Part IA: Mathematics for Natural Sciences A
Examples Sheet 7: Taylor series, and Newton-Raphson iteration

Model Solutions

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Taylor series

1. Carefully state *Taylor's theorem*, giving Lagrange's formula for the remainder term. Hence, obtain the first three non-zero terms in the Taylor series of $\log(x)$ about $x = 1$ by direct differentiation. Using this expansion, together with Lagrange's form of the remainder, show that:

$$|\log(3/2) - 5/12| \leq 1/64,$$

and hence give an approximation of $\log(3/2)$ valid to one decimal place.

Solution: Taylor's theorem states that for a real function f which is $(n+1)$-times differentiable about the point x_0, we have:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_{n+1},$$

where the remainder R_{n+1} is given by:

$$R_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

for some point ξ between x and x_0.

For $\log(x)$, we have:

$$\frac{d}{dx}\log(x) = \frac{1}{x}, \qquad \frac{d^2}{dx^2}\log(x) = -\frac{1}{x^2}, \qquad \frac{d^3}{dx^3}\log(x) = \frac{2}{x^3}, \qquad \frac{d^4}{dx^4}\log(x) = -\frac{6}{x^4}.$$

Hence the Taylor series about $x = 1$, up to the third non-zero term, is given by:

$$\begin{aligned}\log(x) &= \log(1) + \frac{1}{1}(x-1) - \frac{1}{1^2 \cdot 2!}(x-1)^2 + \frac{2}{1^3 \cdot 3!}(x-1)^3 + R_4 \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + R_4,\end{aligned}$$

where the remainder term is given by:

$$R_4 = -\frac{1}{4\xi^4}(x-1)^4,$$

for some ξ between x and 1.

Put $x = 3/2$ in the above formula. Then:

$$\log\left(\frac{3}{2}\right) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 + R_4 = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} + R_4 = \frac{5}{12} + R_4.$$

It follows that:

$$\left|\log\left(\frac{3}{2}\right) - \frac{5}{12}\right| = |R_4|.$$

But note that for ξ satisfying $1 < \xi < 3/2$, we have $3/2 < 1/\xi < 1$, so:

$$|R_4| = \frac{1}{4\xi^4} \cdot \frac{1}{2^4} \leq \frac{1}{4(1)^4} \cdot \frac{1}{2^4} = \frac{1}{64},$$

as required. To one decimal place, $5/12$ is 0.4, which is within $1/64$ of $\log(3/2)$, so is a correct approximation to $\log(3/2)$ to within one decimal place.

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2. Write down the Taylor series about $x = 0$ for the following functions, stating their range of convergence in each case:

(a) e^x, (b) $\log(1+x)$, (c) $\sin(x)$, (d) $\cos(x)$, (e) $\sinh(x)$, (f) $\cosh(x)$, (g) $(1+x)^a$.

What happens when a is a non-negative integer? Learn these series off by heart, and get your supervision partner to test you on them.

Solution:

(a) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Converges for all $x \in \mathbb{R}$.

(b) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$. Converges for $-1 < x \leq 1$.

(c) $\sin(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Converges for all $x \in \mathbb{R}$.

(d) $\cos(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. Converges for all $x \in \mathbb{R}$.

(e) $\sinh(x) = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$. Converges for all $x \in \mathbb{R}$.

(f) $\cosh(x) = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$. Converges for all $x \in \mathbb{R}$.

(g) $(1+x)^a = 1 + ax + \dfrac{a(a-1)}{2!}x^2 + \dfrac{a(a-1)(a-2)}{3!}x^3 + \cdots = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{a(a-1)(a-2)...(a-n+1)}{n!}x^n$.
Converges for $|x| < 1$.

When a is a non-negative integer, the series in the final part terminates after finitely many terms (all terms afterwards are zero). This correspond to the case of the standard binomial expansion of $(1+x)^a$.

3. *Without differentiating*, find the first three terms in the Taylor series of the following functions. [*Note: there are lots of examples from past papers here to practise with, but if you are getting bored, we can do some in the supervision together. The next few questions, 12-17, have more of a problem-solving element.*]

(a) $\dfrac{1}{\sqrt{1+x}}$ about $x=0$;

(b) $\dfrac{1}{(x^2+2)^{3/2}}$ about $x=0$;

(c) $\tan(x)$ about $x=0$;

(d) $\log(\cos(x))$ about $x=0$;

(e) $\arcsin(x)$ about $x=0$;

(f) $\arctan(x)$ about $x=1$;

(g) $(\cosh(x))^{-1/2}$ about $x=0$;

(h) $e^{\sin(x)}$ about $x=\pi/2$;

(i) $x\sinh(x^2)$ about $x=0$;

(j) $\log(1+\log(1+x))$ about $x=0$;

(k) $\sin^6(x)$ about $x=0$;

(l) $\dfrac{\cosh(x)}{\cos(x)}$ about $x=0$;

(m) $\cosh(\log(x))$ about $x=2$;

(n) $\log(2-e^x)$ about $x=0$;

(o) $\dfrac{\sin(x)}{\sinh(x)}$ about $x=0$;

(p) $\sinh(\log(x))$ about $x=1$;

(q) $\sin\left(\dfrac{\pi e^x}{2}\right)$ about $x=0$;

(r) $\dfrac{\sinh(x+1)}{x+2}$ about $x=-1$;

(s) $\dfrac{\log(1+x^3)}{\cosh(x)}$ about $x=0$;

(t) $\dfrac{\cosh(x)}{\sqrt{1+x^2}}$ about $x=0$;

(u) $\dfrac{e^{-x^2}}{\cosh(x)}$ about $x=0$;

(v) $\dfrac{\log(2+x)}{2-x}$ about $x=0$;

(w) $\log(\cosh(x))$ about $x=0$;

(x) $\cosh(\sqrt{x})$ about $x=2$;

(y) $\dfrac{\sin(x)}{(1+x)^2}$ about $x=0$;

(z) $\dfrac{x\sin(x)}{\log(1+x^2)}$ about $x=0$;

(a′) $\cos\left(\sqrt{\dfrac{\pi^2}{16}+x}\right)$ about $x=0$;

(b′) $\log((2+x)^3)$ about $x=0$.

➻ Solution:

(a) Using the binomial expansion, we have:

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^2 + \cdots = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots .$$

(b) Using the binomial expansion, we have:

$$\begin{aligned}\frac{1}{(x^2+2)^{3/2}} = \frac{1}{2^{3/2}}\left(1+\frac{x^2}{2}\right)^{-3/2} &= \frac{1}{2^{3/2}}\left(1 - \frac{3x^2}{4} + \frac{1}{2!}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x^2}{2}\right)^2 + \cdots\right)\\ &= \frac{1}{2^{3/2}}\left(1 - \frac{3x^2}{4} + \frac{15}{32}x^4 + \cdots\right)\\ &= \frac{1}{2^{3/2}} - \frac{3x^2}{4\cdot 2^{3/2}} + \frac{15}{32\cdot 2^{3/2}}x^4 + \cdots .\end{aligned}$$

(c) Observe that:

$$
\begin{aligned}
\tan(x) = \frac{\sin(x)}{\cos(x)} &= \frac{x - x^3/3! + x^5/5! - \cdots}{1 - x^2/2! + x^4/4! - \cdots} \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^{-1} \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(-\frac{x^2}{2!} + \cdots\right)^2 + \cdots\right) \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{5}{24}x^4 + \cdots\right) \\
&= x + \left(\frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(\frac{5}{24} - \frac{1}{2\cdot 6} + \frac{1}{5!}\right)x^5 + \cdots \\
&= x + \frac{1}{3}x^3 + \left(\frac{25}{120} - \frac{10}{120} + \frac{1}{120}\right)x^5 + \cdots \\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots\,.
\end{aligned}
$$

(d) Observe that:

$$
\begin{aligned}
\log(\cos(x)) &= \log\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \\
&= \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 + \frac{1}{3}\left(-\frac{x^2}{2!} + \cdots\right)^3 \\
&= -\frac{1}{2}x^2 + \left(\frac{1}{4!} - \frac{1}{8}\right)x^4 + \left(-\frac{1}{6!} + \frac{2}{2\cdot 2!\cdot 4!} - \frac{1}{3\cdot 2^3}\right)x^6 + \cdots \\
&= -\frac{1}{2}x^2 - \frac{1}{12}x^4 + \left(-\frac{1}{720} + \frac{1}{48} - \frac{1}{24}\right)x^6 + \cdots \\
&= -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 + \cdots\,.
\end{aligned}
$$

(e) This one involves a bit of a trick. Since $\arcsin(x)$ differentiates to $1/\sqrt{1-x^2}$ (we can prove this using the reciprocal rule, for example, just as you did on Examples Sheet 4), we have:

$$
\begin{aligned}
\arcsin(x) = \int \frac{dx}{\sqrt{1-x^2}} &= \int \left(1 - x^2\right)^{-1/2}\, dx \\
&= \int \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + \cdots\right) dx \\
&= x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots\,.
\end{aligned}
$$

There is no constant of integration, because $\arcsin(0) = 0$.

(f) This is similar to the previous part, but we need to be careful because the expansion is around $x = 1$ instead of around $x = 0$. We have:

$$\arctan(x) = \int \frac{dx}{1+x^2}.$$

Rewriting the integrand in terms of $x - 1$, the small quantity we wish to expand around, we have:

$$\frac{1}{1+x^2} = \frac{1}{1+(x-1)^2+2x-1} = \frac{1}{(x-1)^2+2x} = \frac{1}{(x-1)^2+2(x-1)+2} = \frac{1}{2}\left(1+(x-1)+\frac{(x-1)^2}{2}\right)^{-1}.$$

Performing the binomial expansion of the integrand, we have:

$$\begin{aligned}
\frac{1}{2}\left(1+(x-1)+\frac{(x-1)^2}{2}\right)^{-1} &= \frac{1}{2}\left(1-\left((x-1)-\frac{(x-1)^2}{2}+\cdots\right)+((x-1)+\cdots)^2+\cdots\right) \\
&= \frac{1}{2}\left(1-(x-1)+\frac{3}{2}(x-1)^2+\cdots\right) \\
&= \frac{1}{2}-\frac{(x-1)}{2}+\frac{3(x-1)^2}{4}+\cdots .
\end{aligned}$$

Integrating term by term, we have:

$$\frac{(x-1)}{2}-\frac{(x-1)^2}{4}+\frac{3(x-1)^3}{12}+\cdots ,$$

up to a constant of integration. At $x = 1$, the left hand side is $\arctan(1) = \pi/4$, so the constant of integration must be $\pi/4$. This gives:

$$\arctan(x) = \frac{\pi}{4}+\frac{1}{2}(x-1)-\frac{1}{4}(x-1)^2+\cdots .$$

(g) Observe that:

$$\begin{aligned}
(\cosh(x))^{-1/2} &= \left(1+\frac{x^2}{2}+\frac{x^4}{4!}+\cdots\right)^{-1/2} \\
&= \left(1-\frac{1}{2}\left(\frac{x^2}{2}+\frac{x^4}{4!}+\cdots\right)+\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x^2}{2}+\cdots\right)^2+\cdots\right) \\
&= 1-\frac{1}{4}x^2+\left(-\frac{1}{2\cdot 4!}+\frac{3}{2!\cdot 2^2\cdot 2^2}\right)x^4+\cdots \\
&= 1-\frac{1}{4}x^2+\frac{7}{96}x^4+\cdots .
\end{aligned}$$

(h) Note that $\sin(x) = \sin(x - \pi/2 + \pi/2) = \sin(x - \pi/2)\cos(\pi/2) + \cos(x - \pi/2)\sin(\pi/2) = \cos(x - \pi/2)$. Hence we have:

$$\begin{aligned}
e^{\sin(x)} = e^{\cos(x-\pi/2)} &= \exp\left(1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + \cdots\right) \\
&= e \cdot \exp\left(-\frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + \cdots\right) \\
&= e\left(1 + \left(-\frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + \cdots\right) + \frac{1}{2!}\left(-\frac{(x-\pi/2)^2}{2!} + \cdots\right)^2 + \cdots\right) \\
&= e\left(1 - \frac{(x-\pi/2)^2}{2} + \left(\frac{1}{4!} + \frac{1}{8}\right)(x-\pi/2)^4 + \cdots\right) \\
&= e - \frac{e}{2}(x-\pi/2)^2 + \frac{e}{6}(x-\pi/2)^4 + \cdots .
\end{aligned}$$

Note that in the second line, we needed to factor out e, because we know the expansion of e^u where u is small, but we do not know the expansion of e^{1+u} where u is small.

(i) We have:

$$x\sinh(x^2) = x\left(x^2 + \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} + \cdots\right) = x^3 + \frac{1}{6}x^7 + \frac{1}{120}x^{11} + \cdots .$$

(j) Observe that:

$$\begin{aligned}
\log(1 + \log(1+x)) &= \log\left(1 + x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) \\
&= \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) - \frac{1}{2}\left(x - \frac{x^2}{2} + \cdots\right)^2 + \frac{1}{3}(x + \cdots)^3 + \cdots \\
&= x + \left(-\frac{1}{2} - \frac{1}{2}\right)x^2 + \left(\frac{1}{3} + \frac{2}{4} + \frac{1}{3}\right)x^3 + \cdots \\
&= x - x^2 + \frac{7}{6}x^3 + \cdots .
\end{aligned}$$

(k) Combining the expansion of $\sin(x)$ with the binomial expansion, we have:

$$\begin{aligned}
\sin^6(x) &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^6 \\
&= x^6\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)^6 \\
&= x^6\left(1 + 6\left(-\frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) + 15\left(-\frac{x^2}{3!} + \cdots\right)^2 + \cdots\right) \\
&= x^6 - x^8 + \left(\frac{6}{5!} + \frac{15}{(3!)^2}\right)x^{10} + \cdots \\
&= x^6 - x^8 + \frac{7}{15}x^{10} + \cdots .
\end{aligned}$$

(l) Observe that:

$$
\begin{aligned}
\frac{\cosh(x)}{\cos(x)} &= \frac{1 + x^2/2! + x^4/4! + \cdots}{1 - x^2/2! + x^4/4! + \cdots} \\
&= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^{-1} \\
&= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(-\frac{x^2}{2!} + \cdots\right)^2 + \cdots\right) \\
&= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{5}{24}x^4 + \cdots\right) \\
&= 1 + x^2 + \left(\frac{5}{24} + \frac{1}{4} + \frac{1}{4!}\right)x^4 + \cdots \\
&= 1 + x^2 + \frac{1}{2}x^4 + \cdots .
\end{aligned}
$$

(m) In this problem, we want to expand around $x = 2$. Thus we should rewrite everything in terms of the 'small' quantity $x - 2$. We shall call this $u = x - 2$ for convenience. Note that:

$$
\begin{aligned}
\cosh(\log(x)) = \cosh(\log(x - 2 + 2)) &= \cosh(\log(u + 2)) \\
&= \cosh(\log(2) + \log(1 + u/2)) \\
&= \cosh\left(\log(2) + \frac{u}{2} - \frac{u^2}{8} + \frac{u^3}{24} + \cdots\right).
\end{aligned}
$$

This gets even more messy now. We want to expand something of the form $\cosh(\log(2) + v)$, where v is small. Using hyperbolic compound angle identities, we have:

$$
\cosh(\log(2) + v) = \cosh(\log(2))\cosh(v) + \sinh(\log(2))\sinh(v).
$$

For simplicity, also note that $\cosh(\log(2)) = \frac{1}{2}(2+1/2) = 5/4$ and $\sinh(\log(2)) = \frac{1}{2}(2-1/2) = 3/4$. Expanding, we now have:

$$
\begin{aligned}
\cosh(\log(x)) &= \frac{5}{4}\cosh\left(\frac{u}{2} - \frac{u^2}{8} + \frac{u^3}{24} + \cdots\right) + \frac{3}{4}\sinh\left(\frac{u}{2} - \frac{u^2}{8} + \frac{u^3}{24} + \cdots\right) \\
&= \frac{5}{4}\left(1 + \frac{1}{2}\left(\frac{u}{2} - \frac{u^2}{8}\cdots\right)^2 + \cdots\right) + \frac{3}{4}\left(\left(\frac{u}{2} - \frac{u^2}{8} + \cdots\right) + \frac{1}{3!}\left(\frac{u}{2} + \cdots\right)^3 + \cdots\right) \\
&= \frac{5}{4} + \frac{3u}{8} + \left(\frac{5}{4}\cdot\frac{1}{8} - \frac{3}{32}\right)u^2 + \cdots \\
&= \frac{5}{4} + \frac{3}{8}(x - 2) + \frac{1}{16}(x - 2)^2 + \cdots .
\end{aligned}
$$

(n) Expanding the exponential first, we have:

$$\begin{aligned}
\log(2-e^x) &= \log\left(2-1-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\cdots\right) \\
&= \log\left(1-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\cdots\right) \\
&= \left(-x-\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right)-\frac{1}{2}\left(-x-\frac{x^2}{2!}+\cdots\right)^2+\frac{1}{3}\left(-x+\cdots\right)^3+\cdots \\
&= -x+\left(-\frac{1}{2}-\frac{1}{2}\right)x^2+\left(-\frac{1}{3!}-\frac{2}{2\cdot 2}-\frac{1}{3}\right)x^3+\cdots \\
&= -x-x^2-x^3+\cdots .
\end{aligned}$$

(o) We have:

$$\begin{aligned}
\frac{\sin(x)}{\sinh(x)} &= \frac{x-x^3/3!+x^5/5!+\cdots}{x+x^3/3!+x^5/5!+\cdots} \\
&= \left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)^{-1} \\
&= \frac{1}{x}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)\left(1+\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)^{-1} \\
&= \left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)\left(1-\left(\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)+\left(\frac{x^2}{3!}+\cdots\right)^2+\cdots\right) \\
&= \left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)\left(1-\frac{x^2}{3!}+\left(\frac{1}{3!^2}-\frac{1}{5!}\right)x^4+\cdots\right) \\
&= \left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)\left(1-\frac{x^2}{3!}+\frac{7}{360}x^4+\cdots\right) \\
&= 1-\frac{2}{3!}x^2+\left(\frac{7}{360}+\frac{1}{36}+\frac{1}{120}\right)x^4+\cdots \\
&= 1-\frac{1}{3}x^2+\frac{1}{18}x^4+\cdots .
\end{aligned}$$

(p) In this problem, we want to expand around $x = 1$. Thus we should rewrite everything in terms of the 'small' quantity $x-1$. We shall call this $u = x-1$ for convenience. Note that:

$$\begin{aligned}\sinh(\log(x)) = \sinh(\log(x-1+1)) &= \sinh(\log(1+u)) \\ &= \sinh\left(u - \frac{u^2}{2} + \frac{u^3}{3} + \cdots\right) \\ &= \left(u - \frac{u^2}{2} + \frac{u^3}{3} + \cdots\right) + \frac{1}{3!}\left(u - \frac{u^2}{2} + \cdots\right)^3 + \frac{1}{5!}\left(u + \cdots\right)^5 \\ &= u - \frac{u^2}{2} + \left(\frac{1}{3} + \frac{1}{6}\right)u^3 + \cdots \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \cdots .\end{aligned}$$

(q) Expanding the exponential first, we have:

$$\sin\left(\frac{\pi e^x}{2}\right) = \sin\left(\frac{\pi}{2} + \frac{\pi}{2}x + \frac{\pi}{4}x^2 + \frac{\pi}{12}x^3 + \cdots\right).$$

Now, we need to expand an expression of the form $\sin(\pi/2 + u)$, where u is a small quantity. By the compound angle formula, we have:

$$\sin(\pi/2 + u) = \sin(\pi/2)\cos(u) + \cos(\pi/2)\sin(u) = \cos(u).$$

Hence, we can rewrite the above as:

$$\begin{aligned}\sin\left(\frac{\pi e^x}{2}\right) &= \cos\left(\frac{\pi}{2}x + \frac{\pi}{4}x^2 + \frac{\pi}{12}x^3 + \cdots\right) \\ &= 1 - \frac{1}{2!}\left(\frac{\pi}{2}x + \frac{\pi}{4}x^2 + \cdots\right)^2 + \frac{1}{4!}\left(\frac{\pi}{2}x + \cdots\right)^4 + \cdots \\ &= 1 - \frac{\pi^2}{8}x^2 - \frac{\pi^2}{8}x^3 + \cdots .\end{aligned}$$

(r) In this problem, we want to expand around $x = -1$. Thus we should rewrite everything in terms of the 'small' quantity $x+1$. We shall call this $u = x+1$ for convenience. Note that:

$$\begin{aligned}\frac{\sinh(x+1)}{x+2} &= \frac{\sinh(u)}{1+u} \\ &= \sinh(u)(1+u)^{-1} \\ &= \left(u + \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots\right)\left(1 - u + u^2 - u^3 + \cdots\right) \\ &= u - u^2 + \frac{7}{6}u^3 + \cdots \\ &= (x+1) - (x+1)^2 + \frac{7}{6}(x+1)^3 + \cdots .\end{aligned}$$

(s) Observe that:

$$\begin{aligned}
\frac{\log(1+x^3)}{\cosh(x)} &= \frac{x^3 - x^6/2 + x^9/3 + \cdots}{1 + x^2/2! + x^4/4! + \cdots} \\
&= \left(x^3 - \frac{x^6}{2} + \frac{x^9}{3} + \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^{-1} \\
&= \left(x^3 - \frac{x^6}{2} + \frac{x^9}{3} + \cdots\right)\left(1 - \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(\frac{x^2}{2!} + \cdots\right)^2 + \cdots\right) \\
&= \left(x^3 - \frac{x^6}{2} + \frac{x^9}{3} + \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots\right) \\
&= x^3 - \frac{1}{2}x^5 - \frac{1}{2}x^6 + \cdots .
\end{aligned}$$

(t) Observe that:

$$\begin{aligned}
\frac{\cosh(x)}{\sqrt{1+x^2}} = \cosh(x)(1+x^2)^{-1/2} &= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \cdots\right) \\
&= 1 + \left(\frac{3}{8} - \frac{1}{4} + \frac{1}{4!}\right)x^4 + \left(-\frac{5}{16} + \frac{3}{16} - \frac{1}{4!\cdot 2} + \frac{1}{6!}\right)x^6 + \cdots \\
&= 1 + \frac{1}{6}x^4 - \frac{13}{90}x^6 + \cdots .
\end{aligned}$$

(u) Observe that:

$$\begin{aligned}
\frac{e^{-x^2}}{\cosh(x)} &= \frac{1 - x^2 + x^4/2! - x^6/3! + \cdots}{1 + x^2/2! + x^4/4! + \cdots} \\
&= \left(1 - x^2 + \frac{x^4}{2!} + \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^{-1} \\
&= \left(1 - x^2 + \frac{x^4}{2!} + \cdots\right)\left(1 - \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 + \cdots\right) \\
&= \left(1 - x^2 + \frac{x^4}{2!} + \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots\right) \\
&= 1 - \frac{3}{2}x^2 + \left(\frac{5}{24} + \frac{1}{2} + \frac{1}{2}\right)x^4 + \cdots \\
&= 1 - \frac{3}{2}x^2 + \frac{29}{24}x^4 + \cdots .
\end{aligned}$$

(v) Observe that:

$$\frac{\log(2+x)}{2-x} = \frac{\log(2)+\log(1+x/2)}{2(1-x/2)} = \frac{1}{2}\left(\log(2)+\frac{x}{2}-\frac{x^2}{4}+\cdots\right)\left(1-\frac{x}{2}\right)^{-1}$$
$$= \frac{1}{2}\left(\log(2)+\frac{x}{2}-\frac{x^2}{4}+\cdots\right)\left(1+\frac{x}{2}+\frac{x^2}{4}+\cdots\right)$$
$$= \frac{1}{2}\left(\log(2)+\left(\frac{1}{2}+\frac{1}{2}\log(2)\right)x+\left(\frac{1}{4}\log(2)+\frac{1}{4}-\frac{1}{4}\right)x^2+\cdots\right)$$
$$= \frac{1}{2}\log(2)+\frac{(1+\log(2))}{4}x+\frac{\log(2)}{8}x^2+\cdots.$$

(w) We have:

$$\log(\cosh(x)) = \log\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)$$
$$= \left(\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)-\frac{1}{2}\left(\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2+\frac{1}{3}\left(\frac{x^2}{2!}+\cdots\right)^3+\cdots$$
$$= \frac{1}{2}x^2+\left(\frac{1}{24}-\frac{1}{8}\right)x^4+\left(\frac{1}{6!}-\frac{1}{2!4!}+\frac{1}{3\cdot 8}\right)x^6+\cdots$$
$$= \frac{1}{2}x^2-\frac{1}{12}x^4+\frac{1}{45}x^6+\cdots.$$

(x) In this problem, we want to expand around $x=2$. Thus we should rewrite everything in terms of the 'small' quantity $x-2$. We shall call this $u=x-2$ for convenience. Note that:

$$\cosh(\sqrt{x}) = \cosh(\sqrt{2+x-2}) = \cosh(\sqrt{2+u}) = \cosh\left(\sqrt{2}(1+u/2)^{1/2}\right)$$

Expanding using the binomial theorem, we have:

$$\cosh(\sqrt{2}(1+u/2)^{1/2}) = \cosh\left(\sqrt{2}\left(1+\frac{u}{4}-\frac{u^2}{32}+\cdots\right)\right).$$

We don't the expansion of $\cosh(\sqrt{2}+v)$ for small v, so we now use the hyperbolic compound angle identities to give:

$$\cosh(\sqrt{2}+v) = \cosh(\sqrt{2})\cosh(v)+\sinh(\sqrt{2})\sinh(v).$$

Applying this to the above, we have:

$$\cosh\left(\sqrt{2}(1+u/2)^{1/2}\right) = \cosh(\sqrt{2})\cosh\left(\frac{\sqrt{2}u}{4}-\frac{\sqrt{2}u^2}{32}+\cdots\right)+\sinh(\sqrt{2})\sinh\left(\frac{\sqrt{2}u}{4}-\frac{\sqrt{2}u^2}{32}+\cdots\right)$$
$$= \cosh(\sqrt{2})\left(1-\frac{1}{2!}\left(\frac{\sqrt{2}u}{4}\right)^2+\cdots\right)+\sinh(\sqrt{2})\left(\frac{\sqrt{2}u}{4}-\frac{\sqrt{2}u^2}{32}+\cdots\right)$$
$$= \cosh(\sqrt{2})+\frac{\sqrt{2}\sinh(\sqrt{2})}{4}u-\left(\frac{\sqrt{2}\sinh(\sqrt{2})}{32}+\frac{\cosh(\sqrt{2})}{16}\right)u^2+\cdots$$
$$= \cosh(\sqrt{2})+\frac{\sqrt{2}\sinh(\sqrt{2})}{4}(x-2)-\left(\frac{\sqrt{2}\sinh(\sqrt{2})}{32}+\frac{\cosh(\sqrt{2})}{16}\right)(x-2)^2+\cdots.$$

(y) We have:

$$\frac{\sin(x)}{(1+x)^2} = \sin(x)(1+x)^{-2} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\left(1 - 2x + 3x^2 + \cdots\right)$$
$$= x - 2x^2 + \left(3 - \frac{1}{3!}\right)x^3 + \cdots$$
$$= x - 2x^2 + \frac{17}{6}x^3 + \cdots .$$

(z) We have:

$$\frac{x\sin(x)}{\log(1+x^2)} = \frac{x(x - x^3/3! + x^5/5! + \cdots)}{x^2 - x^4/2 + x^6/3 + \cdots}$$
$$= \frac{1 - x^2/3! + x^4/5! + \cdots}{1 - x^2/2 + x^4/3 + \cdots}$$
$$= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{3} + \cdots\right)^{-1}$$
$$= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)\left(1 - \left(-\frac{x^2}{2} + \frac{x^4}{3} + \cdots\right) + \left(-\frac{x^2}{2} + \cdots\right)^2 + \cdots\right)$$
$$= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)\left(1 + \frac{x^2}{2} - \frac{x^4}{12} + \cdots\right)$$
$$= 1 + \frac{1}{3}x^2 + \left(-\frac{1}{12} - \frac{1}{12} + \frac{1}{120}\right)x^4 + \cdots$$
$$= 1 + \frac{1}{3}x^2 - \frac{19}{120}x^4 + \cdots .$$

(a′) Using the binomial expansion first, we have:

$$\cos\left(\sqrt{\frac{\pi^2}{16} + x}\right) = \cos\left(\frac{\pi}{4}\left(1 + \frac{16x}{\pi^2}\right)^{1/2}\right)$$
$$= \cos\left(\frac{\pi}{4}\left(1 + \frac{8x}{\pi^2} - \frac{32x^2}{\pi^4} + \cdots\right)\right)$$
$$= \cos\left(\frac{\pi}{4} + \frac{2x}{\pi} - \frac{8x^2}{\pi^3} + \cdots\right).$$

We don't know the expansion of $\cos(\pi/4 + u)$, where u is small, so we now use the compound angle identities for the trigonometric functions:

$$\cos(\pi/4 + u) = \cos(\pi/4)\cos(u) - \sin(\pi/4)\sin(u) = \frac{1}{\sqrt{2}}\left(\cos(u) - \sin(u)\right).$$

Applying this to the above expansion, we have:

$$
\begin{aligned}
\cos\left(\sqrt{\frac{\pi^2}{16}+x}\right) &= \frac{1}{\sqrt{2}}\left(\cos\left(\frac{2x}{\pi}-\frac{8x^2}{\pi^3}+\cdots\right)-\sin\left(\frac{2x}{\pi}-\frac{8x^2}{\pi^3}\right)\right) \\
&= \frac{1}{\sqrt{2}}\left(1-\frac{1}{2!}\left(\frac{2x}{\pi}+\cdots\right)^2-\left(\frac{2x}{\pi}-\frac{8x^2}{\pi^3}+\cdots\right)+\cdots\right) \\
&= \frac{1}{\sqrt{2}}\left(1-\frac{2x}{\pi}+\left(\frac{2}{\pi^3}-\frac{8}{\pi^2}\right)x^2+\cdots\right) \\
&= \frac{1}{\sqrt{2}}-\frac{\sqrt{2}x}{\pi}+\frac{\sqrt{2}(4-\pi)}{\pi^3}x^2+\cdots .
\end{aligned}
$$

(b′) Finally, we have:

$$
\begin{aligned}
\log((2+x)^3) = 3\log(2+x) &= 3\log(2)+3\log(1+x/2) \\
&= 3\log(2)+3\left(\frac{x}{2}-\frac{x^2}{8}+\cdots\right) \\
&= 3\log(2)+\frac{3x}{2}-\frac{3x^2}{8}+\cdots .
\end{aligned}
$$

We finished on an easy one!

4. *Without differentiating*, find the value of the thirty-second derivative of $\cos(x^4)$ at $x = 0$.

➾ **Solution:** Using the standard Taylor series for cosine about $x = 0$, we have:

$$
\begin{aligned}
\cos(x^4) &= 1 - \frac{(x^4)^2}{2!} + \frac{(x^4)^4}{4!} - \frac{(x^4)^6}{6!} + \frac{(x^4)^8}{8!} - \cdots \\
&= 1 - \frac{x^8}{2!} + \frac{x^{16}}{4!} + \frac{x^{24}}{6!} + \frac{x^{32}}{8!} - \cdots .
\end{aligned}
$$

But recall that the coefficient of x^{32} in the Taylor expansion of $\cos(x^4)$ about $x = 0$ is given by:

$$
\frac{1}{32!}\frac{d^{32}}{dx^{32}}\cos(x^4)\bigg|_{x=0}.
$$

Hence the value of the required derivative is:

$$
\frac{d^{32}}{dx^{32}}\cos(x^4)\bigg|_{x=0} = \frac{32!}{8!}.
$$

---

5. Find the first three non-zero terms in a series approximation of $\log(1 + x + 2x^2) - \log(x^2)$ valid for $x \to \infty$.

➾ **Solution:** When $x \to \infty, 1/x \to 0$. Hence we should write everything in terms of $1/x$, and expand assuming that $1/x$ is close to zero.

We have:

$$
\begin{aligned}
\log(1 + x + 2x^2) - \log(x^2) &= \log(2x^2) + \log\left(1 + \frac{1}{2x} + \frac{1}{2x^2}\right) - \log(x^2) \\
&= \log(2) + \log\left(1 + \frac{1}{2x} + \frac{1}{2x^2}\right) \\
&= \log(2) + \left(\frac{1}{2x} + \frac{1}{2x^2}\right) - \frac{1}{2}\left(\frac{1}{2x} + \frac{1}{2x^2}\right)^2 + \cdots \quad \text{(Taylor series for } \log(1+u) \text{ about } u = 0\text{)} \\
&= \log(2) + \frac{1}{2x} + \frac{1}{2x^2} - \frac{1}{2}\left(\frac{1}{2x}\right)^2 + \cdots \\
&= \log(2) + \frac{1}{2x} + \frac{3}{8x^2} + \cdots .
\end{aligned}
$$

6. Let $f(x)$ be a function which can be expanded as a Taylor series. Find the first two terms in the Taylor series of the function $\log(1+f(x))$, assuming that $1+f(0)>0$, $f'(0)\neq 0$ and $f''(0)(1+f(0))\neq (f'(0))^2$. Why are these conditions necessary?

➾ **Solution:** The Taylor series of $f(x)$ about $x=0$ is given by:

$$f(x)=f(0)+f'(0)x+\frac{f^{(2)}(0)}{2!}x^2+\cdots.$$

Inserting this into $\log(1+f(x))$, we have:

$$\log\left(1+f(x)\right)=\log\left(1+f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots\right).$$

We know the Taylor series of $\log(1+u)$ about $u=0$. So we need to rewrite our logarithm in this form, where u is a quantity close to 0. To do so, we factor out $1+f(0)$ from the argument of the logarithm, giving:

$$\log\left(1+f(0)\right)+\log\left(1+\frac{f'(0)}{1+f(0)}x+\frac{f''(0)}{2!(1+f(0))}x^2+\cdots\right).$$

We can now expand the second logarithmic term in a Taylor series, since it is of the form $\log(1+u)$ where u is a quantity close to 0. We have:

$$\begin{aligned}
&\log\left(1+f(0)\right)+\left(\frac{f'(0)}{1+f(0)}x+\frac{f''(0)}{2!(1+f(0))}x^2+\cdots\right)-\frac{1}{2}\left(\frac{f'(0)}{1+f(0)}x+\cdots\right)^2+\cdots\\
&=\log(1+f(0))+\frac{f'(0)}{1+f(0)}x+\left(\frac{f''(0)}{2!(1+f(0))}-\frac{1}{2}\frac{(f'(0))^2}{(1+f(0))^2}\right)x^2+\cdots\\
&=\log(1+f(0))+\frac{f'(0)}{1+f(0)}x+\frac{1}{2}\left(\frac{f''(0)(1+f(0))-(f'(0))^2}{(1+f(0))^2}\right)x^2+\cdots.
\end{aligned}$$

We need $1+f(0)>0$ for the logarithm to exist. We need $f'(0)\neq 0$, else the first term in the expansion would vanish (and we would have to calculate to higher order!). Similarly, we need $f''(0)(1+f(0))\neq(f'(0))^2$, else the second term in the expansion would vanish (and we would have to calculate to higher order).

7. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ be the Taylor series of $f(x)$ about $x = 0$, with $a_0 > 0$, $a_1 \neq 0$, $a_1^2 \neq a_2a_0$ and $a_1^2 \neq 4a_2a_0$. Find the first three terms in the Taylor series of (a) $1/f(x)$ about $x = 0$; (b) $\sqrt{f(x)}$ about $x = 0$. Explain where you used the assumptions on the a_n in your answer.

➼ **Solution:** (a) We have:

$$\begin{aligned}
\frac{1}{f(x)} &= \frac{1}{a_0 + a_1x + a_2x^2 + \cdots} \\
&= \frac{1}{a_0}\left(1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \cdots\right)^{-1} \\
&= \frac{1}{a_0}\left(1 - \left(\frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \cdots\right) + \left(\frac{a_1}{a_0}x + \cdots\right)^2 + \cdots\right) \\
&= \frac{1}{a_0}\left(1 - \frac{a_1}{a_0}x + \left(\frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}\right)x^2 + \cdots\right) \\
&= \frac{1}{a_0} - \frac{a_1}{a_0^2}x + \frac{(a_1^2 - a_2a_0)}{a_0^3}x^2 + \cdots .
\end{aligned}$$

The coefficients are all non-zero since $a_0 > 0$, $a_1 \neq 0$ and $a_1^2 \neq a_2a_0$.

(b) We have:

$$\begin{aligned}
\sqrt{f(x)} &= \left(a_0 + a_1x + a_2x^2 + \cdots\right)^{1/2} \\
&= a_0^{1/2}\left(1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \cdots\right)^{1/2} \\
&= a_0^{1/2}\left(1 + \frac{1}{2}\left(\frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \cdots\right) + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{a_1}{a_0}x + \cdots\right)^2 + \cdots\right) \\
&= a_0^{1/2}\left(1 + \frac{a_1}{2a_0}x + \left(\frac{a_2}{2a_0} - \frac{a_1^2}{8a_0^2}\right)x^2 + \cdots\right) \\
&= a_0^{1/2} + \frac{a_1}{2a_0^{1/2}}x + \frac{(4a_2a_0 - a_1^2)}{8a_0^{3/2}}x^2 + \cdots .
\end{aligned}$$

The coefficients are all non-zero since $a_0 > 0$ (indeed, this implies the square root of a_0 exists!), $a_1 \neq 0$ and $a_1^2 \neq 4a_2a_0$.

Newton-Raphson root finding

8. Give an explanation of the Newton-Raphson algorithm for root finding, including an appropriate sketch. Under what general conditions is it guaranteed that Newton-Raphson will converge to the root of interest? Prove that, when it converges to the root of interest, the Newton-Raphson method enjoys *quadratic convergence*.

Solution: Suppose that we want to find a specific root of the equation $f(x) = 0$. Let $x = x^*$ be an exact root. We might start with some guess $x = x_0$ which is close to the root of interest, $x = x^*$. Then:

$$0 = f(x^*) = f(x_0) + f'(x_0)(x^* - x_0) + \cdots$$

Assuming that we can neglect higher order terms, this suggests that a more accurate guess for x^* is given by:

$$f(x_0) + f'(x_0)(x^* - x_0) \approx 0 \qquad \Leftrightarrow \qquad x^* \approx x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This suggests the definition of an iterative process:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

which gets closer to x^* on each step. In general, it can be shown that the process will converge if: (i) $f'(x^*) \neq 0$; (ii) we start sufficiently close to the true root of interest.

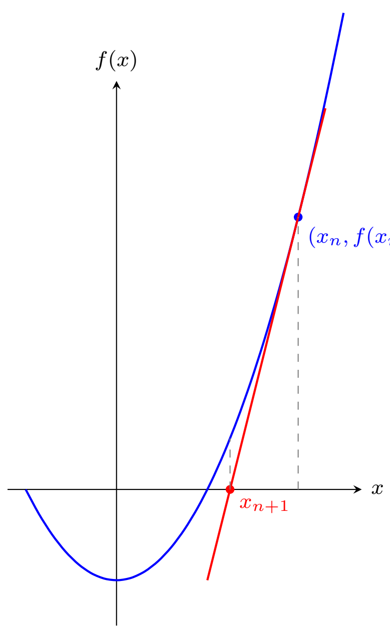
This iterative algorithm has a nice geometric interpretation. Imagine we are at the point x_n, with corresponding function value $f(x_n)$. The tangent to the graph at this point is then:

$$y - f(x_n) = f'(x_n)(x - x_n).$$

This implies that the tangent crosses the x-axis at the point satisfying:

$$-f(x_n) = f'(x_n)(x - x_n) \qquad \Leftrightarrow \qquad x = x_n - \frac{f(x_n)}{f'(x_n)},$$

which geometrically we hope is closer to a root of $f(x) = 0$. This is displayed graphically in the figure below.



Let us now define $\epsilon_{n+1} = x_{n+1} - x^*$ to be the difference between the $(n+1)$th Newton-Raphson iterate, and the true root. Then:

$$\epsilon_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - x^* = \epsilon_n - \frac{f(x_n)}{f'(x_n)}.$$

Performing a Taylor expansion of $f(x_n) = f(x_n - x^* + x^*) = f(\epsilon_n + x^*)$ and $f'(x_n) = f'(x_n - x^* + x^*) = f'(\epsilon_n + x^*)$, assuming that ϵ_n is small, we have:

$$\begin{aligned}
\epsilon_{n+1} &= \epsilon_n - \frac{\epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*) + \cdots}{f'(x^*) + \epsilon_n f''(x^*) + \cdots} && \text{(since } f(x^*) = 0\text{)} \\
&= \epsilon_n - \left(\epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*) + \cdots\right)\left(f'(x^*) + \epsilon_n f''(x^*) + \cdots\right)^{-1} \\
&= \epsilon_n - \frac{1}{f'(x^*)}\left(\epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*) + \cdots\right)\left(1 + \frac{\epsilon_n f''(x^*)}{f'(x^*)} + \cdots\right)^{-1} \\
&= \epsilon_n - \frac{1}{f'(x^*)}\left(\epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*) + \cdots\right)\left(1 - \frac{\epsilon_n f''(x^*)}{f'(x^*)} + \cdots\right) \\
&= \frac{1}{2}\epsilon_n^2 \frac{f''(x^*)}{f'(x^*)} + \cdots .
\end{aligned}$$

Hence, we see that $\epsilon_{n+1} \propto \epsilon_n^2$, so the error in the algorithm converges quadratically fast to zero.

---

9. (a) Find the value of the first iterate of Newton-Raphson iteration for the function $f(x) = x - 2 + \log(x)$ with a starting guess of $x_0 = 1$.

   (b) Find the value of the first and second iterates of Newton-Raphson iteration, valid to two decimal places, for the function $f(x) = x^2 - 2$ with a starting guess of $x_0 = 1$.

[*Both parts of this question are based on old (short) tripos questions, so try doing them*without *a calculator!*]

Solution:

(a) Note that $f'(x) = 1 + 1/x$. Hence, we have:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1 - 2 + \log(1)}{1 + 1/1} = \frac{3}{2}.$$

(b) Note that $f'(x) = 2x$. Hence, we have:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^2 - 2}{2} = \frac{3}{2}.$$

Similarly, we have:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{3}{2} - \frac{(3/2)^2 - 2}{3} = \frac{17}{12}.$$

10. [*You may use a calculator for this question, but remember that you* won't *be able to use a calculator in the exam. Newton-Raphson questions will be more theoretical in the exams, like the next question, or involve easy calculations, like the previous question.*]

(a) Sketch the graph of $f(x) = x^3 - 3x^2 + 2$, indicating the coordinates of the turning points and the coordinates of the intersections with the x-axis.

(b) Use Newton-Raphson with an initial guess of $x_0 = 2.5$ to find an estimate of the largest root of the equation $f(x) = 0$, accurate to 5 decimal places. Draw a sketch showing the progress of the algorithm.

(c) To which roots (if any) does the algorithm converge if we instead start at: (i) $x_0 = 1.5$; (ii) $x_0 = 1.9$; (iii) $x_0 = 2$?

Solution: (a) The given function is a positive cubic. The stationary points occur when:

$$0 = f'(x) = 3x^2 - 6x = 3x(x-2) \qquad \Leftrightarrow \qquad x = 0, 2.$$

The point of inflection of the graph occurs when $0 = f''(x) = 6x - 6$, which is $x = 1$.

The graph intersects with the x-axis when $f(x) = 0$. Guessing a solution, we see that $x = 1$ works. This allows us to factorise the equation $f(x) = 0$ as:

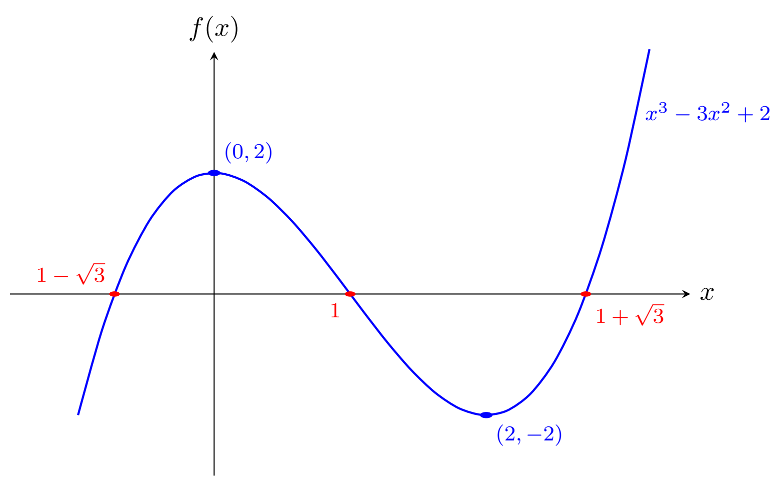
$$f(x) = (x-1)(x^2 - 2x - 2).$$

The quadratic factor can be further reduced by finding its roots using the quadratic formula. We have:

$$x_{\pm} = \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}.$$

Since $\sqrt{3} > 1$, we have that one of these roots is positive and one is negative. They occur symmetrically around $x = 1$.

We now have enough information to draw a fairly accurate graph:



(b) Let's now apply Newton-Raphson iteration to this function, starting with an initial guess of $x_0 = 2.5$. The iterative algorithm is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n^2 + 2}{3x_n^2 - 6x_n}.$$

Iterating using a calculator, we have:

$$x_1 = 2.8$$

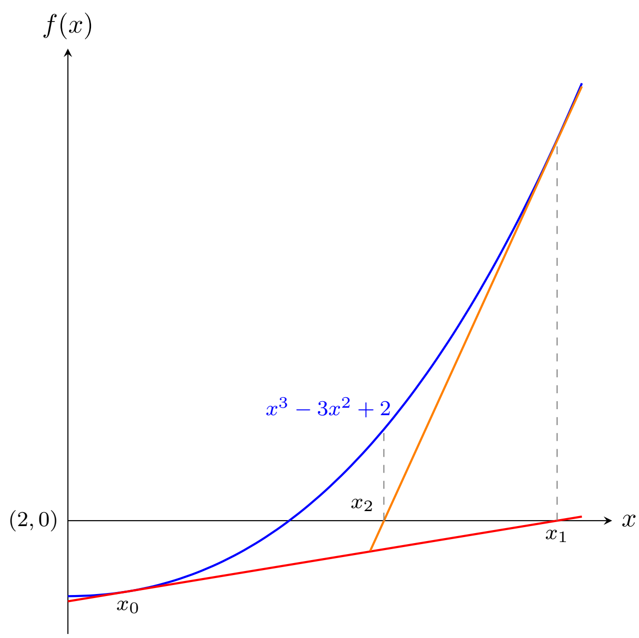
$$x_2 = 2.7357142857...$$

$$x_3 = 2.7320623734...$$

$$x_4 = 2.7320508076...$$

$$x_5 = 2.7320508075...$$

This shows that the correct root to 5 decimal places is 2.73205, which corresponds to the root $1 + \sqrt{3}$. A sketch showing the first two steps of the algorithm is given below:



(c) In this part of the question, we experiment with some other starting values. In the case of $x_0 = 1$, the closest root is 1, because we just start at a root. In the case of $x_0 = 1.9$, the closest root is actually $1+\sqrt{3}$, since $1+\sqrt{3}-1.9 \approx 0.83$. In the case of $x_0 = 2$, the closest root is $1+\sqrt{3}$, but this is also a stationary point of the function. We shall see that the behaviour is not what we might expect in (ii)!

(i) Starting at $x_0 = 1$, we already have $f(x_0) = 0$. Hence the second term in the Newton-Raphson iteration formula gives zero, and all of our iterates are the same: $x_0 = 1, x_1 = 1, x_2 = 1, ...$. This is what we would expect for an exact root.

(ii) Applying the algorithm in this case, we have:

$$x_1 = -1.5578947...$$

$$x_2 = -1.0129154...$$

$$x_3 = -0.7816615...$$

$$x_4 = -0.7340488...$$

$$x_5 = -0.7320542...$$

$$x_6 = -0.7320508...$$

This shows that the algorithm is convergent to $1-\sqrt{3}$ in this case. We were not expecting this, because this is actually the furthest root from our starting point $x_0 = 1.9$! This demonstrates the *chaotic* nature of the Newton-Raphson algorithm - we will only get to a particular root if we start sufficiently close.

(iii) Starting at $x_0 = 2$, the algorithm fails completely. This is because $f'(x_0) = 0$ here, so the denominator in the Newton-Raphson iteration formula is zero, giving a singular result. We cannot use the algorithm with this starting guess.

11. The real function f is defined by $f(x) = x^2 - 2\epsilon x - 1$, where ϵ is a small positive parameter ($0 < \epsilon \ll 1$). Let x_i be the ith Newton-Raphson iterate, with a starting guess of $x_0 = 1$, and let x_* be the unique positive root satisfying $f(x_*) = 0$. By Taylor expansion, show that $|x_i - x_*| \propto \epsilon^{n_i}$, where: (a) $n_0 = 1$; (b) $n_1 = 2$; (c) $n_2 = 4$.

➾ **Solution:** First, let's find the unique positive root of $f(x) = 0$. Using the quadratic equation, the positive root will be:

$$x_* = \frac{2\epsilon + \sqrt{4\epsilon^2 + 4}}{2} = \epsilon + \sqrt{1 + \epsilon^2}.$$

Expanding for small ϵ, we have:

$$x_* = \epsilon + 1 + \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 + \cdots = 1 + \epsilon + \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 + \cdots .$$

(a) The zeroth iterate is $x_0 = 1$. Hence:

$$x_0 - x_* = 1 - \epsilon - \sqrt{1 + \epsilon^2} = -\epsilon - \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 - \cdots .$$

This is indeed proportional to $\epsilon^{n_0} = \epsilon$ as required.

(b) The first iterate is:

$$x_1 = x_0 - \frac{x_0^2 - 2\epsilon x_0 - 1}{2(x_0 - \epsilon)} = 1 + \frac{\epsilon}{1 - \epsilon} = \frac{1}{1 - \epsilon}.$$

Taylor expanding for small ϵ, we have:

$$x_1 = (1 - \epsilon)^{-1} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \cdots .$$

Consequently, we have:

$$x_1 - x_* = \frac{1}{2}\epsilon^2 + \epsilon^3 + \frac{9}{8}\epsilon^4 + \cdots .$$

This is indeed proportional to $\epsilon^{n_1} = \epsilon^2$ as required.

(c) The second iterate is:

$$x_2 = x_1 - \frac{x_1^2 - 2\epsilon x_1 - 1}{2(x_1 - \epsilon)}$$

Substituting $x_1 = 1/(1 - \epsilon)$, we have:

$$\begin{aligned} x_2 &= \frac{1}{1 - \epsilon} - \frac{1/(1 - \epsilon)^2 - 2\epsilon/(1 - \epsilon) - 1}{2(1/(1 - \epsilon) - \epsilon)} \\ &= \frac{1}{1 - \epsilon} - \frac{1 - 2\epsilon(1 - \epsilon) - (1 - \epsilon)^2}{2(1 - \epsilon - \epsilon(1 - \epsilon)^2)} \\ &= \frac{1}{1 - \epsilon} - \frac{\epsilon^2}{2(1 - 2\epsilon + 2\epsilon^2 - \epsilon^3)} \\ &= (1 - \epsilon)^{-1} - \frac{1}{2}\epsilon^2 \left(1 - 2\epsilon + 2\epsilon^2 - \epsilon^3\right)^{-1} . \end{aligned}$$

Now, expanding using a Taylor expansion for small ϵ, we have:

$$\begin{aligned} x_2 &= 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + ... - \frac{1}{2}\epsilon^2 \left(1 + \left(2\epsilon - 2\epsilon^2 + \epsilon^3\right) + \left(2\epsilon - 2\epsilon^2 + \epsilon^3\right)^2 + ...\right) \\ &= 1 + \epsilon + \frac{1}{2}\epsilon^2 + 0 \cdot \epsilon^3 + 0 \cdot \epsilon^4 + \cdots , \end{aligned}$$

where we notice the terms in ϵ^3, ϵ^4 vanish. Hence $x_2 - x_* = \epsilon^4/8 + \cdots$. This implies $|x_2 - x_*| \propto \epsilon^4$, so $n_2 = 4$, as required.