Part IA: Mathematics for Natural Sciences B
Examples Sheet 2: Further vector geometry, triple products, vector area, and polar coordinate systems

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Questions marked with a $(*)$ are difficult and should not be attempted at the expense of the other questions.

More on the equation of a line

1. (a) Explain why the line through the points with positions vectors $\mathbf{a}$, $\mathbf{b}$ is $(\mathbf{r}-\mathbf{a})\times(\mathbf{b}-\mathbf{a}) = \mathbf{0}$. Show using properties of the vector product that an equivalent representation of this line is $\mathbf{r}\times(\mathbf{b}-\mathbf{a}) = \mathbf{a}\times\mathbf{b}$. What is the geometric significance of the quantity $|\mathbf{a}\times\mathbf{b}|/|\mathbf{b}-\mathbf{a}|$ here?

   (b) Express the line $\mathbf{r} = (1,0,1)+\lambda(3,-1,0)$ in the form $\mathbf{r}\times\mathbf{c} = \mathbf{d}$.

2. (a) Show that the shortest distance between the point $\mathbf{p}$ and the line $\mathbf{r} = \mathbf{a}+\lambda\mathbf{b}$ can be written as $\left|\hat{\mathbf{b}}\times(\mathbf{p}-\mathbf{a})\right|$.

   (b) $(*)$ Does this formula agree with the one you derived in terms of the scalar product in Question 10(c) of Sheet 1? [*Hint: Try squaring the formula in part (a), and using properties of the scalar triple product - see later in the sheet!*]

   (c) Find the shortest distance from a vertex of a unit cube to a diagonal excluding that vertex using both the formula in (a), and the formula from Question 10(c) of Sheet 1, and check that your answers agree.

More on the equation of a plane

3. (a) Explain why the plane through the points with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is $(\mathbf{r}-\mathbf{a})\cdot((\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})) = 0$. Show using properties of the vector product, and the result from Question 22 of Sheet 1, that this may equivalently be written in the more symmetric form $\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.

   (b) Find an equation of the form $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n} = 0$ for the plane passing through $(1,1,1)$, $(1,2,3)$ and $(0,0,4)$.

4. You need to drill a hole in a piece of metal starting at a right angle to a flat surface passing through the points $A = (1,0,0)$, $B = (1,1,1)$ and $C = (0,2,0)$, with the hole emerging at the point $D = (2,1,0)$. How long a drill must you use and where (in the plane ABC) must you start drilling?

5. Determine whether:

   (a) the points $\mathbf{P}_1 = (0,0,2)$, $\mathbf{P}_2 = (0,1,3)$, $\mathbf{P}_3 = (1,2,3)$, $\mathbf{P}_4 = (2,3,4)$ are coplanar;

   (b) the points $\mathbf{Q}_1 = (-2,1,1)$, $\mathbf{Q}_2 = (-1,2,2)$, $\mathbf{Q}_3 = (-3,3,2)$, $\mathbf{Q}_4 = (-2,4,3)$ are coplanar.

Shortest distances

6. *Without using a formula*, find the shortest distance between the lines $\mathbf{r}_1 = (1,0,1)+\lambda(2,-1,3)$ and $\mathbf{r}_2 = (0,1,-2)+\mu(1,0,2)$, justifying the steps you take. [*Sometimes, it is better to understand a method, than to quote a formula.*]

7. So far, we have developed formulae for the shortest distance from points to lines, and from points to planes. Now, using the scalar and vector products, establish formulae for the following:

   (a) the shortest distance from the line $\mathbf{r}_1 = \mathbf{v}_1+\lambda\mathbf{w}_1$ to the line $\mathbf{r}_2 = \mathbf{v}_2+\mu\mathbf{w}_2$; [*Hint: Take care when the lines are parallel!*]

   (b) the shortest distance from the line $\mathbf{r} = \mathbf{v}+\lambda\mathbf{w}$ to the plane $(\mathbf{r}-\mathbf{a})\cdot\mathbf{b} = 0$;

   (c) the shortest distance from the plane $(\mathbf{r}_1-\mathbf{a}_1)\cdot\mathbf{b}_1 = 0$ to the plane $(\mathbf{r}_2-\mathbf{a}_2)\cdot\mathbf{b}_2 = 0$.

The vector triple product, and vector equations

8. (a) By expanding in terms of the standard basis vectors, prove *Lagrange's formula* for the vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

Think of a way of remembering this formula off by heart - it is very useful!

(b) Hence, construct an example of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

(c) ($*$) Prove the vector triple product using a geometric argument. [*Hint:* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ *is the projection of* $\mathbf{a}$ *into the plane through the origin perpendicular to* $\mathbf{b} \times \mathbf{c}$*, rotated by* $\frac{1}{2}\pi$*, and scaled by the magnitude of* $\mathbf{b} \times \mathbf{c}$.]

9. Prove the *Jacobi identity*, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$.

10. Two vector operators, $P_{\hat{\mathbf{u}}} : \mathbb{R}^3 \to \mathbb{R}^3$ and $R_{\hat{\mathbf{u}}} : \mathbb{R}^3 \to \mathbb{R}^3$ are defined by $P_{\hat{\mathbf{u}}}(\mathbf{r}) = (\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}$ and $R_{\hat{\mathbf{u}}}(\mathbf{r}) = \hat{\mathbf{u}} \times (\mathbf{r} \times \hat{\mathbf{u}})$ respectively. Interpret these operators geometrically, and hence explain why $P_{\hat{\mathbf{u}}}(\mathbf{r}) + R_{\hat{\mathbf{u}}}(\mathbf{r}) = \mathbf{r}$ for all vectors $\mathbf{r}$. Also explain why $P_{\hat{\mathbf{u}}}^2 = P_{\hat{\mathbf{u}}}$ and $R_{\hat{\mathbf{u}}}^2 = R_{\hat{\mathbf{u}}}$.

11. Solve the following vector equations, and give geometric interpretations of their solutions:

(a) $\mathbf{a} \times \mathbf{r} + \lambda \mathbf{r} = \mathbf{c}$, where $\lambda \neq 0$, and $\mathbf{a}, \mathbf{c} \in \mathbb{R}^3$ are arbitrary 3-vectors;

(b) $\mathbf{r} \times \mathbf{a} = \mathbf{b}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are arbitrary 3-vectors, and $\mathbf{a} \neq \mathbf{0}$;

(c) $\mathbf{r} = \mathbf{a} + (\mathbf{b} \cdot \mathbf{r})\mathbf{c}$, where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ are arbitrary 3-vectors;

(d) $2\mathbf{r} + \hat{\mathbf{n}} \times \mathbf{r} + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{r})^2 = \mathbf{a}$, where $\hat{\mathbf{n}}$ is a unit vector, and $\hat{\mathbf{n}} \cdot \mathbf{a} = -1$.

The scalar triple product, and non-orthonormal bases

12. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ be 3-vectors.

(a) Give the definition of the *scalar triple product* $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of the 3-vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. Hence show that the volume of the parallelepiped defined by the positions vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is $|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|$. Why is the modulus necessary?

(b) Using the relation between the scalar triple product and a parallelepiped, explain why:

(i) the scalar triple product is antisymmetric on odd permutations of its entries, and symmetric on even permutations of its entries;

(ii) the condition $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$ implies that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are not coplanar, and thus form a basis.

(c) Compute the volume of a parallelepiped defined by the three position vectors $\mathbf{a} = (0, \frac{1}{2}, \frac{1}{2})$, $\mathbf{b} = (\frac{1}{2}, 0, \frac{1}{2})$, $\mathbf{c} = (\frac{1}{2}, \frac{1}{2}, 0)$, and comment on whether these vectors form a basis.

13. Simplify the scalar triple products $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} + \mathbf{c}) \times (\mathbf{c} + \mathbf{a})$ and $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$.

14. Let $\mathbf{0}, \mathbf{a}, \mathbf{b}, \mathbf{c}$ form the vertices of a tetrahedron, with $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$. Write down conditions in terms of the scalar triple product for the vector $\mathbf{r}$ to lie inside the tetrahedron.

15. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ be 3-vectors.

(a) If these vectors form an orthonormal basis, derive expressions for the coefficients α, β, γ in the formula $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$. Hence express $(2, 3, 4)$ in terms of the basis $\{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$.

(b) If instead these vectors do *not* form an orthonormal basis, derive expressions for the coefficients α, β, γ in the formula $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$. [*Hint: consider scalar triple products.*] Hence express $(1, 1, 1)$ in terms of the basis $\{(1, 2, 1), (0, 0, 1), (2, -1, 1)\}$.

(c) We define the *reciprocal vectors* to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to be the vectors:

$$\mathbf{A} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \qquad \mathbf{B} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \qquad \mathbf{C} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}.$$

Show that $\mathbf{A} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{b} = \mathbf{C} \cdot \mathbf{c} = 1$, and $\mathbf{A} \cdot \mathbf{b} = \mathbf{A} \cdot \mathbf{c} = \mathbf{B} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{c} = \mathbf{C} \cdot \mathbf{a} = \mathbf{C} \cdot \mathbf{b} = 0$. Hence, by comparing to part (b), explain how the reciprocal basis can be used to express a general vector $\mathbf{d}$ in terms of a non-orthonormal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. Compute the reciprocal basis to the basis $\{(1, 2, 1), (0, 0, 1), (2, -1, 1)\}$.

Vector area

16. (a) Define the *vector area* **A** of a surface composed of k flat faces with areas $A_1, ..., A_k$ and unit normals $\hat{\mathbf{n}}_1, ..., \hat{\mathbf{n}}_k$. What are the conventions usually used when choosing the unit normal(s)?

    (b) In terms of the position vectors **a**, **b** determine the vector areas of: (i) the parallelogram defined by **a**, **b**; (ii) the triangle defined by **a**, **b**. Hence, given points $O = (0, 0, 0)$, $A = (1, 0, 0)$, $B = (1, 1, 1)$, $C = (0, 2, 0)$, compute the vector area of the triangle OAB (with vertices taken in that order), and the vector area of the surface bounded by the loop $OABC$ comprising the two triangular surfaces OAB and BCO.

17. (a) Give a very general explanation of how the idea of vector area could be extended to *curved surfaces*, and hence explain why we expect the vector area of any *closed* surface to be **0**.

    (b) Compute the vector area of the square with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(2, 2, 0)$, $(0, 2, 0)$, taken in that order. Hence, compute the vector area of the pyramid extending this square with the point $(1, 1, 1)$, excluding its square face.

    (c) Compute the vector area of the curved surface of a truncated hollow cone, bounded by a horizontal circle of radius 4 units and a horizontal circle of radius 3 units at some height above the first (note the result is independent of the height!).

18. (a) Let **S** be the vector area of the surface S. Prove that the area of the projection of the surface S onto the plane with unit normal $\hat{\mathbf{m}}$ is $|\mathbf{S} \cdot \hat{\mathbf{m}}|$. [*Hint: consider joining the surface to its 'shadow' on the plane to create a closed surface.*]

    (b) Compute the vector area of the projection of the square with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(2, 2, 0)$, $(0, 2, 0)$ onto the plane with unit normal $\hat{\mathbf{m}} = (0, -1, 1)/\sqrt{2}$.

    (c) By projecting areas onto the yz, xz, and xy planes, compute the vector area of the surface comprised of the two triangles OAB, BCO with vertices $O = (0, 0, 0)$, $A = (1, 0, 0)$, $B = (1, 1, 1)$, $C = (0, 2, 0)$, taken in that order. [*Your answer should match your answer to Question 17(b)!*] What is the area of the surface projected onto: (i) the plane with normal $(0, -1, 1)$; (ii) the plane that maximises the projected area?

Polar coordinate systems

19. Draw (convincing) diagrams defining plane, cylindrical, and spherical polar coordinates. In each case, derive the coordinate transform laws from polars to Cartesians, and from Cartesians to polars. Hence, find the cylindrical polar and spherical polar coordinates of the point $(3, 4, 5)$.

20. (a) In 2D Cartesian coordinates, a circle is specified by $(x - 1)^2 + y^2 = 1$. Find its equation in polar coordinates.

    (b) In 3D Cartesian coordinates, a sphere is specified by $(x - 1)^2 + y^2 + z^2 = 1$. Find its equation in spherical polar coordinates.

21. Let $a > 0$ be a constant. Describe the following loci:

    (a) (i) $\phi = a$; (ii) $r = \phi$, in plane polar coordinates.

    (b) (i) $z = a$; (ii) $r = a$; (iii) $r = a$ and $z = \phi$, in cylindrical polar coordinates.

    (c) (i) $\theta = a$; (ii) $\phi = a$; (iii) $r = a$; (iv) $r = \theta = a$, in spherical polar coordinates.

22. Consider a point with position vector $\hat{\mathbf{n}}$ on the unit sphere S.

    (a) Explain why $\hat{\mathbf{n}} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, where θ, ϕ are the spherical coordinates of $\hat{\mathbf{n}}$.

    (b) Show that the vector area $d\mathbf{S}$ of a small patch near $\hat{\mathbf{n}}$, subtending a small angle $d\theta$ in the θ direction, and a small angle $d\phi$ in the ϕ direction, is given *approximately* by $d\mathbf{S} = \hat{\mathbf{n}}\sin(\theta)d\theta d\phi$.

    (c) ($*$) Hence, by integrating $d\mathbf{S}$ first over ϕ whilst keeping θ constant, then over θ, show that the vector area of the sphere is zero. [*Hint: what are the limits on ϕ, θ?*] You have now performed your first **surface integral**, a topic we shall cover properly in Lent. In fact, it is possible to use surface integration to show that the vector area of *any* closed shape is zero through a theorem called *Stokes' theorem*, which we shall also see in Lent.

(d) ($*$) Without computing it, what is the value of the surface integral $\int_S \hat{\mathbf{n}} \cdot d\mathbf{S}$?