Section 4.4 General Lebesgue Integral

Definition. Define

$$\begin{aligned} f^+(x) &= \text{ non-negative part of } f \\ &= \max\{f(x), 0\}. \end{aligned}$$

and

$$\begin{aligned} f^-(x) &= \text{ non-positive part of } \ f \\ &= \max\{-f(x), 0\}. \end{aligned}$$

Note that then

$$f(x) = f^+(x) - f^-(x)$$

and

$$|f(x)| = f^+(x) + f^-(x).$$

Definition. A measurable function f is said to be **Lebesgue integrable** over E if f^+ and f^- are integrable. In this case, then

$$\int_E = \int_E f^+ - \int_E f^-.$$

Proposition (4.15). Let f, g be integrable functions over E. Then

i. cf are integrable for all $c \in \mathbb{R}$ over E.

ii. $f + g$ is integrable and
$$\int_E f + g = \int_E f + \int_E g$$

iii. If $f \leq g$ a.e, then
$$\int_E f \leq \int_E g.$$

iv. If $A, B \subset E$ and $A \cap B = \emptyset$, then
$$\int_{A \cup B} f = \int_A f + \int_{B.}$$

Proposition (4.16, Lebesgue Convergence Theorem). Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions. Suppose $f_n \to f$ pointwise almost everywhere and $|f_n| \leq g$ on E. Then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. Since $g - f_n \geq 0$ for all $n \in \mathbb{N}$, Fatou's lemma, $|f_n| \leq g$ on E

$$\begin{aligned}\int g - \int f = \int g - f &\leq \varliminf_{n\to\infty} \int g - f_n \\ &= \varliminf_{n\to\infty} \int f - \varliminf_{n\to\infty} \int f_n \\ &= \int g - \varliminf_{n\to\infty} \int f_n\end{aligned}$$

and so

$$\int g - \int f \leq \int g - \varliminf_{n\to\infty} \int f_n$$

which implies that

$$\int f \geq \varliminf_{n\to\infty} \int f_n.$$

Note that $g + f_n \geq 0$ as well. Then

$$\begin{aligned}\int g + \int f_n = \int g + f_n &\leq \varliminf_{n\to\infty} \int g + f_n \\ &= \varliminf_{n\to\infty} \int g + \varliminf_{n\to\infty} \int f_n \\ &= \int g + \varliminf_{n\to\infty} \int f_n\end{aligned}$$

implying that

$$\int f \leq \varliminf_{n\to\infty} \int f_n.$$

Because $\{f_n\}$ converges, we know that $\varliminf_{n\to\infty} f_n = \lim_{n\to\infty} f_n$ and so the result follows. $\square$

Proposition (4.17, Lebesgue Generalized Dominant Convergent Theorem). Let $\{g_n\}$ be a sequence of integrable functions and $g_n \to g$ pointwise a.e with g integrable. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g_n$ and $\{f_n\} \to f$ pointwise a.e. If

$$\int g = \lim_{n\to\infty} \int g_n,$$

then

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. [1] This proof is also similar to Proposition 4.16 but use g_n instead of g.

$\square$

Problem 1 (4.15). Properties of function f being integrable.

(a) Let f be integrable over E. Then for all $\varepsilon > 0$, there exists a simple function ϕ such that

$$\int_E |f - \phi| < \varepsilon.$$

[1]Proof is on page 92 of Royden.

(b) Let f be integrable over E. Then for all $\varepsilon > 0$, there exists a step function ψ such that

$$\int_E |f - \psi| < \varepsilon.$$

(c) Let f be integrable over E. Then for all $\varepsilon > 0$, there exists a continuous function g such that

$$\int_E |f - g| < \varepsilon.$$