Construction of the Real Numbers, $\mathbb{R}$

- We first start from $\mathbb{N} \cup \{0\}$ and add numbers together subsequently (i.e. 1, $\underbrace{1+1}_{2}$, $\underbrace{1+1+1}_{3}$, ...)
- To construct the integers $\mathbb{Z}$, we take the set difference with the natural numbers so that we have
$$\mathbb{Z} = \mathbb{N} \cup \{0\} \setminus \mathbb{N}.$$
- Then the rationals numbers, $\mathbb{Q}$, can be constructed from the integers and are defined by the set
$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \right\}.$$
- To construct the irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$, we can use the dedekind cut to do this. However, this is convoluted and we can go about this in a different way.

Axioms of the Real Numbers

A. The Field Axioms: For all real numbers $x, y \in \mathbb{R}$ we have:

A1. x + y = y + x

A2. (x + y) + z = x + (y + z)

A3. There exists $0 \in \mathbb{R}$ such that $x + 0 =$ for all $x \in \mathbb{R}$.
[**Identity element under addition**]

A4. For each $x \in \mathbb{R}$ there is a $w \in \mathbb{R}$ such that $x + w = 0$.
[**Inverse element under addition**]

A5. xy = yx

A6. (xy)z = x(yz)

A7. There exists $1 \in \mathbb{R}$ such that $1 \neq 0$ and $x \cdot 1 = x$ for all $x \in \mathbb{R}$.

A8. For each $x \in \mathbb{R}$ different from 0 there is $w \in \mathbb{R}$ such that $xw = 1$.

A9. x(y + z) = xy + xz.

We can prove some properties now:

Proposition 1. The additive inverse is unique.

Proof. Let $x \in \mathbb{R}$. Suppose we have two numbers $w_1, w_2 \in \mathbb{R}$ such that $x + w_1 = 0 = x + w_2$. Using the axioms and our assumption, we can show the following:

$$\begin{aligned} w_1 &= w_1 + 0 && \text{Axiom A3} \\ &= w_1 + x + w_2 && \text{Assumption of } 0 = x + w_2 \\ &= w_2 + xw_1 && \text{Axiom A1} \\ &= w_2 \end{aligned}$$

which completes the proof. □

B. **The Axioms of Order:** The subset P of positive real numbers satisfies the following:

B1. If $x, y \in P$, then $x + y \in P$.

B2. If $x, y \in P$, then $xy \in P$.

B3. If $x \in P$, then $-x \notin P$.

B4. If $x \in \mathbb{R}$, then $x = 0$ or $x \in P$ or $-x \in P$.

Note that any system which satisfies the axioms of groups A and B is called an **ordered field**.

Definition. We can give definitions of the ordered operations $<$, $\leq$, $>$ and $\geq$.

- $x < y$ means that $y - x \in P$.
- $x \leq y$ means that $y - x \in P \cup \{0\}$. Or, this means that $x < y$ or $x = y$.
- $x > y$ means that $x - y \in P$.
- $x \geq y$ means that $x - y \in P \cup \{0\}$. Or, this means that $x > y$ or $x = y$.

From this, we can deduce and prove some which is any set which satisfies the axioms of group A and B.

Definition. Let $x, y \in \mathbb{R}$ and define the absolute value as

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

Proposition 2. Let $a, b, c \in \mathbb{R}$.

1. $a < b$ if and only if $-b < -a$.
2. If $a < b$ and $b < c$, then $a < c$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. For $a, b \in \mathbb{R}$, then only one is true $a = b$, $a > b$ and $a < b$.
5. If $x \neq 0$, then $x^2 = x \cdot x > 0$; in particular, $1 > 0$.
6. If $x, y \in \mathbb{R}$, then $|x + y| \leq |x| + |y|$.

Definition. Let $S \subset \mathbb{R}$. The number $b \in \mathbb{R}$ is an **upper bound** for S if for each $x \in S$, we have $x \leq b$.

Similarly, a number $x \in \mathbb{R}$ is the **least upper bound** for S if it is an upper bound for (S) and if $x \leq b$ for each upper bound b of S. We then call x the **supremum** of S and denote this $x = \sup S$.

Definition. Let $S \subset \mathbb{R}$. The number $l \in \mathbb{R}$ is an **lower bound** for S if for each $x \in S$, we have $l \leq x$.

Similarly, a number $x \in \mathbb{R}$ is the **greatest lower bound** for S if it is a lower bound for S and if $x \leq l$ for each lower bound l of S. We then call x the **infimum** of S and denote this $x = \inf S$.

C. **Completeness Axiom:** Every nonempty set $S \subset \mathbb{R}$ which has an upper bound has a least upper bound.

Proposition 3. Let $L, U \subset \mathbb{R}$ be nonempty subsets with $R = L \cup U$ and such that for each $l \in L$ and each $u \in U$ we have $l < u$. Then either L has a greatest element or L has a least element.

Proposition 4 (Approximation Property.)**.** Let $S \subset \mathbb{R}$ be a nonempty. If $u = \sup S$, then for all $\gamma > 0$, there exists $Sr \in S$ such that $u - r < Sr < u$.

Theorem (2.3, **Axiom of Archimedes**)**.** If $x \in \mathbb{R}$ is any real number, then there exists $n \in \mathbb{N}$ such that $x < n$.

Proof. We can break this into two cases

1. Let $x < 1$. If so, then simply choose $x = 1$.
2. Let $x \geq 1$. Define the set $S = \{n \in N : n \leq x\}$. Then since this set is bounded above, by the Completeness Axiom, $\sup S = y$ exists. Because x is an upper bound S, by definition of the supremum, we have that $y \leq x$. Let $r = \frac{1}{2}$. Then we can find $k \in S$ such that $y - \frac{1}{2} < k \leq y$. But then we have that $y < y + \frac{1}{2} < k + 1 \leq y + 1$. Then this means $k + 1 \notin S$ and so $x < k + 1$, completing this case.

Having exhausted all cases, this completes the proof. □

Proposition 1 (**Well-Ordering Principle**)**.** Every nonempty subset $S \subset \mathbb{N}$ has a minimum.

Proposition 2 (**Density of the Rational Numbers**)**.** Let $x, y \in \mathbb{R}$. Then if $x < y$, there exists $q \in \mathbb{Q}$ such that $x < \gamma < y$

Section 2.4, Sequences in $\mathbb{R}$

Definition. We define a **sequence** of real numbers to be a function that maps each each natural number n into the real number x. That is, a sequence is a function $s : \mathbb{N} \to A$ for $A \subset \mathbb{R}$. This is written as $\{x_n\}$ or $\{x_n\}_{n=1}^{\infty}$.

Definition (**Convergence of a Sequence**)**.** A sequence converges to the real number $l \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - l| < \varepsilon.$$

Definition (**Cauchy Sequence**)**.** A sequence $\{x_n\}$ in $\mathbb{R}$ is **Cauchy** sequence if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$|a_n - a_m| < \varepsilon.$$

Theorem. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then $\{x_n\}$ is Cauchy if and only $\{x_n\}$ is Cauchy.

Definition. The number $l \in \mathbb{R}$ is called a **cluster point** of $\{x_n\}$ if there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $x_{n_m} \to l$.

We can define this in another way. The number $l \in \mathbb{R}$ is called **cluster point** of $\{x_n\}$ if for all $\varepsilon > 0$ and for all $N \in \mathbb{N}$, there exists $n \geq N$ such that $|x_n - l| < \varepsilon$.

Definition. We define the **limit superior** of a sequence $\{x_n\}$ in $\mathbb{R}$ to be

$$\overline{\lim}_{n\to\infty} x_n = \inf_n \sup_{k \geq n} x_k.$$

This is also denoted as $\limsup$.

Theorem. A number $l \in \mathbb{R}$ is the **limit superior** of the sequence $\{x_n\}$ if and only if

(i) For all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, $x_k < l + \varepsilon$

(ii) For all $\varepsilon > 0$ and for all $n \in \mathbb{N}$, there exists $k \geq n$ such that $x_k > l - \varepsilon$.

Definition. We define the **limit inferior** of a sequence $\{x_n\}$ in $\mathbb{R}$ to be

$$\underline{\lim}_{n\to\infty} x_n = \sup_n \inf_{k\geq n} x_k.$$

This is also denoted as lim inf.

Theorem. A number $l \in \mathbb{R}$ is the **limit inferior** of the sequence $\{x_n\}$ if and only if

(i) For all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, $x_k > l - \varepsilon$

(ii) For all $\varepsilon > 0$ and for all $n \in \mathbb{N}$, there exists $k \geq n$ such that $x_k < l + \varepsilon$.

Proposition 3. From the last two definitions, we have the following property.

- $\overline{\lim}_{n\to\infty}$ is the largest cluster point.
- $\underline{\lim}_{n\to\infty}$ is the smallest cluster point.

Section 2.5, Open and Closed Sets in $\mathbb{R}$

Definition. The set $O \subset \mathbb{R}$ is called an **open** set if for all $x \in O$, there exists $\delta > 0$ such that $x - \delta, x + \delta$.

Equivalently, O is an **open** set if for all $x \in O$, there is a $\delta > 0$ such that each y with $|x - y| < \delta$ belongs to O.

Proposition 4. From this above, we have the following properties:

1. The set $\bigcup_{\alpha} O_\alpha$ is open.
2. The set $\bigcup_{n=1}^{n} O_m$ is open.

Theorem (Lindelof Theorem). Every open set in $\mathbb{R}$ is a disjoint union of countable union of open intervals.

Proof. This proof is contained on page 42 of Royden. □

Definition. A real number $x \in \mathbb{R}$ is called **point of closure** of a set $E \subset \mathbb{R}$ if for every $\delta > 0$ there exists a $y \in E$ such that $|x - y| < \delta$.

The set of points of closure of E is denoted $\overline{E}$.

Proposition 5. If $A \subset B \subset \mathbb{R}$, then $\overline{A} \subset \overline{B}$. Additionally, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. The proof of this is on page 43 of Royden. □

Definition. A set $F \subset \mathbb{R}$ is called a **closed** set if $\overline{F} = F$.

Note that because $F \subset \overline{F}$ always, a set F is closed if $\overline{F} \subset F$—that is, F contains all of its points of closure.

Proposition 6. For any set E, the set $\overline{E}$ is closed; that is $\overline{\overline{E}} = \overline{E}$.

Proposition 7. Let $E \subset \mathbb{R}$. Then E is open if and only if $E^{\mathcal{C}}$ is closed.

Definition. We say that a collection of sets $\mathcal{C}$ is a **cover** of a set F if

$$F \subset \bigcup_{O \in \mathcal{C}} O.$$

The collection $\mathcal{C}$ is a covering of the set F.

Theorem (**Heine-Borel**)**.** Let $E \subset \mathbb{R}$ be set. Then E is compact if and only if E is closed and bounded.

Compactness

Theorem. Let $E \subset \mathbb{R}$. Then E is compact if and only E is sequentially compact. That is, for every $\{x_n\}$ in E, there exists a convergent subsequence $x_{n_m} \to x_0$ in E.

Theorem. Let $\{I_n\}$ be a sequence of closed intervals such that $I_{n+1} \subset I_n$. Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

If $[a_n, b_n]$ is an interval and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} = a$, then $\bigcap_{n=1}^{\infty}$.

Section 2.6, Continuous Functions

Definition. Let $E \subset \mathbb{R}$, and let $f : E \to \mathbb{R}$ be a real-valued function. Then f is **continuous** at the point $x = a \in E$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in E$ with $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

Note that we can have continuity in terms of sequences. I will state it as a theorem here even though it was not in lecture because it is important to be able to use on its own.

Theorem. Let $f : E \to \mathbb{R}$ be a function with $E \subset \mathbb{R}$. Let $x \in E$ be any point. Then f is continuous at a if and only for every sequence $\{x_n\}$ in E converging to a, the sequence $\{f(x_n)\}$ in $f(E)$ (the image of E) converges to $f(a)$.

Proposition. Let $E \subset \mathbb{R}$ be compact. Let $f : E \to \mathbb{R}$ be continuous real-valued function. Then $f(E)$ is a compact set.

Proof. Let $\subset \mathbb{R}$ be a compact and suppose the function $f : E \to \mathbb{R}$ is continuous. To show that $f(E)$ is compact, we will use the Heine-Borel theorem and show that it is closed and bounded. To show that $f(E)$ is closed, suppose we have any sequence $\{f(x_n)\}$ converging to the point $f(a) \in \mathbb{R}$. Additionally, let $\{x_n\}$ be any sequence in E. Because E is compact, there exists a subsequence $\{x_{n_m}\}$ which converges to a point $x_0 \in E$. Since f is continuous, by the preceding theorem this means that the sequence $\{f(x_{n_m})\}$ converges to $f(x_0) \in f(E)$. □

Proposition (2.17, Extreme Value Theorem). Let $E \subset \mathbb{R}$ be a compact set, and let $f : E \to \mathbb{R}$ be a continuous function. Then there exists $x_1, x_2 \in E$ such that

$$f(x_1) \leq f(x) \leq (x_2), \text{ for all x} \in E.$$

Proposition (2.18). Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is **continuous** if and only if $f^{-1}(O)$ is open for all open sets $O \subset \mathbb{R}$.

Proposition (2.19). Let $E \subset \mathbb{R}$, and let $f : E \to \mathbb{R}$ be continuous. Without loss of generality, suppose that $f(a) \leq f(b)$. Then for all $\gamma \in [f(a), f(b)]$, there exists $c \in [a, b]$ such that $f(c) = \gamma$.

Definition (Uniform Continuity). Let $E \subset \mathbb{R}$. A function $f : E \to \mathbb{R}$ is **uniformly continuous** if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

Proposition (2.20). Let $E \subset \mathbb{R}$ be a compact set. If $f : E \to \mathbb{R}$ is a continuous function on E, then f is uniformly continuous on E.

Definition. Let $f_n : E \to \mathbb{R}$ be a sequence of functions, and let $f : E \to \mathbb{R}$.

1. The sequence $\{f_n\}$ **converges pointwise** on E to f if for all $\varepsilon > 0$ and for all $x \in E$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f(x) - f_n(x)| < \varepsilon$.
2. The sequence $\{f_n\}$ **converges uniformly** if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in E$ and for all $n \geq N$, $|f(x) - f_n(z)| < \varepsilon$.

Section 3.1, Lebesgue Measure

[Perhaps finish these notes another time...]

Section 3.2, Outer Measure

Definition. The **outer measure** $m^*(A)$ of a set $A \subset \mathbb{R}$ is given

$$m^*(A) = \inf_{A \subset \bigcup I_n} \sum_n l(I_n)$$

where $\{I_n\}$ is a countable collection of open intervals that cover A.

Note that from this definition, we get that

1. $m^*(\emptyset) = 0$
2. If $A \subset B$, $m^*(A) < m^*(B)$.
3. m^* does not satisfy disjoint additivity.

Proposition (3.1). The outer measure of an interval is its length; that is, $m^*(I) = l(I)$ where $I = [a, b], (a, b), [a, b)$, or $(a, b]$.

Proof. It is sufficient to show that $m^*([a, b]) = l([a, b])$ since every other interval is a subset of $[a, b]$. Let $\varepsilon > 0$. Then $[a, b] \subset \left[a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right]$ which implies, by the definition of the outer measure,

$$m^*([a, b]) \leq l\left(\left[a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right]\right) = b - a + \varepsilon.$$

Because ε was fixed, this means that $m^* \leq b - a$.

Now we must show that $m^* \geq b - a$. Because $[a, b]$ is compact, for any collection $\{I_n\}$ of open intervals covering $[a, b]$, there exists a finite collection of intervals $\{I_1, \ldots, I_k\}$ so that

$$[a, b] \subset \bigcup_{n=1}^{k} I_n.$$

This gives us that

$$\sum_n I_n \geq \sum_{n=1}^{k} I_n \geq b - a$$

and so $b - a$ is a lower bound. But since m^* as the greatest lower bound of all such sums, we have that $m^* \geq b - a$.

Therefore, $m^*([a, b]) = l([a, b]) = b - a$.

□

Proposition (3.2, Subadditivity). Let $\{A_n\}$ be a countable collection of sets on $\mathbb{R}$. Then

$$m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n).$$

Proof. Proof on page 57.

□

Corollary (3.3). If A is a countable set, then $m^*(A) = 0$.

Proof. Proof is on the end of page 57. $\square$

Section 3.3, Measurable Sets and Lebesgue Measure

Definition. A set $E \subset \mathbb{R}$ is (Lebesgue) **measurable** if for all sets A, we have that

$$m^*(A) = m^*(A \cup E) + m^*(A \cup E^{\complement}).$$

Lemma (3.6). If $m^*(E) = 0$, then E is measurable.

Proof. Let A be any chosen set. Because $A \cap E \subset E$ and $m^*(E) = 0$,

$$m^*(A \cap E) \leq m(E) = 0.$$

Note that $A \cap E^{\complement} \subset A$ and so $m^*(A) \leq m^*(A \cap E^{\complement})$ and so it suffices to show that $m^*(A) \geq m^*(A \cap E^{\complement})$. Using this, we can show that

$$m^*(A) \geq m^*(A \cap E^{\complement}) + 0 = m^*(A \cap E^{\complement}) = m^*(A \cap E)$$

giving us the desired result. $\square$

Definition. Let $\mathcal{M}$ be the set of measurable sets in $\mathbb{R}$

Lemma (3.7). If E_1 and E_2 are measurable sets, then so is $E_1 \cup E_2$.

Proof. Proof on top of page 57. $\square$

Corollary (3.8). The family $\mathcal{M}$ of measurable sets if an algebra of sets. In other words, if $E \in \mathcal{M}$, then $E^{\complement} \in \mathcal{M}$. Further, if $E_1, E_2 \in \mathcal{M}$, then $E_1 \cup E_2 \in \mathcal{M}$.

Lemma (3.9). Let A be any set, and $E_1, \ldots, E_n$ be a finite sequence of sets such that $E_i \cap E_j$ for all $i \neq j$. Then

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Proof. We proceed by induction. For $n = 1$, we have the set E_1 and the equality holds. Suppose that we have $n = k$ sets $E_1, \ldots, E_k$ with $E_i \cap E_j \neq \emptyset$ for all $i \neq j$ so that

$$m^*\left(A \cap \left[\bigcup_{i=1}^{k} E_i\right]\right) = \sum_{i=1}^{k} m^*(A \cap E_i).$$

Consider $n = k + 1$. Because each E_i is disjoint,

$$A \cap \left(\bigcup_{i=1}^{k+1} E_i\right) \cap E_{k+1} = A \cap E_{k+1};$$
$$A \cap \left(\bigcup_{i=1}^{k+1} E_i\right) \cap E_{k+1}^{\complement} = A \cap \bigcup_{i=1}^{k} E_i.$$

Because the E_i's are measurable,

$$\begin{aligned} m^*\left(A \cap \bigcup_{i=1}^{k+1} E_i\right) &= m^*\left(A \cap E_{k+1}\right) + m^*\left(A \cap \bigcup_{i=1}^{k} E_i\right) \\ &= m^*\left(A \cap E_{k+1}\right) + \sum_{i=1}^{k} m^*(A \cap E_i) \qquad \text{Induction Hypothesis} \\ &= \sum_{i=1}^{k+1} m^*(A \cap E_i) \end{aligned}$$

which, by induction, completes the proof. □

Theorem (3.10). $\mathcal{M}$ is a σ-algebra. In other words, in addition to being an algebra of sets, if $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

Proof. [1]

□

Lemma (3.11). The interval (a, ∞) is measurable for all $a \in \mathbb{R}$.

Proof. [2]

□

[1] Proof on bottom of page 59 and top of page 60.

[2] Proof on the bottom of page 60 through the middle of page 61.

Theorem (3.12). Every Borel set is measurable. In particular, each open set and each closed set is measurable.

Proof. [3]

□

Definition. Let $E \in \mathcal{M}$. We define $m(E) := m^*(E)$ to be the **Lebesgue measure** of $E/$

Proposition (3.13, Countable Additivity). Let $\{E_i\}_{i=1}^n$ be a sequence of measurable sets. Then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{n} m(E_i).$$

If, in addition, $E_i \cap E_j$ for all $i \neq j$. then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{n} m(E_i).$$

Proposition (3.14). Let $\{E_i\} \subset \mathcal{M}$ be a decreasing sequence (i.e., $E_{i+1} \subset E_i$). Let $m(E_1) < \infty$. Then

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n).$$

Proposition (3.15). Let E be any given set. Then the following are equivalent:

(i) E is measurable.

(ii) For all $\varepsilon > 0$, there is an open set $O \supset E$ with $m^*(O \setminus E) < \varepsilon$.

(iii) For all $\varepsilon > 0$, there is a closed set $F \subset E$ with $m^*(E \setminus F) < \varepsilon$.

(iv) There is a $G \in G_\delta$ with $E \subset G$ such that $m^*(G \setminus E) = 0$.

(v) There is a $F \in F_\sigma$ with $F \subset E$ such that $m^*(E \setminus F) = 0$.

If $m^*(E) < \infty$, the above statements are equivalent:

(vi) For all $\varepsilon > 0$, there is a finite union U of open intervals such that $m^*(U \Delta E) < \varepsilon$.

[3] Proof on the bottom of page 61.

Section 3.5, Measurable Functions

Proposition (3.18). Let $E \subset \mathbb{R}$, and Let $f : E \to [-\infty, \infty]$ be an extended real-valued function whose domain is measurable. Let $\alpha \in \mathbb{R}$ be any real number. Then the following statements are equivalent:

(i) The set $\{x : f(x) > \alpha\}$ is measurable.

(ii) The set $\{x : f(x) \geq \alpha\}$ is measurable.

(iii) The set $\{x : f(x) < \alpha\}$ is measurable.

(iv) The set $\{x : f(x) \leq \alpha\}$ is measurable.

All together, these imply

(v) The set $\{x : f(x) = \alpha\}$ is measurable.

Proof. [1]

□

Definition. An extended real-valued function $f : E \to [-\infty, \infty]$ is **(Lebesgue) measurable** if its domain is measurable and satisfies one of the first four statements of Proposition 18.

Note that this means that any continuous function is measurable since then the pre-image of any open set is still an open set.

Proposition (3.19). Let f and g be two measurable functions defined on the same domain, and let $c \in \mathbb{R}$. Then the functions $f + c$, cf, $f + g$, $g - f$, and fg are measurable.

Proof. Let $\alpha \in \mathbb{R}$ be any real number. Fix $c \in \mathbb{R}$. For $f(x) + c$, note that

$$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}$$

and $\alpha - c$ is a real number, this set is still measurable i.e., $f + c$ is measurable. A similar argument shows that cf is measurable as well.

Take the set

$$\{x : f(x) + g(x) < \alpha\}. \tag{1}$$

Observe that $f(x) + g(x) < \alpha \Leftrightarrow f(x) < \alpha - g(x)$. By the density of $\mathbb{Q}$, there exists $r \in \mathbb{Q}$ such that $f(x) < r < \alpha - g(x)$. So we can write Equation (1) as

$$\{x : f(x) + g(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} \left(\{x : f(x) < r\} \cap \{x : g(x) < \alpha - r\}\right).$$

Because this set is countable, this set is measurable and thus $f + g$ is measurable.

To show that fg is measurable, we can show that f^2 is measurable since

$$fg = \frac{1}{2}\left((f + g)^2 - f^2 - g^2\right).$$

[1] Proof is on page 67.

Take the set

$$\{x : f^2(x) < \alpha\}. \tag{2}$$

For $\alpha \geq 0$, note that $f^2 < \alpha$ is the same as saying $f(x) > \sqrt{\alpha}$ and $f(x) < -\sqrt{\alpha}$. Thus, Equation (2) can rewritten as

$$\{x : f^2(x) < \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}.$$

This is a measurable set which completes the proof. □

Theorem (3.20, Limit of Measurable Functions is Measurable). [2] Let $\{f_n\}$ be a sequence of measurable functions with the same domain. Then the functions $\sup\{f_1(x), \ldots, f_n(x)\}$, $\inf\{f_1(x), \ldots, f_n(x)\}$, $\sup_n f_n$, $\inf_n f_n$, $\overline{\lim} f_n$, and $\underline{\lim} f_n$ are measurable.

Proof. Let $\{f_n\}$ be a sequence of measurable functions. Let $h(x) = \sup\{f_1(x), \ldots, f_n(x)\}$ and we so must show that $\{x : h(x) < \alpha\}$ for all $\alpha \in \mathbb{R}$. To that end, let $\alpha \in \mathbb{R}$ be chosen. Then

$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$$

which, because the right-hand side is a union of measurable sets from the f_i's being measurable, means that the set $\{x : h(x) < \alpha\}$ is also measurable.

Let $g(x) = \sup_n f_n$. By a similar argument as above,

$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i(x) > \alpha\}$$

is a countable set so $\{x : g(x) < \alpha\}$ is measurable. □

Definition. A property is said to hold **almost everywhere** (a.e) if the set of points where it fails to hold is a set of measure zero. Thus $f = g$ a.e if f and g have the same domain and $m\{x : f(x) \neq g(x)\}$.

Proposition (3.21). [3]

If f is measurable and $f = g$ a.e, then g is measurable.

Proof. Let $E = \{x : f(x) \neq g(x)\}$.

This is equivalent to saying that

$$\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha\}$$

□

The following proposition essentially says that measurable functions are nearly continuous; or. in other words, we can "nicely" approximate measurable functions.

[2] Proof is on bottom of page 68 and top of page 69

[3] Proof is on middle of page 69.

Proposition (3.22, Littlewood's 2nd Principle). Let $f : [a, b] \to E$ be a measurable function with $E \subset \mathbb{R}$ and is equal to $\pm\infty$ only on sets with measure zero. Then for all $\varepsilon > 0$, there exist a step function g and a continuous function f such

$$|f - g| < \varepsilon \quad \text{and} \quad |f - h| < \varepsilon$$

except on set of measure less than ε; i.e., $m\{x : |f(x) - g(x)| \geq \varepsilon\} < \varepsilon$ and $m\{x : |f(x) - h(x)| \geq \varepsilon\} < \varepsilon$. If in addition $m \leq f \leq M$, then we may choose the functions g and h so that $m \leq g \leq M$ and $m \leq h \leq M$.

Proposition (3.23, (Weak) Egonoff's Theorem). Let E be a measurable set of finite measure, and $\{f_n\}$ be a sequence of measurable functions defined on E. Let f be real-valued function such for each $x \in E$ we have $f_n(x) \to f(x)$. Then for all $\varepsilon > 0$ and all $\delta > 0$, there is measurable set $A \subset E$ with $m(A) < \delta$ and $N \in \mathbb{N}$ such that for all $x \notin A$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Proof. [1]

Let $\varepsilon > 0$ be chosen. Define the set

$$G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}$$

and also

$$E_N = \bigcup_{n=N}^{\infty} G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N\}.$$

Because $\{E_N\}$ is a decreasing sequence and $f_n(x) \to f(x)$ pointwise, for all $x \in E$, $|f_n(x) - f(x)| < \varepsilon$ and so $\bigcup_{i=1}^{\infty} E_N = \emptyset$. Thus, by Proposition 3.14,

$$\begin{aligned} E_N = \emptyset \implies m(E_N) &= 0 \\ &= m\left(\bigcup_{N=1}^{\infty} E_N\right) \\ &= \lim_{N\to\infty} E_N. \end{aligned}$$

So for any $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $m(E_N) < \delta$. Now take $A = E_N$ for any $N > N_0$ and so $m(A) < \delta$ and also

$$A^{\complement} = \{x \in E : x \notin E\} = \{x \in E : |f_n(x) - f(x)| < \varepsilon \text{ for all } n > N_0\}.$$

$\square$

Section 4.1 Riemann Integration

Definition. Let $f : [a, b] \to \mathbb{R}$ be a bounded real-valued function and let

$$P = \{a = \xi_0 < \xi_1 < \cdots < \xi_n = b\}$$

be a subdivision (partition) of $[a, b]$. We can define the **upper sum**, S and **lower sum**, s, respectively, as

$$S = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) M_i \qquad \text{and} \qquad s = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) m_i$$

where

$$M_i = \sup\{f(x) : x \in [\xi_{i-1}, \xi_i]\} \qquad \text{and} \qquad m_i = \inf\{f(x) : x \in [\xi_{i-1}, \xi_i]\}.$$

[1] Proof on page 72-73.

Definition. Let $f : [a, b] \to \mathbb{R}$ be a bounded real-valued function. Define the **upper Riemann integral** of f as

$$R\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf\{S : P \text{ is a partition of } [a, b]\}$$

and the **lower Riemann integral** of f as

$$R\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup\{s : P \text{ is a partition of } [a, b]\}.$$

Definition. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if

$$R\underline{\int_a^b} f(x)\,\mathrm{d}x = R\int_a^b f(x)\,\mathrm{d}x = R\overline{\int_a^b} f(x)\,\mathrm{d}x.$$

In other words, if the upper and lower Riemann integrals are equal to each other.

Note this theorem is *NOT* from Royden but from my real analysis course in undergraduate. It comes from the definition of the upper and lower Riemann integral definitions by ends up being a useful way to show Riemann integration.

Theorem. Let f be a bounded function on $[a, b]$. Then f is Riemann integrable if and only if for all $\varepsilon > 0$, there exists a subdivision (partition) P of $[a, b]$ such that

$$S - s < \varepsilon.$$

Section 4.2 The Lebesgue Integral

Definition. The **characteristic function** of E is defined as

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

A linear combination

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

is called a **simple function** if the sets $\{E_1, \ldots, E_n\}$ are measurable. Note that ϕ is simple if and only if it is measurable and only assumes a finite number of values.

The **canonical representation** of ϕ is such that

$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$$

where $A_i = \{x : \phi(x) = a_i\}$ and where the A_i's are disjoint and the a_i's are distinct and nonzero.

Definition. Let
$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$$
be a simple function which vanishes outside of a set of finite measure. Then the integral of ϕ is defined as
$$\int \phi = \sum_{i=1}^{n} a_i \cdot m(A_i).$$

Lemma (4.1). Let $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ with $E_i \cap E_j = \emptyset$ for all $i \neq j$ where $E_i \in \mathfrak{M}$ and $m(E_i) < \infty$ for each $i = 1, \ldots n$. Then
$$\int \phi = \sum_{i=1}^{n} a_i \cdot m(E_i).$$

Proposition (4.2). Let ϕ, ψ be simple functions vanishing outside of a set with finite measure. Then
$$\int (a\phi + b\psi) = a \int \phi + b \int \psi.$$
If $\phi \geq \psi$ almost everywhere,
$$\int \phi \geq \int \psi.$$

Proposition (4.3). Let $E \in \mathfrak{M}$, and let $f : E \to \mathbb{R}$ with $m(E) < \infty$ be a bounded, measurable function. Let ϕ, ψ be simple functions. Then the **Lebesgue integral** of f if
$$\int_E f = \inf \left\{ \int \psi : \psi \geq f \right\} = \sup \left\{ \int \phi : \phi \leq f \right\}.$$

Proof. [2]

□

[2]Proof is on pages 79-80.

Proposition (4.3). Let $E \in \mathfrak{M}$, and let $f : E \to \mathbb{R}$ with $m(E) < \infty$ be a bounded, measurable function. Let ϕ, ψ be simple functions. Then the **Lebesgue integral** of f if

$$\int_E f = \inf \left\{ \int \psi : \psi \geq f \right\} = \sup \left\{ \int \phi : \phi \leq f \right\}.$$

Proof. [1]

$\square$

[1] Proof is on pages 79-80.

filler text.