Problem 1 (2.9)**.** Properties of sequences in $\mathbb{R}$.

(a.) Show that $\limsup x_n$ and $\liminf x_n$ are the largest and smallest cluster points of the sequence $\{x_n\}$.

Proof. Let $\{x_n\}$ be any sequence in $\mathbb{R}$. First to show that $l = \limsup x_n$ is indeed a cluster point, let $\varepsilon > 0$ be chosen. Because l is the limit superior, there exists $n_1 \in \mathbb{N}$ such that $x_{k_1} < l + \varepsilon$ for all $k_1 \geq n_1$. Additionally, there are infinitely many value of this value n_1 such that $x_k > l - \varepsilon$ for some $k_1 \geq n_1$, which together with the last sentence implies that $|x_{k_1} - l| < \varepsilon$. To inductively create a subsequence, let $n_1, \ldots, n_j$ and $x_{k_1}, \ldots, x_{k_j}$ be arbitrary. Let n_{j+1} be chosen such that $n_{j+1} > \max\{k_1, \ldots, k_j\}$. Then, because l is the limit superior $x_k < l + \varepsilon$ for any $k \geq n_{j+1}$. Further, for sufficiently large n_{j+1}, there exists $k_{j+1} \geq n_{j+1}$ such that $x_{k_{j+1}} > l - \varepsilon$, which gives us that $|x_{k_{j+1}} - l| < \varepsilon$. Because we can always choose the next point in the subsequence in this manner, this means that the subsequence $\{x_{n_j}\}$ converges to l. By Problem 2.8, this means that l is a cluster point of $\{x_n\}$.

By way of contradiction, suppose that l is not the largest cluster point of the sequence. That is, there exists a cluster point y of $\{x_n\}$ such that $y > l$. Note that by Problem 2.8, this means that there exists a subsequence $\{x_{n_j}\}$ which converges to y. Because l is the limit superior of the sequence, for any $\varepsilon > 0$, we can find $n \in \mathbb{N}$ such that $x_k < l + \varepsilon$ whenever $k \geq n$. Since this is true for any $\varepsilon > 0$, we can choose $\varepsilon > 0$ small enough so that we have

$$l < l + \varepsilon < y - \varepsilon < y.$$

This means that there are a finite number of terms of $\{x_n\}$ contained within the interval $(y - \varepsilon, y + \varepsilon)$. In other words, there does not exist a subsequence $\{x_{n_j}\}$ which converges to y as we would necessarily need an infinite number of terms within ε of y—a contradiction. Therefore, l is the largest cluster point.

By a reverse argument, we can show that $\liminf x_n$ is a cluster point of $\{x_n\}$ as well as the smallest cluster point. □

(b.) Show that every bounded sequence has a convergent subsequence.

Proof. Let $\{x_n\}$ be a bounded sequence. In other words, $\sup x_n$ is a finite real number. By definition of the limit superior, $\limsup x_n \leq \sup x_n$. From part (a), because *lim* $\sup x_n$ is a cluster point, we can always construct a convergent subsequence (as well as for the limit inferior of the sequence) and so this completes this part. □

Problem 2 (2.43)**.** Let f be defined as

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ p \sin\left(\frac{1}{q}\right) & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

At what points is f continuous? (Please justify your answer.)

Proof. I claim that f is not continuous at the rational numbers. To that end, let $x \in Q$ and choose $\varepsilon = x - f(x)$. Fix $\delta > 0$. Note that we can always find an irrational number $y \in (x, x + \delta)$. Because y is irrational, by definition of the function, $f(y) - f(x) = y - f(x)$. But then $y - f(x) > x - f(x) = \varepsilon$.

For $x = 0$, fix $\varepsilon > 0$. Choose $\delta = \varepsilon$, and a pick a point $y \in \mathbb{R}$ such that $|x - y| = |y - 0| = |y| < \delta$. Now because $\sin(1/q) < 1/q$ for any $q \in \mathbb{N}$, we know that

$$\begin{aligned} |f(y) - f(0)| &\leq |y - 0| \\ &< \delta \\ &= \varepsilon \end{aligned}$$

so f is continuous at 0.

f is also continuous at the irrationals. This is because we can if we pick any point x in the irrationals, we can find sufficiently large q so a rational number $y = \frac{p}{q}$ is close to x (i.e, for a fixed ε, choose δ to be smaller than $f(y) - y$ for this to work). Then this would allow us to bound $|f(y) - f(x)|$ leveraging that we can put the rational numbers in lowest terms

$\square$

Problem 3. Show that $F \subset \mathbb{R}$ is a closed set if and only if $F^{\complement}$ is open.

Proof. To complete this proof, we will need a forward and backwards implication.

($\Rightarrow$) Suppose $F \subset \mathbb{R}$ is a closed set. Because we desire to show that $F^{\complement}$ is open, let $x \in F^{\complement}$ be a point. This means that $x \notin F$. Since F is a closed set (i.e., $F = \overline{F}$) and $x \notin F$, we know x is not a point of closure of F. So there exists $\delta > 0$ such that for all $y \in F$, we do not have $|x - y| < \delta$. But then if $|x - y| < \delta$, this must mean that $y \in F^{\complement}$, and so F is an open set.

($\Leftarrow$) Conversely, suppose that the set $F^{\complement}$ is open. Let $x \in F^{\complement}$. Then there exists $\delta > 0$ such that if $|x - y| < \delta$, then $y \in F^C$. This means that there is no $y \in F$ such that $|x - y| < \delta$ and so x cannot be a point of closure of F. Thus, because x is arbitrary, F necessarily contains all its points of closure; in other words, $F = \overline{F}$ and thus F must be closed, completing this direction.

Having completed both implications, this completes the proof. $\square$