Part IA: Mathematics for Natural Sciences A and B
Examples Sheet 0: Basic skills

Model Solutions

Please send all comments and corrections to jmm232@cam.ac.uk.

Making friends

1. Speak to your supervision partner, and, if you don't know already, find out: (a) where they are from; (b) their favourite food; (c) what they like to do to relax; (d) what part of the maths course they are most excited about this year.

Solution: I'll tell you my answers, for fun:

(a) I'm from a town called Lowestoft, in Suffolk. It's the most easterly place in Britain, and has a rather nice beach!

(b) It's hard to say, but I do love trifle as a dessert.

(c) I like playing piano, reading (mostly science-fiction), cooking and baking, and sometimes running or bouldering.

(d) The part of the first-year maths course that I like the most is the limits and continuity topic in Maths B - it's unusual to see this in a science course, and I think it helps to build an appreciation of what more formal mathematics is about.

So, what was the point of this question? Like all of your courses this year, the maths course is *hard*. It is worth making some friends who are doing the course, so you can discuss the problem sheets together. If you can't do a question, ask someone else in your cohort if they have done it. If someone else in your cohort asks you how to do a problem, try to give them some help if you have time. You should treat the other Natural Scientists (and Chemical Engineers) as team members - try to work together and help each other through the year!

Writing mathematics

2. Read Gareth Wilkes' document 'A Brief Guide to Mathematical Writing', available at: `https://www.dpmms.cam.ac.uk/~grw46/Writing_Guide.pdf`. [*You can ignore Section 4 for now - but we will study quantifiers in the Maths B course, if you are taking it.*] Hence, make a list of things that you should consider when writing a solution to a mathematics problem.

Solution: Some things we should think about are the following:

- **Rewrite your work.** Often, you won't produce a perfect answer to a problem on a first try. Rewriting your work can help organise your ideas so that they are clearer for the reader (I hasten to add, obviously this is something you can't do in an exam setting though!).
- **Write your work as a linear document.** Avoid having lots of arrows on the page, saying 'see this part', or little boxes with extra additional calculations. Your work should flow in a natural way, particularly for very long calculations!
- **State results you are using, and check that they apply.** If you are quoting a result from the course, say what result it is (for example, saying '*using the standard expansion of* $\sinh(x)$ *around* $x = 0$' or saying '*by Stokes' theorem*'). Also, make sure to check the conditions apply (for example, saying '*L'Hôpital's rule applies here because we have an indeterminate form* $0/0$').

- **Diagrams are exceptionally useful.** A lot of parts of the course we will study involve lots of geometry. Having useful diagrams can help you, help the reader, and help the examiner at the end of the year!
- **Use words, where appropriate.** Mathematics should not just be a list of equations one after the other. It can be *extremely* helpful to yourself and the reader to make small remarks telling them *what you are doing*. When you are proving something is true, you can explain why things are true ('*because* ***x***, ***y*** *are not linearly independent*', '*because* $\sqrt{2}$ *is irrational*', etc) and you can state your assumptions clearly ('*if* A *is an* $n \times n$ *matrix, then* A^T *is an* $n \times n$ *matrix*', etc). In a long calculation, putting notes in the margin can be very useful ('*integrating by parts*', '*making the substitution* $u = x^2$', etc).
- **Format your work sensibly.** Try to separate things using paragraphs and indentation. Don't let everything become one large continuous paragraph! Equations, in particular, should be separated from text by a line break. When trying to prove something, it can also be exceptionally useful to use the format:

  Claim: ...

  Proof: ... . □

  The little box □ means the proof has been completed (you can also write QED if you want to seem more like a 17th century mathematician). This format makes it extremely clear *what* you are trying to prove, what the assumptions are in the proof, and clearly gives space for you to write the proof itself.

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3. Write down your best and most presentable model solution to the following problem:

Let L be a line passing through the origin with gradient k. Let C be a circle centred on $(2, 0)$ with radius 1. Determine the values of k for which L and C intersect at zero points, one point, or two points: (a) using an algebraic method; (b) using a geometric method.

Compare your solution with your supervision partner, and give each other advice and feedback.

Solution: Here is a model solution to this problem.

(a) Algebraic solution. The line L has Cartesian equation $y = kx$, and the circle C has Cartesian equation $(x-2)^2 + y^2 = 1$. To find their intersections algebraically, we solve these equations simultaneously. Substituting $y = kx$ into the equation of the circle, we have:

$$(x-2)^2 + k^2x^2 = 1 \quad \Rightarrow \quad x^2 - 4x + 4 + k^2x^2 = 1$$
$$\Rightarrow \quad (1+k^2)x^2 - 4x + 3 = 0. \qquad (*)$$

To determine the number of solutions to this equation, we consider the discriminant of this quadratic, given by:

$$\begin{aligned} 16 - 12(1+k^2) &= 4 - 12k^2 \\ &= 4(1 - 3k^2). \end{aligned}$$

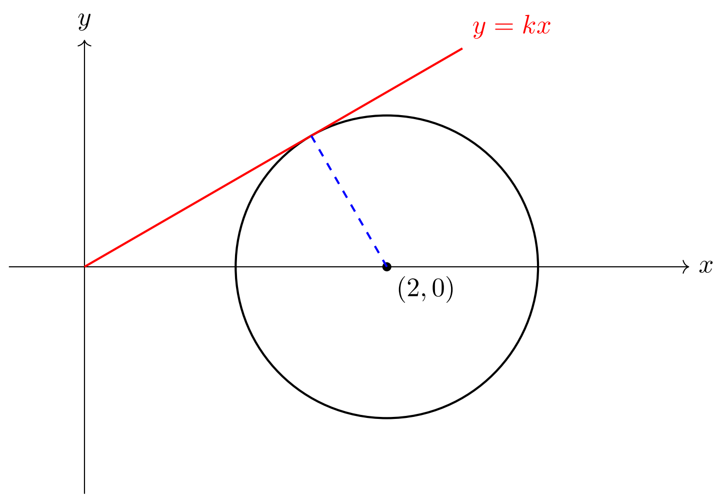
Now, we consider the possible cases:

- Case 1. If the discriminant is negative, there are no solutions to the quadratic equation $(*)$, and hence no intersections. This occurs if $|k| > 1/3$.

- Case 2. If the discriminant is zero, there is exactly one (repeated) solution to the quadratic equation ($*$), and hence one value of x which corresponds to an intersection. Substituting this value into $y = kx$ will give exactly one corresponding value of y, and hence a single intersection. This occurs if $|k| = 1/3$.
- Case 3. If the discriminant is negative, there are two solutions to the quadratic equation ($*$), which each correspond to a single value of y from the equation $y = kx$. This gives exactly two intersections. This occurs if $|k| < 1/3$.

In summary, we have no intersections if $|k| > 1/3$, one intersection if $|k| = 1/3$, and two intersections if $|k| < 1/3$.

(b) Geometric method. To solve this problem geometrically, it will first be useful to produce a diagram. Below, we show the circle and the line $y = kx$ in the case where it is tangent to the circle and $k > 0$ (there is also a case where $k < 0$ and the line is tangent to the circle).



In the case where the line $y = kx$ is tangent to the circle, and $k > 0$, the points $(0, 0)$, $(2, 0)$ and the point of tangency form a right-angled triangle. The length from the origin to the point of tangency is given by Pythagoras' theorem as:

$$\sqrt{2^2 - 1^2} = \sqrt{3}.$$

Hence, using the definition of $\tan(\theta)$ for a right-angled triangle, the angle θ formed by the tangent and the x-axis is given by:

$$\tan(\theta) = \frac{1}{\sqrt{3}}.$$

This implies that the gradient of the line $y = kx$ is $1/\sqrt{3}$. Hence there is precisely one intersection when $|k| = 1/\sqrt{3}$ (the case $k = -1/\sqrt{3}$ follows by the reflectional symmetry in the x-axis). When $|k| < 1/\sqrt{3}$, there are two intersections from the diagram, and when $|k| > 1/\sqrt{3}$ there are no intersections from the diagram.

4. Learn all the letters of the Greek alphabet, and get your supervision partner to test you on them.

Solution: The letters of the Greek alphabet, together with their names, are given below:

- α (uppercase A). This letter is called *alpha*.
- β (uppercase B). This letter is called *beta*.
- γ (uppercase Γ). This letter is called *gamma*.
- δ (uppercase Δ). This letter is called *delta*.
- ϵ (uppercase E). This letter is called *epsilon*.
- ζ (uppercase Z). This letter is called *zeta*.
- η (uppercase H). This letter is called *eta*.
- θ (uppercase Θ). This letter is called *theta*.
- ι (uppercase I). This letter is called *iota*.
- κ (uppercase K). This letter is called *kappa*.
- λ (uppercase Λ). This letter is called *lambda*.
- μ (uppercase M). This letter is called *mu*.
- ν (uppercase N). This letter is called *nu*.
- ξ (uppercase Ξ). This letter is called *xi*.
- o (uppercase O). This letter is called *omicron*.
- π (uppercase Π). This letter is called *pi*.
- ρ (uppercase P). This letter is called *rho*.
- σ (uppercase Σ). This letter is called *sigma*.
- τ (uppercase T). This letter is called *tau*.
- υ (uppercase Υ). This letter is called *upsilon*.
- ϕ (uppercase Φ). This letter is called *phi*.
- χ (uppercase X). This letter is called *chi*.
- ψ (uppercase Ψ). This letter is called *psi*.
- ω (uppercase Ω). This letter is called *omega*.

Basic logic

5. Explain the meaning of the logical symbols $\Rightarrow$, $\Leftarrow$ and $\Leftrightarrow$. [*If you haven't seen them before, look them up online! In general, you should feel free to look up terms you don't understand on an examples sheet.*] Decide which of the following are true:

(a) $x^2 \leq 1 \;\Rightarrow\; x \leq 1$;

(b) $x^2 \leq 1 \;\Leftarrow\; x \leq 1$;

(c) $x^2 \leq 1 \;\Leftrightarrow\; x \leq 1$.

Explain also the meanings of the terms *necessary condition* and *sufficient condition*. Decide which of the following are true:

(d) $|x| = 1$ is sufficient for $x = 1$;

(e) $|x| = 1$ is necessary for $x = 1$;

(f) $|x| = 1$ is necessary and sufficient for $x = 1$.

Solution: Let's start with the definition of the logical symbols:

- The symbol $\Rightarrow$ means 'implies'. If we write $A \Rightarrow B$, then it means that 'if statement A is true, then statement B is true'.
- The symbol $\Leftarrow$ means 'is implied by'. If we write $A \Leftarrow B$, then it means that 'if statement B is true, then statement A is true'.
- The symbol $\Leftrightarrow$ means 'implies and is implied by'. If we write $A \Leftrightarrow B$, then it means that 'statement A is true if and only if statement B is true'. Importantly, the logical implication needs to go both ways!

Now, let's examine the given statements:

(a) The first statement is telling us that if $x^2 \leq 1$, then $x \leq 1$. This is true: the quadratic $x^2 - 1$ is less than or equal to zero in the region $-1 \leq x \leq 1$, which implies $x \leq 1$ as required.

(b) The second statement is telling us that if $x \leq 1$, then $x^2 \leq 1$. This isn't true: if $x = -2$, then $x^2 = 4$, which is not less than or equal to one.

(c) The third statement is telling us that $x \leq 1$ if and only if $x^2 \leq 1$, i.e. these statements are logically equivalent. We have just seen that this isn't the case!

Let A, B be two logical statements. We say that A is *necessary* for B if B implies A, that is, B cannot be true without A being true. In logical symbols, we write $B \to A$. On the other hand, we say that A is *sufficient* for B if A implies B, that is, provided that A is true, it follows that B is also true. In logical symbols, we write $A \to B$.

(d) For the first statement, it is not sufficient that $|x| = 1$ for $x = 1$. That is because given $|x| = 1$, we could have $x = -1$, which does not imply $x = 1$.

(e) For the second statement, it is true that $|x| = 1$ is necessary for $x = 1$. It cannot be the case that $x = 1$ without $|x| = 1$ being true too.

(f) Since $|x| = 1$ is not sufficient for $x = 1$, we see that $|x| = 1$ is not necessary *and* sufficient for $x = 1$.

6. What is the error in the following argument?

> *'Suppose we wish to solve the equation $x-1=2$. We begin by squaring both sides, to obtain $(x-1)^2=4$. Expanding the left hand side we have:*
>
> $$x^2-2x+1=4 \qquad \Leftrightarrow \qquad x^2-2x-3=0 \qquad \Leftrightarrow \qquad (x+1)(x-3)=0.$$
>
> *Since one of these factors must be zero, it follows that there are two solutions to the equation, $x=-1$ or $x=3$.'*

➼ **Solution:** The problem lies in the very first step, where the implication only goes in one direction! Writing the whole argument in logical symbols, we have:

$$\begin{aligned} x-1=2 \quad &\Rightarrow \quad (x-1)^2=4 \\ &\Leftrightarrow \quad x^2-2x+1=4 \\ &\Leftrightarrow \quad x^2-2x-3=0 \\ &\Leftrightarrow \quad (x+1)(x-3)=0 \\ &\Leftrightarrow \quad x=-1, \text{ or } x=3. \end{aligned}$$

The logical symbol in the first line does *not* reverse. This shows that *if* the equation holds, *then* x must take one of the values in the final line. We have *not* shown that *if* x takes one of the values in the final line, *then* the equation holds.

Importantly, this shows us that when we are solving an equation, if we make a non-reversible step in our argument, we need to check that our solutions satisfy the original equation.

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7. What is meant by *proof by contradiction*? Prove by contradiction that $\sqrt{2}$ and $\sqrt{3}$ are irrational numbers.

➼ **Solution:** A 'proof by contradiction' is a sequence of logical steps $A \Rightarrow B \Rightarrow C \Rightarrow \ldots \Rightarrow Z$, where Z is *false*. This implies that A must be false too!

Let's use proof by contradiction to show that $\sqrt{2}$ is irrational. We suppose that $\sqrt{2}$ is rational, so that we can write $\sqrt{2}=p/q$, for some integers p, q. We may assume that the fraction p/q is written in its lowest terms, so that p, q are not both even. Then squaring both sides, we obtain:

$$2=p^2/q^2 \qquad \Rightarrow \qquad 2q^2=p^2.$$

Now, the left hand side is even, so p^2 must be even. Thus, p must be even, since the square of any odd number is odd. Thus, we can write $p=2r$ for some integer r. Substituting into the above equation, we have:

$$2q^2=4r^2 \qquad \Rightarrow \qquad q^2=2r^2.$$

But then, by the same argument, q must be even. This contradicts our original assumption that p/q was written in its lowest terms. Hence $\sqrt{2}$ is irrational.

We can perform a similar proof for $\sqrt{3}$. We suppose that $\sqrt{3}$ is rational, so that we can write $\sqrt{3} = p/q$, for some integers p, q. We again assume that the fraction p/q is written in its lowest terms, so that p, q are not both multiples of three. Then squaring both sides, we obtain:

$$3 = p^2/q^2 \qquad \Rightarrow \qquad 3q^2 = p^2.$$

Now, the left hand side is a multiple of three, so p^2 must be a multiple of three. Thus, p must be a multiple of three. Then, we can write $p = 3r$ for some integer r. Substituting into the above equation, we have:

$$3q^2 = 9r^2 \qquad \Rightarrow \qquad q^2 = 3r^2.$$

But then, by the same argument, q must be a multiple of three. This contradicts our original assumption that p/q was written in its lowest terms. Hence $\sqrt{3}$ is irrational.

Sets and functions

8. State what is meant by the sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$. Decide which of the following statements are true:

(a) $\pi \in \mathbb{Q}$, (b) $3 \notin \mathbb{R}$, (c) $\mathbb{Z} \subseteq \mathbb{Q}$, (d) $\mathbb{Q} \supset \mathbb{C}$.

Solution: The definitions of the given sets are the following:

- The set $\mathbb{Z}$ is the set of all integers, positive and negative:

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}.$$

- The set $\mathbb{Q}$ is the set of all rational numbers, i.e. numbers which are the ratio of two integers, i.e. numbers which are expressible as fractions.
- The set $\mathbb{R}$ is the set of all real numbers, i.e. it includes both fractions, and numbers with an infinite, non-repeating decimal expansion.
- The set $\mathbb{C}$ is the set of all complex numbers, i.e. numbers of the form $x + iy$, where x, y are real numbers. We will study complex numbers a lot more later in Michaelmas term!

The symbol $\in$ means 'is a member of the set', and the symbol $\notin$ means 'is not a member of the set'. The symbol $\subseteq$ means 'is a subset of, or is equal to, the set'. The symbol $\supset$ means 'is a superset of'. Hence:

(a) $\pi \in \mathbb{Q}$ is false, because π is not rational - it can't be written as a fraction!

(b) $3 \notin \mathbb{R}$ is false, because 3 is a real number.

(c) $\mathbb{Z} \subseteq \mathbb{Q}$ is true, because every integer is also a rational number (we can write the integer p as a fraction $p/1$, if you are worried!).

(d) $\mathbb{Q} \supset \mathbb{C}$ is false, because not every complex number is a ratio of integers. For example, i is not a ratio of two integers!

9. State what is meant by the sets $[a, b]$, (a, b) and $(a, b]$, where a, b are real numbers. Decide which of the following statements are true:

(a) $1 \in [0, 1)$, (b) $3 \notin (3, 4)$, (c) $[2, 3] \subset (2, 5]$, (d) $(-1, 0) \subset [-1, 0]$.

Solution: These sets are *intervals* of real numbers. They have the following definitions:

- $[a, b]$ is the set of all real numbers x in the range $a \leq x \leq b$, so it *includes* the endpoints.
- (a, b) is the set of all real numbers x in the range $a < x < b$, so it *excludes* the endpoints.
- $(a, b]$ is the set of all real numbers x in the range $a < x \leq b$, so it excludes the endpoint a, but includes the endpoint b.

Out of the given statements, we have:

(a) $1 \in [0, 1)$ is false, because the curved bracket means that the endpoint of the interval is excluded.

(b) $3 \notin (3, 4)$ is true, because the endpoint 3 is excluded from the interval in this case.

(c) $[2, 3] \subset (2, 5]$ is false, because 2 is in $[2, 3]$ but not in $(2, 5]$.

(d) $(-1, 0) \subset [-1, 0]$ is true, because every real number satisfying $-1 < x < 0$ also satisfies $-1 \leq x \leq 0$.

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10. A function is a mapping from a set to another set. We write $f : A \to B$ to denote the function (or just f when the sets are implied), and we write $f(x)$ for the value of the function at the point $x \in A$. Which of the following define functions, and why?

(a) $f : [0, \infty) \to \mathbb{R}$, given by $f(x) = \sqrt{x}$;

(b) $f : \mathbb{Z} \to \mathbb{Z}$, given by $f(x) = \frac{1}{2}x$;

(c) $f : \mathbb{R} \to \mathbb{R}$, where $f(x)$ is implicitly defined by $2^{f(x)} = x$;

(d) $f : [0, \infty) \to \mathbb{R}$, where $f(x)$ is implicitly defined by $f(x)^2 = x$.

Solution:

(a) This defines a function, because for every value of $x \in [0, \infty)$, the function uniquely gives a value $\sqrt{x} \in \mathbb{R}$. Note that we don't hit *every* real number with this function, because the square root is always positive - that doesn't matter though, as long as the result is a member of the set of real numbers.

(b) This does *not* define a function, because if $x = 1$, then $f(1) = 1/2$ is not in the specified range of the function, which is supposed to be the integers. We could fix this by saying the function is a mapping $f : \mathbb{Z} \to \mathbb{Q}$ or $f : \mathbb{Z} \to \mathbb{R}$.

(c) This does not define a function, because for $x = -1$ for example, there is no real number $f(x)$ such that $2^{f(x)} = -1$. This is because exponentiation of a positive real number always gives a positive result. This could be fixed by considering the function to be a mapping $f : (0, \infty) \to \mathbb{R}$; the function that this defines is then just $f(x) = \log_2(x)$.

(d) This does not define a function, because it is multi-valued. For example, take $x = 1$. Then $f(x)^2 = 1$ has two roots, $f(x) = \pm 1$. We don't know which one to pick! If we instead said the function was a mapping $f : [0, \infty) \to [0, \infty)$, we would know to pick the positive root, which then makes the function equivalent to the one in part (a).

Sequences and series

11. Prove the following results using (i) induction; (ii) a direct argument:

(a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, (b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, (c) $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

[*Hint: for the direct arguments, consider summing* $(k+1)^r - k^r$ *for an appropriate integer* r *in each case; note this sum telescopes.*] Is it more useful to prove a result by induction, or by a direct argument? Why?

➾ **Solution:** (a) (i) We first prove the result using induction. The base case, $n = 1$, is obvious since:

$$\sum_{k=1}^{1} k = 1, \qquad \frac{1(1+1)}{2} = 1.$$

Assume the result is true for $n = m$, and consider the case $n = m + 1$. We have:

$$\begin{aligned}
\sum_{k=1}^{m+1} k &= (m+1) + \sum_{k=1}^{m} k \\
&= (m+1) + \frac{m(m+1)}{2} && \text{(induction hypothesis)} \\
&= (m+1)\left(1 + \frac{m}{2}\right) \\
&= \frac{(m+1)(m+2)}{2}.
\end{aligned}$$

Hence, the result is true for $n = m + 1$ if the induction hypothesis holds. So we're done by the principle of mathematical induction.

(a) (ii) To prove the result directly, note the identity:

$$(k+1)^2 - k^2 = 2k + 1.$$

Summing both sides from $k = 1$ to $k = n$, we have:

$$(1+1)^2 - 1^2 + (2+1)^2 - 2^2 + (3+1)^2 - 3^2 + \cdots + (n+1)^2 - n^2 = 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1.$$

Simplifying both sides, noticing that the left hand side forms a telescoping series, we have:

$$(n+1)^2 - 1 = 2\sum_{k=1}^{n} k + n.$$

Rearranging, we have:

$$\sum_{k=1}^{n} k = \frac{(n+1)^2 - n - 1}{2} = \frac{n(n+1)}{2},$$

as required.

(b) (i) We first prove the result using induction. The base case, $n = 1$, is obvious since:

$$\sum_{k=1}^{1} k^2 = 1, \qquad \frac{1(1+1)(2+1)}{6} = 1.$$

Assume the result is true for $n = m$, and consider the case $n = m + 1$. We have:

$$\begin{aligned}
\sum_{k=1}^{m+1} k^2 &= (m+1)^2 + \sum_{k=1}^{m} k^2 \\
&= (m+1)^2 + \frac{m(m+1)(2m+1)}{6} && \text{(induction hypothesis)} \\
&= (m+1)\left(m+1+\frac{m(2m+1)}{6}\right) \\
&= \frac{(m+1)(6m+6+2m^2+m)}{6} \\
&= \frac{(m+1)(2m^2+7m+6)}{6} \\
&= \frac{(m+1)(m+2)(2m+3)}{6}.
\end{aligned}$$

Hence, the result is true for $n = m + 1$ if the induction hypothesis holds. So we're done by the principle of mathematical induction.

(b) (ii) To prove the result directly, note the identity:

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1$$

Summing both sides from $k = 1$ to $k = n$, we have:

$$(1+1)^3 - 1^3 + (2+1)^3 - 2^3 + (3+1)^3 - 3^3 + \cdots + (n+1)^3 - n^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1.$$

Simplifying both sides, noticing that the left hand side forms a telescoping series, we have:

$$(n+1)^3 - 1 = 3\sum_{k=1}^{n} k^2 + \frac{3n(n+1)}{2} + n.$$

Rearranging, we have:

$$\sum_{k=1}^{n} k^2 = \frac{(n+1)^3 - 1 - n - 3n(n+1)/2}{3} = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6},$$

as required.

(c) (i) We first prove the result using induction. The base case, $n = 1$, is obvious since:

$$\sum_{k=1}^{1} k^3 = 1, \qquad \frac{1^2(1+1)^2}{4} = 1.$$

Assume the result is true for $n = m$, and consider the case $n = m + 1$. We have:

$$\begin{aligned}
\sum_{k=1}^{m+1} k^3 &= (m+1)^3 + \sum_{k=1}^{m} k^3 \\
&= (m+1)^3 + \frac{m^2(m+1)^2}{4} && \text{(induction hypothesis)} \\
&= (m+1)^2 \left(m + 1 + \frac{m^2}{4} \right) \\
&= \frac{(m+1)^2(m^2+4m+4)}{4} \\
&= \frac{(m+1)^2(m+2)^2}{4}.
\end{aligned}$$

Hence, the result is true for $n = m + 1$ if the induction hypothesis holds. So we're done by the principle of mathematical induction.

(c) (ii) To prove the result directly, note the identity:

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$$

Summing both sides from $k = 1$ to $k = n$, we have:

$$(1+1)^4 - 1^4 + (2+1)^4 - 2^4 + (3+1)^4 - 3^4 + \cdots + (n+1)^4 - n^4 = 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1.$$

Simplifying both sides, noticing that the left hand side forms a telescoping series, we have:

$$(n+1)^4 - 1 = 4\sum_{k=1}^{n} k^3 + n(n+1)(2n+1) + 2n(n+1) + n$$

Rearranging, we have:

$$\sum_{k=1}^{n} k^3 = \frac{(n+1)^4 - n(n+1)(2n+1) - 2n(n+1) - n}{4} = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2(n+1)^2}{4},$$

as required.

Often, it is much more useful to prove a result by a direct argument than by induction. This is because induction requires us to know that the result is true in advance! The proof is often simpler through induction, but usually less enlightening about *why* the result is true in the first place.

❈ **Comments:** You might have spotted a very interesting curiosity: the sum of the first n cubes is equal to the sum of the first n natural numbers squared. There is a beautiful visual proof of this fact, which is demonstrated very nicely in this video, if you're interested: https://www.youtube.com/watch?v=NxOcT_VKQR0.

12. What is meant by an *arithmetic sequence* with first term a and common difference d? Prove that the sum of the first n terms of an arithmetic sequence is:

$$S_n = \frac{1}{2}n\left(2a + (n-1)d\right).$$

Hence, find the sum of the series $2, 5, 8, 11, ..., 32$.

➼ **Solution:** The arithmetic sequence with first term a and common difference d is:

$$a, \qquad a+d, \qquad a+2d, \qquad a+3d, \qquad ...$$

The sum of the first n terms is given by:

$$\sum_{k=0}^{n-1}(a+kd) = na + d\sum_{k=0}^{n-1}k = na + \frac{d(n-1)n}{2} = \frac{1}{2}n\left(2a+(n-1)d\right).$$

The given sequence $2, 5, 8, 11, ..., 32$ is an arithmetic sequence with first term $a = 2$ and common difference $d = 3$. The final term is the $(32-2)/3+1 = 11$th term in the series. Hence the sum is:

$$\frac{11}{2}\left(4 + 10\cdot 3\right) = \frac{11\cdot 34}{2} = 11\cdot 17 = 187.$$

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13. What is meant by a *geometric sequence* with first term a and common ratio r? Prove that the sum of the first n terms of a geometric sequence is:

$$S_n = \frac{a(1-r^n)}{1-r}.$$

What happens if $r = 1$? What is the behaviour of this sum in the limit as $n \to \infty$? Hence, find the sum of the infinite series $2, 2/3, 2/9, 2/27, ...$

➼ **Solution:** The geometric sequence with first term a and common ratio r is:

$$a, \qquad ar, \qquad ar^2, \qquad ar^3, \qquad ar^4, \qquad ...$$

The sum of the first n terms is:

$$S_n = a + ar + ... + ar^{n-1}.$$

Multiplying by $1-r$, we have:

$$S_n(1-r) = (a + ar + ... + ar^{n-1})(1-r) = a + ar + ... + ar^{n-1} - ar - ar^2 - ... - ar^n = a - ar^n.$$

Hence the sum is indeed given by:

$$S_n = \frac{a(1-r^n)}{1-r},$$

as required. Observe that this formula *only* applies if $r \neq 1$; in the case where $r = 1$, the sum to n terms is just na (it's just $a + a + ... + a$, but n times). Whenever we have a situation where we are dividing by zero in maths in some special case, we should return to the first place where we tried to divided by zero and see what happens if we are in that special case.

If $|r| < 1$, we have $r^n \to 0$ as $n \to \infty$. Hence if $|r| < 1$, we can sum a geometric series to infinity, giving the result:

$$S_\infty = \frac{a}{1-r}.$$

In the case of the series $2, 2/3, 2/9, 2/27, ...$ the first term is $a = 2$ and the common ratio is $r = 1/3$, so the sum to infinity is:

$$S_\infty = \frac{2}{1-1/3} = \frac{2}{2/3} = 3.$$

14. Using the formula for the sum of an infinite geometric series, find a formula for the sum of the infinite series:

$$\sum_{k=1}^{\infty} kr^k,$$

where $|r| < 1$. [*Hint: differentiation!*] Hence determine:

$$\frac{2}{3} + 2\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^3 + 4\left(\frac{2}{3}\right)^4 + \cdots .$$

➾ **Solution:** In the previous question, we proved that:

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}.$$

Multiplying both sides of this formula by r, we have:

$$\sum_{k=1}^{\infty} ar^k = \frac{ar}{1-r}.$$

Differentiating both sides of this formula with respect to r, we have:

$$\sum_{k=1}^{\infty} akr^{k-1} = \frac{d}{dr}\left(\frac{ar}{1-r}\right) = \frac{a(1-r)+ar}{(1-r)^2} = \frac{a}{(1-r)^2}.$$

To finish, we set $a = 1$ and multiplying through by r, giving:

$$\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}.$$

The given series' value is:

$$\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k = \frac{(2/3)}{(1-2/3)^2} = \frac{2/3}{1/9} = 6.$$

Trigonometric functions

15. Define the trigonometric functions $\sin(x), \cos(x), \tan(x)$ in terms of the side lengths of an appropriate right-angled triangle. Define also the reciprocal trigonometric functions $\operatorname{cosec}(x), \sec(x), \cot(x)$. Hence, prove each of the following trigonometric identities:

(a) *The Pythagorean identities:*

$$\sin^2(x) + \cos^2(x) = 1, \qquad \tan^2(x) + 1 = \sec^2(x), \qquad \cot^2(x) + 1 = \operatorname{cosec}^2(x).$$

(b) *The compound angle formulae:*

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \sin(y)\cos(x), \qquad \cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y),$$

$$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}.$$

(c) *The double angle formulae:*

$$\sin(2x) = 2\sin(x)\cos(x), \qquad \cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x).$$

(d) *The power reduction formulae:*

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x)).$$

(e) *The product to sum formulae:*

$$\sin(x)\sin(y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right), \qquad \sin(x)\cos(y) = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right),$$

$$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right).$$

(f) *The sum to product formulae:*

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right), \qquad \cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),$$

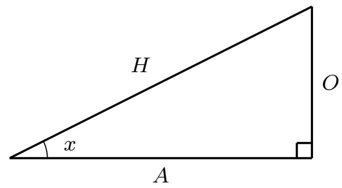
$$\cos(x) - \cos(y) = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

Learn all of these identities off by heart, and get your supervision partner to test you on them.

➻ **Solution:** There are lots of ways to define the trigonometric functions $\sin(x), \cos(x), \tan(x)$. For example, these functions can be defined in terms of complex exponentials, Taylor series, or as solutions of certain differential equations (we shall study all of these options later in the course). Here, however, we will use the most elementary definition, in terms of right-angled triangles.

Given a right-angled triangle with sides O, A, H, and one given angle x, where H is the length of the hypotenuse and O is the length of the sides opposite the angle x, we shall define the trigonometric functions via:

$$\sin(x) := \frac{O}{H}, \qquad \cos(x) := \frac{A}{H}, \qquad \tan(x) := \frac{O}{A} = \frac{O}{H} \cdot \frac{H}{A} = \frac{\sin(x)}{\cos(x)}.$$



The *reciprocal trigonometric functions* are then defined as:

$$\operatorname{cosec}(x) := \frac{1}{\sin(x)}, \qquad \sec(x) := \frac{1}{\cos(x)}, \qquad \cot(x) := \frac{1}{\tan(x)}.$$

Notice that this definition *only* works for x in the range $x \in [0, \pi/2)$, because we need it to be an acute angle. We extend the definition by requiring the standard periodicity properties $\sin(x) = \sin(\pi - x)$ (extending to $[0, \pi]$), $\sin(-x) = -\sin(x)$ (further extending to $[-\pi, \pi]$), $\sin(x) = \sin(x + 2\pi)$ (extending to all real numbers), and $\cos(x) = -\cos(\pi - x)$ (extending to $[0, \pi]$), $\cos(x) = \cos(-x)$ (extending to $[-\pi, \pi]$, $\cos(x) = \cos(x + 2\pi)$ (extending to all real numbers), which allow all of these functions to be defined for all real numbers x.

Notice also the *cofunction identities* which follow immediately from the right-angled triangle definitions:

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x), \qquad \cos\left(\frac{\pi}{2} - x\right) = \sin(x),$$

because the remaining angle in a right-angled triangle, with angle x, is $\pi/2 - x$.

These definitions allow us to prove each of the given identities:

(a) *The Pythagorean identities.* These identities are called the Pythagorean identities because they come from Pythagoras' theorem for the right-angled triangle:

$$O^2 + A^2 = H^2.$$

Dividing by H^2, we have:

$$\left(\frac{O}{H}\right)^2 + \left(\frac{A}{H}\right)^2 = 1 \qquad \Leftrightarrow \qquad \sin^2(x) + \cos^2(x) = 1.$$

Next, we divide this identity by $\cos^2(x)$ to get:

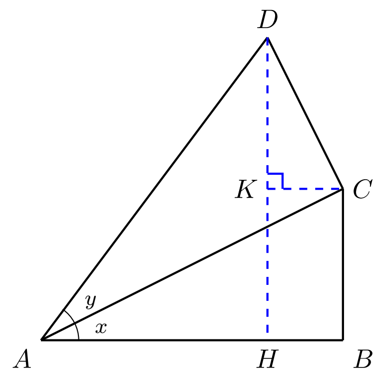
$$\frac{\sin^2(x)}{\cos^2(x)} + 1 = \frac{1}{\cos^2(x)} \qquad \Leftrightarrow \qquad \tan^2(x) + 1 = \sec^2(x).$$

On the other hand, if we had instead divided by $\sin^2(x)$, we would have gotten:

$$1 + \frac{\cos^2(x)}{\sin^2(x)} = \frac{1}{\sin^2(x)} \qquad \Leftrightarrow \qquad 1 + \cot^2(x) + \operatorname{cosec}^2(x),$$

which completes our derivation of the three Pythagorean identities.

(b) *The compound angle formulae.* Consider the following diagram, involving two right-angled triangles ABC and ACD stacked on top of one another. One has an acute angle x at the vertex A, and one has an acute angle y at the vertex A (note that we assume that $x + y$ is acute).



Observe that x and the angle KCA are alternating, so KCA is equal to x. This implies that the triangles DKC and ABC are similar triangles.

Now by the definition of sine, we have (writing AB for the length of the line segment from A to B, etc):

$$\begin{aligned}
\sin(x+y) &= \frac{DH}{AD} \\
&= \frac{DK}{AD} + \frac{KH}{AD} && \text{(since } DH = DK + KH\text{)} \\
&= \frac{DK}{AD} + \frac{CB}{AD} && \text{(since } KH = CB\text{)} \\
&= \frac{DK}{DC} \cdot \frac{DC}{AD} + \frac{CB}{AC} \cdot \frac{AC}{AD} \\
&= \cos(x)\sin(y) + \sin(x)\cos(y) && \text{(definitions of sine, cosine and using similar triangles)}
\end{aligned}$$

This establishes the formula in the case where $x + y, x, y$ are acute; the formula is readily extended to all real values of the angle via the various properties we outlined above.

In particular, this implies we can replace $y \mapsto -y$, which gives:

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y),$$

which gives the full formula $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$. We can also replace $x \mapsto \pi/2 - x$ in the formula giving:

$$\sin\left(\frac{\pi}{2} - x \pm y\right) = \sin\left(\frac{\pi}{2} - x\right)\cos(y) \pm \cos\left(\frac{\pi}{2} - x\right)\sin(y).$$

Using the cofunction identities, this implies:

$$\cos(x \mp y) = \cos(x)\cos(y) \pm \sin(x)\sin(y),$$

as required.

Finally, dividing the formula for $\sin(x \pm y)$ by the formula for $\cos(x \pm y)$, we obtain the tangent compound angle formula:

$$\begin{aligned}\tan(x \pm y) &= \frac{\sin(x \pm y)}{\cos(x \pm y)} \\ &= \frac{\sin(x)\cos(y) \pm \cos(x)\sin(y)}{\cos(x)\cos(y) \mp \sin(x)\sin(y)} \\ &= \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)} && \text{(divide numerator and denominator by } \cos(x)\cos(y)\text{)}\end{aligned}$$

(c) *The double angle formulae.* These follow immediately from the compound angle formulae by taking $x = y$. We have:

$$\sin(2x) = \sin(x + x) = \sin(x)\cos(x) + \cos(x)\sin(x) = 2\sin(x)\cos(x),$$

and similarly:

$$\cos(2x) = \cos(x + x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2(x) - \sin^2(x).$$

To get the alternative forms of the cosine double angle formulae, we use one of the Pythagorean identities in the following forms: $\cos^2(x) = 1 - \sin^2(x)$ and $\sin^2(x) = 1 - \cos^2(x)$. This gives:

$$\cos(2x) = \cos^2(x) - \sin^2(x) = (1 - \sin^2(x)) - \sin^2(x) = 1 - 2\sin^2(x),$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2(x) - (1 - \cos^2(x)) = 2\cos^2(x) - 1.$$

(d) *The power reduction formulae.* These follow immediately from the double angle formulae for $\cos(2x)$, simply by rearrangement:

$$\cos(2x) = 1 - 2\sin^2(x) \qquad \Rightarrow \qquad \sin^2(x) = \frac{1}{2}(1 - \cos(2x)),$$

$$\cos(2x) = 2\cos^2(x) - 1 \qquad \Rightarrow \qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x)).$$

(e) *The product to sum formulae.* We can obtain these formulae by taking sums and differences of the compound angle formuale for $\sin(x+y), \sin(x-y)$ and $\cos(x+y), \cos(x-y)$. We have:

$$\sin(x+y) + \sin(x-y) = \sin(x)\cos(y) + \cos(x)\sin(y) + \sin(x)\cos(y) - \cos(x)\sin(y) = 2\sin(x)\cos(y) \tag{†}$$

and similarly:

$$\cos(x+y) + \cos(x-y) = \cos(x)\cos(y) - \sin(x)\sin(y) + \cos(x)\cos(y) + \sin(x)\sin(y) = 2\cos(x)\cos(y),$$

$$\cos(x+y) - \cos(x-y) = \cos(x)\cos(y) - \sin(x)\sin(y) - \cos(x)\cos(y) - \sin(x)\sin(y) = -2\sin(x)\sin(y).$$

Rearrangement in each case produces the required identities.

(f) *The sum to product formulae.* These formulae are just the product to sum formulae in reverse. If we take $x + y = A$ and $x - y = B$ in the above, we have $x = (A+B)/2$ and $y = (A-B)/2$. This gives:

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right),$$

when inserted into (†), for example. The other identities follow similarly by substitution into the remaining cosine formulae.

17. Prove the trigonometric inequalities:

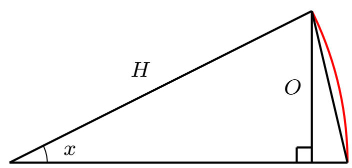
    (a) $|\sin(x)| \leq |x|$, for all real x;

    (b) $\cos(x) \geq 1 - x^2/2$, for all real x.

➻ **Solution:** (a) Since the hypotenuse of a right-angled triangle can never be shorter than its opposite side, we immediately have that $\sin(x) \leq 1$ for acute angles $0 \leq x < \pi/2$. By the extension to negative angles, this gives $-1 \leq \sin(x) \leq 1$ for $-\pi/2 < x < \pi/2$, which then implies by the periodicity properties that $|\sin(x)| \leq 1$ for all x.

For $|x| \geq \pi/2$, since $\pi/2 > 1$, we immediately have $|\sin(x)| \leq |x|$.

For $|x| \leq \pi/2$, we can use the right-angled triangle to prove that $|\sin(x)| \leq |x|$. We consider constructing a circular arc with radius equal to the length of the hypotenuse of the right-angled triangle, as shown below. We also insert an additional chord, as shown in the figure.



From this figure, we see that the area of the triangle contained in the circular sector is:

$$\frac{1}{2}OH = \frac{1}{2}\sin(x)H^2.$$

This is evidently less than the area of the circular sector, which is:

$$\frac{1}{2}xH^2.$$

The inequality $\sin(x) \leq x$ follows for acute x, which implies $-x \leq -\sin(x) = \sin(-x)$ for x in the range $-\pi/2 < x \leq 0$. So we're done!

(b) For the second inequality, we have:

$$\cos(x) = 1 - 2\sin^2\left(\frac{x}{2}\right) \geq 1 - 2\left(\frac{x}{2}\right)^2 = 1 - \frac{x^2}{2},$$

using the result from part (a).

The boring stuff: exams and coursework

18. Look up the format of the first-year maths exams. How much is each exam worth? How long does each exam last? How many questions do you have to answer, and you should you split your time in the exams?

➡ **Solution:** There are two first-year maths exams, which are each worth $6/13 \approx 46\%$ of the your final grade in maths (accounting for a total of $12/13 \approx 92\%$ of your final grade). Each exam lasts 3 hours. Each exam has the following structure:

- **Section A.** The exam begins with 10 short questions, each worth 2 marks, totalling 20 marks. These questions are designed to be answered quickly and credit is always awarded to the correct answer, even if full reasoning is not given. This section is compulsory.

- **Section B.** The second part of the exam contains 10 long questions, which are each worth 20 marks. You must choose **5 questions** out of the **10 questions** to answer. These questions usually award marks for explaining yourself well, in addition to getting the right answers. Out of the 10 questions, 2 questions are only accessible to students who have done the Maths B course, but these will be clearly marked to avoid confusion.

In total, the exam is out of 120 marks, with 20 marks coming from Section A and 5 questions contributing 20 marks each in Section B. This means that you should spend at most 30 minutes on Section A and at most 30 minutes on each Section B question.

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19. Look up the first-year maths coursework. How much is it worth?

➡ **Solution:** The first-year maths coursework is a scientific computing course that is run by the Cavendish Laboratory (**not** by the maths department). It involves 8 practical sessions, each 90 minutes long, 4 of which you attend in Michaelmas and 4 of which you attend in Lent. Each practical involves submission of a small piece of computing work which contributes to your final grade. The total contribution is $1/13 \approx 6\%$ of your grade, so that each piece of work is worth $1/104 \approx 1\%$ of your final grade.