Theorem (2.3, **Axiom of Archimedes**). If $x \in \mathbb{R}$ is any real number, then there exists $n \in \mathbb{N}$ such that $x < n$.

Proof. We can break this into two cases

1. Let $x < 1$. If so, then simply choose $x = 1$.
2. Let $x \geq 1$. Define the set $S = \{n \in N : n \leq x\}$. Then since this set is bounded above, by the Completeness Axiom, $\sup S = y$ exists. Because x is an upper bound S, by definition of the supremum, we have that $y \leq x$. Let $r = \frac{1}{2}$. Then we can find $k \in S$ such that $y - \frac{1}{2} < k \leq y$. But then we have that $y < y + \frac{1}{2} < k + 1 \leq y + 1$. Then this means $k + 1 \notin S$ and so $x < k + 1$, completing this case.

Having exhausted all cases, this completes the proof. $\square$

Proposition 1 (**Well-Ordering Principle**). Every nonempty subset $S \subset \mathbb{N}$ has a minimum.

Proposition 2 (**Density of the Rational Numbers**). Let $x, y \in \mathbb{R}$. Then if $x < y$, there exists $q \in \mathbb{Q}$ such that $x < \gamma < y$

Section 2.4, Sequences in $\mathbb{R}$

Definition. We define a **sequence** of real numbers to be a function that maps each each natural number n into the real number x. That is, a sequence is a function $s : \mathbb{N} \to A$ for $A \subset \mathbb{R}$. This is written as $\{x_n\}$ or $\{x_n\}_{n=1}^{\infty}$.

Definition (**Convergence of a Sequence**). A sequence converges to the real number $l \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - l| < \varepsilon.$$

Definition (**Cauchy Sequence**). A sequence $\{x_n\}$ in $\mathbb{R}$ is **Cauchy** sequence if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$|a_n - a_m| < \varepsilon.$$

Theorem. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then $\{x_n\}$ is Cauchy if and only $\{x_n\}$ is Cauchy.

Definition. The number $l \in \mathbb{R}$ is called a **cluster point** of $\{x_n\}$ if there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $x_{n_m} \to l$.

We can define this in another way. The number $l \in \mathbb{R}$ is called **cluster point** of $\{x_n\}$ if for all $\varepsilon > 0$ and for all $N \in \mathbb{N}$, there exists $n \geq N$ such that $|x_n - l| < \varepsilon$.

Definition. We define the **limit superior** of a sequence $\{x_n\}$ in $\mathbb{R}$ to be

$$\overline{\lim}_{n\to\infty} x_n = \inf_n \sup_{k \geq n} x_k.$$

This is also denoted as $\limsup$.

Theorem. A number $l \in \mathbb{R}$ is the **limit superior** of the sequence $\{x_n\}$ if and only if

(i) For all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, $x_k < l + \varepsilon$

(ii) For all $\varepsilon > 0$ and for all $n \in \mathbb{N}$, there exists $k \geq n$ such that $x_k > l - \varepsilon$.

Definition. We define the **limit inferior** of a sequence $\{x_n\}$ in $\mathbb{R}$ to be

$$\underline{\lim}_{n\to\infty} x_n = \sup_n \inf_{k\geq n} x_k.$$

This is also denoted as $\liminf$.

Theorem. A number $l \in \mathbb{R}$ is the **limit inferior** of the sequence $\{x_n\}$ if and only if

(i) For all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, $x_k > l - \varepsilon$

(ii) For all $\varepsilon > 0$ and for all $n \in \mathbb{N}$, there exists $k \geq n$ such that $x_k < l + \varepsilon$.

Proposition 3. From the last two definitions, we have the following property.

- $\overline{\lim}_{n\to\infty}$ is the largest cluster point.
- $\underline{\lim}_{n\to\infty}$ is the smallest cluster point.

Section 2.5, Open and Closed Sets in $\mathbb{R}$

Definition. The set $O \subset \mathbb{R}$ is called an **open** set if for all $x \in O$, there exists $\delta > 0$ such that $x - \delta, x + \delta$.

Equivalently, O is an **open** set if for all $x \in O$, there is a $\delta > 0$ such that each y with $|x - y| < \delta$ belongs to O.

Proposition 4. From this above, we have the following properties:

1. The set $\bigcup_\alpha O_\alpha$ is open.
2. The set $\bigcup_{n=1}^{n} O_m$ is open.

Theorem (Lindelof Theorem). Every open set in $\mathbb{R}$ is a disjoint union of countable union of open intervals.

Proof. This proof is contained on page 42 of Royden. □

Definition. A real number $x \in \mathbb{R}$ is called **point of closure** of a set $E \subset \mathbb{R}$ if for every $\delta > 0$ there exists a $y \in E$ such that $|x - y| < \delta$.

The set of points of closure of E is denoted $\overline{E}$.

Proposition 5. If $A \subset B \subset \mathbb{R}$, then $\overline{A} \subset \overline{B}$. Additionally, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. The proof of this is on page 43 of Royden. □

Definition. A set $F \subset \mathbb{R}$ is called a **closed** set if $\overline{F} = F$.

Note that because $F \subset \overline{F}$ always, a set F is closed if $\overline{F} \subset F$—that is, F contains all of its points of closure.

Proposition 6. For any set E, the set $\overline{E}$ is closed; that is $\overline{\overline{E}} = \overline{E}$.

Proposition 7. Let $E \subset \mathbb{R}$. Then E is open if and only if $E^{\mathcal{C}}$ is closed.

Definition. We say that a collection of sets $\mathcal{C}$ is a **cover** of a set F if

$$F \subset \bigcup_{O \in \mathcal{C}} O.$$

The collection $\mathcal{C}$ is a covering of the set F.

Theorem (Heine-Borel). Let $E \subset \mathbb{R}$ be set. Then E is compact if and only if E is closed and bounded.