Construction of the Real Numbers, \mathbb{R}

- We first start from $\mathbb{N} \cup \{0\}$ and add numbers together subsequently (i.e. $1, \underbrace{1+1}_{2}, \underbrace{1+1+1}_{3}, \dots)$
- ullet To construct the integers \mathbb{Z} , we take the set difference with the natural numbers so that we have

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \setminus \mathbb{N}.$$

• Then the rationals numbers, \mathbb{Q} , can be constructed from the integers and are defined by the set

 $\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \right\}.$

• To construct the irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$, we can use the dedekind cut to do this. However, this is convoluted and we can go about this in a different way.

Axioms of the Real Numbers

A. The Field Axioms: For all real numbers $x, y \in \mathbb{R}$ we have:

A1. x + y = y + x

A2. (x + y) + z = x + (y + z)

- A3. There exists $0 \in \mathbb{R}$ such that $x + 0 = \text{for all } x \in \mathbb{R}$. [Identity element under addition]
- A4. For each $x \in \mathbb{R}$ there is a $w \in \mathbb{R}$ such that x + w = 0. [Inverse element under addition]

A5. xy = yx

A6. (xy)z = x(yz)

A7. There exists $1 \in \mathbb{R}$ such that $1 \neq 0$ and $x \cdot 1 = x$ for all $x \in \mathbb{R}$.

A8. For each $x \in \mathbb{R}$ different from 0 there is $w \in \mathbb{R}$ such that xw = 1.

A9. x(y + z) = xy + xz.

We can prove some properties now:

Proposition 1. The additive inverse is unique.

Proof. Let $x \in \mathbb{R}$. Suppose we have two numbers $w_1, w_2 \in \mathbb{R}$ such that $x + w_1 = 0 = x + w_2$. Using the axioms and our assumption, we can show the following:

$$w_1 = w_1 + 0$$
 Axiom A3
 $= w_1 + x + w_2$ Assumption of $0 = x + w_2$
 $= w_2 + xw_1$ Axiom A1
 $= w_2$

which completes the proof.

- **B.** The Axioms of Order: The subset *P* of positive real numbers satisfies the following:
 - B1. If $x, y \in P$, then $x + y \in P$.
 - B2. If $x, y \in P$, then $xy \in P$.
 - B3. If $x \in P$, then $-x \notin P$.
 - B4. If $x \in \mathbb{R}$, then x = 0 or $x \in P$ or $-x \in P$.

Note that any system which satisfies the axioms of groups A and B is called an **ordered** field.

Definition. We can give definitions of the ordered operations <, \le , > and \ge .

- x < y means that $y x \in P$.
- $x \leq y$ means that $y x \in P \cup \{0\}$. Or, this means that x < y or x = y.
- x > y means that $x y \in P$.
- $x \ge y$ means that $x y \in P \cup \{0\}$. Or, this means that x > y or x = y.

From this, we can deduce and prove some which is any set which satisfies the axioms of group A and B.

Definition. Let $x, y \in \mathbb{R}$ and define the absolute value as

$$|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0. \end{cases}$$

Proposition 2. Let $a, b, c \in \mathbb{R}$.

- 1. a < b if and only if -b < -a.
- 2. If a < b and b < c, then a < c.
- 3. If a < b and c > 0, then ac < bc.
- 4. For $a, b \in \mathbb{R}$, then only one is true a = b, a > b and a < b.
- 5. If $x \neq 0$, then $x^2 = x \cdot x > 0$; in particular, 1 > 0.
- 6. If $x, y \in \mathbb{R}$, then $|x + y| \le |x| + |y|$.

Definition. Let $S \subset \mathbb{R}$. The number $b \in \mathbb{R}$ is an **upper bound** for S if for each $x \in S$, we have $x \leq b$.

Similarly, a number $x \in \mathbb{R}$ is the **least upper bound** for S if it is an upper bound for S and if S if it is an upper bound for S and denote this S if it is an upper bound S and denote this S if it is an upper bound for S and denote this S if it is an upper bound for S and denote this S if it is an upper bound for S and denote this S if it is an upper bound for S and denote this S if it is an upper bound for S and S if it is an upper bound for S and S if it is an upper bound for S and S if it is an upper bound for S and S if it is an upper bound for S and S if it is an upper bound for S if it is an upper bound for S and S if it is an upper bound for S if it is an upper bound for S and S if it is an upper bound for S if it is an upper bound for S and S if it is an upper bound for S is an upper bound for S if it is an upper bound for S if

Definition. Let $S \subset \mathbb{R}$. The number $l \in \mathbb{R}$ is an **lower bound** for S if for each $x \in S$, we have $l \leq x$.

Similarly, a number $x \in \mathbb{R}$ is the **greatest lower bound** for S if it is a lower bound for S and if $x \leq l$ for each lower bound l of S. We then call x the **infimum** of S and denote this $x = \inf S$.

C. Completeness Axiom: Every nonempty set $S \subset \mathbb{R}$ which has an upper bound has a least upper bound.

Proposition 3. Let $L, U \subset \mathbb{R}$ be nonempty subsets with $R = L \cup U$ and such that for each $l \in L$ and each $u \in U$ we have l < u. Then either L has a greatest element or L has a least element.

Proposition 4 (Approximation Property.). Let $S \subset \mathbb{R}$ be a nonempty. If $u = \sup S$, then for all $\gamma > 0$, there exists $Sr \in S$ such that u - r < Sr < u.