

Part IA: Mathematics for Natural Sciences B

Examples Sheet 11: Linear ordinary differential equations

Please send all comments and corrections to jmm232@cam.ac.uk.

Questions marked with a (*) are difficult and should not be attempted at the expense of the other questions.

Basic definitions

1. Consider the general linear n th-order ordinary differential equation:

$$\alpha_n(x) \frac{d^n y}{dx^n} + \alpha_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + \alpha_1(x) \frac{dy}{dx} + \alpha_0(x)y = f(x).$$

where $\alpha_n(x) \neq 0$.

- (a) Give the definitions of the following terms: (i) homogeneous equation; (ii) coefficient functions; (iii) forcing.
(b) Define a *complementary function* for this equation. How many arbitrary constants feature in the complementary function for this equation?
(c) Define a *particular integral* for this equation. Is a particular integral for this equation unique?
(d) Show that if y_{CF} is the complementary function for this equation, and y_{PI} is a particular integral, then the sum $y = y_{\text{CF}} + y_{\text{PI}}$ solves the equation.
(e) Suppose that we now seek a particular solution of this equation satisfying certain boundary conditions. How many boundary conditions are needed to fully specify a particular solution?
2. By direct differentiation, verify that the following ordinary differential equations have the given complementary functions:
 - (a) $y_{\text{CF}} = Ax + Be^x$ is the complementary function for $(x - 1)y'' - xy' + y = 0$;
 - (b) $y_{\text{CF}} = A + B \log(x)$ is the complementary function for $xy'' + y' = \cos(x)e^{x^2}$;
 - (c) $y_{\text{CF}} = Ax + B \sin(x)$ is the complementary function for $(1 - x \cot(x))y'' - xy' + y = x$;
 - (d) $y_{\text{CF}} = A + Bx + Ce^x$ is the complementary function for $y''' - y'' = x$.
3. By direct differentiation, verify that the following ordinary differential equations have the given particular integrals:
 - (a) $y_{\text{PI}} = \cos(x)$ is a particular integral for $-y'' + y = 2 \cos(x)$;
(b) $y_{\text{PI}} = x^2$ is a particular integral for $xy'' + y' = 4x$;
(c) $y_{\text{PI}} = e^{x^2}$ is a particular integral for $y''' - 2xy'' - 2y' - y = (4x - 1)e^{x^2}$;
(d) $y_{\text{PI}} = \sin(x)/x$ is a particular integral for $xy^{(4)} + 4y^{(3)} + xy^{(2)} + 2y^{(1)} + xy = \sin(x)$.
4. Verify that the equation:

$$(3 + x)y'' + (2 + x)y' - y = x^2$$

has complementary function $y_{\text{CF}}(x) = Ae^{-x} + B(x + 2)$. Hence, by finding a particular integral of the form

$$y_{\text{PI}}(x) = \alpha x^2 + \beta x + \gamma,$$

determine the full solution to the equation subject to the boundary conditions $y(0) = 0$ and $y'(0) = 1$.

Constant coefficient equations

5. Consider the linear second-order ordinary differential equation with *constant coefficients*:

$$\alpha \frac{d^2y}{dx^2} + \beta \frac{dy}{dx} + \gamma y = f(x),$$

where α, β, γ are *constants*, with $\alpha \neq 0$.

- (a) Show that the equation may be rewritten in the ‘factorised’ form:

$$\alpha \left(\frac{d}{dx} - \omega_1 \right) \left(\frac{d}{dx} - \omega_2 \right) y = f(x),$$

where ω_1, ω_2 are the roots of the *auxiliary equation* $\alpha\mu^2 + \beta\mu + \gamma = 0$.

- (b) Deduce that the complementary function of this equation is:

$$y_{CF}(x) = \begin{cases} Ae^{\omega_1 x} + Be^{\omega_2 x}, & \text{if } \omega_1 \neq \omega_2, \\ (A + Bx)e^{\omega x}, & \text{if } \omega_1 = \omega_2 = \omega. \end{cases}$$

How does this result generalise to an n th order differential equation of this form?

- (c) (*) Deduce also that we may construct an analytic particular integral, given by:

$$y_{PI}(x) = \frac{1}{\alpha} e^{\omega_2 x} \int_{x_0}^x \left(e^{(\omega_1 - \omega_2)\eta} \int_{\eta_0}^{\eta} e^{-\omega_1 \xi} f(\xi) d\xi \right) d\eta,$$

where x_0, η_0 are arbitrary constants. By setting $\eta_0 = x_0$ and changing the order of integration in the double integral, deduce the simpler form:

$$y_{PI}(x) = \begin{cases} \frac{1}{\alpha(\omega_1 - \omega_2)} \int_{x_0}^x \left(e^{\omega_1(x-\xi)} - e^{\omega_2(x-\xi)} \right) f(\xi) d\xi, & \text{if } \omega_1 \neq \omega_2, \\ \frac{1}{\alpha} \int_{x_0}^x (x - \xi) e^{\omega(x-\xi)} f(\xi) d\xi, & \text{if } \omega_1 = \omega_2 = \omega. \end{cases}$$

[In practice, it is often just easier to guess a particular integral rather than use this formula, though!]

6. Determine the solutions of the following differential equations:

- (a) $y'' + 6y' + 5y = 0$; (b) $y'' + 3y' + 4y = 0$;
(c) $y'' + 4y = x$; (d) $y'' - 2y' + 2y = 2x^2$;
(e) $y'' + y = |x|$; (f) $y'' + 3y' + 2y = e^{-x}$;
(g) $y'' - 2y' + 5y = e^x \cos(2x)$; (h) $y'' + 2y' + y = 2xe^{-x}$.

7. Determine the solutions of the following differential equations subject to the given constraints:

- (a) $y'' - 4y' + 13y = 0$, subject to $y(0) = \pi$ and $y(-\pi/2) = 1$;
(b) $y'' - 4y' + 5y = 125x^2$, subject to $y(0) = 1$ and $y\left(\frac{\pi}{2}\right) = \frac{25\pi^2}{4} + 20\pi + 22$;
(c) $y'' + 7y' + 12y = 6$, subject to $y(0) = 0$ and $y\left(\frac{1}{3}\right) = \frac{1-e^{-1}}{2}$;
(d) $y'' + 7y' + 12y = 2e^{-3x}$, subject to $y(0) = 1$ and $y'(0) = 0$.

8. Find the value of a for which the complementary function of the ODE:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + ay = 0,$$

is given by $y_{CF} = Axe^{-2x} + Be^{-2x}$.

9. Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + y = \cos(kx),$$

where k is a real number.

10. The differential operator \mathcal{L} is defined by:

$$\mathcal{L} = \frac{d^2}{dx^2} + \sqrt{3}\frac{d}{dx} + 3.$$

Solve the equation $\mathcal{L}y = 0$, and hence solve the equations:

- (a) $\mathcal{L}y = e^{-\sqrt{3}x}$;
- (b) $\mathcal{L}y = x$.

Without further calculation, state the general solution of $\mathcal{L}y = 2x + e^{-\sqrt{3}x}$. Find also the solution of this equation satisfying the boundary conditions:

$$y(0) = 0, \quad y(\pi) = \frac{e^{-\sqrt{3}\pi}}{3} - \frac{2}{3\sqrt{3}}.$$

Harmonic oscillators

11. Consider the constant coefficient linear second-order ordinary differential equation:

$$\frac{d^2y}{dt^2} + 2\gamma\frac{dy}{dt} + \omega_0^2y = f(t),$$

modelling an oscillating system which depends on time t . The coefficients γ, ω_0 are positive.

- (a) What is the physical interpretation of the constant γ ? What is the physical interpretation of the function $f(t)$?
- (b) Find the complementary function of this equation. Discuss the different forms the complementary function can take (in particular, defining the terms *underdamping*, *critical damping*, and *overdamping*), and how this relates to the *transient* behaviour of the oscillator.
- (c) In the underdamped case, find the long-term behaviour of the oscillator in the case of resonant forcing:

$$f(t) = e^{-\gamma t} \sin\left(t\sqrt{\omega_0^2 - \gamma^2}\right).$$

Coupled systems of differential equations

[This section is labelled ‘non-examinable’ in the lecture notes, but has appeared on Tripos papers – see e.g. 2023 Paper 2 or 2021 Paper 2.]

12. (a) Consider the system of differential equations:

$$\frac{dx}{dt} = ax + by + p, \quad \frac{dy}{dt} = cx + dy + q,$$

for the variables $x(t), y(t)$, where a, b, c, d, p, q are constants. Show that:

$$\frac{d^2x}{dt^2} = (a + d)\frac{dx}{dt} + (bc - ad)x + bq - pd.$$

- (b) Hence:

- (i) Find the general solution of the system:

$$\frac{dx}{dt} = 4y + 2, \quad \frac{dy}{dt} = x.$$

- (ii) Solve the system:

$$\frac{dx}{dt} = 3x - y, \quad \frac{dy}{dt} = x + y,$$

subject to the initial conditions $x(0) = 0$ and $y(0) = 1$.

- (iii) Solve the system:

$$\frac{dx}{dt} = -3x + y, \quad \frac{dy}{dt} = -5x + y,$$

subject to the initial conditions $x(0) = 1, y(0) = 1$.

(*) Equidimensional equations

[This section is not lectured, but is very useful if you choose to do Part IB Mathematics in second-year.]

13. Consider a linear second-order ordinary differential equation with non-constant coefficients:

$$\alpha x^2 \frac{d^2y}{dx^2} + \beta x \frac{dy}{dx} + \gamma y = f(x),$$

where α, β, γ are constants, with $\alpha \neq 0$. This type of equation is called an *equidimensional equation*. If you are doing Part IA Physics, suggest a reason for this name.

- (a) Show that the equation may be written in the form:

$$\alpha \left(x \frac{d}{dx} - \omega_1 \right) \left(x \frac{d}{dx} - \omega_2 \right) = f(x),$$

where ω_1, ω_2 are the roots of the *auxiliary equation* $\alpha\mu(\mu - 1) + \beta\mu + \gamma = 0$.

- (b) Deduce that the complementary function of this equation is:

$$y_{CF}(x) = \begin{cases} Ax^{\omega_1} + Bx^{\omega_2}, & \text{if } \omega_1 \neq \omega_2, \\ (A + B \log(x))x^\omega, & \text{if } \omega_1 = \omega_2 = \omega. \end{cases}$$

How does this result generalise to an n th order differential equation of this form?

14. Using the results of Question 13, determine the solutions of the following differential equations:

- (a) $x^2y'' - 2xy' + y = 0$, subject to the initial data $y(1) = 1, y'(1) = 0$;
(b) $x^2y'' - xy' + y = x^2$, subject to the initial data $y(1) = 2, y'(1) = 3$;
(c) $x^2y'' - xy' + y = x \log(x)$, subject to the initial data $y(1) = 0, y'(1) = 1$.