Lemma (5.13). If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ almost everywhere, then f is constant.

Proof. Let $E \subset [a, c]$ be the set such that for all $x \in E$, $f'(x) = 0$. We claim that for all $c \in [a, b]$, $f(a) = f(c)$. Let $\varepsilon, \eta > 0$ be chosen. For every $x \in E$, this means there exists $h > 0$ such that $(x, x + h) \subset [a, c]$ and $|f(x + h) - f(x)| < \eta \cdot h$ since η is arbitrary. By Vitali covering lemma, for any $\delta > 0$, there exists a finite cover $\{x_k, y_k\}$ of nonoverlapping intervals contained in $[a, c]$ such that

$$\sum_{k=0}^{n} |x_{k+1} - y_k| < \delta.$$

So then

$$\sum_{k=1}^{n} |f(y_k) - f(x_k)| \leq \eta n \sum_{k=1}^{n} (y_k - x_k) \leq \eta(c - a).$$

Moreover, by the absolute continuity, there exists a $\delta > 0$ for our fixed ε such that

$$\sum_{k=0}^{n} |f(x)_{k+1} - f(y_k)| < \varepsilon.$$

Then

$$\begin{aligned} |f(x) - f(a)| &= \left| \sum_{k=0}^{n} f(x_{k+1} - f(y_k)) + \sum_{k=0}^{n} f(y_k) - f(x_k) \right| \\ &\leq \left| \sum_{k=0}^{n} f(x_{k+1} - f(y_k)) \right| + \left| \sum_{k=0}^{n} f(y_k) - f(x_k) \right| \\ &< \varepsilon + \eta(c - a). \end{aligned}$$

Because ε and η are arbitrary, $|f(c) - f(a)| = 0$ and so $f(c) = f(a)$ (i.e., f is constant). □

Theorem (5.14). A function F is an indefinite integral if and only F is absolutely continuous.

Proof. We will show two directions.

(⇒) Suppose F is an indefinite integral i.e.,

$$F(x) = \int_a^c f(t)\,\mathrm{d}t.$$

Fix $\varepsilon > 0$. By Proposition 4.14, if $f \geq 0$ and $f \in L^1$, then there exists $\delta > 0$ such that if $m(A) < \delta$, then

$$\int_A f < \varepsilon.$$

Thus absolute continuity follows from this proposition.

($\Leftarrow$) Now suppose F is absolutely continuous. Because absolute continuity implies a function is of bounded variation, we know that $F'(x)$ exists almost everywhere. Additionally, this means it is the subtraction of two monotone increasing functions i.e., $F(x) = F_1(x) - F_2(x)$. So then

$$|F'(x)| \leq F_1'(x) + F_2'(x).$$

This implies that

$$\begin{aligned} \int_a^b |F'(x)| &\leq \int_a^b F_1'(x) + \int_a^b F_2'(x) \\ &\leq (F_1(b) - F_2(a)) + (F_2(b) - F_2(a)) \end{aligned}$$

This means that $F'(x)$ is integrable on (a, b). Consider the function

$$G(x) = \int_a^x F'(t)\,\mathrm{d}t.$$

So $G(x)$ is absolutely continuous on $[a, b]$ and

$$\begin{aligned} (G(x) - F(x))' &= G'(x) - F'(x) \\ &= 0 \end{aligned}$$

almost everywhere. By the previous lemma, we know that $G(x) - F(x) = x = F(a)$ and so take $x = a$. Therefore,

$$F(x) = \int_a^x F'(t)\,\mathrm{d}t + F(a).$$

$\square$

Section 6.5 Bounded Linear Functionals on the L^p Space

Definition. Let $(X, \|\cdot\|)$ be a normed linear space. A **linear functional** is a map $F : X \to \mathbb{R}$ such that

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$$

for all $f, g \in X$ and for $\alpha, \beta \in \mathbb{R}$.

A linear functional F is **bounded** if there exists $M > 0$ such that

$$|F(f)| \leq M \cdot \|f\| \text{ for all } f \in X.$$

Finally, we can define the norm of F by

$$\|F\| = \sup_{f \in X} \frac{|F(f)|}{\|f\|}.$$

For sake of clarity, $X = L^p$ and $\|\cdot\| = \|\cdot\|_p$ and $p, q \in \mathbb{R}$ always satisfy the equation $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Remark. If $g \in L^q$, define

$$F_g(f) = \int f \cdot g \text{ for all } f \in L^p.$$

By Hölder's inequality

$$\int f \cdot g \leq \|f\|_p \cdot \|g\|_g$$

and so this implies that

$$\|Fg\| \leq \|g\|_q .$$

The linear functional $Fg : L^p \to \mathbb{R}$ is a bounded linear functional.

Proposition (6.11).

$$\|Fg\| = \|g\|_q .$$

Proof. We claim there exists $f \in L^p$ such $F(f) = \|f\|_p \cdot \|g\|_q$. Set $f = |g|^{q/p}$. □

We know that $Fg = \displaystyle\int fg$ defines a bounded linear functional and also vise versa—that is, all bounded, linear functionals can take the form

$$F(f) = \int f \cdot g$$

for some $g \in L^q$.

Lemma (6.12). Let g be an integrable function over $[0, 1]$. Suppose there exists $M > 0$ such that

$$\left| \int fg \right| \leq M \cdot \|f\|_p$$

for all f which are bounded, measurable functions. Then $g \in L^q$ and $\|g\|_q \leq M$.