Theorem (6.6, Riesz-Fisher). L^p is complete for $p \in [1, \infty]$.

Proof. Note $p = \infty$ is an exercise. We want to show every absolute summable series is summable. Let $\{f_n\} \subset L^p$ such that

$$\sum_{n=1}^{\infty} \|f_n\| < M.$$

Consider the function $g_n(x) = \sum_{k=1}^{n} f_k(x)$ and want to show that g_n converges some function g (i.e., the limit exists.) By the triangle inequality,

$$\|g_n(x)\| \leq \sum_{k=1}^{n} \|f_k(x)\| < M$$

and so we know that $\int |g_n|^p < M^p$. We know we want to do the following:

1. Find a limit of g_n.
2. Then show it is in L^p.

For each fixed $x \in L^P$, $\{g_n(x)\}$ is monotonically increasing. By the Monotone Convergence Theorem, there exists $g(x) \in \overline{\mathbb{R}}$ (extended real numbers) and $g_n(x) \to g(x)$. Because $g_n \geq 0$ is measurable, $g(x) \geq 0$ is measurable as well an dso

$$\int g^p \leq M^p$$

and so this implies that $g \in L^p$. Notice that for each x such that $g(x)$ is finite, $\sum_{k=1}^{\infty} f_k(x)$ is absolutely summable over $\mathbb{R}$. Since $(\mathbb{R}, |\cdot|)$ is complete, we know that $S_n(x) = \sum_{k=1}^{n} f_k(x)$ is summable i.e.,

$$S_n(x) \to S(x) = \sum_{k=1}^{f_k(x)}$$

over $\mathbb{R}$. We can look at just the limit $S(x)$ because we are looking at only where $g(x)$ has finite measure and defined as

$$\widetilde{S} = \begin{cases} S_n(x) & g(x) < \infty \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $S(x)$ is measurable since $S(x) \neq \sum_{k=1}^{\infty} f_k(x)$ only on a measure zero set. Moreover, $|S_n(x)| \leq |g(x)|$ for all $n \in \mathbb{N}$ and so $|\widetilde{S}(x)| \leq |g(x)|$ and therefore $\widetilde{S}(x) \in L^p$. We claim that

$$\left\| \sum_{k=1}^{n} f_n(x) - \widetilde{S}_n(x) \right\|_p \to 0 \text{ as } n \to \infty.$$

We know that because $S \in L^p$,

$$\left|S_n(x) - \widetilde{S}_n(x)\right|^p \leq 2^p \cdot |g(x)|^p.$$

So by the Lebesgue Dominated Convergence, we know that $\|s_n - \widetilde{s}_n\|^p \to 0$ and so $\|s_n - \widetilde{s}_n\| \to 0$. Thus the sum s from $\{f_n\}$ is in L^p. $\square$

Section 6.4 Approximation in L^p

Our goal with approximation properties is to approximate functions in $f \in L^p$ spaces by step functions ϕ and continuous functions g. That is, for any $\varepsilon > 0$, there exists functions ϕ and g such that

$$\|f - g\|_p < \varepsilon \quad \text{and} \quad \|f - \phi\|_p < \varepsilon.$$

Lemma (6.7). Given $f \in L^p$, $p \in [1, \infty]$, and any $\varepsilon > 0$, there is a bounded measurable functions f_M with $|f_m| \leq M$ and $\|f - f_M\| < \varepsilon$.

Proof. Consider the function

$$f_N(x) = \begin{cases} N & N \leq f(x) \\ f(x) & -N \leq f(x) \leq N \\ -N & f(x) \leq -N. \end{cases}$$

$\square$

Note that $|f_n| \leq N$ and so is measurable for each $n \in \mathbb{N}$. Then $f_N(x) \to f(x)$ converges a.e (there may be unbounded points but they are of measure zero) which implies that $|f_N(x) - f(x)|^p \to 0$ a.e. Thus,

$$|f_N(X) - f(x)|^p < 2|f(x)|^p$$

almost everywhere. So by the Lebesgue Dominated Convergence Theorem,

$$\int |f_N(x) - f(x)|^p \to 0$$

or, equivalently, $\|f_N - f(x)\|_p \to 0$ and so $f_n(x) \to f(x)$ is in L^p.

Proposition (6.8). Given $f \in L^p$, $p \in [1, \infty)$ and any $\varepsilon > 0$, there is a step function ϕ and a continuous function g such that

$$\|f - g\|_p < \varepsilon \quad \text{and} \quad \|f - \phi\|_p < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be chosen. For step functions, we note that by Lemma 6.7, there exists an f_M such that $\|f - f_M\| < \dfrac{\varepsilon}{2}$. By Theorem 3.22 (Littlewood's 2nd Principle), there exists a step function ϕ such that $|f_M - \phi| < \frac{\varepsilon}{4}$ except on a set E of measure less than

$$\delta = \left(\frac{\varepsilon}{8M}\right)^p.$$

Then

$$\begin{aligned}
\|f_M - \phi\|_p &= \int |f_M - \phi|^p \\
&= \int_{[0,1]\setminus E} \int |f_M - \phi|^p + \int_E |f_M - \phi|^p \\
&\leq \int_{[0,1]} \left(\frac{\varepsilon}{4}\right)^p + \int_E |f_M - \phi|^p \\
&= \int_{[0,1]} \left(\frac{\varepsilon}{4}\right)^p = (2M)^p \cdot m(E) \\
&\leq \int_{[0,1]} \left(\frac{\varepsilon}{4}\right) + (2M)^p \cdot \left(\frac{\varepsilon}{8M}\right)
\end{aligned}$$

and taking the pth root, we know that

$$\|f_M - \phi\|_p \leq \frac{\varepsilon}{2}.$$

Doing the same thing with a continuous functions f than step function, we can show that

$$\|f_M - g\| < \frac{\varepsilon}{2}.$$

$\square$

This means that "step functions" and "continuous functions" are "dense" in L^p (i.e., we can always use step and continuous functions in the limit to approximate functions $f \in L^p$).