Chapter 6 Banach Spaces

Section 6.1 L^p Spaces

Definition. A measurable function $f : [0,1] \to \mathbb{R}$ is said to be in the space $L^p = L^p([0,1])$ if

$$\int_a^b |f|^p < \infty.$$

Note the following

(1) L^1 is the space of integrable functions.

(2) L^p is closed under + and under scalar multiplication i.e., if $f, g \in L^p$ then $f + cg \in L^p$ for all $c \in \mathbb{R}$.This implies that L^p is a linear (vector) space.

Definition. The L^p-norm on a L^p space for all $f \in L^p$ is given by

$$\|f\| = \|f\|_p = \left(\int_0^1 |f|^p\right)^{1/p}.$$

In order for $\|\cdot\|$ to be a norm over a vector space V, the following properties must be satisfied for all $v \in V$:

(1) $\|v\| = 0$ if and only $v = 0$.

(2) For all $\alpha \in \mathbb{R}$, $\|\alpha v\| = |\alpha|\|v\|$.

(3) $\|v + w\| \leq \|v\| + \|w\|$.

In terms of L^p spaces, this is what we currently have for all $f \in L^p$:

(1) $\|f\| = 0$ if and only if $f = 0$ a.e.

(2) $\|\alpha f\| = |\alpha|\|f\|$ for all $\alpha \in \mathbb{R}$.

but we do not have the triangle inequality (third property from above) for norms of L^p spaces since $\|f\| = 0$ implies $f = 0$ almost everywhere rather than strict equality.

However, if we consider equivalence classes of L^p where functions are equal almost everywhere, we can define norms on these spaces. That is, define the relation $\sim$ and

$$\tilde{L^p} = L^P / \sim$$

where $f \sim g \Leftrightarrow f = g$ a.e. In other words, if mod out by functions that are equal almost everywhere, we can get a "nice" normed linear space!!

Definition. The L^p-norm on a L^p space is defined as

$$||f||_p := \left(\int_0^1 |f|^p\right) \text{ for all } p \in (0, \infty).$$

If $p \in (0,1)$, then $||f+g||_p \leq ||f||_p + ||g||_p$. We want to show that $||f+g||_p \leq ||f||_p + ||g||_p$ for $p \in [1, \infty]$.

Definition. For $p = \infty$, the space L^∞ is the set of bounded measurable functions for $f \in L^\infty$. Then

$$\begin{aligned} ||f||_\infty &= \operatorname{ess\,sup} |f(x)| \\ &= \inf \{M : m\{t : f(t) > M\} = 0\}. \end{aligned}$$

Note that $|| \cdot ||_\infty$ is the limit of $|| \cdot ||_p$ i.e.,

$$f \in L^\infty, ||f||_p \to ||f||_\infty.$$

Section 5.5 Convex Functions

Definition. A function $\phi : [a, b] \to \mathbb{R}$ is **convex** if for all $x, y \in [a, b]$ and for all $\lambda \in (0, 1)$, we have that

$$\phi(\lambda x + (1 - \lambda)y) \leq \lambda\phi(x) + (1 - \lambda)\phi(y)$$

Proposition (5.17).

If ϕ is convex on $[a, b]$ then

(1) - (don't care about this)

(2) Right-hand and left-hand derivatives are equal except on a countable set.

(3) The left- and right-hand derivatives are monotone increasing functions, and at each point the left-hand derivative is less than or equal to the right-hand derivative.

Corollary (5.19). If ϕ is twice-differentiable, then ϕ is convex if and only $\phi''(x) > 0$.

Corollary (5.20, Jensen's Inequality**).** Let ϕ be a convex function on $(-\infty, \infty)$ and f be an integrable function $[0, 1]$. Then

$$\int_0^1 \phi(f(t))\, \mathrm{d}t \geq \phi \left[\int_0^1 f(t)\, \mathrm{d}t \right].$$

An example of this is $\phi(x) = x^p$. For any $p \in (1, \infty)$, this function is twice-differentiable. Applying Jensen's inequality, we get

$$\int_0^1 |f(x)|^p\, \mathrm{d}x \geq \left(\int_0^1 |f(x)|\, \mathrm{d}x \right).$$

If $f \in L^p$, then $f \in L^1$ i.e., $L^p \subset L^1$.

Theorem (6.1, Minkowski Inequality). If $f, g \in L^p$ with $p \in [1, \infty]$, then $f + g \in L^p$ and

$$||f + g||_p \leq ||f||_p + \ ||g||_p.$$

If $p \in (1, \infty)$, then the equality can hold only if and only if there exists $\alpha, \beta \geq 0$ such that $\beta f = \alpha g$.

Proof. We leave $p = \infty$ as exercise so suppose p is finite. Let $p \in [1, \infty]$. We normalize f and g i.e., there exists two functions $f_0, g_0 \in L^p$ such that $|f| = \alpha \cdot f_0$ and $|g| = \beta \cdot g_0$ with $||f_0|| = ||g_0|| = 1$. Let $\lambda = \dfrac{\alpha}{\alpha+\beta}$ and $1 - \lambda = \dfrac{\beta}{\alpha+\beta}$. By the convexity of $\phi(t) = t^p$ for $p \in [1, \infty]$, we have that

$$\begin{aligned}
|f(x) + g(x)|^p &\leq (|f(x)| + |g(x))^p \\
&= (\alpha f_0 + \beta g_0)^p \\
&= (\alpha + \beta)^p \left(\frac{\alpha}{\alpha+\beta} f_0 + \frac{\beta}{\alpha+\beta} g_0 \right)^p \\
&\leq (\alpha + \beta)^p \left(\lambda f_0 + (1 - \lambda) g_0\right)^p
\end{aligned}$$

Now take the integrals of these guys (Jensen's inequality or something like that). In other words,

$$\begin{aligned}
||f + g||_p^p &\leq (\alpha + \beta)^p \cdot \left(\lambda ||f_0||_p^p + (1 - \lambda)||g_0||_p^p\right) \\
&= \left(||f||_p^p + ||g||_p^p\right) \cdot 1 && \text{because } f_0 = 1 = g_0.
\end{aligned}$$

Taking the pth root,

$$||f + g||_p \leq |f||_p + ||g||_p.$$

□

This gives us the last norm-space requirement (triangle inequality of normed spaces).

Lemma (6.3). Let $p \in [1, \infty]$. Then for $a, b, t \geq 0$, we have

$$(a + tb)^p \geq a^p + ptba^{p-1}.$$

Proof. Define the function

$$\phi(t) = (a + tb)^p - a^p - ptba^{p-1}.$$

We know $\phi(0) = 0$. Take the derivative of this thing and this is greater than zero because

$$\begin{aligned}
\phi'(x) &= p(a + tb)^{p-1} + b - pba^{p-1} \\
&= pb\left((a + bt)^{p-1} - a^{p-1}\right)
\end{aligned}$$

and so ϕ is increasing. □

Theorem (6.4, Holder Inequality). [1] If p and q are nonnegative extended real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and if $f \in L^p$ and $g \in L^q$, then $f \cdot g \in L^1$ and

$$\int |fg| \leq ||f||_p \cdot ||g||_q.$$

[1] If $p, q = 2$, then this just reduces to the Cauchy-Schwarz inequality.