Proposition (3.23, (Weak) Egonoff's Theorem). Let E be a measurable set of finite measure, and $\{f_n\}$ be a sequence of measurable functions defined on E. Let f be real-valued function such for each $x \in E$ we have $f_n(x) \to f(x)$. Then for all $\varepsilon > 0$ and all $\delta > 0$, there is measurable set $A \subset E$ with $m(A) < \delta$ and $N \in \mathbb{N}$ such that for all $x \notin A$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Proof. [1]

Let $\varepsilon > 0$ be chosen. Define the set

$$G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}$$

and also

$$E_N = \bigcup_{n=N}^{\infty} G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N\}.$$

Because $\{E_N\}$ is a decreasing sequence and $f_n(x) \to f(x)$ pointwise, for all $x \in E$, $|f_n(x) - f(x)| < \varepsilon$ and so $\bigcup_{i=1}^{\infty} E_N = \emptyset$. Thus, by Proposition 3.14,

$$\begin{aligned} E_N = \emptyset \implies m(E_N) &= 0 \\ &= m\left(\bigcup_{N=1}^{\infty} E_N\right) \\ &= \lim_{N\to\infty} E_N. \end{aligned}$$

So for any $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $m(E_N) < \delta$. Now take $A = E_N$ for any $N > N_0$ and so $m(A) < \delta$ and also

$$A^{\complement} = \{x \in E : x \notin E\} = \{x \in E : |f_n(x) - f(x)| < \varepsilon \text{ for all } n > N_0\}.$$

□

Section 4.1 Riemann Integration

Definition. Let $f : [a, b] \to \mathbb{R}$ be a bounded real-valued function and let

$$P = \{a = \xi_0 < \xi_1 < \cdots < \xi_n = b\}$$

be a subdivision (partition) of $[a, b]$. We can define the **upper sum**, S and **lower sum**, s, respectively, as

$$S = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) M_i \qquad \text{and} \qquad s = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) m_i$$

where

$$M_i = \sup\{f(x) : x \in [\xi_{i-1}, \xi_i]\} \qquad \text{and} \qquad m_i = \inf\{f(x) : x \in [\xi_{i-1}, \xi_i]\}.$$

[1] Proof on page 72-73.

Definition. Let $f : [a, b] \to \mathbb{R}$ be a bounded real-valued function. Define the **upper Riemann integral** of f as

$$R\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf\{S : P \text{ is a partition of } [a, b]\}$$

and the **lower Riemann integral** of f as

$$R\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup\{s : P \text{ is a partition of } [a, b]\}.$$

Definition. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if

$$R\underline{\int_a^b} f(x)\,\mathrm{d}x = R\int_a^b f(x)\,\mathrm{d}x = R\overline{\int_a^b} f(x)\,\mathrm{d}x.$$

In other words, if the upper and lower Riemann integrals are equal to each other.

Note this theorem is *NOT* from Royden but from my real analysis course in undergraduate. It comes from the definition of the upper and lower Riemann integral definitions by ends up being a useful way to show Riemann integration.

Theorem. Let f be a bounded function on $[a, b]$. Then f is Riemann integrable if and only if for all $\varepsilon > 0$, there exists a subdivision (partition) P of $[a, b]$ such that

$$S - s < \varepsilon.$$

Section 4.2 The Lebesgue Integral

Definition. The **characteristic function** of E is defined as

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

A linear combination

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

is called a **simple function** if the sets $\{E_1, \ldots, E_n\}$ are measurable. Note that ϕ is simple if and only if it is measurable and only assumes a finite number of values.

The **canonical representation** of ϕ is such that

$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$$

where $A_i = \{x : \phi(x) = a_i\}$ and where the A_i's are disjoint and the a_i's are distinct and nonzero.

Definition. Let
$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$$
be a simple function which vanishes outside of a set of finite measure. Then the integral of ϕ is defined as
$$\int \phi = \sum_{i=1}^{n} a_i \cdot m(A_i).$$

Lemma (4.1). Let $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ with $E_i \cap E_j = \emptyset$ for all $i \neq j$ where $E_i \in \mathfrak{M}$ and $m(E_i) < \infty$ for each $i = 1, \ldots n$. Then
$$\int \phi = \sum_{i=1}^{n} a_i \cdot m(E_i).$$

Proposition (4.2). Let ϕ, ψ be simple functions vanishing outside of a set with finite measure. Then
$$\int (a\phi + b\psi) = a \int \phi + b \int \psi.$$
If $\phi \geq \psi$ almost everywhere,
$$\int \phi \geq \int \psi.$$

Proposition (4.3). Let $E \in \mathfrak{M}$, and let $f : E \to \mathbb{R}$ with $m(E) < \infty$ be a bounded, measurable function. Let ϕ, ψ be simple functions. Then
$$inf \left\{ \int \psi : \psi \geq f \right\} = \sup \left\{ \int \phi : \phi \leq f \right\}.$$
if and only if f is measurable.

Proof. [2]

□

[2] Proof is on pages 79-80.