Construction of the Real Numbers, $\mathbb{R}$

- We first start from $\mathbb{N} \cup \{0\}$ and add numbers together subsequently (i.e. 1, $\underbrace{1+1}_{2}$, $\underbrace{1+1+1}_{3}$, ...)
- To construct the integers $\mathbb{Z}$, we take the set difference with the natural numbers so that we have
$$\mathbb{Z} = \mathbb{N} \cup \{0\} \setminus \mathbb{N}.$$
- Then the rationals numbers, $\mathbb{Q}$, can be constructed from the integers and are defined by the set
$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \right\}.$$
- To construct the irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$, we can use the dedekind cut to do this. However, this is convoluted and we can go about this in a different way.

Axioms of the Real Numbers

A. The Field Axioms: For all real numbers $x, y \in \mathbb{R}$ we have:

A1. x + y = y + x

A2. (x + y) + z = x + (y + z)

A3. There exists $0 \in \mathbb{R}$ such that $x + 0 =$ for all $x \in \mathbb{R}$.
[**Identity element under addition**]

A4. For each $x \in \mathbb{R}$ there is a $w \in \mathbb{R}$ such that $x + w = 0$.
[**Inverse element under addition**]

A5. xy = yx

A6. (xy)z = x(yz)

A7. There exists $1 \in \mathbb{R}$ such that $1 \neq 0$ and $x \cdot 1 = x$ for all $x \in \mathbb{R}$.

A8. For each $x \in \mathbb{R}$ different from 0 there is $w \in \mathbb{R}$ such that $xw = 1$.

A9. x(y + z) = xy + xz.

We can prove some properties now:

Proposition 1. The additive inverse is unique.

Proof. Let $x \in \mathbb{R}$. Suppose we have two numbers $w_1, w_2 \in \mathbb{R}$ such that $x + w_1 = 0 = x + w_2$. Using the axioms and our assumption, we can show the following:

$$\begin{aligned} w_1 &= w_1 + 0 && \text{Axiom A3} \\ &= w_1 + x + w_2 && \text{Assumption of } 0 = x + w_2 \\ &= w_2 + xw_1 && \text{Axiom A1} \\ &= w_2 \end{aligned}$$

which completes the proof. □

B. The Axioms of Order: The subset P of positive real numbers satisfies the following:

B1. If $x, y \in P$, then $x + y \in P$.

B2. If $x, y \in P$, then $xy \in P$.

B3. If $x \in P$, then $-x \notin P$.

B4. If $x \in \mathbb{R}$, then $x = 0$ or $x \in P$ or $-x \in P$.

Note that any system which satisfies the axioms of groups A and B is called an **ordered field**.

Definition. We can give definitions of the ordered operations $<$, $\leq$, $>$ and $\geq$.

- $x < y$ means that $y - x \in P$.
- $x \leq y$ means that $y - x \in P \cup \{0\}$. Or, this means that $x < y$ or $x = y$.
- $x > y$ means that $x - y \in P$.
- $x \geq y$ means that $x - y \in P \cup \{0\}$. Or, this means that $x > y$ or $x = y$.

From this, we can deduce and prove some which is any set which satisfies the axioms of group A and B.

Definition. Let $x, y \in \mathbb{R}$ and define the absolute value as

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

Proposition 2. Let $a, b, c \in \mathbb{R}$.

1. $a < b$ if and only if $-b < -a$.
2. If $a < b$ and $b < c$, then $a < c$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. For $a, b \in \mathbb{R}$, then only one is true $a = b$, $a > b$ and $a < b$.
5. If $x \neq 0$, then $x^2 = x \cdot x > 0$; in particular, $1 > 0$.
6. If $x, y \in \mathbb{R}$, then $|x + y| \leq |x| + |y|$.

Definition. Let $S \subset \mathbb{R}$. The number $b \in \mathbb{R}$ is an **upper bound** for S if for each $x \in S$, we have $x \leq b$.

Similarly, a number $x \in \mathbb{R}$ is the **least upper bound** for S if it is an upper bound for (S) and if $x \leq b$ for each upper bound b of S. We then call x the **supremum** of S and denote this $x = \sup S$.

Definition. Let $S \subset \mathbb{R}$. The number $l \in \mathbb{R}$ is an **lower bound** for S if for each $x \in S$, we have $l \leq x$.

Similarly, a number $x \in \mathbb{R}$ is the **greatest lower bound** for S if it is a lower bound for S and if $x \leq l$ for each lower bound l of S. We then call x the **infimum** of S and denote this $x = \inf S$.

C. **Completeness Axiom:** Every nonempty set $S \subset \mathbb{R}$ which has an upper bound has a least upper bound.

Proposition 3. Let $L, U \subset \mathbb{R}$ be nonempty subsets with $R = L \cup U$ and such that for each $l \in L$ and each $u \in U$ we have $l < u$. Then either L has a greatest element or L has a least element.

Proposition 4 (Approximation Property.)**.** Let $S \subset \mathbb{R}$ be a nonempty. If $u = \sup S$, then for all $\gamma > 0$, there exists $Sr \in S$ such that $u - r < Sr < u$.

Theorem (2.3, **Axiom of Archimedes**). If $x \in \mathbb{R}$ is any real number, then there exists $n \in \mathbb{N}$ such that $x < n$.

Proof. We can break this into two cases

1. Let $x < 1$. If so, then simply choose $x = 1$.
2. Let $x \geq 1$. Define the set $S = \{n \in N : n \leq x\}$. Then since this set is bounded above, by the Completeness Axiom, $\sup S = y$ exists. Because x is an upper bound S, by definition of the supremum, we have that $y \leq x$. Let $r = \frac{1}{2}$. Then we can find $k \in S$ such that $y - \frac{1}{2} < k \leq y$. But then we have that $y < y + \frac{1}{2} < k + 1 \leq y + 1$. Then this means $k + 1 \notin S$ and so $x < k + 1$, completing this case.

Having exhausted all cases, this completes the proof. □

Proposition 1 (**Well-Ordering Principle**). Every nonempty subset $S \subset \mathbb{N}$ has a minimum.

Proposition 2 (**Density of the Rational Numbers**). Let $x, y \in \mathbb{R}$. Then if $x < y$, there exists $q \in \mathbb{Q}$ such that $x < \gamma < y$

Section 2.4, Sequences in $\mathbb{R}$

Definition. We define a **sequence** of real numbers to be a function that maps each each natural number n into the real number x. That is, a sequence is a function $s : \mathbb{N} \to A$ for $A \subset \mathbb{R}$. This is written as $\{x_n\}$ or $\{x_n\}_{n=1}^{\infty}$.

Definition (**Convergence of a Sequence**). A sequence converges to the real number $l \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - l| < \varepsilon.$$

Definition (**Cauchy Sequence**). A sequence $\{x_n\}$ in $\mathbb{R}$ is **Cauchy** sequence if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$|a_n - a_m| < \varepsilon.$$

Theorem. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then $\{x_n\}$ is Cauchy if and only $\{x_n\}$ is Cauchy.

Definition. The number $l \in \mathbb{R}$ is called a **cluster point** of $\{x_n\}$ if there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $x_{n_m} \to l$.

We can define this in another way. The number $l \in \mathbb{R}$ is called **cluster point** of $\{x_n\}$ if for all $\varepsilon > 0$ and for all $N \in \mathbb{N}$, there exists $n \geq N$ such that $|x_n - l| < \varepsilon$.

Definition. We define the **limit superior** of a sequence $\{x_n\}$ in $\mathbb{R}$ to be

$$\overline{\lim}_{n\to\infty} x_n = \inf_n \sup_{k \geq n} x_k.$$

This is also denoted as $\limsup$.

Theorem. A number $l \in \mathbb{R}$ is the **limit superior** of the sequence $\{x_n\}$ if and only if

(i) For all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, $x_k < l + \varepsilon$

(ii) For all $\varepsilon > 0$ and for all $n \in \mathbb{N}$, there exists $k \geq n$ such that $x_k > l - \varepsilon$.

Definition. We define the **limit inferior** of a sequence $\{x_n\}$ in $\mathbb{R}$ to be

$$\underline{\lim}_{n\to\infty} x_n = \sup_n \inf_{k\geq n} x_k.$$

This is also denoted as lim inf.

Theorem. A number $l \in \mathbb{R}$ is the **limit inferior** of the sequence $\{x_n\}$ if and only if

(i) For all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, $x_k > l - \varepsilon$

(ii) For all $\varepsilon > 0$ and for all $n \in \mathbb{N}$, there exists $k \geq n$ such that $x_k < l + \varepsilon$.

Proposition 3. From the last two definitions, we have the following property.

- $\overline{\lim}_{n\to\infty}$ is the largest cluster point.
- $\underline{\lim}_{n\to\infty}$ is the smallest cluster point.

Section 2.5, Open and Closed Sets in $\mathbb{R}$

Definition. The set $O \subset \mathbb{R}$ is called an **open** set if for all $x \in O$, there exists $\delta > 0$ such that $x - \delta, x + \delta$.

Equivalently, O is an **open** set if for all $x \in O$, there is a $\delta > 0$ such that each y with $|x - y| < \delta$ belongs to O.

Proposition 4. From this above, we have the following properties:

1. The set $\bigcup_\alpha O_\alpha$ is open.
2. The set $\bigcup_{n=1}^{n} O_m$ is open.

Theorem (Lindelof Theorem). Every open set in $\mathbb{R}$ is a disjoint union of countable union of open intervals.

Proof. This proof is contained on page 42 of Royden. □

Definition. A real number $x \in \mathbb{R}$ is called **point of closure** of a set $E \subset \mathbb{R}$ if for every $\delta > 0$ there exists a $y \in E$ such that $|x - y| < \delta$.

The set of points of closure of E is denoted $\overline{E}$.

Proposition 5. If $A \subset B \subset \mathbb{R}$, then $\overline{A} \subset \overline{B}$. Additionally, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. The proof of this is on page 43 of Royden. □

Definition. A set $F \subset \mathbb{R}$ is called a **closed** set if $\overline{F} = F$.

Note that because $F \subset \overline{F}$ always, a set F is closed if $\overline{F} \subset F$—that is, F contains all of its points of closure.

Proposition 6. For any set E, the set $\overline{E}$ is closed; that is $\overline{\overline{E}} = \overline{E}$.

Proposition 7. Let $E \subset \mathbb{R}$. Then E is open if and only if $E^{\mathcal{C}}$ is closed.

Definition. We say that a collection of sets $\mathcal{C}$ is a **cover** of a set F if

$$F \subset \bigcup_{O \in \mathcal{C}} O.$$

The collection $\mathcal{C}$ is a covering of the set F.

Theorem (Heine-Borel). Let $E \subset \mathbb{R}$ be set. Then E is compact if and only if E is closed and bounded.

Compactness

Theorem. Let $E \subset \mathbb{R}$. Then E is compact if and only E is sequentially compact. That is, for every $\{x_n\}$ in E, there exists a convergent subsequence $x_{n_m} \to x_0$ in E.

Theorem. Let $\{I_n\}$ be a sequence of closed intervals such that $I_{n+1} \subset I_n$. Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

If $[a_n, b_n]$ is an interval and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} = a$, then $\bigcap_{n=1}^{\infty}$.

Section 2.6, Continuous Functions

Definition. Let $E \subset \mathbb{R}$, and let $f : E \to \mathbb{R}$ be a real-valued function. Then f is **continuous** at the point $x = a \in E$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in E$ with $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

Note that we can have continuity in terms of sequences. I will state it as a theorem here even though it was not in lecture because it is important to be able to use on its own.

Theorem. Let $f : E \to \mathbb{R}$ be a function with $E \subset \mathbb{R}$. Let $x \in E$ be any point. Then f is continuous at a if and only for every sequence $\{x_n\}$ in E converging to a, the sequence $\{f(x_n)\}$ in $f(E)$ (the image of E) converges to $f(a)$.

Proposition. Let $E \subset \mathbb{R}$ be compact. Let $f : E \to \mathbb{R}$ be continuous real-valued function. Then $f(E)$ is a compact set.

Proof. Let $\subset \mathbb{R}$ be a compact and suppose the function $f : E \to \mathbb{R}$ is continuous. To show that $f(E)$ is compact, we will use the Heine-Borel theorem and show that it is closed and bounded. To show that $f(E)$ is closed, suppose we have any sequence $\{f(x_n)\}$ converging to the point $f(a) \in \mathbb{R}$. Additionally, let $\{x_n\}$ be any sequence in E. Because E is compact, there exists a subsequence $\{x_{n_m}\}$ which converges to a point $x_0 \in E$. Since f is continuous, by the preceding theorem this means that the sequence $\{f(x_{n_m})\}$ converges to $f(x_0) \in f(E)$. □

Proposition (2.17, Extreme Value Theorem). Let $E \subset \mathbb{R}$ be a compact set, and let $f : E \to \mathbb{R}$ be a continuous function. Then there exists $x_1, x_2 \in E$ such that

$$f(x_1) \leq f(x) \leq (x_2), \text{ for all x} \in E.$$

Proposition (2.18). Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is **continuous** if and only if $f^{-1}(O)$ is open for all open sets $O \subset \mathbb{R}$.

Proposition (2.19). Let $E \subset \mathbb{R}$, and let $f : E \to \mathbb{R}$ be continuous. Without loss of generality, suppose that $f(a) \leq f(b)$. Then for all $\gamma \in [f(a), f(b)]$, there exists $c \in [a, b]$ such that $f(c) = \gamma$.

Definition (Uniform Continuity). Let $E \subset \mathbb{R}$. A function $f : E \to \mathbb{R}$ is **uniformly continuous** if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

Proposition (2.20). Let $E \subset \mathbb{R}$ be a compact set. If $f : E \to \mathbb{R}$ is a continuous function on E, then f is uniformly continuous on E.

Definition. Let $f_n : E \to \mathbb{R}$ be a sequence of functions, and let $f : E \to \mathbb{R}$.

1. The sequence $\{f_n\}$ **converges pointwise** on E to f if for all $\varepsilon > 0$ and for all $x \in E$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f(x) - f_n(x)| < \varepsilon$.
2. The sequence $\{f_n\}$ **converges uniformly** if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in E$ and for all $n \geq N$, $|f(x) - f_n(z)| < \varepsilon$.

Section 3.1, Lebesgue Measure

[Perhaps finish these notes another time...]

Section 3.2, Outer Measure

Definition. The **outer measure** $m^*(A)$ of a set $A \subset \mathbb{R}$ is given

$$m^*(A) = \inf_{A \subset \bigcup I_n} \sum_n l(I_n)$$

where $\{I_n\}$ is a countable collection of open intervals that cover A.

Note that from this definition, we get that

1. $m^*(\emptyset) = 0$
2. If $A \subset B$, $m^*(A) < m^*(B)$.
3. m^* does not satisfy disjoint additivity.

Proposition (3.1). The outer measure of an interval is its length; that is, $m^*(I) = l(I)$ where $I = [a, b], (a, b), [a, b)$, or $(a, b]$.

Proof. It is sufficient to show that $m^*([a, b]) = l([a, b])$ since every other interval is a subset of $[a, b]$. Let $\varepsilon > 0$. Then $[a, b] \subset \left[a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right]$ which implies, by the definition of the outer measure,

$$m^*([a, b]) \leq l\left(\left[a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right]\right) = b - a + \varepsilon.$$

Because ε was fixed, this means that $m^* \leq b - a$.

Now we must show that $m^* \geq b - a$. Because $[a, b]$ is compact, for any collection $\{I_n\}$ of open intervals covering $[a, b]$, there exists a finite collection of intervals $\{I_1, \ldots, I_k\}$ so that

$$[a, b] \subset \bigcup_{n=1}^{k} I_n.$$

This gives us that

$$\sum_n I_n \geq \sum_{n=1}^{k} I_n \geq b - a$$

and so $b - a$ is a lower bound. But since m^* as the greatest lower bound of all such sums, we have that $m^* \geq b - a$.

Therefore, $m^*([a, b]) = l([a, b]) = b - a$.

□

Proposition (3.2, Subadditivity). Let $\{A_n\}$ be a countable collection of sets on $\mathbb{R}$. Then

$$m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n).$$

Proof. Proof on page 57.

□

Corollary (3.3). If A is a countable set, then $m^*(A) = 0$.

Proof. Proof is on the end of page 57. $\square$

Section 3.3, Measurable Sets and Lebesgue Measure

Definition. A set $E \subset \mathbb{R}$ is (Lebesgue) **measurable** if for all sets A, we have that

$$m^*(A) = m^*(A \cup E) + m^*(A \cup E^{\complement}).$$

Lemma (3.6). If $m^*(E) = 0$, then E is measurable.

Proof. Let A be any chosen set. Because $A \cap E \subset E$ and $m^*(E) = 0$,

$$m^*(A \cap E) \leq m(E) = 0.$$

Note that $A \cap E^{\complement} \subset A$ and so $m^*(A) \leq m^*(A \cap E^{\complement})$ and so it suffices to show that $m^*(A) \geq m^*(A \cap E^{\complement})$. Using this, we can show that

$$m^*(A) \geq m^*(A \cap E^{\complement}) + 0 = m^*(A \cap E^{\complement}) = m^*(A \cap E)$$

giving us the desired result. $\square$

Definition. Let $\mathcal{M}$ be the set of measurable sets in $\mathbb{R}$

Lemma (3.7). If E_1 and E_2 are measurable sets, then so is $E_1 \cup E_2$.

Proof. Proof on top of page 57. $\square$

Corollary (3.8). The family $\mathcal{M}$ of measurable sets if an algebra of sets. In other words, if $E \in \mathcal{M}$, then $E^{\complement} \in \mathcal{M}$. Further, if $E_1, E_2 \in \mathcal{M}$, then $E_1 \cup E_2 \in \mathcal{M}$.

Lemma (3.9). Let A be any set, and $E_1, \ldots, E_n$ be a finite sequence of sets such that $E_i \cap E_j$ for all $i \neq j$. Then

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Proof. We proceed by induction. For $n = 1$, we have the set E_1 and the equality holds. Suppose that we have $n = k$ sets $E_1, \ldots, E_k$ with $E_i \cap E_j \neq \emptyset$ for all $i \neq j$ so that

$$m^*\left(A \cap \left[\bigcup_{i=1}^{k} E_i\right]\right) = \sum_{i=1}^{k} m^*(A \cap E_i).$$

Consider $n = k + 1$. Because each E_i is disjoint,

$$A \cap \left(\bigcup_{i=1}^{k+1} E_i\right) \cap E_{k+1} = A \cap E_{k+1};$$
$$A \cap \left(\bigcup_{i=1}^{k+1} E_i\right) \cap E_{k+1}^{\complement} = A \cap \bigcup_{i=1}^{k} E_i.$$

Because the E_i's are measurable,

$$\begin{aligned} m^*\left(A \cap \bigcup_{i=1}^{k+1} E_i\right) &= m^*\left(A \cap E_{k+1}\right) + m^*\left(A \cap \bigcup_{i=1}^{k} E_i\right) \\ &= m^*\left(A \cap E_{k+1}\right) + \sum_{i=1}^{k} m^*(A \cap E_i) \qquad \text{Induction Hypothesis} \\ &= \sum_{i=1}^{k+1} m^*(A \cap E_i) \end{aligned}$$

which, by induction, completes the proof. □

Theorem (3.10). $\mathcal{M}$ is a σ-algebra. In other words, in addition to being an algebra of sets, if $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

Proof. [1]

□

Lemma (3.11). The interval (a, ∞) is measurable for all $a \in \mathbb{R}$.

Proof. [2]

□

[1] Proof on bottom of page 59 and top of page 60.
[2] Proof on the bottom of page 60 through the middle of page 61.

Theorem (3.12). Every Borel set is measurable. In particular, each open set and each closed set is measurable.

Proof. [3]

□

Definition. Let $E \in \mathcal{M}$. We define $m(E) := m^*(E)$ to be the **Lebesgue measure** of $E/$

Proposition (3.13, Countable Additivity). Let $\{E_i\}_{i=1}^n$ be a sequence of measurable sets. Then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{n} m(E_i).$$

If, in addition, $E_i \cap E_j$ for all $i \neq j$. then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{n} m(E_i).$$

Proposition (3.14). Let $\{E_i\} \subset \mathcal{M}$ be a decreasing sequence (i.e., $E_{i+1} \subset E_i$). Let $m(E_1) < \infty$. Then

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} m(E_n).$$

Proposition (3.15). Let E be any given set. Then the following are equivalent:

(i) E is measurable.

(ii) For all $\varepsilon > 0$, there is an open set $O \supset E$ with $m^*(O \setminus E) < \varepsilon$.

(iii) For all $\varepsilon > 0$, there is a closed set $F \subset E$ with $m^*(E \setminus F) < \varepsilon$.

(iv) There is a $G \in G_\delta$ with $E \subset G$ such that $m^*(G \setminus E) = 0$.

(v) There is a $F \in F_\sigma$ with $F \subset E$ such that $m^*(E \setminus F) = 0$.

If $m^*(E) < \infty$, the above statements are equivalent:

(vi) For all $\varepsilon > 0$, there is a finite union U of open intervals such that $m^*(U \Delta E) < \varepsilon$.

[3] Proof on the bottom of page 61.

Section 3.5, Measurable Functions

Proposition (3.18). Let $E \subset \mathbb{R}$, and Let $f : E \to [-\infty, \infty]$ be an extended real-valued function whose domain is measurable. Let $\alpha \in \mathbb{R}$ be any real number. Then the following statements are equivalent:

(i) The set $\{x : f(x) > \alpha\}$ is measurable.

(ii) The set $\{x : f(x) \geq \alpha\}$ is measurable.

(iii) The set $\{x : f(x) < \alpha\}$ is measurable.

(iv) The set $\{x : f(x) \leq \alpha\}$ is measurable.

All together, these imply

(v) The set $\{x : f(x) = \alpha\}$ is measurable.

Proof. [1]

□

Definition. An extended real-valued function $f : E \to [-\infty, \infty]$ is **(Lebesgue) measurable** if its domain is measurable and satisfies one of the first four statements of Proposition 18.

Note that this means that any continuous function is measurable since then the pre-image of any open set is still an open set.

Proposition (3.19). Let f and g be two measurable functions defined on the same domain, and let $c \in \mathbb{R}$. Then the functions $f + c$, cf, $f + g$, $g - f$, and fg are measurable.

Proof. Let $\alpha \in \mathbb{R}$ be any real number. Fix $c \in \mathbb{R}$. For $f(x) + c$, note that

$$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}$$

and $\alpha - c$ is a real number, this set is still measurable i.e., $f + c$ is measurable. A similar argument shows that cf is measurable as well.

Take the set

$$\{x : f(x) + g(x) < \alpha\}. \tag{1}$$

Observe that $f(x) + g(x) < \alpha \Leftrightarrow f(x) < \alpha - g(x)$. By the density of $\mathbb{Q}$, there exists $r \in \mathbb{Q}$ such that $f(x) < r < \alpha - g(x)$. So we can write Equation (1) as

$$\{x : f(x) + g(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} \left(\{x : f(x) < r\} \cap \{x : g(x) < \alpha - r\}\right).$$

Because this set is countable, this set is measurable and thus $f + g$ is measurable.

To show that fg is measurable, we can show that f^2 is measurable since

$$fg = \frac{1}{2}\left((f + g)^2 - f^2 - g^2\right).$$

[1] Proof is on page 67.

Take the set

$$\{x : f^2(x) < \alpha\}. \tag{2}$$

For $\alpha \geq 0$, note that $f^2 < \alpha$ is the same as saying $f(x) > \sqrt{\alpha}$ and $f(x) < -\sqrt{\alpha}$. Thus, Equation (2) can rewritten as

$$\{x : f^2(x) < \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}.$$

This is a measurable set which completes the proof. □

Theorem (3.20, Limit of Measurable Functions is Measurable). [2]

Proof. For $f(x) + c$, note that

$$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}.$$

□

Theorem (3.20, Limit of Measurable Functions is Measurable). Let $\{f_n\}$ be a sequence of measurable functions with the same domain. Then the functions $\sup\{f_1(x), \ldots, f_n(x)\}$, $\inf\{f_1(x), \ldots, f_n(x)\}$, $\sup_n f_n$, $\inf_n f_n$, $\overline{\lim} f_n$, and $\underline{\lim} f_n$ are measurable.

Proof. Let $\{f_n\}$ be a sequence of measurable functions. Let $h(x) = \sup\{f_1(x), \ldots, f_n(x)\}$ and we so must show that $\{x : h(x) < \alpha\}$ for all $\alpha \in \mathbb{R}$. To that end, let $\alpha \in \mathbb{R}$ be chosen. Then

$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$$

which, because the right-hand side is a union of measurable sets from the f_i's being measurable, means that the set $\{x : h(x) < \alpha\}$ is also measurable.

Let $g(x) = \sup_n f_n$. By a similar argument as above,

$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i(x) > \alpha\}$$

is a countable set so $\{x : g(x) < \alpha\}$ is measurable. □

Definition. A property is said to hold **almost everywhere** (a.e) if the set of points where it fails to hold is a set of measure zero. Thus $f = g$ a.e if f and g have the same domain and $m\{x : f(x) \neq g(x)\}$.

Proposition (3.21). If f is measurable and $f = g$ a.e, then g is measurable.

Proof. [3] Let $E = \{x : f(x) \neq g(x)\}$.

This is equivalent to saying that

Let $\{x : g(x) > \alpha\}$. This is equivalent to saying that

$$\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha\}$$

□

[2] Proof is on bottom of page 68 and top of page 69

[3] Proof is on middle of page 69.

The following proposition essentially says that measurable functions are nearly continuous; or. in other words, we can "nicely" approximate measurable functions.

Proposition (3.22, Littlewood's 2nd Principle). Let $f : [a, b] \to E$ be a measurable function with $E \subset \mathbb{R}$ and is equal to $\pm\infty$ only on sets with measure zero. Then for all $\varepsilon > 0$, there exist a step function g and a continuous function f such

$$|f - g| < \varepsilon \quad \text{and} \quad |f - h| < \varepsilon$$

except on set of measure less than ε; i.e., $m\{x : |f(x) - g(x)| \geq \varepsilon\} < \varepsilon$ and $m\{x : |f(x) - h(x)| \geq \varepsilon\} < \varepsilon$. If in addition $m \leq f \leq M$, then we may choose the functions g and h so that $m \leq g \leq M$ and $m \leq h \leq M$.

Proposition (3.23, (Weak) Egonoff's Theorem). Let E be a measurable set of finite measure, and $\{f_n\}$ be a sequence of measurable functions defined on E. Let f be real-valued function such for each $x \in E$ we have $f_n(x) \to f(x)$. Then for all $\varepsilon > 0$ and all $\delta > 0$, there is measurable set $A \subset E$ with $m(A) < \delta$ and $N \in \mathbb{N}$ such that for all $x \notin A$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Proposition (3.23, (Weak) Egonoff's Theorem). Let E be a measurable set of finite measure, and $\{f_n\}$ be a sequence of measurable functions defined on E. Let f be real-valued function such for each $x \in E$ we have $f_n(x) \to f(x)$. Then for all $\varepsilon > 0$ and all $\delta > 0$, there is measurable set $A \subset E$ with $m(A) < \delta$ and $N \in \mathbb{N}$ such that for all $x \notin A$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Proof. [1]

Let $\varepsilon > 0$ be chosen. Define the set

$$G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}$$

and also

$$E_N = \bigcup_{n=N}^{\infty} G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N\}.$$

Because $\{E_N\}$ is a decreasing sequence and $f_n(x) \to f(x)$ pointwise, for all $x \in E$, $|f_n(x) - f(x)| < \varepsilon$ and so $\displaystyle\bigcup_{i=1}^{\infty} E_N = \emptyset$. Thus, by Proposition 3.14,

$$\begin{aligned} E_N = \emptyset \implies m(E_N) &= 0 \\ &= m\left(\bigcup_{N=1}^{\infty} E_N\right) \\ &= \lim_{N\to\infty} E_N. \end{aligned}$$

So for any $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $m(E_N) < \delta$. Now take $A = E_N$ for any $N > N_0$ and so $m(A) < \delta$ and also

$$A^{\complement} = \{x \in E : x \notin E\} = \{x \in E : |f_n(x) - f(x)| < \varepsilon \text{ for all } n > N_0\}.$$

$\square$

Section 4.1 Riemann Integration

Definition. Let $f : [a, b] \to \mathbb{R}$ be a bounded real-valued function and let

$$P = \{a = \xi_0 < \xi_1 < \cdots < \xi_n = b\}$$

be a subdivision (partition) of $[a, b]$. We can define the **upper sum**, S and **lower sum**, s, respectively, as

$$S = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) M_i \qquad \text{and} \qquad s = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) m_i$$

where

$$M_i = \sup\{f(x) : x \in [\xi_{i-1}, \xi_i]\} \qquad \text{and} \qquad m_i = \inf\{f(x) : x \in [\xi_{i-1}, \xi_i]\}.$$

[1] Proof on page 72-73.

Definition. Let $f : [a,b] \to \mathbb{R}$ be a bounded real-valued function. Define the **upper Riemann integral** of f as

$$R\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf\{S : P \text{ is a partition of } [a,b]\}$$

and the **lower Riemann integral** of f as

$$R\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup\{s : P \text{ is a partition of } [a,b]\}.$$

Definition. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if

$$R\underline{\int_a^b} f(x)\,\mathrm{d}x = R\int_a^b f(x)\,\mathrm{d}x = R\overline{\int_a^b} f(x)\,\mathrm{d}x.$$

In other words, if the upper and lower Riemann integrals are equal to each other.

Note this theorem is *NOT* from Royden but from my real analysis course in undergraduate. It comes from the definition of the upper and lower Riemann integral definitions by ends up being a useful way to show Riemann integration.

Theorem. Let f be a bounded function on $[a,b]$. Then f is Riemann integrable if and only if for all $\varepsilon > 0$, there exists a subdivision (partition) P of $[a,b]$ such that

$$S - s < \varepsilon.$$

Section 4.2 The Lebesgue Integral

Definition. The **characteristic function** of E is defined as

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

A linear combination

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

is called a **simple function** if the sets $\{E_1, \dots, E_n\}$ are measurable. Note that ϕ is simple if and only if it is measurable and only assumes a finite number of values.

The **canonical representation** of ϕ is such that

$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$$

where $A_i = \{x : \phi(x) = a_i\}$ and where the A_i's are disjoint and the a_i's are distinct and nonzero.

Definition. Let
$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$$
be a simple function which vanishes outside of a set of finite measure. Then the integral of ϕ is defined as
$$\int \phi = \sum_{i=1}^{n} a_i \cdot m(A_i).$$

Lemma (4.1). Let $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ with $E_i \cap E_j = \emptyset$ for all $i \neq j$ where $E_i \in \mathfrak{M}$ and $m(E_i) < \infty$ for each $i = 1, \ldots n$. Then
$$\int \phi = \sum_{i=1}^{n} a_i \cdot m(E_i).$$

Proposition (4.2). Let ϕ, ψ be simple functions vanishing outside of a set with finite measure. Then
$$\int (a\phi + b\psi) = a \int \phi + b \int \psi.$$
If $\phi \geq \psi$ almost everywhere,
$$\int \phi \geq \int \psi.$$

Proposition (4.3). Let $E \in \mathfrak{M}$, and let $f : E \to \mathbb{R}$ with $m(E) < \infty$ be a bounded, measurable function. Let ϕ, ψ be simple functions. Then
$$inf \left\{ \int \psi : \psi \geq f \right\} = \sup \left\{ \int \phi : \phi \leq f \right\}.$$
if and only if f is measurable.

Proof. [2]

□

[2]Proof is on pages 79-80.

Proposition (4.3). Let $E \in \mathfrak{M}$, and let $f : E \to \mathbb{R}$ with $m(E) < \infty$ be a bounded, measurable function. Let ϕ, ψ be simple functions. Then

$$\inf\left\{\int \psi : \psi \geq f\right\} = \sup\left\{\int \phi : \phi \leq f\right\}.$$

if and only if f is measurable.

Proof. [1] We will need to show two implications.

($\Leftarrow$) First, suppose that f is measurable. Fix any $n \in \mathbb{N}$ and define the set

$$E_k = \left\{x : \frac{k-1}{n}M < f(x) \leq \frac{kM}{n}\right\}$$

with $k \in [-n, n]$ and $|f(x)| < M$. Note that because f is measurable, each E_k is a measurable set and also we have that $\bigcup_{k=-n}^{n} E_K = E$. Define the upper and lower sequence of simple functions, $\{\psi_n\}$ and $\{\phi_n\}$, respectively, as

$$\psi_n(x) = \sum_{k=-n}^{n} \frac{M}{n} \cdot k \cdot \chi_{E_k}(x) \quad \text{and} \quad \phi_n = \sum_{k=-n}^{n} \frac{M}{n} \cdot (k-1) \cdot \chi_{E_k}(x).$$

So for any $x \in E$, $\phi(x) \leq f(x) \leq \psi(x)$. Thus,

$$\inf_{\psi \geq f} \int_E \psi \leq \int_E \psi = \frac{M}{n} \sum_{k=-n}^{n} k \cdot m(E_k)$$

and

$$\sup_{\phi \leq f} \int_E \phi \geq \int_E \phi = \frac{M}{n} \sum_{k=-n}^{n} (k-1) \cdot m(E_k).$$

Putting these two inequalities together, we can show that

$$\begin{aligned}\inf_{\psi \geq f} \int_E \psi - \sup_{\phi \leq f} \int_E \phi &\leq \int_E \psi - \phi \\ &= \sum_{k=-n}^{n} (\psi_n - \phi_n)\, m(E_k) \\ &= \frac{M}{n} m(E).\end{aligned}$$

Since $n \in \mathbb{N}$ is fixed, this quantity is zero. Thus

$$\inf_{\psi \geq f} \int_E \psi - \sup_{\phi \leq f} \int_E \phi = 0 \Rightarrow \inf_{\psi \geq f} \int_E \psi = \sup_{\phi \leq f} \int_E \phi$$

completing this direction.

[1] Proof is on pages 79-80.

($\Rightarrow$) Conversely, suppose that

$$\inf_{\psi \geq f} \int_E \psi = \sup_{\phi \leq f} \int_E \phi.$$

By definition of the infimum and supremum, there exists sequences of simple functions $\{\phi_n\}$ and $\{\psi_n\}$ such that $\phi_n \leq f \leq \psi_n$ for all $n \in \mathbb{N}$ with

$$\int_E \phi_n - \psi_n < \frac{1}{n}.$$

Define $\phi^* = \sup_n \phi_n$ and $\psi^* = \inf_n \psi(n)$. Since simple functions are measurable functions, by Proposition 3.20, ϕ^* and ψ^* are measurable as well and $\phi_n \leq f \leq \psi_n$.

We claim that $f = \phi^*$ a.e. Consider the set

$$\Delta = \{x : \phi^*(x) < \psi^*(x)\}.$$

Let $\nu \in \mathbb{N}$ and let

$$\Delta_\nu = \left\{x : \phi^*(x) < \psi^*(x) - \frac{1}{\nu}\right\}$$

which means that

$$\Delta = \bigcup_{\nu \in \mathbb{N}} \Delta_\nu.$$

For any $n \in \mathbb{N}$,

$$\Delta_\nu \subset \left\{x : \phi(x) < \psi(x) - \frac{1}{\nu}\right\}.$$

Thus, we have that, for any $n \in \mathbb{N}$,

$$\begin{aligned} m\left(\Delta_\nu\right) &= \int \chi_{\Delta_\nu} = \nu \int \frac{1}{\nu} \cdot \chi_{\Delta_\nu} \\ &\leq \nu \int_{\Delta_\nu} (\psi_n - \phi_n) \\ &< \mu \int_E \frac{1}{n} \\ &= \frac{\nu}{n} m(E). \end{aligned}$$

Because ν is fixed and n is arbitrary, $m(\Delta_\nu) = 0$ which implies that $m(\Delta) = 0$. So then $\phi^* = \psi^*$ except on a set of measure zero, and $\phi^* = f$ except on a set of measure zero i.e., $f = \phi^*$ a.e. implying that f is measurable by Proposition 3.21.

Having completed both directions, this completes the proof and shows the well-definedness of Lebesgue integration. $\square$

Proposition (4.4). Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable, then it is measureble and

$$R \int_a^b f(x)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x.$$

Proof. This proof is on page 82 of Royden (very simple proof, in fact). $\square$

Proposition (4.5). If f and g are bounded measurable functions defined on a set E of finite measure, then:

i. For any $a, b \in \mathbb{R}$,
$$\int_E (af + bg) = a \int_E f + b \int_E g.$$

ii. If $f = g$ a.e., then
$$\int_E f = \int_E g.$$

iii. If $f \leq g$ almost everywhere. then
$$\int_E f \leq \int_E g.$$

Hence
$$\left| \int_E f \right| \leq \int_E |f|.$$

iv. If $A \leq f(x) \leq B$, then
$$Am(E) \leq \int_E f \leq Bm(E).$$

v. If A and B are disjoint measurable sets of finite measure

Proposition (4.6, Bounded Convergence Theorem). Let $\{f_n\}$ be a sequence of measurable functions defined over a measurable set E of finite measure. Suppose there is $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for all $x \in E$ and for all $n \in \mathbb{N}$. If $f_n(x) \to f(x)$ pointwise (i.e., $\lim_{n\to\infty} f_n = f(x)$), then

$$\int_E f_n \to \int_E f \Leftrightarrow \lim_{n\to\infty} \int_E f_n(x) = \int_E f.$$

Proof. Let $\varepsilon > 0$ be chosen. By Proposition 3.23 (Weak Ergoroff's Theorem), there exists $N \in \mathbb{N}$ and $A \subset E$ with
$$m(A) = \tilde{\delta} < \frac{\varepsilon}{4M}$$
such that that for all $x \in E \setminus A$ and for all $n > N$,
$$|f_n(x) - f(x)| < \frac{\varepsilon}{2 \cdot m(E)} = \tilde{\varepsilon}.$$

Thus, we can show the following:

$$
\begin{aligned}
\left|\int_E f_n - \int_E f\right| = \left|\int_E f_n - f\right| &\leq \int_E |f_n - f| \\
&= \int_A |f_n - f| + \int_{E\setminus A} |f_n - f| \\
&\leq 2M \cdot m(A) + \int_{E\setminus A} |f_n - f| \\
&\leq 2M \cdot m(A) + \frac{\varepsilon \cdot m(E \setminus A)}{2 \cdot m(E)} \\
&< 2M \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon \cdot m(E \setminus A)}{2 \cdot m(E)} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

and so we are done!! $\square$

Proposition (4.7). Let f be bounded on $[a, b]$. Then f is Riemann integrable if and only the set of discontinuities has measure zero.

Section 4.3 Integral of Nonnegative Functions

Definition. Let $f \geq 0$ be measurable, and E be a measurable set. The **Lebesgue integral** of f over is defined by

$$\int_E f := \sup_{h \leq f} \int_E h$$

where h is a bounded measurable function and $m\{x : h(x) \neq 0\} < \infty$.

Proposition (4.8). If f and g are nonnegative measurable functions, then:

i. For all $c > 0$,
$$\int_E cf = c \int_E f.$$

ii.
$$\int_E f + g = \int_E f + \int_E g.$$

iii. If $f \leq g$ a.e, then
$$\int_E f \leq \int_E g.$$

Proof. [1] Note that (i) and (iii) come from Proposition 4.5. So for (ii), we first claim that

$$\int_E f + g \leq \int_E f + \int_E g.$$

Let $h \leq f$ be a bounded, measurable function with $m\{x : h \neq 0\} < \infty$, and let $k \leq g$ be a bounded, measurable function with $m\{x : k \neq 0\} < \infty$. Then $h + k \leq f + g$ and

$$\{x : h + k \neq 0\} = \{x : h \neq 0\} \cup \{x : k \neq 0\}$$

and so $m\{x : h + k \neq 0\} < \infty$. By definition of the Lebesgue integral (which is a sup), we have the following:

$$\begin{aligned}
\int_E f + g \geq \int_E h + k &= \int_E h + \int_E k \\
&\geq \int_E h + \int_E g \\
&\geq \int_E h + \int_E k \\
&\geq \int_E f + \int_E g.
\end{aligned}$$

For the other direction, let $l \leq f + g$ be a bounded, measurable function and $m\{x : l(x) \neq 0\} < \infty$. Define $h(x) = \min\{f(x), l(x)\} \leq l(x)$ and so $h(x)$ is bounded as well. Then

$$\int_E f + \int_E g \geq \int_E + \int_E k = \int_E h + k = \int_E l$$

[1] Proof is on page 86 of Royden.

which, by definition of the Lebesgue integral,

$$\int_E f + \int_E g \leq \int_E f + g.$$

□

Theorem (4.9, Fatou's Lemma). If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ pointwise almost everywhere on a set E, then

$$\int_E f \leq \varliminf_{n\to\infty} \int_E f_n.$$

Proof. Without loss of generality, suppose that $f_n(x) \to f(x)$ on E (because the integrals over sets of measure zero are zero.) Suppose that $h \leq f$ is a bounded, measurable function and define $E' = \{x : h(x) \neq 0\}$ and so $m(E') < \infty$. Define $h_n(x) = \min\{h(x), f_n(x)\}$ and so $h_n(x) \to h(x)$ pointwise on E' and $h_n \leq h \leq f_n \leq f$ and so $\{h_n\}$ is a bounded sequence. So, by the Bounded Convergence Theorem,

$$\begin{aligned} \int_E h &= \int_{E'} h \\ &= \lim_{n\to\infty} \int_E h_n && \text{Bounded Convergence Theorem} \\ &\leq \varliminf_{n\to\infty} \int_E f_n. \end{aligned}$$

Taking the sup over h,[2]

$$\int_E f \leq \varliminf_{n\to\infty} \int_E f_n.$$

□

Theorem (4.10, Monotone Convergence Theorem). Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions, and let $f = \lim_{n\to\infty} f_n$ almost everywhere. Then

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. By Fatou's Lemma,

$$\int f \leq \varliminf_{n\to\infty} \int f_n.$$

So we just need the other direction for equality. Because $\{f_n\}$ is increasing and converges to f, $f_n \leq f$ for each $n \in \mathbb{N}$ and thus

$$\int f_n \leq \int f.$$

But then the limit inferior is less than than this quantity i.e.,

$$\varliminf_{n\to\infty} \int f_n \leq f$$

[2] Wait, clarify what this means...

and so

$$\varliminf_{n\to\infty} \int f_n = \int f.$$

□

Corollary (4.11). Let $\{u_n\}$ be a sequence of nonnegative measurable functions, and let $f(x) = \sum_{i=1}^{n} u_n(x)$. Then

$$\int f = \sum_{i=1}^{n} \int u_n.$$

Proposition (4.12). Let f be a nonnegative function and $\{E_i\}$ a disjoint sequence of measurable sets. Let $E = \bigcup_{i=1}^{\infty} E_i$. Then

$$\int_E f = \sum_{i=1}^{\infty} \int_{E_i} f.$$

Definition. Let $f \geq 0$ be a nonnegative measurable function. We say that f is **Lebesgue measurable** over E if

$$\int_E f \leq \infty.$$

Proposition (4.13). Let f and g be two nonnegative measurable functions. If f is integrable over E and $g(x) \leq f(x)$ on E, then g is also integrable on E and,

$$\int_E f - g = \int_E f - \int_E g.$$

Proof. Note that $f - g \geq 0$ on E so we can write this as the sum of two nonnegative functions i.e., $f = (f - g) + g$. Thus, since this is the sum of two nonnegative functions, by Proposition 4.8 part (ii), we have that

$$\int_E f = \int_E (f - g) + \int_E g$$

.

Because the integral of f is finite, the right-hand side must also be finite and so g is measurable.[3] □

Proposition (4.14). Let f be a nonnegative function which is integrable over E. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta$, we have that

$$\int_A f < \varepsilon.$$

[3] This proof does not show the explicit formula, though?

Proof. Let $\varepsilon > 0$ be chosen. If f is bounded, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$. So set $\delta = \dfrac{\varepsilon}{M}$ and estimate $\displaystyle\int_A f$.

If is unbounded then define

$$f_n(x) = \min\{n, f(x)\}.$$

Then $\{f_n\}$ is an increasing sequence and $f_n \to f$ pointwise (i.e, $f_n \uparrow f$ pointwise). By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Thus there exists $N \in \mathbb{N}$ such that if $n \geq N$,

$$\int_E f - \lim_{N\to\infty} \int_E f_N < \frac{\varepsilon}{2}$$

and thus implying that

$$\lim_{N\to\infty} \int_E f_N > \int_E f + \frac{\varepsilon}{2}.$$

Set $\delta = \dfrac{\varepsilon}{2N}$. Choose a set $A \subset E$ such that $m(A) < \delta$. Then

$$\begin{aligned}
\int_A f &= \int_A f - f_N + \int_A f_N \\
&< \frac{\varepsilon}{2} + \int_A f_N \\
&= \frac{\varepsilon}{2} + N \cdot m(A) \\
&< \frac{\varepsilon}{2} + N \cdot \frac{\varepsilon}{2N} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

□

Section 4.4 General Lebesgue Integral

Definition. Define

$$\begin{aligned} f^+(x) &= \text{ non-negative part of } f \\ &= \max\{f(x), 0\}. \end{aligned}$$

and

$$\begin{aligned} f^-(x) &= \text{ non-positive part of } \ f \\ &= \max\{-f(x), 0\}. \end{aligned}$$

Note that then

$$f(x) = f^+(x) - f^-(x)$$

and

$$|f(x)| = f^+(x) + f^-(x).$$

Definition. A measurable function f is said to be **Lebesgue integrable** over E if f^+ and f^- are integrable. In this case, then

$$\int_E = \int_E f^+ - \int_E f^-.$$

Proposition (4.15). Let f, g be integrable functions over E. Then

i. cf are integrable for all $c \in \mathbb{R}$ over E.

ii. $f + g$ is integrable and
$$\int_E f + g = \int_E f + \int_E g$$

iii. If $f \leq g$ a.e, then
$$\int_E f \leq \int_E g.$$

iv. If $A, B \subset E$ and $A \cap B = \emptyset$, then
$$\int_{A \cup B} f = \int_A f + \int_{B.}$$

Proposition (4.16, Lebesgue Convergence Theorem). Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions. Suppose $f_n \to f$ pointwise almost everywhere and $|f_n| \leq g$ on E. Then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. Since $g - f_n \geq 0$ for all $n \in \mathbb{N}$, Fatou's lemma, $|f_n| \leq g$ on E

$$\begin{aligned}\int g - \int f = \int g - f &\leq \varliminf_{n\to\infty} \int g - f_n \\ &= \varliminf_{n\to\infty} \int f - \varliminf_{n\to\infty} \int f_n \\ &= \int g - \varliminf_{n\to\infty} \int f_n\end{aligned}$$

and so

$$\int g - \int f \leq \int g - \varliminf_{n\to\infty} \int f_n$$

which implies that

$$\int f \geq \varliminf_{n\to\infty} \int f_n.$$

Note that $g + f_n \geq 0$ as well. Then

$$\begin{aligned}\int g + \int f_n = \int g + f_n &\leq \varliminf_{n\to\infty} \int g + f_n \\ &= \varliminf_{n\to\infty} \int g + \varliminf_{n\to\infty} \int f_n \\ &= \int g + \varliminf_{n\to\infty} \int f_n\end{aligned}$$

implying that

$$\int f \leq \varliminf_{n\to\infty} \int f_n.$$

Because $\{f_n\}$ converges, we know that $\varliminf_{n\to\infty} f_n = \lim_{n\to\infty} f_n$ and so the result follows. □

Proposition (4.17, Lebesgue Generalized Dominant Convergent Theorem). Let $\{g_n\}$ be a sequence of integrable functions and $g_n \to g$ pointwise a.e with g integrable. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g_n$ and $\{f_n\} \to f$ pointwise a.e. If

$$\int g = \lim_{n\to\infty} \int g_n,$$

then

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. [1] This proof is also similar to Proposition 4.16 but use g_n instead of g.

□

Problem 1 (4.15). Properties of function f being integrable.

(a) Let f be integrable over E. Then for all $\varepsilon > 0$, there exists a simple function ϕ such that

$$\int_E |f - \phi| < \varepsilon.$$

[1]Proof is on page 92 of Royden.

(b) Let f be integrable over E. Then for all $\varepsilon > 0$, there exists a step function ψ such that
$$\int_E |f - \psi| < \varepsilon.$$

(c) Let f be integrable over E. Then for all $\varepsilon > 0$, there exists a continuous function g such that
$$\int_E |f - g| < \varepsilon.$$

Section 4.5 Convergence in Measure

Definition. Let $\{f_n\}$ be a sequence of measurable functions. We say $\{f_n\}$ **converges to f in measure**, $f_n \overset{m}{\to} f$, if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such for all $n > N$,

$$m\{x : |f_n(x) - f(x)| \geq \varepsilon\} < \varepsilon.$$

Note two things from this definition:

(1) If $f_n \to f$ pointwisely over E with $m(E) < \infty$, then $f \overset{m}{\to} f$.

(2) So there exists examples with $f_n \overset{m}{\to} f$ but $f_n \not\to f$.

Proposition (4.18). Let $\{f_n\}$ be a sequence of measurable functions. Suppose $f_n \overset{m}{\to} f$. Then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ almost everywhere.

Proof. Suppose $f_n \overset{m}{\to} f$. Then given $\nu \in \mathbb{N}$, there exists $n_\nu \in \mathbb{N}$ such that for all $n > n_\nu$,

$$m\left\{x : |f_n(x) - f(x)| \geq \frac{1}{2^\nu}\right\} < \frac{1}{2^\nu} \text{ for all } \nu > k.$$

Define the set

$$E_\nu = \left\{x : |f_n(x) - f(x)| \geq \frac{1}{2^\nu}\right\}.$$

Then if $x \notin \bigcup_{\nu=k}^{\infty} E_\nu$ which implies that

$$|f_{\nu_v}(x) - f(x)| < \frac{1}{2^\nu} \text{ for all } \nu > k.$$

Then $f_n(x) \to f(x)$ pointwise for all $x \notin A = \bigcap_{k=1}^{\infty} \bigcup_{\nu=k}^{\infty} E_\nu$. Because we are taking the intersection over all k,

$$m(A) \leq m\left(\bigcup_{\nu=k}^{\infty} E_\nu\right) \leq \sum_{\nu=k}^{\infty} m(E_\nu) \leq 2^{-\nu-1}.$$

Because $\nu \in \mathbb{N}$ is given, $m(A) = 0$ and so $f_n(x) \to f(x)$ almost everywhere. $\square$

Corollary (4.19). Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure. Then $f_n \overset{m}{\to} f$ if and only if every subsequence of $\{f_n\}$ has a subsequence that converges almost everywhere to f.

The result aboves follows almost right away from Problem 2.12. Also note that by Problem 4.20 the following is true:

Let $\{f_n\}$ be a sequence of measurable functions. If $f_n \overset{m}{\to} f$, then every subsequence $\{x_{n_k}\} \overset{m}{\to} f$.

Proposition (4.20). Fatou's lemma, the Monotone Convergence Theorem, and the Lebesgue Dominated Convergence Theorem remain valid if $f_n \to f$ almost everywhere is replaced by $f_n \overset{m}{\to} f$.

(1) **Fatuo's Lemma**

Proof. Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \overset{m}{\to} f$. Let us pick a subsequence $\{x_{n_k}\}$ such

$$\int f_{n_k} \to \varliminf_{n\to\infty} \int f_n$$

which follows by the definition of the limit inferior. Since $f_{n_k} \overset{m}{\to} f$, by Problem 4.20, there exists a subsequence $\{f_{n_{k_p}}\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_p}} \overset{p\to\infty}{\to} f$ almost everywhere by Proposition 4.18. Then by applying Fatuo's lemma,

$$\begin{aligned}\int f = \lim_{p\to\infty} f_{n_{k_p}} &\leq \varliminf_{n\to\infty} \int f_{n_{k_p}} \\ &= \varliminf_{k\to\infty} \int f_{n_k} \\ &= \varliminf_{nto\infty} \int f_n\end{aligned}$$

and so the result holds! $\square$

(2) **Lebesgue Dominated Convergence Theorem** Suppose $|f_n| \leq g$ and $f_n \overset{m}{\to} f$. Then

$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof. We claim that to show this result, we must show that we can any subsequence $\displaystyle\int f_{n_k}$ of $\int f_n$ which then implies that

$$\lim_{k\to\infty} \int f_{n_k} = \int f.$$

Because $f_{n_k} \overset{m}{\to} f$, there exists a subsequence $\{f_{n_{k_p}}\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_p}} \to f$ almost everywhere. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{p\to\infty} \int f_{n_{k_p}} = \lim_{k\to\infty} \int f_{n_k} = \int f.$$

Then by Problem 2.12,

$$\lim_{n\to\infty} f_n = \int f$$

which is what we desired to show. $\square$

Section 6.1 L^p Spaces

Definition. A measurable function $f : [0,1] \to \mathbb{R}$ is said to be in the space $L^p = L^p([0,1])$ if

$$\int_a^b |f|^p < \infty.$$

Note the following

(1) L^1 is the space of integrable functions

(2) L^p is closed under $+$ and under scalar multiplication i.e.,

The L^p is defined as equivalance classes as follows:

$$\left\{ f : \text{ measurable and } \int |f|^p < \infty \right\} / \sim f = g \text{ a.e.}$$

(mod out by functions that are equal almost everywhere)

Definition. ()

Definition. The L^p-norm on L^p space is defined as

$$||f||_p := \left(\int_0^1 |f|^p \right).$$

If $p \in (0,1)$, then $||f+g||_p \leq ||f||_p + ||g||_p$. We want to show that $||f+g||_p \leq ||f||_p + ||g||_p$ for $p \in [1, \infty]$.

Definition. For $p = \infty$, the space L^∞ is the set of bounded measurable functions for $f \in L^\infty$. Then

$$\begin{aligned} ||f||_\infty &= \operatorname{ess\,sup} |f(x)| \\ &= \inf \{M : m\{t : f(t) > M\} = 0\}. \end{aligned}$$

Note that $|| \cdot ||_\infty$ is the limit of $|| \cdot ||_p$ i.e.,

$$f \in L^\infty, ||f||_p \to ||f||_\infty.$$

Section 5.5 Convex Functions

Definition. A function $\phi : [a,b] \to \mathbb{R}$ is **convex** if for all $x, y \in [a,b]$ and for all $\lambda \in (0,1)$, we have that

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$$

Proposition (5.17).

If ϕ is convex on $[a,b]$ then

(1) - (don't care about this)

(2) Right-hand and left-hand derivatives are equal except on a countable set.

(3) The left- and right-hand derivatives are monotone increasing functions, and at each point the left-hand derivative is less than or equal to the right-hand derivative.

Corollary (5.19). If ϕ is twice-differentiable, then ϕ is convex if and only $\phi''(x) > 0$.

Corollary (5.20, Jensen's Inequality**).** Let ϕ be a convex function on $(-\infty, \infty)$ and f be an integrable function $[0, 1]$. Then

$$\int_0^1 \phi(f(t))\,\mathrm{d}t \geq \phi\left[\int_0^1 f(t)\,\mathrm{d}t\right].$$

An example of this is $\phi(x) = x^p$. For any $p \in (1, \infty)$, this function is twice-differentiable. Applying Jensen's inequality, we get

$$\int_0^1 |f(x)|^p\,\mathrm{d}x \geq \left(\int_0^1 |f(x)|\,\mathrm{d}x\right).$$

If $f \in L^p$, then $f \in L^1$ i.e., $L^p \subset L^1$.

Theorem (6.1, Minkowski Inequality). If $f, g \in L^p$ with $p \in [1, \infty]$, then $f + g \in L^p$ and

$$||f + g||_p \leq ||f||_p + \ ||g||_p.$$

If $p \in (1, \infty)$, then the equality can hold only if and only if there exists $\alpha, \beta \geq 0$ such that $\beta f = \alpha g$.

Proof. We leave $p = \infty$ as exercise so suppose p is finite. Let $p \in [1, \infty]$. We normalize f and g i.e., there exists two functions $f_0, g_0 \in L^p$ such that $|f| = \alpha \cdot f_0$ and $|g| = \beta \cdot g_0$ with $||f_0|| = ||g_0|| = 1$. Let $\lambda = \dfrac{\alpha}{\alpha + \beta}$ and $1 - \lambda = \dfrac{\beta}{\alpha + \beta}$. By the convexity of $\phi(t) = t^p$ for $p \in [1, \infty]$, we have that

$$\begin{aligned}
|f(x) + g(x)|^p &\leq (|f(x)| + |g(x))^p \\
&= (\alpha f_0 + \beta g_0)^p \\
&= (\alpha + \beta)^p \left(\frac{\alpha}{\alpha + \beta} f_0 + \frac{\beta}{\alpha + \beta} g_0\right)^p \\
&\leq (\alpha + \beta)^p \left(\lambda f_0 + (1 - \lambda) g_0\right)^p
\end{aligned}$$

Now take the integrals of these guys (Jensen's inequality or something like that). In other words,

$$\begin{aligned}
||f + g||_p^p &\leq (\alpha + \beta)^p \cdot \left(\lambda ||f_0||_p^p + (1 - \lambda)||g_0||_p^p\right) \\
&= \left(||f||_p^p + ||g||_p^p\right) \cdot 1 && \text{because } f_0 = 1 = g_0.
\end{aligned}$$

Taking the pth root,

$$||f + g||_p \leq |f||_p + ||g||_p.$$

□

This gives us the last norm-space requirement (triangle inequality of normed spaces).

Lemma (6.3). Let $p \in [1, \infty]$. Then for $a, b, t \geq 0$, we have

$$(a + tb)^p \geq a^p + ptba^{p-1}.$$

Proof. Define the function

$$\phi(t) = (a + tb)^p - a^p - ptba^{p-1}.$$

We know $\phi(0) = 0$. Take the derivative of this thing and this is greater than zero because

$$\begin{aligned} \phi'(x) &= p(a + tb)^{p-1} + b - pba^{p-1} \\ &= pb\left((a + bt)^{p-1} - a^{p-1}\right) \end{aligned}$$

and so ϕ is increasing. □

Theorem (6.4, Holder Inequality). [1] If p and q are nonnegative extended real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and if $f \in L^p$ and $g \in L^q$, then $f \cdot g \in L^1$ and

$$\int |fg| \leq ||f||_p \cdot ||g||_q.$$

[1] If $p, q = 2$, then this just reduces to the Cauchy-Schwarz inequality.

Theorem (6.4, Holder Inequality). [1] If p and q are nonnegative extended real numbers such that

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$$\int |fg| \leq ||f||_p \cdot ||g||_q.$$

Proof. There are two cases. (i) ($p=1,\ q=\infty$) ...add notes on this. (ii) $p,q \in (1,\infty)$. Without loss of generality, suppose $f,g \geq 0$; otherwise, just take the absolute value. Set

$$h(x)=g(x)^{q-1}=g(x)^{q/p}$$

and

$$g(x)=h(x)^{p-1}=h(x)^{p/q}.$$

Then

$$\begin{aligned} p \cdot t \cdot f(x) \cdot g(x) &= p \cdot t \cdot f(x) \cdot h(x) \\ &\leq (h(x)+tf(x))^p - h(x)^p. && \text{Lemma 6.3} \end{aligned}$$

Taking the integral of both sides, (pulling out constants),

$$\begin{aligned} p \cdot t \int f(x)g(x) &\leq \int ||h(x)+tf(x)|\,|_p^p - \int ||h|\,|_p^p \\ &\leq (||h(x)|\,|_p + t||f(x)||_p)^p - ||h(x)||_p^p && \text{Triangle inequality} \end{aligned}$$

Dividing by t,

$$p\int f(x)g(x) \leq \frac{(||h(x)|\,|_p + t||f(x)||_p)^p - ||h(x)||_p^p}{t}$$

which the right-hand side is derivative of $\phi(t)=(||h||_p+t||f||_p)^p$. Taking the derivative with respect to t at $t=0$, we get that

$$p\int f(x)g(x) \leq p\left(\|h(x)\|_p^{p-1} + \|f(x)\|_p\right)^{p-1} = p\|f(x)\|\|g(x)\|$$

and so we are done! □

Section 6.3 Convergence and Completeness

Recall that if $(X,||\cdot||)$ is a norm space (naturally a metric space), then (X,d) is a metric space where

$$d(f,g) := ||f-g||$$

so the norm is the metric of the space.

Definition. We $\{f_n\} \in L^p$ converges to an element $f \in L^p$ in L^p norm if

$$||f_n - f||_p \to 0.$$

That is, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $||f-f_n||_p < \varepsilon$.

[1]If $p,q=2$, then this just reduces to the Cauchy-Schwarz inequality.

Definition. A normed space $(X, ||\cdot||)$ is called a **complete** space if every Cauchy sequence of X is convergent.

- Note that a completed normed space is a called a **Banach space**.

Our goal will be to show that L^p for $p \geq 1$ is a Banach space.

Definition. A sequence $f_n \subset X$ for any normed space X is **summable** to a sum s in X if the partial sum converges, i.e.,

$$\left\| s - \sum_{k=1}^{n} f_k \right\| \to 0.$$

- A sequence is **absolute summable** if

$$\sum_{i=1}^{\infty} \|f_n\| < \infty.$$

Proposition (6.5). A normed linear space X is complete if and only if every absolutely summable series is summable.

Proof. We will need to complete two directions.

($\Rightarrow$) Let X be a Banach space and let $\{f_n\}$ be an absolute summable sequence. This means we have that

$$\sum_{n=1}^{\infty} \|f_n\| < M.$$

Our goal will be show that the partial sums are Cauchy sequence (then convergent by the completeness of a Banach space) i.e.,

$$S_n = \sum_{i=1}^{n} f_i$$

is Cauchy. Then suppose $n > m$ and so

$$\|S_n - S_m\| = \left\| \sum_{k=m}^{n} f_k \right\| \leq \sum_{k=m}^{n} \|f_k\| < \underbrace{\sum_{k=m}^{\infty} \|f_k\|}_{< \varepsilon}$$

for any $\varepsilon > 0$ because $\{f_n\}$ is absolutely summable and therefore convergent. Thus, the partial sums are Cauchy and so convergent.

($\Leftarrow$) Now suppose every absolutely summable series is summable. We will construct a series from the Cauchy sequence. Let $\{f_n\}$ be a Cauchy sequence. Pick $\frac{\varepsilon}{2^k}$, and then pick the subsequence $\{f_{n_k}\}$ such that

$$\|f_{n_{k+1}} - f_{n_k}\| < \frac{1}{2^k}$$

which we can do because $\{f_n\}$ is Cauchy. Consider the series $g_k = f_{n_k} - f_{n_{k-1}}$, which is summable because the sequence is decreasing by construction. By assumption, then $\{g_k\}$ must be absolutely summable; i.e., the sum

$$S_m = \sum_{k=1}^{m} g_k$$

has a limit. Note that S_m is a telescoping series by construction again thus $S_m = -f_{n_1} + f_{n_m}$. This implies that $\{f_{n_k}\}$ converges to f for some $f \in X$ as $k \to \infty$. Since $\{f_n\}$ is Cauchy,

$$\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\|.$$

Then use the fact that $\{f_n\}$ is Cauchy and $\{f_{n_k}\}$ is convergent, pick $\frac{\varepsilon}{2}$ for each thing and so the result follows.

$\square$

Theorem (6.6, Riesz-Fisher). L^p is complete for $p \in [1, \infty]$.