Problem 1 (11.34)**.** Let μ, ν, and λ be σ-finite. Show that the Radon-Nikodym derivative $[\mathrm{d}\nu/\mathrm{d}\mu]$ has the following properties:

a. If $\nu \ll \mu$ and f is nonnegative measurable function, then

$$\int f \,\mathrm{d}\nu = \int f \left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right] \mathrm{d}\mu.$$

Proof. Let $v \ll u$ and suppose f is a nonnegative measurable function over a set E. We can break this down into two cases: (i) f is a simple function or (ii) f is a non-simple, measurable function. If f is a simple function, then

$$f = \sum_{i=1}^{n} a_i \chi_{E_i}$$

for $a_i \in \mathbb{R}$. Then, using properties of simple functions and integration, we have the following:

$$\begin{aligned}
\int_E f \,\mathrm{d}\nu &= \sum_{i=1}^{n} a_i \nu(E_i) \\
&= \sum_{i=1}^{n} a_i \left(\int_{E_i} f \,\mathrm{d}\mu\right) \\
&= \sum_{i=1}^{n} a_i \left(\int_{E_i} \left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right] \mathrm{d}\mu\right) \\
&= \int_E \sum_{i=1}^{n} a_i \chi_{E_i} \left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right] \mathrm{d}\mu \\
&= \int f \left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right] \mathrm{d}\mu.
\end{aligned}$$

Turning to case (ii), suppose that f is a non-simple measurable function. Then there exists a sequence of increasing simple functions $\{\phi_n\}$ such that $\phi_n \to f$ pointwise. Using the Monotone Convergence Theorem, we have

$$\begin{aligned}
\int_E f \,\mathrm{d}\nu &= \lim_{n\to\infty} \int_E \phi_n \,\mathrm{d}\nu \\
&= \lim_{n\to\infty} \int_E \phi_n \left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right] \mathrm{d}\mu \\
&= \int_E f \left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right] \mathrm{d}\mu.
\end{aligned}$$

Having exhausted all cases, this completes the proof. □

b. Not assigned.

c. If $\nu \ll \mu \ll \lambda$, then

$$\left[\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right] = \left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right]\left[\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right].$$

Proof. Let $\nu \ll \mu \ll \lambda$, and let E any measurable set. Then, by definition,

$$\nu(E) = \int_E \left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right] \mathrm{d}\mu.$$

Because $\mu \ll \lambda$, we also have that (from part (a))

$$\int_E f \,\mathrm{d}\mu = \int_E f \left[\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right] \mathrm{d}\lambda.$$

Combining these two, we have that

$$\begin{aligned}\nu(E) &= \int_E \left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right] \mathrm{d}\mu. \\ &= \int_E \left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right]\left[\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right] \mathrm{d}\lambda\end{aligned}$$

and so it follows that

$$\left[\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right] = \left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right]\left[\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right]$$

which completes the proof. □

Problem 2 (11.45)**.** For $g \in L^q$, let F be the linear functional on L^p defined by

$$F(f) = \int fg \,\mathrm{d}\mu.$$

Show that $\|F\| = \|g\|_q$.

Proof. Let $g \in L^q$, and let F be the linear functional on L^p be defined by

$$F(f) = \int fg \,\mathrm{d}\mu.$$

From the Hölder inequality, we have

$$\begin{aligned}|F(f)| &= \left|\int fg \,\mathrm{d}\mu\right| \\ &\le \int |fg| \,\mathrm{d}\mu \\ &\le \|f\|_p \cdot \|g\|_q.\end{aligned}$$

So dividing over by $\|f\|_p$, we have

$$\frac{|F(f)|}{\|f\|_p} \le \|g\|_q$$

which, by taking the supremum of the left-hand side over all $f \in L^p$, implies that

$$\|F\| \le \|g\|_q.$$

To show the other inequality, for $p \in (1,\infty)$, we have three cases: (i) $p \in (1,\infty)$ and $q \in (1,\infty)$; (ii) $p = \infty$ and $q = 1$; or (iii) $p = 1$ and $q = \infty$.

First, suppose $p, q \in (1, \infty)$. Define the function f by

$$f = |g|^{q/p} \cdot \operatorname{sgn} g.$$

Then $|f|^p = |f|^q = fg$ meaning $f \in L^p$. This implies

$$\|f\|_p = \|g\|_p^{p/q}$$

and so

$$\begin{aligned}
|F(f)| &= \left| \int fg \, \mathrm{d}\mu \right| \\
&= \int |g|^q \, \mathrm{d}\mu \\
&= \|g\|_q^q \\
&= \|g\|_q \|f\|_p
\end{aligned}$$

and by definition of the norm of the linear functional,

$$\|F\| \geq \|g\|_q .$$

Suppose $p = \infty$ and $q = 1$ meaning $\|g\|_1$. Without loss of generality, assume $\|g\|_1$. Let $f = \operatorname{sgn} g$. Then $f \in L^\infty$ and so $\|f\|_\infty = 1$ from how we defined f. So

$$\begin{aligned}
|F(f)| &= \left| \int fg \, \mathrm{d}\mu \right| \\
&= \int |g| \, \mathrm{d}\mu \\
&= \|g\|_1 \\
&= \|g\|_1 \|f\|_\infty
\end{aligned}$$

and by a similar argument as in case (i),

$$\|F\| \geq \|g\|_q .$$

Finally, suppose $p = 1$ but $q = \infty$, and let $\varepsilon > 0$ be chosen. Define the set $E = \{x : g(x) > \|g\|_\infty - \varepsilon\}$ and define $f = \chi_E$. This means $f \in L^1$ and

$$\|f\|_1 = \int |f| \, \mathrm{d}\mu = \mu(E).$$

Also,

$$\begin{aligned}
\|F(f)\| &= \left| \int fg \, \mathrm{d}\mu \right| \\
&= \left| \int_E g \, \mathrm{d}\mu \right| \\
&\leq (\|g\|_\infty - \varepsilon) \|f\|_1
\end{aligned}$$

and by moving $\|f\|_1$ to the other side, we have that

$$\|F(f)\| \geq \|g\|_1 .$$

Therefore, having exhausted all possible cases, this completes the proof. □