Problem 1 (4.2). (a) Let f be a bounded function on $[a,b]$, and let h be the upper envelope of f (cf. Problem 2.51). Then $R\overline{\int_a^b} f = \int_a^b h$.

Proof. Let f be a bounded function on $[a,b]$ with $h(y) = \inf_{\delta>0} \sup_{|x-y|<\delta} f(x)$ for all $x \in [a,b]$ be the upper envelope of f. Because f is bounded, by Problem 2.51 part (b), h is lower semicontinuous. To show equality, we will show that

$$R\overline{\int_a^b} f \leq \int_a^b h \quad \text{and} \quad R\overline{\int_a^b} f \geq \int_a^b h.$$

Let ϕ be a step function on $[a,b]$ such that $\phi \geq f$. Then for any $x \in [a,b]$, $h(x) \leq f(x) \leq \phi(x)$, except at the defined partition points of ϕ. Thus, by using the definition of Lebesgue integration, we have that

$$\int_a^b h(x)\,\mathrm{d}x \leq \int_a^b f(x)\,\mathrm{d}x = \inf \int_a^b \phi(x)\,\mathrm{d}x \leq R\overline{\int_a^b} f(x)\,\mathrm{d}x.$$

For the other inequality, we note that because h is upper semicontinuous, by Problem 2.51 part (g), there exists a monotonically decreasing sequence of step functions $\{\phi_n\}$ such that $\phi_n \to h$ pointwise. Because f is bounded, we have that for all $x \in [a,b]$, there exists some $M > 0$ such that

$$|\phi_n| \leq |h| \leq |f| \leq M \text{ for all } n \in \mathbb{N}.$$

Thus using the Bounded Convergence Theorem and properties of the upper Riemann integral,

$$\lim_{n\to\infty} \int_a^b \phi_n(x)\,\mathrm{d}x = \int_a^b h(x)\,\mathrm{d}x \leq R\int_a^b f(x)\,\mathrm{d}x \leq R\overline{\int_a^b} f(x)\,\mathrm{d}x.$$

Therefore,

$$R\overline{\int_a^b} f = \int_a^b h$$

which is the desired result. □

(b) Use part (a) to prove Proposition 7 which is stated as follows

Proposition (4.7). A bounded function f on $[a,b]$ is Riemann integrable if and only if the set of points at which f is discontinuous has measure zero.

Proof. Let f be a bounded function on $[a,b]$. We will need to show a forward and backwards implication to complete this proof. For simplicity, define E to be the set of discontinuities of f. Additionally, let $g(y) = \sup_{\delta>0} \inf_{|x-y|<\delta} f(x)$ be the lower envelope of f.

($\Leftarrow$) First, suppose $m(E) = 0$. Since g is the lower envelope of f, there exists a monotonically increasing sequence of step functions $\{\phi_n\} \to g$ pointwise. Thus, by a similar but reverse argument to part (a),

$$\int_a^b g(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x. \tag{1}$$

So because f is continuous everywhere except on the set E—namely, continuous on $[a,b]\setminus E$— by Problem 2.51, $g(x)=h(x)$ is continuous on the set $[a,b]\setminus E$. But since $m(E)=0$, this means $g=h$ almost everywhere and thus, using part (a) of this problem and Equation (1),

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x = \int_a^b h(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x.$$

Thus, f is Riemann integrable.

($\Rightarrow$) Now suppose f is Riemann integrable. Thus, the lower and upper integrals of f are equal to each other and so using Equation (1)

$$\int_a^b h(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x.$$

Consider the set $A_n = \left\{x : |g(x)-h(x)| > \frac{1}{n}\right\}$ for all $n\in\mathbb{N}$. Because the integrals of g and h are equal,

$$\int_a^b |h(x)-g(x)|\,\mathrm{d}x = 0.$$

So for any fixed $n\in\mathbb{N}$,

$$\int_a^b |h(x)-g(x)|\,\mathrm{d}x \geq m(A_n).$$

So $h(x)=g(x)$ almost everywhere and so by Problem 2.51 part(a), we must that f is continuous almost everywhere as well. But then this means that the measure of discontinuities is zero—that is, $m(E)=0$ which is what we wanted to show.

Having completed both directions, we have shown the desired equivalence. □

Problem 2 (4.3)**.** Let f be a nonnegative measurable function. Show that $\displaystyle\int f = 0$ implies $f=0$ almost everywhere.

Proof. Let $f\geq 0$ be a measurable function, and suppose that $\displaystyle\int f = 0$. We want to show that the set $E = x : f(x)\neq 0 = \{x : f(x)>0\}$ has measure 0. Define the set

$$E_n = \left\{x : f(x)\geq \frac{1}{n}\right\} \text{ for all } n\in\mathbb{N}.$$

Note that $\displaystyle\bigcup_{n=1}^{\infty} E_n = E$. Fix $n\in\mathbb{N}$. Because the integral of f is equal to 0,

$$0 = \int_{E_n} \geq \int_{E_n} \frac{1}{n} = m(E_n)\cdot\frac{1}{n} \geq 0.$$

Thus, because $n\in\mathbb{N}$ was fixed, $m\left(\displaystyle\bigcup_{n=1}^{\infty} E_n\right) = 0 = m(E)$, and therefore $f=0$ almost everywhere. □

Problem 3 (4.8)**.** Prove the following generalization of Fatuo's Lemma: If f_n is a sequence of nonnegative functions then

$$\int \varliminf_{n\to\infty} f_n \leq \varliminf_{n\to\infty} \int f_n.$$

Proof. Let $\{f_n\} \geq 0$ for each $n \in \mathbb{N}$ on any set E. Define $h_n = \inf_{k\geq n} f_k$ for all $n \in \mathbb{N}$. Note that as $n \to \infty$, $h_n \to \varliminf_{n\to\infty} f_n$ (i.e., h_n converges pointwise on E to the limit inferior of f_n). Thus, by Fatou's lemma, we have that

$$\int_E \varliminf_{n\to\infty} f \leq \int_E \varliminf_{n\to\infty} h_n.$$

But since h_n is the infimum of the f_n's, this implies that $h_n \leq f_n$ for all $n \in \mathbb{N}$ and so

$$\int_E h_n \leq \int_E f_n$$

and thus

$$\varliminf_{n\to\infty} \int_E h_n \leq \varliminf_{n\to\infty} \int_E f_n.$$

By combining all three inequalities, we get that

$$\int_E \varliminf_{n\to\infty} f \leq \varliminf_{n\to\infty} \int_E h_n \leq \varliminf_{n\to\infty} \int_E f_n$$

which then completes the proof. $\square$