Lemma (6.12). Let g be an integrable function over $[0,1]$. Suppose there exists $M > 0$ such that

$$\left| \int fg \right| \leq M \cdot \|f\|_p$$

for all f which are bounded, measurable functions. Then $g \in L^q$ and $\|g\|_q \leq M$.

Proof. We claim that $\|g\|_q \leq M$. We are done if we prove this because then $g \in L^q$. First, assume that $p \in (1, \infty)$ and so q is over the same region as well. Define the function g_n by

$$g_n(x) = \begin{cases} g(x) & |g(x)| \leq n \\ 0 & |g(x)| > n. \end{cases}$$

Now define $f_n(x) = |g_n|^{q/p} \cdot (\text{sgn } g_n)$. Now,

$$\|f_n\|_p = \left(\|g_n\|_q\right)^{q/p}$$

and also

$$|g_n|^q = f_n \cdot g_n = f_n \cdot g.$$

Hence,

$$\left(\|g_n\|_q\right)^q = \int f_n \cdot g \leq M \|f_n\|_P = M \left(\|g_n\|_q\right)^{q/p}.$$

Because $q - \dfrac{p}{q} = 1$, then for all $n \in N$,

$$\|g_n\|_q \leq M.$$

Note that as $n \to \infty$, $|g_n|^q \to |g|^q$ almost everywhere and so the above expression implies by Fatuo's lemma

$$\int |g_n|^q \leq \lim_{n \to \infty} \int |g_n|^q \leq M^q.$$

□

Theorem (6.13, Riesz Representation Theorem). Let F be a bounded linear functional on L^p. Then there exists $g \in L^q$ such that

$$F(f) = \int fg.$$

We also have $\|F\| = \|g\|_q$.

Proof. Our outline of the proof will be as follows:

Step I: Find g which is integrable.

Step II: Upgrade g to L^q.

Step III: Verify for all $f \in L^p$.

Let $\chi_s = 1_{[0,s]}$ and $\Phi(s) = F(\chi_x)$. We claim that $\Phi(s)$ is absolutely continuous. This means that there exists g integrable such that

$$\Phi(s) - \underbrace{\Phi(0)}_{=0} = \int_0^s g(t)\,\mathrm{d}t = \int_0^1 g \cdot \chi_s.$$

Let $\varepsilon > 0$ be chosen. Let $\{(s_i, s_i')\}$ be a disjoint collection of intervals over $[0,1]$ with

$$\sum_{i=1}^k s_i' - s_i < \delta.$$

We want to show that this implies that

$$\sum_{i=1}^k |\Phi(s_i') - \Phi(s)| < \varepsilon.$$

Note that

$$\sum_{i=1}^k |\Phi(s_i') - \Phi(s)| = F(f)$$

where

$$\begin{aligned} f &= \sum_{i=1}^k \left(\chi_{s_i'} - \chi_{s_i}\right) \cdot \operatorname{sgn}\left(\Phi(s_i') - \Phi(s)\right) \\ &\leq \sum_{i=1}^k (s_i' - s_i) \\ &< \delta \end{aligned}$$

and so this implies that $\|f\|_p < \delta$. By definition of Φ and F being a bounded linear functional, we know that

$$\begin{aligned} \sum_{i=1}^k |\Phi(s_i') - \Phi(s)| &= F(f) \\ &\leq M \cdot \|f\|_p \\ &< M \cdot \delta \\ &< \varepsilon. \end{aligned}$$

Thus Φ is absolutely continuous if we choose $\delta = \dfrac{\varepsilon}{M}$. So this tell us

$$\Phi(s) = \int_0^1 g \cdot \chi_s$$

for some integrable function g. Since every step function ϕ on $[0,1]$ is a linear combination of ϕ_s''s, we know that

$$F(\phi) = \int_0^1 g\phi$$

for every step function ϕ.

Now we claim that this function g is in L^q. To that end, let f be a bounded measurable function on $[0,1]$. Then there exists bounded step functions ϕ_n such that $\phi_n \to f$ almost everywhere. Thus the sequence $\{|f-\phi_n|^p\}$ is uniformly bounded and converges to 0 almost everywhere. This tells us that $\|f-\phi_n\|_p \to 0$. Since F is a bounded linear functional, we know that

$$\begin{aligned} |F(f)-F(\phi_n)| &= |F(f-\phi_n)| \\ &\leq M \cdot \|f-\phi_n\|_p, \end{aligned}$$

and so we know that $F(f) = \lim_{n\to\infty} F(\phi_n)$. Letting $\tilde{M}$ be the uniform bound of the ϕ_n's, we have $g\phi_n < |g| \cdot \tilde{M}$. By the Lebesgue dominated convergence theorem,

$$\begin{aligned} \int f \cdot g &= \lim_{n\to\infty} \int g \cdot \phi_n \\ &= \lim_{n\to\infty} F(\phi_n) \\ &= F(f) \end{aligned}$$

for all f which are bounded measurable functions. Thus, by the Hölder inequality,

$$\left| \int fg \right| = |F(f)| \leq \|F\| \cdot \|f\|_p .$$

By Lemma 6.12, $g \in L^q$, finishing our second claim.

Finally, we claim that

$$F(f) = \int g \cdot f$$

for all $f \in L^p$.

Let $f \in L^p$ be any function and fix $\varepsilon > 0$. By the density of step functions in L^p, there exists a step function ϕ such that

$$\|f-\phi\|_p < \varepsilon$$

and so

$$F(\phi) = \int \phi g.$$

Hence, we have by the linearity of F,

$$\begin{aligned} \left| F(f) - \int f \cdot g \right| &= \left| F(f) - F(\phi) + F(\phi) - \int f \cdot g \right| \\ &= \left| F(f-\phi) + \int g(\phi - f) \right| \\ &\leq |F(f-\phi)| + \left| \int g(\phi-f) \right| \\ &\leq M \cdot \|f-\phi\|_p + \left| \int g(\phi - f) \right| \\ &\leq M \cdot \underbrace{\|f-\phi\|_p}_{<\varepsilon} + \|g\|_q \cdot \underbrace{\|f-\phi\|_p}_{<\varepsilon} \qquad \text{Hölder Inequality} \end{aligned}$$

This tells us that

$$F(f) = \int fg.$$

$\square$

Chapter 11 Measure Spaces

The set up will be that we have any set X and $\mathcal{B}$ which is a σ-algebra i.e., a collection of subsets of X such that

1. $\emptyset \in \mathcal{B}$
2. If $A \in \mathcal{B}$, then $A^{\complement} \in \mathcal{B}$.
3. Closed under countable union.

Define $\mu : \mathcal{B} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ to be a set function such that $E \mapsto \mu(E) \in \overline{\mathbb{R}}$.

Definition. A couple $(X, \mathcal{B})$ is called a measurable space consisting of a set X and a σ-algebra $\mathcal{B}$ subsets of X. A set $E \subset X$ is called **measurable** if $E \in \mathcal{B}$.

Definition. A set function $\mu : \mathcal{B} \to \overline{\mathbb{R}}$ is called a **measure** if we have the following:

1. $\mu(\emptyset) = 0$
2. Countable additivity i.e., $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i-1}^{\infty} \mu\left(E_i\right)$ for any $E_i \cap E_j = 0$ with $i \neq j$.

We then call the triple $(X, \mathcal{B}, \mu)$ a **measure space**.

Remark. We know that countable additivity implies finite additivity. An example of this is the triple $(\mathbb{R}, \mathcal{M}, m)$ where $\mathcal{M}$ is all Lebesgue measurable sets and m is the Lebesgue measure.