Part IA: Mathematics for Natural Sciences B
Examples Sheet 2: Further vector geometry, triple products, vector area, and polar coordinate systems

Model Solutions

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More on the equation of a line

1. (a) Explain why the line through the points with positions vectors $\mathbf{a}$, $\mathbf{b}$ is $(\mathbf{r}-\mathbf{a}) \times (\mathbf{b}-\mathbf{a}) = \mathbf{0}$. Show using properties of the vector product that an equivalent representation of this line is $\mathbf{r} \times (\mathbf{b}-\mathbf{a}) = \mathbf{a} \times \mathbf{b}$. What is the geometric significance of the quantity $|\mathbf{a} \times \mathbf{b}|/|\mathbf{b}-\mathbf{a}|$ here?

   (b) Express the line $\mathbf{r} = (1,0,1) + \lambda(3,-1,0)$ in the form $\mathbf{r} \times \mathbf{c} = \mathbf{d}$.

Solution: (a) The direction of the line is $\mathbf{b}-\mathbf{a}$. For any point on the line $\mathbf{r}$, we must have $\mathbf{r}-\mathbf{a}$ parallel to this direction. This happens if and only if $(\mathbf{r}-\mathbf{a}) \times (\mathbf{b}-\mathbf{a}) = \mathbf{0}$, as required. Hence, this is an alternative way of writing the equation of a line.

Using the distributive property of the vector product, we can expand this to give $\mathbf{r} \times (\mathbf{b}-\mathbf{a}) - \mathbf{a} \times (\mathbf{b}-\mathbf{a}) = \mathbf{0}$. Rearranging, and using the distributive property again, we have:

$$\begin{aligned} \mathbf{r} \times (\mathbf{b}-\mathbf{a}) &= \mathbf{a} \times (\mathbf{b}-\mathbf{a}) \\ &= \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{a} \\ &= \mathbf{a} \times \mathbf{b} && (\text{since } \mathbf{a} \times \mathbf{a} = \mathbf{0}) \end{aligned}$$

We can interpret the quantity $|\mathbf{a} \times \mathbf{b}|/|\mathbf{b}-\mathbf{a}|$ in a similar way to Question 13(a) from Sheet 1. Writing the left hand side of this equation as $|\mathbf{r}||\mathbf{b}-\mathbf{a}|\sin(\theta)\hat{\mathbf{n}}$, then taking the length of both sides, we have:

$$|\mathbf{r}|\sin(\theta) = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}-\mathbf{a}|}.$$

This tells us that $|\mathbf{r}|$ is minimised when $\sin(\theta)$ is maximised, i.e. when $\sin(\theta) = 1$ occurring at $\theta = \pi/2$. This tells us that the shortest distance between the origin and the line is given by:

$$\frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}-\mathbf{a}|}.$$

(b) The given line goes through the point $(1,0,1)$ and has direction $(3,-1,0)$. Hence, it has the equation:

$$(\mathbf{r} - (1,0,1)) \times (3,-1,0) = \mathbf{0}.$$

Rearranging, we have:

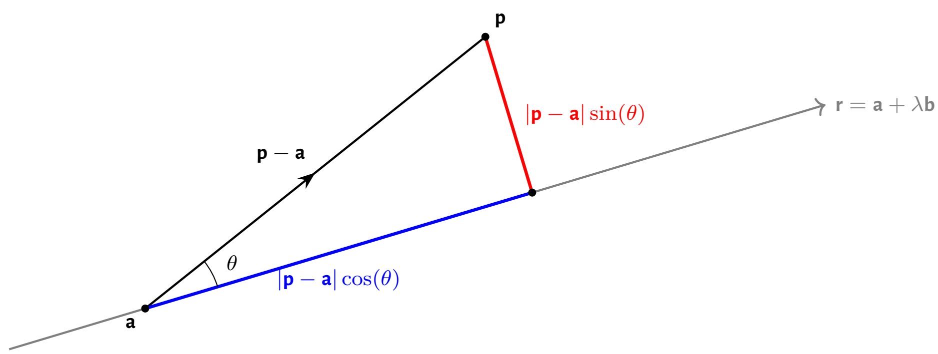
$$\mathbf{r} \times \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}.$$

2. (a) Show that the shortest distance between the point $\mathbf{p}$ and the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ can be written as $\left|\hat{\mathbf{b}} \times (\mathbf{p} - \mathbf{a})\right|$.

   (b) ($*$) Does this formula agree with the one you derived in terms of the scalar product in Question 10(c) of Sheet 1? [*Hint: Try squaring the formula in part (a), and using properties of the scalar triple product - see later in the sheet!*]

   (c) Find the shortest distance from a vertex of a unit cube to a diagonal excluding that vertex using both the formula in (a), and the formula from Question 10(c) of Sheet 1, and check that your answers agree.

Solution: (a) Consider the diagram below:



The shortest distance between the line and the point is $|\mathbf{p}-\mathbf{a}|\sin(\theta)$, which is the magnitude of the vector product $\hat{\mathbf{b}}\times(\mathbf{p}-\mathbf{a})$, as required.

(b) Yes, this formula agrees with the one we derived in Question 10(c) of Sheet 1. To see this, we square the formula from part (a):

$$\begin{aligned}
\left|\hat{\mathbf{b}} \times (\mathbf{p} - \mathbf{a})\right|^2 &= \left(\hat{\mathbf{b}} \times (\mathbf{p} - \mathbf{a})\right) \cdot \left(\hat{\mathbf{b}} \times (\mathbf{p} - \mathbf{a})\right) \\
&= \hat{\mathbf{b}} \cdot \left((\mathbf{p} - \mathbf{a}) \times (\hat{\mathbf{b}} \times (\mathbf{p} - \mathbf{a}))\right) && \text{(property of scalar triple product)} \\
&= \hat{\mathbf{b}} \cdot \left(\hat{\mathbf{b}}|\mathbf{p} - \mathbf{a}|^2 - (\mathbf{p} - \mathbf{a})\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\right) && \text{(Lagrange's formula)} \\
&= |\mathbf{p} - \mathbf{a}|^2 - (\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a}))^2.
\end{aligned}$$

Taking the square root, we obtain the Pythagorean formula we obtained in terms of the scalar product on Sheet 1.

(c) Let the unit cube have vertices $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ and $(1,1,1)$. A diagonal of the cube is then $\lambda(1,1,1)$ (this is the easiest one to pick!). A separate vertex that does not lie on this diagonal is $(0,0,1)$. Hence using the formula from (a), the shortest distance is the magnitude of:

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\0\end{pmatrix},$$

which is $\sqrt{2/3}$.

Using the formula from Question 10(c) of Sheet 1, the shortest distance is instead given by:

$$\sqrt{1^2 - \left(\frac{(1,1,1)}{\sqrt{3}} \cdot (0,0,1)\right)^2} = \sqrt{1 - \frac{1}{3}} = \sqrt{\frac{2}{3}},$$

in perfect agreement.

More on the equation of a plane

3. (a) Explain why the plane through the points with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is $(\mathbf{r}-\mathbf{a}) \cdot ((\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})) = 0$. Show using properties of the vector product, and the result from Question 22 of Sheet 1, that this may equivalently be written in the more symmetric form $\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

   (b) Find an equation of the form $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ for the plane passing through $(1,1,1)$, $(1,2,3)$ and $(0,0,4)$.

➾ **Solution:** (a) Two vectors contained in the plane are $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$. Taking their cross product, we produce a vector orthogonal to the plane, $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$. Hence, using the standard equation of a plane from Sheet 1, we have that $(\mathbf{r} - \mathbf{a}) \cdot ((\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})) = 0$ is indeed the equation of the plane.

The result from Question 22 of Sheet 1 gives $(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}) = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$. Hence the equation can be rewritten as:

$$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = 0.$$

Using the distributive property of the vector product, and rearranging, we have:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \cdot (\mathbf{c} \times \mathbf{a}).$$

Finally, the terms $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ and $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{a})$ vanish, because $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{a}$ are orthogonal to $\mathbf{a}$ by definition.

(b) We follow the procedure outline in part (a). Two vectors contained in the plane are $(1,2,3) - (1,1,1) = (0,1,2)$ and $(0,0,4) - (1,1,1) = (0,0,3)$. Taking their cross product, we have:

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}.$$

Hence the equation of the plane is:

$$\left(\mathbf{r} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) \cdot \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = 0.$$

4. You need to drill a hole in a piece of metal starting at a right angle to a flat surface passing through the points $A = (1, 0, 0)$, $B = (1, 1, 1)$ and $C = (0, 2, 0)$, with the hole emerging at the point $D = (2, 1, 0)$. How long a drill must you use and where (in the plane ABC) must you start drilling?

➾ **Solution:** Two vectors contained in the piece of metal are $\overrightarrow{AB} = (1, 1, 1) - (1, 0, 0) = (0, 1, 1)$ and $\overrightarrow{AC} = (0, 2, 0) - (1, 0, 0) = (-1, 2, 0)$. Hence a vector orthogonal to the plane is:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$

To obtain the length of the drill, we must find the shortest distance from the point D to the plane of metal. A vector from the plane to the point is $\overrightarrow{AD} = (2, 1, 0) - (1, 0, 0) = (1, 1, 0)$. The length of the projection onto the direction normal to the metal is:

$$\left| (1, 1, 0) \cdot \frac{(-2, -1, 1)}{\sqrt{4 + 1 + 1}} \right| = \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}.$$

Hence, we must use a drill that is $\sqrt{3/2}$ units long.

To answer where in the plane must we start drilling, we must add to the point $A = (1, 0, 0)$ the projection of $\overrightarrow{AD}$ parallel to the plane. Subtracting the component orthogonal to the plane gives us the parallel component:

$$\overrightarrow{AD}_{\parallel} = (1, 1, 0) + \sqrt{\frac{3}{2}} \cdot \frac{(-2, -1, 1)}{\sqrt{6}} = (1, 1, 0) + (-1, -1/2, 1/2) = (0, 1/2, 1/2).$$

Adding this to the point A, we have $(1, 0, 0) + (0, 1/2, 1/2) = (1, 1/2, 1/2)$.

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5. Determine whether:

(a) the points $\mathbf{P}_1 = (0, 0, 2)$, $\mathbf{P}_2 = (0, 1, 3)$, $\mathbf{P}_3 = (1, 2, 3)$, $\mathbf{P}_4 = (2, 3, 4)$ are coplanar;

(b) the points $\mathbf{Q}_1 = (-2, 1, 1)$, $\mathbf{Q}_2 = (-1, 2, 2)$, $\mathbf{Q}_3 = (-3, 3, 2)$, $\mathbf{Q}_4 = (-2, 4, 3)$ are coplanar.

➾ **Solution:** In each case, we construct the planes going through the first three points, then check if the fourth point lies in the plane. We have:

(a) Two vectors parallel to the plane containing the first three points are:

$$\mathbf{P}_2 - \mathbf{P}_1 = (0, 1, 3) - (0, 0, 2) = (0, 1, 1), \qquad \mathbf{P}_3 - \mathbf{P}_1 = (1, 2, 3) - (0, 0, 2) = (1, 2, 1).$$

Taking the cross product, a vector orthogonal to the plane is:

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

Hence, the equation of the plane is $(\mathbf{r} - (0, 0, 2)) \cdot (-1, 1, -1) = 0$. Substituting $\mathbf{r} = \mathbf{P}_4 = (2, 3, 4)$, we have $(2, 3, 4) = (0, 0, 2) = (2, 3, 2)$, but $(2, 3, 2) \cdot (-1, 1, -1) = -1 \neq 0$, hence these points are not coplanar.

(b) Two vectors parallel to the plane containing the first three points are:

$$\mathbf{Q}_2 - \mathbf{Q}_1 = (-1,2,2)-(-2,1,1) = (1,1,1), \qquad \mathbf{Q}_3 - \mathbf{Q}_1 = (-3,3,2)-(-1,2,2) = (-2,2,0).$$

Taking the cross product, a vector orthogonal to the plane is:

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}-2\\2\\0\end{pmatrix} = \begin{pmatrix}-2\\-2\\4\end{pmatrix}.$$

Hence, the equation of the plane is $(\mathbf{r}-(-2,1,1))\cdot(-1,-1,2)=0$ (scaling the normal). Substituting $\mathbf{r}=\mathbf{Q}_4=(-2,4,3)$, we have $(-2,4,3)-(-2,1,1)=(0,3,2)$, but $(0,3,2)\cdot(-1,-1,2)=1\neq 0$, hence these points are not coplanar.

Shortest distances

6. *Without using a formula*, find the shortest distance between the lines $\mathbf{r}_1=(1,0,1)+\lambda(2,-1,3)$ and $\mathbf{r}_2=(0,1,-2)+\mu(1,0,2)$, justifying the steps you take. [*Sometimes, it is better to understand a method, than to quote a formula*.]

➭ **Solution:** If we wanted to connect the lines in the shortest way possible, we would draw a line that was *orthogonal* to both lines. Thus we would connect them along the direction:

$$\begin{pmatrix}2\\-1\\3\end{pmatrix} \times \begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}-2\\-1\\1\end{pmatrix}.$$

Now, let's pick any vector joining the two lines, say $(0,1,-2)-(1,0,1)=(-1,1,-3)$. If we project this vector along the orthogonal direction joining both lines, we will get the shortest distance between the two. We have:

$$(-1,1,-3)\cdot\frac{(-2,-1,1)}{\sqrt{4+1+1}} = -\frac{2}{\sqrt{6}}.$$

Taking the modulus, we obtain the shortest length $\sqrt{2/3}$.

7. So far, we have developed formulae for the shortest distance from points to lines, and from points to planes. Now, using the scalar and vector products, establish formulae for the following:

(a) the shortest distance from the line $\mathbf{r}_1 = \mathbf{v}_1 + \lambda\mathbf{w}_1$ to the line $\mathbf{r}_2 = \mathbf{v}_2 + \mu\mathbf{w}_2$; [*Hint: Take care when the lines are parallel!*]

(b) the shortest distance from the line $\mathbf{r} = \mathbf{v} + \lambda\mathbf{w}$ to the plane $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$;

(c) the shortest distance from the plane $(\mathbf{r}_1 - \mathbf{a}_1) \cdot \mathbf{b}_1 = 0$ to the plane $(\mathbf{r}_2 - \mathbf{a}_2) \cdot \mathbf{b}_2 = 0$.

Solution: (a) Following the procedure we outlined in the previous question, we note that a direction orthogonal to both lines is $\mathbf{w}_1 \times \mathbf{w}_2$. An arbitrary vector joining the lines is $\mathbf{v}_2 - \mathbf{v}_1$. Projecting this vector in the direction orthogonal to both lines, we get the shortest distance:

$$\left| \frac{(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{w}_1 \times \mathbf{w}_2)}{|\mathbf{w}_1 \times \mathbf{w}_2|} \right| .$$

This is fine unless $|\mathbf{w}_1 \times \mathbf{w}_2| = 0$, when the lines are parallel. In this case, the lines both have direction $\mathbf{w} \equiv \mathbf{w}_1 = \mathbf{w}_2$. We then need the projection of $\mathbf{v}_2 - \mathbf{v}_1$ orthogonal to $\mathbf{w}$, which is simply given by:

$$|(\mathbf{v}_2 - \mathbf{v}_1) \times \hat{\mathbf{w}}| .$$

(b) If $\mathbf{w}$ is not parallel to the plane, then the line and the plane must intersect, giving the shortest distance zero. This occurs if $\mathbf{w} \cdot \mathbf{b} \neq 0$.

In the case where $\mathbf{w} \cdot \mathbf{b} = 0$, then we need the projection of a vector $\mathbf{v} - \mathbf{a}$ perpendicular to the plane, which is just $|(\mathbf{v} - \mathbf{a}) \cdot \hat{\mathbf{b}}|$ (this agrees with our standard point-to-plane formula, because any point on the line is equally acceptable).

(c) If the planes are not parallel, that is $\mathbf{b}_1, \mathbf{b}_2$ are not parallel, then the planes intersect, giving the shortest distance zero.

In the case where $\mathbf{b}_1, \mathbf{b}_2$ are parallel, then we can just take the projection of $\mathbf{a}_1 - \mathbf{a}_2$ parallel to $\mathbf{b}_1$, say, giving $|(\mathbf{a}_1 - \mathbf{a}_2) \cdot \hat{\mathbf{b}}_1|$ (again, this agrees with our standard point-to-plane formula, because any pair of points on the planes is equally acceptable).

The vector triple product, and vector equations

8. (a) By expanding in terms of the standard basis vectors, prove *Lagrange's formula* for the vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

Think of a way of remembering this formula off by heart - it is very useful!

(b) Hence, construct an example of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

(c) ($*$) Prove the vector triple product using a geometric argument. [*Hint:* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ *is the projection of* $\mathbf{a}$ *into the plane through the origin perpendicular to* $\mathbf{b} \times \mathbf{c}$*, rotated by* $\frac{1}{2}\pi$*, and scaled by the magnitude of* $\mathbf{b} \times \mathbf{c}$.]

Solution: (a) Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$. Then:

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \left(\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right) \\
&= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix} \\
&= \begin{pmatrix} a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3 \\ a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1 \\ a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 \end{pmatrix} \\
&= \begin{pmatrix} b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3) \\ b_2(a_1c_1 + a_2c_2 + a_3c_3) - c_2(a_1b_1 + a_2b_2 + a_3b_3) \\ b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3(a_1b_1 + a_2b_2 + a_3b_3) \end{pmatrix} \\
&= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).
\end{aligned}$$

We can remember this formula using the phrase 'BACk of the CAB', which tells us which order the vectors come in.

(b) An easy example can be constructed using the Cartesian unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. We have:

$$\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_2) = -\mathbf{e}_2,$$

but:

$$(\mathbf{e}_1 \times \mathbf{e}_1) \times \mathbf{e}_2 = \mathbf{0},$$

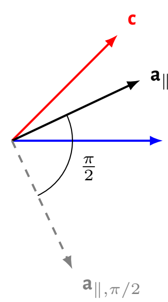
since $\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{0}$.

(c) This part is rather tricky, and certainly non-examinable, hence it is 'starred'! Using the hint, we observe that:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

is the projection of the vector $\mathbf{a}$ parallel to the plane with normal $\mathbf{b} \times \mathbf{c}$, rotated by $\pi/2$, and scaled by the magnitude of $\mathbf{b} \times \mathbf{c}$.

Alternatively, we can produce the same vector as follows. Let $\mathbf{a}_{||}$ be the projection of $\mathbf{a}$ into the plane perpendicular to $\mathbf{b} \times \mathbf{c}$. Let θ be the angle between $\mathbf{b}$, $\mathbf{c}$, and let ϕ be the angle between $\mathbf{a}_{||}$, $\mathbf{b}$.



Since $\mathbf{a}_{||,\pi/2}$ is in the plane spanned by $\mathbf{b}$, $\mathbf{c}$, we can write:

$$\mathbf{a}_{||,\pi/2} = \beta\mathbf{b} + \gamma\mathbf{c}.$$

for some coefficients β, γ. To obtain the coefficients, we can compute:

$$\mathbf{a}_{||,\pi/2} \times \mathbf{c} = \beta\mathbf{b} \times \mathbf{c} \qquad \Leftrightarrow \qquad \beta = \frac{|\mathbf{a}_{||,\pi/2}||\mathbf{c}|\sin(\pi/2 + (\theta - \phi))}{|\mathbf{b}||\mathbf{c}|\sin(\theta)} = \frac{|\mathbf{a}_{||}||\mathbf{c}|\cos(\theta - \phi)}{|\mathbf{b}||\mathbf{c}|\sin(\theta)} = \frac{\mathbf{a}_{||} \cdot \mathbf{c}}{|\mathbf{b} \times \mathbf{c}|}$$

Similarly, we have:

$$\mathbf{a}_{||,\pi/2} \times \mathbf{b} = \gamma\mathbf{c} \times \mathbf{b} \qquad \Leftrightarrow \qquad \gamma = -\frac{|\mathbf{a}_{||,\pi/2}||\mathbf{b}|\sin(\pi/2 - \phi)}{|\mathbf{b}||\mathbf{c}|\sin(\theta)} = -\frac{|\mathbf{a}_{||}||\mathbf{b}|\cos(\phi)}{|\mathbf{b}||\mathbf{c}|\sin(\theta)} = -\frac{\mathbf{a}_{||} \cdot \mathbf{b}}{|\mathbf{b} \times \mathbf{c}|}.$$

Overall then, if we scale $\mathbf{a}_{||,\pi/2}$ by $|\mathbf{b} \times \mathbf{c}|$, we get:

$$|\mathbf{b} \times \mathbf{c}|\mathbf{a}_{||,\pi/2} = (\mathbf{a}_{||} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a}_{||} \cdot \mathbf{b})\mathbf{c}.$$

In the final line, we may replace $\mathbf{a}_{||}$ by $\mathbf{a}$ because the perpendicular component of $\mathbf{a}$ is orthogonal to both $\mathbf{b}$ and $\mathbf{c}$ by construction. Hence, we're done!

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9. Using the vector triple product, prove the *Jacobi identity*, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$.

Solution: We have:

$$\begin{aligned}\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) + \mathbf{c}(\mathbf{b} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) \\ &= 0,\end{aligned}$$

as required.

10. Two vector operators, $P_{\hat{\mathbf{u}}} : \mathbb{R}^3 \to \mathbb{R}^3$ and $R_{\hat{\mathbf{u}}} : \mathbb{R}^3 \to \mathbb{R}^3$ are defined by $P_{\hat{\mathbf{u}}}(\mathbf{r}) = (\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}$ and $R_{\hat{\mathbf{u}}}(\mathbf{r}) = \hat{\mathbf{u}} \times (\mathbf{r} \times \hat{\mathbf{u}})$ respectively. Interpret these operators geometrically, and hence explain why $P_{\hat{\mathbf{u}}}(\mathbf{r}) + R_{\hat{\mathbf{u}}}(\mathbf{r}) = \mathbf{r}$ for all vectors $\mathbf{r}$. Also explain why $P_{\hat{\mathbf{u}}}^2 = P_{\hat{\mathbf{u}}}$ and $R_{\hat{\mathbf{u}}}^2 = R_{\hat{\mathbf{u}}}$.

Solution: The operator $P_{\hat{\mathbf{u}}}$ gives the projection of a vector in the $\hat{\mathbf{u}}$ direction. Using the vector triple product, we have:

$$R_{\hat{\mathbf{u}}}(\mathbf{r}) = \hat{\mathbf{u}} \times (\mathbf{r} \times \hat{\mathbf{u}}) = \mathbf{r} - (\hat{\mathbf{u}} \cdot \mathbf{r})\hat{\mathbf{u}},$$

hence this operator removes the component of a vector in the $\hat{\mathbf{u}}$ direction. Hence, it gives the projection of a vector perpendicular to the $\hat{\mathbf{u}}$ direction. This immediately implies that:

$$P_{\hat{\mathbf{u}}}(\mathbf{r}) + R_{\hat{\mathbf{u}}}(\mathbf{r}) = \mathbf{r}$$

for all vectors $\mathbf{r}$, as required.

It is also straightforward that $P_{\hat{\mathbf{u}}}^2 = P_{\hat{\mathbf{u}}}$, because it we apply the projection operator twice, after the first application, the resulting vector already points in the $\hat{\mathbf{u}}$ direction, so the second application does nothing. We can check this explicitly using some algebra:

$$P_{\hat{\mathbf{u}}}^2(\mathbf{r}) = P_{\hat{\mathbf{u}}}((\hat{\mathbf{u}} \cdot \mathbf{r})\hat{\mathbf{u}}) = (\hat{\mathbf{u}} \cdot \hat{\mathbf{u}})(\hat{\mathbf{u}} \cdot \mathbf{r})\hat{\mathbf{u}} = (\hat{\mathbf{u}} \cdot \mathbf{r})\hat{\mathbf{u}}.$$

The same is true for $R_{\hat{\mathbf{u}}}^2 = R_{\hat{\mathbf{u}}}$, because after one application of the projection operator, we are already pointing in a direction orthogonal to $\hat{\mathbf{u}}$.

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11. Solve the following vector equations, and give geometric interpretations of their solutions:

(a) $\mathbf{a} \times \mathbf{r} + \lambda \mathbf{r} = \mathbf{c}$, where $\lambda \neq 0$, and $\mathbf{a}, \mathbf{c} \in \mathbb{R}^3$ are arbitrary 3-vectors;

(b) $\mathbf{r} \times \mathbf{a} = \mathbf{b}$, where $\mathbf{a} \in \mathbb{R}^3$ is an arbitrary non-zero 3-vector;

(c) $\mathbf{r} = \mathbf{a} + (\mathbf{b} \cdot \mathbf{r})\mathbf{c}$, where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ are arbitrary 3-vectors;

(d) $2\mathbf{r} + \hat{\mathbf{n}} \times \mathbf{r} + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{r})^2 = \mathbf{a}$, where $\hat{\mathbf{n}}$ is a unit vector, and $\hat{\mathbf{n}} \cdot \mathbf{a} = -1$.

Solution:

(a) Begin by taking the scalar product of the equation with $\mathbf{a}$ (this hopefully gives us information parallel to $\mathbf{a}$). We then have:

$$\lambda \mathbf{r} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{a},$$

giving us the component of $\mathbf{r}$ parallel to $\mathbf{a}$. Now consider taking the vector product of the equation with $\mathbf{a}$ (this hopefully gives us information perpendicular to $\mathbf{a}$). We then have:

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{r}) + \lambda \mathbf{a} \times \mathbf{r} = \mathbf{a} \times \mathbf{c} \qquad \Leftrightarrow \qquad \mathbf{a}(\mathbf{a} \cdot \mathbf{r}) - |\mathbf{a}|^2 \mathbf{r} + \lambda \mathbf{a} \times \mathbf{r} = \mathbf{a} \times \mathbf{c}.$$

To finish, we substitute for $\mathbf{a} \cdot \mathbf{r}$ using the expression we found earlier when computing the scalar product. We also substitute for $\mathbf{a} \times \mathbf{r}$ using the original equation, $\mathbf{a} \times \mathbf{r} = \mathbf{c} - \lambda \mathbf{r}$. This gives:

$$\frac{(\mathbf{a} \cdot \mathbf{c})\mathbf{a}}{\lambda} - |\mathbf{a}|^2 \mathbf{r} + \lambda \mathbf{c} - \lambda^2 \mathbf{r} = \mathbf{a} \times \mathbf{c}.$$

Rearranging for $\mathbf{r}$, we have:

$$\mathbf{r} = \frac{(\mathbf{a} \cdot \mathbf{c})\mathbf{a} - \lambda \mathbf{a} \times \mathbf{c} + \lambda^2 \mathbf{c}}{\lambda(|\mathbf{a}|^2 + \lambda^2)}.$$

This is a single point.

(b) First, let's take the scalar product with **a** to give us information parallel to **a**. We find:

$$0 = \mathbf{a} \cdot \mathbf{b}.$$

Hence, we see the equation has no solutions unless $\mathbf{a} \cdot \mathbf{b}$. Next, we take the vector product with **a** to give us information perpendicular to **a**. We have:

$$\mathbf{a} \times (\mathbf{r} \times \mathbf{a}) = \mathbf{a} \times \mathbf{b} \qquad \Leftrightarrow \qquad |\mathbf{a}|^2 \mathbf{r} - \mathbf{a}(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a} \times \mathbf{b}.$$

Rearranging, we have:

$$\mathbf{r} = \frac{\mathbf{a}(\mathbf{a} \cdot \mathbf{r})}{|\mathbf{a}|^2} + \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}|^2}.$$

It appears we cannot get any more information on $\mathbf{a} \cdot \mathbf{r}$, because if we add any vector parallel to **a** to **r**, then this just gets annihilated. So the equation must have many solutions, of the form:

$$\mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}|^2} + \lambda \mathbf{a},$$

for any $\lambda \in \mathbb{R}$. We have so far only proved that solutions of this form are *necessary* (if we assume the equation, this is the form that the solutions must take). We must also prove that they are *sufficient*, by showing that these actually solve the equation in practice. We have:

$$\mathbf{r} \times \mathbf{a} = \frac{(\mathbf{a} \times \mathbf{b}) \times \mathbf{a}}{|\mathbf{a}|^2} = -\frac{\mathbf{a}(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{a}|^2} + \frac{|\mathbf{a}|^2 \mathbf{b}}{|\mathbf{a}|^2} = \mathbf{b},$$

provided that $\mathbf{a} \cdot \mathbf{b} = 0$. So we're done!

Summarising: the equation has no solutions if $\mathbf{a} \cdot \mathbf{b} \neq 0$, but it has many solutions of the form:

$$\mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}|^2} + \lambda \mathbf{a}$$

otherwise. This is a line through the point $(\mathbf{a} \times \mathbf{b})/|\mathbf{a}|^2$ parallel to the vector **a**.

(c) Taking the scalar product with **b**, we aim to get information parallel to **b**:

$$\mathbf{r} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + (\mathbf{r} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{b}). \qquad (*)$$

Rearranging, we have:

$$\mathbf{r} \cdot \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{1 - \mathbf{b} \cdot \mathbf{c}},$$

provided that $\mathbf{b} \cdot \mathbf{c} \neq 1$. Hence, if $\mathbf{b} \cdot \mathbf{c}$, we get the solution:

$$\mathbf{r} = \mathbf{a} + \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{c}}{1 - \mathbf{b} \cdot \mathbf{c}},$$

which is just a single point.

On the other hand, if $\mathbf{b} \cdot \mathbf{c} = 1$, we see that equation $(*)$ implies $\mathbf{a} \cdot \mathbf{b} = 0$. Thus there are no solutions unless $\mathbf{a} \cdot \mathbf{b} = 0$ too. We get no further information on $\mathbf{r} \cdot \mathbf{b}$, so we guess that this is a free parameter and the solution is of the form:

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{c}.$$

Certainly the solution is *necessarily* of this form, because the equation looks like this in the first place! We also must check it is *sufficient* by substituting back into the original equation. We have:

$$\mathbf{a} + (\mathbf{b} \cdot \mathbf{r})\mathbf{c} = \mathbf{a} + (\mathbf{b} \cdot \mathbf{a} + \lambda \mathbf{b} \cdot \mathbf{c})\mathbf{c} = \mathbf{a} + \lambda \mathbf{c} = \mathbf{r},$$

since $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 1$. Thus this is indeed the general solution in this case.

Summarising: if $\mathbf{b} \cdot \mathbf{c} \neq 1$, the solution is a point $\mathbf{a} + (\mathbf{a} \cdot \mathbf{b})/(1 - \mathbf{b} \cdot \mathbf{c})$; if $\mathbf{b} \cdot \mathbf{c} = 1$ and $\mathbf{a} \cdot \mathbf{b} \neq 0$, there are no solutions; if $\mathbf{b} \cdot \mathbf{c} = 1$ and $\mathbf{a} \cdot \mathbf{b} = 0$, the solution is a line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{c}$.

(d) Take the scalar product of the equation with $\hat{\mathbf{n}}$ to get information parallel to $\hat{\mathbf{n}}$. Then:

$$2\mathbf{r}\cdot\hat{\mathbf{n}} + (\hat{\mathbf{n}}\cdot\mathbf{r})^2 = \mathbf{a}\cdot\hat{\mathbf{n}} = -1 \qquad \Leftrightarrow \qquad (\hat{\mathbf{n}}\cdot\mathbf{r})^2 + 2\hat{\mathbf{n}}\cdot\mathbf{r} + 1 = 0.$$

This is a quadratic equation for $\hat{\mathbf{n}}\cdot\mathbf{r}$; it has a repeated root:

$$\hat{\mathbf{n}}\cdot\mathbf{r} = -1.$$

Substituting this back into the original equation, we have:

$$2\mathbf{r} + \hat{\mathbf{n}}\times\mathbf{r} = \mathbf{a} - \hat{\mathbf{n}}. \tag{$**$}$$

Next, we take the vector product of the equation with $\hat{\mathbf{n}}$ to get information perpendicular to $\hat{\mathbf{n}}$. We have:

$$2\hat{\mathbf{n}}\times\mathbf{r} + \hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{r}) = \hat{\mathbf{n}}\times\mathbf{a} \qquad \Leftrightarrow \qquad 2\hat{\mathbf{n}}\times\mathbf{r} + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r}) - \mathbf{r} = \hat{\mathbf{n}}\times\mathbf{a}.$$

Substituting for $\hat{\mathbf{n}}\cdot\mathbf{r} = -1$ and for $\hat{\mathbf{n}}\times\mathbf{r}$ using $(**)$, we have:

$$2\left(\mathbf{a} - \hat{\mathbf{n}} - 2\mathbf{r}\right) - \hat{\mathbf{n}} - \mathbf{r} = \hat{\mathbf{n}}\times\mathbf{a}.$$

Rearranging for $\mathbf{r}$, we have:

$$\mathbf{r} = \frac{2\mathbf{a} - 3\hat{\mathbf{n}} - \hat{\mathbf{n}}\times\mathbf{a}}{5}.$$

This is a single point.

The scalar triple product, and non-orthonormal bases

12. Let **a**, **b**, **c** $\in \mathbb{R}^3$ be 3-vectors.

(a) Give the definition of the *scalar triple product* $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of the 3-vectors **a**, **b**, **c** $\in \mathbb{R}^3$. Hence show that the volume of the parallelepiped defined by the positions vectors **a**, **b**, **c** is $|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|$. Why is the modulus necessary?

(b) Using the relation between the scalar triple product and a parallelepiped, explain why:

(i) the scalar triple product is antisymmetric on odd permutations of its entries, and symmetric on even permutations of its entries;

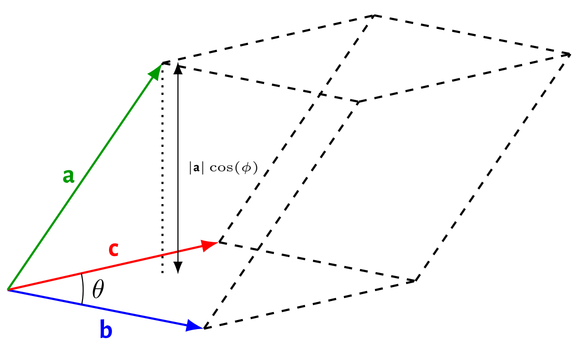
(ii) the condition $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$ implies that **a**, **b**, **c** are not coplanar, and thus form a basis.

(c) Compute the volume of a parallelepiped defined by the three position vectors $\mathbf{a} = (0, \frac{1}{2}, \frac{1}{2})$, $\mathbf{b} = (\frac{1}{2}, 0, \frac{1}{2})$, $\mathbf{c} = (\frac{1}{2}, \frac{1}{2}, 0)$, and comment on whether these vectors form a basis.

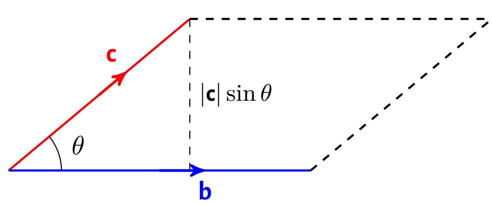
➽ **Solution:** (a) The scalar triple product is defined by:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] := \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

If the vectors are arranged in a right-handed way, then **a** forms an acute angle ϕ with $\mathbf{b} \times \mathbf{c}$, so $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \equiv [\mathbf{a}, \mathbf{b}, \mathbf{c}] > 0$. Suppose that the vectors **b**, **c** make an angle θ.



The height of the parallelogram formed by **a**, **b**, **c** is given by $|\mathbf{a}| \cos(\phi)$, as shown in the figure above. The area of the base can be computed by considering the figure below:



We see that the height of the parallelogram base is $|\mathbf{c}| \sin(\theta)$, so that its area is $|\mathbf{b}||\mathbf{c}| \sin(\theta)$, which is equal to the magnitude of $\mathbf{b} \times \mathbf{c}$. Therefore, the volume of the parallelepiped is $|\mathbf{a}||\mathbf{b}||\mathbf{c}| \sin(\theta) \cos(\phi)$, which is equal to $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \equiv [\mathbf{a}, \mathbf{b}, \mathbf{c}]$, as required.

In the case where **a**, **b**, **c** is left-handed, we have that **a** makes an *obtuse* angle ϕ with $\mathbf{b} \times \mathbf{c}$. We still have that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is equal in magnitude to the volume, but it now has a relative negative sign. Hence $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is the volume of the parallelepiped in general, explaining the need for the modulus.

(b) Recall from part (a) that the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is the signed volume of the parallelepiped formed by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$; it is positive if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is right-handed and negative if left-handed. Hence, we have:

(i) Swapping the arguments of $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ an even number of times retains the right-handedness of the parallelepiped; hence, the value is unchanged. Swapping the arguments of $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ an odd number of times changes the handedness of the parallelepiped to a left-handed orientation; hence, the value acquires a minus sign. The result follows.

(ii) Since $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is the signed volume of a parallelepiped, if it is non-zero, then the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ do not lie in the same plane (in this case, the volume degenerates to zero). Thus, they form a basis for three-dimensional space, as required.

(c) The volume is:

$$\begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \times \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} -1/4 \\ 1/4 \\ 1/4 \end{pmatrix} = \frac{1}{4} \neq 0,$$

hence these vectors do indeed form a basis.

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13. Simplify the scalar triple products $(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{b}+\mathbf{c}) \times (\mathbf{c}+\mathbf{a})$ and $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$.

Solution: In the first case, we have:

$$(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{b}+\mathbf{c}) \times (\mathbf{c}+\mathbf{a}) = (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = 2\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \equiv 2[\mathbf{a}, \mathbf{b}, \mathbf{c}].$$

In the second case, we have:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] &= (\mathbf{a} \times \mathbf{b}) \cdot [\mathbf{c}((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}) - \mathbf{a}((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c})] && \text{(Lagrange's formula)} \\ &= (\mathbf{a} \times \mathbf{b}) \cdot [\mathbf{c}[\mathbf{a}, \mathbf{b}, \mathbf{c}]] \\ &= [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2.\end{aligned}$$

14. Let **0**, **a**, **b**, **c** form the vertices of a tetrahedron, with $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$. Write down conditions in terms of the scalar triple product for the vector **r** to lie inside the tetrahedron.

➼ **Solution:** First, observe that since $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$, we have that the vectors are right-handed.

Now, the vector **r** must lie on the correct side of each of the four planes bounding the tetrahedron. Beginning with the side spanned by **a**, **b**, the normal $\mathbf{a} \times \mathbf{b}$ points in the direction of the tetrahedron. Hence:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) > 0$$

is needed to ensure that **r** is on the correct side of the plane. The same is true for the other pairs (taking care to consider handedness), giving:

$$\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) > 0, \qquad \mathbf{r} \cdot (\mathbf{c} \times \mathbf{a}) > 0.$$

Finally, we consider the plane that goes through points with position vectors **a**, **b**, **c**. We saw in Question 3(a) that this plane can be written in the form:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

The normal $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ is outward-pointing from the tetrahedron, so the side contained inside the tetrahedron satisfies:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{c}) < \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

Overall then, the four conditions are:

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) > 0, \qquad \mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) > 0, \qquad \mathbf{r} \cdot (\mathbf{c} \times \mathbf{a}) > 0,$$

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{c}) < \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

15. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ be 3-vectors.

(a) If these vectors form an orthonormal basis, derive expressions for the coefficients α, β, γ in the formula $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$. Hence express $(2,3,4)$ in terms of the basis $\{(1,1,0),(1,-1,0),(0,0,1)\}$.

(b) If instead these vectors do *not* form an orthonormal basis, derive expressions for the coefficients α, β, γ in the formula $\mathbf{d} = \alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}$. [*Hint: consider scalar triple products.*] Hence express $(1,1,1)$ in terms of the basis $\{(1,2,1),(0,0,1),(2,-1,1)\}$.

(c) We define the *reciprocal vectors* to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to be the vectors:

$$\mathbf{A} = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a},\mathbf{b},\mathbf{c}]}, \qquad \mathbf{B} = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a},\mathbf{b},\mathbf{c}]}, \qquad \mathbf{C} = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a},\mathbf{b},\mathbf{c}]}.$$

Show that $\mathbf{A}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{b} = \mathbf{C}\cdot\mathbf{c} = 1$, and $\mathbf{A}\cdot\mathbf{b} = \mathbf{A}\cdot\mathbf{c} = \mathbf{B}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{c} = \mathbf{C}\cdot\mathbf{a} = \mathbf{C}\cdot\mathbf{b} = 0$. Hence, by comparing to part (b), explain how the reciprocal basis can be used to express a general vector $\mathbf{d}$ in terms of a non-orthonormal basis $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$. Compute the reciprocal basis to the basis $\{(1,2,1),(0,0,1),(2,-1,1)\}$.

Solution: (a) If the vectors are orthonormal, we have $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} = 0$ and $\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = 1$. Hence given:

$$\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c},$$

we can take the scalar product with the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in turn to find the coefficients. Taking the scalar product with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in turn, we have:

$$\mathbf{a}\cdot\mathbf{d} = \alpha, \qquad \mathbf{b}\cdot\mathbf{d} = \beta, \qquad \mathbf{c}\cdot\mathbf{d}\gamma.$$

To express $(2,3,4)$ in terms of the basis $\{(1,1,0),(1,-1,0),(0,0,1)\}$, we cannot immediately use the formula above, because this is not an *orthonormal basis*. It is orthogonal however, since $(1,1,0)\cdot(1,-1,0) = 0$, $(1,1,0)\cdot(0,0,1) = 0$ and $(1,-1,0)\cdot(0,0,1) = 0$. To make it orthonormal, we consider the following related basis:

$$\{(1,1,0)/\sqrt{2},(1,-1,0)/\sqrt{2},(0,0,1)\}.$$

Using the formulae for the coefficients from above, we therefore have:

$$\begin{aligned}(2,3,4) &= \left((2,3,4)\cdot\frac{(1,1,0)}{\sqrt{2}}\right)\frac{(1,1,0)}{\sqrt{2}} + \left((2,3,4)\cdot\frac{(1,-1,0)}{\sqrt{2}}\right)\frac{(1,-1,0)}{\sqrt{2}} + \left((2,3,4)\cdot(0,0,1)\right)(0,0,1)\\ &= \frac{5}{2}(1,1,0) - \frac{1}{2}(1,-1,0) + 4(0,0,1),\end{aligned}$$

which is an expression for $(2,3,4)$ in terms of the desired basis.

(b) Now, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a *non*-orthonormal basis. We still would like to find the coefficients α, β, γ in:

$$\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c},$$

but it is no longer so easy. Can we use the same trick, to somehow take the scalar product with something, leaving only one coefficient leftover?

The answer is *yes*, if we dot with something perpendicular to two of the basis vectors. For example, to get the coefficient α, we consider taking the scalar product with $\mathbf{b}\times\mathbf{c}$. Then:

$$\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c}) = \alpha\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \qquad \Leftrightarrow \qquad \alpha = \frac{\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}.$$

Similarly, we have:

$$\beta = \frac{\mathbf{d} \cdot (\mathbf{c} \times \mathbf{a})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}, \qquad \gamma = \frac{\mathbf{d} \cdot (\mathbf{a} \times \mathbf{b})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}.$$

To calculate the coefficients for the given example then, we first compute the cross products:

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix}, \qquad \begin{pmatrix}2\\-1\\1\end{pmatrix} \times \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}-3\\-1\\5\end{pmatrix}, \qquad \begin{pmatrix}1\\2\\1\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\0\end{pmatrix}.$$

The scalar triple product is:

$$\begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}2\\-1\\1\end{pmatrix} = 5.$$

Hence we have:

$$\alpha = \frac{1}{5}\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\2\\0\end{pmatrix} = \frac{3}{5},$$

$$\beta = \frac{1}{5}\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}-3\\-1\\5\end{pmatrix} = \frac{1}{5},$$

$$\gamma = \frac{1}{5}\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\0\end{pmatrix} = \frac{1}{5}.$$

That is, we have:

$$(1,1,1) = \frac{3}{5}(1,2,1) + \frac{1}{5}(0,0,1) + \frac{1}{5}(2,-1,1).$$

(c) The properties $\mathbf{A}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{b} = \mathbf{C}\cdot\mathbf{c} = 1$ are obvious from the permutation properties of the scalar triple product. Similarly $\mathbf{A}\cdot\mathbf{b} = \mathbf{A}\cdot\mathbf{c} = \mathbf{B}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{c} = \mathbf{C}\cdot\mathbf{a} = \mathbf{C}\cdot\mathbf{b} = 0$ are all obvious from the fact that the scalar triple product vanishes when two of its arguments are equal.

We have shown that if $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$, then the coefficients α, β, γ can be written as:

$$\alpha = \mathbf{d}\cdot\mathbf{A}, \qquad \beta = \mathbf{d}\cdot\mathbf{B}, \qquad \gamma = \mathbf{d}\cdot\mathbf{C}.$$

Hence if we use the reciprocal basis **A**, **B**, **C**, we can happily dot as if we were dealing with an orthonormal basis. An orthonormal basis has the special property that it is its own reciprocal basis.

For the given basis, we already computed in part (b) that:

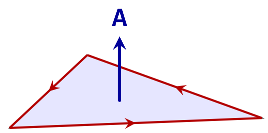
$$\mathbf{A} = \frac{1}{5}\begin{pmatrix}1\\2\\0\end{pmatrix}, \qquad \mathbf{B} = \frac{1}{5}\begin{pmatrix}-3\\-1\\5\end{pmatrix}, \qquad \mathbf{C} = \frac{1}{5}\begin{pmatrix}2\\-1\\0\end{pmatrix}.$$

Vector area

16.(a) Define the *vector area* **A** of a surface composed of k flat faces with areas A_1, ..., A_k and unit normals $\hat{\mathbf{n}}_1$, ..., $\hat{\mathbf{n}}_k$. What are the conventions usually used when choosing the unit normal(s)?

(b) In terms of the position vectors **a**, **b** determine the vector areas of: (i) the parallelogram defined by **a**, **b**; (ii) the triangle defined by **a**, **b**. Hence, given points $O = (0, 0, 0)$, $A = (1, 0, 0)$, $B = (1, 1, 1)$, $C = (0, 2, 0)$, compute the vector area of the triangle OAB (with vertices taken in that order), and the vector area of the surface bounded by the loop $OABC$ comprising the two triangular surfaces OAB and BCO.

Solution: The *vector area* of a flat surface A is the vector **A** with magnitude A and direction normal to the surface. The choice of which normal is a matter of convention, but if the orientation of the boundary of **A** is specified, then the normal is usually taken in a right-handed sense. This means that if you fingers curl round the orientation of the boundary, then your thumb points in the direction of the normal.

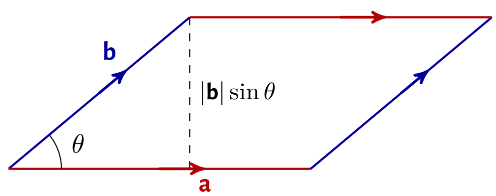


For a surface composed of multiple flat faces with area A_1, ..., A_k and unit normals $\hat{\mathbf{n}}_1$, ..., $\hat{\mathbf{n}}_k$, the total vector area is defined to be the sum of the vector areas of the individual faces:

$$\mathbf{A} = \sum_{i=1}^{k} A_i \hat{\mathbf{n}}_i.$$

(b) First, we compute the vector areas of the basic shapes asked for in the question:

(i) The parallelogram formed by **a**, **b** having an angle θ between the vectors takes the form:



Evidently, the area of the parallelogram is $|\mathbf{a}||\mathbf{b}|\sin(\theta) = |\mathbf{a} \times \mathbf{b}|$.

Now, if we choose an orientation of the parallelogram where we follow the vector **a** first, then the vector area will be a unit vector pointing out of the page scaled by the area of the parallelogram. That is, the vector area will just be the *cross product*, $\mathbf{a} \times \mathbf{b}$.

(ii) For a triangle, we just have half the area of the parallelogram above. So assuming the same orientation (following **a** first), the vector area is $\frac{1}{2}\mathbf{a} \times \mathbf{b}$.

Now, consider the explicit coordinates we are given. First, we are asked for the vector area of the triangle OAB, which by the above work is simply:

$$\frac{1}{2}\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\-1\\1\end{pmatrix}.$$

The vector area of the surface comprised of the triangle OAB together with the triangle BCO is the sum of the previous vector area, together with:[1]

$$\frac{1}{2}\overrightarrow{BC} \times \overrightarrow{BO} = \frac{1}{2}\begin{pmatrix}-1\\1\\-1\end{pmatrix} \times \begin{pmatrix}-1\\-1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-2\\0\\2\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

Hence, the total vector area is:

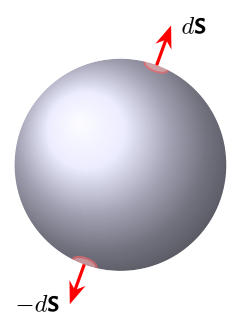
$$\frac{1}{2}\begin{pmatrix}0\\-1\\1\end{pmatrix} + \begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\-1/2\\3/2\end{pmatrix}$$

17.(a) Give a very general explanation of how the idea of vector area could be extended to *curved surfaces*, and hence explain why we expect the vector area of any *closed* surface to be $\mathbf{0}$.

(b) Compute the vector area of the square with vertices $(0,0,0)$, $(2,0,0)$, $(2,2,0)$, $(0,2,0)$, taken in that order. Hence, compute the vector area of the pyramid extending this square with the point $(1,1,1)$, excluding its square face.

(c) Compute the vector area of the curved surface of a truncated hollow cone, bounded by a horizontal circle of radius 4 units and a horizontal circle of radius 3 units at some height above the first (note the result is independent of the height!).

Solution: (a) For a closed surface, we could split the surface into lots of small *approximately* flat pieces, calculate each of their vector areas, then add them all up. This amounts to performing an *integral* over the surface. We will study surface integration properly in Lent term.



For a *closed* surface, i.e. one with no boundary, we expect each little flat piece that we split the surface up into to have a neighbour on the opposite side of the surface, but with an equal and opposite vector area. These will exactly cancel giving the vector area of a closed surface as zero. Note, this is not a proof, just a hopeful argument - we will see a proof properly in Lent when we study Stokes' theorem.

(b) The square clearly has area 4, so has vector area $(0,0,4)$, since $(0,0,1)$ is a unit normal to the square.

Let $\mathbf{A}$ be the vector area of the top part of the square-based pyramid. Then since the vector area of a closed surface is always zero, we must have $\mathbf{A} + (0,0,4) = \mathbf{0}$, so $\mathbf{A} = -(0,0,4)$. This assumes a certain orientation of the upper part of the square-based pyramid; if we had used a different orientation, we would have obtained $(0,0,4)$.

[1] The solution to Question 18(c) has a good diagram for this part!

(c) Let the axis of the cone be along the z-axis. Then if the complete shape has an outward pointing normal everywhere, the smaller top disc has vector area $\pi(3^2)\cdot(0,0,1)=(0,0,9\pi)$. The larger bottom disc has vector area $\pi(4^2)\cdot(0,0,-1)=(0,0,-16\pi)$. Let the vector area of the curved surface be **A**. Then:

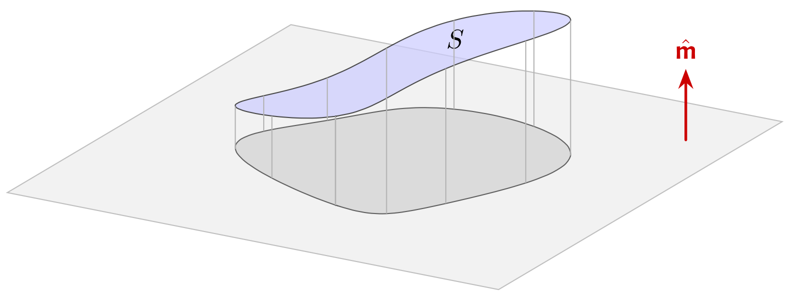
$$\mathbf{A}+(0,0,9\pi)+(0,0,-16\pi)=\mathbf{0} \qquad \Leftrightarrow \qquad \mathbf{A}=(0,0,7\pi).$$

18.(a) Let **S** be the vector area of the surface S. Prove that the area of the projection of the surface S onto the plane with unit normal $\hat{\mathbf{m}}$ is $|\mathbf{S}\cdot\hat{\mathbf{m}}|$. [*Hint: consider joining the surface to its 'shadow' on the plane to create a closed surface.*]

(b) Compute the vector area of the projection of the square with vertices $(0,0,0)$, $(2,0,0)$, $(2,2,0)$, $(0,2,0)$ onto the plane with unit normal $\hat{\mathbf{m}}=(0,-1,1)/\sqrt{2}$.

(c) By projecting areas onto the yz, xz, and xy planes, compute the vector area of the surface comprised of the two triangles OAB, BCO with vertices $O=(0,0,0)$, $A=(1,0,0)$, $B=(1,1,1)$, $C=(0,2,0)$, taken in that order. [*Your answer should match your answer to Question 17(b)!*] What is the area of the surface projected onto: (i) the plane with normal $(0,-1,1)$; (ii) the plane that maximises the projected area?

Solution: (a) This part is a very cute argument, but it is tricky if you have never seen it before! We imagine joining the shadow of S to the plane with normal $\hat{\mathbf{m}}$ via some prism-like surface, as shown in the figure.



Let S_{proj} be the area of the projection of the surface S onto the plane with normal $\hat{\mathbf{m}}$. Let $\mathbf{S}_\perp$ be the *vector* area of the curved surface joining S to the plane, which is necessarily perpendicular to $\hat{\mathbf{m}}$. Then since we now have a closed surface, we must have:

$$\mathbf{S}+\mathbf{S}_\perp-S_{\text{proj}}\hat{\mathbf{m}}=\mathbf{0}.$$

Taking the scalar product with $\hat{\mathbf{m}}$, and using the fact $\mathbf{S}_\perp\cdot\hat{\mathbf{m}}=0$, we immediately get the result $\mathbf{S}\cdot\hat{\mathbf{m}}=S_{\text{proj}}$, as required.

This argument assumed that **S** pointed *out* of the volume created, so that the appropriate vector area for the shadow was $-S_{\text{proj}}\hat{\mathbf{m}}$. If instead **S** pointed *into* the volume created, the appropriate vector area for the shadow is instead $S_{\text{proj}}\hat{\mathbf{m}}$ (this ensures that the normal is continuous across the surface, if we imagine moving it around). This implies the need for the modulus sign if we are calculating area in general: $|\mathbf{S}\cdot\hat{\mathbf{m}}|$. This will be an important point in part (c) of this question.

(b) The square has vector area $(0,0,4)$. Projecting onto the plane with unit normal $\hat{\mathbf{m}}$, the area of the projection is:

$$(0,0,4)\cdot(0,-1,1)/\sqrt{2}=\frac{4}{\sqrt{2}}=2\sqrt{2}.$$

Therefore, the *vector area* of the projection is $2\sqrt{2}\hat{\mathbf{m}}=(0,-2,2)$.

(c) The vector area may be expressed as:

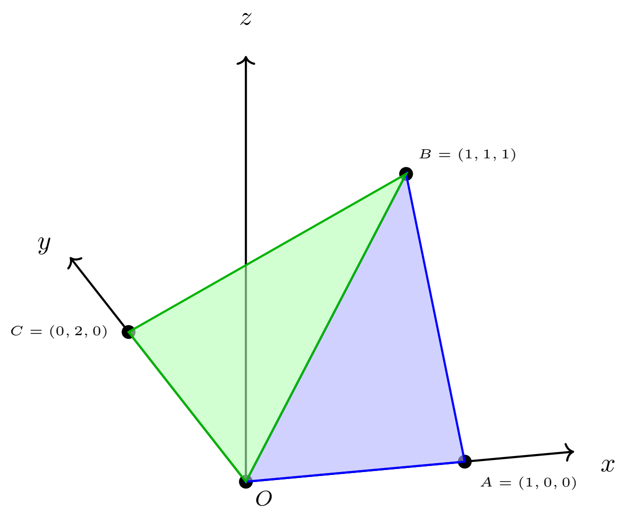
$$\mathbf{S} = S_x \hat{\mathbf{e}}_x + S_y \hat{\mathbf{e}}_y + S_z \hat{\mathbf{e}}_z,$$

where S_x, S_y, S_z are the coefficients in the expansion in an orthonormal basis, hence are given by:

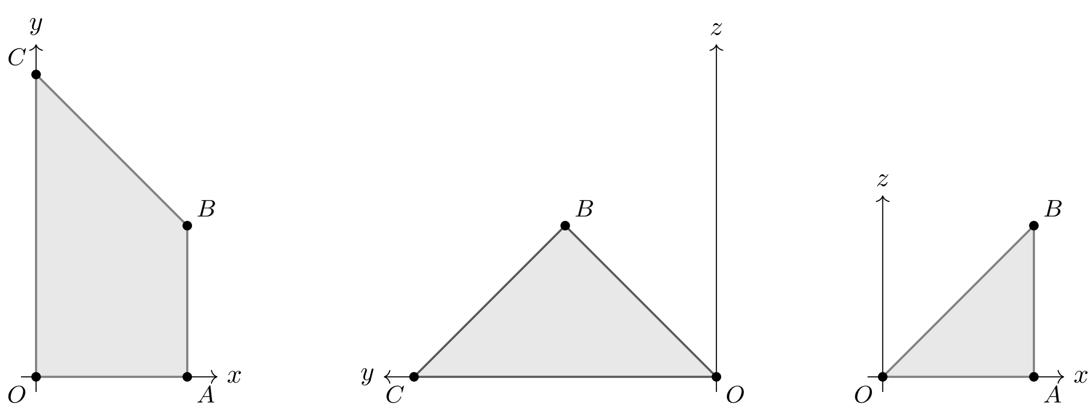
$$S_x = \mathbf{S} \cdot \hat{\mathbf{e}}_x, \qquad S_y = \mathbf{S} \cdot \hat{\mathbf{e}}_y, \qquad S_z = \mathbf{S} \cdot \hat{\mathbf{e}}_z.$$

In particular, the coefficients in the expansion are the (signed) projected areas of the surface onto the planes with normals $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ respectively, i.e. the coordinate planes.

Drawing a figure of the full 3D situation first, we have:



Now, imagine the projections of the surface onto each of the coordinate planes. We have:



Calculating the areas of each of the projections, we immediately see that:

$$|S_z| = \frac{3}{2}, \qquad |S_x| = 1, \qquad |S_y| = \frac{1}{2}.$$

To work out the signs correctly, we do the following. Following the argument presented in (a), we imagine joining up the surface $OABC$ to its shadow in a coordinate plane with normal $\hat{\mathbf{n}}$ (where $\hat{\mathbf{n}} = \hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$). If the vector area points *into* the volume that is created, then for consistency the vector area from the coordinate plane is the projection multiplied by $\hat{\mathbf{n}}$, giving:

$$\mathbf{S} + S_{\text{proj}}\hat{\mathbf{n}} + \mathbf{S}_{\perp} = \mathbf{0}.$$

This implies that $\mathbf{S} \cdot \hat{\mathbf{n}} = -S_{\text{proj}}$, so we get a negative sign. On the other hand, if the vector area points *out* of the volume that is created, then there is no change of sign.

In the xy plane, the orientation of $OABC$ implies the vector area points out of the volume it would create with the xy-plane. This means that the sign of S_z is positive, $S_z = 3/2$.

On the other hand, in both the yz and the xz planes, the orientation of $OABC$ implies that the vector area points *into* the volume it would create with the corresponding planes; hence $S_x = -1$ and $S_y = -1/2$. Overall we have:

$$\mathbf{S} = (-1, -1/2, 3/2),$$

consistent with our findings earlier on in the sheet.

To finish, we compute the projected area of $OABC$ onto two planes.

(i) Projecting onto the plane with normal $(0, -1, 1)$, we get the area:

$$\left|\mathbf{S} \cdot \frac{(0,-1,1)}{\sqrt{2}}\right| = \frac{1/2 + 3/2}{\sqrt{2}} = \sqrt{2}.$$

(ii) We now seek the plane which would maximise the projected area. If we have a plane with normal $\hat{\mathbf{m}}$, then the projected area is $|\mathbf{S} \cdot \hat{\mathbf{m}}| = |\mathbf{S}||\hat{\mathbf{m}}||\cos(\theta)|$, where θ is the angle between the vectors $\mathbf{S}$, $\hat{\mathbf{m}}$. Hence, the plane that maximises the projected area is the one with unit normal $\hat{\mathbf{S}}$. The projected area in this direction is:

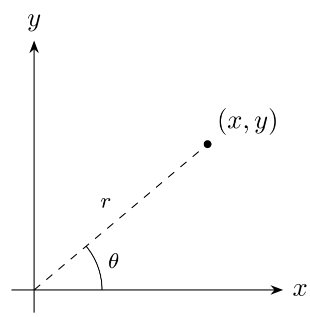
$$\left|\mathbf{S} \cdot \hat{\mathbf{S}}\right| = |\mathbf{S}| = \sqrt{1 + \frac{1}{4} + \frac{9}{4}} = \sqrt{\frac{7}{2}}.$$

Polar coordinate systems

19. Draw (convincing) diagrams defining plane, cylindrical, and spherical polar coordinates. In each case, derive the coordinate transform laws from polars to Cartesians, and from Cartesians to polars. Hence, find the cylindrical polar and spherical polar coordinates of the point $(3, 4, 5)$.

Solution: We address each coordinate system in turn.

- **Plane polar coordinates.** This system is defined by the diagram below:



From basic trigonometry, we have that:

$$x = r\cos(\theta), y = r\sin(\theta).$$

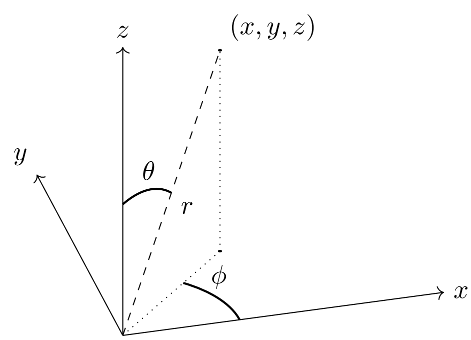
On the other hand, r is the length from the origin to the point (x, y), so:

$$r = \sqrt{x^2 + y^2}.$$

Finally, the angle θ satisfies $\tan(\theta) = x/y$. Depending on conventions, the root of this equation might be chosen to lie in the range $[0, 2\pi)$ or $(-\pi, \pi)$ (or indeed, other ranges).

- **Cylindrical polar coordinates.** These are the same as plane polars, just with an added z-direction. The formula for plane polars still hold when transforming $(r, \theta, z) \leftrightarrow (x, y, z)$.

- **Spherical polar coordinates.** Spherical polar coordinates for three dimensions are defined according to a *radial coordinate* r, which tells us how far from the origin we are, a *polar angle* θ, which tells us the angle between the radius from the origin to the point of interest and the z-axis (called the *polar axis*), and an *azimuthal angle* ϕ, which tells us the angle between the projection of the radius onto the xy-plane and the x-axis.



Some basic trigonometry tells us that:

$$x = r\sin(\theta)\cos(\phi), \qquad y = r\sin(\theta)\sin(\phi), \qquad z = r\cos(\theta).$$

Inverting these formula, we have:

$$r = \sqrt{x^2+y^2+z^2}, \qquad \tan(\phi) = \frac{x}{y}, \qquad \cos(\theta) = \frac{z}{\sqrt{x^2+y^2+z^2}},$$

where the coordinate θ is chosen to lie in the range $[0, \pi/2]$, and the coordinate ϕ is usually chosen to lie either in the range $[0, 2\pi)$ or $[-\pi, \pi)$.

Applying these formulae to the point $(3, 4, 5)$, we have:

- In cylindrical polar coordinates, the coordinates of the point are:

$$(r, \theta, z) = \left(\sqrt{3^2+4^2}, \arctan\left(\frac{4}{3}\right), 5\right) = \left(5, \arctan\left(\frac{4}{3}\right), 5\right).$$

- In spherical polar coordinates, the coordinates of the point are:

$$(r, \theta, \phi) = \left(\sqrt{3^2+4^2+5^2}, \arccos\left(\frac{5}{\sqrt{3^2+4^2+5^2}}\right), \arctan\left(\frac{4}{3}\right)\right) = \left(5\sqrt{2}, \frac{\pi}{4}, \arctan\left(\frac{4}{3}\right)\right).$$

20(a) In 2D Cartesian coordinates, a circle is specified by $(x-1)^2 + y^2 = 1$. Find its equation in polar coordinates.

(b) In 3D Cartesian coordinates, a sphere is specified by $(x-1)^2 + y^2 + z^2 = 1$. Find its equation in spherical polar coordinates.

Solution: (a) In plane polar coordinates, we have $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Hence the equation in polar coordinates is:

$$(r\cos(\theta) - 1)^2 + r^2\sin^2(\theta) = 1.$$

Simplifying, we have:

$$1 = r^2\cos^2(\theta) - 2r\cos(\theta) + 1 + r^2\sin^2(\theta) = r^2 - 2r\cos(\theta) + 1,$$

which can be rearranged to $r^2 = 2r\cos(\theta)$. Dividing by r, we have $r = 2\cos(\theta)$. This allows for the case $r = 0$ when $\theta = \pi/2$, so we have not lost anything by dividing through by r.

(b) In spherical polar coordinates, we have $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$ and $z = r\cos(\theta)$. Hence the equation in polar coordinates is:

$$\begin{aligned} 1 &= (r\sin(\theta)\cos(\phi) - 1)^2 + r^2\sin^2(\theta)\sin^2(\phi) + r^2\cos^2(\theta) \\ &= r^2\sin^2(\theta)\cos^2(\phi) - 2r\sin(\theta)\cos(\phi) + 1 + r^2\sin^2(\theta)\sin^2(\phi) + r^2\cos^2(\theta) \\ &= r^2 - 2r\sin(\theta)\cos(\phi) + 1. \end{aligned}$$

This can be rearranged to $r^2 = 2r\sin(\theta)\cos(\phi)$, which similarly can be reduced to $r = 2\sin(\theta)\cos(\phi)$.

21. Let $a > 0$ be a constant. Describe the following loci:

(a) (i) $\phi = a$; (ii) $r = \phi$, in plane polar coordinates.

(b) (i) $z = a$; (ii) $r = a$; (iii) $r = a$ and $z = \phi$, in cylindrical polar coordinates.

(c) (i) $\theta = a$; (ii) $\phi = a$; (iii) $r = a$; (iv) $r = \theta = a$, in spherical polar coordinates.

➾ **Solution:** We have:

(a) (i) $\phi = a$ corresponds to a half-line emanating from the origin, at an angle a to the x-axis. (ii) $r = \phi$ corresponds to a spiral, starting out at the origin then as ϕ increases, r increases too - it crosses the y-axis at $\pi/2$, then the negative x-axis at $-\pi$, then the negative y-axis at $-3\pi/2$. Eventually it gets back to the positive x-axis at the point 2π.

(b) (i) $z = a$ is a plane with normal $(0, 0, 1)$ a distance a from the origin. (ii) $r = a$ is a cylinder of radius a, with axis along the z-axis. (iii) The locus $r = a$ says that we on the cylinder of radius a, with axis along the z-axis. The equation $z = \phi$ says the angle and z-coordinate are related; in particular, the higher we are along the z-axis, the greater the angle with the x-axis. Thus this shape is a *helix* (a corkscrew type shape).

(c) (i) $\theta = a$ is a semi-infinite cone, with opening angle a, and axis along the z-axis. Its point is at the origin. (ii) $\phi = a$ is an infinite half-plane, inclined at an angle a to the x-axis, with its edge along the z-axis. (iii) $r = a$ is a sphere of radius a centred on the origin. (iv) $r = \theta = a$ is the intersection of the figures in (i) and (iii); this is a circle, at a height $a\cos(a)$ from the origin and radius $a\sin(a)$.

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22. Consider a point with position vector $\hat{\mathbf{n}}$ on the unit sphere S.

(a) Explain why $\hat{\mathbf{n}} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, where θ, ϕ are the spherical coordinates of $\hat{\mathbf{n}}$.

(b) Show that the vector area $d\mathbf{S}$ of a small patch near $\hat{\mathbf{n}}$, subtending a small angle $d\theta$ in the θ direction, and a small angle $d\phi$ in the ϕ direction, is given *approximately* by $d\mathbf{S} = \hat{\mathbf{n}}\sin(\theta)d\theta d\phi$.

(c) $(*)$ Hence, by integrating $d\mathbf{S}$ first over ϕ whilst keeping θ constant, then over θ, show that the vector area of the sphere is zero. [*Hint: what are the limits on ϕ, θ?*] You have now performed your first **surface integral**, a topic we shall cover properly in Lent. In fact, it is possible to use surface integration to show that the vector area of *any* closed shape is zero through a theorem called *Stokes' theorem*, which we shall also see in Lent.

(d) $(*)$ Without computing it, what is the value of the surface integral $\displaystyle\int_S \hat{\mathbf{n}} \cdot d\mathbf{S}$?

➾ **Solution:** (a) In general, a position vector in spherical polar coordinates is given by:

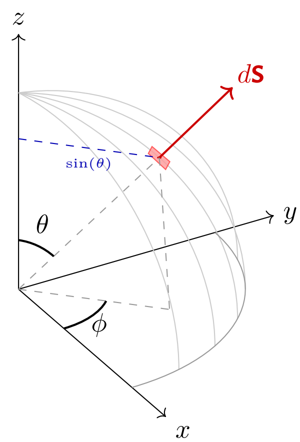
$$\mathbf{r} = (x, y, z) = (r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta)).$$

If a vector lies on the surface of the unit sphere, the only condition that needs to be satisfied is $r = 1$. This gives the general position vector on the surface of the unit sphere to be:

$$\hat{\mathbf{n}} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)),$$

as required.

(b) Consider the following diagram:



We are interested in obtaining the approximate size of the patch; its direction will be $\hat{\mathbf{n}}$, which allows us to form the vector area. Note that the length of the arc in the θ direction defining the vertical extent of the patch is just $d\theta$, corresponding to the fact that we have a unit sphere (recall the standard formula 'radius times angle' gives arc length). On the other hand, the length of the arc in the ϕ direction depends on the height we are above the xy-plane; at an angle θ, the arc is part of a circle of radius $\sin(\theta)$, so the length of the arc is approximately $\sin(\theta)d\phi$. Thus the area of the patch is approximately $\sin(\theta)d\theta d\phi$, and so the vector area is approximately $d\mathbf{S} = \sin(\theta)d\theta d\phi\hat{\mathbf{n}}$, as required.

(c) $(*)$ Now for the fun part! We compute the integral:

$$\int_{\text{sphere}} d\mathbf{S} = \int_{\text{sphere}} \hat{\mathbf{n}}\sin(\theta)d\theta d\phi,$$

to give the total vector area of the sphere. We need to perform a double integral over both the ranges $0 \leq \theta\pi$ and $0 \leq \phi \leq 2\pi$. We are told in the question that the way to do this is to keep one variable fixed, and integrate with respect to the other. Then, we integrate with respect to the remaining variable we kept fixed in the first step.

Carrying out this procedure, and treating each part of the vector in the integrand as a separate integral, we have:

$$\int_0^\pi \left[\int_0^{2\pi} \hat{\mathbf{n}}d\phi\right]\sin(\theta)d\theta = \int_0^\pi \left[\int_0^{2\pi} \begin{pmatrix}\sin(\theta)\cos(\phi)\\ \sin(\theta)\sin(\phi)\\ \cos(\theta)\end{pmatrix} d\phi\right]\sin(\theta)d\theta = \int_0^\pi \begin{pmatrix}0\\ 0\\ 2\pi\cos(\theta)\end{pmatrix}\sin(\theta)\,d\theta = \int_0^\pi \begin{pmatrix}0\\ 0\\ \pi\sin(2\theta)\end{pmatrix} d\theta = \mathbf{0},$$

so indeed, we find the vector area is zero!

(d) $(*)$ In the last part, observe that $\hat{\mathbf{n}} \cdot d\mathbf{S} = \sin(\theta)d\theta d\phi$ is just the area of a small patch on the surface of the sphere, not the vector area. Hence, if we integrate over the entire sphere, we must just get the surface area of the sphere:

$$\int_{\text{sphere}} \hat{\mathbf{n}} \cdot d\mathbf{S} = 4\pi.$$

You can check this for yourself, if you are keen, by performing a similar multiple integral calculation to part (c).