Problem 1 (4.14)**.** Some sequence and integral convergence problems.

(a) Show that under the hypotheses of Theorem 4.17 we have

$$\int |f_n - f| \to 0.$$

Proof. Let $\{g_n\}$ be a sequence of integrable functions such that $g_n \to g$ pointwise with g integrable. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ and $f_n \to f$ pointwise almost everywhere. Suppose that

$$\int g = \lim_{n\to\infty} \int g_n.$$

For any $n \in \mathbb{N}$ we have $|f_n| \leq g$ and so because $f_n \to f$ and $g_n \to g$, $|f| \leq g$. Thus, we have that

$$\begin{aligned} |f_n| - |f| \leq |f_n - f| &\leq |f_n + f| \\ &\leq |f_n| + |f| \\ &\leq g_n + g. \end{aligned}$$

This means that the sequence defined by $\{(g_n + g) - |f_n - f|\}$ is a nonnegative sequence. So by Fatou's lemma and properties of lim inf and lim sup,

$$\begin{aligned} 0 \leq \int (g_n + g) - |f_n - f| &\leq \varliminf_{n\to\infty} \int (g_n + g) - |f_n - f| \\ &\leq \int (g_n + g) + \varliminf_{n\to\infty} \int -|f_n - f| \\ &= \int (g_n + g) - \varlimsup_{n\to\infty} \int |f_n - f|. \end{aligned}$$

But then this implies that

$$\varlimsup_{n\to\infty} \int |f_n - f| \leq 0 \leq \varliminf_{n\to\infty} \int |f_n - f|$$

and so we have

$$\lim_{n\to\infty} \int |f_n - f| = 0.$$

□

(b) Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ almost everywhere with f integrable. Then $\int |f - f_n| \to 0$ if and only if $\int |f_n| \to \int |f|$.

Proof. We will show two directions to complete this proof.

($\Rightarrow$) First, suppose that

$$\lim_{n\to\infty} \int |f_n - f| = 0.$$

Then, by using the reverse triangle inequality, can show that

$$\begin{aligned} \left| \lim_{n\to\infty} \int |f_n| - \int |f| \right| &\leq \lim_{n\to\infty} \int |f_n| - \int |f| \\ &\leq \lim_{n\to\infty} \int |f_n - f| \\ &= 0. \end{aligned}$$

Because $|\cdot| \geq 0$ always, we know that

$$0 \leq \lim_{n\to\infty} |f_n| \leq \int |f| \leq 0$$

and so

$$\lim_{n\to\infty} \int |f_n| = \int |f|.$$

($\Leftarrow$) Conversely, suppose that

$$\lim_{n\to\infty} \int |f_n| = \int |f|.$$

Because $f_n \to f$ a.e, $|f_n| \leq f$ for all $n \in \mathbb{N}$. By a similar argument to part (a),

$$\begin{aligned} |f_n| - |f| \leq |f_n - f| &\leq |f_n + f| \\ &\leq |f_n| + |f| \end{aligned}$$

Then the sequence $\{(|f_n| + |f|) - |f_n - |\}$ is a nonnegative sequence. Again, similar to part (a), using Fatou's lemma we have

$$\begin{aligned} 0 \leq \int (|f_n| + |f|) - |f_n - f| &\leq \varliminf_{n\to\infty} \int (|f_n| + |f|) - |f_n - f| \\ &\leq \int (|f_n| + |f|) + \varliminf_{n\to\infty} \int -|f_n - f| \\ &= \int (|f_n| + |f|) - \varlimsup_{n\to\infty} \int |f_n - f|. \end{aligned}$$

So we again that

$$\varlimsup_{n\to\infty} \int |f_n - f| \leq 0 \leq \varliminf_{n\to\infty} \int |f_n - f|$$

and so we have

$$\lim_{n\to\infty} \int |f_n - f| = 0.$$

finishing the proof.

Thus, having showed both directions, $\int |f - f_n| \to 0$ if and only if $\int |f_n| \to \int |f|$. ☐

Problem 2 (4.16)**.** Establish the *Riemann-Lebesgue Theorem*: If f is an integrable function on $(-\infty, \infty)$, then $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\cos(nx)\,\mathrm{d}x = 0$. [Hint: The theorem is easy if f is a step function. Use Problem 15.]

Proof. Let f be an integrable function $(-\infty, \infty)$. Let $\varepsilon > 0$ be chosen. By Problem 15 part (b), there exists a step function ψ such that

$$\int |f - \psi| < \frac{\varepsilon}{2}.$$

Turning to $\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos(nx)\,\mathrm{d}x$, we can note that following:

$$\begin{aligned}
\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos(nx)\,\mathrm{d}x &= \left|\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos(nx)\,\mathrm{d}x\right| \\
&\leq \lim_{n\to\infty}\int_{-\infty}^{\infty} |f(x)\cos(nx)|\,\mathrm{d}x \\
&\leq \lim_{n\to\infty}\int_{-\infty}^{\infty} |f(x)\cos(nx)|\,\mathrm{d}x \\
&\leq \lim_{n\to\infty}\int_{-\infty}^{\infty} |(f(x)-\psi(x))\cos(nx)|\,\mathrm{d}x + \lim_{n\to\infty}\int_{-\infty}^{\infty} |\psi(x)\cos(nx)|\,\mathrm{d}x \\
&< \frac{\varepsilon}{2} + \lim_{n\to\infty}\int_{-\infty}^{\infty} |\psi(x)\cos(nx)|\,\mathrm{d}x.
\end{aligned}$$

Because $\psi(x)$ is a step function, we can integrate the right-hand side integral in the last inequality over $(-\infty,\infty)$ in each interval which $\psi(x)$ is constant. So then because $\phi(x)$ is fixed over these intervals, as $n\to\infty$, the antiderivative of $|\cos(nx)|$ goes to zero i.e.,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} |\psi(x)\cos(nx)|\,\mathrm{d}x = 0.$$

Since this integral converges to 0, there exists $N\in\mathbb{N}$ such for all $n>N$, we have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} |\psi(x)\cos(nx)| < \frac{\varepsilon}{2}.$$

Thus, we have that

$$\begin{aligned}
\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos(nx)\,\mathrm{d}x &< \frac{\varepsilon}{2} + \lim_{n\to\infty}\int_{-\infty}^{\infty} |\psi(x)\cos(nx)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Since ε was chosen arbitrarily,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos(nx)\,\mathrm{d}x = 0$$

which was our desired result. $\square$

Problem 3 (4.25). A sequence $\{f_n\}$ of measurable functions is said to be a Cauchy sequence in measure if given $\varepsilon>0$, there is $N\in\mathbb{N}$ such that for all $m,n\geq N$ we have

$$m\left\{x : |f_n(x)-f_m(x)|\geq\varepsilon\right\} < \varepsilon.$$

Show that if $\{f_n\}$ is a Cauchy sequence in measure, then there is a function f to which the sequence $\{f_n\}$ converges in measure.

Proof. Let $\{f_n\}$ be a sequence of measurable functions which is Cauchy in measure. Fix $\nu \in \mathbb{N}$. Choose $n_{\nu+1} > v_\nu$ such that

$$m\left\{x : \left|f_{n_{\nu+1}}(x) - f_{n_\nu}(x)\right| \geq \frac{1}{2^\nu}\right\} < \frac{1}{2^\nu}.$$

We claim that the series

$$S_n(x) = \sum_{\nu=1}^{\infty} \left(f_{n_{\nu+1}}(x) - f_{n_\nu}(x)\right)$$

converges almost everywhere to a function g. Define the set

$$E_\nu = \left\{x : \left|f_{n_{\nu+1}}(x) - f_{n_\nu}(x)\right| \geq \frac{1}{2^\nu}\right\}.$$

If $x \notin A_k = \bigcup\limits_{\nu=k}^{\infty} E_\nu$, then

$$\left|f_{n_{\nu+1}}(x) - f_{n_\nu}(x)\right| < \frac{1}{2^v} \text{ for all } \nu > k.$$

Taking the intersection over all k for A would mean that this set would be contained in A i.e.,

$$\bigcap_{k=1}^{\infty} A_k \subset A_k$$

and so

$$m\left(\bigcap_{k=1}^{\infty} A_k\right) \leq m\left(A_k\right) \leq \sum_{k=1}^{\infty} m(A_k) < \frac{1}{2^{\nu-1}}.$$

Because ν is fixed, $m\left(\bigcap\limits_{k=1}^{\infty} A_k\right) = 0$. Thus $S_n(x) \to g(x)$ almost everywhere.

Let $f = g + f_{n_1}$ be a sequence. By construction, the partial sums of this sequence are telescoping i.e., for any $\nu \in \mathbb{N}$, the partials sums of f are of the form $f_{n_\nu} - f_{n_1}$. Thus $f_{n_\nu} \overset{m}{\to} f$. Now let $\varepsilon > 0$ be chosen. Because the sequence $\{f_n\}$ is Cauchy in measure, there exists $N_1 \in \mathbb{N}$ such for all $m, n \geq N_1$,

$$m\left\{x : |f_n(x) - f_m(x)| \geq \frac{\varepsilon}{2}\right\} < \frac{\varepsilon}{2}.$$

Since $f_{n_\nu} \overset{m}{\to} f$, there exists $N_2 \in \mathbb{N}$ such that for all $k > N_2$

$$m\left\{x : |f_{n_k} - f(x)| \geq \frac{\varepsilon}{2}\right\} < \frac{\varepsilon}{2}.$$

Set $N = \max\{N_1, N_2\}$. So for any $n, k > N$, we know

$$\begin{aligned} m\left\{x : |f_n(x) - f(x)| \geq \varepsilon\right\} &\leq m\left\{x : |f_{n_k} - f_n(x)| \geq \frac{\varepsilon}{2}\right\} + m\left\{x : |f(x) - f_{n_k}(x)| \geq \frac{\varepsilon}{2}\right\} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Having satisfied the definition of convergence of measure, $f_n \overset{m}{\to} f$ which completes the proof. □

Problem 4. Compute $\lim_{n\to\infty}\int_0^1 (1+nx^2)(1+x^2)^{-n}\,\mathrm{d}x$. Justify your answer.

Proof. Note that we can rewrite this integral as

$$\lim_{n\to\infty}\int_0^1 \frac{1+nx^2}{(1+x^2)^n}\,\mathrm{d}x$$

We can interchange the limit operation and the integral because the sequence of functions $f_n(x) = \left\{\frac{1+nx^2}{(1+x^2)^n}\right\}$ is uniformly convergent and, in fact, this sequence is uniformly convergent to 0. To that end fix $\varepsilon > 0$. Take the derivative of $f(x) = \frac{1+nx^2}{(1+x^2)^n}$ with respect to x as we want to find where this function is maximized over $[0,1]$. It can be shown that (saving showing all of the algebra),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1+nx^2}{(1+x^2)^n}\right) = -2(n-1)nx^3(x^2+1)^{-n-1}.$$

For any $x \in [0,1]$, as $n \to \infty$, this quantity goes to 0 i.e., $f(x)$ is maximized when $x = 0$. So then $f(0) = \frac{1}{1^n} = 1$ for all $n \in \mathbb{N}$. Thus choose $N \in \mathbb{N}$ large enough so that $\frac{1}{N} < \varepsilon$. So for any $n > N$,

$$\left|\frac{1+nx^2}{(1+x^2)^n}\right| \leq \frac{1}{n} < \varepsilon.$$

Thus $f_n(x) \to 0$ uniformly and so

$$\lim_{n\to\infty}\int_0^1 \frac{1+nx^2}{(1+x^2)^n}\,\mathrm{d}x = \int_0^1 \lim_{n\to\infty}\frac{1+nx^2}{(1+x^2)^n} = \int_0^1 0\,\mathrm{d}x = 0.$$

□