Part IA: Mathematics for Natural Sciences A
Examples Sheet 5: Differential calculus, Riemann sums, and basic integrals

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Limit definition of the derivative

1. Let $y \equiv y(x)$ be a function of x. Define the *derivative* dy/dx of y as a limit. Using the limit definition:
   (a) show that differentiation is a linear operation;
   (b) find the derivative of $y(x) = x^n$, for $n = 0, 1, 2, 3, ...$

   Hence obtain the derivative of $ax + bx^2 \sin(\theta)$, where a, b, θ are real constants.

2. (a) Using *only* the limit definition, show that for $a > 0$, the derivative of $y(x) = a^x$ is proportional to a^x.
   (b) One definition of the number e is the value of a for which the proportionality constant in the previous part is 1. Using *only* this definition, show that the derivative of a^x is given by $\log(a)a^x$.

Rules of differentiation

3. State the chain rule, the product rule, the quotient rule, and the reciprocal rule, respectively. Make sure you know them off by heart!

4. Using the rules you stated in the previous question, compute the derivatives of:

   (a) $\log(x)$, (b) 3^{x^2}, (c) $\dfrac{e^x}{x^3 - 1}$, (d) $x^3 \log(x^2 - 7)$, (e) $\sqrt{x^3 - e^x \log(x)}$.

5. By writing each of the following trigonometric and hyperbolic functions in terms of exponentials, compute their derivatives: (a) $\cos(x)$; (b) $\sin(x)$; (c) $\cosh(x)$; (d) $\sinh(x)$; (e) $\tan(x)$; (f) $\tanh(x)$. Learn these derivatives off by heart.

6. Using: (i) the logarithmic formulae for the inverse hyperbolic functions you derived on Sheet 4; (ii) the reciprocal rule, compute the derivatives of: (a) $\cosh^{-1}(x)$; (b) $\sinh^{-1}(x)$; (c) $\tanh^{-1}(x)$. Learnt these derivatives off by heart.

7. If $y \equiv y(x)$ is a function of x, show that $\dfrac{d^3x}{dy^3} = -\left(\dfrac{dy}{dx}\right)^{-4} \dfrac{d^3y}{dx^3} + 3\left(\dfrac{dy}{dx}\right)^{-5} \left(\dfrac{d^2y}{dx^2}\right)^2$. Verify this when $y = e^{2x}$.

8. What is *implicit differentiation*, and why is it called implicit? Using: (a) implicit differentiation; (b) the reciprocal rule, find dy/dx given $y + e^y \sin(y) = 1/x$, and make sure that your answers agree.

Curve-sketching

9. State what it means for a function to be *even* and for a function to be *odd*, and explain the geometric significance of these definitions. Hence, decide whether the following functions are even, odd, both, or neither:

   (a) x, (b) $\sin(x)$, (c) e^x, (d) $\sin(\frac{\pi}{2} - x)$, (e) $|x|\cos(x)$, (f) $\sqrt{x}$, (g) 2, (h) 0, (i) $\log\left|\dfrac{1+x}{1-x}\right|$.

10. Write down a list of things you should consider when sketching the graph of a function. Compare with your supervision partner before the supervision, and exchange ideas!

11. Sketch the graphs of the following functions, explaining your reasoning in each case:

    (a) $(x-3)^3 + 2x$, (b) $\dfrac{x}{1+x^2}$, (c) $\dfrac{x^2+3}{x-1}$, (d) xe^x, (e) $\dfrac{\log(x)}{1+x}$, (f) $\dfrac{1}{1-e^x}$, (g) $e^x \cos(x)$.

Riemann sums and the definition of the integral

12. Explain what is meant by a *Riemann sum* for a function $f : [a, b] \to \mathbb{R}$ using a *partition* $P = (x_0, ..., x_n)$ (with $x_0 = a, x_n = b$) and *tagging* $T = (t_1, ..., t_n)$. By choosing appropriate partitions and taggings in each case, use sequences of Riemann sums to evaluate the definite integrals of the following functions on $[0, 1]$ from first principles:

    (a) x, (b) x^2, (c) x^3, (d) $\sqrt{x}$

    [*Hint: For part (d), consider a non-uniform tagging*.]

13. Using a non-uniform tagging, use a sequence of Riemann sums to evaluate the integral $\int_1^\infty \frac{dx}{x^{1+\alpha}}$, where $\alpha > 0$. [*Hint: To take the required limit, it might be useful to use the binomial expansion* $(1+\epsilon)^{-\alpha} = 1 - \alpha\epsilon + ...$]

14. Show by considering Riemann sums that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt{n^2 - k^2}}{n^2} = \frac{\pi}{4}$.

Basic integrals

15. *Write down* the indefinite integrals of each of the following functions, where $a \neq 0, \alpha \neq -1$, and f is any (differentiable, non-zero) function:

    (a) $(ax+b)^\alpha$, (b) e^{ax+b}, (c) $(ax+b)^{-1}$, (d) $\sin(ax+b)$, (e) $\cos(ax+b)$,

    (f) $\sec^2(ax+b)$, (g) $\mathrm{cosec}^2(ax+b)$, (h) $\sinh(ax+b)$, (i) $\cosh(ax+b)$, (j) $f'(x)f(x)^\alpha$,

    (k) $f'(x)/f(x)$.

    Learn these integrals off by heart, and get your supervision partner to test you on them.

16. Using the results of the previous question, evaluate the definite integrals:

    (a) $\int_0^2 (x-1)^2\, dx$, (b) $\int_0^\pi e^{i\theta}\, d\theta$, (c) $\int_0^\pi \cos(x)\, dx$, (d) $\int_{-\pi/4}^{\pi/4} \sec^2(x)\, dx$, (e) $\int_0^1 \frac{2x+4}{x^2+4x+1}\, dx$.

17. By writing $\cos(bx)$ as the real part of a complex exponential, determine the indefinite integral of $e^{ax}\cos(bx)$. Similarly, determine the indefinite integrals of $e^x(\sin(x) - \cos(x))$ and $e^x(\sin(x) + \cos(x))$.