Part IA: Mathematics for Natural Sciences B
Examples Sheet 12: Partial differentiation, differentials, and the single-variable chain rule with multivariable functions

Model Solutions

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Partial differentiation: definitions and basic examples

1. Let $f \equiv f(x,y)$ be a function of x and y.

   (a) Define the *partial derivatives* $\partial f/\partial x$ and $\partial f/\partial y$ in terms of limits. Define also the gradient $\nabla f(x,y)$.

   (b) Determine the gradient of the following functions:

   $$\text{(i) } f = x^3 - 2x^2y + 3xy^3 - 4y^3, \quad \text{(ii) } f = \exp(-x^2y^2), \quad \text{(iii) } f = \exp(-x/y), \quad \text{(iv) } f = \sin(x+y).$$

   (c) For each of the functions in part (b), compute the four possible second partial derivatives. Verify that in each case we have symmetry of the mixed partial derivatives.

Solution: (a) The partial derivatives at the point (x,y) are defined by:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\left[\frac{f(x+h,y)-f(x,y)}{h}\right], \qquad \frac{\partial f}{\partial y} = \lim_{h\to 0}\left[\frac{f(x,y+h)-f(x,y)}{h}\right].$$

The *gradient* is defined to be the vector whose entries are the partial derivatives:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

(b) For the given functions, we have:

(i) $\nabla f = (3x^2 - 4xy + 3y^3, -2x^2 + 9xy^2 - 12y^2)$.

(ii) $\nabla f = (-2xy^2 e^{-x^2y^2}, -2yx^2 e^{-x^2y^2})$.

(iii) $\nabla f = (-\frac{1}{y}e^{-x/y}, \frac{x}{y^2}e^{-x/y})$.

(iv) $\nabla f = (\cos(x+y), \cos(x+y))$.

(c) We now take further partial derivatives.

(i) For the first function,

$$\frac{\partial^2 f}{\partial x^2} = 6x - 4y, \qquad \frac{\partial^2 f}{\partial x \partial y} = -4x + 9y^2, \qquad \frac{\partial^2 f}{\partial y^2} = 18xy - 24y.$$

We can check that:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x},$$

as we can check in all the other cases.

(ii) For the second function, we have:

$$\frac{\partial^2 f}{\partial x^2} = -2y^2 e^{-x^2y^2} + 4x^2y^4 e^{-x^2y^2}, \qquad \frac{\partial^2 f}{\partial x \partial y} = -4xye^{-x^2y^2} + 4xy^3 e^{-x^2y^2}, \qquad \frac{\partial^2 f}{\partial y^2} = -2x^2 e^{-x^2y^2} + 4y^2x^2 e^{-x^2y^2}.$$

(iii) For the third function, we have:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{y^2} e^{-x/y}, \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{1}{y^2} e^{-x/y} - \frac{x}{y^3} e^{-x/y}, \qquad \frac{\partial^2 f}{\partial y^2} = -\frac{2x}{y^3} e^{-x/y} + \frac{x}{y^4} e^{-x/y}.$$

(iv) For the final function, we have:

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x+y), \qquad \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y), \qquad \frac{\partial^2 f}{\partial y^2} = -\sin(x+y).$$

2. Show that the function:

$$w(x,y) = \frac{1}{360}\left(15x^4y^2 - x^6 + 15x^2y^4 - y^6\right)$$

is a solution of the equation:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = x^2y^2.$$

➭ **Solution:** We have:

$$\frac{\partial w}{\partial x} = \frac{1}{360}\left(60x^3y^2 - 6x^5 + 30xy^4\right), \qquad \frac{\partial^2 w}{\partial x^2} = \frac{1}{360}\left(180x^2y^2 - 30x^4 + 30y^4\right).$$

Similarly, since the function is symmetric in x and y, we have:

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{360}\left(180x^2y^2 - 30y^4 + 30x^4\right).$$

Adding these together, we have:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{360}\left(360x^2y^2\right) = x^2y^2,$$

as required.

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3. Show that the function:

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\sigma^2 t}}\exp\left(-\frac{(x-x_0)^2}{4\sigma^2 t}\right),$$

where $t > 0, x_0, \sigma$ are real positive constants, and $\sigma^2 \neq 0$, is a solution of the equation:

$$\frac{\partial \phi}{\partial t} = \sigma^2 \frac{\partial^2 \phi}{\partial x^2}.$$

➭ **Solution:** We have:

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2\sqrt{4\pi\sigma^2}t^{3/2}}\exp\left(-\frac{(x-x_0)^2}{4\sigma^2 t}\right) + \frac{(x-x_0)^2}{\sqrt{4\pi\sigma^2 t}\cdot 4\sigma^2 t^2}\exp\left(-\frac{(x-x_0)^2}{4\sigma^2 t}\right).$$

We also have:

$$\frac{\partial \phi}{\partial x} = -\frac{2(x-x_0)}{\sqrt{4\pi\sigma^2 t}\cdot 4\sigma^2 t}\exp\left(-\frac{(x-x_0)^2}{4\sigma^2 t}\right),$$

and hence:

$$\begin{aligned}\sigma^2\frac{\partial^2 \phi}{\partial x^2} &= -\frac{2}{\sqrt{4\pi\sigma^2 t}\cdot 4t}\exp\left(-\frac{(x-x_0)^2}{4\sigma^2 t}\right) + \frac{4(x-x_0)^2}{\sqrt{4\pi\sigma^2 t}16\sigma^2 t^2}\exp\left(-\frac{(x-x_0)^2}{4\sigma^2 t}\right)\\ &= -\frac{1}{2\sqrt{4\pi\sigma^2}t^{3/2}}\exp\left(-\frac{(x-x_0)^2}{4\sigma^2 t}\right) + \frac{(x-x_0)^2}{\sqrt{4\pi\sigma^2 t}\cdot 4\sigma^2 t^2}\exp\left(-\frac{(x-x_0)^2}{4\sigma^2 t}\right),\end{aligned}$$

hence indeed the equation is solved by this function.

4. $(*)$ Show that the mixed partial derivatives of the function:

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{for } (x,y) \neq (0,0), \\ 0, & \text{for } (x,y) = (0,0), \end{cases}$$

are not symmetric at the point $(0,0)$. Why is this allowed to occur here?

➾ **Solution:** Away from $(0,0)$, we have by the quotient rule:

$$\frac{\partial f}{\partial x}(x,y) = \frac{(3x^2y - y^3)(x^2+y^2) - 2x^2y(x^2-y^2)}{(x^2+y^2)^2}.$$

At $(0,0)$, we have, by the definition of the partial derivative:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\left[\frac{f(h,0)-f(0,0)}{h}\right] = \lim_{h\to 0}\frac{0}{h} = 0.$$

The mixed partial derivative evaluated at the origin is, by the definition of the partial derivative, given by:

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h\to 0}\left[\frac{\partial f/\partial x(0,h) - \partial f/\partial x(0,0)}{h}\right] = \lim_{h\to 0}\frac{1}{h}\cdot\frac{-h^3\cdot h^2}{(h^2)^2} = -1.$$

On the other hand, away from $(0,0)$, we have by the quotient rule:

$$\frac{\partial f}{\partial y}(x,y) = \frac{(x^3 - 3xy^2)(x^2+y^2) - 2xy^2(x^2-y^2)}{(x^2+y^2)^2}.$$

At $(0,0)$, we have, by the definition of the partial derivative:

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\left[\frac{f(0,h)-f(0,0)}{h}\right] = \lim_{h\to 0}\frac{0}{h} = 0.$$

The mixed partial derivative evaluated at the origin is, by the definition of the partial derivative, given by:

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h\to 0}\left[\frac{\partial f/\partial y(h,0) - \partial f/\partial y(0,0)}{h}\right] = \lim_{h\to 0}\frac{1}{h}\cdot\frac{h^5}{h^4} = 1.$$

Oops! $1 \neq -1$, so the mixed partial derivatives are indeed not symmetric at the origin.

In general, *Clairaut's theorem* guarantees that the mixed partial derivatives will be symmetric provided that the second partial derivatives are all *continuous*. This does not hold for this function.

Integration and basic partial differential equations

5. Let $f \equiv f(x,y)$ be a function of x and y. Find the general solution of the following partial differential equations:

$$\text{(a) } \frac{\partial f}{\partial x} = xy^2 + \cos(x), \qquad \text{(b) } \frac{\partial f}{\partial y} = y^2 - xe^y, \qquad \text{(c) } \frac{\partial^2 f}{\partial x^2} + y^2 f = x, \qquad \text{(d) } \frac{\partial^2 f}{\partial x \partial y} = 0.$$

Solution: The basic idea here is that if we integrate with respect to a partial derivative, we must remember that we might have lost an arbitrary *function* of the remaining variable. For each example:

(a) Here, integrating directly with respect to x we have:

$$f(x,y) = \frac{1}{2}x^2y^2 + \sin(x) + g(y),$$

where $g(y)$ is an arbitrary *function* of y, that we might have dropped.

(b) Integrating directly, we have:

$$f(x,y) = \frac{1}{3}y^3 - xe^y + g(x),$$

where $g(x)$ is an arbitrary *function* of x, that we might have dropped.

(c) Here, treat y as a constant. Then the equation is just a second-order differential equation. The complementary function is:

$$f_{\mathsf{CF}} = A(y)\sin(yx) + B(y)\cos(yx),$$

where the coefficients could, in principle, depend on y. Guess a particular integral $f_{\mathsf{PI}} = Cx$. Then:

$$y^2(Cx) = x \qquad \Rightarrow \qquad C = \frac{1}{y^2}.$$

Hence, the general solution is:

$$f(x,y) = A(y)\sin(xy) + B(y)\cos(xy) + \frac{x}{y^2},$$

where $A(y), B(y)$ are arbitrary functions of y.

(d) Integrate first with respect to x. Then we have:

$$\frac{\partial f}{\partial y} = g(y),$$

for some arbitrary function $g(y)$. Integrating now with respect to y again, we have:

$$f(x,y) = g_1(y) + g_2(x),$$

where $g_1(y)$ is the integral of $g(y)$, which remember is arbitrary, so is also arbitrary; $g_1(x)$ is another arbitrary function of x. So the general solution is just any sum of a function of x and a function of y. Try it!

6. Let $f \equiv f(x,y)$ be a function of x and y. Find the solution of the following partial differential equations, subject to the given boundary conditions:

(a) $\dfrac{\partial f}{\partial x} = xy^2$, where $f(0,y) = y^3$, (b) $y^3\dfrac{\partial f}{\partial y} = x$, where $\lim_{y\to\infty} f(x,y) = e^x$.

➾ Solution:

(a) Integrating directly, we have:

$$f(x,y) = \frac{1}{2}x^2y^2 + g(y),$$

where $g(y)$ is an arbitrary function of y. Imposing the boundary data $f(0,y) = y^3$, we have $f(0,y) = g(y) = y^3$. Thus the solution satisfying the boundary condition is $f(x,y) = \frac{1}{2}x^2y^2 + y^3$.

(b) Rearranging, and integrating directly, we have:

$$\frac{\partial f}{\partial y} = \frac{x}{y^3} \qquad \Rightarrow \qquad f(x,y) = g(x) - \frac{x}{2y^2}.$$

We require that as $y \to \infty$, the limit of f is e^x. This fixes $g(x) = e^x$, giving the specific solution satisfying the boundary conditions as:

$$f(x,y) = e^x - \frac{x}{2y^2}.$$

Differentials

7. In lectures, differentials are introduced as 'infinitesimal quantities'; however, there is no need for this, and the concept can easily be made mathematically precise. In real multivariable calculus, we can view the differential as alternative notation for the gradient, $df \equiv \nabla f$.

(a) Using this definition, show that $d(x^2+y^2) = 2xdx + 2ydy$ and $d(x^2y) = 2xydx + x^2dy$.

(b) Generalising your argument, show that for any smooth function $f(x,y)$, we have:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy,$$

as stated in lectures.

➾ Solution:

(a) Since $df \equiv \nabla f$, we note that $dx = \nabla(x) = (1,0)$ and $dy = \nabla(y) = (0,1)$ as vectors. Hence:

$$d(x^2+y^2) = \nabla(x^2+y^2) = (2x, 2y) = 2x(1,0) + 2y(0,1) = 2xdx + 2ydy.$$

Similarly, we have:

$$d(x^2y) = \nabla(x^2y) = (2xy, x^2) = 2xy(1,0) + x^2(0,1) = 2xydx + x^2dy,$$

as required.

(b) More generally, we have $df = \nabla f = (\partial f/\partial x, \partial f/\partial y)$, so:

$$df = \frac{\partial f}{\partial x}(1,0) + \frac{\partial f}{\partial y}(0,1) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy,$$

as required.

8. By computing the partial derivatives, determine the differentials of each of the following functions in terms of the differentials of x and y:

(a) $\exp(-1/(x+y))$, (b) $\sinh(x)/\sinh(y)$, (c) $\sqrt{x^2+y^2}$, (d) $\arctan(y/x)$, (e) x^y.

➾ **Solution:** We use the formula:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

throughout. For each of the given functions:

(a) We have:

$$d(e^{-1/(x+y)}) = \frac{1}{(x+y)^2}e^{-1/(x+y)}dx + \frac{1}{(x+y)^2}e^{-1/(x+y)}dy.$$

(b) We have:

$$d\left(\frac{\sinh(x)}{\sinh(y)}\right) = \frac{\cosh(x)}{\sinh(y)}dx - \frac{\sinh(x)\cosh(y)}{\sinh^2(y)}dy.$$

(c) We have:

$$d\left(\sqrt{x^2+y^2}\right) = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy.$$

(d) We have:

$$d\left(\arctan\left(\frac{y}{x}\right)\right) = -\frac{y/x^2}{1+(y/x)^2}dx + \frac{1/x}{1+(y/x)^2}dy = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

(e) We have:

$$d(x^y) = yx^{y-1}dx + \ln(x)x^y dy.$$

9. Let f, g be functions of (x, y), let a, b be constants, and let $F : \mathbb{R} \to \mathbb{R}$ be any differentiable single-variable function. Prove the following basic properties of differentials:

(a) $d(af + bg) = adf + bdg$, (b) $d(fg) = fdg + gdf$, (c) $d(F(f)) = F'(f)df$.

Hence, without computing partial derivatives, show that if $f(x, y) = \log(xy^2)$, we have:

$$df = \frac{dx}{x} + \frac{2\,dy}{y}.$$

Now, verify that your result is correct by computing the partial derivatives of $f(x, y)$.

Solution: Proving the basic properties of differentials, we have:

(a) Note that:

$$\begin{aligned} d(af + bg) &= \frac{\partial}{\partial x}(af + bg)dx + \frac{\partial}{\partial y}(af + bg)dy \\ &= \left(a\frac{\partial f}{\partial x} + b\frac{\partial g}{\partial x}\right) dx + \left(a\frac{\partial f}{\partial y} + b\frac{\partial g}{\partial y}\right) dy \\ &= a\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) + b\left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}\right) dy \\ &= adf + bdg, \end{aligned}$$

as required.

(b) Note that:

$$\begin{aligned} d(fg) &= \frac{\partial}{\partial x}(fg)dx + \frac{\partial}{\partial y}(fg)dy \\ &= \left(\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}\right) dy \\ &= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) g + f\left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right) \\ &= df \cdot g + f \cdot dg, \end{aligned}$$

as required.

(c) Finally, note that:

$$\begin{aligned} d(F(f)) &= \frac{\partial}{\partial x}[F(f(x,y))]dx + \frac{\partial}{\partial y}[F(f(x,y))]dy \\ &= \frac{\partial f}{\partial x}F'(f)dx + \frac{\partial f}{\partial y}F'(f)dy \\ &= F'(f)df. \end{aligned}$$

Using the above properties, we have:

$$d\left(\log(xy^2)\right) = \frac{1}{xy^2}d(xy^2) = \frac{1}{xy^2}\left(y^2dx + xd(y^2)\right) = \frac{dx}{x} + \frac{2\,dy}{y}.$$

On the other hand, directly using partial derivatives, we have:

$$d\left(\log(xy^2)\right) = \frac{\partial}{\partial x}\left(\log(xy^2)\right)\,dx + \frac{\partial}{\partial y}\left(\log(xy^2)\right)\,dy = \frac{y^2}{xy^2}dx + \frac{2xy}{xy^2}dy = \frac{dx}{x} + \frac{2\,dy}{y}.$$

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10. The period T of a simple pendulum can be approximated by the formula:

$$T = 2\pi\sqrt{\frac{l}{g}},$$

where l is the length of the pendulum, and g is gravitational acceleration.

(a) By taking logarithms, show that:

$$\frac{dT}{T} = \frac{dl}{2l} - \frac{dg}{2g}.$$

(b) Hence, estimate the percentage change in the period of a pendulum if: (i) the length is increased by 0.1%; (ii) gravitational acceleration increased by 0.2%.

Solution:

(a) Taking logarithms, we have:

$$\log(T) = \log(2\pi) + \frac{1}{2}\log(l) - \frac{1}{2}\log(g).$$

Taking the differential of both sides, and using the properties we proved in the previous question, we have:

$$\frac{dT}{T} = \frac{dl}{2l} - \frac{dg}{2g},$$

as required.

(b) (i) If the length is increased by 0.1%, whilst keeping gravitational acceleration constant, we have $dl/l \approx 0.1$ (because the *relative* change in l is 0.1, $(l + dl)/l = 1.01$). Using the equation from part (a), this implied $dT/T \approx 0.05$, implying the period increases by roughly 0.05%.

(ii) Similarly, if the gravitational acceleration increased by 0.2%, we would see that the period would fall by roughly 0.1%.

11. The magnitude of the gravitational force between two points masses m_1, m_2 which are separated by a distance $r > 0$ in three dimensional space is given by:

$$F(r, m_1, m_2) = \frac{Gm_1m_2}{r^2},$$

where G is a positive constant. Find dF in terms of dr, dm_1 and dm_2. Hence compute the (approximate) fractional change in distance if there is no change in the force, and the masses of both particles increase by 1%.

➾ **Solution:** Similarly to Question 10, taking logarithms is a great idea. We have:

$$\log(F) = \log(G) + \log(m_1) + \log(m_2) - 2\log(r).$$

Taking the differential, we have:

$$\frac{dF}{F} = \frac{dm_1}{m_1} + \frac{dm_2}{m_2} - \frac{2dr}{r},$$

since G is a constant. Rearranging, we have:

$$\frac{dr}{r} = \frac{dm_1}{2m_1} + \frac{dm_2}{2m_2} - \frac{dF}{2F}.$$

If there is no change in the force, and the masses of both particles increase by 1%, then the approximate fractional change in the distance is:

$$\frac{dr}{r} \approx \frac{1}{2} + \frac{1}{2} = 1,$$

i.e. 1% too.

12. The energy, $E(m, v)$, of a relativistic particle of rest mass m and speed v is given by:

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}},$$

where c, the speed of light, is a constant.

(a) Find dE in terms of dm, dv.

(b) Two particles, A, B, have equal energy and move at 90% and 91% of the speed of light respectively. Particle A has rest mass m_A. What is the (approximate) difference in the rest masses of the particles, in terms of m_A? Which particle has the larger rest mass?

➾ **Solution:**

(a) Once again, taking logarithms is a fantastic idea. We have:

$$\log(E) = \log(m) + \log(c^2) - \frac{1}{2}\log\left(1 - \frac{v^2}{c^2}\right).$$

Taking the differential, we have:

$$\frac{dE}{E} = \frac{dm}{m} + \frac{2v/c^2}{2(1 - v^2/c^2)}dv = \frac{dm}{m} + \frac{v}{c^2 - v^2}dv.$$

(b) Particle A's velocity is $90c/100$ and particle B's velocity is $91c/100$. Hence the difference in velocities is:

$$dv = \frac{c}{100}.$$

Since the particles have equal energy, $dE = E$. Hence the difference in masses relative to A is approximately:

$$\frac{m_B - m_A}{m_A} = \frac{dm}{m_A} = -\left(\frac{v_A}{c^2 - v_A^2}\right)dv = -\frac{9c/10}{c^2 - (81c^2/100)} \cdot \frac{c}{100} = -\frac{9/10}{19} = -\frac{9}{190}.$$

Thus the approximate difference in rest masses is $9m_A/190$. It follows that the rest mass of particle B is given approximately by:

$$m_B \approx m_A\left(1 - \frac{9}{190}\right) = \frac{181m_A}{190}.$$

The rest mass of B is smaller.

13. The differential of the volume V of a geometrical figure is given by:

$$dV = 2\pi rhdr + \pi r^2 dh,$$

where r and h are non-negative parameters and the volume vanishes when these parameters are zero. Find an expression for the fractional change in volume dV/V for fractional changes in the parameters dr/r and dh/h. Find dV/V if r increases by 1% and h increases by 2%.

➼ **Solution:** The volume must depend on r, h, so must be a function $V(r, h)$. We are essentially given its partial derivatives:

$$\frac{\partial V}{\partial r} = 2\pi rh, \qquad \frac{\partial V}{\partial h} = \pi r^2.$$

Integrating the first equation directly, we have:

$$V(r, h) = \pi r^2 h + g(h),$$

for some arbitrary function $g(h)$. However, this must be consistent with the second equation. Differentiating, the solution we just found, we have:

$$\frac{\partial V}{\partial h} = \pi r^2 + g'(h),$$

so we see that we need $g'(h) = 0$. Thus, $g(h)$ is actually a *constant*, completely independent of h too. We see that:

$$V(r, h) = \pi r^2 h + c.$$

We are given the volume vanishes when both parameters are zero, which fixes the volume as $V(r, h) = \pi r^2 h$ (this is in fact the volume of a cylinder with radius r and height h).

Now, take logarithms of $V(r, h) = \pi r^2 h$ to get:

$$\log(V) = \log(\pi) + 2\log(r) + \log(h).$$

Taking the differential, we have:

$$\frac{dV}{V} = \frac{2dr}{r} + \frac{dh}{h},$$

which is the required expression. If r increases by 1% and h increases by 2%, this equation shows that V increases by approximately 4%.

(†) Multivariable Taylor series, and error propagation

14. Find, up to and including terms of quadratic order, the Taylor series of the functions:

(a) $f(x,y) = \sin(x+2y)$ about the point $(x,y) = (\pi/2, 0)$;

(b) $f(x,y) = e^x \cos(y)$ about the point $(x,y) = (0,0)$.

Solution: To find a multivariable Taylor series, we simply Taylor expand in both arguments of the function. Fixing y, and Taylor expanding in x about $x = x_0$, we have:

$$f(x,y) = f(x_0,y) + (x-x_0)\frac{\partial f}{\partial x}(x_0,y) + \frac{1}{2}(x-x_0)^2\frac{\partial^2 f}{\partial x}(x_0,y) + \cdots .$$

Now expanding each of the functions on the right hand side using a Taylor expansion in y about $y = y_0$, we have:

$$\begin{aligned} f(x,y) = f(x_0,y_0) &+ (x-x_0)\frac{\partial f}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(x_0,y_0) \\ &+ \frac{1}{2}(x-x_0)^2\frac{\partial^2 f}{\partial x^2}(x_0,y_0) + (x-x_0)(y-y_0)\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) + \frac{1}{2}(y-y_0)^2\frac{\partial^2 f}{\partial y^2}(x_0,y_0)\cdots , \end{aligned}$$

ignoring terms that are higher than quadratic in the product of $(x-x_0)$ and $(y-y_0)$ factors. Applying this to the given functions, we have:

(a) For this function,

$$\frac{\partial f}{\partial x} = \cos(x+2y), \qquad \frac{\partial^2 f}{\partial x^2} = -\sin(x+2y),$$

and:

$$\frac{\partial f}{\partial y} = 2\cos(x+2y), \qquad \frac{\partial^2 f}{\partial y^2} = -4\cos(x+2y),$$

and finally:

$$\frac{\partial^2 f}{\partial x \partial y} = -2\sin(x+2y).$$

Evaluating all these at the point $(x,y) = (\pi/2, 0)$, we obtain the multivariable Taylor series:

$$\sin(x+2y) = 1 - \frac{1}{2}(x-\pi/2)^2 - 2y^2 + \cdots .$$

(b) For the second function, we can just multiply the normal Taylor expansions. We have:

$$e^x\cos(y) = \left(1 + x + \frac{1}{2}x^2 + \cdots\right)\left(1 - \frac{1}{2}y^2 + \cdots\right) = 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2 + \cdots .$$

15. Let $f(X,Y)$ be a function of the independent random variables X and Y, and let $\mathbb{E}[X] = \mu_X$, $\mathbb{E}[Y] = \mu_Y$. Using properties of variance, and multivariable Taylor series, show that:

$$\mathrm{Var}(f(X,Y)) \approx \left(\frac{\partial f}{\partial X}\right)^2 \mathrm{Var}(X) + \left(\frac{\partial f}{\partial Y}\right)^2 \mathrm{Var}(Y),$$

where the partial derivatives are evaluated at the mean $(X,Y) = (\mu_X, \mu_Y)$. Deduce the standard formula for error propagation ('adding errors in quadrature'):

$$\Delta f(X,Y) = \sqrt{\left(\frac{\partial f}{\partial X}\right)^2 (\Delta X)^2 + \left(\frac{\partial f}{\partial Y}\right)^2 (\Delta Y)^2}.$$

($*$) If you are taking Part IA Physics, check that this agrees with the results stated therein when $f(X,Y) = X+Y$ and $f(X,Y) = X/Y$.

➾ **Solution:** Using the multivariable Taylor expansion of $f(X,Y)$ about $(X,Y) = (\mu_X,\mu_Y)$, we have to linear order:

$$f(X,Y) \approx f(\mu_X,\mu_Y) + (X-\mu_X)\frac{\partial f}{\partial X}(\mu_X,\mu_Y) + (Y-\mu_Y)\frac{\partial f}{\partial Y}(\mu_X,\mu_Y).$$

Taking the variance of both sides, we note that $\mathrm{Var}(Z+c) = \mathrm{Var}(Z)$ for any constant c and any random variable Z, since a linear shift in the random variable values does not affect their spread. Hence:

$$\mathrm{Var}(f(X,Y)) \approx \mathrm{Var}\left(X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y}\right).$$

Next, we use that $\mathrm{Var}(aX+bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$ for independent random variables. This gives:

$$\mathrm{Var}(f(X,Y)) \approx \left(\frac{\partial f}{\partial X}\right)^2 \mathrm{Var}(X) + \left(\frac{\partial f}{\partial Y}\right)^2 \mathrm{Var}(Y),$$

as required. The standard formula for error propagation follows immediately, because by definition the standard deviation (or *error* in the physics literature) is the square root of the variance, $\Delta X = \sqrt{\mathrm{Var}(X)}$.

The formula for $f(X,Y) = X+Y$ thus takes the form:

$$\Delta(X+Y)^2 \approx (\Delta X)^2 + (\Delta Y)^2,$$

that is, for a sum the absolute error is given by adding the absolute errors in quadrature.

The formula for $f(X,Y) = X/Y$ similarly takes the form:

$$\Delta\left(\frac{X}{Y}\right)^2 \approx \frac{1}{\mu_Y^2}(\Delta X)^2 + \frac{\mu_X^2}{\mu_Y^4}(\Delta Y)^2.$$

Rearranging, we have:

$$\frac{\Delta(X/Y)^2}{(\mu_X/\mu_Y)^2} \approx \frac{(\Delta X)^2}{\mu_X^2} + \frac{(\Delta Y)^2}{\mu_Y^2}.$$

That is, for a ratio, the *relative* error is given by adding the *relative* errors in quadrature.

16. ($*$) If you are taking Part IA Physics, use the formula for propagation of error to determine the error in gravitational acceleration as determined from the period of a simple pendulum when the relative error in the string length is 0.1% and the relative error in the period is 0.2%.

➡ **Solution:** The period of a simple pendulum is $T = 2\pi\sqrt{l/g}$. Rearranging, we have:

$$g = \frac{4\pi^2 l}{T^2}.$$

Hence:

$$(\Delta g)^2 = \frac{16\pi^4}{T^4}(\Delta l)^2 + \frac{64\pi^4 l^2}{T^6}(\Delta T)^2.$$

Rearranging, we have:

$$\frac{(\Delta g)^2}{g^2} = \frac{(\Delta l)^2}{l^2} + \frac{4(\Delta T)^2}{T^2}.$$

It follows that the relative error in the gravitational acceleration is:

$$\sqrt{0.1^2 + 4(0.2)^2} = \sqrt{0.01 + 4(0.04)} = \sqrt{0.17}.$$

The single-variable chain rule, with multivariable functions

17. Let $z(x,y)$ be a function defined implicitly by the equation:

$$x - \alpha z = \phi(y - \beta z),$$

where α, β are real constants, and ϕ is an arbitrary differentiable function. Show that z satisfies the partial differential equation:

$$\alpha \frac{\partial z}{\partial x} + \beta \frac{\partial z}{\partial y} = 1.$$

[*Hint: you can still use the normal single-variable chain rule here when taking each of the partial derivatives! Why?*]

➾ **Solution:** We begin by taking the partial derivative with respect to x. Note that we are keeping y fixed, so differentiating $\phi(y - \beta z(x,y))$ is essentially just like differentiating something like $\phi(2 - \beta z(x))$ - we can use the ordinary chain rule on this kind of thing! We have:

$$1 - \alpha \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\phi(y - \beta z)\right) = -\beta \frac{\partial z}{\partial x}\phi'(y - \beta z).$$

Rearranging, we have:

$$\alpha \frac{\partial z}{\partial x} = \frac{\alpha}{\alpha - \beta\phi'(y - \beta z)}.$$

On the other hand, taking the derivative with respect to y, we have:

$$-\alpha \frac{\partial z}{\partial y} = \left(1 - \beta \frac{\partial z}{\partial y}\right)\phi'(y - \beta z).$$

Rearranging, we have:

$$\beta \frac{\partial z}{\partial y} = -\frac{\beta\phi'(y - \beta z)}{\alpha - \beta\phi'(y - \beta z)}$$

Summing our results, we get the equation in the question.

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18. Consider the function $u(x,y) = x\phi(y/x)$, where ϕ is a differentiable function of its argument and $x \neq 0$. Show that u satisfies:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$$

➾ **Solution:** Similarly to the previous question, we note that the derivative of u with respect to x is:

$$\frac{\partial u}{\partial x} = \phi(y/x) - \frac{y}{x}\phi'(y/x).$$

The derivative with respect to y is:

$$\frac{\partial u}{\partial y} = \phi'(y/x).$$

Hence we have:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\phi(y/x) - y\phi'(y/x) + y\phi'(y/x) = x\phi(y/x) = u,$$

as required.

19. If $u(x,y) = \phi(xy) + \sqrt{xy}\psi(y/x)$, where ϕ and ψ are twice-differentiable functions of their arguments, show that u satisfies the partial differential equation:

$$x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

➾ **Solution:** Taking the x-derivatives of u, we have:

$$\frac{\partial u}{\partial x} = y\phi'(xy) + \frac{1}{2}\sqrt{\frac{y}{x}}\psi(y/x) - \frac{y\sqrt{xy}}{x^2}\psi'(y/x).$$

Taking a second x-derivative, we have:

$$\frac{\partial^2 u}{\partial x^2} = y^2\phi''(xy) - \frac{1}{4}\frac{\sqrt{y}}{x^{3/2}}\psi(y/x) + \frac{y\sqrt{y}}{2x^{5/2}}\psi'(y/x) + \frac{y^2\sqrt{xy}}{x^4}\psi''(y/x).$$

On the other hand taking the y-derivative of u, we have:

$$\frac{\partial u}{\partial y} = x\phi'(xy) + \frac{1}{2}\sqrt{\frac{x}{y}}\psi(y/x) + \frac{\sqrt{xy}}{x}\psi'(y/x)$$

Taking a second y-derivative, we have:

$$\frac{\partial^2 u}{\partial y^2} = x^2\phi''(xy) - \frac{1}{4}\frac{\sqrt{x}}{y^{3/2}}\psi(y/x) + \frac{1}{2\sqrt{xy}}\psi'(y/x) + \frac{\sqrt{xy}}{x^2}\psi''(y/x).$$

Putting everything together, we have:

$$\begin{aligned} x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} &= x^2y^2\phi''(xy) - y^2x^2\phi''(xy) - \frac{1}{4}\sqrt{xy}\psi(y/x) + \frac{1}{4}\sqrt{xy}\psi(y/x) \\ &\quad + \frac{1}{2}x^{-1/2}y^{3/2}\psi'(y/x) - \frac{1}{2}x^{-1/2}y^{3/2}\psi'(y/x) + \frac{y^2\sqrt{xy}}{x^2}\psi''(y/x) - \frac{y^2\sqrt{xy}}{x^2}\psi''(y/x) = 0, \end{aligned}$$

as required.

20. Consider the partial differential equation:

$$2y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = xy(2y^2 - x^2).$$

(a) Show that $u(x,y) = \phi(x^2 + 2y^2)$ is a solution of the homogeneous version of this equation, where ϕ is an arbitrary differentiable function.

(b) By considering $u_p(x,y) = Ax^m y^n$ for some constants A, m, n, find a particular integral for this equation.

(c) Hence, find the complete solution of the equation subject to the boundary condition $u(x,1) = x^2$.

➺ Solution:

(a) We have:

$$\frac{\partial u}{\partial x} = 2x\phi'(x^2 + 2y^2), \qquad \frac{\partial u}{\partial y} = 4y\phi'(x^2 + y^2).$$

Hence:

$$2y \cdot 2x\phi'(x^2 + 2y^2) - x \cdot 4y\phi'(x^2 + y^2) = 0,$$

indeed solves the the homogeneous version of the differential equation.

(b) Let $u_p(x,y) = Ax^m y^n$. Then inserting into the PDE, we have:

$$2y \cdot Amx^{m-1}y^n - x \cdot nAx^m y^{n-1} = xy(2y^2 - x^2).$$

Collecting like terms on the left hand side, we have:

$$Ax^{m-1}y^{n-1}\left(2my^2 - nx^2\right),$$

so we should take $m = 2, n = 2$ and $A = 1/2$. The particular integral is then $u_p = \frac{1}{2}x^2y^2$.

(c) The general solution is therefore $u = \phi(x^2 + 2y^2) + \frac{1}{2}x^2y^2$. Imposing the boundary condition $u(x,1) = x^2$, we have:

$$x^2 = \phi(x^2 + 2) + \frac{1}{2}x^2.$$

Rearranging, we see that:

$$\phi(x^2 + 2) = \frac{1}{2}x^2.$$

Let $z = x^2 + 2$. Then $x^2 = z - 2$, giving:

$$\phi(z) = \frac{1}{2}(z - 2).$$

This shows the general solution obeying this boundary condition is:

$$u(x,y) = \frac{1}{2}(x^2 + 2y^2 - 2) + \frac{1}{2}x^2y^2.$$

21. Consider the partial differential equation:

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2},$$

where $\lambda > 0$.

(a) Show that $u(x,t) = (t+a)^{-1/2} v(y)$, where $y = (t+a)^{-1/2}(x+b)$, solves the equationif and only if v satisfies the ordinary differential equation:

$$-\frac{1}{2}\left(v + y\frac{dv}{dy}\right) = \lambda \frac{d^2 v}{dy^2}. \tag{$*$}$$

(b) Verify that ($*$) has a solution of the form $v(y) = e^{-cy^2}$ for appropriately chosen c.

(c) Using parts (a) and (b), find the solution of the original partial differential equation subject to the boundary condition:

$$u(x,0) = \exp\left(-(x+1)^2\right) + \exp\left(-(x-1)^2\right).$$

Solution:

(a) We have:

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{1}{2}(t+a)^{-3/2} v(y) + (t+a)^{-1/2}\frac{\partial y}{\partial t} v'(y) \\ &= -\frac{1}{2}(t+a)^{-3/2} v(y) - \frac{1}{2}(t+a)^{-2}(x+b) v'(y)\end{aligned}$$

On the other hand, we also have:

$$\frac{\partial u}{\partial x} = (t+a)^{-1/2}\frac{\partial y}{\partial x} v'(y) = (t+a)^{-1} v'(y),$$

and then:

$$\frac{\partial^2 u}{\partial x^2} = (t+a)^{-1}\frac{\partial y}{\partial x} v''(y) = (t+a)^{-3/2} v''(y).$$

Inserting into the equation, we have:

$$-\frac{1}{2}(t+a)^{-3/2} v(y) - \frac{1}{2}(t+a)^{-2}(x+b) v'(y) = \lambda (t+a)^{-3/2} v''(y)$$

Simplifying, we have:

$$-\frac{1}{2}\left(v(y) + (t+a)^{-1/2}(x+b) v'(y)\right) = \lambda v''(y),$$

which on using $y = (t+a)^{-1/2}(x+b)$ gives:

$$-\frac{1}{2}\left(v + y\frac{dv}{dy}\right) = \lambda \frac{d^2 v}{dy^2},$$

as required.

(b) Inserting $v = e^{-cy^2}$, we have:

$$-\frac{1}{2}\left(e^{-cy^2} - 2cy^2 e^{-cy^2}\right) = \lambda\left(-2ce^{-cy^2} + 4c^2 y^2 e^{-cy^2}\right).$$

Comparing coefficients, we see that $-\frac{1}{2} = -2\lambda c$, and $c = 4\lambda c^2$. Both of these are consistent, and give $c = 1/4\lambda$.

(c) The solution we have derived is:

$$u(x,t) = (t+a)^{-1/2} \exp\left(-\frac{(x+b)^2}{4\lambda(t+a)}\right),$$

using parts (a) and (b). This cannot hope to satisfy the initial condition at $t = 0$, because it is of the incorrect form. *However* the equation is linear, so we can take the linear combination of two solutions of this form easily:

$$u(x,t) = C_1(t+a)^{-1/2} \exp\left(-\frac{(x+b)^2}{4\lambda(t+a)}\right) + C_2(t+a')^{-1/2} \exp\left(-\frac{(x+b')^2}{4\lambda(t+a')}\right),$$

where we are allowed to pick the constants C_1, C_2, a, b, a', b' in each term to try to satisfy the initial data. This is still a solution by linearity.

At $t = 0$, we need $u(x,0) = \exp(-(x+1)^2) + \exp(-(x-1)^2)$. This suggests choosing $b = 1, b' = -1$, and $a = 1/4\lambda, a' = 1/4\lambda$. Further, we see that we should choose $C_1 = 1/\sqrt{4\lambda}$ and $C_2 = 1/\sqrt{4\lambda}$. Overall the solution takes the form:

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\lambda}}\left(\frac{1}{\sqrt{t+1/4\lambda}} \exp\left(-\frac{(x+1)^2}{4\lambda(t+1/4\lambda)}\right) + \frac{1}{\sqrt{t+1/4\lambda}} \exp\left(-\frac{(x-1)^2}{4\lambda(t+1/4\lambda)}\right)\right) \\ &= \frac{1}{\sqrt{4\lambda t+1}} \exp\left(-\frac{(x+1)^2}{4\lambda t+1}\right) + \frac{1}{\sqrt{4\lambda t+1}} \exp\left(-\frac{(x-1)^2}{4\lambda t+1}\right). \end{aligned}$$