The goal of this lecture is to derive a different version of the fundamental theorem of calculus.

Definition. Let $f : [a, b] \to R$, and let $P = \{x_0 = a, x_1, \ldots, x_k = b\}$ be a partition of $[a, b]$. Then define

$$p = \sum_{i=1}^{k} \left(f(x_i) - f(x_{i-1})\right)^+$$
$$n = \sum_{i=1}^{k} \left(f(x_i) - f(x_{i-1})\right)^- .$$

Recall that

$$f^+(x) = \max\{f(x), 0\}$$
$$f^-(x) = \max\{-f(x), 0\}$$
$$f(x) = f^+(x) - f^-(x)$$
$$|f(x)| = f^+(x) + f^-(x).$$

Then

$$t = n + p = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|.$$

Further, define

$$P = \sup_P p$$
$$N = \sup_P n$$
$$T = \sup_P t$$

over all partitions P of $[a, b]$. Then P is the **positive variation** of f, N is the **negative variation** of f, and T is the **total variation** of f.

Note that $f(b) - f(a) = p - n$. Also, for each partition of $[a, b]$, $p \leq T \leq p + n$.

Definition. Using the same structure of the definition above, f is a function of **bounded variation** if

$$T = T_f < \infty.$$

This tells us that the function is not "wiggling" that much (an example of a function that is not of bounded variation is $f(x) = \sin(1/x)$.)

Lemma (5.4). If $f : [a, b] \to \mathbb{R}$ is a function of bounded variation on $[a, b]$, then

$$T_a^b = P_a^b + N_a^b$$

and

$$f(b) - f(a) = P_a^b - N_a^b.$$

Proof. For any partition of $[a, b]$,

$$p = n + f(b) - f(a);$$

in other words, for any partition P of $[a, b]$. Taking the supremum over any fixed partition,

$$p = N + f(b) - f(a).$$

Further,

$$\begin{aligned} t = p + n &= p + (p - f(b) + f(a)) \\ &= 2p - f(b) + f(a). \end{aligned}$$

Taking the supremum over all partitions again,

$$T = 2P - f(b) - f(a) = P + N$$

and so we are done! □

Theorem (5.5). A function f is of bounded variation on $[a, b]$ if and only if f is the difference of two monotone (increasing) real-valued functions $[a, b]$.

Proof. We will show two directions to complete this proof.

($\Rightarrow$) First, we will note that the functions P_a^x, N_a^x, and T_a^x are increasing functions in x. We also know that $0 \leq P_a^x \leq T_a^x \leq T_a^b < \infty$ and $0 \leq N_a^x \leq T_a^x \leq T_a^b < \infty$. Set $g(x) = P_a^x$ and $h(x) = N_a^x$. By our remark, g and h are increasing and so

$$f(x) - f(a) = P_a^x - N_a^x = g(x) - h(x)$$

which follows by Lemma 5.4.

($\Leftarrow$) Let $f = g - h$ and suppose g, h are increasing on $[a, b]$. Then for any partition of $[a, b]$,

$$\begin{aligned} \sum_{i=1}^{k} |f(x_i - f(x_{i-1}))| &\leq \sum_{i=1}^{k} g(x_i) - g(x_{i-1}) - \sum_{i=1}^{k} h(x_i) - h(x_{i-1}) \\ &= (g(b) - g(a)) + (h(b) - h(a)) \end{aligned}$$

which does not depend on the total variation of f. Taking the suprema over partitions,

$$T_a^b \leq (g(b) - g(a)) + (h(b) - h(a)).$$

Having shown a forward and backwards implication, this completes the proof. □

Corollary (5.6). If f is of bounded variation on $[a, b]$, then $f'(x)$ exist almost everywhere on $[a, b]$.

Note that if f is of bounded variation, this implies that f is bounded. This is because

$$f(x) \leq |f(x) - f(a)| - f(x) \leq t - f(x)$$

and so f is bounded.

Section 5.3: Differentiation of an Integral

Definition. Let f be an integrable function $[a, b]$. Define

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

for all $x \in [a, b]$ is called the **indefinite integral** of f over $[a, b]$.

Our goal is to show that $F'(x) = f(x)$ almost everywhere provided that f is integrable.

Lemma (5.7). If f is integrable on $[a, b]$, then then function F defined by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is a continuous function of bounded variation.

Proof. Let $f \geq 0$ and let $f \in L^1[a, b]$ (an integrable function). Fix $\varepsilon > 0$. Then by Proposition 4.14, there exists $\delta > 0$ such that $A \subset [a, b]$ with $m(A) < \delta$ implies that

$$\int_A f < \varepsilon.$$

THen we have that

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f - \int_a^x f \\ &= \int_x^{x+h} f \\ &\leq \int_A f \\ &< \varepsilon \end{aligned}$$

and so F is continuous. To show bounded variation, fix any partition $P = \{x_0 = a, x_1, \ldots, x_k = b\}$ of $[a, b]$. Then

$$\begin{aligned} \sum_{i=1}^k |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^k \left| \int_{x_{i-1}}^{x_i} f(t)\,\mathrm{d}t \right| \\ &\leq \sum_{i=1}^k \int_{x_{i-1}}^{x_i} |f(t)|\,\mathrm{d}t \\ &= \int_a^b |f(t)|\,\mathrm{d}t \\ &< \infty \end{aligned}$$

and so we are done! □

Lemma (5.8). If f is integrable on $[a, b]$ and

$$\int_a^x f(t)\,\mathrm{d}t = 0$$

for all $\in [a, b]$, then $f(t) = 0$ almost everywhere on $[a, b]$.

Proof. By way of contradiction, suppose $f(t) \neq 0$ almost everywhere in $[a,b]$. Let $E = \{x : f(x) > 0\}$ and suppose $m(E) > 0$. By Littlewood's first principle, there exists a closed set $K \subset E$ such that $m(K) > 0$. Let $O = [a,b] \setminus K$ and so is an open set. Then we know that

$$0 = \int_a^b f = \underbrace{\int_K f}_{>0} + \int_O f$$

which is true because if $g \geq 0$ and $m(A) > 0$, then $g = 0$ if and only if $g = 0$ almost everywhere. Thus $\displaystyle\int_O f \neq 0$ as long as O is an open set. By Lindelof's lemma,

$$O = \bigcup_{n=1}^{\infty} (a_n, b_n)$$

where $(a_n, b_n) \cap (a_m, b_m) = \emptyset$ for all $n \neq m$. So

$$0 \neq \int_O f = \sum_{i=1}^{\infty} \int_{a_n}^{b_n} f(t)\,\mathrm{d}t.$$

So there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} 0 \neq \int_{a_n}^{b_n} f(x)\,\mathrm{d}t &= \int_b^{b_n} f(t)\,\mathrm{d}t - \int_a^{a_n} f(t)\,\mathrm{d}t \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

which comes from assumption. But this implies that $m(E) = 0$, which is a contradiction. By a similar argument

$$m\left(\{x : f(x) < 0\}\right).$$

□

Lemma (5.9). If f is bounded and measurable on $[a,b]$, and

$$F(x) = \int_a^x f(t)\,\mathrm{d}t,$$

then $F'(x) = f(x)$ for almost all $x \in [a,b]$.

Proof. By Lemma 5.7, since F is integrable, F is a function of bounded variation and so $F'(x)$ exists almost everywhere on $[a,b]$. Let $|f| < K$. Then we write

$$f_n(x) = \frac{F\left(x + \frac{1}{n}\right) - F(x)}{\frac{1}{n}}$$

which as $n \to \infty$, $f_n(x) \to F'(x)$. So we have that

$$\begin{aligned} f_n(x) &= \frac{F\left(x + \frac{1}{n}\right) - F(x)}{\frac{1}{n}} \\ &= n \cdot \int_x^{x+\frac{1}{n}} F(t)\,\mathrm{d}t. \end{aligned}$$

Also $|f_n(x)| \leq K$. Because $f_n(x) \to F'(x)$ almost everywhere $f_n(x)$ is bounded, by the Bounded Convergence Theorem, for all $c \in [a, b]$,

$$\begin{aligned}
\int_a^c F'(t)\,\mathrm{d}t &= \lim_{n\to\infty} \int_a^c f_n(t)\,\mathrm{d}t \\
&= \lim_{n\to\infty} n \int_a^c \left(F\left(x+\frac{1}{n}\right) - F(x)\right) \\
&= \lim_{n\to\infty} n \left(\int_a^c F\left(x+\frac{1}{n}\right)\mathrm{d}x - \int_a^c F(x)\,\mathrm{d}x\right) \quad \int_{a+1/n}^{b+1/n} = \int_a^c F\left(x+\frac{1}{n}\right) - \int_a^c F(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} n \left(\int_c^{c+\frac{1}{n}} F(x)\,\mathrm{d}x - \int_a^{a+\frac{1}{n}} F(x)\,\mathrm{d}x\right) \\
&= F(c) - F(a) \\
&= \int_a^c f(x)\,\mathrm{d}x.
\end{aligned}$$

$\square$