# Part IA: Mathematics for Natural Sciences B
# Examples Sheet 6: Single-variable integration

## Model Solutions

*Please send all comments and corrections to jmm232@cam.ac.uk.*

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**Riemann sums and the definition of the integral**

1. Explain what is meant by a *Riemann sum* for a function $f : [a, b] \to \mathbb{R}$ using a *partition* $P = (x_0, ..., x_n)$ (with $x_0 = a, x_n = b$) and *tagging* $T = (t_1, ..., t_n)$. By choosing appropriate partitions and taggings in each case, use sequences of Riemann sums to evaluate the definite integrals of the following functions on $[0, 1]$ from first principles:

(a) $x$, (b) $x^2$, (c) $x^3$, (d) $\sqrt{x}$, (e) $\cos(x)$.

[*Hint: For part (d), consider a non-uniform tagging. For part (e), consider the integral of* $\mathrm{Re}(e^{ix})$ *instead of* $\cos(x)$.]

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**Solution:** If we want to approximate the integral of the function $f(x)$ on the interval $[a, b]$, we can do so by considering a sum of *rectangles*. We split the interval $[a, b]$ into a *partition* $P = (x_0, ..., x_n)$ which satisfies:

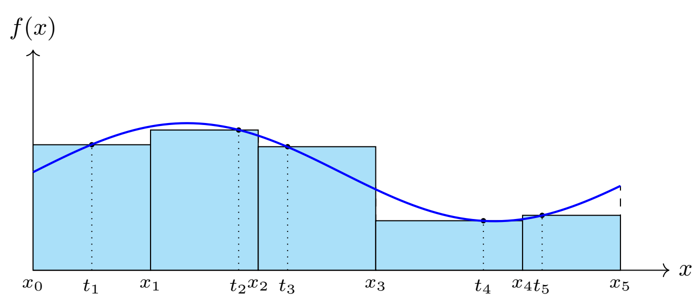
$$a = x_0 < x_1 < ... < x_n = b,$$

where $[x_0, x_1]$ is the base of the first rectangle, $[x_1, x_2]$ is the base of the second rectangle, etc. Notice we start at zero, because then the highest index $n$ matches the number of rectangles we get!

Within each rectangle, we then pick a point called a *tag* for the rectangle. This produces a *tagging* $T = (t_1, ..., t_n)$ which satisfies:

$$x_0 \le t_1 \le x_1 \le t_2 \le x_2 \le ... \le x_{n-1} \le t_n \le x_n.$$

At the tags, we evaluate the function to give us the heights of the rectangles in our approximation, $f(t_1), ..., f(t_n)$. This is illustrated in the diagram below.




The *Riemann sum* for $f : [a, b] \to \mathbb{R}$ that results from this partition and tagging is defined to be the area of the rectangles:

$$R = \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}).$$

We say that $f$ is a *Riemann integrable function*, with *integral* $I$, if for all sequences of partitions $P_n$ and taggings $T_n$, such that the width of the largest sub-interval in the partition $P_n$ tends to zero as $n \to \infty$, we have that the associated sequences of Riemann sums $R_n$ all converge to $I$.

*Proving* that a function is Riemann integrable is very hard, because it requires us to consider all possible partitions and taggings in one go.[1] However, if we already know that a function is Riemann integrable, we can find its integral by just considering *one* sequence of partitions and taggings; that is what we do in this question. In general, any continuous function is integrable, so this is completely fine in this case!

Finding integrals in this way is not straightforward, and is definitely not the best approach - however, it is the most fundamental, and is similar to the 'first principles' limit approach we used when we first introduced differentiation. Integration in practice involves knowing a standard set of integrals, and a set of techniques (namely integration by substitution and integration by parts), which allows us to determine integrals of most common functions.

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(a) Use the uniform partition $(0, 1/n, .., n/n)$ with right-handed tagging $(1/n, 2/n, ..., n/n)$. Then we have the sequence of Riemann sums:

$$R_n = \sum_{k=1}^{n} \left(\frac{k}{n}\right) \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{n(n+1)}{2n^2} = \frac{n+1}{2n}$$

Taking the limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1+1/n}{2} = \frac{1}{2}.$$

This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}.$$

(b) Use the uniform partition $(0, 1/n, ..., n/n)$ with right-handed tagging $(1/n, 2/n, ..., n/n)$. Then we have the sequence of Riemann sums:

$$R_n = \sum_{k=1}^{n} \left(\frac{k^2}{n^2}\right) \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{(n+1)(2n+1)}{6n^2}.$$

Taking the limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{(1+1/n)(2+1/n)}{6} = \frac{2}{6} = \frac{1}{3}.$$

This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}.$$

[1] It is possible to do this for very simple functions though. You might like to show that a constant function is integrable from first principles, if you are feeling adventurous!

(c) Use the uniform partition $(0, 1/n, ..., n/n)$ with right-handed tagging $(1/n, 2/n, ..., n/n)$. Then we have the sequence of Riemann sums:

$$R_n = \sum_{k=1}^{n} \left(\frac{k^3}{n^3}\right) \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^4} \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4n^4} = \frac{(n+1)^2}{4n^2}.$$

Taking the limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)^2}{4n^2} = \lim_{n\to\infty} \frac{(1+1/n)^2}{4} = \frac{1}{4}.$$

This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 x^3 \, dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}.$$

(d) This time, things get more exciting, because using a uniform partition won't work. Instead, let's use a *quadratically spaced* partition, to try to clear the square root. We use the partition $(0, 1/n^2, 4/n^2, 9/n^2, ..., n^2/n^2)$ with right-handed tagging $(1/n^2, 4/n^2, ..., n^2/n^2)$. Then we have the sequence of Riemann sums:

$$R_n = \sum_{k=1}^{n} \sqrt{\frac{k^2}{n^2}} \cdot \left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2}\right) = \frac{1}{n^3} \sum_{k=1}^{n} k\left(k^2 - (k^2 - 2k + 1)\right) = \frac{1}{n^3} \sum_{k=1}^{n} \left(2k^2 - k\right).$$

Performing the sum, we have:

$$\frac{1}{n^3} \sum_{k=1}^{n} \left(2k^2 - k\right) = \frac{2}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2n^3} = \frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2}.$$

Taking the limit as $n \to \infty$, the second term vanishes, and the first term gives $2/3$. This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 \sqrt{x} \, dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}.$$

(e) This is an even more exciting problem, because we get to use complex numbers. Observe that:

$$\int_0^1 \cos(x) \, dx = \operatorname{Re}\left[\int_0^1 e^{ix} \, dx\right],$$

so instead of constructing a Riemann sum for $\cos(x)$, we will construct a Riemann sum for $e^{ix}$, as hinted at in the question. Choose a uniform partition $(0, 1/n, ..., n/n)$ with right-handed tagging $(1/n, 2/n, ..., n/n)$. Then we have the sequence of Riemann sums for the complex integral:

$$R_n = \sum_{k=1}^{n} e^{ik/n} \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n} \sum_{k=1}^{n} e^{ik/n}.$$

This is a geometric progression with first term $e^{i/n}$ and common ratio $e^{i/n}$. Hence the sum is:

$$R_n = \frac{e^{i/n}(1 - e^i)}{n(1 - e^{i/n})}.$$

We now need to take the limit of this expression as $n \to \infty$. The numerator approaches $1 - e^i$, but the denominator is of the form $\infty \cdot 0$, so is an indeterminate form. We could use L'Hôpital's rule to evaluate this:

$$\lim_{n\to\infty} n(1 - e^{i/n}) = \lim_{n\to\infty} \frac{1 - e^{i/n}}{1/n} = \lim_{n\to\infty} \frac{ie^{i/n}/n^2}{-1/n^2} = -i.$$

Alternatively, we can use a Taylor series expansion for $e^{i/n}$ (see the next sheet, if you are unfamiliar!). We have:

$$\lim_{n\to\infty} n(1 - e^{i/n}) = \lim_{n\to\infty} n\left(1 - 1 - \frac{i}{n} - \frac{1}{2!}\left(\frac{i}{n}\right)^2 + \cdots\right) = -i.$$

Hence we have:

$$\lim_{n\to\infty} \frac{e^{i/n}(1 - e^i)}{n(1 - e^{i/n})} = \frac{1 - e^i}{-i} = i - ie^i = i - i\cos(1) + \sin(1).$$

Taking the real part, this leaves $\sin(1)$. This agrees with what we would expect from our knowledge of integration:

$$\int_0^1 \cos(x)\,dx = [\sin(x)]_0^1 = \sin(1).$$

In fact, this proof also shows (by taking imaginary parts), that the integral of $\sin(x)$ over $[0, 1]$ is given by $1 - \cos(1)$. This also agrees with what we would expect from our knowledge of integration:

$$\int_0^1 \sin(x)\,dx = [-\cos(x)]_0^1 = 1 - \cos(1).$$

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2. Using a non-uniform tagging, use a sequence of Riemann sums to evaluate the integral $\displaystyle\int_1^\infty \frac{dx}{x^{1+\alpha}}$, where $\alpha > 0$.

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**Solution:** Let us define $r_n = 1 + 1/n$ to be a ratio larger than one. Then, we use a geometric partition $(1, r_n, r_n^2, r_n^3, ...)$ with right-handed tagging $(r_n, r_n^2, r_n^3, ...)$. The corresponding sequence of Riemann sums is:

$$R_n = \sum_{k=0}^{\infty} \frac{1}{(r_n^k)^{1+\alpha}} \left(r_n^{k+1} - r_n^k\right) = (r_n - 1) \sum_{k=0}^{\infty} r_n^{-k\alpha} = \frac{r_n - 1}{1 - r_n^{-\alpha}}.$$

Here, we could take the sum to infinity since $r_n > 1$, so $r_n^{-\alpha} < 1$. We now take the limit as $n \to \infty$, which is equivalent to the limit $r_n \to 1$. To do this, we can use L'Hôpital's rule:

$$\lim_{n\to\infty} R_n = \lim_{r\to 1} \frac{r - 1}{1 - r^{-\alpha}} = \lim_{r\to 1} \frac{1}{\alpha r^{-\alpha}} = \frac{1}{\alpha}.$$

Observe that this agrees with the expected result, since:

$$\int_1^\infty \frac{dx}{x^{1+\alpha}} = \left[-\frac{1}{\alpha x^\alpha}\right]_1^\infty = \frac{1}{\alpha}.$$

3. Show by considering Riemann sums that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt{n^2-k^2}}{n^2} = \frac{\pi}{4}$.

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➡ **Solution:** Observe that:

$$\sum_{k=1}^{n} \frac{\sqrt{n^2-k^2}}{n^2} = \sum_{k=1}^{n} \sqrt{1-\left(\frac{k}{n}\right)^2} \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right)$$

is a sequence of Riemann sums for the function $\sqrt{1-x^2}$ on the interval $[0,1]$ using the partition $0 = 0/n < 1/n < \ldots < n/n = 1$, with the (right handed) tagging $1/n, 2/n, \ldots, n/n$. Hence:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt{n^2-k^2}}{n^2} = \int_0^1 \sqrt{1-x^2}\, dx.$$

To evaluate this integral, we could use a substitution. Alternatively, we could note it is the area under the graph $y = \sqrt{1-x^2}$ between $0$ and $1$. Rearranging this equation, we see that $x^2 + y^2 = 1$, which tells us the graph is a *circle*. It immediately follows that the integral is just the area of quarter of a circle, hence:

$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{4},$$

as required.

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4. ($*$) If a sequence of Riemann sums for a function $f : [a,b] \to \mathbb{R}$ converges, must the function be integrable?

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➡ **Solution:** The answer is *no*; just because one sequence of Riemann sums converges, it does not mean that all sequences of Riemann sums will converge to the same value. Consider, for example, the function $f : [0,1] \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational,} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Then using the uniform partition $(0, 1/n, 2/n, \ldots, n/n)$ with right-handed tagging $(1/n, 2/n, \ldots, n/n)$, we have a sequence of Riemann sums:

$$R_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{k}{n} - \frac{k-1}{n}\right) = 0,$$

since $f(k/n) = 0$, since $k/n$ is rational for each $k$. This suggests that the integral of $f(x)$ on $[0,1]$ is zero.

*However*, if we instead use a tagging consisting of *irrational* numbers $(s_1, \ldots, s_n)$, chosen such that $0 = 0/n < s_1 < 1/n < s_2 < 2/n < \ldots < s_n < n/n = 1$, then we have a sequence of Riemann sums:

$$R_n' = \sum_{k=1}^{n} f(s_k)\left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{n}{n} = 1.$$

This suggests that the integral of $f(x)$ on $[0,1]$ is one. In particular, we see that the function is *not* integrable, because two different sequences of Riemann sums for $f(x)$ on $[0,1]$ converge to two different values.

### Basic integrals

5. *Write down* the indefinite integrals of each of the following functions, where $a \neq 0, \alpha \neq -1$, and $f$ is any (differentiable, non-zero) function:

(a) $(ax+b)^\alpha$, (b) $e^{ax+b}$, (c) $(ax+b)^{-1}$, (d) $\sin(ax+b)$, (e) $\cos(ax+b)$, (f) $\sec^2(ax+b)$, (g) $\operatorname{cosec}^2(ax+b)$, (h) $\sinh(ax+b)$, (i) $\cosh(ax+b)$, (j) $f'(x)f(x)^\alpha$, (k) $f'(x)/f(x)$.

Learn these integrals off by heart, and get your supervision partner to test you on them.

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**Solution:** These are all standard integrals. We have (where $c$ is an arbitrary constant in each case):

(a) $\displaystyle\int (ax+b)^\alpha \, dx = \frac{(ax+b)^{\alpha+1}}{a(\alpha+1)} + c$, if $a \neq 0$ and $\alpha \neq -1$.

(b) $\displaystyle\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c$, if $a \neq 0$.

(c) $\displaystyle\int \frac{1}{ax+b} \, dx = \frac{\log(ax+b)}{a} + c$, if $a \neq 0$.

(d) $\displaystyle\int \sin(ax+b) \, dx = -\frac{\cos(ax+b)}{a} + c$, if $a \neq 0$.

(e) $\displaystyle\int \cos(ax+b) \, dx = \frac{\sin(ax+b)}{a} + c$, if $a \neq 0$.

(f) $\displaystyle\int \sec^2(ax+b) \, dx = \frac{\tan(ax+b)}{a} + c$, if $a \neq 0$.

(g) $\displaystyle\int \operatorname{cosec}^2(ax+b) \, dx = -\frac{\cot(ax+b)}{a} + c$, if $a \neq 0$.

(h) $\displaystyle\int \sinh(ax+b) \, dx = \frac{\cosh(ax+b)}{a} + c$, if $a \neq 0$.

(i) $\displaystyle\int \cosh(ax+b) \, dx = \frac{\sinh(ax+b)}{a} + c$, if $a \neq 0$.

(j) $\displaystyle\int f'(x) f(x)^\alpha \, dx = \frac{f(x)^{\alpha+1}}{\alpha+1}$, if $\alpha \neq -1$.

(k) $\displaystyle\int \frac{f'(x)}{f(x)} \, dx = \log(f(x)) + c.$

6. Using the results of the previous question, evaluate the definite integrals:

(a) $\displaystyle\int_0^2 (x-1)^2\,dx$, (b) $\displaystyle\int_0^\pi e^{i\theta}\,d\theta$, (c) $\displaystyle\int_0^\pi \cos(x)\,dx$, (d) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx$, (e) $\displaystyle\int_0^1 \frac{2x+4}{x^2+4x+1}\,dx$.

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➡ **Solution:** We have:

(a) $\displaystyle\int_0^2 (x-1)^2\,dx = \left[\frac{(x-1)^3}{3}\right]_0^2 = \frac{1}{3} - -\frac{1}{3} = \frac{2}{3}.$

(b) $\displaystyle\int_0^\pi e^{i\theta}\,d\theta = \left[\frac{e^{i\theta}}{i}\right]_0^\pi = -\frac{1}{i} - \frac{1}{i} = -\frac{2}{i} = 2i.$

(c) $\displaystyle\int_0^\pi \cos(x)\,dx = [\sin(x)]_0^\pi = \sin(\pi) - \sin(0) = 0$. Alternatively, spot that $\cos(x)$ is rotationally symmetric about $x = \pi/2$, and the positive and negative contributions to the integral from $0 < x < \pi/2$ and $\pi/2 < x < \pi$ therefore exactly cancel out.

(d) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx = [\tan(x)]_{-\pi/4}^{\pi/4} = 1 - -1 = 2.$

(e) $\displaystyle\int_0^1 \frac{2x+4}{x^2+4x+1}\,dx = \left[\log(x^2+4x+1)\right]_0^1 = \log(6) - \log(1) = \log(6).$

7. By writing $\cos(bx)$ as the real part of a complex exponential, determine the indefinite integral of $e^{ax}\cos(bx)$. Similarly, determine the indefinite integrals of $e^x(\sin(x)-\cos(x))$ and $e^x(\sin(x)+\cos(x))$.

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➻ **Solution:** We have:

$$\begin{aligned}
\int e^{ax}\cos(bx)\,dx &= \mathrm{Re}\left[\int e^{ax}e^{ibx}\,dx\right] \\
&= \mathrm{Re}\left[\int e^{(a+ib)x}\,dx\right] \\
&= \mathrm{Re}\left[\frac{e^{(a+ib)x}}{a+ib}+c\right] \\
&= \mathrm{Re}\left[\frac{e^{ax}(a-ib)(\cos(bx)+i\sin(bx))}{a^2+b^2}+c\right] \\
&= \frac{e^{ax}(a\cos(bx)+b\sin(bx)}{a^2+b^2}+c,
\end{aligned}$$

where $c$ is a real constant of integration.

Similarly, we have:

$$\int e^{ax}\sin(bx)\,dx = \mathrm{Im}\left[\int e^{ax}e^{ibx}\,dx\right] = \frac{e^{ax}(a\sin(bx)-b\cos(bx))}{a^2+b^2}+c,$$

using the penultimate line of the calculation from above, but just taking the imaginary part instead of the real part. It follows that:

$$\int e^x(\sin(x)-\cos(x))\,dx = \frac{e^x(\sin(x)-\cos(x))}{2}-\frac{e^x(\cos(x)+\sin(x))}{2}+c = -e^x\cos(x)+c,$$

for a real constant of integration $c$. Similarly, we have:

$$\int e^x(\sin(x)+\cos(x))\,dx = \frac{e^x(\sin(x)-\cos(x))}{2}+\frac{e^x(\cos(x)+\sin(x))}{2}+c = e^x\sin(x)+c,$$

for a real constant of integration $c$.

**Integration by substitution**

8. By means of an appropriate substitution in each case, determine the indefinite integrals of the following functions:

(a) $\dfrac{1}{\sqrt{1-x^2}}$, (b) $\dfrac{1}{\sqrt{x^2-1}}$, (c) $\dfrac{1}{\sqrt{1+x^2}}$, (d) $\dfrac{1}{1+x^2}$, (e) $\dfrac{1}{1-x^2}$

Learn these integrals off by heart, and get your supervision partner to test you on them.

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➾ **Solution:** Each of these integrals can be performed by making an appropriate trigonometric or hyperbolic substitution.

(a) Use the substitution $x = \sin(\theta)$, so that $dx = \cos(\theta)d\theta$. This is appropriate here, since for the square root to make sense, we need $-1 < x < 1$. This is covered by the range of $\sin(\theta)$, so the substitution is valid. We then have:

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos(\theta)d\theta}{\sqrt{1-\sin^2(\theta)}} = \int d\theta = \theta + c = \arcsin(x) + c.$$

(b) Use the substitution $x = \cosh(\theta)$, so that $dx = \sinh(\theta)d\theta$. This is appropriate here, since for the square root to make sense, we need $x > 1$ or $x < -1$. Since the range is disjoint, we can focus on $x > 1$, which is covered by the range of $\cosh(\theta)$ (we could have chosen to make the substitution $x = -\cosh(\theta)$ if $x < -1$ was the range of interest). We then have:

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sinh(\theta)d\theta}{\sqrt{\cosh^2(\theta)-1}} = \int d\theta = \theta + c = \operatorname{arcosh}(x) + c.$$

(c) Use the substitution $x = \sinh(\theta)$, so that $dx = \cosh(\theta)d\theta$. This is appropriate here, since $x$ can take any value, and $\sinh(\theta)$ has range $\mathbb{R}$. We then have:

$$\int \frac{dx}{\sqrt{x^2+1}} = \int \frac{\cosh(\theta)d\theta}{\sqrt{\sinh^2(\theta)+1}} = \int d\theta = \theta + c = \operatorname{arsinh}(x) + c.$$

(d) Use the substitution $x = \tan(\theta)$, so that $dx = \sec^2(\theta)d\theta$. This is appropriate here, since $x$ can take any value, and $\sinh(\theta)$ has range $\mathbb{R}$. We then have:

$$\int \frac{dx}{1+x^2} = \int \frac{\sec^2(\theta)d\theta}{1+\tan^2(\theta)} = \int d\theta = \theta + c = \arctan(x) + c.$$

(e) Use the substitution $x = \tanh(\theta)$, so that $dx = \operatorname{sech}^2(\theta)d\theta$. This is appropriate here since $x$ takes values in $x < -1$, $-1 < x < 1$ and $x > 1$. These ranges are disjoint, so if we are interested in $-1 < x < 1$, we are safe to make this substitution. We then have:

$$\int \frac{dx}{1-x^2} = \int \frac{\operatorname{sech}^2(\theta)d\theta}{1-\tanh^2(\theta)} = \int d\theta = \theta + c = \operatorname{artanh}(x) + c.$$

9. Using the results of the previous question, determine: (a) $\displaystyle\int \frac{dx}{\sqrt{x^2+x+1}}$; (b) $\displaystyle\int \frac{8-2x}{\sqrt{6x-x^2}}\,dx$.

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**Solution:**

(a) We complete the square in the denominator, $x^2+x+1=(x+1/2)^2+3/4$. Then:

$$\int \frac{dx}{\sqrt{(x+1/2)^2+3/4}} = \frac{2}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1}}.$$

This is now a linear function of one of our standard integrals from the previous question. Hence the integral is just given by:

$$\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2}\mathrm{arsinh}\left(\frac{2x+1}{\sqrt{3}}\right)+c=\mathrm{arsinh}\left(\frac{2x+1}{\sqrt{3}}\right)+c.$$

To get the constants that multiply the inverse hyperbolic function correct, it can be useful to think about what happens when you differentiate the final expression. By the chain rule, a $2/\sqrt{3}$ will pop out, so we need to cancel that when we integrate!

(b) This is more complicated, because even completing the square on the denominator will not give us an integral of the form of the previous question. Instead, we observe that the derivative of $6x-x^2$ is $6-2x$, so the integral is close to being of the form $f'(x)f(x)^\alpha$. If we rewrite the integrand as:

$$\frac{8-2x}{\sqrt{6x-x^2}}=\frac{6-2x}{\sqrt{6x-x^2}}+\frac{2}{\sqrt{6x-x^2}}=(6-2x)\left(6x-x^2\right)^{-1/2}+\frac{2}{\sqrt{6x-x^2}},$$

then the first term is now of the form $f'(x)f(x)^\alpha$ and can be directly integrated, whilst the second term can be transformed to one of our standard forms studied in the previous question.

Completing the square in the denominator of the second term, we have $6x-x^2=9-(x-3)^2$. Hence we have:

$$\begin{aligned}\int\frac{8-2x}{\sqrt{6x-x^2}}\,dx &= \int(6-2x)(6x-x^2)^{-1/2}\,dx+\frac{2}{3}\int\frac{dx}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}}\\ &= 2\sqrt{6x-x^2}+\frac{2}{3}\cdot 3\cdot\arcsin\left(\frac{x-3}{3}\right)+c\\ &= 2\sqrt{6x-x^2}+2\arcsin\left(\frac{x-3}{3}\right)+c.\end{aligned}$$

10. By means of an appropriate substitution in each case, determine the indefinite integrals of the following functions:

(a) $x\sqrt{x+3}$, (b) $\tan(x)\sqrt{\sec(x)}$, (c) $\dfrac{e^x}{\sqrt{1-e^{2x}}}$, (d) $\dfrac{1}{x\sqrt{x^2-1}}$.

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**➻ Solution:**

(a) Consider the substitution $u = x+3$, trying to clear the square root. We have $du = dx$, so that the integral can be rewritten as:

$$\int x\sqrt{x+3}\,dx = \int (u-3)u^{1/2}\,du = \int \left(u^{3/2} - 3u^{1/2}\right)\,du = \frac{2}{5}u^{5/2} - 2u^{3/2} + c.$$

Hence the integral is given by:

$$\frac{2}{5}(x+1)^{5/2} + 2(x+1)^{3/2} + c.$$

(b) Consider the substitution $u = \sec(x)$. This is a good choice, because the derivative of $\sec(x)$ is $\sec(x)\tan(x)$, so we will be able to clear the $\tan(x)$ and leave only $\sec(x)$ terms behind. We have $du = \sec(x)\tan(x)dx$, so:

$$\int \tan(x)\sqrt{\sec(x)}\,dx = \int \frac{du}{\sqrt{u}} = 2u^{1/2} + c = 2\sqrt{\sec(x)} + c.$$

(c) Here, the obvious substitution is $u = e^x$. We have $du = e^x\,dx$, so:

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \arcsin(u) + c = \arcsin(e^x) + c.$$

(d) Consider a trigonometric substitution $x = \sec(\theta)$, since this will clear the square root ($\sqrt{\sec^2(\theta)-1} = \tan(\theta)$), but also give us $dx = \sec(\theta)\tan(\theta)d\theta$, so that the remaining $\sec(\theta)$ on the denominator will cancel. We have:

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec(\theta)\tan(\theta)d\theta}{\sec(\theta)\tan(\theta)} = \int d\theta = \theta + c = \operatorname{arcsec}(x) + c.$$

Another way of writing $\operatorname{arcsec}(x)$ in terms of more standard function is as $\arccos(1/x)$ - to see this, note that:

$$y = \operatorname{arcsec}(x) \qquad \Rightarrow \qquad \sec(y) = x \qquad \Rightarrow \qquad \cos(y) = \frac{1}{x} \qquad \Rightarrow \qquad y = \arccos\left(\frac{1}{x}\right).$$

11. This question shows that any trigonometric integral can be turned into an algebraic integral through the use of the powerful *half-tangent substitution*.

(a) Show that if $t = \tan\left(\frac{1}{2}x\right)$, then $\sin(x) = 2t/(1+t^2)$, $\cos(x) = (1-t^2)/(1+t^2)$ and $dx/dt = 2/(1+t^2)$. Deduce that for any function $f$, we have:

$$\int f\left(\sin(x), \cos(x)\right)\, dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2dt}{1+t^2}.$$

(b) Using the method derived in (a), find the indefinite integrals of the following functions:

(i) $\operatorname{cosec}(x)$, (ii) $\sec(x)$, (iii) $\dfrac{1}{2+\cos(x)}$.

---

**Solution:** (a) We have:

$$\sin(x) = 2\sin(x/2)\cos(x/2) = \frac{2\sin(x/2)\cos(x/2)}{\cos^2(x/2)+\sin^2(x/2)} = \frac{2\tan(x/2)}{1+\tan^2(x/2)} = \frac{2t}{1+t^2}$$

and:

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2) = \frac{\cos^2(x/2)-\sin^2(x/2)}{\cos^2(x/2)+\sin^2(x/2)} = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-t^2}{1+t^2},$$

as required. We also have:

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{1}{2}x\right) = \frac{1}{2}\left(1+t^2\right) \qquad \Leftrightarrow \qquad \frac{dx}{dt} = \frac{2}{1+t^2}.$$

The required equality then follows immediately from the substitution $t = \tan(x/2)$.

---

(b) Applying the substitution in each of the given cases, we have:

(i) $$\int \operatorname{cosec}(x)\, dx = \int \frac{dx}{\sin(x)} = \int \frac{2(1+t^2)dt}{2t(1+t^2)} = \int \frac{dt}{t} = \log(t) + c = \log(\tan(x/2)) + c.$$

(ii) $$\int \sec(x)\, dx = \int \frac{dx}{\cos(x)} = \int \frac{2(1+t^2)dt}{(1-t^2)(1+t^2)} = 2\int \frac{dt}{1-t^2} = 2\operatorname{artanh}\left(\tan(x/2)\right) + c.$$

(iii) $$\int \frac{dx}{2+\cos(x)} = \int \frac{2dt}{(2(1+t^2)+(1-t^2))} = \int \frac{2dt}{3+t^2} = \frac{2}{3}\int \frac{dt}{1+\left(\frac{t}{\sqrt{3}}\right)^2} = \frac{2}{\sqrt{3}}\arctan\left(\frac{\tan(x/2)}{\sqrt{3}}\right) + c.$$

**Partial fractions and rational functions**

12. Explain the general strategy that one should adopt when integrating a rational function. Hence, determine the indefinite integrals of the following rational functions by decomposing into partial fractions:

(a) $\dfrac{1}{1-x^2}$, (b) $\dfrac{3x}{2x^2+x-1}$, (c) $\dfrac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}$.

Compare your answer to (a) with your answer to Question 7(e), where you evaluated the same integral using a substitution. Are your results compatible?

---

➾ **Solution:** Consider the rational function $p(x)/q(x)$, where $q(x)$ can be factorised in the form:

$$q(x) = (x-a_1)^{j_1}...(x-a_m)^{j_m}(x^2+b_1x+c_1)^{k_1}...(x^2+b_nx+c_n)^{k_n},$$

where the quadratic factors have no real roots. Then $p(x)/q(x)$ can be decomposed in the form:

$$\frac{p(x)}{q(x)} = r(x) + \sum_{i=1}^{m}\sum_{r=1}^{j_i}\frac{A_{ir}}{(x-a_i)^r} + \sum_{i=1}^{n}\sum_{r=1}^{k_i}\frac{B_{ir}x+C_{ir}}{(x^2+b_ix+c_i)^r},$$

where $r(x)$ is a polynomial, which is called the *partial fraction decomposition* of the rational function. This gives us a general strategy for integrating a rational function:

- First, perform the partial fraction decomposition.
- The polynomial term $r(x)$ can be integrated straightforwardly.
- The terms in the partial fraction decomposition involving the real roots can be integrated straightforwardly via:

$$\int \frac{A_{ir}}{(x-a_i)^r} = \begin{cases} -\dfrac{A_{ir}}{(r-1)(x-a_i)^{r-1}}, & \text{if } r \neq 1, \\ A_{ir}\log(x-a_i), & \text{if } r = 1. \end{cases}$$

- For the terms in the partial fraction decomposition involving a simple quadratic factor, i.e. a quadratic factor with $k_i = 1$, we can write:

$$\frac{B_{i1}x+C_{i1}}{x^2+b_{k_i}x+c_i} = \frac{B_{i1}}{2}\frac{2x+b_{k_i}}{x^2+b_{k_i}x+c_i} + \frac{C_{i1}-B_{i1}b_{k_i}/2}{x^2+b_{k_i}x+c_i}.$$

  The first term is now a *logarithmic derivative*, and can be integrated directly. Meanwhile, the remaining term is a constant multiplied by the reciprocal of a quadratic; by completing the square in the denominator, this can be made into a derivative of an *arctangent* or *hyperbolic arctangent*.

- For terms in the partial fraction decomposition involving non-simple quadratic factors, i.e. quadratic factors with $k_i > 1$, things are more complicated. These functions are all integrable, through an arctangent or hyperbolic arctangent substitution, but it is unlikely that integrals of this kind will be in your standard arsenal.

---

We shall now apply this technique in the cases of three given rational functions.

(a) We have:

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right).$$

Hence, integrating, we have:

$$\int \frac{dx}{1-x^2} = -\frac{1}{2}\log(1-x) + \frac{1}{2}\log(1+x) + c = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) + c.$$

This is perfectly consistent with Question 7(e), because $\tanh^{-1}(x) = \frac{1}{2}\log((1+x)/(1-x))$.

(b) Factorising, we have $2x^2 + x - 1 = (2x-1)(x+1)$. Hence decomposing into partial fractions, we have:

$$\frac{3x}{2x^2+x-1} = \frac{1}{2x-1} + \frac{1}{x+1}.$$

Hence, integrating, we have:

$$\int \frac{3x}{2x^2+x-1}\,dx = \frac{1}{2}\log(2x-1) + \log(x+1) + c.$$

(c) First, we perform polynomial division:

$$\frac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4} = -1 + \frac{9+6x-x^2-2x^3}{3+2x-2x^2-2x^3-x^4}.$$

We can now decompose the second term into partial fractions. First, we need to factorise the denominator. We spot that $x=1$ is a factor, so:

$$3+2x-2x^2-2x^3-x^4 = (1-x)\left(3+5x+3x^2+x^3\right)$$

We spot that $x=-1$ is a factor of the second bracket, so:

$$(1-x)(x^3+3x^2+5x+3) = (1-x)(1+x)(x^2+2x+3).$$

The final factor has discriminant $4 - 4\cdot 3 = -8 < 0$, hence there are no more real roots. Thus the partial fractions take the form:

$$\frac{9+6x-x^2-2x^3}{3+2x-2x^2-2x^3-x^4} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{x^2+2x+3}.$$

Multiplying up, we have:

$$9+6x-x^2-2x^3 = A(1+x)(x^2+2x+3) + B(1-x)(x^2+2x+3) + (Cx+D)(1-x^2). \qquad (\dagger)$$

Setting $x=1$ in (†), we have:

$$12A = 12 \qquad \Rightarrow \qquad A = 1.$$

Setting $x=-1$ in (†), we have:

$$4B = 4 \qquad \Rightarrow \qquad B = 1.$$

Setting $x=0$ in (†), we have:

$$9 = 3A + 3B + D \qquad \Rightarrow \qquad D = 9-3-3 = 3.$$

Finally, comparing coefficients of $x^3$ on both sides of (†), we have:

$$-2 = A - B - C \qquad \Rightarrow \qquad C = A - B + 2 = 2.$$

Hence, the partial fractions for the original rational function are:

$$-1 + \frac{1}{1-x} + \frac{1}{1+x} + \frac{2x+3}{x^2+2x+3}.$$

To integrate, we need to split the final term into a logarithmic derivative, and a constant divided by a quadratic. We have:

$$\frac{2x+3}{x^2+2x+3} = \frac{2x+2}{x^2+2x+3} + \frac{1}{x^2+2x+3} = \frac{2x+2}{x^2+2x+3} + \frac{1}{(x+1)^2+2} = \frac{2x+2}{x^2+2x+3} + \frac{1}{2}\frac{1}{((x+1)/\sqrt{2})^2+1}$$

Therefore, integrating the original rational function, we have:

$$\begin{aligned}\int \frac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}\,dx &= \int \left(-1 + \frac{1}{1-x} + \frac{1}{1+x} + \frac{2x+2}{x^2+2x+3} + \frac{1}{2}\frac{1}{((x+1)/\sqrt{2})^2+1}\right)dx \\ &= -x - \log(1-x) + \log(1+x) + \log(2x^2+2x+3) + \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c.\end{aligned}$$

**Integration by parts**

13. Using integration by parts, determine the following integrals:

(a) $\displaystyle\int_{-\pi/2}^{\pi/2} x\sin(2x)\,dx$, (b) $\displaystyle\int_0^\infty xe^{-2x}\,dx$, (c) $\displaystyle\int_0^1 x\log\left(\frac{1}{x}\right)\,dx$, (d) $\displaystyle\int_0^\infty x^3e^{-x^2}\,dx$.

---

➾ **Solution:**

(a) We have:

$$\int_{-\pi/2}^{\pi/2} x\sin(2x)\,dx = \left[-\frac{1}{2}x\cos(2x)\right]_{-\pi/2}^{\pi/2} + \frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos(2x)\,dx$$
$$= \frac{\pi}{2} + \frac{1}{2}\left[\frac{1}{2}\sin(2x)\right]_{-\pi/2}^{\pi/2}$$
$$= \frac{\pi}{2}.$$

(b) We have:

$$\int_0^\infty xe^{-2x}\,dx = \left[-\frac{1}{2}xe^{-2x}\right]_0^\infty + \frac{1}{2}\int_0^\infty e^{-2x}\,dx$$
$$= \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_0^\infty$$
$$= \frac{1}{4}.$$

(c) We have:

$$\int_0^1 x\log\left(\frac{1}{x}\right)\,dx = -\int_0^1 x\log(x)\,dx = -\left[\frac{1}{2}x^2\log(x)\right]_0^1 + \frac{1}{2}\int_0^1 x\,dx$$
$$= \frac{1}{2}\left[\frac{1}{2}x^2\right]_0^1$$
$$= \frac{1}{4}.$$

Here, we used the fact that $x^2\log(x)\to 0$ as $x\to 0$. This is because the polynomial approaches zero faster than the logarithm approaches negative infinity. This is a general phenomena, as we proved using L'Hôpital's rule earlier in the course.

(d) Observe that the derivative of $e^{-x^2}$ is $-2xe^{-x^2}$, so that the integral of $xe^{-x^2}$ is $-\frac{1}{2}e^{-x^2}$. Hence, we have:

$$\begin{aligned}\int_0^\infty x^3 e^{-x^2}\,dx &= \left[-\frac{1}{2}x^2 e^{-x^2}\right]_0^\infty + \frac{1}{2}\int_0^\infty (2x)e^{-x^2}\,dx \\ &= \left[-\frac{1}{2}e^{-x^2}\right]_0^\infty \\ &= \frac{1}{2}.\end{aligned}$$

---

14. By writing each of the following functions $f(x)$ in the form $1 \cdot f(x)$, and using integration by parts, determine their indefinite integrals:

(a) $\log(x)$, (b) $\log^3(x)$, (c) $\cosh^{-1}(x)$, (d) $\tanh^{-1}(x)$, (e) $\sin(\log(x))$.

---

**Solution:**

(a) $\displaystyle\int 1 \cdot \log(x)\,dx = x\log(x) - \int x \cdot (1/x)\,dx = x\log(x) - x + c.$

(b) Here, we use integration by parts multiple times. We have:

$$\begin{aligned}\int 1 \cdot \log^3(x)\,dx &= x\log^3(x) - 3\int \log^2(x)\,dx \\ &= x\log^3(x) - 3x\log^2(x) + 6\int \log(x)\,dx \\ &= x\log^3(x) - 3x\log^2(x) + 6x\log(x) - 6x + c.\end{aligned}$$

(c) First, observe that:

$$\int 1 \cdot \cosh^{-1}(x)\,dx = x\cosh^{-1}(x) - \int \frac{x}{\sqrt{x^2-1}}\,dx.$$

Since the derivative of $x^2$ is $2x$, the remaining integral is of the form $f'(x)f(x)^\alpha$, hence can be directly integrated. We have:

$$x\cosh^{-1}(x) - \sqrt{x^2-1} + c.$$

(d) $\displaystyle\int 1 \cdot \tanh^{-1}(x)\,dx = x\tanh^{-1}(x) - \int \frac{x}{1-x^2}\,dx = x\tanh^{-1}(x) + \frac{1}{2}\log(x^2-1) + c.$

(e) We begin by performing one integration by parts:

$$\int 1 \cdot \sin(\log(x))\,dx = x\sin(\log(x)) - \int \cos(\log(x))\,dx.$$

We now iterate, performing a second integration by parts:

$$\int \sin(\log(x))\,dx = x\sin(\log(x)) - x\cos(\log(x)) - \int \sin(\log(x))\,dx.$$

But notice that the new integral is the same as the original one - hence, rearranging this equation, we see that:

$$\int \sin(\log(x))\,dx = \frac{x}{2}\left(\sin(\log(x)) - \cos(\log(x))\right) + c.$$

### Reduction formulae

15.(a) Show that for $n \geq 1$, we have:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx,$$

Hence, evaluate $\int \sin^6(x)\,dx$.

(b) Using (a), show that the integral $I_n = \int\limits_0^{\pi/2} \sin^n(x)\,dx$ satisfies $I_n = (n-1)I_n/n$. Hence, evaluate $I_2$ and $I_4$.

---

➾ **Solution:** (a) Integrating by parts, we have:

$$\int \sin^n(x)\,dx = \int \sin(x)\cdot\sin^{n-1}(x)\,dx = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x)\,dx.$$

Expanding $\cos^2(x) = 1 - \sin^2(x)$, we can rearrange this to read:

$$\int \sin^n(x)\,dx = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx.$$

Rearranging, we then have:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx,$$

as required.

This allows us to evaluate the given integral, by applying the recurrence relation repeatedly:

$$\begin{aligned}
\int \sin^6(x)\,dx &= -\frac{1}{6}\cos(x)\sin^5(x) + \frac{5}{6}\int \sin^4(x)\,dx \\
&= -\frac{1}{6}\cos(x)\sin^5(x) + \frac{5}{6}\left(-\frac{1}{4}\cos(x)\sin^3(x) + \frac{3}{4}\int \sin(x)\,dx\right) \\
&= -\frac{1}{6}\cos(x)\sin^5(x) - \frac{5}{24}\cos(x)\sin^3(x) - \frac{5}{8}\cos(x) + c.
\end{aligned}$$

---

(b) Simply inserting the limits into our recurrence relation, we have:

$$\int\limits_0^{\pi/2} \sin^n(x)\,dx = \left[-\frac{1}{n}\cos(x)\sin^{n-1}(x)\right]_0^{\pi/2} + \frac{n-1}{n}\int\limits_0^{\pi/2} \sin^{n-2}(x)\,dx = \frac{n-1}{n}\int\limits_0^{\pi/2} \sin^{n-2}(x)\,dx.$$

This immediately gives $I_n = (n-1)I_{n-2}/n$. To evaluate the given integrals, we note:

$$I_0 = \int\limits_0^{\pi/2} 1\,dx = \frac{\pi}{2}.$$

Hence:

$$I_2 = \frac{1}{2}I_0 = \frac{\pi}{4}, \qquad I_4 = \frac{3}{4}I_2 = \frac{3\pi}{8}.$$

16. Establish reduction formulae for each of the following parametric integrals:

$$\text{(a) } I_n = \int_0^\infty x^n e^{-x^2}\,dx, \quad \text{(b) } J_n = \int_0^\pi x^{2n}\cos(x)\,dx, \quad \text{(c) } K_n = \int_0^\infty x^{n-1}e^{-x}\,dx, \quad \text{(d) } L_n = \int_0^\infty \frac{dx}{(1+x^2)^n}.$$

Hence: (i) evaluate $I_3, I_5$; (ii) evaluate $J_3$; (iii) establish a general formula for $K_n$; (iv) evaluate $L_4$. ($*$) Using part (c), suggest a reasonable definition of $z!$ where $z$ is a complex number. Will this work for all complex numbers?

---

**Solution:**

(a) Since the derivative of $e^{-x^2}$ is $-2xe^{-x^2}$, we can integrate $xe^{-x^2}$ to get $-\frac{1}{2}e^{-x^2}$. Hence:

$$I_n = \int_0^\infty x^n e^{-x^2}\,dx = \left[-\frac{1}{2}x^{n-1}e^{-x^2}\right]_0^\infty + \frac{(n-1)}{2}\int_0^\infty x^{n-2}e^{-x^2}\,dx = \frac{(n-1)}{2}I_{n-2},$$

assuming that $n \geq 2$. To answer (i), we note that this implies:

$$I_3 = I_1 = \int_0^\infty xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^\infty = \frac{1}{2}.$$

We also have:

$$I_5 = 2I_3 = 1.$$

(b) Integrating by parts twice, we have:

$$J_n = \int_0^\pi x^{2n}\cos(x)\,dx = \left[x^{2n}\sin(x)\right]_0^\pi - 2n\int_0^\pi x^{2n-1}\sin(x)\,dx$$

$$= 2n\left[x^{2n-1}\cos(x)\right]_0^\pi - 2n(2n-1)\int_0^\pi x^{2n-2}\cos(x)\,dx$$

$$= -2n\pi^{2n-1} - 2n(2n-1)J_{n-1}.$$

To answer (ii), we note that this implies:

$$J_3 = -6\pi^5 - 30J_2 = -6\pi^5 - 30\left(-4\pi^3 - 12J_1\right) = -6\pi^5 + 120\pi^3 + 360\left(-2\pi\right) = -6\pi^5 + 120\pi^3 - 720\pi,$$

since $J_0 = 0$.

(c) Integrating by parts, we have:

$$K_n = \int_0^\infty x^{n-1}e^{-x}\,dx = \left[-x^{n-1}e^{-x}\right]_0^\infty + (n-1)\int_0^\infty x^{n-2}e^{-x}\,dx = (n-1)K_{n-1}.$$

In particular, to answer (iii), we can use this relation iteratively to give:

$$K_n = (n-1)K_{n-1} = (n-1)(n-2)K_{n-2} = (n-1)(n-2)(n-3)K_{n-3} = \cdots = (n-1)!K_1.$$

But we have:

$$K_1 = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty = 1.$$

Hence, $K_n = (n-1)!$ for all positive integers $n$.

(d) Here, we can only integrate $1$ and differentiate the integrand. We have:

$$L_n = \int_0^\infty \frac{dx}{(1+x^2)^n} = \left[\frac{x}{(1+x^2)^n}\right]_0^\infty + 2n\int_0^\infty \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

$$= 2n\int_0^\infty \frac{(1+x^2)-1}{(1+x^2)^{n+1}}\,dx$$

$$= 2n\int_0^\infty \frac{1}{(1+x^2)^n}\,dx - 2n\int_0^\infty \frac{1}{(1+x^2)^{n+1}}\,dx.$$

This can be written as $L_n = 2nL_n - 2nL_{2n+1}$, which can be rearranged to read:

$$L_{n+1} = \frac{2n-1}{2n}L_n,$$

for all $n \geq 1$. To answer (iv), and evaluate $L_4$, we note:

$$L_4 = \frac{5}{6}L_3 = \frac{5}{6}\cdot\frac{3}{4}L_2 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}L_1.$$

But now we have:

$$L_1 = \int_0^\infty \frac{dx}{1+x^2} = [\arctan(x)]_0^\infty = \frac{\pi}{2}.$$

Hence:

$$L_4 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}.$$

---

From part (c), we know that:

$$(n-1)! = \int_0^\infty x^{n-1}e^{-x}\,dx.$$

This suggests that a possible definition of $z!$ for $z$ a complex number is:

$$z! = \int_0^\infty x^z e^{-x}\,dx.$$

Importantly, for this definition to make sense, the integral needs to converge. For large values of $x$, the integrand is exponentially suppressed, so will be convergent. For small values of $x$, the integrand resembles $x^z e^{-z} \approx x^z$, which gives a contribution to the integral:

$$\int_0^\epsilon x^z\,dx = \left[\frac{x^{z+1}}{z+1}\right]_0^\epsilon.$$

This is finite only if $\mathrm{Re}(z)+1>0$, hence this definition only makes sense if $\mathrm{Re}(z) > -1$.

We can extend the definition by *analytic continuation* however, a tool from complex analysis, which you will learn about in Part IB Mathematics for Natural Sciences.

**Miscellaneous integrals**

17. Evaluate the following integrals, using the most efficient method in each case:

(a) $\displaystyle\int_4^9 \frac{dx}{\sqrt{x}-1}$

(b) $\displaystyle\int_{\pi/3}^{\pi/4} \frac{1+\tan^2(x)}{(1+\tan(x))^2}\,dx$

(c) $\displaystyle\int \frac{e^{2x}-2e^x}{e^{2x}+1}\,dx$

(d) $\displaystyle\int \frac{dx}{1+3\cos^2(x)}$

(e) $\displaystyle\int_2^3 \frac{2x+1}{x(x+1)}\,dx$

(f) $\displaystyle\int \frac{1}{2\sqrt{x}}e^{\sqrt{x}}\,dx$

(g) $\displaystyle\int x^3 e^{-x^4}\,dx$

(h) $\displaystyle\int \left(\frac{\sin(2x)}{\sin^2(x)+\log(x)} + \frac{1}{x(\sin^2(x)+\log(x))}\right)dx$

(i) $\displaystyle\int x\sqrt{3-2x}\,dx$

(j) $\displaystyle\int \frac{\sin(x)}{\cos^2(x)-5\cos(x)+6}\,dx$

(k) $\displaystyle\int \frac{\log(x)}{x^4}\,dx$

(l) $\displaystyle\int \sqrt{1-x^2}\,dx$

(m) $\displaystyle\int_{\pi/3}^{\pi/2} \tan(x)\cos^4(x)\,dx$

(n) $\displaystyle\int_1^5 x^2\log(x)\,dx$

(o) $\displaystyle\int e^x\sinh(3x)\,dx$

(p) $\displaystyle\int \frac{\arctan(x)}{x^2}\,dx$

(q) $\displaystyle\int_{e^3}^{e^4} \frac{3\log(x)-4}{x\log^2(x)-3x\log(x)+2x}\,dx$

(r) $\displaystyle\int_0^{\pi/6} x\sin(3x)\,dx$

(s) $\displaystyle\int \sin(2x)e^{\sin^2(x)}\,dx$

(t) $\displaystyle\int \frac{dx}{\cos^2(x)(\tan^3(x)-\tan(x))}$

(u) $\displaystyle\int_{-1/\pi}^{1/\pi} \sin^2(3x^3+2x)\log\left[\frac{1-x^5}{1+x^5}\right]dx$

(v) $\displaystyle\int \sin(2x)\cos(x)\,dx$

(w) $\displaystyle\int x\log(x)\,dx$

(x) $\displaystyle\int \frac{dx}{x\log(x)}$

(y) $\displaystyle\int \frac{\sinh^3(x)}{\cosh^2(x)}\,dx$

(z) $\displaystyle\int \frac{1}{\sin^2(3x+1)}\,dx$

---

**Solution:**

(a) Let $u=\sqrt{x}$, to clear the square root. Then $u^2=x$, so that $2udu=dx$. Thus we have:

$$\int_4^9 \frac{dx}{\sqrt{x}-1} = \int_2^3 \frac{2udu}{u-1} = \int_2^3 \left(\frac{2(u-1)}{u-1}+\frac{2}{u-1}\right)du = 2+2\left[\log(u-1)\right]_2^3 = 2+2\log(2).$$

(b) Note that $1+\tan^2(x) = \sec^2(x)$, so the numerator is just $\sec^2(x)$ in disguise. This suggests the obvious substitution $u = \tan(x)$, which gives $du = \sec^2(x)dx$ and the limits change from $[\pi/3, \pi/4] \mapsto [\sqrt{3}, 1]$. Hence we have:

$$\int\limits_{\pi/3}^{\pi/4} \frac{1+\tan^2(x)}{(1+\tan(x))^2}\, dx = \int\limits_{\sqrt{3}}^{1} \frac{du}{(1+u)^2} = \left[-\frac{1}{1+u}\right]_{\sqrt{3}}^{1} = \frac{1}{1+\sqrt{3}} - \frac{1}{2}.$$

(c) An obvious substitution is $u = e^x$. We have $du = e^x dx$, so:

$$\begin{aligned}\int \frac{e^{2x} - 2e^x}{e^{2x}+1}\, dx &= \int \frac{u-2}{u^2+1}\, du \\ &= \int \left(\frac{u}{u^2+1} - \frac{2}{u^2+1}\right) du \\ &= \frac{1}{2}\log(u^2+1) - 2\arctan(u) + c \\ &= \frac{1}{2}\log(e^{2x}+1) - 2\arctan(e^x) + c.\end{aligned}$$

(d) This is quite sneaky. Multiply the numerator and denominator by $\sec^2(x)$, to give:

$$\int \frac{\sec^2(x)dx}{\sec^2(x)+3} = \int \frac{\sec^2(x)dx}{\tan^2(x)+4}.$$

Now, the obvious substitution is $u = \tan(x)$. We have $du = \sec^2(x)dx$, which gives:

$$\int \frac{du}{u^2+4} = \frac{1}{4}\int \frac{du}{(u/2)^2+1} = \frac{2}{4}\arctan\left(\frac{u}{2}\right) + c = \frac{1}{2}\arctan\left(\frac{1}{2}\tan(x)\right) + c.$$

(e) Note that:

$$\int\limits_{2}^{3} \frac{2x+1}{x(x+1)}\, dx = \int\limits_{2}^{3} \frac{2x+1}{x^2+x}\, dx = \left[\log(x^2+x)\right]_2^3 = \log(12) - \log(6) = \log(2).$$

(f) Note that the derivative of $\sqrt{x}$ is $1/2\sqrt{x}$. This gives:

$$\int \frac{1}{2\sqrt{x}} e^{\sqrt{x}}\, dx = e^{\sqrt{x}} + c,$$

by thinking about the reverse chain rule.

(g) Note that the derivative of $-x^4$ is $-4x^3$. Hence:

$$\int x^3 e^{-x^4}\, dx = -\frac{1}{4}e^{-x^4} + c,$$

by thinking about the reverse chain rule.

(h) We have:

$$\begin{aligned}\int \left(\frac{\sin(2x)}{\sin^2(x)+\log(x)} + \frac{1}{x(\sin^2(x)+\log(x))}\right) dx &= \int \left(\frac{2\sin(x)\cos(x) + 1/x}{\sin^2(x)+\log(x)}\right) dx \\ &= \log\left(\sin^2(x)+\log(x)\right) + c.\end{aligned}$$

(i) Make the substitution $u = 3 - 2x$, to clear the square root. Then $du = -2dx$, and $x = \frac{3}{2} - \frac{1}{2}u$. Hence:

$$\begin{aligned}\int x\sqrt{3-2x}\,dx &= -\frac{1}{2}\int\left(\frac{3}{2}-\frac{1}{2}u\right)u^{1/2}\,du\\ &= \frac{1}{4}\int\left(u^{3/2}-3u^{1/2}\right)\,du\\ &= \frac{1}{4}\left(\frac{2}{5}u^{5/2}-2u^{3/2}\right)+c\\ &= \frac{1}{10}(3-2x)^{5/2}-\frac{1}{2}(3-2x)^{3/2}+c.\end{aligned}$$

(j) Evidently, the substitution $u = \cos(x)$ will work. We have $du = -\sin(x)dx$, hence:

$$\int \frac{\sin(x)}{\cos^2(x)-5\cos(x)+6}\,dx = -\int\frac{du}{u^2-5u+6} = -\int\frac{du}{(u-5/2)^2-1/4} = 4\int\frac{du}{1-(2u-5)^2}.$$

Now, using a standard inverse hyperbolic integral, we have:

$$4\int\frac{du}{1-(2u-5)^2} = 2\tanh^{-1}(2u-5)+c = 2\tanh^{-1}(2\cos(x)-5)+c.$$

(k) We use integration by parts. We have:

$$\int\frac{\log(x)}{x^4}\,dx = -\frac{\log(x)}{3x^3}+\frac{1}{3}\int\frac{1}{x^4}\,dx = -\frac{\log(x)}{3x^3}-\frac{1}{9}\frac{1}{x^3}+c.$$

(l) Here, a trigonometric substitution is appropriate. Let $x = \cos(\theta)$. Then $dx = -\sin(\theta)d\theta$, which gives:

$$\int\sqrt{1-x^2}\,dx = -\int\sqrt{1-\cos^2(\theta)}\sin(\theta)\,d\theta = -\int\sin^2(\theta)\,d\theta.$$

Using the standard trick for integrating $\sin^2(\theta)$, we have:

$$-\int\sin^2(\theta)\,d\theta = \frac{1}{2}\int\left(\cos(2\theta)-1\right)\,d\theta = \frac{1}{2}\left(\frac{\sin(2\theta)}{2}-\theta\right)+c = \frac{1}{4}\sin(2\arcsin(x))-\frac{1}{2}\arcsin(x)+c.$$

We can slightly simplify this by using $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\sin(\theta)\sqrt{1-\sin^2(\theta)}$. This leaves:

$$\frac{1}{2}x\sqrt{1-x^2}-\frac{1}{2}\arcsin(x)+c.$$

(m) We have:

$$\int_{\pi/3}^{\pi/2}\tan(x)\cos^4(x)\,dx = \int_{\pi/3}^{\pi/2}\sin(x)\cos^3(x)\,dx = \left[-\frac{1}{4}\cos^4(x)\right]_{\pi/3}^{\pi/2} = \frac{1}{4}\cdot\left(\frac{1}{2}\right)^4 = \frac{1}{64}.$$

(n) We have:

$$\int_1^5 x^2 \log(x)\, dx = \left[\frac{1}{3}x^3 \log(x)\right]_1^5 - \frac{1}{3}\int_1^5 x^2\, dx = \frac{125}{3}\log(5) - \frac{1}{9}(125-1) = \frac{125}{3}\log(5) - \frac{124}{9}.$$

(o) Writing the hyperbolic function in terms of exponentials, we have:

$$\int e^x \sinh(3x)\, dx = \frac{1}{2}\int \left(e^{4x} - e^{-2x}\right)\, dx = \frac{1}{2}\left(\frac{1}{4}e^{4x} - \frac{1}{2}e^{-2x}\right) + c = \frac{1}{8}e^{4x} - \frac{1}{4}e^{-2x} + c.$$

Alternatively, this can be done by parts.

(p) Here, we can integrate by parts:

$$\int \frac{\arctan(x)}{x^2}\, dx = -\frac{\arctan(x)}{x} + \int \frac{1}{x(1+x^2)}\, dx.$$

Using partial fractions, we have:

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} \qquad \Rightarrow \qquad 1 = A(1+x^2) + Bx^2 + Cx.$$

Here, $A = 1, B = -1, C = 0$. Thus we have:

$$-\frac{\arctan(x)}{x} + \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right)\, dx = -\frac{\arctan(x)}{x} + \log(x) - \frac{1}{2}\log(1+x^2) + c.$$

(q) Evidently, a sensible substitution is $u = \log(x)$; note that we have $du = dx/x$, and the limits transform as $[e^3, e^4] \mapsto [3, 4]$. Thus:

$$\begin{aligned}
\int_{e^3}^{e^4} \frac{3\log(x) - 4}{x\log^2(x) - 3x\log(x) + 2x}\, dx &= \int_3^4 \frac{3u-4}{u^2-3u+2}\, du \\
&= \frac{3}{2}\int_3^4 \frac{2u-3}{u^2-3u+2}\, du + \frac{1}{2}\int_3^4 \frac{1}{u^2-3u+2}\, du \\
&= \frac{3}{2}\left[\log\left(u^2-3u+2\right)\right]_3^4 + \frac{1}{2}\int_3^4 \frac{1}{(u-3/2)^2 - 1/4}\, du \\
&= \frac{3}{2}\left(\log(6) - \log(2)\right) - 2\int_3^4 \frac{du}{1-(2u-3)^2}\, du \\
&= \frac{3}{2}\log(3) - \left[\operatorname{artanh}(2u-3)\right]_3^4 \\
&= \frac{3}{2}\log(3) - \operatorname{artanh}(5) + \operatorname{artanh}(3).
\end{aligned}$$

If we really want to tidy everything up, we could simplify the two inverse hyperbolic functions. We let:

$$t = \operatorname{artanh}(5) - \operatorname{artanh}(3).$$

Now by the hyperbolic compound angle identity, we have:

$$\tanh(t) = \frac{5-3}{1-15} = -\frac{2}{14} = -\frac{1}{7}.$$

This leaves us with:

$$\frac{3}{2}\log(3) + \operatorname{artanh}\left(\frac{1}{7}\right).$$

(r) This can be done easily with integration by parts:

$$\begin{aligned}\int_0^{\pi/6} x\sin(3x)\,dx &= \left[-\frac{1}{3}x\cos(3x)\right]_0^{\pi/6} + \frac{1}{3}\int_0^{\pi/6}\cos(3x)\,dx \\ &= \frac{1}{3}\left[\frac{1}{3}\sin(3x)\right]_0^{\pi/6} \\ &= \frac{1}{9}.\end{aligned}$$

(s) Observe that the derivative of $\sin^2(x)$ is $2\sin(x)\cos(x) = \sin(2x)$. Hence:

$$\int \sin(2x)e^{\sin^2(x)}\,dx = e^{\sin^2(x)} + c.$$

(t) The integral in question can be rewritten in the form:

$$\int \frac{\sec^2(x)}{\tan^3(x)-\tan(x)}\,dx.$$

This suggests the substitution $u = \tan(x)$, which gives $du = \sec^2(x)dx$. Hence we have:

$$\int \frac{du}{u^3-u} = \int \frac{du}{u(u-1)(u+1)}.$$

We decompose the integrand into partial fractions via:

$$\frac{1}{u(u-1)(u+1)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1}.$$

This implies:

$$1 = A(u-1)(u+1) + Bu(u+1) + Cu(u-1).$$

Taking $u=0$ gives $A=-1$. Taking $u=1$ gives $B=1/2$. Taking $u=-1$ gives $C=-1/2$. Thus we have:

$$\int \left(-\frac{1}{u} + \frac{1}{2(u-1)} - \frac{1}{2(u+1)}\right)\,du = -\log(u) + \frac{1}{2}\log(u-1) - \frac{1}{2}\log(u+1) + c.$$

Thus the final integral is:

$$\frac{1}{2}\log\left(\frac{u-1}{u^2(u+1)}\right) + c = \frac{1}{2}\log\left(\frac{\tan(x)-1}{\tan^2(x)(\tan(x)+1)}\right) + c.$$

(u) This is a trick question - it is an odd function integrated over a symmetric domain, so the integral is just zero.

(v) We have:

$$\int \sin(2x)\cos(x)\,dx = 2\int \sin(x)\cos^2(x)\,dx = -\frac{2}{3}\cos^3(x) + c.$$

(w) Using integration by parts, we have:

$$\int x\log(x)\,dx = \frac{1}{2}x^2\log(x) - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\log(x) - \frac{1}{4}x^2 + c.$$

(x) Here, an obvious substitution is $u = \log(x)$, so that $du = dx/x$. We then have:

$$\int \frac{dx}{x\log(x)} = \int \frac{du}{u} = \log(u) + c = \log(\log(x)) + c.$$

(y) This integral can be rewritten as:

$$\int \frac{\sinh^3(x)}{\cosh^2(x)}\,dx = \int \frac{(\cosh^2(x) - 1)}{\cosh^2(x)}\sinh(x)\,dx,$$

suggesting the substitution $u = \cosh(x)$, which gives $du = \sinh(x)dx$. We then have:

$$\int \frac{u^2 - 1}{u^2}\,du = \int \left(1 - \frac{1}{u^2}\right)\,du = u + \frac{1}{u} + c = \cosh(x) + \operatorname{sech}(x) + c.$$

(z) An easy one to finish! This is a standard integral, from the Basic Integrals section of the start of the sheet. We have:

$$\int \frac{1}{\sin^2(3x+1)}\,dx = \int \operatorname{cosec}^2(3x+1)\,dx = -\frac{1}{3}\cot(3x+1) + c.$$

**The fundamental theorem of calculus**

18. State both parts of the *fundamental theorem of calculus*. Use the fundamental theorem of calculus to evaluate the following derivatives:

(a) $\displaystyle\frac{d}{dx}\int_1^x \frac{\log(t)\sin^2(t)}{t^2+7}\,dt$, (b) $\displaystyle\frac{d}{dx}\left[\sum_{n=0}^{N}\binom{N}{n}\int_n^x \sin(y^2+y^6)\,dy\right]$, (c) $\displaystyle\frac{d}{dx}\left[\sin(x)\int_x^0 \sin(\cos(t))\,dt\right]$.

**Solution:** The fundamental theorem of calculus states two things:

- Integration reverses differentiation, in the sense that:

$$\int_a^b \frac{df}{dx}\,dx = f(b)-f(a).$$

- Differentiation reverses integration, in the sense that:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Applying this result to the given derivatives:

(a) Straightforwardly, this derivative is given by: $\dfrac{\log(x)\sin^2(x)}{x^2+7}$.

(b) Since differentiation is linear, we have:

$$\begin{aligned}\frac{d}{dx}\left[\sum_{n=0}^{N}\binom{N}{n}\int_n^x \sin(y^2+y^6)\,dy\right] &= \sum_{n=0}^{N}\binom{N}{n}\frac{d}{dx}\int_n^x \sin(y^2+y^6)\,dy\\ &= \sum_{n=0}^{N}\binom{N}{n}\sin(x^2+x^6)\\ &= \sin(x^2+x^6)\sum_{n=0}^{N}\binom{N}{n}.\end{aligned}$$

Here's a fun thing: the sum of the binomial coefficients is always $2^N$. To see why, study the identity (which is just the binomial expansion):

$$2^N = (1+1)^N = \sum_{n=0}^{N}\binom{N}{n}1^n\cdot 1^{N-n} = \sum_{n=0}^{N}\binom{N}{n}.$$

Hence, the derivative simplifies to $2^N\sin(x^2+x^6)$.

(c) First, we have:

$$\int_x^0 \sin(\cos(t))\,dt = -\int_0^x \sin(\cos(t))\,dt.$$

Hence, using the product rule, we have:

$$\frac{d}{dx}\left[\sin(x)\int_x^0 \sin(\cos(t))\,dt\right] = -\frac{d}{dx}\left[\sin(x)\int_0^x \sin(\cos(t))\,dt\right] = -\cos(x)\int_0^x \sin(\cos(t))\,dt - \sin(x)\sin(\cos(x)).$$

This cannot be further simplified (we cannot perform the integral in terms of elementary functions).

19. Without evaluating the integrals, determine the local extrema of the functions $F_1, F_2$ defined by:

$$\text{(a) } F_1(x) = \int_0^x t^2 \sin^2(t)\, dt, \qquad \text{(b) } F_2(x) = \int_{-\infty}^x e^{-t^2}\, dt.$$

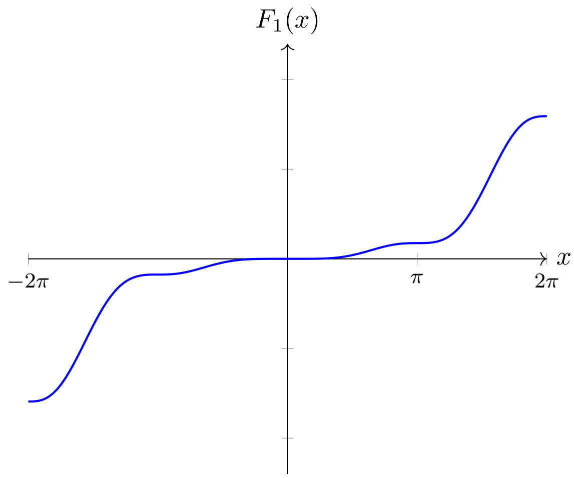
Hence, sketch the graphs of the functions $F_1, F_2$. [*Note:* $F_2(x) \to \sqrt{\pi}$ *as* $x \to \infty$*; see Question 23!*]

---

➾ **Solution:** (a) We have:

$$F_1'(x) = x^2 \sin^2(x).$$

Hence, the stationary points of the function occur at $x = n\pi$, for $n$ an integer. The function is zero at $x = 0$, and is strictly increasing (because as $x$ increases, more area under the function $t^2 \sin^2(t)$ contributes!). Therefore, the function looks like a series of inflection points.

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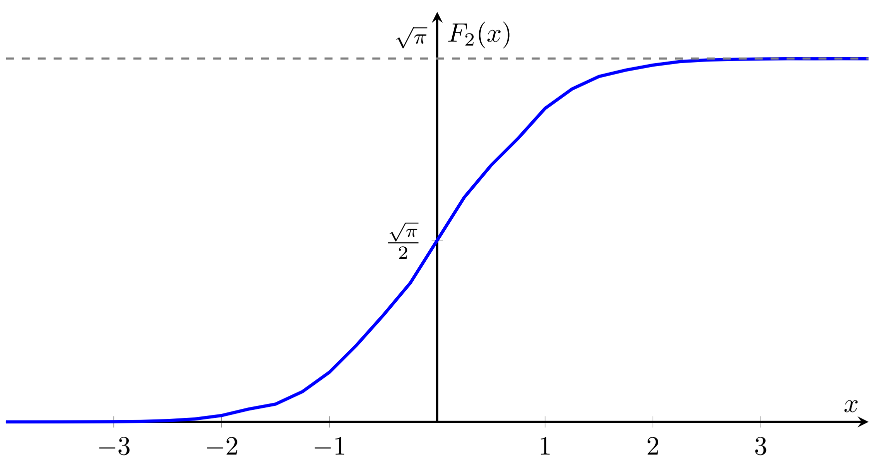
(b) We have:

$$F_2'(x) = e^{-x^2},$$

so this function has no stationary points. The function is again strictly increasing. We are given that $F_2(x) \to \sqrt{\pi}$ as $x \to \infty$, and we note that $F_2(x) \to 0$ as $x \to -\infty$. The function is everywhere positive. Hence the graph looks like:



**(†) Leibniz's integral rule**

20. Using the multivariable chain rule (we'll study it properly next term!), derive *Leibniz's integral rule*:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = f(x,b(x))\frac{db}{dx} - f(x,a(x))\frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt.$$

Give a geometric explanation for the rule in terms of changing areas. Verify that the rule holds in the following cases:

(a) $a(x) = 0$, $b(x) = 1 + x$, and $f(x,t) = t(x-t)$;

(b) $a(x) = \pi x^2$, $b(x) = x$, and $f(x,t) = 2x^2 t + x\sin(t)$.

---

➬ **Solution:** Define a multivariable function:

$$F(a,b,x) = \int_a^b f(x,t)\,dt.$$

Consider the case where the arguments $a, b$ depend on $x$, so that we are considering $F(a(x), b(x), x)$. To take the derivative of this function with respect to $x$, we apply the chain rule separately to each argument, and then sum the results; this is the *multivariable chain rule*, which we shall study properly in Lent term. We have:

$$\frac{d}{dx}F(a(x),b(x),x) = \frac{\partial F}{\partial a}a'(x) + \frac{\partial F}{\partial b}b'(x) + \frac{\partial F}{\partial x}.$$

Observe that, by the fundamental theorem of calculus:

$$\frac{\partial F}{\partial b} = \frac{\partial}{\partial b}\int_a^b f(x,t)\,dt = f(x,b).$$

Similarly, observe that:

$$\frac{\partial F}{\partial a} = \frac{\partial}{\partial a}\int_a^b f(x,t)\,dt = -\frac{\partial}{\partial a}\int_b^a f(x,t)\,dt = -f(x,a).$$

Finally, observe that the partial derivative of $F$ with respect to its third argument is just:

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial f}{\partial x}(x,t)\,dt,$$

since the arguments $a, b$ are kept constant. Putting everything together, this gives:

$$\frac{d}{dx}F(a(x),b(x),x) = f(x,b(x))\frac{db}{dx} - f(x,a(x))\frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)\,dt,$$

as required.

This has a very nice interpretation in terms of changing areas. The integral:

$$\int_{a(x)}^{b(x)} f(x,t)\,dt$$

represents the area under the curve $f(x,t)$, *considered as a function of* $t$, between $t=a(x)$ and $t=b(x)$. When we ask about the derivative with respect to $x$, we ask how this area changes as a function of $x$. There are three ways that this can happen:

- The function itself can change. If $x$ changes by an amount $\Delta x$, resulting in a change $\Delta f$ in $f$, then the change in area is:
  $$\Delta A = \int_a^b \Delta f(x,t)\,dt.$$
  Thus the rate of change of area is given by:
  $$\int_a^b \frac{\partial f}{\partial x}(x,t)\,dt.$$
- The upper limit can change. If $x$ changes by an amount $\Delta x$, resulting in a change $\Delta b$ in $b$, then the change in area is:
  $$\Delta A = \int_a^{b+\Delta b} f(x,t)\,dt - \int_a^b f(x,t)\,dt = \int_b^{b+\Delta b} f(x,t)\,dt \approx f(x,b)\Delta b.$$
  Hence the rate of change of area is given by:
  $$\frac{db}{dx} f(x,b(x)).$$
- Finally, the lower limit can change. If $x$ changes by an amount $\Delta x$, resulting in a change $\Delta a$ in $a$, then the change in area is:
  $$\Delta A = \int_{a+\Delta a}^{b} f(x,t)\,dt - \int_a^b f(x,t)\,dt = -\int_a^{a+\Delta a} f(x,t)\,dt \approx -f(x,a)\Delta a.$$
  Hence the rate of change of area is given by:
  $$-\frac{da}{dx} f(x,a(x)).$$

Putting together all these possible ways of changing the area, we get the Leibniz integral rule.

---

(a) For $a(x)=0, b(x)=1+x$ and $f(x,t)=t(x-t)$, the integral in question is:

$$\int_0^{1+x} t(x-t)\,dt = \int_0^{1+x} \left(tx-t^2\right)\,dt = \left[\frac{1}{2}xt^2 - \frac{1}{3}t^3\right]_0^{1+x} = \frac{1}{2}x(1+x)^2 - \frac{1}{3}(1+x)^3 = \frac{1}{6}x^3 - \frac{1}{2}x - \frac{1}{3}.$$

Taking the derivative, we see that:

$$\frac{d}{dx}\int_0^{1+x} t(x-t)\,dt = \frac{1}{2}x^2 - \frac{1}{2}.$$

Compare with the result from using the Leibniz integral rule. We have:

$$\frac{d}{dx}\int_0^{1+x} t(x-t)\,dt = (1+x)(x-(1+x))\frac{d}{dx}(1+x) + \int_0^{1+x} t\,dt = -(1+x) + \frac{1}{2}(1+x)^2 = \frac{1}{2}x^2 - \frac{1}{2}.$$

Thus, the rule works.

---

(b) For $a(x) = \pi x^2, b(x) = x$ and $f(x,t) = 2x^2t + x\sin(t)$, the integral in question is:

$$\begin{aligned}\int_{\pi x^2}^{x} \left(2x^2t + x\sin(t)\right)\,dt &= \left[x^2t^2 - x\cos(t)\right]_{\pi x^2}^{x} \\ &= x^4 - x\cos(x) - \pi^2x^6 + x\cos(\pi x^2).\end{aligned}$$

Taking the derivative, we see that:

$$\frac{d}{dx}\int_{\pi x^2}^{x} \left(2x^2t + x\sin(t)\right)\,dt = 4x^3 - \cos(x) + x\sin(x) - 6\pi^2x^5 + \cos(\pi x^2) - 2\pi x^2\sin(\pi x^2).$$

Compare with the result from using the Leibniz integral rule. We have:

$$\begin{aligned}\frac{d}{dx}\int_{\pi x^2}^{x} \left(2x^2t + x\sin(t)\right)\,dt &= \left(2x^3 + x\sin(x)\right) - \left(2\pi x^4 + x\sin(\pi x^2)\right)(2\pi x) + \int_{\pi x^2}^{x} \left(4xt + \sin(t)\right)\,dt \\ &= 2x^3 + x\sin(x) - 4\pi^2x^5 - 2\pi x^2\sin(\pi x^2) + \left[2xt^2 - \cos(t)\right]_{\pi x^2}^{x} \\ &= 2x^3 + x\sin(x) - 4\pi^2x^5 - 2\pi x^2\sin(\pi x^2) + 2x^3 - \cos(x) - 2\pi^2x^5 + \cos(\pi x^2) \\ &= 4x^3 - \cos(x) + x\sin(x) - 6\pi^2x^5 + \cos(\pi x^2) - 2\pi x^2\sin(\pi x^2).\end{aligned}$$

Thus, the rule works again!

---

21. Evaluate the limit $\displaystyle\lim_{x\to\infty}\frac{d}{dx}\int_{\sin(1/x)}^{\sqrt{x}} \frac{2t^4+1}{(t-2)(t^2+3)}\,dt.$

---

➾ **Solution:** We start by evaluating the derivative using the Leibniz integral rule. We have:

$$\frac{d}{dx}\int_{\sin(1/x)}^{\sqrt{x}} \frac{2t^4+1}{(t-2)(t^2+3)}\,dt = \frac{2x^2+1}{(\sqrt{x}-2)(x+3)}\cdot\frac{1}{2\sqrt{x}} - \frac{2\sin^4(1/x)+1}{(\sin(1/x)-2)(\sin^2(1/x)+3)}\left(-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right).$$

We now take the limit as $x\to\infty$. The second term vanishes, because $1/x^2 \to 0$ as $x\to\infty$, and $\sin(1/x), \cos(1/x)$ are bounded between $-1$ and $1$.

The first term gives:

$$\lim_{x\to\infty}\frac{2x^2+1}{(\sqrt{x}-2)(x+3)}\cdot\frac{1}{2\sqrt{x}} = \lim_{x\to\infty}\frac{2+1/x^2}{2(1-2/\sqrt{x})(1+3/x)} = 1.$$

Hence, the limit is $1$.

22. For all values of $x$, evaluate the integrals:

$$\text{(a) } f(x) = \int_0^1 \frac{t^x - 1}{\log(t)}\, dt, \qquad \text{(b) } g(x) = \int_0^\infty \frac{\log(1+x^2t^2)}{1+t^2}\, dt, \qquad \text{(c) } h(x) = \int_0^1 \frac{\sin(x\log(t))}{\log(t)}\, dt,$$

by considering the derivatives $f'(x), g'(x), h'(x)$. This method is sometimes called *Feynman's trick* for integration.

---

➾ **Solution:**

(a) Taking the derivative, by differentiating under the integral sign, we have:

$$f'(x) = \int_0^1 \frac{\partial}{\partial x}\left(\frac{t^x - 1}{\log(t)}\right) dt.$$

To take the derivative, observe that $t^x = e^{x\log(t)}$. Hence we have:

$$f'(x) = \int_0^1 t^x \, dt = \left[\frac{t^{x+1}}{x+1}\right]_0^1 = \frac{1}{x+1},$$

provided that $x > -1$ (else the derivative does not exist). Integrating this expression, we have:

$$f(x) = \log(x+1) + c.$$

To evaluate the value of the constant, observe that $f(0) = 0$, since $t^0 = 1$. Hence $f(x) = \log(x+1)$.

(b) Taking the derivative, by differentiating under the integral sign, we have:

$$g'(x) = \int_0^\infty \frac{2xt^2}{(1+t^2)(1+x^2t^2)}\, dt.$$

Splitting the integrand into partial fractions, we note:

$$\frac{2xt^2}{(1+t^2)(1+x^2t^2)} = \frac{2x}{x^2-1}\left(\frac{1}{1+t^2} - \frac{1}{1+x^2t^2}\right).$$

Hence we can integrate directly to give:

$$g'(x) = \frac{2x}{x^2-1}\left[\arctan(t) - \frac{1}{x}\arctan(xt)\right]_0^\infty = \frac{2x}{x^2-1}\left(1 - \frac{1}{x}\right)\frac{\pi}{2} = \frac{\pi}{x+1}.$$

Integrating this expression, we have:

$$g(x) = \pi\log(x+1) + c.$$

To evaluate the value of the constant, observe that $g(0) = 0$, since $\log(1) = 0$. Hence $g(x) = \pi\log(x+1)$.

(c) Taking the derivative, by differentiating under the integral sign, we have:

$$h'(x) = \int_0^1 \cos(x \log(t))\, dt.$$

Integrating by parts, by writing $1 \cdot \cos(x \log(t))$, we have:

$$\int_0^1 \cos(x \log(t))\, dt = [t \cos(x \log(t))]_0^1 + x \int_0^1 \sin(x \log(t))\, dt = 1 + x \int_0^1 \sin(x \log(t))\, dt.$$

Integrating by parts again, we have:

$$\int_0^1 \sin(x \log(t))\, dt = [t \sin(x \log(t))]_0^1 - x \int_0^1 \cos(x \log(t))\, dt = -x \int_0^1 \cos(x \log(t))\, dt.$$

Hence, we have shown that:

$$h'(x) = 1 + x(-xh'(x)) = 1 - x^2 h'(x).$$

Rearranging, we have:

$$h'(x) = \frac{1}{1 + x^2}.$$

Integrating, we have:

$$h(x) = \arctan(x) + c,$$

for a constant $c$. Taking $x = 0$, we see that $h(0) = 0$. Hence $c = 0$, and it follows that $h(x) = \arctan(x)$.

23. This question determines the Gaussian integral in a different way to the lectures (you will use a transformation to polar coordinates on the next sheet!). Define:

$$f(x) = \left( \int_0^x e^{-t^2} \, dt \right)^2, \qquad \text{and} \qquad g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{1+t^2} \, dt.$$

Show that $f'(x) + g'(x) = 0$, and hence deduce that $f(x) + g(x) = \pi/4$. Conclude that $\int_0^\infty e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2}$.

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**Solution:** By the chain rule, and the fundamental theorem of calculus, we have:

$$f'(x) = 2e^{-x^2} \int_0^x e^{-t^2} \, dt.$$

On the other hand, by differentiating under the integral sign, we have:

$$g'(x) = -2x \int_0^1 e^{-x^2(t^2+1)} \, dt = -2xe^{-x^2} \int_0^1 e^{-x^2t^2} \, dt.$$

Substituting $u = xt$, so that $du = xdt$, in the final integral in the expression for $g'(x)$, we have:

$$g'(x) = -2e^{-x^2} \int_0^x e^{-u^2} \, du.$$

Hence $f'(x) + g'(x) = 0$, as required.

We immediately deduce that $f(x) + g(x)$ is a constant. To evaluate the value of the constant, we consider $x = 0$. Then $f(0) = 0$, whilst:

$$g(0) = \int_0^1 \frac{dt}{1+t^2} = [\arctan(t)]_0^1 = \frac{\pi}{4}.$$

So we see that indeed $f(x) + g(x) = \pi/4$, as required.

To finish, take the limit of the equation $f(x) + g(x) = \pi/4$ as $x \to \infty$. Inside the integrand of $g(x)$, we have $e^{-x^2(t^2+1)} \to 0$ for all $t$, so that the contribution from $g(x)$ vanishes. Meanwhile,

$$f(x) \to \left( \int_0^\infty e^{-t^2} \, dt \right)^2.$$

The right hand side of the equation is a constant in $x$, so is unchanged by taking the limit. Hence we have shown:

$$\left( \int_0^\infty e^{-t^2} \, dt \right)^2 = \frac{\pi}{4} \qquad \Rightarrow \qquad \int_0^\infty e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2},$$

as required.

**(†) Integral inequalities**

24. Using a sketch, show that $\sin(x) \geq 2x/\pi$ for $0 \leq x \leq \pi/2$. Hence show that $\displaystyle\int_0^{\pi/2} \frac{x^2}{1+\sin^2(x)}\,dx < \frac{\pi^3}{8}\left(1-\frac{\pi}{4}\right)$.

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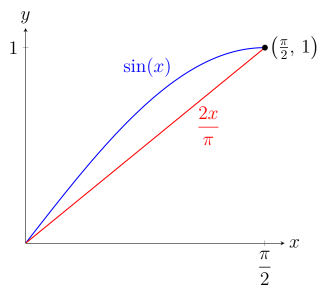
**Solution:** Plotting the graphs of $\sin(x)$ and $2x/\pi$, we have:




Hence, we see from the figure that $\sin(x) \geq 2x/\pi$ for $x \in [0, \pi/2]$. Applying this to the integral, we have:

$$\int_0^{\pi/2} \frac{x^2}{1+\sin^2(x)}\,dx \leq \int_0^{\pi/2} \frac{x^2}{1+4x^2/\pi^2}\,dx = \frac{\pi^2}{4}\int_0^{\pi/2}\left(1-\frac{1}{1+4x^2/\pi^2}\right)dx = \frac{\pi^2}{4}\left(\frac{\pi}{2}-\left[\frac{\pi}{2}\arctan\left(\frac{2x}{\pi}\right)\right]_0^{\pi/2}\right) = \frac{\pi^3}{8}\left(1-\frac{\pi}{4}\right),$$

as required.

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25. State and prove *Schwarz's inequality* for integrals. Use it to show that $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{x^2+1}}\,dx < \sqrt{\frac{\pi}{4}\arctan\left(\frac{\pi}{2}\right)}$.

---

**Solution:** *Schwarz's inequality* states $\displaystyle\left(\int_a^b f(x)g(x)\,dx\right)^2 \leq \int_a^b f(x)^2\,dx \cdot \int_a^b g(x)^2\,dx$. To prove this, let $\lambda \in \mathbb{R}$, and consider:

$$0 \leq \int_a^b (f(x)+\lambda g(x))^2\,dx = \lambda^2\int_a^b g(x)^2\,dx + 2\lambda\int_a^b f(x)g(x)\,dx + \int_a^b f(x)^2.$$

The right hand side is a quadratic in $\lambda$. But this quadratic is positive for *all* values of $\lambda$, hence it must have discriminant less than or equal to zero. The discriminant is:

$$4\left(\int_a^b f(x)g(x)\right)^2 - 4\int_a^b f(x)^2\,dx \cdot \int_a^b g(x)^2\,dx \leq 0,$$

which on rearrangement produces Schwarz's inequality.

Applying Schwarz's inequality to the given integral, we have:

$$\left(\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{x^2+1}}\,dx\right)^2 \leq \int_0^{\pi/2}\sin^2(x)\,dx\int_0^{\pi/2}\frac{1}{x^2+1}\,dx = \left(\frac{1}{2}\int_0^{\pi/2}(1-\cos(x))\,dx\right)[\arctan(x)]_0^{\pi/2} = \frac{\pi}{4}\arctan\left(\frac{\pi}{2}\right).$$

Taking the square root, we get the desired result.