Part IA: Mathematics for Natural Sciences
2014 Paper 1 (Unofficial) Mark Scheme

Section A

1. (a) Differentiate:

$$\frac{1}{x^2+4}$$

with respect to x. [1]

Solution: We use the chain rule:

$$\frac{d}{dx}\frac{1}{x^2+4} = -\frac{2x}{(x^2+4)^2}.$$

[*1 mark for correct answer.*]

(b) Differentiate:

$$e^{\sin(x)}$$

with respect to x. [1]

Solution: We again use the the chain rule:

$$\frac{d}{dx}e^{\sin(x)} = \cos(x)e^{\sin(x)}.$$

[*1 mark for correct answer.*]

2. (a) Differentiate a^{-x} with respect to x, where a is a constant which satisfies $a > 0$ and $a \neq 1$. [1]

Solution: Write $a^{-x} = e^{-x\log(a)}$ (this is okay because $a > 0$). We can now differentiate straightforwardly to give:

$$\log(a)e^{-x\log(a)} = \log(a)a^{-x}.$$

This answer still works when $a = 1$, because $a^{-x} = 1$ in that case, and $\log(a) = 0$. [*1 mark for correct answer.*]

(b) Evaluate the indefinite integral: [1]

$$\int \frac{\ln(\ln(x))}{x}\,dx.$$

Solution: Consider the substitution $u = \ln(x)$. Then $du = dx/x$, hence the integral becomes:

$$\int \ln(u)\,du = \int 1\cdot\ln(u)\,du = u\ln(u) - \int u\cdot\frac{1}{u}\,du = u\ln(u) - u + c,$$

using integration by parts to determine the integral of $\ln(u)$. Inserting $u = \ln(x)$, we see the integral is:

$$\ln(x)\ln(\ln(x)) - \ln(x) + c.$$

[*1 mark for correct answer.*]

3. (a) Evaluate the definite integral: [1]

$$\int_0^{\frac{\pi}{4}} \tan(x)\,dx.$$

Solution: We have:

$$\int_0^{\frac{\pi}{4}} \tan(x)\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos(x)}\,dx = -\left[\ln(\cos(x))\right]_0^{\pi/4} = -\left(\ln(1/\sqrt{2})\right) = \ln(\sqrt{2}).$$

[*1 mark for correct answer.*]

(b) Evaluate the definite integral: [1]

$$\int_{-2}^{-1} \frac{dx}{x}.$$

Solution: We have:

$$\int_{-2}^{-1} \frac{dx}{x} = \left[\ln|x|\right]_{-2}^{-1} = -\ln(2).$$

Crucially, the modulus is needed here! [*1 mark for correct answer.*]

4. (a) Find the general solution of the differential equation:

$$\frac{dy}{dx} = \cos^2(y)\sin(x)$$

for $-\pi/2 < y < \pi/2$. [1]

Solution: This equation is separable. We have:

$$\int \sec^2(y)\,dy = \int \sin(x)\,dx \qquad \Leftrightarrow \qquad \tan(y) = \sin(x) + c.$$

Rearranging, we have:

$$y = \arctan(\sin(x) + c),$$

which is okay because we are given that $-\pi/2 < y < \pi/2$, so we don't need to use any periodicity to get any other solutions. [*1 mark for correct answer, in simplified form.*]

(b) Find the solution of the differential equation:

$$\frac{dy}{dx} = 3y$$

such that $y = 3$ when $x = 0$. [1]

Solution: There are various ways of solving this (e.g. using an integrating factor, separating variables, or using the auxiliary equation), however we can just spot the general answer $y = Ae^{3x}$. Requiring $y = 3$ when $x = 0$, we get $y = 3e^{3x}$. [*1 mark for correct answer.*]

5. (a) Find all pairs of coordinates where the curve defined by:

$$7x^2 - y^2 = 7$$

meets the straight line $y = x + 1$. [1]

Solution: Substituting $y = x + 1$ into the quadratic, we have:

$$7x^2 - (x+1)^2 = 7 \qquad \Rightarrow \qquad 6x^2 - 2x - 8 = 0.$$

Simplifying, this becomes:

$$0 = 3x^2 - x - 4 = (3x-4)(x+1),$$

hence the solutions are $(x, y) = (\frac{4}{3}, \frac{7}{3}), (-1, 0)$. [*1 mark for both intersections correct.*]

(b) Sketch the curve defined by the equation $(x-1)^2 + 2y^2 = 3$. [1]

Solution: This curve is an ellipse centred on $(x, y) = (1, 0)$. It extends a radial distance $\sqrt{3}$ either side of $x = 1$, and a radial distance of $\sqrt{3/2}$ either side of $y = 0$. [*1 mark for convincing sketch, clearly labelled with centre and lengths of minor and major axes.*]

6. (a) A sphere has radius 10 m. Draw on its surface a circular patch which has area 10 m^2. What fraction of the surface of the sphere does this cover? [1]

Solution: Consider spherical coordinates (r, θ, ϕ) on the surface of the sphere, with $r = 10$ m. A circle on the surface of the sphere centred on the z-axis, subtending an angle θ_0 from the z-axis, has area:

$$A(\theta) = \int_0^{2\pi} \int_0^{\theta_0} r^2 \sin(\theta)\, d\theta d\phi = 2\pi r^2 [-\cos(\theta)]_0^{\theta_0} = 2\pi r^2 (1 - \cos(\theta_0)).$$

In particular, if we need a circular patch of area 10 m^2, we require r m $= 2\pi r^2 (1 - \cos(\theta_0))$, which implies $\cos(\theta_0) = 1 - 1/20\pi$. This is very close to 1, so the angle θ_0 is very small. This allows us to provide a sketch. The fraction of the area is:

$$\frac{10}{4\pi(10)^2} = \frac{1}{400\pi}.$$

[*1 mark for sketch of very small circular area on surface of sphere, AND fraction of area correct.*]

(b) A square pyramid has all of its sides of length 1 m. What is its volume? [1]

Solution: The volume of a pyramid is a third the area of its base, multiplied by its height. The area of the base is 1 m^2 in this case. The height is given by h, satisfying:

$$1^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + h^2,$$

using three-dimensional Pythagoras. Hence $h = \frac{1}{2}$, and it follows the volume is $\frac{1}{6}$ m^3. [*1 mark for correct answer.*]

7. (a) Sketch the graph of: [1]

$$y = \frac{1}{1 + \tan(x)}.$$

Solution: The graph is periodic with period π. It has asymptotes at $\tan(x) = -1$, i.e. at $x = 3\pi/4 + n\pi$ where n is an integer. The graph passes zero when $x = \pi/2 + n\pi$ (since there, $\tan(x)$ is singular). In between, the graph looks like an inverted tangent graph. [*1 mark for correct sketch.*]

(b) Sketch the graph of $y = e^{-x^3}$. [1]

Solution: Observe that as $x \to \infty$, $y \to 0$. Additionally, as $x \to -\infty$, $y \to +\infty$. The graph is everywhere positive and passes through the origin at $x = 0$. Finally, note that:

$$\frac{d}{dx} e^{-x^3} = -3x^2 e^{-x^3},$$

so the derivative is everywhere negative; there is a point of inflection at $x = 0$. The graph looks like a decaying exponential graph, just flatter at the origin $x = 0$. [*1 mark for correct sketch.*]

8. (a) Find the values of x at the stationary points of the function: [1]

$$y = x^3 - 2x^2 - 7x + 6.$$

Solution: The stationary points satisfy:

$$0 = \frac{dy}{dx} = 3x^2 - 4x - 7 = (3x - 7)(x + 1),$$

hence occur at $x = 7/3$ and $x = -1$. [*1 mark for correct answer.*]

(b) The function $y = x^3 - 3x + 7$ has a stationary point at $x = 1$. Is this point a maximum, minimum or a point of inflection? [1]

Solution: Taking two derivatives, we have:

$$\frac{d^2y}{dx^2} = 6x.$$

Hence at $x = 1$, there is a minimum (second derivative is positive). [*1 mark for correct answer.*]

9. (a) What is the area bounded by the curve $y = x^2 - 3x + 2$, the positive half of the x-axis, the positive half of the y-axis, and the line $x = 1/2$? [1]

(b) Sketch the curve(s) defined by the relation $y^2 = x^3$. [1]

Solution: (a) Observe that $0 = x^2 - 3x + 2 = (x-2)(x-1)$, so that there are turning points of the function at $x = 1$ and $x = 2$. Therefore, for $0 < x < 1/2$ the function lies entirely above the x-axis. Therefore, this question is just asking us to compute the integral:

$$\int_0^{1/2} (x^2 - 3x + 2)\, dx = \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_0^{1/2} = \frac{1}{24} - \frac{3}{8} + 1 = \frac{16}{24} = \frac{2}{3}.$$

[1 *mark for correct answer.*]

(b) This is the square root of the graph of $y = x^3$. Hence the graph does not exist for $x < 0$, and for $x > 0$, the graph looks like it has a cusp at $x = 0$, and is mirror-symmetric about the line $y = 0$. [1 *mark for correct graph.*]

10. (a) The straight line L is defined by the equation $y = 2x + 3$. Find the equation of the line L' that is perpendicular to L and intersects L where L crosses the x-axis. [1]

Solution: A perpendicular line is $y = -\frac{1}{2}x + c$. The place where L crosses the x-axis is $(x, y) = (-3/2, 0)$. Hence for the perpendicular to intersect there, we require $0 = 3/4 + c$, which gives:

$$y = -\frac{1}{2}x - \frac{3}{4},$$

as the required equation of the line. [1 *mark for correct answer.*]

(b) Express:

$$\frac{13(x+1)}{(x-4)(x+9)}$$

as a sum of partial fractions. [1]

Solution: We have:

$$\frac{13(x+1)}{(x-4)(x+9)} = \frac{A}{x-4} + \frac{B}{x+9} \qquad \Rightarrow \qquad 13(x+1) = A(x+9) + B(x-4).$$

Set $x = 4$, which gives $13 \cdot 5 = 13A$, which implies $A = 5$. Set $x = -9$, then $13 \cdot (-8) = -13B$, hence $B = 8$. Thus the partial fractions are:

$$\frac{13(x+1)}{(x-4)(x+9)} = \frac{5}{x-4} + \frac{8}{x+9}.$$

[1 *mark for correct answer.*]

Section B

11. (a) Calculate $\mathrm{Det}(A)$ and $\mathrm{Tr}(A)$ where:

$$A = \begin{pmatrix} 1 & -4 & 7 \\ -4 & 4 & -4 \\ 7 & -4 & 1 \end{pmatrix}.$$

From the value of $\mathrm{Det}(A)$ make a deduction about the eigenvalues of A. [4]

Solution: The trace is $\mathrm{Tr}(A) = 1+4+1 = 6$. [*1 mark.*] The determinant can be found more simply by considering equivalent row operations on the matrix. Adding the second row to the first, and adding the second row to the third, we have:

$$\mathrm{Det}(A) = \mathrm{Det}\begin{pmatrix} 1 & -4 & 7 \\ -4 & 4 & -4 \\ 7 & -4 & 1 \end{pmatrix} = \mathrm{Det}\begin{pmatrix} -3 & 0 & 3 \\ -4 & 4 & -4 \\ 3 & 0 & -3 \end{pmatrix}.$$

But note that now the first and last rows are scalar multiples of one another, so the determinant vanishes, $\mathrm{Det}(A) = 0$. [*1 mark for partial progress to calculation of determinant, 1 mark for correct determinant.*] The determinant is the product of the eigenvalues, so at least one eigenvalue must be 0 (the other two must sum to 6 because the trace is the sum of the eigenvalues). [*1 mark for saying at least one eigenvalue is zero.*]

(b) Calculate the eigenvalues and the corresponding normalised eigenvectors of A. Verify that the eigenvectors are mutually orthogonal. [10]

Solution: The eigenvalues satisfy the following equation:

$$\begin{aligned} 0 &= \mathrm{Det}\begin{pmatrix} 1-\lambda & -4 & 7 \\ -4 & 4-\lambda & -4 \\ 7 & -4 & 1-\lambda \end{pmatrix} \\ &= (1-\lambda)((4-\lambda)(1-\lambda)-16) + 4(-4(1-\lambda)+28) + 7(16-7(4-\lambda)) \\ &= (1-\lambda)(-12-5\lambda+\lambda^2) + 4(24+4\lambda) + 7(7\lambda-12) \\ &= -12+7\lambda+6\lambda^2-\lambda^3+96+16\lambda-84+49\lambda \\ &= -\lambda^3+6\lambda^2+72\lambda \\ &= -\lambda(\lambda^2-6\lambda-72) \\ &= -\lambda(\lambda-12)(\lambda+6). \end{aligned}$$

Hence the eigenvalues are $\lambda = 0$, $\lambda = -6$ and $\lambda = 12$, consistent with the results of the first part. [*1 mark for correct determinant, 1 mark for correct reduction to cubic, 1 mark for factoring cubic correctly and finding the roots.*] For the eigenvectors, observe that for $\lambda = 0$, we require:

$$\begin{pmatrix} 1 & -4 & 7 \\ -4 & 4 & -4 \\ 7 & -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,$$

which is satisfied by any scalar multiple of:

$$\begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ -3 \end{pmatrix},$$

so a normalised eigenvector is $(1, 2, 1)/\sqrt{6}$. [*1 mark for some correct working, 1 mark for correct eigenvector.*]

For $\lambda = -6$, we require:

$$\begin{pmatrix} 7 & -4 & 7 \\ -4 & 10 & -4 \\ 7 & -4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,$$

which is satisfied by any scalar multiple of:

$$\begin{pmatrix} 7 \\ -4 \\ 7 \end{pmatrix} \times \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} -27 \\ 0 \\ 27 \end{pmatrix},$$

so a normalised eigenvector is $(1, 0, -1)/\sqrt{2}$. [*1 mark for some correct working, 1 mark for correct eigenvector.*]

For $\lambda = 12$, we require:

$$\begin{pmatrix} -11 & -4 & 7 \\ -4 & -8 & -4 \\ 7 & -4 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

which is satisfied by any scalar multiple of:

$$\begin{pmatrix} -11 \\ -4 \\ 7 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -18 \\ 18 \\ -18 \end{pmatrix},$$

so a normalised eigenvector is $(1, -1, 1)/\sqrt{3}$. [*1 mark for some correct working, 1 mark for correct eigenvector.*]

Finally, we check that they are mutually orthogonal. We have $(1, 2, 1) \cdot (1, 0, -1) = 1 - 1 = 0$, $(1, 2, 1) \cdot (1, -1, 1) = 1 - 2 + 1 = 0$ and $(1, 0, -1) \cdot (1, -1, 1) = 1 - 1 = 0$. [*1 mark for verifying mutually orthogonal.*]

(c) By expressing an arbitrary vector $\mathbf{r}$ in terms of the eigenvectors or otherwise, show that a non-zero vector $\mathbf{e}$ exists such that $A\mathbf{r} \cdot \mathbf{e} = 0$ for all $\mathbf{r}$. [4]

Solution: Let the eigenvectors be $\mathbf{e}_0, \mathbf{e}_{-6}, \mathbf{e}_{12}$ respectively. Then if $\mathbf{r} = \alpha\mathbf{e}_0 + \beta\mathbf{e}_{-6} + \gamma\mathbf{e}_{12}$ [*1 mark for expressing* $\mathbf{r}$ *in this form.*], we have:

$$A\mathbf{r} = -6\beta\mathbf{e}_{-6} + 12\gamma\mathbf{e}_{12}.$$

[*2 marks for correct action of A.*] Since the eigenvectors are mutually orthogonal, observe that as a result we have $A\mathbf{r} \cdot \mathbf{e}_0 = 0$. Hence there exists a non-zero vector $\mathbf{e} = \mathbf{e}_0$ such that $A\mathbf{r} \cdot \mathbf{e} = 0$ for all $\mathbf{r}$. [*1 mark for correctly identifying $\mathbf{e} = \mathbf{e}_0$.*]

(d) Describe in words the action of A on an arbitrary non-zero vector. [2]

Solution: A projects vectors into the plane with normal $\mathbf{e}_0 = (1, 2, 1)$. It scales the vector by a factor of 12 in the $\mathbf{e}_{12}$ direction, and by a factor of -6 in the $\mathbf{e}_{-6}$ direction. [*1 mark for saying projection into plane, 1 mark for saying scaling behaviour in remaining directions.*]

12. (a) Express the cube roots of $i-1$ in terms of their modulus and argument. [4]

Solution: We begin by writing $i-1$ in modulus argument form. Its modulus is $\sqrt{2}$ and its argument is $3\pi/4$ (think of an Argand diagram), hence:

$$i-1=\sqrt{2}e^{3i\pi/4}.$$

[*1 mark for correct modulus, 1 mark for correct argument.*] It follows that the cube roots are:

$$\sqrt{2}e^{i\pi/4}, \qquad \sqrt{2}e^{i\pi/4+2\pi/3}=\sqrt{2}e^{11\pi i/12}, \qquad \sqrt{2}e^{i\pi/4+4\pi/3}=\sqrt{2}e^{19\pi i/12}=\sqrt{2}e^{-5i\pi/12}.$$

[*1 mark for some correct cube roots, and an additional 1 mark for all correct cube roots.*]

(b) Find all solutions to the equation $\tanh(z)=-i$. [5]

Solution: Writing $\tanh(z)$ in terms of exponentials, we have:

$$\frac{e^z-e^{-z}}{e^z+e^{-z}}=-i.$$

[*1 mark for expressing $\tanh(z)$ in terms of exponentials.*] Multiplying up, then multiplying through by e^z, we have:

$$e^{2z}-1=-ie^{2z}-i.$$

Rearranging, we have:

$$e^{2z}=\frac{1-i}{1+i}=\frac{(1-i)^2}{2}=-i.$$

[*1 mark for rearranging to quadratic equation.*] Hence:

$$2z=\ln(-i)=-\frac{i\pi}{2}+2ni\pi,$$

[*1 mark for correctly taking logarithm using argument of $-i$, additional 1 mark for remembering many solutions.*] from which it follows that:

$$z=-\frac{i\pi}{4}+in\pi.$$

[*1 mark for correct final answer (including many solutions).*]

(c) Given that $z = 2 + i$ solves the equation:

$$z^3 - (4+2i)z^2 + (4+5i)z - (1+3i) = 0,$$

find the remaining solutions. [6]

Solution: We need to factorise the left hand side. Observe that:

$$\frac{1+3i}{2+i} = \frac{(1+3i)(2-i)}{5} = \frac{5+5i}{5} = 1+i.$$

Hence, we have:

$$z^3 - (4+2i)z^2 + (4+5i)z - (1+3i) = (z-(2+i))(z^2-(2+i)z+(1+i)).$$

[*Up to 3 marks for correctly factorising using given factor; award some marks for working if appropriate.*] Using the quadratic formula on the second factor, we have:

$$z = \frac{2+i \pm \sqrt{(2+i)^2 - 4(1+i)}}{2} = \frac{2+i \pm \sqrt{-1}}{2} = \frac{2+i \pm i}{2} = 1+i, \text{ or } 1.$$

Thus the solutions are:

$$2+i, \qquad 1, \qquad 1+i.$$

[*Up to 3 marks for solution of quadratic; 1 mark for using quadratic formula, 1 mark for correctly simplifying, 1 mark for complete correct statement of roots.*]

(d) Use complex numbers to show that $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$. [5]

Solution: We have, by De Moivre's theorem:

$$\begin{aligned}
\cos(4\theta) &= \mathrm{Re}\left(\cos(4\theta) + i\sin(4\theta)\right) \\
&= \mathrm{Re}\left((\cos(\theta) + i\sin(\theta))^4\right) \\
&= \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta) \\
&= \cos^4(\theta) - 6\cos^2(\theta)(1-\cos^2(\theta)) + (1-\cos^2(\theta))^2 \\
&= \cos^4(\theta) - 6\cos^2(\theta) + 6\cos^4(\theta) + \left(1 - 2\cos^2(\theta) + \cos^4(\theta)\right) \\
&= 8\cos^4(\theta) - 8\cos^2(\theta) + 1,
\end{aligned}$$

as required. [*1 mark for writing as real part, 1 mark for using De Moivre's theorem, 1 mark for correct binomial expansion, 1 mark for some simplification using trigonometric identities, 1 mark for correct final answer.*]

13. (a) Solve the following differential equations for $y(x)$ subject to the listed boundary conditions, making your answer explicit for y.

(i) $\dfrac{dy}{dx} + 3y = 8$, with $y(0) = 4$. [3]

(ii) $\dfrac{dy}{dx} - y\cos(x) = \dfrac{1}{2}\sin(2x)$, with $y(0) = 0$. [7]

Solution: (a) This is a linear differential equation with constant coefficients, so can be solved using the auxiliary equation approach. The auxiliary equation is $\lambda + 3 = 0$ with roots $\lambda = -3$, hence the complementary function is $y_c = Ae^{-3x}$. A particular integral is clearly $y_p = 8/3$, hence the complete general solution is:

$$y = Ae^{-3x} + \frac{8}{3}.$$

Imposing $y(0) = 4$, we have $A + 8/3 = 4 = 12/3$, hence $A = 4/3$. Thus we have:

$$y = \frac{4}{3}\left(e^{-3x} + 2\right).$$

(b) This equation does not have constant coefficients, but is still linear. It can be solved instead using an integrating factor; an appropriate one is clearly $e^{-\sin(x)}$. We get:

$$\frac{d}{dx}\left(ye^{-\sin(x)}\right) = \frac{1}{2}e^{-\sin(x)}\sin(2x) = \sin(x)\cos(x)e^{-\sin(x)}.$$

Integrating both sides, we have:

$$ye^{-\sin(x)} = \int \sin(x)\cos(x)e^{-\sin(x)}\,dx.$$

To do the integral, consider making the substitution $u = \sin(x)$, so that $du = \cos(x)\,dx$. Then we get:

$$\int ue^{-u}\,du = -ue^{-u} + \int e^{-u}\,du = -ue^{-u} - e^{-u} + c.$$

Hence the solution of the differential equation is:

$$y = ce^{\sin(x)} - \sin(x) - 1.$$

Imposing the initial data, we require $y(0) = 0 = c - 1$, so that $c = 1$. Thus the particular solution is:

$$y = e^{\sin(x)} - \sin(x) - 1.$$

(b) The function $y(x)$ satisfies the differential equation:

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 2e^{-3x}.$$

Solve the equation for $y(x)$ subject to $y(0) = 1$ and $\left.\frac{dy}{dx}\right|_{x=0} = 0.$ [10]

Solution: The auxiliary equation is $0 = \lambda^2 + 7\lambda + 12 = (\lambda + 4)(\lambda + 3)$, hence the complementary function is:

$$y_c = Ae^{-4x} + Be^{-3x}.$$

Since e^{-3x} is already contained in the complementary function, we posit $y_p = \alpha x e^{-3x}$ as a potential particular integral. Then:

$$y_p' = \alpha e^{-3x} - 3\alpha x e^{-3x}, \qquad y_p'' = -6\alpha e^{-3x} + 9\alpha x e^{-3x}.$$

Hence we require:

$$-6\alpha e^{-3x} + 9\alpha x e^{-3x} + 7\alpha e^{-3x} - 21\alpha x e^{-3x} + 12\alpha x e^{-3x} = 2e^{-3x}.$$

Comparing coefficients, we see $\alpha = 2$. Thus the complete solution is:

$$y = Ae^{-4x} + Be^{-3x} + 2xe^{-3x}.$$

It remains to impose the initial data. We have $y(0) = 1$, so $A + B = 1$. We also have $y'(0) = 0$, so we require:

$$-4A - 3B + 2 = 0 \qquad \Leftrightarrow \qquad 4A + 3B = 2.$$

Solving these equations simultaneously, we have:

$$A + 3(A+B) = 2 \qquad \Rightarrow \qquad A + 3 = 2 \qquad \Rightarrow \qquad A = -1, B = 2.$$

Thus the particular solution is:

$$y = 2e^{-3x} - e^{-4x} + 2xe^{-3x}.$$

14. (a) An arbitrary point along a straight line in a three-dimensional space can be written as $\mathbf{r}_1 = \mathbf{a} + \lambda\hat{\mathbf{b}}$, where λ is a scale parameter and $\hat{\mathbf{b}}$ is a unit vector. Obtain a formula for the minimum distance between $\mathbf{r}_1$ and $\mathbf{r}_2 = \mathbf{c} + \mu\hat{\mathbf{d}}$, where $\hat{\mathbf{d}}$ is a unit vector, assuming that the two lines are not parallel. [5]

Solution: A vector orthogonal to both lines is $\hat{\mathbf{b}} \times \hat{\mathbf{d}}$ [1 *mark for explaining this.*]. A vector that joins both lines is $\mathbf{c} - \mathbf{a}$ [1 *mark for explaining this.*]. Projecting this vector onto the orthogonal direction between the lines gives the shortest distance [1 *mark for recognising this.*]; we have:

$$\frac{|(\mathbf{c} - \mathbf{a}) \cdot \hat{\mathbf{b}} \times \hat{\mathbf{d}}|}{|\hat{\mathbf{b}} \times \hat{\mathbf{d}}|}.$$

[1 *mark for correct numerator, 1 mark for normalising correctly.*] Note that $\hat{\mathbf{b}}$, $\hat{\mathbf{d}}$ unit vectors does *not* imply that $\hat{\mathbf{b}} \times \hat{\mathbf{d}}$ is a unit vector!

(b) Find all vectors $\mathbf{x}$ that obey the equation $\mathbf{x} \cdot \mathbf{p} = k$, where $\mathbf{p}$ is a fixed non-zero vector in a three-dimensional space and k is a fixed scalar. [7]

[*Hint: Your answer should contain an arbitrary non-zero vector $\boldsymbol{q}$ which can be taken to be non-collinear with $\boldsymbol{p}$, i.e. $\boldsymbol{p} \times \boldsymbol{q} \neq \boldsymbol{0}$. You should treat the cases $\boldsymbol{p} \cdot \boldsymbol{q} \neq 0$ and $\boldsymbol{p} \cdot \boldsymbol{q} = 0$ separately.*]

[*Notation: $\boldsymbol{u} \times \boldsymbol{v}$ is equivalent to $\boldsymbol{u} \wedge \boldsymbol{v}$.*]

Solution: This is a rather poorly phrased question, so don't worry about it too much if it seemed hard.

Suppose we are given an arbitrary non-zero vector $\mathbf{q}$, which we assume is non-collinear to $\mathbf{p}$. Then $\{\mathbf{p}, \mathbf{q}, \mathbf{p} \times \mathbf{q}\}$ form a (possibly non-orthogonal basis) for $\mathbb{R}^3$, and hence we may express $\mathbf{x}$ in the form:

$$\mathbf{x} = a\mathbf{p} + b\mathbf{q} + c\mathbf{p} \times \mathbf{q}.$$

Imposing the condition $\mathbf{x} \cdot \mathbf{p} = k$, we have:

$$k = a\mathbf{p} \cdot \mathbf{p} + b\mathbf{p} \cdot \mathbf{q}.$$

Rearranging to obtain a, we have:

$$a = \frac{k - b\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}|^2}.$$

Hence we can express $\mathbf{x}$ as:

$$\mathbf{x} = \frac{k - b\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}|^2}\mathbf{p} + b\mathbf{q} + c\mathbf{p} \times \mathbf{q}.$$

There is no need to distinguish the cases $\mathbf{p} \cdot \mathbf{q} = 0$ and $\mathbf{p} \cdot \mathbf{q} \neq 0$ as the question claims.

(c) A particle moves along a path on which the position coordinate in terms of a parameter t is given by:

$$x = \frac{\cos(t)}{\sqrt{1+t^2}}, \qquad y = \frac{\sin(t)}{\sqrt{1+t^2}}, \qquad z = \frac{t}{\sqrt{1+t^2}}.$$

Express the equation for a point on the path in spherical polar coordinates $r(t), \theta(t), \phi(t)$. [8]

Solution: We have:

$$r(t) = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{\cos^2(t)+\sin^2(t)+t^2}{1+t^2}} = \sqrt{\frac{1+t^2}{1+t^2}} = 1.$$

[*1 mark for correct formula for* r*. 1 mark for correct substitution. 1 mark for correct simplified form.*] Similarly, we have:

$$\theta(t) = \arccos\left(\frac{z}{r}\right) = \arccos\left(\frac{t}{\sqrt{1+t^2}}\right).$$

[*1 mark for correct formula for* θ*. 1 mark for correct substitution. 1 mark for correct simplified form.*] Finally, we have:

$$\phi(t) = \arctan\left(\frac{y}{x}\right) = \arctan\left(\tan(t)\right) = t.$$

[*1 mark for correct formula for* ϕ*. 1 mark for correct substitution and simplification.*]

15. (a) The area of integration, D, is defined in plane polar coordinates (r, ϕ) by the inequality $r_2 \le r \le r_1$ where $r_1 = 1 + \cos(\phi)$ and $r_2 = 3/2$.

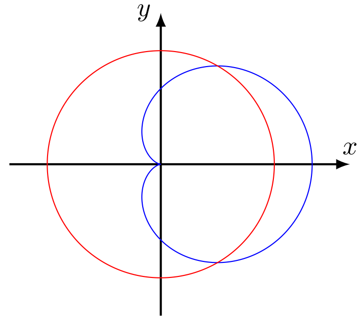
(i) Sketch the area of integration. [4]

(ii) Calculate the value of the area D. [6]

(iii) Evaluate the following integral over this area: [5]

$$\iint_D \frac{x + y + xy}{x^2 + y^2}\, dxdy.$$

Solution: (i) The region of integration is between a circle and a *cardioid*, as shown in the figure below.



This is because the cardioid ranges between 0 and 2, so there are intersections between the circle $r = 3/2$ and the cardioid $r = 1 + \cos(\phi)$. There is an intersection when $3/2 = 1 + \cos(\phi)$, which gives $\cos(\phi) = 1/2$, and hence $\phi = \pm\pi/3$. [*1 mark for circle drawn correctly. 2 marks for cardioid drawn correctly (1 mark for partially). 1 mark for intersection angle computed.*]

(ii) The area of the region D is:

$$\iint_D dA = \int_{-\pi/3}^{\pi/3} \int_{3/2}^{1+\cos(\phi)} r dr d\phi$$

$$= \frac{1}{2} \int_{-\pi/3}^{\pi/3} \left[r^2\right]_{3/2}^{1+\cos(\phi)} d\phi$$

$$= \frac{1}{2} \int_{-\pi/3}^{\pi/3} \left((1 + \cos(\phi))^2 - \frac{9}{4}\right) d\phi$$

$$= \frac{1}{2} \int_{-\pi/3}^{\pi/3} \left(\cos^2(\phi) + 2\cos(\phi) - \frac{5}{4}\right) d\phi$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(\frac{1}{2}\cos(2\phi) + 2\cos(\phi) - \frac{3}{4}\right)\, d\phi$$

$$= \frac{1}{2}\left[\frac{1}{4}\sin(2\phi) + 2\sin(\phi) - \frac{3}{4}\phi\right]_{-\pi/3}^{\pi/3}$$

$$= \frac{1}{2}\left(\frac{1}{4}\sqrt{3} + 2\sqrt{3} - \frac{\pi}{2}\right)$$

$$= \frac{9\sqrt{3}}{8} - \frac{\pi}{4}$$

[*1 mark for correct polar measure $rdrd\phi$. 1 mark for integral written correctly with correct limits. 1 mark for correct evaluation of radial integral. 1 mark for simplification. 1 mark for correct angular integral. 1 mark for final answer.*]

(iii) The integrand can be expressed in terms of polar coordinates as:

$$\frac{x+y+xy}{x^2+y^2}dxdy = \frac{r\cos(\phi) + r\sin(\phi) + r^2\cos(\phi)\sin(\phi)}{r^2} rdrd\phi$$

$$= (\cos(\phi) + \sin(\phi) + r\sin(\phi)\cos(\phi))\, drd\phi.$$

Hence we have:

$$\iint_D \frac{x+y+xy}{x^2+y^2}dxdy = \int_{-\pi/3}^{\pi/3}\int_{3/2}^{1+\cos(\phi)} (\cos(\phi) + \sin(\phi) + r\sin(\phi)\cos(\phi))\, drd\phi$$

$$= \int_{-\pi/3}^{\pi/3} \left[r(\cos(\phi) + \sin(\phi)) + \frac{1}{2}r^2\cos(\phi)\sin(\phi)\right]_{3/2}^{1+\cos(\phi)} d\phi$$

$$= \int_{-\pi/3}^{\pi/3} \Big((1+\cos(\phi))(\cos(\phi)+\sin(\phi)) + \frac{1}{2}(1+2\cos(\phi)+\cos^2(\phi))\cos(\phi)\sin(\phi)$$

$$- \frac{3}{2}(\cos(\phi)+\sin(\phi)) - \frac{9}{8}\cos(\phi)\sin(\phi)\Big)\, d\phi$$

Note that terms involving a single sine will cancel because they are odd functions integrated over an even range. Further, $\cos(\phi)\sin(\phi) = \frac{1}{2}\sin(2\phi)$, so these terms will cancel too. Hence this simplifies to:

$$= \int_{-\pi/3}^{\pi/3} \left(\cos(\phi) + \cos^2(\phi) - \frac{3}{2}\cos(\phi) \right) d\phi$$

$$= \frac{1}{2} \int_{-\pi/3}^{\pi/3} \left(1 + \cos(2\phi) - \cos(\phi)\right)\, d\phi$$

$$= \frac{1}{2} \left[\phi + \frac{1}{2}\sin(2\phi) - \sin(\phi) \right]_{-\pi/3}^{\pi/3}$$

$$= \frac{1}{2} \left[\frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \sqrt{3} \right]$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

[*1 mark for integrand written correctly in polar coordinates. 1 mark for integral written correctly with correct limits. 1 mark for radial integral done correctly. 1 mark for correct simplification of angular integrand. 1 mark for correct angular integral and final answer.*]

(b) Evaluate the triple integral:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \exp\left[-\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{a^2}\right] dxdydz,$$

where $a > 0, x_0, y_0$ and z_0 are real constants. [5]

Solution: Observe that we can separate the integral into three separate factors:

$$\left(\int_{-\infty}^{\infty} x \exp\left(-\frac{(x-x_0)^2}{a^2}\right) dx\right) \cdot \left(\int_{-\infty}^{\infty} \exp\left(-\frac{(y-y_0)^2}{a^2}\right) dy\right) \cdot \left(\int_{-\infty}^{\infty} \exp\left(-\frac{(z-z_0)^2}{a^2}\right) dz\right).$$

The second two factors are Gaussian integrals. In the first factor, let $u = (y-y_0)/a$. Then $du = dy/a$, and hence:

$$\int_{-\infty}^{\infty} \exp\left(-\frac{(y-y_0)^2}{a^2}\right) dy = a\int_{-\infty}^{\infty} e^{-u^2}\, du = a\sqrt{\pi}.$$

This is independent of y_0, hence the final factor is also $a\sqrt{\pi}$. The first factor is:

$$\int_{-\infty}^{\infty} x \exp\left(-\frac{(x-x_0)^2}{a^2}\right) dx = \int_{-\infty}^{\infty} (x-x_0) \exp\left(-\frac{(x-x_0)^2}{a^2}\right) dx + x_0\int_{-\infty}^{\infty} \exp\left(-\frac{(x-x_0)^2}{a^2}\right) dx.$$

Note that the first term has an integrand which is rotationally symmetric about $x = x_0$, so it is like an 'odd' function about the line $x = x_0$. Therefore, it is zero. The remaining term is $x_0 a\sqrt{\pi}$. Thus the final integral is:

$$x_0 a^3 \pi^{3/2}.$$

[*1 mark for noticing separation of integrals. 1 mark for correct substitution in Gaussian integral and 1 mark for correct evaluation of Gaussian integral. 1 mark for correct evaluation of non-Gaussian integral through whatever method. 1 mark for final combination into correct answer.*]

16. (a) A function f of two variables x and y is defined as:

$$f(x, y) = 2x^3 + 6xy^2 - 3y^3 - 150x.$$

Determine the positions of the stationary points of f and their characters (maximum, minimum or saddle point).

[8]

Solution: Differentiating, we have:

$$\nabla f = \left(6x^2 + 6y^2 - 150, 12xy - 9y^2\right).$$

In particular, we have a stationary point wherever $\nabla f = (0, 0)$. The second equation implies that $3y(4x-3y) = 0$, hence either $y = 0$ or $y = \frac{4}{3}x$. Inserting into the first equation, we have two cases:

- If $y = 0$, then $6x^2 = 150$, and hence $x = \pm 5$. Thus we have two stationary points $(x, y) = (\pm 5, 0)$.
- If $y = \frac{4}{3}x$, then $6x^2 + 32x^2/3 = 150$, which implies $50x^2/3 = 150$. Thus $x^2 = 9$, and hence $x = \pm 3$. It follows that $(x, y) = (\pm 3, \pm 4)$ are stationary points.

[*1 mark for correctly evaluating gradient, 1 mark for solving equations for some of stationary points, 1 mark for getting all of stationary points.*]

To do the classification, we notice there is no significant symmetry or boundedness properties of $f(x, y)$ that can help. Thus we will need to do the second derivative tests. We first compute:

$$f_{xx} = 12x, \qquad f_{xy} = 12y, \qquad f_{yy} = 12x - 18y,$$

hence at each of the stationary points:

- At $(x, y) = (\pm 5, 0)$, we have $f_{xx} = \pm 60$, $f_{xy} = 0$, $f_{yy} = \pm 60$. Thus $f_{xx}f_{yy} - f_{xy}^2 = 60^2 > 0$. Further, for $(x, y) = (5, 0)$ we have $f_{xx} + f_{yy} > 0$ and for $(x, y) = (-5, 0)$ we have $f_{xx} + f_{yy} < 0$. It follows that $(5, 0)$ is a minimum and $(-5, 0)$ is a maximum.
- At $(x, y) = (\pm 3, \pm 4)$, we have $f_{xx} = \pm 36$, $f_{xy} = \pm 48$, $f_{yy} = \pm(36-72) = \mp 36$. Hence $f_{xx}+f_{yy} = 0$ and $f_{xx}f_{yy} - f_{xy}^2 = -36^2 - 48^2 < 0$. It follows that both of these points are saddles.

[*1 mark for correctly evaluating second derivatives, 1 mark for applying correct test for $(\pm 5, 0)$, 1 mark for correct conclusion in terms of maximum/minimum, 1 mark for applying correct test for $(\pm 3, \pm 4)$, 1 mark for correct conclusion of saddles.*]

(b) A function g of two variables x and y is defined as:

$$g(x,y) = x^4 + y^4 - 36xy.$$

Sketch the contours of g in the x-y plane, indicating on the sketch the positions and characters of all the stationary points. [12]

Solution: Let's begin by finding the stationary points and their character, since we need to do that already. We have:

$$\nabla g = (4x^3 - 36y, 4y^3 - 36x).$$

Equating these to zero, we have $x^3 = 9y$ and $y^3 = 9x$. Substituting the first equation into the second, we have $(x^3/9)^3 = 9x$, and hence $x^9 = 9^4 x$. Thus either $x = 0$ or $x^8 = 9^4 = 3^8$, and hence $x = \pm 3$. It follows that the stationary points are:

$$(0,0), \qquad (\pm 3, \pm 3).$$

[*1 mark for correctly evaluating gradient of g. 1 mark for partial progress towards positions of stationary points. 1 mark for finding all stationary points correctly.*]

We could do the second derivative test to work out the character of the stationary points (it is not very hard here). However, we shall use some other properties to work this out. Near $(0,0)$, the function is dominated by the $-36xy$ part because the $x^4 + y^4$ part is very small. Hence the constant contours look like contours of $-36xy = C$. In particular, they look like $y \propto 1/x$ graphs. Further when $C = 0$ we get $xy = 0$ so the contours look like straight lines passing through the origin. This type of contour implies that $(0,0)$ is a saddle. [*1 mark for correct argument at origin, second derivative test or otherwise. 1 mark for correctly concluding saddle.*]

For very large values of x, y, the function looks like $x^4 + y^4$, which dominates over the $-36xy$ part. Therefore the contours look like $x^4 + y^4 = C$. This is *almost* a circle, but a bit more squished; for example if we fix $C = 1$, then the circle $x^2 + y^2 = 1$ and the shape $x^4 + y^4 = 1$ have on their locus the points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(1/2^{1/4}, 1/2^{1/4})$ respectively. Since $1/\sqrt{2} < 1/2^{1/4}$, the shape $x^4 + y^4 = 1$ is like a 'stretched circle' instead. [*1 mark for identifying that contours are closed at large distances, and 1 mark for any argument about their 'stretched circle' shape.*]

There is also symmetry in the line $y = x$ since swapping $x \leftrightarrow y$ keeps the function the same. [*1 mark for spotting this.*]

We can now just about construct the plot. The plot is shown in the figure below. [*1 mark for partially correct plot, 1 mark for fully correct plot.*]

To finish, we just need to classify $(\pm 3, \pm 3)$. Considering the information we have above, the only possibility is that $(3,3)$ is a minimum and $(-3,-3)$ is also a minimum. [*1 mark for identifying $(3,3)$ minimum, 1 mark for identifying $(3,3)$ minimum.*]

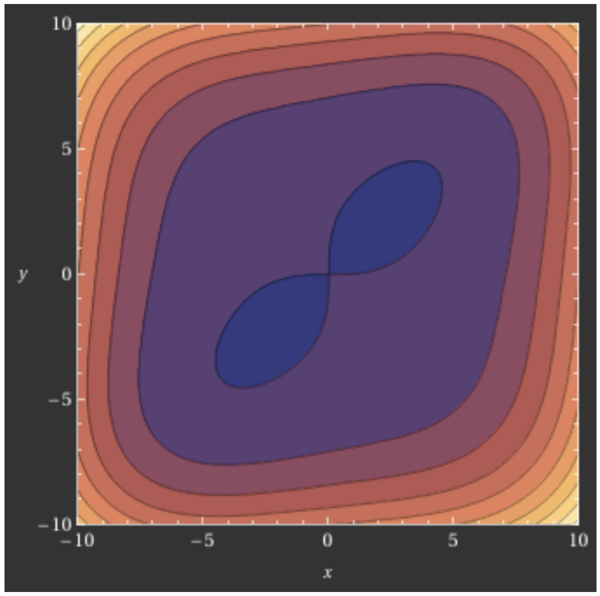


Figure 1: *The contour plot, courtesy of Wolfram Alpha.*

17. (a) Suppose $f(x)$ is a periodic function with period 2π. Write down its Fourier series and give expressions for the coefficients that appear in it. [3]

Solution: The Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

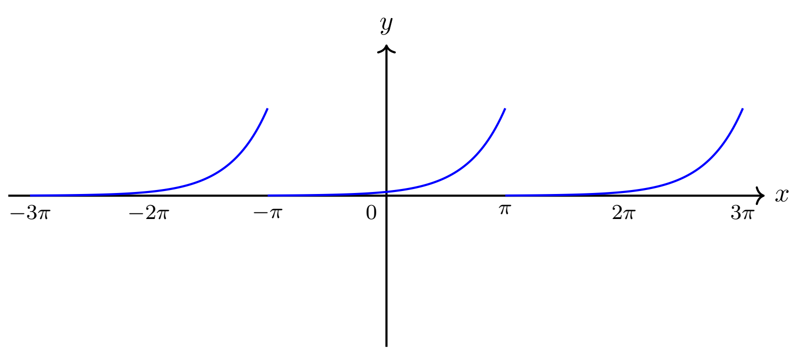
[1 *mark*] The coefficients are given by:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx.$$

[*1 mark for some coefficients correctly given, 1 additional mark for all coefficients correctly given.*]

(b) The function $g(x) = e^x$ is defined on the interval $-\pi \leq x < \pi$. Sketch the periodic continuation of $g(x)$ with period 2π, between $x = -3\pi$ and $x = 3\pi$. If we were to calculate the Fourier series of this periodic continuation of $g(x)$, what value would it take at the point $x = \pi$? [4]

Solution: The periodic continuation is pictured below.



[*1 mark for correct sketch of* e^x. *1 mark for correct periodic extension (repeated three times).*]

At a point of discontinuity, the Fourier series of a function converges to the *average* value across the discontinuity. Thus it would take the value $\frac{1}{2}(e^{\pi} + e^{-\pi})$. [*1 mark for knowing convergence to average. 1 mark for correct value.*]

(c) Consider the function $h(x) = x(\pi - x)$ defined on the interval $0 \leq x < \pi$. By considering the appropriate periodic continuation of $h(x)$ over the real line, show that the half-range sine series for $h(x)$ is: [10]

$$h(x) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin\left[(2n+1)x\right].$$

Hence demonstrate that: [3]

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

Solution: We extend with an odd extension, so that $h(x) = -h(-x)$. [*1 mark for saying odd extension.*] Then we obtain a sine series. In particular, all the cosine coefficients vanish by symmetry [*1 mark for saying this*] and $a_0 = 0$ by symmetry too [*1 mark for saying this*]. The remaining coefficients are given by:

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} h(x) \sin(nx)\, dx \\
&= \frac{2}{\pi} \int\limits_{0}^{\pi} x(\pi - x) \sin(nx)\, dx \\
&= \frac{2}{\pi} \left(\left[-\frac{x(\pi - x)}{n} \cos(nx) \right]_0^{\pi} + \frac{1}{n} \int\limits_0^{\pi} (\pi - 2x) \cos(nx)\, dx \right) \\
&= \frac{2}{n\pi} \left(\left[\frac{(\pi - 2x)}{n} \sin(nx) \right]_0^{\pi} + \frac{2}{n} \int\limits_0^{\pi} \sin(nx)\, dx \right) \\
&= \frac{4}{n^2\pi} \left[-\frac{\cos(nx)}{n} \right]_0^{\pi} \\
&= \frac{4}{n^3\pi}(1 - (-1)^n).
\end{aligned}$$

[*Up to 4 marks for correct evaluation of coefficients using integration by parts, allocated according to how successful integration is.*] It follows that the even terms vanish, and the odd terms have coefficients $8/n^3\pi$. Thus the Fourier series is:

$$h(x) = \sum_{p=0}^{\infty} \frac{8}{(2p+1)^3 \pi} \sin((2p+1)x),$$

as required. [*1 mark for saying odd terms important, and using $n = 2p + 1$. 1 mark for saying even terms vanish. 1 mark for convincing derivation of final answer.*] Evaluating this series at the point $x = \pi/2$ [*1 mark*], we have $\sin((2p+1)\pi/2) = (-1)^p$ [*1 mark*]. Thus:

$$\sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)^3} = \frac{\pi}{8} h(\pi/2) = \frac{\pi}{8} \cdot \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^3}{32},$$

as required. [*1 mark for convincing rearrangement to given answer.*]

18. (a) Suppose X is a discrete random variable that takes the integer values $0, 1, 2, ..., N$. Its normalised probability distribution is denoted by $P(X)$.

Write down expressions for the mean μ, the variance σ^2, and the probability $P(X < Y)$, where Y is a fixed positive integer less than N. [3]

Solution: We have:

$$\mu = \sum_{n=0}^{N} nP(X = n), \qquad \sigma^2 = \sum_{n=0}^{N} n^2 P(X = n) - \mu^2, \qquad P(X < Y) = \sum_{n=0}^{Y-1} P(X = n).$$

[*1 mark for mean, 1 mark for variance (or equivalent correct form), 1 mark for probability.*]

(b) A pond contains K trout and $N - K$ carp. Bruce goes fishing at the pond and in one day catches M fish. Note that Bruce never returns a fish to the pond once it is caught.

Calculate the number of ways M fish (of any species) can be caught from a pond of N fish. Suppose Bruce catches X trout out of his haul of M fish. Show that the number of ways of catching X trout out of the haul of M fish is:

$$\binom{K}{X}\binom{N-K}{M-X}.$$

Assuming that trout and carp are equally likely to be caught, show that the probability that Bruce catches X trout in one day is: [5]

$$P(X) = \binom{K}{X}\binom{N-K}{M-X} \Big/ \binom{N}{M}.$$

Solution: The number of ways of catching M fish from a pond of N fish is:

$$\frac{N(N-1)...(N-M+1)}{M!} = \frac{N!}{M!(N-M)!} = \binom{N}{M},$$

since there are N different ways of catching the first fish, there are $N - 1$ different ways of catching the second fish, etc. Since we don't care about the order of the M fish caught, we divide by $M!$. The result is the binomial coefficient. [*1 mark for convincing explanation, and 1 mark for reduction to binomial coefficient.*]

Now if Bruce catches X trout out of his M fish, we require X of the fish to be chosen from the K trout in the pond, and $M - X$ of the fish to be chosen from the remaining $N - K$ carp in the pond. Thus the total number of ways is:

$$\binom{K}{X}\binom{N-K}{M-X}.$$

[*1 mark for convincing explanation (note answer is already given!).*] If any of the fish are equally likely to be caught, then all the ways of catching the fish are equally likely. Hence the probability of any one haul is $1/\binom{N}{M}$ [*1 mark for words to this effect.*], which gives the required probability of getting X trout in the haul:

$$P(X) = \binom{K}{X}\binom{N-K}{M-X} \Big/ \binom{N}{M}.$$

[*1 mark for combining previous answers (again, note answer is already given!).*]

(c) Suppose the pond contains 2 trout and 8 carp, and Bruce catches 2 fish in total. What is the probability that of these two fish (i) none are trout, (ii) one is a trout, and (iii) both are trout? Verify that the three probabilities sum to 1. Consequently, determine the mean and variance of the probability distribution $P(X)$ for this case. (You may leave you answers in reduced fractional form.) [10]

[*Recall that* $\binom{N}{n} = N!/(n!(N-n)!)$.]

Solution: (i) The probability of no trout is:

$$P(0) = \binom{2}{0}\binom{8}{2}\Big/\binom{10}{2} = \frac{(8\cdot 7)/2}{(9\cdot 10)/2} = \frac{56}{90} = \frac{28}{45}.$$

[*1 mark for some correct evaluation of binomial coefficients, 1 mark for complete reduction to simplified form*.]

(ii) The probability of one trout is:

$$P(1) = \binom{2}{1}\binom{8}{1}\Big/\binom{10}{2} = \frac{2\cdot 8}{(9\cdot 10)/2} = \frac{16}{45}.$$

[*1 mark for some correct evaluation of binomial coefficients, 1 mark for complete reduction to simplified form*.]

(iii) The probability of two trouts is:

$$P(2) = \binom{2}{2}\binom{8}{0}\Big/\binom{10}{2} = \frac{1}{45}.$$

[*1 mark for some correct evaluation of binomial coefficients, 1 mark for complete reduction to simplified form*.]

Observe that:

$$\frac{28}{45} + \frac{16}{45} + \frac{1}{45} = \frac{45}{45} = 1,$$

as expected. [*1 mark for verification*.] The mean is:

$$\mu = \frac{16}{45} + \frac{2}{45} = \frac{18}{45} = \frac{2}{5}.$$

[*1 mark for complete mean calculation*.] Meanwhile, the variance is:

$$\sigma^2 = \frac{16}{45} + \frac{4}{45} - \frac{4}{25} = \frac{20}{45} - \frac{4}{25} = \frac{4}{9} - \frac{4}{25} = \frac{100-36}{225} = \frac{64}{225}.$$

[*1 mark for partial computation of variance, 1 mark for complete variance calculation and simplified*.]

(d) The next day Bruce goes fishing at a large lake that contains only trout and carp but in enormous quantities, i.e. both K and $N-K$ are much larger than M. In this limit $P(X)$ approaches the binomial distribution. Give a qualitative explanation for why this is so. [2]

Solution: A variable X is binomially distributed if it is the number of successful trials in a process where there are (i) only two possible outcomes (in this case, either get a trout or do not get a trout), (ii) the total number of trials is fixed (in this case, Bruce obtains M fish), (iii) the probability of success at each trial is fixed (this is the condition which changes here - since there are so many fish in the pond, the probability of getting a trout is approximately constant, and unchanged by taking a trout out of the pond in the first place). Thus we indeed expect X to be binomially distributed in the large fish limit. [*1 mark for verifying some conditions of binomial, 1 mark for fully convincing explanation*.]

19. (a) A differentiable function $f(x)$ is expanded using the Maclaurin series. Derive an expression which determines the interval for x within which the series is absolutely convergent. [3]

Use your result to determine the interval for x within which the Maclaurin series for $f(x) = \ln(2+x)$ converges absolutely. Determine whether or not the series converges at the end points of the interval. [4]

Solution: The Maclaurin series is given by:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

[*1 mark*] This is absolutely convergent by the ratio test if:

$$1 > \lim_{n\to\infty} \left| \frac{f^{(n+1)}(0) x^{n+1} n!}{f^{(n)}(0) x^n (n+1)!} \right| = |x| \lim_{n\to\infty} \left| \frac{f^{(n+1)}(0)}{f^{(n)}(0)} \frac{1}{n+1} \right|,$$

[*1 mark for applying ratio test.*] which can be rearranged to give:

$$|x| < \lim_{n\to\infty} \left| \frac{f^{(n)}(0)(n+1)}{f^{(n+1)}(0)} \right|.$$

[*1 mark for correct rearrangement to give radius of convergence of series.*] The series diverges if $|x|$ is bigger than the right hand side, hence this is the 'radius of convergence' of the series.

For $f(x) = \ln(2+x)$, to use the above result, we will need the nth derivative. We have $f^{(0)}(0) = \ln(2)$. For $n \geq 1$, we have:

$$f^{(1)}(x) = \frac{1}{2+x}, \qquad f^{(2)}(x) = -\frac{1}{(2+x)^2}, \qquad f^{(3)}(x) = \frac{2}{(2+x)^3}, \qquad \cdots$$

$$f^{(n)}(x) = \frac{(n-1)!(-1)^{n+1}}{(2+x)^n}.$$

Hence, we must consider:

$$\lim_{n\to\infty} \left| \frac{(n+1)\cdot(n-1)!\cdot 2^{n+1}}{2^n \cdot n!} \right| = 2 \lim_{n\to\infty} \left| \frac{n+1}{n} \right| = 2.$$

Thus the radius of convergence is 2, i.e. $|x| < 2$. [*1 mark for correctly differentiating* $\ln(x+2)$ *to give general term. 1 mark for correctly using criterion to deduce radius of convergence* 2.]

We are also asked to determine whether the series converges at the end points of the interval. We have:

$$f(x) = \ln(2) + \sum_{n=1}^{\infty} \frac{(n-1)!(-1)^{n+1}}{n!\cdot 2^n} x^n = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n n} x^n.$$

At $x = 2$, we have:

$$f(2) = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

This is convergent by the alternating series test, since $1/n > 0$, $1/n \to 0$ as $n \to \infty$, and $1/n$ is monotonically decreasing. [*1 mark for saying convergent at* $x = 2$.] On the other hand, when $x = -2$, we have:

$$f(-2) = \ln(2) + \sum_{n=1}^{\infty} \frac{-1}{n},$$

which shows that the Taylor series is proportional to the harmonic series, which diverges. Hence the series is not convergent at this point. [*1 mark for saying divergent at* $x = -2$.]

(b) Find the first three terms in the Maclaurin series for: [3]

$$f(x) = e^{-x}(1+x)^{-1/2}.$$

Solution: We just compute the product of the series for e^{-x} and $(1+x)^{-1/2}$. We have:

$$\begin{aligned} f(x) &= \left(1 - x + \frac{x^2}{2} + \cdots\right)\left(1 - \frac{x}{2} + \frac{3x^2}{8} + \cdots\right) \\ &= 1 - \frac{3x}{2} + \left(\frac{1}{2} + \frac{1}{2} + \frac{3}{8}\right)x^2 + \cdots \\ &= 1 - \frac{3x}{2} + \frac{11x^2}{8} + \cdots. \end{aligned}$$

[*1 mark for correct exponential series, 1 mark for correct binomial series, 1 mark for correct multiplication AND reduction to simplified form.*]

(c) Establish the convergence or divergence of the series $\sum_{n=1}^{\infty} u_n$ whose nth terms, u_n, are:

(i) $2^n/(n\ln(n))$, [2]

(ii) $\sin(n)/n^2$. [2]

Solution: (i) Using the ratio test, we consider:

$$\lim_{n\to\infty} \frac{2^{n+1} n \ln(n)}{2^n (n+1)\ln(n+1)} = 2\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\cdot\left(\frac{\ln(n)}{\ln(n+1)}\right) = 2\lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)}.$$

To take the limit here, we can use L'Hôpital's rule. We have:

$$2\lim_{n\to\infty}\frac{1/n}{1/(n+1)} = 2.$$

This is greater than 1, so the series diverges. [*1 mark for using the ratio test convincingly, 1 mark for conclusion of divergent.*]

(ii) Note that:

$$0 \leq \left|\frac{\sin(n)}{n^2}\right| \leq \frac{1}{n^2},$$

hence the absolute version of the series is strictly less than $1/n^2$. The sum of $1/n^2$ converges, since it is a standard p-series. As a result, the series converges absolutely by the comparison test. Absolute convergence implies convergence, so the original series also converges. [*1 mark for correct use of comparison test - MUST use positive absolute series here. 1 mark for correct conclusion that series converges.*]

(d) Sum the series: [6]

$$S(x) = \frac{x^4}{3(0!} + \frac{x^5}{4(1!)} + \frac{x^6}{5(2!)} + \cdots .$$

Solution: Observe that:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{S(x)}{x}\right) &= \frac{d}{dx}\left(\frac{x^3}{3(0!)} + \frac{x^4}{4(1!)} + \frac{x^5}{5(2!)} + \cdots\right) \\
&= \frac{x^2}{0!} + \frac{x^3}{1!} + \frac{x^4}{2!} + \cdots \\
&= x^2\left(\frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right) \\
&= x^2 e^x.
\end{aligned}$$

[*1 mark for dividing by x. 1 mark for taking the derivative. 1 mark for spotting exponential series.*] Integrating both sides, we have:

$$\frac{S(x)}{x} = \int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx = x^2 e^x - 2xe^x + 2\int e^x \, dx = (x^2 - 2x + 2)e^x + c.$$

[*1 mark for correct integration by parts.*] Hence:

$$S(x) = (x^2 - 2x + 2)xe^x + cx.$$

Notice that:

$$\lim_{x \to 0} \frac{S(x)}{x} = 0.$$

[*1 mark for noticing this boundary condition.*] Hence, we require:

$$2 + c = 0,$$

and thus:

$$S(x) = (x^2 - 2x + 2)xe^x - 2x.$$

[*1 mark for fully correct answer.*]

20. The point (a, b) is a stationary point of the function $f(x, y)$ subject to the constraint $g(x, y) = 0$. Using the method of Lagrange multipliers, show that: [4]

$$\begin{vmatrix} \frac{\partial f}{\partial x}(a,b) & \frac{\partial g}{\partial x}(a,b) \\ \frac{\partial f}{\partial y}(a,b) & \frac{\partial g}{\partial y}(a,b) \end{vmatrix} = 0.$$

Solution: Consider the Lagrangian $L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$. Taking the gradient of the Lagrangian, we have:

$$\nabla L = (\nabla f + \lambda \nabla g, g),$$

hence stationary points occur when $g = 0$ and when:

$$\nabla f = -\lambda \nabla g.$$

This implies that at the point (a, b), we have that the vectors ∇f and ∇g are parallel. Thus the area of the parallelogram formed by them is zero, so the determinant of the matrix whose columns are ∇f and ∇g must be zero. The result in the question follows. [*1 mark for Lagrangian correctly written down. 1 mark for correctly deducing conditions for stationary points. 1 mark for noticing ∇f, ∇g parallel. 1 mark for determinant argument.*]

(a) By considering the function $f(x, y) = x^2 + y^2$, use the method of Lagrange multipliers to find the maximum distance from the origin to the curve $x^2 + y^2 + xy - 4 = 0$. [4]

Solution: The Lagrangian is:

$$L(x, y, \lambda) = x^2 + y^2 + \lambda(x^2 + y^2 + xy - 4).$$

We have just seen that stationary points of this function satisfy:

$$\begin{vmatrix} 2x & 2x + y \\ 2y & 2y + x \end{vmatrix} = 0, \qquad x^2 + y^2 + xy = 4.$$

Multiplying out the determinant, we have:

$$0 = 2x(2y + x) - 2y(2x + y) = 4xy + 2x^2 - 4xy - 2y^2 = 2x^2 - 2y^2,$$

so that either $x = y$ or $x = -y$. In the first case, the constraint implies:

$$3y^2 = 4 \qquad \Rightarrow \qquad y = \pm 4/3.$$

In the second case, the constraint implies:

$$y^2 = 4 \qquad \Rightarrow \qquad y = \pm 2.$$

Thus the stationary points are $\pm(4/3, 4/3)$ and $\pm(2, -2)$. The point at maximum distance from the origin to the curve is therefore one of the following, which we can find by substitution:

- $\pm(4/3, 4/3)$ gives squared distance $16/9 + 16/9 = 32/9$.
- $\pm(2, -2)$ gives squared distance $4 + 4 = 8$, which is strictly greater than $32/9$.

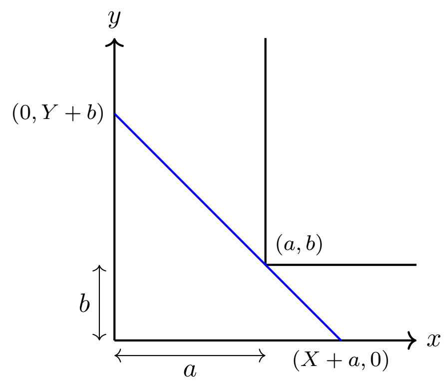
Thus the furthest distance is $\sqrt{8} = 2\sqrt{2}$. [*1 mark for correct simultaneous equations from first part. 1 mark for obtaining one set of solutions. 1 mark for obtaining other set of solutions. 1 mark for substituting into distances and deciding that $2\sqrt{2}$ is the greatest distance.*]

(b) In a school, two horizontal corridors, $0 \leq x \leq a, y \geq 0$ and $x \geq 0, 0 \leq y \leq b$ meet at right angles. The caretaker wishes to know the maximum possible length, L, of a ladder that may be carried horizontally around the corner. Regarding the ladder as a stick, use the method of Lagrange multipliers to calculate L by first placing the ends of the ladder at the points $(a+X, 0)$ and $(0, b+Y)$ and imposing the condition that the corner (a, b) be on the ladder. Then show that at the constrained stationary point, the value of X satisfies the equation:

$$(X^3 - ab^2)(X + a) = 0,$$

and hence show that $L = (a^{2/3} + b^{2/3})^{3/2}$. [12]

Solution: In the ladder's critical position, it is just about to turn around the corner:



Therefore, if we place the end points of the ladder at $(a+X, 0)$ and $(0, b+Y)$, we must now impose the condition that the corner (a, b) is on the ladder. [*1 mark for a good diagram, 1 mark for a convincing explanation of why we need the ladder to be on the corner at the critical point of rotation.*] Hence, we want to maximise the length $(a+X)^2 + (b+Y)^2$ subject to the condition that the ladder passes through the corner (a, b) (any smaller distance will be able to rotate round after that).

The condition that the line through $(a+X, 0)$ and $(0, b+Y)$ passes through the point (a, b) is equivalent to:

$$b = \frac{b+Y}{a+X}(a+X-a) \qquad \Leftrightarrow \qquad b(a+X) = (b+Y)X \qquad \Leftrightarrow \qquad ab = XY.$$

[*1 mark for correctly using line through $(a+X, 0)$, $(0, b+Y)$, and 1 mark for imposing that (a, b) is on the line. 1 mark for simplifying to form $ab = XY$.*] The Lagrangian is therefore:

$$\mathcal{L} = (X+a)^2 + (Y+b)^2 + \lambda(XY - ab).$$

[*1 mark for correct Lagrangian.*] Taking the gradient, we have:

$$\nabla\mathcal{L} = (2(X+a) + \lambda Y, 2(Y+b) + \lambda X, XY - ab).$$

[*1 mark for correctly taking gradient.*] At stationary points, we have:

$$2(X+a) = -\lambda Y, \qquad 2(Y+b) = -\lambda X, \qquad XY = ab.$$

The third equation implies that $Y = ab/X$. Dividing the first equation by the second, we have:

$$\frac{X+a}{Y+b} = \frac{Y}{X} \qquad \Leftrightarrow \qquad \frac{X+a}{ab/X+b} = \frac{ab}{X^2}.$$

[*Up to 2 marks for eliminating Y, λ to get an equation for X.*] Multiplying up, we have:

$$(X+a)X^2 = ab(ab/X+b) = \frac{ab^2}{X}(X+a) \qquad \Leftrightarrow \qquad (X+a)(X^3-ab^2)=0,$$

as required. [*1 mark for simplifying to required equation.*] We require that $X > 0$, hence $X \neq -a$. Thus $X = (ab^2)^{1/3}$, and it follows that $Y = ab/X = (a^3b^3)^{1/3}/(ab^2)^{1/3} = (a^2b)^{1/3}$. Thus the maximum length of the ladder is:

$$\begin{aligned} L &= \left(\left((ab^2)^{1/3}+a\right)^2 + \left((a^2b)^{1/3}+b\right)^2\right)^{1/2} \\ &= \left(a^{2/3}(b^{2/3}+a^{2/3})^2 + b^{2/3}(a^{2/3}+b^{2/3})^2\right)^{1/2} \\ &= (a^{2/3}+b^{2/3}) \cdot (a^{2/3}+b^{2/3})^{1/2} \\ &= (a^{2/3}+b^{2/3})^{3/2}. \end{aligned}$$

[*1 mark for obtaining correct X, 1 mark for simplifying to final form (note the answer is already given!).*]