Part IA: Physical Natural Sciences
Preparatory Mathematics Examples Sheet Solutions

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1. Find the derivatives of the following functions:

(a) e^x, (b) $xe^{3x} - 2$, (c) e^{x^2+3}, (d) $\dfrac{e^{x^2}}{x^2+1}$, (e) 2^x, (f) x^{x^3}.

Solution: Where possible, you should be able to do these derivatives in your head without needing to write much down.

(a) The derivative of e^x is e^x, which is one of the exponential's most fundamental properties. In fact, we can *define* the exponential function to be the unique function $f : \mathbb{R} \to \mathbb{R}$ satisfying:

$$\frac{df}{dx}(x) = f(x), \qquad f(0) = 1.$$

(b) This question combines the product rule (used to differentiate xe^{3x}), the chain rule (to take the derivative of e^{3x}), and the linearity of the derivative (so that we can differentiate term-by-term). The resulting derivative is:

$$\frac{d}{dx}\left(xe^{3x} - 2\right) = e^{3x} + 3xe^{3x} = (3x+1)e^{3x}.$$

(c) This question uses the chain rule. Since the derivative of x^2+3 is $2x$, and the derivative of e^x is e^x, we can combine these facts to find the derivative:

$$\frac{d}{dx}e^{x^2+3} = 2xe^{x^2+3}.$$

(d) This question combines the chain rule (to differentiate e^{x^2}) and the quotient rule (or the product rule - whichever you prefer). We have:

$$\frac{d}{dx}\frac{e^{x^2}}{x^2+1} = \frac{2xe^{x^2}}{x^2+1} - \frac{2xe^{x^2}}{(x^2+1)^2} = \frac{(2x(x^2+1) - 2x)e^{x^2}}{(x^2+1)^2} = \frac{2x^3e^{x^2}}{(x^2+1)^2}.$$

(e) This question requires us to rewrite 2^x as an exponential to base e, then differentiate using the chain rule. We have $2^x = (e^{\log(2)})^x = e^{\log(2)x}$. Thus the derivative is:

$$\frac{d}{dx}2^x = \frac{d}{dx}e^{\log(2)x} = \log(2)e^{\log(2)x} = \log(2)\cdot 2^x.$$

(f) This question again requires us to rewrite x^{x^3} as an exponential to base e, then differentiate using the chain rule (but the use of the chain rule here is a bit more complicated, since we additionally need the product rule). Since $x^{x^3} = (e^{\log(x)})^{x^3} = e^{x^3\log(x)}$, we have:

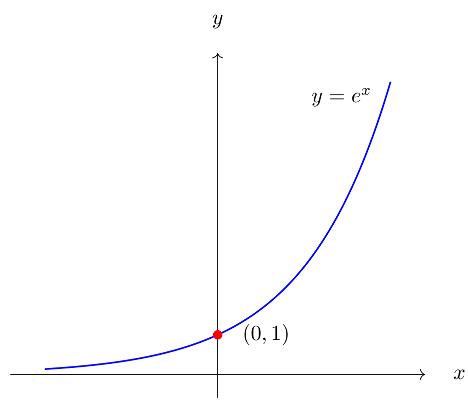
$$\frac{d}{dx}x^{x^3} = \frac{d}{dx}e^{x^3\log(x)} = e^{x^3\log(x)}\cdot\frac{d}{dx}(x^3\log(x)) = e^{x^3\log(x)}\cdot\left(3x^2\log(x) + x^2\right) = x^2x^{x^3}\left(3\log(x) + 1\right).$$

2. Sketch the graphs of the following functions:

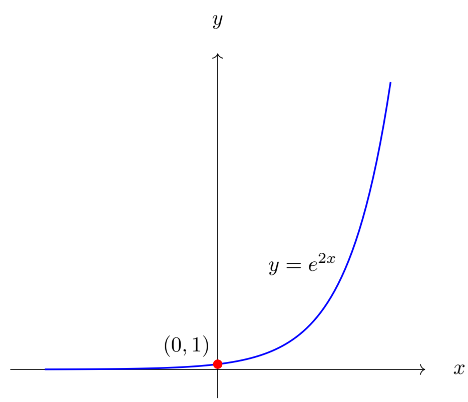
(a) e^x, (b) e^{2x}, (c) e^{-x}, (d) xe^{-x}, (e) $\dfrac{e^{-x}}{x}$, (f) e^{-x^2}.

Solution: The first three graphs are standard; the remaining ones require some thinking.

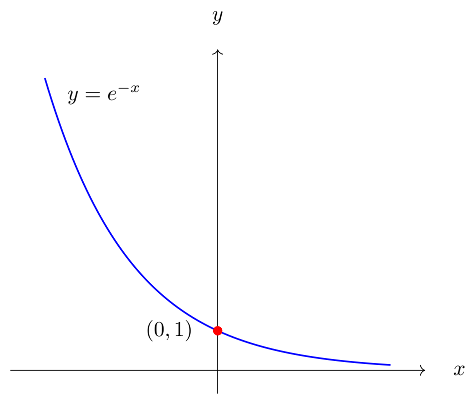
(a) This is just the graph of the exponential function, which is standard.



(b) The graph of $y = e^{2x}$ looks the same as the graph of $y = e^x$, except it increases much faster (it is the previous graph stretched by a factor of $1/2$ in the x-direction).



(c) This graph is the same as the graph of e^x, but reflected in the y-axis.

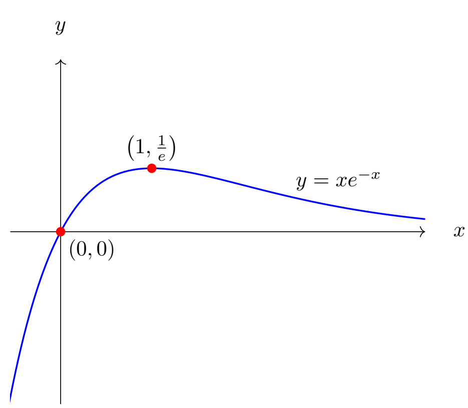


(d) The graph of $y = xe^{-x}$ is not a simple transformation of an exponential graph, on the other hand. We notice:

- The y-intercept is $y = 0$. The x-intercepts occur when $xe^{-x} = 0$, i.e. $x = 0$. Thus the graph crosses the axes only when $x = 0$.
- As $x \to \infty$, we can write the product xe^{-x} as:
$$\frac{x}{e^x}.$$
Since the exponential grows much more quickly than x grows, the quotient here converges to *zero*. Note, this is not a strictly rigorous proof - we shall learn later in the course how to evaluate this limit properly using L'Hôpital's rule.

  As $x \to -\infty$, we have that $e^{-x} \to \infty$ and $x \to -\infty$, so $xe^{-x} \to -\infty$.
- Taking the derivative of the function xe^{-x}, we can search for stationary points. We have:
$$\frac{d}{dx}(xe^{-x}) = e^{-x} - xe^{-x} = (1 - x)e^{-x}.$$
Hence there is a unique stationary point occurring at $x = 1, y = 1/e$.

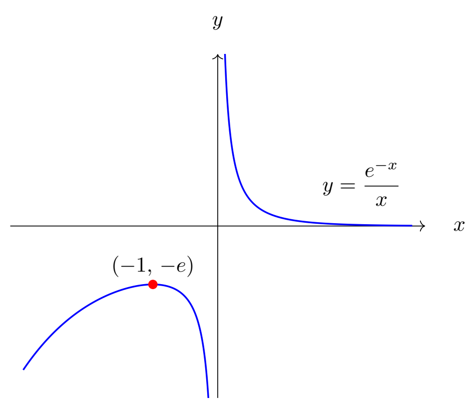
If we think about the shape of the graph, the only way that the graph can remain continuous and tend to zero at ∞, and to $-\infty$ at $-\infty$, is if this point is a *maximum*. This allows us to sketch the graph.



(e) The graph of $y = e^{-x}/x$ is also more interesting than just a standard transformation of an existing exponential graph. We notice:

- There is a singularity of the graph at the point $x = 0$. This gives a vertical asymptote. For $x > 0$, both e^{-x} and x are positive which means the asymptote comes down from $+\infty$ for $x > 0$. For $x < 0$, e^{-x} is positive and x is negative which means the asymptote comes up from $-\infty$ for $x < 0$.
- Since $x = 0$ is a singularity, there is no y-intercept. On the other hand, if $0 = e^{-x}/x$, then there are no values of x satisfying this equation, implying there are no x-intercepts either.
- As $x \to \infty$, we can write the function as:
$$y = \frac{1}{xe^x},$$
which shows it converges to zero.
As $x \to -\infty$, the exponential e^{-x} grows to infinity, and x grows to $-\infty$. The exponential in the numerator grows faster, so overall the fraction tends to $-\infty$.
- Finally, we search for stationary points by taking the derivative. We use the quotient rule, giving:
$$\frac{d}{dx}\left(\frac{e^{-x}}{x}\right) = \frac{-xe^{-x} - e^{-x}}{x^2}.$$
This is zero if and only if $x = -1$, so there is a stationary point at $x = -1, y = -e$. By considering the continuity of the graph in the region $x < 0$, the only possibility is that this point is a maximum.

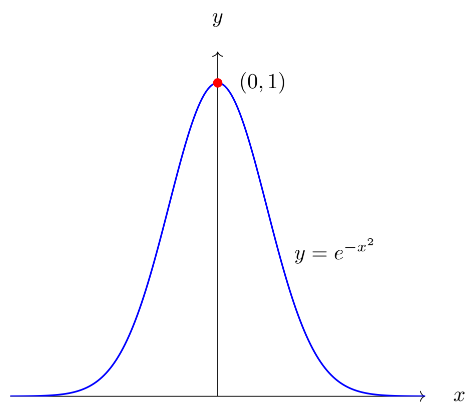
We now have enough information to sketch the graph.



(f) The last graph is also quite exciting - it is a *Gaussian* function, which will be quite important later in the course. We notice:

- The y-intercept occurs when $x = 0$, giving $y = 1$. There are no x-intercepts, since the exponential function is never zero.
- As $x \to \pm\infty$, we have that $e^{-x^2} \to 0$. Thus $y = 0$ is a horizontal asymptote as the graph approaches positive and negative infinity.
- We could check for stationary points by taking the derivative. Here though, we could just think about the function. For $x > 0$, the graph of e^{-x^2} is decreasing because x^2 is increasing. Similarly, for $x < 0$, the graph of e^{-x^2} is increasing because x^2 is decreasing in this region. Hence there is a unique maximum at $x = 0$, with value $y = 1$.

We now have enough information to sketch the graph.



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3. Show that the x-coordinates of the points of intersection between the normal to the graph of $y = e^x$ at $x = 1$ and the graph of $y = e^{-x}$ satisfy:

$$x = e^2 + 1 - e^{1-x}.$$

How many solutions do you expect to this equation? Using a numerical method of your choice, show that the unique positive solution to this equation is $x \approx 8.39$.

Solution: First, we compute the normal to the graph of $y = e^x$ at $x = 1$. The derivative of the graph is:

$$\frac{dy}{dx} = e^x,$$

so the gradient of its tangent at $x = 1$ is e. Therefore, the gradient of its normal at $x = 1$ is the negative reciprocal of e, namely $-1/e$. It follows that the normal has the equation:

$$y - e = -\frac{1}{e}(x - 1) \qquad \Leftrightarrow \qquad y = e - \frac{1}{e}(x - 1).$$

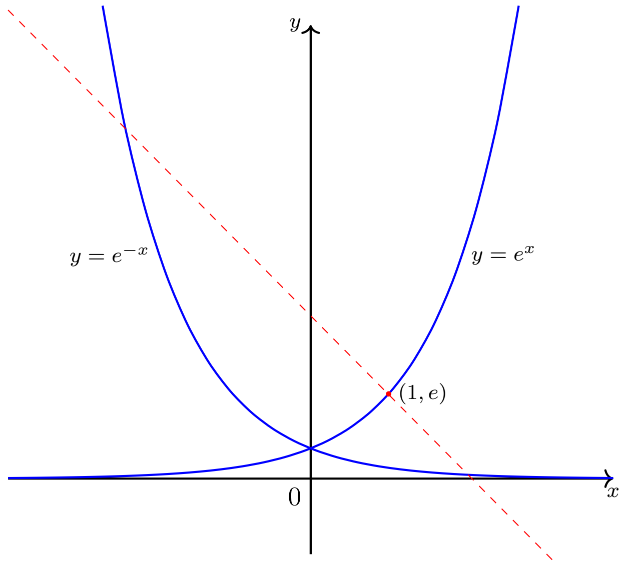
We would like to obtain the x-coordinates of the intersections with the graph $y = e^{-x}$; therefore, we simply solve the equation of the normal and the equation of the graph $y = e^{-x}$ simultaneously for x. We have:

$$e^{-x} = e - \frac{1}{e}(x - 1).$$

Multiplying through by e and rearranging, we obtain the desired equation:

$$x = e^2 + 1 - e^{1-x}.$$

We can find the number of solutions we expect to this equation by drawing the graph of $y = e^{-x}$, and also the normal to the graph of $y = e^x$ at $x = 1$.



We can use multiple numerical methods to solve the equation $x = e^2 + 1 - e^{1-x}$. One method we shall learn later in the course is *Newton-Raphson iteration*, but this is a bit complicated here - it is easier to instead notice that this is a relation of the form $x = f(x)$, so we are searching for a *fixed point* of the function $f(x) = e^2 + 1 - e^{1-x}$.

How do we search for fixed points? If we make an initial guess x_0, we can approximate the function $f(x)$ near x_0 using a linear approximation of the form $f(x) \approx f(x_0) + (x - x_0)f'(x_*)$ (compare with *Taylor expansion* if you know it!). Rearranging this approximation, and taking the modulus, we have:

$$|f(x) - f(x_0)| \approx |x - x_0||f'(x_0)|.$$

This suggests that if we take $x_1 = f(x_0)$, then:

$$|f(x_1) - x_1| = |f(x_1) - f(x_0)| \approx |x_1 - x_0||f'(x_0)| = |f(x_0) - x_0||f'(x_0)|.$$

In particular, if the *derivative is small*, then $x_1 = f(x_0)$ is a better guess for the fixed point than our initial guess! This suggests the iterative method:

$$x_{n+1} = f(x_n).$$

In our case, we have $f'(x) = e^{1-x}$. If we take an initial guess $x_0 = 8$, this is certainly very small, so we expect convergence to a root. Applying the iteration, we have:

$$x_0 = 8, \qquad x_1 = 8.38814..., \qquad x_2 = 8.38843..., \qquad x_3 = 8.38843...$$

so we have rapid convergence to the root $x \approx 8.39$, as required.

4. Find the indefinite integrals of the following functions:

(a) e^x, (b) $3e^{7x} - 4$, (c) 2^x, (d) $3xe^{x^2}$, (e) $\dfrac{e^{-1/x}}{x^2}$, (f) xe^x.

➾ **Solution:** The indefinite integrals in this question should mostly be found by using known integrals, and inspection. The final integral requires us to use integration by parts. In all cases, we should include an arbitrary constant of integration, c.

(a) Since e^x has derivative e^x, it follows that e^x has indefinite integral $e^x + c$.

(b) Using linearity of the integral, we have:

$$\int (3e^{7x} - 4)\, dx = 3\int e^{7x}\, dx - 4x + c = \frac{3}{7}e^{7x} - 4x + c.$$

(c) Here, it is easiest to write $2^x = (e^{\log(2)})^x = e^{x\log(2)}$. Integrating, we now obtain:

$$\frac{1}{\log(2)}e^{x\log(2)} + c = \frac{2^x}{\log(2)} + c.$$

(d) A substitution is possible here, but it is much easier just to spot that the chain rule must have been used here. Note that the derivative of e^{x^2} is $2xe^{x^2}$ by the chain rule. This is very close to our function, just out by a factor. Hence we note:

$$\int (3xe^{x^2})\, dx = \frac{3}{2}\int (2xe^{x^2})\, dx = \frac{3}{2}e^{x^2} + c.$$

This can be a very fast method, if you get confident with it!

(e) Similar to the previous part, we could use a substitution, but it is quicker to spot that the derivative of $e^{-1/x}$ is $e^{-1/x}/x^2$ by the chain rule. Hence the indefinite integral is $e^{-1/x} + c$.

(f) This part requires integration by parts. We have:

$$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + c = (x-1)e^x + c.$$

5. For non-negative integers n, we define the integral I_n to be:

$$I_n := \int_0^\infty x^n e^{-x}\, dx.$$

Using integration by parts, show that $I_n = nI_{n-1}$. Hence or otherwise, show that $I_n = n!$.

($*$) Hence, suggest a possible definition of $x!$ for x a *real* number. ($**$) Using your suggested definition, determine the value of $\frac{1}{2}!$. *[Hint: look up the Gaussian integral.]*

Solution: The formula we establish in this question is known as a *reduction formula*, and you may have seen these in A-level Further Maths. We just do as the question instructs, and perform integration by parts:

$$I_n = \int_0^\infty x^n e^{-x}\, dx = \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\, dx = nI_{n-1}.$$

Here, we should notice that $x^n e^{-x}$ tends to zero as $x \to \infty$; this is because the exponential decay occurs more rapidly than the polynomial growth for all values of n.

We can easily compute the value of the integral when $n = 0$, since:

$$I_0 = \int_0^\infty e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty = 0 - -1 = 1,$$

using the fact that the limit of e^{-x} as $x \to \infty$ is zero. To obtain values of I_n for larger values of n, we can use the recurrence we just proved:

$$I_n = nI_{n-1} = n(n-1)I_{n-2} = n(n-1)(n-2)I_{n-3} = ... = n(n-1)(n-2)...2 \cdot 1 \cdot I_0 = n!,$$

using the value of I_0 we obtained above.

($*$) Usually we define factorials using the products of natural numbers. However, this questions suggests we could extend the definition to *real* numbers by defining:

$$x! := \int_0^\infty u^x e^{-u}\, du.$$

Note we have replaced the dummy integration variable x with a u now instead, to avoid confusing ourselves!

($**$) Assuming our definition was valid, we can now attempt to compute $\frac{1}{2}!$, which is rather exciting! We have:

$$\tfrac{1}{2}! = \int_0^\infty x^{1/2} e^{-x}\, dx.$$

The hint in the question suggests we look up the *Gaussian integral* to help us; this integral comes up all the time in the physical natural sciences, so it is well-worth having a look at it sometime. It is given by:

$$\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}. \qquad (\dagger)$$

Since the Gaussian integral is an integral of e^{-x^2}, this suggests that maybe we should make a change of variables in the above integral (†). We let $x = u^2$ (this is valid because the integration range is over *positive* values of x only), so that $dx = 2udu$ and the limits transform as $[0,\infty) \mapsto [0,\infty)$. Thus the integral transforms to:

$$2\int_0^\infty u^2 e^{-u^2}\,du,$$

which is still quite scary. However, noticing that:

$$\frac{d}{du}e^{-u^2} = -2ue^{-u^2},$$

we see that we can actually make progress by *integrating by parts*. We have:

$$2\int_0^\infty u^2 e^{-u^2}\,du = \left[-ue^{-u^2}\right]_0^\infty + \int_0^\infty e^{-u^2}\,du.$$

The first term vanishes since $e^{-u^2} \to 0$ as $u \to \infty$. This leaves us with a Gaussian integral over *half range*. But, since e^{-u^2} is an *even* function, we have:

$$\int_0^\infty e^{-u^2}\,du = \frac{1}{2}\int_{-\infty}^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}.$$

Phew! It follows that:

$$\tfrac{1}{2}! = \frac{\sqrt{\pi}}{2},$$

which is a remarkable result!

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6. Using any method you know, solve the differential equation:

$$\frac{dy}{dx} = \alpha y + \beta, \qquad y(0) = y_0,$$

where α, β are constants. What happens when $\alpha = 0$? What happens when $y_0 = -\beta/\alpha$? Taking $\alpha < 0$ and $\beta > 0$, sketch some solution curves for different values of y_0, on the same set of axes.

Solution: One easy method is separation of variables. We have:

$$\int \frac{dy}{\alpha y + \beta} = \int dx,$$

which implies:

$$\frac{1}{\alpha}\log|\alpha y + \beta| = x + c,$$

provided $\alpha \neq 0$. Rearranging for y, we obtain:

$$y = Ae^{\alpha x} - \frac{\beta}{\alpha},$$

for some constant A. Imposing the boundary condition $y(0) = y_0$, we have:

$$y_0 = A - \frac{\beta}{\alpha},$$

which yields the complete solution:

$$y = \left(y_0 + \frac{\beta}{\alpha}\right) e^{\alpha x} - \frac{\beta}{\alpha}. \tag{†}$$

If $\alpha = 0$, this solution does not work. However, the solution is easy, since in this case we can just integrate directly:

$$y = \beta x + C,$$

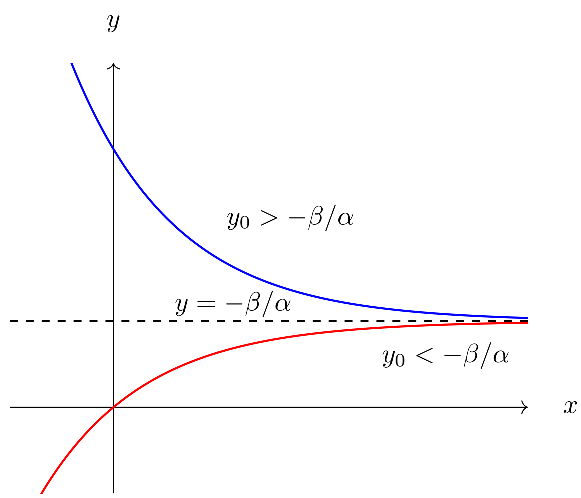
for some constant of integration C. Imposing the boundary data $y(0) = y_0$ gives the complete solution:

$$y = y_0 + \beta x.$$

On the other hand, if $y_0 = -\beta/\alpha$, then the solution (†) reduces to a constant solution $y(x) = -\beta/\alpha$ for all time.

If $\alpha < 0$, and $\beta > 0$, then the solutions are decaying exponentials (except for the initial data $y_0 = -\beta/\alpha$, when we get a constant solution). All the solutions converge to the line $y = -\beta/\alpha$ as $x \to \infty$. Further, $-\beta/\alpha > 0$, so the constant solution we found earlier is above $x = 0$.

Hence, the graphs of the solutions to this ODE take the form:



7. ($*$) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying:

$$f(x)f(y) = f(x+y)$$

for all $x, y \in \mathbb{R}$. Show that $f(x) = f(1)^x$. *[Hint: take logarithms, then differentiate with respect to x, keeping y constant.]* Hence, we have shown that the property $f(x)f(y) = f(x+y)$ uniquely characterises exponential functions.

($**$) What if we *only* assume that f is *continuous*, instead of *differentiable* - must we have $f(x) = f(1)^x$?

➺ **Solution:** ($*$) We follow the hint in the question, and take logarithms:

$$\log(f(x)) + \log(f(y)) = \log(f(x+y)).$$

Differentiating with respect to x while keeping y constant, we have (using the chain rule):

$$\frac{f'(x)}{f(x)} = \frac{f'(x+y)}{f(x+y)}.$$

This holds for all x, y, so in particular, y can be varied independently of x on the right hand side. This shows that the quantity:

$$\frac{f'(z)}{f(z)}$$

must be a *constant*, call it α. Then, using the result of Question 6, we have:

$$\frac{f'(z)}{f(z)} = \alpha \qquad \Leftrightarrow \qquad f(z) = Ae^{\alpha z},$$

for some constant A.

It remains to determine the constants A, α. Observe that the equation $f(x)f(y) = f(x+y)$ with $x = y = 0$ gives $f(0)^2 = f(0)$, which is a quadratic equation implying that either $f(0) = 0$ or $f(0) = 1$.

If $f(0) = 0$, then $A = 0$, so that $f(x) = 0$ for all values of x. It follows that indeed we have $f(x) = f(1)^x$ trivially (taking the definition $0^0 = 0$ in this case). On the other hand, if $f(0) = 1$, we have $A = 1$, which yields:

$$f(x) = e^{\alpha x} = (e^\alpha)^x = B^x,$$

for some constant $B = e^\alpha$. But $f(1) = B^1 = B$, which shows that $f(x) = f(1)^x$ as required.

($**$) In fact, we do not need to assume that f is differentiable for this functional equation to imply that $f(x) = f(1)^x$; we *only* need continuity, a property we shall investigate in the Mathematics B course.

First, note that if n is a positive integer, the defining equation tells us that:

$$f(n) = f(1 + n - 1) = f(1)f(n-1).$$

Applying this recurrence relation n times, we have $f(n) = f(1)^n$, so the proposed formula is true for positive integers. For negative integers $-n$, we have:

$$f(n+1)f(-n) = f(n+1-n) = f(1),$$

so that:

$$f(-n) = \frac{f(1)}{f(n+1)} = \frac{f(1)}{f(1)^{n+1}} = f(1)^{-n},$$

so the proposed formula is true for negative integers.

Next, consider a rational number p/q, with p, q integers. In this case, we can use the defining equation repeatedly to give:

$$f\left(\underbrace{\frac{p}{q} + \frac{p}{q} + \cdots + \frac{p}{q}}_{q \text{ times}}\right) = \underbrace{f\left(\frac{p}{q}\right) f\left(\frac{p}{q}\right) ... f\left(\frac{p}{q}\right)}_{q \text{ times}} = f\left(\frac{p}{q}\right)^q.$$

The left hand side can be written in a different way, however, simply as $f(p)$. But p is an integer, so we have:

$$f(p) = f\left(\frac{p}{q}\right)^q \qquad \Leftrightarrow \qquad f(1)^p = f\left(\frac{p}{q}\right)^q \qquad \Leftrightarrow \qquad f\left(\frac{p}{q}\right) = f(1)^{p/q}.$$

Hence, the formula works for rational numbers too!

So far, we have not used continuity. We shall see later in the course that the definition of a continuous function is that it satisfies the relation:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

for all values of x_0. To finish the question then, we observe that any real number r can be approximated as the limit of a sequence of rational numbers $r_1, r_2, ...$ Thus by continuity we have:

$$f(r) = \lim_{n \to \infty} f(r_n) = \lim_{n \to \infty} f(1)^{r_n} = f(1)^r,$$

using the fact that $f(r_n) = f(1)^{r_n}$ for rational r_n in the second equality, and using the fact that exponential functions themselves are continuous in the final step.