Problem 1 (5.10)**.** (a) Let f be defined by

$$f(x) = \begin{cases} 0 & x = 0 \\ x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0. \end{cases}$$

Is f of bounded variation on $[-1, 1]$?

Proof. It will suffice to show that the function is not of bounded variation on $[0, 1]$ as the function is symmetric across the y-axis and thus the same argument holds for $[-1, 1]$. Consider the following subdivision/partition $\mathcal{P}$ of $[0, 1]$:

$$0 < \sqrt{\frac{1}{\pi n}} < \cdots < \sqrt{\frac{2}{\pi(1+4n)}} < 1$$

for any $n \in \mathbb{N}$. Note that we picked these points because

$$f\left(\sqrt{\frac{1}{\pi n}}\right) = 0 \quad \text{and} \quad f\left(\sqrt{\frac{2}{\pi(1+4n)}}\right) = 1$$

since $\sin\left(\frac{1}{x^2}\right) = 1$ when $x = \sqrt{\frac{1}{\pi n}}$ and $\sin\left(\frac{1}{x^2}\right) = 1$ when $x = \frac{2}{\pi(1+4n)}$.
Additionally, note that the range of f is $[0, 1]$. When a point $x \in [0, 1]$ can be written as $\sqrt{\frac{1}{\pi n}}$ for some $n \in \mathbb{N}$, the variation of f across all of these points is 0. The maximum of f is 1 and so this means the total variation of f is determined by x^2 where $\sin\left(\frac{1}{x^2}\right) = 1$. So we have that

$$T_f = \sum_{n=1}^{k} \left| \left(\sqrt{\frac{2}{\pi(1+4n)}}\right)^2 \right| = \sum_{n=1}^{k} \frac{2}{\pi(1+4n)} = \frac{2}{\pi} \sum_{i=1}^{k} \frac{1}{1+4n}.$$

The series on the right-hand side is of a form similar to the harmonic series so as $k \to \infty$, this means $T_f \to \infty$. Therefore, the function f is not of bounded variation. □

(b) Not assigned.

Problem 2 (5.15)**.** The Cantor ternary function (Problem 2.48) is continuous and monotone but not absolutely continuous.

Proof. By Problem 2.48, the Cantor function is continuous and monotone on $[0, 1]$. Thus, we must show that the Cantor function f is not absolutely continuous. By way of contradiction, suppose that f is indeed absolutely continuous on $[0, 1]$. By Theorem 5.14, for any $x \in [0, 1]$, f can be written as an indefinite integral i.e.,

$$f(x) = \int_0^x f'(t)\,\mathrm{d}t + f(0)$$

or, equivalently,

$$f(x) - f(0) = \int_0^x f'(t)\,\mathrm{d}t.$$

First, we will show that the Cantor set has measure 0. Let $\{C_n\}$ represent the sequence of Cantor sets where $C_0 = [0, 1]$, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, and so on. Note that the Cantor is represented by $C = \bigcap_{n=1}^{\infty} C_n$. We can see that the measure of the n^{th} cantor set is $m(C_n) = \left(\frac{2}{3}\right)^n$, and that $\{C_n\}$ is a decreasing sequence of measurable sets. So we can apply Proposition 3.14 and say that

$$\lim_{n\to\infty} m(C_n) = 0 = m\left(\bigcap_{n=1}^{\infty} C_n\right) = m(C)$$

and so the measure of the Cantor set is 0. By definition of the function, f is constant if an element is not in the Cantor set. Thus f is constant on $[0, 1]$ almost everywhere and so $f'(x) = 0$ almost everywhere as well (i.e., for all $x \in [0, 1] \setminus C$). However, suppose that $x = 1$. Then we have that

$$f(1) - f(0) = 1 \neq \int_0^x f'(t)\,\mathrm{d}t = 0$$

which contradicts Theorem 5.14. Thus the Cantor function f is not absolutely continuous. □

Problem 3 (5.20)**.** A function f is said to satisfy a Lipschitz condition on an interval if there is a constant M such that $|f(x) - f(y)| \leq M\,|x - y|$ for all x and y in the interval.

(a) Show that a function satisfying a Lipschitz condition is absolutely continuous.

Proof. Not assigned. □

(b) Show that an absolutely continuous function f satisfies a Lipschitz condition if and only $|f'|$ is bounded.

Proof. We will show a forward and reverse implication. Let f be an absolutely continuous function on an interval $I \subset \mathbb{R}$.

($\Rightarrow$) First, suppose that f satisfies a Lipschitz condition. Note that the definition of the derivative says that

$$f'(x) = \lim_{y\to x} \frac{f(y) - f(x)}{y - x}.$$

Because f satisfies a Lipschitz condition, there exists $M \in \mathbb{R}$ such that

$$|f(x) - f(y)| \leq M\,|x - y|$$

or

$$\frac{|f(x) - f(y)|}{|x - y|} \leq M.$$

It can be shown readily that

$$|f'(x)| = \lim_{y\to x} \left|\frac{f(y) - f(x)}{y - x}\right|$$

and so we can conclude that

$$|f'(x)| \leq M$$

and therefore f' is bounded.

($\Leftarrow$) We proceed by contraposition, and so first suppose that f does not satisfy the Lipschitz condition. So for any $M > 0$, there exists $x, y \in I$ such that $|f(x) - f(y)| > M\,|x - y|$. equivalently,

$$\left|\frac{f(x) - f(y)}{x - y}\right| > M.$$

Because f is absolutely continuous and therefore continuous on I, we can apply the Mean Value Theorem. Thus, there exists $c \in I$ such that $|f'(c)| > M$ which completes the proof.

$\square$