Section 4.5 Convergence in Measure

Definition. Let $\{f_n\}$ be a sequence of measurable functions. We say $\{f_n\}$ **converges to f in measure**, $f_n \overset{m}{\to} f$, if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such for all $n > N$,

$$m\{x : |f_n(x) - f(x)| \geq \varepsilon\} < \varepsilon.$$

Note two things from this definition:

(1) If $f_n \to f$ pointwisely over E with $m(E) < \infty$, then $f \overset{m}{\to} f$.

(2) So there exists examples with $f_n \overset{m}{\to} f$ but $f_n \not\to f$.

Proposition (4.18). Let $\{f_n\}$ be a sequence of measurable functions. Suppose $f_n \overset{m}{\to} f$. Then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ almost everywhere.

Proof. Suppose $f_n \overset{m}{\to} f$. Then given $\nu \in \mathbb{N}$, there exists $n_\nu \in \mathbb{N}$ such that for all $n > n_\nu$,

$$m\left\{x : |f_n(x) - f(x)| \geq \frac{1}{2^\nu}\right\} < \frac{1}{2^\nu} \text{ for all } \nu > k.$$

Define the set

$$E_\nu = \left\{x : |f_n(x) - f(x)| \geq \frac{1}{2^\nu}\right\}.$$

Then if $x \notin \bigcup_{\nu=k}^{\infty} E_\nu$ which implies that

$$|f_{\nu_v}(x) - f(x)| < \frac{1}{2^\nu} \quad \text{for all} \quad \nu > k.$$

Then $f_n(x) \to f(x)$ pointwise for all $x \notin A = \bigcap_{k=1}^{\infty} \bigcup_{\nu=k}^{\infty} E_\nu$. Because we are taking the intersection over all k,

$$m(A) \leq m\left(\bigcup_{\nu=k}^{\infty} E_\nu\right) \leq \sum_{\nu=k}^{\infty} m(E_\nu) \leq 2^{-\nu-1}.$$

Because $\nu \in \mathbb{N}$ is given, $m(A) = 0$ and so $f_n(x) \to f(x)$ almost everywhere. □

Corollary (4.19). Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure. Then $f_n \overset{m}{\to} f$ if and only if every subsequence of $\{f_n\}$ has a subsequence that converges almost everywhere to f.

The result aboves follows almost right away from Problem 2.12. Also note that by Problem 4.20 the following is true:

Let $\{f_n\}$ be a sequence of measurable functions. If $f_n \overset{m}{\to} f$, then every subsequence $\{x_{n_k}\} \overset{m}{\to} f$.

Proposition (4.20). Fatou's lemma, the Monotone Convergence Theorem, and the Lebesgue Dominated Convergence Theorem remain valid if $f_n \to f$ almost everywhere is replaced by $f_n \overset{m}{\to} f$.

(1) **Fatou's Lemma**

Proof. Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \overset{m}{\to} f$. Let us pick a subsequence $\{x_{n_k}\}$ such

$$\int f_{n_k} \to \varliminf_{n\to\infty} \int f_n$$

which follows by the definition of the limit inferior. Since $f_{n_k} \overset{m}{\to} f$, by Problem 4.20, there exists a subsequence $\{f_{n_{k_p}}\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_p}} \overset{p\to\infty}{\to} f$ almost everywhere by Proposition 4.18. Then by applying Fatuo's lemma,

$$\begin{aligned} \int f = \lim_{p\to\infty} f_{n_{k_p}} &\leq \varliminf_{n\to\infty} \int f_{n_{k_p}} \\ &= \varliminf_{k\to\infty} \int f_{n_k} \\ &= \varliminf_{n\to\infty} \int f_n \end{aligned}$$

and so the result holds! □

(2) **Lebesgue Dominated Convergence Theorem** Suppose $|f_n| \leq g$ and $f_n \overset{m}{\to} f$. Then

$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof. We claim that to show this result, we must show that we can construct any subsequence $\int f_{n_k}$ such that $\int f_n$ which then implies that

$$\lim_{k\to\infty} \int f_{n_k} = \int f.$$

Because $f_{n_k} \overset{m}{\to} f$, there exists a subsequence $\{f_{n_{k_p}}\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_p}} \to f$ almost everywhere. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{p\to\infty} \int f_{n_{k_p}} = \lim_{k\to\infty} \int f_{n_k} = \int f.$$

Then by Problem 2.12,

$$\lim_{n\to\infty} f_n = \int f$$

which is what we desired to show. □