Problem 1 (6.2)**.** Let f be a bounded measurable function on $[0,1]$. Then $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

Proof. First, I will note that $\lim_{p\to\infty} \|f\|_p \leq \|f\|_\infty$ follows pretty readily from the definition of $\|\cdot\|$. This is because

$$\|f\|_p = \left\{\int_0^1 |f|^p\right\}^{1/p} \leq \left\{\int_0^1 \|f\|_\infty^p\right\}^{1/p} = \|f\|_\infty$$

and so as we take the limit of $\|f\|_p$ as $p \to \infty$, we get $\lim_{p\to\infty} \|f\|_p \leq \|f\|_\infty$. Now we must also show that $\lim_{p\to\infty} \|f\|_p \geq \|f\|_\infty$. To that end, let $\varepsilon > 0$ be chosen. Define the set $A = \{x \in [0,1] : |f(x)| > \|f\|_\infty - \varepsilon\}$. Then we have that

$$\begin{aligned}\|f\|_p = \left\{\int_0^1 |f|^p\right\}^{1/p} &\geq \left\{\int_A |f|^p\right\}^{1/p} \\ &\geq \left\{\int_A (\|f\|_\infty - \varepsilon)^p\right\}^{1/p} \\ &= (\|f\|_\infty - \varepsilon)^p \cdot m(A).\end{aligned}$$

This implies that

$$\|f\|_\infty - \varepsilon \cdot (m(A))^p \leq \|f\|_p.$$

Because $\|f\|_\infty$ is the essential supremum (i.e. the smallest, greatest value not on a set of measure zero), we know that $m(A) > 0$. Thus, taking the limit of both sides as $p \to \infty$, we get that

$$\lim_{p\to\infty} \|f\|_\infty - \varepsilon \cdot (m(A))^p = \|f\|_\infty - \varepsilon \leq \lim_{p\to\infty} \|f\|_p,$$

Since ε is arbitrary, then $\lim_{p\to\infty} \|f\|$ is a superior bound i.e., $\|f\|_\infty \leq \lim_{p\to\infty} \|f\|$. Thus we get $\|f\|_\infty = \lim_{p\to\infty} \|f\|$. □

Problem 2 (6.8)**.** Young's Inequality

(a) Let $a, b \geq 0$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Establish Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Not assigned. □

(b) Use Young's inequality to give a proof of the Hölder inequality.

Proof. Let p and q be nonnegative extended real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Suppose $f \in L^p$ and $g \in L^q$. Without loss of generality, assume that $\|f\|, \|g\| \geq 0$. With $a = \frac{|f|}{\|f\|_p}$ and $b = \frac{|g|}{\|g\|_q}$, by Young's Inequality from part (a), we have that

$$\frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q} \leq \frac{|f|^p}{p\,\|f\|_p^p}\frac{|g|_p^q}{q\,\|g\|_q^q}.$$

From the monotonocity of integrals, this implies that

$$\int \frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q} \leq \int \frac{|f|^p}{p\,\|f\|_p^p} + \frac{|g|^q}{q\,\|g\|_q^q} = \frac{1}{p\,\|f\|_p^p}\int |f|^p + \frac{1}{q\,\|g\|_q^q}\int |g|^q.$$

However, the the integral of $|f|^p$ is the same as $\|f\|_p^p$ and the same argument for $|g|^p$. So by cancelling out $\|f\|_p^p$ and $\|g\|_q^q$, we get that

$$\int \frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1.$$

Multiplying both sides by $\|f\|_p \cdot \|g\|_q$ and so

$$\int |fg| \leq \|f\|_p \cdot \|g\|_q\,.$$

Young's inequality is equality if and only $a^p = b^q$ and so the Hölder inequality is equality if and only if $\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q}$. Thus there exists $\alpha, \beta \neq 0$ such that $\alpha|f|^p = \beta|g|^q$ almost everywhere and so this completes the proof. □

Problem 3 (6.10)**.** Let $\{f_n\}$ be a sequence of functions in L^∞. Prove that $\{f_n\}$ converges to f in L^∞ if and only if there is a set E of measure zero such that f_n converges to f uniformly on $E^\complement$.

Proof. We will need to complete two directions and so let $\{f_n\}$ be a sequence of functions in L^∞.

($\Rightarrow$) First, suppose that $\{f_n\} \to f$, and let $\varepsilon > 0$ be chosen. Because $f_n \to f$ in L^∞, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|f_n - f\|_\infty = \inf\{M : m\{t : |f_n(t) - f(t| > M\} = 0\} < \varepsilon.$$

Let $E = \{t : |f_n(t) - f| \geq \varepsilon\}$. Per the above expression, for any $n \geq N$, we have that $m(E) = 0$ and $\|f_n - f\|_\infty < \varepsilon$ on the set $L^\infty \setminus E = E^\complement$. Thus, since $\varepsilon > 0$ is arbitrary, f_n converges uniformly to f on $E^\complement$.

($\Leftarrow$) Conversely, suppose there exists a set E with $m(E) = 0$ such that $f_n \to f$ uniformly on $E^\complement$. Let $\varepsilon > 0$ be chosen. Since $f_n \to f$ uniformly on $E^\complement$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $t \in E^\complement$,

$$|f_n(t) - f(t)| < \frac{\varepsilon}{2}.$$

But then this means that the set $\left\{t : f_n(t) - f(t) > \frac{\varepsilon}{2}\right\} \subset E$. By the definition of the infimum, for our fixed $\varepsilon > 0$ and any $n \geq N$,

$$\inf\{M : m\{t : |f_n(t) - f(t)| > M\} = 0\} < \varepsilon.$$

This is the essential supremum and so this means $\|f_n - f\|_\infty < \varepsilon$. Therefore, since ε is arbitrary, $\|f_n - f\| < \varepsilon$ and which implies that $f_n \to f$ pointwise on L^∞.

Thus, having completed the forward and backwards implication, this completes the proof. □