

Problem 1 (5.10). (a) Let f be defined by

$$f(x) = \begin{cases} 0 & x = 0 \\ x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0. \end{cases}$$

Is f of bounded variation on $[-1, 1]$?

Proof. No, this function is not of bounded variation. □

(b) Not assigned.

Problem 2 (5.15). The Cantor ternary function (Problem 2.48) is continuous and monotone but not absolutely continuous.

Proof. By Problem 2.48, the Cantor function is continuous and monotone on $[0, 1]$. Thus, we must show that the Cantor function f is not absolutely continuous. By way of contradiction, suppose that f is indeed absolutely continuous on $[0, 1]$. By Theorem 5.14, for any $x \in [0, 1]$, f can be written as an indefinite integral i.e.,

$$f(x) = \int_0^x f'(t) dt + f(0)$$

or, equivalently,

$$f(x) - f(0) = \int_0^x f'(t) dt.$$

First, we will show that the Cantor set has measure 0. Let $\{C_n\}$ represent the sequence of Cantor sets where $C_0 = [0, 1]$, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, and so on. Note that the Cantor set is represented by $C = \bigcap_{n=1}^{\infty} C_n$. We can see that the measure of the n^{th} Cantor set is $m(C_n) = \left(\frac{2}{3}\right)^n$, and that $\{C_n\}$ is a decreasing sequence of measurable sets. So we can apply Proposition 3.14 and say that

$$\lim_{n \rightarrow \infty} m(C_n) = 0 = m\left(\bigcap_{n=1}^{\infty} C_n\right) = m(C)$$

and so the measure of the Cantor set is 0. By definition of the function, f is constant if an element is not in the Cantor set. Thus f is constant on $[0, 1]$ almost everywhere and so $f'(x) = 0$ almost everywhere as well (i.e., for all $x \in [0, 1] \setminus C$). However, suppose that $x = 1$. Then we have that

$$f(1) - f(0) = 1 \neq \int_0^1 f'(t) dt = 0$$

which contradicts Theorem 5.14. Thus the Cantor function f is not absolutely continuous. □

Problem 3 (5.20). A function f is said to satisfy a Lipschitz condition on an interval if there is a constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all x and y in the interval.

- (a) Show that a function satisfying a Lipschitz condition is absolutely continuous.

Proof. Not assigned. □

- (b) Show that an absolutely continuous function f satisfies a Lipschitz condition if and only $|f'|$ is bounded.

Proof. We will show a forward and reverse implication. Let f be an absolutely continuous function on an interval $I \subset \mathbb{R}$.

- (\Rightarrow) First, suppose that f satisfies a Lipschitz condition. Note that the definition of the derivative says that

$$f'(x) = \lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x}.$$

Because f satisfies a Lipschitz condition, there exists $M \in \mathbb{R}$ such that

$$|f(x) - f(y)| \leq M |x - y|$$

or

$$\frac{|f(x) - f(y)|}{|x - y|} \leq M.$$

It can be shown readily that

$$|f'(x)| = \lim_{y \rightarrow x} \left| \frac{f(y) - f(x)}{y - x} \right|$$

and so we can conclude that

$$|f'(x)| \leq M$$

and therefore f' is bounded.

- (\Leftarrow) We proceed by contraposition, and so first suppose that f does not satisfy the Lipschitz condition. So for any $M > 0$, there exists $x, y \in I$ such that $|f(x) - f(y)| > M |x - y|$. equivalently,

$$\left| \frac{f(x) - f(y)}{x - y} \right| > M.$$

Because f is absolutely continuous and therefore continuous on I , we can apply the Mean Value Theorem. Thus, there exists $c \in I$ such that $|f'(c)| > M$ which completes the proof. □