

**Theorem (2.3, Axiom of Archimedes).** If  $x \in \mathbb{R}$  is any real number, then there exists  $n \in \mathbb{N}$  such that  $x < n$ .

*Proof.* We can break this into two cases

1. Let  $x < 1$ . If so, then simply choose  $x = 1$ .
2. Let  $x \geq 1$ . Define the set  $S = \{n \in \mathbb{N} : n \leq x\}$ . Then since this set is bounded above, by the Completeness Axiom,  $\sup S = y$  exists. Because  $x$  is an upper bound  $S$ , by definition of the supremum, we have that  $y \leq x$ . Let  $r = \frac{1}{2}$ . Then we can find  $k \in S$  such that  $y - \frac{1}{2} < k \leq y$ . But then we have that  $y < y + \frac{1}{2} < k + 1 \leq y + 1$ . Then this means  $k + 1 \notin S$  and so  $x < k + 1$ , completing this case.

Having exhausted all cases, this completes the proof.  $\square$

**Proposition 1 (Well-Ordering Principle).** Every nonempty subset  $S \subset \mathbb{N}$  has a minimum.

**Proposition 2 (Density of the Rational Numbers).** Let  $x, y \in \mathbb{R}$ . Then if  $x < y$ , there exists  $q \in \mathbb{Q}$  such that  $x < q < y$

## Section 2.4, Sequences in $\mathbb{R}$

**Definition.** We define a **sequence** of real numbers to be a function that maps each natural number  $n$  into the real number  $x$ . That is, a sequence is a function  $s : \mathbb{N} \rightarrow \mathbb{R}$  for  $A \subset \mathbb{R}$ . This is written as  $\{x_n\}$  or  $\{x_n\}_{n=1}^{\infty}$ .

**Definition (Convergence of a Sequence).** A sequence converges to the real number  $l \in \mathbb{R}$  if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$|a_n - l| < \varepsilon.$$

**Definition (Cauchy Sequence).** A sequence  $\{x_n\}$  in  $\mathbb{R}$  is **Cauchy** sequence if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n, m \geq N$ ,

$$|a_n - a_m| < \varepsilon.$$

**Theorem.** Let  $\{x_n\}$  be a sequence in  $\mathbb{R}$ . Then  $\{x_n\}$  is Cauchy if and only if  $\{x_n\}$  is convergent.

**Definition.** The number  $l \in \mathbb{R}$  is called a **cluster point** of  $\{x_n\}$  if there exists a subsequence  $\{x_{n_m}\}$  of  $\{x_n\}$  such that  $x_{n_m} \rightarrow l$ .

We can define this in another way. The number  $l \in \mathbb{R}$  is called **cluster point** of  $\{x_n\}$  if for all  $\varepsilon > 0$  and for all  $N \in \mathbb{N}$ , there exists  $n \geq N$  such that  $|x_n - l| < \varepsilon$ .

**Definition.** We define the **limit superior** of a sequence  $\{x_n\}$  in  $\mathbb{R}$  to be

$$\overline{\lim}_{n \rightarrow \infty} x_n = \inf_n \sup_{k \geq n} x_k.$$

This is also denoted as  $\limsup$ .

**Theorem.** A number  $l \in \mathbb{R}$  is the **limit superior** of the sequence  $\{x_n\}$  if and only if

- (i) For all  $\varepsilon > 0$ , there exists  $n \in \mathbb{N}$  such that for all  $k \geq n$ ,  $x_k < l + \varepsilon$
- (ii) For all  $\varepsilon > 0$  and for all  $n \in \mathbb{N}$ , there exists  $k \geq n$  such that  $x_k > l - \varepsilon$ .

**Definition.** We define the **limit inferior** of a sequence  $\{x_n\}$  in  $\mathbb{R}$  to be

$$\underline{\lim}_{n \rightarrow \infty} x_n = \sup_n \inf_{k \geq n} x_k.$$

This is also denoted as  $\liminf$ .

**Theorem.** A number  $l \in \mathbb{R}$  is the **limit inferior** of the sequence  $\{x_n\}$  if and only if

- (i) For all  $\varepsilon > 0$ , there exists  $n \in \mathbb{N}$  such that for all  $k \geq n$ ,  $x_k > l - \varepsilon$
- (ii) For all  $\varepsilon > 0$  and for all  $n \in \mathbb{N}$ , there exists  $k \geq n$  such that  $x_k < l + \varepsilon$ .

**Proposition 3.** From the last two definitions, we have the following property.

- $\overline{\lim}_{n \rightarrow \infty}$  is the largest cluster point.
- $\underline{\lim}_{n \rightarrow \infty}$  is the smallest cluster point.

## Section 2.5, Open and Closed Sets in $\mathbb{R}$

**Definition.** The set  $O \subset \mathbb{R}$  is called an **open** set if for all  $x \in O$ , there exists  $\delta > 0$  such that  $x - \delta, x + \delta$ .

Equivalently,  $O$  is an **open** set if for all  $x \in O$ , there is a  $\delta > 0$  such that each  $y$  with  $|x - y| < \delta$  belongs to  $O$ .

**Proposition 4.** From this above, we have the following properties:

1. The set  $\bigcup_{\alpha} O_{\alpha}$  is open.
2. The set  $\bigcup_{n=1}^{\infty} O_n$  is open.

**Theorem (Lindelof Theorem).** Every open set in  $\mathbb{R}$  is a disjoint union of countable union of open intervals.

*Proof.* This proof is contained on page 42 of Royden. □

**Definition.** A real number  $x \in \mathbb{R}$  is called **point of closure** of a set  $E \subset \mathbb{R}$  if for every  $\delta > 0$  there exists a  $y \in E$  such that  $|x - y| < \delta$ .

The set of points of closure of  $E$  is denoted  $\overline{E}$ .

**Proposition 5.** If  $A \subset B \subset \mathbb{R}$ , then  $\overline{A} \subset \overline{B}$ . Additionally,  $\overline{A \cup B} = \overline{A} \cup \overline{B}$ .

*Proof.* The proof of this is on page 43 of Royden. □

**Definition.** A set  $F \subset \mathbb{R}$  is called a **closed** set if  $\overline{F} = F$ .

Note that because  $F \subset \overline{F}$  always, a set  $F$  is closed if  $\overline{F} \subset F$ —that is,  $F$  contains all of its points of closure.

**Proposition 6.** For any set  $E$ , the set  $\overline{E}$  is closed; that is  $\overline{\overline{E}} = \overline{E}$ .

**Proposition 7.** Let  $E \subset \mathbb{R}$ . Then  $E$  is open if and only if  $E^c$  is closed.

**Definition.** We say that a collection of sets  $\mathcal{C}$  is a **cover** of a set  $F$  if

$$F \subset \bigcup_{O \in \mathcal{C}} O.$$

The collection  $\mathcal{C}$  is a covering of the set  $F$ .

**Theorem (Heine-Borel).** Let  $E \subset \mathbb{R}$  be set. Then  $E$  is compact if and only if  $E$  is closed and bounded.