Proposition (4.3). Let $E \in \mathfrak{M}$, and let $f : E \to \mathbb{R}$ with $m(E) < \infty$ be a bounded, measurable function. Let ϕ, ψ be simple functions. Then

$$\inf\left\{\int \psi : \psi \geq f\right\} = \sup\left\{\int \phi : \phi \leq f\right\}.$$

if and only if f is measurable.

Proof. [1] We will need to show two implications.

($\Leftarrow$) First, suppose that f is measurable. Fix any $n \in \mathbb{N}$ and define the set

$$E_k = \left\{x : \frac{k-1}{n}M < f(x) \leq \frac{kM}{n}\right\}$$

with $k \in [-n, n]$ and $|f(x)| < M$. Note that because f is measurable, each E_k is a measurable set and also we have that $\bigcup_{k=-n}^{n} E_K = E$. Define the upper and lower sequence of simple functions, $\{\psi_n\}$ and $\{\phi_n\}$, respectively, as

$$\psi_n(x) = \sum_{k=-n}^{n} \frac{M}{n} \cdot k \cdot \chi_{E_k}(x) \quad \text{and} \quad \phi_n = \sum_{k=-n}^{n} \frac{M}{n} \cdot (k-1) \cdot \chi_{E_k}(x).$$

So for any $x \in E$, $\phi(x) \leq f(x) \leq \psi(x)$. Thus,

$$\inf_{\psi \geq f} \int_E \psi \leq \int_E \psi = \frac{M}{n} \sum_{k=-n}^{n} k \cdot m(E_k)$$

and

$$\sup_{\phi \leq f} \int_E \phi \geq \int_E \phi = \frac{M}{n} \sum_{k=-n}^{n} (k-1) \cdot m(E_k).$$

Putting these two inequalities together, we can show that

$$\begin{aligned}
\inf_{\psi \geq f} \int_E \psi - \sup_{\phi \leq f} \int_E \phi &\leq \int_E \psi - \phi \\
&= \sum_{k=-n}^{n} (\psi_n - \phi_n)\, m(E_k) \\
&= \frac{M}{n} m(E).
\end{aligned}$$

Since $n \in \mathbb{N}$ is fixed, this quantity is zero. Thus

$$\inf_{\psi \geq f} \int_E \psi - \sup_{\phi \leq f} \int_E \phi = 0 \Rightarrow \inf_{\psi \geq f} \int_E \psi = \sup_{\phi \leq f} \int_E \phi$$

completing this direction.

[1] Proof is on pages 79-80.

($\Rightarrow$) Conversely, suppose that

$$\inf_{\psi \geq f} \int_E \psi = \sup_{\phi \leq f} \int_E \phi.$$

By definition of the infimum and supremum, there exists sequences of simple functions $\{\phi_n\}$ and $\{\psi_n\}$ such that $\phi_n \leq f \leq \psi_n$ for all $n \in \mathbb{N}$ with

$$\int_E \phi_n - \psi_n < \frac{1}{n}.$$

Define $\phi^* = \sup_n \phi_n$ and $\psi^* = \inf_n \psi(n)$. Since simple functions are measurable functions, by Proposition 3.20, ϕ^* and ψ^* are measurable as well and $\phi_n \leq f \leq \psi_n$. We claim that $f = \phi^*$ a.e. Consider the set

$$\Delta = \{x : \phi^*(x) < \psi^*(x)\}.$$

Let $\nu \in \mathbb{N}$ and let

$$\Delta_\nu = \left\{x : \phi^*(x) < \psi^*(x) - \frac{1}{\nu}\right\}$$

which means that

$$\Delta = \bigcup_{\nu \in \mathbb{N}} \Delta_\nu.$$

For any $n \in \mathbb{N}$,

$$\Delta_\nu \subset \left\{x : \phi(x) < \psi(x) - \frac{1}{\nu}\right\}.$$

Thus, we have that, for any $n \in \mathbb{N}$,

$$\begin{aligned} m\left(\Delta_\nu\right) &= \int \chi_{\Delta_\nu} = \nu \int \frac{1}{\nu} \cdot \chi_{\Delta_\nu} \\ &\leq \nu \int_{\Delta_\nu} (\psi_n - \phi_n) \\ &< \mu \int_E \frac{1}{n} \\ &= \frac{\nu}{n} m(E). \end{aligned}$$

Because ν is fixed and n is arbitrary, $m(\Delta_\nu) = 0$ which implies that $m(\Delta) = 0$. So then $\phi^* = \psi^*$ except on a set of measure zero, and $\phi^* = f$ except on a set of measure zero i.e., $f = \phi^*$ a.e. implying that f is measurable by Proposition 3.21.

Having completed both directions, this completes the proof and shows the well-definedness of Lebesgue integration. $\square$

Proposition (4.4). Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable, then it is measureble and

$$R \int_a^b f(x) \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x.$$

Proof. This proof is on page 82 of Royden (very simple proof, in fact). $\square$

Proposition (4.5). If f and g are bounded measurable functions defined on a set E of finite measure, then:

i. For any $a, b \in \mathbb{R}$,
$$\int_E (af + bg) = a \int_E f + b \int_E g.$$

ii. If $f = g$ a.e., then
$$\int_E f = \int_E g.$$

iii. If $f \leq g$ almost everywhere. then
$$\int_E f \leq \int_E g.$$
Hence
$$\left| \int_E f \right| \leq \int_E |f|.$$

iv. If $A \leq f(x) \leq B$, then
$$Am(E) \leq \int_E f \leq Bm(E).$$

v. If A and B are disjoint measurable sets of finite measure

Proposition (4.6, Bounded Convergence Theorem). Let $\{f_n\}$ be a sequence of measurable functions defined over a measurable set E of finite measure. Suppose there is $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for all $x \in E$ and for all $n \in \mathbb{N}$. If $f_n(x) \to f(x)$ pointwise (i.e., $\lim_{n\to\infty} f_n = f(x)$), then
$$\int_E f_n \to \int_E f \Leftrightarrow \lim_{n\to\infty} \int_E f_n(x) = \int_E f.$$

Proof. Let $\varepsilon > 0$ be chosen. By Proposition 3.23 (Weak Ergoroff's Theorem), there exists $N \in \mathbb{N}$ and $A \subset E$ with
$$m(A) = \tilde{\delta} < \frac{\varepsilon}{4M}$$
such that that for all $x \in E \setminus A$ and for all $n > N$,
$$|f_n(x) - f(x)| < \frac{\varepsilon}{2 \cdot m(E)} = \tilde{\varepsilon}.$$

Thus, we can show the following:

$$
\begin{aligned}
\left|\int_E f_n - \int_E f\right| = \left|\int_E f_n - f\right| &\leq \int_E |f_n - f| \\
&= \int_A |f_n - f| + \int_{E\setminus A} |f_n - f| \\
&\leq 2M \cdot m(A) + \int_{E\setminus A} |f_n - f| \\
&\leq 2M \cdot m(A) + \frac{\varepsilon \cdot m(E \setminus A)}{2 \cdot m(E)} \\
&< 2M \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon \cdot m(E \setminus A)}{2 \cdot m(E)} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

and so we are done!! $\square$

Proposition (4.7). Let f be bounded on $[a, b]$. Then f is Riemann integrable if and only the set of discontinuities has measure zero.