Section 5.1 Differentiation of Monotone Functions

Our goal will be to show if f is monotone increasing over an interval $[a, b]$, then f is differentiable a.e.

Definition (Vitali Cover). Let $E \subset \mathbb{R}$ and Γ is a collection of intervals. We call Γ a **Vitali cover** of E if for all $x \in E$ and all $\varepsilon > 0$, there $I \in \Gamma$ such that $x \in I$ and $0 < |I| < \varepsilon$.

Here is an example. Let $E = [0, 1]$. Then

$$\Gamma_1 = \left\{ \left(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2} \right) \right\}$$

[there was more to the above... finish this after]

Lemma (5.1). Let $E \subset \mathbb{R}$, $m^*(E) < \infty$, and Γ is a Vitali cover of E. Then for all $\varepsilon > 0$ there a finite disjoint collection $\{I_1, \ldots, I_n\} \subset \Gamma$ such that

$$m^* \left(E \setminus \bigcup_{i=1}^{N} I_i \right) < \varepsilon.$$

Definition (Dini Derivatives). We define the **Dini Derivatives** as follows:

$$D^+ f(x) = \varlimsup_{h \to 0+} \frac{f(x+h) - f(x)}{h}$$
$$D^- f(x) = \varlimsup_{h \to 0+} \frac{f(x) - f(x-h)}{h}$$
$$D_+ f(x) = \overline{\lim_{h \to 0+}} \frac{f(x+h) - f(x)}{h}$$
$$D_- f(x) = \overline{\lim_{h \to 0+}} \frac{f(x) - f(x-h)}{h}$$

We can remark by definition the following

1. $$D^+ f(x) \geq D_+ f(x)$$
   $$D^- f(x) \geq D_- f(x).$$

2. Also, f is differentiable if and only if $D_+^+ = D_-^-$.

Lemma. Let $f : [a, b] \to \mathbb{R}$ be an increasing function, and let $\gamma < R$. Then

$$E = E_{\gamma, R} = \left\{ x \in (a, b) : D_- f(x) < \gamma < R < D^+ f(x) \right\}.$$

has measure zero.

Proof. Write $m^* E = s < \infty$. Let $\varepsilon > 0$ be chosen. By definition of outer measure, there exists an open set $E \subset O$ such that $m(O) < s + \varepsilon$. Let $x \in E$. Then $D_- f(x) < \gamma$ which implies that for any $\delta > 0$, there exists $h \in (0, \delta)$ with

$$\frac{f(x) - f(x-h)}{h} < \gamma$$

coming from the definition of $\liminf$. Then the collection $[x-h,x]$ forms a Vitali cover of E. By the definition of Vitalia covering Lemma (**Lemma 5.1**), for our fixed $\varepsilon > 0$, there exists a finite collection $\{I_1 = [x_1 - h, x_1], \ldots, I_N = [x_N - h, x_N]\}$ such that

$$m^*\left(E \setminus \bigcup_{k=1}^{N} I_k\right) < \varepsilon.$$

Let $A = E \cap \bigcup_{k=1}^{N} I_k$. Then $m^*(A) > s - \varepsilon$ (recalling $s = m^*(E)$.) This implies that

$$\begin{aligned} 0 \le \sum_{k=1}^{n} f(x_k) - f(x_k - h_k) &< \gamma \sum_{k=1}^{N} h_k \\ &= m^*\left(\bigcup_{k=1}^{N} I_k\right) \\ &< \gamma \cdot (s+\varepsilon) \qquad \text{because } \bigcup_{k=1}^{N} I_k \subset O \end{aligned}$$

and so we are done with this part.

Let $y \in A$, and by $\limsup$ we know that $D^+ f(x) > R$. Then there exists an arbitrarily small $k > 0$ such that $[y, y+k] \subset I_n$ for some $n \in \mathbb{N}$ such that

$$f(y+k) - f(y) > R \cdot k.$$

Then the collection formed from $[y, y+k]$ forms a Vitali cover on A. Again, by the Vitalia cover lemma (**Lemma 5.1**), there exists a collection

$$J = \{J_1 = [y_1, y_1 + k_1], \ldots, J_M = [y_M, y_M + k_M]\}$$

such that

$$m^*\left(A \setminus \bigcup_{j}^{M} J_j\right) < \varepsilon.$$

This implies that

$$m^*\left(A \cap \setminus \bigcup_{j}^{M} J_j\right) > s - 2\varepsilon.$$

So we can sum across these intervals and get that

$$\begin{aligned} \sum_{j=1}^{M} f(y_j + k_j) - f(y_j) &> R \sum_{j=1}^{M} k_j \\ &> R(s - 2\varepsilon). \end{aligned}$$

Putting this all together and noting that f is an increasing function

$$\begin{aligned} \sum_{n=1}^{N} f(x_n) - f(x_n - h_n) &\ge \sum_{j=1}^{M} f(y_j + k_j) - f(y_j) \\ &> R(s - 2\varepsilon) \end{aligned}$$

and so we get that

$$\gamma(s+\varepsilon) < R(s-2\varepsilon).$$

Thus $\gamma s \geq Rs$ and implies that $s = 0$ (noting that $R > \gamma$). Now we are done! $\square$

Theorem (5.3). Let $f : [a,b] \to \mathbb{R}$ be monotonic increasing. Then f is differentiable almost everywhere, f' is measurable, and

$$\int_a^b f'(x)\,\mathrm{d}\,x \leq f(b) - f(a).$$

Proof. Let $E = \{x : D_- f(x) < D^+ f(x)\}$. Then

$$E = \bigcup_{r,R\in\mathbb{Q}} E_{r,R}$$

has a measure zero by our lemma above. This tells us that f is differentiable almost everywhere because we showed that the sets where any two derivates are not equal have measure zero. Thus

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists almost everywhere. Define the function

$$f(x) = \begin{cases} f(b) & x \geq b \\ f(x) & x \in [a,b] \end{cases}$$

which extends the function f to the right. Define the sequence $\{G_n\}$ by

$$G_n(x) = \frac{f\left(x + \frac{1}{n}\right) - f(x)}{\frac{1}{n}} \geq 0.$$

This sequence converges a.e. to $f'(x)$ as $n \to \infty$. So by Fatou's lemma,

$$\begin{aligned}
\int_a^b f'(x)\,\mathrm{d}x &\leq \varliminf_{n\to\infty} \int_a^b G_n(x)\,\mathrm{d}x \\
&= \varliminf_{n\to\infty} \int_a^b n\left(f\left(x + \frac{1}{n}\right) - f(x)\right)\mathrm{d}x \\
&= \int_{a+1/n}^{b+1/n} f(y)\,\mathrm{d}y \\
&= \varliminf_{n\to\infty} n \int_{a+1/n}^{b+1/n} f(y)\,\mathrm{d}x - \int_a^{a+1/n} f(x)\,\mathrm{d}x \\
&= \varliminf_{n\to\infty} n \left(\int_b^{b+1/n} f(y)\,\mathrm{d}x - \int_a^{a+1/n} f(x)\,\mathrm{d}x\right) \\
&= \varliminf_{n\to\infty} n \cdot f(b) \cdot \frac{1}{n} - \varliminf_{n\to\infty} \int_a^{a+1/n} f(x)\,\mathrm{d}x \\
&\leq f(b) - \varliminf_{n\to\infty} \int_a^{a+1/n} f(x)\,\mathrm{d}x \\
&\leq f(b) - f(a) && \text{noting that } f(x) \geq \frac{1}{n}
\end{aligned}$$

$\square$

The next thing that will be covered is functions of bounded variation in Section 5.2.

Definition. Let $f : [a,b] \to \mathbb{R}$. Then f is a function of **bounded variation** if $\|f\|_{TV[a,b]} < \infty$ where $TV[a,b]$ is the total variation of $[a,b]$ and

$$\|f\|_{TV[a,b]} = \sup\left\{\sum_{i=0}^{n} |f(x_{i+1}) - f(x)| : P \text{ is a partition of } [a,b]\right\}.$$