Remark. Note that if $f \geq 0$,

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is increasing. This implies F is a monotone function and so

$$\int_a^b F'(x)\,\mathrm{d}x \leq F(b) - F(a).$$

Theorem (5.10). Let f be an integrable function on $[a, b]$ and suppose that

$$F(x) = F(a) + \int_a^x f(t)\,\mathrm{d}t.$$

Then $F'(x) = f(x)$ for almost all $x \in [a, b]$.

Proof. Using the remark above, without loss of generality, suppose $f \geq 0$. To use the previous lemma (which supposes f is bounded), define

$$f_n(x) = \begin{cases} f(x) & f(x) \leq n \\ n & \text{otherwise.} \end{cases}$$

Then $f - f_n \geq 0$ for all $n \in \mathbb{N}$. Now define

$$G_n(x) = \int_a^x f - f_n$$

which is an increasing function since $f - f_n$ is nonnegative. So $G_n'(x)$ exists almost everywhere and $G_n'(x) \geq 0$. By Lemma 5.9, since $f_n(x)$ is a bounded function,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,\mathrm{d}t\right) = f_n(x)$$

almost everywhere. Then

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} G_n + \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f_n$$

implies that $F'(x) \geq f_n(x)$ almost everywhere. From the beginning remark (so that F is monotonic), this gives that

$$\begin{aligned} \int_a^b F'(x)\,\mathrm{d}x &\leq F(b) - F(a) \\ &= \int_a^b f(x)\,\mathrm{d}x \end{aligned}$$

and so we get that

$$\int_a^b \Bigg(\underbrace{F'(x) - f(x)}_{\geq 0}\Bigg)\,\mathrm{d}x = 0$$

implying that $F'(x) = f(x)$ almost everywhere. ∎

Section 5.4 Absolute Continuity

Definition. A real-valued function f on $[a, b]$ is said to be **absolutely continuous** on $[a, b]$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{i=}^{n} |f(x_i') - f(x_i')|$$

for every finite collections of $\{(x_i, x_i')\}$ of nonoverlapping intervals with

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta.$$

Lemma (5.11). If f is absolutely continuous on $[a, b]$, then it is of bounded variation on $[a, b]$.

Proof. Let $\varepsilon = 1$. Then there exists $\delta > 0$ for the absolutely continuity property. Let $K = \left\lceil \frac{b-a}{\delta} + 1 \right\rceil$ For this partition of $[a, b]$, we group the intervals into K sets of intervals each with total length less than δ. □

Lemma (5.13). If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ almost everywhere, then f is constant.

Theorem (5.14). A function F is an indefinite integral if and only F is absolutely continuous.

Remark. The above theorem tells us that there exists an integrable function f such that

$$F(x) = F(a) + \int_a^x f(t)\,\mathrm{d}t.$$