Part IB: Mathematical Methods
Examples Sheet 1 Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

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1. Let $c_n(\theta) = \cos(2\pi n\theta/L)$ and $s_n(\theta) = \sin(2\pi n\theta/L)$. Verify the orthogonality relations:

$$\langle c_n, c_m\rangle = \langle s_n, s_m\rangle = \frac{1}{2}L\delta_{mn}, \qquad \langle 1, c_n\rangle = \langle 1, s_m\rangle = \langle c_n, s_m\rangle = 0, \qquad m, n \geq 1,$$

where $\langle f, g\rangle = \int\limits_0^L f(\theta)\overline{g(\theta)}\,d\theta$. This confirms $\{1, c_n, s_n\}_{n=1}^{\infty}$ are orthogonal, as stated in lectures.

Solution: We can either prove these relations from first principles, or we can use some of the results from lectures on the orthogonality of exponentials. Let's see both methods.

- <u>Direct proof.</u> The key trigonometric identities we need for a direct proof are the *product-to-sum formulae*, namely:

$$\begin{aligned}
\cos\alpha\cos\beta &= \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right], \\
\sin\alpha\sin\beta &= \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right], \\
\sin\alpha\cos\beta &= \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right], \\
\cos\alpha\sin\beta &= \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right].
\end{aligned}$$

To prove the required cosine-cosine orthogonality relation, for example, we simply deploy one of these identities at an appropriate point of the calculation (note since $n, m \geq 1$, we have $n + m > 0$):

$$\begin{aligned}
\langle c_n, c_m\rangle &= \int\limits_0^L \cos\left(\frac{2\pi n\theta}{L}\right)\cos\left(\frac{2\pi m\theta}{L}\right)\,d\theta \\
&= \frac{1}{2}\int\limits_0^L \left(\cos\left(\frac{2\pi(n+m)\theta}{L}\right) + \cos\left(\frac{2\pi(n-m)\theta}{L}\right)\right)\,d\theta \\
&= \frac{1}{2}\cdot\begin{cases} \left[\dfrac{L}{2\pi(n+m)}\sin\left(\dfrac{2\pi(n+m)\theta}{L}\right) + \dfrac{L}{2\pi(n-m)}\sin\left(\dfrac{2\pi(n-m)\theta}{L}\right)\right]_0^L & \text{if } n \neq m, \\ \left[\dfrac{L}{2\pi(n+m)}\sin\left(\dfrac{2\pi(n+m)\theta}{L}\right) + \theta\right]_0^L & \text{if } n = m \end{cases} \\
&= \frac{1}{2}\begin{cases} 0 & \text{if } n \neq m, \\ L & \text{if } n = m \end{cases} \\
&= \frac{1}{2}L\delta_{mn},
\end{aligned}$$

as required. The other relations follow similarly.

- <u>Proof using complex exponentials.</u> In lectures, you already proved that the set of complex exponentials $\{e_n\}_{n=-\infty}^{\infty}$, where:
$$e_n(\theta) = \exp\left(\frac{2\pi i n\theta}{L}\right),$$
is orthogonal with respect to the stated inner product. The proof is simple and doesn't require us to remember any complicated trigonometric identities:
$$\begin{aligned}
\langle e_n, e_m \rangle &= \int_0^L e^{2\pi i(n-m)\theta/L}\, d\theta \\
&= \begin{cases} \left[\dfrac{Le^{2\pi i(n-m)\theta/L}}{2\pi i(n-m)}\right]_0^L & \text{if } n \neq m, \\ [\theta]_0^L & \text{if } n = m \end{cases} \\
&= \begin{cases} 0 & \text{if } n \neq m, \\ L & \text{if } n = m \end{cases} \\
&= L\delta_{mn}.
\end{aligned}$$
Now, simply relating complex exponentials to trigonometric functions via Euler's formula, $e_n = c_n + is_n$ and $e_{-n} = c_n - is_n$, we have:
$$L\delta_{mn} = \langle e_n, e_m\rangle = \langle c_n + is_n, c_m + is_m\rangle = \langle c_n, c_m\rangle + \langle s_n, s_m\rangle + i(\langle s_n, c_m\rangle - \langle c_n, s_m\rangle),$$
$$L\delta_{-m,n} = \langle e_n, e_{-m}\rangle = \langle c_n + is_n, c_m - is_m\rangle = \langle c_n, c_m\rangle - \langle s_n, s_m\rangle + i(\langle s_n, c_m\rangle + \langle c_n, s_m\rangle),$$
using linearity in the first argument and anti-linearity in the second argument in both cases. Taking real and imaginary parts, we deduce that:
$$\langle c_n, c_m\rangle + \langle s_n, s_m\rangle = L\delta_{mn}, \qquad \langle s_n, c_m\rangle = \langle c_n, s_m\rangle,$$
$$\langle c_n, c_m\rangle - \langle s_n, s_m\rangle = L\delta_{-m,n}, \qquad \langle s_n, c_m\rangle = -\langle c_n, s_m\rangle.$$
Solving these simultaneous equations, we deduce that:
$$\langle c_n, c_m\rangle = \frac{1}{2}L(\delta_{mn} + \delta_{-m,n}), \qquad \langle s_n, s_m\rangle = \frac{1}{2}L(\delta_{mn} - \delta_{-m,n}),$$
$$\langle s_n, c_m\rangle = \langle c_n, s_m\rangle = 0.$$
These relations hold in the more general case where $n, m \in \mathbb{Z}$. Restricting to $n, m \geq 1$ results in the required relations in the question, when we can ignore the delta function $\delta_{-m,n}$. The relations for $\langle 1, 1\rangle$, $\langle 1, c_n\rangle$ and $\langle 1, s_n\rangle$ also follow from these general formulae.

2. Consider the 2-periodic function $f:\mathbb{R}\to\mathbb{R}$ with $f(\theta)=(1-\theta^2)^2$ when $\theta\in[-1,1)$. Show that it has Fourier series:

$$f(\theta)\sim\frac{8}{15}+\frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\cos(n\pi\theta).$$

Can we replace '$\sim$' with '$=$' in this case? Sketch the graph of f and comment on the number of continuous derivatives it has, and the relation to the decay of the Fourier coefficients.

➫ **Solution:** Since f is even, we know that its Fourier series takes the form:

$$f(\theta)\sim\sum_{n=0}^{\infty}a_n\cos(n\pi\theta). \tag{$*$}$$

Now, recall that to obtain the *definition* of the Fourier coefficients, we should take the inner product of f with appropriate orthogonal functions in $(*)$, and replace $\sim$ with equality. We can then decide at a later time whether the resulting series that we construct to form the right hand side of $(*)$ is indeed *equal* to the left hand side of $(*)$. Thus, taking the inner product with $c_m=\cos(m\pi\theta)$ on both sides of $(*)$, and replacing $\sim$ with equality, we obtain:

$$\langle c_m,f\rangle=\sum_{n=0}^{\infty}a_n\langle c_m,c_n\rangle=a_m,$$

using the result $\langle c_m,c_n\rangle=\frac{1}{2}L\delta_{mn}$ from Question 1, with $L=2$. This works for $m\geq 1$; on the other hand, in the case where $m=0$, we have:

$$\langle 1,f\rangle=\sum_{n=0}^{\infty}a_0\langle 1,c_n\rangle=2a_0,\qquad\text{since}\qquad\int_{-1}^{1}1\,d\theta=2,$$

and using the result $\langle 1,c_n\rangle=0$ for $n\geq 1$ from Question 1, with $L=2$. Hence the Fourier coefficients are given by:

$$a_n=\langle c_n,f\rangle=\int_{-1}^{1}(1-\theta^2)^2\cos(n\pi\theta)\,d\theta,\qquad\text{for } n>0,$$

$$a_0=\frac{1}{2}\langle 1,f\rangle=\frac{1}{2}\int_{-1}^{1}(1-\theta^2)^2\,d\theta.$$

Thus to compute the Fourier coefficients, we split into two cases: $n=0$ and $n>0$.

In the case $n=0$, we can ignore the cosine and simply integrate directly:

$$a_0=\frac{1}{2}\int_{-1}^{1}(1-\theta^2)^2\,d\theta=\frac{1}{2}\int_{-1}^{1}(1-2\theta^2+\theta^4)\,d\theta=\frac{1}{2}\left[\theta-\frac{2}{3}\theta^3+\frac{1}{5}\theta^5\right]_{-1}^{1}=1-\frac{2}{3}+\frac{1}{5}=\frac{8}{15},$$

as required.

In the case $n \neq 0$, we use repeated integration by parts to handle the cosine and obtain the remaining Fourier coefficients:

$$
\begin{aligned}
a_n &= \int_{-1}^{1} (1-\theta^2)^2 \cos(n\pi\theta)\, d\theta \\
&= \underbrace{\left[\frac{1}{n\pi}(1-\theta^2)^2 \sin(n\pi\theta)\right]_{-1}^{1}}_{=0} - \frac{1}{n\pi}\int_{-1}^{1} 2(-2\theta)(1-\theta^2)\sin(n\pi\theta)\, d\theta \\
&= \underbrace{-\left[\frac{4}{n^2\pi^2}\theta(1-\theta^2)\cos(n\pi\theta)\right]_{-1}^{1}}_{=0} + \frac{4}{n^2\pi^2}\int_{-1}^{1}(1-3\theta^2)\cos(n\pi\theta)\, d\theta \\
&= \underbrace{\left[\frac{4}{n^3\pi^3}(1-3\theta^2)\sin(n\pi\theta)\right]_{-1}^{1}}_{=0} + \frac{24}{n^3\pi^3}\int_{-1}^{1}\theta\sin(n\pi\theta)\, d\theta \\
&= \left[-\frac{24}{n^4\pi^4}\theta\cos(n\pi\theta)\right]_{-1}^{1} + \frac{24}{n^4\pi^4}\underbrace{\int_{-1}^{1}\cos(n\pi\theta)\, d\theta.}_{=0} \\
&= \frac{48}{n^4\pi^4}(-1)^{n+1},
\end{aligned}
$$

as required.

Next, we are asked to return to the issue of whether we can replace $\sim$ with $=$ in this case. We recall the following result from the lectures:

Theorem: Let $f : \mathbb{R} \to \mathbb{C}$ be an L-periodic function such that on $[0, L)$ we have:

(i) f is continuous on $[0, L)$ away from a finite number of jump discontinuities;

(ii) f has a finite number of minima and maxima on $[0, L)$.

Then for each $\theta \in [0, L)$, we have:

$$\frac{f(\theta_+) + f(\theta_-)}{2} = \lim_{N\to\infty}(S_N f)(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{2\pi i n\theta/L},$$

where $f(\theta_\pm) = \lim_{\epsilon\to 0} f(\theta \pm \epsilon)$, and the partial Fourier sums are defined by:

$$S_N f = \sum_{n=-N}^{N} \hat{f}_n e^{2\pi i n\theta/L}.$$

In other words, at points of continuity the Fourier series converges to the original function since $f(\theta) = f(\theta_+) = f(\theta_-)$. On the other hand, at points of discontinuity, the Fourier series will converge to the *average* of the original function across the two sides of the discontinuity.

Proof: In lectures, we only prove the case where f is smooth (infinitely differentiable); the proof is harder when f is not smooth, and hence is *non-examinable* in this case.

Thus, assume that f is smooth from now on. The partial Fourier sums are given by:

$$(S_N f)(\theta_0) = \sum_{n=-N}^{N} \hat{f}_n e^{2\pi i n\theta_0/L} = \frac{1}{L}\int_0^L \left[\sum_{n=-N}^{N} e^{2\pi i n(\theta_0-\theta)/L}\right] f(\theta)\,d\theta.$$

When $\theta \neq \theta_0$, the sum in square brackets is a geometric progression with first term $e^{-2\pi i N(\theta_0-\theta)/L}$ and common ratio $e^{2\pi i(\theta_0-\theta)/L}$ (with total number of terms $2N+1$), hence it has sum:

$$\begin{aligned}\frac{e^{-2\pi i N(\theta_0-\theta)/L}\left(1-e^{2\pi i(2N+1)(\theta_0-\theta)/L}\right)}{1-e^{2\pi i(\theta_0-\theta)/L}} &= \frac{e^{\pi i(\theta_0-\theta)/L}\left(e^{-2\pi i(N+1/2)(\theta_0-\theta)/L}-e^{2\pi i(N+1/2)(\theta_0-\theta)/L}\right)}{e^{\pi i(\theta_0-\theta)/L}\left(e^{-\pi i(\theta_0-\theta)/L}-e^{\pi i(\theta_0-\theta)/L}\right)}\\ &= \frac{\sin\left(\frac{2\pi}{L}\left(N+\frac{1}{2}\right)(\theta_0-\theta)\right)}{\sin\left(\frac{\pi}{L}(\theta_0-\theta)\right)}.\end{aligned}$$

Otherwise, when $\theta = \theta_0$, the sum in the brackets is simply $2N+1$. We define the *Dirichlet kernel* to be the function:

$$D_N(\theta) = \begin{cases} \dfrac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}, & \text{if } \theta \notin 2\pi\mathbb{Z}, \\ 2N+1, & \text{if } \theta \in 2\pi\mathbb{Z}. \end{cases}$$

Then we have shown that the partial Fourier sums take the form:

$$(S_N f)(\theta_0) = \frac{1}{L}\int_0^L D_N\left(\frac{2\pi}{L}(\theta-\theta_0)\right) f(\theta)\,d\theta.$$

Observe that the Dirichlet kernel is both continuous (for example, consider using L'Hôpital's rule to fill in the gaps), and 2π-periodic; hence $D_N(\frac{2\pi}{L}(\theta-\theta_0))$ is L-periodic. Further, observe that if we had summed the Dirichlet kernel in a different way, we could have written it as 1 plus some cosines. Since cosines integrate to sines, it follows that:

$$\int_0^{2\pi} D_N(\theta)\,d\theta = 2\pi \qquad \Leftrightarrow \qquad \int_0^L D_N\left(\frac{2\pi}{L}(\theta-\theta_0)\right)\,d\theta = L,$$

using the substitution $\phi = 2\pi(\theta-\theta_0)/L$, then the periodicity of the kernel to translate the integration range in the final step.

In particular, this allows us to write:

$$
\begin{aligned}
(S_N f)(\theta_0) - f(\theta_0) &= \frac{1}{L}\int\limits_0^L D_N\left(\frac{2\pi}{L}(\theta-\theta_0)\right)\left(f(\theta)-f(\theta_0)\right)\,d\theta \\
&= \frac{1}{L}\int\limits_{\theta_0}^{L+\theta_0} D_N\left(\frac{2\pi}{L}\theta\right)\left(f(\theta+\theta_0)-f(\theta_0)\right)\,d\theta \\
&= \frac{1}{L}\int\limits_0^L D_N\left(\frac{2\pi}{L}\theta\right)\left(f(\theta+\theta_0)-f(\theta_0)\right)\,d\theta \\
&= \frac{1}{L}\int\limits_{-L/2}^{L/2} \sin\left(\frac{2\pi}{L}\left(N+\frac{1}{2}\right)\theta\right)F(\theta,\theta_0)\,d\theta,
\end{aligned}
$$

where we define:

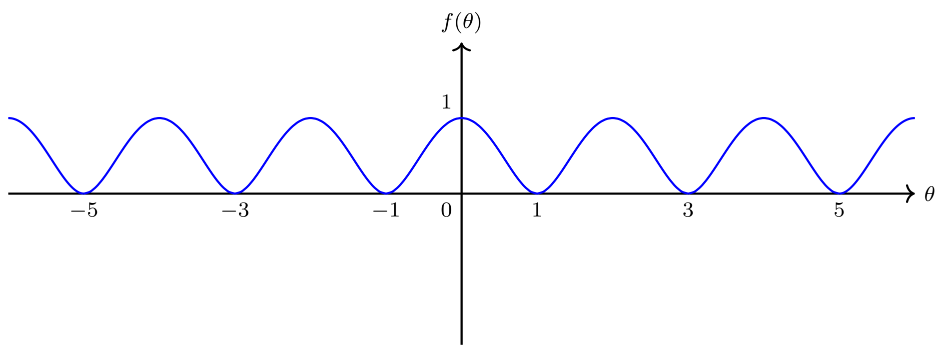
$$F(\theta,\theta_0) := \frac{\theta}{\sin\left(\pi\theta/L\right)}\frac{f(\theta+\theta_0)-f(\theta)}{\theta}.$$

Since f is smooth, it follows that F as a function of θ must be smooth everywhere in $[-L/2, L/2]$. The only problem point is $\theta = 0$, but this is a removable singularity of $\theta/\sin(\pi\theta/L)$, which we can prove using L'Hôpital's rule for example. In particular, we can integrate by parts to give:

$$(S_N f)(\theta_0) - f(\theta_0) = \frac{1}{2\pi}\frac{1}{N+1/2}\int\limits_{-L/2}^{L/2}\cos\left(\frac{2\pi}{L}\left(N+\frac{1}{2}\right)\theta\right)F'(\theta,\theta_0)\,d\theta \to 0,$$

as $N\to\infty$. This implies that $(S_N f)(\theta_0)\to f(\theta_0)$ as $N\to\infty$ (indeed, the convergence is uniform in this case). $\square$

Since $f(\theta)$ is clearly continuous in this case (see the following sketch if you remain unconvinced), we can indeed replace $\sim$ with $=$ in this case. The required sketch is:



Finally, we are asked to comment on the number of continuous derivatives that $f(\theta)$ has in relation to the decay of the Fourier coefficients. We again recall the important result from lectures:

Proposition: If $f : \mathbb{R} \to \mathbb{R}$ is a k-times continuously differentiable L-periodic function, then its complex Fourier coefficients obey:

$$\hat{f}_n = o\left(\frac{1}{n^k}\right) \qquad \text{as} \qquad |n| \to \infty.$$

Proof: The proof is by integration by parts. The nth Fourier coefficient is given by:

$$\hat{f}_n = \frac{1}{L}\int_0^L f(\theta)e^{-2\pi i n\theta/L}\, d\theta.$$

Integrating by parts k times, we have:

$$\hat{f}_n = \frac{1}{L}\left(\frac{L}{2\pi i n}\right)^k \int_0^L f^{(k)}(\theta)e^{-2\pi i n\theta/L}\, d\theta.$$

In each case, the boundary terms cancel since at the ends of the interval, we have $f(0) = f(L)$, and $f'(0) = f'(L)$, etc, since f is k-times continuously differentiable. In particular, the continuity of the kth derivative ensures that if f is periodic, its kth derivative is additionally L-periodic.

We now use the *Riemann-Lebesgue lemma* (which is *non-examinable*), which tells us that if $g(\theta)$ is integrable on $[a, b]$, we have:

$$\int_a^b g(\theta)e^{-i\theta\lambda}\, d\theta \to 0 \qquad \text{as} \qquad |\lambda| \to \infty.$$

This implies that:

$$\frac{\hat{f}_n}{1/n^k} \to 0 \qquad \text{as} \qquad |n| \to \infty,$$

which yields the result. $\square$

In particular, we have learned that the more regular our periodic function is, the faster the decay of its Fourier coefficients. In our case, our function $f(\theta) = (1-\theta^2)^2$ has derivatives:

$$f'(\theta) = -4\theta(1-\theta^2), \qquad f''(\theta) = -4 + 12\theta^2, \qquad f'''(\theta) = 24\theta.$$

on the interval $(-1, 1)$. Thus, $f(-1) = f(1)$, $f'(-1) = f'(1)$, $f''(-1) = f''(1)$, but $f'''(-1) \neq f'''(1)$. It follows that f is twice-continuously differentiable, which implies its Fourier coefficients should obey:

$$\hat{f}_n = o\left(\frac{1}{n^2}\right) \qquad \text{as} \qquad |n| \to \infty,$$

according to the result above. Checking this in practice, we have Fourier coefficients:

$$\frac{\hat{f}_n}{1/n^2} = \frac{48}{n^2\pi^4}(-1)^{n+1} \to 0 \qquad \text{as} \qquad |n| \to \infty.$$

Thus we do indeed have that the coefficients are $o(1/n^2)$.

However, it appears that our coefficients are actually slightly faster-decaying than that even! We observe that in fact:

$$\frac{\hat{f}_n}{1/n^3} \to 0, \qquad \frac{\hat{f}_n}{1/n^4} \not\to 0 \qquad \text{as } |n| \to \infty.$$

In particular, we have that $\hat{f}_n = o(1/n^3)$, but $\hat{f}_n \neq o(1/n^4)$. Our result that we proved using the Riemann-Lebesgue lemma appears not to be strong enough to handle this example!

This is simply due to the fact that $f(\theta)$ is in fact *differentiable* three times with an integrable derivative, just *continuously differentiable* only twice. We prove the following result:

Proposition: If $f : \mathbb{R} \to \mathbb{R}$ is a $(k+1)$-times differentiable L-periodic function with an integrable derivative,[1] then its complex Fourier coefficients obey:

$$\hat{f}_n = o\left(\frac{1}{n^{k+1}}\right) \qquad \text{as} \qquad |n| \to \infty.$$

Proof: Again, the proof is by integration by parts. The function is k-times continuously differentiable, so we have that:

$$\hat{f}_n = \frac{1}{L}\left(\frac{L}{2\pi i n}\right)^k \int\limits_0^L f^{(k)}(\theta) e^{-2\pi i n\theta/L}\, d\theta,$$

using the steps of the previous proof. Integrating by parts one further time, we have:

$$\begin{aligned}\hat{f}_n &= \frac{1}{L}\left(\frac{L}{2\pi i n}\right)^{k+1}\left(-\left[f^{(k)}(\theta)e^{-2\pi i n\theta/L}\right]_0^L + \int\limits_0^L f^{(k+1)}(\theta)e^{-2\pi i n\theta/L}\, d\theta\right)\\ &= \frac{1}{L}\left(\frac{L}{2\pi i n}\right)^{k+1}\int\limits_0^L f^{(k+1)}(\theta)e^{-2\pi i n\theta/L}\, d\theta,\end{aligned}$$

since $f^{(k)}$ is continuous, so the boundary term vanishes. This is legitimate, because the function is $(k+1)$-times differentiable, and the derivative itself is integrable. The Riemann-Lebesgue lemma now gives the result. □

This simple extension of the result from lectures gives the decay behaviour we see in this question, namely $\hat{f}_n = o(1/n^3)$ as required.

[1] Functions can be differentiable, but their derivative might not be integrable! For example, the function $f(x) = x^2 \sin(1/x^2)$, with $f(0) = 0$, is differentiable on $[-1, 1]$ but its derivative is unbounded. Hence, its derivative is not Riemann integrable.

3. Suppose $f(\theta) = \theta^2$ when $\theta \in [0, \pi)$.

(i) Construct (a) the sine series for f, and (b) the cosine series for f, each having period 2π. Sketch the 2π-periodic functions obtained in (a) and (b) in the range $\theta \in [-6\pi, 6\pi)$.

(ii) If the series in (a) and (b) are formally differentiated term-by-term, are the resulting series related to the Fourier series for 2π-periodic function $g, h : \mathbb{R} \to \mathbb{R}$ for which $g(\theta) = 2\theta$ and $h(\theta) = 2|\theta|$ when $\theta \in [-\pi, \pi)$?

➡ **Solution:** (i) (a) The odd extension of θ^2 is:

$$f(\theta) = \begin{cases} \theta^2 & \text{for } \theta \in [0, \pi) \\ -\theta^2 & \text{for } \theta \in (-\pi, 0] \end{cases}$$

This is now a 2π-periodic function. Thus its Fourier series takes the form:

$$f(\theta) \sim \sum_{n=1}^{\infty} a_n \sin(n\theta) = \sum_{n=1}^{\infty} a_n s_n,$$

in the notation of Question 1. To obtain the Fourier coefficients, we note:

$$\langle s_m, f \rangle = \pi a_m,$$

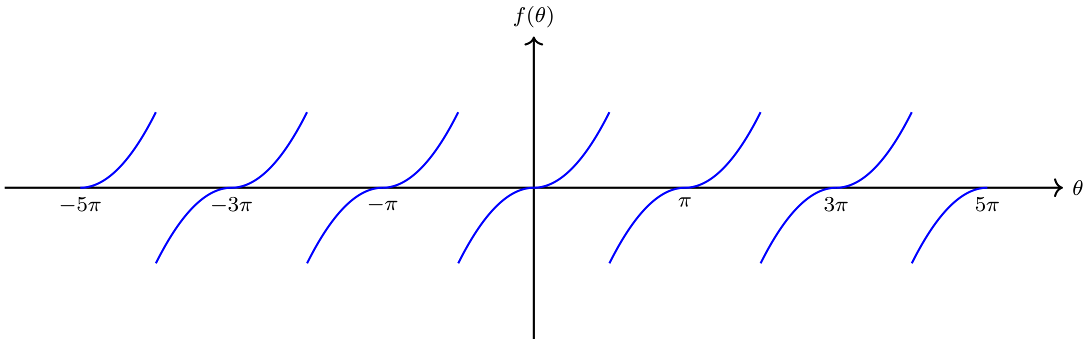
using the result $\langle s_m, s_n \rangle = \frac{1}{2} L \delta_{mn}$ with $L = 2\pi$. Hence the Fourier coefficients are:

$$\begin{aligned} a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(m\theta) f(\theta) \, d\theta \\ &= \frac{2}{\pi} \int_0^{\pi} \theta^2 \sin(m\theta) \, d\theta \\ &= \frac{2}{\pi} \left[-\frac{1}{m} \theta^2 \cos(m\theta) \right]_0^{\pi} + \frac{4}{\pi m} \int_0^{\pi} \theta \cos(m\theta) \, d\theta \\ &= \frac{2\pi}{m} (-1)^{m+1} + \frac{4}{\pi m} \left[\frac{1}{m} \theta \sin(m\theta) \right]_0^{\pi} - \frac{4}{\pi m^2} \int_0^{\pi} \sin(m\theta) \, d\theta \\ &= \frac{2\pi}{m} (-1)^{m+1} + \frac{4}{\pi m^3} \left((-1)^m - 1 \right). \end{aligned}$$

Hence the series is:

$$f(\theta) \sim \sum_{n=1}^{\infty} \left(\frac{2\pi}{n} (-1)^{n+1} + \frac{4}{\pi n^3} \left((-1)^n - 1 \right) \right) \sin(n\theta).$$

The required sketch is:



(b) The even extension of θ^2 is:

$$f(\theta) = \theta^2,$$

for $\theta \in [-\pi, \pi)$. This is now a 2π-periodic function, and its Fourier series takes the form:

$$f(\theta) \sim \sum_{n=0}^{\infty} a_n \cos(n\theta) = a_0 + \sum_{n=1}^{\infty} a_n c_n$$

in the notation of Question 1. Therefore, the coefficients are given by:

$$\langle c_m, f\rangle = \pi a_m, \qquad \text{for } m \geq 1,$$

and:

$$\langle 1, f\rangle = 2\pi a_0.$$

Computing the Fourier coefficients, we have:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta^2 \, d\theta = \frac{1}{\pi}\int_{0}^{\pi} \theta^2 \, d\theta = \frac{\pi^2}{3},$$

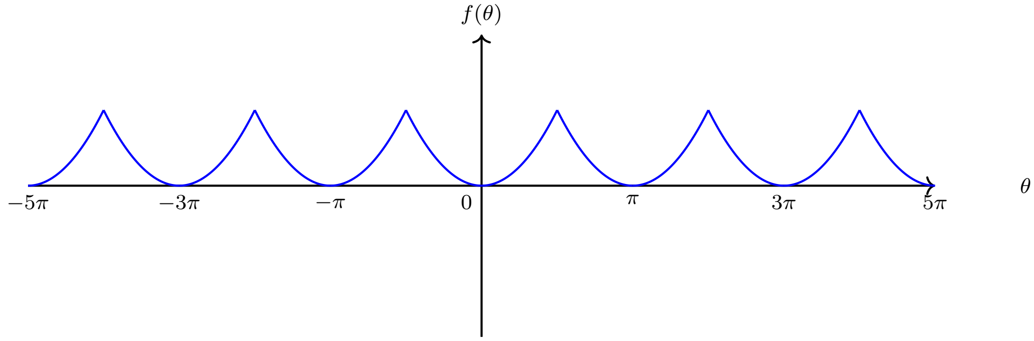
and for $m \geq 1$,

$$\begin{aligned} a_m &= \frac{2}{\pi}\int_0^{\pi} \theta^2 \cos(m\theta)\, d\theta \\ &= \frac{2}{\pi}\left[\frac{1}{m}\theta^2 \sin(m\theta)\right]_0^{\pi} - \frac{4}{\pi m}\int_0^{\pi} \theta \sin(m\theta)\, d\theta \\ &= \frac{4}{\pi m^2}\left[\theta \cos(m\theta)\right]_0^{\pi} - \frac{4}{\pi m^2}\int_0^{\pi} \cos(m\theta)\, d\theta \\ &= \frac{4}{m^2}(-1)^m. \end{aligned}$$

Thus the series is:

$$f(\theta) \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(n\theta).$$

The required sketch is:



(ii) Recall from lectures that provided f is continuous across the endpoints of its interval of definition, then the Fourier series can be differentiated term-by-term.

Proposition: Suppose that $f : \mathbb{R} \to \mathbb{C}$ is L-periodic and continuously differentiable on $[0, L)$. If f is continuous, then the Fourier series of the derivative f' can be obtained by term-wise differentiation of the Fourier series of f.

Proof: The complex Fourier coefficients of the derivative $g = f'$ on $[0, L)$ are given by the (improper) integral:

$$\begin{aligned}
\hat{g}_n &= \frac{1}{L} \lim_{\epsilon \to 0^+} \int_0^{L-\epsilon} f'(\theta) e^{-2\pi i n\theta/L}\, d\theta \\
&= \frac{1}{L} \lim_{\epsilon \to 0^+} \Big[f(\theta) e^{-2\pi i n\theta/L} \Big]_0^{L-\epsilon} + \frac{2\pi i n}{L^2} \int_0^{L} f(\theta) e^{-2\pi i n\theta/L}\, d\theta \\
&= \lim_{\epsilon \to 0^+} \left[\frac{f(L-\epsilon) e^{2\pi i n\epsilon/L} - f(0)}{L} \right] + \frac{2\pi i n}{L} \hat{f}_n.
\end{aligned}$$

Hence if f is continuous, the first term vanishes. We therefore find the Fourier coefficients of $g = f'$ are $\hat{g}_n = 2\pi i \hat{f}_n / L$, which is what we would have obtained by term-wise differentiation. $\square$

Now, we turn to the question. In (i)(b), the extension of the function $f(\theta)$ is continuous, and hence the Fourier series of its derivative on $[-\pi, \pi)$ can be obtained by term-wise differentiation. The derivative is $g(\theta) = 2\theta$, which has the same Fourier series.

On the other hand, in (i)(a), the extension of the function $f(\theta)$ is discontinuous and hence the Fourier series of its derivative on $[-\pi, \pi)$ *cannot* be obtained by term-wise differentiation.

Checking more explicitly, the Fourier coefficients of the even function $h(\theta) = 2|\theta|$ can be calculated directly as:

$$a_0 = \frac{1}{\pi} \int_0^\pi 2\theta \, d\theta = \pi,$$

and:

$$\begin{aligned} a_m &= \frac{2}{\pi} \int_0^\pi 2\theta \cos(m\theta) \, d\theta \\ &= \frac{4}{m\pi} \left[\theta \sin(m\theta)\right]_0^\pi - \frac{4}{m\pi} \int_0^\pi \sin(m\theta) \, d\theta \\ &= \frac{4}{m^2\pi} \left((-1)^m - 1\right). \end{aligned}$$

Thus the series is:

$$h(\theta) \sim \pi + \sum_{n=1}^{\infty} \frac{4}{n^2\pi} \left((-1)^n - 1\right) \cos(n\theta),$$

whereas differentiating term-by-term the Fourier series of the odd extension of θ^2, we end up getting:

$$\sum_{n=1}^{\infty} \left(2\pi(-1)^{n+1} + \frac{4}{n^2\pi} \left((-1)^n - 1\right) \right) \cos(n\theta).$$

4. Find the complex Fourier series for the 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ for which $f(\theta) = e^{\theta}$ when $\theta \in [-\pi, \pi)$. Using Parseval's theorem, deduce that:

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2}(\pi \coth(\pi) - 1).$$

Obtain the same result by evaluating the complex Fourier series at an appropriate point in $[-\pi, \pi)$.

➭ **Solution:** The complex Fourier series takes the form:

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$

As usual, to define the Fourier coefficients, we take the inner product with $e_m(\theta) = e^{im\theta}$ and replace $\sim$ with $=$:

$$\langle e_m, f \rangle = \sum_{n=-\infty}^{\infty} a_n \langle e_m, e_n \rangle = 2\pi a_m,$$

using $\langle e_m, e_n \rangle = L\delta_{mn}$ with $L = 2\pi$. Thus the Fourier coefficients are:

$$\begin{aligned} a_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\theta} e^{in\theta} \, d\theta \\ &= \frac{1}{2\pi} \left[\frac{e^{(in+1)\theta}}{in+1} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \left[\frac{(-1)^n e^{\pi}}{in+1} - \frac{(-1)^n e^{-\pi}}{in+1} \right] \\ &= \frac{(-1)^n \sinh(\pi)}{\pi(in+1)}. \end{aligned}$$

Next, we recall *Parseval's theorem* from lectures:

Parseval's theorem: Let $f : \mathbb{R} \to \mathbb{C}$ be an L-periodic function whose complex Fourier series is given by:

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}_n e_n(\theta),$$

where $e_n(\theta) = \exp(2\pi in\theta/L)$. Then:

$$\frac{1}{L} \int_0^L |f(\theta)|^2 \, d\theta = \sum_{n=-\infty}^{\infty} |\hat{f}_n|^2.$$

Proof: The left hand side is $\langle f, f \rangle$; assuming we can replace f with its Fourier series and freely exchange sums and integrals everywhere, we have:

$$\langle f, f \rangle = \sum_{n,m=-\infty}^{\infty} \hat{f}_n \overline{\hat{f}_m} \langle e_n, e_m \rangle = L \sum_{n=-\infty}^{\infty} |\hat{f}_n|^2,$$

using $\langle e_n, e_m \rangle = L\delta_{mn}$. The result follows immediately. □

Now in our case, applying Parseval's theorem, we find:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2\theta}\,d\theta = \frac{\sinh^2(\pi)}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{n^2+1}. \tag{$*$}$$

Evaluating the integral on the left hand side of $(*)$, we have:

$$\int_{-\pi}^{\pi} e^{2\theta}\,d\theta = \left[\frac{e^{2\theta}}{2}\right]_{-\pi}^{\pi} = \sinh(2\pi) = 2\sinh(\pi)\cosh(\pi).$$

Further, note that the sum on the right hand side of $(*)$ can be rewritten in the form:

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+1} = 1 + 2\sum_{n=1}^{\infty}\frac{1}{n^2+1}.$$

Inserting these two identities this back into $(*)$ and rearranging, we have:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2+1} &= \frac{1}{2}\left[\frac{\pi^2}{\sinh^2(\pi)}\cdot\frac{2\sinh(\pi)\cosh(\pi)}{2\pi} - 1\right] \\ &= \frac{1}{2}\left(\pi\coth(\pi) - 1\right),\end{aligned}$$

as required.

Finally, we are asked to obtain the same result by evaluating the complex Fourier series at an appropriate point in $[-\pi,\pi)$. We choose the point $\theta = -\pi$, and use the 'mean-value property' of Fourier series at discontinuities, so that:

$$\frac{e^{\pi}+e^{-\pi}}{2} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n\sinh(\pi)}{\pi(in+1)}e^{-in\pi}. \tag{$\dagger$}$$

The left hand simplifies to $\cosh(\pi)$. The right hand side can be simplified too:

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^n\sinh(\pi)}{\pi(in+1)}e^{-in\pi} &= \frac{\sinh(\pi)}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{in+1} \\ &= \frac{\sinh(\pi)}{\pi}\sum_{n=-\infty}^{\infty}\frac{1-in}{n^2+1} \\ &= \frac{\sinh(\pi)}{\pi}\left[1 + 2\sum_{n=1}^{\infty}\frac{1}{n^2+1} - i\sum_{n=-\infty}^{\infty}\frac{n}{n^2+1}\right].\end{aligned}$$

Inserting this back into $(\dagger)$ and taking real parts (or note that the sum from $-\infty$ to ∞ of $n/(n^2+1)$ is a sum over an odd summand), we have:

$$\cosh(\pi) = \frac{\sinh(\pi)}{\pi}\left[1 + 2\sum_{n=1}^{\infty}\frac{1}{n^2+1}\right],$$

which on rearrangement produces the desired result:

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1} = \frac{1}{2}\left(\pi\coth(\pi) - 1\right).$$

5. We say a sequence $\{r_n\}_{n\in\mathbb{Z}}$ *decays rapidly* if $|n|^k r_n \to 0$ as $|n| \to \infty$ for every $k \geq 0$.

(i) Let f be a smooth, L-periodic function. Show that the complex Fourier coefficients $\{\hat{f}_n\}$ decay rapidly.

(ii) Construct an L-periodic function with rapidly decaying, non-zero complex Fourier coefficients.

Solution: (i) First, observe that if f is a smooth L-periodic function, then the kth derivative $f^{(k)}$ is also a smooth L-periodic function. To see this, observe that the relation:

$$f(x+L) = f(x)$$

can be differentiated k times to obtain:

$$f^{(k)}(x+L) = f^{(k)}(x),$$

the condition for L-periodicity of the kth derivative (and of course, for a smooth function all derivatives are smooth).

Now, the complex Fourier coefficients which we wish to show are rapidly decaying are given by:

$$\hat{f}_n = \frac{1}{L}\int_0^L f(\theta)e^{-2\pi i n\theta/L}\,d\theta.$$

Given any $k \in \mathbb{Z}$, and ignoring $n = 0$ (a case which we don't care about, because it doesn't affect the behaviour of the sequence of Fourier coefficients as $|n| \to \infty$), we integrate the complex Fourier coefficients by parts $k+1$ times:

$$\begin{aligned}
\hat{f}_n &= \frac{1}{L}\underbrace{\left[-\frac{L}{2\pi i n}f(\theta)e^{-2\pi i n\theta/L}\right]_0^L}_{=0} + \frac{1}{2\pi i n}\int_0^L f'(\theta)e^{-2\pi i n\theta/L}\,d\theta \\
&= \frac{1}{2\pi i n}\underbrace{\left[-\frac{L}{2\pi i n}f'(\theta)e^{-2\pi i n\theta/L}\right]_0^L}_{=0} + \frac{L}{(2\pi i n)^2}\int_0^L f''(\theta)e^{-2\pi i n\theta/L}\,d\theta \\
&\ \vdots \\
&= \frac{1}{L}\left(\frac{L}{2\pi i n}\right)^{k+1}\int_0^L f^{(k+1)}(\theta)e^{-2\pi i n\theta/L}\,d\theta,
\end{aligned}$$

where in each case the periodicity of f and its derivatives ensures that the boundary terms vanish. Now providing an estimate on $\hat{f}_n|n|^k$, we have:

$$\begin{aligned}
|\hat{f}_n|n|^k| &= \frac{L^k}{(2\pi)^{k+1}|n|}\left|\int_0^L f^{(k+1)}(\theta)e^{-2\pi i n\theta/L}\,d\theta\right| \\
&\leq \frac{L^k}{(2\pi)^{k+1}|n|}\int_0^L \left|f^{(k+1)}(\theta)\right|\,d\theta \\
&\leq \frac{L^{k+1}}{(2\pi)^{k+1}|n|}\max_{\theta\in[0,L]} f^{(k+1)}(\theta).
\end{aligned}$$

Note that the maximum of $f^{(k+1)}(\theta)$ exists in $[0,L]$, since $[0,L]$ is a closed interval and $f^{(k+1)}$ is smooth so continuous (see Part IA Analysis). Thus taking the limit as $|n| \to \infty$, we have that $|\hat{f}_n|n|^k| \to 0$. It follows that $\hat{f}_n|n|^k \to 0$ as $|n| \to \infty$, as required. $\square$

(ii) There are multiple ways of doing this, but part (i) of the question suggests that any smooth, L-periodic function will work. To ensure we have non-zero Fourier coefficients however, we can't go for something like an exponential or a trigonometric function, because most of the Fourier coefficients will vanish.

An easy choice that makes it clear that the coefficients *won't* vanish on the other hand is:

$$f(\theta) = \exp\left(e^{2\pi i\theta/L}\right) + \exp\left(e^{-2\pi i\theta/L}\right).$$

This is evidently L-periodic and smooth. Furthermore, the Fourier series can be found simply by Taylor expansion in this case:

$$f(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} e^{2\pi in\theta/L} + \sum_{n=0}^{\infty} \frac{1}{n!} e^{-2\pi in\theta/L},$$

which shows that all the coefficients are non-zero in the complex Fourier series.

6. By considering the Sturm-Liouville problem for $y = y(x)$:

$$y'' + \lambda y = 0, \qquad 0 < x < L, \qquad y'(0) = 0, \qquad y'(L) = 0,$$

re-derive the cosine series representation for any $f \in C^2[0, L]$ with $f'(0) = f'(L) = 0$.

Solution:
Now, let's apply this theory to the given problem. The equation $y'' + \lambda y = 0$ can be recast as an eigenvalue problem:

$$\mathcal{L}y = \lambda y, \qquad \mathcal{L} = -\frac{d^2}{dx^2},$$

where the operator $\mathcal{L}$ is a Sturm-Liouville operator with $w = 1, p = 1$ and $q = 0$. The space of functions it is acting on is the space of functions on the interval $[0, L]$ with boundary conditions $y'(0) = 0$ and $y'(L) = 0$, which are certainly homogeneous real boundary conditions. It follows that $\mathcal{L}$ is self-adjoint on this space, so the space has a basis of eigenfunctions of $\mathcal{L}$.

Now, when $\lambda \neq 0$, the solution to the Sturm-Liouville problem takes the form:

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Differentiating, we have:

$$y'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x).$$

Imposing the boundary data, we have $y'(0) = 0$ implies that either $\lambda = 0$ or $B \equiv 0$. But we assumed that $\lambda \neq 0$ in deriving the form of the solution, so $B \equiv 0$. Imposing the remaining boundary data $y'(L) = 0$ implies that:

$$0 = \sin(\sqrt{\lambda}L),$$

and hence $\sqrt{\lambda}L = \pi n$ for some $n \in \mathbb{Z}$. It follows that $\lambda = \pi^2 n^2/L^2$, and hence solutions take the form:

$$y_n(x) = \cos\left(\frac{\pi n x}{L}\right),$$

where $n = 0, 1, \ldots$. Since the eigenfunctions form a basis, it follows that we can express any function $f : [0, L] \to \mathbb{C}$ as:

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{\pi n x}{L}\right).$$

❊ **Comments:** The complete Fourier series representation can be shown to result from the Sturm-Liouville problem $-y'' = \lambda y$, with boundary conditions $y(0) = y(L)$ and $y'(0) = y'(L)$ (though note that these are *not* of the usual form stated in the lectures). Indeed, the general solution to the differential equation is:

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x),$$

which gives the boundary conditions:

$$A = A\cos(\sqrt{\lambda}L) + B\sin(\sqrt{\lambda}L)$$

$$B\sqrt{\lambda} = -A\sqrt{\lambda}\sin(\sqrt{\lambda}L) + B\sqrt{\lambda}\cos(\sqrt{\lambda}L).$$

If $\sqrt{\lambda} = 0$, then any A, B solve these equations; thus the solution $y_0(x) = a_0$ for a_0 a constant is an eigenfunction with eigenvalue $\lambda = 0$.

Otherwise, divide the second equation by $\sqrt{\lambda}$. Then we can rearrange to:

$$B = -\frac{A\sin(\sqrt{\lambda}L)}{1-\cos(\sqrt{\lambda}L)},$$

except in the case where $\cos(\sqrt{\lambda}L) = 1$. Assuming that this is not the case, substituting into the first equation yields $A = -A$ so that $A = 0$; solving the second equation then yields $B = 0$ since $\cos(\sqrt{\lambda}L) \neq 1$. Thus we find the zero function is a solution, which is trivial.

In the only remaining case, we have $\cos(\sqrt{\lambda}L) = 1$ which implies $\sqrt{\lambda} = 2\pi n/L$ for $n \in \mathbb{Z}$. In this case, we get A, B arbitrary, and hence the remaining eigenfunctions with eigenvalues $\lambda = 4\pi^2 n^2/L^2$ are:

$$y_n(x) = a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right),$$

which yields the familiar Fourier series representation for a periodic function:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{2n\pi}{L}\right) + b_n\sin\left(\frac{2n\pi}{L}\right)\right).$$

In fact, this holds for any square integrable function on $[0, L]$ by virtue of the spectral theorem.

Compared to the standard Sturm-Liouville problems you saw in lectures, which did *not* include the class of problems with periodic conditions, observe that the eigenspaces for this problem are two-dimensional and are spanned by $\cos(2n\pi/L)$ and $\sin(2n\pi/L)$; the theory *still* applies in this case, but it is a bit more subtle, hence excluded from the general theorem stated in lectures.

7. Prove that the boundary value problem for $y = y(x)$:

$$y'' + \lambda y = 0, \qquad 0 < x < 1, \qquad y(0) = 0, \qquad y(1) + y'(1) = 0,$$

has infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$, and indicate roughly the behaviour of λ_n as $n \to \infty$.

Solution: The operator in question is self-adjoint, as we proved in Question 6. The general solution to the problem is:

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

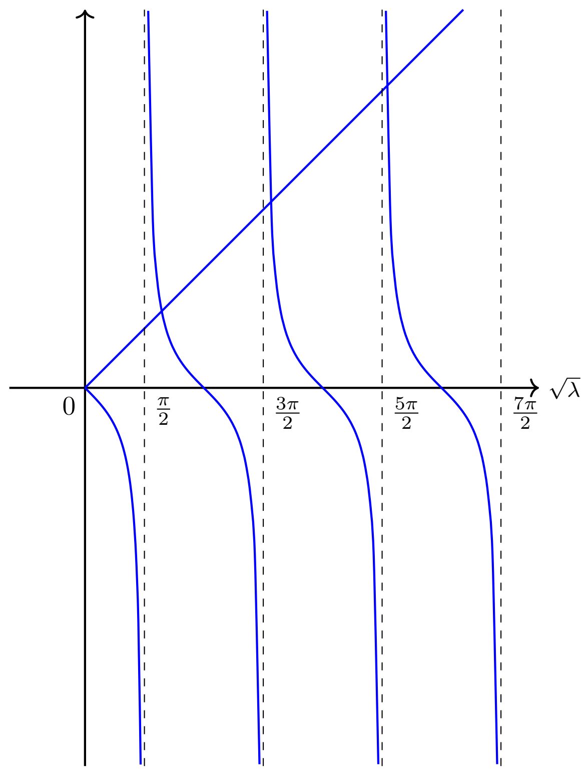
Imposing the condition $y(0) = 0$, we see that $A \equiv 0$ identically. Imposing the remaining boundary condition, we have:

$$0 = y(1) + y'(1) = B\sin(\sqrt{\lambda}) + B\sqrt{\lambda}\cos(\sqrt{\lambda}).$$

Assuming $B \neq 0$ (the zero function is not an eigenfunction), this is satisfied if and only if:

$$-\tan(\sqrt{\lambda}) = \sqrt{\lambda}.$$

Sketching the graphs of $-\tan(\sqrt{\lambda})$ and $\sqrt{\lambda}$ as functions of $\sqrt{\lambda}$ (clearly we need only consider $\sqrt{\lambda} > 0$ - this quantity must be non-negative, and cannot be zero, else we get the zero function as our solution), we obtain the following:



Hence we see that there there are infinitely many eigenvalues λ_n, with:

$$\frac{(2n-1)\pi}{2} < \sqrt{\lambda_n} < \frac{(2n+1)\pi}{2} \qquad \Leftrightarrow \qquad \frac{(2n-1)^2\pi^2}{4} < \lambda_n < \frac{(2n+1)^2\pi^2}{4}.$$

for $n = 1, 2, \ldots$. We see from the graph that as $n \to \infty$, we have the asymptotic behaviour:

$$\lambda_n \sim \frac{(2n-1)^2\pi^2}{4}.$$

8. Express the following eigenvalue problem as Sturm-Liouville problems on $[-1,1]$ and $[0,1]$, respectively:

(i) $(1-x^2)y'' - 2xy' + \lambda y = 0$,

(ii) $x(1-x)y'' - (ax-b)y' + \lambda y = 0$,

where $a > b > 0$ are constant, and λ is constant. Are either of these problems singular?

Now find the eigenvalues and eigenfunctions of the boundary value problem for $y = y(x)$:

(iii) $y'' + 4y' + (4+\lambda)y = 0, \quad 0 < x < 1, \quad y(0) = 0, \quad y(1) = 0.$

What is the orthogonality relation for these eigenfunctions?

Solution: Recall from lectures that a general second-order differential equation:

$$\alpha(x)y'' + \beta(x)y' + \gamma(x)y + \lambda y = 0$$

can be reduced to Sturm-Liouville form as follows. Begin by dividing through by $\alpha(x)$ (which we assume is not identically zero) to yield:

$$y'' + \frac{\beta(x)}{\alpha(x)}y' + \frac{\gamma(x)}{\alpha(x)}y = -\frac{\lambda}{\alpha(x)}y.$$

Now multiplying through by the *integrating factor* $I(x) = \exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx\right)$, we can rewrite the equation as:

$$-\frac{d}{dx}\left[\exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx\right)\frac{d}{dx}\right]y - \frac{\gamma(x)}{\alpha(x)}\exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx\right)y = \frac{\lambda}{\alpha(x)}\exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx\right)y.$$

This is of Sturm-Liouville form with:

$$p(x) = \exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx\right), \qquad q(x) = -\frac{\gamma(x)}{\alpha(x)}\exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx\right), \qquad w(x) = \frac{1}{\alpha(x)}\exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx\right).$$

For this to constitute a Sturm-Liouville equation, we require $\alpha(x) > 0$ throughout the open interval (a,b) on which the equation is defined, in order to guarantee that the weight-function $w(x)$ is positive throughout this open interval.

(i) In this case, we don't really need the integrating factor. By inspection, we can rearrange the first equation into Sturm-Liouville form as:

$$-\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right]y = \lambda y.$$

This is *Legendre's equation* from lectures. It is a Sturm-Liouville problem with $p(x) = 1-x^2$, $q(x) = 0$ and $w(x) = 1$. This problem is singular at both endpoints, since at the endpoints we have $p(1) = p(-1) = 0$.

(ii) In this case, the integrating factor is:

$$\exp\left(-\int \frac{ax-b}{x(1-x)}\,dx\right) = \exp\left(\int\left(\frac{b}{x} + \frac{b-a}{1-x}\right)\,dx\right) = \exp\left(b\log|x| - (b-a)\log|1-x|\right) = x^b(1-x)^{a-b},$$

using partial fractions as appropriate. From the general calculation then, we see that the Sturm-Liouville form is:

$$-\frac{d}{dx}\left[x^b(1-x)^{a-b}\frac{d}{dx}\right]y = \lambda x^{b-1}(1-x)^{a-b-1}y,$$

which has $p(x) = x^b(1-x)^{a-b}$ and $w(x) = x^{b-1}(1-x)^{a-b-1}$. This is fine on $[0,1]$, because $p, w > 0$ in $(0,1)$. However, the problem is singular at both endpoints, since at the endpoints we have $p(1) = p(-1) = 0$.

(iii) As usual, we can recast the given equation in Sturm-Liouville form; here, by inspection we have:

$$-\frac{d}{dx}\left[e^{4x}\frac{d}{dx}\right]y - 4e^{4x}y = \lambda e^{4x}y,$$

with $p(x) = e^{4x}$, $q(x) = -4e^{4x}$ and $w(x) = e^{4x}$.

Now let us solve the problem to find the eigenvalues and eigenfunctions. The characteristic equation of $y''+4y'+(4+\lambda)y = 0$ is:

$$\mu^2 + 4\mu + (4+\lambda) = 0,$$

which by the quadratic formula, has roots:

$$\mu_{\pm} = -2 \pm i\sqrt{\lambda}.$$

This implies that the general solution takes the form:

$$y(x) = Ae^{-2x}\sin(\sqrt{\lambda}x) + Be^{-2x}\cos(\sqrt{\lambda}x).$$

Imposing the boundary condition $y(0) = 0$, we see that $B = 0$ identically. Imposing the boundary condition $y(1) = 0$, we have:

$$0 = Ae^{-2}\sin(\sqrt{\lambda}),$$

so either $A = 0$, yielding the zero solution, or $\sin(\sqrt{\lambda}) = 0$ implying $\sqrt{\lambda} = n\pi$ for $n \in \mathbb{Z}$. Thus we see the eigenvalues are $\lambda_n = n^2\pi^2$ for $n = 1, 2, ...$ with corresponding eigenfunctions:

$$y_n(x) = e^{-2x}\sin(n\pi x).$$

The orthogonality relation is:

$$\int_0^1 y_n(x)y_m(x)e^{4x}\,dx = \alpha_n\delta_{nm}$$

for some constants α, inserting the appropriate weight function $w(x) = e^{4x}$. To work out the values of α_n, we take $n = m$:

$$\alpha_n = \int_0^1 \sin^2(n\pi x)\,dx = \frac{1}{2}\int_0^1 (1-\cos(2n\pi x))\,dx = \frac{1}{2}.$$

Thus the complete orthogonality relation is:

$$\int_0^1 y_n(x)y_m(x)e^{4x}\,dx = \frac{1}{2}\delta_{nm}.$$

9. Define the functions:

$$q_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

for $n = 1, 2, ...$

(a) Show that:

(i) q_n is a polynomial of degree n;

(ii) $q_n(1) = 1$ for all n;

(iii) q_n satisfies Legendre's equation.

(b) Deduce that:

(i) $q_n = P_n$;

(ii) $\displaystyle\int_{-1}^{1} P_n(x)^2\,dx = \frac{2}{2n+1}$;

(iii) $\displaystyle\int_{-1}^{1} x^m P_n(x)\,dx = 0$ if $m < n$.

Hint: for (a)(iii) show $u_n = (x^2-1)^n$ satisfies $(x^2-1)u_n' - 2nxu_n = 0$ and differentiate further.

Solution: This question is about the *Legendre polynomials*, which we defined in lectures. Using the method of series solutions of differential equations from Part IA Differential Equations, you showed that:

Proposition: The Sturm-Liouville problem:

$$-\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right]y = \lambda y,$$

known as *Legendre's equation*, has bounded solutions in $C^2[-1,1]$ only when $\lambda = n(n+1)$ for $n = 0, 1, 2....$[2] There is a unique such solution P_n satisfying $P_n(1) = 1$ for $\lambda = n(n+1)$, which is a polynomial of degree n; we call P_n the *nth Legendre polynomial*. Further, $P_n(x)$ is even when n is even, and odd when n is odd.

(a) (i) $q_n(x)$ is defined to be the nth derivative of $(x^2-1)^n/2^n n!$. This function is a polynomial whose highest degree term is $x^{2n}/2^n n!$. After taking the nth derivative then, the result will still be a polynomial, but now with highest degree term $(2n)(2n-1)...(2n-n+1)x^{2n-n}/2^n n! = (2n)!x^n/2^n(n!)^2$. Hence $q_n(x)$ is indeed a polynomial of degree n.

(a) (ii) By Leibniz's formula for the nth derivative, we have:

$$\begin{aligned} q_n(x) &= \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x-1)^n(x+1)^n\right) \\ &= \frac{1}{2^n n!}\sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}(x-1)^n\frac{d^{n-k}}{dx^{n-k}}(x+1)^n = \frac{1}{2^n n!}\sum_{k=0}^{n}\binom{n}{k}\frac{n!}{(n-k)!}(x-1)^{n-k}\cdot\frac{n!}{k!}(x+1)^k. \end{aligned}$$

[2] There are *unbounded* solutions too, called Legendre functions of the second kind, but they are not discussed in this course.

Evaluating at $x = 1$, only one term in the sum survives, namely $n = k$. This leaves us with:

$$q_n(1) = \frac{1}{2^n n!} \cdot n!(1+1)^n = 1,$$

as required.

(a) (iii) We aim to use the hint. Defining $u_n = (x^2 - 1)^n$, we observe that:

$$u_n' = 2xn(x^2-1)^{n-1}.$$

Hence:

$$(x^2-1)u_n' - 2xnu_n = 0.$$

as required. Differentiating $n+1$ times we have:

$$\frac{d^{n+1}}{dx^{n+1}}(x^2-1)u_n' - 2n\frac{d^{n+1}}{dx^{n+1}}xu_n = 0.$$

Using Leibniz's formula for repeated differentiation of a product, this can be expanded as:

$$\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{d^k}{dx^k}(x^2-1)\frac{d^{n+1-k}}{dx^{n+1-k}}u_n' - 2n\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{d^k}{dx^k}x\frac{d^{n+1-k}}{dx^{n+1-k}}u_n = 0.$$

Taking the derivatives, we end up with:

$$(x^2-1)u_n^{(n+2)} + 2(n+1)xu_n^{(n+1)} + (n+1)nu_n^{(n)} - 2nxu_n^{(n+1)} - 2n(n+1)u_n^{(n)} = 0.$$

Simplifying, and recalling $q_n = u_n^{(n)}/2^n n!$, we have:

$$(x^2-1)q_n'' + 2xq_n' - n(n+1)q_n = 0,$$

which is exactly Legendre's equation, as required.

(b) (i) Follows immediately from the definition of P_n as the unique bounded solution of Legendre's equation satisfying $P_n(1) = 1$.

(ii) Using integration by parts n times, we have (since $(x^2-1)^n$ contains a factor of $x = \pm 1$ whenever it is differentiated less than n times):

$$\int_{-1}^{1} P_n(x)^2\,dx = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\frac{d^{2n}}{dx^{2n}}(x^2-1)^n\,dx.$$

Now $(x^2-1)^n$ is a $2n$th degree polynomial with leading term x^{2n}. So its $2n$th derivative is simply the constant $(2n)!$. Overall this leaves:

$$\int_{-1}^{1} P_n(x)^2\,dx = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\,dx$$

To perform the remaining integral, we can use a recurrence relation. Let:

$$I_n = \int_{-1}^{1} (x^2-1)^n \, dx.$$

Integrating by parts, for $n > 0$, we have:

$$I_n = \left[x(x^2-1)^n\right]_{-1}^{1} - \int_{-1}^{1} x \frac{d}{dx}(x^2-1)^n \, dx$$

$$= -2n \int_{-1}^{1} x^2 (x^2-1)^{n-1} \, dx$$

$$= -2n \int_{-1}^{1} (x^2-1)^n \, dx - 2n \int_{-1}^{1} (x^2-1)^{n-1} \, dx$$

$$= -2nI_n - 2nI_{n-1}.$$

Rearranging, we have:

$$I_n = -\frac{2n}{2n+1} I_{n-1} = ... = (-1)^n \frac{(2n)(2(n-1))...(2(2))(2(1))}{(2n+1)(2n-1)(2n-3)...(2(2)+1)(2(1)+1)} I_0$$

$$= \frac{(-1)^n 2^{2n} (n!)^2}{(2n+1)!} I_0 = \frac{(-1)^n 2^{2n+1} (n!)^2}{(2n+1)!}.$$

Overall then, putting everything together, we have:

$$\int_{-1}^{1} P_n(x)^2 \, dx = \frac{2}{2n+1},$$

as required.

(iii) There are a couple of alternative ways we can prove this result.

- Direct proof. We simply use integration by parts repeatedly. We have:

$$\int_{-1}^{1} x^m P_n(x) \, dx = \frac{1}{2^n n!} \int_{-1}^{1} x^m \frac{d^n}{dx^n} (x^2-1)^n \, dx$$

$$= \frac{(-1)^m}{2^n n!} \int_{-1}^{1} \frac{d^m}{dx^m} x^m \frac{d^{n-m}}{dx^{n-m}} (x^2-1)^n \, dx,$$

  noting that the boundary terms always vanish, because taking less than the n derivative of $(x^2-1)^n$ leaves factors of $x-1$ and $x+1$. As a result, we have:

$$\int_{-1}^{1} x^m P_n(x) \, dx = \frac{(-1)^m m!}{2^n n!} \int_{-1}^{1} \frac{d^{n-m}}{dx^{n-m}} (x^2-1)^n \, dx = 0,$$

  since $n > m$.

- PROOF BY ORTHOGONALITY. Since $(P_0, P_1, ..., P_n)$ is an orthogonal basis (we showed this directly earlier in this question), and x^m is a polynomial of degree $m < n$, we have $x^m \in \operatorname{span}\{P_0, P_1, ..., P_m\}$. As a result, we must have:

  $$\int_{-1}^{1} x^m P_n(x)\, dx = 0,$$

  by orthogonality.

10. Recall from lectures that if $y_\alpha(r) = J_m(\alpha r)$, where J_m is the mth order Bessel function of the first kind, then:

$$-\frac{d}{dr}\left(r\frac{dy_\alpha}{dr}\right) + \frac{m^2}{r}y_\alpha = \alpha^2 r y_\alpha, \qquad r \in (0,1).$$

Show that if $y_\beta(r) = J_m(\beta r)$ then $[r(y_\alpha y_\beta' - y_\beta y_\alpha')]' = (\alpha^2 - \beta^2) r y_\alpha y_\beta$. Deduce that:

$$\int_0^1 J_m(\alpha r) J_m(\beta r) r\, dr = \frac{\beta J_m(\alpha) J_m'(\beta) - \alpha J_m(\beta) J_m'(\alpha)}{\alpha^2 - \beta^2}, \qquad \alpha \neq \beta.$$

Use this result to show that:

$$\int_0^1 J_m(j_{mk} r) J_m(j_{ml} r) r\, dr = \frac{1}{2}[J_m'(j_{mk})]^2\, \delta_{kl},$$

where $J_m(j_{mk}) = 0$ for $k = 1, 2, \ldots$.

➼ **Solution:** The equation can be rewritten as $-ry_\alpha'' - y_\alpha' + m^2 y_\alpha / r = \alpha^2 r y_\alpha$. Thus we have:

$$\begin{aligned}
[r(y_\alpha y_\beta' - y_\beta y_\alpha')]' &= y_\alpha y_\beta' + r y_\alpha' y_\beta' + r y_\alpha y_\beta'' - y_\beta y_\alpha' - r y_\beta' y_\alpha' - r y_\beta y_\alpha'' \\
&= y_\alpha y_\beta' + r y_\alpha' y_\beta' + y_\alpha\left(-y_\beta' + \frac{m^2}{r}y_\beta - \beta^2 r y_\beta\right) - y_\beta y_\alpha' - r y_\beta' y_\alpha' - y_\beta\left(-y_\alpha' + \frac{m^2}{r}y_\alpha - \alpha^2 r y_\alpha\right) \\
&= (\alpha^2 - \beta^2) r y_\alpha y_\beta,
\end{aligned}$$

as required. Integrating both sides, we have:

$$\begin{aligned}
\int_0^1 r y_\alpha y_\beta &= \frac{1}{\alpha^2 - \beta^2}\left[r(y_\alpha y_\beta' - y_\beta y_\alpha')\right]_0^1 \\
&= \frac{\beta J_m(\alpha) J_m'(\beta) - \alpha J_m(\beta) J_m'(\alpha)}{\alpha^2 - \beta^2},
\end{aligned}$$

as required. Inserting $\alpha = j_{mk}$ and $\beta = j_{ml}$ for $k \neq l$, we immediately get:

$$\int_0^1 J_m(j_{mk} r) J_m(j_{ml} r) r\, dr = 0,$$

where $k \neq l$. On the other hand, let $\alpha = j_{mk}$ and consider $\beta \neq \alpha$, but the limit as $\beta \to j_{mk}$. Then we have:

$$\begin{aligned}
\int_0^1 J_m(j_{mk} r) J_m(j_{mk} r) r\, dr &= \lim_{\beta \to j_{mk}} \int_0^1 J_m(j_{mk} r) J_m(\beta r) r\, dr \\
&= \lim_{\beta \to j_{mk}} \left[\frac{\beta J_m(j_{mk}) J_m'(\beta) - j_{mk} J_m(\beta) J_m'(j_{mk})}{j_{mk}^2 - \beta^2}\right] \\
&= \lim_{\beta \to j_{mk}} \frac{j_{mk} J_m'(\beta) J_m'(j_{mk})}{2\beta} \\
&= \frac{1}{2}\left[J_m'(j_{mk})\right]^2,
\end{aligned}$$

using L'Hôpital's rule, as required.

11. $(*)$ Let f be the 2π-periodic square wave for which $f(\theta) = 1$ on $[0, \pi)$ and $f(\theta) = 0$ on $[\pi, 2\pi)$.

(i) Sketch the graph of f and show that:

$$f(\theta) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)\theta]}{2n-1}.$$

(ii) Let $S_N f$ denote the partial Fourier series for f. By considering:

$$\sum_{n=1}^{N} \cos[(2n-1)\theta],$$

or otherwise, show that:

$$(S_N f)(\theta) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\theta} \frac{\sin(2N\phi)}{\sin(\phi)} \, d\phi.$$

(iii) Deduce that $(S_N f)(\theta)$ has a local extrema at $\theta = 2\pi m/2N$, $m \in \mathbb{Z} \backslash 2N\mathbb{Z}$ and that for large N:

$$(S_N f)\left(\frac{\pi}{2N}\right) \approx \frac{1}{2} + \frac{1}{\pi} \int_0^{\pi} \frac{\sin(u)}{u} \, du = \frac{1}{2} + \int_0^{\pi/2} \frac{\sin(u)}{u(\pi - u)} \, du \geq 1.08.$$

Hint for lower bound: $\sin(u) \geq u - u^3/3!$. Comment on the accuracy of partial Fourier series at discontinuities.

Solution: The complex Fourier expansion takes the form:

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$

hence the Fourier coefficients are:

$$\begin{aligned} a_m = \frac{1}{2\pi} \langle e^{im\theta}, f(\theta) \rangle &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{im\theta} \, d\theta \\ &= \frac{1}{2\pi} \int_0^{\pi} e^{im\theta} \, d\theta \\ &= \frac{1}{2\pi} \begin{cases} \dfrac{(-1)^m - 1}{m}, & \text{if } m \neq 0, \\ \pi, & \text{if } m = 0. \end{cases} \end{aligned}$$

Hence the Fourier series takes the form:

$$f(\theta) \sim \frac{1}{2} + \frac{1}{2\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(\frac{(-1)^n - 1}{n} \right) e^{in\theta}.$$

Only the odd terms survive from the sum, so we can rewrite it as:

$$f(\theta) \sim \frac{1}{2} - \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \left(e^{(2n-1)i\theta} - e^{-(2n-1)i\theta} \right) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\theta)}{2n-1},$$

as required.

(ii) We follow the advice of the question, and consider the partial sums of $\cos[(2n-1)\theta]$. Observe:

$$\begin{aligned}
\sum_{n=1}^{N} \cos[(2n-1)\theta] &= \mathrm{Re}\left[\sum_{n=1}^{N} e^{(2n-1)i\theta}\right] \\
&= \mathrm{Re}\left[\frac{e^{i\theta}(1-e^{2Ni\theta})}{1-e^{2i\theta}}\right] \\
&= \mathrm{Re}\left[\frac{e^{i\theta}e^{Ni\theta}(e^{Ni\theta}-e^{-Ni\theta})}{e^{i\theta}(e^{i\theta}-e^{-i\theta})}\right] \\
&= \frac{\cos(N\theta)\sin(N\theta)}{\sin(\theta)} \\
&= \frac{\sin(2N\theta)}{2\sin(\theta)}
\end{aligned}$$

Now, observe that the partial sums of f are given by:

$$(S_N f)(\theta) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{N} \frac{\sin[(2n-1)\theta]}{2n-1} = \frac{1}{2} + \frac{2}{\pi}\int_0^{\theta} \sum_{n=1}^{N} \cos[(2n-1)\phi]\, d\phi.$$

Inserting the formula for the partial sums, we have:

$$(S_N f)(\theta) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\phi} \frac{\sin(2N\phi)}{\sin(\phi)}\, d\phi,$$

as required.

(iii) Taking the derivative, we have:

$$\frac{d}{d\theta}(S_N f)(\theta) = \frac{\sin(2N\theta)}{\sin(\theta)},$$

which has local extrema at $\sin(2N\theta) = 0$ provided $\sin(\theta) \neq 0$, i.e. $\theta = 2\pi m/2N$ for $m \in \mathbb{Z}\backslash 2N\mathbb{Z}$. If additionally $\sin(\theta) = 0$, then we have removable singularities (with limit $2N$ at the relevant points).

For large N, we have:

$$(S_N f)\left(\frac{\pi}{2N}\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi/2N} \frac{\sin(2N\phi)}{\sin(\phi)}\, d\phi.$$

Making the substitution $u = 2N\phi$, we have $du = 2N d\phi$, and the limits transform as $[0, \pi/2N] \mapsto [0, \pi]$. As a result, we have:

$$
\begin{aligned}
(S_N f)\left(\frac{\pi}{2N}\right) &= \frac{1}{2} + \frac{1}{\pi}\int_0^\pi \frac{\sin(u)}{2N\sin(u/2N)}\,du \\
&= \frac{1}{2} + \frac{1}{\pi}\int_0^\pi \frac{\sin(u)}{u\left(1-\frac{u^2}{3!(2N)^2}+\cdots\right)}\,du \\
&= \frac{1}{2} + \frac{1}{\pi}\int_0^\pi \frac{\sin(u)}{u}\,du + O\left(\frac{1}{N^2}\right),
\end{aligned}
$$

by the binomial theorem. To finish, we perform the manipulation:

$$
\begin{aligned}
\int_0^\pi \frac{\sin(u)}{u}\,du &= \int_0^{\pi/2} \frac{\sin(u)}{u}\,du + \int_{\pi/2}^{\pi} \frac{\sin(u)}{u}\,du \\
&= \int_0^{\pi/2} \frac{\sin(u)}{u}\,du - \int_{\pi/2}^{0} \frac{\sin(\pi-u)}{\pi-u}\,du \\
&= \pi\int_0^{\pi/2} \frac{\sin(u)}{u(\pi-u)}\,du.
\end{aligned}
$$

Thus we have shown:

$$
(S_N f)\left(\frac{\pi}{2N}\right) = \frac{1}{2} + \int_0^{\pi/2} \frac{\sin(u)}{u(\pi-u)}\,du,
$$

as required.

Finally, using the hint, we note that:

$$
\begin{aligned}
(S_N f)\left(\frac{\pi}{2N}\right) &\geq \frac{1}{2} + \frac{1}{6}\int_0^{\pi/2} \frac{6u - u^3}{u(\pi - u)}\, du \\
&= \frac{1}{2} + \frac{1}{6}\int_0^{\pi/2} \frac{6 - u^2}{\pi - u}\, du \\
&= \frac{1}{2} + \frac{1}{6}\int_0^{\pi/2} \frac{6 - \pi^2 + (\pi - u)(\pi + u)}{\pi - u}\, du \\
&= \frac{1}{2} + \frac{6 - \pi^2}{6}\left[-\log(\pi - u)\right]_0^{\pi/2} + \frac{1}{6}\int_0^{\pi/2} (\pi + u)\, du \\
&= \frac{1}{2} + \left(\frac{6 - \pi^2}{6}\right)\log(2) + \frac{1}{6}\frac{\pi^2}{2} + \frac{\pi^2}{48} \\
&= \frac{1}{2} + \left(\frac{6 - \pi^2}{6}\right)\log(2) + \frac{5\pi^2}{48} \\
&\approx 1.08104...
\end{aligned}
$$

This implies that no matter how many terms we take in the partial Fourier series, there is always a point close to the discontinuity at 0 which suffers an error of at least $0.68104...$. In particular, this implies that even though the Fourier series converges to the average value at the discontinuity, it does so in a *non-uniform* fashion.

This phenomenon is actually completely generic...

12. Set $V = \{y \in C^2[a,b] : y(a) = y(b) = 0\}$ (i.e. Dirichlet conditions) and let $L = \frac{1}{w}\left[-\frac{d}{dx}\left(p\frac{d}{dx}\right) + q\right]$ be a Sturm-Liouville operator with p, q, w smooth and $p, w > 0$ on $[a,b]$. Consider the *Rayleigh quotient*:

$$R[y] = \frac{\int_a^b \left[p(y')^2 + qy^2\right]\,dx}{\int_a^b wy^2\,dx}, \qquad y \in V.$$

(a) By considering $\langle Ly, y\rangle_w$, show that if $y \in V$ satisfies $Ly = \lambda y$, then $\lambda = R[y]$.

(b) Let $\lambda_1 = \inf_{y \in V\setminus\{0\}} R[y]$ and suppose that there exists a $y_1 \in V$ such that $R[y_1] = \lambda_1$. If we set:

$$F(\epsilon) = R[y_1 + \epsilon\eta],$$

where $\eta \in V$, explain why $F'(0) = 0$. Hence show that[3] $Ly_1 = \lambda_1 y_1$. Comment on this result in relation to finding the smallest eigenvalue of L. How might you try to find the second smallest? (*Hint: orthogonality.*)

(c) Take $[a,b] = [0,1]$ and $L = -d^2/dx^2$. Compute $R[y]$ where $y(x) = x(1-x)$ and deduce $\lambda_{\text{min}} = \pi^2 \le 10$.

➡ **Solution:** (a) We have:

$$\langle Ly, y\rangle_w = \lambda\langle y, y\rangle_w = \int_a^b wy^2\,dx.$$

On the other hand, we also have:

$$\begin{aligned}\langle Ly, y\rangle_w &= \int_a^b \left(-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x)\right)y(x)\,dx \\ &= -\left[p(x)\frac{dy}{dx}y(x)\right]_a^b + \int_a^b \left[p(y')^2 + qy^2\right]\,dx \\ &= \int_a^b \left[p(y')^2 + qy^2\right]\,dx,\end{aligned}$$

using integration by parts and the Dirichlet boundary conditions. It follows that:

$$\lambda = \frac{\langle Ly, y\rangle_w}{\langle y, y\rangle_w} = R[y],$$

as required.

---

[3] You may assume that if $\int_a^b f\eta\,dx = 0$ for all $\eta \in V$, then $f = 0$.

(b) Note that:

$$F(\epsilon) = R[y_1 + \epsilon\eta] \geq \inf_{y \in V \setminus \{0\}} R[y] = \lambda_1 = R[y_1] = F(0).$$

Thus $\epsilon = 0$ is a local minimum of $F(\epsilon)$, and it follows that $F'(0) = 0$. To show that $Ly_1 = \lambda_1 y_1$, we note that:

$$\begin{aligned}
R[y_1 + \epsilon\eta] &= \left(\int_a^b \left[p(y_1' + \epsilon\eta')^2 + q(y_1 + \epsilon\eta)^2 \right] \, dx \right) \left(\int_a^b w(y_1 + \epsilon\eta)^2 \, dx \right)^{-1} \\
&= \left(\int_a^b \left[p(y_1')^2 + q(y_1)^2 \right] \, dx + 2\epsilon \int_a^b p y_1' \eta' + q y_1 \eta \, dx + O(\epsilon^2) \right) \left(\int_a^b w y_1^2 \, dx + 2\epsilon \int_a^b w y_1 \eta \, dx + O(\epsilon^2) \right)^{-1} \\
&= \left(\int_a^b \left[p(y_1')^2 + q(y_1)^2 \right] \, dx + 2\epsilon \int_a^b (p y_1' \eta' + q y_1 \eta) \, dx + O(\epsilon^2) \right) \\
&\quad \cdot \left(\left(\int_a^b w y_1^2 \, dx \right)^{-1} - 2\epsilon \left(\int_a^b w y_1 \eta \, dx \right) \left(\int_a^b w y_1^2 \, dx \right)^{-2} + O(\epsilon^2) \right) \\
&= R[y_1] + 2\epsilon \Bigg(\left(\int_a^b (p y_1' \eta' + q y_1 \eta) \, dx \right) \left(\int_a^b w y_1^2 \, dx \right)^{-1} \\
&\quad - \left(\int_a^b \left[p(y_1')^2 + q(y_1)^2 \right] \, dx \right) \left(\int_a^b w y_1 \eta \, dx \right) \left(\int_a^b w y_1^2 \, dx \right)^{-2} \Bigg).
\end{aligned}$$

Since $F'(0) = 0$, we know that the term in ϵ must vanish. Thus we have:

$$\int_a^b (p y_1' \eta' + q y_1 \eta) \, dx = R[y_1] \left(\int_a^b w y_1 \eta \, dx \right) = R[y_1] \left\langle y_1, \eta \right\rangle_w .$$

Integrating the left hand side by parts, we have:

$$R[y_1] \left\langle y_1, \eta \right\rangle_w = \int_a^b \left(-\frac{d}{dx} \left(p \frac{dy_1}{dx} \right) + q y_1 \right) \eta \, dx = \left\langle L y_1, \eta \right\rangle_w .$$

In particular, by linearity of the inner product, we have $\left\langle L y_1 - R[y_1] y_1, \eta \right\rangle_w = 0$ for all η. It follows that $L y_1 = R[y_1] y_1 = \lambda_1 y_1$, as required. This tells us that the minimiser of the Rayleigh quotient is precisely the solution to the eigenvalue problem of L with the smallest eigenvalue. To find the next smallest eigenvalue-eigenfunction pair, we should determine:

$$\lambda_2 = \inf_{\substack{y \in V \setminus \{0\} \\ y \perp y_1}} R[y].$$

(c) From lectures, we know $\lambda_n = (n\pi)^2$, so $\pi^2 \leq R[y]$ for any y. Taking $y = x(1-x)$, we also have:

$$\begin{aligned}
R[x(1-x)] &= \left(\int_0^1 (1-2x)^2 \, dx \right) \left(\int_0^1 x^2(1-x)^2 \, dx \right)^{-1} \\
&= \left[-\frac{(1-2x)^3}{6} \right]_0^1 \left(\left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \right]_0^1 \right)^{-1} \\
&= \frac{1}{3} \cdot \left(\frac{1}{30} \right)^{-1} \\
&= 10 \geq \pi^2.
\end{aligned}$$

Part IB: Mathematical Methods
Examples Sheet 2 Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

1. The function $\phi = \phi(x, y, z)$ satisfies the Laplace equation $\Delta\phi = 0$ on the cuboid $(x, y, z) \in (0, a) \times (0, b) \times (0, c)$ such that $\phi = 1$ on the side $z = 0$ and $\phi = 0$ on all other sides. Show that:

$$\phi(x, y, z) = \frac{16}{\pi^2} \sum_{p,q=0}^{\infty} \frac{\sinh(l_{p,q}(c-z)) \sin((2p+1)\pi x/a) \sin((2q+1)\pi y/b)}{(2p+1)(2q+1)\sinh(cl_{p,q})}$$

where $l_{p,q}^2 = (2p+1)^2\pi^2/a^2 + (2q+1)^2\pi^2/b^2$. Discuss the behaviour of the solution as $c \to \infty$.

Solution: Let $\phi(x, y, z) = X(x)Y(y)Z(z)$, and substitute into Laplace's equation $\Delta\phi = 0$. We obtain:

$$X''YZ + Y''XZ + Z''XY = 0 \qquad \Leftrightarrow \qquad \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0.$$

Since each of these summands can be varied independently of the others, we must be able to *separate variables*, allowing us to write:

$$-X'' = \lambda_x X, \qquad -Y'' = \lambda_y Y, \qquad -Z'' = \lambda_z Z,$$

where $\lambda_x, \lambda_y, \lambda_z$ are constants satisfying $\lambda_x + \lambda_y + \lambda_z = 0$. Let us solve each of these equations separately, imposing the boundary data as appropriate:

- In the x-direction, we have:

  $$X(x) = A\sin(\sqrt{\lambda_x}x) + B\cos(\sqrt{\lambda_x}x).$$

  Imposing the boundary condition $X(0) = 0$, we have $B \equiv 0$. Imposing the boundary condition $X(a) = 0$, we have $\sin(\sqrt{\lambda_x}a) = 0$, which implies $\lambda_x = n^2\pi^2/a^2$ for $n = 1, 2, \ldots$. Thus we possible values for X:

  $$X_n(x) = \sin(n\pi x/a).$$

  Observe that in this case, $-X'' = \lambda_x X$ is a Sturm-Liouville equation with $p(x) = 1$, $q(x) = 0$ and $w(x) = 1$, on the interval $[0, a]$; thus the X_n are orthogonal with the relation:

  $$\int_0^a X_n(x)X_m(x)\,dx = \alpha_n\delta_{nm},$$

  for some constants α_n. Note that for $n = m$, we have:

  $$\alpha_n = \int_0^a \sin^2(n\pi x/a)\,dx = \frac{1}{2}\int_0^a (1 - \cos(2n\pi x/a))\,dx = \frac{a}{2},$$

  hence we have:

  $$\int_0^a X_n(x)X_m(x)\,dx = \frac{a}{2}\delta_{nm}.$$

- The y-direction has the same analysis, yielding the possible values for Y:

  $$Y_n(y) = \sin(n\pi y/b).$$

  Similarly to $X_n(x)$, we have:

  $$\int_0^b Y_n(y)Y_m(y)\,dy = \frac{b}{2}\delta_{nm}.$$

- Finally, the z-direction has $\lambda_z = -\lambda_x - \lambda_y < 0$, so solutions in this direction take the form:

$$Z(z) = A\sinh(\sqrt{\lambda_z}(c-z)) + B\cosh(\sqrt{\lambda_z}(c-z)).$$

Imposing the boundary condition $Z(c) = 0$, we have $B \equiv 0$. Inserting the expressions for λ_x, λ_y we find the family of solutions in the z-direction:

$$Z_{n,m}(z) = \sinh\left(\sqrt{\frac{\pi^2 n^2}{a^2} + \frac{\pi^2 m^2}{b^2}}(c-z)\right).$$

Combining these solutions as a set of basis functions, we see that any solution of Laplace's equation that satisfies $\phi = 0$ on all sides of the cube except $z = 0$ can be expressed in the form:

$$\phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{n,m} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sinh\left(\sqrt{\frac{\pi^2 n^2}{a^2} + \frac{\pi^2 m^2}{b^2}}(c-z)\right).$$

It remains to impose the condition $\phi = 1$ on the side $z = 0$, which requires the appropriate determination of the constants $A_{n,m}$. We require:

$$1 = \sum_{n,m=1}^{\infty} A_{n,m} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sinh\left(\sqrt{\frac{\pi^2 n^2}{a^2} + \frac{\pi^2 m^2}{b^2}}c\right).$$

Integrating against $\sin(k\pi x/a)\sin(l\pi y/b)$ for $k, l = 1, 2, ...$, we can apply the orthogonality relations from above:

$$\int_0^a \sin\left(\frac{k\pi x}{a}\right)\,dx \int_0^b \sin\left(\frac{l\pi y}{b}\right)\,dy = \frac{ab}{4}A_{k,l}\sinh\left(\sqrt{\frac{\pi^2 k^2}{a^2} + \frac{\pi^2 l^2}{b^2}}c\right).$$

The left hand side evaluates to:

$$\left[-\frac{a}{k\pi}\cos\left(\frac{k\pi x}{a}\right)\right]_0^a \cdot \left[-\frac{b}{l\pi}\cos\left(\frac{l\pi y}{b}\right)\right]_0^b = \frac{ab}{kl\pi^2}\left((-1)^k - 1\right)\left((-1)^l - 1\right).$$

This is only non-zero if both k, l are odd; let $k = 2p+1$ and $l = 2q+1$, for $p, q = 0, 1, 2...$; in this case it evaluates to:

$$\frac{4ab}{(2p+1)(2q+1)\pi^2}.$$

Hence the constants $A_{k,l}$ are zero unless both k, l are odd, in which case they are given by:

$$A_{2p+1,2q+1} = \frac{16}{(2p+1)(2q+1)\pi^2}\frac{1}{\sinh(cl_{p,q})},$$

where $l_{p,q}^2 = (2p+1)^2\pi^2/a^2 + (2q+1)^2\pi^2/b^2$. Therefore, we derive the formula stated in the question:

$$\phi(x,y,z) = \frac{16}{\pi^2}\sum_{p,q=0}^{\infty}\frac{\sinh(l_{p,q}(c-z))\sin((2p+1)\pi x/a)\sin((2q+1)\pi y/a)}{(2p+1)(2q+1)\sinh(cl_{p,q})}$$

As $c \to \infty$, we have:

$$\lim_{c\to\infty}\frac{\sinh(l_{p,q}(c-z))}{\sinh(cl_{p,q})} = \lim_{c\to\infty}\frac{e^{l_{p,q}(c-z)} - e^{-l_{p,q}(c-z)}}{e^{cl_{p,q}} - e^{-cl_{p,q}}} = e^{-l_{p,q}z}.$$

Thus the solution reduces to:

$$\phi(x,y,z) = \frac{16}{\pi^2}\sum_{p,q=0}^{\infty}\frac{\sin((2p+1)\pi x/a)\sin((2q+1)\pi y/a)}{(2p+1)(2q+1)}e^{-l_{p,q}z}.$$

This satisfies $\Delta\phi = 0$ with $\phi = 0$ on all sides of $(0,a)\times(0,b)\times(0,\infty)$ except $z = 0$, where $\phi = 1$.

2. The function $\phi = \phi(r,\theta)$ satisfies the Laplace equation $\Delta\phi = 0$ on the unit disc $(r,\theta) \in [0,1) \times [0,2\pi)$ such that $\phi(1,\theta) = \pi/2$ on $0 \leq \theta < \pi$ and $\phi(1,\theta) = -\pi/2$ on $\pi \leq \theta \leq 2\pi$. Show that:

$$\phi(r,\theta) = 2\sum_{n\,\text{odd}} \frac{r^n \sin(n\theta)}{n}.$$

Sum the series using the substitution $z = re^{i\theta}$. [Your solution can be interpreted geometrically as the angle between the lines to the two points on the boundary where the data jumps.]

➾ **Solution:** First, we shall repeat the argument from lectures to obtain the general solution to Laplace's equation on an annulus, $r_1 \leq |\mathbf{x}| \leq r_2$ in the plane $\mathbb{R}^2$. Laplace's equation in polar coordinates takes the form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

Separating variables by writing $\phi = R(r)\Theta(\theta)$, we have:

$$\frac{r(rR')'}{R} + \frac{\Theta''}{\Theta} = 0,$$

so that:

$$\Theta'' + \lambda\Theta = 0, \qquad r(rR')' - \lambda R = 0$$

for some constant λ. Solving the first of these equations, we have:

$$\Theta(\theta) = A\sin(\sqrt{\lambda}\theta) + B\cos(\sqrt{\lambda}\theta).$$

The only boundary condition we need to enforce here is $\Theta(\theta+2\pi) = \Theta(\theta)$, to ensure that the angular part is single-valued. This requires:

$$A\sin(\sqrt{\lambda}\theta) + B\cos(\sqrt{\lambda}\theta) = A\sin(\sqrt{\lambda}(\theta+2\pi)) + B\cos(\sqrt{\lambda}(\theta+2\pi))$$

for all values of θ, which is sufficient to force $2\pi\sqrt{\lambda} = 2\pi n$ for n an integer, and hence $\lambda_n = n^2$ for $n = 0, 1, 2, \ldots$ and so on.

The radial part of the equation is now:

$$r(rR')' - n^2 R = 0 \qquad \Leftrightarrow \qquad r^2R'' + rR' - n^2R = 0.$$

This is a linear equidimensional equation of the type studied in Part IA Differential Equations, and hence has solutions of the form r^α. Trialling a solution of this form gives the characteristic equation:

$$\alpha(\alpha-1) + \alpha - n^2 = 0,$$

which implies that $\alpha = \pm n$. When $n = 0$, this only yields on solution; the other solution is logarithmic and is of the form $\log(r)$. Thus we obtain the complete solution to Laplace's equation on an annulus:

$$\phi(r,\theta) = A + B\log(r) + \sum_{n=1}^{\infty}\left(A_n r^n + \frac{B_n}{r^n}\right)\left(C_n\sin(n\theta) + D_n\cos(n\theta)\right).$$

We can now begin the question proper. Since we are working on the unit disk, we require $B = 0$, $B_n = 0$ for the function ϕ to be regular at the origin. We can then take $A_n = 1$ without loss of generality. We now impose our boundary data at $r = 1$. We have:

$$\phi(1,\theta) = A + \sum_{n=1}^{\infty}\left(C_n\sin(n\theta) + D_n\cos(n\theta)\right).$$

This looks just like obtaining the Fourier coefficients from Sheet 1!

We observe that, with the inner product defined on $[0, 2\pi]$, and with $n \geq 1$, we have:

$$\langle 1, \phi(1,\theta) \rangle = 2\pi A, \qquad \langle c_n, \phi(1,\theta) \rangle = \pi D_n, \qquad \langle s_n, \phi(1,\theta) \rangle = \pi C_n,$$

where $c_n = \cos(n\theta)$, $s_n = \sin(n\theta)$. Moreover, shifting the function so that it is $-\pi/2$ on $-\pi \leq \theta < 0$, we see that it is an *odd* function; thus all the even Fourier coefficients vanish (and so does the constant term). In particular, we can obtain the remaining coefficients as:

$$C_n = \frac{2}{\pi} \int_0^\pi \frac{\pi}{2} \sin(n\theta) \, d\theta = \left[-\frac{1}{n} \cos(n\theta) \right]_0^\pi = \frac{1 - (-1)^n}{n}.$$

This yields the solution:

$$\phi(r,\theta) = \sum_{n=1}^{\infty} \frac{r^n (1 - (-1)^n) \sin(n\theta)}{n} = 2 \sum_{n \text{ odd}} \frac{r^n \sin(n\theta)}{n},$$

as required.

Finally, we are asked to sum this series using the substitution $z = re^{i\theta}$. We have:

$$\begin{aligned}
\phi(r,\theta) &= \operatorname{Im}\left[2 \sum_{n \text{ odd}} \frac{r^n e^{in\theta}}{n} \right] \\
&= \operatorname{Im}\left[2 \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1} \right] \\
&= \operatorname{Im}\left[2 \int_0^z dz \sum_{n=0}^{\infty} z^{2n} \right] \\
&= \operatorname{Im}\left[2 \int_0^z dz \left(\frac{1}{1 - z^2} \right) \right] \\
&= \operatorname{Im}\left[\int_0^z dz \left(\frac{1}{1-z} + \frac{1}{1+z} \right) \right] \\
&= \operatorname{Im}\left[-\log(1-z) + \log(1+z) \right] \\
&= \operatorname{Im}\left[\log|1+z| + i \arg(1+z) - \log|1-z| - i \arg(1-z) \right] \\
&= \arg(1+z) - \arg(1-z) \\
&= \arg(1 + re^{i\theta}) - \arg(1 - re^{i\theta}).
\end{aligned}$$

3. The function $\phi = \phi(r,\theta)$ satisfies the Laplace equation $\Delta\phi = 0$ on the unit ball $(r,\theta,\phi) \in [0,1) \times [0,\pi] \times [0,2\pi)$ such that $\phi(1,\theta) = 1$ on $0 \leq \theta < \pi/2$ and $\phi(1,\theta) = -1$ on $\pi/2 \leq \theta \leq \pi$. Show that:

$$\phi(r,\theta) = \sum_{n=0}^{\infty} \alpha_n r^n P_n(\cos(\theta)),$$

where α_n are constants you should determine in terms of the Legendre polynomials. It will be useful to note that $P'_{n+1}(z) - P'_{n-1}(z) = (2n+1)P_n(z)$ and:

$$\int_{-1}^{1} P_n(z)P_m(z)\,dz = \frac{2\delta_{mn}}{2n+1}.$$

➻ **Solution:** Similarly, we shall recap from lectures how to perform separation of variables on a ball. Suppose that we wish to solve Laplace's equation on the thick shell $a \leq |\mathbf{x}| \leq b$ in $\mathbb{R}^3$, given that the solution is axisymmetric (i.e. symmetric about the z-axis). Laplace's equation in spherical polar coordinates then takes the form:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\phi}{\partial\theta}\right) = 0,$$

since the solution $\phi = \phi(r,\theta)$ is independent of the polar angle about the z-axis. Trialling a separable solution $\phi = R(r)\Theta(\theta)$ gives the equation:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Theta}{\partial\theta}\right) = 0,$$

which implies that we can separate variables to get:

$$(r^2R')' - \lambda R = 0, \qquad (\sin(\theta)\Theta')' + \lambda\sin(\theta)\Theta = 0.$$

for some λ.

In the second equation, make the substitution $x = \cos(\theta)$. Then:

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin(\theta)\frac{d}{dx}.$$

Hence the equation can be recast as:

$$\sin(\theta)\frac{d}{dx}\left(\sin^2(\theta)\frac{d\Theta}{dx}\right) + \lambda\sin(\theta)\Theta = 0 \qquad \Leftrightarrow \qquad -\frac{d}{dx}\left((1-x^2)\frac{d\theta}{dx}\right) = \lambda\Theta.$$

This is precisely Legendre's equation! It has eigenfunctions $\Theta = P_n(x)$ and corresponding eigenvalues $\lambda_n = n(n+1)$ for $n = 0,1,2,...$, where the P_n are the Legendre polynomials. Hence undoing the substitution, we have the solutions $\Theta(\theta) = P_n(\theta)$ with $\lambda_n = n(n+1)$ for $n = 0,1,2,...$.

Turning to the radial equation, we have:

$$r^2R'' + 2rR' - n(n+1)R = 0,$$

which is a linear equidimensional equation of the type encountered in Part IA Differential Equations. Thus trialling a solution $R = r^\alpha$, we arrive at the characteristic equation:

$$\alpha(\alpha-1) + 2\alpha - n(n+1) = 0 \qquad \Leftrightarrow \qquad \alpha^2 + \alpha - n(n+1) = 0 \qquad \Leftrightarrow \qquad (\alpha - n)(\alpha + (n+1)) = 0.$$

Thus $\alpha = n, -n-1$ are the solutions (which are always distinct). It follows that the general solution to Laplace's equation is:

$$\phi(r,\theta) = \sum_{n=0}^{\infty} P_n(\cos(\theta))\left(\alpha_n r^n + \frac{\beta_n}{r^{n+1}}\right).$$

Now we can start on the problem of interest. We wish to solve Laplace's equation on the unit ball, so regularity at the origin requires $\beta_n = 0$. This leaves us with:

$$\phi(r,\theta) = \sum_{n=0}^{\infty} \alpha_n r^n P_n(\cos(\theta)).$$

Imposing the boundary data on $r = 1$, we require:

$$\phi(1,\theta) = \sum_{n=0}^{\infty} \alpha_n P_n(\cos(\theta)). \tag{$*$}$$

Let $x = \cos(\theta)$. Then we have $\phi(1,\theta) = 1$ if $0 < x \le 1$ and $\phi(1,\theta) = -1$ if $-1 \le x \le 0$. Thus multiplying by $P_m(x)$ and integrating, we have:

$$\begin{aligned}\frac{2\alpha_m}{2m+1} &= \int_{-1}^{1} P_m(x)\phi(1,\theta)\,dx \\ &= \int_{0}^{1} P_m(x)\,dx - \int_{-1}^{0} P_m(x)\,dx.\end{aligned}$$

Using the hint, we have:

$$\int P_m(x)\,dx = \frac{1}{2m+1}\int \left(P'_{m+1}(x) - P'_{m-1}(x)\right)\,dx = \frac{1}{2m+1}\left(P_{m+1}(x) - P_{m-1}(x)\right) + \text{constant},$$

hence applying it to the above integrals, we have:

$$\begin{aligned}2\alpha_m &= P_{m+1}(1) - P_{m-1}(1) - P_{m+1}(0) + P_{m-1}(0) - P_{m+1}(0) + P_{m-1}(0) + P_{m+1}(-1) - P_{m-1}(-1) \\ &= 2\left(P_{m-1}(0) - P_{m+1}(0)\right) + P_{m+1}(-1) - P_{m-1}(-1),\end{aligned}$$

since $P_n(1) = 1$ for all Legendre polynomials. Further, note that when m is even, $P_m(x)$ is an even polynomial, and when m is even $P_m(x)$ is an odd polynomial. Thus in the case where m is even, we have $P_{m-1}(x), P_{m+1}(x)$ odd, hence they are zero at zero, and satisfy $P_{m+1}(-1) = -P_{m+1}(1) = -1$; it follows that $\alpha_m = 0$ in this case. On the other hand when m is odd, we have $P_{m-1}(x), P_{m+1}(x)$ even, hence $P_{m+1}(-1) = P_{m+1}(1) = 1$; it follows that:

$$\alpha_m = P_{m-1}(0) - P_{m+1}(0).$$

To go any further, we need to be able to evaluate the even Legendre polynomials at $x = 0$. To do so, start from Rodrigue's formula from Sheet 1:

$$\begin{aligned}P_{2n}(0) &= \frac{1}{2^{2n}(2n)!}\frac{d^{2n}}{dx^{2n}}(x^2-1)^{2n}\bigg|_{x=0} = \frac{1}{2^{2n}2n!}\frac{d^{2n}}{dx^{2n}}\left(\sum_{k=0}^{2n}(-1)^{2n-k}\binom{2n}{k}x^{2k}\right)\bigg|_{x=0} \\ &= \frac{(-1)^n}{2^{2n}(2n)!}\binom{2n}{n}\cdot(2n)! \\ &= \frac{(-1)^n}{2^{2n}}\binom{2n}{n}.\end{aligned}$$

Overall then, when $m = 2k + 1$ is odd, we have:

$$\begin{aligned}
\alpha_{2k+1} = P_{2k}(0) - P_{2k+2}(0) &= \frac{(-1)^k}{2^{2k}}\binom{2k}{k} - \frac{(-1)^{k+1}}{2^{2k+2}}\binom{2k+2}{k+1} \\
&= \frac{(-1)^k}{2^{2k}}\left(\binom{2k}{k} + \frac{1}{2}\binom{2k+2}{k+1}\right) \\
&= \frac{(-1)^k}{2^{2k}}\left(\frac{(2k)!}{k!^2} + \frac{1}{2}\frac{(2k+2)!}{(k+1)!^2}\right) \\
&= \frac{(-1)^k}{2^{2k}}\left(1 + \frac{2k+1}{k+1}\right)\binom{2k}{k}.
\end{aligned}$$

4. A uniform string of mass per unit length μ and tension τ undergoes small transverse vibrations of amplitude $y = y(x,t)$. The string is fixed at $x = 0$ and $x = L$ and satisfies the initial conditions:

$$y(x,0) = 0, \qquad \frac{\partial y}{\partial t}(x,0) = \frac{4V}{L^2}x(L-x), \qquad \text{for } 0 < x < L.$$

Using the fact that y satisfies the wave equation with speed c where $c^2 = \tau/\mu$, find the amplitudes of the normal modes and deduce the kinetic and potential energies of the string at time t. Hence show that:

$$\sum_{n \text{ odd}} \frac{1}{n^6} = \frac{\pi^6}{960}.$$

➾ **Solution:** The wave equation satisfied by y is:

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}.$$

Employing separation of variables, we write $y = X(x)T(t)$, so that X, T satisfy the equations:

$$T''(t) + \lambda c^2 T = 0, \qquad X''(x) + \lambda X = 0,$$

where λ is a constant. The string is fixed at $x = 0$ and $x = L$, implying that $y(0,t) = y(L,t) = 0$, and hence $X(0) = X(L) = 0$. Thus solving the second equation, we obtain:

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Imposing the boundary condition $X(0) = 0$, we have $A = 0$. Imposing the boundary condition $X(L) = 0$, we have $\sqrt{\lambda}L = n\pi$ for $n \in \mathbb{Z}$, which implies the eigenvalues are $\lambda_n = n^2\pi^2/L^2$.

Turning to the first equation, we have:

$$T''(t) + \lambda_n c^2 T = 0,$$

which has solutions:

$$T(t) = C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right).$$

The boundary condition $y(x,0) = 0$ implies that $T(0) = 0$, thus $C \equiv 0$. It follows that the general solution to the wave equation with these boundary conditions is:

$$y(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

To obtain the coefficients α_n, we need to impose the boundary condition on $\partial y/\partial t(x,0)$. We have:

$$\frac{4V}{L^2}x(L-x) = \sum_{n=1}^{\infty} \frac{n\pi c\alpha_n}{L}\sin\left(\frac{n\pi x}{L}\right). \tag{$*$}$$

From Question 1 of Examples Sheet 1, we have:

$$\int_{-L}^{L} \sin\left(\frac{2n\pi x}{2L}\right)\sin\left(\frac{2m\pi x}{2L}\right)\,dx = L\delta_{mn},$$

which by evenness of the integrand on the left hand side implies:

$$\int_{0}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = \frac{L}{2}\delta_{mn}. \tag{$\dagger$}$$

Thus taking the integral of $(*)$ against $\sin(m\pi x/L)$ from 0 to L, we obtain:

$$\begin{aligned}
\frac{m\pi c\alpha_m}{2} &= \int_0^L \frac{4V}{L^2}x(L-x)\sin\left(\frac{m\pi x}{L}\right)\,dx \\
&= \frac{4V}{L^2}\left(\left[-\frac{L}{m\pi}x(L-x)\cos\left(\frac{m\pi x}{L}\right)\right]_0^L + \frac{L}{m\pi}\int_0^L (L-2x)\cos\left(\frac{m\pi x}{L}\right)\,dx\right) \\
&= \frac{4V}{L^2}\left(\left[\frac{L^2}{m^2\pi^2}(L-2x)\sin\left(\frac{m\pi x}{L}\right)\right]_0^L + \frac{2L^2}{m^2\pi^2}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\,dx\right) \\
&= \frac{4V}{L^2}\cdot\frac{2L^2}{m^2\pi^2}\cdot\frac{L}{m\pi}(1-(-1)^m) \\
&= \frac{8VL}{m^3\pi^3}(1-(-1)^m).
\end{aligned}$$

Hence:

$$\alpha_m = \begin{cases} \dfrac{32VL}{m^4\pi^4 c}, & \text{if } m \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

It follows that the complete solution is:

$$y(x,t) = \sum_{n=0}^{\infty} \frac{32VL}{(2n+1)^4\pi^4 c}\sin\left(\frac{(2n+1)\pi x}{L}\right)\sin\left(\frac{(2n+1)\pi ct}{L}\right).$$

The kinetic energy of the solution is:

$$\begin{aligned}
\text{KE} &= \frac{\mu}{2}\int_0^L \left(\frac{\partial y}{\partial t}\right)^2\,dx \\
&= \frac{\mu}{2}\cdot\frac{32^2V^2L^2}{\pi^8c^2}\int_0^L\left(\sum_{n=0}^{\infty}\frac{\pi c}{(2n+1)^3L}\sin\left(\frac{(2n+1)\pi x}{L}\right)\cos\left(\frac{(2n+1)\pi ct}{L}\right)\right)^2\,dx \\
&= \frac{256\mu V^2L}{\pi^6}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6}\cos^2\left(\frac{(2n+1)\pi ct}{L}\right),
\end{aligned}$$

by the orthogonality relation (†) we proved for sine above. Similarly, the potential energy of the solution is:

$$\begin{aligned}
\text{PE} &= \frac{\tau}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx \\
&= \frac{\tau}{2}\cdot\frac{32^2V^2L^2}{\pi^8c^2}\int_0^L \left(\sum_{n=0}^{\infty}\frac{\pi}{(2n+1)^3L}\cos\left(\frac{(2n+1)\pi x}{L}\right)\sin\left(\frac{(2n+1)\pi ct}{L}\right)\right)^2 dx \\
&= \frac{256\tau V^2L}{\pi^6c^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6}\sin^2\left(\frac{(2n+1)\pi ct}{L}\right) \\
&= \frac{256\mu V^2L}{\pi^6}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6}\sin^2\left(\frac{(2n+1)\pi ct}{L}\right).
\end{aligned}$$

Thus the total energy in the string is given by:

$$E = \frac{256\mu V^2L}{\pi^6}\sum_{n\text{ odd}}\frac{1}{n^6},$$

using the identity $\sin^2(\theta)+\cos^2(\theta)=1$.

To make the final deduction, we use energy conservation. The initial energy of the string is entirely kinetic, and is given by:

$$\begin{aligned}
E &= \frac{\mu}{2}\int_0^L \frac{16V^2}{L^4}x^2(L-x)^2\,dx \\
&= \frac{8\mu V^2}{L^4}\int_0^L \left(L^2x^2-2Lx^3+x^4\right)\,dx \\
&= \frac{8\mu V^2}{L^4}\left[\frac{L^2x^3}{3}-\frac{Lx^4}{2}+\frac{x^5}{5}\right]_0^L \\
&= 8\mu V^2L\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right) \\
&= \frac{4\mu V^2L}{15}.
\end{aligned}$$

This implies:

$$\frac{4\mu V^2L}{15} = \frac{256\mu V^2L}{\pi^6}\sum_{n\text{ odd}}\frac{1}{n^6},$$

which on rearrangement produces:

$$\sum_{n\text{ odd}}\frac{1}{n^6} = \frac{\pi^6}{960},$$

as required.

5. The displacement $y = y(x,t)$ of a uniform string stretched between $x = 0$ and $x = L$ satisfies the wave equation with the boundary conditions $y(0,t) = y(L,t) = 0$. For $t < 0$ the string oscillates in the fundamental mode $y(x,t) = A\sin(\pi x/L)\sin(\pi ct/L)$. A musician strikes the midpoint of the string impulsively at time $t = 0$ so that the change in $\partial y/\partial t$ at $t = 0$ is $\lambda\delta(x - \frac{1}{2}L)$. Find $y = y(x,t)$ for $t > 0$.

➾ **Solution:** We already know from the separation of variables we carried out in Question 4 that the general solution with boundary conditions $y(0,t) = y(L,t) = 0$ takes the form, for $t > 0$:

$$y(x,t) = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right). \tag{$*$}$$

We are now given the further boundary conditions, regarding impulse:

$$\lim_{\epsilon\to 0^+}\left[\frac{\partial y}{\partial t}(x,\epsilon) - \frac{\partial y}{\partial t}(x,-\epsilon)\right] = \lambda\delta\left(x - \frac{1}{2}L\right),$$

and regarding the string's behaviour for $t < 0$, which by continuity demands:

$$\lim_{\epsilon\to 0^+}\left[y(x,\epsilon) + A\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi c\epsilon}{L}\right)\right] = 0.$$

Imposing the second condition on ($*$), we obtain:

$$\sum_{n=0}^{\infty} A_n\sin\left(\frac{n\pi x}{L}\right) = 0,$$

which implies $A_n = 0$ for all n, by linear independence, leaving us with:

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

Imposing the jump condition in the derivative, we have:

$$\sum_{n=1}^{\infty}\frac{n\pi c}{L}B_n\sin\left(\frac{n\pi x}{L}\right) - \frac{A\pi c}{L}\sin\left(\frac{\pi x}{L}\right) = \lambda\delta\left(x - \frac{1}{2}L\right).$$

Multiplying by $\sin(m\pi x/L)$ and integrating both sides from 0 to L (using the orthogonality relation from Question 4), we obtain:

$$\frac{m\pi c}{2}B_m - \frac{1}{2}A\pi c\delta_{m1} = \lambda\sin\left(\frac{1}{2}m\pi\right)$$

Hence the coefficients are given by:

$$B_m = A\delta_{m1} + \frac{2\lambda}{m\pi c}\sin\left(\frac{1}{2}m\pi\right).$$

Noting that $\sin(\frac{1}{2}m\pi) = 0$ if $m = 2k$ is even, and $\sin(\frac{1}{2}m\pi) = (-1)^k$ if $m = 2k+1$ is odd, we can write the final solution as:

$$y(x,t) = \left(A + \frac{2\lambda}{\pi c}\right)\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi ct}{L}\right) + \frac{2\lambda}{\pi c}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}\sin\left(\frac{(2k+1)\pi x}{L}\right)\sin\left(\frac{(2k+1)\pi ct}{L}\right).$$

6. Consider a uniform stretched string of length L, mass per unit length μ, tension $\tau = \mu c^2$ and ends fixed.

(i) The string undergoes oscillations in a resistive medium that produces a resistive force per unit length of $-2k\mu y_t$, where $y = y(x,t)$ is the transverse displacement and $k = \pi c/L$. Derive the equation of motion:

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} - \frac{2k}{c^2}\frac{\partial y}{\partial t}.$$

Find $y = y(x,t)$ if $y(x,0) = A\sin(\pi x/L)$ and $y_t(x,0) = 0$.

(ii) If an extra transverse force $F\sin(\pi x/L)\cos(\pi ct/L)$ per unit length is applied to the string, find the associated particular integral. Discuss the behaviour of the full solution as $t \to \infty$.

➾ **Solution:** (i) Consider applying Newton's second law to the second of string between x_A and x_B, with $x_B - x_A = \delta x$. The tension τ is constant throughout the string, and the oscillations are transverse, so resolving in the x-direction we have:

$$\tau\cos(\theta_A) \approx \tau\cos(\theta_B),$$

since δx is very small. This is self-consistent, because:

$$\cos(\theta_A) = \frac{1}{\sqrt{1+\tan^2(\theta_A)}} = \frac{1}{\sqrt{1+(\partial y/\partial x)^2}} \approx 1,$$

by the binomial theorem, assuming $|\partial y/\partial x| \ll 1$ (since each part of the string moves transversely, this must be very small).

On the other hand, resolving the y-direction and using Newton's second law, we have:

$$\mu\delta x\frac{\partial^2 y}{\partial t^2} = \tau\sin(\theta_B) - \tau\sin(\theta_A) - 2k\mu\delta x\frac{\partial y}{\partial t},$$

where the term on the right hand side is added because we are dealing with a resistive medium that produces a force per unit length of $-2k\mu y_t$. Dividing through by $\delta x\tau \approx \delta x\tau\cos(\theta_B) \approx \delta x\tau\cos(\theta_A)$, we have:

$$\frac{\mu}{\tau}\frac{\partial^2 y}{\partial t^2} = \frac{\tan(\theta_B)-\tan(\theta_A)}{\delta x} - \frac{2k\mu}{\tau}\frac{\partial y}{\partial t}.$$

Noting:

$$\tan(\theta_B) - \tan(\theta_A) = \left.\frac{\partial y}{\partial x}\right|_{x=x_B} - \left.\frac{\partial y}{\partial x}\right|_{x=x_A},$$

we have in the limit where $\delta x \to 0$ (using $c^2 = \tau/\mu$):

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} - \frac{2k}{c^2}\frac{\partial y}{\partial t},$$

as required.

Next, we are asked to determine $y = y(x,t)$ if $y(x,0) = A\sin(\pi x/L)$ and $y_t(x,0) = 0$. Instead of going straight for full separation of variables, note that $\sin(\pi x/L)$ is already an eigenfunction of $\partial^2/\partial x^2$, so we can just try:

$$y(x,t) = T(t)\sin\left(\frac{\pi x}{L}\right).$$

Inserting into the equation, we get:

$$\frac{1}{c^2}\sin\left(\frac{\pi x}{L}\right)T''(t) = -\frac{\pi^2 T}{L^2}\sin\left(\frac{\pi x}{L}\right) - \frac{2k}{c^2}T'(t)\sin\left(\frac{\pi x}{L}\right),$$

which allows us to cancel the x-dependence (as expected), leaving only:

$$T''(t) + 2kT' + \frac{\pi^2 c^2}{L^2} T = 0 \qquad \Leftrightarrow \qquad T''(t) + 2kT' + k^2 T = 0.$$

The characteristic equation of this ODE is $0 = \lambda^2 + 2k\lambda + k^2 = (\lambda + k)^2$, which has the double root $\lambda = -k$. Thus the general solution is:

$$T(t) = (C + Dt)e^{-kt}.$$

The complete solution to the partial differential equation is therefore:

$$y(x,t) = (C + Dt)e^{-kt} \sin\left(\frac{\pi x}{L}\right).$$

Imposing the boundary data, at $t = 0$ we have $C = A$. Taking the derivative with respect to t, we have:

$$y_t(x,t) = (De^{-kt} - k(C + Dt)e^{-kt}) \sin\left(\frac{\pi x}{L}\right),$$

which inserting $t = 0$ gives $D - kC \equiv 0$, i.e. $D = kA$. Overall then, the complete solution satisfying the boundary conditions is:

$$y(x,t) = A(1 + kt)e^{-kt} \sin\left(\frac{\pi x}{L}\right).$$

(ii) In the second part of the question, we add an extra transverse force per unit length, which modifies the equation to (by going back through the derivation):

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} - \frac{2k}{c^2}\frac{\partial y}{\partial t} + \frac{F}{\mu c^2} \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi c t}{L}\right),$$

We wish to find a particular integral $y_{\mathsf{PI}}(x,t)$ with zero initial conditions; we can then add it to the homogeneous solution that has non-zero initial conditions. In particular, the particular integral must satisfy the above equation with:

$$y_{\mathsf{PI}}(x,0) = 0, \qquad \frac{\partial y_{\mathsf{PI}}}{\partial t}(x,0) = 0.$$

Let's trial a solution of the form $y_{\mathsf{PI}}(t) = T(t)\sin(\pi x/L)$ again, since the forcing is proportional to $\sin(\pi x/L)$, and we know that $\sin(\pi x/L)$ is an eigenfunction of $\partial^2/\partial x^2$. Inserting this into the given equation, we get:

$$\frac{1}{c^2} T'' \sin\left(\frac{\pi x}{L}\right) = -\frac{\pi^2}{L^2} T - \frac{2k}{c^2} T' \sin\left(\frac{\pi x}{L}\right) + \frac{F}{\mu c^2} \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi c t}{L}\right).$$

Again, we see that the $\sin(\pi x/L)$ dependence cancels throughout, yielding:

$$T''(t) + 2kT' + k^2 T = \frac{F}{\mu} \cos(kt).$$

Using the solution from the previous part of the question, we know that the homogeneous solution of this ODE will be $T(t) = (\alpha + \beta t)e^{-kt}$ for some α, β. To obtain a particular integral, trial $T = a\cos(kt) + b\sin(kt)$. Then:

$$-ak^2\cos(kt) - bk^2\sin(kt) + 2k(-ak\sin(kt) + bk\cos(kt)) + k^2 a\cos(kt) + k^2 b\sin(kt) = \frac{F}{\mu}\cos(kt).$$

Comparing coefficients, we see that:

$$2bk^2 = \frac{F}{\mu c^2}, \qquad -2ak^2 = 0,$$

which implies that $a = 0$ and $b = F/2\mu k^2$. Thus the complete solution of the ODE is:

$$T(t) = (\alpha + \beta t)e^{-kt} + \frac{F}{2\mu k^2}\sin(kt).$$

Imposing the initial condition $T(0) = 0$, we have $\alpha = 0$. Then taking the derivative, we have:

$$\dot{T}(t) = \beta e^{-kt} - k\beta t e^{-kt} + \frac{F}{2\mu k}\cos(kt).$$

Imposing the initial condition $\dot{T}(0) = 0$, we have $\beta = -F/2\mu k$, which implies the full solution to the ODE with boundary conditions imposed is:

$$T(t) = \frac{F}{2\mu k}\left(\frac{1}{k}\sin(kt) - te^{-kt}\right).$$

Therefore the full particular integral is:

$$y_{\text{PI}}(x,t) = \frac{F}{2\mu k}\left(\frac{1}{k}\sin(kt) - te^{-kt}\right)\sin\left(\frac{\pi x}{L}\right),$$

and hence the full solution to the new inhomogeneous equation is:

$$y(x,t) = A(1+kt)e^{-kt}\sin\left(\frac{\pi x}{L}\right) + \frac{F}{2\mu k}\left(\frac{1}{k}\sin(kt) - te^{-kt}\right)\sin\left(\frac{\pi x}{L}\right).$$

Finally, we are asked to discuss the behaviour of the full solution as $t \to \infty$. As $t \to \infty$, the solution tends to:

$$y(x,t) \approx \frac{F}{2\mu k^2}\sin(kt)\sin\left(\frac{\pi x}{L}\right),$$

i.e. the forcing dominates in the long term regime, as expected.

7. A string of uniform density is stretched along the x-axis under tension τ. It undergoes small transverse oscillations so that the displacement $y = y(x,t)$ satisfies the wave equation.

(i) Show that if a mass M is fixed to the string at $x = 0$ then its equation of motion can be written:

$$M \frac{\partial^2 y}{\partial t^2}\bigg|_{x=0} = \tau \left[\frac{\partial y}{\partial x}\right]_{x=0^-}^{x=0^+}. \qquad (\star)$$

(ii) A wave of the form $(x,t) \mapsto \exp[i\omega(t - x/c)]$ is incident from $x \to -\infty$ giving rise to a solution of the wave equation of the form:

$$y(x,t) = \begin{cases} e^{i\omega(t-x/c)} + Re^{i\omega(t+x/c)}, & x < 0, \\ Te^{i\omega(t-x/c)} & x > 0. \end{cases}$$

Using ($\star$) and an appropriate continuity condition at $x = 0$, find expressions for $T = T(\lambda)$ and $R = R(\lambda)$ where $\lambda = M\omega c/\tau$. Discuss the limiting behaviour of R and T when λ is large or small.

➾ **Solution:** (i) Consider a small length of string δx passing through $x = 0$, with a length of $\delta x/2$ on either side. Suppose that the string makes an angle $\theta+$ with the horizontal to the right, and makes an angle θ_- with the horizontal to the left. Then as before, we have $\tau \approx \tau\cos(\theta_+) \approx \tau\cos(\theta_-)$ by force balance in the horizontal direction, and by Newton's second law applied in the vertical direction we have:

$$(M + \mu\delta x)\frac{\partial^2 y}{\partial t^2}\bigg|_{x=0} = \tau\sin(\theta_+) - \tau\sin(\theta_-),$$

where the mass is the sum of the mass M and the string's mass $\mu\delta x$. Dividing by $\cos(\theta_+) \approx \cos(\theta_-) \approx 1$, we have:

$$(M + \mu\delta x)\frac{\partial^2 y}{\partial t^2}\bigg|_{x=0} = \tau\tan(\theta_+) - \tau\tan(\theta_-) \approx \tau\frac{\partial y}{\partial x}\bigg|_{x=0^+} - \tau\frac{\partial y}{\partial x}\bigg|_{x=0^-}.$$

Taking the limit as $\delta x \to 0$, this equation approaches:

$$M \frac{\partial^2 y}{\partial t^2}\bigg|_{x=0} = \tau \left[\frac{\partial y}{\partial x}\right]_{x=0^-}^{x=0^+},$$

as required.

(ii) Certainly $y(x,t)$ should be continuous as a function of x for each fixed t. This means that at $x = 0$, we require:

$$e^{i\omega t} + Re^{i\omega t} = Te^{i\omega t},$$

or in other words, $1 + R = T$.

To obtain another condition on R, T, consider:

$$\frac{\partial^2 y}{\partial t^2} = \begin{cases} (i\omega)^2 e^{i\omega(t-x/c)} + (i\omega)^2 Re^{i\omega(t+x/c)}, & x < 0, \\ (i\omega)^2 Te^{i\omega(t-x/c)}, x > 0. \end{cases}$$

This is also continuous at $x = 0$, assuming the continuity of $y(x,t)$ at $x = 0$, taking the value $(i\omega)^2(1+R)e^{i\omega t} = (i\omega)^2 Te^{i\omega t}$. Thus the equation of motion ($\star$) that we derive in (i) implies:

$$-\omega^2 MTe^{i\omega t} = \tau\left(-\frac{i\omega}{c}Te^{i\omega t} + \frac{i\omega}{c}e^{i\omega t} - \frac{i\omega}{c}Re^{i\omega t}\right).$$

Dividing through by $e^{i\omega t}$, we have:

$$c\omega MT = i\tau(T - 1 + R) = 2i\tau R,$$

using the continuity equation in the last equality. It follows that $T = 2\tau iR/M\omega c = 2iR/\lambda$, and hence:

$$1 + R = \frac{2iR}{\lambda} \qquad \Leftrightarrow \qquad R\left(\frac{2i}{\lambda} - 1\right) = 1 \qquad \Leftrightarrow \qquad R = \frac{\lambda}{2i - \lambda} = \frac{-i\lambda}{2 + i\lambda}.$$

We also find:

$$T = \frac{2i}{2i - \lambda} = \frac{2}{2 + i\lambda}.$$

As $\lambda \to \infty$, we have $R \to -1$ and $T \to 0$, implying complete reflection. This corresponds to the case where M is very large, i.e. the mass is very heavy.

If $|\lambda| \ll 1$, we have $R \approx 0$ and $T \approx 1$, implying complete transmission. This corresponds to the case where M is very small, i.e. the mass is negligible.

8. Here we solve the heat equation on an interval with *non-zero* boundary data. Let $\phi = \phi(x,t)$ satisfy:

$$\begin{cases} \phi_t - \phi_{xx} = 0, & (x,t) \in (0,1)\times(0,\infty), \\ \phi(0,t) = 0, & t > 0, \\ \phi(1,t) = 1, & t > 0, \\ \phi(x,0) = x^2, & x \in (0,1). \end{cases}$$

By considering a suitable function of the form $\Phi(x,t) = \phi(x,t) - (Ax+B)$ with A, B constant, reduce the problem to one for Φ with homogeneous boundary data. Hence find $\phi(x,t)$ and discuss its behaviour as $t \to \infty$.

Solution: Note that $\Phi_t = \phi_t$ and $\Phi_{xx} = \phi_{xx}$, so the function Φ satisfies the heat equation $\Phi_t - \Phi_{xx} = 0$ as required. Further, we have:

$$\Phi(0,t) = \phi(0,t) - B = -B, \qquad \Phi(1,t) = \phi(1,t) - (A+B) = 1 - A - B, \qquad \Phi(x,0) = \phi(x,0) - (Ax+B) = x^2 - Ax - B,$$

so that if we choose $B = 0$, $A = 1$, the problem reduces to having homogeneous boundary conditions:

$$\Phi(0,t) = 0, \qquad \Phi(1,t) = 0, \qquad \Phi(x,0) = x^2 - x = x(x-1).$$

We can now proceed to solve for $\Phi(x,t)$ using separation of variables. Let $\Phi(x,t) = X(x)T(t)$. Then:

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda,$$

yielding the ODEs $X'' + \lambda X = 0$ and $T' = -\lambda T$. The first has the solution:

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x),$$

which on imposing the boundary conditions $X(0) = 0$ and $X(1) = 0$ gives $A = 0$ and $\sqrt{\lambda} = n\pi$ for $n \in \mathbb{Z}$, i.e. eigenvalues $\lambda_n = n^2\pi^2$. Substituting into the second equation, we have:

$$T(t) = A_n e^{-\lambda_n t}.$$

Overall, this gives the solution:

$$\Phi(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \sin(n\pi x).$$

To obtain the coefficients A_n, we use Fourier decomposition. We have:

$$x(x-1) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = \sum_{n=1}^{\infty} A_n s_n,$$

where $s_n = \sin(n\pi x)$. By Question 1 of Sheet 1, we have:

$$\langle s_n, s_m \rangle = \int_{-1}^{1} \sin(n\pi x)\sin(m\pi x)\,dx = \delta_{mn},$$

which using the fact that the integrand is even implies:

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \frac{1}{2}\delta_{mn}.$$

Thus we have:

$$
\begin{aligned}
A_m &= 2\int_0^1 x(x-1)\sin(m\pi x)\,dx \\
&= 2\left(\left[-\frac{1}{m\pi}x(x-1)\cos(m\pi x)\right]_0^1 + \frac{1}{m\pi}\int_0^1 (2x-1)\cos(m\pi x)\,dx\right) \\
&= 2\left(\frac{1}{m^2\pi^2}\left[(2x-1)\sin(m\pi x)\right]_0^1 - \frac{2}{m^2\pi^2}\int_0^1 \sin(m\pi x)\,dx\right) \\
&= -\frac{4}{m^2\pi^2}\left(-\frac{1}{m\pi}((-1)^m-1)\right) \\
&= \frac{4}{m^3\pi^3}((-1)^m-1).
\end{aligned}
$$

Hence overall we get:

$$
\Phi(x,t) = -\sum_{n=0}^{\infty}\frac{8}{(2n+1)^3\pi^3}e^{-\pi^2(2n+1)^2 t}\sin\left((2n+1)\pi x\right).
$$

Hence the solution to the original equation was:

$$
\phi(x,t) = x - \sum_{n=0}^{\infty}\frac{8}{(2n+1)^3\pi^3}e^{-\pi^2(2n+1)^2 t}\sin\left((2n+1)\pi x\right).
$$

As $t\to\infty$, we have $\phi\to x$. This is the unique steady state solution of the heat equation given the boundary conditions (i.e. $\phi_t = 0, \phi_{xx} = 0$ implies $\phi = Ax + B$. The boundary data with $\phi = 0$ at 0 and $\phi = 1$ at 1 gives $\phi = x$; note the initial condition $\phi = x^2$ at $t = 0$ is completely forgotten about).

9. Let $f = f(\theta)$ be a 2π-periodic function and consider the periodic initial value problem for the heat equation $\phi_t = \phi_{\theta\theta}$ with $\phi(\theta, 0) = f(\theta)$, and $\phi(\theta + 2\pi, t) = \phi(\theta, t)$ for each (θ, t). Using an appropriate Fourier series, solve for ϕ and write it in the form:

$$\phi(\theta, t) = \int_0^{2\pi} \vartheta_t(\theta - \phi) f(\phi)\, d\phi,$$

where $\vartheta_t(\theta)$ is a function you should determine.

Solution: Suppose that:

$$\phi(t, \theta) = \sum_{n=-\infty}^{\infty} a_n(t) e^{in\theta},$$

as a sum of Fourier modes. Then the periodic boundary condition is necessarily already satisfied. Substituting into the heat equation, we obtain:

$$\sum_{n=-\infty}^{\infty} \dot{a}_n(t) e^{in\theta} = \sum_{n=-\infty}^{\infty} (-n^2 a_n) e^{in\theta},$$

which comparing modes (recall they are orthogonal) implies:

$$\dot{a}_n(t) = -n^2 a_n.$$

As a result, we have $a_n(t) \propto e^{-n^2 t}$, which implies the general solution takes the form:

$$\phi(t, \theta) = \sum_{n=-\infty}^{\infty} A_n e^{-n^2 t} e^{in\theta} = \sum_{n=-\infty}^{\infty} A_n e^{-n^2 t} e_n,$$

where A_n are constants and $e_n(\theta) = e^{in\theta}$. We know from lectures (or the solution to Question 1 of Sheet 1) that:

$$\langle e_n, e_m \rangle = \int_0^{2\pi} e^{in\theta} e^{-im\theta}\, d\theta = 2\pi \delta_{mn},$$

and hence for $t = 0$, we get:

$$f(\theta) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta} \qquad \Rightarrow \qquad A_m = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} f(\theta)\, d\theta.$$

It follows that:

$$\phi(t, \theta) = \int_0^{2\pi} \left(\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\theta - \phi)} e^{-n^2 t} \right) f(\phi)\, d\phi,$$

which reveals that:

$$\vartheta_t(\theta - \phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\theta - \phi)} e^{-n^2 t}.$$

10. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain and $(\mathbf{x}, t) \in \Omega \times (0, \infty)$. We will be concerned with the following initial-boundary value problems for the heat and wave equations, respectively:

$$\text{(A)} \begin{cases} \phi_t - \kappa \Delta \phi = 0, & \text{in } \Omega \times (0,\infty), \\ \phi = f, & \text{on } \Omega \times \{t = 0\}, \\ \phi = 0, & \text{on } \partial\Omega \times [0,\infty), \end{cases} \qquad \text{(B)} \begin{cases} \phi_{tt} - c^2 \Delta \phi = 0, & \text{in } \Omega \times (0,\infty), \\ \phi = g, & \text{on } \Omega \times \{t = 0\}, \\ \phi_t = h, & \text{on } \Omega \times \{t = 0\}, \\ \phi = 0, & \text{on } \partial\Omega \times [0,\infty). \end{cases}$$

You may *assume* the following: there is a collection $\{(\psi_n, \lambda_n)\}_{n=1}^{\infty}$ of real eigenfunction-eigenvalue pairs such that (a) $-\Delta \psi_n = \lambda_n \psi_n$ in Ω; (b) $\psi_n = 0$ on $\partial\Omega$; (c) each eigenvalue has finite multiplicity; (d) $\{\psi_n\}$ are complete on Ω. The latter means for $f : \Omega \to \mathbb{R}$ satisfying $f = 0$ on $\partial\Omega$, we can write $f = \sum_n \alpha_n \psi_n$ for some $\{\alpha_n\}$.

(i) Show that $\lambda_n > 0$ for each n and:

$$\int_\Omega \psi_n \psi_m \, dV = 0, \qquad \text{for } \lambda_n \neq \lambda_m.$$

(ii) Explain why we can assume, without loss of generality, that:

$$\int_\Omega \psi_n \psi_m \, dV = 0, \qquad \text{for } n \neq m.$$

(iii) Using separation of variables, show that the solution to (A) is given by:

$$\phi(\mathbf{x}, t) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \kappa t} \psi_n(\mathbf{x}), \qquad \text{where} \qquad \alpha_n = \frac{\displaystyle\int_\Omega f \psi_n \, dV}{\displaystyle\int_\Omega \psi_n^2 \, dV}.$$

Explain why this might be formally interpreted as $\phi(\mathbf{x}, t) = e^{\kappa t \Delta} \phi(\mathbf{x}, 0)$ where:

$$e^{\kappa t \Delta} = \sum_{p=0}^{\infty} \frac{(\kappa t)^p}{p!} \Delta^p.$$

(iv) Solve (B), again using separation of variables. Relate your answer to the formal expression:

$$\phi(\mathbf{x}, t) = \frac{\sin\left(ct\sqrt{-\Delta}\right)}{c\sqrt{-\Delta}} \phi_t(\mathbf{x}, 0) + \cos\left(ct\sqrt{-\Delta}\right) \phi(\mathbf{x}, 0).$$

Solution: (i) Define a bilinear map $\langle \cdot, \cdot \rangle$ via:

$$\langle \psi, \phi \rangle = \int_\Omega \psi(\mathbf{x}) \phi(\mathbf{x}) \, dV.$$

Observe that:

$$\langle -\Delta \psi, \psi \rangle = -\int_\Omega (\nabla^2 \psi) \psi \, dV.$$

Noting that $\nabla \cdot (\psi \nabla \psi) = |\nabla \psi|^2 + \psi \nabla^2 \psi$, we can use the divergence theorem to recast this integral as:

$$\langle -\Delta\psi, \psi\rangle = \int\limits_{\Omega} |\nabla \psi|^2 \, dV - \int\limits_{\partial\Omega} \psi \nabla \psi \cdot d\mathbf{S} = \int\limits_{\Omega} |\nabla \psi|^2 \, dV \geq 0,$$

if ψ solves either (A) or (B), because it vanishes on the boundary. Equality occurs if and only if $\nabla \psi = 0$, which occurs if and only if $\psi = 0$ if ψ is assumed to solve (A) or (B). Thus if ψ_n satisfies $-\Delta \psi_n = \lambda_n \psi_n$ and satisfies either (A) or (B), we have:

$$\lambda_n = \frac{\langle \lambda_n \psi_n, \psi_n\rangle}{\langle \psi_n, \psi_n\rangle} = \frac{\langle -\Delta\psi_n, \psi_n\rangle}{\langle \psi_n, \psi_n\rangle} > 0,$$

as required.

To show the orthogonality, note that:

$$\langle -\Delta\psi, \phi\rangle = -\int\limits_{\Omega} \phi \nabla^2 \psi \, dV.$$

Since $\nabla \cdot (\phi \nabla \psi) = \nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi$, we can rearrange this using the divergence theorem and the vanishing on the boundary to give:

$$\langle -\Delta\psi, \phi\rangle = \int\limits_{\Omega} \nabla \phi \cdot \nabla \psi \, dV.$$

But then applying $\nabla \cdot (\psi \nabla \phi) = \nabla \psi \cdot \nabla \phi + \psi \nabla^2 \phi$, we have:

$$\langle -\Delta\psi, \phi\rangle = -\int\limits_{\Omega} \psi \nabla^2 \phi \, dV = \langle \psi, -\Delta\phi\rangle,$$

so $-\Delta$ is a symmetric operator on this space. The proof of orthogonality is now trivial:

$$\lambda_n \langle \psi_n, \psi_m\rangle = \langle -\Delta\psi_n, \psi_m\rangle = \langle \psi_n, -\Delta\psi_m\rangle = \lambda_m \langle \psi_n, \psi_m\rangle,$$

so $\lambda_n \neq \lambda_m$ implies $\langle \psi_n, \psi_m\rangle = 0$ as required.

(ii) We have just shown that distinct eigenspaces are orthogonal. If an eigenspace has multiplicity greater than one, we are still given it has finite multiplicity by (c); hence it can be made orthogonal by the Gram-Schmidt procedure. Thus without loss of orthogonality, we may assume that all eigenfunctions are orthogonal.

(iii) Let $\phi = X(\mathbf{x})T(t)$. Then inserting into the given equation, we have:

$$\frac{T'(t)}{\kappa T(t)} - \frac{\Delta X(\mathbf{x})}{X(\mathbf{x})} = 0,$$

so separating variables we have:

$$T' = -\mu\kappa T, \qquad -\Delta X(\mathbf{x}) = \mu X(\mathbf{x}),$$

for some constant μ. The solution to the second equation is the solution to the eigenvalue problem, which is $\mu = \lambda_1, \lambda_2, \ldots$, $X(\mathbf{x}) = \psi_1(\mathbf{x}), \ldots$ etc. The solution to the second equation for $\mu = \lambda_n$ is given by $T(t) = \alpha_n e^{-\lambda_n \kappa t}$. Thus putting everything together, the general solution is:

$$\phi(\mathbf{x}, t) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \kappa t} \psi_n(\mathbf{x}).$$

To determine the coefficients α_n, we impose the boundary data at $t = 0$. We have:

$$f(\mathbf{x}) = \sum_{n=1}^{\infty} \alpha_n \psi_n(\mathbf{x}) \qquad \Rightarrow \qquad \int\limits_{\Omega} \psi_m(\mathbf{x}) f(\mathbf{x})\, dV = \alpha_m \int\limits_{\Omega} \psi_m(\mathbf{x})^2\, dV,$$

using orthogonality of the eigenfunctions in the implication. Thus we deduce that:

$$\alpha_n = \frac{\int\limits_{\Omega} f\psi_n\, dV}{\int\limits_{\Omega} \psi_n^2\, dV},$$

as required.

To understand the formal interpretation, we note that:

$$\begin{aligned} e^{\kappa t\Delta}\phi(\mathbf{x}, 0) &= e^{\kappa t \Delta}\left(\sum_{n=1}^{\infty} \alpha_n \psi_n(\mathbf{x})\right) \\ &= \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \kappa t}\psi_n(\mathbf{x}) \\ &= \phi(\mathbf{x}, t), \end{aligned}$$

using the fact that $e^{\kappa t\Delta}\psi_n(\mathbf{x}) = e^{-\kappa\lambda_n t}\psi_n(\mathbf{x})$ in the second step.

(iv) This time, we let $\phi = T(t)X(\mathbf{x})$ resulting in the equation:

$$\frac{T''}{c^2 T} - \frac{\Delta X(\mathbf{x})}{X(\mathbf{x})} = 0.$$

Separating variables, we have for some constant μ:

$$T'' + \mu c^2 T = 0, \qquad -\Delta X = \mu X.$$

Just as before, we require $\mu = \lambda_1, \lambda_2, ...$ and $X = \psi_1, \psi_2, ...$ to solve the second equation. Turning to the first equation, we then have $T'' + \lambda_n c^2 T = 0$ implies:

$$T = \alpha_n \sin(c\sqrt{\lambda_n} t) + \beta_n \cos(c\sqrt{\lambda_n t}).$$

Hence the general solution is:

$$\phi(\mathbf{x}, t) = \sum_{n=1}^{\infty} \left(\alpha_n \sin(c\sqrt{\lambda_n} t) + \beta_n \cos(c\sqrt{\lambda_n t})\right)\psi_n(\mathbf{x}).$$

At $t = 0$, we require $\phi(\mathbf{x}, 0) = g$, which gives:

$$g(\mathbf{x}) = \sum_{n=1}^{\infty} \beta_n \psi_n(\mathbf{x}) \qquad \Rightarrow \qquad \beta_n = \frac{\int\limits_{\Omega} g\psi_n\, dV}{\int\limits_{\Omega} \psi_n^2\, dV}.$$

At $t = 0$, we additionally require $\phi_t(\mathbf{x}, 0) = h$, which gives:

$$h(\mathbf{x}) = \sum_{n=1}^{\infty} c\sqrt{\lambda_n}\alpha_n\psi_n(\mathbf{x}) \qquad \Rightarrow \qquad \alpha_n = \frac{\displaystyle\int_\Omega h\psi_n\, dV}{c\sqrt{\lambda_n}\displaystyle\int_\Omega \psi_n^2\, dV}.$$

In the same way, this is evidently related to the formal solution:

$$\phi(t, \mathbf{x}) = \frac{\sin\left(c\sqrt{-\Delta}t\right)}{c\sqrt{-\Delta}}\phi_t(\mathbf{x}, 0) + \cos\left(c\sqrt{-\Delta}t\right)\phi(\mathbf{x}, 0).$$

In both cases, the formal solution is immensely interesting, because e.g. imagine solving the heat equation:

$$\phi_t - \kappa\Delta\phi = 0$$

treating $\kappa\Delta$ as a 'constant'. Then the solution is:

$$\phi(\mathbf{x}, t) = e^{\kappa t\Delta}\phi(\mathbf{x}, 0),$$

which is precisely the solution we found above!

Part IB: Mathematical Methods
Examples Sheet 3 Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

1. Let $\{\delta_n\}_{n=1}^{\infty}$ denote the sequence of functions that approximates the Dirac Delta, as seen in lectures. Recall the properties: $x \mapsto \delta_n(x)$ is smooth and non-negative; $\delta_n(x) = 0$ if $|nx| > 1$; and $\forall \epsilon > 0 \, \exists N > 0$ such that $\forall n > N$,

$$\int_{-\epsilon}^{\epsilon} \delta_n(x) \, dx = 1.$$

Using this definition, show that for any smooth function $f = f(x)$:

$$\text{(i)} \int \delta(x-a) f(x) \, dx = \lim_{n\to\infty} \int \delta_n(x-a) f(x) \, dx = f(a).$$

For the rest of the sheet treat $\delta(x)$ as an ordinary function, as we do in lectures. Establish the results:

$$\text{(ii)} \int \delta'(x) f(x) \, dx = -f'(0), \qquad \text{(iii)} \lim_{h\to 0} \int \left[\frac{\delta(x+h) - \delta(x)}{h} \right] f(x) \, dx = -f'(0).$$

Deduce that:

$$\lim_{h\to 0} \left[\frac{\delta(x+h) - \delta(x)}{h} \right] = \delta'(x).$$

Solution: (i) Before starting, it is useful to notice that the third property can be replaced by the condition:

$$\int_{-\infty}^{\infty} \delta_n(x) \, dx = 1,$$

for all n. This follows from the fact that given any $\epsilon > 0$, from the second property we have that if $n > 1/\epsilon$, we have $\delta_n(x) = 0$. Thus we see a possible choice of N is $1/\epsilon$.

We now begin the proof. The first equality is the definition of the δ function. Therefore, we need only show that:

$$\lim_{n\to\infty} \int \delta_n(x-a) f(x) \, dx = f(a).$$

We observe that:

$$\begin{aligned}
\left| \int_{-\infty}^{\infty} \delta_n(x-a) \, dx - f(a) \right| &= \left| \int_{-\infty}^{\infty} \delta_n(x-a) f(x) \, dx - \int_{-\infty}^{\infty} \delta_n(x-a) f(a) \, dx \right| \\
&= \left| \int_{-\infty}^{\infty} \delta_n(x-a)(f(x) - f(a)) \, dx \right| \\
&= \left| \int_{a-1/n}^{a+1/n} \delta_n(x-a)(f(x) - f(a)) \, dx \right|,
\end{aligned}$$

where in the last step we used the fact that δ_n vanishes outside the interval $(-1/n, 1/n)$ by the second property stated in the question. Bounding this, we note that by the continuity of f, if we are given $\epsilon > 0$, there exists a δ such that $|x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. Choosing N such that $\frac{1}{N} < \delta$, we have for all $n \geq N$ that $|x-a| < 1/n$ implies $|f(x)-f(a)| < \epsilon$. Thus:

$$\left| \int\limits_{a-1/n}^{a+1/n} \delta_n(x-a)(f(x)-f(a))\,dx \right| \leq \int\limits_{a-1/n}^{a+1/n} \delta_n(x-a)|f(x)-f(a)|\,dx < \epsilon \int\limits_{a-1/n}^{a+1/n} \delta_n(x-a) = \epsilon.$$

This is the definition of convergence. $\square$

For the rest of the examples sheet, we treat δ as a normal function that can be manipulated straightforwardly. We can now establish the remaining results.

(ii) We use integration by parts. We have:

$$\int \delta'(x)f(x)\,dx = [\delta(x)f(x)]_{-\infty}^{\infty} - \int \delta(x)f'(x)\,dx = -f'(0),$$

since $\delta(x)$ vanishes at $\pm\infty$, and integrating a function against $\delta(x)$ gives its value at zero.

(iii) We have the following sequence of equalities:

$$\begin{aligned} \int \frac{\delta(x+h)-\delta(x)}{h} f(x)\,dx &= \frac{1}{h}\int \delta(x+h)f(x)\,dx - \frac{1}{h}\int \delta(x)f(x)\,dx \\ &= \frac{1}{h}f(-h) - \frac{1}{h}f(0) \\ &= -\frac{f(-h)-f(0)}{-h}. \end{aligned}$$

In the limit as $h \to 0$, this tends to $-f'(0)$.

Putting (ii) and (iii) together, we see that for all functions f, we have:

$$\int \delta'(x)f(x)\,dx = \lim_{h\to 0} \int \frac{\delta(x+h)-\delta(x)}{h} f(x)\,dx,$$

and hence:

$$\delta'(x) = \lim_{h\to 0}\left[\frac{\delta(x+h)-\delta(x)}{h}\right],$$

just like an ordinary function.

2. Let $\phi : [a, b] \to \mathbb{R}$ be monotone increasing with a simple zero $\phi(c) = 0, \phi'(c) \neq 0$ for some $c \in (a, b)$. Show that:

$$\int_a^b f(x)\delta[\phi(x)]\,dx = \frac{f(c)}{|\phi'(c)|}.$$

Show that the same formula holds if ϕ is monotone decreasing, and hence derive a formula for general ϕ, provided that the zeroes are simple. Deduce that $\delta(at) = \delta(t)/|a|$ for $a \neq 0$, and also establish the identity:

$$\delta(x^2 - y^2) = \frac{\delta(x - y) + \delta(x + y)}{2|y|}.$$

Solution: Note that in this question, monotone increasing should be read as *strictly* increasing. To show the first result, we substitute $u = \phi(x)$ in the integral; this substitution makes sense because ϕ is monotone increasing and hence invertible on the range of interest. We have $du = \phi'(x)dx = \phi'(\phi^{-1}(u))dx$, and the limits transform to $\phi(a), \phi(b)$, hence the entire integral transforms to:

$$\int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(u))\delta(u) \cdot \frac{du}{\phi'(\phi^{-1}(u))} = \frac{f(\phi^{-1}(0))}{\phi'(\phi^{-1}(0))} = \frac{f(c)}{\phi'(c)},$$

as required (note that $\phi(a) < 0 < \phi(b)$, since $a < c < b$), since $\phi'(c) > 0$ because the function is monotone increasing.

If the function is monotone *decreasing*, we have that $-\phi$ is monotone *increasing*, so the previous result tells us that:

$$\int_a^b f(x)\delta(\phi(x))\,dx = \int_a^b f(x)\delta(-\phi(x))\,dx = \frac{f(c)}{(-\phi)'(c)} = \frac{f(c)}{|\phi'(c)|},$$

since the δ function is even. It follows that the formula holds for ϕ monotone increasing and monotone decreasing.

3. By constructing an appropriate Green's function, find the general solution to the boundary value problem:

$$y'' - 2y' + y = f(x), \quad 0 < x < 1, \qquad\qquad y(0) = y(1) = 0.$$

➡ **Solution:** Consider the related equation:

$$y'' - 2y' + y = \delta(x - \xi), \qquad y(0) = y(1) = 0.$$

We require that y is continuous at $x = \xi$, but the derivative of y experiences a discontinuity:

$$\lim_{\epsilon \to 0^+} [y'(\epsilon) - y'(-\epsilon)] = 1.$$

This is enough to solve the equation. We know that when $x \neq \xi$, we have that y satisfies:

$$y'' - 2y' + y = 0,$$

which has characteristic equation $0 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$. Hence the solution takes the general form:

$$y(x) = \begin{cases} Ae^x + Bxe^x, & x < \xi, \\ Ce^x + Dxe^x, & x > \xi. \end{cases}$$

Imposing the condition $y(0) = 0$, we have $A = 0$. Imposing the condition $y(1) = 0$, we have $Ce + De = 0$, and hence $C = -D$. Thus the form of the solution reduces to (with a redefinition of the constants):

$$y(x) = \begin{cases} Bxe^{x-\xi}, & x < \xi, \\ D(x-1)e^{x-\xi}, & x > \xi. \end{cases}$$

Continuity at $x = \xi$ implies that $B\xi = D(\xi - 1)$. The jump condition at $x = \xi$ implies that:

$$1 = D + D(\xi - 1) - B - B\xi = D\xi - B(\xi + 1).$$

Solving these equations simultaneously, we have $B = D(\xi - 1)/\xi$, and hence:

$$\xi = D\xi^2 - D(\xi^2 - 1) = D,$$

and $B = \xi - 1$. Thus overall we have:

$$y(x;\xi) = \begin{cases} (\xi - 1)xe^{x-\xi}, & x \leq \xi, \\ \xi(x-1)e^{x-\xi}, & x > \xi. \end{cases}$$

Thus the general solution to the given equation is:

$$y(x) = (x-1)e^x \int_0^x f(\xi)\xi e^{-\xi}\, d\xi + xe^x \int_x^1 f(\xi)(\xi - 1)e^{-\xi}\, d\xi.$$

4. Let $\lambda \in \mathbb{R}\backslash\{0\}$ be given. Obtain the Dirichlet Green's function for the operator:

$$L = -\frac{d^2}{dx^2} + \lambda^2, \qquad 0 < x < 1,$$

i.e. find $G = G(x, \xi)$ such that for each $0 < \xi < 1$ we have $LG = \delta(x - \xi)$ and $G(0, \xi) = G(1, \xi) = 0$. Hence show that the solution to the equation $Ly = f$ with $y(0) = y(1) = 0$ is given by:

$$y(x) = \frac{1}{\lambda \sinh(\lambda)} \left[\sinh(\lambda x) \int_x^1 f(\xi) \sinh[\lambda(1-\xi)]\, d\xi + \sinh[\lambda(1-x)] \int_0^x f(\xi) \sinh(\lambda\xi)\, d\xi \right].$$

Does this solution hold in the limit $\lambda \to 0$? Why does this formula break down when $\lambda = m\pi i, m \in \mathbb{N}$? *Hint: when are there non-trivial solutions to $Ly = 0$ satisfying $y(0) = y(1) = 0$?*

Solution: We wish to solve:

$$-G'' + \lambda^2 G = \delta(x - \xi). \tag{$*$}$$

The characteristic equation away from $x = \xi$ is just $\mu - \lambda^2 = 0$, so $\mu = \pm\lambda$. Then, the solution is standard:

$$G(x, \xi) = \begin{cases} A\cosh(\lambda x) + B\sinh(\lambda x), & x < \xi \\ C\cosh(\lambda(x-1)) + D\sinh(\lambda(x-1)), & x > \xi. \end{cases}$$

Imposing the boundary conditions, we have $G(0, \xi) = 0$, which implies $A = 0$. Similarly, $G(1, \xi) = 0$ implies $C = 0$. Continuity at $x = \xi$ requires:

$$B\sinh(\lambda\xi) = D\sinh(\lambda(\xi - 1)).$$

Finally, from integrating the initial equation ($*$), the jump condition is:

$$\lim_{\epsilon \to 0} [G'(\xi + \epsilon, \xi) - G'(\xi - \epsilon, \xi)] = -1.$$

Inserting the expression for G', we have:

$$\lambda D\cosh(\lambda(\xi - 1)) - \lambda B\cosh(\lambda\xi) = -1.$$

Substituting $B = D\sinh(\lambda(\xi - 1))/\sinh(\lambda\xi)$ from above, we have:

$$\lambda D\cosh(\lambda(\xi - 1)) - \lambda D\sinh(\lambda(\xi - 1))\coth(\lambda\xi) = -1,$$

which on rearrangement produces:

$$D = \frac{-\sinh(\lambda\xi)}{\lambda\cosh(\lambda(\xi-1))\sinh(\lambda\xi) - \lambda\sinh(\lambda(\xi-1))\cosh(\lambda\xi)} = \frac{\sinh(\lambda\xi)}{\lambda\sinh(\lambda(\xi-1) - \lambda\xi)} = -\frac{\sinh(\lambda\xi)}{\lambda\sinh(\lambda)}.$$

Using the formula for B, this gives:

$$B = \frac{D\sinh(\lambda(\xi - 1)}{\sinh(\lambda\xi)} = \frac{\sinh(\lambda(1-\xi))}{\lambda\sinh(\lambda)}.$$

Overall, the Green's function is therefore:

$$G(x, \xi) = \begin{cases} \dfrac{\sinh(\lambda(1-\xi))}{\lambda\sinh(\lambda)}\sinh(\lambda x), & x \leq \xi, \\ \dfrac{\sinh(\lambda\xi)}{\lambda\sinh(\lambda)}\sinh(\lambda(1-x)), & x \geq \xi. \end{cases}$$

Hence the general solution to the given equation $Ly = f$ is:

$$y(x) = \frac{\sinh(\lambda(1-x))}{\lambda\sinh(\lambda)} \int_0^x f(\xi)\sinh(\lambda\xi)\, d\xi + \frac{\sinh(\lambda x)}{\lambda\sinh(\lambda)} \int_x^1 f(\xi)\sinh(\lambda(1-\xi))\, d\xi,$$

as required.

In the limit where $\lambda \to 0$, we have:

$$\frac{\sinh(\lambda k)}{\lambda} = \frac{\lambda k + \frac{\lambda^3 k^3}{3!} + ...}{\lambda} = k + O(\lambda^2) \qquad \Rightarrow \qquad \lim_{\lambda \to 0} \frac{\sinh(\lambda k)}{\lambda} = k.$$

Hence the given solution becomes:

$$y(x) = x \int_x^1 f(\xi)(1-\xi)\, d\xi + (1-x) \int_0^x f(\xi)\xi \, d\xi.$$

Finish off

5. The function $\theta = \theta(t)$ measures the displacement of a damped oscillator. It satisfies the initial value problem:

$$\ddot{\theta} + 2p\dot{\theta} + (p^2+q^2)\theta = f(t), \quad t > 0, \qquad \theta(0) = \dot{\theta}(0) = 0,$$

where p, q are real constants with $p > 0$ and $q \neq 0$. By constructing an appropriate Green's function, show that:

$$\theta(t) = \frac{1}{q}\int_0^t e^{-p(t-\tau)}\sin[q(t-\tau)]f(\tau)\,d\tau.$$

Obtain the same result using the Fourier transform.

Solution: Let G be the Green's function, satisfying:

$$\ddot{G} + 2p\dot{G} + (p^2+q^2)G = \delta(t-\tau).$$

When $t \neq \tau$, the characteristic equation for this ODE is $\lambda^2 + 2p\lambda + (p^2+q^2) = 0$. Thus:

$$\lambda_\pm = -p \pm iq.$$

Hence:

$$G(t,\tau) = \begin{cases} Ae^{-pt}\cos(qt) + Be^{-pt}\sin(qt), & t < \tau, \\ Ce^{-p(t-\tau)}\cos(q(t-\tau)) + De^{-p(t-\tau)}\sin(q(t-\tau)), & t > \tau. \end{cases}$$

Imposing the initial conditions, we have $G(0,\tau) = 0$ implies $A = 0$. Similarly, $\dot{G}(0,\tau) = 0$ implies that:

$$\left[-pBe^{-pt}\sin(qt) + qBe^{-pt}\cos(qt)\right]_{t=0} = qB = 0,$$

and since $q \neq 0$, we have $B = 0$. Imposing continuity at $t = \tau$, we deduce that $C = 0$. Finally, the jump condition for this equation is:

$$\lim_{\epsilon\to 0}\left[G'(\tau+\epsilon,\tau) - G'(\tau-\epsilon,\tau)\right] = 1.$$

Applying this condition, we have:

$$1 = \left[-pDe^{-p(t-\tau)}\sin(q(t-\tau)) + qDe^{-p(t-\tau)}\cos(q(t-\tau))\right]_{t=\tau} = qD,$$

which implies $D = 1/q$. Thus the complete Green's function for the equation is:

$$G(t,\tau) = \begin{cases} 0, & t < \tau, \\ \dfrac{1}{q}e^{-p(t-\tau)}\sin(q(t-\tau)), & t > \tau. \end{cases}$$

It follows that the the solution to the equation is:

$$\theta(t) = \frac{1}{q}\int_0^t e^{-p(t-\tau)}\sin(q(t-\tau))f(\tau)\,d\tau,$$

as required.

Taking the Fourier transform, the equation becomes:

$$(i\omega)^2\hat{\theta} + 2p(i\omega)\hat{\theta} + (p^2+q^2)\hat{\theta} = \hat{f},$$

which on rearrangement gives:

$$\hat{\theta} = \frac{\hat{f}}{-\omega^2 + 2pi\omega + p^2 + q^2}.$$

Using partial fractions (we can essentially guess what they are going to be from the work we did earlier in this question), we can expand the right hand side as:

$$\hat{\theta} = \frac{1}{2q}\left(\frac{\hat{f}}{\omega - ip + q} - \frac{\hat{f}}{\omega - ip - q}\right).$$

Now recall the convolution theorem, $\mathcal{F}[(f*g)(x)] = \hat{f}(\omega)\hat{g}(\omega)$. This implies that:

$$\theta(t) = \frac{1}{2q}\int\limits_0^\infty f(\tau)\mathcal{F}^{-1}\left[\frac{1}{\omega - ip + q} - \frac{1}{\omega - ip - q}\right](t-\tau)\,d\tau.$$

Note the lower limit because $f(\tau) = 0$ when $\tau < 0$.

It remains to compute the inverse Fourier transform. We recall from lectures that:

$$\mathcal{F}^{-1}\left[\frac{1}{i\omega + \sigma}\right](x) = H(x)e^{-\sigma x},$$

where $H(x)$ is the Heaviside step function. Hence in our case, we have:

$$\begin{aligned}
\mathcal{F}^{-1}\left[\frac{1}{\omega - ip + q} - \frac{1}{\omega - ip - q}\right] &= i\mathcal{F}^{-1}\left[\frac{1}{i\omega + p + iq} - \frac{1}{i\omega + p - iq}\right] \\
&= i\left(H(t)e^{-pt}e^{-iqt} - H(t)e^{-pt}e^{iqt}\right) \\
&= 2H(t)e^{-pt}\sin(qt).
\end{aligned}$$

Putting everything together, we obtain the required solution:

$$\theta(t) = \frac{1}{q}\int\limits_0^t f(\tau)e^{-p(t-\tau)}\sin(q(t-\tau))\,d\tau,$$

where the upper limit comes from imposing the Heaviside step function.

You might be worried that at no point did we impose any of the initial data in this solution by Fourier transforms. Indeed, in general, the Fourier transform can *only* give us the particular integral in the presence of zero initial conditions; the remaining pieces of the complementary function that impose non-zero initial conditions are not Fourier-transformable, and hence cannot be cast in this form.

6. Calculate the Fourier transforms of the functions that are zero on $|x| > c$, and otherwise defined on $|x| \leq c$ by:

(i) $f(x) = 1,$ (ii) $f(x) = e^{iax},$ (iii) $f(x) = \sin(ax),$ (iv) $f(x) = \cos(ax).$

➾ **Solution:** (i) The Fourier transform is:

$$\int_{-c}^{c} e^{-i\omega x}\,dx = \left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-c}^{c} = \frac{e^{i\omega c} - e^{-i\omega c}}{i\omega} = \frac{2\sin(\omega c)}{\omega}.$$

(ii) Using the shift property, $\mathcal{F}[e^{-iax}f(x)] = \hat{f}(\omega + a)$, the Fourier transform can be immediately computed from (i), giving:

$$\frac{2\sin(c(\omega - a))}{\omega - a}.$$

(iii) Note $f(x) = \sin(ax) = \frac{1}{2i}(e^{iax} - e^{-iax})$. So by linearity, the Fourier transform must be:

$$\frac{1}{2i}\left(\frac{2\sin(c(\omega - a))}{\omega - a} - \frac{2\sin(c(\omega + a))}{\omega + a}\right) = \frac{(\omega + a)\sin(c(\omega - a)) - (\omega - a)\sin(c(\omega + a))}{i(\omega^2 - a^2)}$$

Expanding, this may also be expressed as:

$$\frac{(\omega + a)\left(\sin(c\omega)\cos(ca) - \sin(ca)\cos(c\omega)\right) - (\omega - a)\left(\sin(c\omega)\cos(ca) + \sin(ca)\cos(c\omega)\right)}{i(\omega^2 - a^2)}$$

$$= \frac{2a\sin(c\omega)\cos(ca) - 2\omega\sin(ca)\cos(c\omega)}{i(\omega^2 - a^2)}.$$

(iv) Again by linearity, the Fourier transform must be:

$$\frac{1}{2}\left(\frac{2\sin(c(\omega - a))}{\omega - a} + \frac{2\sin(c(\omega + a))}{\omega + a}\right) = \frac{(\omega + a)\sin(c(\omega - a)) + (\omega - a)\sin(c(\omega + a))}{\omega^2 - a^2}$$

Expanding, this may also be expressed as:

$$\frac{(\omega + a)\left(\sin(c\omega)\cos(ca) - \sin(ca)\cos(c\omega)\right) + (\omega - a)\left(\sin(c\omega)\cos(ca) + \sin(ca)\cos(c\omega)\right)}{\omega^2 - a^2}$$

$$= \frac{2\omega\sin(c\omega)\cos(ca) - 2a\sin(ca)\cos(c\omega)}{\omega^2 - a^2}.$$

7. Calculate the Fourier transforms of the following functions in terms of the Dirac delta function:

$$\text{(i) } f(x) = 1, \qquad \text{(ii) } f(x) = e^{iax}, \qquad \text{(iii) } f(x) = \sin(ax), \qquad \text{(iv) } f(x) = \cos(ax).$$

Compare your answers to the previous question and comment on the results.

➻ **Solution:** (i) Note that:

$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\delta(\omega)\,d\omega = \frac{1}{2\pi},$$

hence taking the Fourier transform of both sides, we obtain:

$$\mathcal{F}[1] = 2\pi\delta(\omega).$$

(ii) By the shift property, and part (i), the Fourier transform is $2\pi\delta(\omega - a)$.

(iii) By linearity and part (ii), the Fourier transform of $f(x) = \sin(ax) = \frac{1}{2i}\left(e^{iax} - e^{-iax}\right)$ is given by:

$$\frac{1}{2i}\left(2\pi\delta(\omega - a) - 2\pi\delta(\omega + a)\right) = \pi i\left(\delta(\omega + a) - \delta(\omega - a)\right).$$

(iv) Similarly, by linearity and part (ii), the Fourier transform of $f(x) = \cos(ax) = \frac{1}{2}\left(e^{iax} + e^{-iax}\right)$ is given by:

$$\frac{1}{2}\left(2\pi\delta(\omega - a) + 2\pi\delta(\omega + a)\right) = \pi\left(\delta(\omega + a) + \delta(\omega - a)\right).$$

The point of comparing this question and the last is that our results agree in a distributional sense. For example, in (i) we have, for any sufficiently nice function f:

$$\lim_{c\to\infty}\int_{-\infty}^{\infty}\frac{2\sin(\omega c)}{\omega}f(\omega)\,d\omega = \lim_{c\to\infty}\int_{-\infty}^{\infty}\frac{2\sin(u)}{u}f\left(\frac{u}{c}\right)\,du = f(0)\int_{-\infty}^{\infty}\frac{2\sin(u)}{u}\,du.$$

The remaining integral is a classic (see Part IB Complex Methods or Analysis for a quick way of evaluating it). We saw in Part IA Differential Equations, Sheet 1, Question 13, that the result of the integral (called the *Dirichlet integral*) is:

$$\int_{-\infty}^{\infty}\frac{\sin(u)}{u}\,du = \pi.$$

Overall then, we have shown:

$$\lim_{c\to\infty}\int_{-\infty}^{\infty}\frac{2\sin(\omega c)}{\omega}f(\omega)\,d\omega = 2\pi f(0) = 2\pi\int_{-\infty}^{\infty}\delta(\omega)f(\omega)\,d\omega,$$

for any function f. Thus in a distributional sense, we have:

$$\lim_{c\to\infty}\frac{2\sin(\omega c)}{\omega} = 2\pi\delta(\omega).$$

The other results follow similarly.

8. Compute the discrete Fourier transform of the sequence $X_n = n$ for $n = 0, 1, ..., N-1$. Hence show that:

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{k=1}^{N-1} \frac{1}{\sin^2(\pi k/N)} = \frac{1}{3}.$$

➺ **Solution:** When $k = 0$, the discrete Fourier transform is:

$$\hat{X}_0 = \sum_{n=0}^{N-1} n = \frac{(N-1)N}{2}.$$

For $k \neq 0$, the discrete Fourier transform is:

$$\hat{X}_k = \sum_{n=0}^{N-1} n e^{-2\pi i n k/N} = -\frac{N}{2\pi i}\frac{d}{dk}\sum_{n=0}^{N-1} e^{-2\pi i n k/N} = -\frac{N}{2\pi i}\frac{d}{dk}\left(\frac{1-e^{-2\pi i k}}{1-e^{-2\pi i k/N}}\right)$$
$$= -\frac{N}{2\pi i}\frac{2\pi i e^{-2\pi i k}(1-e^{-2\pi i k/N}) - 2\pi i(1-e^{-2\pi i k})e^{-2\pi i k/N}/N}{(1-e^{-2\pi i k/N})^2}.$$

Now, remembering that k is an integer, we have $e^{2\pi i k} = 1$. Thus the formula reduces to:

$$\hat{X}_k = -\frac{N}{2\pi i}\frac{2\pi i}{1-e^{-2\pi i k/N}} = -\frac{N}{1-e^{-2\pi i k/N}} = -\frac{N e^{-\pi i k/N}}{e^{\pi i k/N} - e^{-\pi i k/N}} = \frac{N i e^{-\pi i k/N}}{2\sin(\pi k/N)}.$$

Now recall *Parseval's theorem* from lectures:

$$\frac{1}{N}\sum_{k=0}^{N-1} |\hat{X}_k|^2 = \sum_{n=0}^{N-1} |X_n|^2.$$

The left hand side is:

$$\frac{1}{N}\sum_{k=0}^{N-1} |\hat{X}_k|^2 = \frac{N(N-1)^2}{4} + \frac{N}{4}\sum_{k=0}^{N-1}\frac{1}{\sin^2(\pi k/N)}.$$

The right hand side is:

$$\sum_{n=0}^{N-1} |X_n|^2 = \sum_{n=0}^{N-1} n^2 = \frac{(N-1)N(2N-1)}{6}.$$

In particular, we have:

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{k=1}^{N-1}\frac{1}{\sin^2(\pi k/N)} = \lim_{N\to\infty}\frac{4}{N^3}\left(\frac{(N-1)N(2N-1)}{6} - \frac{N(N-1)^2}{4}\right) = \frac{4}{3} - 1 = \frac{1}{3},$$

as required.

9. By considering the Fourier transform of the function $f(x) = \cos(x)$ when $|x| < \pi/2$ and $f(x) = 0$ when $|x| \geq \pi/2$, and the Fourier transform of its derivative, show that:

$$\int_0^\infty \frac{\cos^2(\pi t/2)}{(1-t^2)^2}\,dt = \int_0^\infty \frac{t^2\cos^2(\pi t/2)}{(1-t^2)^2}\,dt = \frac{\pi^2}{8}.$$

➾ **Solution:** We saw the Fourier transform of $f(x)$ is given by:

$$\hat{f}(\omega) = \frac{2\cos(\omega\pi/2)}{1-\omega^2}.$$

in Question 6(iv). Parseval's theorem then gives:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\cos^2(\omega\pi/2)}{(1-\omega^2)^2}\,d\omega = \int_{-\pi/2}^{\pi/2} \cos^2(x)\,dx.$$

Rearranging, and noting the integral on the left hand side is an integral of an even function on a symmetric domain, we immediately obtain the result:

$$\int_0^\infty \frac{\cos^2(\omega\pi/2)}{(1-\omega^2)^2}\,d\omega = \frac{\pi}{4}\int_{-\pi/2}^{\pi/2} \cos^2(x)\,dx = \frac{\pi^2}{8}.$$

To establish the second result, we note the Fourier transform of the derivative is $\hat{f}'(\omega) = (i\omega)\hat{f}(\omega)$. Thus by Parseval's theorem, we have:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\omega^2\cos^2(\omega\pi/2)}{(1-\omega^2)^2}\,d\omega = \int_{-\pi/2}^{\pi/2} \sin^2(x)\,dx.$$

Exactly the same argument produces:

$$\int_0^\infty \frac{\omega^2\cos^2(\omega\pi/2)}{(1-\omega^2)^2}\,d\omega = \frac{\pi^2}{8}.$$

10. By choosing an appropriate function in the Poisson summation formula, establish the identity:

$$\sqrt{\alpha}\left[\frac{1}{2}+\sum_{n=1}^{\infty}e^{-\alpha^2n^2/2}\right]=\sqrt{\beta}\left[\frac{1}{2}+\sum_{n=1}^{\infty}e^{-\beta^2n^2/2}\right],$$

where $\alpha\beta=2\pi$. Deduce that:

$$\sum_{n=1}^{\infty}(4\pi n^2-1)e^{-\pi n^2}=\frac{1}{2}.$$

➺ **Solution:** Recall that the Poisson summation formula states:

$$\sum_n f(x+n)=\sum_n \hat{f}(2\pi n)e^{2\pi inx}.$$

To establish the identity in question, we take $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\alpha^2x^2/2}$. Then $\hat{f}(x)=\frac{1}{\alpha}e^{-x^2/2\alpha^2}$, from lectures; hence evaluating the Poisson summation at $x=0$ for this function we have:

$$\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}e^{-\alpha^2n^2/2}=\frac{1}{\alpha}\sum_{n=-\infty}^{\infty}e^{-(2\pi/\alpha)^2n^2/2},$$

which can be rearranged to:

$$\sqrt{\alpha}\left[1+2\sum_{n=1}^{\infty}e^{-\alpha^2n^2/2}\right]=\sqrt{\frac{2\pi}{\alpha}}\left[1+2\sum_{n=1}^{\infty}e^{-(2\pi/\alpha)^2n^2/2}\right]$$

Choosing $\beta=2\pi/\alpha$, the formula follows immediately.

To make the deduction, we rearrange the given formula to:

$$\sum_{n=1}^{\infty}\frac{\alpha e^{-\alpha^2n^2/2}-\sqrt{2\pi}e^{-(2\pi/\alpha)^2n^2/2}}{\sqrt{2\pi}-\alpha}=\frac{1}{2}.$$

To finish, we take the limit $\alpha\to\sqrt{2\pi}$. Using L'Hôpital's rule, we have:

$$\sum_{n=1}^{\infty}\lim_{\alpha\to\sqrt{2\pi}}\left[\frac{e^{-\alpha^2n^2/2}-\alpha^2n^2e^{-\alpha^2n^2/2}-\sqrt{2\pi}(2\pi)^2n^2e^{-(2\pi/\alpha)^2n^2/2}/\alpha^3}{-1}\right]=\frac{1}{2},$$

from which we have the final result:

$$\sum_{n=1}^{\infty}\left(4\pi n^2-1\right)e^{-\pi^2n^2}=\frac{1}{2},$$

as required.

11. Show that:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x^2+y^2)\delta'(x^2+y^2-1)\delta(x^2-y^2)\,dxdy = f(1)-f'(1)$$

in two different ways: (a) using the identity derived in Question 2; (b) using plane polar coordinates.

➻ **Solution:** (a) The identity we proved in Question 2 was:

$$\delta(x^2-y^2) = \frac{\delta(x+y)+\delta(x-y)}{2|y|},$$

which implies we must evaluate:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x^2+y^2)\delta'(x^2+y^2-1)\left[\frac{\delta(x+y)+\delta(x-y)}{2|y|}\right]\,dxdy.$$

Performing the inner x integral, we have:

$$\int_{-\infty}^{\infty} \frac{f(2y^2)\delta'(2y^2-1)}{|y|}\,dy.$$

Now make the substitution $u=2y^2$. We have $du = 4ydy$, and hence the integral reduces to:

$$\int_{-\infty}^{\infty} \frac{f(u)\delta'(u-1)}{\sqrt{u/2}}\frac{1}{4\sqrt{u/2}}\,du = \int_{-\infty}^{\infty}\frac{f(u)}{u}\delta'(u-1)\,du = -\frac{d}{du}\frac{f(u)}{u}\bigg|_{u=1} = -\left(\frac{f'(1)}{1}-\frac{f(1)}{1^2}\right) = f(1)-f'(1),$$

as required.

(b) Changing variables to plane polar coordinates, we have $r^2 = x^2+y^2$, and $x = r\cos(\theta), y = r\sin(\theta)$. Thus we must evaluate the integral:

$$\int_0^{\infty}\int_0^{2\pi} f(r^2)\delta'(r^2-1)\delta(r^2\cos(2\theta))\,rd\theta dr$$

To perform the θ integral, we evaluate where $r^2\cos(2\theta)=0$ on the range $[0,2\pi]$. We know this occurs at $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$, and hence we have:

$$\begin{aligned}\delta(r^2\cos(2\theta)) &= \frac{\delta(\theta-\pi/4)}{|2r^2\sin(\pi/2)|}+\frac{\delta(\theta-3\pi/4)}{|2r^2\sin(3\pi/2)|}+\frac{\delta(\theta-5\pi/4)}{|2r^2\sin(5\pi/2)|}+\frac{\delta(\theta-7\pi/4)}{|2r^2\sin(7\pi/2)|}\\ &= \frac{1}{2r^2}\left(\delta(\theta-3\pi/4)+\delta(\theta-5\pi/4)+\delta(\theta-5\pi/4)+\delta(\theta-7\pi/4)\right).\end{aligned}$$

Hence the integral reduces to:

$$\int_0^{\infty}\frac{2f(r^2)\delta'(r^2-1)}{r}\,dr.$$

Making the substitution $u=r^2$, we have $du = 2rdr = 2\sqrt{u}dr$, and hence the integral transforms to:

$$\int_0^{\infty}\frac{f(u)\delta'(u-1)}{u}\,du = -\frac{d}{du}\frac{f(u)}{u}\bigg|_{u=1} = f(1)-f'(1),$$

as we obtained before.

12. By constructing a suitable Green's function, find the solution to the initial value problem:

$$y^{(n)} = f, \quad t > 0, \qquad\qquad y(0) = \ldots = y^{(n-1)}(0) = 0.$$

Hence prove Taylor's theorem with integral remainder for n-times continuously differentiable functions.

Solution: Straightforward

Part IB: Mathematical Methods
Examples Sheet 4 Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

1. Using the method of characteristics, find the solution to each of the initial value problems:

(i) $u_x + yu_y = 0, u(0, y) = y^3$;

(ii) $u_x + u_y + u = e^{x+2y}, u(x, 0) = 0$.

Solution: (i) We rewrite the equation as:

$$(1, y) \cdot \nabla u = 0,$$

which shows that u is constant along the curves whose tangent is $(1, y)$. In particular, to find these curves we solve:

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = y,$$

which yields:

$$x(t) = t + x_0, \qquad y(t) = Ae^t.$$

The initial data is $u(0, s) = s^3$, imposing this boundary data, we have $x(t) = t, y(t) = se^t$.

Now consider taking the derivative along the curves. We have:

$$\frac{du}{dt} = 0$$

by construction, so that:

$$u(x(t), y(t)) = u_0.$$

When $t = 0$, this gives $u(0, s) = s^3$, hence $u_0 = s^3$. It follows that:

$$u(t, se^t) = s^3.$$

Inverting the relationship between (x, y) and (s, t) gives the solution in the original coordinates (we observe $y = se^t$ and $x = t$ implies that $s = ye^{-t} = ye^{-x}$), which is:

$$u(x, y) = y^3 e^{-3x}.$$

(ii) In this example, we consider:

$$(1, 1) \cdot \nabla u + u = e^{x+2y}.$$

The characteristic curves in this case must satisfy:

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = 1 \qquad \Rightarrow \qquad x(t) = t + x_0, \qquad y(t) = t + y_0.$$

Initially, we want $x(0) = s$ and $y(0) = 0$, hence $x(t) = t + s$ and $y(t) = t$ is the general form of the characteristics. If we differentiate along the characteristics, we end up getting:

$$\frac{du}{dt} + u = e^{t+s+2t} = e^{3t+s}.$$

It remains to solve this differential equation. This can be done with an integrating factor; observe:

$$\frac{d}{dt}(e^t u) = e^{4t+s} \qquad \Rightarrow \qquad e^t u = \frac{e^{4t+s}}{4} + C \qquad \Rightarrow \qquad u = \frac{1}{4}e^{3t+s} + Ce^{-t}.$$

Imposing the initial condition at $t = 0$, we have $u(x(0), y(0)) = u(s, 0) = 0$, and hence:

$$0 = \frac{1}{4}e^s + C \qquad \Rightarrow \qquad C = -\frac{1}{4}e^s.$$

It follows that:

$$u(x(t), y(t)) = \frac{1}{4}e^{3t+s} - \frac{1}{4}e^{s-t}.$$

Inverting the relationships, we recall that $x(t) = t + s$ and $y(t) = t$, hence $t = y$ and $s = x - t = x - y$. Thus overall we have:

$$u(x, y) = \frac{1}{4}e^{3y+(x-y)} - \frac{1}{4}e^{(x-y)-y} = \frac{1}{4}e^{2y+x} - \frac{1}{4}e^{x-2y} = \frac{1}{2}e^x \sinh(2y).$$

2. Tricomi's equation in $\mathbb{R}^2$ is $u_{xx} + xu_{yy} = 0$.

(i) Determine the regions in $\mathbb{R}^2$ where Tricomi's equation is (a) elliptic; (b) parabolic; (c) hyperbolic.

(ii) For the hyperbolic region, determine the characteristic curves. Hence put Tricomi's equation in canonical form.

➾ **Solution:** (i) Recall that the differential equation:

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0$$

is said to be *elliptic* on a region if $b^2 - ac < 0$, *parabolic* on a region if $b^2 - ac = 0$, and *hyperbolic* on a region if $b^2 - ac > 0$. In the given example, we have $a = 1$, $b = 0$ and $c = x$, hence:

$$b^2 - ac = -x.$$

Thus Tricomi's equation is elliptic in the region $x > 0$, parabolic on $x = 0$, and hyperbolic on the region $x < 0$.

(ii) The characteristic curves are defined by:

$$\frac{dy}{dx} = -\frac{-b \pm \sqrt{b^2 - ac}}{a} = \mp\sqrt{-x}.$$

This has solution:

$$y_\pm(x) = c \pm \frac{2}{3}(-x)^{3/2},$$

which are the two families of characteristic curves. It follows that we should introduce coordinates ξ, η such that:

$$\xi(x, y) = y + \frac{2}{3}(-x)^{3/2}, \qquad \eta(x, y) = y - \frac{2}{3}(-x)^{3/2}.$$

Inverting these relationships, we have:

$$y = \frac{\xi + \eta}{2}, \qquad x = -\left(\frac{3}{4}\right)^{2/3}(\xi - \eta)^{2/3}.$$

We can obtain the canonical form by repeated differentiation. We have:

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x = -\sqrt{-x}u_\xi + \sqrt{-x}u_\eta,\\
u_{xx} &= \frac{1}{2\sqrt{-x}}u_\xi - \sqrt{-x}(u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x) - \frac{1}{2\sqrt{-x}}u_\eta + \sqrt{-x}(u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x)\\
&= \frac{1}{2\sqrt{-x}}u_\xi - xu_{\xi\xi} + 2xu_{\xi\eta} - \frac{1}{2\sqrt{-x}}u_\eta - xu_{\eta\eta}.
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
u_y &= u_\xi \xi_y + u_\eta \eta_y = u_\xi + u_\eta,\\
u_{yy} &= u_{\xi y} + u_{\eta y} = u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.
\end{aligned}$$

In particular, we see that the canonical form is given by:

$$0 = u_{xx} + xu_{yy} = \frac{1}{2\sqrt{-x}}u_\xi + 4xu_{\xi\eta} - \frac{1}{2\sqrt{-x}}u_\eta.$$

Rewriting this in terms of ξ, η, we have:

$$u_{\xi\eta} = \frac{u_\eta - u_\xi}{8x\sqrt{-x}} = \frac{u_\xi - u_\eta}{8(-x)^{3/2}} = \frac{u_\xi - u_\eta}{6(\xi - \eta)}.$$

3. Reduce the equation $u_{xx} + yu_{yy} + \frac{1}{2}u_y = 0$ to the canonical form $U_{\eta\xi} = 0$ in the hyperbolic region. Deduce that the general solution to the original equation is $u(x,y) = f(x + 2\sqrt{-y}) + g(x - 2\sqrt{-y})$ for arbitrary functions f, g.

➾ **Solution:** The coefficients are $a = 1, b = 0$ and $c = y$. Hence $b^2 - ac = -y$, which reveals that the equation is hyperbolic only in the region $-y > 0$, i.e. $y < 0$.

In this region, the characteristic curves are given by:

$$\frac{dy}{dx} = \mp\sqrt{-y} \qquad \Rightarrow \qquad \int \mp\frac{dy}{\sqrt{-y}} = \int dx \qquad \Rightarrow \qquad x = c \pm 2\sqrt{-y}.$$

Thus we should define canonical coordinates via:

$$\xi(x,y) = x + 2\sqrt{-y}, \qquad \eta = x - 2\sqrt{-y}.$$

Inverting these relationships, we have:

$$x = \frac{\xi + \eta}{2}, \qquad y = -\left(\frac{\xi - \eta}{4}\right)^2$$

Using these coordinates, we can transform the equation to canonical form using repeated differentiation. We have:

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta,$$
$$u_{xx} = u_{\xi x} + u_{\eta x} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Similarly, we have:

$$u_y = u_\xi \xi_y + u_\eta \eta_y = -\frac{1}{\sqrt{-y}}u_\xi + \frac{1}{\sqrt{-y}}u_\eta,$$
$$u_{yy} = -\frac{1}{2(-y)^{3/2}}u_\xi - \frac{1}{\sqrt{-y}}\left(-\frac{1}{\sqrt{-y}}u_{\xi\xi} + \frac{1}{\sqrt{-y}}u_{\xi\eta}\right) + \frac{1}{2(-y)^{3/2}}u_\eta + \frac{1}{\sqrt{-y}}\left(-\frac{1}{\sqrt{-y}}u_{\eta\xi} + \frac{1}{\sqrt{-y}}u_{\eta\eta}\right)$$
$$= -\frac{1}{2(-y)^{3/2}}u_\xi + \frac{1}{2(-y)^{3/2}}u_\eta - \frac{1}{y}u_{\xi\xi} + \frac{2}{y}u_{\xi\eta} - \frac{1}{y}u_{\eta\eta}.$$

As a result, the equation is transformed to:

$$0 = u_{xx} + yu_{yy} + \frac{1}{2}u_y$$
$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} - \frac{y}{2(-y)^{3/2}}u_\xi + \frac{y}{2(-y)^{3/2}}u_\eta - u_{\xi\xi} + 2u_{\xi\eta} - u_{\eta\eta} - \frac{1}{2\sqrt{-y}}u_\xi + \frac{1}{2\sqrt{-y}}u_\eta$$
$$= 4u_{\xi\eta}.$$

Hence the canonical form is indeed $u_{\xi\eta} = 0$, which has the general solution $u(\xi,\eta) = f(\xi) + g(\eta)$ (this is D'Alembert's solution of the wave equation), for general functions f, g. Restoring the original variables, we have:

$$u(x,y) = f(x + 2\sqrt{-y}) + g(x - 2\sqrt{-y}),$$

as required.

4. Let $F, G : \mathbb{R}^n \to \mathbb{C}$ be smooth functions that decay rapidly as $|\mathbf{x}| \to \infty$. Using the identity:

$$\delta(\mathbf{x}-\mathbf{y}) = \frac{1}{(2\pi)^n} \int e^{i\boldsymbol{\lambda}\cdot(\mathbf{x}-\mathbf{y})}\, d^n\boldsymbol{\lambda},$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, establish Parseval's theorem for the Fourier transform on $\mathbb{R}^n$:

$$\frac{1}{(2\pi)^n} \int \hat{F}(\boldsymbol{\lambda})\overline{\hat{G}(\boldsymbol{\lambda})}\, d^n\boldsymbol{\lambda} = \int F(\mathbf{x})\overline{G(\mathbf{x})}\, d^n\mathbf{x}, \qquad \text{hence} \qquad \frac{1}{(2\pi)^n} \int |\hat{F}(\boldsymbol{\lambda})|^2\, d^n\boldsymbol{\lambda} = \int |F(\mathbf{x})|^2\, d^n\mathbf{x}.$$

Solution: We can do this directly. Recall that the definition of the Fourier transform in $\mathbb{R}^n$ is given by:

$$\hat{F}(\boldsymbol{\lambda}) = \int e^{-i\boldsymbol{\lambda}\cdot\mathbf{x}} F(\mathbf{x})\, d^n\mathbf{x}.$$

Inserting this into the left hand side of the given result (and assuming we can freely change the order of integration everywhere), we have:

$$\begin{aligned}
\frac{1}{(2\pi)^n} \int d^n\boldsymbol{\lambda} \int d^n\mathbf{x} \int d^n\mathbf{y}\, e^{-i\boldsymbol{\lambda}\cdot\mathbf{x}} e^{i\boldsymbol{\lambda}\cdot\mathbf{y}} F(\mathbf{x})\overline{G(\mathbf{y})} &= \frac{1}{(2\pi)^n} \int d^n\mathbf{x} \int d^n\mathbf{y} F(\mathbf{x})\overline{G(\mathbf{y})} \int d^n\boldsymbol{\lambda}\, e^{i\boldsymbol{\lambda}\cdot(\mathbf{y}-\mathbf{x})} \\
&= \int d^n\mathbf{x} \int d^n\mathbf{y} F(\mathbf{x})\overline{G(\mathbf{y})}\delta(\mathbf{x}-\mathbf{y}),
\end{aligned}$$

using the delta function identity given in the question. Hence we can simplify everything to:

$$\int d^n\mathbf{x} F(\mathbf{x})\overline{G(\mathbf{x})},$$

thus proving the result. Taking $G = F$ proves the second form.

5. Consider the initial value problem for the heat equation on $\mathbb{R}^n$:

$$\begin{cases} u_t - \kappa\Delta u = F(\mathbf{x}, t), & (\mathbf{x}, t) \in \mathbb{R}^n \times (0, \infty) \\ u(\mathbf{x}, 0) = f(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n. \end{cases} \tag{†}$$

Let u_i denote the solution to (†) that has initial data f_i. Using the Fourier transform and Parseval's theorem, show:

$$||u_1(\cdot, t) - u_2(\cdot, t)|| \leq C||f_1 - f_2|| \qquad \text{where} \qquad ||u(\cdot, t)||^2 = \int |u(\mathbf{x}, t)|^2 d^n\mathbf{x}, \qquad \text{etc.}$$

for some constant $C > 0$ which you should determine. Deduce that (†) is well-posed with respect to the $||\cdot||$ norm.

➡ **Solution:** By linearity, if u_1, u_2 solve the problem with initial data f_1, f_2, it follows that $u_1 - u_2$ solves the unforced problem ($F = 0$) with initial data $f_1 - f_2$. Hence, as we saw in the lecture notes, the solution to the problem is:

$$(u_1 - u_2)(\mathbf{x}, t) = K_t * (f_1 - f_2)(\mathbf{x}),$$

where K_t is the *heat kernel*, defined by:

$$K_t(\mathbf{x}) = \frac{1}{(4\pi\kappa t)^{n/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4\kappa t}\right).$$

Taking the Fourier transform of this equation, we have:

$$\hat{u}_1(\boldsymbol{\lambda}, t) - \hat{u}_2(\boldsymbol{\lambda}, t) = \hat{K}_t(\boldsymbol{\lambda})(\hat{f}_1(\boldsymbol{\lambda}) - \hat{f}_2(\boldsymbol{\lambda})) = e^{-\kappa t|\boldsymbol{\lambda}|^2}(\hat{f}_1(\boldsymbol{\lambda}) - \hat{f}_2(\boldsymbol{\lambda})),$$

again using the results from the lecture notes. It follows using Parseval's theorem that:

$$||u_1(\cdot, t) - u_2(\cdot, t)|| = (2\pi)^{-n/2}||\hat{u}_1(\cdot, t) - \hat{u}_2(\cdot, t)|| = (2\pi)^{-n/2}\left(\int |e^{-\kappa t|\boldsymbol{\lambda}|^2}(\hat{f}_1(\boldsymbol{\lambda}) - \hat{f}_2(\boldsymbol{\lambda}))|^2 \, d^n\boldsymbol{\lambda}\right)^{1/2} \leq (2\pi)^{-n/2}||\hat{f}_1 - \hat{f}_2||,$$

using $e^{-\kappa t|\boldsymbol{\lambda}|^2} \leq 1$. One more application of Parseval's theorem gives the result:

$$||u_1(\cdot, t) - u_2(\cdot, t)|| \leq ||f_1 - f_2||,$$

i.e. the given estimate holds with $C = 1$. We never stated an exact definition of well-posedness in the course, but this seems pretty reasonable here - if the initial data are 'close' in the $||\cdot||$ norm, then the solutions are 'close' in the $||\cdot||$ norm.

6. Show that the heat kernel satisfies the semi-group property $K_{t+s}(\mathbf{x}) = (K_t * K_s)(\mathbf{x})$.

➾ **Solution:** Ashton proves directly, but can do with Fourier convolution theorem By definition, we have:

$$\begin{aligned}(K_t * K_s)(\mathbf{x}) &= \int d^n\mathbf{y}\, K_t(\mathbf{x}-\mathbf{y})K_s(\mathbf{y}) \\ &= \int d^n\mathbf{y}\, \frac{1}{(4\pi\kappa t)^{n/2}}\frac{1}{(4\pi\kappa s)^{n/2}}\exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\kappa t}\right)\exp\left(-\frac{|\mathbf{y}|^2}{4\kappa s}\right).\end{aligned}$$

Our plan is to complete the square in the exponential, to convert this to a product of Gaussians. We have:

$$\begin{aligned}\frac{|\mathbf{x}-\mathbf{y}|^2}{4\kappa t} + \frac{|\mathbf{y}|^2}{4\kappa s} &= \left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)|\mathbf{y}|^2 - \frac{\mathbf{x}\cdot\mathbf{y}}{2\kappa t} + \frac{|\mathbf{x}|^2}{4\kappa t} \\ &= \left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)\left[|\mathbf{y}|^2 - \left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)^{-1}\frac{\mathbf{x}\cdot\mathbf{y}}{2\kappa t}\right] + \frac{|\mathbf{x}|^2}{4\kappa t} \\ &= \left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)\left[\left|\mathbf{y} - \left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)^{-1}\frac{\mathbf{x}}{4\kappa t}\right|^2 - \left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)^{-2}\frac{|\mathbf{x}|^2}{16\kappa^2 t^2}\right] + \frac{|\mathbf{x}|^2}{4\kappa t} \\ &= \left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)\left|\mathbf{y} - \left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)^{-1}\frac{\mathbf{x}}{4\kappa t}\right|^2 + \frac{|\mathbf{x}|^2}{4\kappa t}\left(1 - \frac{1}{4\kappa t}\left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)^{-1}\right) \\ &= \left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)\left|\mathbf{y} - \left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)^{-1}\frac{\mathbf{x}}{4\kappa t}\right|^2 + \frac{|\mathbf{x}|^2}{4\kappa t}\left(1 - \left(1+\frac{t}{s}\right)^{-1}\right) \\ &= \left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)\left|\mathbf{y} - \left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)^{-1}\frac{\mathbf{x}}{4\kappa t}\right|^2 + \frac{|\mathbf{x}|^2}{4\kappa(s+t)}.\end{aligned}$$

Hence, we have:

$$(K_t*K_s)(\mathbf{x}) = \frac{1}{(4\pi\kappa s)^{n/2}(4\pi\kappa t)^{n/2}}\exp\left(-\frac{|\mathbf{x}|^2}{4\kappa(s+t)}\right)\int d^n\mathbf{y}\,\exp\left(-\left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)\left|\mathbf{y} - \left(\frac{1}{4\kappa t}+\frac{1}{4\kappa s}\right)^{-1}\frac{\mathbf{x}}{4\kappa t}\right|^2\right)$$

Recognising the integral as the product of n Gaussians, and using the Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-a(x-b)^2}\,dx = \sqrt{\frac{\pi}{a}},$$

we arrive at:

$$\begin{aligned}(K_t * K_s)(\mathbf{x}) &= \frac{1}{(16\pi^2\kappa^2 st)^{n/2}}\cdot\frac{\pi^{n/2}}{(1/4\kappa s + 1/4\kappa t)^{n/2}}\exp\left(-\frac{|\mathbf{x}|^2}{4\kappa(s+t)}\right) \\ &= \frac{1}{(4\pi\kappa(s+t))^{n/2}}\exp\left(-\frac{|\mathbf{x}|^2}{4\kappa(s+t)}\right) \\ &= K_{s+t}(\mathbf{x}),\end{aligned}$$

as required.

7. Suppose $\mathcal{G} = \mathcal{G}(\mathbf{x};\mathbf{y})$ is the Dirichlet Green's function for the Laplacian on a domain $\Omega \subseteq \mathbb{R}^n$, i.e. for each $\mathbf{y} \in \Omega$:

$$\begin{cases} \Delta\mathcal{G} = \delta(\mathbf{x}-\mathbf{y}), & \mathbf{x} \in \Omega \\ \mathcal{G}(\mathbf{x};\mathbf{y}) = 0, & \mathbf{x} \in \partial\Omega. \end{cases}$$

Using Green's second identity, show that if $\Delta u = 0$ and $u = f$ on $\partial\Omega$ then for $\mathbf{y} \in \Omega$:

$$u(\mathbf{y}) = \int\limits_{\partial\Omega} f(\mathbf{x}) \frac{\partial \mathcal{G}}{\partial \mathbf{n}}(\mathbf{x};\mathbf{y})\, dS(\mathbf{x}). \tag{‡}$$

➭ **Solution:** Green's second identity, from vector calculus, states that:

$$\int\limits_{\Omega} \left(\psi\nabla^2\phi - \phi\nabla^2\psi\right)\, d^n\mathbf{x} = \int\limits_{\partial\Omega} (\psi\nabla\phi - \phi\nabla\psi)\cdot d\mathbf{S}.$$

Choosing $\phi(\mathbf{x}) = \mathcal{G}(\mathbf{x};\mathbf{y})$ and $\psi(\mathbf{x}) = u(\mathbf{x})$, we have $\nabla^2\phi = \delta(\mathbf{x}-\mathbf{y})$, and hence:

$$\begin{aligned} u(\mathbf{y}) = \int\limits_{\Omega} u(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\, d^n\mathbf{x} &= \int\limits_{\Omega} \mathcal{G}(\mathbf{x};\mathbf{y})\nabla^2 u\, d^n\mathbf{x} + \int\limits_{\partial\Omega} (u\nabla\mathcal{G} - \mathcal{G}\nabla u)\cdot d\mathbf{S} \\ &= \int\limits_{\partial\Omega} f(\mathbf{x})\frac{\partial\mathcal{G}}{\partial\mathbf{n}}(\mathbf{x};\mathbf{y})\, dS(\mathbf{x}), \end{aligned}$$

since $\nabla^2 u = 0$ inside the volume, so the first term disappears, and $\mathcal{G} = 0$, $u = f$ on the boundary. The result follows immediately.

An immediate consequence of this result is that if we have the Dirichlet Green's function for the forced Laplace equation on a region Ω, with zero boundary conditions, then we can obtain the Dirichlet Green's function for the unforced Laplace equation on a region Ω, with non-zero boundary conditions. The general solution of Poisson's equation on the region can thus be obtained.

8. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and consider the boundary value problem:

$$\begin{cases} \Delta u = 0, & (x,y) \in \Omega, \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u(x,y) \to 0, & \text{rapidly as } |x| + |y| \to \infty. \end{cases}$$

(a) Use the method of images to construct the Dirichlet Green's function for this problem and use (‡) to show:

$$u(x,y) = \frac{y}{\pi} \int \frac{f(\xi)}{(x-\xi)^2 + y^2} \, d\xi.$$

(b) Obtain the same result by first taking the Fourier transform (with respect to x) of $\Delta u = 0$ and $u(x, 0) = f(x)$.

➾ **Solution:** (a) Let the Dirichlet Green's function for free 2D space, $\mathbb{R}^2$, be $\mathcal{G}(\mathbf{x}; \mathbf{x}')$. Then the Green's function satisfies:

$$\nabla^2 \mathcal{G}(\mathbf{x}; \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$$

everywhere in $\mathbb{R}^2$. Given a point $\mathbf{x}' = (x', y') \in \Omega$, we can introduce an image point $\mathbf{x}'_0 = (x', -y')$ such that:

$$\nabla^2 (\mathcal{G}(\mathbf{x}; \mathbf{x}') - \mathcal{G}(\mathbf{x}; \mathbf{x}'_0)) = \delta(\mathbf{x} - \mathbf{x}') - \delta(\mathbf{x} - \mathbf{x}'_0) = \delta(\mathbf{x} - \mathbf{x}')$$

in the region Ω, since $\mathbf{x}'_0 \notin \Omega$. Furthermore, on the boundary $\partial\Omega$ we have:

$$\mathcal{G}((x,0); \mathbf{x}') - \mathcal{G}((x,0); \mathbf{x}'_0) = \frac{1}{2\pi} \log \left| \frac{|(x,0) - \mathbf{x}'|}{|(x,0) - \mathbf{x}'_0|} \right| = \frac{1}{2\pi} \log \left| \frac{\sqrt{(x-x')^2 + (y')^2}}{\sqrt{(x-x')^2 + (-y')^2}} \right| = \frac{1}{2\pi} \log(1) = 0.$$

Thus this combination of Green's functions satisfies the boundary condition that $\mathcal{G}(\mathbf{x}; \mathbf{x}') - \mathcal{G}(\mathbf{x}; \mathbf{x}'_0)$ vanishes on the $y = 0$. This combination also decays rapidly to zero as $(x, y) \to \infty$ in the upper half plane. It follows that the Dirichlet Green's function for the region Ω is given by:

$$\mathcal{G}_\Omega(\mathbf{x}; \mathbf{x}') = \frac{1}{4\pi} \log\left((x-x')^2 + (y-y')^2\right) - \frac{1}{4\pi} \log\left((x-x')^2 + (y+y')^2\right) = \frac{1}{4\pi} \log \left| \frac{(x-x')^2 + (y-y')^2}{(x-x')^2 + (y+y')^2} \right|.$$

Using the result of the previous question, it follows that the solution to the given problem is (using the *outward pointing normal*, $\hat{\mathbf{n}} = -\hat{\mathbf{e}}_y$, from the region Ω):

$$\begin{aligned} u(x', y') &= \int_{-\infty}^{\infty} d\xi \, f(\xi) \cdot \left(-\frac{\partial \mathcal{G}_\Omega}{\partial y} \right) ((\xi, 0); (x', y')) \\ &= -\int_{-\infty}^{\infty} d\xi \, f(\xi) \cdot \left(\frac{1}{4\pi} \frac{2(y-y')}{(x-x')^2 + (y-y')^2} - \frac{1}{4\pi} \frac{2(y-y')}{(x-x')^2 + (y+y')^2} \right) \Bigg|_{(x,y)=(\xi,0)} \\ &= \frac{y'}{\pi} \int_{-\infty}^{\infty} d\xi \frac{f(\xi)}{(\xi - x')^2 + (y')^2}, \end{aligned}$$

as required.

(b) Instead, taking the Fourier transform of $0 = \nabla^2 u = u_{xx} + u_{yy}$ with respect to x, we have:

$$-\lambda^2 \hat{u}(\lambda, y) + \hat{u}_{yy} = 0.$$

Similarly, taking the Fourier transform of the boundary conditions we have $\hat{u}(\lambda, 0) = \hat{f}(\lambda)$ and $\hat{u}(\lambda, y) \to 0$ as $|\lambda|, |y| \to \infty$. This ODE has the general solution:

$$\hat{u}(\lambda, y) = e^{-\lambda y} A(\lambda) + e^{\lambda y} B(\lambda).$$

Requiring that $\hat{u} \to 0$ in the region $y > 0$ (which is the region under consideration in the question) implies that:

$$\hat{u}(\lambda, y) = \begin{cases} e^{-\lambda y} A(\lambda), & \text{for } \lambda > 0, \\ e^{\lambda y} B(\lambda), & \text{for } \lambda < 0. \end{cases}$$

In particular, applying the other boundary condition, we see that the general solution is:

$$\hat{u}(\lambda, y) = e^{-|\lambda| y} f(\lambda).$$

in the region $y > 0$. Taking an inverse Fourier transform, and using the convolution theorem, we have:

$$u(x, y) = (\mathcal{F}^{-1}[e^{-|\lambda| y}] * f)(x).$$

Computing the correct Fourier transform, we arrive at:

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(\xi - x)^2 + y^2} \, dx,$$

as required.

9. Find the Dirichlet Green's function for the Laplacian on the unit ball $\Omega = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq 1\}$. [*Hint: only one external charge is needed. Guess which line this external charge should lie on and go from there.*]

➾ **Solution:** We conjecture that given the point $\mathbf{x}'$, a sensible position for the external charge is probably along the line from the origin to this point. Therefore, we suggest that an image charge for this point should be placed at:

$$\mathbf{x}_0' = p\mathbf{x}',$$

for some scale factor $p > 1/|\mathbf{x}'|$ (so that $|p\mathbf{x}'| > 1$, and the point lies outside the volume Ω). We also let the strength of the charge be arbitrary, say q. Then the conjectured form of the Dirichlet Green's function for the problem is:

$$\mathcal{G}_\Omega(\mathbf{x};\mathbf{x}') = \mathcal{G}(\mathbf{x};\mathbf{x}') + q\mathcal{G}(\mathbf{x};p\mathbf{x}'),$$

where $\mathcal{G}$ is the free-space Green's function. Clearly this satisfies Laplace's equation on Ω, since $p\mathbf{x}' \notin \Omega$. To satisfy the boundary condition that the Green's function vanish on the boundary of the ball, we need:

$$\mathcal{G}(\mathbf{x};\mathbf{x}') = -q\mathcal{G}(\mathbf{x};p\mathbf{x}'),$$

when $|\mathbf{x}| = 1$. There are two cases depending on the dimension of the problem.

- When $n > 2$, we have:

  $$-\frac{1}{(n-2)A(S^{n-1})}|\mathbf{x}-\mathbf{x}'|^{2-n} = \frac{q}{(n-2)A(S^{n-1})}|\mathbf{x}-p\mathbf{x}'|^{2-n},$$

  where A is the surface area of the sphere S^{n-1}. Rearranging, we have:

  $$-|\mathbf{x}-\mathbf{x}'|^{2-n} = q|\mathbf{x}-p\mathbf{x}'|^{2-n} \qquad \Rightarrow \qquad \left(|\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{x}' + |\mathbf{x}'|^2\right)^{2-n} = q^2\left(|\mathbf{x}|^2 - 2p\mathbf{x}\cdot\mathbf{x}' + p^2|\mathbf{x}'|^2\right)^{2-n}.$$

  Taking the $(2-n)$th root, and bringing everything to one side, we have:

  $$|\mathbf{x}|^2\left(1-q^{2/(2-n)}\right) - 2\mathbf{x}\cdot\mathbf{x}'\left(1-pq^{2/(2-n)}\right) + |\mathbf{x}'|^2\left(1-p^2q^{2/(2-n)}\right) = 0.$$

  This holds for all $\mathbf{x}$ such that $|\mathbf{x}| = 1$. Hence, writing $\mathbf{x}\cdot\mathbf{x}' = |\mathbf{x}||\mathbf{x}'|\cos(\theta) = |\mathbf{x}'|\cos(\theta)$, where θ is the angle between $\mathbf{x}, \mathbf{x}'$, we obtain:

  $$\left(1-q^{2/(2-n)}\right) - 2|\mathbf{x}'|\cos(\theta)\left(1-pq^{2/(2-n)}\right) + |\mathbf{x}'|^2\left(1-p^2q^{2/(2-n)}\right) = 0.$$

  Since this must hold for all values of θ, we can compare coefficients of $\cos(\theta)$ to yield:

  $$1-pq^{2/(2-n)} = 0 \qquad \Rightarrow \qquad q^{2/(2-n)} = 1/p.$$

  The coefficient of the constant term then becomes:

  $$1-\frac{1}{p}+|\mathbf{x}'|^2\left(1-\frac{p^2}{p}\right) = 0 \qquad \Rightarrow \qquad \frac{p-1}{p}+(1-p)|\mathbf{x}'|^2 = 0 \qquad \Rightarrow \qquad |\mathbf{x}'|^2 = \frac{1}{p}.$$

  Hence we have:

  $$p = \frac{1}{|\mathbf{x}'|^2}, \qquad q = \pm|\mathbf{x}'|^{2-n};$$

  evidently we require the minus root to satisfy the original equation. It follows that the required Green's function is:

  $$\mathcal{G}_\Omega(\mathbf{x};\mathbf{x}') = -\frac{1}{(n-2)A(S^{n-1})}\left[\frac{1}{|\mathbf{x}-\mathbf{x}'|^{n-2}} - \frac{1}{|\mathbf{x}'|^{n-2}}\frac{1}{|\mathbf{x}-\mathbf{x}'/|\mathbf{x}'|^2|^2}\right]$$

- When $n = 2$, we do something very *slightly* different; we choose to consider a Green's function of the form:

$$\mathcal{G}(\mathbf{x};\mathbf{x}') = \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}'| - \frac{1}{2\pi}\log\left(q|\mathbf{x}-p\mathbf{x}'|\right).$$

Evidently both terms satisfy Poisson's equation, since the second term can be expressed as $\log(q)/2\pi$ plus a solution of Poisson' equation (by the multiplicative/additive property of logarithms). Hence we require on $|\mathbf{x}| = 1$:

$$|\mathbf{x}-\mathbf{x}'| = q|\mathbf{x}-p\mathbf{x}'|,$$

which is the same problem as before, but with the sign changed, and effectively $n = 1$. Thus we should choose:

$$p = \frac{1}{|\mathbf{x}'|}, \qquad q = |\mathbf{x}'|.$$

We can check this works by squaring, giving:

$$1 - 2\mathbf{x}\cdot\mathbf{x}' + |\mathbf{x}'|^2 = q^2 - 2q^2 p\mathbf{x}\cdot\mathbf{x}' + q^2p^2|\mathbf{x}'|^2,$$

which generates the simultaneous equations:

$$1 + |\mathbf{x}'|^2 = q^2 + q^2p^2|\mathbf{x}'|^2, \qquad 1 = q^2 p.$$

The second equation implies $p = 1/q^2$. Substituting into the first, we have:

$$1 + |\mathbf{x}'|^2 = q^2 + \frac{1}{q^2}|\mathbf{x}'|^2 \qquad \Rightarrow \qquad 0 = q^4 - (1+|\mathbf{x}'|^2)^2 q^2 + |\mathbf{x}'|^2 = (q^2-1)(q^2-|\mathbf{x}'|^2).$$

The solution we are clearly interested in is $q = |\mathbf{x}'|$ and $p = 1/|\mathbf{x}'|^2$, as we expected. The result is:

$$\mathcal{G}(\mathbf{x};\mathbf{x}') = \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}'| - \frac{1}{2\pi}\log\left(|\mathbf{x}'|\left|\mathbf{x}-\mathbf{x}'/|\mathbf{x}'|^2\right|\right).$$

10. An infinite string, at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity $V\delta(x - x_0)$, where V is constant. Derive the position of the string for $t > 0$.

➾ **Solution:** Let $u(x,t)$ be the displacement of the string. Then u satisfies the following problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} & = 0, \\ u(x,0) & = 0, \\ \dfrac{\partial u}{\partial t}(x,0) & = V\delta(x - x_0). \end{cases}$$

This problem was solved in the lectures through the use of a Green's function. The result is:

$$u(x,t) = (\Phi_t * V\delta_{x_0})(x),$$

where $\Phi_t(x) = \frac{1}{2c}H(x+ct) - \frac{1}{2c}H(x-ct)$ and $\delta_{x_0}(x) = \delta(x - x_0)$. The result is:

$$\begin{aligned} u(x,t) &= \int\limits_{-\infty}^{\infty} \left(\frac{1}{2c}H(y+ct) - \frac{1}{2c}H(y-ct) \right) V\delta(x - x_0 - y) \\ &= \frac{V}{2c}H(x - x_0 + ct) - \frac{V}{2c}H(x - x_0 - ct). \end{aligned}$$

Alternatively, we could do this with D'Alembert's solution. We know the general solution of the wave equation is:

$$u(x,t) = f(x+ct) + g(x-ct),$$

for functions f, g. The requirement that $u(x,0) = 0$ implies that $f(x) + g(x) \equiv 0$, so $f = -g$. The other requirement is:

$$V\delta(x - x_0) = \frac{\partial u}{\partial t}(x,0) = cf'(x) - cg'(x) = 2cf'(x).$$

Thus:

$$f'(x) = \frac{V}{2c}\delta(x - x_0) \qquad \Rightarrow \qquad f(x) - f(-\infty) = \frac{V}{2c}\begin{cases} 1, & \text{if } x > x_0, \\ 0, & \text{if } x < x_0. \end{cases} = \frac{V}{2c}H(x - x_0).$$

Therefore:

$$u(x,t) = \frac{V}{2c}H(x - x_0 + ct) + f(\infty) - \frac{V}{2c}H(x - x_0 - ct) - f(\infty) = \frac{V}{2c}\left(H(x - x_0 + ct) - H(x - x_0 - ct)\right),$$

as we derived above.

11. A semi-infinite string, fixed for all times at zero at $x = 0$ and at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V\delta(x - x_0)$, where V is constant and $x_0 > 0$. Derive the position of the string for $t > 0$ and compare the solution to the infinite case in the previous question.

➼ **Solution:** This time, the setup of the problem is:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} & = 0, \\ u(0,t) & = 0, \\ u(x,0) & = 0, \\ \dfrac{\partial u}{\partial t}(x,0) & = V\delta(x - x_0), \end{cases}$$

where all of these relationships hold on $x \geq 0$ only, and $u : [0,\infty) \times [0,\infty) \to \mathbb{R}$. Again, the solution is given in the lectures by:

$$u(x,t) = (\Phi_t * (V\delta_{x_0})_{\mathsf{odd}})(x),$$

where $(V\delta_{x_0})_{\mathsf{odd}}$ is the odd extension of the function $V\delta_{x_0}$. The odd extension is evidently:

$$(V\delta_{x_0})_{\mathsf{odd}} = V\delta(x - x_0) - V\delta(x + x_0).$$

Hence the solution is given by:

$$\begin{aligned} u(x,t) &= \frac{V}{2c}\int_{-\infty}^{\infty} \left(H(y+ct) - H(y-ct)\right)\left(\delta(x - x_0 - y) - \delta(x + x_0 - y)\right) \\ &= \frac{V}{2c}\left(H(x - x_0 + ct) - H(x - x_0 - ct) - H(x + x_0 + ct) + H(x + x_0 - ct)\right). \end{aligned}$$

Checking this satisfies the data (for fun), we have:

- $u(x,0) = 0$;
- $\dfrac{\partial u}{\partial t}(x,0) = V\delta(x - x_0) - V\delta(x + x_0)$. Since $x_0 > 0$, we have for $x > 0$, $\delta(x + x_0) = 0$, so the condition is satisfied.
- $u(0,t) = \dfrac{V}{2c}\left(H(ct - x_0) - H(-x_0 - ct) - H(x_0 + ct) + H(x_0 - ct)\right)$. Note that $H(-x) = 1 - H(x)$. Hence we have:

  $$H(ct - x_0) - H(-x_0 - ct) - H(x_0 + ct) + H(x_0 - ct) = 1 - H(x_0 - ct) - (1 - H(x_0 + ct)) - H(x_0 + ct) + H(x_0 - ct) = 0,$$

  and the boundary condition is satisfied.
- All of these functions satisfy the wave equation by construction.

We see that the solution is equivalent to if an infinite string was given velocity V at $x = x_0$, but also additionally velocity $-V$ at $x = -x_0$. This ensures that the string is always stationary at $x = 0$ (the waves all cancel each other out at $x = 0$).

12. Give a construction for the Dirichlet Green's function for the Laplacian on $\Omega = \{\mathbf{x} \in \mathbb{R}^n : x_1 > 0, ..., x_n > 0\}$.

➾ **Solution:** Method of images. Writing maths is worst bit.

13. Suppose that $u_{tt} - c^2\Delta u = 0$ on $\mathbb{R}^n \times (0,\infty)$. Fix $(\mathbf{x}_0, t_0)$ and suppose that $u(\mathbf{x},0) = u_t(\mathbf{x},0) = 0$ for $\mathbf{x}$ in the ball $B_0 = \{\mathbf{x} : |\mathbf{x}-\mathbf{x}_0| \leq ct_0\}$. By considering the *energy* in the ball $B_t = \{\mathbf{x} : |\mathbf{x}-\mathbf{x}_0| \leq c(t_0 - t)\}$,

$$E(t) = \frac{1}{2}\int\limits_{B_t}\left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2|\nabla u|^2\right] d^n\mathbf{x},$$

show that $u(\mathbf{x},t) = 0$ inside the *backward light cone* $\Sigma_{t_0}(\mathbf{x}_0) = \{(\mathbf{x},t) : 0 \leq t \leq t_0 \text{ and } |\mathbf{x}-\mathbf{x}_0| \leq c(t_0-t)\}$. This shows that the solution to the wave equation at $(\mathbf{x}_0, t_0)$ depends only on the initial data in the region B_0.

Comment on the uniqueness of the solution to the initial value problem for the wave equation on $\mathbb{R}^n$.

➾ **Solution:** We use the following identity:

$$\frac{d}{d\epsilon}\int\limits_{|\mathbf{x}|\leq\epsilon} f(\mathbf{x},\epsilon)dV = \frac{d}{d\epsilon}\int_0^\epsilon\int\limits_{S^{n-1}} f(r\mathbf{x},\epsilon)r^{n-1}\,dr dS^{n-1} = \int\limits_{|\mathbf{x}|\leq\epsilon}\frac{\partial f}{\partial\epsilon}(\mathbf{x},\epsilon)\,dV + \int\limits_{S^{n-1}} f(\epsilon\mathbf{x})\epsilon^{n-1}dS = \int\limits_{|\mathbf{x}|\leq\epsilon}\frac{\partial f}{\partial\epsilon}(\mathbf{x},\epsilon)\,dV + \int\limits_{|\mathbf{x}|=\epsilon} f(\mathbf{x})\,dS.$$

Using this identity, we can compute the derivative of the energy. Without loss of generality, we may assume $\mathbf{x}_0 = 0$. Then we have:

$$\frac{dE}{d(t_0-t)} = -\int\limits_{B_t}\left(u_t u_{tt} + c^2\nabla u\cdot\nabla u_t\right)\,dV + \frac{c}{2}\int\limits_{\partial B_t}\left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2|\nabla u|^2\right]\,dS$$

Hence, we have:

$$\begin{aligned}
\frac{dE}{dt} &= \int\limits_{B_t}\left(u_t u_{tt} + c^2\nabla u\cdot\nabla u_t\right)\,dV - \frac{c}{2}\int\limits_{\partial B_t}\left[u_t^2 + c^2|\nabla u|^2\right]\,dS \\
&= \int\limits_{B_t}\left(c^2 u_t\nabla^2 u + c^2\nabla u\cdot\nabla u_t\right)\,dV - \frac{c}{2}\int\limits_{\partial B_t}\left[u_t^2 + c^2|\nabla u|^2\right]\,dS \\
&= \int\limits_{B_t} c^2\nabla\cdot\left(u_t\nabla u\right)\,dV - \frac{c}{2}\int\limits_{\partial B_t}\left[u_t^2 + c^2|\nabla u|^2\right]\,dS \\
&= \int\limits_{\partial B_t}\left[cu_t(c\nabla u\cdot\hat{\mathbf{n}}) - \frac{c}{2}u_t^2 - \frac{c^3}{2}|\nabla u|^2\right]\,dS.
\end{aligned}$$

But now, using the Cauchy-Schwarz inequality (in the form $2ab \leq a^2 + b^2$ in the first instance), we have $u_t(c\nabla u\cdot\hat{\mathbf{n}}) \leq \frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u\cdot\hat{\mathbf{n}}|^2 \leq \frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2$, since $|\nabla u\cdot\hat{\mathbf{n}}| \leq |\nabla u|$. It follows that:

$$\frac{dE}{dt} \leq 0.$$

But $E(t)$ is a non-negative quantity, and we are given initially that $u_t(\mathbf{x},0) = 0$ and $u(\mathbf{x},0) = 0$ which implies $\nabla u(\mathbf{x},0) = 0$. Hence $E(0) = 0$, and thus $E(t) = 0$ for all time. Hence we must have:

$$\left(\frac{\partial u}{\partial t}\right)^2 + c^2|\nabla u|^2 = 0$$

everywhere in the backwards light cone B_t. As a result, we have $\nabla u(\mathbf{x},t) = 0$ in the backwards light cone, and $u_t(\mathbf{x},t) = 0$ in the backwards light cone. The boundary data on B_0 imply that $u(\mathbf{x},t) = 0$ everywhere in the backwards light cone.