

# Construction of the Real Numbers, $\mathbb{R}$

- We first start from  $\mathbb{N} \cup \{0\}$  and add numbers together subsequently (i.e.  $1, \underbrace{1+1}_2, \underbrace{1+1+1}_3, \dots$ )

- To construct the integers  $\mathbb{Z}$ , we take the set difference with the natural numbers so that we have

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \setminus \mathbb{N}.$$

- Then the rational numbers,  $\mathbb{Q}$ , can be constructed from the integers and are defined by the set

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \right\}.$$

- To construct the irrational numbers,  $\mathbb{R} \setminus \mathbb{Q}$ , we can use the dedekind cut to do this. However, this is convoluted and we can go about this in a different way.

## Axioms of the Real Numbers

**A. The Field Axioms:** For all real numbers  $x, y \in \mathbb{R}$  we have:

A1.  $x + y = y + x$

A2.  $(x + y) + z = x + (y + z)$

A3. There exists  $0 \in \mathbb{R}$  such that  $x + 0 = x$  for all  $x \in \mathbb{R}$ .  
[Identity element under addition]

A4. For each  $x \in \mathbb{R}$  there is a  $w \in \mathbb{R}$  such that  $x + w = 0$ .  
[Inverse element under addition]

A5.  $xy = yx$

A6.  $(xy)z = x(yz)$

A7. There exists  $1 \in \mathbb{R}$  such that  $1 \neq 0$  and  $x \cdot 1 = x$  for all  $x \in \mathbb{R}$ .

A8. For each  $x \in \mathbb{R}$  different from 0 there is  $w \in \mathbb{R}$  such that  $xw = 1$ .

A9.  $x(y + z) = xy + xz$ .

We can prove some properties now:

**Proposition 1.** The additive inverse is unique.

*Proof.* Let  $x \in \mathbb{R}$ . Suppose we have two numbers  $w_1, w_2 \in \mathbb{R}$  such that  $x + w_1 = 0 = x + w_2$ . Using the axioms and our assumption, we can show the following:

$$\begin{aligned} w_1 &= w_1 + 0 && \text{Axiom A3} \\ &= w_1 + x + w_2 && \text{Assumption of } 0 = x + w_2 \\ &= w_2 + xw_1 && \text{Axiom A1} \\ &= w_2 \end{aligned}$$

which completes the proof. □

**B. The Axioms of Order:** The subset  $P$  of positive real numbers satisfies the following:

- B1. If  $x, y \in P$ , then  $x + y \in P$ .
- B2. If  $x, y \in P$ , then  $xy \in P$ .
- B3. If  $x \in P$ , then  $-x \notin P$ .
- B4. If  $x \in \mathbb{R}$ , then  $x = 0$  or  $x \in P$  or  $-x \in P$ .

Note that any system which satisfies the axioms of groups A and B is called an **ordered field**.

**Definition.** We can give definitions of the ordered operations  $<$ ,  $\leq$ ,  $>$  and  $\geq$ .

- $x < y$  means that  $y - x \in P$ .
- $x \leq y$  means that  $y - x \in P \cup \{0\}$ . Or, this means that  $x < y$  or  $x = y$ .
- $x > y$  means that  $x - y \in P$ .
- $x \geq y$  means that  $x - y \in P \cup \{0\}$ . Or, this means that  $x > y$  or  $x = y$ .

From this, we can deduce and prove some which is any set which satisfies the axioms of group A and B.

**Definition.** Let  $x, y \in \mathbb{R}$  and define the absolute value as

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

**Proposition 2.** Let  $a, b, c \in \mathbb{R}$ .

1.  $a < b$  if and only if  $-b < -a$ .
2. If  $a < b$  and  $b < c$ , then  $a < c$ .
3. If  $a < b$  and  $c > 0$ , then  $ac < bc$ .
4. For  $a, b \in \mathbb{R}$ , then only one is true  $a = b$ ,  $a > b$  and  $a < b$ .
5. If  $x \neq 0$ , then  $x^2 = x \cdot x > 0$ ; in particular,  $1 > 0$ .
6. If  $x, y \in \mathbb{R}$ , then  $|x + y| \leq |x| + |y|$ .

**Definition.** Let  $S \subset \mathbb{R}$ . The number  $b \in \mathbb{R}$  is an **upper bound** for  $S$  if for each  $x \in S$ , we have  $x \leq b$ .

Similarly, a number  $x \in \mathbb{R}$  is the **least upper bound** for  $S$  if it is an upper bound for  $S$  and if  $x \leq b$  for each upper bound  $b$  of  $S$ . We then call  $x$  the **supremum** of  $S$  and denote this  $x = \sup S$ .

**Definition.** Let  $S \subset \mathbb{R}$ . The number  $l \in \mathbb{R}$  is an **lower bound** for  $S$  if for each  $x \in S$ , we have  $l \leq x$ .

Similarly, a number  $x \in \mathbb{R}$  is the **greatest lower bound** for  $S$  if it is a lower bound for  $S$  and if  $x \leq l$  for each lower bound  $l$  of  $S$ . We then call  $x$  the **infimum** of  $S$  and denote this  $x = \inf S$ .

**C. Completeness Axiom:** Every nonempty set  $S \subset \mathbb{R}$  which has an upper bound has a least upper bound.

**Proposition 3.** Let  $L, U \subset \mathbb{R}$  be nonempty subsets with  $R = L \cup U$  and such that for each  $l \in L$  and each  $u \in U$  we have  $l < u$ . Then either  $L$  has a greatest element or  $L$  has a least element.

**Proposition 4** (Approximation Property.). Let  $S \subset \mathbb{R}$  be a nonempty. If  $u = \sup S$ , then for all  $\gamma > 0$ , there exists  $Sr \in S$  such that  $u - r < Sr < u$ .

**Theorem (2.3, Axiom of Archimedes).** If  $x \in \mathbb{R}$  is any real number, then there exists  $n \in \mathbb{N}$  such that  $x < n$ .

*Proof.* We can break this into two cases

1. Let  $x < 1$ . If so, then simply choose  $x = 1$ .
2. Let  $x \geq 1$ . Define the set  $S = \{n \in \mathbb{N} : n \leq x\}$ . Then since this set is bounded above, by the Completeness Axiom,  $\sup S = y$  exists. Because  $x$  is an upper bound  $S$ , by definition of the supremum, we have that  $y \leq x$ . Let  $r = \frac{1}{2}$ . Then we can find  $k \in S$  such that  $y - \frac{1}{2} < k \leq y$ . But then we have that  $y < y + \frac{1}{2} < k + 1 \leq y + 1$ . Then this means  $k + 1 \notin S$  and so  $x < k + 1$ , completing this case.

Having exhausted all cases, this completes the proof.  $\square$

**Proposition 1 (Well-Ordering Principle).** Every nonempty subset  $S \subset \mathbb{N}$  has a minimum.

**Proposition 2 (Density of the Rational Numbers).** Let  $x, y \in \mathbb{R}$ . Then if  $x < y$ , there exists  $q \in \mathbb{Q}$  such that  $x < q < y$

## Section 2.4, Sequences in $\mathbb{R}$

**Definition.** We define a **sequence** of real numbers to be a function that maps each natural number  $n$  into the real number  $x$ . That is, a sequence is a function  $s : \mathbb{N} \rightarrow \mathbb{R}$  for  $A \subset \mathbb{R}$ . This is written as  $\{x_n\}$  or  $\{x_n\}_{n=1}^{\infty}$ .

**Definition (Convergence of a Sequence).** A sequence converges to the real number  $l \in \mathbb{R}$  if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$|a_n - l| < \varepsilon.$$

**Definition (Cauchy Sequence).** A sequence  $\{x_n\}$  in  $\mathbb{R}$  is **Cauchy** sequence if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n, m \geq N$ ,

$$|a_n - a_m| < \varepsilon.$$

**Theorem.** Let  $\{x_n\}$  be a sequence in  $\mathbb{R}$ . Then  $\{x_n\}$  is Cauchy if and only if  $\{x_n\}$  is convergent.

**Definition.** The number  $l \in \mathbb{R}$  is called a **cluster point** of  $\{x_n\}$  if there exists a subsequence  $\{x_{n_m}\}$  of  $\{x_n\}$  such that  $x_{n_m} \rightarrow l$ .

We can define this in another way. The number  $l \in \mathbb{R}$  is called **cluster point** of  $\{x_n\}$  if for all  $\varepsilon > 0$  and for all  $N \in \mathbb{N}$ , there exists  $n \geq N$  such that  $|x_n - l| < \varepsilon$ .

**Definition.** We define the **limit superior** of a sequence  $\{x_n\}$  in  $\mathbb{R}$  to be

$$\overline{\lim}_{n \rightarrow \infty} x_n = \inf_n \sup_{k \geq n} x_k.$$

This is also denoted as  $\limsup$ .

**Theorem.** A number  $l \in \mathbb{R}$  is the **limit superior** of the sequence  $\{x_n\}$  if and only if

- (i) For all  $\varepsilon > 0$ , there exists  $n \in \mathbb{N}$  such that for all  $k \geq n$ ,  $x_k < l + \varepsilon$
- (ii) For all  $\varepsilon > 0$  and for all  $n \in \mathbb{N}$ , there exists  $k \geq n$  such that  $x_k > l - \varepsilon$ .

**Definition.** We define the **limit inferior** of a sequence  $\{x_n\}$  in  $\mathbb{R}$  to be

$$\underline{\lim}_{n \rightarrow \infty} x_n = \sup_n \inf_{k \geq n} x_k.$$

This is also denoted as  $\liminf$ .

**Theorem.** A number  $l \in \mathbb{R}$  is the **limit inferior** of the sequence  $\{x_n\}$  if and only if

- (i) For all  $\varepsilon > 0$ , there exists  $n \in \mathbb{N}$  such that for all  $k \geq n$ ,  $x_k > l - \varepsilon$
- (ii) For all  $\varepsilon > 0$  and for all  $n \in \mathbb{N}$ , there exists  $k \geq n$  such that  $x_k < l + \varepsilon$ .

**Proposition 3.** From the last two definitions, we have the following property.

- $\overline{\lim}_{n \rightarrow \infty}$  is the largest cluster point.
- $\underline{\lim}_{n \rightarrow \infty}$  is the smallest cluster point.

## Section 2.5, Open and Closed Sets in $\mathbb{R}$

**Definition.** The set  $O \subset \mathbb{R}$  is called an **open** set if for all  $x \in O$ , there exists  $\delta > 0$  such that  $x - \delta, x + \delta$ .

Equivalently,  $O$  is an **open** set if for all  $x \in O$ , there is a  $\delta > 0$  such that each  $y$  with  $|x - y| < \delta$  belongs to  $O$ .

**Proposition 4.** From this above, we have the following properties:

1. The set  $\bigcup_{\alpha} O_{\alpha}$  is open.
2. The set  $\bigcup_{n=1}^n O_n$  is open.

**Theorem (Lindelof Theorem).** Every open set in  $\mathbb{R}$  is a disjoint union of countable union of open intervals.

*Proof.* This proof is contained on page 42 of Royden. □

**Definition.** A real number  $x \in \mathbb{R}$  is called **point of closure** of a set  $E \subset \mathbb{R}$  if for every  $\delta > 0$  there exists a  $y \in E$  such that  $|x - y| < \delta$ .

The set of points of closure of  $E$  is denoted  $\overline{E}$ .

**Proposition 5.** If  $A \subset B \subset \mathbb{R}$ , then  $\overline{A} \subset \overline{B}$ . Additionally,  $\overline{A \cup B} = \overline{A} \cup \overline{B}$ .

*Proof.* The proof of this is on page 43 of Royden. □

**Definition.** A set  $F \subset \mathbb{R}$  is called a **closed** set if  $\overline{F} = F$ .

Note that because  $F \subset \overline{F}$  always, a set  $F$  is closed if  $\overline{F} \subset F$ —that is,  $F$  contains all of its points of closure.

**Proposition 6.** For any set  $E$ , the set  $\overline{E}$  is closed; that is  $\overline{\overline{E}} = \overline{E}$ .

**Proposition 7.** Let  $E \subset \mathbb{R}$ . Then  $E$  is open if and only if  $E^c$  is closed.

**Definition.** We say that a collection of sets  $\mathcal{C}$  is a **cover** of a set  $F$  if

$$F \subset \bigcup_{O \in \mathcal{C}} O.$$

The collection  $\mathcal{C}$  is a covering of the set  $F$ .

**Theorem (Heine-Borel).** Let  $E \subset \mathbb{R}$  be set. Then  $E$  is compact if and only if  $E$  is closed and bounded.

## Compactness

**Theorem.** Let  $E \subset \mathbb{R}$ . Then  $E$  is compact if and only if  $E$  is sequentially compact. That is, for every  $\{x_n\}$  in  $E$ , there exists a convergent subsequence  $x_{n_m} \rightarrow x_0$  in  $E$ .

**Theorem.** Let  $\{I_n\}$  be a sequence of closed intervals such that  $I_{n+1} \subset I_n$ . Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

If  $[a_n, b_n]$  is an interval and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = a$ , then  $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$ .

## Section 2.6, Continuous Functions

**Definition.** Let  $E \subset \mathbb{R}$ , and let  $f : E \rightarrow \mathbb{R}$  be a real-valued function. Then  $f$  is **continuous** at the point  $x = a \in E$  if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $y \in E$  with  $|x - y| < \delta$  implies that  $|f(x) - f(y)| < \varepsilon$ .

Note that we can have continuity in terms of sequences. I will state it as a theorem here even though it was not in lecture because it is important to be able to use on its own.

**Theorem.** Let  $f : E \rightarrow \mathbb{R}$  be a function with  $E \subset \mathbb{R}$ . Let  $x \in E$  be any point. Then  $f$  is continuous at  $a$  if and only if for every sequence  $\{x_n\}$  in  $E$  converging to  $a$ , the sequence  $\{f(x_n)\}$  in  $f(E)$  (the image of  $E$ ) converges to  $f(a)$ .

**Proposition.** Let  $E \subset \mathbb{R}$  be compact. Let  $f : E \rightarrow \mathbb{R}$  be continuous real-valued function. Then  $f(E)$  is a compact set.

*Proof.* Let  $E \subset \mathbb{R}$  be a compact and suppose the function  $f : E \rightarrow \mathbb{R}$  is continuous. To show that  $f(E)$  is compact, we will use the Heine-Borel theorem and show that it is closed and bounded. To show that  $f(E)$  is closed, suppose we have any sequence  $\{f(x_n)\}$  converging to the point  $f(a) \in \mathbb{R}$ . Additionally, let  $\{x_n\}$  be any sequence in  $E$ . Because  $E$  is compact, there exists a subsequence  $\{x_{n_m}\}$  which converges to a point  $x_0 \in E$ . Since  $f$  is continuous, by the preceding theorem this means that the sequence  $\{f(x_{n_m})\}$  converges to  $f(x_0) \in f(E)$ .  $\square$

**Proposition (2.17, Extreme Value Theorem).** Let  $E \subset \mathbb{R}$  be a compact set, and let  $f : E \rightarrow \mathbb{R}$  be a continuous function. Then there exists  $x_1, x_2 \in E$  such that

$$f(x_1) \leq f(x) \leq f(x_2), \text{ for all } x \in E.$$

**Proposition (2.18).** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. Then  $f$  is **continuous** if and only if  $f^{-1}(O)$  is open for all open sets  $O \subset \mathbb{R}$ .

**Proposition (2.19).** Let  $E \subset \mathbb{R}$ , and let  $f : E \rightarrow \mathbb{R}$  be continuous. Without loss of generality, suppose that  $f(a) \leq f(b)$ . Then for all  $\gamma \in [f(a), f(b)]$ , there exists  $c \in [a, b]$  such that  $f(c) = \gamma$ .

**Definition (Uniform Continuity).** Let  $E \subset \mathbb{R}$ . A function  $f : E \rightarrow \mathbb{R}$  is **uniformly continuous** if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $x, y \in E$  with  $|x - y| < \delta$  implies that  $|f(x) - f(y)| < \varepsilon$ .

**Proposition (2.20).** Let  $E \subset \mathbb{R}$  be a compact set. If  $f : E \rightarrow \mathbb{R}$  is a continuous function on  $E$ , then  $f$  is uniformly continuous on  $E$ .

**Definition.** Let  $f_n : E \rightarrow \mathbb{R}$  be a sequence of functions, and let  $f : E \rightarrow \mathbb{R}$ .

1. The sequence  $\{f_n\}$  **converges pointwise** on  $E$  to  $f$  if for all  $\varepsilon > 0$  and for all  $x \in E$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|f(x) - f_n(x)| < \varepsilon$ .
2. The sequence  $\{f_n\}$  **converges uniformly** if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $x \in E$  and for all  $n \geq N$ ,  $|f(x) - f_n(x)| < \varepsilon$ .

## Section 3.1, Lebesgue Measure

[Perhaps finish these notes another time...]



## Section 3.2, Outer Measure

**Definition.** The **outer measure**  $m^*(A)$  of a set  $A \subset \mathbb{R}$  is given

$$m^*(A) = \inf_{A \subset \bigcup I_n} \sum_n l(I_n)$$

where  $\{I_n\}$  is a countable collection of open intervals that cover  $A$ .

Note that from this definition, we get that

1.  $m^*(\emptyset) = 0$
2. If  $A \subset B$ ,  $m^*(A) \leq m^*(B)$ .
3.  $m^*$  does not satisfy disjoint additivity.

**Proposition (3.1).** The outer measure of an interval is its length; that is,  $m^*(I) = l(I)$  where  $I = [a, b]$ ,  $(a, b)$ ,  $[a, b)$ , or  $(a, b]$ .

*Proof.* It is sufficient to show that  $m^*([a, b]) = l([a, b])$  since every other interval is a subset of  $[a, b]$ . Let  $\varepsilon > 0$ . Then  $[a, b] \subset [a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}]$  which implies, by the definition of the outer measure,

$$m^*([a, b]) \leq l\left([a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}]\right) = b - a + \varepsilon.$$

Because  $\varepsilon$  was fixed, this means that  $m^* \leq b - a$ .

Now we must show that  $m^* \geq b - a$ . Because  $[a, b]$  is compact, for any collection  $\{I_n\}$  of open intervals covering  $[a, b]$ , there exists a finite collection of intervals  $\{I_1, \dots, I_k\}$  so that

$$[a, b] \subset \bigcup_{n=1}^k I_n.$$

This gives us that

$$\sum_n l(I_n) \geq \sum_{n=1}^k l(I_n) \geq b - a$$

and so  $b - a$  is a lower bound. But since  $m^*$  is the greatest lower bound of all such sums, we have that  $m^* \geq b - a$ .

Therefore,  $m^*([a, b]) = l([a, b]) = b - a$ .

□

**Proposition (3.2, Subadditivity).** Let  $\{A_n\}$  be a countable collection of sets on  $\mathbb{R}$ . Then

$$m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n).$$

*Proof.* Proof on page 57.

□

**Corollary (3.3).** If  $A$  is a countable set, then  $m^*(A) = 0$ .

*Proof.* Proof is on the end of page 57. □

## Section 3.3, Measurable Sets and Lebesgue Measure

**Definition.** A set  $E \subset \mathbb{R}$  is (Lebesgue) **measurable** if for all sets  $A$ , we have that

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

**Lemma (3.6).** If  $m^*(E) = 0$ , then  $E$  is measurable.

*Proof.* Let  $A$  be any chosen set. Because  $A \cap E \subset E$  and  $m^*(E) = 0$ ,

$$m^*(A \cap E) \leq m^*(E) = 0.$$

Note that  $A \cap E^c \subset A$  and so  $m^*(A) \leq m^*(A \cap E^c)$  and so it suffices to show that  $m^*(A) \geq m^*(A \cap E^c)$ . Using this, we can show that

$$m^*(A) \geq m^*(A \cap E^c) + 0 = m^*(A \cap E^c) = m^*(A \cap E)$$

giving us the desired result. □

**Definition.** Let  $\mathcal{M}$  be the set of measurable sets in  $\mathbb{R}$

**Lemma (3.7).** If  $E_1$  and  $E_2$  are measurable sets, then so is  $E_1 \cup E_2$ .

*Proof.* Proof on top of page 57. □

**Corollary (3.8).** The family  $\mathcal{M}$  of measurable sets is an algebra of sets. In other words, if  $E \in \mathcal{M}$ , then  $E^c \in \mathcal{M}$ . Further, if  $E_1, E_2 \in \mathcal{M}$ , then  $E_1 \cup E_2 \in \mathcal{M}$ .

**Lemma (3.9).** Let  $A$  be any set, and  $E_1, \dots, E_n$  be a finite sequence of sets such that  $E_i \cap E_j = \emptyset$  for all  $i \neq j$ . Then

$$m^* \left( A \cap \left[ \bigcup_{i=1}^n E_i \right] \right) = \sum_{i=1}^n m^*(A \cap E_i).$$

*Proof.* We proceed by induction. For  $n = 1$ , we have the set  $E_1$  and the equality holds. Suppose that we have  $n = k$  sets  $E_1, \dots, E_k$  with  $E_i \cap E_j = \emptyset$  for all  $i \neq j$  so that

$$m^* \left( A \cap \left[ \bigcup_{i=1}^k E_i \right] \right) = \sum_{i=1}^k m^*(A \cap E_i).$$

Consider  $n = k + 1$ . Because each  $E_i$  is disjoint,

$$\begin{aligned} A \cap \left( \bigcup_{i=1}^{k+1} E_i \right) \cap E_{k+1} &= A \cap E_{k+1}; \\ A \cap \left( \bigcup_{i=1}^{k+1} E_i \right) \cap E_{k+1}^c &= A \cap \bigcup_{i=1}^k E_i. \end{aligned}$$

Because the  $E_i$ 's are measurable,

$$\begin{aligned} m^* \left( A \cap \bigcup_{i=1}^{k+1} E_i \right) &= m^*(A \cap E_{k+1}) + m^* \left( A \cap \bigcup_{i=1}^k E_i \right) \\ &= m^*(A \cap E_{k+1}) + \sum_{i=1}^k m^*(A \cap E_i) \quad \text{Induction Hypothesis} \\ &= \sum_{i=1}^{k+1} m^*(A \cap E_i) \end{aligned}$$

which, by induction, completes the proof.  $\square$

**Theorem (3.10).**  $\mathcal{M}$  is a  $\sigma$ -algebra. In other words, in addition to being an algebra of sets, if  $\{E_i\}_{i=1}^\infty \subset \mathcal{M}$ , then  $\bigcup_{i=1}^\infty E_i \in \mathcal{M}$ .

*Proof.* <sup>1</sup>

$\square$

**Lemma (3.11).** The interval  $(a, \infty)$  is measurable for all  $a \in \mathbb{R}$ .

*Proof.* <sup>2</sup>

$\square$

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<sup>1</sup>Proof on bottom of page 59 and top of page 60.

<sup>2</sup>Proof on the bottom of page 60 through the middle of page 61.

**Theorem (3.12).** Every Borel set is measurable. In particular, each open set and each closed set is measurable.

*Proof.* <sup>3</sup>

□

**Definition.** Let  $E \in \mathcal{M}$ . We define  $m(E) := m^*(E)$  to be the **Lebesgue measure** of  $E$ .

**Proposition (3.13, Countable Additivity).** Let  $\{E_i\}_{i=1}^n$  be a sequence of measurable sets. Then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^n m(E_i).$$

If, in addition,  $E_i \cap E_j = \emptyset$  for all  $i \neq j$ , then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^n m(E_i).$$

**Proposition (3.14).** Let  $\{E_i\} \subset \mathcal{M}$  be a decreasing sequence (i.e.,  $E_{i+1} \subset E_i$ ). Let  $m(E_1) < \infty$ . Then

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} m(E_n).$$

**Proposition (3.15).** Let  $E$  be any given set. Then the following are equivalent:

- (i)  $E$  is measurable.
- (ii) For all  $\varepsilon > 0$ , there is an open set  $O \supset E$  with  $m^*(O \setminus E) < \varepsilon$ .
- (iii) For all  $\varepsilon > 0$ , there is a closed set  $F \subset E$  with  $m^*(E \setminus F) < \varepsilon$ .
- (iv) There is a  $G \in G_\delta$  with  $E \subset G$  such that  $m^*(G \setminus E) = 0$ .
- (v) There is a  $F \in F_\sigma$  with  $F \subset E$  such that  $m^*(E \setminus F) = 0$ .

If  $m^*(E) < \infty$ , the above statements are equivalent:

- (vi) For all  $\varepsilon > 0$ , there is a finite union  $U$  of open intervals such that  $m^*(U \Delta E) < \varepsilon$ .

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<sup>3</sup>Proof on the bottom of page 61.

## Section 3.5, Measurable Functions

**Proposition (3.18).** Let  $E \subset \mathbb{R}$ , and Let  $f : E \rightarrow [-\infty, \infty]$  be an extended real-valued function whose domain is measurable. Let  $\alpha \in \mathbb{R}$  be any real number. Then the following statements are equivalent:

- (i) The set  $\{x : f(x) > \alpha\}$  is measurable.
- (ii) The set  $\{x : f(x) \geq \alpha\}$  is measurable.
- (iii) The set  $\{x : f(x) < \alpha\}$  is measurable.
- (iv) The set  $\{x : f(x) \leq \alpha\}$  is measurable.

All together, these imply

- (v) The set  $\{x : f(x) = \alpha\}$  is measurable.

*Proof.* <sup>1</sup>

□

**Definition.** An extended real-valued function  $f : E \rightarrow [-\infty, \infty]$  is **(Lebesgue) measurable** if its domain is measurable and satisfies one of the first four statements of Proposition 18.

Note that this means that any continuous function is measurable since then the pre-image of any open set is still an open set.

**Proposition (3.19).** Let  $f$  and  $g$  be two measurable functions defined on the same domain, and let  $c \in \mathbb{R}$ . Then the functions  $f + c$ ,  $cf$ ,  $f + g$ ,  $g - f$ , and  $fg$  are measurable.

*Proof.* Let  $\alpha \in \mathbb{R}$  be any real number. Fix  $c \in \mathbb{R}$ . For  $f(x) + c$ , note that

$$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}$$

and  $\alpha - c$  is a real number, this set is still measurable i.e.,  $f + c$  is measurable. A similar argument shows that  $cf$  is measurable as well.

Take the set

$$\{x : f(x) + g(x) < \alpha\}. \tag{1}$$

Observe that  $f(x) + g(x) < \alpha \Leftrightarrow f(x) < \alpha - g(x)$ . By the density of  $\mathbb{Q}$ , there exists  $r \in \mathbb{Q}$  such that  $f(x) < r < \alpha - g(x)$ . So we can write Equation (1) as

$$\{x : f(x) + g(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} (\{x : f(x) < r\} \cap \{x : g(x) < \alpha - r\}).$$

Because this set is countable, this set is measurable and thus  $f + g$  is measurable.

To show that  $fg$  is measurable, we can show that  $f^2$  is measurable since

$$fg = \frac{1}{2} ((f + g)^2 - f^2 - g^2).$$

---

<sup>1</sup>Proof is on page 67.

Take the set

$$\{x : f^2(x) < \alpha\}. \quad (2)$$

For  $\alpha \geq 0$ , note that  $f^2 < \alpha$  is the same as saying  $f(x) > \sqrt{\alpha}$  and  $f(x) < -\sqrt{\alpha}$ . Thus, Equation (2) can be rewritten as

$$\{x : f^2(x) < \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}.$$

This is a measurable set which completes the proof.  $\square$

**Theorem (3.20, Limit of Measurable Functions is Measurable).** <sup>2</sup>

*Proof.* For  $f(x) + c$ , note that

$$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}.$$

$\square$

**Theorem (3.20, Limit of Measurable Functions is Measurable).** Let  $\{f_n\}$  be a sequence of measurable functions with the same domain. Then the functions  $\sup\{f_1(x), \dots, f_n(x)\}$ ,  $\inf\{f_1(x), \dots, f_n(x)\}$ ,  $\sup_n f_n$ ,  $\inf_n f_n$ ,  $\overline{\lim} f_n$ , and  $\underline{\lim} f_n$  are measurable.

*Proof.* Let  $\{f_n\}$  be a sequence of measurable functions. Let  $h(x) = \sup\{f_1(x), \dots, f_n(x)\}$  and we so must show that  $\{x : h(x) < \alpha\}$  for all  $\alpha \in \mathbb{R}$ . To that end, let  $\alpha \in \mathbb{R}$  be chosen. Then

$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^n \{x : f_i(x) > \alpha\}$$

which, because the right-hand side is a union of measurable sets from the  $f_i$ 's being measurable, means that the set  $\{x : h(x) < \alpha\}$  is also measurable.

Let  $g(x) = \sup_n f_n$ . By a similar argument as above,

$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i(x) > \alpha\}$$

is a countable set so  $\{x : g(x) < \alpha\}$  is measurable.  $\square$

**Definition.** A property is said to hold **almost everywhere** (a.e) if the set of points where it fails to hold is a set of measure zero. Thus  $f = g$  a.e if  $f$  and  $g$  have the same domain and  $m\{x : f(x) \neq g(x)\} = 0$ .

**Proposition (3.21).** If  $f$  is measurable and  $f = g$  a.e, then  $g$  is measurable.

*Proof.* <sup>3</sup> Let  $E = \{x : f(x) \neq g(x)\}$ .

This is equivalent to saying that

Let  $\{x : g(x) > \alpha\}$ . This is equivalent to saying that

$$\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha\}$$

$\square$

---

<sup>2</sup>Proof is on bottom of page 68 and top of page 69

<sup>3</sup>Proof is on middle of page 69.

The following proposition essentially says that measurable functions are nearly continuous; or, in other words, we can “nicely” approximate measurable functions.

**Proposition (3.22, Littlewood’s 2nd Principle).** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a measurable function with  $E \subset \mathbb{R}$  and is equal to  $\pm\infty$  only on sets with measure zero. Then for all  $\varepsilon > 0$ , there exist a step function  $g$  and a continuous function  $h$  such

$$|f - g| < \varepsilon \quad \text{and} \quad |f - h| < \varepsilon$$

except on set of measure less than  $\varepsilon$ ; i.e.,  $m\{x : |f(x) - g(x)| \geq \varepsilon\} < \varepsilon$  and  $m\{x : |f(x) - h(x)| \geq \varepsilon\} < \varepsilon$ . If in addition  $m \leq f \leq M$ , then we may choose the functions  $g$  and  $h$  so that  $m \leq g \leq M$  and  $m \leq h \leq M$ .

**Proposition (3.23, (Weak) Egonoff’s Theorem).** Let  $E$  be a measurable set of finite measure, and  $\{f_n\}$  be a sequence of measurable functions defined on  $E$ . Let  $f$  be real-valued function such for each  $x \in E$  we have  $f_n(x) \rightarrow f(x)$ . Then for all  $\varepsilon > 0$  and all  $\delta > 0$ , there is measurable set  $A \subset E$  with  $m(A) < \delta$  and  $N \in \mathbb{N}$  such that for all  $x \notin A$  and all  $n \geq N$ ,

$$|f_n(x) - f(x)| < \varepsilon.$$

**Proposition (3.23, (Weak) Egonoff's Theorem).** Let  $E$  be a measurable set of finite measure, and  $\{f_n\}$  be a sequence of measurable functions defined on  $E$ . Let  $f$  be real-valued function such for each  $x \in E$  we have  $f_n(x) \rightarrow f(x)$ . Then for all  $\varepsilon > 0$  and all  $\delta > 0$ , there is measurable set  $A \subset E$  with  $m(A) < \delta$  and  $N \in \mathbb{N}$  such that for all  $x \notin A$  and all  $n \geq N$ ,

$$|f_n(x) - f(x)| < \varepsilon.$$

*Proof.*<sup>1</sup>

Let  $\varepsilon > 0$  be chosen. Define the set

$$G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}$$

and also

$$E_N = \bigcup_{n=N}^{\infty} G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N\}.$$

Because  $\{E_N\}$  is a decreasing sequence and  $f_n(x) \rightarrow f(x)$  pointwise, for all  $x \in E$ ,  $|f_n(x) - f(x)| < \varepsilon$  and so  $\bigcup_{i=1}^{\infty} E_N = \emptyset$ . Thus, by Proposition 3.14,

$$\begin{aligned} E_N = \emptyset &\implies m(E_N) = 0 \\ &= m\left(\bigcup_{N=1}^{\infty} E_N\right) \\ &= \lim_{N \rightarrow \infty} E_N. \end{aligned}$$

So for any  $\delta > 0$ , there exists  $N_0 \in \mathbb{N}$  such that for all  $n > N_0$ ,  $m(E_N) < \delta$ . Now take  $A = E_N$  for any  $N > N_0$  and so  $m(A) < \delta$  and also

$$A^c = \{x \in E : x \notin E\} = \{x \in E : |f_n(x) - f(x)| < \varepsilon \text{ for all } n > N_0\}.$$

□

## Section 4.1 Riemann Integration

**Definition.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded real-valued function and let

$$P = \{a = \xi_0 < \xi_1 < \cdots < \xi_n = b\}$$

be a subdivision (partition) of  $[a, b]$ . We can define the **upper sum**,  $S$  and **lower sum**,  $s$ , respectively, as

$$S = \sum_{i=1}^n (\xi_i - \xi_{i-1}) M_i \quad \text{and} \quad s = \sum_{i=1}^n (\xi_i - \xi_{i-1}) m_i$$

where

$$M_i = \sup \{f(x) : x \in [\xi_{i-1}, \xi_i]\} \quad \text{and} \quad m_i = \inf \{f(x) : x \in [\xi_{i-1}, \xi_i]\}.$$

---

<sup>1</sup>Proof on page 72-73.



**Definition.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded real-valued function. Define the **upper Riemann integral** of  $f$  as

$$\overline{R} \int_a^b f(x) \, dx = \inf \{S : P \text{ is a partition of } [a, b]\}$$

and the **lower Riemann integral** of  $f$  as

$$\underline{R} \int_a^b f(x) \, dx = \sup \{s : P \text{ is a partition of } [a, b]\}.$$

**Definition.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Then  $f$  is Riemann integrable if

$$\underline{R} \int_a^b f(x) \, dx = R \int_a^b f(x) \, dx = \overline{R} \int_a^b f(x) \, dx.$$

In other words, if the upper and lower Riemann integrals are equal to each other.

Note this theorem is *NOT* from Royden but from my real analysis course in undergraduate. It comes from the definition of the upper and lower Riemann integral definitions by ends up being a useful way to show Riemann integration.

**Theorem.** Let  $f$  be a bounded function on  $[a, b]$ . Then  $f$  is Riemann integrable if and only if for all  $\varepsilon > 0$ , there exists a subdivision (partition)  $P$  of  $[a, b]$  such that

$$S - s < \varepsilon.$$

## Section 4.2 The Lebesgue Integral

**Definition.** The **characteristic function** of  $E$  is defined as

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

A linear combination

$$\phi(x) = \sum_{i=1}^n a_i \chi_{E_i}(x)$$

is called a **simple function** if the sets  $\{E_1, \dots, E_n\}$  are measurable. Note that  $\phi$  is simple if and only if it is measurable and only assumes a finite number of values.

The **canonical representation** of  $\phi$  is such that

$$\phi = \sum_{i=1}^n a_i \chi_{A_i}$$

where  $A_i = \{x : \phi(x) = a_i\}$  and where the  $A_i$ 's are disjoint and the  $a_i$ 's are distinct and nonzero.

**Definition.** Let

$$\phi = \sum_{i=1}^n a_i \chi_{A_i}$$

be a simple function which vanishes outside of a set of finite measure. Then the integral of  $\phi$  is defined as

$$\int \phi = \sum_{i=1}^n a_i \cdot m(A_i).$$

**Lemma (4.1).** Let  $\phi = \sum_{i=1}^n a_i \chi_{A_i}$  with  $E_i \cap E_j = \emptyset$  for all  $i \neq j$  where  $E_i \in \mathfrak{M}$  and  $m(E_i) < \infty$  for each  $i = 1, \dots, n$ . Then

$$\int \phi = \sum_{i=1}^n a_i \cdot m(E_i).$$

**Proposition (4.2).** Let  $\phi, \psi$  be simple functions vanishing outside of a set with finite measure. Then

$$\int (a\phi + b\psi) = a \int \phi + b \int \psi.$$

If  $\phi \geq \psi$  almost everywhere,

$$\int \phi \geq \int \psi.$$

**Proposition (4.3).** Let  $E \in \mathfrak{M}$ , and let  $f : E \rightarrow \mathbb{R}$  with  $m(E) < \infty$  be a bounded, measurable function. Let  $\phi, \psi$  be simple functions. Then

$$\inf \left\{ \int \psi : \psi \geq f \right\} = \sup \left\{ \int \phi : \phi \leq f \right\}.$$

if and only if  $f$  is measurable.

*Proof.* <sup>2</sup>

□

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<sup>2</sup>Proof is on pages 79-80.

**Proposition (4.3).** Let  $E \in \mathfrak{M}$ , and let  $f : E \rightarrow \mathbb{R}$  with  $m(E) < \infty$  be a bounded, measurable function. Let  $\phi, \psi$  be simple functions. Then

$$\inf \left\{ \int \psi : \psi \geq f \right\} = \sup \left\{ \int \phi : \phi \leq f \right\}.$$

if and only if  $f$  is measurable.

*Proof.* <sup>1</sup> We will need to show two implications.

( $\Leftarrow$ ) First, suppose that  $f$  is measurable. Fix any  $n \in \mathbb{N}$  and define the set

$$E_k = \left\{ x : \frac{k-1}{n}M < f(x) \leq \frac{kM}{n} \right\}$$

with  $k \in [-n, n]$  and  $|f(x)| < M$ . Note that because  $f$  is measurable, each  $E_k$  is a measurable set and also we have that  $\bigcup_{k=-n}^n E_k = E$ . Define the upper and lower sequence of simple functions,  $\{\psi_n\}$  and  $\{\phi_n\}$ , respectively, as

$$\psi_n(x) = \sum_{k=-n}^n \frac{M}{n} \cdot k \cdot \chi_{E_k}(x) \quad \text{and} \quad \phi_n(x) = \sum_{k=-n}^n \frac{M}{n} \cdot (k-1) \cdot \chi_{E_k}(x).$$

So for any  $x \in E$ ,  $\phi(x) \leq f(x) \leq \psi(x)$ . Thus,

$$\inf_{\psi \geq f} \int_E \psi \leq \int_E \psi = \frac{M}{n} \sum_{k=-n}^n k \cdot m(E_k)$$

and

$$\sup_{\phi \leq f} \int_E \phi \geq \int_E \phi = \frac{M}{n} \sum_{k=-n}^n (k-1) \cdot m(E_k).$$

Putting these two inequalities together, we can show that

$$\begin{aligned} \inf_{\psi \geq f} \int_E \psi - \sup_{\phi \leq f} \int_E \phi &\leq \int_E \psi - \phi \\ &= \sum_{k=-n}^n (\psi_n - \phi_n) m(E_k) \\ &= \frac{M}{n} m(E). \end{aligned}$$

Since  $n \in \mathbb{N}$  is fixed, this quantity is zero. Thus

$$\inf_{\psi \geq f} \int_E \psi - \sup_{\phi \leq f} \int_E \phi = 0 \Rightarrow \inf_{\psi \geq f} \int_E \psi = \sup_{\phi \leq f} \int_E \phi$$

completing this direction.

---

<sup>1</sup>Proof is on pages 79-80.

( $\Rightarrow$ ) Conversely, suppose that

$$\inf_{\psi \geq f} \int_E \psi = \sup_{\phi \leq f} \int_E \phi.$$

By definition of the infimum and supremum, there exists sequences of simple functions  $\{\phi_n\}$  and  $\{\psi_n\}$  such that  $\phi_n \leq f \leq \psi_n$  for all  $n \in \mathbb{N}$  with

$$\int_E \phi_n - \psi_n < \frac{1}{n}.$$

Define  $\phi^* = \sup_n \phi_n$  and  $\psi^* = \inf_n \psi_n$ . Since simple functions are measurable functions, by Proposition 3.20,  $\phi^*$  and  $\psi^*$  are measurable as well and  $\phi_n \leq f \leq \psi_n$ .

We claim that  $f = \phi^*$  a.e. Consider the set

$$\Delta = \{x : \phi^*(x) < \psi^*(x)\}.$$

Let  $\nu \in \mathbb{N}$  and let

$$\Delta_\nu = \left\{x : \phi^*(x) < \psi^*(x) - \frac{1}{\nu}\right\}$$

which means that

$$\Delta = \bigcup_{\nu \in \mathbb{N}} \Delta_\nu.$$

For any  $n \in \mathbb{N}$ ,

$$\Delta_\nu \subset \left\{x : \phi(x) < \psi(x) - \frac{1}{\nu}\right\}.$$

Thus, we have that, for any  $n \in \mathbb{N}$ ,

$$\begin{aligned} m(\Delta_\nu) &= \int \chi_{\Delta_\nu} = \nu \int \frac{1}{\nu} \cdot \chi_{\Delta_\nu} \\ &\leq \nu \int_{\Delta_\nu} (\psi_n - \phi_n) \\ &< \mu \int_E \frac{1}{n} \\ &= \frac{\nu}{n} m(E). \end{aligned}$$

Because  $\nu$  is fixed and  $n$  is arbitrary,  $m(\Delta_\nu) = 0$  which implies that  $m(\Delta) = 0$ . So then  $\phi^* = \psi^*$  except on a set of measure zero, and  $\phi^* = f$  except on a set of measure zero i.e.,  $f = \phi^*$  a.e. implying that  $f$  is measurable by Proposition 3.21.

Having completed both directions, this completes the proof and shows the well-definedness of Lebesgue integration.  $\square$

**Proposition (4.4).** Let  $f$  be a bounded function defined on  $[a, b]$ . If  $f$  is Riemann integrable, then it is measurable and

$$R \int_a^b f(x) \, dx = \int_a^b f(x) \, dx.$$

*Proof.* This proof is on page 82 of Royden (very simple proof, in fact).  $\square$

**Proposition (4.5).** If  $f$  and  $g$  are bounded measurable functions defined on a set  $E$  of finite measure, then:

i. For any  $a, b \in \mathbb{R}$ ,

$$\int_E (af + bg) = a \int_E f + b \int_E g.$$

ii. If  $f = g$  a.e., then

$$\int_E f = \int_E g.$$

iii. If  $f \leq g$  almost everywhere. then

$$\int_E f \leq \int_E g.$$

Hence

$$\left| \int_E f \right| \leq \int_E |f|.$$

iv. If  $A \leq f(x) \leq B$ , then

$$Am(E) \leq \int_E f \leq Bm(E).$$

v. If  $A$  and  $B$  are disjoint measurable sets of finite measure

**Proposition (4.6, Bounded Convergence Theorem).** Let  $\{f_n\}$  be a sequence of measurable functions defined over a measurable set  $E$  of finite measure. Suppose there is  $M \in \mathbb{R}$  such that  $|f_n(x)| \leq M$  for all  $x \in E$  and for all  $n \in \mathbb{N}$ . If  $f_n(x) \rightarrow f(x)$  pointwise (i.e.,  $\lim_{n \rightarrow \infty} f_n = f(x)$ ), then

$$\int_E f_n \rightarrow \int_E f \Leftrightarrow \lim_{n \rightarrow \infty} \int_E f_n(x) = \int_E f.$$

*Proof.* Let  $\varepsilon > 0$  be chosen. By Proposition 3.23 (Weak Ergoroff's Theorem), there exists  $N \in \mathbb{N}$  and  $A \subset E$  with

$$m(A) = \tilde{\delta} < \frac{\varepsilon}{4M}$$

such that that for all  $x \in E \setminus A$  and for all  $n > N$ ,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2 \cdot m(E)} = \tilde{\varepsilon}.$$

Thus, we can show the following:

$$\begin{aligned} \left| \int_E f_n - \int_E f \right| &= \left| \int_E f_n - f \right| \leq \int_E |f_n - f| \\ &= \int_A |f_n - f| + \int_{E \setminus A} |f_n - f| \\ &\leq 2M \cdot m(A) + \int_{E \setminus A} |f_n - f| \\ &\leq 2M \cdot m(A) + \frac{\varepsilon \cdot m(E \setminus A)}{2 \cdot m(E)} \\ &< 2M \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon \cdot m(E \setminus A)}{2 \cdot m(E)} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

and so we are done!!

□

**Proposition (4.7).** Let  $f$  be bounded on  $[a, b]$ . Then  $f$  is Riemann integrable if and only if the set of discontinuities has measure zero.

## Section 4.3 Integral of Nonnegative Functions

**Definition.** Let  $f \geq 0$  be measurable, and  $E$  be a measurable set. The **Lebesgue integral** of  $f$  over  $E$  is defined by

$$\int_E f := \sup_{h \leq f} \int_E h$$

where  $h$  is a bounded measurable function and  $m\{x : h(x) \neq 0\} < \infty$ .

**Proposition (4.8).** If  $f$  and  $g$  are nonnegative measurable functions, then:

i. For all  $c > 0$ ,

$$\int_E cf = c \int_E f.$$

ii.

$$\int_E f + g = \int_E f + \int_E g.$$

iii. If  $f \leq g$  a.e, then

$$\int_E f \leq \int_E g.$$

*Proof.*<sup>1</sup> Note that (i) and (iii) come from Proposition 4.5. So for (ii), we first claim that

$$\int_E f + g \leq \int_E f + \int_E g.$$

Let  $h \leq f$  be a bounded, measurable function with  $m\{x : h \neq 0\} < \infty$ , and let  $k \leq g$  be a bounded, measurable function with  $m\{x : k \neq 0\} < \infty$ . Then  $h + k \leq f + g$  and

$$\{x : h + k \neq 0\} = \{x : h \neq 0\} \cup \{x : k \neq 0\}$$

and so  $m\{x : h + k \neq 0\} < \infty$ . By definition of the Lebesgue integral (which is a sup), we have the following:

$$\begin{aligned} \int_E f + g &\geq \int_E h + k = \int_E h + \int_E k \\ &\geq \int_E h + \int_E g \\ &\geq \int_E h + \int_E k \\ &\geq \int_E f + \int_E g. \end{aligned}$$

For the other direction, let  $l \leq f + g$  be a bounded, measurable function and  $m\{x : l(x) \neq 0\} < \infty$ . Define  $h(x) = \min\{f(x), l(x)\} \leq l(x)$  and so  $h(x)$  is bounded as well. Then

$$\int_E f + \int_E g \geq \int_E h + \int_E k = \int_E h + k = \int_E l$$

---

<sup>1</sup>Proof is on page 86 of Royden.

which, by definition of the Lebesgue integral,

$$\int_E f + \int_E g \leq \int_E f + g.$$

□

**Theorem (4.9, Fatou's Lemma).** If  $\{f_n\}$  is a sequence of nonnegative measurable functions and  $f_n(x) \rightarrow f(x)$  pointwise almost everywhere on a set  $E$ , then

$$\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n.$$

*Proof.* Without loss of generality, suppose that  $f_n(x) \rightarrow f(x)$  on  $E$  (because the integrals over sets of measure zero are zero.) Suppose that  $h \leq f$  is a bounded, measurable function and define  $E' = \{x : h(x) \neq 0\}$  and so  $m(E') < \infty$ . Define  $h_n(x) = \min\{h(x), f_n(x)\}$  and so  $h_n(x) \rightarrow h(x)$  pointwise on  $E'$  and  $h_n \leq h \leq f_n \leq f$  and so  $\{h_n\}$  is a bounded sequence. So, by the Bounded Convergence Theorem,

$$\begin{aligned} \int_E h &= \int_{E'} h \\ &= \lim_{n \rightarrow \infty} \int_{E'} h_n && \text{Bounded Convergence Theorem} \\ &\leq \liminf_{n \rightarrow \infty} \int_E f_n. \end{aligned}$$

Taking the sup over  $h$ ,<sup>2</sup>

$$\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n.$$

□

**Theorem (4.10, Monotone Convergence Theorem).** Let  $\{f_n\}$  be an increasing sequence of nonnegative measurable functions, and let  $f = \lim_{n \rightarrow \infty} f_n$  almost everywhere. Then

$$\int f = \lim_{n \rightarrow \infty} \int f_n.$$

*Proof.* By Fatou's Lemma,

$$\int f \leq \liminf_{n \rightarrow \infty} \int f_n.$$

So we just need the other direction for equality. Because  $\{f_n\}$  is increasing and converges to  $f$ ,  $f_n \leq f$  for each  $n \in \mathbb{N}$  and thus

$$\int f_n \leq \int f.$$

But then the limit inferior is less than than this quantity i.e.,

$$\liminf_{n \rightarrow \infty} \int f_n \leq \int f$$

---

<sup>2</sup>Wait, clarify what this means...



and so

$$\lim_{n \rightarrow \infty} \int f_n = \int f.$$

□

**Corollary (4.11).** Let  $\{u_n\}$  be a sequence of nonnegative measurable functions, and let  $f(x) = \sum_{i=1}^n u_n(x)$ . Then

$$\int f = \sum_{i=1}^n \int u_n.$$

**Proposition (4.12).** Let  $f$  be a nonnegative function and  $\{E_i\}$  a disjoint sequence of measurable sets. Let  $E = \bigcup_{i=1}^{\infty} E_i$ . Then

$$\int_E f = \sum_{i=1}^{\infty} \int_{E_i} f.$$

**Definition.** Let  $f \geq 0$  be a nonnegative measurable function. We say that  $f$  is **Lebesgue measurable** over  $E$  if

$$\int_E f \leq \infty.$$

**Proposition (4.13).** Let  $f$  and  $g$  be two nonnegative measurable functions. If  $f$  is integrable over  $E$  and  $g(x) \leq f(x)$  on  $E$ , then  $g$  is also integrable on  $E$  and,

$$\int_E f - g = \int_E f - \int_E g.$$

*Proof.* Note that  $f - g \geq 0$  on  $E$  so we can write this as the sum of two nonnegative functions i.e.,  $f = (f - g) + g$ . Thus, since this is the sum of two nonnegative functions, by Proposition 4.8 part (ii), we have that

$$\int_E f = \int_E (f - g) + \int_E g$$

Because the integral of  $f$  is finite, the right-hand side must also be finite and so  $g$  is measurable.<sup>3</sup> □

**Proposition (4.14).** Let  $f$  be a nonnegative function which is integrable over  $E$ . Then for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for every set  $A \subset E$  with  $m(A) < \delta$ , we have that

$$\int_A f < \varepsilon.$$

---

<sup>3</sup>This proof does not show the explicit formula, though?

*Proof.* Let  $\varepsilon > 0$  be chosen. If  $f$  is bounded, there exists  $M > 0$  such that  $|f(x)| \leq M$  for all  $x \in E$ . So set  $\delta = \frac{\varepsilon}{M}$  and estimate  $\int_A f$ .

If  $f$  is unbounded then define

$$f_n(x) = \min\{n, f(x)\}.$$

Then  $\{f_n\}$  is an increasing sequence and  $f_n \rightarrow f$  pointwise (i.e,  $f_n \uparrow f$  pointwise). By the Monotone Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f.$$

Thus there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ ,

$$\int_E f - \lim_{N \rightarrow \infty} \int_E f_N < \frac{\varepsilon}{2}$$

and thus implying that

$$\lim_{N \rightarrow \infty} \int_E f_N > \int_E f + \frac{\varepsilon}{2}.$$

Set  $\delta = \frac{\varepsilon}{2N}$ . Choose a set  $A \subset E$  such that  $m(A) < \delta$ . Then

$$\begin{aligned} \int_A f &= \int_A f - f_N + \int_A f_N \\ &< \frac{\varepsilon}{2} + \int_A f_N \\ &= \frac{\varepsilon}{2} + N \cdot m(A) \\ &< \frac{\varepsilon}{2} + N \cdot \frac{\varepsilon}{2N} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

□

## Section 4.4 General Lebesgue Integral

**Definition.** Define

$$\begin{aligned} f^+(x) &= \text{non-negative part of } f \\ &= \max\{f(x), 0\}. \end{aligned}$$

and

$$\begin{aligned} f^-(x) &= \text{non-positive part of } f \\ &= \max\{-f(x), 0\}. \end{aligned}$$

Note that then

$$f(x) = f^+(x) - f^-(x)$$

and

$$|f(x)| = f^+(x) + f^-(x).$$

**Definition.** A measurable function  $f$  is said to be **Lebesgue integrable** over  $E$  if  $f^+$  and  $f^-$  are integrable. In this case, then

$$\int_E f = \int_E f^+ - \int_E f^-.$$

**Proposition (4.15).** Let  $f, g$  be integrable functions over  $E$ . Then

i.  $cf$  are integrable for all  $c \in \mathbb{R}$  over  $E$ .

ii.  $f + g$  is integrable and

$$\int_E f + g = \int_E f + \int_E g$$

iii. If  $f \leq g$  a.e, then

$$\int_E f \leq \int_E g.$$

iv. If  $A, B \subset E$  and  $A \cap B = \emptyset$ , then

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

**Proposition (4.16, Lebesgue Convergence Theorem).** Let  $g$  be integrable over  $E$  and let  $\{f_n\}$  be a sequence of measurable functions. Suppose  $f_n \rightarrow f$  pointwise almost everywhere and  $|f_n| \leq g$  on  $E$ . Then

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f.$$

*Proof.* Since  $g - f_n \geq 0$  for all  $n \in \mathbb{N}$ , Fatou's lemma,  $|f_n| \leq g$  on  $E$

$$\begin{aligned} \int g - \int f &= \int g - f \leq \liminf_{n \rightarrow \infty} \int g - f_n \\ &= \liminf_{n \rightarrow \infty} \int f - \liminf_{n \rightarrow \infty} \int f_n \\ &= \int g - \liminf_{n \rightarrow \infty} \int f_n \end{aligned}$$

and so

$$\int g - \int f \leq \int g - \liminf_{n \rightarrow \infty} \int f_n$$

which implies that

$$\int f \geq \liminf_{n \rightarrow \infty} \int f_n.$$

Note that  $g + f_n \geq 0$  as well. Then

$$\begin{aligned} \int g + \int f_n &= \int g + f_n \leq \liminf_{n \rightarrow \infty} \int g + f_n \\ &= \liminf_{n \rightarrow \infty} \int g + \liminf_{n \rightarrow \infty} \int f_n \\ &= \int g + \liminf_{n \rightarrow \infty} \int f_n \end{aligned}$$

implying that

$$\int f \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Because  $\{f_n\}$  converges, we know that  $\liminf_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} f_n$  and so the result follows.  $\square$

**Proposition (4.17, Lebesgue Generalized Dominant Convergent Theorem).** Let  $\{g_n\}$  be a sequence of integrable functions and  $g_n \rightarrow g$  pointwise a.e with  $g$  integrable. Let  $\{f_n\}$  be a sequence of measurable functions such that  $|f_n| \leq g_n$  and  $\{f_n\} \rightarrow f$  pointwise a.e. If

$$\int g = \lim_{n \rightarrow \infty} \int g_n,$$

then

$$\int f = \lim_{n \rightarrow \infty} \int f_n.$$

*Proof.* <sup>1</sup> This proof is also similar to Proposition 4.16 but use  $g_n$  instead of  $g$ .  $\square$

**Problem 1 (4.15).** Properties of function  $f$  being integrable.

- (a) Let  $f$  be integrable over  $E$ . Then for all  $\varepsilon > 0$ , there exists a simple function  $\phi$  such that

$$\int_E |f - \phi| < \varepsilon.$$

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<sup>1</sup>Proof is on page 92 of Royden.

- (b) Let  $f$  be integrable over  $E$ . Then for all  $\varepsilon > 0$ , there exists a step function  $\psi$  such that

$$\int_E |f - \psi| < \varepsilon.$$

- (c) Let  $f$  be integrable over  $E$ . Then for all  $\varepsilon > 0$ , there exists a continuous function  $g$  such that

$$\int_E |f - g| < \varepsilon.$$

## Section 4.5 Convergence in Measure

**Definition.** Let  $\{f_n\}$  be a sequence of measurable functions. We say  $\{f_n\}$  **converges to  $f$  in measure**,  $f_n \xrightarrow{m} f$ , if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such for all  $n > N$ ,

$$m \{x : |f_n(x) - f(x)| \geq \varepsilon\} < \varepsilon.$$

Note two things from this definition:

- (1) If  $f_n \rightarrow f$  pointwisely over  $E$  with  $m(E) < \infty$ , then  $f \xrightarrow{m} f$ .
- (2) So there exists examples with  $f_n \xrightarrow{m} f$  but  $f_n \not\rightarrow f$ .

**Proposition (4.18).** Let  $\{f_n\}$  be a sequence of measurable functions. Suppose  $f_n \xrightarrow{m} f$ . Then there exists a subsequence  $\{f_{n_k}\}$  such that  $f_{n_k} \rightarrow f$  almost everywhere.

*Proof.* Suppose  $f_n \xrightarrow{m} f$ . Then given  $\nu \in \mathbb{N}$ , there exists  $n_\nu \in \mathbb{N}$  such that for all  $n > n_\nu$ ,

$$m \left\{ x : |f_n(x) - f(x)| \geq \frac{1}{2^\nu} \right\} < \frac{1}{2^\nu} \text{ for all } \nu > k.$$

Define the set

$$E_\nu = \left\{ x : |f_n(x) - f(x)| \geq \frac{1}{2^\nu} \right\}.$$

Then if  $x \notin \bigcup_{\nu=k}^{\infty} E_\nu$  which implies that

$$|f_{n_\nu}(x) - f(x)| < \frac{1}{2^\nu} \text{ for all } \nu > k.$$

Then  $f_{n_\nu}(x) \rightarrow f(x)$  pointwise for all  $x \notin A = \bigcap_{k=1}^{\infty} \bigcup_{\nu=k}^{\infty} E_\nu$ . Because we are taking the intersection over all  $k$ ,

$$m(A) \leq m \left( \bigcup_{\nu=k}^{\infty} E_\nu \right) \leq \sum_{\nu=k}^{\infty} m(E_\nu) \leq 2^{-\nu-1}.$$

Because  $\nu \in \mathbb{N}$  is given,  $m(A) = 0$  and so  $f_{n_\nu}(x) \rightarrow f(x)$  almost everywhere. □

**Corollary (4.19).** Let  $\{f_n\}$  be a sequence of measurable functions defined on a measurable set  $E$  of finite measure. Then  $f_n \xrightarrow{m} f$  if and only if every subsequence of  $\{f_n\}$  has a subsequence that converges almost everywhere to  $f$ .

The result above follows almost right away from Problem 2.12. Also note that by Problem 4.20 the following is true:

Let  $\{f_n\}$  be a sequence of measurable functions. If  $f_n \xrightarrow{m} f$ , then every subsequence  $\{f_{n_k}\} \xrightarrow{m} f$ .

**Proposition (4.20).** Fatou's lemma, the Monotone Convergence Theorem, and the Lebesgue Dominated Convergence Theorem remain valid if  $f_n \rightarrow f$  almost everywhere is replaced by  $f_n \xrightarrow{m} f$ .

**(1) Fatuo's Lemma**

*Proof.* Let  $\{f_n\}$  be a sequence of measurable functions such that  $f_n \xrightarrow{m} f$ . Let us pick a subsequence  $\{x_{n_k}\}$  such

$$\int f_{n_k} \rightarrow \underline{\lim}_{n \rightarrow \infty} \int f_n$$

which follows by the definition of the limit inferior. Since  $f_{n_k} \xrightarrow{m} f$ , by Problem 4.20, there exists a subsequence  $\{f_{n_{k_p}}\}$  of  $\{f_{n_k}\}$  such that  $f_{n_{k_p}} \xrightarrow{p \rightarrow \infty} f$  almost everywhere by Proposition 4.18. Then by applying Fatuo's lemma,

$$\begin{aligned} \int f &= \lim_{p \rightarrow \infty} \int f_{n_{k_p}} \leq \underline{\lim}_{n \rightarrow \infty} \int f_{n_{k_p}} \\ &= \underline{\lim}_{k \rightarrow \infty} \int f_{n_k} \\ &= \underline{\lim}_{n \rightarrow \infty} \int f_n \end{aligned}$$

and so the result holds! □

**(2) Lebesgue Dominated Convergence Theorem** Suppose  $|f_n| \leq g$  and  $f_n \xrightarrow{m} f$ . Then

$$\lim_{n \rightarrow \infty} \int f_n = \int f.$$

*Proof.* We claim that to show this result, we must show that we can any subsequence  $\int f_{n_k}$  of  $\int f_n$  which then implies that

$$\lim_{k \rightarrow \infty} \int f_{n_k} = \int f.$$

Because  $f_{n_k} \xrightarrow{m} f$ , there exists a subsequence  $\{f_{n_{k_p}}\}$  of  $\{f_{n_k}\}$  such that  $f_{n_{k_p}} \rightarrow f$  almost everywhere. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{p \rightarrow \infty} \int f_{n_{k_p}} = \lim_{k \rightarrow \infty} \int f_{n_k} = \int f.$$

Then by Problem 2.12,

$$\lim_{n \rightarrow \infty} \int f_n = \int f$$

which is what we desired to show. □