Lemma (3.9). Let A be any set, and $E_1, \ldots, E_n$ be a finite sequence of sets such that $E_i \cap E_j$ for all $i \neq j$. Then

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Proof. We proceed by induction. For $n = 1$, we have the set E_1 and the equality holds. Suppose that we have $n = k$ sets $E_1, \ldots, E_k$ with $E_i \cap E_j \neq \emptyset$ for all $i \neq j$ so that

$$m^*\left(A \cap \left[\bigcup_{i=1}^{k} E_i\right]\right) = \sum_{i=1}^{k} m^*(A \cap E_i).$$

Consider $n = k + 1$. Because each E_i is disjoint,

$$A \cap \left(\bigcup_{i=1}^{k+1} E_i\right) \cap E_{k+1} = A \cap E_{k+1};$$
$$A \cap \left(\bigcup_{i=1}^{k+1} E_i\right) \cap E_{k+1}^{\complement} = A \cap \bigcup_{i=1}^{k} E_i.$$

Because the E_i's are measurable,

$$\begin{aligned} m^*\left(A \cap \bigcup_{i=1}^{k+1} E_i\right) &= m^*\left(A \cap E_{k+1}\right) + m^*\left(A \cap \bigcup_{i=1}^{k} E_i\right) \\ &= m^*\left(A \cap E_{k+1}\right) + \sum_{i=1}^{k} m^*(A \cap E_i) \qquad \text{Induction Hypothesis} \\ &= \sum_{i=1}^{k+1} m^*(A \cap E_i) \end{aligned}$$

which, by induction, completes the proof. □

Theorem (3.10). $\mathcal{M}$ is a σ-algebra. In other words, in addition to being an algebra of sets, if $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

Proof. [1]

□

Lemma (3.11). The interval (a, ∞) is measurable for all $a \in \mathbb{R}$.

Proof. [2]

□

[1] Proof on bottom of page 59 and top of page 60.
[2] Proof on the bottom of page 60 through the middle of page 61.

Theorem (3.12). Every Borel set is measurable. In particular, each open set and each closed set is measurable.

Proof. [3]

□

Definition. Let $E \in \mathcal{M}$. We define $m(E) := m^*(E)$ to be the **Lebesgue measure** of $E/$

Proposition (3.13, Countable Additivity). Let $\{E_i\}_{i=1}^n$ be a sequence of measurable sets. Then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{n} m(E_i).$$

If, in addition, $E_i \cap E_j$ for all $i \neq j$. then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{n} m(E_i).$$

Proposition (3.14). Let $\{E_i\} \subset \mathcal{M}$ be a decreasing sequence (i.e., $E_{i+1} \subset E_i$). Let $m(E_1) < \infty$. Then

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} m(E_n).$$

Proposition (3.15). Let E be any given set. Then the following are equivalent:

(i) E is measurable.

(ii) For all $\varepsilon > 0$, there is an open set $O \supset E$ with $m^*(O \setminus E) < \varepsilon$.

(iii) For all $\varepsilon > 0$, there is a closed set $F \subset E$ with $m^*(E \setminus F) < \varepsilon$.

(iv) There is a $G \in G_\delta$ with $E \subset G$ such that $m^*(G \setminus E) = 0$.

(v) There is a $F \in F_\sigma$ with $F \subset E$ such that $m^*(E \setminus F) = 0$.

If $m^*(E) < \infty$, the above statements are equivalent:

(vi) For all $\varepsilon > 0$, there is a finite union U of open intervals such that $m^*(U \Delta E) < \varepsilon$.

[3] Proof on the bottom of page 61.