Problem 1 (11.10)**.** Prove Proposition 11.7 which is stated follows:

Proposition (11.7)**.** Let f be an nonnegative measurable function. Then there is a sequence $\{\phi_n\}$ of simple functions with $\phi_{n+1} \geq \phi_n$ such that $f = \lim_{n\to\infty} \phi_n$ at each point of X. If f is defined on a σ-finite measure space, then we may choose the functions ϕ_n so that each vanishes outside a set of finite measure.

Proof. Let f be a nonnegative measurable function. Per the hint, for every pair of integers (n, k), let

$$E_{n,k} = \{x : k2^{-n} \leq f(x) < (k+1)2^{-n}\}, \text{ and set } \phi_n = 2^{-n} \sum_{k=0}^{2^{2n}} k\chi_{E_{n,k}}.$$

Let (n, k) be any arbitrary pair of each integers. Then because f is measurable, each $E_{n,k}$ is a measurable set and so ϕ_n is a simple function defined on each $E_{n,k}$. First, we will note that

$$E_{n,k} = E_{n+1,2k} \cup E_{n+1,2k+1}.$$

Let $x \in E_{n,k}$. This means $\phi_n(x) = k2^{-n}$. Now suppose $x \in E_{n+1,2k}$. Then we know that

$$\phi_{n+1}(x) = (2k)2^{-(n+1)} = k2^{-n} = \phi_n(x).$$

Lastly, suppose that $x \in E_{n+1,2k+1}$. Then we know that

$$\phi_{n+1}(x) = (2k+1)2^{-(n+1)} > (2k)2^{-(n+1)}\phi_n(x).$$

Thus, in all cases, $\phi_n(x) \leq \phi_{n+1}(x)$.

To prove pointwise convergence, let $x \in X$ be any point. This brings two cases: either (i) $f(x) < \infty$ or (ii) $f(x) = \infty$. First, assume that $f(x) < \infty$. Because of how we defined ϕ_n and $E_{n,k}$, we know that

$$|f(x) - \phi_n(x)| \leq 2^{-n}$$

will always exist with $n \in \mathbb{N}$ large enough. But because (n, k) are chosen arbitrarily, we have that $f = \lim_{n\to\infty} \phi_n$. Now, suppose that $f(x) = \infty$. Then

$$\phi_n(x) = (2^{2n} + 1)2^{-n} = 2^n + \frac{1}{2^n} > 2^n.$$

So as $n \to \infty$, $\phi_n \to \infty$ as well and so we still have $f = \lim_{n\to\infty} \phi_n$. Therefore, in all cases, we have pointwise convergence.

Suppose f is defined on a σ-finite measure space. Then $X = \bigcup_n X_n$ with $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$. Define $E_{n,k}$ the same as above but define ϕ_n on the set $E_{n,k} \cap \bigcup_{m=1}^{n} X_m$ i.e.,

$$\phi_n = 2^{-n} \sum_{k=0}^{2^{2n}} k\chi_{E_{n,k} \cap \bigcup_{m=1}^{n} X_m}.$$

So, by a similar argument to above, $\phi_{n+1} \geq \phi_n$ and $f = \lim_{n\to\infty} \phi_n$. However, each simple function will vanish outside of the set of finite measure, $\bigcup_{m=1}^{n} X_m$. This completes the proof. □

Problem 2 (11.22)**.** (a) Let $(X, \mathcal{B}, \mu)$ be a measure space and g a nonnegative measurable function on X. Set $\nu(E) = \displaystyle\int_E g \, \mathrm{d}\mu$. Show that ν is a measure on $\mathcal{B}$.

Proof. Let g be a nonnegative measurable function on the measure space $(X, \mathcal{B}, \mu)$. Set $\nu(E) = \displaystyle\int g \, \mathrm{d}$. Let $E = \emptyset$. Then certainly

$$\int_E g = 0$$

and so $\nu(\emptyset) = 0$.

To prove countable additivity, let $\{E_n\}$ be a sequence of sets with $E_i \cap E_j = \emptyset$ for any $i \neq j$. Thus, we have then that

$$\begin{aligned}
\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \int_{\bigcup_{n=1}^{\infty} E_n} g \, \mathrm{d}\mu &= \int g \chi_{\bigcup_{n=1}^{\infty} E_n} \, \mathrm{d}\mu \\
&= \int \sum_{n=1}^{\infty} g \chi_{E_n} \, \mathrm{d}\mu \\
&= \sum_{n=1}^{\infty} \int g \chi_{E_n} \, \mathrm{d}\mu \\
&= \sum_{n=1}^{\infty} \int_E g \, \mathrm{d}\mu \\
&= \sum_{n=1}^{\infty} \nu\left(E_n\right)
\end{aligned}$$

which completes this proof. □

(b) Let f be a nonnegative measurable function on X. Then

$$\int f \, \mathrm{d}v = \int f g \, \mathrm{d}\mu.$$

Proof. Let f be a nonnegative measurable function on X. We will work through two cases: (i) f is a simple function and (ii) f is any other measurable function. Suppose f is a simple function i.e.,

$$f = \sum_{n=1}^{\infty} c_i \chi_{E_i}.$$

Using properties of simple function, we can show the following:

$$\begin{aligned}
\int f \, \mathrm{d}\mu = \sum_{i=1}^{n} c_i \nu(E_i) = \sum_{i=1}^{n} c_i \int_{E_i} g \, \mathrm{d}\mu &\\
&= \sum_{i=1}^{n} c_i \int g \chi_{E_i} \, \mathrm{d}\mu \\
&= \int_E \sum_{i=1}^{n} c_i g \chi_{E_i} \, \mathrm{d}\mu \\
&= \int_E f g.
\end{aligned}$$

Now, suppose f is any measurable but not simple function. Because f is non-negative, there exists an increasing sequence of simple functions $\{\phi_n\}$ such that $f = \lim_{n\to\infty} \phi_n$. Now take the sequence $\{\phi_n g\}$ at each point on X. We have g as non-negative and so $\{\phi_n g\}$ is also an increasing sequence of functions and converges with $fg = \lim_{n\to\infty} \phi_n g$. Thus, having satisfied the properties of the Monotone Convergence Theorem, we have that

$$\int fg \, \mathrm{d}\mu = \lim_{n\to\infty} \int \phi_n g \, \mathrm{d}\mu = \lim_{n\to\infty} \int \phi_n \, \mathrm{d}\nu = \int f \, \mathrm{d}\nu.$$

Therefore, having exhausted all cases, this completes the proof. $\square$