Section 3.5, Measurable Functions

Proposition (3.18). Let $E \subset \mathbb{R}$, and Let $f : E \to [-\infty, \infty]$ be an extended real-valued function whose domain is measurable. Let $\alpha \in \mathbb{R}$ be any real number. Then the following statements are equivalent:

(i) The set $\{x : f(x) > \alpha\}$ is measurable.

(ii) The set $\{x : f(x) \geq \alpha\}$ is measurable.

(iii) The set $\{x : f(x) < \alpha\}$ is measurable.

(iv) The set $\{x : f(x) \leq \alpha\}$ is measurable.

All together, these imply

(v) The set $\{x : f(x) = \alpha\}$ is measurable.

Proof. [1]

□

Definition. An extended real-valued function $f : E \to [-\infty, \infty]$ is **(Lebesgue) measurable** if its domain is measurable and satisfies one of the first four statements of Proposition 18.

Note that this means that any continuous function is measurable since then the pre-image of any open set is still an open set.

Proposition (3.19). Let f and g be two measurable functions defined on the same domain, and let $c \in \mathbb{R}$. Then the functions $f + c$, cf, $f + g$, $g - f$, and fg are measurable.

Proof. Let $\alpha \in \mathbb{R}$ be any real number. Fix $c \in \mathbb{R}$. For $f(x) + c$, note that

$$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}$$

and $\alpha - c$ is a real number, this set is still measurable i.e., $f + c$ is measurable. A similar argument shows that cf is measurable as well.

Take the set

$$\{x : f(x) + g(x) < \alpha\}. \tag{1}$$

Observe that $f(x) + g(x) < \alpha \Leftrightarrow f(x) < \alpha - g(x)$. By the density of $\mathbb{Q}$, there exists $r \in \mathbb{Q}$ such that $f(x) < r < \alpha - g(x)$. So we can write **??** as

$$\{x : f(x) + g(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} \left(\{x : f(x) < r\} \cap \{x : g(x) < \alpha - r\}\right).$$

Because this set is countable, this set is measurable and thus $f + g$ is measurable.

To show that fg is measurable, we can show that f^2 is measurable since

$$fg = \frac{1}{2}\left((f + g)^2 - f^2 - g^2\right).$$

[1]Proof is on page 67.

Take the set
$$\{x : f^2(x) < \alpha\}. \tag{2}$$
For $\alpha \geq 0$, note that $f^2 < \alpha$ is the same as saying $f(x) > \sqrt{\alpha}$ and $f(x) < -\sqrt{\alpha}$. Thus, **??** can rewritten as
$$\{x : f^2(x) < \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}.$$
This is a measurable set which completes the proof. □

Theorem (3.20, Limit of Measurable Functions is Measurable). [2]

Proof. For $f(x) + c$, note that
$$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}.$$
□

Theorem (3.20, Limit of Measurable Functions is Measurable). Let $\{f_n\}$ be a sequence of measurable functions with the same domain. Then the functions $\sup\{f_1(x), \ldots, f_n(x)\}$, $\inf\{f_1(x), \ldots, f_n(x)\}$, $\sup_n f_n$, $\inf_n f_n$, $\overline{\lim} f_n$, and $\underline{\lim} f_n$ are measurable.

Proof. Let $\{f_n\}$ be a sequence of measurable functions. Let $h(x) = \sup\{f_1(x), \ldots, f_n(x)\}$ and we so must show that $\{x : h(x) < \alpha\}$ for all $\alpha \in \mathbb{R}$. To that end, let $\alpha \in \mathbb{R}$ be chosen. Then
$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$$
which, because the right-hand side is a union of measurable sets from the f_i's being measurable, means that the set $\{x : h(x) < \alpha\}$ is also measurable.

Let $g(x) = \sup_n f_n$. By a similar argument as above,
$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i(x) > \alpha\}$$
is a countable set so $\{x : g(x) < \alpha\}$ is measurable. □

Definition. A property is said to hold **almost everywhere** (a.e) if the set of points where it fails to hold is a set of measure zero. Thus $f = g$ a.e if f and g have the same domain and $m\{x : f(x) \neq g(x)\}$.

Proposition (3.21). If f is measurable and $f = g$ a.e, then g is measurable.

Proof. [3] Let $E = \{x : f(x) \neq g(x)\}$.

This is equivalent to saying that

Let $\{x : g(x) > \alpha\}$. This is equivalent to saying that

$$\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha\}$$
□

[2] Proof is on bottom of page 68 and top of page 69

[3] Proof is on middle of page 69.

The following proposition essentially says that measurable functions are nearly continuous; or. in other words, we can "nicely" approximate measurable functions.

Proposition (3.22, Littlewood's 2nd Principle). Let $f : [a, b] \to E$ be a measurable function with $E \subset \mathbb{R}$ and is equal to $\pm\infty$ only on sets with measure zero. Then for all $\varepsilon > 0$, there exist a step function g and a continuous function f such

$$|f - g| < \varepsilon \quad \text{and} \quad |f - h| < \varepsilon$$

except on set of measure less than ε; i.e., $m\{x : |f(x) - g(x)| \geq \varepsilon\} < \varepsilon$ and $m\{x : |f(x) - h(x)| \geq \varepsilon\} < \varepsilon$. If in addition $m \leq f \leq M$, then we may choose the functions g and h so that $m \leq g \leq M$ and $m \leq h \leq M$.

Proposition (3.23, (Weak) Egonoff's Theorem). Let E be a measurable set of finite measure, and $\{f_n\}$ be a sequence of measurable functions defined on E. Let f be real-valued function such for each $x \in E$ we have $f_n(x) \to f(x)$. Then for all $\varepsilon > 0$ and all $\delta > 0$, there is measurable set $A \subset E$ with $m(A) < \delta$ and $N \in \mathbb{N}$ such that for all $x \notin A$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$