Part IA: Mathematics for Natural Sciences
Supervisions in Mathematics: FAQs

1. What are the topics we will study?

The topics we will study are listed below. Unless otherwise indicated, each topic corresponds to exactly one exam question at the end of the year. Some of the topics are Maths B topics only, which are indicated with a (†).

Maths A Topics

Michaelmas

- Vector geometry
- Complex numbers and hyperbolic functions
- Single-variable integration
- Taylor series
- Probability (x2 exam questions)

Lent

- First-order ODEs
- Second-order ODEs
- Partial differentiation
- Contour sketching in 2D
- Multivariable integration and Gaussian integrals
- Vector fields and line integrals
- Surface integrals and integral theorems

Easter

- Matrix algebra: basics, determinants, eigenvalues and eigenvectors, symmetric and orthogonal matrices (x2 exam questions)
- Fourier series

Maths B Topics

Michaelmas

- Vector geometry
- Complex numbers and hyperbolic functions
- (†) Formal real analysis (limits, continuity, big-O notation, and convergence of series)
- Taylor series
- Single-variable integration
- (†) Advanced integration (the Leibniz integral rule and integral inequalities)
- Multivariable integration and Gaussian integrals
- Probability (x2 exam questions)

Lent

- First-order ODEs
- Second-order ODEs
- Partial differentiation
- Contour sketching in 2D
- (†) Conditional optimisation
- Vector fields and line integrals
- Surface integrals and integral theorems
- Fourier series

Easter

- Matrix algebra: basics, determinants, eigenvalues and eigenvectors, symmetric and orthogonal matrices (x2 exam questions)
- (†) Solution of PDEs

2. How should I present my work?

Your work should be legibly handwritten on A4 lined paper (*not* square), secured with a paperclip or staple. More specific to the mathematics, you should note the following:

- Your work should not just be a series of equations, but should explain how to do the problems - this involves writing short *sentences* (gasp!), showing that you understand what you are doing and why.
- You should draw clear, labelled diagrams, where relevant. A lot of problems in the course involve some complicated geometry, and a diagram will help you, me, and the examiner at the end of the year!
- When you are doing lots of algebra, try to start each equality on a new line; it is much more complicated to follow a calculation if it is all done on one line! You can also enhance this layout by writing short comments in brackets at the end of a line, explaining what you did on each line (e.g. 'substitute $u = x^2$', if you substituted for x^2).

For more advice, I would highly recommend you read Kevin Houston's document, *How to Write Mathematics*, available here: https://www.maths.ox.ac.uk/system/files/attachments/How%20to%20write%20mathematics.pdf. He gives an excellent example of how to improve a student's proof of the cosine rule; as practice, you might like to try Exercises 3.2(i) and (ii) from the text.

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3. How long should I spend on the problem sheets?

You have four subjects in first-year Natural Sciences. Hence, to make up a 48 hour working week (assuming 8 hours of work each day Monday-Saturday), you should be spending:

$$\frac{48 \text{ hours}}{4 \text{ subjects}} = 12 \text{ hours per subject.}$$

For your science subjects, this will involve 3 hours of lectures, plus practicals - but, maths does not have practicals, it has some small pieces of computing coursework instead. The computing coursework sessions are each 1.5 hours long, and there are 4 per term, giving a total of 6 hours split amongst 8 weeks - that is, 0.75 hours per week. Thus you should be spending *at most* around

$$12 \text{ hours} - 3 \text{ hours of lectures} - 0.75 \text{ hours of computing practical} = 8.25 \text{ hours} \approx 8 \text{ hours}$$

on the problem sheets each week. If you feel that you are repeatedly unable to complete the problem sheets adequately in that amount of time, tell me, and we can make a plan of what to do. If you are doing course B, this will likely involve me telling you to switch to course A.

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4. I'm stuck on a problem, and not sure what to do.

Mathematics is a difficult subject, and often you will get stuck on problems. This is not a bad thing, it is an opportunity to learn! Things you can try are the following:

- Read through the lecture notes again - perhaps there is a relevant example of the type of problem you are solving given there.
- Leave the problem for a while, focus on another subject, or go for a walk, and come back to it later with fresh eyes.
- Chat with a friend, for example your supervision partner, and see if they have done the problem - perhaps you can learn from each other, trading for a problem they have struggled with.
- Send an email to me, and ask for a hint.

Please make sure you have had a good go at each of the problems *before* coming to a supervision - figuring things out for yourself can be much more educational than me just telling you the answer!

5. I didn't have time to do all the problems this week, sorry!

You did - see point 3 above! I know that you will have busy weeks from time to time, but it is really important you have a serious attempt at the sheets. The best way to learn mathematics, and hence to do well in the end-of-year exams, is to do lots of problems; please try to make sure that this is a priority throughout the term!

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6. I did the question, but was out by a factor of 7, can you find where it went?

No, but I'm sure that if you looked through your calculation once more, *you* could! Work you hand in for supervisions should be your best try, not a rough first attempt. Part of learning mathematics is also learning how to spot your mistakes, and as discussed above, it is much more educational for *you* to have had a serious attempt at figuring things out yourself, rather than me just telling you where you went wrong. (That said, I probably can find the odd factor of 7 for you, if you're really struggling!)

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7. Can I use ChatGPT to do my work for me?

Well, you can try! When setting the problems on these sheets, I did try a few of them in ChatGPT, and it rarely gave me the correct answer - most of the time, it got something fairly fundamental wrong. Maybe AI will improve drastically in the next few years, but for now, you're probably safer trying to learn mathematics the 'old-fashioned way' - you might even learn how to spot ChatGPT's mistakes! It's also worth mentioning that ChatGPT won't be there to help you in the end-of-year exams...

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8. I love maths so much! What should I do?

Much as it pains me to write, deep down, you are a *scientist*, not a mathematician - you are studying Natural Sciences, and hence maths is just a means to an end, not the end itself. However, if you really love the Part IA Maths course, you could:

- Consider taking the Part IB Maths course - just as much fun as IA, with lots more interesting and enjoyable content (particularly complex analysis and group theory, two of my favourite areas of maths!).
- Consider taking the Part IB Quantitative Environmental Science course - a bit of a different flavour to the first-year maths course, but still a very interesting maths-oriented course, applied to important real-world issues.
- Consider taking Physics A or Physics B in second year - these options make lots of use of the mathematics we learn in first year, particularly vector calculus (in Electromagnetism), transform theory (in Experimental Methods), and matrix algebra (in Quantum Physics).
- Consider switching to the Maths Tripos. Lucy Cavendish currently imposes a strong barrier to doing this however; you will need to either:
  - Sit the STEP Mathematics entrance exams in the summer, and re-enter Cambridge as a first-year maths student, provided that you are successful.
  - Sit the first-year Maths Tripos exams in addition to the Natural Sciences Tripos exams, and achieve first-class results in both sets of exams.

  If this is something you are seriously considering, you will need to let me know as soon as possible (ideally no later than Michaelmas), so we can prepare you for either of these options.

Any more questions? Feel free to drop me an email at `jmm232@cam.ac.uk`.

Part IA: Mathematics for Natural Sciences
Examples Sheet 0: Basic skills

Please send all comments and corrections to jmm232@cam.ac.uk.

Making friends

1. Speak to your supervision partner, and, if you don't know already, find out: (a) where they are from; (b) their favourite food; (c) what they like to do to relax; (d) what part of the maths course they are most excited about this year.

Writing mathematics

2. Read Gareth Wilkes' document 'A Brief Guide to Mathematical Writing', available at: https://www.dpmms.cam.ac.uk/~grw46/Writing_Guide.pdf. [*You can ignore Section 4 for now - but we will study quantifiers in the Maths B course, if you are taking it.*] Hence, make a list of things that you should consider when writing a solution to a mathematics problem.

3. Write down your best and most presentable model solution to the following problem:

   Let L be a line passing through the origin with gradient k. Let C be a circle centred on $(2, 0)$ with radius 1. Determine the values of k for which L and C intersect at zero points, one point, or two points: (a) using an algebraic method; (b) using a geometric method.

   Compare your solution with your supervision partner, and give each other advice and feedback.

4. Learn all the letters of the Greek alphabet, and get your supervision partner to test you on them.

Basic logic

5. Explain the meaning of the logical symbols $\Rightarrow$, $\Leftarrow$ and $\Leftrightarrow$. [*If you haven't seen them before, look them up online! In general, you should feel free to look up terms you don't understand on an examples sheet.*] Decide which of the following are true:

   (a) $x^2 \leq 1 \;\Rightarrow\; x \leq 1$, (b) $x^2 \leq 1 \;\Leftarrow\; x \leq 1$, (c) $x^2 \leq 1 \;\Leftrightarrow\; x \leq 1$.

   Explain also the meanings of the terms *necessary condition* and *sufficient condition*. Decide which of the following are true:

   (d) $|x| = 1$ is sufficient for $x = 1$
   (e) $|x| = 1$ is necessary for $x = 1$
   (f) $|x| = 1$ is necessary and sufficient for $x = 1$.

6. What is the error in the following argument?

   'Suppose we wish to solve the equation $x - 1 = 2$. We begin by squaring both sides, to obtain $(x-1)^2 = 4$. Expanding the left hand side we have:

   $$x^2 - 2x + 1 = 4 \qquad \Leftrightarrow \qquad x^2 - 2x - 3 = 0 \qquad \Leftrightarrow \qquad (x+1)(x-3) = 0.$$

   Since one of these factors must be zero, it follows that there are two solutions to the equation, $x = -1$ or $x = 3$.'

7. What is meant by *proof by contradiction*? Prove by contradiction that $\sqrt{2}$ and $\sqrt{3}$ are irrational numbers.

Sets and functions

8. State what is meant by the sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$. Decide which of the following statements are true:

   (a) $\pi \in \mathbb{Q}$, (b) $3 \notin \mathbb{R}$, (c) $\mathbb{Z} \subseteq \mathbb{Q}$, (d) $\mathbb{Q} \supset \mathbb{C}$.

9. State what is meant by the sets $[a, b]$, (a, b) and $(a, b]$, where a, b are real numbers. Decide which of the following statements are true:

   (a) $1 \in [0, 1)$, (b) $3 \notin (3, 4)$, (c) $[2, 3] \subset (2, 5]$, (d) $(-1, 0) \subset [-1, 0]$.

10. A function is a mapping from a set to another set. We write $f : A \to B$ to denote the function (or just f when the sets are implied), and we write $f(x)$ for the value of the function at the point $x \in A$. Which of the following define functions, and why?

    (a) $f : [0, \infty) \to \mathbb{R}$, given by $f(x) = \sqrt{x}$;
    (b) $f : \mathbb{Z} \to \mathbb{Z}$, given by $f(x) = \frac{1}{2}x$;
    (c) $f : \mathbb{R} \to \mathbb{R}$, where $f(x)$ is implicitly defined by $2^{f(x)} = x$;
    (d) $f : [0, \infty) \to \mathbb{R}$, where $f(x)$ is implicitly defined by $f(x)^2 = x$.

Sequences and series

11. Prove the following results using (i) induction; (ii) a direct argument:

    (a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, (b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, (c) $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

    [*Hint: for the direct arguments, consider summing $(k+1)^r - k^r$ for an appropriate integer r in each case; note this sum* ***telescopes****. If you haven't heard of telescoping sums before, look them up.*] Is it more useful to prove a result by induction, or by a direct argument? Why?

12. What is meant by an *arithmetic sequence* with first term a and common difference d? Prove that the sum of the first n terms of an arithmetic sequence is:
    $$S_n = \frac{1}{2}n\left(2a + (n-1)d\right).$$
    Hence, find the sum of the series $2, 5, 8, 11, ..., 32$.

13. What is meant by a *geometric sequence* with first term a and common ratio r? Prove that the sum of the first n terms of a geometric sequence is:
    $$S_n = \frac{a(1-r^n)}{1-r}.$$
    What happens if $r = 1$? What is the behaviour of this sum in the limit as $n \to \infty$? Hence, find the sum of the infinite series $2, 2/3, 2/9, 2/27, ...$

14. Using the formula for the sum of an infinite geometric series, find a formula for the sum of the infinite series:
    $$\sum_{k=1}^{\infty} kr^k,$$
    where $|r| < 1$. [*Hint: differentiation!*] Hence determine:
    $$\frac{2}{3} + 2\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^3 + 4\left(\frac{2}{3}\right)^4 + \cdots.$$

Trigonometric functions and their identities

15. Define the trigonometric functions $\sin(x), \cos(x), \tan(x)$ in terms of the side lengths of an appropriate right-angled triangle. Define also the reciprocal trigonometric functions $\operatorname{cosec}(x), \sec(x), \cot(x)$. Hence, prove each of the following trigonometric identities:

    (a) *The Pythagorean identities:*

    $$\sin^2(x) + \cos^2(x) = 1, \qquad \tan^2(x) + 1 = \sec^2(x), \qquad \cot^2(x) + 1 = \operatorname{cosec}^2(x).$$

    (b) *The compound angle formulae:*

    $$\sin(x \pm y) = \sin(x)\cos(y) \pm \sin(y)\cos(x), \qquad \cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y),$$

    $$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}.$$

    (c) *The double angle formulae:*

    $$\sin(2x) = 2\sin(x)\cos(x), \qquad \cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x).$$

    (d) *The power reduction formulae:*

    $$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x)).$$

    (e) *The product to sum formulae:*

    $$\sin(x)\sin(y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right), \qquad \sin(x)\cos(y) = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right),$$

    $$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right).$$

    (f) *The sum to product formulae:*

    $$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right), \qquad \cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),$$

    $$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

    Learn all of these identities off by heart, and get your supervision partner to test you on them. [*Having a good knowledge of trigonometric identities is exceptionally useful; in particular, you will use the product-to-sum identities extremely frequently when studying Fourier series later in the course.*]

16. Prove the trigonometric inequalities:

    (a) $|\sin(x)| \leq |x|$, for all real x, (b) $\cos(x) \geq 1 - x^2/2$, for all real x.

The boring stuff: exams and coursework

17. Look up the format of the first-year maths exams. How much is each exam worth? How long does each exam last? How many questions do you have to answer, and how should you split your time in the exams?

18. Look up the first-year maths coursework. How much is it worth?