since $y = mx$ has only one point of contact, it must be a tangent line. We therefore require:

$$mx = e^x, \qquad m = \frac{dy}{dx} = e^x,$$

at the point of contact (x, y). Equating coefficients, we see that $mx = m$, and hence $x = 1$. Thus it follows that $m = e$ and the point of contact is $(1, e)$. [*1 mark for value of e, 1 mark for point of contact.*]

3. In the (x, y) plane sketch and label the locus defined by:

$$y^2 + x^2 - 2x = 3.$$

Solution: Completing the square, we have:

$$(x-1)^2 + y^2 = 4.$$

Hence this is a circle centred on $(1, 0)$ with radius 2. [*1 mark for correctly identifying circle, 1 mark for correct centre AND radius.*]

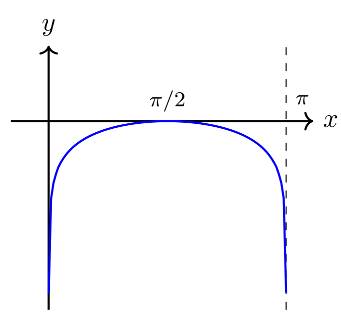
4. (a) Calculate the second derivative with respect to x of the real function $y = \ln(\sin(x))$. [1]

Solution: We have:
$$\frac{dy}{dx} = \frac{\cos(x)}{\sin(x)} = \cot(x), \qquad \frac{d^2y}{dx^2} = -\mathrm{cosec}^2(x).$$

[*1 mark.*]

(b) Sketch the function $y = \ln(\sin(x))$ in the range $0 < x < \pi$. [1]

Solution: In the range $0 < x < \pi/2$, $\sin(x)$ is increasing between 0 and 1, and $\ln(\sin(x))$ is therefore increasing from $-\infty$ to 0. In the range $\pi/2 < x < \pi$, $\sin(x)$ is decreasing from 1 to 0, and hence $\ln(\sin(x))$ is decreasing from 0 to $-\infty$. Thus the graph looks like:



5. Calculate the indefinite integral:

$$\int x^3 \cos(x^2)\, dx$$

and the definite integral: [2]

$$\int_0^{\sqrt{\pi/2}} x^3 \cos(x^2)\, dx.$$

Solution: We can obtain the indefinite integral by integration by parts:

$$\int x^3 \cos(x^2)\, dx = \frac{x^2 \sin(x^2)}{2} - \int x \sin(x^2)\, dx = \frac{x^2 \sin(x^2)}{2} + \frac{1}{2}\cos(x^2) + c.$$

[1 *mark*] The definite integral is therefore:

$$\frac{\pi \sin(\pi/2)}{4} + \frac{1}{2}\cos(\pi/2) - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}.$$

[1 *mark*]

6. Given $x = \cos(t)$ and $y = \sin(3t)$, where $0 \le t < 2\pi$,

(a) calculate the maximum value of y and the corresponding value of values of x, and [1]

Solution: The maximum value of y is 1, which occurs when $3t = (4n+1)\pi/2$ for n an integer, giving $t = (4n+1)\pi/6$. Thus:

$$x = \cos\left(\frac{(4n+1)\pi}{6}\right) = \cos\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right).$$

This is periodic, so if we just compute the values of $n = 0, 1, 2$, we will have all possible values of x. Thus the values are $\cos(\pi/6), \cos(5\pi/6), \cos(9\pi/6) = \cos(3\pi/2)$. These are $\sqrt{3}/2, -\sqrt{3}/2$ and 0. [*1 mark for value of y AND the three values of x.*]

(b) using the chain rule, calculate $\dfrac{dy}{dx}$ as a function of t. [1]

Solution: We have:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1} = 3\cos(3t)\,(-\sin(t))^{-1} = -\frac{3\cos(3t)}{\sin(t)}.$$

[1 *mark*.]

7. (a) Write down the first three non-zero terms of the expansion about $x = 0$ of $y = \sqrt{1 - 3x}$. [1]

Solution: We have:

$$(1-3x)^{1/2} = 1 - \frac{3x}{2} + \frac{(1/2)(-1/2)}{2}(-3x)^2 + \cdots = 1 - \frac{3x}{2} - \frac{9x}{8} + \cdots.$$

[*1 mark*.]

(b) For what range of values of x is this expansion valid? [1]

Solution: The expansion is valid when $|3x| < 1$, i.e. $|x| < 1/3$. [*1 mark*.]

8. Solve the following equation for x in the range $-\pi < x < \pi$, with $x \neq 0$: [2]

$$\cot(x) = \sin(2x).$$

Solution: Using some trigonometric identities, we have:

$$\frac{\cos(x)}{\sin(x)} = 2\sin(x)\cos(x).$$

Multiplying up by $\sin(x)$, we have:

$$0 = (2\sin^2(x) - 1)\cos(x).$$

Hence either $\cos(x) = 0$ or $0 = 2\sin^2(x) - 1 = \cos(2x)$. In the range $-\pi < x < \pi$, we have $\cos(x) = 0$ if and only if $x = -\pi/2, \pi/2$. We have $\cos(2x) = 0$ if and only if $2x = -\pi/2, \pi/2, -3\pi/2, 3\pi/2$, which gives:

$$x = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}.$$

All of these solutions are valid, since sine is non-zero for all of these values. [*2 marks for all correct solutions; 1 mark for some correct solutions.*]

9. (a) Find the gradient $\frac{dy}{dx}$ at the point where $x > 0$ and $y = 1$ for the curve defined by: [1]

$$y^2 + 2x^2 = 4.$$

Solution: Differentiating implicitly, we have:

$$2y\frac{dy}{dx} + 4x = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{2x}{y}.$$

When $y = 1$, we have $2x^2 = 3$, which gives $x = \sqrt{3/2}$ if $x > 0$. Hence the gradient is:

$$\frac{dy}{dx} = -\frac{2\sqrt{3/2}}{1} = -\sqrt{6}.$$

[1 *mark*.]

(b) Find the equation of the normal to the curve at this point. [1]

Solution: The normal is of the form $y = x/\sqrt{6} + c$. It goes through the point $(\sqrt{3/2}, 1)$, which gives:

$$1 = \sqrt{3/2}/\sqrt{6} + c = \frac{1}{2} + c \qquad \Rightarrow \qquad c = \frac{1}{2}.$$

Thus we have the normal $y = x/\sqrt{6} + 1/2$. [1 *mark*.]

10. Evaluate:

(a) $N_1 = 1 + 2 + 3 + \cdots + 999 + 1000$, [1]

Solution: The sum is:

$$N_1 = \frac{1000(1000+1)}{2} = 500 \cdot 1001 = 500,500.$$

[1 *mark*.]

(b) $N_2 = 2 + 4 + 8 + \cdots + 1024 + 2048$. [1]

Solution: The sum is a geometric series with first term 2, common ratio 2, and final term $2^{11} = 2048$. Thus:

$$N_2 = 2^1 + 2^2 + ... + 2^{11} = \frac{2(1-2^{11})}{1-2} = 2(2^{11}-1) = 2(2048-1) = 2(2047) = 4094.$$

[1 *mark*.]

Section B

11. (a) Let z be a complex number and let $w = \dfrac{z-i}{z+i}$.

(i) Evaluate w when $z = 0$, and when $z = 1$.

(ii) Let $z = \lambda$ where λ is real. Show that for any such z the corresponding w always has unit modulus. [4]

Solution: (i) When $z = 0$, we have:

$$w = \frac{-i}{i} = -1.$$

[1 *mark*.] When $z = 1$, we have:

$$w = \frac{1-i}{1+i} = \frac{(1-i)^2}{2} = \frac{1-2i-1}{2} = -i.$$

[1 *mark*.]

(ii) We have:

$$|w| = \left|\frac{\lambda - i}{\lambda + i}\right| = \frac{|\lambda - i|}{|\lambda + i|} = \frac{\sqrt{\lambda^2+1}}{\sqrt{\lambda^2+1}} = 1.$$

Hence w has unit modulus. [*1 mark for using modulus of quotient is quotient of moduli. 1 mark for convincing remaining rest of proof.*]

(b) Let $z(t) = re^{i\theta}$, where both $r(t)$ and $\theta(t)$ are functions of a real parameter t. Denote the derivatives with respect to the parameter t with a dot. Find $\dot{z}(t)$ and $\ddot{z}(t)$ in terms of $r(t)$ and $\theta(t)$ and their derivatives with respect to t. In each case write your answer in the form $(a(t) + ib(t))e^{i\theta(t)}$. [4]

Solution: We have:

$$\dot{z}(t) = \dot{r}(t)e^{i\theta} + ir\dot{\theta}e^{i\theta} = (\dot{r} + ir\dot{\theta})e^{i\theta}.$$

[2 *marks*] We also have:

$$\begin{aligned}\ddot{z}(t) &= \left(\ddot{r} + i\dot{r}\dot{\theta} + ir\ddot{\theta}\right)e^{i\theta} + \left(i\dot{r}\dot{\theta} - r\dot{\theta}^2\right)e^{i\theta} \\ &= \left(\ddot{r} - r\dot{\theta}^2 + i(2\dot{r}\dot{\theta} + r\ddot{\theta})\right)e^{i\theta}.\end{aligned}$$

[2 *marks*]

(c) Assume that a point moves in the Argand plane and has the position $z(t)$ at a time t. The velocity and acceleration are then $\dot{z}(t)$ and $\ddot{z}(t)$, respectively.

(i) By simplifying the equation $|z = 4| = 2|z-1|$ identify this locus and sketch it in the Argand plane. [2]

(ii) Assume that the point's path $z(t)$ is on the locus $|z-4| = 2|z-1|$. For an arbitrary position on this locus, sketch a vector representing the velocity of the point moving in the anticlockwise direction on the locus. What can you deduce about $\dot{r}$ and $\ddot{r}$? Write down expressions for $\dot{z}(t)$ and $\ddot{z}(t)$, as derived in (b), for this particular motion. [5]

(iii) Assume, instead, that the point moves in the Argand plane with an arbitrary path $z(t)$. Write down the radial and transverse components of the velocity and acceleration with respect to the origin. (The transverse velocity is perpendicular to the line joining the origin to the moving point.) [*Hint: Think of the significance of the two vectors in the Argand plane which can be represented by* $e^{i\theta}$ *and* $ie^{i\theta}$.] [5]

Solution: (i) Let $z = x + iy$. Then $|z-4| = 2|z-1|$ can be reduced to:

$$(x-4)^2 + y^2 = 4(x-1)^2 + 4y^2.$$

Rearranging, we have:

$$0 = 4x^2 - 8x + 4 + 4y^2 - x^2 + 8x - 16 - y^2 = 3x^2 + 3y^2 - 12,$$

which can be simplified to $x^2 + y^2 = 4$. Hence this is the equation of a circle with centre 0 in the complex plane and radius 2. [2 *marks; 1 mark for partial progress, 1 mark for correct locus.*]

(ii) The velocity is always tangential to the circle, so any tangential arrow is appropriate. [1 *mark for appropriate sketch*.] We can therefore deduce that $\dot{r} = 0$ and $\ddot{r} = 0$, since the distance from the origin is not changing as the particle moves around the circle. [2 *marks for stating this.*] For this motion, we have:

$$\dot{z}(t) = 2i\dot{\theta}e^{i\theta}, \qquad \ddot{z}(t) = \left(-2\dot{\theta}^2 + 2i\ddot{\theta}\right)e^{i\theta},$$

where we simply set $r = 2$ and $\dot{r} = \ddot{r} = 0$ everywhere. [2 *marks for this.*]

(iii) The vector $e^{i\theta}$ is a radial unit vector, whilst $ie^{i\theta}$ is a radial unit vector rotated by $\pi/2$. Hence $e^{i\theta}$ is the radial direction, and $ie^{i\theta}$ is the transverse direction. [1 *mark for identifying this.*] It follows that the radial component of velocity and the transverse component of velocity are, respectively,

$$\dot{r}, \qquad r\dot{\theta}.$$

[2 *marks, one each.*] The radial component of acceleration and the transverse component of acceleration are, respectively,

$$\ddot{r} - r\dot{\theta}^2, \qquad 2\dot{r}\dot{\theta} + r\ddot{\theta}.$$

[2 *marks, one each.*]

12. (a) Determine whether the following differential forms are exact. For each one that is exact, find a function f such that the differential form is equal to df.

(i) $\exp(x+y)dx + \exp(x+y)dy$,

(ii) $\sin(x)\sin(y)dx + \cos(x)\cos(y)dy$,

(iii) $2xy^3z^4dx + 3x^2y^2z^4dy + 4x^2y^3z^3dz$. [9]

Solution: A differential form $Pdx + Qdy$ is exact if and only if $\partial P/\partial y = \partial Q/\partial x$. [*1 mark for knowing this.*]

(i) We have:

$$\frac{\partial}{\partial y}\exp(x+y) = \exp(x+y), \qquad \frac{\partial}{\partial x}\exp(x+y) = \exp(x+y),$$

so this differential form is exact. [*1 mark*] A potential is $f = \exp(x+y)$, by inspection. [*2 marks*]

(ii) We have:

$$\frac{\partial}{\partial y}\sin(x)\sin(y) = \sin(x)\cos(y), \qquad \frac{\partial}{\partial x}\cos(x)\cos(y) = -\sin(x)\cos(y).$$

Hence this form is not exact. [*2 marks*]

(iii) This is a three-dimensional form, so the standard P, Q test fails here. Instead, by inspection note that the form is exact, with:

$$f = x^2y^3z^4 + c$$

[*3 marks*] You might wonder if there is a test that works here too. Suppose that we are considering a form:

$$Pdx + Qdy + Udz.$$

Then we require some function f satisfying:

$$\frac{\partial f}{\partial x} = P, \qquad \frac{\partial f}{\partial y} = Q, \qquad \frac{\partial f}{\partial z} = U.$$

Symmetry of mixed partial derivatives then implies:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}, \qquad \text{etc.}$$

This gives rise to three conditions:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial U}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial U}{\partial y},$$

which are actually necessary and sufficient for a three-dimensional differential form to be exact (on a simply-connected domain, anyway).

(b) Find and classify all the stationary points of the function:

$$g(x,y) = 1 - \cos(x) + \frac{1}{2}y^2,$$

and calculate the stationary values of g.

Sketch the contours of $g(x,y)$ in the region $-2\pi < x < 2\pi$, $-3 < y < 3$, paying particular attention to any contour lines that pass through the stationary points, and labelling the important features of the plot. [11]

Solution: Taking the gradient, we have:

$$\nabla g = (\sin(x), y)\,.$$

[*1 mark*] Hence the stationary points occur at $\sin(x) = 0$ and $y = 0$. Thus $x = n\pi$ for n an integer. The stationary points are therefore $(n\pi, 0)$ for each value of n an integer. [*1 mark*] The corresponding stationary values are $g(n\pi, 0) = 1 - (-1)^n$. [*1 mark*]

For each fixed x, the graph looks like a positive parabola in the y-direction. Hence the graph increases in the y-direction away from the line $y = 0$, i.e. away from the x-axis. On the other hand, in the x-direction, for each fixed y, the graph looks like a negative cosine graph.

This implies that when cosine has a minimum, the function $g(x,y)$ has a saddle. This implies that $(n\pi, 0)$ is a saddle for each n odd. On the other hand, when cosine has a maximum, the function $g(x,y)$ has a minimum (since negative cosine has a minimum). This implies that $(n\pi, 0)$ is a minimum for each n even. [*4 marks for any sort of analysis that concludes that $(n\pi, 0)$ are minima for n even, $(n\pi, 0)$ are saddles for n odd.*]

We are also asked to pay *special* attention to the contour lines going through the stationary points, so let's do a good job of this. Near $(2n\pi, 0)$, we may approximate $g(x,y)$ by Taylor expansion:

$$\begin{aligned} g(x,y) &= 1 - \cos(x) + \frac{1}{2}y^2 \\ &= 1 - \cos\left(x - 2n\pi + 2n\pi\right) + \frac{1}{2}y^2 \\ &= 1 - \cos\left(x - 2n\pi\right) + \frac{1}{2}y^2 \\ &= 1 - \left(1 - \frac{(x-2n\pi)^2}{2} + \cdots\right) + \frac{1}{2}y^2 \\ &= \frac{(x-2n\pi)^2}{2} + \frac{1}{2}y^2 + \cdots\,. \end{aligned}$$

Hence to a quadratic approximation, the contours about the minima are exactly *circles*. [*1 mark for this analysis*]

Near $((2n+1)\pi, 0)$, we can similar approximate $g(x, y)$ by Taylor expansion:

$$\begin{aligned} g(x,y) &= 1 - \cos\left(x - (2n+1)\pi + (2n+1)\pi\right) + \frac{1}{2}y^2 \\ &= 1 + \cos\left(x - (2n+1)\pi\right) + \frac{1}{2}y^2 \\ &= 2 - \frac{(x-(2n+1)\pi)^2}{2} + \frac{1}{2}y^2. \end{aligned}$$

Hence the constant contours in the vicinity of $((2n+1)\pi, 0)$ look like *square hyperbolae*, $(x-(2n+1)\pi)^2 - y^2 =$ *constant*, and through the stationary points, they look like orthogonal lines $x - (2n+1)\pi = \pm y$. [*2 marks for this analysis*]

Finally, note that for very large values of (x, y), the function looks like $g(x, y) = \frac{1}{2}y^2$, so the contour lines become $y \approx C$. Thus the contours are essentially horizontal for large values of y. [*1 mark*]

Putting this all together, we get a sketch like the one below.

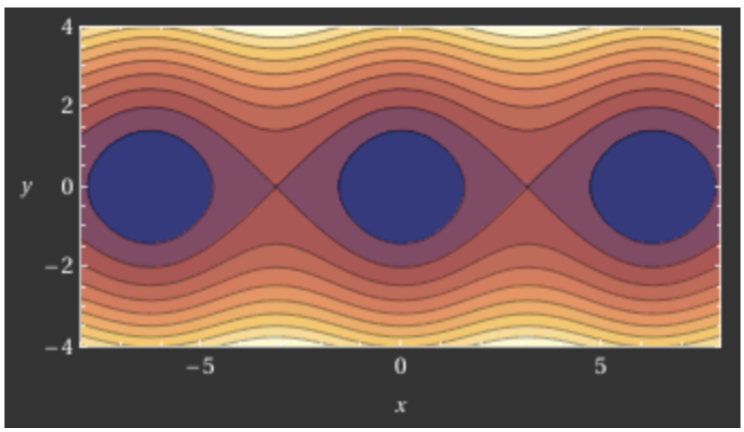


Figure 1: *The contour plot, courtesy of WolramAlpha.*

13. The function $u(x,t)$ satisfies the partial differential equation:

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2} \tag{†}$$

in $-\infty < x < \infty$, where λ is a positive constant.

(a) Given that equation (†) has a solution of the form:

$$u(x,t) = (t+a)^{-1/2} v(y)$$

valid for $t > -a$, where:

$$y = (t+a)^{-1/2}(x+b),$$

and a and b are arbitrary constants, show that the function $v(y)$ satisfies the ordinary differential equation: [6]

$$-\frac{1}{2}\left(v + y\frac{dv}{dy}\right) = \lambda \frac{d^2 v}{dy^2}.$$

Solution: By the chain rule, we have:

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{1}{2}(t+a)^{-3/2}v(y) + (t+a)^{-1/2}v'(y)\frac{\partial y}{\partial t} \\ &= -\frac{1}{2}(t+a)^{-3/2}v(y) - \frac{1}{2}(t+a)^{-1/2}\cdot(t+a)^{-3/2}(x+b)v'(y).\end{aligned}$$

[*Up to 3 marks for correct $\partial u/\partial t$ calculation.*] Similarly, we have:

$$\begin{aligned}\frac{\partial u}{\partial x} &= (t+a)^{-1/2}v'(y)\frac{\partial y}{\partial x} = (t+a)^{-1}v'(y) \\ \frac{\partial^2 u}{\partial x^2} &= (t+a)^{-1}v''(y)\frac{\partial y}{\partial x} = (t+a)^{-3/2}v''(y).\end{aligned}$$

[*Up to 2 marks for correct $\partial u/\partial x$ calculation.*] Equating these expressions, we have:

$$-\frac{1}{2}\left(v + (t+a)^{-1/2}(x+b)v'\right) = \lambda v'' \qquad \Rightarrow \qquad -\frac{1}{2}\left(v + y\frac{dv}{dy}\right) = \lambda\frac{d^2 v}{dy^2},$$

as required. [*1 mark for equating and simplifying to required form.*]

(b) Verify that this ordinary differential equation has a solution of the form:

$$v(y) = \exp\left(-cy^2\right)$$

if the constant c is chosen appropriately. Write the corresponding function $u(x,t)$ explicitly. [4]

Solution: With this $v(y)$, we have:

$$v'(y) = -2cy\exp\left(-cy^2\right), \qquad \Rightarrow \qquad v''(y) = \left(-2c + 4c^2y^2\right)\exp\left(-cy^2\right).$$

[*1 mark for correct first derivative, 1 mark for correct second derivative.*] Hence we have:

$$\lambda v'' + \frac{1}{2}\left(v + yv'\right) = \lambda(4c^2y^2 - 2c)e^{-cy^2} + \frac{1}{2}\left(e^{-cy^2} - 2cy^2e^{-cy^2}\right)$$

$$= \left(4\lambda c^2 - c\right)y^2e^{-cy^2} + \left(\frac{1}{2} - 2\lambda c\right).$$

Hence we see that if both $4\lambda c^2 = c$ and $4\lambda c = 1$, the equation will be satisfied. [*1 mark for deriving these conditions by substitution.*] If $c = 0$, the second equation is not satisfied, so in fact both equations are identical. We see that we should choose $c = 1/4\lambda$ (note λ is a positive constant, so is non-zero). The corresponding solution is:

$$u(x,t) = \frac{1}{\sqrt{t+a}}\exp\left(-\frac{(x+b)^2}{4\lambda(t+a)}\right).$$

[*1 mark for solving equations correctly, and obtaining correct final answer, written in explicit form.*]

(c) Which properties of equation (†) allow the principle of superposition to be applied? Find (in terms of x, t and λ) the solution of equation (†) for $t > 0$ given the initial condition:

$$u(x,0) = \exp\left(-(x+1)^2\right) + \exp\left(-(x-1)^2\right).$$

For what range of $t < 0$ is the solution also valid? [10]

Solution: The equation (†) is a *linear* equation which allows the principle of superposition to be applied. In more detail, if u_1, u_2 are both solutions of (†), then for constants α, β we have:

$$\frac{\partial}{\partial t}(\alpha u_1 + \beta u_2) = \alpha\frac{\partial u_1}{\partial t} + \beta\frac{\partial u_2}{\partial t} = \alpha\lambda\frac{\partial^2 u_1}{\partial x^2} + \beta\lambda\frac{\partial^2 u_2}{\partial x^2} = \lambda\frac{\partial^2}{\partial x^2}\left(\alpha u_1 + \beta u_2\right).$$

Hence $\alpha u_1 + \beta u_2$ additionally solves the equation. [*1 mark for saying linear equation. 1 mark for explaining what this means, and showing that linear combinations solve the equation.*]

From the previous part, the solution:

$$u(x,t) = \frac{1}{\sqrt{t+a}}\exp\left(-\frac{(x+b)^2}{4\lambda(t+a)}\right)$$

is initially given by:

$$u(x,0) = \frac{1}{\sqrt{a}}\exp\left(-\frac{(x+b)^2}{4\lambda a}\right).$$

Thus we see that if we choose $b = 1$ and $a = 1/4\lambda$, then:

$$u(x,0) = \sqrt{4\lambda}\exp\left(-(x+1)^2\right) \qquad \Rightarrow \qquad \frac{u(x,0)}{\sqrt{4\lambda}} = \exp\left(-(x+1)^2\right).$$

[*Up to 2 marks for finding a, b appropriately in this case.*] Similarly, we can obtain a solution which is initially $\exp(-(x-1)^2)$ by choosing $b=-1$ and $a = 1/4\lambda$. [*Up to 2 marks for finding a, b appropriately in this case.*] In particular, this implies that the solution with the given initial data is given by:

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\lambda}\sqrt{t+1/4\lambda}} \exp\left(-\frac{(x+1)^2}{4\lambda(t+1/4\lambda)}\right) + \frac{1}{\sqrt{4\lambda}\sqrt{t+1/4\lambda}} \exp\left(-\frac{(x-1)^2}{4\lambda(t+1/4\lambda)}\right) \\
&= \frac{1}{\sqrt{4\lambda t+1}} \exp\left(-\frac{(x+1)^2}{4\lambda t+1}\right) + \frac{1}{\sqrt{4\lambda t+1}} \exp\left(-\frac{(x-1)^2}{4\lambda t+1}\right).
\end{aligned}
$$

[*Up to 2marks for stating solution correctly, using superposition principle.*] This solution is still valid for $t < 0$ provided that the arguments of the square roots do not become non-positive. Thus we require $4\lambda t+1 > 0$, or equivalently, $t > -1/4\lambda$. [*1 mark for noticing solution is still valid if arguments of square roots do not become non-positive. 1 mark for deriving correct condition.*]

14. A curve C is defined in terms of the parameter t as:

$$x = a(t - \sin(t)), \qquad y = a(1 - \cos(t)),$$

where a is a positive constant and $0 < t < 2\pi$.

(a) Determine $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ as functions of t. [3]

Solution: We have:

$$\frac{dx}{dt} = a(1 - \cos(t)), \qquad \frac{dy}{dt} = a\sin(t),$$

[*1 mark for both derivatives correct.*] so:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{a\sin(t)}{a(1-\cos(t))} = \frac{\sin(t)}{1-\cos(t)} = \frac{2\sin(\frac{1}{2}t)\cos(\frac{1}{2}t)}{2\sin^2(\frac{1}{2}t)} = \cot\left(\frac{1}{2}t\right).$$

[*1 mark for attempting to divide expressions, 1 mark for correct division (could be unsimplified).*]

(b) Sketch the curve in the (x, y) plane. [4]

Solution: This curve is an example of a *cycloid*, which is the shape that is traced out by a particle stuck to the surface of a wheel as it moves along the ground (this is because a linear x-motion is added to a circular motion). We can get the sketch like this, or we can do a more careful analysis.

For example, note that there are stationary points if and only if $\cot(t/2) = 0$, which occurs precisely when $t/2 = \pi/2 + n\pi$, i.e. $t = \pi + 2n\pi$. Thus there is exactly one stationary point at $t = \pi$, which has corresponding x-coordinate $x = a\pi$, and y-coordinate $y = 2a$. Otherwise, the curve is increasing on $0 < t < \pi/2$, and the curve is decreasing on $\pi/2 < t < \pi$ (by looking at dy/dx). This corresponds to the curve increasing on the domain $0 < x < a\pi$ and decreasing on $a\pi < x < 2a\pi$.

We also observe that the graph is symmetric about the line $x = a\pi$. This is because if we replace $t \mapsto 2\pi - t$, we have $x = a(2\pi - t + \sin(t))$, $y = a(1 - \cos(t))$, which gives $x = 2\pi a - (t - \sin(t))$. So the (x, y) coordinates are related via $(x, y) \mapsto (2\pi a - x, y)$. This implies that points to the left of πa get mapped to the right of πa, and vice-versa, giving the symmetry about $x = a\pi$. The final sketch is given below.

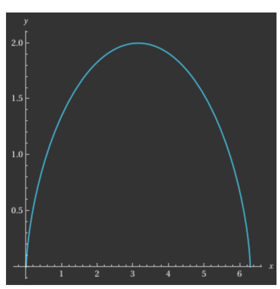


Figure 2: *The final plot for $a = 1$, courtesy of WolframAlpha.*

[*Up to 2 marks for correct gradient (increasing/decreasing/stationary point) analysis. 1 mark for spotting symmetry. 1 mark for special values (correct start and end point, etc).*]

(c) Find the point (x, y) on the curve at which the tangent to the curve has slope $\sqrt{3}$. [3]

Solution: Here, we require $\cot(\frac{1}{2}t) = \sqrt{3}$, which equivalently gives $\tan(\frac{1}{2}t) = 1/\sqrt{3}$. Hence $\frac{1}{2}t = \pi/6 + n\pi$, and thus $t = \pi/3 + 2n\pi/$. Thus the unique point with gradient $\sqrt{3}$ is the point with $t = \pi/3$ (since t is in the range 0 to 2π). [*Up to 2 marks for obtaining correct t-value.*] As a result, we have:

$$(x, y) = \left(a\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right), \frac{a}{2}\right).$$

[*1 mark for obtaining correct (x, y) values.*]

(d) Determine the area between the curve and the x axis. [5]

[*Hint: You may wish to express the area as an integral with respect to t.*]

Solution: The area can be expressed as:

$$\int_0^{2\pi a} y(x)dx = \int_0^{2\pi} y(x(t))\frac{dx}{dt}dt = \int_0^{2\pi} a^2(1-\cos(t))^2\,dt$$

$$= a^2\int_0^{2\pi}\left(1 - 2\cos(t) + \frac{1}{2} + \frac{1}{2}\cos(2t)\right)\,dt = \frac{3}{2}a^2\cdot(2\pi) = 3\pi a^2.$$

[*1 mark for correctly identifying region. 1 mark for correctly parametrising integral in terms of t. 1 mark for completely correct 1-variable integral in t. 1 mark for evaluation of integral. 1 mark for fully correct final answer.*]

(e) Determine the length of the curve. [5]

Solution: The infinitesimal arclength of the curve is given by $\sqrt{dx^2 + dy^2}$ for a small section of the curve. Noting:

$$\sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt,$$

[*1 mark for knowing this.*] we see that the required arclength is given by:

$$\int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi}\sqrt{a^2(1-\cos(t))^2 + a^2\sin^2(t)}\,dt = a\int_0^{2\pi}\sqrt{2-2\cos(t)}\,dt.$$

[*1 mark for obtaining this integral.*] Now, spot that $2 - 2\cos(t) = 4\sin^2(\frac{1}{2}t)$, so we have:

$$2a\int_0^{2\pi}\left|\sin\left(\frac{1}{2}t\right)\right|\,dt = 2a\left[-2\cos(t/2)\right]_0^{2\pi} = 2a\,(2 - -2) = 8a.$$

[*1 mark for simplification with trigonometric identity, 1 mark for evaluating integral, 1 mark for correct final answer.*] Note that we need to be careful here; we are taking the square root of sine squared so need the modulus, but fortunately $\sin(t/2)$ is positive in the range $t \in [0, 2\pi]$.

15. (a) State Taylor's theorem for the expansion about $x = a$ of a function that is differentiable n times, giving the first three terms explicitly, together with an expression for the remainder term R_n after n terms. [4]

Solution: Taylor's theorem states that for an n-times differentiable function, we have:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n,$$

[*1 mark for correct first three terms, 1 mark for correct general term.*] where the remainder is given by:

$$R_n = \frac{(x-a)^n}{n!}f^{(n)}(\xi)$$

for some ξ in the interval between x and a. [*Up to 2 marks for correct remainder.*]

(b) Find, by any method, the first three non-zero terms in the expansion about $x = 2$ of the function: [6]

$$\cosh(\sqrt{x}).$$

Solution: Note that:

$$\sqrt{x} = \sqrt{2 + (x-2)} = \sqrt{2}\left(1 + \frac{x-2}{2}\right)^{1/2}.$$

Expanding using the binomial theorem, we have:

$$\sqrt{2}\left(1 + \frac{x-2}{4} - \frac{(x-2)^2}{32} + \cdots\right).$$

Now if we are expanding around $x = 2$, we are expanding hyperbolic cosine around $\sqrt{2}$. We have:

$$\begin{aligned}\cosh(y) &= \cosh(\sqrt{2} + y - \sqrt{2}) = \cosh(\sqrt{2})\cosh(y - \sqrt{2}) + \sinh(\sqrt{2})\sinh(y - \sqrt{2}) \\ &= \cosh(\sqrt{2})\left(1 + \frac{1}{2}(y - \sqrt{2})^2 + \cdots\right) + \sinh(\sqrt{2})\left((y - \sqrt{2}) + \cdots\right).\end{aligned}$$

Substituting $y - \sqrt{2}$ from above, we have:

$$\cosh(\sqrt{x}) = \cosh(\sqrt{2}) + \frac{\sqrt{2}\sinh(\sqrt{2})}{4}(x-2) + \left(\frac{\cosh(\sqrt{2})}{16} - \frac{\sqrt{2}\sinh(\sqrt{2})}{32}\right)(x-2)^2 + \cdots$$

[*1 mark for correct first term, 2 marks for correct second term, 3 marks for correct third term, obtained via any method (may use repeated differentiation.*]

(c) Find, by any method, the first three non-zero terms in the expansions about $x = 0$ of:

(i) $\dfrac{\sin(x)}{(1+x)^2}$, [4]

(ii) $\dfrac{x\sin(x)}{\ln(1+x^2)}$, [6]

[You may quote standard power series without proof.]

Solution: We use standard Taylor series. We have:

$$\sin(x)(1+x)^{-2} = \left(x - \frac{x^3}{3!} + \cdots\right)\left(1 - 2x + 3x^2 + \cdots\right)$$
$$= x - 2x^2 + \left(3 - \frac{1}{3!}\right)x^3 + \cdots$$
$$= x - 2x^2 + \frac{17}{6}x^3 + \cdots.$$

[*1 mark for knowing* $\sin(x)$ *power series, 1 mark for knowing binomial series, 1 mark for attempt to multiply, 1 mark for completely correct final answer.*] Similarly, we have:

$$x\sin(x)\left(\ln(1+x^2)\right)^{-1} = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} + \cdots\right)^{-1}$$
$$= \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{3} + \cdots\right)^{-1}$$
$$= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)\left(1 + \left(\frac{x^2}{2} - \frac{x^4}{3} + \cdots\right) + \left(\frac{x^2}{2} - \frac{x^4}{3} + \cdots\right)^2 + \cdots\right)$$
$$= 1 + \left(\frac{1}{2} - \frac{1}{3!}\right)x^2 + \left(\frac{1}{4} - \frac{1}{3} - \frac{1}{2\cdot 3!} + \frac{1}{5!}\right)x^4 + \cdots$$
$$= 1 + \frac{1}{3}x^2 - \frac{19}{120}x^4 + \cdots.$$

[*1 mark for knowing* $\log(1+x)$ *power series, 1 mark for applying binomial expansion to* $\log(1+x)$ *power series, 1 mark for attempt to multiply everything, up to 2 marks for working, 1 mark for fully correct final answer.*]

16. (a) A box contains 3 white (W) and 4 black (B) balls. Balls are take randomly from the box without replacement. Using the notations $P(W_i)$ for the probability to withdraw a white ball at the ith withdrawal and $P(B_j)$ for the probability to withdraw a black ball at the jth withdrawal, find:

(i) $P(W_1), P(B_2), P(W_3)$, [4]

(ii) $P(B_1 \cap B_2)$, [1]

(iii) $P(B_2 \cap B_3), P(B_2|B_3), P(B_3|B_2)$, [5]

(iv) $P(B_1 \cup B_2)$, [2]

(v) the expectation value of the number of black balls $\mathbb{E}[N(B_3)]$, taken from the box after three withdrawals. [3]

(i) On the first withdrawal, there are 3 white balls of a total of 7 balls, hence the chance of drawing a white ball is $P(W_1) = 3/7$. [*1 mark*]

For the second withdrawal, the probability of drawing a black ball is:

$$P(B_2) = P(B_2|W_1)P(W_1) + P(B_2|B_1)P(B_1) = \frac{4}{6} \cdot \frac{3}{7} + \frac{3}{6} \cdot \frac{4}{7} = \frac{2}{7} + \frac{2}{7} = \frac{4}{7},$$

where we have split into cases depending on whether a white ball was drawn first or a black ball. [*1 mark*]

For the third withdrawal, we have:

$$\begin{aligned} P(W_3) &= P(W_3 \cap B_2 \cap B_1) + P(W_3 \cap B_2 \cap W_1) + P(W_3 \cap W_2 \cap B_1) + P(W_3 \cap W_2 \cap W_1) \\ &= \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{3}{5} + \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{2}{5} + \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} + \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{1}{5} \\ &= \frac{6}{35} + \frac{4}{35} + \frac{4}{35} + \frac{1}{35} \\ &= \frac{15}{35} \\ &= \frac{3}{7}. \end{aligned}$$

splitting into the four cases of possible first two withdrawals: (black, black), (white, black), (black, white) and (white, white). [*2 marks*]

Curiously, we see that $P(W_1) = P(W_3)$ - why is this the case? This is because if we draw three balls, it doesn't matter which of them we call the 'first' and which we call the 'third', so the probability that the first is white is the same as the probability that the third is white. Similarly for $P(B_1) = P(B_2)$. In general, the probability of drawing x white balls in a total of n draws is given by a *hypergeometric distribution*.

(ii) The probability of drawing a black ball on the first draw, and on the second draw, is:

$$P(B_1 \cap B_2) = \frac{4}{7} \cdot \frac{3}{6} = \frac{2}{7}.$$

[*1 mark*]

(iii) We have:

$$P(B_2 \cap B_3) = P(B_2 \cap B_3|B_1)P(B_1) + P(B_2 \cap B_3|W_1)P(W_1)$$
$$= \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} + \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{3}{5} = \frac{4}{35} + \frac{6}{35} = \frac{10}{35} = \frac{2}{7}.$$

This is similar to the argument we made in (i), where $P(B_1 \cap B_2) = P(B_2 \cap B_3)$. [*2 marks*]

For the next two, we can use the definition of conditional probability:

$$P(B_2|B_3) = \frac{P(B_2 \cap B_3)}{P(B_3)} = \frac{2/7}{4/7} = \frac{1}{2}.$$

[*1 mark for knowing definition of conditional probability, 1 mark for correct answer.*] Similarly, we have:

$$P(B_3|B_2) = \frac{P(B_2 \cap B_3)}{P(B_2)} = \frac{2/7}{4/7} = \frac{1}{2}.$$

[*1 mark for knowing definition of conditional probability, 1 mark for correct answer.*]

(iv) For this part, we have:

$$P(B_1 \cup B_2) = P(B_1) + P(B_2) - P(B_1 \cap B_2) = \frac{4}{7} + \frac{4}{7} - \frac{2}{7} = \frac{6}{7}.$$

[*1 mark for using law for probability of intersection of events that are not mutually exclusive (or equivalent), 1 mark for final answer.*]

(v) To calculate the expected number of black balls in the first three draws, observe:

- The probability of getting no black balls is $\frac{3}{7} \cdot \frac{2}{6} \cdot \frac{1}{5}$.
- The probability of getting one black ball is:

$$\frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} + \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{2}{5} + \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{4}{5} = \frac{12}{35}.$$

- The probability of getting two black balls is:

$$\frac{4}{7} \cdot \frac{3}{6} \cdot \frac{3}{5} + \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{3}{5} + \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{3}{5} = \frac{18}{35}.$$

- The probability of getting three black balls is:

$$\frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} = \frac{4}{35}.$$

Hence the expectation is:

$$\frac{12}{35} + \frac{36}{35} + \frac{12}{35} = \frac{60}{35} = \frac{12}{7}.$$

[*Up to 3 marks for calculation; award marks for partial progress and for correct final answer.*]

(b) A box contains $N > 1$ white and black balls, of which N_B ($0 < N_B \leq N$) are black. Again, balls are taken randomly from the box without replacement. Find $P(B_n)$ for $1 \leq n \leq N$. [5]

Solution: Suppose that all the balls are numbered, $1, ..., N$, and assume that we make N draws. Then the probability of drawing any sequence of balls $(n_1, ..., n_N)$ is the same in each case. There are $N!$ such sequences, and the probability of finding a black ball in position n of the sequence is given by $N_B \cdot (N-1)!$, i.e. N_B choices of which black ball to put at position n, and then $(N-1)!$ choices of arranging any of the remaining balls. Thus the probability is:

$$P(B_n) = \frac{N_B \cdot (N-1)!}{N!} = \frac{N_B}{N}.$$

Notice, this argument demonstrates why the draws have the same probabilities in part (a)! [2 *marks for correct final answer. Up to 3 marks for any convincing argument or derivation.*]

17. (a) Evaluate from first principles, by considering elementary areas, the integral:

$$I = \int_0^b x^2 \, dx.$$

[You may assume that $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.] [6]

Solution: The integral can be obtained as the limit of a sequence of Riemann sums. Consider partitioning $[0, b]$ with n equal intervals, with endpoints:

$$P_n = (0, b/n, 2b/n, ..., (n-1)b/n, b),$$

and choose to use the right-handed tags for the interval at which to evaluate x^2:

$$T_n = (b/n, 2b/n, ..., (n-1)b/n, b).$$

Then a Riemann sum for the integral is:

$$\begin{aligned} R(x^2, P_n, T_n) &= \sum_{k=1}^{n} \left(\frac{kb}{n}\right)^2 \left(\frac{kb}{n} - \frac{(k-1)b}{n}\right) = \frac{b^3}{n^3}\sum_{k=1}^{n} k^2 = \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= \frac{b^3}{3}\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right). \end{aligned}$$

Taking the limit as $n \to \infty$, we have:

$$R(x^2, P_n, T_n) \to \frac{b^3}{3},$$

which shows that:

$$\int_0^b x^2 \, dx = \frac{b^3}{3},$$

as expected. [*1 mark for using stating a partition that will work. 1 mark for writing down an appropriate Riemann sum which is partially correct, 1 mark for writing down an appropriate Riemann sum that is fully correct. 1 mark for simplifying. 1 mark for attempting to take limit, 1 additional mark for correctly taking limit and obtaining answer.*]

(b) Evaluate:

(i) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$ [3]

(ii) $\displaystyle\int \frac{(x+1)dx}{x^2+4x+8}$. [5]

Solution: (i) We have:

$$\int \frac{dx}{\sqrt{4-x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-(x/2)^2}} = \arcsin\left(\frac{x}{2}\right) + c,$$

using knowledge of our standard inverse trigonometric integrals. [*Up to 3 marks for answer, award marks for partial progress.*]

(ii) This integral is more complicated. We do the standard trick of splitting into a logarithmic derivative term, and a term which is a constant divided by a quadratic. [*1 mark for any attempt to do this.*] Note that the denominator has derivative $2x+4$. Hence we write:

$$\int \frac{(x+1)dx}{x^2+4x+8} = \frac{1}{2}\int \frac{(2x+4)dx}{x^2+4x+8} - \int \frac{dx}{x^2+4x+8} = \frac{1}{2}\ln(x^2+4x+8) - \int \frac{dx}{(x+2)^2+4}.$$

[*1 mark for correctly separating into logarithmic derivative and constant over quadratic. 1 mark for correctly obtaining logarithmic integral.*] To do the remaining integral, we note:

$$\int \frac{dx}{(x+2)^2+4} = \frac{1}{4}\int \frac{dx}{((x+2)/2)^2+1} = \frac{1}{2}\arctan\left(\frac{x+2}{2}\right) + c,$$

using our standard inverse trigonometric integrals. Thus we have:

$$\frac{1}{2}\ln(x^2+4x+8) - \frac{1}{2}\arctan\left(\frac{x+2}{2}\right) + c,$$

for some arbitrary constant c. [*Up to 2 marks for evaluation of remaining constant divided by quadratic integral, award marks for partial progress.*]

(c) Find the recurrence relation between $F(k)$ and $F(k-2)$, valid for $k \geq 1$, where:

$$F(k) = \int_0^\pi x^k \sin(x)\, dx,$$

and use it to calculate $F(k)$ for the case $k = 5$. [6]

Solution: We integrate by parts twice. [*1 mark for any attempt to do this.*] We have:

$$\int_0^\pi x^k \sin(x)\, dx = \left[-x^k \cos(x)\right]_0^\pi + k \int_0^\pi x^{k-1} \cos(x)\, dx$$

$$= \pi^k + k\left[x^{k-1}\sin(x)\right]_0^\pi - k(k-1)\int_0^\pi x^{k-2}\sin(x)\, dx$$

$$= \pi^k - k(k-1)F(k-2).$$

[*Up to 2 marks for correctly obtaining recurrence, award marks for partial progress.*] Hence, we have:

$$F(5) = \pi^5 - 20F(3) = \pi^5 - 20(\pi^3 - 6F(1)) = \pi^5 - 20\pi^3 + 120F(1).$$

[*1 mark for using recurrence to get $F(5)$ in terms of $F(1)$.*] It remains to find $F(1)$, which we can do with integration by parts:

$$\int_0^\pi x\sin(x)\, dx = [-x\cos(x)]_0^\pi + \int_0^\pi \cos(x)\, dx = \pi + [\sin(x)]_0^\pi = \pi.$$

[*1 mark for evaluating $F(1)$.*] Hence the result is:

$$F(5) = \pi^5 - 20\pi^3 + 120\pi.$$

[*1 mark for fully correct final answer.*]

18. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and the unit vector

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

(a) Calculate the effect of applying **A** once, twice and three times to **x**. [3]

Solution: Recall that a matrix maps $(1, 0, 0)$ to its first column, $(0, 1, 0)$ to its second column, and $(0, 0, 1)$ to its third column. Hence we have:

$$\mathbf{Ax} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{A}^2\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{A}^3\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{x}.$$

[*1 mark for each correct evaluation.*]

(b) Calculate the transpose of **A**, and the effect of applying it once, twice and three times to **x**.

Solution: We have:

$$\mathbf{A}^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

[*1 mark.*] Hence:

$$\mathbf{A}^T\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad (\mathbf{A}^T)^2\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad (\mathbf{A}^T)^3\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{x}.$$

[*Up to 2 marks for remaining evaluations.*]

(c) Considering the observed effects of **A** and $\mathbf{A}^T$, why does it follow that $\mathbf{A}^T\mathbf{A} = \mathbf{I}$ and $\mathbf{A}\mathbf{A}^T = \mathbf{I}$?

Solution: We have shown that **A** cycles the basis elements $\mathbf{e}_1 \to \mathbf{e}_2 \to \mathbf{e}_3 \to \mathbf{e}_1$. On the other hand, $\mathbf{A}^T$ cycles the basis elements in the opposite order $\mathbf{e}_1 \to \mathbf{e}_3 \to \mathbf{e}_2 \to \mathbf{e}_1$. Therefore, applying **A**, $\mathbf{A}^T$ after one another in either order must do nothing. [*Up to 2 marks for a convincing explanation.*]

(d) Calculate the eigenvalues of **A** in complex polar form, and the eigenvectors. [4]

[The eigenvalues and eigenvectors of a non-symmetric matrix are calculated in the same way as those of a symmetric matrix, but may be complex. You are not required to normalise the eigenvectors.]

Solution: The eigenvalues satisfy the characteristic equation:

$$0 = \mathrm{Det} \begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} = -\lambda \cdot \lambda^2 + 1 \cdot 1 = -\lambda^3 + 1.$$

It follows that $\lambda^3 = 1$, so that the eigenvalues are the cube roots of unity. They are given by:

$$1, \qquad e^{2\pi i/3}, \qquad e^{-2\pi i/3}.$$

The corresponding eigenvectors satisfy:

$$\begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

for each λ. In the case of $\lambda = 1$, this results in an eigenvector:

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

In the case of $\lambda = e^{\pm 2\pi i/3}$, this results in:

$$\begin{pmatrix} -e^{\pm 2\pi i/3} \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -e^{\pm 2\pi i/3} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{\pm 2\pi i/3} \\ 1 \\ e^{\pm 4\pi i/3} \end{pmatrix} = e^{\pm 2\pi i/3} \begin{pmatrix} 1 \\ e^{\mp 2\pi i/3} \\ e^{\pm 2\pi i/3} \end{pmatrix},$$

so we can choose eigenvectors:

$$\mathbf{v}_{\pm} = \begin{pmatrix} 1 \\ e^{\mp 2\pi i/3} \\ e^{\pm 2\pi i/3} \end{pmatrix}.$$

[*1 mark for correctly obtaining characteristic equation. 1 mark for correct eigenvalues. 1 mark for correct eigenvector for eigenvalue* 1. *1 mark for correct eigenvectors for eigenvalues* $e^{\pm 2\pi i/3}$.]

(e) Calculate the eigenvalues of $\mathbf{A}^T$ in complex polar form, and the eigenvectors. [4]

Solution: Notice that in part (c), we showed that $\mathbf{A}^T$ was the inverse of $\mathbf{A}$. If $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with non-zero eigenvalue λ, then:

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \rightarrow \qquad \frac{1}{\lambda}\mathbf{v} = \mathbf{A}^{-1}\mathbf{v}.$$

In particular, in our case this shows us that the eigenvectors of the transpose matrix are the same, and the eigenvalues are:

$$1, \qquad \frac{1}{e^{\pm 2\pi i/3}} = e^{\mp 2\pi i/3}.$$

So, the eigenvector corresponding to 1 is unchanged, but the eigenvectors corresponding to $e^{\pm 2\pi i/3}$ are swapped.

[*1 mark for correct eigenvalue* 1. *1 mark for correct eigenvalues* $e^{\mp 2\pi i/3}$. *1 mark for correct eigenvector corresponding to* 1. *1 mark for correct eigenvectors corresponding to eigenvalues.*]

(f) How is the symmetric part of the matrix $\mathbf{A}$ defined in terms of $\mathbf{A}$ and $\mathbf{A}^T$? [1]

Solution: The symmetric part is $\frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$. [*1 mark.*]

(g) Use the eigenvalues of **A** and $\mathbf{A}^T$ to calculate the eigenvalues of the symmetric part of **A** without calculating the symmetric part of **A** explicitly. [3]

Solution: Let $1, e^{\pm 2\pi i/3}$ have corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_\pm$ for **A**. Then the corresponding eigenvectors for $\mathbf{A}^T$ are $\mathbf{v}_1, \mathbf{v}_\mp$. Therefore, for the symmetric part we have:

$$\frac{1}{2}(\mathbf{A} + \mathbf{A}^T)\mathbf{v}_1 = \frac{1}{2}(1+1)\mathbf{v}_1 = \mathbf{v}_1.$$

So $\mathbf{v}_1$ is an eigenvector of the symmetric part with eigenvalue 1. We also have:

$$\frac{1}{2}(\mathbf{A} + \mathbf{A}^T)\mathbf{v}_\pm = \frac{1}{2}\left(e^{\pm 2\pi i/3} + e^{\mp 2\pi i/3}\right)\mathbf{v}_\pm = \cos\left(\pm\frac{2\pi}{3}\right)\mathbf{v}_\pm = \cos\left(\frac{2\pi}{3}\right)\mathbf{v}_\pm.$$

Thus $\mathbf{v}_\pm$ are eigenvectors of the symmetric part with eigenvalues $-1/2, -1/2$ each.

[*1 mark for each correctly determined eigenvalue, so up to 3 in total*.]

19. (a) Find the sum of the first N terms of the following series. Deduce that the infinite series converges, and determine its value. [7]

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n}$$

Solution: This is *almost* a geometric series, so we use the standard 'differentiation' trick. Consider the sum of the standard geometric series:

$$\sum_{n=1}^{N} ar^n = \frac{ar(1-r^N)}{1-r} = \frac{a(r-r^{N+1})}{1-r}.$$

[*1 mark for knowing this fact.*] Differentiating both sides with respect to r, we have:

$$\sum_{n=1}^{N} anr^{n-1} = \frac{a(1-(N+1)r^N)(1-r) + a(r-r^{N+1})}{(1-r)^2},$$

[*Up to 2 marks for correct differentiation.*] which on multiplication by r produces:

$$\sum_{n=1}^{N} anr^n = \frac{a(r-(N+1)r^{N+1})(1-r) + ar(r-r^{N+1})}{(1-r)^2}.$$

For our specific case, we can take $a = -1$, and $r = -1/3$. [*1 mark for taking correct values in general sum*] Therefore, the sum is given by:

$$\sum_{n=1}^{N} \frac{(-1)^{n+1} n}{3^n} = \frac{\frac{4}{3}(\frac{1}{3} + (N+1)(-\frac{1}{3})^{N+1}) - \frac{1}{3}(\frac{1}{3} + (\frac{1}{3})^{N+1})}{(\frac{4}{3})^2}.$$

In the limit as $N \to \infty$, the power terms converge to zero. [*1 mark for recognising this*] Hence we are left with just:

$$\frac{4/9 - 1/9}{16/9} = \frac{3}{16}.$$

[*Up to 2 marks for taking limit correctly to obtain correct value of sum.*]

(b) Use de Moivre's theorem to show that: [8]

$$\cos(\theta) + \cos(\theta+\alpha) + \cdots + \cos(\theta + n\alpha) = \frac{\sin\left[\frac{1}{2}(n+1)\alpha\right]}{\sin\left(\frac{1}{2}\alpha\right)} \cos\left(\theta + \frac{1}{2}n\alpha\right).$$

Solution: De Moivre's theorem is not really appropriate here, so the question is a bit misleading. Recall it states $\cos(n\theta) + i\sin(n\theta) = (\cos(\theta) + i\sin(\theta))^n$ (this was maybe worth a mark for stating this). Now observe:

$$\begin{aligned}\sum_{k=0}^{n} \cos(\theta + k\alpha) &= \mathrm{Re}\left[\sum_{k=0}^{n} e^{i(\theta+k\alpha)}\right] \\ &= \mathrm{Re}\left[\frac{e^{i\theta}(1-e^{i(n+1)\alpha})}{1-e^{i\alpha}}\right]\end{aligned}$$

$$= \mathrm{Re}\left[\frac{e^{i\theta + i(n+1)\alpha/2}(e^{-i(n+1)\alpha/2} - e^{i(n+1)\alpha/2})}{e^{i\alpha/2}(e^{-i\alpha/2} - e^{i\alpha/2})}\right]$$

$$= \mathrm{Re}\left[\frac{e^{i\theta + in\alpha/2}\sin(\frac{1}{2}(n+1)\alpha)}{\sin(\frac{1}{2}\alpha)}\right]$$

$$= \frac{\sin\left[\frac{1}{2}(n+1)\alpha\right]}{\sin\left(\frac{1}{2}\alpha\right)}\cos\left(\theta + \frac{1}{2}n\alpha\right),$$

as required.

[*1 mark for knowing cosine is the real part of a complex exponential, 1 mark for rewriting sum as real part of sum of complex exponentials, 1 mark for spotting geometric series, 1 mark for successfully summing geometric series of complex exponentials, 1 mark for attempt to factorise numerator and denominator, 1 mark for correctly factorising numerator and denominator, 1 mark for rewriting in terms of sines, 1 mark for taking real part correctly and concluding*.]

(c) Show that the series:

$$\sum_{n=2}^{\infty} \ln\left[\frac{n + (-1)^n}{n}\right]$$

is only conditionally convergent. [5]

Solution: First, we show it is convergent. Clearly, the factor of $(-1)^n$ indicates that the difference between odd and even terms will be important. If n is even, we have:

$$\ln\left(\frac{n + (-1)^n}{n}\right) = \ln\left(\frac{n+1}{n}\right),$$

whilst if n is odd, we have:

$$\ln\left(\frac{n + (-1)^n}{n}\right) = \ln\left(\frac{n-1}{n}\right).$$

Using the fact that the sum of logarithms is the logarithm of a product, this implies that the partial sum up to N is given by:

$$\sum_{n=2}^{N} \ln\left[\frac{n + (-1)^n}{n}\right] = \ln\left(\frac{2+1}{2}\cdot\frac{3-1}{3}\cdot\frac{4+1}{4}\cdot\frac{5-1}{5}\cdot \ldots \cdot \frac{N + (-1)^N}{N}\right).$$

Observe that the terms cancel pairwise, since:

$$\frac{2n+1}{2n}\cdot\frac{(2n+1)-1}{2n+1} = 1.$$

Hence, we are left with:

$$\sum_{n=2}^{N} \ln\left[\frac{n + (-1)^n}{n}\right] = \begin{cases} \ln(1) = 0, & \text{if } N \text{ is odd,} \\ \ln\left((N+1)/N\right), & \text{if } N \text{ is even.} \end{cases}$$

But observe $(N+1)/N \to 1$ as $N \to \infty$, so overall we must have that the partial sums converge to zero as $N \to \infty$. Thus the series is convergent, with sum zero. [*Up to 3 marks for any valid proof of convergence.*]

On the other hand, we must show that:

$$\sum_{n=2}^{\infty}\left|\ln\left(\frac{n+(-1)^n}{n}\right)\right|$$

is divergent. We observe that for n odd, we have:

$$\frac{n+(-1)^n}{n}=1-\frac{1}{n}$$

which is less than 1, and for n even, we have:

$$\frac{n+(-1)^n}{n}=1+\frac{1}{n},$$

which is greater than 1. Hence the partial sums up to an odd term must look like:

$$\sum_{n=2}^{2N+1}\left|\ln\left(\frac{n+(-1)^n}{n}\right)\right|=\sum_{k=1}^{N}\ln\left(1+\frac{1}{2k}\right)-\ln\left(1-\frac{1}{2k+1}\right).$$

Now observe that:

$$\ln\left(1-\frac{1}{2k+1}\right)=-\frac{1}{2k+1}-\frac{1}{2(2k+1)^2}+\cdots\leq-\frac{1}{2k+1},$$

which implies:

$$-\ln\left(1-\frac{1}{2k+1}\right)\geq\frac{1}{2k+1}.$$

Hence we have a sum of positive terms where the odd terms in the series grow at least as quickly as the odd terms in the harmonic series. Since:

$$\sum_{k=0}^{\infty}\frac{1}{2k+1}=1+\frac{1}{3}+\frac{1}{5}+\cdots\geq\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots=\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\right),$$

we have by comparison that the odd terms of the harmonic series diverge. So by the comparison test, the original series diverges. [*Up to 2 marks for any valid proof of divergence.*]

20. The velocity field of a body of incompressible liquid takes the form:

$$\mathbf{v} = -\alpha yz\mathbf{i} + \alpha xz\mathbf{j},$$

where x, y and z are Cartesian coordinates, **i**, **j** and **k** are the corresponding unit vectors, and α is a positive constant.

(a) Sketch a vector diagram showing the spatial three-dimensional form of the flow, and label any salient features. [3]

[*Hint: Consider the vector field in a plane* $z =$ *constant.*]

Solution: When $z =$ constant, the flow looks like it is proportional to $-y\mathbf{i} + x\mathbf{j}$. We already sketched this on the examples sheet, and it looks like a swirl as pictured below (you can get this by just plotting some arrows for a selection of points).

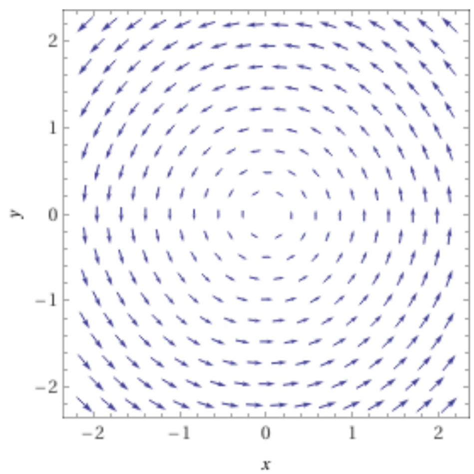


Figure 3: *A cross-section for constant z, courtesy of WolframAlpha.*

[*2 marks for plotting something swirly like this.*] As z varies, the swirls become stronger and the arrows become larger when z is larger. When $z = 0$, everything goes to zero. When z is negative, the direction of the swirls reverses. [*1 mark for correct behaviour along the z-direction (plot doesn't really need to be 3D, could say in words).*]

(b) State the divergence theorem, and by calculating the divergence of **v**, show that there is no net flow across any closed surface within the liquid. [4]

Solution: The divergence theorem states that for a vector field **v**, we have:

$$\int_V \nabla \cdot \mathbf{v}\, dV = \int_S \mathbf{v} \cdot d\mathbf{S},$$

where V is a bounded volume, S is the surface of the volume, and the vector area $d\mathbf{S}$ always points outward from the volume. [*Up to 2 marks for statement of divergence theorem, including careful details about normal.*]

The divergence is given by:

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(-\alpha yz) + \frac{\partial}{\partial y}(\alpha xz) + \frac{\partial}{\partial z}(0) = 0.$$

[*1 mark for correctly obtaining this.*] Hence by the divergence theorem, we have:

$$\int_S \mathbf{v} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{v} \, dV = 0,$$

across any surface S, which implies no net flow across that surface. [*1 mark for saying this.*]

(c) Calculate the curl of $\mathbf{v}$ at a general point and express the result in terms of the unit vector $\mathbf{k}$ and a radial unit vector $\hat{\boldsymbol{\rho}}$ that is perpendicular to $\mathbf{k}$ and directed away from the z-axis. [4]

Solution: We have:

$$\nabla \times \mathbf{v} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} -\alpha yz \\ \alpha xz \\ 0 \end{pmatrix} = \begin{pmatrix} -\alpha x \\ -\alpha y \\ 2\alpha z \end{pmatrix} = -\alpha \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + 2\alpha z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

[*Up to 2 marks for correct evaluation of curl.*] The vector $(x, y, 0)$ is radial, but not a unit vector. It has length $r = \sqrt{x^2 + y^2}$, so $(x, y, 0) = r\hat{\boldsymbol{\rho}}$. [*1 mark for rewriting in this form for radial part.*] Thus we have:

$$\nabla \times \mathbf{v} = -\alpha r \hat{\boldsymbol{\rho}} + 2\alpha z \mathbf{k}.$$

[*1 mark for correct final answer.*]

(d) Consider a closed cylindrical surface, $\mathcal{S}$, having radius R and centred on the z axis. If the end of the cylinder are at $z = \pm h/2$, show explicitly that: [5]

$$\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = 0.$$

Solution: We interpret 'show explicitly' to mean that we shouldn't use Stokes' theorem. We can divide the surface integral up into three pieces: the integral over the top surface of the cylinder, the integral over the bottom surface of the cylinder, and the integral over the curved surface. Over the curved surface, we have unit normal $\hat{\boldsymbol{\rho}}$, so that $d\mathbf{S} = \hat{\boldsymbol{\rho}} dS$, so the integral becomes:

$$\int_{\mathcal{S}_{\text{curved}}} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = -\alpha R \int_{\mathcal{S}_{\text{curved}}} dS = -\alpha R (2\pi R \cdot h) = -2\pi \alpha h R^2.$$

[*Up to 2 marks for evaluation of integral on curved side.*] On the top of the cylinder, we have $d\mathbf{S} = \mathbf{k} dS$, so we get the contribution:

$$\int_{\mathcal{S}_{\text{top}}} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \alpha h \int_{\mathcal{S}_{\text{top}}} dS = \alpha h \cdot (\pi R^2).$$

We get a similar contribution from the bottom, where we have $d\mathbf{S} = -\mathbf{k} dS$. [*Up to 2 marks for evaluation of integrals on top and bottom.*] Thus overall we have:

$$-2\pi \alpha h R^2 + \pi \alpha h R^2 + \pi \alpha h R^2 = 0,$$

as required. [*1 mark for verifying contributions sum to zero.*]

(e) An arbitrarily shaped hole is opened up on the curved surface of the cylinder, such that an area A of the curved surface is removed. State Stokes' theorem and thereby derive an expression for the line integral of the velocity field around the edge of the hole. [4]

Solution: Stokes' theorem states that for a vector field $\mathbf{v}$, we have:

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \oint_C \mathbf{v} \cdot d\mathbf{x},$$

where S is a bounded surface, C is its boundary, and the orientation of the line integral is given in a right-handed sense with respect to the direction of the vector area $d\mathbf{S}$. [*Up to 2 marks for clear statement of Stokes' theorem, with discussion of normals and orientations.*]

The line integral around the hole is therefore:

$$\oint_C \mathbf{v} \cdot d\mathbf{x} = \int_{S_{\text{hole}}} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = -\alpha R \int_{S_{\text{hole}}} dS = -\alpha R A,$$

using the fact that $d\mathbf{S} = \hat{\boldsymbol{\rho}} dS$ for the hole, as it is on the curved surface. [*1 mark for correct use of Stokes' theorem, 1 mark for final expression*.]

Part IA: Mathematics for Natural Sciences
2015 Paper 2 (Unofficial) Mark Scheme

Section A

1. The point A with position vector $(1,1,1)$ lies in a plane. The vectors $\mathbf{u} = (1,1,2)$ and $\mathbf{v} = (0,2,-1)$ are parallel to the same plane. Find $\hat{\mathbf{n}}$, the unit normal to the plane. Find the perpendicular distance, p, from the plane to the origin. [2]

Solution: The normal is:

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix}.$$

Normalising, we have:

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{30}} \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix}.$$

[1 *mark*] The perpendicular distance is the projection of $(1,1,1)$ onto this unit vector, which is:

$$\left| -\frac{5}{\sqrt{30}} + \frac{1}{\sqrt{30}} + \frac{2}{\sqrt{30}} \right| = \frac{2}{\sqrt{30}}.$$

[1 *mark*]

2. Find all the roots of the equation $z^3 = -8$. [2]

Solution: Write the right hand side as:

$$z^3 = 2^3 e^{i\pi}.$$

Then the roots are:

$$z = 2e^{i\pi/3}, \qquad 2e^{i\pi} = -2, \qquad 2e^{-i\pi/3}.$$

[*1 mark for some roots correct, 1 mark for all roots correct.*]

3. Write down the first non-zero term of the Taylor series of the function $f(x) = \ln(x^2+1)$ about the origin, $x = 0$. Hence, or otherwise, state the type of stationary point at the origin. [2]

[You may quote the Taylor series expansions for standard functions.]

Solution: We have $\ln(1+x^2) = x^2 + \cdots$. [1 *mark*] Hence the point $x = 0$ is a minimum of the function. [1 *mark*]

4. Find the eigenvalues and normalised eigenvectors of the matrix:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Solution: The eigenvalues satisfy:

$$0 = \mathrm{Det}\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1),$$

so the eigenvalues are $\lambda = 3$ and $\lambda = -1$. [*1 mark*] The eigenvectors satisfy:

$$\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $\lambda = 3$ has $-2x + 2y = 0$, which gives normalised eigenvector:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

On the other hand, $\lambda = -1$ has $2x + 2y = 0$, which gives normalised eigenvector:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

[*1 mark for both eigenvectors correct.*]

5. Let **F** be the gradient of $\Phi(x, y, z) = x\cos(y^5)\sinh(z)$. Find an expression for **F**. What is the curl of **F**?

Solution: We have:

$$\mathbf{F} = \nabla\Phi = \left(\cos(y^5)\sinh(z), -5y^4 x\sin(y^5)\sinh(z), x\cos(y^5)\cosh(z)\right).$$

[*1 mark*] The curl is zero, because the curl of a gradient is always zero (since **F** is a conservative vector field!). [*1 mark*]

6. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector. Evaluate:

   (a) $\nabla(\mathbf{r}\cdot\mathbf{r})$, [1]

   Solution: We have $\nabla(x^2 + y^2 + z^2) = (2x, 2y, 2z) = 2\mathbf{r}$. [*1 mark*]

   (b) $\nabla\cdot(a\mathbf{r} - \mathbf{b})$, where a is a constant real number and **b** is a constant vector. [1]

   Solution: We have:

   $$\nabla\cdot(a\mathbf{r} - \mathbf{b}) = a + a + a = 3a.$$

   [*1 mark*]

7. Solve the differential equation:

$$\frac{dy}{dx} - \frac{y}{x} = x$$

for $x \geq 1$, given that $y = 1$ when $x = 1$. [2]

> *Solution:* This is a first-order linear equation, so can be solved with an integrating factor. The integrating factor is:
>
> $$\exp\left(-\int \frac{1}{x}\,dx\right) = \exp\left(-\ln(x)\right) = \frac{1}{x}.$$
>
> Hence we have:
>
> $$\frac{d}{dx}\left(\frac{y}{x}\right) = 1 \qquad \Rightarrow \qquad \frac{y}{x} = x + c,$$
>
> which gives:
>
> $$y = x^2 + cx.$$
>
> [1 *mark*] When $x = 1$, we have $1 = 1 + c$, hence $c = 0$. Thus the solution is $y = x^2$. [1 *mark*]

8. Given $\mathbf{F} = y^2x\mathbf{i} + x^2y\mathbf{j} + \frac{1}{3}z^3\mathbf{k}$, evaluate $\iiint \nabla \cdot \mathbf{F}\,dV$ inside a sphere of radius R, centred at the origin. [2]

[*Hint: You may find it helpful to work in spherical polar coordinates.*]

> *Solution:* Hence we have:
>
> $$\nabla \cdot \mathbf{F} = y^2 + x^2 + z^2 = r^2,$$
>
> in spherical polar coordinates. [1 *mark*] Thus we have:
>
> $$\int_V \nabla \cdot \mathbf{F}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \cdot r^2 \sin(\theta) dr d\theta d\phi = \frac{4\pi R^5}{5}.$$
>
> [1 *mark*]

9. Find the Fourier sine series for $f(x) = \sin(x)(1 + 4\cos(x))$ defined on $-\pi < x < \pi$. [2]

> *Solution:* Note that $4\sin(x)\cos(x) = 2\sin(2x)$. [1 *mark*] Hence we can write:
>
> $$f(x) = \sin(x) + 2\sin(2x).$$
>
> This is the Fourier sine series (since it is of the form $a_1 \sin(x) + a_2 \sin(2x) + a_3 \sin(3x) + \cdots$). [1 *mark*]

10. Ten fair coins are tossed simultaneously. Find expressions (which need not be evaluated) for:

   (a) the probability that ten heads are obtained,

   (b) the probability that at least two coins give tails. [2]

Solution: The probability that ten heads are obtained is $1/2^{10}$. [*1 mark*] The probability that at least two coins give tails is the same as one minus the probability that there is either one tail or no tails. This is given by:

$$1 - \frac{1}{2^{10}} - \frac{10}{2^{10}} = 1 - \frac{11}{2^{10}},$$

where $1/2^{10}$ is the probability of getting all heads, and there are 10 ways of getting exactly one tail all of which have probability $1/2^{10}$. [*1 mark*]

Section B

11. (a) Two lines are defined by:

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = 6\mathbf{j} + 4\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} + \mathbf{k}),$$

where **r** is the position vector, **i**, **j** and **k** are the Cartesian unit vectors, and λ and μ are real parameters. Find the position vector, **p**, of the point of intersection of the two lines, and the values of λ and μ at the point of intersection. [4]

Solution: Equating coefficients at the point of intersection, we have:

$$2 + \lambda = -\mu, \qquad -1 + \lambda = 6 + 2\mu, \qquad \lambda = 4 + \mu.$$

Inserting equation (3) into equation (1), we have: $6 + \mu = -\mu$, which gives $2\mu = -6$, and thus $\mu = -3$. Therefore, $\lambda = 1$. It follows that the position vector for the point of intersection is:

$$\mathbf{p} = 3\mathbf{i} + \mathbf{k}.$$

[*1 mark for equating lines and comparing coefficients. 1 mark for obtaining λ. 1 mark for obtaining μ. 1 mark for obtaining **p**.*]

(b) Solve the vector equation:

$$\mathbf{r} + (\mathbf{a} \cdot \mathbf{r})\mathbf{b} = \mathbf{c}$$

for the vector **r**, where **a**, **b** and **c** are constant vectors, in each of the following cases:

(i) $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, [4]

(ii) $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{c} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$, [7]

(iii) $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{c} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$. [2]

and give a geometrical interpretation for each case. [3]

Solution: We will solve the equation generally, then look at the specific cases individually later. Observe that if we dot the equation with **a**, we have:

$$\mathbf{a} \cdot \mathbf{r} + (\mathbf{a} \cdot \mathbf{r})(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{c},$$

which can be rearranged to give:

$$\mathbf{a} \cdot \mathbf{r} = \frac{\mathbf{a} \cdot \mathbf{c}}{1 + \mathbf{a} \cdot \mathbf{b}},$$

provided that $\mathbf{a} \cdot \mathbf{b} \neq -1$. In that case, we then have:

$$\mathbf{r} = \mathbf{c} - (\mathbf{a} \cdot \mathbf{r})\mathbf{b} = \mathbf{c} - \frac{(\mathbf{a} \cdot \mathbf{c})\mathbf{b}}{1 + \mathbf{a} \cdot \mathbf{b}}.$$

In the case where $\mathbf{a} \cdot \mathbf{b} = -1$, we additionally require $\mathbf{a} \cdot \mathbf{c} = 0$ for any solutions to the equation. We then pose $\mathbf{r} = \mathbf{c} + \lambda\mathbf{b}$, which solves the original equation for any λ since:

$$\mathbf{c} + \lambda\mathbf{b} + \mathbf{a} \cdot (\mathbf{c} + \lambda\mathbf{b})\mathbf{b} = \mathbf{c} + \lambda\mathbf{b} - \lambda\mathbf{b} = \mathbf{c}.$$

Thus in the case $\mathbf{a} \cdot \mathbf{b} = -1$, if $\mathbf{a} \cdot \mathbf{c} = 0$, the solution is a *line* $\mathbf{r} = \mathbf{c} + \lambda\mathbf{b}$.

In the specific cases that are presented to us in the question, we have:

(i) $\mathbf{a} \cdot \mathbf{b} = 5 \neq -1$, hence the solution is uniquely given by the point:

$$\mathbf{r} = (3,1,2) - \frac{6(2,1,2)}{1+5} = (3,1,2) - (2,1,2) = (1,0,0).$$

(ii) $\mathbf{a} \cdot \mathbf{b} = -1$, and $\mathbf{a} \cdot \mathbf{c} = 0$, hence the solution is given by a line:

$$\mathbf{r} = (-2,1,1) + \lambda(1,-1,-1).$$

(iii) $\mathbf{a} \cdot \mathbf{b} = -1$, and $\mathbf{a} \cdot \mathbf{c} = 1 \neq 0$, hence there are no solutions to the original equation.

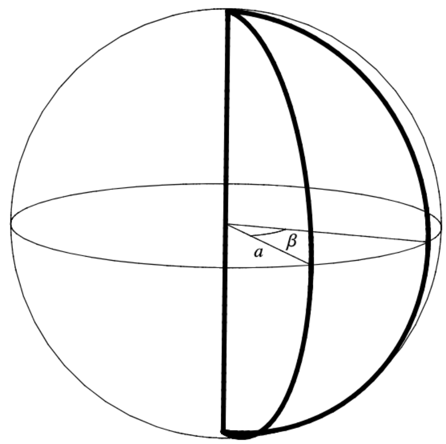
Stating the geometry of 'point' and 'line' in the first and second cases is enough to get all the marks for the interpretation. [*3 marks*]

[*For the remaining marks, it is a bit difficult to allocate, so use your best judgment. Award up to 4 marks for getting to the correct answer for the first case, where the solution is a point. Award up to 7 marks for getting to the correct answer for the second case, where the solution is a line. Award up to 2 marks for the final case, where there are no solutions.*]

12. The position vector of the centre of mass of a homogeneous solid body occupying a volume V is:

$$\bar{\mathbf{x}} = \frac{1}{V}\int\limits_V \mathbf{x}\, dV.$$

Let V be a wedge of angle β ($0 < \beta < 2\pi$) taken from a solid sphere of radius a:



(a) Show that the centre of mass of V is located at a distance $af(\beta)$ from the centre of the sphere, where: [10]

$$f(\beta) = \frac{3\pi}{8\beta}\sin\left(\frac{1}{2}\beta\right).$$

Sketch the graph of $f(\beta)$ for $0 < \beta < 2\pi$. [3]

Solution: Work in spherical polar coordinates, where:

$$\mathbf{x} = (r\cos(\phi)\sin(\theta), r\sin(\phi)\sin(\theta), r\cos(\theta)),$$

where the x-axis passes through the centre of the wedge. [*1 mark for any attempt to use spherical coordinates.*] Then the volume of the wedge is:

$$\frac{\beta}{2\pi}\cdot\frac{4\pi a^3}{3} = \frac{2\beta a^3}{3},$$

where $\beta/2\pi$ is the fraction of the total angle that the wedge subtends. [*1 mark for computing volume of wedge.*] Then the integral is given by:

$$\bar{\mathbf{x}} = \frac{3}{2\beta a^3}\int\limits_{-\beta/2}^{\beta/2}\int\limits_0^{\pi}\int\limits_0^{a}\begin{pmatrix} r\cos(\phi)\sin(\theta) \\ r\sin(\phi)\sin(\theta) \\ r\cos(\theta)\end{pmatrix} r^2\sin(\theta)\, dr d\theta d\phi.$$

[*2 marks for constructing correct integrals.*] Observe that the integral of $\sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$ over 0 to π is the same as the integral of sine over a full range, hence the z-component vanishes. Further, sine is an odd function integrated over a symmetric range in the ϕ integral for the y-component, hence that vanishes too. [*1 mark for observing vanishing components.*] For the remaining components, observe that:

$$\int\limits_0^{\pi}\sin^2(\theta)\, d\theta = \frac{\pi}{2}, \qquad \int\limits_0^{a} r^3\, dr = \frac{a^4}{4}, \qquad \int\limits_{-\beta/2}^{\beta/2}\cos(\phi)\, d\phi = 2\sin(\beta/2).$$

Hence, we are left with:

$$\overline{\overline{x}} = \frac{3}{2\beta a^3} \cdot \frac{a^4}{4} \cdot \frac{\pi}{2} \cdot 2\sin(\beta/2)\hat{\mathbf{e}}_x = \frac{3\pi a}{8\beta}\sin\left(\frac{1}{2}\beta\right)\hat{\mathbf{e}}_x,$$

as required. [2 *marks for calculating integral, and getting given result.*]

To draw the graph of $f(\beta)$, we first identify that it appears to be singular at $\beta = 0$. But $\sin(\beta/2) \approx \beta/2$ for small β (consider the Taylor series), which gives:

$$\lim_{\beta\to 0^+} f(\beta) = \frac{3\pi}{8} \cdot \frac{1}{2} = \frac{3\pi}{16}.$$

[1 *mark*] The stationary points occur when:

$$0 = f'(\beta) = -\frac{3\pi}{8\beta^2}\sin\left(\frac{1}{2}\beta\right) + \frac{3\pi}{16\beta}\cos\left(\frac{1}{2}\beta\right).$$

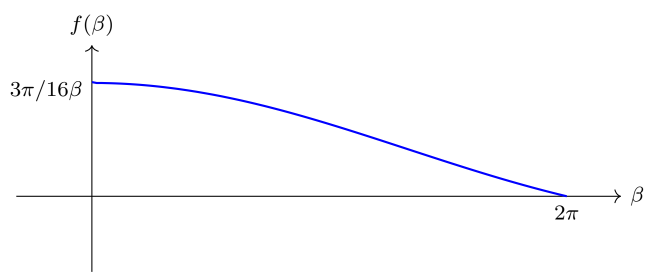
Rearranging, this gives:

$$\tan\left(\frac{1}{2}\beta\right) = \frac{1}{2}\beta.$$

This equation is the intersection of a line of gradient one with a tangent graph, which occurs when $\frac{1}{2}\beta = 0$ and at one other point (which is not expressible in elementary terms) in the range $0 < \beta < 2\pi$. Near $\beta = 0$, we have:

$$\frac{3\pi}{8\beta}\sin\left(\frac{1}{2}\beta\right) \approx \frac{3\pi}{8\beta}\left(\frac{1}{2}\beta - \frac{1}{48}\beta^3\right),$$

which suggests the graph is *decreasing* near $\beta = 0$. Furthermore, when $\beta = 2\pi$, $f(2\pi) = 0$. But the value of $f(0)$ is strictly greater than the value of $f(2\pi)$, and hence the stationary point in the middle of the range must be a point of *inflection*.



[2 *marks for correct sketch, labelling points. May not get the point of inflection part, not necessary for the marks.*]

(b) Calculate the vector area of the curved part of the surface of V. [7]

Solution: Recall that the vector area of a closed surface is zero. [*1 mark for knowing this.*] Therefore, we can compute the vector area of the two flat semicircular parts instead. The area of each semicircle is:

$$\frac{1}{2}\pi a^2.$$

[*1 mark for semicircular area.*] Working in Cartesians with the x-axis along the centre of the spherical segment, we have that the vector area of the respective pieces are:

$$\frac{1}{2}\pi a^2(-\sin(\beta/2), \cos(\beta/2), 0), \qquad \frac{1}{2}\pi a^2(-\sin(\beta/2), -\cos(\beta/2), 0)$$

[*3 marks for getting these vectors.*] Adding these together, we have:

$$\pi a^2(-\sin(\beta/2), 0, 0),$$

[*1 mark for combining correctly.*] so that the vector area is:

$$\pi a^2 \sin(\beta/2)\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is an outward-pointing radial unit vector. [*1 mark for completely correct final answer.*]

13. The force fields **F** and **G** are given by:

$$\mathbf{F} = \begin{pmatrix} xy\cosh(z) \\ x^2\cosh(z) \\ x^2y\sinh(z) \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} 2xy\cosh(z) \\ x^2\cosh(z) \\ x^2y\sinh(z) \end{pmatrix}.$$

(a) For each of the vector fields **F** and **G**, determine whether the vector field is conservative, and, if so, find a function Φ such that it is equal to $-\nabla\Phi$. [6]

> *Solution:* An easy way to check is to take the curl, because a vector field **F** is conservative if $\nabla \times \mathbf{F} = \mathbf{0}$. [*1 mark for knowing this.*] In the case of **F**, we have:
>
> $$\nabla \times \mathbf{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} xy\cosh(z) \\ x^2\cosh(z) \\ x^2y\sinh(z) \end{pmatrix} = \begin{pmatrix} x^2\sinh(z) - x^2\sinh(z) \\ -2xy\sinh(z) + xy\sinh(z) \\ 2x\cosh(z) - x\cosh(z) \end{pmatrix} = \begin{pmatrix} 0 \\ -xy\sinh(z) \\ x\cosh(z) \end{pmatrix} \neq \mathbf{0}.$$
>
> Hence **F** is not conservative. [*1 mark for taking curl correctly, 1 mark for correct conclusion of not conservative.*]
>
> In the case of **G**, we have:
>
> $$\nabla \times \mathbf{G} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} 2xy\cosh(z) \\ x^2\cosh(z) \\ x^2y\sinh(z) \end{pmatrix} = \begin{pmatrix} x^2\sinh(z) - x^2\sinh(z) \\ -2xy\sinh(z) + 2xy\sinh(z) \\ 2x\cosh(z) - 2x\cosh(z) \end{pmatrix} = \mathbf{0}.$$
>
> Hence **G** is conservative. [*1 mark for taking curl correctly, 1 mark for correct conclusion of conservative.*] It remains to find a potential function Φ such that $\mathbf{G} = -\nabla\Phi$. By inspection, we guess $\Phi = -x^2y\cosh(z)$, which satisfies all the required properties. [*1 mark for potential.*]

(b) Evaluate $\int \mathbf{F}\cdot d\mathbf{x}$ and $\int \mathbf{G}\cdot d\mathbf{x}$ along the path consisting of straight lines $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$. [6]

> *Solution:* Since **G** is conservative, by the gradient theorem we have:
>
> $$\int \mathbf{G}\cdot d\mathbf{x} = \int (-\nabla\Phi)\cdot d\mathbf{x} = \Phi(0,0,0) - \Phi(1,1,1) = \cosh(1).$$
>
> [*Up to 2 marks for this answer, evaluated any way; award marks for partial progress. Note we are not explicitly told to avoid using the gradient theorem here!*]
>
> Since **F** is not conservative, we must evaluate the integral piece by piece. Let the sections of the curve be C_1, C_2, C_3, which are parametrised as:
>
> $$\begin{aligned} C_1 &: \quad \mathbf{x}(t) = (t,0,0) & 0 \leq t \leq 1, \\ C_2 &: \quad \mathbf{x}(t) = (1,t,0) & 0 \leq t \leq 1, \\ C_3 &: \quad \mathbf{x}(t) = (1,1,t) & 0 \leq t \leq 1. \end{aligned}$$

Then we have:

$$\int \mathbf{F}\cdot d\mathbf{x} = \int_{C_1} \mathbf{F}\cdot d\mathbf{x} + \int_{C_2} \mathbf{F}\cdot d\mathbf{x} + \int_{C_3} \mathbf{F}\cdot d\mathbf{x}$$

$$= \int_0^1 (0,t^2,0)\cdot(1,0,0)\,dt + \int_0^1 (t,1,0)\cdot(0,1,0)\,dt + \int_0^1 (\cosh(t),\cosh(t),\sinh(t))\cdot(0,0,1)\,dt$$

$$= 1 + [\cosh(t)]_0^1$$

$$= \cosh(1).$$

[*Up to 4 marks for this calculation, award marks for partial progress.*]

(c) Evaluate $\int \mathbf{F}\cdot d\mathbf{x}$ and $\int \mathbf{G}\cdot d\mathbf{x}$ along the straight line from $(0,0,0)$ to $(1,1,1)$. [8]

Solution: The **G**-integral is the same as before, giving $\cosh(1)$. [*Up to 2 marks for this.*]

The **F**-integral's path of integration can be parametrised as (t,t,t) with $0 \le t \le 1$ [*1 mark for correct parametrisation*], which gives:

$$\int \mathbf{F}\cdot d\mathbf{x} = \int_0^1 (t^2\cosh(t), t^2\cosh(t), t^3\sinh(t))\cdot(1,1,1)\,dt$$

$$= \int_0^1 \left(2t^2\cosh(t) + t^3\sinh(t)\right)\,dt.$$

It remains to do this integral. [*Up to 2 marks for obtaining this integral.*] We first spot that it looks like the derivative of a product almost, since $\cosh(t)$ differentiates to $\sinh(t)$ and t^3 differentiates to $3t^2$. Thus we can rewrite the integral as:

$$= \int_0^1 \left(\frac{d}{dt}\left(t^3\cosh(t)\right) - t^2\cosh(t)\right)\,dt$$

$$= \left[t^3\cosh(t)\right]_0^1 - \int_0^1 t^2\cosh(t)\,dt$$

$$= \cosh(1) - \int_0^1 t^2\cosh(t)\,dt.$$

To do the remaining integral, we can use integration by parts. We have:

$$\begin{aligned}
\int_0^1 t^2 \cosh(t)\, dt &= \left[t^2 \sinh(t)\right]_0^1 - 2\int_0^1 t \sinh(t)\, dt \\
&= \sinh(1) - 2\left[t\cosh(t)\right]_0^1 + 2\int_0^1 \cosh(t)\, dt \\
&= \sinh(1) - 2\cosh(1) + 2\sinh(1) \\
&= 3\sinh(1) - 2\cosh(1).
\end{aligned}$$

Putting everything together, we have:

$$\int \mathbf{F} \cdot d\mathbf{x} = 3\cosh(1) - 3\sinh(1) = \frac{3}{e}.$$

[*Up to 3 marks for correctly performing integral*.]

14. A device consists of two blocks. The time of failure of the first block, t_1, is uniformly distributed in the interval $0 < t_1 < T_1$. For $t \geq T_1$, the first block has certainly failed. The time of failure of the second block, t_2, is linearly distributed in the interval $0 < t_2 < T_2$ according to the probability density function $f_2(t_2) = At_2$. For $t \geq T_2$, the second block has certainly failed.

(a) What is the probability density function, $f_1(t_1)$, for the time of failure of the first block? [2]

Solution: The block fails uniformly on $0 < t_1 < T_1$. Hence the density is:

$$f_1(t_1) = \begin{cases} 0, & t_1 < 0, \\ \dfrac{1}{T_1}, & 0 \leq t_1 \leq T_1, \\ 0, & t_1 > T_1, \end{cases}$$

where we have a constant in the section $0 < t_1 < T_1$ which is chosen to be $1/T_1$ to normalise the distribution appropriately. [*Up to 2 marks for correct density.*]

(b) Find A and sketch $f_2(t_2)$ for $0 < t_2 < \infty$. [2]

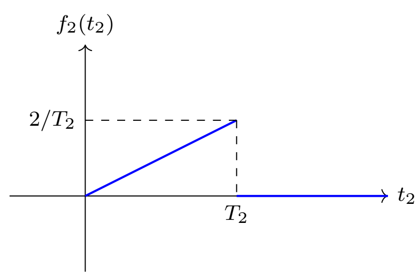
Solution: We require:

$$1 = \int_0^{T_2} At_2 \, dt_2 = A\left[\frac{1}{2}t_2^2\right]_0^{T_2} = \frac{1}{2}AT_2^2.$$

Hence A is given by:

$$A = \frac{2}{T_2^2}.$$

[*1 mark for correct value of A.*] The sketch is straightforward:



[*1 mark for fully correct sketch, with labels.*]

(c) Find and sketch $P_1(t)$, the probability that the first block fails at any time less than t, where $0 < t < \infty$. [3]

Solution: If $0 < t < T_1$, we have:

$$P_1(t) = \int_{-\infty}^{t} f_1(t_1)\, dt_1 = \int_{0}^{t} \frac{1}{T_1}\, dt_1 = \frac{t}{T_1}.$$

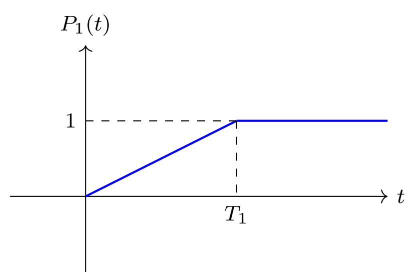
If $t > T_1$, we have:

$$P_1(t) = \int_{-\infty}^{t} f_1(t_1)\, dt_1 = \int_{0}^{T_1} \frac{1}{T_1}\, dt_1 = 1.$$

Hence we have:

$$P_1(t) = \begin{cases} t/T_1, & 0 < t < T_1, \\ 1, & t \geq T_1. \end{cases}$$

[*Up to 2 marks for obtaining* $P_1(t)$.] The sketch is again straightforward:



[*1 mark for fully correct sketch, with labels.*]

(d) Find and sketch $P_2(t)$, the probability that the second block fails at any time less than t, where $0 < t < \infty$. [3]

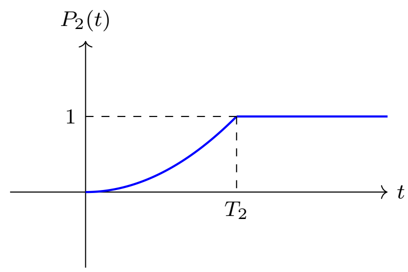
Solution: If $0 < t < T_2$, we have:

$$P_2(t) = \frac{2}{T_2^2} \int_0^t t_2 \, dt_2 = \frac{t^2}{T_2^2},$$

and if $t > T_2$ we have $P_2(t) = 1$. Hence:

$$P_2(t) = \begin{cases} t^2/T_2^2, & 0 < t < T_2, \\ 0, & t > T_2. \end{cases}$$

[2 *marks*] The sketch is straightforward:



[*1 mark for fully correct sketch, with labels.*]

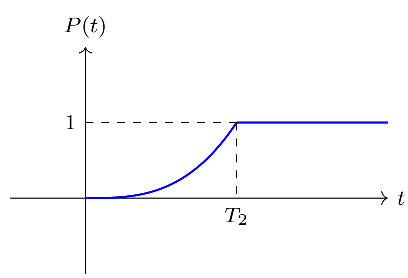
Assume from now on that $T_1 = T_2 = T$.

(e) Find and sketch $P(t)$, the probability that both blocks have failed by time t, where $0 < t < \infty$. [4]

Solution: We won't be able to do this question if we don't assume that the blocks fail independently, so we'll assume that here. We have:

$$P(t) = P_1(t)P_2(t) = \begin{cases} t^3/T^3, & 0 < t < T, \\ 1, & t > T, \end{cases}$$

[*1 mark for multiplying probabilities. 1 mark for correctly obtaining piecewise product.*] The sketch is, once again, straightforward, but now looks like a cubic than a quadratic:



[*2 marks for fully correct sketch, with labels.*]

(f) Find and sketch $R(t)$, the probability that at least one of the blocks has failed by time t, where $0 < t < \infty$. Mark on your graph the inflexion point(s) (if any) and calculate their coordinates.

Solution: The probability that *at least one* fails by time t is one minus the probability that neither of them have failed. [*1 mark for recognising this.*]

$$\begin{aligned} R(t) &= 1 - (1 - P_1(t))(1 - P_2(t)) \\ &= \begin{cases} 1 - (1 - t/T)(1 - t^2/T^2), & 0 < t < T, \\ 0, & t > T \end{cases} \\ &= \begin{cases} t/T + t^2/T^2 - t^3/T^3, & 0 < t < T, \\ 1, & t > T. \end{cases} \end{aligned}$$

[*2 marks for correct $R(t)$ computed by any method.*] This is the graph of a (negative) cubic in the region $0 < t < T$. The turning points occur in the region $0 < t < T$ if:

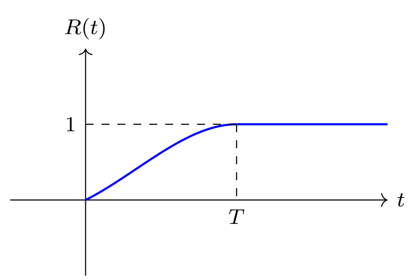
$$\frac{1}{T} + \frac{2t}{T^2} - \frac{3t^2}{T^3} = 0 \quad \Rightarrow \quad 0 = 3t^2 - 2Tt - T^2 = (3t + T)(t - T),$$

so the turning points are at $t = T$ and $t = -T/3$. The only one in the relevant range is $t = T$, and this is evidently a maximum (since probabilities cannot exceed 1). [*1 mark for calculation of stationary points.*]

Inflection points occur when:

$$\frac{2}{T^2} - \frac{6t}{T^3} = 0 \quad \Rightarrow \quad t = \frac{T}{3},$$

which corresponds to the point $(T/3, 1/3 + 1/9 - 1/27) = (T/3, 11/27)$. [*1 mark for this calculation, completely correct.*] The sketch is straightforward:



[*1 mark for correct sketch, labelled.*]

15. (a) The function $f(t)$ satisfies the differential equation:

$$\frac{d^2f}{dt^2} + 8\frac{df}{dt} + 12f = 12e^{-4t}.$$

For the following sets of boundary conditions determine whether the equation has solutions consistent with all three conditions and, if so, find those solutions. [10]

(i) $f(0) = 0$, $\frac{df}{dt}(0) = 0$, $f(\ln(\sqrt{2})) = 0$,

(ii) $f(0) = 0$, $\frac{df}{dt}(0) = -2$, $f(\ln(\sqrt{2})) = 0$.

Solution: This is a constant coefficient second-order differential equation, so can be solved using the auxiliary equation approach. The auxiliary equation is:

$$0 = \lambda^2 + 8\lambda + 12 = (\lambda+6)(\lambda+2),$$

which has roots $\lambda = -6$ and $\lambda = -2$, hence the complementary function is:

$$f_c(t) = Ae^{-6t} + Be^{-2t}.$$

[*Up to 2 marks for obtaining this correctly.*] Trial $f_p(t) = \alpha e^{-4t}$ for a particular integral to get:

$$16\alpha e^{-4t} - 32\alpha e^{-4t} + 12\alpha e^{-4t} = 12e^{-4t} \qquad \Leftrightarrow \qquad \alpha = -3.$$

Hence the complete solution is:

$$f(t) = Ae^{-6t} + Be^{-2t} - 3e^{-4t}.$$

[*Up to 2 marks for obtaining particular integral correctly.*]

Next, we impose the boundary data as stated in the question. Both sets of boundary data have $f(0) = 0$ and $f(\ln(\sqrt{2})) = 0$, so let's impose those and see if the third condition is consistent or not. Imposing these two pieces of data, and observing that $e^{-n\ln(\sqrt{2})} = 2^{-n/2}$, we have:

$$A + B - 3 = 0, \qquad \frac{A}{8} + \frac{B}{2} - \frac{3}{4} = 0.$$

Simplifying, we have:

$$A + B = 3, \qquad A + 4B = 6.$$

Solving simultaneously, we have $B = 1$ and $A = 2$. Thus the solution subject to the boundary conditions $f(0) = 0$ and $f(\ln(\sqrt{2})) = 0$ is given by:

$$f(t) = 2e^{-6t} + e^{-2t} - 3e^{-4t}.$$

We now check that this is consistent with the third condition. We have:

$$\frac{df}{dt} = -12e^{-6t} - 2e^{-2t} + 12e^{-4t},$$

which is equal to -2 at $t = 0$. Thus the solution exists only in the case (ii), and do not exist in the case (i). [*Up to 4 marks for imposing boundary conditions correctly and deriving the solution, up to 2 marks for deciding which cases are consistent and which are inconsistent.*]

(b) A solution of the differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 4e^{-x}$$

takes the value 1 when $x = 0$ and the value e^{-1} when $x = 1$. What is its value when $x = 2$? [10]

Solution: The auxiliary equation is now $0 = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$. This has a repeated root $\lambda = -1$, which gives complementary function:

$$y_c = Ae^{-x} + Bxe^{-x}.$$

[*Up to 3 marks for obtaining the complementary function.*] Next, we trial $y_p = \alpha x^2 e^{-x}$ for the particular integral (since e^{-x} and xe^{-x} are already part of the complementary function). This gives:

$$4e^{-x} = \alpha x^2 e^{-x} + 2\left(2\alpha x e^{-x} - \alpha x^2 e^{-x}\right) + \left(2\alpha e^{-x} - 4\alpha x e^{-x} + \alpha x^2 e^{-x}\right) = 2\alpha e^{-x},$$

which gives $\alpha = 2$. Thus the complete solution is:

$$y = Ae^{-x} + Bxe^{-x} + 2x^2 e^{-x}.$$

[*Up to 4 marks for obtaining particular integral correctly.*] Imposing the data, we have:

$$1 = A, \qquad \frac{1}{e} = \frac{A}{e} + \frac{B}{e} + \frac{2}{e}.$$

This implies $A = 1$ and $B = -2$. Thus the solution with boundary data imposed is:

$$y(x) = \left(1 - 2x + 2x^2\right) e^{-x}.$$

[*Up to 2 marks for obtaining the specific solution.*] Thus the value at $x = 2$ is given by:

$$\frac{5}{e^2}.$$

[*1 mark for correct final value.*]

16. (a) State the condition on the partial derivatives of P and Q for the differential form:

$$P(x,y)dx + Q(x,y)dy$$

to be exact. If this condition is not satisfied, show that the differential form can be made exact by multiplying by an integrating factor of the form $\mu(x)$, provided that:

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

is a function of x only. What is the corresponding condition for the differential form to have an integrating factor of the form $\mu(y)$? [5]

Solution: The condition is:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

[1 *mark*] Multiplying by μ, we require that:

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q) \qquad \Rightarrow \qquad \mu(x)\frac{\partial P}{\partial y} = \mu'(x)Q + \mu(x)\frac{\partial Q}{\partial x}.$$

Rearranging, we obtain a separable equation:

$$\frac{\mu'}{\mu} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right),$$

which implies that we can find an appropriate μ only if the right hand side is a function of x only, as required. Such a μ can be found by integrating this separable equation. [*Up to 3 marks for this argument made correctly.*] The corresponding condition for the equation to have an integrating factor of the form $\mu(y)$ is:

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q) \qquad \Rightarrow \qquad \mu'(y)P + \mu(y)\frac{\partial P}{\partial y} = \mu(y)\frac{\partial Q}{\partial x} \qquad \Rightarrow \qquad \frac{\mu'}{\mu} = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

must be a function of y only. [1 *mark*]

Solve the following differential equations using an integrating factor:

(i) $2x\frac{dy}{dx} + 3x + y = 0,$ [5]

(ii) $(\cos^2(x) + y\sin(2x))\frac{dy}{dx} + y^2 = 0,$ [5]

Solution: The trick to this question is to know which type of integrating factor they are talking about.

(i) In the first case, we have a linear first order equation of the form:

$$\frac{dy}{dx} + \frac{y}{2x} = -\frac{3}{2}.$$

This has integrating factor (not of the type in (a)) given by:

$$\exp\left(\int \frac{1}{2x}\,dx\right) = \exp\left(\frac{1}{2}\log(x)\right) = \sqrt{x}.$$

[2 *marks for integrating factor obtained correctly.*] Hence we have:

$$-\frac{3}{2}\sqrt{x} = \sqrt{x}\frac{dy}{dx} + \frac{1}{2\sqrt{x}}y = \frac{d}{dx}\left(\sqrt{x}y\right).$$

Integrating both sides, we have:

$$\sqrt{x}y = A - x^{3/2} \qquad \Rightarrow \qquad y = \frac{A}{\sqrt{x}} - x,$$

for an arbitrary constant A. [*Up to 3 marks for using integrating factor to find general solution.*]

This question *can* be done using the exact approach discussed in the earlier part of the question, but it is easier not to use that method!

(ii) Here, the equation is non-linear and we cannot use the 'normal' integrating factor method. Instead, we need to use the 'exact' integrating factor method. Multiplying up by dx, we see that we wish to consider the differential form:

$$(\cos^2(x) + y\sin(2x))dy + y^2 dx,$$

where $P = y^2$ and $Q = \cos^2(x) + y\sin(2x)$. This differential form is evidently not exact, since:

$$\frac{\partial P}{\partial y} = 2y, \qquad \frac{\partial Q}{\partial x} = -2\cos(x)\sin(x) + 2y\cos(2x) = 2y\cos(2x) - \sin(2x)$$

are unequal. Therefore, we consider the quantity:

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 2y\left(1 - \cos(2x)\right) + \sin(2x).$$

Now, dividing by P will be unhelpful, as we will still have a function of x afterwards (where we would need a function of y). Instead, we should consider dividing by Q to get a function of x exclusively. We have:

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{2y(1-\cos(2x)) + \sin(2x)}{\cos^2(x) + y\sin(2x)}.$$

Note that $1 - \cos(2x) = 2\sin^2(x)$, $\sin(2x) = 2\sin(x)\cos(x)$. This gives:

$$\frac{4y\sin^2(x) + 2\sin(x)\cos(x)}{\cos^2(x) + 2y\sin(x)\cos(x)} = 2\tan(x)\frac{2y\sin(x) + \cos(x)}{\cos(x) + 2y\sin(x)} = 2\tan(x).$$

[*3 marks for this argument, it is pretty hard though!*] Thus indeed this is a function of x exclusively. We have:

$$\frac{\mu'}{\mu} = 2\tan(x) \qquad \Rightarrow \qquad \ln(\mu) = A - 2\ln(\cos(x)) \qquad \Rightarrow \qquad \mu(x) \propto \sec^2(x).$$

Hence multiplying the equation by the integrating factor, we have:

$$(1 + y\sec^2(x)\sin(2x))dy + y^2\sec^2(x)dx = 0 \qquad \Rightarrow \qquad (1 + 2y\tan(x))dy + y^2\sec^2(x)dx = 0.$$

The left hand side is now exact, and an obvious function $f(x, y)$ such that df is equal to the left hand side is simply $f = y + y^2\tan(x) + c$. Thus the general solution is $y + y^2\tan(x) =$ constant. [*Up to 2 marks for final answer.*]

(b) Use the change of variables $y(x) = u(x)x$ to solve the differential equation: [5]

$$(y - x)\frac{dy}{dx} + 2x + 3y = 0.$$

Solution: Observe that:

$$\frac{dy}{dx} = \frac{du}{dx}x + u,$$

so that the equation becomes:

$$(ux - x)\left(\frac{du}{dx}x + u\right) + 2x + 3ux = 0.$$

Rearranging, we have:

$$x\frac{du}{dx} = \frac{u^2 + 2u + 2}{1 - u}.$$

[*Up to 2 marks for reaching this point.*] This is separable. We have:

$$\int \frac{1 - u}{u^2 + 2u + 2}\, du = \int \frac{dx}{x}$$

The integral on the left can be done by splitting into a logarithmic term and a constant divided by a quadratic. We observe that the derivative of the bottom is $2u + 2$, hence:

$$\begin{aligned} \int \frac{1 - u}{u^2 + 2u + 2}\, du &= -\frac{1}{2}\int \frac{2u + 2}{u^2 + 2u + 2}\, du + 2\int \frac{1}{u^2 + 2u + 2}\, du \\ &= -\frac{1}{2}\ln(u^2 + 2u + 2) + 2\int \frac{1}{(u + 1)^2 + 1}\, du \\ &= -\frac{1}{2}\ln(u^2 + 2u + 2) + 2\arctan(u + 1) + c. \end{aligned}$$

[*Up to 2 marks for this integral done correctly.*] The right hand side integral is trivial, so we get:

$$c - \frac{1}{2}\ln(u^2 + 2u + 2) + 2\arctan(u + 1) = \ln(x).$$

Back-substituting, we obtain the final solution:

$$c - \frac{1}{2}\ln(y^2/x^2 + 2y/x + 2) + 2\arctan(y/x + 1) = \ln(x).$$

[*1 mark for final answer (there is no simple form).*]

17. (a) In the context of matrices, describe what is meant by an eigenvector equation. [3]

> *Solution:* Let A be a matrix. An *eigenvector* **v** of A with *eigenvalue* λ obeys the equation $A\mathbf{v} = \lambda\mathbf{v}$, with $\mathbf{v} \neq \mathbf{0}$. This equation is referred to as an *eigenvector equation*. [*1 mark for correct equation, 1 mark for saying vector should be non-zero, 1 mark for defining eigenvalue.*]

Four springs, each having stiffness constant k, are used to hold three masses m in a line between two rigid supports. It can be shown that the dynamical behaviour of the system is described by the following three coupled differential equations:

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1),$$

$$m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2),$$

$$m\ddot{x}_3 = -k(x_3 - x_2) - kx_3,$$

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the positions of the masses, along the line of the springs, relative to their static positions, and the double dot denotes the second derivative with respect to time t.

(b) Show that the substitutions $x_i = a_i \cos(\omega t)$, $i = 1, 2, 3$, where a_i is independent of t, transform the differential equations into a set of simultaneous algebraic equations. [2]

> *Solution:* Making the substitution, we have:
>
> $$-ma_1\omega^2 \cos(\omega t) = -ka_1 \cos(\omega t) + k(a_2 - a_1)\cos(\omega t),$$
>
> $$-ma_2\omega^2 \cos(\omega t) = -k(a_2 - a_1)\cos(\omega t) + k(a_3 - a_2)\cos(\omega t),$$
>
> $$-ma_3\omega^2 \cos(\omega t) = -k(a_3 - a_2)\cos(\omega t) - ka_3 \cos(\omega t).$$
>
> Dividing by $\cos(\omega t)$, we have:
>
> $$-ma_1\omega^2 = -ka_1 + k(a_2 - a_1),$$
>
> $$-ma_2\omega^2 = -k(a_2 - a_1) + k(a_3 - a_2),$$
>
> $$-ma_3\omega^2 = -k(a_3 - a_2) - ka_3,$$
>
> which is indeed a set of simultaneous algebraic equations. [*2 marks for arriving at these equations.*]

(c) Write down a single matrix equation, in the form of an eigenvector equation, that describes the system. [3]

> *Solution:* Rearranging the equation from above, we have:
>
> $$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = -\frac{m\omega^2}{k} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},$$
>
> which is indeed an eigenvector equation. [*1 mark for getting matrix correct, 1 mark for casting in correct form, 1 mark for vector (a_1, a_2, a_3) as eigenvector.*]

(d) Find the eigenvalues and normalised eigenvectors that satisfy this equation. [6]

Solution: The characteristic equation of the matrix from above is:

$$0 = \begin{vmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} - (-2-\lambda) = (-2-\lambda)^3 - 2(-2-\lambda).$$

This is a cubic equation for $-2-\lambda$, which can be factorised as:

$$0 = (\lambda+2)((\lambda+2)^2 - 2),$$

giving roots $\lambda = -2$ and $\lambda = -2 \pm \sqrt{2}$. [*1 mark for correct characteristic equation for eigenvalues, 1 mark for solving correctly.*] The eigenvectors (x, y, z) for the eigenvalue λ satisfy:

$$\begin{pmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first two of these equations will be independent, but the third will depend on the first two. The first two equations can be solved by producing a vector orthogonal to the first two rows, i.e. by taking the cross product:

$$\begin{pmatrix} -2-\lambda \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2-\lambda \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2+\lambda \\ (2+\lambda)^2 - 1 \end{pmatrix}.$$

This gives the eigenvectors:

- In the case $\lambda = -2$, we have $(1, 0, -1)$, which normalises to:

$$\frac{(1, 0, -1)}{\sqrt{2}}.$$

- In the case $\lambda = -2 \pm \sqrt{2}$, we have $(1, \pm\sqrt{2}, 1)$, which normalises to:

$$\frac{(1, \pm\sqrt{2}, 1)}{2}$$

In particular, this implies that we should take:

$$\omega = \sqrt{\frac{2k}{m}}, \qquad \sqrt{\frac{k(2 \pm \sqrt{2})}{m}},$$

with corresponding (a_1, a_2, a_3) vectors in each case given by:

$$(a_1, a_2, a_3) = \frac{(1, 0, -1)}{\sqrt{2}}, \qquad \frac{(1, \pm\sqrt{2}, 1)}{2}.$$

Probably, we should normalise (a_1, a_2, a_3) further by dimensional analysis, but this question is long enough as it is, and this isn't a physics course. [*Up to 4 marks for obtaining the correct NORMALISED eigenvectors.*]

(e) Show that the eigenvectors are orthogonal. [3]

Solution: We can just check this directly. We have:

$$(1, 0, -1) \cdot (1, \pm\sqrt{2}, 1) = 1 - 1 = 0,$$

and:

$$(1, \sqrt{2}, 1) \cdot (1, -\sqrt{2}, 1) = 1 - 2 + 1 = 0,$$

as required. [*Not sure why this is three marks - but award up to 3 marks for the correct verification here.*]

(f) For the solution with the largest value of $|\omega|$, sketch x_1, x_2 and x_3 versus time. [3]

Solution: The solution with the largest value of ω has:

$$\omega = \sqrt{\frac{k(2+\sqrt{2})}{m}}.$$

The corresponding solutions are:

$$(x_1, x_2, x_3) \propto \left(1, \sqrt{2}, 1\right) \cos\left(\frac{k(2+\sqrt{2})}{m} t\right).$$

Hence the sketch should be three cosine curves. Two curves are identical (x_1 and x_3), and the second curve is the same as the other two but with an amplitude that is a factor of $\sqrt{2}$ larger. [*Up to 3 marks for sketching, drawn reasonably accurately.*]

18. A certain electrical circuit produces a voltage waveform $V_1(t)$ that is periodic in time. The waveform can be written in terms of the Fourier series:

$$V_1(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} \sin(n\omega t). \qquad (\dagger)$$

(a) If the period of the waveform is T, write down an expression for ω. [1]

Solution: $\omega = 2\pi/T$. [*1 mark*]

(b) Starting with (†), demonstrate that the coefficients a_n and b_n are given by: [6]

$$a_n = \frac{2}{T}\int_0^T V_1(t)\cos(n\omega t)\,dt,$$

$$b_n = \frac{2}{T}\int_0^T V_1(t)\sin(n\omega t)\,dt.$$

Solution: We now assume that $\omega = 2\pi/T$ everywhere. Observe that:

$$\int_0^T \cos(n\omega t)\cos(m\omega t)\,dt = \frac{1}{2}\int_0^T \left[\cos\left((n-m)\omega t\right) + \cos\left((n+m)\omega t\right)\right]\,dt = \begin{cases} 0, & n \neq m, \\ \dfrac{T}{2}, & n = m. \end{cases}$$

Similarly, we have:

$$\int_0^T \sin(n\omega t)\sin(m\omega t)\,dt = \frac{1}{2}\int_0^T \left[\cos\left((n-m)\omega t\right) - \cos\left((n+m)\omega t\right)\right]\,dt = \begin{cases} 0, & n \neq m, \\ \dfrac{T}{2}, & n = m, \end{cases}$$

and:

$$\int_0^T \sin(n\omega t)\cos(m\omega t)\,dt = \frac{1}{2}\int_0^T \left[\sin\left((n+m)\omega t\right) + \sin\left((n-m)\omega t\right)\right]\,dt = 0.$$

[*Up to 2 marks for proving these identities.*] Hence, we have:

$$\begin{aligned} &\int_0^T V_1(t)\cos(m\omega t)\,dt \\ &= \frac{1}{2}a_0\int_0^T \cos(m\omega t)\,dt + \sum_{n=1}^{\infty} a_n \int_0^T \cos(n\omega t)\cos(m\omega t)\,dt + \sum_{n=1}^{\infty} b_n \int_0^T \sin(n\omega t)\cos(m\omega t)\,dt \\ &= \frac{a_m T}{2}, \end{aligned}$$

so rearranging we have:

$$a_m = \frac{2}{T}\int_0^T V_1(t)\cos(m\omega t)\,dt,$$

as required. [*2 marks for obtaining these coefficients.*]

Similarly, we have:

$$\int_0^T V_1(t)\sin(m\omega t)\,dt$$

$$= \frac{1}{2}a_0\int_0^T \sin(m\omega t)\,dt + \sum_{n=1}^{\infty} a_n \int_0^T \cos(n\omega t)\sin(m\omega t)\,dt + \sum_{n=1}^{\infty} b_n \int_0^T \cos(n\omega t)\sin(m\omega t)\,dt$$

$$= \frac{b_m T}{2},$$

so rearranging we have:

$$b_m = \frac{2}{T}\int_0^T V_1(t)\sin(m\omega t)\,dt,$$

as required. [2 *marks for obtaining these coefficients.*]

(c) Suppose that the periodic voltage has the form $V_1(t) = V_0\sin(\omega t)$ for $0 \le t \le T/2$ and $V_1(t) = 0$ for $T/2 < t < T$. Derive expressions for the Fourier coefficients, and write out the first 5 non-zero terms (in order of increasing frequency) of the series expansion explicitly. [8]

Solution: We have:

$$a_n = \frac{2}{T}\int_0^{T/2} V_0\sin(\omega t)\cos(n\omega t)\,dt$$

$$= \frac{V_0}{T}\int_0^{T/2} \left[\sin((n+1)\omega t) - \sin((n-1)\omega t)\right]\,dt$$

$$= \frac{V_0}{T}\left[-\frac{\cos((n+1)\omega t)}{(n+1)\omega} + \frac{\cos((n-1)\omega t)}{(n-1)\omega}\right]_0^{T/2}$$

$$= \frac{V_0}{T\omega}\left[-\frac{\cos(2\pi(n+1)t/T}{(n+1)} + \frac{\cos(2\pi(n-1)t/T)}{n-1}\right]_0^{T/2}$$

$$= \frac{V_0}{2\pi}\left(\frac{1-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}-1}{n-1}\right).$$

in the case where $n \neq 1$. Cross-multiplying and combining, this becomes:

$$\frac{V_0}{\pi}\left(\frac{(-1)^{n+1}-1}{n^2-1}\right) = \begin{cases} 0, & \text{if } n \text{ odd,} \\ \dfrac{2V_0}{\pi(1-n^2)}, & \text{if } n \text{ even.} \end{cases}$$

In the case where $n = 1$, the second part of the integral vanishes since $\sin(0) = 0$. The remaining part also vanishes since $1-(-1)^2 = 0$. Thus the formula still holds.

For the b_n, we instead have:

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_0^{T/2} V_0 \sin(\omega t)\sin(n\omega t)\,dt \\
&= \frac{V_0}{T}\int_0^{T/2} \left[\cos((n-1)\omega t) - \cos((n+1)\omega t)\right]\,dt \\
&= \frac{V_0}{T}\left[\frac{\sin((n-1)\omega t)}{(n-1)\omega} - \frac{\sin((n+1)\omega t)}{(n+1)\omega}\right]_0^{T/2} \\
&= \frac{V_0}{T\omega}\left[\frac{\sin(2\pi(n-1)t/T}{(n-1)} + \frac{\sin(2\pi(n+1)t/T)}{n+1}\right]_0^{T/2} \\
&= 0,
\end{aligned}
$$

if $n \neq 1$. In the case where $n = 1$, we have:

$$
b_1 = \frac{2V_0}{T}\int_0^{T/2} \sin^2(\omega t)\,dt = \frac{V_0}{T}\int_0^{T/2} (1-\cos(2\omega t))\,dt = \frac{V_0}{T}\left[t - \frac{\sin(2\omega t)}{2\omega}\right]_0^{T/2} = \frac{V_0}{2}.
$$

Thus the complete Fourier series is:

$$
\frac{V_0}{\pi} + \frac{V_0}{2}\sin(\omega t) + \frac{2V_0}{\pi}\sum_{p=1}^{\infty}\frac{1}{1-4p^2}\cos(2p\omega t).
$$

Writing the terms explicitly, we have:

$$
\frac{V_0}{\pi} + \frac{V_0}{2}\sin(\omega t) - \frac{2V_0}{3\pi}\cos(2\omega t) - \frac{2V_0}{15\pi}\cos(4\omega t) - \frac{2V_0}{35\pi}\cos(6\omega t) + \cdots .
$$

(d) Create a new function $V_2(t)$ by shifting $V_1(t)$ in time by $T/2$. By noting that $V_1(t) - V_2(t) = V_0 \sin(\omega t)$, and without evaluating the Fourier integrals explicitly, write out the first 5 non-zero terms of the series expansion of $V_2(t)$.

Solution: We have that:

$$V_2(t) = \begin{cases} 0, & 0 \leq t \leq T/2, \\ V_0 \sin(\omega(t - T/2)), & T/2 \leq t \leq T. \end{cases}$$

But $\omega = 2\pi/T$, so that $\sin(\omega t - \omega T/2) = \sin(\omega t - \pi) = -\sin(\omega t)$. Hence we have:

$$V_1(t) - V_2(t) = \begin{cases} V_0 \sin(\omega t), & 0 \leq t \leq T/2, \\ V_0 \sin(\omega t), & T/2 \leq t \leq T = V_0 \sin(\omega t). \end{cases}$$

[*Up to 3 marks for a careful proof of this fact.*] Therefore, we have:

$$V_2(t) = V_1(t) - V_0 \sin(\omega t) = \frac{V_0}{\pi} - \frac{V_0}{2} \sin(\omega t) - \frac{2V_0}{3\pi} \cos(2\omega t) - \frac{2V_0}{15\pi} \cos(4\omega t) - \frac{2V_0}{35\pi} \cos(6\omega t) + \cdots$$

[*Up to 2 marks for final correct formula.*]

19. (a) Explain (without proof) how the method of Lagrange multipliers is used to find the stationary points of the function $f(x, y)$ subject to the constraint $g(x, y) = c$, where c is a constant. How is the method generalised to handle a function of more than two variables subject to one or more constraints? [6]

Solution: The method of Lagrange multipliers involves the introduction of a new function, $L(x, y, \lambda)$, called the *Lagrangian*, which is defined by:

$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c).$$

[1 *mark*] The stationary points of this function, (x, y, λ), viewed as a function of x, y and λ (a new parameter called the Lagrangian multiplier) then correspond to the stationary points of the original constrained problem (x, y). [1 *mark*] Explicitly, we require:

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}, \qquad g(x, y) = c$$

at the stationary points, which corresponds to the gradients ∇f and ∇g being parallel, and the constraint being satisfied. [1 *mark for an explanation of gradient vectors being parallel*, 1 *mark for saying constraint enforced by stationary in λ-direction*.]

For multiple variables $(x_1, ..., x_n)$ and constraints $g_1(x_1, ..., x_n) = c_1, g_2(x_1, ..., x_n) = c_2, ..., g_m(x_1, ..., x_n) = c_m$ the Lagrangian is generalised to:

$$L(x_1, ..., x_n, \lambda_1, ..., \lambda_m) = f(x_1, ..., x_n) - \lambda_1(g_1(x_1, ..., x_n) - c_1) - ... - \lambda_m(g_m(x_1, ..., x_n) - c_m).$$

That is, there are m multipliers (one for each constraint). [2 *marks for convincing generalisation to multiple variables and multiple constraints*.]

(b) The function $f(x_1, x_2, x_3, ..., x_n)$ of n variables is defined by:

$$f = -\sum_{i=1}^{n} x_i \ln(x_i),$$

where the variables x_i are positive and subject to the constraint:

$$\sum_{i=1}^{n} x_i = 1.$$

Show that the stationary point of f subject to this constraint is located where $x_i = 1/n$ for each i, and calculate the stationary value of f. [7]

Solution: The Lagrangian is:

$$L(x_1, ..., x_n, \lambda) = -\sum_{i=1}^{n} x_i \ln(x_i) - \lambda \left(\sum_{i=1}^{n} x_i - 1 \right).$$

[2 *marks for constructing this correctly*.] Taking derivatives, we have:

$$\frac{\partial L}{\partial x_k} = -\ln(x_k) - 1 - \lambda, \qquad \frac{\partial L}{\partial \lambda} = 1 - \sum_{i=1}^{n} x_i.$$

[2 *marks for correct derivatives*.] At a stationary point then, we have $-\ln(x_k) - 1 - \lambda = 0$, which implies that $x_k = e^{-1-\lambda}$, i.e. all the x_k take the same value. [1 *mark for deducing this*.] They must also sum to 1, which implies that $x_i = 1/n$ for each i, as required. [*1 mark*]

The stationary value of f is given by:

$$f(1/n, 1/n, ..., 1/n) = -\sum_{i=1}^{n} \frac{1}{n} \ln\left(\frac{1}{n}\right) = -\frac{1}{n} \ln\left((1/n)^n\right) = \ln(n).$$

[*1 mark*]

(c) If a further constraint:

$$\sum_{i=1}^{n} x_i y_i = Y$$

is applied, where $y_1, y_2, y_3, ..., y_n$ and Y are given constants, show that the stationary point of the same function f is located instead where $x_i = a\exp(-by_i)$, where a and b are constants. Write down two equations that determine the values of a and b. [7]

Solution: The new Lagrangian is:

$$L(x_1, x_2, ..., x_n, \lambda, \mu) = -\sum_{i=1}^{n} x_i \ln(x_i) - \lambda\left(\sum_{i=1}^{n} x_i - 1\right) - \mu\left(\sum_{i=1}^{n} x_i y_i - Y\right).$$

[2 *marks for constructing Lagrangian correctly.*] Taking derivatives, we have:

$$\frac{\partial L}{\partial x_k} = -\ln(x_k) - 1 - \lambda - \mu y_k, \qquad \frac{\partial L}{\partial \lambda} = 1 - \sum_{i=1}^{n} x_i, \qquad \frac{\partial L}{\partial \mu} = Y - \sum_{i=1}^{n} x_i y_i.$$

[2 *marks for taking derivative correctly.*] At a stationary point, we therefore have:

$$0 = -\ln(x_k) - 1 - \lambda - \mu y_k \qquad \Rightarrow \qquad x_k = e^{-1-\lambda-\mu y_k} = ae^{-by_k},$$

where $a = e^{-1-\lambda}$, $b = \mu$. So the constants a, b in the question are related to the Lagrange multipliers. This is the required form. [2 *marks*]

The equations which determine the values of a and b are the constraint equations. We have:

$$\sum_{i=1}^{n} ae^{-by_i} = 1, \qquad \sum_{i=1}^{n} ay_i e^{-by_i} = Y.$$

[*1 mark*]

20. A fluid flows with velocity $u(x,t)$ along a channel bounded by walls at $x = 0$ and $x = 1$. The fluid velocity satisfies the partial differential equation:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + G\sin(n\pi x), \qquad (\dagger)$$

where $\nu > 0$ and G are real constants and $n \geq 1$ is an integer. It also satisfies the boundary conditions $u(0,t) = u(1,t) = 0$ and the initial condition $u(x,0) = 0$.

(a) For sufficiently large time t the fluid velocity tends to a steady-state solution $u_s(x)$ independent of t, that satisfies equation (†) and the same boundary conditions. Find $u_s(x)$. [3]

Solution: The steady state solution satisfies:

$$0 = \nu u_s'' + G\sin(n\pi x) \qquad \Rightarrow \qquad u_s'' = -\frac{G}{\nu}\sin(n\pi x).$$

[1 *mark for correct equation*.] Integrating twice, we have:

$$u_s = \frac{G}{\nu n^2\pi^2}\sin(n\pi x) + Ax + B.$$

[1 *mark for general solution*.] Imposing the boundary conditions, we have $u_s(0) = 0$ and $u_s(1) = 0$. Hence $A = B = 0$, and thus:

$$u_s(x) = \frac{G}{\nu n^2\pi^2}\sin(n\pi x)$$

is the complete steady-state solution. [1 *mark for imposing boundary data*.]

(b) Now consider:

$$\tilde{u}(x,t) = u(x,t) - u_s(x).$$

Find the partial differential equation, analogous to equation (†), satisfied by $\tilde{u}(x,t)$ and show that this equation is independent of G. What are the boundary conditions that $\tilde{u}$ must satisfy, and what is the initial condition for $\tilde{u}$? [4]

Solution: We have:

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial u}{\partial t}.$$

[1 *mark*] On the other hand, we have:

$$\nu\frac{\partial^2 \tilde{u}}{\partial x^2} = \nu\frac{\partial^2 u}{\partial x^2} - \nu u_s'' = \nu\frac{\partial^2 u}{\partial x^2} + G\sin(n\pi x).$$

[1 *mark*] Hence we have:

$$\frac{\partial \tilde{u}}{\partial t} = \nu\frac{\partial^2 \tilde{u}}{\partial x^2}.$$

[1 *mark*] Since $u(0,t) = u(1,t) = 0$, we have that $\tilde{u}(0,t) = u(0,t) - u_s(0) = 0$ and $\tilde{u}(1,t) = u(1,t) - u_s(1) = 0$. The initial condition is given by:

$$\tilde{u}(x,0) = u(x,0) - u_s(x) = -\frac{G}{\nu n^2\pi^2}\sin(n\pi x).$$

[1 *mark*]

(c) By means of separation of variables, find a solution for $\tilde{u}$ of the form:

$$\tilde{u}(x,t) = f(t)g(x)$$

where $\lim_{t\to\infty} f(t) = 0$. [10]

Solution: Pose $\tilde{u} = f(t)g(x)$ in the equation we derived in (b). Then:

$$f'(t)g(x) = f(t)g''(x) \qquad \Rightarrow \qquad \frac{f'(t)}{f(t)} = \frac{g''(x)}{g(x)} = -\lambda,$$

for some constant λ. [*Up to 3 marks for separating variables correctly.*] Solving the equation for $g(x)$, we have:

$$g(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

[*1 mark*] But, we require that $\tilde{u}(0,t) = \tilde{u}(1,t) = 0$, which implies that $g(0) = g(1) = 0$. Hence, we have $A = 1$ [*1 mark*], and we require that:

$$\sin(\sqrt{\lambda}) = 0 \qquad \Rightarrow \qquad \lambda = n^2\pi^2$$

for $n \geq 1$ an integer. [*1 mark for getting correct values of* λ.] Thus we have a family of solutions $g_n(x) = B_n\sin(n\pi x)$. [*1 mark*] On the other hand, solving the $f(t)$ equation we have:

$$f(t) = Ce^{-\lambda t} \qquad f_n(t) = C_n e^{-n^2\pi^2 t}$$

is the corresponding family of solutions. [*1 mark*] Thus:

$$\tilde{u}_n(x,t) = C_n e^{-n^2\pi^2 t}\sin(n\pi x).$$

However, imposing the boundary condition $\tilde{u}(x,0) = -G\sin(n\pi x)/\nu n^2\pi^2$, we see that we have precisely one solution given by:

$$\tilde{u}(x,t) = -\frac{G}{\nu n^2\pi^2}e^{-n^2\pi^2 t}\sin(n\pi x).$$

[*2 marks for producing final correct solution.*]

(d) Hence determine the fluid velocity $u(x,t)$ and the total flow rate: [3]

$$Q(t) = \int_0^1 u(x,t)\,dt.$$

Solution: The fluid velocity is:

$$u(x,t) = \frac{G}{\nu n^2\pi^2}\sin(n\pi x) - \frac{G}{\nu n^2\pi^2}e^{-n^2\pi^2 t}\sin(n\pi x) = \frac{G}{\nu n^2\pi^2}\sin(n\pi x)\left(1 - e^{-n^2\pi^2 t}\right).$$

[*1 mark*] The total flow rate is:

$$Q(t) = \frac{G}{\nu n^2\pi^2}(1 - e^{-n^2\pi^2 t}) \int_0^1 \sin(n\pi x)\, dx = \frac{G}{\nu n^2\pi^2}(1 - e^{-n^2\pi^2 t}) \left[-\frac{\cos(n\pi x)}{n\pi}\right]_0^1$$

$$= \frac{G}{\nu n^3\pi^3}(1 - e^{-n^2\pi^2 t}\,(1 - (-1)^n)\,.$$

[*2 marks*]