Part IA: Mathematics for Natural Sciences
Examples Sheet 0: Basic skills

Please send all comments and corrections to jmm232@cam.ac.uk.

Making friends

1. Speak to your supervision partner, and, if you don't know already, find out: (a) where they are from; (b) their favourite food; (c) what they like to do to relax; (d) what part of the maths course they are most excited about this year.

Writing mathematics

2. Read Gareth Wilkes' document 'A Brief Guide to Mathematical Writing', available at: https://www.dpmms.cam.ac.uk/~grw46/Writing_Guide.pdf. [*You can ignore Section 4 for now - but we will study quantifiers in the Maths B course, if you are taking it.*] Hence, make a list of things that you should consider when writing a solution to a mathematics problem.

3. Write down your best and most presentable model solution to the following problem:

   Let L be a line passing through the origin with gradient k. Let C be a circle centred on $(2, 0)$ with radius 1. Determine the values of k for which L and C intersect at zero points, one point, or two points: (a) using an algebraic method; (b) using a geometric method.

   Compare your solution with your supervision partner, and give each other advice and feedback.

4. Learn all the letters of the Greek alphabet, and get your supervision partner to test you on them.

Basic logic

5. Explain the meaning of the logical symbols $\Rightarrow$, $\Leftarrow$ and $\Leftrightarrow$. [*If you haven't seen them before, look them up online! In general, you should feel free to look up terms you don't understand on an examples sheet.*] Decide which of the following are true:

   (a) $x^2 \leq 1 \;\Rightarrow\; x \leq 1,$ (b) $x^2 \leq 1 \;\Leftarrow\; x \leq 1,$ (c) $x^2 \leq 1 \;\Leftrightarrow\; x \leq 1.$

   Explain also the meanings of the terms *necessary condition* and *sufficient condition*. Decide which of the following are true:

   (d) $|x| = 1$ is sufficient for $x = 1$
   (e) $|x| = 1$ is necessary for $x = 1$
   (f) $|x| = 1$ is necessary and sufficient for $x = 1$.

6. What is the error in the following argument?

   'Suppose we wish to solve the equation $x - 1 = 2$. We begin by squaring both sides, to obtain $(x-1)^2 = 4$. Expanding the left hand side we have:

   $$x^2 - 2x + 1 = 4 \qquad \Leftrightarrow \qquad x^2 - 2x - 3 = 0 \qquad \Leftrightarrow \qquad (x+1)(x-3) = 0.$$

   Since one of these factors must be zero, it follows that there are two solutions to the equation, $x = -1$ or $x = 3$.'

7. What is meant by *proof by contradiction*? Prove by contradiction that $\sqrt{2}$ and $\sqrt{3}$ are irrational numbers.

Sets and functions

8. State what is meant by the sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$. Decide which of the following statements are true:

   (a) $\pi \in \mathbb{Q}$, (b) $3 \notin \mathbb{R}$, (c) $\mathbb{Z} \subseteq \mathbb{Q}$, (d) $\mathbb{Q} \supset \mathbb{C}$.

9. State what is meant by the sets $[a, b]$, (a, b) and $(a, b]$, where a, b are real numbers. Decide which of the following statements are true:

   (a) $1 \in [0, 1)$, (b) $3 \notin (3, 4)$, (c) $[2, 3] \subset (2, 5]$, (d) $(-1, 0) \subset [-1, 0]$.

10. A function is a mapping from a set to another set. We write $f : A \to B$ to denote the function (or just f when the sets are implied), and we write $f(x)$ for the value of the function at the point $x \in A$. Which of the following define functions, and why?

    (a) $f : [0, \infty) \to \mathbb{R}$, given by $f(x) = \sqrt{x}$;
    (b) $f : \mathbb{Z} \to \mathbb{Z}$, given by $f(x) = \frac{1}{2}x$;
    (c) $f : \mathbb{R} \to \mathbb{R}$, where $f(x)$ is implicitly defined by $2^{f(x)} = x$;
    (d) $f : [0, \infty) \to \mathbb{R}$, where $f(x)$ is implicitly defined by $f(x)^2 = x$.

Sequences and series

11. Prove the following results using (i) induction; (ii) a direct argument:

    (a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, (b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, (c) $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

    [*Hint: for the direct arguments, consider summing* $(k+1)^r - k^r$ *for an appropriate integer* r *in each case; note this sum* ***telescopes***. *If you haven't heard of telescoping sums before, look them up.*] Is it more useful to prove a result by induction, or by a direct argument? Why?

12. What is meant by an *arithmetic sequence* with first term a and common difference d? Prove that the sum of the first n terms of an arithmetic sequence is:
    $$S_n = \frac{1}{2}n\left(2a + (n-1)d\right).$$
    Hence, find the sum of the series $2, 5, 8, 11, ..., 32$.

13. What is meant by a *geometric sequence* with first term a and common ratio r? Prove that the sum of the first n terms of a geometric sequence is:
    $$S_n = \frac{a(1-r^n)}{1-r}.$$
    What happens if $r = 1$? What is the behaviour of this sum in the limit as $n \to \infty$? Hence, find the sum of the infinite series $2, 2/3, 2/9, 2/27, ...$.

14. Using the formula for the sum of an infinite geometric series, find a formula for the sum of the infinite series:
    $$\sum_{k=1}^{\infty} kr^k,$$
    where $|r| < 1$. [*Hint: differentiation!*] Hence determine:
    $$\frac{2}{3} + 2\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^3 + 4\left(\frac{2}{3}\right)^4 + \cdots.$$

Trigonometric functions and their identities

15. Define the trigonometric functions $\sin(x), \cos(x), \tan(x)$ in terms of the side lengths of an appropriate right-angled triangle. Define also the reciprocal trigonometric functions $\operatorname{cosec}(x), \sec(x), \cot(x)$. Hence, prove each of the following trigonometric identities:

    (a) *The Pythagorean identities:*

    $$\sin^2(x) + \cos^2(x) = 1, \qquad \tan^2(x) + 1 = \sec^2(x), \qquad \cot^2(x) + 1 = \operatorname{cosec}^2(x).$$

    (b) *The compound angle formulae:*

    $$\sin(x \pm y) = \sin(x)\cos(y) \pm \sin(y)\cos(x), \qquad \cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y),$$

    $$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}.$$

    (c) *The double angle formulae:*

    $$\sin(2x) = 2\sin(x)\cos(x), \qquad \cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x).$$

    (d) *The power reduction formulae:*

    $$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x)).$$

    (e) *The product to sum formulae:*

    $$\sin(x)\sin(y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right), \qquad \sin(x)\cos(y) = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right),$$

    $$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right).$$

    (f) *The sum to product formulae:*

    $$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right), \qquad \cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),$$

    $$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

    Learn all of these identities off by heart, and get your supervision partner to test you on them. [*Having a good knowledge of trigonometric identities is exceptionally useful; in particular, you will use the product-to-sum identities extremely frequently when studying Fourier series later in the course.*]

16. Prove the trigonometric inequalities:

    (a) $|\sin(x)| \leq |x|$, for all real x, (b) $\cos(x) \geq 1 - x^2/2$, for all real x.

The boring stuff: exams and coursework

17. Look up the format of the first-year maths exams. How much is each exam worth? How long does each exam last? How many questions do you have to answer, and how should you split your time in the exams?

18. Look up the first-year maths coursework. How much is it worth?