Theorem (6.4, Holder Inequality). [1] If p and q are nonnegative extended real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and if $f \in L^p$ and $g \in L^q$, then $f \cdot g \in L^1$ and

$$\int |fg| \leq ||f||_p \cdot ||g||_q.$$

Proof. There are two cases. (i) ($p = 1$, $q = \infty$) ...add notes on this. (ii) $p, q \in (1, \infty)$. Without loss of generality, suppose $f, g \geq 0$; otherwise, just take the absolute value. Set

$$h(x) = g(x)^{q-1} = g(x)^{q/p}$$

and

$$g(x) = h(x)^{p-1} = h(x)^{p/q}.$$

Then

$$\begin{aligned} p \cdot t \cdot f(x) \cdot g(x) &= p \cdot t \cdot f(x) \cdot h(x) \\ &\leq (h(x) + tf(x))^p - h(x)^p. && \text{Lemma 6.3} \end{aligned}$$

Taking the integral of both sides, (pulling out constants),

$$\begin{aligned} p \cdot t \int f(x)g(x) &\leq \int ||h(x) + tf(x)| \, |_p^p - \int ||h| \, |_p^p \\ &\leq (||h(x)| \, |_p + t||f(x)||_p)^p - ||h(x)||_p^p && \text{Triangle inequality} \end{aligned}$$

Dividing by t,

$$p \int f(x)g(x) \leq \frac{(||h(x)| \, |_p + t||f(x)||_p)^p - ||h(x)||_p^p}{t}$$

which the right-hand side is derivative of $\phi(t) = (||h||_p + t||f||_p)^p$. Taking the derivative with respect to t at $t = 0$, we get that

$$p \int f(x)g(x) \leq p \left(\|h(x)\|_p^{p-1} + \|f(x)\|_p \right)^{p-1} = p\|f(x)\|\|g(x)\|$$

and so we are done! ∎

Section 6.3 Convergence and Completeness

Recall that if $(X, || \cdot ||)$ is a norm space (naturally a metric space), then (X, d) is a metric space where

$$d(f, g) := ||f - g||$$

so the norm is the metric of the space.

Definition. We $\{f_n\} \in L^p$ converges to an element $f \in L^p$ in L^p norm if

$$||f_n - f||_p \to 0.$$

That is, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $||f - f_n||_p < \varepsilon$.

[1] If $p, q = 2$, then this just reduces to the Cauchy-Schwarz inequality.

Definition. A normed space $(X, ||\cdot||)$ is called a **complete** space if every Cauchy sequence of X is convergent.

- Note that a completed normed space is a called a **Banach space**.

Our goal will be to show that L^p for $p \geq 1$ is a Banach space.

Definition. A sequence $f_n \subset X$ for any normed space X is **summable** to a sum s in X if the partial sum converges, i.e.,

$$\left\| s - \sum_{k=1}^{n} f_k \right\| \to 0.$$

- A sequence is **absolute summable** if

$$\sum_{i=1}^{\infty} \|f_n\| < \infty.$$

Proposition (Proposition 6.5). A normed linear space X is complete if and only if every absolutely summable series is summable.

Proof. We will need to complete two directions.

($\Rightarrow$) Let X be a Banach space and let $\{f_n\}$ be an absolute summable sequence. This means we have that

$$\sum_{n=1}^{\infty} \|f_n\| < M.$$

Our goal will be show that the partial sums are Cauchy sequence (then convergent by the completeness of a Banach space) i.e.,

$$S_n = \sum_{i=1}^{n} f_i$$

is Cauchy. Then suppose $n > m$ and so

$$\|S_n - S_m\| = \left\| \sum_{k=m}^{n} f_k \right\| \leq \sum_{k=m}^{n} \|f_k\| < \underbrace{\sum_{k=m}^{\infty} \|f_k\|}_{< \varepsilon}$$

for any $\varepsilon > 0$ because $\{f_n\}$ is absolutely summable and therefore convergent. Thus, the partial sums are Cauchy and so convergent.

($\Leftarrow$) Now suppose every absolutely summable series is summable. We will construct a series from the Cauchy sequence. Let $\{f_n\}$ be a Cauchy sequence. Pick $\frac{\varepsilon}{2^k}$, and then pick the subsequence $\{f_{n_k}\}$ such that

$$\|f_{n_{k+1}} - f_{n_k}\| < \frac{1}{2^k}$$

which we can do because $\{f_n\}$ is Cauchy. Consider the series $g_k = f_{n_k} - f_{n_{k-1}}$, which is summable because the sequence is decreasing by construction. By assumption, then $\{g_k\}$ must be absolutely summable; i.e., the sum

$$S_m = \sum_{k=1}^{m} g_k$$

has a limit. Note that S_m is a telescoping series by construction again thus $S_m = -f_{n_1} + f_{n_m}$. This implies that $\{f_{n_k}\}$ converges to f for some $f \in X$ as $k \to \infty$. Since $\{f_n\}$ is Cauchy,

$$\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\|.$$

Then use the fact that $\{f_n\}$ is Cauchy and $\{f_{n_k}\}$ is convergent, pick $\frac{\varepsilon}{2}$ for each thing and so the result follows.

$\square$

Theorem (6.6, Riesz-Fisher). L^p is complete for $p \in [1, \infty]$.