8.2 Vector Loops–Constraints

MOTIVATION: *Figure 8.27 shows two rigid links connected by the pin at point c. The system*

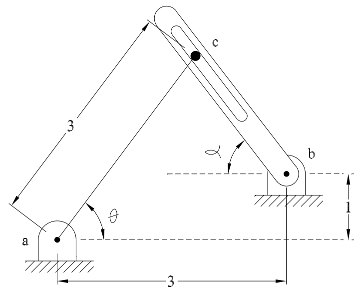

Figure 8.27: A System with Two Rotating Links.

has one degree of freedom. Suppose we know that link ac is rotating a known constant speed and we want to know if the pin at c will break. To find out, we would apply Newton's laws to each link and compute the forces applied to the pin. To apply Newton's laws, we will eventually have to compute the accelerations of points on each link. To do this, we will need to know something about how quickly each link is rotating. For link ac this is not a problem since its rotation rate is known, however how can we determine information about the angle α? Since the links have one degree of freedom, it ought to be possible, but how? For this simple system the reader is possibly thinking about the law of sines or cosines studied in geometry. Good, they can be used if desired. Unfortunately we often run across systems in which our geometric intuition fails. For example, suppose we want to determine the position of body 5 (see figure 8.28) given the sizes of links and

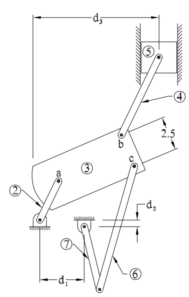

Figure 8.28: A Diagram of a Variable Stoke Engine.

angular positions of links 2 and 7? Such a problem is not trivial and is a good example of why we need to develop a systematic method for relating positions.

Many of the problems in this lesson, and those that follow, can be solved with some intuition and simple geometric formulae. We will, however, always use a systematic solution technique so experience and skills are gained before we have to conquer really tough geometric challenges in the real world.

Lesson Objectives

In this lesson we try to establish a simple method for writing relative position equations introduced in the lesson 8.1. In this lesson the reader will learn to do the following:

1. Understand why the technique is called "vector loops" and know how vector loops can be used.

2. Write valid and useful vector loop equations.

3. Count the number of unknowns present in a vector loop equation and determine when more loop equations can/should be written.

Application

We have experienced in previous lessons that it is sometimes convenient to use more position variables than absolutely necessary. In the motivation section of this lesson we saw another example of such a problem. When we do use more position variables than the degree of freedom, or if we use variables that are kinematically dependent, we soon discover that these variables must be related to each other.

When we have a system that contains ropes and pulleys we know that we can express these motion relationships by writing an equation for the length of the ropes, see lesson 4.1. When the system contains a series of interconnected bodies, it is often convenient to express vector loop equations. The relative positions discussed in lesson 8.1 can also be used (many experts do). However, loop equations are often easier for students to understand and use.

The reason the method is called "vector loops" is that we start from any point, fixed or moving, and write relative positions from point to point until we arrive back at the starting point. We form a loop. For the loop to be really useful we should attempt to follow a small set of rules.

Four rules to follow

1. When writing vector loops only use special points. If we use a nonspecial point, we usually introduce variables that are irrelevant to the problem. If we get extra variables it becomes more difficult to obtain a solution to our equations. Therefore, keep the loops as simple as possible. On the flip side, we will not always need to use all the special points in the problem.

2. Never backtrack. For example if we move from a to b to c do not backtrack from c to b to a. Basically if we backtrack we simply are adding quantities in then pulling them back out again; all backtracking does is cancel terms. If we discover an urge to backtrack consider starting the loop at a different point or consider using more than one loop.

3. Step in directions that have minimal unknowns. Often we will have multiple choices in how to get from a to c. For example, we might go directly from a to c, or we can go from a to b to c. The option we choose should generally be the one that includes the smaller number of variables.

4. Step in directions that bring in the variables of interest. Before writing a vector loop, we should have a basic idea of what variables we are trying to relate. When we are choosing a step in the loop, try to step in a direction that brings one (or more) of the desired variables into the equation.

Concept	**Begin 8.2.1**	Check

Consider the two rotating link system shown in figure 8.27. With standard assumptions like the links are rigid, the two links have one degree of freedom. Since there is one degree of freedom, given one motion variable, such as θ as a function of time, we can determine all other motion variables. Suppose we need to determine the angle α given θ. As usual, take $\hat{\imath}$ to the right and $\hat{\jmath}$ toward the top of the page.

Since the system consists of links, we imagine that one or more vector loops will be useful. There are three special points in this system, a, b and c. Arbitrarily starting at point a, we can step to c or b, which way should we go? Going either way introduces no unknown variables (remember θ is known); so we will move to c. Now standing at c where now? Well we cannot backtrack so the only way to go is to b. From b we go directly back to a ending the loop.

Mathematically, the loop is written as:

$$ {}^{a}\vec{r}^{\,c} + {}^{c}\vec{r}^{\,b} + {}^{b}\vec{r}^{\,a} = 0 \tag{8.7}$$

Putting details in we have:

$$3\left(\cos\theta\hat{\imath}+\sin\theta\hat{\jmath}\right)+L\left(\cos\alpha\hat{\imath}-\sin\alpha\hat{\jmath}\right)-3\hat{\imath}-1\hat{\jmath}=0$$

In this equation, we have three variables (θ, α, L), two are unknown. We can equate the $\hat{\imath}$ and $\hat{\jmath}$ components to get two scalar equations and solve for both unknowns.

Concept	**End 8.2.1**	Check

To prepare for future material, consider drawing two new vectors onto the links as shown in Figure 8.29. The important thing to understand about these vectors is that they move when the

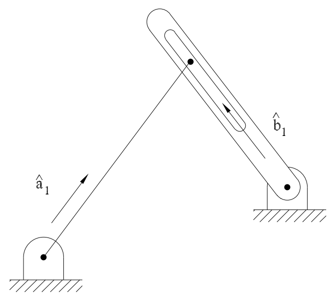

Figure 8.29: Two Unit Vectors are Painted onto the Links.

links move. They are rotating vectors, no big deal. Now, using these vectors, it is easy to write equation 8.7 as:

$$3\hat{a}_1-L\hat{b}_1-3\hat{\imath}-\hat{\jmath}=0$$

Although this may not look useful at this point, it will come in handy when we compute derivatives.

Often linkages contain slots and grooves in which pegs slide. When such a situation occurs in a vector loop problem, we should generally step from point to point on the slotted link taking advantage of the fact that the slot constrains the peg. The following problem demonstrates this idea.

Concept	**Begin 8.2.2**	Check

Figure 8.30 shows two rigid links connected together. The links have one degree of freedom

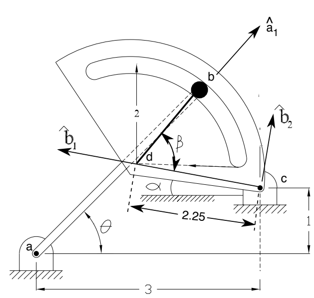

Figure 8.30: Two Links with One Curved.

therefore we can relate all motion variables to a single variable. Suppose we wish to compute α given θ. We decide to form a loop between the four special points a, b, c and d. The reason d is special is because it is the center of curvature of the curved slot, hence it is special because it can be used to describe the shape of the link.

Start at point a and step to b. If we tried to step from b directly to c we would introduce two variables, neither of which is α. Now this is okay except we really want to calculate α. Thinking ahead, one loop is going to give us two equations allowing us to solve for only two unknowns. If we have introduced two unknowns going from b to c then we cannot bring in a third unknown variable α and still expect to solve for anything. We conclude, therefore, that stepping from b directly to c is not a good idea.

How about stepping from b to d then to c. Going from b to d involves only one unknown variable β. Going from d to c brings in α. Hence we have made it to c with only two unknowns, one of which is what we want to compute. Now from c we shoot back over to a finishing the loop.

Mathematically, the loop can be expressed as:

$${}^{a}\vec{r}^{\,b} + {}^{b}\vec{r}^{\,d} + {}^{d}\vec{r}^{\,c} + {}^{c}\vec{r}^{\,a} = 0$$

Substituting in specific values we can write:

$$3\hat{a}_1 - 2\left(\cos\beta\hat{b}_1 + \sin\beta\hat{b}_2\right) - 2.25\hat{b}_1 - \hat{\jmath} - 3\hat{\imath} = 0$$

If we substitute values for $\hat{b}_1$ and $\hat{b}_2$, the variable α shows up. For example $\hat{b}_1 = -\cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$ and $\hat{b}_2 = \sin\alpha\hat{\jmath} + \cos\alpha\hat{\imath}$.

Concept	**End 8.2.2**	Check

This last example demonstrates how to "use the slot geometry wisely" to avoid introducing too many unknown variables. In the next example, we show a similar situation.

Concept	**Begin 8.2.3**	Check

For the slider crank mechanism shown in Figure 8.31, given the angles θ and β, find the distance

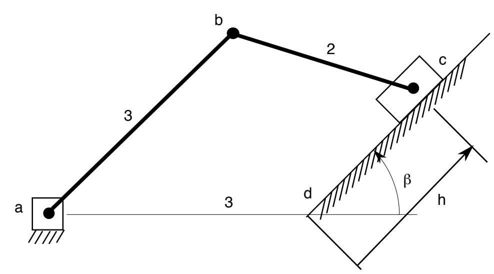

Figure 8.31: A Slider Crank Mechanism.

h. Since this problems asked us to relate one motion variable to another and because the system contains links, this looks like a vector loop problem.

Notice that the system has one degree of freedom and four special points. Since we want to use the slot to "constrain" the slider, we introduce a non-obvious special point (like in the last example), d. Point d is special because we know it is at the "end" of the slot. We like it to be at the end of the slot so we can step from it to c using one variable, h, the one we want to determine.

A useful loop is from c to d to a to b then back to c. The loop can start from any point. Mathematically the loop is written as:

$${}^{c}\vec{r}^{\,d} + {}^{d}\vec{r}^{\,a} + {}^{a}\vec{r}^{\,b} + {}^{b}\vec{r}^{\,c} = 0$$

Using numbers:

$$h\left(-\cos\beta\hat{\imath} - \sin\beta\hat{\jmath}\right) - 3\hat{\imath} + 3\left(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}\right) + 2\left(\cos\alpha\hat{\imath} - \sin\alpha\hat{\jmath}\right) = 0 \tag{8.8}$$

Now in these two equations, the ($\hat{\imath}$ and $\hat{\jmath}$) components, there are two variables, h and α. We leave the solution of these equations as an exercise.

Concept	**End 8.2.3**	Check

To determine whether we need more vector loop equations than presently written, do the following. If the equations do not include all the variables of interest, we probably should write another vector loop. If the number of unknown variables plus the degree of freedom is greater than 3 times[7] the number of loops, then:

1. A loop has been incorrectly written. Make sure all the steps taken in the loop introduce the least number of variables possible.

2. Or, we need to write another vector loop equation. This is not an uncommon requirement.

3. Or, we need to find some other kinematic constraint such as a rope or gearbox constraints.

In appendix E, lesson 2, *Mathematica* is used to manipulate vector loop equations. When the expressions become complicated it is useful to use tools such as *Mathematica* to assist in the analysis.

Problems

1. Solve equation 8.8 for h and α.

2. Figure 8.32 shows a slider crank mechanism. For the mechanism, plot the position of the slider b in terms of the crank angle θ. Does the solution exist for all angles θ?

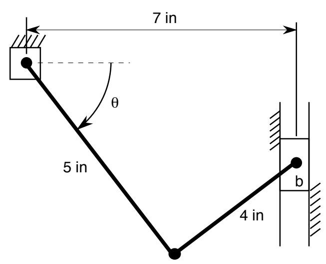

Figure 8.32:

3. Figure 8.33 shows a four bar mechanism. For each link, determine the angle between the link and a horizontal line as a function of the angle θ. In other words assume θ is a known function of time. Note that your equations will be very hard to solve so simply write them down and quit. Let θ be the angle between the horizontal and the upper link of length L.

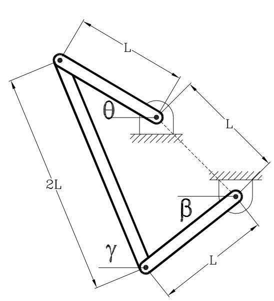

Figure 8.33:

[7]Two times in the case of two dimensional problems.

4. For problem 3, solve the equations and plot the angles of each link as a function of θ.

5. For the mechanism in figure 8.34, determine the position of the peg p (that holds the two links together) as a function of as few angles as possible.

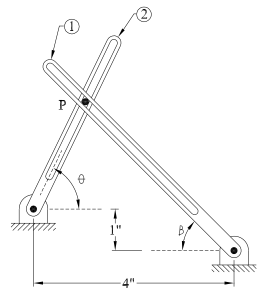

Figure 8.34:

6. The mechanism shown in figure 8.35 is for operating an aircraft landing gear. Express equations for determining the angular position of body 1 in terms of the lengths of the links and the distance d. Be sure to define whatever angles you need.

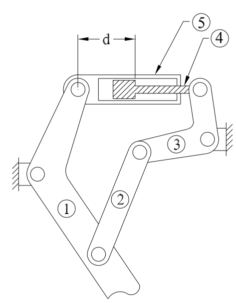

Figure 8.35:

7. Figure 8.36 shows a variable stroke internal combustion engine. Express equations that could be used to find the position of the piston (body 5) in terms of the angle of links 2 and 7.

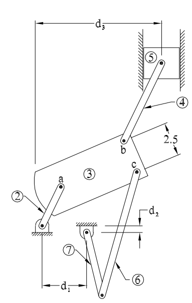

Figure 8.36:

Review

A vector loop is a vector expression that expresses relative positions of several points. The expression is a loop since it ends at the same point it starts. When writing loop equations pay attention

When writing vector loop equations, consider the following rules: 1. When writing vector loops only use special points. However, not all the special points are needed. 2. Never backtrack. If an urge to backtrack arises, consider starting the loop at a different point or consider using more than one loop. 3. Step in directions that have minimal unknowns. Stepping along straight slots, or to the center of curvature of curved slots, helps achieve this rule. 4. Step in directions that bring in the variables of interest.

Table 8.3: Rules for Writing Vector Loops Equations.

to the four basic steps listed in table 8.3.