

Part IA: Mathematics for Natural Sciences A

Examples Sheet 12: Partial differentiation, and the chain rule for multivariable functions

Model Solutions

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Partial differentiation: basic examples and properties

1. Let $f \equiv f(x, y)$ be a function of x and y .

(a) Determine the partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ of the following functions.

(i) $f = x^3 - 2x^2y + 3xy^3 - 4y^3$, (ii) $f = \exp(-x^2y^2)$, (iii) $f = \exp(-x/y)$, (iv) $f = \sin(x + y)$.

(b) For each of the functions in part (a), compute the four possible second partial derivatives. Verify that in each case we have symmetry of the mixed partial derivatives.

♦♦ **Solution:** (a) For the given functions, we have:

(i) $(\partial f / \partial x, \partial f / \partial y) = (3x^2 - 4xy + 3y^3, -2x^2 + 9xy^2 - 12y^2)$.

(ii) $(\partial f / \partial x, \partial f / \partial y) = (-2xy^2e^{-x^2y^2}, -2yx^2e^{-x^2y^2})$.

(iii) $(\partial f / \partial x, \partial f / \partial y) = (-\frac{1}{y}e^{-x/y}, \frac{x}{y^2}e^{-x/y})$.

(iv) $(\partial f / \partial x, \partial f / \partial y) = (\cos(x + y), \cos(x + y))$.

(b) We now take further partial derivatives.

(i) For the first function,

$$\frac{\partial^2 f}{\partial x^2} = 6x - 4y, \quad \frac{\partial^2 f}{\partial x \partial y} = -4x + 9y^2, \quad \frac{\partial^2 f}{\partial y^2} = 18xy - 24y.$$

We can check that:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x},$$

as we can check in all the other cases.

(ii) For the second function, we have:

$$\frac{\partial^2 f}{\partial x^2} = -2y^2e^{-x^2y^2} + 4x^2y^4e^{-x^2y^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = -4xye^{-x^2y^2} + 4xy^3e^{-x^2y^2}, \quad \frac{\partial^2 f}{\partial y^2} = -2x^2e^{-x^2y^2} + 4y^2x^2e^{-x^2y^2}.$$

(iii) For the third function, we have:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{y^2}e^{-x/y}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{1}{y^2}e^{-x/y} - \frac{x}{y^3}e^{-x/y}, \quad \frac{\partial^2 f}{\partial y^2} = -\frac{2x}{y^3}e^{-x/y} + \frac{x}{y^4}e^{-x/y}.$$

(iv) For the final function, we have:

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x + y), \quad \frac{\partial^2 f}{\partial x \partial y} = -\sin(x + y), \quad \frac{\partial^2 f}{\partial y^2} = -\sin(x + y).$$

2. Show that the function:

$$w(x, y) = \frac{1}{360} (15x^4y^2 - x^6 + 15x^2y^4 - y^6)$$

is a solution of the equation:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = x^2y^2.$$

♦♦ Solution: We have:

$$\frac{\partial w}{\partial x} = \frac{1}{360} (60x^3y^2 - 6x^5 + 30xy^4), \quad \frac{\partial^2 w}{\partial x^2} = \frac{1}{360} (180x^2y^2 - 30x^4 + 30y^4).$$

Similarly, since the function is symmetric in x and y , we have:

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{360} (180x^2y^2 - 30y^4 + 30x^4).$$

Adding these together, we have:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{360} (360x^2y^2) = x^2y^2,$$

as required.

3. Show that the function:

$$\phi(x, t) = \frac{1}{\sqrt{4\pi\sigma^2t}} \exp\left(-\frac{(x-x_0)^2}{4\sigma^2t}\right),$$

where $t > 0$, x_0 , σ are real positive constants, and $\sigma^2 \neq 0$, is a solution of the equation:

$$\frac{\partial \phi}{\partial t} = \sigma^2 \frac{\partial^2 \phi}{\partial x^2}.$$

♦♦ Solution: We have:

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2\sqrt{4\pi\sigma^2t^{3/2}}} \exp\left(-\frac{(x-x_0)^2}{4\sigma^2t}\right) + \frac{(x-x_0)^2}{\sqrt{4\pi\sigma^2t} \cdot 4\sigma^2t^2} \exp\left(-\frac{(x-x_0)^2}{4\sigma^2t}\right).$$

We also have:

$$\frac{\partial \phi}{\partial x} = -\frac{2(x-x_0)}{\sqrt{4\pi\sigma^2t} \cdot 4\sigma^2t} \exp\left(-\frac{(x-x_0)^2}{4\sigma^2t}\right),$$

and hence:

$$\begin{aligned} \sigma^2 \frac{\partial^2 \phi}{\partial x^2} &= -\frac{2}{\sqrt{4\pi\sigma^2t} \cdot 4t} \exp\left(-\frac{(x-x_0)^2}{4\sigma^2t}\right) + \frac{4(x-x_0)^2}{\sqrt{4\pi\sigma^2t} \cdot 16\sigma^2t^2} \exp\left(-\frac{(x-x_0)^2}{4\sigma^2t}\right) \\ &= -\frac{1}{2\sqrt{4\pi\sigma^2t^{3/2}}} \exp\left(-\frac{(x-x_0)^2}{4\sigma^2t}\right) + \frac{(x-x_0)^2}{\sqrt{4\pi\sigma^2t} \cdot 4\sigma^2t^2} \exp\left(-\frac{(x-x_0)^2}{4\sigma^2t}\right), \end{aligned}$$

hence indeed the equation is solved by this function.

The single-variable chain rule, with multivariable functions

4. Let $z(x, y)$ be a function defined implicitly by the equation:

$$x - \alpha z = \phi(y - \beta z),$$

where α, β are real constants, and ϕ is an arbitrary differentiable function. Show that z satisfies the partial differential equation:

$$\alpha \frac{\partial z}{\partial x} + \beta \frac{\partial z}{\partial y} = 1.$$

[Hint: you can still use the normal single-variable chain rule here when taking each of the partial derivatives! Why?]

♦♦ **Solution:** We begin by taking the partial derivative with respect to x . Note that we are keeping y fixed, so differentiating $\phi(y - \beta z(x, y))$ is essentially just like differentiating something like $\phi(2 - \beta z(x))$ - we can use the ordinary chain rule on this kind of thing! We have:

$$1 - \alpha \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (\phi(y - \beta z)) = -\beta \frac{\partial z}{\partial x} \phi'(y - \beta z).$$

Rearranging, we have:

$$\alpha \frac{\partial z}{\partial x} = \frac{\alpha}{\alpha - \beta \phi'(y - \beta z)}.$$

On the other hand, taking the derivative with respect to y , we have:

$$-\alpha \frac{\partial z}{\partial y} = \left(1 - \beta \frac{\partial z}{\partial y}\right) \phi'(y - \beta z).$$

Rearranging, we have:

$$\beta \frac{\partial z}{\partial y} = -\frac{\beta \phi'(y - \beta z)}{\alpha - \beta \phi'(y - \beta z)}$$

Summing our results, we get the equation in the question.

5. Consider the function $u(x, y) = x\phi(y/x)$, where ϕ is a differentiable function of its argument and $x \neq 0$. Show that u satisfies:

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u.$$

♦♦ **Solution:** Similarly to the previous question, we note that the derivative of u with respect to x is:

$$\frac{\partial u}{\partial x} = \phi(y/x) - \frac{y}{x} \phi'(y/x).$$

The derivative with respect to y is:

$$\frac{\partial u}{\partial y} = \phi'(y/x).$$

Hence we have:

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x\phi(y/x) - y\phi'(y/x) + y\phi'(y/x) = x\phi(y/x) = u,$$

as required.

6. If $u(x, y) = \phi(xy) + \sqrt{xy}\psi(y/x)$, where ϕ and ψ are twice-differentiable functions of their arguments, show that u satisfies the partial differential equation:

$$x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

♦♦ **Solution:** Taking the x -derivatives of u , we have:

$$\frac{\partial u}{\partial x} = y\phi'(xy) + \frac{1}{2}\sqrt{\frac{y}{x}}\psi(y/x) - \frac{y\sqrt{xy}}{x^2}\psi'(y/x).$$

Taking a second x -derivative, we have:

$$\frac{\partial^2 u}{\partial x^2} = y^2\phi''(xy) - \frac{1}{4}\frac{\sqrt{y}}{x^{3/2}}\psi(y/x) + \frac{y\sqrt{y}}{2x^{5/2}}\psi'(y/x) + \frac{y^2\sqrt{xy}}{x^4}\psi''(y/x).$$

On the other hand taking the y -derivative of u , we have:

$$\frac{\partial u}{\partial y} = x\phi'(xy) + \frac{1}{2}\sqrt{\frac{x}{y}}\psi(y/x) + \frac{\sqrt{xy}}{x}\psi'(y/x)$$

Taking a second y -derivative, we have:

$$\frac{\partial^2 u}{\partial y^2} = x^2\phi''(xy) - \frac{1}{4}\frac{\sqrt{x}}{y^{3/2}}\psi(y/x) + \frac{1}{2\sqrt{xy}}\psi'(y/x) + \frac{\sqrt{xy}}{x^2}\psi''(y/x).$$

Putting everything together, we have:

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} &= x^2 y^2 \phi''(xy) - y^2 x^2 \phi''(xy) - \frac{1}{4}\sqrt{xy}\psi(y/x) + \frac{1}{4}\sqrt{xy}\psi(y/x) \\ &\quad + \frac{1}{2}x^{-1/2}y^{3/2}\psi'(y/x) - \frac{1}{2}x^{-1/2}y^{3/2}\psi'(y/x) + \frac{y^2\sqrt{xy}}{x^2}\psi''(y/x) - \frac{y^2\sqrt{xy}}{x^2}\psi''(y/x) = 0, \end{aligned}$$

as required.

7. Consider the partial differential equation:

$$2y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = xy(2y^2 - x^2).$$

- (a) Show that $u(x, y) = \phi(x^2 + 2y^2)$ is a solution of the homogeneous version of this equation, where ϕ is an arbitrary differentiable function.
- (b) By considering $u_p(x, y) = Ax^m y^n$ for some constants A, m, n , find a particular integral for this equation.
- (c) Hence, find the complete solution of the equation subject to the boundary condition $u(x, 1) = x^2$.
-

♦♦ Solution:

- (a) We have:

$$\frac{\partial u}{\partial x} = 2x\phi'(x^2 + 2y^2), \quad \frac{\partial u}{\partial y} = 4y\phi'(x^2 + y^2).$$

Hence:

$$2y \cdot 2x\phi'(x^2 + 2y^2) - x \cdot 4y\phi'(x^2 + y^2) = 0,$$

indeed solves the the homogeneous version of the differential equation.

- (b) Let $u_p(x, y) = Ax^m y^n$. Then inserting into the PDE, we have:

$$2y \cdot Amx^{m-1}y^n - x \cdot nAx^m y^{n-1} = xy(2y^2 - x^2).$$

Collecting like terms on the left hand side, we have:

$$Ax^{m-1}y^{n-1}(2my^2 - nx^2),$$

so we should take $m = 2, n = 2$ and $A = 1/2$. The particular integral is then $u_p = \frac{1}{2}x^2y^2$.

- (c) The general solution is therefore $u = \phi(x^2 + 2y^2) + \frac{1}{2}x^2y^2$. Imposing the boundary condition $u(x, 1) = x^2$, we have:

$$x^2 = \phi(x^2 + 2) + \frac{1}{2}x^2.$$

Rearranging, we see that:

$$\phi(x^2 + 2) = \frac{1}{2}x^2.$$

Let $z = x^2 + 2$. Then $x^2 = z - 2$, giving:

$$\phi(z) = \frac{1}{2}(z - 2).$$

This shows the general solution obeying this boundary condition is:

$$u(x, y) = \frac{1}{2}(x^2 + 2y^2 - 2) + \frac{1}{2}x^2y^2.$$

8. Consider the partial differential equation:

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2},$$

where $\lambda > 0$.

- (a) Show that $u(x, t) = (t + a)^{-1/2}v(y)$, where $y = (t + a)^{-1/2}(x + b)$, solves the equation if and only if v satisfies the ordinary differential equation:

$$-\frac{1}{2} \left(v + y \frac{dv}{dy} \right) = \lambda \frac{d^2 v}{dy^2}. \quad (*)$$

- (b) Verify that $(*)$ has a solution of the form $v(y) = e^{-cy^2}$ for appropriately chosen c .

- (c) Using parts (a) and (b), find the solution of the original partial differential equation subject to the boundary condition:

$$u(x, 0) = \exp(-(x+1)^2) + \exp(-(x-1)^2).$$

♦ Solution:

- (a) We have:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{1}{2}(t+a)^{-3/2}v(y) + (t+a)^{-1/2}\frac{\partial y}{\partial t}v'(y) \\ &= -\frac{1}{2}(t+a)^{-3/2}v(y) - \frac{1}{2}(t+a)^{-2}(x+b)v'(y) \end{aligned}$$

On the other hand, we also have:

$$\frac{\partial u}{\partial x} = (t+a)^{-1/2}\frac{\partial y}{\partial x}v'(y) = (t+a)^{-1}v'(y),$$

and then:

$$\frac{\partial^2 u}{\partial x^2} = (t+a)^{-1}\frac{\partial y}{\partial x}v''(y) = (t+a)^{-3/2}v''(y).$$

Inserting into the equation, we have:

$$-\frac{1}{2}(t+a)^{-3/2}v(y) - \frac{1}{2}(t+a)^{-2}(x+b)v'(y) = \lambda(t+a)^{-3/2}v''(y)$$

Simplifying, we have:

$$-\frac{1}{2} \left(v(y) + (t+a)^{-1/2}(x+b)v'(y) \right) = \lambda v''(y),$$

which on using $y = (t+a)^{-1/2}(x+b)$ gives:

$$-\frac{1}{2} \left(v + y \frac{dv}{dy} \right) = \lambda \frac{d^2 v}{dy^2},$$

as required.

- (b) Inserting $v = e^{-cy^2}$, we have:

$$-\frac{1}{2} \left(e^{-cy^2} - 2cy^2 e^{-cy^2} \right) = \lambda \left(-2ce^{-cy^2} + 4c^2 y^2 e^{-cy^2} \right).$$

Comparing coefficients, we see that $-\frac{1}{2} = -2\lambda c$, and $c = 4\lambda c^2$. Both of these are consistent, and give $c = 1/4\lambda$.

(c) The solution we have derived is:

$$u(x, t) = (t + a)^{-1/2} \exp\left(-\frac{(x + b)^2}{4\lambda(t + a)}\right),$$

using parts (a) and (b). This cannot hope to satisfy the initial condition at $t = 0$, because it is of the incorrect form. However the equation is linear, so we can take the linear combination of two solutions of this form easily:

$$u(x, t) = C_1(t + a)^{-1/2} \exp\left(-\frac{(x + b)^2}{4\lambda(t + a)}\right) + C_2(t + a')^{-1/2} \exp\left(-\frac{(x + b')^2}{4\lambda(t + a')}\right),$$

where we are allowed to pick the constants C_1, C_2, a, b, a', b' in each term to try to satisfy the initial data. This is still a solution by linearity.

At $t = 0$, we need $u(x, 0) = \exp(-(x + 1)^2) + \exp(-(x - 1)^2)$. This suggests choosing $b = 1, b' = -1$, and $a = 1/4\lambda, a' = 1/4\lambda$. Further, we see that we should choose $C_1 = 1/\sqrt{4\lambda}$ and $C_2 = 1/\sqrt{4\lambda}$. Overall the solution takes the form:

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{4\lambda}} \left(\frac{1}{\sqrt{t + 1/4\lambda}} \exp\left(-\frac{(x + 1)^2}{4\lambda(t + 1/4\lambda)}\right) + \frac{1}{\sqrt{t + 1/4\lambda}} \exp\left(-\frac{(x - 1)^2}{4\lambda(t + 1/4\lambda)}\right) \right) \\ &= \frac{1}{\sqrt{4\lambda t + 1}} \exp\left(-\frac{(x + 1)^2}{4\lambda t + 1}\right) + \frac{1}{\sqrt{4\lambda t + 1}} \exp\left(-\frac{(x - 1)^2}{4\lambda t + 1}\right). \end{aligned}$$

The multivariable chain rule

9. Suppose that $f(x, y)$ is a function of two variables x, y . State the forms of the multivariable chain rule if:

- x, y are both functions of a single-variable t , $x \equiv x(t), y \equiv y(t)$;
 - y is a function of x , $y \equiv y(x)$;
 - x, y are both functions of two variables (u, v) , $x \equiv x(u, v), y \equiv y(u, v)$.
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♦♦ Solution:

- Here, we have a function $f(x(t), y(t))$. The multivariable chain rule then takes the form:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

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- Here, we have a function $f(x(u, v), y(u, v))$. The multivariable chain rule then takes the form:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u},$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

10. Using the multivariable chain rule, show that if $f(x, y) = xy$ and $y = x \cos(x)$, we have:

$$\frac{df}{dx} = x(2\cos(x) - x\sin(x)).$$

Check your result by writing f in terms of x first, then taking partial derivatives.

♦ Solution: We have:

$$\begin{aligned}\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} \\ &= y + x \cdot \frac{d}{dx}(x \cos(x)) \\ &= x \cos(x) + x(\cos(x) - x \sin(x)) \\ &= x(2\cos(x) - x\sin(x)),\end{aligned}$$

as required. To check our answer, we note that:

$$f(x, y(x)) = x \cdot (x \cos(x)) = x^2 \cos(x).$$

Taking the x -derivative directly, we have:

$$\frac{df}{dx} = 2x \cos(x) - x^2 \sin(x),$$

in perfect agreement.

11. Using the multivariable chain rule, show that if $f(u, v) = u^2 + v^2$, and $u(x, y) = x^3 - 2y$, $v(x, y) = 3y - 2x^2$, we have:

$$\frac{\partial f}{\partial x} = 2x(3x^4 - 6xy - 12y + 8x^2), \quad \frac{\partial f}{\partial y} = 2(13y - 6x^2 - 2x^3).$$

Check your results by writing f in terms of x, y first, then taking partial derivatives.

♦ Solution: We have:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} \\&= 2u \cdot 3x^2 + 2v \cdot (-4x) \\&= 2(x^3 - 2y) \cdot 3x^2 + 2(3y - 2x^2) \cdot (-4x) \\&= 6x^5 - 12x^2y - 24xy + 16x^3 \\&= 2x(3x^4 - 6xy - 12y + 8x^2).\end{aligned}$$

Checking the answer, we note that:

$$f(x, y) = (x^3 - 2y)^2 + (3y - 2x^2)^2 = x^6 - 4x^3y + 13y^2 - 12yx^2 + 4x^4.$$

Differentiating, we have:

$$\frac{\partial f}{\partial x} = 6x^5 - 12x^2y - 24xy + 16x^3 = 2x(3x^4 - 6xy - 12y + 8x^2),$$

as expected.

For the y -derivative, we have:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} \\&= 2u \cdot (-2) + 2v \cdot (3) \\&= -4(x^3 - 2y) + 6(3y - 2x^2) \\&= 2(13y - 6x^2 - 2x^3),\end{aligned}$$

as required. Checking the answer, using the expression for $f(x, y)$ from above, we have:

$$\frac{\partial f}{\partial y} = -4x^3 + 26y - 12x^2 = 2(13y - 6x^2 - 2x^3),$$

as expected.

12. Let (x, y) be plane Cartesian coordinates, and let (r, θ) be plane polar coordinates. Let $f \equiv f(x, y)$ be a multivariable function whose expression in terms of Cartesian coordinates is $f(x, y) = e^{-xy}$.

- (a) Compute $\partial f / \partial x$ and $\partial f / \partial y$.
 - (b) Compute $\partial f / \partial r$ and $\partial f / \partial \theta$, by: (i) writing f in terms of polar coordinates; (ii) using the multivariable chain rule.
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♦ Solution:

- (a) We have:

$$\frac{\partial f}{\partial x} = -ye^{-xy}, \quad \frac{\partial f}{\partial y} = -xe^{-xy}.$$

- (b) (i) By first writing f in polar coordinates using $x = r \cos(\theta)$, $y = r \sin(\theta)$, we have $f(r, \theta) = e^{-r^2 \sin(\theta) \cos(\theta)} = e^{-\frac{1}{2}r^2 \sin(2\theta)}$. Thus:

$$\frac{\partial f}{\partial r} = -r \sin(2\theta) e^{-\frac{1}{2}r^2 \sin(2\theta)}, \quad \frac{\partial f}{\partial \theta} = -r^2 \cos(2\theta) e^{-\frac{1}{2}r^2 \sin(2\theta)}.$$

- (ii) Instead using the multivariable chain rule, we have:

$$\begin{aligned} \frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} \\ &= -ye^{-xy} \cos(\theta) - xe^{-xy} \sin(\theta) \\ &= -2r \sin(\theta) \cos(\theta) e^{-\frac{1}{2}r^2 \sin(2\theta)} \\ &= -r \sin(2\theta) e^{-\frac{1}{2}r^2 \sin(2\theta)}, \end{aligned}$$

which agrees with the previous result. Similarly, we have:

$$\begin{aligned} \frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} \\ &= -ye^{-xy} \cdot (-r \sin(\theta)) - xe^{-xy} \cdot (r \cos(\theta)) \\ &= r^2 (\sin^2(\theta) - \cos^2(\theta)) e^{-\frac{1}{2}r^2 \sin(2\theta)} \\ &= -r^2 \cos(2\theta) e^{-\frac{1}{2}r^2 \sin(2\theta)}, \end{aligned}$$

which agrees with the previous result.

13. (**Reciprocity and the cyclic relation**) Three variables x, y, z are related by the implicit equation $f(x, y, z) = 0$ where f is some multivariable function.

- (a) Using the multivariable chain rule, derive the *reciprocity relation* and the *cyclic relation* for the partial derivatives:

$$\left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial x}{\partial y}\right)_z = 1, \quad \left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial y}\right)_x = -1.$$

- (b) Verify that these relationships hold if: (i) $f(x, y, z) = xyz + x^3 + y^4 + z^5$; (ii) $f(x, y, z) = xyz - \sinh(x + z)$.

» Solution:

- (a) For the reciprocity relation, note that an implicit equation $f(x, y, z) = 0$ allows us to consider y to be a function of x and z . Thus, we have $f(x, y(x, z), z) = 0$. Taking the partial derivative with respect to x , we obtain:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} \Big|_z = 0.$$

Rearranging, we see that:

$$\frac{\partial y}{\partial x} \Big|_z = -\frac{\partial f / \partial x}{\partial f / \partial y}.$$

On the other hand, we could also view x to be a function of y and z . Thus, we have $f(x(y, z), y, z) = 0$ being the defining the relation. Taking the partial derivative with respect to y , we obtain:

$$\frac{\partial f}{\partial x} \frac{\partial x}{\partial y} \Big|_z + \frac{\partial f}{\partial y} = 0.$$

Rearranging, we see that:

$$\frac{\partial x}{\partial y} \Big|_z = -\frac{\partial f / \partial y}{\partial f / \partial x}.$$

Multiplying these expressions together, we see that the reciprocity relation indeed holds.

We can similarly show that the cyclic chain rule holds. Considering the implicit relation between the variables through $f(x, y, z(x, y)) = 0$, $f(x, y(x, z), z) = 0$ and $f(x(y, z), y, z) = 0$ and differentiating with respect to y, x, z respectively, we have:

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} \Big|_x = 0,$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} \Big|_z = 0,$$

$$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} \Big|_y = 0.$$

Rearranging each of these equations, we obtain:

$$\frac{\partial z}{\partial y} \Big|_x = -\frac{\partial f / \partial y}{\partial f / \partial z}, \quad \frac{\partial y}{\partial x} \Big|_z = -\frac{\partial f / \partial x}{\partial f / \partial y}, \quad \frac{\partial x}{\partial z} \Big|_y = -\frac{\partial f / \partial z}{\partial f / \partial x}.$$

Multiplying these results together, we obtain the cyclic chain rule, as required.

(b) We now verify that both relationships hold in two special cases:

- (i) First, we have $f(x, y, z) = xyz + x^3 + y^4 + z^5 = 0$. We begin by viewing y as a function of x, z ; then, taking the partial derivative with respect to x keeping z constant, we have:

$$yz + x \frac{\partial y}{\partial x} \Big|_z z + 3x^2 + 4y^3 \frac{\partial y}{\partial x} \Big|_z = 0.$$

Rearranging, we have:

$$\frac{\partial y}{\partial x} \Big|_z = -\frac{yz + 3x^2}{xz + 4y^3}.$$

Similarly, we can view x as a function of y, z ; then, taking the partial derivative with respect to y keeping z constant, we have:

$$\frac{\partial x}{\partial y} \Big|_z yz + xz + 3x^2 \frac{\partial x}{\partial y} \Big|_z + 4y^3 = 0.$$

Rearranging, we have:

$$\frac{\partial x}{\partial y} \Big|_z = -\frac{xz + 4y^3}{yz + 3x^2}.$$

Thus, multiplying these expressions together we see that the reciprocity relation indeed holds.

Now regarding $xyz + x^3 + y^4 + z^5 = 0$ as defined y as a function of x, z , then x as a function of y, z , and then z as a function of x, y , we obtain the following derivatives by implicit differentiation:

$$yz + xz \frac{\partial y}{\partial x} \Big|_z + 3x^2 + 4y^3 \frac{\partial y}{\partial x} \Big|_z = 0,$$

$$yz \frac{\partial x}{\partial z} \Big|_y + xy + 3x^2 \frac{\partial x}{\partial z} \Big|_y + 5z^4 = 0,$$

$$xz + xy \frac{\partial z}{\partial y} \Big|_x + 4y^3 + 5z^4 \frac{\partial z}{\partial y} \Big|_x = 0.$$

Rearranging, we have:

$$\frac{\partial y}{\partial x} \Big|_z = -\frac{3x^2 + yz}{xz + 4y^3}, \quad \frac{\partial x}{\partial z} \Big|_y = -\frac{xy + 5z^4}{3x^2 + yz}, \quad \frac{\partial z}{\partial y} \Big|_x = -\frac{xz + 4y^3}{xy + 5z^4}.$$

Multiplying these together, we obtain -1 , verifying the cyclic chain rule.

- (ii) For $f(x, y, z) = xyz - \sinh(x + z) = 0$, it's a similar story. Calculating the necessary derivatives, we have:

$$yz + xz \frac{\partial y}{\partial x} \Big|_z - \cosh(x + z) = 0,$$

$$yz \frac{\partial x}{\partial y} \Big|_z + xz - \cosh(x + z) \frac{\partial x}{\partial y} \Big|_z = 0,$$

$$yz \frac{\partial x}{\partial z} \Big|_y + xy - \cosh(x + z) \frac{\partial x}{\partial z} \Big|_y - \cosh(x + z) = 0,$$

$$xz + xy \frac{\partial z}{\partial y} \Big|_x - \cosh(x + z) \frac{\partial z}{\partial y} \Big|_x = 0.$$

Rearranging the first two relations, we have:

$$\frac{\partial y}{\partial x} \Big|_z = -\frac{yz - \cosh(x+z)}{xz}, \quad \frac{\partial x}{\partial y} \Big|_z = -\frac{xz}{yz - \cosh(x+z)},$$

which indeed multiply to give one, verifying the reciprocity relation. On the other hand, rearranging the first, third and fourth relations, we have:

$$\frac{\partial y}{\partial x} \Big|_z = -\frac{yz - \cosh(x+z)}{xz}, \quad \frac{\partial x}{\partial z} \Big|_y = -\frac{xy - \cosh(x+z)}{yz - \cosh(x+z)}, \quad \frac{\partial z}{\partial y} \Big|_x = -\frac{xz}{xy - \cosh(x+z)}.$$

These multiply to give -1 , verifying the cyclic chain rule in this case.

The multivariable chain rule for second-order derivatives

14. Let $f(u, v) = u^2 \sinh(v)$, and let $u = x, v = x + y$.

(a) By differentiating with respect to u , compute $\partial^2 f / \partial u^2$.

(b) Using the multivariable chain rule, show that:

$$\frac{\partial^2 f}{\partial u^2} = \frac{\partial^2 f}{\partial x^2} - 2 \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2},$$

Hence compute the derivative in (a) by writing f in terms of x, y , differentiating, and using this relationship.

(c) Repeat this exercise for the derivatives $\partial^2 f / \partial v^2$ and $\partial^2 f / \partial u \partial v$.

♦♦ Solution:

(a) We obviously have:

$$\frac{\partial^2 f}{\partial u^2} = 2 \sinh(v).$$

(b) First, note that $x = u$ and $y = v - x = v - u$. Hence, taking the first derivative, we have by the multivariable chain rule:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}.$$

Now we treat $\partial f / \partial u$ as another function to which we can apply the multivariable chain rule (this is where the confusion usually starts). We have by the multivariable chain rule:

$$\begin{aligned} \frac{\partial^2 f}{\partial u^2} &= \frac{\partial}{\partial u} \left(\frac{\partial f}{\partial u} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial u} \right) \frac{\partial x}{\partial u} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial u} \right) \frac{\partial y}{\partial u} \\ &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial u} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial u} \right). \end{aligned}$$

Now, inserting the expression we already found for the first partial u -derivative in terms of (x, y) , we have:

$$\begin{aligned} \frac{\partial^2 f}{\partial u^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial u} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial u} \right) \\ &= \frac{\partial^2 f}{\partial x^2} - 2 \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}, \end{aligned}$$

as required.

Writing f in terms of (x, y) then, we have $f(x, y) = x^2 \sinh(x + y)$. Thus:

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 2 \sinh(x + y) + 4x \cosh(x + y) + x^2 \sinh(x + y), & \frac{\partial^2 f}{\partial x \partial y} &= 2x \cosh(x + y) + x^2 \sinh(x + y), \\ \frac{\partial^2 f}{\partial y^2} &= x^2 \sinh(x + y). \end{aligned}$$

Thus:

$$\begin{aligned} \frac{\partial^2 f}{\partial u^2} &= 2 \sinh(x + y) + 4x \cosh(x + y) + x^2 \sinh(x + y) - 4x \cosh(x + y) - 2x^2 \sinh(x + y) + x^2 \sinh(x + y) \\ &= 2 \sinh(x + y) \\ &= 2 \sinh(v), \end{aligned}$$

agreeing with what we found previously.

(c) We are now asked to laboriously repeat this exercise for the remaining two second derivatives, as useful practice. We first have:

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} = \frac{\partial f}{\partial y}.$$

Taking a second derivative, we have:

$$\begin{aligned}\frac{\partial^2 f}{\partial v^2} &= \frac{\partial}{\partial v} \left(\frac{\partial f}{\partial v} \right) \\&= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial v} \right) \frac{\partial x}{\partial v} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial v} \right) \frac{\partial y}{\partial v} \\&= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial v} \right) \\&= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \\&= \frac{\partial^2 f}{\partial y^2}.\end{aligned}$$

Comparing the derivatives computed both ways, we note for $f(u, v) = u^2 \sinh(v) = x^2 \sinh(x + y)$, we have:

$$\frac{\partial^2 f}{\partial v^2} = u^2 \sinh(v), \quad \frac{\partial^2 f}{\partial y^2} = x^2 \sinh(x + y),$$

which indeed agree.

On the other hand, for the mixed derivative, we have:

$$\begin{aligned}\frac{\partial^2 f}{\partial u \partial v} &= \frac{\partial}{\partial u} \left(\frac{\partial f}{\partial v} \right) \\&= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \frac{\partial x}{\partial u} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \frac{\partial y}{\partial u} \\&= \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y^2}.\end{aligned}$$

Comparing the derivatives computed both ways, we note for $f(u, v) = u^2 \sinh(v) = x^2 \sinh(x + y)$, we have:

$$\begin{aligned}\frac{\partial^2 f}{\partial u \partial v} &= 2u \cosh(v) = 2x \cosh(x + y), \\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y^2} &= 2x \cosh(x + y) + x^2 \sinh(x + y) - x^2 \sinh(x + y) = 2x \cosh(x + y),\end{aligned}$$

in perfect agreement.

15. Let $f(u, v)$ be a multivariable function of $u(x, y) = 1 + x^2 + y^2, v(x, y) = 1 + x^2y^2$, where (x, y) are plane Cartesian coordinates.

- (a) Calculate $\partial f / \partial x, \partial f / \partial y, \partial^2 f / \partial x^2, \partial^2 f / \partial y^2, \partial^2 f / \partial x \partial y$ in terms of the derivatives of f with respect to u, v .
- (b) For $f(u, v) = \log(uv)$, find $\partial^2 f / \partial x \partial y$ by: (i) using the expression derived in part (a); (ii) first expressing f in terms of x, y and then differentiating directly. Verify that your results agree.

♦♦ Solution:

- (a) Using the multivariable chain rule, we have:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = 2x \frac{\partial f}{\partial u} + 2xy^2 \frac{\partial f}{\partial v}, \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} = 2y \frac{\partial f}{\partial u} + 2yx^2 \frac{\partial f}{\partial v}.\end{aligned}$$

The second derivatives can be computed similarly. We have:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \\ &= \frac{\partial}{\partial x} \left(2x \frac{\partial f}{\partial u} + 2xy^2 \frac{\partial f}{\partial v} \right) \\ &= 2 \frac{\partial f}{\partial u} + 2x \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial u} \right) + 2y^2 \frac{\partial f}{\partial v} + 2xy^2 \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial v} \right).\end{aligned}$$

Note that $\partial f / \partial u, \partial f / \partial v$ are functions of $(u, v) = (u(x, y), v(x, y))$ so we can use the multivariable chain rule again on them. We get:

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial u} \right) = \frac{\partial^2 f}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial u \partial v} \frac{\partial v}{\partial x} = 2x \frac{\partial^2 f}{\partial u^2} + 2xy^2 \frac{\partial^2 f}{\partial u \partial v}.$$

Similarly, we have:

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial v} \right) = \frac{\partial^2 f}{\partial u \partial v} \frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v^2} \frac{\partial v}{\partial x} = 2x \frac{\partial^2 f}{\partial u \partial v} + 2xy^2 \frac{\partial^2 f}{\partial v^2}.$$

Putting everything together, we have:

$$\frac{\partial^2 f}{\partial x^2} = 4x^2 \frac{\partial^2 f}{\partial u^2} + 8x^2y^2 \frac{\partial^2 f}{\partial u \partial v} + 4x^2y^4 \frac{\partial^2 f}{\partial v^2} + 2 \frac{\partial f}{\partial u} + 2y^2 \frac{\partial f}{\partial v}.$$

We now repeat the calculation for the second y -derivative. We have:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \\ &= \frac{\partial}{\partial y} \left(2y \frac{\partial f}{\partial u} + 2yx^2 \frac{\partial f}{\partial v} \right) \\ &= 2 \frac{\partial f}{\partial u} + 2y \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial u} \right) + 2x^2 \frac{\partial f}{\partial v} + 2yx^2 \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial v} \right).\end{aligned}$$

Using the multivariable chain rule to calculate the y -derivatives of $\partial f / \partial u$ and $\partial f / \partial v$, we have:

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial u} \right) = \frac{\partial^2 f}{\partial u^2} \frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial u \partial v} \frac{\partial v}{\partial y} = 2y \frac{\partial^2 f}{\partial u^2} + 2yx^2 \frac{\partial^2 f}{\partial u \partial v},$$

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial v} \right) = \frac{\partial^2 f}{\partial u \partial v} \frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2} \frac{\partial v}{\partial y} = 2y \frac{\partial^2 f}{\partial u \partial v} + 2yx^2 \frac{\partial^2 f}{\partial v^2}.$$

Putting everything together, we have:

$$\frac{\partial^2 f}{\partial y^2} = 4y^2 \frac{\partial^2 f}{\partial u^2} + 8y^2 x^2 \frac{\partial^2 f}{\partial u \partial v} + 4y^2 x^4 \frac{\partial^2 f}{\partial v^2} + 2 \frac{\partial f}{\partial u} + 2x^2 \frac{\partial f}{\partial v}$$

Finally, we repeat the calculation for the mixed derivative. We have:

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \\ &= \frac{\partial}{\partial x} \left(2y \frac{\partial f}{\partial u} + 2yx^2 \frac{\partial f}{\partial v} \right) \\ &= 2y \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial u} \right) + 4xy \frac{\partial f}{\partial v} + 2yx^2 \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial v} \right). \end{aligned}$$

Using the expressions we calculated previously for the x -derivative of $\partial f / \partial u$, $\partial f / \partial v$, we have:

$$\frac{\partial^2 f}{\partial x \partial y} = 4xy \frac{\partial^2 f}{\partial u^2} + 8xy^3 \frac{\partial^2 f}{\partial u \partial v} + 4x^3 y^3 \frac{\partial^2 f}{\partial v^2} + 4xy \frac{\partial f}{\partial v}.$$

(b) Here, we have $f(u, v) = \log(uv)$. Hence,

$$\frac{\partial f}{\partial u} = \frac{1}{u}, \quad \frac{\partial f}{\partial v} = \frac{1}{v}, \quad \frac{\partial^2 f}{\partial u^2} = -\frac{1}{u^2}, \quad \frac{\partial^2 f}{\partial u \partial v} = 0, \quad \frac{\partial^2 f}{\partial v^2} = -\frac{1}{v^2}.$$

Inserting these expressions into the formula for $\partial^2 f / \partial x \partial y$ that we derived above, we have:

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{4xy}{u^2} - \frac{4x^3 y^3}{v^2} + \frac{4xy}{v} = -\frac{4xy}{(1+x^2+y^2)^2} - \frac{4x^3 y^3}{(1+x^2 y^2)^2} + \frac{4xy}{1+x^2 y^2}.$$

Alternatively, we can first write $f(u, v) = f(u(x, y), v(x, y)) = \log(1 + x^2 + y^2) + \log(1 + x^2 y^2)$. This gives:

$$\frac{\partial f}{\partial x} = \frac{2x}{1+x^2+y^2} + \frac{2xy^2}{1+x^2 y^2},$$

and hence:

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{4xy}{(1+x^2+y^2)^2} - \frac{4x^3 y^3}{(1+x^2 y^2)^2} + \frac{4xy}{1+x^2 y^2},$$

verifying our calculation.

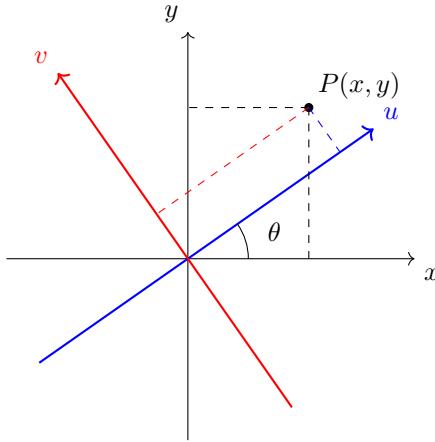
16. Let (x, y) be plane Cartesian coordinates, and let (u, v) be plane Cartesian coordinates which are rotated an angle θ anticlockwise about the origin relative to the (x, y) coordinates. Let f be an arbitrary multivariable function of either (x, y) or (u, v) . Show that:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}.$$

Solution: Using some geometry, we see that the coordinate systems are related via the transformation:

$$u = \cos(\theta)x + \sin(\theta)y,$$

$$v = -\sin(\theta)x + \cos(\theta)y.$$



One way of easily getting this transformation is to imagine what the new coordinates of the points $(1, 0)$ and the points $(0, 1)$ will be in the (u, v) system. By linearity (rotations are linear maps), this generates the entire transformation.

This relationship implies:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = \cos(\theta) \frac{\partial f}{\partial u} - \sin(\theta) \frac{\partial f}{\partial v},$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} = \sin(\theta) \frac{\partial f}{\partial u} + \cos(\theta) \frac{\partial f}{\partial v}.$$

Taking a second derivative, we have:

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\cos(\theta) \frac{\partial f}{\partial u} - \sin(\theta) \frac{\partial f}{\partial v} \right) \\ &= \cos(\theta) \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial u} \right) - \sin(\theta) \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial v} \right) \\ &= \cos(\theta) \left(\frac{\partial^2 f}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial u \partial v} \frac{\partial v}{\partial x} \right) - \sin(\theta) \left(\frac{\partial^2 f}{\partial u \partial v} \frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v^2} \frac{\partial v}{\partial x} \right) \\ &= \cos(\theta) \left(\cos(\theta) \frac{\partial^2 f}{\partial u^2} - \sin(\theta) \frac{\partial^2 f}{\partial u \partial v} \right) - \sin(\theta) \left(\cos(\theta) \frac{\partial^2 f}{\partial u \partial v} - \sin(\theta) \frac{\partial^2 f}{\partial v^2} \right) \\ &= \cos^2(\theta) \frac{\partial^2 f}{\partial u^2} - 2 \sin(\theta) \cos(\theta) \frac{\partial^2 f}{\partial u \partial v} + \sin^2(\theta) \frac{\partial^2 f}{\partial v^2}. \end{aligned}$$

Similarly, we have:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left(\sin(\theta) \frac{\partial f}{\partial u} + \cos(\theta) \frac{\partial f}{\partial v} \right) \\&= \sin(\theta) \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial u} \right) + \cos(\theta) \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial v} \right) \\&= \sin(\theta) \left(\frac{\partial^2 f}{\partial u^2} \frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial u \partial v} \frac{\partial v}{\partial y} \right) + \cos(\theta) \left(\frac{\partial^2 f}{\partial u \partial v} \frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2} \frac{\partial v}{\partial y} \right) \\&= \sin^2(\theta) \frac{\partial^2 f}{\partial u^2} + 2 \sin(\theta) \cos(\theta) \frac{\partial^2 f}{\partial u \partial v} + \cos^2(\theta) \frac{\partial^2 f}{\partial v^2}.\end{aligned}$$

Summing the results we have obtained previously, we obtain the identity:

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

as required.

17. Let (x, y) be plane Cartesian coordinates, and let (r, θ) be plane polar coordinates. Let f be a multivariable function. Show that:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}.$$

Hence determine all solutions of the partial differential equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

which are rotationally symmetric about the origin.

♦♦ **Solution:** Note that $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$. Hence we have:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial x} \\&= \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial f}{\partial r} + \frac{(-y/x^2)}{1 + (y/x)^2} \frac{\partial f}{\partial \theta} \\&= \cos(\theta) \frac{\partial f}{\partial r} - \frac{\sin(\theta)}{r} \frac{\partial f}{\partial \theta}.\end{aligned}$$

Observe that we can take the second derivative in exactly the same way, giving:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \cos(\theta) \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial x} \right) - \frac{\sin(\theta)}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial x} \right) \\&= \cos(\theta) \frac{\partial}{\partial r} \left(\cos(\theta) \frac{\partial f}{\partial r} - \frac{\sin(\theta)}{r} \frac{\partial f}{\partial \theta} \right) - \frac{\sin(\theta)}{r} \frac{\partial}{\partial \theta} \left(\cos(\theta) \frac{\partial f}{\partial r} - \frac{\sin(\theta)}{r} \frac{\partial f}{\partial \theta} \right) \\&= \cos^2(\theta) \frac{\partial^2 f}{\partial r^2} + \frac{\sin(\theta) \cos(\theta)}{r^2} \frac{\partial f}{\partial \theta} - \frac{\sin(\theta) \cos(\theta)}{r} \frac{\partial^2 f}{\partial r \partial \theta} + \frac{\sin^2(\theta)}{r} \frac{\partial f}{\partial r} - \frac{\sin(\theta) \cos(\theta)}{r} \frac{\partial^2 f}{\partial r \partial \theta} \\&\quad + \frac{\sin(\theta) \cos(\theta)}{r^2} \frac{\partial f}{\partial \theta} + \frac{\sin^2(\theta)}{r^2} \frac{\partial^2 f}{\partial \theta^2} \\&= \cos^2(\theta) \frac{\partial^2 f}{\partial r^2} + \frac{2 \sin(\theta) \cos(\theta)}{r^2} \frac{\partial f}{\partial \theta} - \frac{2 \sin(\theta) \cos(\theta)}{r} \frac{\partial^2 f}{\partial r \partial \theta} + \frac{\sin^2(\theta)}{r} \frac{\partial f}{\partial r} + \frac{\sin^2(\theta)}{r^2} \frac{\partial^2 f}{\partial \theta^2}.\end{aligned}$$

Proceeding in a similar way for the y -derivatives, we have:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial y} \\&= \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial f}{\partial r} + \frac{(1/x)}{1 + (y/x)^2} \frac{\partial f}{\partial \theta} \\&= \sin(\theta) \frac{\partial f}{\partial r} + \frac{\cos(\theta)}{r} \frac{\partial f}{\partial \theta}.\end{aligned}$$

Taking the second derivative, we have:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \sin(\theta) \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial y} \right) + \frac{\cos(\theta)}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial y} \right) \\&= \sin(\theta) \frac{\partial}{\partial r} \left(\sin(\theta) \frac{\partial f}{\partial r} + \frac{\cos(\theta)}{r} \frac{\partial f}{\partial \theta} \right) + \frac{\cos(\theta)}{r} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial f}{\partial r} + \frac{\cos(\theta)}{r} \frac{\partial f}{\partial \theta} \right) \\&= \sin^2(\theta) \frac{\partial^2 f}{\partial r^2} - \frac{\sin(\theta) \cos(\theta)}{r^2} \frac{\partial f}{\partial \theta} + \frac{\sin(\theta) \cos(\theta)}{r} \frac{\partial^2 f}{\partial r \partial \theta} + \frac{\cos^2(\theta)}{r} \frac{\partial f}{\partial r} \\&\quad + \frac{\sin(\theta) \cos(\theta)}{r} \frac{\partial^2 f}{\partial r \partial \theta} - \frac{\sin(\theta) \cos(\theta)}{r} \frac{\partial f}{\partial \theta} + \frac{\cos^2(\theta)}{r^2} \frac{\partial^2 f}{\partial \theta^2} \\&= \sin^2(\theta) \frac{\partial^2 f}{\partial r^2} - \frac{2 \sin(\theta) \cos(\theta)}{r^2} \frac{\partial f}{\partial \theta} + \frac{2 \sin(\theta) \cos(\theta)}{r} \frac{\partial^2 f}{\partial r \partial \theta} + \frac{\cos^2(\theta)}{r} \frac{\partial f}{\partial r} + \frac{\cos^2(\theta)}{r^2} \frac{\partial^2 f}{\partial \theta^2}.\end{aligned}$$

Summing the second derivatives, we obtain:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2},$$

as requested.

When $f(x, y) = f(r, \theta)$ is rotationally symmetric about the origin, it has no θ dependence. Hence $f \equiv f(r)$. Therefore, using the polar form of the given equation, we have:

$$f''(r) + \frac{f'(r)}{r} = 0.$$

This is a first-order equation for $f'(r)$, with integrating factor $e^{\ln(r)} = r$. Hence:

$$\frac{d}{dr} (r f'(r)) = 0 \quad \Rightarrow \quad r f'(r) = c \quad \Rightarrow \quad f(r) = c \ln(r) + d.$$

Thus all rotationally symmetric solutions are given by $f(r, \theta) = A \ln(r) + B$ for constants A, B .

18. Consider a function $z(x, y)$ that satisfies $z(\lambda x, \lambda y) = \lambda^n z(x, y)$ for any real λ and a fixed integer n . Show that:

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz,$$

and

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$

♦♦ **Solution:** Take the λ partial derivative of both sides. Then using the multivariable chain rule on the left, we have:

$$\frac{\partial z}{\partial x}(\lambda x, \lambda y) \frac{\partial(\lambda x)}{\partial \lambda} + \frac{\partial z}{\partial y}(\lambda x, \lambda y) \frac{\partial(\lambda y)}{\partial \lambda} = n\lambda^{n-1}z(x, y).$$

Simplifying, we have:

$$x \frac{\partial z}{\partial x}(\lambda x, \lambda y) + y \frac{\partial z}{\partial y}(\lambda x, \lambda y) = n\lambda^{n-1}z(x, y). \quad (*)$$

Taking $\lambda = 1$ gives the result.

Now taking a second derivative with respect to λ of $(*)$, we have (using the multivariable chain rule again on the left hand side):

$$x \frac{\partial}{\partial \lambda} \left(\frac{\partial z}{\partial x}(\lambda x, \lambda y) \right) + y \frac{\partial}{\partial \lambda} \left(\frac{\partial z}{\partial y}(\lambda x, \lambda y) \right) = n(n-1)\lambda^{n-2}z(x, y)$$

$$x \left(\frac{\partial^2 z}{\partial x^2}(\lambda x, \lambda y) \frac{\partial(\lambda x)}{\partial \lambda} + \frac{\partial^2 z}{\partial x \partial y}(\lambda x, \lambda y) \frac{\partial(\lambda y)}{\partial \lambda} \right) + y \left(\frac{\partial^2 z}{\partial x \partial y}(\lambda x, \lambda y) \frac{\partial(\lambda x)}{\partial \lambda} + \frac{\partial^2 z}{\partial y^2}(\lambda x, \lambda y) \frac{\partial(\lambda y)}{\partial \lambda} \right) = n(n-1)\lambda^{n-2}z(x, y)$$

$$x^2 \frac{\partial^2 z}{\partial x^2}(\lambda x, \lambda y) + 2xy \frac{\partial^2 z}{\partial x \partial y}(\lambda x, \lambda y) + y^2 \frac{\partial^2 z}{\partial y^2}(\lambda x, \lambda y) = n(n-1)\lambda^{n-2}z(x, y).$$

Taking $\lambda = 1$, we establish:

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z(x, y),$$

as required.