Part IA: Mathematics for Natural Sciences A
Examples Sheet 1: Basics of vector geometry, and the scalar product

Model Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

Basics of vector algebra

1. Let $A = (1, 3, 4)$, $B = (-1, 2, 4)$, and $C = (2, 2, 3)$. Which of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ is the longest?

Solution: The three-dimensional version of Pythagoras' theorem gives the lengths of the vectors as:

$$|\overrightarrow{AB}| = \sqrt{(-1-1)^2 + (2-3)^2 + (4-4)^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$|\overrightarrow{BC}| = \sqrt{(2--1)^2 + (2-2)^2 + (3-4)^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$$

$$|\overrightarrow{AC}| = \sqrt{(2-1)^2 + (2-3)^2 + (3-4)^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

Hence the longest vector is $\overrightarrow{BC}$.

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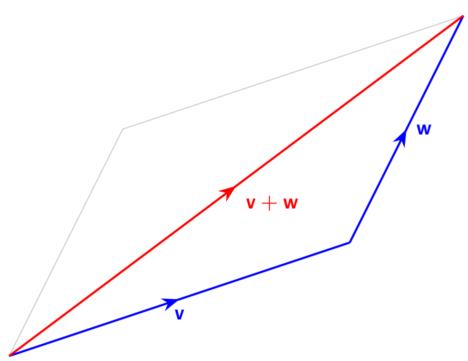
2. (a) State the definition of $\mathbf{v}+\mathbf{w}$, given the vectors $\mathbf{v}$, $\mathbf{w}$. Using this definition, show that $\mathbf{v}+\mathbf{w} = \mathbf{w}+\mathbf{v}$, and $(\mathbf{v}+\mathbf{w})+\mathbf{u} = \mathbf{v} + (\mathbf{w} + \mathbf{u})$ for any vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$.

   (b) Suppose that an aeroplane's engine produces a velocity 125 km h^{-1} due North. If there is a wind travelling at a velocity 80 km h^{-1} at a bearing $60°$ West of North, use trigonometry to determine how fast the aeroplane travels across the Earth, and the bearing of its direction of travel from North.

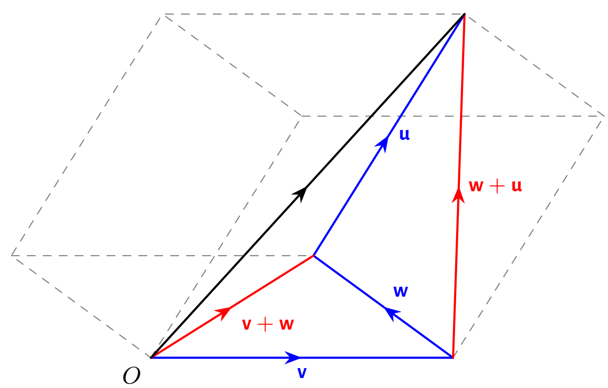
Solution: (a) To produce the vector $\mathbf{v} + \mathbf{w}$ from the vectors $\mathbf{v}$, $\mathbf{w}$, we put the start of the vector $\mathbf{w}$ at the end of the vector $\mathbf{v}$, as shown in the diagram below. The directed line segment from the start of the vector $\mathbf{v}$ to the end of the vector $\mathbf{w}$ in this configuration is then defined to be the vector $\mathbf{v} + \mathbf{w}$.



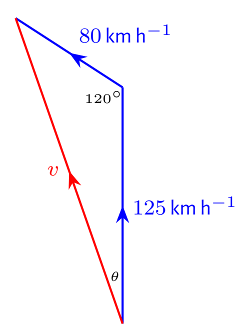
This definition immediately shows that $\mathbf{v}+\mathbf{w} = \mathbf{w}+\mathbf{v}$. This is because the if we started with the vector $\mathbf{w}$, then put the start of the vector $\mathbf{v}$ at the end of the vector $\mathbf{w}$, we produce a parallelogram (shown in grey in the above figure). The diagonal of the parallelogram is both equal to $\mathbf{v}+\mathbf{w}$ and $\mathbf{w}+\mathbf{v}$, which shows that it does not matter in which order we add vectors (In fancy language, we say that vector addition is a *commutative operation*).

Next, we are asked to show the property $(\mathbf{v}+\mathbf{w})+\mathbf{u} = \mathbf{v}+(\mathbf{w}+\mathbf{u})$. We shall do this in three dimensions, which is the more general case.

The best proof is just a 'proof by picture'. The vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$ form a parallelepiped, as shown in the diagram below. In red, we show the vectors $\mathbf{v}+\mathbf{w}$ and $\mathbf{w}+\mathbf{u}$, which are diagonals on the faces of the parallelepiped. Finally, the vectors $(\mathbf{v}+\mathbf{w})+\mathbf{u}$ and $\mathbf{v}+(\mathbf{w}+\mathbf{u})$ both correspond to the black diagonal of the parallelepiped shown in the figure below; hence, they are equal.



(b) To find the resultant velocity of the aircraft, we need to add the two given velocities (which are, of course, vector quantities). Here, a good diagram is helpful!



By the cosine rule, we have that the resultant speed of the aircraft is:

$$v = \sqrt{80^2 + 125^2 - 2\cdot 80\cdot 125\cos(120°)}\ \text{km h}^{-1} = 5\sqrt{1281}\ \text{km h}^{-1}.$$

By the sine rule, we have that the angle θ in the diagram is given by:

$$\frac{\sin(\theta)}{80\ \text{km h}^{-1}} = \frac{\sin(120°)}{v} \qquad \Rightarrow \qquad \theta = \arcsin\left(\frac{80\ \text{km h}^{-1}\sin(120°)}{v}\right) = \arcsin\left(\frac{40\sqrt{3}}{5\sqrt{1281}}\right) = \arcsin\left(\frac{8}{\sqrt{427}}\right).$$

Despite the rather nasty numbers, in general, exact answers are preferred, since calculators are *not* permitted in the first-year mathematics exams.

3. (a) Define a *basis* of vectors.

(b) Let **v** $= (1, 2)$, $\mathbf{e}_1 = (1, -1)$ and $\mathbf{e}_2 = (2, 3)$. Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbb{R}^2$, and determine the components of **v** with respect to this basis.

(c) Let $\mathbf{w}_1 = (1, 2, 3)$ with respect to the basis $\{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$, and let $\mathbf{w}_2 = (3, 2, 1)$ with respect to the basis $\{(0, 1, 2), (2, 1, 0), (0, 1, -2)\}$. Find $\mathbf{w}_1 - 3\mathbf{w}_2$ with respect to the standard basis of $\mathbb{R}^3$.

Solution: (a) A *basis of vectors in n dimensions* is a collection of n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ satisfying two properties:

(i) SPANNING. Any vector **v** can be written as a linear combination of the vectors in the basis; that is, for some scalar coefficients $\alpha_1, \ldots, \alpha_n$ we can write:

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n.$$

(ii) LINEAR INDEPENDENCE. If we have:

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

for some coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$, then we must have $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. This property is telling us that no vectors in the basis are redundant - if there *were* some non-zero coefficients that satisfied this equation, we would be able to rearrange this equation to write one of the basis vectors in terms of the others.

It is actually the case that if we have n vectors that span $\mathbb{R}^n$, they must be linearly independent. Similarly, if we have n vectors that are linearly independent in $\mathbb{R}^n$, they must span. The proof is difficult and beyond the scope of the course (see the Maths B Solutions, if you are interested).

(b) First, we are asked to show that $\{(1, -1), (2, 3)\}$ is a basis for $\mathbb{R}^2$. We show each of the properties separately (although as commented above, we actually only need to show one of these properties, and the other is guaranteed to follow!):

(i) SPANNING. Let (v, w) be any vector in $\mathbb{R}^2$. We need to find coefficients α, β such that:

$$\begin{pmatrix} v \\ w \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \alpha + 2\beta \\ -\alpha + 3\beta \end{pmatrix}.$$

Hence, we have a system of two linear simultaneous equations for the variables α, β. Adding the first equation to the second, we obtain $5\beta = v + w$. Thus $\beta = \frac{1}{5}(v + w)$. Substituting back into the first equation, we have:

$$v = \alpha + \frac{2}{5}(v + w) \qquad \Rightarrow \qquad \alpha = \frac{1}{5}\left(3v - 2w\right).$$

It follows that we can write:

$$\begin{pmatrix} v \\ w \end{pmatrix} = \frac{1}{5}(3v - 2w) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{5}(v + w) \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

for any vector $(v, w) \in \mathbb{R}^2$.

(ii) LINEAR INDEPENDENCE. In the calculation for spanning, we have just shown that if $(0, 0) = \alpha(1, -1) + \beta(2, 3)$, then the coefficients α, β are given by:

$$\alpha = \frac{1}{5}(3 \cdot 0 - 2 \cdot 0) = 0, \qquad \beta = \frac{1}{5}(0 + 0) = 0.$$

Hence we have linear independence.

As mentioned above, we know that it must be the case that if we have already shown the vectors span $\mathbb{R}^2$, they must be linearly independent - this is why the calculations are so similar.

Finally, we are asked the components of $\mathbf{v} = (1, 2)$ with respect to this basis. By the above calculation, we have:

$$\begin{pmatrix}1\\2\end{pmatrix} = \frac{1}{5}(3 \cdot 1 - 2 \cdot 2)\begin{pmatrix}1\\-1\end{pmatrix} + \frac{1}{5}(1+2)\begin{pmatrix}2\\3\end{pmatrix} = -\frac{1}{5}\begin{pmatrix}1\\-1\end{pmatrix} + \frac{3}{5}\begin{pmatrix}2\\3\end{pmatrix}.$$

Hence the components are $-1/5, 3/5$.

(c) We are told that:

$$\mathbf{w}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} + 2\begin{pmatrix}1\\-1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\-1\\3\end{pmatrix},$$

and:

$$\mathbf{w}_2 = 3\begin{pmatrix}0\\1\\2\end{pmatrix} + 2\begin{pmatrix}2\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}4\\6\\8\end{pmatrix}.$$

Hence with respect to the standard basis of $\mathbb{R}^3$, we have:

$$\mathbf{w}_1 - 3\mathbf{w}_2 = \begin{pmatrix}3\\-1\\3\end{pmatrix} - 3\begin{pmatrix}4\\6\\9\end{pmatrix} = \begin{pmatrix}-9\\-19\\-24\end{pmatrix}.$$

The equation of a line

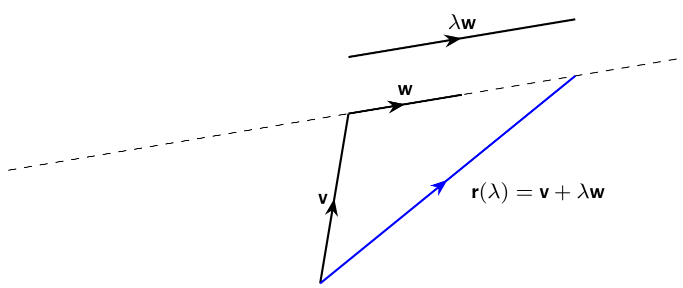
4. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ be 3-vectors, and suppose that $\mathbf{w} \neq \mathbf{0}$.

   (a) Explain why the equation $\mathbf{r} = \mathbf{v} + \lambda\mathbf{w}$, as $\lambda \in \mathbb{R}$ varies, represents a line, and summarise its properties. Why is the condition $\mathbf{w} \neq \mathbf{0}$ necessary?

   (b) If $\mathbf{v} = (x_0, y_0, z_0)$ and $\mathbf{w} = (a, b, c)$, where $a, b, c \neq 0$, show that the same line may be equivalently described through the system of equations:
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$
   What is the corresponding system of equations in the cases where one or more of a, b, c are zero?

   (c) Show that the position vectors $(1, 0, 1)$, $(1, 1, 0)$ and $(1, -3, 4)$ lie on a straight line, and find both its vector form, as in (a), and its Cartesian form, as in (b).

Solution: (a) The equation $\mathbf{r} = \mathbf{v} + \lambda\mathbf{w}$ represents the dashed line shown in the figure below. The dashed line is parallel to the vector $\mathbf{w}$, and the point with position vector $\mathbf{v}$ lies on the line. Therefore, to get to any point on the line, we can first follow the position vector $\mathbf{v}$ from the origin onto the line, then follow a scaled version of the vector $\mathbf{w}$ to get to any other point on the line.



In particular, $\mathbf{r} = \mathbf{v} + \lambda\mathbf{w}$ describes a line through the point with position vector $\mathbf{v}$, parallel to $\mathbf{w}$. The condition $\mathbf{w} \neq \mathbf{0}$ is needed, else the equation $\mathbf{r} = \mathbf{v} + \lambda\mathbf{w} = \mathbf{v}$ would describe a single point instead of a line.

(b) We insert $\mathbf{r} = (x, y, z)$, $\mathbf{v} = (x_0, y_0, z_0)$ and $\mathbf{w} = (a, b, c)$ into the vector equation to get:
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_0 + \lambda a \\ y_0 + \lambda b \\ z_0 + \lambda c \end{pmatrix}.$$
This is a system of three equations, $x = x_0 + \lambda a$, $y = y_0 + \lambda b$, $z = z_0 + \lambda c$. Rearranging each equation for λ, and equating them, we have:
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$
as required.

If one of a, b, c is zero, we cannot perform the division in the solution of the equations. For example, if $a = 0$, then we have the equations $x = x_0$, $y = y_0 + \lambda b$, $z = z_0 + \lambda c$. This means that the resulting equation of the line, eliminating λ, is given by:
$$x = x_0, \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$
That is, the x-direction equation is 'decoupled' from the equations in the y-direction and z-direction. Similar conclusions hold when $b = 0$ or $c = 0$.

(c) Observe that $(1,1,0)-(1,0,1) = (0,1,-1)$ and $(1,-3,4)-(1,1,0) = (0,-4,4) = -4(0,1,-1)$. Hence the vectors joining the points $(1,0,1)$ to $(1,1,0)$, and then $(1,1,0)$ to $(1,-3,4)$, are parallel; it follows that all three points lie on a straight line.

In vector form, one possible equation of the line is:

$$\mathbf{r} = (1,1,0) + \lambda(0,1,-1),$$

but other forms are possible. Setting $\mathbf{r} = (x,y,z)$, we have:

$$(x,y,z) = (1,1,0) + \lambda(0,1,-1) \qquad \Rightarrow \qquad \{x=1,\ y-1=\lambda,\ -z=\lambda\}.$$

Therefore, eliminating λ, the Cartesian equation of the line is:

$$\{x=1,\ 1-y=z\}.$$

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5. Show that the solution of the linear system $x-2y+3z=0, 3x-2y+z=0$ is a line that is equally inclined to the x and z-axes, and makes an angle $\arccos(\sqrt{2/3})$ with the y-axis.

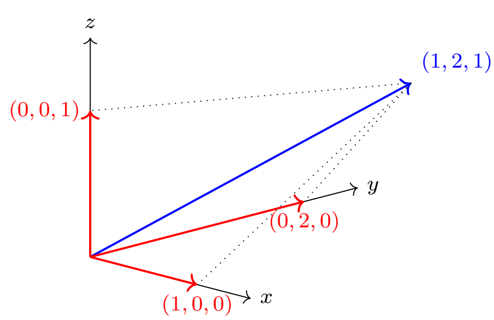
Solution: This is a linear system of equations with two equations, but three free variables. Thus we expect that one variable will be unconstrained. Therefore, we should try to write two of the variables in terms of the third variable.

Subtracting the second equation from the first, we obtain $-2x+2z=0$, which on rearrangement gives $x=z$. Substituting into the first equation, we have $z-2y+3z=0$, which on rearrangement gives $y=2z$. Thus the solution of the system can be written as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 2z \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

where z is a free real parameter. We recognise that this is the equation of a line going through the origin, with direction vector $(1,2,1)$.

To obtain the angles, we do some trigonometry (or, we use the *scalar product* - see later in the sheet).



We see from the figure that the angle the vector makes with both the x-axis and the z-axis is the same, given by:

$$\arccos\left(\frac{1}{\sqrt{1^2+2^2+1^2}}\right) = \arccos\left(\frac{1}{\sqrt{6}}\right).$$

The angle the vector makes with the y-axis is given by:

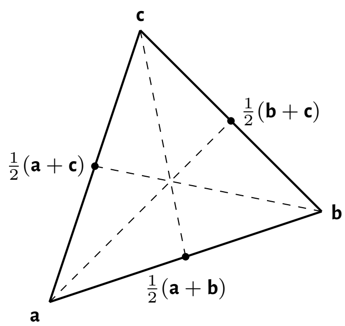
$$\arccos\left(\frac{2}{\sqrt{1^2+2^2+1^2}}\right) = \arccos\left(\frac{2}{\sqrt{6}}\right) = \arccos\left(\sqrt{\frac{2}{3}}\right),$$

as required.

6. (a) A *median* of a triangle is a line joining a vertex to the midpoint of its opposite edge. Prove that the three medians of a triangle are concurrent (the point at which they meet is called the *centroid* of the triangle).

   (b) Similarly, prove that in any tetrahedron, the lines joining the midpoints of opposite edges are concurrent.

Solution: (a) Let the vertices of the triangle have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively, as shown in the figure below.



The midpoints of the edges of the triangle are:

$$\frac{\mathbf{a}+\mathbf{b}}{2}, \qquad \frac{\mathbf{b}+\mathbf{c}}{2}, \qquad \frac{\mathbf{a}+\mathbf{c}}{2}.$$

The vector equations of the lines going the midpoints and the opposite vertices are:

$$\mathbf{r}_1(\lambda) = \mathbf{c} + \lambda\left(\frac{\mathbf{a}+\mathbf{b}}{2} - \mathbf{c}\right), \qquad \mathbf{r}_2(\mu) = \mathbf{a} + \mu\left(\frac{\mathbf{b}+\mathbf{c}}{2} - \mathbf{a}\right), \qquad \mathbf{r}_3(\nu) = \mathbf{b} + \nu\left(\frac{\mathbf{a}+\mathbf{c}}{2} - \mathbf{b}\right).$$

Collecting like terms, these equations can be written as:

$$\mathbf{r}_1(\lambda) = \frac{\lambda}{2}\mathbf{a} + \frac{\lambda}{2}\mathbf{b} + (1-\lambda)\mathbf{c}, \qquad \mathbf{r}_2(\mu) = (1-\mu)\mathbf{a} + \frac{\mu}{2}\mathbf{b} + \frac{\mu}{2}\mathbf{c}, \qquad \mathbf{r}_3(\nu) = \frac{\nu}{2}\mathbf{a} + (1-\nu)\mathbf{b} + \frac{\nu}{2}\mathbf{c}.$$

Now, we would like to set $\mathbf{r}_1(\lambda) = \mathbf{r}_2(\mu) = \mathbf{r}_3(\nu)$, and find values of λ, μ, ν that solve these equations. *However*, in general $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are *not* linearly independent in two dimensions so we cannot compare coefficients!

We can be a bit sneaky here: if we imagine that the triangle is not planar - that is, we imagine that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are *three-dimensional vectors* - we *can* assume they are linearly independent. Hence, comparing coefficients is completely okay here!

In the first equality, $\mathbf{r}_1(\lambda) = \mathbf{r}_2(\mu)$, we see that $\lambda = \mu$ and $\lambda/2 = 1 - \mu$, which gives $\lambda = \mu = 2/3$. This gives the point of intersection of $\mathbf{r}_1, \mathbf{r}_2$ as:

$$\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Symmetrically, we see that $\mathbf{r}_2, \mathbf{r}_3$ and $\mathbf{r}_1, \mathbf{r}_3$ intersect at the same point, so all three lines intersect at the same point.

(b) For the second part, let the vertices of the tetrahedron have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. Then the midpoints of the edges are:

$$\frac{\mathbf{a}+\mathbf{b}}{2}, \qquad \frac{\mathbf{a}+\mathbf{c}}{2}, \qquad \frac{\mathbf{a}+\mathbf{d}}{2}, \qquad \frac{\mathbf{b}+\mathbf{c}}{2}, \qquad \frac{\mathbf{b}+\mathbf{d}}{2}, \qquad \frac{\mathbf{c}+\mathbf{d}}{2}.$$

The lines joining the midpoints of opposite edges are:

$$\mathbf{r}_1(\lambda) = \frac{\mathbf{a}+\mathbf{b}}{2} + \lambda\left(\frac{\mathbf{a}+\mathbf{b}}{2} - \frac{\mathbf{c}+\mathbf{d}}{2}\right), \qquad \mathbf{r}_2(\mu) = \frac{\mathbf{a}+\mathbf{c}}{2} + \mu\left(\frac{\mathbf{a}+\mathbf{c}}{2} - \frac{\mathbf{b}+\mathbf{d}}{2}\right), \qquad \mathbf{r}_3(\nu) = \frac{\mathbf{a}+\mathbf{d}}{2} + \nu\left(\frac{\mathbf{a}+\mathbf{d}}{2} - \frac{\mathbf{b}+\mathbf{c}}{2}\right).$$

Collecting like terms, these equations can be rewritten as:

$$\mathbf{r}_1(\lambda) = \frac{(1+\lambda)}{2}\mathbf{a} + \frac{(1+\lambda)}{2}\mathbf{b} - \frac{\lambda}{2}\mathbf{c} - \frac{\lambda}{2}\mathbf{d}, \qquad \mathbf{r}_2(\mu) = \frac{(1+\mu)}{2}\mathbf{a} - \frac{\mu}{2}\mathbf{b} + \frac{(1+\mu)}{2}\mathbf{c} - \frac{\mu}{2}\mathbf{d},$$

$$\mathbf{r}_3(\nu) = \frac{(1+\nu)}{2}\mathbf{a} - \frac{\nu}{2}\mathbf{b} - \frac{\nu}{2}\mathbf{c} + \frac{(1+\nu)}{2}\mathbf{d}.$$

As in part (a), we can compare coefficients here (even though $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are not linearly independent in three dimensions, we could embed the tetrahedron in a four-dimensional space if we wanted!). The equation $\mathbf{r}_1(\lambda) = \mathbf{r}_2(\mu)$ gives $\mu = \lambda$ and $1 + \lambda = -\mu$, which tells us that $\lambda = \mu = -1/2$. This gives the point of intersection between the first two lines:

$$\frac{1}{4}\left(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}\right).$$

The remaining pairs of lines have the same intersection, by symmetry of the calculation, so we're done.

The scalar product

7. Explain how we can use the two different formulae for the scalar product to determine the angles between vectors. Hence:

(a) determine the angles AOB and OAB, where the points A, B have coordinates $(0,3,4)$, $(3,2,1)$ respectively;

(b) find the acute angle at which two diagonals of a cube intersect.

Solution: From the lectures, we have two formulae for the scalar product:

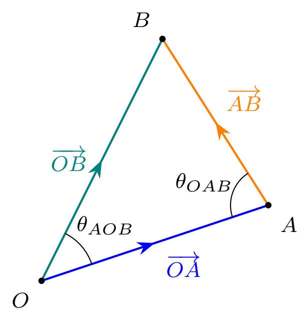
$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos(\theta) = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

Rearranging the second equality, we have:

$$\cos(\theta) = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{|\mathbf{v}||\mathbf{w}|},$$

which allows us to compute the angle between two vectors easily.

(a) In the first case, we want the angle between the vector $\overrightarrow{OA}$ and $\overrightarrow{OB}$ (remember that the angle we get from this formula is the smaller angle produced when the vectors start at the same point - see the diagram below).



The relevant vectors are:

$$\overrightarrow{OA} = (0,3,4), \qquad \overrightarrow{OB} = (3,2,1).$$

Hence we have:

$$\cos(\theta_{AOB}) = \frac{0 \cdot 3 + 3 \cdot 2 + 4 \cdot 1}{\sqrt{3^2+4^2}\sqrt{3^2+2^2+1^2}} = \frac{10}{5\sqrt{14}} = \sqrt{\frac{2}{7}}.$$

Thus the angle is:

$$\theta_{AOB} = \arccos\sqrt{\frac{2}{7}}.$$

In the second case, we want the angle between the vector $\overrightarrow{AO}$ and $\overrightarrow{AB}$. The relevant vectors are:

$$\overrightarrow{AO} = (0,-3,-4), \qquad \overrightarrow{AB} = (3,2,1) - (0,3,4) = (3,-1,-3).$$

Hence we have:

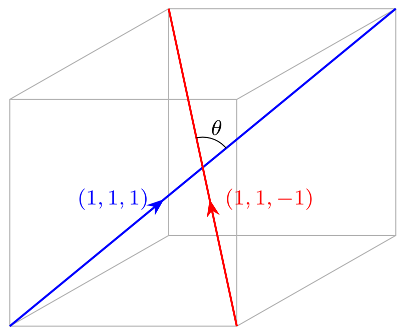
$$\cos(\theta_{OAB}) = \frac{0 \cdot 3 + (-3)\cdot(-1) + (-4)\cdot(-3)}{\sqrt{3^2+4^2}\sqrt{3^2+1^2+3^2}} = \frac{15}{5\sqrt{19}} = \frac{3}{\sqrt{19}}.$$

Thus the angle is:

$$\theta_{OAB} = \arccos\frac{3}{\sqrt{19}}.$$

(b) Without loss of generality, we may assume that we are working with a unit cube, of side length 1, since the angles are unchanged by scaling the cube up or down. Let's put the vertices of the cube at the points:

$$(0,0,0), \quad (0,0,1), \quad (0,1,1), \quad (0,1,0), \quad (1,0,0), \quad (1,0,1), \quad (1,1,1), \quad (1,1,0).$$



The vector from the origin to the opposite vertex is $(1,1,1)$. On the other hand, the vector from the vertex $(1,0,0)$ to the opposite vertex at $(0,1,1)$ is given by $(0,1,1)-(1,0,0)=(-1,1,1)$. The required angle θ therefore satisfies:

$$\cos(\theta) = \frac{1\cdot(-1)+1\cdot 1+1\cdot 1}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+1^2+1^2}} = \frac{1}{3}.$$

Hence the angle is:

$$\theta = \arccos\frac{1}{3}.$$

8. Consider the line with vector equation $\mathbf{r} = (-1,0,1) + \lambda(3,2,1)$, where λ is a real parameter.

(a) Using the scalar product, compute the projection of the vector $(1,2,3)$ in the direction $(3,2,1)$.

(b) Hence, determine the point on the line which is closest to the point $(0,2,4)$, and the shortest distance from the line to the point $(0,2,4)$.

(c) Now, generalise your result: find a formula for the point on the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ which is closest to the point with position vector $\mathbf{p}$, and a formula for the the shortest distance from the line to the point.

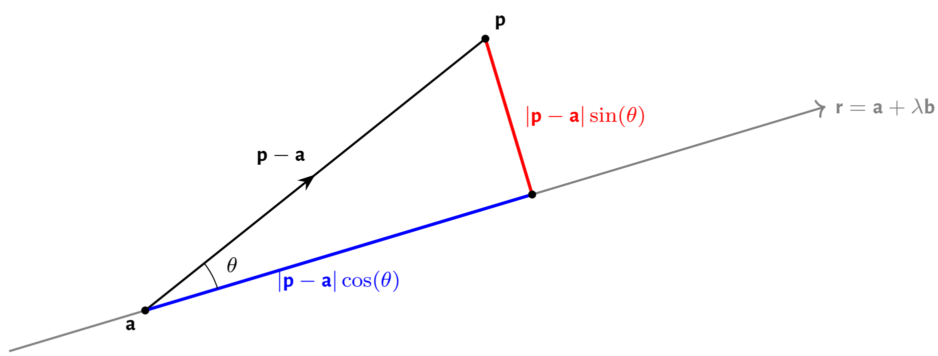
Solution: (a) As we learned in Question 7, the projection is:

$$(1,2,3) \cdot \frac{(3,2,1)}{\sqrt{3^2+2^2+1^2}} = \frac{10}{\sqrt{14}}.$$

(b) Our strategy will be the following:

- Find a vector joining the point to another special point on the line.
- Compute the projection of our vector onto the line.
- Move the length of the projection along the line, from the special point, to find the closest point.

This can be visualised with the diagram below, where $\mathbf{p} = (0,2,4)$ and $\mathbf{a} = (-1,0,1)$. The direction of the line is $\mathbf{b} = (3,2,1)$.



A vector joining the line to the point is $(0,2,4)-(-1,0,1) = (1,2,3)$. The projection of the vector onto the line is therefore:

$$(1,2,3) \cdot \frac{(3,2,1)}{|(3,2,1)|} = \frac{10}{\sqrt{14}},$$

from part (a). Therefore, the closest point on the line to the point is:

$$(-1,0,1) + \frac{10}{\sqrt{14}}\frac{(3,2,1)}{\sqrt{14}} = (-1,0,1) + (30/14, 20/14, 10/14) = (16/14, 20/14, 24/14) = \frac{1}{7}(8,10,12).$$

The shortest distance between the point and the line is therefore:

$$\left|(0,2,4) - \frac{1}{7}(8,10,12)\right| = \frac{1}{7}\left|(-8,4,16)\right| = \frac{4}{7}\left|(-2,1,4)\right| = \frac{4}{7}\sqrt{4+1+16} = \frac{4\sqrt{21}}{7} = 4\sqrt{\frac{3}{7}}.$$

(c) Now, we do the general argument. Following the diagram above, the vector joining the point to the line is $\mathbf{p}-\mathbf{a}$. The projection of the point onto the direction of the line is $\hat{\mathbf{b}}\cdot(\mathbf{p}-\mathbf{a})$. Hence the closest point on the line to the point is:

$$\mathbf{a}+(\hat{\mathbf{b}}\cdot(\mathbf{p}-\mathbf{a}))\hat{\mathbf{b}}.$$

The shortest distance between the point and the line is therefore:

$$\left|\mathbf{p}-\mathbf{a}-\hat{\mathbf{b}}\cdot(\mathbf{p}-\mathbf{a})\hat{\mathbf{b}}\right|.$$

Alternatively, the shortest distance can be computed by Pythagoras:

$$\sqrt{|\mathbf{p}-\mathbf{a}|^2-(|\mathbf{p}-\mathbf{a}|\cos(\theta))^2}=\sqrt{|\mathbf{p}-\mathbf{a}|^2-(\hat{\mathbf{b}}\cdot(\mathbf{p}-\mathbf{a}))^2}.$$

These expressions are the same, because:

$$\begin{aligned}\left|\mathbf{p}-\mathbf{a}-\hat{\mathbf{b}}\cdot(\mathbf{p}-\mathbf{a})\hat{\mathbf{b}}\right| &= \sqrt{\left(\mathbf{p}-\mathbf{a}-\hat{\mathbf{b}}\cdot(\mathbf{p}-\mathbf{a})\hat{\mathbf{b}}\right)\cdot\left(\mathbf{p}-\mathbf{a}-\hat{\mathbf{b}}\cdot(\mathbf{p}-\mathbf{a})\hat{\mathbf{b}}\right)}\\ &= \sqrt{|\mathbf{p}-\mathbf{a}|^2-2\left(\hat{\mathbf{b}}\cdot(\mathbf{p}-\mathbf{a})\right)^2+\left(\hat{\mathbf{b}}\cdot(\mathbf{p}-\mathbf{a})|\right)^2}\\ &= \sqrt{|\mathbf{p}-\mathbf{a}|^2-\left(\hat{\mathbf{b}}\cdot(\mathbf{p}-\mathbf{a})\right)^2}.\end{aligned}$$

It is not very useful to remember either version of this formula - but it *is* useful to know how to derive it. Knowing the method here is much more important than remembering a formula and being able to substitute things into it!

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9. Show that if four points A,B,C,D are such that $AD\perp BC$ and $BD\perp AC$, then $CD\perp AB$.

Solution: Let the position vectors of the points be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$. We are given that $AD\perp BC$ and $BD\perp AC$, which in terms of vectors become:

$$(\mathbf{d}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{b})=0,\qquad(\mathbf{d}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{a})=0.$$

We want to show that:

$$(\mathbf{d}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{a})=0.$$

Expanding all of these conditions using the properties of the scalar product, we have:

$$\mathbf{d}\cdot\mathbf{c}-\mathbf{d}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{b}=0\ (\dagger1),\qquad\mathbf{d}\cdot\mathbf{c}-\mathbf{d}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{a}=0\ (\dagger2),$$

and we want to show that:

$$\mathbf{d}\cdot\mathbf{b}-\mathbf{d}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{a}=0.\tag{$*$}$$

But notice that equation $(*)$ is just equation $(\dagger2)$ subtract equation $(\dagger1)$. So we are done!

10. Using the scalar product, prove that for any tetrahedron, the sum of the squares of the lengths of the edges equals four times the sum of the squares of the lengths of the lines joining the mid-points of opposite edges.

➾ **Solution:** Let the vertices of the tetrahedron have position vectors **a**, **b**, **c**, **d**. Then the sum of the squares of the lengths of the edges of the tetrahedron is:

$$(\mathbf{b}-\mathbf{a})^2+(\mathbf{c}-\mathbf{a})^2+(\mathbf{d}-\mathbf{a})^2+(\mathbf{c}-\mathbf{b})^2+(\mathbf{d}-\mathbf{b})^2+(\mathbf{d}-\mathbf{c})^2. \tag{†}$$

The mid-points of the edges are:

$$\frac{\mathbf{a}+\mathbf{b}}{2},\qquad \frac{\mathbf{a}+\mathbf{c}}{2},\qquad \frac{\mathbf{a}+\mathbf{d}}{2},\qquad \frac{\mathbf{b}+\mathbf{c}}{2},\qquad \frac{\mathbf{b}+\mathbf{d}}{2},\qquad \frac{\mathbf{c}+\mathbf{d}}{2}.$$

The sum of the squares of the lengths joining the opposite mid-points is therefore:

$$\left(\frac{\mathbf{c}+\mathbf{d}}{2}-\frac{\mathbf{a}+\mathbf{b}}{2}\right)^2+\left(\frac{\mathbf{b}+\mathbf{d}}{2}-\frac{\mathbf{a}+\mathbf{c}}{2}\right)^2+\left(\frac{\mathbf{a}+\mathbf{d}}{2}-\frac{\mathbf{b}+\mathbf{c}}{2}\right)^2.$$

We need to make this look like (†). We can do this by organising the terms as follows:

$$\left(\frac{\mathbf{c}+\mathbf{d}}{2}-\frac{\mathbf{a}+\mathbf{b}}{2}\right)^2=\frac{1}{4}\left((\mathbf{c}-\mathbf{a})+(\mathbf{d}-\mathbf{b})\right)^2=\frac{1}{4}\left((\mathbf{c}-\mathbf{a})^2+2(\mathbf{c}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{b})+(\mathbf{d}-\mathbf{b})^2\right),$$

$$\left(\frac{\mathbf{b}+\mathbf{d}}{2}-\frac{\mathbf{a}+\mathbf{c}}{2}\right)^2=\frac{1}{4}\left((\mathbf{b}-\mathbf{c})+(\mathbf{d}-\mathbf{a})\right)^2=\frac{1}{4}\left((\mathbf{b}-\mathbf{c})^2+2(\mathbf{b}-\mathbf{c})\cdot(\mathbf{d}-\mathbf{a})+(\mathbf{d}-\mathbf{a})^2\right),$$

$$\left(\frac{\mathbf{a}+\mathbf{d}}{2}-\frac{\mathbf{b}+\mathbf{c}}{2}\right)^2=\frac{1}{4}\left((\mathbf{a}-\mathbf{b})+(\mathbf{d}-\mathbf{c})\right)^2=\frac{1}{4}\left((\mathbf{a}-\mathbf{b})^2+2(\mathbf{a}-\mathbf{b})\cdot(\mathbf{d}-\mathbf{c})+(\mathbf{d}-\mathbf{c})^2\right).$$

We see that the sum of the right hand side contains all the terms we need, apart from the cross-terms; we just need to show that these all cancel. We have:

$$\begin{aligned}&2(\mathbf{c}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{b})+2(\mathbf{b}-\mathbf{c})\cdot(\mathbf{d}-\mathbf{a})+2(\mathbf{a}-\mathbf{b})\cdot(\mathbf{d}-\mathbf{c})\\&\quad=2\left(\mathbf{c}\cdot\mathbf{d}-\mathbf{c}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{d}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{d}-\mathbf{b}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{d}+\mathbf{c}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{d}-\mathbf{a}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{d}+\mathbf{b}\cdot\mathbf{c}\right)\\&\quad=0,\end{aligned}$$

and hence the result follows.

11.(a) Using the geometric definition of the scalar product, prove the *Cauchy-Schwarz inequality* $|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

(b) From the Cauchy-Schwarz inequality, deduce the *triangle inequality* $|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$. What is the geometrical significance of this inequality? Learn this equality off by heart; it will be useful later when we study limits!

(c) From the triangle inequality, deduce the *reverse triangle inequality* $||\mathbf{a}| - |\mathbf{b}|| \leq |\mathbf{a}-\mathbf{b}|$.

➻ **Solution:** (a) Since $|\cos(\theta)| \leq 1$, we have:

$$|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos(\theta)| \leq |\mathbf{a}||\mathbf{b}|,$$

which proves the Cauchy-Schwarz inequality, as required.

(b) We just proved an inequality to do with the scalar product, so to prove the triangle inequality, we should try to relate things to a scalar product. To do so, consider the square of $|\mathbf{a}+\mathbf{b}|$. We have:

$$\begin{aligned}
|\mathbf{a}+\mathbf{b}|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) && (\text{since } |\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}) \\
&= \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} \\
&= |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 \\
&\leq |\mathbf{a}|^2 + 2|\mathbf{a}\cdot\mathbf{b}| + |\mathbf{b}|^2 && (\text{since for any } x, \text{ we have } x \leq |x|) \\
&\leq |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 && (\text{Cauchy-Schwarz}) \\
&= (|\mathbf{a}| + |\mathbf{b}|)^2.
\end{aligned}$$

To finish, we take the square root of both sides, which gives:

$$|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|,$$

which is the triangle inequality as required.

If we consider the triangle formed by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{a}-\mathbf{b}$, then the triangle inequality tells us that the lengths of these vectors satisfy $|\mathbf{a}-\mathbf{b}| \leq |\mathbf{a}| + |-\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$. In particular, it tells us that the third side of a triangle cannot be longer than the sum of the lengths of the other two sides.

(c) Using the triangle inequality, we have:

$$|\mathbf{a}| = |\mathbf{a}-\mathbf{b}+\mathbf{b}| \leq |\mathbf{a}-\mathbf{b}| + |\mathbf{b}| \qquad \Rightarrow \qquad |\mathbf{a}| - |\mathbf{b}| \leq |\mathbf{a}-\mathbf{b}|.$$

Additionally, we have:

$$|\mathbf{b}| = |\mathbf{b}-\mathbf{a}+\mathbf{a}| \leq |\mathbf{a}-\mathbf{b}| + |\mathbf{a}| \qquad \Rightarrow \qquad |\mathbf{b}| - |\mathbf{a}| \leq |\mathbf{a}-\mathbf{b}|.$$

Putting these together, we see that:

$$||\mathbf{a}| - |\mathbf{b}|| \leq |\mathbf{a}-\mathbf{b}|,$$

as required. This inequality tells us that the length of the third side of a triangle is always at least the difference of the lengths of the other two sides.

The equation of a plane

12. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ be fixed 3-vectors, with $\mathbf{b} \neq \mathbf{0}$.

(a) Explain why the equation $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$ represents a plane, and summarise its properties. Show using properties of the scalar product that an equivalent representation of this plane is $\mathbf{r} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$. What is the geometric significance of the quantity $|\mathbf{a} \cdot \mathbf{b}|/|\mathbf{b}|$ here?

(b) By writing $\mathbf{r} = (x, y, z)$, $\mathbf{b} = (l, m, n)$, and $\mathbf{a} \cdot \mathbf{b} = d$, show that the equation of a plane may equivalently be written in the Cartesian form $lx + my + nz = d$.

(c) Find the equation of the plane containing the point $(3, 2, 1)$ with normal $(1, 2, 3)$ in both the vector form, as in (a), and the Cartesian form, as in (b). What is the shortest distance from the origin to the plane?

Solution: (a) The equation $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$ states that the vector from the point **a** to the point **r** is orthogonal to the vector **b**. Thus this equation represents a plane which is normal to the vector **b**, and passes through the point **a**. Using distributivity of the scalar product, we have $0 = (\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = \mathbf{r} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b}$. Rearranging, we obtain $\mathbf{r} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ as required.

Let θ be the angle between **r** and **b**. Then the equation of the plane may be written as:

$$|\mathbf{r}||\mathbf{b}| \cos(\theta) = \mathbf{a} \cdot \mathbf{b}.$$

Since $\mathbf{a} \cdot \mathbf{b}$ is fixed, $|\mathbf{r}|$ is minimised when the magnitude of $\cos(\theta)$ is maximised (the sign of $\cos(\theta)$ must match the sign of $\mathbf{a} \cdot \mathbf{b}$). Since the maximum value of the magnitude of $\cos(\theta)$ is ± 1, this shows that *the shortest distance from the origin to the plane* is given by:

$$|\mathbf{r}| = \frac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{b}|}.$$

(b) Writing $d = \mathbf{a} \cdot \mathbf{b}$, $\mathbf{r} = (x, y, z)$ and $\mathbf{b} = (l, m, n)$, the equation $\mathbf{r} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ becomes $(x, y, z) \cdot (l, m, n) = d$, which multiplies out to $lx + my + nz = d$. This is the Cartesian form of a plane.

(c) The equation of the plane in vector form is:

$$\left(\mathbf{r} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\right) \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0.$$

Letting $\mathbf{r} = (x, y, z)$, this multiplies out to:

$$x + 2y + 3z - 3 - 4 - 3 = 0 \qquad \Rightarrow \qquad x + 2y + 3z = 10,$$

which is the Cartesian equation of the plane. By part (a), the shortest distance from the origin to the plane is given by:

$$\frac{10}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{10}{\sqrt{14}}.$$

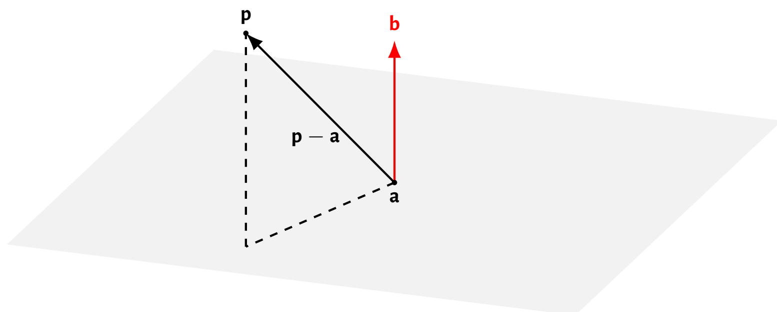
13. Consider the plane with vector equation $(\mathbf{r} - (1,0,1)) \cdot (2,-1,0) = 0$.

(a) Using the scalar product, compute the projection of the vector $(2,0,3)$ in the direction $(2,-1,0)$.

(b) Using the result of part (a), determine the point on the plane which is closest to the point $(3,0,4)$, and the shortest distance from the plane to the point $(3,0,4)$.

(c) Now, generalise your result: find a formula for the point on the plane $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$ which is the closest to the point with position vector $\mathbf{p}$, and a formula for the shortest distance from the plane to this point.

Solution: When doing this question, we should have in mind the following diagram:



(a) For the first part, we have:

$$(2,0,3) \cdot \frac{(2,-1,0)}{\sqrt{2^2+1^2}} = \frac{4}{\sqrt{5}}.$$

(b) Next, we note that a vector joining the point to the plane is $(3,0,4) - (1,0,1) = (2,0,3)$. The length of the projection of this vector in the direction of the normal $(2,-1,0)$ will be the shortest distance to the plane, which by part (a) is $4/\sqrt{5}$.

To get the closest point, we subtract the component of $(2,0,3)$ in the direction normal to the plane, giving us:

$$(2,0,3) - \frac{4}{\sqrt{5}}\frac{(2,-1,0)}{\sqrt{5}} = (2,0,3) - (8/5,-4/5,0) = (2/5,4/5,3).$$

This vector is the planar component of the vector joining the plane and the point. Adding this vector to $(1,0,1)$, we must get the closest point on the plane to the point:

$$(1,0,1) + (2/5,4/5,3) = (7/5,4/5,3).$$

(c) Now, we do the same argument with abstract vectors. A vector joining the plane to the point is $\mathbf{p} - \mathbf{a}$. The length of the projection of this vector in a direction normal to the plane will be the shortest distance to the plane, which is given by:

$$\left|\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\right|$$

The modulus is required, because the projection is a *signed* projection. To finish, the component of the vector $\mathbf{p} - \mathbf{a}$ parallel to the plane is then:

$$\mathbf{p} - \mathbf{a} + (\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a}))\hat{\mathbf{b}}.$$

Adding this to the point $\mathbf{a}$, we get the closest point on the plane to the point:

$$\mathbf{p} + \left(\hat{\mathbf{b}} \cdot (\mathbf{p} - \mathbf{a})\right)\hat{\mathbf{b}}.$$

14. Using the results of Question 13, calculate the shortest distances between the plane $5x+2y-7z+9=0$ and the points $(1,-1,3)$ and $(3,2,3)$. Are the points on the same side of the plane?

Solution: By inspection, a point on the plane is $(-1,-2,0)$, and the normal to the plane is $(5,2,-7)$. Therefore vectors joining the points to the plane are $(1,-1,3)-(-1,-2,0)=(2,1,3)$ and $(3,2,3)-(-1,-2,0)=(4,4,3)$. The shortest distances are then given by:

$$\left|(2,1,3)\cdot\frac{(5,2,-7)}{\sqrt{5^2+2^2+7^2}}\right| = \left|-\frac{9}{\sqrt{78}}\right| = \frac{9}{\sqrt{78}},$$

and:

$$\left|(4,4,3)\cdot\frac{(5,2,-7)}{\sqrt{78}}\right| = \frac{7}{\sqrt{78}}.$$

Importantly, we see that the modulus was relevant in the first case but not in the second. This means that the angle between the normal and the vector joining a point in the plane to the point off the plane is obtuse in the first case, and acute in the second case (think about the signs of the scalar product). Hence, they must be on opposite sides of the plane.

Equations of other 3D surfaces

15. Let k, m be positive constants, with $m<1$. Describe the following surfaces: (a) $|\mathbf{r}|=k$; (b) $\mathbf{r}\cdot\mathbf{u}=m|\mathbf{r}|$.

Solution:

(a) This surface, $|\mathbf{r}|=k$, comprises the set of all vectors whose distance from the origin is equal to k. Hence this is a sphere centred on the origin of radius k.

(b) Write $\mathbf{r}\cdot\mathbf{u}=|\mathbf{r}|\cos(\theta)$ (using the fact that $\mathbf{u}$ is a unit vector). Then the equation can be rewritten as:

$$\cos(\theta)=m.$$

Hence, this surface consists of all vectors which are a constant angle $\arccos(m)$ to the vector $\mathbf{u}$. Thus the surface is a *cone*, with axis along $\mathbf{u}$. The tip of the cone is at the origin, since $\mathbf{r}=\mathbf{0}$ satisfies the equation.

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16. Describe the surface given by the vector equation:

$$|\mathbf{r}-(\mathbf{r}\cdot\mathbf{u})\mathbf{u}|=2,$$

where $\mathbf{u}=\frac{1}{\sqrt{2}}(1,0,1)$. What is the intersection of this surface and the surface $x+z=0$?

Solution: Note that $\mathbf{u}$ is a unit vector, so $\mathbf{r}\cdot\mathbf{u}$ is the length of the projection of $\mathbf{u}$ in the direction of $\mathbf{u}$. Hence the vector $\mathbf{r}-(\mathbf{r}\cdot\mathbf{u})\mathbf{u}$ represents the vector $\mathbf{r}$ with its $\mathbf{u}$ component entirely 'removed'. That is, $\mathbf{r}-(\mathbf{r}\cdot\mathbf{u})\mathbf{u}$ is the *projection of the vector $\mathbf{r}$ orthogonal to the $\mathbf{u}$ direction*.

The equation of the surface tells us that the component of $\mathbf{r}$ orthogonal to $\mathbf{u}$ is constant, and equal to 2. Thus this equation describes a *cylinder of radius* 2, with axis along the vector $\mathbf{u}$.

Since the direction of $\mathbf{u}$ in this question is $(1,0,1)$, this is the same as the direction orthogonal to the plane $x+z=0$. Hence, the intersection of $x+z=0$ and the figure $|\mathbf{r}-(\mathbf{r}\cdot\mathbf{u})\mathbf{u}|=2$ must be a circle of radius 2, centred on the origin, in the plane orthogonal to the vector $(1,0,1)$.

17.(a) Write down a vector equation for the sphere with centre at the point with position vector **a**, and radius $p > 0$.

(b) If there is a second sphere with centre at the point with position vector **b**, and radius $q > 0$, what conditions are required on **a**, **b**, p and q for the two spheres to intersect in a circle?

(c) Show that, if the two spheres do intersect, then the plane in which their intersection occurs is given by the equation $2\mathbf{r} \cdot (\mathbf{b} - \mathbf{a}) = p^2 - q^2 + |\mathbf{b}|^2 - |\mathbf{a}|^2$.

Solution: (a) The vector equation of the sphere is $|\mathbf{r} - \mathbf{a}| = p$, since the distance between any point on the sphere **r** and the sphere's centre **a** must always be equal to the radius p.

(b) If there is a second sphere $|\mathbf{r} - \mathbf{b}| = q$, we need the distance between the centres of the spheres to be less than the sum of the radii of the spheres. That is, we need:

$$|\mathbf{b} - \mathbf{a}| < p + q.$$

We need strict inequality here, because if $|\mathbf{b} - \mathbf{a}| = p + q$, then the spheres just 'touch' at a single point.

(c) To get the plane of intersection, consider squaring the two equations of the spheres and using properties of the scalar product:

$$p^2 = |\mathbf{r} - \mathbf{a}|^2 = (\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{a}) = |\mathbf{r}|^2 - 2\mathbf{r} \cdot \mathbf{a} + |\mathbf{a}|^2$$

$$q^2 = |\mathbf{r} - \mathbf{b}|^2 = (\mathbf{r} - \mathbf{b}) \cdot (\mathbf{r} - \mathbf{b}) = |\mathbf{r}|^2 - 2\mathbf{r} \cdot \mathbf{b} + |\mathbf{b}|^2.$$

Subtracting the second equation from the first, we have:

$$2\mathbf{r} \cdot (\mathbf{b} - \mathbf{a}) + |\mathbf{a}|^2 - |\mathbf{b}|^2 = p^2 - q^2.$$

Rearranging, we have:

$$2\mathbf{r} \cdot (\mathbf{b} - \mathbf{a}) = p^2 - q^2 + |\mathbf{b}|^2 - |\mathbf{a}|^2,$$

as required.