**Problem 1** (2.9)**.** Properties of sequences in $\mathbb{R}$.

(a.) Show that $\limsup x_n$ and $\liminf x_n$ are the largest and smallest cluster points of the sequence $\{x_n\}$.

*Proof.* Let $\{x_n\}$ be any sequence in $\mathbb{R}$. First to show that $l = \limsup x_n$ is indeed a cluster point, let $\varepsilon > 0$ be chosen. Because $l$ is the limit superior, there exists $n_1 \in \mathbb{N}$ such that $x_{k_1} < l + \varepsilon$ for all $k_1 \geq n_1$. Additionally, there are infinitely many value of this value $n_1$ such that $x_k > l - \varepsilon$ for some $k_1 \geq n_1$, which together with the last sentence implies that $|x_{k_1} - l| < \varepsilon$. To inductively create a subsequence, let $n_1, \ldots, n_j$ and $x_{k_1}, \ldots, x_{k_j}$ be arbitrary. Let $n_{j+1}$ be chosen such that $n_{j+1} > \max\{k_1, \ldots, k_j\}$. Then, because $l$ is the limit superior $x_k < l + \varepsilon$ for any $k \geq n_{j+1}$. Further, for sufficiently large $n_{j+1}$, there exists $k_{j+1} \geq n_{j+1}$ such that $x_{k_{j+1}} > l - \varepsilon$, which gives us that $|x_{k_{j+1}} - l| < \varepsilon$. Because we can always choose the next point in the subsequence in this manner, this means that the subsequence $\{x_{n_j}\}$ converges to $l$. By Problem 2.8, this means that $l$ is a cluster point of $\{x_n\}$.

By way of contradiction, suppose that $l$ is not the largest cluster point of the sequence. That is, there exists a cluster point $y$ of $\{x_n\}$ such that $y > l$. Note that by Problem 2.8, this means that there exists a subsequence $\{x_{n_j}\}$ which converges to $y$. Because $l$ is the limit superior of the sequence, for any $\varepsilon > 0$, we can find $n \in \mathbb{N}$ such that $x_k < l + \varepsilon$ whenever $k \geq n$. Since this is true for any $\varepsilon > 0$, we can choose $\varepsilon > 0$ small enough so that we have

$$l < l + \varepsilon < y - \varepsilon < y.$$

This means that there are a finite number of terms of $\{x_n\}$ contained within the interval $(y-\varepsilon, y+\varepsilon)$. In other words, there does not exist a subsequence $\{x_{n_j}\}$ which converges to $y$ as we would necessarily need an infinite number of terms within $\varepsilon$ of $y$—a contradiction. Therefore, $l$ is the largest cluster point.

By a reverse argument, we can show that $\liminf x_n$ is a cluster point of $\{x_n\}$ as well as the smallest cluster point. □

(b.) Show that every bounded sequence has a convergent subsequence.

*Proof.* Let $\{x_n\}$ be a bounded sequence. In other words, $\sup x_n$ is a finite real number. By definition of the limit superior, $\limsup x_n \leq \sup x_n$. From part (a), because $lim \sup x_n$ is a cluster point, we can always construct a convergent subsequence (as well as for the limit inferior of the sequence) and so this completes this part. □

**Problem 2** (2.43)**.** Let $f$ be defined as

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ p \sin\left(\frac{1}{q}\right) & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

At what points is $f$ continuous? (Please justify your answer.)

*Proof.* I claim that $f$ is not continuous at the rational numbers. To that end, let $x \in Q$ and choose $\varepsilon = x - f(x)$. Fix $\delta > 0$. Note that we can always find an irrational number $y \in (x, x+\delta)$. Because $y$ is irrational, by definition of the function, $f(y) - f(x) = y - f(x)$. But then $y - f(x) > x - f(x) = \varepsilon$.

For $x = 0$, fix $\varepsilon > 0$. Choose $\delta = \varepsilon$, and a pick a point $y \in \mathbb{R}$ such that $|x - y| = |y - 0| = |y| < \delta$. Now because $\sin(1/q) < 1/q$ for any $q \in \mathbb{N}$, we know that

$$\begin{aligned} |f(y) - f(0)| &\leq |y - 0| \\ &< \delta \\ &= \varepsilon \end{aligned}$$

so $f$ is continuous at 0.

$f$ is also continuous at the irrationals. This is because we can if we pick any point $x$ in the irrationals, we can find sufficiently large $q$ so a rational number $y = \frac{p}{q}$ is close to $x$ (i.e, for a fixed $\varepsilon$, choose $\delta$ to be smaller than $f(y) - y$ for this to work). Then this would allow us to bound $|f(y) - f(x)|$ leveraging that we can put the rational numbers in lowest terms

□

**Problem 3.** Show that $F \subset \mathbb{R}$ is a closed set if and only if $F^{\complement}$ is open.

*Proof.* To complete this proof, we will need a forward and backwards implication.

($\Rightarrow$) Suppose $F \subset \mathbb{R}$ is a closed set. Because we desire to show that $F^{\complement}$ is open, let $x \in F^{\complement}$ be a point. This means that $x \notin F$. Since $F$ is a closed set (i.e., $F = \overline{F}$ ) and $x \notin F$, we know $x$ is not a point of closure of $F$. So there exists $\delta > 0$ such that for all $y \in F$, we do not have $|x - y| < \delta$. But then if $|x - y| < \delta$, this must mean that $y \in F^{\complement}$, and so $F$ is an open set.

($\Leftarrow$) Conversely, suppose that the set $F^{\complement}$ is open. Let $x \in F^{\complement}$. Then there exists $\delta > 0$ such that if $|x - y| < \delta$, then $y \in F^C$. This means that there is no $y \in F$ such that $|x - y| < \delta$ and so $x$ cannot be a point of closure of $F$. Thus, because $x$ is arbitrary, $F$ necessarily contains all its points of closure; in other words, $F = \overline{F}$ and thus F must be closed, completing this direction.

Having completed both implications, this completes the proof. □

**Problem 1** (3.5)**.** Let $A$ be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a **finite** collection of open intervals covering A. Then $\displaystyle\sum_{n=1}^{k} l(I_n) \geq 1$.

*Proof.* Define the set $A = \mathbb{Q} \cap [0,1]$. Let $\{I_n\}_{n=1}^{k}$ be a **finite** collection of open intervals covering A meaning we have that

$$A \subset \bigcup_{n=1}^{k} I_n$$

We can create the following string of inequalities:

$$\begin{aligned}
1 = l([0,1]) &= m^*([0,1]) \\
&= m^*\left(\overline{A}\right) && \text{Density of } \mathbb{Q} \\
&\leq m^*\left(\overline{\bigcup_{k=1}^{n} I_n}\right) && A \subset B \implies \overline{A} \subset \overline{B} \\
&= m^*\left(\bigcup_{k=1}^{n} \overline{I_n}\right) && \overline{A \cup B} = \overline{A} \cup \overline{B} \\
&\leq \sum_{k=1}^{n} m^*\left(\overline{I_n}\right) && \text{Subadditivity of } m^* \\
&= \sum_{k=1}^{n} l\left(\overline{I_n}\right) \\
&= \sum_{k=1}^{n} l(I_n)
\end{aligned}$$

which shows the desired result, completing the proof. □

**Problem 2** (3.10)**.** Show that if $E_1$ and $E_2$ are measurable, then

$$m(E_1 \cap E_2) + m(E_1 \cup E_2) = mE_1 + mE_2.$$

Recall that if $E$ is measurable, then $mE := m^*E$.

*Proof.* Suppose $E_1$ and $E_2$ are measurable sets. To show the above equality, the goal will be to rewrite sets as disjoint unions and utilize the subadditivity of $m$. So note that

$$E_1 \cup E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1) \cup (E_1 \cap E_2)$$

and so by the subadditivity of $m$,

$$m(E_1 \cup E_2) = m(E_1 \setminus E_2) + m(E_2 \setminus E_1) + m(E_1 \cap E_2).$$

Thus, adding $m(E_1 \cap E_2)$ to the left-hand side

$$\begin{aligned}
m(E_1 \cup E_2) + m(E_1 \cap E_2) &= (m(E_1 \setminus E_2) + m(E_2 \setminus E_1) + m(E_1 \cap E_2)) + m(E_1 \cap E_2) \\
&= (m(E_1 \setminus E_2) + m(E_1 \cap E_2)) + (m(E_2 \setminus E_1) + m(E_1 \cap E_2)).
\end{aligned}$$

Now, let us write $E_1$ and $E_2$ as disjoint unions:

$$E_1 = (E_1 \setminus E_2) \cup (E_1 \cap E_2);$$
$$E_2 = (E_2 \setminus E_1) \cup (E_1 \cap E_2)$$

which, again, by the subadditivity of $m$,

$$m(E_1) = m(E_1 \setminus E_2) + m(E_1 \cap E_2);$$
$$m(E_2) = m(E_2 \setminus E_1) + m(E_1 \cap E_2).$$

Therefore, this gives us that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

showing the desired result. □

**Problem 3** (3.13)**.** Prove Proposition 15 by the following steps which I will state below for the record.

Let $E$ be any given set. Then the following are equivalent:

(i) $E$ is measurable.

(ii) For all $\varepsilon > 0$, there is an open set $O \supset E$ with $m^*(O \setminus E) < \varepsilon$.

(iii) For all $\varepsilon > 0$, there is a closed set $F \subset E$ with $m^*(E \setminus F) < \varepsilon$.

(iv) There is a $G \in G_\delta$ with $E \subset G$ such that $m^*(G \setminus E) = 0$.

(v) There is a $F \in F_\sigma$ with $F \subset E$ such that $m^*(E \setminus F) = 0$.

If $m^*(E) < \infty$, the above statements are equivalent:

(vi) For all $\varepsilon > 0$, there is a finite union $U$ of open intervals such that $m^*(U \Delta E) < \varepsilon$.

a. Show that for $m^*E < \infty$, $(i) \Rightarrow (ii) \Leftrightarrow (vi)$.

*Proof.* To show that the following are equivalent, we will need to create a chain of implications starting with the outline above. For all of these proofs, assume that $m^*(E) < \infty$.

(i) $\Rightarrow$ (ii) Suppose $E$ is a measurable set. Let $\varepsilon > 0$ be chosen. Because $E$ is measurable and thus $m^*(E) = m(E)$, there exists a countable collection of open intervals $\{I_n\}_{n=1}^\infty$ so that

$$m(E) + \varepsilon > \sum_{n=1}^\infty l(I_n).$$

Since $\{I_n\}_{n=1}^\infty$ is open, the set $O = \bigcup_{n=1}^\infty I_n$ is an open set as well. By Proposition 3.1, we know that

$$m(O) = m\left(\bigcup_{n=1}^\infty I_n\right) = \sum_{n=1}^\infty l(I_n).$$

Again, because $E$ is measurable, we know that $E \subset O$. Now it is left to show that $m(O \setminus E)$. Because $O$ and $E$ are disjoint, we have that

$$\begin{aligned} m(O \setminus E) &= m(O) - m(E) \\ &= \sum_{n=1}^{\infty} l(I_n) - m(E) \\ &< (m(E) + \varepsilon) - m(E) \\ &= \varepsilon \end{aligned}$$

which completes this direction.

(ii) $\Rightarrow$ (vi) Let $\varepsilon > 0$ be chosen. Then by our hypothesis, there exists an open set $O$ such that $m^*(O \setminus E) < \dfrac{\varepsilon}{2}$. By the Lindelof Lemma, the set $O$ can be written as countable union of open intervals i.e., there exists a countable collection of intervals $\{I_n\}_{n=1}^{\infty}$ so that $O = \displaystyle\bigcup_{n=1}^{\infty} I_n$. Satisfying the conditions of Proposition 3.5, we can leverage this as suggested and get that

$$\begin{aligned} \sum_{n=1}^{\infty} l(I_n) &= m^* \left( \bigcup_{n=1}^{\infty} I_n \right) \\ &\leq m^*(E) + \frac{\varepsilon}{2}. \end{aligned}$$

This means that there exists $N \in \mathbb{N}$ so that

$$\sum_{n=N+1}^{\infty} l(I_n) = m^* \left( \bigcup_{n=N+1}^{\infty} I_n \right) < \frac{\varepsilon}{2}.$$

Let $\{I_1, \ldots, I_N\}$ be the finite set of collections up to but including $N+1$ and so we let $U = \displaystyle\bigcup_{n=1}^{N} I_n$. We can note that $U \Delta E = (U \setminus E) \cup (E \setminus U)$. Additionally, $U \setminus E \subset O \setminus E$ by construction of $U$ and $E \setminus U \subset O \setminus E$ by hypothesis. Finally, note that

$$O \setminus U = \left( \bigcup_{n=1}^{\infty} I_n \right) \setminus \left( \bigcup_{n=1}^{N} I_n \right) = \bigcup_{n=N+1}^{\infty} I_n.$$

Thus, with of all this, we have that

$$\begin{aligned} m^*(U \Delta E) &= m^*((U \setminus E) \cup (E \setminus U)) \\ &= m^*(U \setminus E) + m^*(E \setminus U) \\ &\leq m^*(O \setminus E) + m^*(O \setminus U) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

which finishes this direction.

(vi) ⇒ (ii) Let $\varepsilon > 0$ be chosen. By assumption, for any set $E$, there exists a finite union $U$ of open intervals so that

$$m^*(U\Delta E) = m((U \setminus E) \cup (E \setminus U)) < \frac{2\varepsilon}{3}.$$

By Proposition 3.5, there exists an open set $O \supset E \setminus U$ so that

$$m^*(O) \leq m(E \setminus U) + \frac{\varepsilon}{3}$$

which is equivalent to saying that

$$m^*(O \setminus (E \setminus U)) < \frac{\varepsilon}{3}.$$

Note that $E \subset O \cup U$ trivially. Thus, we have that

$$\begin{aligned}
m^*(O \setminus E) &\leq m^*((U \cup O) \setminus (E)) \\
&= m^*((U \setminus E) \cup (O \setminus E)) \\
&\leq m((O \setminus (E \setminus U)) \cup (U \setminus E) \cup (E \setminus U)) \\
&= m(O \setminus (E \setminus U)) + m(U \setminus E) + m(E \setminus U) \\
&< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

giving us the desired result.

This completes the first set of our chain of equivalences. □

b. Use part (a) to show that for arbitrary sets, (i) ⇒ (ii) ⇒ (iv) ⇒ $(i)$.

*Proof.* We continue on our journey to a chain of equivalences with this next set! :)

(i) ⇒ (ii) Suppose that $E$ is measurable and since we showed that this direction for $m^*(E) < \infty$, suppose $m^*(E) = \infty$. For any $n \in \mathbb{N}$, define the set $E_n = E \cup [-n, n]$. From part(a), there exists an open set $O_n \supset E_n$ for all $n \in \mathbb{N}$ so that

$$m^*(O_n \setminus E_n) < \frac{\varepsilon}{2^n}.$$

Define the set $O = \bigcup_{n=1}^{\infty} O_n$. Then note that $E \subset O$ and $E \subset \bigcup_{n=1}^{\infty} E_n$. Using this, we can show that

$$\begin{aligned} m^*(O \setminus E) &= m^*\left(\bigcup_{n=1}^{\infty} O_n \setminus E\right) \\ &\le m^*\left(\bigcup_{n=1}^{\infty} O_n \setminus \bigcup_{n=1}^{\infty} E_n\right) \\ &\le m^*\left(\bigcup_{n=1}^{\infty} O_n \setminus E_n\right) \\ &\le \sum_{i=1}^{\infty} m^*(O_n \setminus E_n) \\ &< \sum_{i=1}^{\infty} \frac{\varepsilon}{2^k} \\ &= \varepsilon \end{aligned}$$

which completes the proof.

(ii) $\Rightarrow$ (iv) By assumption, we can choose $n \in \mathbb{N}$ so that the open set $O_n \supset E$ implies that $m^*(E \setminus O_n) < \dfrac{1}{n} < \varepsilon$ for any $\varepsilon > 0$, which is possible using the Archimedes principle. Let $G = \displaystyle\bigcap_{n=1}^{\infty} O_n$, which is thus a countable intersection of open sets (i.e., $G \in G_\delta$). Note that $E \subset G \subset O_n$ and so

$$\begin{aligned} m^*(G \setminus E) &\le m^*(O_n \setminus E) \\ &< \frac{1}{n} \\ &< \varepsilon. \end{aligned}$$

Because we can always find $n \in \mathbb{N}$ for all $\varepsilon < 0$, we have that $m^*(G \setminus E) = 0$. Since we can choose $n \in \mathbb{N}$, certainly $F \subset E$ and $F_n \subset F$ which gives that

$$m^*(E \setminus F) \le m(E \setminus F_n)$$

(iv) $\Rightarrow$ (i) Assume there exists some $G \in G_\delta$ such that $E \subset G$ and $m^*(G \setminus E) = 0$. Because $G \in G_\delta$ and $m^*(G \setminus E) = 0$, this implies that $G \setminus E$ is a measurable set. But then since $G \setminus E$ is a measurable set, $G$ is a measurable set. Thus since $E = G \setminus (G \setminus E)$, it follows that $E$ is measurable.

This completes this chain of implications. □

c. Use part (b) to show that (i) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (i).

*Proof.* Finally, can finish the chain on equivalences and finish proving Proposition 3.15.

(i) $\Rightarrow$ (iii) Suppose $E$ is measurable set (i.e., $E \in \mathcal{M}$). Let $\varepsilon > 0$ be chosen. Because $\mathcal{M}$ is a $\sigma$-algebra and closed under complement, we know that $E^{\complement}$ is a measurable set as well. From part (b) (the infinite case of (i) $\Rightarrow$ (ii)), there exists an open set $O \supset E^{\complement}$ so that $m^*\left(O \setminus E^{\complement}\right) < \varepsilon$. Let $F = O^{\complement}$, which is a closed set because its complement is open. Then $F \subset E$ and noting that $O \setminus E^{\complement} = E \cap O = E \setminus F$, we have that

$$m^*(F \setminus E) = m^*(O \setminus E^{\complement}) = m^*(E \cap O) < \varepsilon$$

which completes the proof.

(iii) $\Rightarrow$ (v) Similar to the approach of (ii) $\Rightarrow$ (iv) in part (b), let us choose $n \in \mathbb{N}$ using the Archimedes principle so that a closed $F_n \subset E$ means that

$$m^*(E \setminus F_n) < \frac{1}{n} < \varepsilon \text{ for all } \varepsilon > 0.$$

Let $F = \bigcup_{n=1}^{\infty} F_n$, which is a countable union of closed sets and so $F \in F_\sigma$. Also $F \subset E$ and $F_n \subset F$ for any $n \in \mathbb{N}$ so we know that

$$\begin{aligned} m(E \setminus F) &\leq m(E \setminus F_n) \\ &< \frac{1}{n} \\ &< \varepsilon. \end{aligned}$$

By the same reasoning as the end of the proof of (ii) $\Rightarrow$ (iv) from part (b), we can conclude that $m(E \setminus F) = 0$.

(v) $\Rightarrow$ (i) Again, from part (b), we will use similar logic as (iv) $\Rightarrow$ (i). Because $F \in F_\sigma$ and $m^*(E \setminus F) = 0$, this implies that $E \setminus F$ is a measurable set. But then since $E \setminus F$ is a measurable set, $F$ is a measurable set. Thus since $E = F \cup (E \setminus F)$, it follows that $E$ is measurable.

Finally, having finished the chain of equivalences, we have shown that all of the statements are equivalent, which completes the proof. $\square$

**Problem 1** (3.23)**.** Prove Proposition 3.22 by the following lemmas:

a. Given a measurable function $f$ on $[a,b]$ that takes the values $\pm\infty$ only on a set of measure zero, and given $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that $|f| \leq M$ except on a set of measure less than $\dfrac{\varepsilon}{3}$.

*Proof.* Suppose $f$ is a measurable function on $[a,b]$ and that $f(x) = \pm\infty$ only on a set of measure zero. Let $\varepsilon > 0$ be chosen. Define the set

$$E_n = \{x \in [a,b] : |f(x)| > n\} \text{ for all } n \in \mathbb{N}.$$

Because the function $f$ is measurable, by definition, this means that each $E_i$ is a measurable set as well. Note that by construction of $E_n$, we have that $E_i \subset E_{i+1}$ and so $\{E_n\}$ is a decreasing sequence. Since $E_1$ is a subset of the inverse image of $f$ which is itself a subset of $[a,b]$ i.e., $E_1 \subset [a,b]$, we have that

$$m(E_1) < m([a,b]) = b - a < \infty.$$

Again, by the construction of $E_n$, we have that

$$\bigcap_{n=1}^{\infty} E_n = \emptyset$$

implying that

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = 0.$$

But having satisfied the conditions of Proposition 3.14, this is the same as saying $E_n \to 0$ as $n \to \infty$ or

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} E_n = 0.$$

Thus, because $\varepsilon$ is fixed, we can always find $M \in \mathbb{N}$ such that

$$m(E_M) = m\{x \in [a,b] : |f(x)| > M\} < \frac{\varepsilon}{3}.$$

But this necessarily implies that $|f(x)| \leq M$ for all $x \in [a,b]$ thereby completing the proof. □

b. Let $f$ be a measurable function on $[a,b]$. Given $\varepsilon > 0$ and $M > 0$, there is a simple function $\phi$ such that $|f(x) - \phi(x)| < \varepsilon$ except where $|f(x)| \geq M$. If $m \leq f \leq M$, then we may take $\phi$ so that $m \leq \phi \leq M$.

*Proof.* Suppose $f$ is a measurable function on $[a,b]$. Let $\varepsilon > 0$ and $M > 0$ be chosen. Because $\varepsilon$ and $M$ are fixed, by the Archimedes principle, we can choose $N \in \mathbb{N}$ large enough so that $\dfrac{M}{N} < \varepsilon$. From this, let us define the set

$$E_k = \left\{k\frac{M}{N} \leq f(x) \leq (k+1)\frac{M}{N}\right\}$$

for $k \in [-N, N]$ (integer-valued). Since $f$ is a measurable function, each $E_i$ is a measurable set as well. Let us define the function $\phi$ by

$$\phi(x) = \sum_{k=-N}^{N} k\left(\frac{M}{N}\right)\chi_{E_k}$$

with $a_i = k\dfrac{M}{N} \in \mathbb{R}$ for each $k \in [-N, N]$. So because $\phi$ is a linear combination of characteristic functions of $E_i$ and each $E_i$ is a measurable set (in fact, the $E_i$'s are pairwise disjoint), $\phi$ is a simple function. Suppose that $|f(x)| < M$. Because $E_i$'s are pairwise disjoint, then for all $x \in [a, b]$, $x \in E_k$ for some $k \in [-N, N]$ which implies that

$$k\frac{M}{N} \leq f(x) \leq (k+1)\frac{M}{N}.$$

Thus, $\phi(x) = k\dfrac{M}{N}$ which gives us that

$$\begin{aligned} |f(x) - \phi(x) &= \left|f(x) - k\frac{M}{N}\right| \\ &< \frac{M}{N} \\ &< \varepsilon. \end{aligned}$$

Now suppose that $f(x) \in [m, M]$ for all $x \in [a, b]$ (i.e., $f$ is a bounded function.) Then the same argument holds as before but instead we have that

$$\begin{aligned} |f(x) - \phi(x) &= \left|f(x) - k\frac{M-m}{N}\right| \\ &< \frac{M-m}{N} \\ &< \varepsilon \end{aligned}$$

meaning for all $x \in [a, b]$, we have $\phi(x) = k\dfrac{M-m}{N}$ implying that $\phi(x) \in [m, M]$. □

c. Given a simple function $\phi$ on $[a, b]$, there is a step function $g$ on $[a, b]$ such that $g(x) = \phi(x)$ except on a set of measure less than $\dfrac{\varepsilon}{3}$.

*Proof.* Let $\phi$ by the simple function defined by

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}$$

for measurable, disjoints sets $E_1, \ldots, E_n$ and $a_i \in \mathbb{R}$ for $i = 1, \ldots n$. Let $\varepsilon > 0$ be chosen. Because each $E_i$ is a measurable set, by Proposition 3.15, for each $i = 1, \ldots n$, there exists a finite union $U_i$ of open intervals $I_i$ such that $m(E_i \Delta U_i) < \dfrac{\varepsilon}{3n}$ with

$$U_i = \sum_{k=1}^{N_i} I_{i,k}.\text{[1]}$$

[1] This is mostly for myself, but $k$ is the index for the number of intervals $N_i$ associated with each $E_i$.

Let $A_i = U_i \setminus \left(\bigcup_{j=1}^{i-1} U_j\right)$.[2] For any $x \in [a,b]$, define the function

$$g(x) = \sum_{i=1}^{n} a_i \chi_{A_i}.$$

Because the $E_i$'s are measurable and the difference of measurable sets is measurable, the set $\{A_1, \dots, A_n\}$ is a set of measurable sets. The $A_i$'s are a subdivision of $[a,b]$ and so $g$ is a step function per the definition on page 76 of Royden. We claim that this function is equal to $\phi(x)$ except on a set of measure less than $\frac{\varepsilon}{3}$. To that end, fix $x \in [a,b]$ so that $\phi(x) \neq g(x)$. Because $\phi$ and $g$ are linear combinations with the same coefficients, this brings two cases: (i) there is some $i = 1, \dots n$ so that $g(x) = a_i$ but $\phi(x) \neq a_i$ or (ii) there is some $i = 1, \dots, n$ so that $g(x) \neq a_i$ but $\phi(x) = a_i$.

For case (i), this means that $x \in A_i \subset U_i \setminus E_i$ for some $i = 1, \dots, n$. For case (ii), we must have that $x \in E_i \subset E_i \setminus U_i$ for some $i = 1, \dots n$. So, combining both results,

$$\{x \in [a,b] : \phi(x) \neq g(x)\} \subset \bigcup_{i=1}^{n} U_i \setminus E_i;$$
$$\{x \in [a,b] : \phi(x) \neq g(x)\} \subset \bigcup_{i=1}^{n} E_i \setminus U_i$$

and thereby implies that

$$\{x \in [a,b] : \phi(x) \neq g(x)\} \subset \bigcup_{i=1}^{n} E_i \Delta U_i.$$

Finally, this allows us to show that

$$\begin{aligned} m\left(\{x \in [a,b] : \phi(x) \neq g(x)\}\right) &\leq m\left(\bigcup_{i=1}^{n} E_i \Delta U_i\right) \\ &= \sum_{i=1}^{n} m\left(E_i \setminus U_i\right) \\ &< \sum_{i=1}^{n} \frac{\varepsilon}{3n} \\ &= n \cdot \frac{\varepsilon}{3n} \\ &= \varepsilon \end{aligned}$$

giving us the desired result. □

**Problem 2** (3.31)**.** Prove Lusin's Theorem: Let $f$ be a measurable real-valued function on an interval $[a,b]$. Then for all $\delta > 0$, there is a continuous function $\phi$ on $[a,b]$ such that $m\{x : f(x) \neq \phi(x)\} < \delta$.

[2] Again, mostly for myself, but for each $U_i$ associated with $E_i$, throw out the preceding $U_i's$.

*Proof.* Let $\delta > 0$ be chosen. Suppose $f$ is a measurable real-valued function on an interval $[a,b]$. Then by Proposition 3.22, there exists a continuous function $h_n$ for all $n \in \mathbb{N}$ such that

$$|f - h_n| < \frac{\delta}{2^{n+2}}$$

with $m\left\{x \in [a,b] : |f - h_n| \geq \frac{\delta}{2^{n+2}}\right\} < \frac{\delta}{2^{n+2}}$. For convenience, define the sets

$$E_n = \left\{x \in [a,b] : |f - h_n| \geq \frac{\delta}{2^{n+2}}\right\}$$

and

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Note for a fixed $x \in [a,b] \setminus E_n$ and any $n \in \mathbb{N}$, we know by how we defined $E_n$ that

$$|f - h_n| < \frac{\delta}{2^{n+2}}.$$

Thus, since $E$ is the union of the $E_n$'s, we have that

$$\begin{aligned} m(E) &= m\left(\bigcup_{n=1}^{\infty} E_n\right) \\ &\leq \sum_{n=1}^{\infty} m(E_n) \\ &< \sum_{n=1}^{\infty} \frac{\delta}{2^{n+2}} \\ &= \frac{\delta}{4}. \end{aligned}$$

So then on the set $[a,b]\setminus E$, the sequence of continuous, and thereby, measurable functions $\{h_n\}$ converges to $f$. Having satisfied the conditions of Egoroff's theorem, there exists a set $A \subset [a,b] \setminus E$ with $m(A) < \frac{\delta}{4}$ such that $h_n$ converges uniformly on $([a,b] \setminus E) \setminus A = [a,b] \setminus (E \cup A)$. Since the uniform limit of continuous functions is a continuous function, the function $f$ is continuous on $[a,b] \setminus (E \cup A)$. Because $m(E)$ and $m(A)$ are less than $\frac{\delta}{4}$, $m(E \cup A) < \frac{\delta}{2}$.

Using Proposition 3.15 part (ii), there exists an open set $O \supset (E \cup A)$ with

$$m(O \setminus (E \cup A)) < \frac{\delta}{2}.$$

Because $[a,b] \setminus (E \cup A) \supset [a,b] \setminus O$ and $[a,b] \setminus O = [a,b] \cap O^{\mathrm{c}}$ (i.e., a closed set), $f$ is continuous on the closed set $[a,b] \setminus O$. Then for any $x \in [a,b] \setminus O$, by Problem 2.40, there exists a continuous function $\phi$ so that $f(x) = \phi(x)$. But then the set $O$ represents the set of points where $\phi$ and $g$ are not equal. In particular, we can show that

$$\begin{aligned} m\{x \in [a,b] : f(x) \neq \phi(x)\} &= m(O) \\ &= m((O \setminus (E \cup A)) \cup (E \cup A)) \\ &= m(O \setminus (E \cup A)) + m(E \cup A) \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta \end{aligned}$$

which finally completes the proof. $\square$

**Problem 1** (4.2). (a) Let $f$ be a bounded function on $[a,b]$, and let $h$ be the upper envelope of $f$ (cf. Problem 2.51). Then $R\overline{\int_a^b} f = \int_a^b h$.

*Proof.* Let $f$ be a bounded function on $[a,b]$ with $h(y) = \inf_{\delta>0} \sup_{|x-y|<\delta} f(x)$ for all $x \in [a,b]$ be the upper envelope of $f$. Because $f$ is bounded, by Problem 2.51 part (b), $h$ is lower semicontinuous. To show equality, we will show that

$$R\overline{\int_a^b} f \leq \int_a^b h \quad \text{and} \quad R\overline{\int_a^b} f \geq \int_a^b h.$$

Let $\phi$ be a step function on $[a,b]$ such that $\phi \geq f$. Then for any $x \in [a,b]$, $h(x) \leq f(x) \leq \phi(x)$, except at the defined partition points of $\phi$. Thus, by using the definition of Lebesgue integration, we have that

$$\int_a^b h(x)\,\mathrm{d}x \leq \int_a^b f(x)\,\mathrm{d}x = \inf \int_a^b \phi(x)\,\mathrm{d}x \leq R\overline{\int_a^b} f(x)\,\mathrm{d}x.$$

For the other inequality, we note that because $h$ is upper semicontinuous, by Problem 2.51 part (g), there exists a monotonically decreasing sequence of step functions $\{\phi_n\}$ such that $\phi_n \to h$ pointwise. Because $f$ is bounded, we have that for all $x \in [a,b]$, there exists some $M > 0$ such that

$$|\phi_n| \leq |h| \leq |f| \leq M \text{ for all } n \in \mathbb{N}.$$

Thus using the Bounded Convergence Theorem and properties of the upper Riemann integral,

$$\lim_{n\to\infty} \int_a^b \phi_n(x)\,\mathrm{d}x = \int_a^b h(x)\,\mathrm{d}x \leq R\int_a^b f(x)\,\mathrm{d}x \leq R\overline{\int_a^b} f(x)\,\mathrm{d}x.$$

Therefore,

$$R\overline{\int_a^b} f = \int_a^b h$$

which is the desired result. □

(b) Use part (a) to prove Proposition 7 which is stated as follows

**Proposition (4.7).** A bounded function $f$ on $[a,b]$ is Riemann integrable if and only if the set of points at which $f$ is discontinuous has measure zero.

*Proof.* Let $f$ be a bounded function on $[a,b]$. We will need to show a forward and backwards implication to complete this proof. For simplicity, define $E$ to be the set of discontinuities of $f$. Additionally, let $g(y) = \sup_{\delta>0} \inf_{|x-y|<\delta} f(x)$ be the lower envelope of $f$.

($\Leftarrow$) First, suppose $m(E) = 0$. Since $g$ is the lower envelope of $f$, there exists a monotonically increasing sequence of step functions $\{\phi_n\} \to g$ pointwise. Thus, by a similar but reverse argument to part (a),

$$\int_a^b g(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x. \tag{1}$$

So because $f$ is continuous everywhere except on the set $E$—namely, continuous on $[a,b] \setminus E$— by Problem 2.51, $g(x) = h(x)$ is continuous on the set $[a,b] \setminus E$. But since $m(E) = 0$, this means $g = h$ almost everywhere and thus, using part (a) of this problem and Equation (1),

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x = \int_a^b h(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x.$$

Thus, $f$ is Riemann integrable.

($\Rightarrow$) Now suppose $f$ is Riemann integrable. Thus, the lower and upper integrals of $f$ are equal to each other and so using Equation (1)

$$\int_a^b h(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x.$$

Consider the set $A_n = \left\{x : |g(x) - h(x)| > \dfrac{1}{n}\right\}$ for all $n \in \mathbb{N}$. Because the integrals of $g$ and $h$ are equal,

$$\int_a^b |h(x) - g(x)|\,\mathrm{d}x = 0.$$

So for any fixed $n \in \mathbb{N}$,

$$\int_a^b |h(x) - g(x)|\,\mathrm{d}x \geq m(A_n).$$

So $h(x) = g(x)$ almost everywhere and so by Problem 2.51 part(a), we must that $f$ is continuous almost everywhere as well. But then this means that the measure of discontinuities is zero—that is, $m(E) = 0$ which is what we wanted to show.

Having completed both directions, we have shown the desired equivalence. □

**Problem 2** (4.3)**.** Let $f$ be a nonnegative measurable function. Show that $\displaystyle\int f = 0$ implies $f = 0$ almost everywhere.

*Proof.* Let $f \geq 0$ be a measurable function, and suppose that $\displaystyle\int f = 0$. We want to show that the set $E = x : f(x) \neq 0 = \{x : f(x) > 0\}$ has measure 0. Define the set

$$E_n = \left\{x : f(x) \geq \frac{1}{n}\right\} \text{ for all } n \in \mathbb{N}.$$

Note that $\displaystyle\bigcup_{n=1}^{\infty} E_n = E$. Fix $n \in \mathbb{N}$. Because the integral of $f$ is equal to 0,

$$0 = \int_{E_n} \geq \int_{E_n} \frac{1}{n} = m(E_n) \cdot \frac{1}{n} \geq 0.$$

Thus, because $n \in \mathbb{N}$ was fixed, $m\left(\displaystyle\bigcup_{n=1}^{\infty} E_n\right) = 0 = m(E)$, and therefore $f = 0$ almost everywhere. □

**Problem 3** (4.8)**.** Prove the following generalization of Fatuo's Lemma: If $f_n$ is a sequence of nonnegative functions then

$$\int \varliminf_{n\to\infty} f_n \leq \varliminf_{n\to\infty} \int f_n.$$

*Proof.* Let $\{f_n\} \geq 0$ for each $n \in \mathbb{N}$ on any set $E$. Define $h_n = \inf_{k\geq n} f_k$ for all $n \in \mathbb{N}$. Note that as $n \to \infty$, $h_n \to \varliminf_{n\to\infty} f_n$ (i.e., $h_n$ converges pointwise on $E$ to the limit inferior of $f_n$). Thus, by Fatou's lemma, we have that

$$\int_E \varliminf_{n\to\infty} f \leq \int_E \varliminf_{n\to\infty} h_n.$$

But since $h_n$ is the infimum of the $f_n$'s, this implies that $h_n \leq f_n$ for all $n \in \mathbb{N}$ and so

$$\int_E h_n \leq \int_E f_n$$

and thus

$$\varliminf_{n\to\infty} \int_E h_n \leq \varliminf_{n\to\infty} \int_E f_n.$$

By combining all three inequalities, we get that

$$\int_E \varliminf_{n\to\infty} f \leq \varliminf_{n\to\infty} \int_E h_n \leq \varliminf_{n\to\infty} \int_E f_n$$

which then completes the proof. $\square$

**Problem 1** (4.14)**.** Some sequence and integral convergence problems.

(a) Show that under the hypotheses of Theorem 4.17 we have

$$\int |f_n - f| \to 0.$$

*Proof.* Let $\{g_n\}$ be a sequence of integrable functions such that $g_n \to g$ pointwise with $g$ integrable. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ and $f_n \to f$ pointwise almost everywhere. Suppose that

$$\int g = \lim_{n\to\infty} \int g_n.$$

For any $n \in \mathbb{N}$ we have $|f_n| \leq g$ and so because $f_n \to f$ and $g_n \to g$, $|f| \leq g$. Thus, we have that

$$\begin{aligned} |f_n| - |f| \leq |f_n - f| &\leq |f_n + f| \\ &\leq |f_n| + |f| \\ &\leq g_n + g. \end{aligned}$$

This means that the sequence defined by $\{(g_n + g) - |f_n - f|\}$ is a nonnegative sequence. So by Fatou's lemma and properties of lim inf and lim sup,

$$\begin{aligned} 0 \leq \int (g_n + g) - |f_n - f| &\leq \varliminf_{n\to\infty} \int (g_n + g) - |f_n - f| \\ &\leq \int (g_n + g) + \varliminf_{n\to\infty} \int -|f_n - f| \\ &= \int (g_n + g) - \varlimsup_{n\to\infty} \int |f_n - f|. \end{aligned}$$

But then this implies that

$$\varlimsup_{n\to\infty} \int |f_n - f| \leq 0 \leq \varliminf_{n\to\infty} \int |f_n - f|$$

and so we have

$$\lim_{n\to\infty} \int |f_n - f| = 0.$$

□

(b) Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ almost everywhere with $f$ integrable. Then $\int |f - f_n| \to 0$ if and only if $\int |f_n| \to \int |f|$.

*Proof.* We will show two directions to complete this proof.

(⇒) First, suppose that

$$\lim_{n\to\infty} \int |f_n - f| = 0.$$

Then, by using the reverse triangle inequality, can show that

$$\begin{aligned} \left| \lim_{n\to\infty} \int |f_n| - \int |f| \right| &\leq \lim_{n\to\infty} \int |f_n| - \int |f| \\ &\leq \lim_{n\to\infty} \int |f_n - f| \\ &= 0. \end{aligned}$$

Because $|\cdot| \geq 0$ always, we know that

$$0 \leq \lim_{n\to\infty} |f_n| \leq \int |f| \leq 0$$

and so

$$\lim_{n\to\infty} \int |f_n| = \int |f|.$$

($\Leftarrow$) Conversely, suppose that

$$\lim_{n\to\infty} \int |f_n| = \int |f|.$$

Because $f_n \to f$ a.e, $|f_n| \leq f$ for all $n \in \mathbb{N}$. By a similar argument to part (a),

$$\begin{aligned} |f_n| - |f| \leq |f_n - f| &\leq |f_n + f| \\ &\leq |f_n| + |f| \end{aligned}$$

Then the sequence $\{(|f_n| + |f|) - |f_n - |\}$ is a nonnegative sequence. Again, similar to part (a), using Fatou's lemma we have

$$\begin{aligned} 0 \leq \int (|f_n| + |f|) - |f_n - f| &\leq \underline{\lim}_{n\to\infty} \int (|f_n| + |f|) - |f_n - f| \\ &\leq \int (|f_n| + |f|) + \underline{\lim}_{n\to\infty} \int -|f_n - f| \\ &= \int (|f_n| + |f|) - \overline{\lim}_{n\to\infty} \int |f_n - f|. \end{aligned}$$

So we again that

$$\overline{\lim}_{n\to\infty} \int |f_n - f| \leq 0 \leq \underline{\lim}_{n\to\infty} \int |f_n - f|$$

and so we have

$$\lim_{n\to\infty} \int |f_n - f| = 0.$$

finishing the proof.

Thus, having showed both directions, $\int |f - f_n| \to 0$ if and only if $\int |f_n| \to \int |f|$. □

**Problem 2** (4.16)**.** Establish the *Riemann-Lebesgue Theorem*: If $f$ is an integrable function on $(-\infty, \infty)$, then $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\cos(nx)\,\mathrm{d}x = 0$. [Hint: The theorem is easy if $f$ is a step function. Use Problem 15.]

*Proof.* Let $f$ be an integrable function $(-\infty, \infty)$. Let $\varepsilon > 0$ be chosen. By Problem 15 part (b), there exists a step function $\psi$ such that

$$\int |f - \psi| < \frac{\varepsilon}{2}.$$

Turning to $\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos(nx)\,\mathrm{d}x$, we can note that following:

$$\begin{aligned}
\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos(nx)\,\mathrm{d}x &= \left|\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos(nx)\,\mathrm{d}x\right| \\
&\leq \lim_{n\to\infty}\int_{-\infty}^{\infty} |f(x)\cos(nx)|\,\mathrm{d}x \\
&\leq \lim_{n\to\infty}\int_{-\infty}^{\infty} |f(x)\cos(nx)|\,\mathrm{d}x \\
&\leq \lim_{n\to\infty}\int_{-\infty}^{\infty} |(f(x)-\psi(x))\cos(nx)|\,\mathrm{d}x + \lim_{n\to\infty}\int_{-\infty}^{\infty} |\psi(x)\cos(nx)|\,\mathrm{d}x \\
&< \frac{\varepsilon}{2} + \lim_{n\to\infty}\int_{-\infty}^{\infty} |\psi(x)\cos(nx)|\,\mathrm{d}x.
\end{aligned}$$

Because $\psi(x)$ is a step function, we can integrate the right-hand side integral in the last inequality over $(-\infty,\infty)$ in each interval which $\psi(x)$ is constant. So then because $\phi(x)$ is fixed over these intervals, as $n\to\infty$, the antiderivative of $|\cos(nx)|$ goes to zero i.e.,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} |\psi(x)\cos(nx)|\,\mathrm{d}x = 0.$$

Since this integral converges to 0, there exists $N\in\mathbb{N}$ such for all $n>N$, we have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} |\psi(x)\cos(nx)| < \frac{\varepsilon}{2}.$$

Thus, we have that

$$\begin{aligned}
\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos(nx)\,\mathrm{d}x &< \frac{\varepsilon}{2} + \lim_{n\to\infty}\int_{-\infty}^{\infty} |\psi(x)\cos(nx)| \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Since $\varepsilon$ was chosen arbitrarily,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos(nx)\,\mathrm{d}x = 0$$

which was our desired result. □

**Problem 3** (4.25)**.** A sequence $\{f_n\}$ of measurable functions is said to be a Cauchy sequence in measure if given $\varepsilon>0$, there is $N\in\mathbb{N}$ such that for all $m,n\geq N$ we have

$$m\{x : |f_n(x)-f_m(x)|\geq\varepsilon\} < \varepsilon.$$

Show that if $\{f_n\}$ is a Cauchy sequence in measure, then there is a function $f$ to which the sequence $\{f_n\}$ converges in measure.

*Proof.* Let $\{f_n\}$ be a sequence of measurable functions which is Cauchy in measure. Fix $\nu \in \mathbb{N}$. Choose $n_{\nu+1} > v_\nu$ such that

$$m\left\{x : \left|f_{n_{\nu+1}}(x) - f_{n_\nu}(x)\right| \geq \frac{1}{2^\nu}\right\} < \frac{1}{2^\nu}.$$

We claim that the series

$$S_n(x) = \sum_{\nu=1}^{\infty} \left(f_{n_{\nu+1}}(x) - f_{n_\nu}(x)\right)$$

converges almost everywhere to a function $g$. Define the set

$$E_\nu = \left\{x : \left|f_{n_{\nu+1}}(x) - f_{n_\nu}(x)\right| \geq \frac{1}{2^\nu}\right\}.$$

If $x \notin A_k = \bigcup_{\nu=k}^{\infty} E_\nu$, then

$$\left|f_{n_{\nu+1}}(x) - f_{n_\nu}(x)\right| < \frac{1}{2^v} \text{ for all } \nu > k.$$

Taking the intersection over all $k$ for $A$ would mean that this set would be contained in $A$ i.e.,

$$\bigcap_{k=1}^{\infty} A_k \subset A_k$$

and so

$$m\left(\bigcap_{k=1}^{\infty} A_k\right) \leq m\left(A_k\right) \leq \sum_{k=1}^{\infty} m(A_k) < \frac{1}{2^{\nu-1}}.$$

Because $\nu$ is fixed, $m\left(\bigcap_{k=1}^{\infty} A_k\right) = 0$. Thus $S_n(x) \to g(x)$ almost everywhere.

Let $f = g + f_{n_1}$ be a sequence. By construction, the partial sums of this sequence are telescoping i.e., for any $\nu \in \mathbb{N}$, the partials sums of $f$ are of the form $f_{n_\nu} - f_{n_1}$. Thus $f_{n_\nu} \xrightarrow{m} f$. Now let $\varepsilon > 0$ be chosen. Because the sequence $\{f_n\}$ is Cauchy in measure, there exists $N_1 \in \mathbb{N}$ such for all $m, n \geq N_1$,

$$m\left\{x : |f_n(x) - f_m(x)| \geq \frac{\varepsilon}{2}\right\} < \frac{\varepsilon}{2}.$$

Since $f_{n_\nu} \xrightarrow{m} f$, there exists $N_2 \in \mathbb{N}$ such that for all $k > N_2$

$$m\left\{x : |f_{n_k} - f(x)| \geq \frac{\varepsilon}{2}\right\} < \frac{\varepsilon}{2}.$$

Set $N = \max\{N_1, N_2\}$. So for any $n, k > N$, we know

$$\begin{aligned} m\left\{x : |f_n(x) - f(x)| \geq \varepsilon\right\} &\leq m\left\{x : |f_{n_k} - f_n(x)| \geq \frac{\varepsilon}{2}\right\} + m\left\{x : |f(x) - f_{n_k}(x)| \geq \frac{\varepsilon}{2}\right\} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Having satisfied the definition of convergence of measure, $f_n \xrightarrow{m} f$ which completes the proof. □

**Problem 4.** Compute $\lim_{n\to\infty}\int_0^1 (1+nx^2)(1+x^2)^{-n}\,\mathrm{d}x$. Justify your answer.

*Proof.* Note that we can rewrite this integral as

$$\lim_{n\to\infty}\int_0^1 \frac{1+nx^2}{(1+x^2)^n}\,\mathrm{d}x$$

We can interchange the limit operation and the integral because the sequence of functions $f_n(x) = \left\{\frac{1+nx^2}{(1+x^2)^n}\right\}$ is uniformly convergent and, in fact, this sequence is uniformly convergent to 0. To that end fix $\varepsilon > 0$. Take the derivative of $f(x) = \frac{1+nx^2}{(1+x^2)^n}$ with respect to $x$ as we want to find where this function is maximized over $[0,1]$. It can be shown that (saving showing all of the algebra),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1+nx^2}{(1+x^2)^n}\right) = -2(n-1)nx^3(x^2+1)^{-n-1}.$$

For any $x \in [0,1]$, as $n \to \infty$, this quantity goes to 0 i.e., $f(x)$ is maximized when $x = 0$. So then $f(0) = \frac{1}{1^n} = 1$ for all $n \in \mathbb{N}$. Thus choose $N \in \mathbb{N}$ large enough so that $\frac{1}{N} < \varepsilon$. So for any $n > N$,

$$\left|\frac{1+nx^2}{(1+x^2)^n}\right| \leq \frac{1}{n} < \varepsilon.$$

Thus $f_n(x) \to 0$ uniformly and so

$$\lim_{n\to\infty}\int_0^1 \frac{1+nx^2}{(1+x^2)^n}\,\mathrm{d}x = \int_0^1 \lim_{n\to\infty}\frac{1+nx^2}{(1+x^2)^n} = \int_0^1 0\,\mathrm{d}x = 0.$$

□

**Problem 1** (6.2)**.** Let $f$ be a bounded measurable function on $[0,1]$. Then $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

*Proof.* First, I will note that $\lim_{p\to\infty} \|f\|_p \le \|f\|_\infty$ follows pretty readily from the definition of $\|\cdot\|$. This is because

$$\|f\|_p = \left\{ \int_0^1 |f|^p \right\}^{1/p} \le \left\{ \int_0^1 \|f\|_\infty^p \right\}^{1/p} = \|f\|_\infty$$

and so as we take the limit of $\|f\|_p$ as $p \to \infty$, we get $\lim_{p\to\infty} \|f\|_p \le \|f\|_\infty$. Now we must also show that $\lim_{p\to\infty} \|f\|_p \ge \|f\|_\infty$. To that end, let $\varepsilon > 0$ be chosen. Define the set $A = \{x \in [0,1] : |f(x)| > \|f\|_\infty - \varepsilon\}$. Then we have that

$$\begin{aligned}\|f\|_p = \left\{ \int_0^1 |f|^p \right\}^{1/p} &\ge \left\{ \int_A |f|^p \right\}^{1/p} \\ &\ge \left\{ \int_A (\|f\|_\infty - \varepsilon)^p \right\}^{1/p} \\ &= (\|f\|_\infty - \varepsilon)^p \cdot m(A).\end{aligned}$$

This implies that

$$\|f\|_\infty - \varepsilon \cdot (m(A))^p \le \|f\|_p.$$

Because $\|f\|_\infty$ is the essential supremum (i.e. the smallest, greatest value not on a set of measure zero), we know that $m(A) > 0$. Thus, taking the limit of both sides as $p \to \infty$, we get that

$$\lim_{p\to\infty} \|f\|_\infty - \varepsilon \cdot (m(A))^p = \|f\|_\infty - \varepsilon \le \lim_{p\to\infty} \|f\|_p,$$

Since $\varepsilon$ is arbitrary, then $\lim_{p\to\infty} \|f\|$ is a superior bound i.e., $\|f\|_\infty \le \lim_{p\to\infty} \|f\|$. Thus we get $\|f\|_\infty = \lim_{p\to\infty} \|f\|$. □

**Problem 2** (6.8)**.** Young's Inequality

(a) Let $a, b \ge 0$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Establish Young's inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

*Proof.* Not assigned. □

(b) Use Young's inequality to give a proof of the Hölder inequality.

*Proof.* Let $p$ and $q$ be nonnegative extended real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Suppose $f \in L^p$ and $g \in L^q$. Without loss of generality, assume that $\|f\|, \|g\| \ge 0$. With $a = \frac{|f|}{\|f\|_p}$ and $b = \frac{|g|}{\|g\|_q}$, by Young's Inequality from part (a), we have that

$$\frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \le \frac{|f|^p}{p\,\|f\|_p^p} \frac{|g|_p^q}{q\,\|g\|_q^q}.$$

From the monotonocity of integrals, this implies that

$$\int \frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q} \leq \int \frac{|f|^p}{p\,\|f\|_p^p} + \frac{|g|^q}{q\,\|g\|_q^q} = \frac{1}{p\,\|f\|_p^p}\int |f|^p + \frac{1}{q\,\|g\|_q^q}\int |g|^q.$$

However, the the integral of $|f|^p$ is the same as $\|f\|_p^p$ and the same argument for $|g|^p$. So by cancelling out $\|f\|_p^p$ and $\|g\|_q^q$, we get that

$$\int \frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1.$$

Multiplying both sides by $\|f\|_p \cdot \|g\|_q$ and so

$$\int |fg| \leq \|f\|_p \cdot \|g\|_q.$$

Young's inequality is equality if and only $a^p = b^q$ and so the Hölder inequality is equality if and only if $\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q}$. Thus there exists $\alpha, \beta \neq 0$ such that $\alpha|f|^p = \beta|g|^q$ almost everywhere and so this completes the proof. □

**Problem 3** (6.10)**.** Let $\{f_n\}$ be a sequence of functions in $L^\infty$. Prove that $\{f_n\}$ converges to $f$ in $L^\infty$ if and only if there is a set $E$ of measure zero such that $f_n$ converges to $f$ uniformly on $E^\complement$.

*Proof.* We will need to complete two directions and so let $\{f_n\}$ be a sequence of functions in $L^\infty$.

($\Rightarrow$) First, suppose that $\{f_n\} \to f$, and let $\varepsilon > 0$ be chosen. Because $f_n \to f$ in $L^\infty$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|f_n - f\|_\infty = \inf\{M : m\{t : |f_n(t) - f(t| > M\} = 0\} < \varepsilon.$$

Let $E = \{t : |f_n(t) - f| \geq \varepsilon\}$. Per the above expression, for any $n \geq N$, we have that $m(E) = 0$ and $\|f_n - f\|_\infty < \varepsilon$ on the set $L^\infty \setminus E = E^\complement$. Thus, since $\varepsilon > 0$ is arbitrary, $f_n$ converges uniformly to $f$ on $E^\complement$.

($\Leftarrow$) Conversely, suppose there exists a set $E$ with $m(E) = 0$ such that $f_n \to f$ uniformly on $E^\complement$. Let $\varepsilon > 0$ be chosen. Since $f_n \to f$ uniformly on $E^\complement$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $t \in E^\complement$,

$$|f_n(t) - f(t)| < \frac{\varepsilon}{2}.$$

But then this means that the set $\left\{t : f_n(t) - f(t) > \frac{\varepsilon}{2}\right\} \subset E$. By the definition of the infimum, for our fixed $\varepsilon > 0$ and any $n \geq N$,

$$\inf\{M : m\{t : |f_n(t) - f(t)| > M\} = 0\} < \varepsilon.$$

This is the essential supremum and so this means $\|f_n - f\|_\infty < \varepsilon$. Therefore, since $\varepsilon$ is arbitrary, $\|f_n - f\| < \varepsilon$ and which implies that $f_n \to f$ pointwise on $L^\infty$.

Thus, having completed the forward and backwards implication, this completes the proof. □

**Problem 1** (6.11)**.** Prove that $L^\infty$ is complete.

*Proof.* To show that $L^\infty$ is complete, we must show every Cauchy sequence converges. To that end, let $\{f_n\}$ be any Cauchy sequence in $L^\infty$ and let $\varepsilon > 0$ be chosen. Since $\{f_n\}$ is Cauchy, there exists $N_1 \in \mathbb{N}$ such that for all $n, m \geq N_1$, we have

$$\|f_n - f_m\|_\infty = \inf \{M : m\{t : |f_n(t) - f_m(t)| > M\} = 0\} < \frac{\varepsilon}{2}.$$

So for any fixed $n, m \geq N_1$, there exists $M < \dfrac{\varepsilon}{2}$ such that $m\{t : |f_n(t) - f_m(t)| > M\} = 0$. implying that $m\left\{t : |f_n(t) - f_m(t)| > \dfrac{\varepsilon}{2}\right\} = 0$. So then on the set $L^\infty \backslash \{t : |f_n(t) - f_m(t)| > \frac{\varepsilon}{2}\}$, the sequence $\{f_n\}$ converges to some function $f$ almost everywhere. We must show that this limit function $f$ is in $L^\infty$

Since $f_n \to f$ almost everywhere, there exists $N_2 \in \mathbb{N}$ such that for any $n > N_2$, $|f_n - f| < \dfrac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then for any fixed $n > N$, we can see that

$$\|f_n - f\|_\infty = \inf \{M : m\{t : |f_n(t) - f(t)| > M\} = 0\} < \varepsilon$$

which, because $\varepsilon$ is arbitrary, means that $\|f_n - f\| \to 0$. Thus $f \in L^\infty$ and so $L^\infty$ is complete. $\square$

**Problem 2** (6.13)**.** Let $C = C[0, 1]$ be the space of all continuous functions on $[0, 1]$ and define $\|f\| = \max |f(x)|$. Show that $C$ is a Banach space.

*Proof.* To show that the space $(C, \|\cdot\|)$ is a Banach space, we must show $\|f\| = \max |f(x)|$ is indeed a norm and that $C$ with this norm is a complete space.

First, we will show that $\|\cdot\|$ defined above on $C$ is a norm. That is, we must show the following the following three properties:

(i) For any $f \in C$, $\|f\| = 0$ if and only if $f = 0$.

(ii) For any $f, g \in C$, $\|f + g\| \leq \|f\| + \|g\|$.

(iii) For any $f \in C$ and for all $\alpha \in \mathbb{R}$, $\|\alpha f\| = |\alpha| \|f\|$.

For (i), first suppose that $\|f\| = 0$. Then $\max |f(x) = 0|$ which is true only if $f(x) = 0$ for any $x \in [0, 1]$. Conversely, suppose $f = 0$. Then for any $x \in [0, 1]$, we have that $\|f\| = \max |f(x)| = \max 0 = 0$.

To prove (ii), fix $f, g \in C$. By the triangle inequality, we know that $|f + g| \leq |f| + |g|$. The max function adheres to the triangle inequality and so

$$\begin{aligned}\|f + g\| &= \max |f + g| \\ &\leq \max |f| + \max |g| \\ &= \|f\| + \|g\|.\end{aligned}$$

Finally, let $\alpha \in \mathbb{R}$ and $f \in C$ be chosen. Then

$$\begin{aligned}\|\alpha f\| = \max |\alpha f| &= \max |\alpha| |f| \\ &= |\alpha| \max |f| \\ &= |\alpha| \|f\|\end{aligned}$$

where the last equality follows since $\alpha$ is a scalar and not dependent upon taking the max over $[0,1]$.

To show $C$ is complete, let $\{f_n\}$ be a Cauchy sequence on $C$. Let $\varepsilon > 0$ be chosen. Since $\{f_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have that

$$\|f_n - f_m\| < \varepsilon.$$

From how $\|\cdot\|$ is defined, this implies that $|f_n(x) - f_m(x)| < \varepsilon$ for any $x \in [0,1]$. But this means we can always find $n > N$ large enough so that $|f_n - f| < \varepsilon$ i.e., $\{f_n\}$ converges to a function $f$ pointwise. Because $x \in [0,1]$, this means that that this convergence is uniform and so $\|f_n - f\| < \varepsilon$ i.e., $\|f_n - f\| \to 0$. Thus, $f \in C$ and so $C$ is a complete space.

Therefore, having shown that $\|f\| = \max|f|$ is a norm and $C$ is a complete space, $C$ is a Banach space. □

**Problem 3** (5.1)**.** Let $f$ be the function defined

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Find $D^+f(0)$, $D_+f(0)$, $D^-f(0)$, and $D_-f(0)$.

*Proof.* First, we will note that

$$D^+(0) = \frac{f(0+h) - f(0)}{h} = \overline{\lim_{h\to 0^+}} \sin\left(\frac{1}{h}\right) = 1.$$

Similarly,

$$D^-(0) = \frac{f(0) - f(0-h)}{h} = \overline{\lim_{h\to 0^+}} \sin\left(\frac{1}{h}\right) = 1.$$

However, this flips once we look at the limit inferior i.e.,

$$D_+(0) = \underline{\lim_{h\to 0^+}} \sin\left(\frac{1}{h}\right) = -1$$

and

$$D_-(0) = \frac{f(0) - f(0-h)}{h} = \overline{\lim_{h\to 0^+}} \sin\left(\frac{1}{h}\right) = -1..$$

□