Definition. Let $(X, \mathcal{B}, \mu)$ be a measure space (and will always be implicitly assumed). Let $f \geq 0$ be a nonnonegative measurable function. Then define

$$\int f \, \mathrm{d}\mu := \sup_{\phi} \left\{ \int \phi \, \mathrm{d}\mu \right\}$$

where $0 \leq \phi \leq f$ is a simple function.

Proposition. Note that the following comes from the definition above:

(1) If $c \geq 0$, then $\displaystyle\int cf = c \int f$.

(2) If $0 \leq f \leq g$, then $\displaystyle\int f \leq \int g$.

Remark. We will prove the linearity of integrals in a general measure space i.e., for any $\alpha, \beta \in \mathbb{R}$, we have that

$$\int \alpha f + \int \beta g = \alpha \int f + \beta \int g.$$

Theorem (11.11, Fatuo's Lemma)**.** Let $\{f_n\}$ be a sequence of nonnegative measurable functions. Suppose that $f_n \to f$ almost everywhere on $E \in \mathcal{B}$. Then

$$\int_E f \, \mathrm{d}\mu \leq \varliminf_{n \to \infty} \int_E f_n.$$

Proof. Without loss of generality, we may assume that $f_n \to f$ everywhere for each $x \in E$. We want to show that for any simple function $0 \leq \phi \leq f$, we have

$$\int_E \phi \, \mathrm{d}\mu \leq \varliminf_{n \to \infty} \int_E f_n \, \mathrm{d}\mu.$$

We can write the simple function ϕ in its canonical representation i.e.,

$$\phi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}$$

for $c_i \in \mathbb{R}$.

We have two cases to show: (i) If $\displaystyle\int_E \phi = \infty$, then $\displaystyle\int_E f_n \to \infty$.

Then this means that there exists $i \in \mathbb{N}$ such that $\mu(E_i \cap E) = \infty$. For notation, let $a = c_i$ and $A = E_i \cap E$. Consider the set

$$A_n = \{x \in E : f_k(x) > a, \text{ for all } k \geq n\}.$$

We can note two things:

(1) The sequence $\{A_n\}$ is an increasing sequence i.e., $A_{n+1} \supset A_n$,

(2) And $A = \displaystyle\bigcup_{n=1}^{\infty} A_n$.

Since $\mu(A) = \infty$ and $\{A_n\}$ is an increasing sequence (i.e., the measure of each subsequence A_n is increasing), we know that $\mu(A_n) \to \infty$. This implies that

$$\int_E f_n \geq a \cdot \mu(A_n)$$

From this, we know that

$$\lim_{n\to\infty} = \infty = \int_E \phi$$

and completes this case.

For case (ii), we will suppose $\int_E \phi < \infty$. Define the set

$$A = \{x \in E : \phi(x) > 0\} \in \mathcal{B}$$

which has $\mu(A) < \infty$ because $A \subset E$ and E has finite measure as well.

Let $\varepsilon > 0$ be chosen, and let M be equal to the max of ϕ. For this fixed ε, we can construct a sequence of sets such that

$$A_n = \{x \in E : f_k(x) > (1-\varepsilon)\phi(x), \text{ for all } k \geq n\}.$$

Since $f_n \to f$ on E, we know that $\{A_n\}$ is an increasing sequence as in case (i), and also $\lim_{n\to\infty} A_n = A \subset \bigcup_{n=1}^{\infty} A_n$. In other words, $A \setminus A_n = A \cap A_n^{\mathsf{c}}$ and the sequence $\{A \setminus A_n\}$ is decreasing which means that

$$\bigcap_{n=1}^{\infty} (A \setminus A_n) = \emptyset.$$

This implies that $\lim_{n\to\infty} \mu(A \setminus A_n) = 0$. So for our fixed ε, there exists $n \in \mathbb{N}$ such that $\mu(A \setminus A_n) < \varepsilon$. Then for all $k \geq n$,

$$\begin{aligned}\int_E f_k \geq \int_{A_k} &\geq \int_{A_k} (1-\varepsilon)\phi(x) \\ &\geq (1-\varepsilon)\int_E \phi - \int_{A\setminus A_k} \phi\end{aligned}$$

and so it follows that

$$\begin{aligned}(1-\varepsilon)\int_{A_k} \phi + \int_{A\setminus A_k} \phi &\geq (1-\varepsilon)\int_A \phi \\ &\geq \int_E \phi - \varepsilon\left(\int_E \phi + M\right).\end{aligned}$$

Thus, from the definition of lim inf,

$$\varliminf_{n\to\infty} \int_E f_n \geq int_E \phi - \varepsilon\left(\int_E \phi + M\right)$$

and since ε was arbitrary,

$$\varliminf_{n\to\infty} \int_E f_n \geq \int_E \phi.$$

$\square$

Theorem (11.12, Monotone Convergence Theorem)**.** Let $\{f_n\}$ be a sequence of nonnegative measurable functions such that $f_n \to f$ almost everywhere. Suppose that for all $n \in \mathbb{N}$, $f_n \leq f$. Then

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. By monotonicity, since $f_n \geq f$,

$$\int f_n \leq \int f.$$

Then by Fatuo's lemma,

$$\overline{\lim_{n\to\infty}} \int f_n \leq \int f \leq \underline{\lim_{n\to\infty}} \int f_n.$$

□

Proposition (11.13, Linearity)**.** If $f, g \geq 0$ and $a, b \geq 0$, then

$$\int af + \int bg = a \int f + b \int g.$$

We have $\displaystyle\int f \geq 0$ with equality if and only if $f = 0$ almost everywhere.

Definition. Let $f \geq 0$.

(1) Then f is called integrable over $E \in \mathcal{B}$ if

$$\int f \,\mathrm{d}\mu < \infty$$

or $f \in L^1(E)$, $f \in L^1(\mu)$, or $f \in L^1(X, \mu)$.

(2) A measurable function f is called integrable if f^+ and $f^-$are integrable. In this case.

$$\int_E f := \int_E f^+ \int_E f^-.$$

Proposition. Let $f, g \in L^1(X, \mu)$. Then we have

(1) $\displaystyle\int_E af + bg = a \int_E f + b \int_E g.$

(2) If $|h| \leq |f|$ and h is measurable, then $h \in L^1(X, \mu)$.

(3) If $f \geq g$ almost everywhere, then $\displaystyle\int f \geq \int g.$

Theorem (11.16, Lesbesgue Convergence Theorem)**.** Let $f \in L^1(X, \mu)$ and suppose $\{f_n\}$ is a sequence of measurable functions such that $|f_n(x) \leq g|$ and such that almost everywhere, $f_n(x) \to f$ for $x \in E$. Then

$$\int_E f = \lim_{n\to\infty} f_n.$$

Section 11.3 General Convergence Theorems

Definition. We say $\{\mu_n\}_{n=1}^{\infty}$ converges to μ setwisely if $\mu_n(A) \to \mu(A)$ for all $A \in \mathcal{B}$.

Proposition. Let $\{\mu_n\}$ be a sequence of measures which converges setwisely to μ, and $\{f_n\}$ be nonnegative and converge to f pointwise. Then

$$\int f \, \mathrm{d}\mu = \underline{\lim} \int f_n \, \mathrm{d}\mu.$$