Part IA: Mathematics for Natural Sciences B
Examples Sheet 5: Infinite series and Taylor series

Model Solutions

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(†) Basics of infinite series

1. State clearly what it means for an infinite series to be: (i) *convergent*; (ii) *absolutely convergent*. If a series is absolutely convergent, must it be convergent? Is the converse true?

➻ Solution: Let:

$$\sum_{n=1}^{\infty} a_n$$

be an infinite series. We say that:

(i) the infinite series is *convergent* if the limit:

$$\lim_{N\to\infty}\sum_{n=1}^{N} a_n$$

of partial sums exists (and is finite); otherwise, the series is *divergent*;

(ii) the infinite series is *absolutely convergent* if the series:

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent.

If a series if absolutely convergent, then it must be convergent (the proof is beyond the scope of the course). The converse is not true; there exist series which are convergent but not absolutely convergent (we shall see examples later on).

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2. By evaluating the partial sums, determine whether $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ is convergent. Is it absolutely convergent?

➻ Solution: We have:

$$\sum_{n=1}^{N}\left(\sqrt{n+1}-\sqrt{n}\right) = (\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+\cdots+(\sqrt{N}-\sqrt{N-1})+(\sqrt{N+1}-\sqrt{N}).$$

We see that this is a *telescoping series*, where most terms cancel out, leaving only:

$$\sum_{n=1}^{N}\left(\sqrt{n+1}-\sqrt{n}\right) = \sqrt{N+1}-1.$$

Hence, this has an infinite limit as $N\to\infty$, so is not convergent. It cannot be absolutely convergent, because it is not convergent (and anything that is absolutely convergent is necessarily convergent too).

3. By evaluating the partial sums, prove that the geometric series $\sum_{n=0}^{\infty} ar^n$ is absolutely convergent for $-1 < r < 1$.

➺ **Solution:** Let:

$$S_{N+1} = |a| + |a||r| + |a||r|^2 + ... + |a||r|^N$$

be the $(N+1)$th partial sum of the absolute values. Multiplying by $1-|r|$, we have:

$$\begin{aligned}(1-|r|)S_{N+1} &= (1-|r|)\left(|a| + |a||r| + |a||r|^2 + ... + |a||r|^N\right)\\ &= |a| + |a||r| + |a||r|^2 + ... + |a||r|^N - |a||r| - |a||r|^2 - |a||r|^3 - ... - |a||r|^{N+1}\\ &= |a|\left(1-|r|^{N+1}\right).\end{aligned}$$

Hence the partial sums are given by:

$$S_{N+1} = \frac{|a|\left(1-|r|^{N+1}\right)}{1-|r|},$$

provided that $|r| \neq 1$. If $|r| < 1$, then as $N \to \infty$ we have $|r|^{N+1} \to 0$. Hence the limit of the partial sums is given by:

$$\frac{|a|}{1-|r|},$$

which is finite. It follows that if $|r| < 1$, the sum is absolutely convergent (and hence convergent).

(†) Tests for convergence

4. (a) Clearly state the *comparison test* for series convergence or divergence.

   (b) Using the comparison test, prove that the harmonic series diverges. Hence, show that the following definition of the Riemann zeta function:

$$\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p},$$

   with p real, converges if and only if $p > 1$. Is it absolutely convergent when $p > 1$?

➺ **Solution:** (a) Let:

$$\sum_{n=1}^{\infty} a_n, \qquad \sum_{n=1}^{\infty} b_n$$

be two series. The *comparison test* states that if there exists some N such that $0 \leq a_n \leq b_n$ for all $n \geq N$, then:

- if the first series diverges, then the second series diverges;
- if the second series converges, then the first series converges.

(b) The harmonic series is:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots .$$

Grouping the terms, we can rewrite the harmonic series as:

$$\begin{aligned}
&1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \cdots\right) + \cdots \\
&> \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots\right) + \cdots \\
&= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots .
\end{aligned}$$

Hence, we see by comparison with the divergent series $1/2 + 1/2 + 1/2 + 1/2 + \ldots$, the harmonic series must be divergent.

Now consider the Riemann zeta function $\zeta(p)$. If $p = 1$, the series diverges because it is the harmonic series. Further, if $p < 1$, we have $n^p \leq n$ for all positive integers n, hence:

$$\begin{aligned}
\zeta(p) &= \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \cdots \\
&> 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots ,
\end{aligned}$$

so by comparison we have that the series diverges for $p < 1$. Finally, observe that if $p > 1$, we have:

$$\begin{aligned}
\zeta(p) &= \frac{1}{1^p} + \frac{1}{2^p} + \left(\frac{1}{3^p} + \frac{1}{4^p}\right) + \left(\frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p}\right) + \cdots \\
&\leq \frac{1}{1^p} + \frac{1}{2^p} + \left(\frac{1}{4^p} + \frac{1}{4^p}\right) + \left(\frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p}\right) + \cdots \\
&= 1 + 2^{-p} + 2 \cdot 4^{-p} + 4 \cdot 8^{-p} + \cdots \\
&= 1 + 2^{-p} + 2^{-2p+1} + 2^{-3p+2} + \cdots \\
&= 1 + \frac{2^{-p}}{1 - 2^{-p+1}},
\end{aligned}$$

since for $p > 1$, we have $2^{-p+1} = 1/2^{p-1} < 1$, so we can bound things above with a convergent geometric series. It follows that the series converges for $p > 1$.

It is absolutely convergent for $p > 1$, since all of the terms are positive already.

5. (a) Clearly state, *and prove*, the *alternating series test* for series convergence or divergence.

(b) Hence, show that the following definition of the Dirichlet eta function:

$$\eta(p) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p},$$

with p real, converges if and only if $p > 0$. Is it absolutely convergent when $p > 0$?

➾ **Solution:** (a) Let:

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

satisfying the following three key properties:

(i) $a_n \geq 0$;

(ii) $a_{n+1} \leq a_n$;

(iii) $a_n \to 0$ as $n \to \infty$.

Then the *alternating series test* states the series converges.

To prove the test, we split the partial sums into even and odd cases. The even partial sums satisfy:

$$\begin{aligned}
\sum_{n=1}^{2N} (-1)^{n+1} a_n &= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2N-1} - a_{2N} \\
&= a_1 + \underbrace{(a_3 - a_2)}_{<0} + \underbrace{(a_5 - a_4)}_{<0} + \cdots + \underbrace{(a_{2N} - a_{2N-1})}_{<0} \\
&\leq a_1.
\end{aligned}$$

Therefore, the even partial sums are bounded above. Furthermore:

$$\sum_{n=1}^{2N} (-1)^{n+1} a_n = \sum_{n=1}^{2(N-1)} (-1)^{n+1} a_n + \underbrace{(a_{2N-1} - a_{2N})}_{>0}.$$

Hence, the even partial sums are increasing. These two facts combined imply that the even partial sums converge.

The odd partial sums satisfy:

$$\begin{aligned}
\sum_{n=1}^{2N+1} (-1)^{n+1} a_n &= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2N-1} - a_{2N} + a_{2N+1} \\
&= \underbrace{(a_1 - a_2)}_{>0} + \underbrace{(a_3 - a_4)}_{>0} + \cdots + \underbrace{(a_{2N-1} - a_{2N})}_{>0} + a_{2N+1} \\
&\geq a_{2N+1} > 0.
\end{aligned}$$

Therefore, the odd partial sums are bounded below. Furthermore:

$$\sum_{n=1}^{2N+1} (-1)^{n+1} a_n = \sum_{n=1}^{2N-1} (-1)^{n+1} a_n + \underbrace{(-a_{2N} + a_{2N+1})}_{<0}.$$

Hence, the odd partial sums are decreasing. These two facts combined imply that the odd partial sums converge.

To see that these sequences of partial sums converge to the same limit, observe that:

$$\lim_{N\to\infty} \left(\sum_{n=1}^{2N+1} (-1)^{n+1} a_n - \sum_{n=1}^{2N} (-1)^{n+1} a_n \right) = \lim_{N\to\infty} a_{2N+1} = 0,$$

by assumption (iii). Hence the two limits must be equal.

(b) For the Dirichlet eta function:

$$\eta(p) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p},$$

we observe that if $p > 0$, we have $1/n^p \to 0$ as $n \to \infty$, and:

$$\frac{1}{(n+1)^p} < \frac{1}{n^p}$$

for all positive integers n. Hence, by the alternating series test, we have convergence for $p > 0$.

When $p = 0$, the series with terms $(-1)^{n+1}$ does not converge, so the nth term of the series does not tend to zero. Hence the series is not convergent.

When $p < 0$, the series with terms $(-1)^{n+1}/n^p$ oscillates between being large and positive, and large and negative, and hence does not converge. Thus, the nth term of the series does not tend to zero. Hence the series is not convergent.

Finally, observe that if we take the absolute value of all the terms, we get $\zeta(p)$ instead. Hence the series is absolutely convergent if and only if $p > 1$, from Question 4(b).

6. Clearly state the *ratio test* for series convergence or divergence. Use it to show that the series:

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

is absolutely convergent. To what value does it converge? [*Hint: If you're unsure, come back after studying Taylor series.*]

➺ **Solution:** Let:

$$\sum_{n=1}^{\infty} a_n$$

be an infinite series of complex numbers a_n, containing only non-zero terms. Suppose also that the limit:

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists in $\mathbb{R} \cup \{\infty\}$ (that is, we allow infinite limits). Then the *ratio test* states:

- if $r > 1$, the series diverges;
- if $r < 1$, the series absolutely converges, and hence is convergent.

If $r = 1$, the test is inconclusive.

Applying this to the series in the question, we have the nth term $a_n = 2^n/n!$. Hence:

$$r = \lim_{n\to\infty} \left| \frac{2^{n+1}/(n+1)!}{2^n/n!} \right| = \lim_{n\to\infty} \left| \frac{2}{n+1} \right| = 0.$$

This is less than 1, hence the series is absolutely convergent, and hence is convergent.

We spot that the series looks like the Taylor series for e^x about $x = 0$, given by:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Comparing with the series in the question, we see that:

$$\sum_{n=1}^{\infty} \frac{2^n}{n!} = e^2 - 1,$$

since the Taylor series starts at the term $n = 0$, but the series in the question starts at the term $n = 1$, so we need to subtract the $n = 0$ term.

7. Another test that was not lectured (but has come up in exams before!) is the *integral test*.

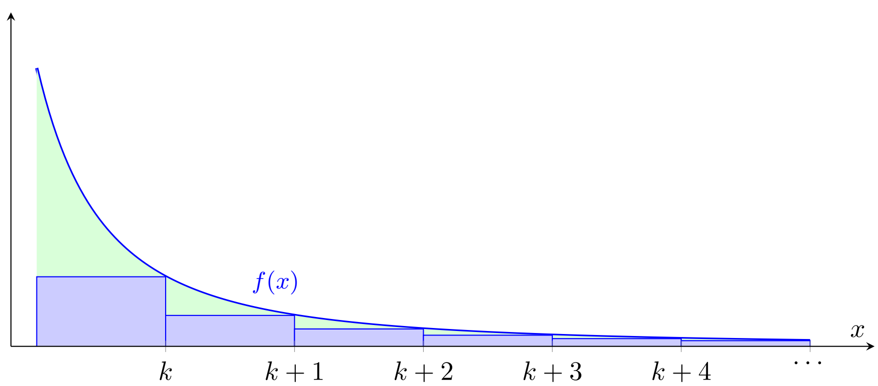
   (a) Suppose that $f : [k, \infty) \to \mathbb{R}$ is a continuous, positive, decreasing function, where k is an integer. By drawing a convincing diagram, show that:

   $$\int_k^\infty f(x)\,dx \text{ converges} \qquad \Rightarrow \qquad \sum_{n=k}^\infty f(n) \text{ converges.}$$

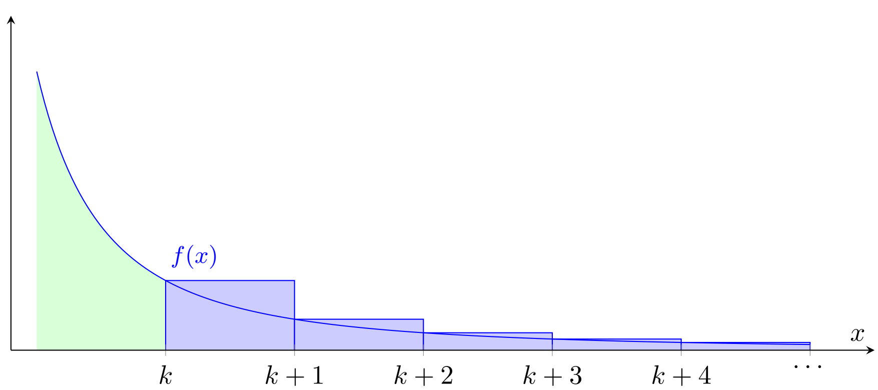
   Show also that if the integral diverges, then the series diverges. This test is called the *integral test*.

   (b) Using the integral test, reanalyse the convergence of the definition of $\zeta(p)$ given in Question 4.

Solution: (a) Since $f(x)$ is continuous, positive, and decreasing, it takes the form shown in the picture below. Inserting rectangles which are of height $f(k), f(k+1), f(k+2)$, etc, under the curve, we see immediately that if the integral converges (i.e. the area under the curve is finite), then the sum must converge too.



On the other hand, we could equally place the rectangles above the curve, by aligning their left edges with the curve instead, as shown in the figure below.



This shows that if the integral diverges (i.e. there is infinite area under the curve), then the sum must diverge too.

(b) For:

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p},$$

we should consider the integral:

$$\int_1^{\infty} \frac{dx}{x^p}.$$

- If $p > 1$, we have:

  $$\int_1^{\infty} \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^{\infty} = 0 - \frac{1}{1-p} = \frac{1}{p-1}.$$

  This is finite, hence the original series is convergent in this case.

- If $p = 1$, we have:

  $$\int_1^{\infty} \frac{dx}{x} = [\log(x)]_1^{\infty} = \infty.$$

  This is infinite, hence the original series is divergent in this case.

- If $p < 1$, we have:

  $$\int_1^{\infty} \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^{\infty} = \infty.$$

  This is infinite, hence the original series is divergent in this case.

Hence, the analysis agrees with the analysis we performed earlier using the comparison test.

(†) Miscellaneous series

8. Applying an appropriate test in each case, determine which of the following series are convergent, and which are absolutely convergent:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{3n^2+4}$;

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$;

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{n!}$;

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$;

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{5 \cdot 8 \cdot 11 \cdot \ldots \cdot (3n+2)}{1 \cdot 5 \cdot 9 \cdot \ldots \cdot (4n-3)}$;

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$;

(g) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2n+5}{3n+1}\right)^n$;

(h) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\log(n)}$;

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{3^n}$;

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}$;

(k) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2\log(n)}$;

(l) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2n}{n^3+3n^2+1}$;

(m) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{1+n^2}$;

(n) $\displaystyle\sum_{n=0}^{\infty} \frac{a^{2n+1}}{2n+1}$, where $a > 0$;

(o) $\displaystyle\sum_{n=1}^{\infty} n^p \sin(\omega n)$ where $\omega > 0$, and $p < -1$;

(p) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi)}{n}$;

(q) $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{n\log(n)}$;

(r) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$;

(s) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{\log^n(2)}$;

(t) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n^4+a^2}-n^2\right)$, for $a > 0$;

(u) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n-1}$;

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3 e^{3n}}{(3n)!}$.

Solution: One important test that we have not mentioned on the sheet so far is the 'nth term test' or the 'divergence test': the series:

$$\sum_{n=1}^{\infty} a_n$$

is divergent if $a_n \not\to 0$ as $n \to \infty$. We will use this in some of the examples.

(a) The divergence test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{n^2+1}{3n^2+4} = \lim_{n\to\infty} \frac{1+1/n^2}{3+4/n^2} = \frac{1}{3} \neq 0.$$

Hence, the series diverges.

(b) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{(n+1)^{10}}{2^{n+1}} \cdot \frac{2^n}{n^{10}} = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{10} = \frac{1}{2} < 1.$$

Hence, the series converges. All terms of the series are positive, so the series converges absolutely.

(c) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{(n+1)^{10}}{(n+1)!} \cdot \frac{n!}{n^{10}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{10} \cdot \lim_{n\to\infty} \frac{1}{n+1} = 1 \cdot 0 = 0 < 1.$$

Hence, the series converges. All terms of the series are positive, so the series converges absolutely.

(d) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \frac{1}{10} \lim_{n\to\infty} (n+1) = \infty.$$

Hence, the series diverges.

(e) The ratio test is appropriate here. Observe that:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{5 \cdot 8 \cdot 11 \cdot \ldots \cdot (3(n+1)+2)}{1 \cdot 5 \cdot 9 \cdot \ldots \cdot (4(n+1)-3)} \cdot \frac{1 \cdot 5 \cdot 9 \cdot \ldots \cdot (4n-3)}{5 \cdot 8 \cdot 11 \cdot \ldots \cdot (3n+2)} \\
&\quad = \lim_{n\to\infty} \frac{3(n+1)+1)}{4(n+1)-3} \\
&\quad = \lim_{n\to\infty} \frac{3n+4}{4n+1} \\
&\quad = \lim_{n\to\infty} \frac{3+4/n}{4+1/n} \\
&\quad = \frac{3}{4}.
\end{aligned}$$

Hence, the series converges. All terms of the series are positive, so the series converges absolutely.

(f) The alternating series test is appropriate here. Observe that: $1/\sqrt{n} > 0$, $1/\sqrt{n} \to 0$ as $n \to \infty$, and:

$$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}.$$

Hence the three conditions of the alternating series test are satisfied, so the series converges.

The series converges absolutely if:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

converges. The comparison test is appropriate here. Note that for all integers $n \geq 1$, we have $\sqrt{n} < n$, so:

$$\frac{1}{\sqrt{n}} > \frac{1}{n}.$$

The harmonic series diverges, so this implies that this series diverges too.

Alternatively, this question can be done by simply spotting that the series is the Dirichlet eta function $\eta(1/2)$, which we already discussed in Question 5(b).

(g) Superficially, it looks like the alternating series test is the right test to use here. Instead though, we can try to use the ratio test to prove absolute convergence, from which we can immediately deduce convergence.

Consider the limit:

$$\lim_{n\to\infty}\left(\frac{2(n+1)+5}{3(n+1)+1}\right)^{n+1}\left(\frac{3n+1}{2n+5}\right)^{n}=\frac{2}{3}\lim_{n\to\infty}\left(\frac{n+1+5/2}{n+1+1/3}\right)^{n+1}\left(\frac{n+1/3}{n+5/2}\right)^{n}.$$

Note that the final limit is the product:

$$\lim_{n\to\infty}\left(\frac{n+1+5/2}{n+1+1/3}\right)^{n+1}\left(\frac{n+1/3}{n+5/2}\right)^{n}=\lim_{n\to\infty}\left(\frac{n+1+5/2}{n+1+1/3}\right)^{n+1}\lim_{n\to\infty}\left(\frac{n+1/3}{n+5/2}\right)^{n}.$$

Making the substitution $m = n + 1$ in the first limit on the right hand side, we immediately see that the two limits on the right hand side are reciprocals of one another. Hence the product is 1, and it follows that:

$$\lim_{n\to\infty}\left(\frac{2(n+1)+5}{3(n+1)+1}\right)^{n+1}\left(\frac{3n+1}{2n+5}\right)^{n}=\frac{2}{3}<1.$$

Hence, the series is absolutely convergent. Absolute convergence implies convergence, so the series is also convergent.

(h) The integral test is appropriate here. Observe that (using the reverse chain rule - or the substitution $u = \log(x)$, if you're worried):

$$\int_2^\infty \frac{dx}{x\log(x)}=[\log(\log(x))]_2^\infty=\infty.$$

Hence, the integral is divergent, so the series is also divergent.

(i) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty}\frac{(n+1)^4}{3^{n+1}}\cdot\frac{3^n}{n^4}=\frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^4=\frac{1}{3}<1.$$

Hence, the series converges. All terms of the series are positive, so the series converges absolutely.

(j) We could use the alternating series test. However, it is actually better to look at the series:

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \tag{†}$$

first. Observe that:

$$\frac{1}{(2n-1)^2}\leq\frac{1}{n^2}$$

for all integers $n \geq 1$. Since the series definition of $\zeta(2)$ converges by Question 4(b), we have by the comparison test that the series (†) converges. Hence the series in the question is absolutely convergent, which additionally implies convergence of the original series.

(k) The comparison test is appropriate here. Note that for all integers $n \geq 2$, we have:

$$\frac{1}{n^2\log(n)}\leq\frac{1}{n^2}.$$

Since the series definition of $\zeta(2)$ converges by Question 4(b), we have by the comparison test that the original series converges. All terms of the series are positive, so the series converges absolutely.

(l) The comparison test is appropriate here. Note that for all positive integers $n \geq 1$, we have:

$$\frac{n^2+2n}{n^3+3n^2+1} \geq \frac{n^2+2n}{n^3+3n^2+n} = \frac{n+2}{n^2+3n+1} \geq \frac{n+2}{n^2+3n+2} = \frac{n+2}{(n+2)(n+1)} = \frac{1}{n+1}.$$

The harmonic series diverges, hence the original series diverges too.

(m) The comparison test is appropriate here. Observe that:

$$\frac{1}{1+n^2} \leq \frac{1}{n^2}.$$

Since the series definition of $\zeta(2)$ converges by Question 4(b), we have by the comparison test that the original series converges. All terms of the series are positive, so the series converges absolutely.

(n) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{a^{2n+3}}{2n+3} \cdot \frac{2n+1}{a^{2n+1}} = a^2 \lim_{n\to\infty} \frac{2+1/n}{2+3/n} = a^2.$$

Hence if $a^2 < 1$, i.e. $a < 1$, we have that the series converges. All terms are positive in this case, so the series converges absolutely too. If $a^2 > 1$, i.e. $a > 1$, we have that the series diverges.

The case $a = 1$ cannot be dealt with by the ratio test. Instead, we use the comparison test. Observe that:

$$\frac{1}{2n+1} \geq \frac{1}{2n+2} = \frac{1}{2} \cdot \frac{1}{n+1}.$$

But the harmonic series diverges, hence the series with $a = 1$ must diverge too.

(o) The comparison test is appropriate here. Since $|\sin(\omega n)| \leq 1$, we have:

$$n^p|\sin(\omega n)| \leq n^p.$$

For $p < -1$, by Question 4(b), the series with terms on the right hand side converges. Hence the original series is absolutely convergent for $p < -1$, and hence convergent.[1]

(p) This is a bit of a trick question; note that $\cos((2n-1)\pi) = -1$. Hence the series is just the negative harmonic series, which diverges.

(q) The divergence test is appropriate here (the numerator of the terms grows exponentially, while the denominator of the terms grows only polynomially). We have:

$$\lim_{n\to\infty} \frac{2^n}{n\log(n)} = \lim_{x\to\infty} \frac{e^{x\log(2)}/x}{\log(x)},$$

since the limit of this sequence must equal the limit of the real function on the right hand side. The numerator on the right hand side approaches ∞, because exponential growth is faster than polynomial growth, a fact we proved on the previous Examples Sheet. The denominator on the right hand side also approaches ∞. Hence by L'Hôpital's rule, this limit equals:

$$\lim_{x\to\infty} \frac{\log(2)e^{x\log(2)}/x - e^{x\log(2)}/x^2}{1/x} = \lim_{x\to\infty}\left(\log(2) - \frac{1}{x}\right)e^{x\log(2)} = \infty.$$

Hence the nth term of the series does not converge, so the original series is divergent.

[1] Note that the case $p = -1$ is actually *very* hard to deal with - it requires another test, called Dirichlet's test, which you can look up online. It turns out that the series converges when $p = -1$, but is not absolutely convergent. The series diverges when $p > -1$.

(r) The comparison test is appropriate here. Observe that:

$$\frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2},$$

since $|\sin(n)| \leq 1$. The series with terms on the right hand side converges by Question 4(b), hence the original series is absolutely convergent. Absolute convergence implies convergence, so the original series is also convergent.

(s) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{(n+1)^3}{\log^{n+1}(2)} \cdot \frac{\log^n(2)}{n^3} = \frac{1}{\log(2)} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3 = \frac{1}{\log(2)} < 1.$$

Hence the series is convergent. All terms of the series are positive, so the series converges absolutely.

(t) We can use a nice trick here. Observe that:

$$\sqrt{n^4 + a^2} - n^2 = \frac{(\sqrt{n^4 + a^2} - n^2)(\sqrt{n^4 + a^2} + n^2)}{\sqrt{n^4 + a^2} + n^2} = \frac{n^4 + a^2 - n^4}{\sqrt{n^4 + a^2} + n^2} = \frac{a^2}{\sqrt{n^4 + a^2} + n^2}.$$

Next, note that:

$$\frac{a^2}{\sqrt{n^4 + a^2} + n^2} \leq \frac{a^2}{\sqrt{n^4} + n^2} = \frac{a^2}{2n^2}.$$

But the series with these terms converges by Question 4(b), hence by the comparison test the original series in this question converges. Note that all the terms of the original series are also positive, so the series converges absolutely.

(u) The ratio test is appropriate here. Observe that:

$$\lim_{n\to\infty} \frac{n+1}{2^{n+1} - 1} \cdot \frac{2^n - 1}{n} = \lim_{n\to\infty} \frac{1/2 - 1/2^{n+1}}{1 - 1/2^{n+1}} \cdot \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \frac{1}{2} < 1.$$

Hence, the series converges. All terms of the series are positive, so the series converges absolutely.

(v) The ratio test is appropriate here. Observe that:

$$\begin{aligned}
\lim_{n\to\infty} \frac{((n+1)!)^3 e^{3(n+1)}}{(3(n+1))!} \cdot \frac{(3n)!}{(n!)^3 e^{3n}} &= e^3 \lim_{n\to\infty} \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \\
&= e^3 \lim_{n\to\infty} \frac{1 + 1/n}{3 + 3/n} \cdot \frac{1 + 1/n}{3 + 2/n} \cdot \frac{1 + 1/n}{3 + 1/n} \\
&= \frac{e^3}{27}.
\end{aligned}$$

Since $e < 3$, this fraction is less than 1. Hence, the series converges. All terms of the series are positive, so the series converges absolutely.

Taylor series

9. Carefully state *Taylor's theorem*, giving Lagrange's formula for the remainder term. Hence, obtain the first three non-zero terms in the Taylor series of $\log(x)$ about $x = 1$ by direct differentiation. Using this expansion, together with Lagrange's form of the remainder, show that:

$$|\log(3/2) - 5/12| \leq 1/64,$$

and hence give an approximation of $\log(3/2)$ valid to one decimal place.

➾ **Solution:** Taylor's theorem states that for a real function f which is $(n+1)$-times differentiable about the point x_0, we have:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_{n+1},$$

where the remainder R_{n+1} is given by:

$$R_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

for some point ξ between x and x_0.

For $\log(x)$, we have:

$$\frac{d}{dx}\log(x) = \frac{1}{x}, \qquad \frac{d^2}{dx^2}\log(x) = -\frac{1}{x^2}, \qquad \frac{d^3}{dx^3}\log(x) = \frac{2}{x^3}, \qquad \frac{d^4}{dx^4}\log(x) = -\frac{6}{x^4}.$$

Hence the Taylor series about $x = 1$, up to the third non-zero term, is given by:

$$\begin{aligned} \log(x) &= \log(1) + \frac{1}{1}(x-1) - \frac{1}{1^2 \cdot 2!}(x-1)^2 + \frac{2}{1^3 \cdot 3!}(x-1)^3 + R_4 \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + R_4, \end{aligned}$$

where the remainder term is given by:

$$R_4 = -\frac{1}{4\xi^4}(x-1)^4,$$

for some ξ between x and 1.

Put $x = 3/2$ in the above formula. Then:

$$\log\left(\frac{3}{2}\right) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 + R_4 = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} + R_4 = \frac{5}{12} + R_4.$$

It follows that:

$$\left|\log\left(\frac{3}{2}\right) - \frac{5}{12}\right| = |R_4|.$$

But note that for ξ satisfying $1 < \xi < 3/2$, we have $3/2 < 1/\xi < 1$, so:

$$|R_4| = \frac{1}{4\xi^4} \cdot \frac{1}{2^4} \leq \frac{1}{4(1)^4} \cdot \frac{1}{2^4} = \frac{1}{64},$$

as required. To one decimal place, $5/12$ is 0.4, which is within $1/64$ of $\log(3/2)$, so is a correct approximation to $\log(3/2)$ to within one decimal place.

10. Write down the Taylor series about $x = 0$ for the following functions, finding their range of convergence by appropriate tests in each case:

(a) e^x, (b) $\log(1+x)$, (c) $\sin(x)$, (d) $\cos(x)$, (e) $\sinh(x)$, (f) $\cosh(x)$, (g) $(1+x)^a$,

where in the final part $a \in \mathbb{R}$ is any real number (ignore endpoints of the range of convergence in the final case, where convergence is subtle). What happens when a is a non-negative integer? Learn these series off by heart, and get your supervision partner to test you on them.

Solution:

(a) We have:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Consider:

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

Hence, by the ratio test, this series is absolutely convergent for all values of x, and hence convergent for all values of x.

(b) We have:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

Consider:

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} \cdot \frac{n}{|x|^n} = |x| \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = |x|.$$

By the ratio test, we see that the series is absolutely convergent for $|x| < 1$. Hence, the series converges for $|x| < 1$.

On the boundaries, we have $x = -1$ is the negative harmonic series $-1 - 1/2 - 1/3 - 1/4 - \ldots$, which diverges. On the other hand, at $x = 1$, we have the alternating series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

which has $1/n \to 0$ as $n \to \infty$, with $1/n$ monotonically decreasing. Hence the alternating series test implies that the series converges at $x = 1$. Thus the range of convergence is $-1 << x \leq 1$.

(c) We have:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Consider:

$$\lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{|x|^{2n+1}} = |x|^2 \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = 0.$$

Hence, by the ratio test this series is absolutely convergent (and thus convergent) for all values of x.

(d) We have:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Consider:

$$\lim_{n\to\infty} \frac{|x|^{2n+2}}{(2n+2)!} \frac{(2n)!}{|x|^{2n}} = |x|^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

Hence, by the ratio test this series is absolutely convergent (and thus convergent) for all values of x.

(e) We have:

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Similarly to part (c), this is convergent for all values of x.

(f) We have:

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Similarly to part (d), this is convergent for all values of x.

(g) We have:

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots = 1 + \sum_{n=1}^{\infty} \frac{a(a-1)(a-2)...(a-n+1)}{n!}x^n.$$

Consider:

$$\lim_{n\to\infty} \frac{|a(a-1)...(a-n)||x|^{n+1}}{(n+1)!} \frac{n!}{|a(a-1)...(a-n+1)||x|^n} = |x| \lim_{n\to\infty} \frac{|a-n|}{n+1} = |x| \lim_{n\to\infty} \frac{|1-a/n|}{1+1/n} = |x|.$$

Hence this series is absolutely convergent, and thus convergent, if $|x| < 1$. It may or may not converge at the end-points (it depends on the value of a, mostly, and is rather complicated - not something we need to worry about usually).

When a is a non-negative integer, the series terminates after finitely many terms (all terms afterwards are zero). This correspond to the case of the standard binomial expansion of $(1+x)^a$.

11. *Without differentiating*, find the first three terms in the Taylor series of the following functions. [*Note: there are lots of examples from past papers here to practise with, but if you are getting bored, we can do some in the supervision together. The next few questions, 12-17, have more of a problem-solving element.*]

(a) $\dfrac{1}{\sqrt{1+x}}$ about $x=0$;

(b) $\dfrac{1}{(x^2+2)^{3/2}}$ about $x=0$;

(c) $\tan(x)$ about $x=0$;

(d) $\log(\cos(x))$ about $x=0$;

(e) $\arcsin(x)$ about $x=0$;

(f) $\arctan(x)$ about $x=1$;

(g) $(\cosh(x))^{-1/2}$ about $x=0$;

(h) $e^{\sin(x)}$ about $x=\pi/2$;

(i) $x\sinh(x^2)$ about $x=0$;

(j) $\log(1+\log(1+x))$ about $x=0$;

(k) $\sin^6(x)$ about $x=0$;

(l) $\dfrac{\cosh(x)}{\cos(x)}$ about $x=0$;

(m) $\cosh(\log(x))$ about $x=2$;

(n) $\log(2-e^x)$ about $x=0$;

(o) $\dfrac{\sin(x)}{\sinh(x)}$ about $x=0$;

(p) $\sinh(\log(x))$ about $x=1$;

(q) $\sin\left(\dfrac{\pi e^x}{2}\right)$ about $x=0$;

(r) $\dfrac{\sinh(x+1)}{x+2}$ about $x=-1$;

(s) $\dfrac{\log(1+x^3)}{\cosh(x)}$ about $x=0$;

(t) $\dfrac{\cosh(x)}{\sqrt{1+x^2}}$ about $x=0$;

(u) $\dfrac{e^{-x^2}}{\cosh(x)}$ about $x=0$;

(v) $\dfrac{\log(2+x)}{2-x}$ about $x=0$;

(w) $\log(\cosh(x))$ about $x=0$;

(x) $\cosh(\sqrt{x})$ about $x=2$;

(y) $\dfrac{\sin(x)}{(1+x)^2}$ about $x=0$;

(z) $\dfrac{x\sin(x)}{\log(1+x^2)}$ about $x=0$;

(a′) $\cos\left(\sqrt{\dfrac{\pi^2}{16}+x}\right)$ about $x=0$;

(b′) $\log((2+x)^3)$ about $x=0$.

Solution:

(a) Using the binomial expansion, we have:

$$\frac{1}{\sqrt{1+x}}=(1+x)^{-1/2}=1-\frac{1}{2}x+\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^2+\cdots=1-\frac{1}{2}x+\frac{3}{8}x^2+\cdots.$$

(b) Using the binomial expansion, we have:

$$\begin{aligned}\frac{1}{(x^2+2)^{3/2}}=\frac{1}{2^{3/2}}\left(1+\frac{x^2}{2}\right)^{-3/2}&=\frac{1}{2^{3/2}}\left(1-\frac{3x^2}{4}+\frac{1}{2!}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x^2}{2}\right)^2+\cdots\right)\\&=\frac{1}{2^{3/2}}\left(1-\frac{3x^2}{4}+\frac{15}{32}x^4+\cdots\right)\\&=\frac{1}{2^{3/2}}-\frac{3x^2}{4\cdot 2^{3/2}}+\frac{15}{32\cdot 2^{3/2}}x^4+\cdots.\end{aligned}$$

(c) Observe that:

$$\begin{aligned}
\tan(x) = \frac{\sin(x)}{\cos(x)} &= \frac{x - x^3/3! + x^5/5! - \cdots}{1 - x^2/2! + x^4/4! - \cdots} \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^{-1} \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(-\frac{x^2}{2!} + \cdots\right)^2 + \cdots\right) \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{5}{24}x^4 + \cdots\right) \\
&= x + \left(\frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(\frac{5}{24} - \frac{1}{2 \cdot 6} + \frac{1}{5!}\right)x^5 + \cdots \\
&= x + \frac{1}{3}x^3 + \left(\frac{25}{120} - \frac{10}{120} + \frac{1}{120}\right)x^5 + \cdots \\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots .
\end{aligned}$$

(d) Observe that:

$$\begin{aligned}
\log(\cos(x)) &= \log\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \\
&= \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 + \frac{1}{3}\left(-\frac{x^2}{2!} + \cdots\right)^3 \\
&= -\frac{1}{2}x^2 + \left(\frac{1}{4!} - \frac{1}{8}\right)x^4 + \left(-\frac{1}{6!} + \frac{2}{2 \cdot 2! \cdot 4!} - \frac{1}{3 \cdot 2^3}\right)x^6 + \cdots \\
&= -\frac{1}{2}x^2 - \frac{1}{12}x^4 + \left(-\frac{1}{720} + \frac{1}{48} - \frac{1}{24}\right)x^6 + \cdots \\
&= -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 + \cdots .
\end{aligned}$$

(e) This one involves a bit of a trick. Since $\arcsin(x)$ differentiates to $1/\sqrt{1-x^2}$ (we can prove this using the reciprocal rule, for example, just as you did on Examples Sheet 4), we have:

$$\begin{aligned}
\arcsin(x) = \int \frac{dx}{\sqrt{1-x^2}} &= \int \left(1 - x^2\right)^{-1/2}\, dx \\
&= \int \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + \cdots\right) dx \\
&= x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots .
\end{aligned}$$

There is no constant of integration, because $\arcsin(0) = 0$.

(f) This is similar to the previous part, but we need to be careful because the expansion is around $x = 1$ instead of around $x = 0$. We have:

$$\arctan(x) = \int \frac{dx}{1+x^2}.$$

Rewriting the integrand in terms of $x-1$, the small quantity we wish to expand around, we have:

$$\frac{1}{1+x^2} = \frac{1}{1+(x-1)^2+2x-1} = \frac{1}{(x-1)^2+2x} = \frac{1}{(x-1)^2+2(x-1)+2} = \frac{1}{2}\left(1+(x-1)+\frac{(x-1)^2}{2}\right)^{-1}.$$

Performing the binomial expansion of the integrand, we have:

$$\begin{aligned}
\frac{1}{2}\left(1+(x-1)+\frac{(x-1)^2}{2}\right)^{-1} &= \frac{1}{2}\left(1-\left((x-1)-\frac{(x-1)^2}{2}+\cdots\right)+((x-1)+\cdots)^2+\cdots\right) \\
&= \frac{1}{2}\left(1-(x-1)+\frac{3}{2}(x-1)^2+\cdots\right) \\
&= \frac{1}{2}-\frac{(x-1)}{2}+\frac{3(x-1)^2}{4}+\cdots.
\end{aligned}$$

Integrating term by term, we have:

$$\frac{(x-1)}{2}-\frac{(x-1)^2}{4}+\frac{3(x-1)^3}{12}+\cdots,$$

up to a constant of integration. At $x = 1$, the left hand side is $\arctan(1) = \pi/4$, so the constant of integration must be $\pi/4$. This gives:

$$\arctan(x) = \frac{\pi}{4}+\frac{1}{2}(x-1)-\frac{1}{4}(x-1)^2+\cdots.$$

(g) Observe that:

$$\begin{aligned}
(\cosh(x))^{-1/2} &= \left(1+\frac{x^2}{2}+\frac{x^4}{4!}+\cdots\right)^{-1/2} \\
&= \left(1-\frac{1}{2}\left(\frac{x^2}{2}+\frac{x^4}{4!}+\cdots\right)+\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x^2}{2}+\cdots\right)^2+\cdots\right) \\
&= 1-\frac{1}{4}x^2+\left(-\frac{1}{2\cdot 4!}+\frac{3}{2!\cdot 2^2\cdot 2^2}\right)x^4+\cdots \\
&= 1-\frac{1}{4}x^2+\frac{7}{96}x^4+\cdots.
\end{aligned}$$

(h) Note that $\sin(x) = \sin(x - \pi/2 + \pi/2) = \sin(x - \pi/2)\cos(\pi/2) + \cos(x - \pi/2)\sin(\pi/2) = \cos(x - \pi/2)$. Hence we have:

$$\begin{aligned}
e^{\sin(x)} = e^{\cos(x-\pi/2)} &= \exp\left(1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + \cdots\right) \\
&= e \cdot \exp\left(-\frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + \cdots\right) \\
&= e\left(1 + \left(-\frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + \cdots\right) + \frac{1}{2!}\left(-\frac{(x-\pi/2)^2}{2!} + \cdots\right)^2 + \cdots\right) \\
&= e\left(1 - \frac{(x-\pi/2)^2}{2} + \left(\frac{1}{4!} + \frac{1}{8}\right)(x-\pi/2)^4 + \cdots\right) \\
&= e - \frac{e}{2}(x-\pi/2)^2 + \frac{e}{6}(x-\pi/2)^4 + \cdots .
\end{aligned}$$

Note that in the second line, we needed to factor out e, because we know the expansion of e^u where u is small, but we do not know the expansion of e^{1+u} where u is small.

(i) We have:

$$x\sinh(x^2) = x\left(x^2 + \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} + \cdots\right) = x^3 + \frac{1}{6}x^7 + \frac{1}{120}x^{11} + \cdots .$$

(j) Observe that:

$$\begin{aligned}
\log(1 + \log(1+x)) &= \log\left(1 + x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) \\
&= \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) - \frac{1}{2}\left(x - \frac{x^2}{2} + \cdots\right)^2 + \frac{1}{3}(x + \cdots)^3 + \cdots \\
&= x + \left(-\frac{1}{2} - \frac{1}{2}\right)x^2 + \left(\frac{1}{3} + \frac{2}{4} + \frac{1}{3}\right)x^3 + \cdots \\
&= x - x^2 + \frac{7}{6}x^3 + \cdots .
\end{aligned}$$

(k) Combining the expansion of $\sin(x)$ with the binomial expansion, we have:

$$\begin{aligned}
\sin^6(x) &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^6 \\
&= x^6\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)^6 \\
&= x^6\left(1 + 6\left(-\frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) + 15\left(-\frac{x^2}{3!} + \cdots\right)^2 + \cdots\right) \\
&= x^6 - x^8 + \left(\frac{6}{5!} + \frac{15}{(3!)^2}\right)x^{10} + \cdots \\
&= x^6 - x^8 + \frac{7}{15}x^{10} + \cdots .
\end{aligned}$$

(l) Observe that:

$$
\begin{aligned}
\frac{\cosh(x)}{\cos(x)} &= \frac{1 + x^2/2! + x^4/4! + \cdots}{1 - x^2/2! + x^4/4! + \cdots} \\
&= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^{-1} \\
&= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(-\frac{x^2}{2!} + \cdots\right)^2 + \cdots\right) \\
&= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{5}{24}x^4 + \cdots\right) \\
&= 1 + x^2 + \left(\frac{5}{24} + \frac{1}{4} + \frac{1}{4!}\right)x^4 + \cdots \\
&= 1 + x^2 + \frac{1}{2}x^4 + \cdots .
\end{aligned}
$$

(m) In this problem, we want to expand around $x = 2$. Thus we should rewrite everything in terms of the 'small' quantity $x - 2$. We shall call this $u = x - 2$ for convenience. Note that:

$$
\begin{aligned}
\cosh(\log(x)) = \cosh(\log(x - 2 + 2)) &= \cosh(\log(u + 2)) \\
&= \cosh(\log(2) + \log(1 + u/2)) \\
&= \cosh\left(\log(2) + \frac{u}{2} - \frac{u^2}{8} + \frac{u^3}{24} + \cdots\right).
\end{aligned}
$$

This gets even more messy now. We want to expand something of the form $\cosh(\log(2) + v)$, where v is small. Using hyperbolic compound angle identities, we have:

$$
\cosh(\log(2) + v) = \cosh(\log(2))\cosh(v) + \sinh(\log(2))\sinh(v).
$$

For simplicity, also note that $\cosh(\log(2)) = \frac{1}{2}(2 + 1/2) = 5/4$ and $\sinh(\log(2)) = \frac{1}{2}(2 - 1/2) = 3/4$. Expanding, we now have:

$$
\begin{aligned}
\cosh(\log(x)) &= \frac{5}{4}\cosh\left(\frac{u}{2} - \frac{u^2}{8} + \frac{u^3}{24} + \cdots\right) + \frac{3}{4}\sinh\left(\frac{u}{2} - \frac{u^2}{8} + \frac{u^3}{24} + \cdots\right) \\
&= \frac{5}{4}\left(1 + \frac{1}{2}\left(\frac{u}{2} - \frac{u^2}{8}\cdots\right)^2 + \cdots\right) + \frac{3}{4}\left(\left(\frac{u}{2} - \frac{u^2}{8} + \cdots\right) + \frac{1}{3!}\left(\frac{u}{2} + \cdots\right)^3 + \cdots\right) \\
&= \frac{5}{4} + \frac{3u}{8} + \left(\frac{5}{4}\cdot\frac{1}{8} - \frac{3}{32}\right)u^2 + \cdots \\
&= \frac{5}{4} + \frac{3}{8}(x - 2) + \frac{1}{16}(x - 2)^2 + \cdots .
\end{aligned}
$$

(n) Expanding the exponential first, we have:

$$\begin{aligned}\log(2-e^x) &= \log\left(2-1-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\cdots\right)\\ &= \log\left(1-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\cdots\right)\\ &= \left(-x-\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right)-\frac{1}{2}\left(-x-\frac{x^2}{2!}+\cdots\right)^2+\frac{1}{3}\left(-x+\cdots\right)^3+\cdots\\ &= -x+\left(-\frac{1}{2}-\frac{1}{2}\right)x^2+\left(-\frac{1}{3!}-\frac{2}{2\cdot 2}-\frac{1}{3}\right)x^3+\cdots\\ &= -x-x^2-x^3+\cdots\,.\end{aligned}$$

(o) We have:

$$\begin{aligned}\frac{\sin(x)}{\sinh(x)} &= \frac{x-x^3/3!+x^5/5!+\cdots}{x+x^3/3!+x^5/5!+\cdots}\\ &= \left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)^{-1}\\ &= \frac{1}{x}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)\left(1+\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)^{-1}\\ &= \left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)\left(1-\left(\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)+\left(\frac{x^2}{3!}+\cdots\right)^2+\cdots\right)\\ &= \left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)\left(1-\frac{x^2}{3!}+\left(\frac{1}{3!^2}-\frac{1}{5!}\right)x^4+\cdots\right)\\ &= \left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)\left(1-\frac{x^2}{3!}+\frac{7}{360}x^4+\cdots\right)\\ &= 1-\frac{2}{3!}x^2+\left(\frac{7}{360}+\frac{1}{36}+\frac{1}{120}\right)x^4+\cdots\\ &= 1-\frac{1}{3}x^2+\frac{1}{18}x^4+\cdots\,.\end{aligned}$$

(p) In this problem, we want to expand around $x = 1$. Thus we should rewrite everything in terms of the 'small' quantity $x - 1$. We shall call this $u = x - 1$ for convenience. Note that:

$$\begin{aligned}
\sinh(\log(x)) = \sinh(\log(x-1+1)) &= \sinh(\log(1+u)) \\
&= \sinh\left(u - \frac{u^2}{2} + \frac{u^3}{3} + \cdots\right) \\
&= \left(u - \frac{u^2}{2} + \frac{u^3}{3} + \cdots\right) + \frac{1}{3!}\left(u - \frac{u^2}{2} + \cdots\right)^3 + \frac{1}{5!}\left(u + \cdots\right)^5 \\
&= u - \frac{u^2}{2} + \left(\frac{1}{3} + \frac{1}{6}\right)u^3 + \cdots \\
&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \cdots.
\end{aligned}$$

(q) Expanding the exponential first, we have:

$$\sin\left(\frac{\pi e^x}{2}\right) = \sin\left(\frac{\pi}{2} + \frac{\pi}{2}x + \frac{\pi}{4}x^2 + \frac{\pi}{12}x^3 + \cdots\right).$$

Now, we need to expand an expression of the form $\sin(\pi/2 + u)$, where u is a small quantity. By the compound angle formula, we have:

$$\sin(\pi/2 + u) = \sin(\pi/2)\cos(u) + \cos(\pi/2)\sin(u) = \cos(u).$$

Hence, we can rewrite the above as:

$$\begin{aligned}
\sin\left(\frac{\pi e^x}{2}\right) &= \cos\left(\frac{\pi}{2}x + \frac{\pi}{4}x^2 + \frac{\pi}{12}x^3 + \cdots\right) \\
&= 1 - \frac{1}{2!}\left(\frac{\pi}{2}x + \frac{\pi}{4}x^2 + \cdots\right)^2 + \frac{1}{4!}\left(\frac{\pi}{2}x + \cdots\right)^4 + \cdots \\
&= 1 - \frac{\pi^2}{8}x^2 - \frac{\pi^2}{8}x^3 + \cdots.
\end{aligned}$$

(r) In this problem, we want to expand around $x = -1$. Thus we should rewrite everything in terms of the 'small' quantity $x + 1$. We shall call this $u = x + 1$ for convenience. Note that:

$$\begin{aligned}
\frac{\sinh(x+1)}{x+2} &= \frac{\sinh(u)}{1+u} \\
&= \sinh(u)(1+u)^{-1} \\
&= \left(u + \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots\right)\left(1 - u + u^2 - u^3 + \cdots\right) \\
&= u - u^2 + \frac{7}{6}u^3 + \cdots \\
&= (x+1) - (x+1)^2 + \frac{7}{6}(x+1)^3 + \cdots.
\end{aligned}$$

(s) Observe that:

$$\begin{aligned}
\frac{\log(1+x^3)}{\cosh(x)} &= \frac{x^3 - x^6/2 + x^9/3 + \cdots}{1 + x^2/2! + x^4/4! + \cdots} \\
&= \left(x^3 - \frac{x^6}{2} + \frac{x^9}{3} + \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^{-1} \\
&= \left(x^3 - \frac{x^6}{2} + \frac{x^9}{3} + \cdots\right)\left(1 - \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(\frac{x^2}{2!} + \cdots\right)^2 + \cdots\right) \\
&= \left(x^3 - \frac{x^6}{2} + \frac{x^9}{3} + \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots\right) \\
&= x^3 - \frac{1}{2}x^5 - \frac{1}{2}x^6 + \cdots .
\end{aligned}$$

(t) Observe that:

$$\begin{aligned}
\frac{\cosh(x)}{\sqrt{1+x^2}} = \cosh(x)(1+x^2)^{-1/2} &= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \cdots\right) \\
&= 1 + \left(\frac{3}{8} - \frac{1}{4} + \frac{1}{4!}\right)x^4 + \left(-\frac{5}{16} + \frac{3}{16} - \frac{1}{4!\cdot 2} + \frac{1}{6!}\right)x^6 + \cdots \\
&= 1 + \frac{1}{6}x^4 - \frac{13}{90}x^6 + \cdots .
\end{aligned}$$

(u) Observe that:

$$\begin{aligned}
\frac{e^{-x^2}}{\cosh(x)} &= \frac{1 - x^2 + x^4/2! - x^6/3! + \cdots}{1 + x^2/2! + x^4/4! + \cdots} \\
&= \left(1 - x^2 + \frac{x^4}{2!} + \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^{-1} \\
&= \left(1 - x^2 + \frac{x^4}{2!} + \cdots\right)\left(1 - \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 + \cdots\right) \\
&= \left(1 - x^2 + \frac{x^4}{2!} + \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots\right) \\
&= 1 - \frac{3}{2}x^2 + \left(\frac{5}{24} + \frac{1}{2} + \frac{1}{2}\right)x^4 + \cdots \\
&= 1 - \frac{3}{2}x^2 + \frac{29}{24}x^4 + \cdots .
\end{aligned}$$

(v) Observe that:

$$\begin{aligned}\frac{\log(2+x)}{2-x} = \frac{\log(2)+\log(1+x/2)}{2(1-x/2)} &= \frac{1}{2}\left(\log(2)+\frac{x}{2}-\frac{x^2}{4}+\cdots\right)\left(1-\frac{x}{2}\right)^{-1} \\ &= \frac{1}{2}\left(\log(2)+\frac{x}{2}-\frac{x^2}{4}+\cdots\right)\left(1+\frac{x}{2}+\frac{x^2}{4}+\cdots\right) \\ &= \frac{1}{2}\left(\log(2)+\left(\frac{1}{2}+\frac{1}{2}\log(2)\right)x+\left(\frac{1}{4}\log(2)+\frac{1}{4}-\frac{1}{4}\right)x^2+\cdots\right) \\ &= \frac{1}{2}\log(2)+\frac{(1+\log(2))}{4}x+\frac{\log(2)}{8}x^2+\cdots.\end{aligned}$$

(w) We have:

$$\begin{aligned}\log(\cosh(x)) &= \log\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right) \\ &= \left(\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)-\frac{1}{2}\left(\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2+\frac{1}{3}\left(\frac{x^2}{2!}+\cdots\right)^3+\cdots \\ &= \frac{1}{2}x^2+\left(\frac{1}{24}-\frac{1}{8}\right)x^4+\left(\frac{1}{6!}-\frac{1}{2!4!}+\frac{1}{3\cdot 8}\right)x^6+\cdots \\ &= \frac{1}{2}x^2-\frac{1}{12}x^4+\frac{1}{45}x^6+\cdots.\end{aligned}$$

(x) In this problem, we want to expand around $x=2$. Thus we should rewrite everything in terms of the 'small' quantity $x-2$. We shall call this $u=x-2$ for convenience. Note that:

$$\cosh(\sqrt{x}) = \cosh(\sqrt{2+x-2}) = \cosh(\sqrt{2+u}) = \cosh\left(\sqrt{2}(1+u/2)^{1/2}\right)$$

Expanding using the binomial theorem, we have:

$$\cosh(\sqrt{2}(1+u/2)^{1/2}) = \cosh\left(\sqrt{2}\left(1+\frac{u}{4}-\frac{u^2}{32}+\cdots\right)\right).$$

We don't the expansion of $\cosh(\sqrt{2}+v)$ for small v, so we now use the hyperbolic compound angle identities to give:

$$\cosh(\sqrt{2}+v) = \cosh(\sqrt{2})\cosh(v)+\sinh(\sqrt{2})\sinh(v).$$

Applying this to the above, we have:

$$\begin{aligned}\cosh\left(\sqrt{2}(1+u/2)^{1/2}\right) &= \cosh(\sqrt{2})\cosh\left(\frac{\sqrt{2}u}{4}-\frac{\sqrt{2}u^2}{32}+\cdots\right)+\sinh(\sqrt{2})\sinh\left(\frac{\sqrt{2}u}{4}-\frac{\sqrt{2}u^2}{32}+\cdots\right) \\ &= \cosh(\sqrt{2})\left(1-\frac{1}{2!}\left(\frac{\sqrt{2}u}{4}\right)^2+\cdots\right)+\sinh(\sqrt{2})\left(\frac{\sqrt{2}u}{4}-\frac{\sqrt{2}u^2}{32}+\cdots\right) \\ &= \cosh(\sqrt{2})+\frac{\sqrt{2}\sinh(\sqrt{2})}{4}u-\left(\frac{\sqrt{2}\sinh(\sqrt{2})}{32}+\frac{\cosh(\sqrt{2})}{16}\right)u^2+\cdots \\ &= \cosh(\sqrt{2})+\frac{\sqrt{2}\sinh(\sqrt{2})}{4}(x-2)-\left(\frac{\sqrt{2}\sinh(\sqrt{2})}{32}+\frac{\cosh(\sqrt{2})}{16}\right)(x-2)^2+\cdots.\end{aligned}$$

(y) We have:

$$\frac{\sin(x)}{(1+x)^2} = \sin(x)(1+x)^{-2} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)(1 - 2x + 3x^2 + \cdots)$$

$$= x - 2x^2 + \left(3 - \frac{1}{3!}\right)x^3 + \cdots$$

$$= x - 2x^2 + \frac{17}{6}x^3 + \cdots .$$

(z) We have:

$$\frac{x\sin(x)}{\log(1+x^2)} = \frac{x(x - x^3/3! + x^5/5! + \cdots)}{x^2 - x^4/2 + x^6/3 + \cdots}$$

$$= \frac{1 - x^2/3! + x^4/5! + \cdots}{1 - x^2/2 + x^4/3 + \cdots}$$

$$= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{3} + \cdots\right)^{-1}$$

$$= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)\left(1 - \left(-\frac{x^2}{2} + \frac{x^4}{3} + \cdots\right) + \left(-\frac{x^2}{2} + \cdots\right)^2 + \cdots\right)$$

$$= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)\left(1 + \frac{x^2}{2} - \frac{x^4}{12} + \cdots\right)$$

$$= 1 + \frac{1}{3}x^2 + \left(-\frac{1}{12} - \frac{1}{12} + \frac{1}{120}\right)x^4 + \cdots$$

$$= 1 + \frac{1}{3}x^2 - \frac{19}{120}x^4 + \cdots .$$

(a′) Using the binomial expansion first, we have:

$$\cos\left(\sqrt{\frac{\pi^2}{16} + x}\right) = \cos\left(\frac{\pi}{4}\left(1 + \frac{16x}{\pi^2}\right)^{1/2}\right)$$

$$= \cos\left(\frac{\pi}{4}\left(1 + \frac{8x}{\pi^2} - \frac{32x^2}{\pi^4} + \cdots\right)\right)$$

$$= \cos\left(\frac{\pi}{4} + \frac{2x}{\pi} - \frac{8x^2}{\pi^3} + \cdots\right).$$

We don't know the expansion of $\cos(\pi/4 + u)$, where u is small, so we now use the compound angle identities for the trigonometric functions:

$$\cos(\pi/4 + u) = \cos(\pi/4)\cos(u) - \sin(\pi/4)\sin(u) = \frac{1}{\sqrt{2}}(\cos(u) - \sin(u)).$$

Applying this to the above expansion, we have:

$$\begin{aligned}
\cos\left(\sqrt{\frac{\pi^2}{16}+x}\right) &= \frac{1}{\sqrt{2}}\left(\cos\left(\frac{2x}{\pi}-\frac{8x^2}{\pi^3}+\cdots\right)-\sin\left(\frac{2x}{\pi}-\frac{8x^2}{\pi^3}\right)\right)\\
&= \frac{1}{\sqrt{2}}\left(1-\frac{1}{2!}\left(\frac{2x}{\pi}+\cdots\right)^2-\left(\frac{2x}{\pi}-\frac{8x^2}{\pi^3}+\cdots\right)+\cdots\right)\\
&= \frac{1}{\sqrt{2}}\left(1-\frac{2x}{\pi}+\left(\frac{2}{\pi^3}-\frac{8}{\pi^2}\right)x^2+\cdots\right)\\
&= \frac{1}{\sqrt{2}}-\frac{\sqrt{2}x}{\pi}+\frac{\sqrt{2}(4-\pi)}{\pi^3}x^2+\cdots.
\end{aligned}$$

(b′) Finally, we have:

$$\begin{aligned}
\log((2+x)^3) = 3\log(2+x) &= 3\log(2)+3\log(1+x/2)\\
&= 3\log(2)+3\left(\frac{x}{2}-\frac{x^2}{8}+\cdots\right)\\
&= 3\log(2)+\frac{3x}{2}-\frac{3x^2}{8}+\cdots.
\end{aligned}$$

We finished on an easy one!

12. *Without differentiating*, find the value of the thirty-second derivative of $\cos(x^4)$ at $x = 0$.

➾ **Solution:** Using the standard Taylor series for cosine about $x = 0$, we have:

$$\begin{aligned}\cos(x^4) &= 1 - \frac{(x^4)^2}{2!} + \frac{(x^4)^4}{4!} - \frac{(x^4)^6}{6!} + \frac{(x^4)^8}{8!} - \cdots \\ &= 1 - \frac{x^8}{2!} + \frac{x^{16}}{4!} + \frac{x^{24}}{6!} + \frac{x^{32}}{8!} - \cdots .\end{aligned}$$

But recall that the coefficient of x^{32} in the Taylor expansion of $\cos(x^4)$ about $x = 0$ is given by:

$$\frac{1}{32!}\frac{d^{32}}{dx^{32}}\cos(x^4)\bigg|_{x=0}.$$

Hence the value of the required derivative is:

$$\frac{d^{32}}{dx^{32}}\cos(x^4)\bigg|_{x=0} = \frac{32!}{8!}.$$

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13. Find the first three non-zero terms in a series approximation of $\log(1 + x + 2x^2) - \log(x^2)$ valid for $x \to \infty$.

➾ **Solution:** When $x \to \infty, 1/x \to 0$. Hence we should write everything in terms of $1/x$, and expand assuming that $1/x$ is close to zero.

We have:

$$\begin{aligned}\log(1 + x + 2x^2) - \log(x^2) &= \log(2x^2) + \log\left(1 + \frac{1}{2x} + \frac{1}{2x^2}\right) - \log(x^2) \\ &= \log(2) + \log\left(1 + \frac{1}{2x} + \frac{1}{2x^2}\right) \\ &= \log(2) + \left(\frac{1}{2x} + \frac{1}{2x^2}\right) - \frac{1}{2}\left(\frac{1}{2x} + \frac{1}{2x^2}\right)^2 + \cdots \quad \text{(Taylor series for } \log(1+u) \text{ about } u = 0\text{)} \\ &= \log(2) + \frac{1}{2x} + \frac{1}{2x^2} - \frac{1}{2}\left(\frac{1}{2x}\right)^2 + \cdots \\ &= \log(2) + \frac{1}{2x} + \frac{3}{8x^2} + \cdots .\end{aligned}$$

14. Let $f(x)$ be a function which can be expanded as a Taylor series. Find the first two terms in the Taylor series of the function $\log(1+f(x))$, assuming that $1+f(0)>0$, $f'(0)\neq 0$ and $f''(0)(1+f(0))\neq (f'(0))^2$. Why are these conditions necessary?

➾ **Solution:** The Taylor series of $f(x)$ about $x=0$ is given by:

$$f(x)=f(0)+f'(0)x+\frac{f^{(2)}(0)}{2!}x^2+\cdots .$$

Inserting this into $\log(1+f(x))$, we have:

$$\log\left(1+f(x)\right)=\log\left(1+f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots\right).$$

We know the Taylor series of $\log(1+u)$ about $u=0$. So we need to rewrite our logarithm in this form, where u is a quantity close to 0. To do so, we factor out $1+f(0)$ from the argument of the logarithm, giving:

$$\log\left(1+f(0)\right)+\log\left(1+\frac{f'(0)}{1+f(0)}x+\frac{f''(0)}{2!(1+f(0))}x^2+\cdots\right).$$

We can now expand the second logarithmic term in a Taylor series, since it is of the form $\log(1+u)$ where u is a quantity close to 0. We have:

$$\begin{aligned}
&\log\left(1+f(0)\right)+\left(\frac{f'(0)}{1+f(0)}x+\frac{f''(0)}{2!(1+f(0))}x^2+\cdots\right)-\frac{1}{2}\left(\frac{f'(0)}{1+f(0)}x+\cdots\right)^2+\cdots\\
&=\log(1+f(0))+\frac{f'(0)}{1+f(0)}x+\left(\frac{f''(0)}{2!(1+f(0))}-\frac{1}{2}\frac{(f'(0))^2}{(1+f(0))^2}\right)x^2+\cdots\\
&=\log(1+f(0))+\frac{f'(0)}{1+f(0)}x+\frac{1}{2}\left(\frac{f''(0)(1+f(0))-(f'(0))^2}{(1+f(0))^2}\right)x^2+\cdots .
\end{aligned}$$

We need $1+f(0)>0$ for the logarithm to exist. We need $f'(0)\neq 0$, else the first term in the expansion would vanish (and we would have to calculate to higher order!). Similarly, we need $f''(0)(1+f(0))\neq (f'(0))^2$, else the second term in the expansion would vanish (and we would have to calculate to higher order).

15. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ be the Taylor series of $f(x)$ about $x = 0$, with $a_0 > 0$, $a_1 \neq 0$, $a_1^2 \neq a_2 a_0$ and $a_1^2 \neq 4a_2 a_0$. Find the first three terms in the Taylor series of (a) $1/f(x)$ about $x = 0$; (b) $\sqrt{f(x)}$ about $x = 0$. Explain where you used the assumptions on the a_n in your answer.

➺ **Solution:** (a) We have:

$$\begin{aligned}
\frac{1}{f(x)} &= \frac{1}{a_0 + a_1 x + a_2 x^2 + \cdots} \\
&= \frac{1}{a_0}\left(1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \cdots\right)^{-1} \\
&= \frac{1}{a_0}\left(1 - \left(\frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \cdots\right) + \left(\frac{a_1}{a_0}x + \cdots\right)^2 + \cdots\right) \\
&= \frac{1}{a_0}\left(1 - \frac{a_1}{a_0}x + \left(\frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}\right)x^2 + \cdots\right) \\
&= \frac{1}{a_0} - \frac{a_1}{a_0^2}x + \frac{(a_1^2 - a_2 a_0)}{a_0^3}x^2 + \cdots .
\end{aligned}$$

The coefficients are all non-zero since $a_0 > 0$, $a_1 \neq 0$ and $a_1^2 \neq a_2 a_0$.

(b) We have:

$$\begin{aligned}
\sqrt{f(x)} &= \left(a_0 + a_1 x + a_2 x^2 + \cdots\right)^{1/2} \\
&= a_0^{1/2}\left(1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \cdots\right)^{1/2} \\
&= a_0^{1/2}\left(1 + \frac{1}{2}\left(\frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \cdots\right) + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{a_1}{a_0}x + \cdots\right)^2 + \cdots\right) \\
&= a_0^{1/2}\left(1 + \frac{a_1}{2a_0}x + \left(\frac{a_2}{2a_0} - \frac{a_1^2}{8a_0^2}\right)x^2 + \cdots\right) \\
&= a_0^{1/2} + \frac{a_1}{2a_0^{1/2}}x + \frac{(4a_2 a_0 - a_1^2)}{8a_0^{3/2}}x^2 + \cdots .
\end{aligned}$$

The coefficients are all non-zero since $a_0 > 0$ (indeed, this implies the square root of a_0 exists!), $a_1 \neq 0$ and $a_1^2 \neq 4a_2 a_0$.

16. (†) By considering a Taylor series expansion in each case, evaluate the limits:

$$\text{(a) } \lim_{x\to 0}\frac{\tan(x)-\tanh(x)}{\sinh(x)-x}, \qquad \text{(b) } \lim_{x\to 0}\left(\frac{\operatorname{cosec}(x)}{x^3}-\frac{\sinh(x)}{x^5}\right).$$

➾ **Solution:** (a) For the first limit, we observe that the denominator has the expansion:

$$\sinh(x)-x = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots - x = \frac{x^3}{3!}+\frac{x^5}{5!}+\cdots .$$

Meanwhile, tangent has the expansion:

$$\begin{aligned}
\tan(x) = \frac{\sin(x)}{\cos(x)} &= \frac{x-x^3/3!+\cdots}{1-x^2/2!+\cdots}\\
&= \left(x-\frac{x^3}{3!}+\cdots\right)\left(1-\frac{x^2}{2!}+\cdots\right)^{-1}\\
&= \left(x-\frac{x^3}{3!}+\cdots\right)\left(1+\frac{x^2}{2!}+\cdots\right)\\
&= x+\left(\frac{1}{2!}-\frac{1}{3!}\right)x^3+\cdots\\
&= x+\frac{1}{3}x^3+\cdots .
\end{aligned}$$

Similarly, hyperbolic tangent has the expansion:

$$\begin{aligned}
\tanh(x) = \frac{\sinh(x)}{\cosh(x)} &= \frac{x+x^3/3!+\cdots}{1+x^2/2!+\cdots}\\
&= \left(x+\frac{x^3}{3!}+\cdots\right)\left(1+\frac{x^2}{2!}+\cdots\right)^{-1}\\
&= \left(x+\frac{x^3}{3!}+\cdots\right)\left(1-\frac{x^2}{2!}+\cdots\right)\\
&= x+\left(\frac{1}{3!}-\frac{1}{2!}\right)x^3+\cdots\\
&= x-\frac{1}{3}x^3+\cdots .
\end{aligned}$$

Putting this altogether, we have:

$$\frac{\tan(x)-\tanh(x)}{\sinh(x)-x} = \frac{x+x^3/3+\cdots - x + x^3/3+\cdots}{x^3/3!+x^5/5!+\cdots} = \frac{2x^3/3+\cdots}{x^3/6+\cdots} = \frac{2/3+\cdots}{1/6+\cdots}.$$

In the limit as $x\to 0$, the terms we have neglected vanish. Thus the limit is:

$$\frac{2}{3}\cdot 6 = 4.$$

(b) We have:

$$\begin{aligned}
\frac{\operatorname{cosec}(x)}{x^3} - \frac{\sinh(x)}{x^5} &= \frac{1}{x^3\sin(x)} - \frac{\sinh(x)}{x^5} \\
&= \frac{1}{x^3(x - x^3/3! + x^5/5! + ...)} - \frac{x + x^3/3! + x^5/5! + ...}{x^5} \\
&= \frac{1}{x^4}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)^{-1} - \frac{1}{x^4}\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) \\
&= \frac{1}{x^4}\left(1 - \left(-\frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) + \left(-\frac{x^2}{3!} + \cdots\right)^2 + \cdots\right) - \frac{1}{x^4}\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) \\
&= \frac{1}{x^4}\left(1 + \frac{x^2}{3!} + \left(\frac{1}{3!^2} - \frac{1}{5!}\right)x^4 + \cdots\right) - \frac{1}{x^4}\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) \\
&= \left(\frac{1}{3!^2} - \frac{1}{5!}\right) - \frac{1}{5!} + \cdots \\
&= \frac{1}{90} + \cdots .
\end{aligned}$$

The terms that are remaining tend to zero as $x \to 0$. Hence the required limit is $1/90$.

17.(a) Using the Taylor series for $\log(1+x)$, show that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots=\log(2)$.

(b) $(*)$ Hence, by an appropriate sequence of transformations of the series in part (a), show that:

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots=\frac{3}{2}\log(2).$$

(c) $(*)$ Comment on this result in relation to absolute convergence. [*Look up the Riemann rearrangement theorem afterwards!*]

Solution: (a) The Taylor series for $\log(1+x)$ is:

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots.$$

Letting $x=1$, we have:

$$\log(2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots,$$

as required.

(b) This is just a trick:

$$\begin{array}{ccccccccc}
1 & -\frac{1}{2} & +\frac{1}{3} & -\frac{1}{4} & +\frac{1}{5} & -\frac{1}{6} & +\frac{1}{7} & -\frac{1}{8}+\cdots & =\log(2) \\
 & 1 & & -\frac{1}{2} & & +\frac{1}{3} & & -\frac{1}{4}+\cdots & =\log(2) \\
 & \frac{1}{2} & & -\frac{1}{4} & & +\frac{1}{6} & & -\frac{1}{8}+\cdots & =\frac{1}{2}\log(2)
\end{array}$$

In the second line, we have just aligned the terms in a slightly different way. In the third line, we have multiplied everything in the second line by $1/2$. Now consider adding the first and third lines:

$$\begin{array}{ccccccccc}
1 & & +\frac{1}{3} & -\frac{1}{2} & +\frac{1}{5} & & +\frac{1}{7} & -\frac{1}{4}+\cdots & =\frac{3}{2}\log(2)
\end{array}$$

Hence, the result follows.

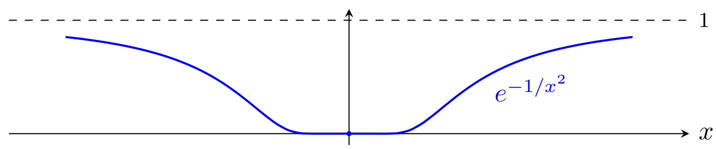
(c) This is allowed to occur here because the series is not *absolutely convergent*. If a series is absolutely convergent, we can rearrange its terms arbitrarily and still obtain the same infinite sum. If a series is not absolutely convergent, but still converges, then the *Riemann rearrangement theorem* says that we can rearrange its terms arbitrarily to obtain *any* value we wish via some rearrangement!

18. ($*$) Sketch the graph of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-1/x^2}$ for $x \neq 0$, and $f(0) = 0$. Show that this function is infinitely differentiable at $x = 0$, and determine its Taylor series at $x = 0$. Comment on the general utility of Taylor series.

➾ **Solution:** As $x \to \pm\infty$, the graph goes to 1, since $-1/x^2 \to 0$. Further, the graph is everywhere positive, and since $-1/x^2 < 0$, we have $e^{-1/x^2} < 1$, so that the graph is everywhere less than 1 too. Stationary points occur when:

$$\frac{d}{dx}e^{-1/x^2} = \frac{2}{x^3}e^{-1/x^2} = 0,$$

which never happens, hence there are none. Thus the graph takes the 'bowl' shape given below:



Away from zero, we claim that the nth derivative of the function is of the form:

$$p_n\left(\frac{1}{x}\right)e^{-1/x^2},$$

where p_n is some polynomial. To prove this, we can use induction. Certainly the function itself is of this form, so the case $n = 0$ works. Assuming the result is true for $n = k$, we consider the case $n = k + 1$:

$$\frac{d}{dx}\left(p_k\left(\frac{1}{x}\right)e^{-1/x^2}\right) = -\frac{1}{x^2}p_k'\left(\frac{1}{x}\right)e^{-1/x^2} + \frac{2}{x^3}p_k\left(\frac{1}{x}\right)e^{-1/x^2} = \left(\frac{2}{x^3}p_k\left(\frac{1}{x}\right) - \frac{1}{x^2}p_k'\left(\frac{1}{x}\right)\right)e^{-1/x^2},$$

so indeed the $(k+1)$th derivative is also of this form, if the kth derivative is. We see that the polynomials that multiply e^{-1/x^2} satisfy the recurrence relation $2y^3p_k(y) - y^2p_k'(y) = p_{k+1}(y)$.

Next, we claim that the nth derivative of the function at $x = 0$ is 0. We shall prove this with induction too. Certainly this is true of the function itself, since $f(0) = 0$, so the case $n = 0$ works. Now assuming the result is true for $n = k$, we note that the $(k+1)$th derivative at $x = 0$ is given by:

$$\lim_{h\to 0}\left[\frac{p_k(1/h)e^{-1/h^2} - 0}{h}\right] = \lim_{h\to 0}\left[\frac{p_k(1/h)/h}{e^{1/h^2}}\right].$$

This limit is zero. To see why, consider making the substitution $u = 1/h$ in the limit, and taking the limit as $u \to \pm\infty$ and as $u \to -\infty$ to give the left and right limits as $h \to 0$. Both of these limits correspond to taking the limit of a polynomial $p_k(u)u$ over an exponential e^{u^2}, which by repeated use of L'Hôpital's rule will result in zero. Hence, the result is proved by induction.

It follows that indeed the function is infinitely differentiable at $x = 0$, as required.

Using the previous results, the Taylor series of this function about $x = 0$ is:

$$0 + 0x + \frac{0x^2}{2!} + \frac{0x^3}{3!} + \cdots \equiv 0.$$

Whoops! The Taylor series is identically equal to zero. This is a *useless* approximation to the function in a neighbourhood of $x = 0$. In general, a Taylor series may converge, but might not converge to the original function (here the Taylor series is convergent everywhere, because it is just a sum of zeroes, but only equals the original function when $x = 0$).

(†) Landau's big-O notation

19. Give the formal definition of Landau's big-O notation, '$f(x) = O(g(x))$ as $x \to x_0$', including the cases where $x_0 = \pm\infty$. Decide which of the following statements are true, justifying your reasoning with careful proofs:

(a) $x = O(x^2)$ as $x \to 0$, (b) $x^2 = O(x)$ as $x \to 0$, (c) $x = O(x^2)$ as $x \to \infty$, (d) $x^2 = O(x)$ as $x \to \infty$.

Solution: We say that $f(x) = O(g(x))$ as $x \to x_0$ if there exist $\delta > 0, M > 0$ such that whenever $0 < |x - x_0| < \delta$, we have:

$$|f(x)| \leq M|g(x)|.$$

This definition is saying that there is a neighbourhood of x_0 where the size of $f(x)$ is at most some constant M times the size of $g(x)$.

In the case of $x_0 = \infty$, we say that $f(x) = O(g(x))$ as $x \to \infty$ if there exist $K > 0, M > 0$ such that whenever $x > K$, we have:

$$|f(x)| \leq M|g(x)|.$$

This definition is saying that if we take x sufficiently large, the size of $f(x)$ is at most some constant M times the size of $g(x)$. A similar definition applies when $x_0 = -\infty$.

(a) As x gets close to zero from above, we have that x^2 is smaller. So we expect the statement to be false. To prove it, note that if x is positive, we have $x < Mx^2$ if and only if:

$$0 < x(Mx - 1) \qquad \Rightarrow \qquad x > \frac{1}{M}.$$

In particular, there is no neighbourhood of $x = 0$ on which the inequality $x < Mx^2$ holds. Hence, the statement is indeed false.

(b) This statement is true. Take $\delta = 1$ and $M = 1$. Then if $|x| < 1$, we have:

$$|x^2| = |x|^2 = |x| \cdot |x| < 1 \cdot |x|.$$

Thus $x^2 = O(x)$ as $x \to 0$.

(c) This statement is true. Take $K = 1$ and $M = 1$. Then if $x > 1$, we have:

$$|x| = 1 \cdot |x| < x \cdot |x| = |x^2|,$$

since $x > 0$.

(d) Similarly to part (a), this statement is false. To prove it, note that if x is positive, we have $x^2 < Mx$ if and only if:

$$x(x - M) < 0 \qquad \Rightarrow \qquad x < M.$$

In particular, for any constant M, there is always a point at which the growth of x^2 surpasses the growth of x (at the point M itself). Hence, the statement is indeed false.

20. Give the leading terms in an approximation to each of the following functions in the given limits, indicating the leading behaviour of the remainder in Landau's big-O notation:

(a) $\dfrac{x^3+x}{x+2}$ as $x \to 0$, (b) $\dfrac{\cos(x)-1}{x^3}$ as $x \to 0$, (c) $\dfrac{1+2x+2x^2}{3x+3}$ as $x \to \infty$.

➾ **Solution:** We will now be less formal with our proofs, and use Landau's big O-notation more intuitively.

(a) Using the binomial expansion around $x = 0$, we have:

$$\begin{aligned}\frac{x^3+x}{x+2} &= \frac{1}{2}(x^3+x)\left(1+\frac{x}{2}\right)^{-1} \\ &= \frac{1}{2}(x^3+x)\left(1-\frac{x}{2}+O(x^2)\right) \\ &= \frac{x}{2}+O(x^2).\end{aligned}$$

(b) Using the Taylor expansion of cosine around $x = 0$, we have:

$$\begin{aligned}\frac{\cos(x)-1}{x^3} &= \frac{1-x^2/2!+x^4/4!+O(x^6)-1}{x^3} \\ &= -\frac{1}{2x}+\frac{x}{4!}+O(x^3) \\ &= -\frac{1}{2x}+O(x).\end{aligned}$$

(c) Here, $1/x$ is the small quantity we should expand in as $x \to \infty$. Using the binomial expansion, we have:

$$\begin{aligned}\frac{1+2x+2x^2}{3x+3} &= \frac{1}{3x}\left(1+2x+2x^2\right)\left(1+\frac{1}{x}\right)^{-1} \\ &= \frac{1}{3x}\left(1+2x+2x^2\right)\left(1-\frac{1}{x}+\frac{1}{x^2}+O\left(\frac{1}{x^3}\right)\right) \\ &= \frac{1}{3x}\left(2x^2+2x-2x+2-2+1+O\left(\frac{1}{x}\right)\right) \\ &= \frac{1}{3x}\left(2x^2+O(1)\right) \\ &= \frac{2x}{3}+O\left(\frac{1}{x}\right).\end{aligned}$$

21. Show that:

$$(x^3+x^2+1)^{1/3}-(x^2+x)^{1/2}=-\frac{1}{6}+\frac{1}{72x}+O\left(\frac{1}{x^2}\right) \qquad \text{as } x\to\infty.$$

➡ **Solution:** Since we are taking the limit as $x \to \infty$, we will expand in powers of $1/x$. We have:

$$\begin{aligned}
(x^3+x^2+1)^{1/3}-(x^2+x)^{1/2} &= x\left(1+\frac{1}{x}+\frac{1}{x^3}\right)^{1/3}-x\left(1+\frac{1}{x}\right)^{1/2}\\
&= x\left(1+\frac{1}{3}\left(\frac{1}{x}+\frac{1}{x^3}\right)-\frac{1}{9}\left(\frac{1}{x}+\frac{1}{x^3}\right)^2+O\left(\frac{1}{x^3}\right)\right)-x\left(1+\frac{1}{2x}-\frac{1}{8x^2}+O\left(\frac{1}{x^3}\right)\right)\\
&= \left(\frac{1}{3}-\frac{1}{2}\right)+\left(-\frac{1}{9}+\frac{1}{8}\right)\frac{1}{x}+O\left(\frac{1}{x^2}\right)\\
&= -\frac{1}{6}+\frac{1}{72x}+O\left(\frac{1}{x^2}\right),
\end{aligned}$$

as required.

Newton-Raphson root finding

22. Give an explanation of the Newton-Raphson algorithm for root finding, including an appropriate sketch. Under what general conditions is it guaranteed that Newton-Raphson will converge to the root of interest? Prove that, when it converges to the root of interest, the Newton-Raphson method enjoys *quadratic convergence*.

Solution: Suppose that we want to find a specific root of the equation $f(x) = 0$. Let $x = x^*$ be an exact root. We might start with some guess $x = x_0$ which is close to the root of interest, $x = x^*$. Then:

$$0 = f(x^*) = f(x_0) + f'(x_0)(x^* - x_0) + \cdots$$

Assuming that we can neglect higher order terms, this suggests that a more accurate guess for x^* is given by:

$$f(x_0) + f'(x_0)(x^* - x_0) \approx 0 \qquad \Leftrightarrow \qquad x^* \approx x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This suggests the definition of an iterative process:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

which gets closer to x^* on each step. In general, it can be shown that the process will converge if: (i) $f'(x^*) \neq 0$; (ii) we start sufficiently close to the true root of interest.

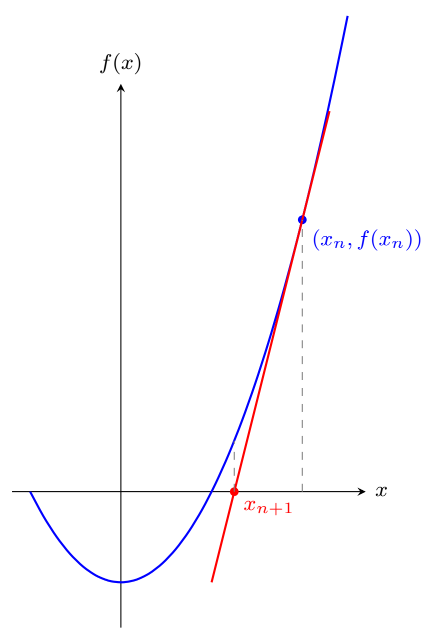
This iterative algorithm has a nice geometric interpretation. Imagine we are at the point x_n, with corresponding function value $f(x_n)$. The tangent to the graph at this point is then:

$$y - f(x_n) = f'(x_n)(x - x_n).$$

This implies that the tangent crosses the x-axis at the point satisfying:

$$-f(x_n) = f'(x_n)(x - x_n) \qquad \Leftrightarrow \qquad x = x_n - \frac{f(x_n)}{f'(x_n)},$$

which geometrically we hope is closer to a root of $f(x) = 0$. This is displayed graphically in the figure below.



Let us now define $\epsilon_{n+1} = x_{n+1} - x^*$ to be the difference between the $(n+1)$th Newton-Raphson iterate, and the true root. Then:

$$\epsilon_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - x^* = \epsilon_n - \frac{f(x_n)}{f'(x_n)}.$$

Performing a Taylor expansion of $f(x_n) = f(x_n - x^* + x^*) = f(\epsilon_n + x^*)$ and $f'(x_n) = f'(x_n - x^* + x^*) = f'(\epsilon_n + x^*)$, assuming that ϵ_n is small, we have:

$$\begin{aligned}
\epsilon_{n+1} &= \epsilon_n - \frac{\epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*) + \cdots}{f'(x^*) + \epsilon_n f''(x^*) + \cdots} && \text{(since } f(x^*) = 0\text{)} \\
&= \epsilon_n - \left(\epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*) + \cdots\right)\left(f'(x^*) + \epsilon_n f''(x^*) + \cdots\right)^{-1} \\
&= \epsilon_n - \frac{1}{f'(x^*)}\left(\epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*) + \cdots\right)\left(1 + \frac{\epsilon_n f''(x^*)}{f'(x^*)} + \cdots\right)^{-1} \\
&= \epsilon_n - \frac{1}{f'(x^*)}\left(\epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*) + \cdots\right)\left(1 - \frac{\epsilon_n f''(x^*)}{f'(x^*)} + \cdots\right) \\
&= \frac{1}{2}\epsilon_n^2 \frac{f''(x^*)}{f'(x^*)} + \cdots .
\end{aligned}$$

Hence, we see that $\epsilon_{n+1} \propto \epsilon_n^2$, so the error in the algorithm converges quadratically fast to zero.

23(a) Find the value of the first iterate of Newton-Raphson iteration for the function $f(x) = x - 2 + \log(x)$ with a starting guess of $x_0 = 1$.

(b) Find the value of the first and second iterates of Newton-Raphson iteration, valid to two decimal places, for the function $f(x) = x^2 - 2$ with a starting guess of $x_0 = 1$.

[*Both parts of this question are based on old (short) tripos questions, so try doing them*without *a calculator!*]

➼ Solution:

(a) Note that $f'(x) = 1 + 1/x$. Hence, we have:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1 - 2 + \log(1)}{1 + 1/1} = \frac{3}{2}.$$

(b) Note that $f'(x) = 2x$. Hence, we have:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^2 - 2}{2} = \frac{3}{2}.$$

Similarly, we have:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{3}{2} - \frac{(3/2)^2 - 2}{3} = \frac{17}{12}.$$

24. [*You may use a calculator for this question, but remember that you* won't *be able to use a calculator in the exam. Newton-Raphson questions will be more theoretical in the exams, like the next question, or involve easy calculations, like the previous question.*]

(a) Sketch the graph of $f(x) = x^3 - 3x^2 + 2$, indicating the coordinates of the turning points and the coordinates of the intersections with the x-axis.

(b) Use Newton-Raphson with an initial guess of $x_0 = 2.5$ to find an estimate of the largest root of the equation $f(x) = 0$, accurate to 5 decimal places. Draw a sketch showing the progress of the algorithm.

(c) To which roots (if any) does the algorithm converge if we instead start at: (i) $x_0 = 1.5$; (ii) $x_0 = 1.9$; (iii) $x_0 = 2$?

Solution: (a) The given function is a positive cubic. The stationary points occur when:

$$0 = f'(x) = 3x^2 - 6x = 3x(x-2) \qquad \Leftrightarrow \qquad x = 0, 2.$$

The point of inflection of the graph occurs when $0 = f''(x) = 6x - 6$, which is $x = 1$.

The graph intersects with the x-axis when $f(x) = 0$. Guessing a solution, we see that $x = 1$ works. This allows us to factorise the equation $f(x) = 0$ as:

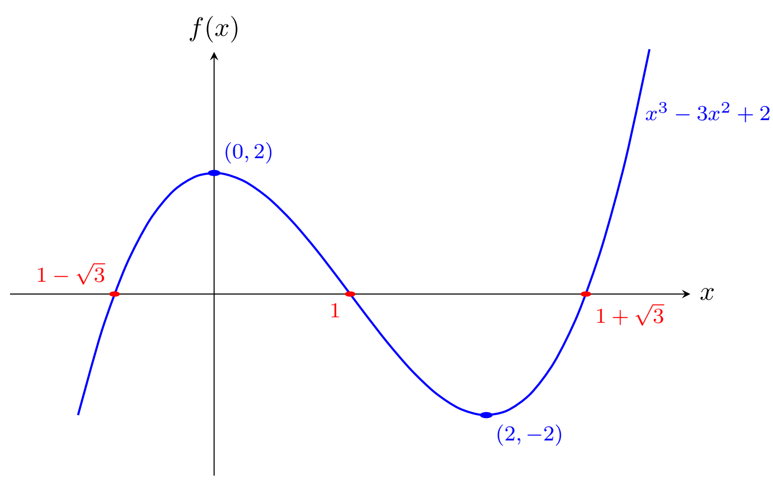
$$f(x) = (x-1)(x^2 - 2x - 2).$$

The quadratic factor can be further reduced by finding its roots using the quadratic formula. We have:

$$x_\pm = \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}.$$

Since $\sqrt{3} > 1$, we have that one of these roots is positive and one is negative. They occur symmetrically around $x = 1$.

We now have enough information to draw a fairly accurate graph:



(b) Let's now apply Newton-Raphson iteration to this function, starting with an initial guess of $x_0 = 2.5$. The iterative algorithm is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n^2 + 2}{3x_n^2 - 6x_n}.$$

Iterating using a calculator, we have:

$$x_1 = 2.8$$

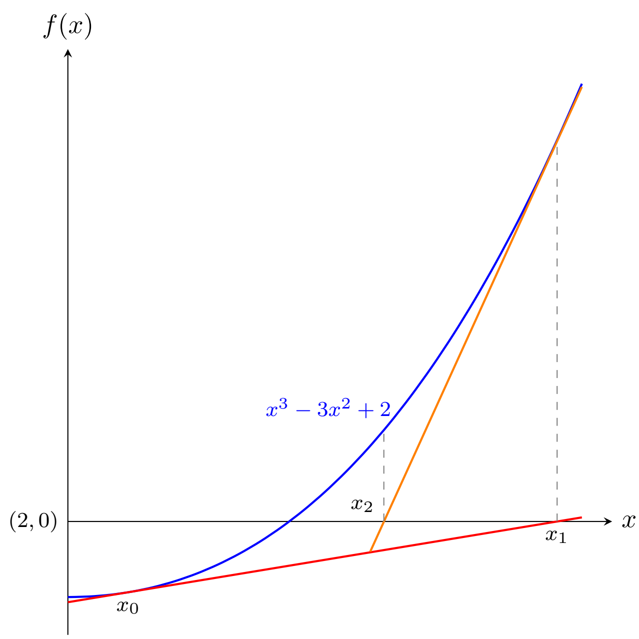
$$x_2 = 2.7357142857...$$

$$x_3 = 2.7320623734...$$

$$x_4 = 2.7320508076...$$

$$x_5 = 2.7320508075...$$

This shows that the correct root to 5 decimal places is 2.73205, which corresponds to the root $1 + \sqrt{3}$. A sketch showing the first two steps of the algorithm is given below:



(c) In this part of the question, we experiment with some other starting values. In the case of $x_0 = 1$, the closest root is 1, because we just start at a root. In the case of $x_0 = 1.9$, the closest root is actually $1+\sqrt{3}$, since $1+\sqrt{3}-1.9 \approx 0.83$. In the case of $x_0 = 2$, the closest root is $1+\sqrt{3}$, but this is also a stationary point of the function. We shall see that the behaviour is not what we might expect in (ii)!

(i) Starting at $x_0 = 1$, we already have $f(x_0) = 0$. Hence the second term in the Newton-Raphson iteration formula gives zero, and all of our iterates are the same: $x_0 = 1, x_1 = 1, x_2 = 1, ...$. This is what we would expect for an exact root.

(ii) Applying the algorithm in this case, we have:

$$x_1 = -1.5578947...$$

$$x_2 = -1.0129154...$$

$$x_3 = -0.7816615...$$

$$x_4 = -0.7340488...$$

$$x_5 = -0.7320542...$$

$$x_6 = -0.7320508...$$

This shows that the algorithm is convergent to $1-\sqrt{3}$ in this case. We were not expecting this, because this is actually the furthest root from our starting point $x_0 = 1.9$! This demonstrates the *chaotic* nature of the Newton-Raphson algorithm - we will only get to a particular root if we start sufficiently close. This is explored in more detail in Question 26.

(iii) Starting at $x_0 = 2$, the algorithm fails completely. This is because $f'(x_0) = 0$ here, so the denominator in the Newton-Raphson iteration formula is zero, giving a singular result. We cannot use the algorithm with this starting guess.

25. The real function f is defined by $f(x) = x^2 - 2\epsilon x - 1$, where ϵ is a small positive parameter ($0 < \epsilon \ll 1$). Let x_i be the ith Newton-Raphson iterate, with a starting guess of $x_0 = 1$, and let x_* be the unique positive root satisfying $f(x_*) = 0$. By Taylor expansion, show that $|x_i - x_*| \propto \epsilon^{n_i}$, where: (a) $n_0 = 1$; (b) $n_1 = 2$; (c) $n_2 = 4$.

➾ **Solution:** First, let's find the unique positive root of $f(x) = 0$. Using the quadratic equation, the positive root will be:

$$x_* = \frac{2\epsilon + \sqrt{4\epsilon^2 + 4}}{2} = \epsilon + \sqrt{1 + \epsilon^2}.$$

Expanding for small ϵ, we have:

$$x_* = \epsilon + 1 + \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 + \cdots = 1 + \epsilon + \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 + \cdots .$$

(a) The zeroth iterate is $x_0 = 1$. Hence:

$$x_0 - x_* = 1 - \epsilon - \sqrt{1 + \epsilon^2} = -\epsilon - \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 - \cdots .$$

This is indeed proportional to $\epsilon^{n_0} = \epsilon$ as required.

(b) The first iterate is:

$$x_1 = x_0 - \frac{x_0^2 - 2\epsilon x_0 - 1}{2(x_0 - \epsilon)} = 1 + \frac{\epsilon}{1 - \epsilon} = \frac{1}{1 - \epsilon}.$$

Taylor expanding for small ϵ, we have:

$$x_1 = (1 - \epsilon)^{-1} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \cdots .$$

Consequently, we have:

$$x_1 - x_* = \frac{1}{2}\epsilon^2 + \epsilon^3 + \frac{9}{8}\epsilon^4 + \cdots .$$

This is indeed proportional to $\epsilon^{n_1} = \epsilon^2$ as required.

(c) The second iterate is:

$$x_2 = x_1 - \frac{x_1^2 - 2\epsilon x_1 - 1}{2(x_1 - \epsilon)}$$

Substituting $x_1 = 1/(1 - \epsilon)$, we have:

$$\begin{aligned} x_2 &= \frac{1}{1 - \epsilon} - \frac{1/(1-\epsilon)^2 - 2\epsilon/(1-\epsilon) - 1}{2(1/(1-\epsilon) - \epsilon)} \\ &= \frac{1}{1 - \epsilon} - \frac{1 - 2\epsilon(1-\epsilon) - (1-\epsilon)^2}{2(1 - \epsilon - \epsilon(1-\epsilon)^2)} \\ &= \frac{1}{1 - \epsilon} - \frac{\epsilon^2}{2(1 - 2\epsilon + 2\epsilon^2 - \epsilon^3)} \\ &= (1-\epsilon)^{-1} - \frac{1}{2}\epsilon^2 \left(1 - 2\epsilon + 2\epsilon^2 - \epsilon^3\right)^{-1} . \end{aligned}$$

Now, expanding using a Taylor expansion for small ϵ, we have:

$$\begin{aligned} x_2 &= 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + ... - \frac{1}{2}\epsilon^2 \left(1 + \left(2\epsilon - 2\epsilon^2 + \epsilon^3\right) + \left(2\epsilon - 2\epsilon^2 + \epsilon^3\right)^2 + ...\right) \\ &= 1 + \epsilon + \frac{1}{2}\epsilon^2 + 0 \cdot \epsilon^3 + 0 \cdot \epsilon^4 + \cdots , \end{aligned}$$

where we notice the terms in ϵ^3, ϵ^4 vanish. Hence $x_2 - x_* = \epsilon^4/8 + \cdots$. This implies $|x_2 - x_*| \propto \epsilon^4$, so $n_2 = 4$, as required.

26. (∗) Consider the cubic equation $x^3 - 2x + 2 = 0$. Perform a numerical investigation (for example, by writing some simple code) to determine the ranges in $\mathbb{R}$ which converge to the various roots, if any. Comment on the sensitivity of Newton-Raphson to the choice of initial guess x_0. [*Afterwards, look up Newton fractals - it is particularly interesting to see the behaviour of Newton-Raphson in the complex plane!*]

Solution: Let $f(x) = x^3 - 2x + 2$. Observe that $f'(x) = 3x^2 - 2$ for this function, so the minimum value occurs at $x = \sqrt{2/3}$. This has corresponding value $f(\sqrt{2/3}) = (2/3)\sqrt{2/3} - 2\sqrt{2/3} + 2 \approx 0.911 > 0$, which implies that the cubic only has one real root.

The Newton-Raphson recurrence relation for the given function is:

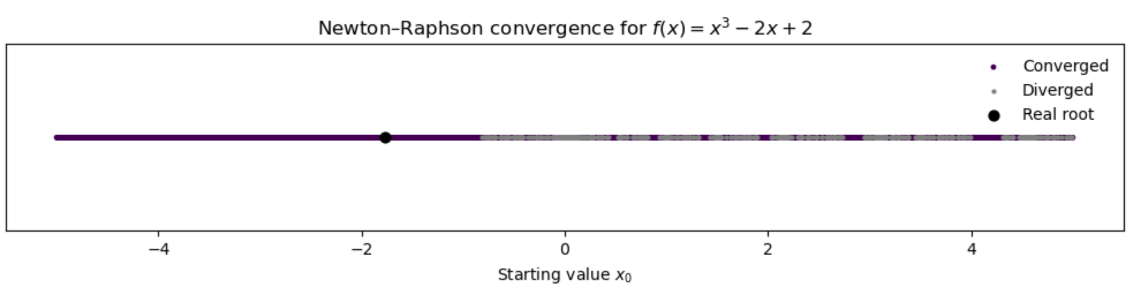
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n + 2}{3x_n^2 - 2}.$$

After finding this recurrence relation, I asked ChatGPT to write some Python code that performs the recurrence on a range of initial values on the real line, to determine which root the algorithm converges to:

```python
import numpy as np
import matplotlib.pyplot as plt

# Define the function and derivative
def f(x):
    return x**3 - 2*x + 2

def df(x):
    return 3*x**2 - 2

# Newton-Raphson iteration that tests true convergence
def newton(x0, max_iter=50, tol=1e-8):
    x = x0
    for _ in range(max_iter):
        dfx = df(x)
        if abs(dfx) < 1e-12:  # avoid division by zero
            return np.nan
        x_new = x - f(x)/dfx
        if abs(f(x_new)) < tol:  # true convergence check
            return x_new
        x = x_new
    return np.nan  # did not converge

# Compute roots (complex, but we only expect one real)
roots = np.roots([1, 0, -2, 2])
print("Roots:", roots)

# Sample starting values on the real line
x_vals = np.linspace(-5, 5, 2000)
final_vals = np.array([newton(x0) for x0 in x_vals])

# Classify convergence
def classify_root(x):
    if np.isnan(x):  # diverged or failed
        return np.nan
    distances = [abs(x - r) for r in roots]
    return np.argmin(distances)
```

```python

colors = np.array([classify_root(x) for x in final_vals])

# Plot setup
plt.figure(figsize=(10, 2))

# Points that converged (coloured)
mask_converged = ~np.isnan(colors)
plt.scatter(x_vals[mask_converged],
            np.zeros_like(x_vals[mask_converged]),
            c=colors[mask_converged],
            cmap='viridis', s=6, label="Converged")

# Points that diverged (gray)
mask_diverged = np.isnan(colors)
plt.scatter(x_vals[mask_diverged],
            np.zeros_like(x_vals[mask_diverged]),
            color='gray', s=4, label="Diverged")

# Axes and labels
plt.title("-NewtonRaphson convergence for $f(x)=x^3 - 2x + 2$")
plt.yticks([])
plt.xlabel("Starting value $x_0$")

# Mark the real root with a small circle
real_root = roots[np.isclose(roots.imag, 0)][0].real
plt.scatter([real_root], [0], color='black', s=40, zorder=5, label="Real root")

plt.legend(frameon=False, loc="upper right")
plt.tight_layout()
plt.show()
```

The resulting plot is the following:



In particular, we see that the purple starting points arrive at the real root, whilst the grey points *diverge*, not converging to any particular root. The behaviour appears a bit random as to whether the algorithm actually converges or not; indeed, we say that the Newton-Raphson method is *chaotic*. The method only converges to a root if we start sufficiently close to it (and how close is close enough can depend strongly on the function we are working with).

This behaviour is even more fascinating in the complex plane; I encourage you to look at the Wikipedia article about *Newton fractals*, which you can find here: https://en.wikipedia.org/wiki/Newton_fractal.