Problem 1 (6.21). (a) Let g be an integrable function on $[0,1]$. Show that there is a bounded measurable function f such that $\|f\| \neq 0$

$$\int fg = \|g\|_1 \cdot \|f\|_\infty .$$

Proof. Let g be an integrable function on $[0,1]$. This brings two cases: (i) $\|g\|_1 = 0$ or (ii) $\|g\|_1 \neq 0$. For case (i), if $\|g\|$, then $g = 0$ almost everywhere. Thus, let $f = 1$ which gives us that

$$\int 1 \cdot g = \int g = 0 = \|g\|_1 \cdot \|f\|_\infty .$$

Now suppose $\|g\|_1 \neq 0$. Define $f = \operatorname{sgn} g$. Then f is a bounded and measurable function, $\|f\|_\infty = 1$, and thus

$$\int fg = \int |g| = \|g\|_1 = \|g\| \, \|f\|_\infty .$$

So having exhausted all cases, this completes the proof. □

(b) Let g be a bounded measurable function. Show that for each $\varepsilon > 0$, there is an integrable function f such that

$$\int fg \geq (\|g\|_\infty - \varepsilon) \, \|f\|_1 .$$

Proof. Let g be a bounded measurable function, and let $\varepsilon > 0$ be chosen. Define the set $E = \{x : g(x) > \|g\|_\infty - \varepsilon\}$ and the function f by $f(x) = \chi_E(x)$. Then we have that

$$\int fg = \int_E g \geq (\|g\|_\infty - \varepsilon) \, m(E) = (\|g\|_\infty - \varepsilon) \cdot \|f\|_1$$

and which completes the proof. □

Problem 2 (11.3). (a) Show that $\mu(E_1 \bigtriangleup E_2) = 0$ implies $\mu(E_1) = \mu(E_2)$ provided that $E_1, E_2 \in \mathcal{B}$.

Proof. Let $E_1, E_2 \in \mathcal{B}$ and suppose that $\mu(E_1 \bigtriangleup E_2) = 0$. This means that

$$\mu(E_1 \setminus E_2) = \mu(E_2 \setminus E_1) = 0.$$

From this, we can write E_1 and E_2 as disjoint unions and show that

$$\mu(E_1) = \mu(E_1 \setminus E_2) + \mu(E_1 \cap E_2) = \mu(E_2 \setminus E_1) + \mu(E_1 \cap E_2) = \mu(E_2)$$

which shows the desired result. □

(b) Not assigned.