# Construction of the Real Numbers, $\mathbb{R}$

- We first start from $\mathbb{N} \cup \{0\}$ and add numbers together subsequently (i.e. 1, $\underbrace{1+1}_{2}$, $\underbrace{1+1+1}_{3}$, ...)

- To construct the integers $\mathbb{Z}$, we take the set difference with the natural numbers so that we have
$$\mathbb{Z} = \mathbb{N} \cup \{0\} \setminus \mathbb{N}.$$

- Then the rationals numbers, $\mathbb{Q}$, can be constructed from the integers and are defined by the set
$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \right\}.$$

- To construct the irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$, we can use the dedekind cut to do this. However, this is convoluted and we can go about this in a different way.

## Axioms of the Real Numbers

**A**. **The Field Axioms:** For all real numbers $x, y \in \mathbb{R}$ we have:

A1.  x + y = y + x

A2.  (x + y) + z = x + (y + z)

A3.  There exists $0 \in \mathbb{R}$ such that $x + 0 =$ for all $x \in \mathbb{R}$.
     [**Identity element under addition**]

A4.  For each $x \in \mathbb{R}$ there is a $w \in \mathbb{R}$ such that $x + w = 0$.
     [**Inverse element under addition**]

A5.  xy = yx

A6.  (xy)z = x(yz)

A7.  There exists $1 \in \mathbb{R}$ such that $1 \neq 0$ and $x \cdot 1 = x$ for all $x \in \mathbb{R}$.

A8.  For each $x \in \mathbb{R}$ different from 0 there is $w \in \mathbb{R}$ such that $xw = 1$.

A9.  x(y + z) = xy + xz.

We can prove some properties now:

**Proposition 1.** The additive inverse is unique.

*Proof.* Let $x \in \mathbb{R}$. Suppose we have two numbers $w_1, w_2 \in \mathbb{R}$ such that $x + w_1 = 0 = x + w_2$. Using the axioms and our assumption, we can show the following:

$$
\begin{aligned}
w_1 &= w_1 + 0 & \text{Axiom A3} \\
&= w_1 + x + w_2 & \text{Assumption of } 0 = x + w_2 \\
&= w_2 + x w_1 & \text{Axiom A1} \\
&= w_2
\end{aligned}
$$

which completes the proof. $\qquad\square$

**B**. **The Axioms of Order:** The subset $P$ of positive real numbers satisfies the following:

    B1. If $x, y \in P$, then $x + y \in P$.

    B2. If $x, y \in P$, then $xy \in P$.

    B3. If $x \in P$, then $-x \notin P$.

    B4. If $x \in \mathbb{R}$, then $x = 0$ or $x \in P$ or $-x \in P$.

Note that any system which satisfies the axioms of groups A and B is called an **ordered field**.

**Definition.** We can give definitions of the ordered operations $<, \leq, >$ and $\geq$.

- $x < y$ means that $y - x \in P$.

- $x \leq y$ means that $y - x \in P \cup \{0\}$. Or, this means that $x < y$ or $x = y$.

- $x > y$ means that $x - y \in P$.

- $x \geq y$ means that $x - y \in P \cup \{0\}$. Or, this means that $x > y$ or $x = y$.

From this, we can deduce and prove some which is any set which satisfies the axioms of group A and B.

**Definition.** Let $x, y \in \mathbb{R}$ and define the absolute value as

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

**Proposition 2.** Let $a, b, c \in \mathbb{R}$.

1. $a < b$ if and only if $-b < -a$.

2. If $a < b$ and $b < c$, then $a < c$.

3. If $a < b$ and $c > 0$, then $ac < bc$.

4. For $a, b \in \mathbb{R}$, then only one is true $a = b$, $a > b$ and $a < b$.

5. If $x \neq 0$, then $x^2 = x \cdot x > 0$; in particular, $1 > 0$.

6. If $x, y \in \mathbb{R}$, then $|x + y| \leq |x| + |y|$.

**Definition.** Let $S \subset \mathbb{R}$. The number $b \in \mathbb{R}$ is an **upper bound** for S if for each $x \in S$, we have $x \leq b$.

Similarly, a number $x \in \mathbb{R}$ is the **least upper bound** for $S$ if it is an upper bound for $(S)$ and if $x \leq b$ for each upper bound $b$ of $S$. We then call $x$ the **supremum** of $S$ and denote this $x = \sup S$.

**Definition.** Let $S \subset \mathbb{R}$. The number $l \in \mathbb{R}$ is an **lower bound** for S if for each $x \in S$, we have $l \leq x$.

Similarly, a number $x \in \mathbb{R}$ is the **greatest lower bound** for $S$ if it is a lower bound for $S$ and if $x \leq l$ for each lower bound $l$ of $S$. We then call $x$ the **infimum** of $S$ and denote this $x = \inf S$.

C. **Completeness Axiom:** Every nonempty set $S \subset \mathbb{R}$ which has an upper bound
   has a least upper bound.

**Proposition 3.** Let $L, U \subset \mathbb{R}$ be nonempty subsets with $R = L \cup U$ and such that for
each $l \in L$ and each $u \in U$ we have $l < u$. Then either $L$ has a greatest element or $L$ has
a least element.

**Proposition 4** (Approximation Property.)**.** Let $S \subset \mathbb{R}$ be a nonempty. If $u = \sup S$,
then for all $\gamma > 0$, there exists $Sr \in S$ such that $u - r < Sr < u$.

**Theorem** (2.3, **Axiom of Archimedes**). If $x \in \mathbb{R}$ is any real number, then there exists $n \in \mathbb{N}$ such that $x < n$.

*Proof.* We can break this into two cases

1. Let $x < 1$. If so, then simply choose $x = 1$.

2. Let $x \geq 1$. Define the set $S = \{n \in N : n \leq x\}$. Then since this set is bounded above, by the Completeness Axiom, $\sup S = y$ exists. Because $x$ is an upper bound $S$, by definition of the supremum, we have that $y \leq x$. Let $r = \frac{1}{2}$. Then we can find $k \in S$ such that $y - \frac{1}{2} < k \leq y$. But then we have that $y < y + \frac{1}{2} < k + 1 \leq y + 1$. Then this means $k + 1 \notin S$ and so $x < k + 1$, completing this case.

Having exhausted all cases, this completes the proof. $\qquad\qquad\square$

**Proposition 1** (**Well-Ordering Principle**). Every nonempty subset $S \subset \mathbb{N}$ has a minimum.

**Proposition 2** (**Density of the Rational Numbers**). Let $x, y \in \mathbb{R}$. Then if $x < y$, there exists $q \in \mathbb{Q}$ such that $x < \gamma < y$

# Section 2.4, Sequences in $\mathbb{R}$

**Definition.** We define a **sequence** of real numbers to be a function that maps each each natural number $n$ into the real number $x$. That is, a sequence is a function $s : \mathbb{N} \to A$ for $A \subset \mathbb{R}$. This is written as $\{x_n\}$ or $\{x_n\}_{n=1}^{\infty}$.

**Definition** (**Convergence of a Sequence**). A sequence converges to the real number $l \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - l| < \varepsilon.$$

**Definition** (**Cauchy Sequence**). A sequence $\{x_n\}$ in $\mathbb{R}$ is **Cauchy** sequence if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$|a_n - a_m| < \varepsilon.$$

**Theorem.** Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then $\{x_n\}$ is Cauchy if and only $\{x_n\}$ is Cauchy.

**Definition.** The number $l \in \mathbb{R}$ is called a **cluster point** of $\{x_n\}$ if there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $x_{n_m} \to l$.

We can define this in another way. The number $l \in \mathbb{R}$ is called **cluster point** of $\{x_n\}$ if for all $\varepsilon > 0$ and for all $N \in \mathbb{N}$, there exists $n \geq N$ such that $|x_n - l| < \varepsilon$.

**Definition.** We define the **limit superior** of a sequence $\{x_n\}$ in $\mathbb{R}$ to be

$$\overline{\lim}_{n \to \infty} x_n = \inf_n \sup_{k \geq n} x_k.$$

This is also denoted as $\limsup$.

**Theorem.** A number $l \in \mathbb{R}$ is the **limit superior** of the sequence $\{x_n\}$ if and only if

(i) For all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, $x_k < l + \varepsilon$

(ii) For all $\varepsilon > 0$ and for all $n \in \mathbb{N}$, there exists $k \geq n$ such that $x_k > l - \varepsilon$.

**Definition.** We define the **limit inferior** of a sequence $\{x_n\}$ in $\mathbb{R}$ to be

$$\underline{\lim}_{n\to\infty} x_n = \sup_n \inf_{k \geq n} x_k.$$

This is also denoted as $\liminf$.

**Theorem.** A number $l \in \mathbb{R}$ is the **limit inferior** of the sequence $\{x_n\}$ if and only if

(i) For all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, $x_k > l - \varepsilon$

(ii) For all $\varepsilon > 0$ and for all $n \in \mathbb{N}$, there exists $k \geq n$ such that $x_k < l + \varepsilon$.

**Proposition 3.** From the last two definitions, we have the following property.

- $\overline{\lim}_{n\to\infty}$ is the largest cluster point.

- $\underline{\lim}_{n\to\infty}$ is the smallest cluster point.

# Section 2.5, Open and Closed Sets in $\mathbb{R}$

**Definition.** The set $O \subset \mathbb{R}$ is called an **open** set if for all $x \in O$, there exists $\delta > 0$ such that $x - \delta, x + \delta$.

Equivalently, $O$ is an **open** set if for all $x \in O$, there is a $\delta > 0$ such that each $y$ with $|x - y| < \delta$ belongs to $O$.

**Proposition 4.** From this above, we have the following properties:

1. The set $\bigcup_{\alpha} O_\alpha$ is open.

2. The set $\bigcup_{n=1}^{n} O_m$ is open.

**Theorem (Lindelof Theorem).** Every open set in $\mathbb{R}$ is a disjoint union of countable union of open intervals.

*Proof.* This proof is contained on page 42 of Royden. $\qquad\square$

**Definition.** A real number $x \in \mathbb{R}$ is called **point of closure** of a set $E \subset \mathbb{R}$ if for every $\delta > 0$ there exists a $y \in E$ such that $|x - y| < \delta$.

The set of points of closure of $E$ is denoted $\overline{E}$.

**Proposition 5.** If $A \subset B \subset \mathbb{R}$, then $\overline{A} \subset \overline{B}$. Additionally, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

*Proof.* The proof of this is on page 43 of Royden. $\qquad\square$

**Definition.** A set $F \subset \mathbb{R}$ is called a **closed** set if $\overline{F} = F$.

Note that because $F \subset \overline{F}$ always, a set $F$ is closed if $\overline{F} \subset F$—that is, $F$ contains all of its points of closure.

**Proposition 6.** For any set $E$, the set $\overline{E}$ is closed; that is $\overline{\overline{E}} = \overline{E}$.

**Proposition 7.** Let $E \subset \mathbb{R}$. Then $E$ is open if and only if $E^{\complement}$ is closed.

**Definition.** We say that a collection of sets $\mathcal{C}$ is a **cover** of a set $F$ if

$$F \subset \bigcup_{O \in \mathcal{C}} O.$$

The collection $\mathcal{C}$ is a covering of the set $F$.

**Theorem** (**Heine-Borel**). Let $E \subset \mathbb{R}$ be set. Then $E$ is compact if and only if $E$ is closed and bounded.

# Compactness

**Theorem.** Let $E \subset \mathbb{R}$. Then $E$ is compact if and only $E$ is sequentially compact. That is, for every $\{x_n\}$ in $E$, there exists a convergent subsequence $x_{n_m} \to x_0$ in E.

**Theorem.** Let $\{I_n\}$ be a sequence of closed intervals such that $I_{n+1} \subset I_n$. Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

If $[a_n, b_n]$ is an interval and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} = a$, then $\bigcap_{n=1}^{\infty}$.

# Section 2.6, Continuous Functions

**Definition.** Let $E \subset \mathbb{R}$, and let $f : E \to \mathbb{R}$ be a real-valued function. Then $f$ is **continuous** at the point $x = a \in E$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in E$ with $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

Note that we can have continuity in terms of sequences. I will state it as a theorem here even though it was not in lecture because it is important to be able to use on its own.

**Theorem.** Let $f : E \to \mathbb{R}$ be a function with $E \subset \mathbb{R}$. Let $x \in E$ be any point. Then $f$ is continuous at $a$ if and only for every sequence $\{x_n\}$ in E converging to $a$, the sequence $\{f(x_n)\}$ in $f(E)$ (the image of $E$) converges to $f(a)$.

**Proposition.** Let $E \subset \mathbb{R}$ be compact. Let $f : E \to \mathbb{R}$ be continuous real-valued function. Then $f(E)$ is a compact set.

*Proof.* Let $\subset \mathbb{R}$ be a compact and suppose the function $f : E \to \mathbb{R}$ is continuous. To show that $f(E)$ is compact, we will use the Heine-Borel theorem and show that it is closed and bounded. To show that $f(E)$ is closed, suppose we have any sequence $\{f(x_n)\}$ converging to the point $f(a) \in \mathbb{R}$. Additionally, let $\{x_n\}$ be any sequence in $E$. Because $E$ is compact, there exists a subsequence $\{x_{n_m}\}$ which converges to a point $x_0 \in E$. Since $f$ is continuous, by the preceding theorem this means that the sequence $\{f(x_{n_m})\}$ converges to $f(x_0) \in f(E)$. $\qquad \square$

**Proposition (2.17, Extreme Value Theorem).** Let $E \subset \mathbb{R}$ be a compact set, and let $f : E \to \mathbb{R}$ be a continuous function. Then there exists $x_1, x_2 \in E$ such that

$$f(x_1) \leq f(x) \leq (x_2), \text{ for all x} \in E.$$

**Proposition (2.18).** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $f$ is **continuous** if and only if $f^{-1}(O)$ is open for all open sets $O \subset \mathbb{R}$.

**Proposition (2.19).** Let $E \subset \mathbb{R}$, and let $f : E \to \mathbb{R}$ be continuous. Without loss of generality, suppose that $f(a) \leq f(b)$. Then for all $\gamma \in [f(a), f(b)]$, there exists $c \in [a, b]$ such that $f(c) = \gamma$.

**Definition (Uniform Continuity).** Let $E \subset \mathbb{R}$. A function $f : E \to \mathbb{R}$ is **uniformly continuous** if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

**Proposition (2.20).** Let $E \subset \mathbb{R}$ be a compact set. If $f : E \to \mathbb{R}$ is a continuous function on $E$, then $f$ is uniformly continuous on $E$.

**Definition.** Let $f_n : E \to \mathbb{R}$ be a sequence of functions, and let $f : E \to \mathbb{R}$.

1. The sequence $\{f_n\}$ **converges pointwise** on $E$ to $f$ if for all $\varepsilon > 0$ and for all $x \in E$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f(x) - f_n(x)| < \varepsilon$.

2. The sequence $\{f_n\}$ **converges uniformly** if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in E$ and for all $n \geq N$, $|f(x) - f_n(z)| < \varepsilon$.

# Section 3.1, Lebesgue Measure

[Perhaps finish these notes another time...]

# Section 3.2, Outer Measure

**Definition.** The **outer measure** $m^*(A)$ of a set $A \subset \mathbb{R}$ is given

$$m^*(A) = \inf_{A \subset \bigcup I_n} \sum_n l(I_n)$$

where $\{I_n\}$ is a countable collection of open intervals that cover $A$.

Note that from this definition, we get that

1. $m^*(\emptyset) = 0$

2. If $A \subset B$, $m^*(A) < m^*(B)$.

3. $m^*$ does not satisfy disjoint additivity.

**Proposition (3.1).** The outer measure of an interval is its length; that is, $m^*(I) = l(I)$ where $I = [a,b], (a,b), [a,b)$, or $(a,b]$.

*Proof.* It is sufficient to show that $m^*([a,b]) = l([a,b])$ since every other interval is a subset of $[a,b]$. Let $\varepsilon > 0$. Then $[a,b] \subset \left[a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right]$ which implies, by the definition of the outer measure,

$$m^*([a,b]) \leq l\left(\left[a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right]\right) = b - a + \varepsilon.$$

Because $\varepsilon$ was fixed, this means that $m^* \leq b - a$.

Now we must show that $m^* \geq b - a$. Because $[a,b]$ is compact, for any collection $\{I_n\}$ of open intervals covering $[a,b]$, there exists a finite collection of intervals $\{I_1, \ldots, I_k\}$ so that

$$[a,b] \subset \bigcup_{n=1}^{k} I_n.$$

This gives us that

$$\sum_n I_n \geq \sum_{n=1}^{k} I_n \geq b - a$$

and so $b - a$ is a lower bound. But since $m^*$ as the greatest lower bound of all such sums, we have that $m^* \geq b - a$.

Therefore, $m^*([a,b]) = l([a,b]) = b - a$.

$\square$

**Proposition (3.2, Subadditivity).** Let $\{A_n\}$ be a countable collection of sets on $\mathbb{R}$. Then

$$m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n).$$

*Proof.* Proof on page 57.

$\square$

**Corollary (3.3).** If $A$ is a countable set, then $m^*(A) = 0$.

*Proof.* Proof is on the end of page 57.

$\square$

# Section 3.3, Measurable Sets and Lebesgue Measure

**Definition.** A set $E \subset \mathbb{R}$ is (Lebesgue) **measurable** if for all sets $A$, we have that

$$m^*(A) = m^*(A \cup E) + m^*(A \cup E^{\complement}).$$

**Lemma (3.6).** If $m^*(E) = 0$, then $E$ is measurable.

*Proof.* Let $A$ be any chosen set. Because $A \cap E \subset E$ and $m^*(E) = 0$,

$$m^*(A \cap E) \leq m(E) = 0.$$

Note that $A \cap E^{\complement} \subset A$ and so $m^*(A) \leq m^*(A \cap E^{\complement})$ and so it suffices to show that $m^*(A) \geq m^*(A \cap E^{\complement})$. Using this, we can show that

$$m^*(A) \geq m^*(A \cap E^{\complement}) + 0 = m^*(A \cap E^{\complement}) = m^*(A \cap E)$$

giving us the desired result. $\square$

**Definition.** Let $\mathcal{M}$ be the set of measurable sets in $\mathbb{R}$

**Lemma (3.7).** If $E_1$ and $E_2$ are measurable sets, then so is $E_1 \cup E_2$.

*Proof.* Proof on top of page 57. $\square$

**Corollary (3.8).** The family $\mathcal{M}$ of measurable sets if an algebra of sets. In other words, if $E \in \mathcal{M}$, then $E^{\complement} \in \mathcal{M}$. Further, if $E_1, E_2 \in \mathcal{M}$, then $E_1 \cup E_2 \in \mathcal{M}$.

**Lemma (3.9).** Let $A$ be any set, and $E_1, \ldots, E_n$ be a finite sequence of sets such that $E_i \cap E_j$ for all $i \neq j$. Then

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

*Proof.* We proceed by induction. For $n = 1$, we have the set $E_1$ and the equality holds. Suppose that we have $n = k$ sets $E_1, \ldots, E_k$ with $E_i \cap E_j \neq \emptyset$ for all $i \neq j$ so that

$$m^*\left(A \cap \left[\bigcup_{i=1}^{k} E_i\right]\right) = \sum_{i=1}^{k} m^*(A \cap E_i).$$

Consider $n = k + 1$. Because each $E_i$ is disjoint,

$$A \cap \left(\bigcup_{i=1}^{k+1} E_i\right) \cap E_{k+1} = A \cap E_{k+1};$$

$$A \cap \left(\bigcup_{i=1}^{k+1} E_i\right) \cap E_{k+1}^{\complement} = A \cap \bigcup_{i=1}^{k} E_i.$$

Because the $E_i$'s are measurable,

$$m^*\left(A \cap \bigcup_{i=1}^{k+1} E_i\right) = m^*(A \cap E_{k+1}) + m^*\left(A \cap \bigcup_{i=1}^{k} E_i\right)$$

$$= m^*(A \cap E_{k+1}) + \sum_{i=1}^{k} m^*(A \cap E_i) \qquad \text{Induction Hypothesis}$$

$$= \sum_{i=1}^{k+1} m^*(A \cap E_i)$$

which, by induction, completes the proof. $\qquad \square$

**Theorem (3.10).** $\mathcal{M}$ is a $\sigma$-algebra. In other words, in addition to being an algebra of sets, if $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

*Proof.* [1]

$\square$

**Lemma (3.11).** The interval $(a, \infty)$ is measurable for all $a \in \mathbb{R}$.

*Proof.* [2]

$\square$

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[1] Proof on bottom of page 59 and top of page 60.
[2] Proof on the bottom of page 60 through the middle of page 61.

**Theorem (3.12).** Every Borel set is measurable. In particular, each open set and each closed set is measurable.

*Proof.* [3]

$\square$

**Definition.** Let $E \in \mathcal{M}$. We define $m(E) := m^*(E)$ to be the **Lebesgue measure** of $E/$

**Proposition (3.13, Countable Additivity).** Let $\{E_i\}_{i=1}^n$ be a sequence of measurable sets. Then

$$m \left( \bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{n} m(E_i).$$

If, in addition, $E_i \cap E_j$ for all $i \neq j$. then

$$m \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{n} m(E_i).$$

**Proposition (3.14).** Let $\{E_i\} \subset \mathcal{M}$ be a decreasing sequence (i.e., $E_{i+1} \subset E_i$). Let $m(E_1) < \infty$. Then

$$m \left( \bigcap_{n=1}^{\infty} E_n \right) = \lim_{n \to \infty} m(E_n).$$

**Proposition (3.15).** Let $E$ be any given set. Then the following are equivalent:

  (i)  $E$ is measurable.

 (ii)  For all $\varepsilon > 0$, there is an open set $O \supset E$ with $m^*(O \setminus E) < \varepsilon$.

(iii)  For all $\varepsilon > 0$, there is a closed set $F \subset E$ with $m^*(E \setminus F) < \varepsilon$.

(iv)  There is a $G \in G_\delta$ with $E \subset G$ such that $m^*(G \setminus E) = 0$.

 (v)  There is a $F \in F_\sigma$ with $F \subset E$ such that $m^*(E \setminus F) = 0$.

       If $m^*(E) < \infty$, the above statements are equivalent:

(vi)  For all $\varepsilon > 0$, there is a finite union $U$ of open intervals such that $m^*(U \Delta E) < \varepsilon$.

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[3]Proof on the bottom of page 61.

# Section 3.5, Measurable Functions

**Proposition (3.18).** Let $E \subset \mathbb{R}$, and Let $f : E \to [-\infty, \infty]$ be an extended real-valued function whose domain is measurable. Let $\alpha \in \mathbb{R}$ be any real number. Then the following statements are equivalent:

(i) The set $\{x : f(x) > \alpha\}$ is measurable.

(ii) The set $\{x : f(x) \geq \alpha\}$ is measurable.

(iii) The set $\{x : f(x) < \alpha\}$ is measurable.

(iv) The set $\{x : f(x) \leq \alpha\}$ is measurable.

All together, these imply

(v) The set $\{x : f(x) = \alpha\}$ is measurable.

*Proof.* [1]

$\square$

**Definition.** An extended real-valued function $f : E \to [-\infty, \infty]$ is **(Lebesgue) measurable** if its domain is measurable and satisfies one of the first four statements of Proposition 18.

Note that this means that any continuous function is measurable since then the pre-image of any open set is still an open set.

**Proposition (3.19).** Let $f$ and $g$ be two measurable functions defined on the same domain, and let $c \in \mathbb{R}$. Then the functions $f + c$, $cf$, $f + g$, $g - f$, and $fg$ are measurable.

*Proof.* Let $\alpha \in \mathbb{R}$ be any real number. Fix $c \in \mathbb{R}$. For $f(x) + c$, note that

$$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}$$

and $\alpha - c$ is a real number, this set is still measurable i.e., $f + c$ is measurable. A similar argument shows that $cf$ is measurable as well.

Take the set

$$\{x : f(x) + g(x) < \alpha\}. \tag{1}$$

Observe that $f(x) + g(x) < \alpha \Leftrightarrow f(x) < \alpha - g(x)$. By the density of $\mathbb{Q}$, there exists $r \in \mathbb{Q}$ such that $f(x) < r < \alpha - g(x)$. So we can write Equation (1) as

$$\{x : f(x) + g(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} \left( \{x : f(x) < r\} \cap \{x : g(x) < \alpha - r\} \right).$$

Because this set is countable, this set is measurable and thus $f + g$ is measurable.

To show that $fg$ is measurable, we can show that $f^2$ is measurable since

$$fg = \frac{1}{2} \left( (f + g)^2 - f^2 - g^2 \right).$$

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[1] Proof is on page 67.

Take the set

$$\{x : f^2(x) < \alpha\}. \tag{2}$$

For $\alpha \geq 0$, note that $f^2 < \alpha$ is the same as saying $f(x) > \sqrt{\alpha}$ and $f(x) < -\sqrt{\alpha}$. Thus, Equation (2) can rewritten as

$$\{x : f^2(x) < \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}.$$

This is a measurable set which completes the proof. □

**Theorem (3.20, Limit of Measurable Functions is Measurable).** [2]

*Proof.* For $f(x) + c$, note that

$$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}.$$

□

**Theorem (3.20, Limit of Measurable Functions is Measurable).** Let $\{f_n\}$ be a sequence of measurable functions with the same domain. Then the functions $\sup\{f_1(x), \ldots, f_n(x)\}$, $\inf\{f_1(x), \ldots, f_n(x)\}$, $\sup_n f_n$, $\inf_n f_n$, $\overline{\lim} f_n$, and $\underline{\lim} f_n$ are measurable.

*Proof.* Let $\{f_n\}$ be a sequence of measurable functions. Let $h(x) = \sup\{f_1(x), \ldots, f_n(x)\}$ and we so must show that $\{x : h(x) < \alpha\}$ for all $\alpha \in \mathbb{R}$. To that end, let $\alpha \in \mathbb{R}$ be chosen. Then

$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$$

which, because the right-hand side is a union of measurable sets from the $f_i$'s being measurable, means that the set $\{x : h(x) < \alpha\}$ is also measurable.

Let $g(x) = \sup_n f_n$. By a similar argument as above,

$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i(x) > \alpha\}$$

is a countable set so $\{x : g(x) < \alpha\}$ is measurable. □

**Definition.** A property is said to hold **almost everywhere** (a.e) if the set of points where it fails to hold is a set of measure zero. Thus $f = g$ a.e if $f$ and $g$ have the same domain and $m\{x : f(x) \neq g(x)\}$.

**Proposition (3.21).** If $f$ is measurable and $f = g$ a.e, then $g$ is measurable.

*Proof.* [3] Let $E = \{x : f(x) \neq g(x)\}$.

This is equivalent to saying that

Let $\{x : g(x) > \alpha\}$. This is equivalent to saying that

$$\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha\}$$

□

---

[2]Proof is on bottom of page 68 and top of page 69
[3]Proof is on middle of page 69.

The following proposition essentially says that measurable functions are nearly continuous; or. in other words, we can "nicely" approximate measurable functions.

**Proposition** (**3.22, Littlewood's 2nd Principle**). Let $f : [a, b] \to E$ be a measurable function with $E \subset \mathbb{R}$ and is equal to $\pm\infty$ only on sets with measure zero. Then for all $\varepsilon > 0$, there exist a step function $g$ and a continuous function $f$ such

$$|f - g| < \varepsilon \quad \text{and} \quad |f - h| < \varepsilon$$

except on set of measure less than $\varepsilon$; i.e., $m\{x : |f(x) - g(x)| \geq \varepsilon\} < \varepsilon$ and $m\{x : |f(x) - h(x)| \geq \varepsilon\} < \varepsilon$. If in addition $m \leq f \leq M$, then we may choose the functions $g$ and $h$ so that $m \leq g \leq M$ and $m \leq h \leq M$.

**Proposition** (**3.23, (Weak) Egonoff's Theorem**). Let $E$ be a measurable set of finite measure, and $\{f_n\}$ be a sequence of measurable functions defined on $E$. Let $f$ be real-valued function such for each $x \in E$ we have $f_n(x) \to f(x)$. Then for all $\varepsilon > 0$ and all $\delta > 0$, there is measurable set $A \subset E$ with $m(A) < \delta$ and $N \in \mathbb{N}$ such that for all $x \notin A$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$

**Proposition (3.23, (Weak) Egonoff's Theorem).** Let $E$ be a measurable set of finite measure, and $\{f_n\}$ be a sequence of measurable functions defined on $E$. Let $f$ be real-valued function such for each $x \in E$ we have $f_n(x) \to f(x)$. Then for all $\varepsilon > 0$ and all $\delta > 0$, there is measurable set $A \subset E$ with $m(A) < \delta$ and $N \in \mathbb{N}$ such that for all $x \notin A$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$

*Proof.* [1]

Let $\varepsilon > 0$ be chosen. Define the set

$$G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}$$

and also

$$E_N = \bigcup_{n=N}^{\infty} G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N\}.$$

Because $\{E_N\}$ is a decreasing sequence and $f_n(x) \to f(x)$ pointwise, for all $x \in E$, $|f_n(x) - f(x)| < \varepsilon$ and so $\bigcup_{i=1}^{\infty} E_N = \emptyset$. Thus, by Proposition 3.14,

$$E_N = \emptyset \implies m(E_N) = 0$$
$$= m\left(\bigcup_{N=1}^{\infty} E_N\right)$$
$$= \lim_{N \to \infty} E_N.$$

So for any $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $m(E_N) < \delta$. Now take $A = E_N$ for any $N > N_0$ and so $m(A) < \delta$ and also

$$A^{\complement} = \{x \in E : x \notin E\} = \{x \in E : |f_n(x) - f(x)| < \varepsilon \text{ for all } n > N_0\}.$$

$\square$

# Section 4.1 Riemann Integration

**Definition.** Let $f : [a, b] \to \mathbb{R}$ be a bounded real-valued function and let

$$P = \{a = \xi_0 < \xi_1 < \cdots < \xi_n = b\}$$

be a subdivision (partition) of $[a, b]$. We can define the **upper sum**, $S$ and **lower sum**, $s$, respectively, as

$$S = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) M_i \qquad \text{and} \qquad s = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) m_i$$

where

$$M_i = \sup \{f(x) : x \in [\xi_{i-1}, \xi_i]\} \qquad \text{and} \qquad m_i = \inf \{f(x) : x \in [\xi_{i-1}, \xi_i]\}.$$

---

[1]Proof on page 72-73.

**Definition.** Let $f : [a, b] \to \mathbb{R}$ be a bounded real-valued function. Define the **upper Riemann integral** of $f$ as

$$R\overline{\int_a^b} f(x)\, \mathrm{d}x = \inf\{S : P \text{ is a partition of } [a, b]\}$$

and the **lower Riemann integral** of $f$ as

$$R\underline{\int_a^b} f(x)\, \mathrm{d}x = \sup\{s : P \text{ is a partition of } [a, b]\}.$$

**Definition.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f$ is Riemann integrable if

$$R\underline{\int_a^b} f(x)\, \mathrm{d}x = R\int_a^b f(x)\, \mathrm{d}x = R\overline{\int_a^b} f(x)\, \mathrm{d}x.$$

In other words, if the upper and lower Riemann integrals are equal to each other.

Note this theorem is *NOT* from Royden but from my real analysis course in undergraduate. It comes from the definition of the upper and lower Riemann integral definitions by ends up being a useful way to show Riemann integration.

**Theorem.** Let $f$ be a bounded function on $[a, b]$. Then $f$ is Riemann integrable if and only if for all $\varepsilon > 0$, there exists a subdivision (partition) $P$ of $[a, b]$ such that

$$S - s < \varepsilon.$$

# Section 4.2 The Lebesgue Integral

**Definition.** The **characteristic function** of $E$ is defined as

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

A linear combination

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

is called a **simple function** if the sets $\{E_1, \ldots, E_n\}$ are measurable. Note that $\phi$ is simple if and only if it is measurable and only assumes a finite number of values.

The **canonical representation** of $\phi$ is such that

$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$$

where $A_i = \{x : \phi(x) = a_i\}$ and where the $A_i$'s are disjoint and the $a_i$'s are distinct and nonzero.

**Definition.** Let

$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$$

be a simple function which vanishes outside of a set of finite measure. Then the integral of $\phi$ is defined as

$$\int \phi = \sum_{i=1}^{n} a_i \cdot m(A_i).$$

**Lemma (4.1).** Let $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ with $E_i \cap E_j = \emptyset$ for all $i \neq j$ where $E_i \in \mathfrak{M}$ and $m(E_i) < \infty$ for each $i = 1, \ldots n$. Then

$$\int \phi = \sum_{i=1}^{n} a_i \cdot m(E_i).$$

**Proposition (4.2).** Let $\phi, \psi$ be simple functions vanishing outside of a set with finite measure. Then

$$\int (a\phi + b\psi) = a \int \phi + b \int \psi.$$

If $\phi \geq \psi$ almost everywhere,

$$\int \phi \geq \int \psi.$$

**Proposition (4.3).** Let $E \in \mathfrak{M}$, and let $f : E \to \mathbb{R}$ with $m(E) < \infty$ be a bounded, measurable function. Let $\phi, \psi$ be simple functions. Then

$$inf \left\{ \int \psi : \psi \geq f \right\} = \sup \left\{ \int \phi : \phi \leq f \right\}.$$

if and only if $f$ is measurable.

*Proof.* [2]

$\square$

---

[2]Proof is on pages 79-80.

**Proposition (4.3).** Let $E \in \mathfrak{M}$, and let $f : E \to \mathbb{R}$ with $m(E) < \infty$ be a bounded, measurable function. Let $\phi, \psi$ be simple functions. Then

$$\inf\left\{\int \psi : \psi \geq f\right\} = \sup\left\{\int \phi : \phi \leq f\right\}.$$

if and only if $f$ is measurable.

*Proof.* [1] We will need to show two implications.

($\Leftarrow$) First, suppose that $f$ is measurable. Fix any $n \in \mathbb{N}$ and define the set

$$E_k = \left\{x : \frac{k-1}{n}M < f(x) \leq \frac{kM}{n}\right\}$$

with $k \in [-n, n]$ and $|f(x)| < M$. Note that because $f$ is measurable, each $E_k$ is a measurable set and also we have that $\bigcup_{k=-n}^{n} E_K = E$. Define the upper and lower sequence of simple functions, $\{\psi_n\}$ and $\{\phi_n\}$, respectively, as

$$\psi_n(x) = \sum_{k=-n}^{n} \frac{M}{n} \cdot k \cdot \chi_{E_k}(x) \quad \text{and} \quad \phi_n = \sum_{k=-n}^{n} \frac{M}{n} \cdot (k-1) \cdot \chi_{E_k}(x).$$

So for any $x \in E$, $\phi(x) \leq f(x) \leq \psi(x)$. Thus,

$$\inf_{\psi \geq f} \int_E \psi \leq \int_E \psi = \frac{M}{n} \sum_{k=-n}^{n} k \cdot m(E_k)$$

and

$$\sup_{\phi \leq f} \int_E \phi \geq \int_E \phi = \frac{M}{n} \sum_{k=-n}^{n} (k-1) \cdot m(E_k).$$

Putting these two inequalities together, we can show that

$$\inf_{\psi \geq f} \int_E \psi - \sup_{\phi \leq f} \int_E \phi \leq \int_E \psi - \phi$$

$$= \sum_{k=-n}^{n} (\psi_n - \phi_n)\, m(E_k)$$

$$= \frac{M}{n} m(E).$$

Since $n \in \mathbb{N}$ is fixed, this quantity is zero. Thus

$$\inf_{\psi \geq f} \int_E \psi - \sup_{\phi \leq f} \int_E \phi = 0 \Rightarrow \inf_{\psi \geq f} \int_E \psi = \sup_{\phi \leq f} \int_E \phi$$

completing this direction.

---

[1] Proof is on pages 79-80.

($\Rightarrow$) Conversely, suppose that

$$\inf_{\psi \geq f} \int_E \psi = \sup_{\phi \leq f} \int_E \phi.$$

By definition of the infimum and supremum, there exists sequences of simple functions $\{\phi_n\}$ and $\{\psi_n\}$ such that $\phi_n \leq f \leq \psi_n$ for all $n \in \mathbb{N}$ with

$$\int_E \phi_n - \psi_n < \frac{1}{n}.$$

Define $\phi^* = \sup_n \phi_n$ and $\psi^* = \inf_n \psi(n)$. Since simple functions are measurable functions, by Proposition 3.20, $\phi^*$ and $\psi^*$ are measurable as well and $\phi_n \leq f \leq \psi_n$. We claim that $f = \phi^*$ a.e. Consider the set

$$\Delta = \{x : \phi^*(x) < \psi^*(x)\}.$$

Let $\nu \in \mathbb{N}$ and let

$$\Delta_\nu = \left\{ x : \phi^*(x) < \psi^*(x) - \frac{1}{\nu} \right\}$$

which means that

$$\Delta = \bigcup_{\nu \in \mathbb{N}} \Delta_\nu.$$

For any $n \in \mathbb{N}$,

$$\Delta_\nu \subset \left\{ x : \phi(x) < \psi(x) - \frac{1}{\nu} \right\}.$$

Thus, we have that, for any $n \in \mathbb{N}$,

$$m(\Delta_\nu) = \int \chi_{\Delta_\nu} = \nu \int \frac{1}{\nu} \cdot \chi_{\Delta_\nu}$$
$$\leq \nu \int_{\Delta_\nu} (\psi_n - \phi_n)$$
$$< \mu \int_E \frac{1}{n}$$
$$= \frac{\nu}{n} m(E).$$

Because $\nu$ is fixed and $n$ is arbitrary, $m(\Delta_\nu) = 0$ which implies that $m(\Delta) = 0$. So then $\phi^* = \psi^*$ except on a set of measure zero, and $\phi^* = f$ except on a set of measure zero i.e., $f = \phi^*$ a.e. implying that $f$ is measurable by Proposition 3.21.

Having completed both directions, this completes the proof and shows the well-definedness of Lebesgue integration. $\qquad\square$

**Proposition (4.4).** Let $f$ be a bounded function defined on $[a, b]$. If $f$ is Riemann integrable, then it is measureble and

$$R \int_a^b f(x)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x.$$

*Proof.* This proof is on page 82 of Royden (very simple proof, in fact). $\qquad\square$

**Proposition (4.5).** If $f$ and $g$ are bounded measurable functions defined on a set $E$ of finite measure, then:

i. For any $a, b \in \mathbb{R}$,
$$\int_E (af + bg) = a \int_E f + b \int_E g.$$

ii. If $f = g$ a.e., then
$$\int_E f = \int_E g.$$

iii. If $f \leq g$ almost everywhere. then
$$\int_E f \leq \int_E g.$$

Hence
$$\left| \int_E f \right| \leq \int_E |f| \,.$$

iv. If $A \leq f(x) \leq B$, then
$$Am(E) \leq \int_E f \leq Bm(E).$$

v. If $A$ and $B$ are disjoint measurable sets of finite measure

**Proposition (4.6, Bounded Convergence Theorem).** Let $\{f_n\}$ be a sequence of measurable functions defined over a measurable set $E$ of finite measure. Suppose there is $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for all $x \in E$ and for all $n \in \mathbb{N}$. If $f_n(x) \to f(x)$ pointwise (i.e., $\lim_{n \to \infty} f_n = f(x)$), then

$$\int_E f_n \to \int_E f \Leftrightarrow \lim_{n \to \infty} \int_E f_n(x) = \int_E f.$$

*Proof.* Let $\varepsilon > 0$ be chosen. By Proposition 3.23 (Weak Ergoroff's Theorem), there exists $N \in \mathbb{N}$ and $A \subset E$ with
$$m(A) = \tilde{\delta} < \frac{\varepsilon}{4M}$$
such that that for all $x \in E \setminus A$ and for all $n > N$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2 \cdot m(E)} = \tilde{\varepsilon}.$$

Thus, we can show the following:

$$\left| \int_E f_n - \int_E f \right| = \left| \int_E f_n - f \right| \leq \int_E |f_n - f|$$

$$= \int_A |f_n - f| + \int_{E \setminus A} |f_n - f|$$

$$\leq 2M \cdot m(A) + \int_{E \setminus A} |f_n - f|$$

$$\leq 2M \cdot m(A) + \frac{\varepsilon \cdot m(E \setminus A)}{2 \cdot m(E)}$$

$$< 2M \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon \cdot m(E \setminus A)}{2 \cdot m(E)}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

and so we are done!!                                                                 $\square$

**Proposition** (**4.7**). Let $f$ be bounded on $[a, b]$. Then $f$ is Riemann integrable if and only the set of discontinuities has measure zero.

# Section 4.3 Integral of Nonnegative Functions

**Definition.** Let $f \geq 0$ be measurable, and $E$ be a measurable set. The **Lebesgue integral** of $f$ over is defined by

$$\int_E f := \sup_{h \leq f} \int_E h$$

where $h$ is a bounded measurable function and $m\{x : h(x) \neq 0\} < \infty$.

**Proposition (4.8).** If $f$ and $g$ are nonnegative measurable functions, then:

i. For all $c > 0$,
$$\int_E cf = c\int_E f.$$

ii.
$$\int_E f + g = \int_E f + \int_E g.$$

iii. If $f \leq g$ a.e, then
$$\int_E f \leq \int_E g.$$

*Proof.* [1] Note that (i) and (iii) come from Proposition 4.5. So for (ii), we first claim that

$$\int_E f + g \leq \int_E f + \int_E g.$$

Let $h \leq f$ be a bounded, measurable function with $m\{x : h \neq 0\} < \infty$, and let $k \leq g$ be a bounded, measurable function with $m\{x : k \neq 0\} < \infty$. Then $h + k \leq f + g$ and

$$\{x : h + k \neq 0\} = \{x : h \neq 0\} \cup \{x : k \neq 0\}$$

and so $m\{x : h + k \neq 0\} < \infty$. By definition of the Lebesgue integral (which is a sup), we have the following:

$$\int_E f + g \geq \int_E h + k = \int_E h + \int_E k$$
$$\geq \int_E h + \int_E g$$
$$\geq \int_E h + \int_E k$$
$$\geq \int_E f + \int_E g.$$

For the other direction, let $l \leq f + g$ be a bounded, measurable function and $m\{x : l(x) \neq 0\} < \infty$. Define $h(x) = \min\{f(x), l(x)\} \leq l(x)$ and so $h(x)$ is bounded as well. Then

$$\int_E f + \int_E g \geq \int_E + \int_E k = \int_E h + k = \int_E l$$

---

[1] Proof is on page 86 of Royden.

which, by definition of the Lebesgue integral,

$$\int_E f + \int_E g \leq \int_E f + g.$$

$\square$

**Theorem (4.9, Fatou's Lemma).** If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ pointwise almost everywhere on a set $E$, then

$$\int_E f \leq \varliminf_{n\to\infty} \int_E f_n.$$

*Proof.* Without loss of generality, suppose that $f_n(x) \to f(x)$ on E (because the integrals over sets of measure zero are zero.) Suppose that $h \leq f$ is a bounded, measurable function and define $E' = \{x : h(x) \neq 0\}$ and so $m(E') < \infty$. Define $h_n(x) = \min\{h(x), f_n(x)\}$ and so $h_n(x) \to h(x)$ pointwise on $E'$ and $h_n \leq h \leq f_n \leq f$ and so $\{h_n\}$ is a bounded sequence. So, by the Bounded Convergence Theorem,

$$\int_E h = \int_{E'} h$$
$$= \lim_{n\to\infty} \int_E h_n \qquad \text{Bounded Convergence Theorem}$$
$$\leq \varliminf_{n\to\infty} \int_E f_n.$$

Taking the sup over $h$,[2]

$$\int_E f \leq \varliminf_{n\to\infty} \int_E f_n.$$

$\square$

**Theorem (4.10, Monotone Convergence Theorem).** Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions, and let $f = \lim_{n\to\infty} f_n$ almost everywhere. Then

$$\int f = \lim_{n\to\infty} \int f_n.$$

*Proof.* By Fatou's Lemma,

$$\int f \leq \varliminf_{n\to\infty} \int f_n.$$

So we just need the other direction for equality. Because $\{f_n\}$ is increasing and converges to $f$, $f_n \leq f$ for each $n \in \mathbb{N}$ and thus

$$\int f_n \leq \int f.$$

But then the limit inferior is less than than this quantity i.e.,

$$\varliminf_{n\to\infty} \int f_n \leq f$$

---

[2]Wait, clarify what this means...

and so

$$\lim_{n\to\infty} \int f_n = \int f.$$

$\square$

**Corollary (4.11).** Let $\{u_n\}$ be a sequence of nonnegative measurable functions, and let $f(x) = \sum_{i=1}^{n} u_n(x)$. Then

$$\int f = \sum_{i=1}^{n} \int u_n.$$

**Proposition (4.12).** Let $f$ be a nonnegative function and $\{E_i\}$ a disjoint sequence of measurable sets. Let $E = \bigcup_{i=1}^{\infty} E_i$. Then

$$\int_E f = \sum_{i=1}^{\infty} \int_{E_i} f.$$

**Definition.** Let $f \geq 0$ be a nonnegative measurable function. We say that $f$ is **Lebesgue measurable** over $E$ if

$$\int_E f \leq \infty.$$

**Proposition (4.13).** Let $f$ and $g$ be two nonnegative measurable functions. If $f$ is integrable over $E$ and $g(x) \leq f(x)$ on $E$, then $g$ is also integrable on $E$ and,

$$\int_E f - g = \int_E f - \int_E g.$$

*Proof.* Note that $f - g \geq 0$ on $E$ so we can write this as the sum of two nonnegative functions i.e., $f = (f - g) + g$. Thus, since this is the sum of two nonnegative functions, by Proposition 4.8 part (ii), we have that

$$\int_E f = \int_E (f - g) + \int_E g$$
.

Because the integral of $f$ is finite, the right-hand side must also be finite and so $g$ is measurable.[3] $\square$

**Proposition (4.14).** Let $f$ be a nonnegative function which is integrable over $E$. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta$, we have that

$$\int_A f < \varepsilon.$$

---

[3]This proof does not show the explicit formula, though?

*Proof.* Let $\varepsilon > 0$ be chosen. If $f$ is bounded, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$. So set $\delta = \dfrac{\varepsilon}{M}$ and estimate $\displaystyle\int_A f$.

If is unbounded then define
$$f_n(x) = \min\{n, f(x)\}.$$

Then $\{f_n\}$ is an increasing sequence and $f_n \to f$ pointwise (i.e, $f_n \uparrow f$ pointwise). By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Thus there exists $N \in \mathbb{N}$ such that if $n \geq N$,

$$\int_E f - \lim_{N\to\infty} \int_E f_N < \frac{\varepsilon}{2}$$

and thus implying that

$$\lim_{N\to\infty} \int_E f_N > \int_E f + \frac{\varepsilon}{2}.$$

Set $\delta = \dfrac{\varepsilon}{2N}$. Choose a set $A \subset E$ such that $m(A) < \delta$. Then

$$
\begin{aligned}
\int_A f &= \int_A f - f_N + \int_A f_N \\
&< \frac{\varepsilon}{2} + \int_A f_N \\
&= \frac{\varepsilon}{2} + N \cdot m(A) \\
&< \frac{\varepsilon}{2} + N \cdot \frac{\varepsilon}{2N} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

$\square$

# Section 4.4 General Lebesgue Integral

**Definition.** Define

$$
\begin{aligned}
f^+(x) = &\ \text{non-negative part of } f \\
= &\ \max\{f(x), 0\}.
\end{aligned}
$$

and

$$
\begin{aligned}
f^-(x) = &\ \text{non-positive part of } f \\
= &\ \max\{-f(x), 0\}.
\end{aligned}
$$

Note that then

$$
f(x) = f^+(x) - f^-(x)
$$

and

$$
|f(x)| = f^+(x) + f^-(x).
$$

**Definition.** A measurable function $f$ is said to be **Lebesgue integrable** over $E$ if $f^+$ and $f^-$ are integrable. In this case, then

$$
\int_E = \int_E f^+ - \int_E f^-.
$$

**Proposition (4.15).** Let $f, g$ be integrable functions over $E$. Then

    i. $cf$ are integrable for all $c \in \mathbb{R}$ over $E$.

    ii. $f + g$ is integrable and

$$
\int_E f + g = \int_E f + \int_E g
$$

    iii. If $f \leq g$ a.e, then

$$
\int_E f \leq \int_E g.
$$

    iv. If $A, B \subset E$ and $A \cap B = \emptyset$, then

$$
\int_{A \cup B} f = \int_A f + \int_B.
$$

**Proposition (4.16, Lebesgue Convergence Theorem).** Let $g$ be integrable over $E$ and let $\{f_n\}$ be a sequence of measurable functions. Suppose $f_n \to f$ pointwise almost everywhere and $|f_n| \leq g$ on $E$. Then

$$
\lim_{n \to \infty} \int_E f_n = \int_E f.
$$

*Proof.* Since $g - f_n \geq 0$ for all $n \in \mathbb{N}$, Fatou's lemma, $|f_n| \leq g$ on $E$

$$\int g - \int f = \int g - f \leq \varliminf_{n \to \infty} \int g - f_n$$

$$= \varliminf_{n \to \infty} \int f - \varliminf_{n \to \infty} \int f_n$$

$$= \int g - \varliminf_{n \to \infty} \int f_n$$

and so

$$\int g - \int f \leq \int g - \varliminf_{n \to \infty} \int f_n$$

which implies that

$$\int f \geq \varliminf_{n \to \infty} \int f_n.$$

Note that $g + f_n \geq 0$ as well. Then

$$\int g + \int f_n = \int g + f_n \leq \varliminf_{n \to \infty} \int g + f_n$$

$$= \varliminf_{n \to \infty} \int g + \varliminf_{n \to \infty} \int f_n$$

$$= \int g + \varliminf_{n \to \infty} \int f_n$$

implying that

$$\int f \leq \varliminf_{n \to \infty} \int f_n.$$

Because $\{f_n\}$ converges, we know that $\varliminf_{n \to \infty} f_n = \varlimsup_{n \to \infty} f_n$ and so the result follows.     □

**Proposition (4.17, Lebesgue Generalized Dominant Convergent Theorem).** Let $\{g_n\}$ be a sequence of integrable functions and $g_n \to g$ pointwise a.e with $g$ integrable. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g_n$ and $\{f_n\} \to f$ pointwise a.e. If

$$\int g = \lim_{n \to \infty} \int g_n,$$

then

$$\int f = \lim_{n \to \infty} \int f_n.$$

*Proof.* [1] This proof is also similar to Proposition 4.16 but use $g_n$ instead of $g$.

□

**Problem 1 (4.15).** Properties of function $f$ being integrable.

(a) Let $f$ be integrable over $E$. Then for all $\varepsilon > 0$, there exists a simple function $\phi$ such that

$$\int_E |f - \phi| < \varepsilon.$$

---

[1]Proof is on page 92 of Royden.

(b) Let $f$ be integrable over $E$. Then for all $\varepsilon > 0$, there exists a step function $\psi$ such that
$$\int_E |f - \psi| < \varepsilon.$$

(c) Let $f$ be integrable over $E$. Then for all $\varepsilon > 0$, there exists a continuous function $g$ such that
$$\int_E |f - g| < \varepsilon.$$

# Section 4.5 Convergence in Measure

**Definition.** Let $\{f_n\}$ be a sequence of measurable functions. We say $\{f_n\}$ **converges to $f$ in measure**, $f_n \xrightarrow{m} f$, if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such for all $n > N$,

$$m\left\{x : |f_n(x) - f(x)| \geq \varepsilon\right\} < \varepsilon.$$

Note two things from this definition:

(1) If $f_n \to f$ pointwisely over $E$ with $m(E) < \infty$, then $f \xrightarrow{m} f$.

(2) So there exists examples with $f_n \xrightarrow{m} f$ but $f_n \not\to f$.

**Proposition (4.18).** Let $\{f_n\}$ be a sequence of measurable functions. Suppose $f_n \xrightarrow{m} f$. Then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ almost everywhere.

*Proof.* Suppose $f_n \xrightarrow{m} f$. Then given $\nu \in \mathbb{N}$, there exists $n_\nu \in \mathbb{N}$ such that for all $n > n_\nu$,

$$m\left\{x : |f_n(x) - f(x)| \geq \frac{1}{2^\nu}\right\} < \frac{1}{2^\nu} \text{ for all } \nu > k.$$

Define the set

$$E_\nu = \left\{x : |f_n(x) - f(x)| \geq \frac{1}{2^\nu}\right\}.$$

Then if $x \notin \bigcup_{\nu=k}^{\infty} E_\nu$ which implies that

$$|f_{\nu_v}(x) - f(x)| < \frac{1}{2^\nu} \quad \text{for all} \quad \nu > k.$$

Then $f_n(x) \to f(x)$ pointwise for all $x \notin A = \bigcap_{k=1}^{\infty} \bigcup_{\nu=k}^{\infty} E_\nu$. Because we are taking the intersection over all $k$,

$$m(A) \leq m\left(\bigcup_{\nu=k}^{\infty} E_\nu\right) \leq \sum_{\nu=k}^{\infty} m(E_\nu) \leq 2^{-\nu-1}.$$

Because $\nu \in \mathbb{N}$ is given, $m(A) = 0$ and so $f_n(x) \to f(x)$ almost everywhere. $\square$

**Corollary (4.19).** Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set $E$ of finite measure. Then $f_n \xrightarrow{m} f$ if and only if every subsequence of $\{f_n\}$ has a subsequence that converges almost everywhere to $f$.

The result aboves follows almost right away from Problem 2.12. Also note that by Problem 4.20 the following is true:

Let $\{f_n\}$ be a sequence of measurable functions. If $f_n \xrightarrow{m} f$, then every subsequence $\{x_{n_k}\} \xrightarrow{m} f$.

**Proposition (4.20).** Fatou's lemma, the Monotone Convergence Theorem, and the Lebesgue Dominated Convergence Theorem remain valid if $f_n \to f$ almost everywhere is replaced by $f_n \xrightarrow{m} f$.

(1) **Fatuo's Lemma**

*Proof.* Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \xrightarrow{m} f$. Let us pick a subsequence $\{x_{n_k}\}$ such

$$\int f_{n_k} \to \varliminf_{n\to\infty} \int f_n$$

which follows by the definition of the limit inferior. Since $f_{n_k} \xrightarrow{m} f$, by Problem 4.20, there exists a subsequence $\{f_{n_{k_p}}\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_p}} \xrightarrow{p\to\infty} f$ almost everywhere by Proposition 4.18. Then by applying Fatuo's lemma,

$$\int f = \lim_{p\to\infty} f_{n_{k_p}} \le \varliminf_{n\to\infty} \int f_{n_{k_p}}$$

$$= \lim_{k\to\infty} \int f_{n_k}$$

$$= \lim_{n to\infty} \int f_n$$

and so the result holds! □

(2) **Lebesgue Dominated Convergence Theorem** Suppose $|f_n| \le g$ and $f_n \xrightarrow{m} f$. Then

$$\lim_{n\to\infty} \int f_n = \int f.$$

*Proof.* We claim that to show this result, we must show that we can any subsequence $\int f_{n_k}$ of $\int f_n$ which then implies that

$$\lim_{k\to\infty} \int f_{n_k} = \int f.$$

Because $f_{n_k} \xrightarrow{m} f$, there exists a subsequence $\{f_{n_{k_p}}\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_p}} \to f$ almost everywhere. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{p\to\infty} \int f_{n_{k_p}} = \lim_{k\to\infty} \int f_{n_k} = \int f.$$

Then by Problem 2.12,

$$\lim_{n\to\infty} f_n = \int f$$

which is what we desired to show. □

# Section 6.1 $L^p$ Spaces

**Definition.** A measurable function $f : [0, 1] \to \mathbb{R}$ is said to be in the space $L^p = L^p([0, 1])$ if

$$\int_a^b |f|^p < \infty.$$

Note the following

  (1) $L^1$ is the space of integrable functions

  (2) $L^p$ is closed under $+$ and under scalar multiplication i.e.,

The $L^p$ is defined as equivalance classes as follows:

$$\left\{ f : \text{ measurable and } \int |f|^p < \infty \right\} / \sim f = g \text{ a.e.}$$

(mod out by functions that are equal almost everywhere)

**Definition.** ()

**Definition.** The $L^p$-norm on $L^p$ space is defined as

$$||f||_p := \left( \int_0^1 |f|^p \right).$$

If $p \in (0, 1)$, then $||f + g||_p \leq ||f||_p + ||g||_p$. We want to show that $||f + g||_p \leq ||f||_p + ||g||_p$ for $p \in [1, \infty]$.

**Definition.** For $p = \infty$, the space $L^\infty$ is the set of bounded measurable functions for $f \in L^\infty$. Then

$$||f||_\infty = \operatorname{ess\,sup} |f(x)|$$
$$= \inf \{M : m\{t : f(t) > M\} = 0\}.$$

Note that $|| \cdot ||_\infty$ is the limit of $|| \cdot ||_p$ i.e.,

$$f \in L^\infty, ||f||_p \to ||f||_\infty.$$


# Section 5.5 Convex Functions

**Definition.** A function $\phi : [a, b] \to \mathbb{R}$ is **convex** if for all $x, y \in [a, b]$ and for all $\lambda \in (0, 1)$, we have that
$$\phi(\lambda x + (1 - \lambda)y) \leq \lambda \phi(x) + (1 - \lambda)\phi(y)$$

**Proposition (5.17).**

If $\phi$ is convex on $[a, b]$ then

  (1) - (don't care about this)

  (2) Right-hand and left-hand derivatives are equal except on a countable set.

(3) The left- and right-hand derivatives are monotone increasing functions, and at each point the left-hand derivative is less than or equal to the right-hand derivative.

**Corollary (5.19).** If $\phi$ is twice-differentiable, then $\phi$ is convex if and only $\phi''(x) > 0$.

**Corollary (5.20**, Jensen's Inequality**).** Let $\phi$ be a convex function on $(-\infty, \infty)$ and $f$ be an integrable function $[0, 1]$. Then

$$\int_0^1 \phi(f(t)) \, \mathrm{d}t \geq \phi \left[ \int_0^1 f(t) \, \mathrm{d}t \right].$$

An example of this is $\phi(x) = x^p$. For any $p \in (1, \infty)$, this function is twice-differentiable. Applying Jensen's inequality, we get

$$\int_0^1 |f(x)|^p \, \mathrm{d}x \geq \left( \int_0^1 |f(x)| \, \mathrm{d}x \right).$$

If $f \in L^p$, then $f \in L^1$ i.e., $L^p \subset L^1$.

**Theorem (6.1, Minkowski Inequality).** If $f, g \in L^p$ with $p \in [1, \infty]$, then $f + g \in L^p$ and
$$||f + g||_p \leq ||f||_p + ||g||_p.$$
If $p \in (1, \infty)$, then the equality can hold only if and only if there exists $\alpha, \beta \geq 0$ such that $\beta f = \alpha g$.

*Proof.* We leave $p = \infty$ as exercise so suppose $p$ is finite. Let $p \in [1, \infty]$. We normalize $f$ and $g$ i.e., there exists two functions $f_0, g_0 \in L^p$ such that $|f| = \alpha \cdot f_0$ and $|g| = \beta \cdot g_0$ with $||f_0|| = ||g_0|| = 1$ . Let $\lambda = \dfrac{\alpha}{\alpha + \beta}$ and $1 - \lambda = \dfrac{\beta}{\alpha + \beta}$. By the convexity of $\phi(t) = t^p$ for $p \in [1, \infty]$, we have that

$$\begin{aligned}
|f(x) + g(x)|^p &\leq (|f(x)| + |g(x)|)^p \\
&= (\alpha f_0 + \beta g_0)^p \\
&= (\alpha + \beta)^p \left( \frac{\alpha}{\alpha + \beta} f_0 + \frac{\beta}{\alpha + \beta} g_0 \right)^p \\
&\leq (\alpha + \beta)^p \left( \lambda f_0 + (1 - \lambda) g_0 \right)^p
\end{aligned}$$

Now take the integrals of these guys (Jensen's inequality or something like that). In other words,

$$\begin{aligned}
||f + g||_p^p &\leq (\alpha + \beta)^p \cdot \left( \lambda ||f_0||_p^p + (1 - \lambda) ||g_0||_p^p \right) \\
&= \left( ||f||_p^p + ||g||_p^p \right) \cdot 1 \qquad\qquad \text{because } f_0 = 1 = g_0.
\end{aligned}$$

Taking the $p$th root,
$$||f + g||_p \leq |f||_p + ||g||_p.$$

$\square$

This gives us the last norm-space requirement (triangle inequality of normed spaces).

**Lemma (6.3).** Let $p \in [1, \infty]$. Then for $a, b, t \geq 0$, we have

$$(a + tb)^p \geq a^p + ptba^{p-1}.$$

*Proof.* Define the function

$$\phi(t) = (a + tb)^p - a^p - ptba^{p-1}.$$

We know $\phi(0) = 0$. Take the derivative of this thing and this is greater than zero because

$$\phi'(x) = p(a + tb)^{p-1} + b - pba^{p-1}$$
$$= pb\left((a + bt)^{p-1} - a^{p-1}\right)$$

and so $\phi$ is increasing. $\qquad\square$

**Theorem (6.4, Holder Inequality).** [1] If $p$ and $q$ are nonnegative extended real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and if $f \in L^p$ and $g \in L^q$, then $f \cdot g \in L^1$ and

$$\int |fg| \leq ||f||_p \cdot ||g||_q.$$

---

[1]If $p, q = 2$, then this just reduces to the Cauchy-Schwarz inequality.

**Theorem (6.4, Holder Inequality).** [1] If $p$ and $q$ are nonnegative extended real numbers such that

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$$\int |fg| \leq ||f||_p \cdot ||g||_q.$$

*Proof.* There are two cases. (i) ($p = 1$, $q = \infty$ ) ...add notes on this. (ii) $p, q \in (1, \infty)$. Without loss of generality, suppose $f, g \geq 0$; otherwise, just take the absolute value. Set

$$h(x) = g(x)^{q-1} = g(x)^{q/p}$$

and

$$g(x) = h(x)^{p-1} = h(x)^{p/q}.$$

Then

$$p \cdot t \cdot f(x) \cdot g(x) = p \cdot t \cdot f(x) \cdot h(x)$$
$$\leq (h(x) + tf(x))^p - h(x)^p. \qquad \text{Lemma 6.3}$$

Taking the integral of both sides, (pulling out constants),

$$p \cdot t \int f(x)g(x) \leq \int ||h(x) + tf(x)||_p^p - \int ||h||_p^p$$
$$\leq (||h(x)||_p + t||f(x)||_p)^p - ||h(x)||_p^p \qquad \text{Triangle inequality}$$

Dividing by $t$,

$$p \int f(x)g(x) \leq \frac{(||h(x)||_p + t||f(x)||_p)^p - ||h(x)||_p^p}{t}$$

which the right-hand side is derivative of $\phi(t) = (||h||_p + t||f||_p)^p$. Taking the derivative with respect to $t$ at $t = 0$, we get that

$$p \int f(x)g(x) \leq p \left( ||h(x)||_p^{p-1} + ||f(x)||_p \right)^{p-1} = p||f(x)|| ||g(x)||$$

and so we are done! $\qquad \square$

# Section 6.3 Convergence and Completeness

Recall that if $(X, || \cdot ||)$ is a norm space (naturally a metric space), then $(X, d)$ is a metric space where

$$d(f, g) := ||f - g||$$

so the norm is the metric of the space.

**Definition.** We $\{f_n\} \in L^p$ converges to an element $f \in L^p$ in $L^p$ norm if

$$||f_n - f||_p \to 0.$$

That is, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $||f - f_n||_p < \varepsilon$.

---

[1] If $p, q = 2$, then this just reduces to the Cauchy-Schwarz inequality.

**Definition.** A normed space $(X, ||\cdot||)$ is called a **complete** space if every Cauchy sequence of $X$ is convergent.

- Note that a completed normed space is a called a **Banach space**.

Our goal will be to show that $L^p$ for $p \geq 1$ is a Banach space.

**Definition.** A sequence $f_n \subset X$ for any normed space $X$ is **summable** to a sum $s$ in $X$ if the partial sum converges, i.e.,

$$\left\| s - \sum_{k=1}^{n} f_k \right\| \to 0.$$

- A sequence is **absolute summable** if

$$\sum_{i=1}^{\infty} \|f_n\| < \infty.$$

**Proposition (6.5).** A normed linear space $X$ is complete if and only if every absolutely summable series is summable.

*Proof.* We will need to complete two directions.

$(\Rightarrow)$ Let $X$ be a Banach space and let $\{f_n\}$ be an absolute summable sequence. This means we have that

$$\sum_{n=1}^{\infty} \|f_n\| < M.$$

Our goal will be show that the partial sums are Cauchy sequence (then convergent by the completeness of a Banach space) i.e.,

$$S_n = \sum_{i=1}^{n} f_i$$

is Cauchy. Then suppose $n > m$ and so

$$\|S_n - S_m\| = \left\| \sum_{k=m}^{n} f_k \right\| \leq \sum_{k=m}^{n} \|f_k\| < \sum_{k=m}^{\infty} \|f_k\|$$
$$< \varepsilon$$

for any $\varepsilon > 0$ because $\{f_n\}$ is absolutely summable and therefore convergent. Thus, the partial sums are Cauchy and so convergent.

$(\Leftarrow)$ Now suppose every absolutely summable series is summable. We will construct a series from the Cauchy sequence. Let $\{f_n\}$ be a Cauchy sequence. Pick $\dfrac{\varepsilon}{2^k}$, and then pick the subsequence $\{f_{n_k}\}$ such that

$$\left\| f_{n_{k+1}} - f_{n_k} \right\| < \frac{1}{2^k}$$

which we can do because $\{f_n\}$ is Cauchy. Consider the series $g_k = f_{n_k} - f_{n_{k-1}}$, which is summable because the sequence is decreasing by construction. By assumption, then $\{g_k\}$ must be absolutely summable; i.e., the sum

$$S_m = \sum_{k=1}^{m} g_k$$

has a limit. Note that $S_m$ is a telescoping series by construction again thus $S_m = -f_{n_1} + f_{n_m}$. This implies that $\{f_{n_k}\}$ converges to $f$ for some $f \in X$ as $k \to \infty$. Since $\{f_n\}$ is Cauchy,

$$\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\|.$$

Then use the fact that $\{f_n\}$ is Cauchy and $\{f_{n_k}\}$ is convergent, pick $\dfrac{\varepsilon}{2}$ for each thing and so the result follows.

$\square$

**Theorem (6.6, Riesz-Fisher).** $L^p$ is complete for $p \in [1, \infty]$.

**Theorem (6.6, Riesz-Fisher).** $L^p$ is complete for $p \in [1, \infty]$.

*Proof.* Note $p = \infty$ is an exercise. We want to show every absolute summable series is summable. Let $\{f_n\} \subset L^p$ such that

$$\sum_{n=1}^{\infty} \|f_n\| < M.$$

Consider the function $g_n(x) = \sum_{k=1}^{n} f_k(x)$ and want to show that $g_n$ converges some function $g$ (i.e., the limit exists.) By the triangle inequality,

$$\|g_n(x)\| \le \sum_{k=1}^{n} \|f_k(x)\| < M$$

and so we know that $\int |g_n|^p < M^p$. We know we want to do the following:

1. Find a limit of $g_n$.

2. Then show it is in $L^p$.

For each fixed $x \in L^P$, $\{g_n(x)\}$ is monotonically increasing. By the Monotone Convergence Theorem, there exists $g(x) \in \overline{\mathbb{R}}$ (extended real numbers) and $g_n(x) \to g(x)$. Because $g_n \ge 0$ is measurable, $g(x) \ge 0$ is measurable as well an dso

$$\int g^p \le M^p$$

and so this implies that $g \in L^p$. Notice that for each $x$ such that $g(x)$ is finite, $\sum_{k=1}^{\infty} f_k(x)$ is absolutely summable over $\mathbb{R}$. Since $(\mathbb{R}, |\cdot|)$ is complete, we know that $S_n(x) = \sum_{k=1}^{n} f_k(x)$ is summable i.e.,

$$S_n(x) \to S(x) = \sum_{k=1}^{f_k(x)} $$

over $\mathbb{R}$. We can look at just the limit $S(x)$ because we are looking at only where $g(x)$ has finite measure and defined as

$$\widetilde{S} = \begin{cases} S_n(x) & g(x) < \infty \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $S(x)$ is measurable since $S(x) \ne \sum_{k=1}^{\infty} f_k(x)$ only on a measure zero set.

Moreover, $|S_n(x)| \le |g(x)|$ for all $n \in \mathbb{N}$ and so $|\widetilde{S}(x)| \le |g(x)|$ and therefore $\widetilde{S}(x) \in L^p$. We claim that

$$\left\| \sum_{k=1}^{n} f_n(x) - \widetilde{S}_n(x) \right\|_p \to 0 \text{ as } n \to \infty.$$

We know that because $S \in L^p$,

$$\left|S_n(x) - \widetilde{S}_n(x)\right|^p \leq 2^p \cdot |g(x)|^p.$$

So by the Lebesgue Dominated Convergence, we know that $\|s_n - \widetilde{s}_n\|^p \to 0$ and so $\|s_n - \widetilde{s}_n\| \to 0$. Thus the sum $s$ from $\{f_n\}$ is in $L^p$. $\qquad\square$

# Section 6.4 Approximation in $L^p$

Our goal with approximation properties is to approximate functions in $f \in L^p$ spaces by step functions $\phi$ and continuous functions $g$. That is, for any $\varepsilon > 0$, there exists functions $\phi$ and $g$ such that

$$\|f - g\|_p < \varepsilon \quad \text{and} \quad \|f - \phi\|_p < \varepsilon.$$

**Lemma (6.7).** Given $f \in L^p$, $p \in [1, \infty]$, and any $\varepsilon > 0$, there is a bounded measurable functions $f_M$ with $|f_m| \leq M$ and $\|f - f_M\| < \varepsilon$.

*Proof.* Consider the function

$$f_N(x) = \begin{cases} N & N \leq f(x) \\ f(x) & -N \leq f(x) \leq N \\ -N & f(x) \leq -N. \end{cases}$$

$\qquad\square$

Note that $|f_n| \leq N$ and so is measurable for each $n \in \mathbb{N}$. Then $f_N(x) \to f(x)$ converges a.e (there may be unbounded points but they are of measure zero) which implies that $|f_N(x) - f(x)|^p \to 0$ a.e. Thus,

$$|f_N(X) - f(x)|^p < 2|f(x)|^p$$

almost everywhere. So by the Lebesgue Dominated Convergence Theorem,

$$\int |f_N(x) - f(x)|^p \to 0$$

or, equivalently, $\|f_N - f(x)\|_p \to 0$ and so $f_n(x) \to f(x)$ is in $L^p$.

**Proposition (6.8).** Given $f \in L^p$, $p \in [1, \infty)$ and any $\varepsilon > 0$, there is a step function $\phi$ and a continuous function $g$ such that

$$\|f - g\|_p < \varepsilon \quad \text{and} \quad \|f - \phi\|_p < \varepsilon.$$

*Proof.* Let $\varepsilon > 0$ be chosen. For step functions, we note that by Lemma 6.7, there exists an $f_M$ such that $\|f - f_M\| < \dfrac{\varepsilon}{2}$. By Theorem 3.22 (Littlewood's 2nd Principle), there exists a step function $\phi$ such that $|f_M - \phi| < \frac{\varepsilon}{4}$ except on a set $E$ of measure less than

$$\delta = \left(\frac{\varepsilon}{8M}\right)^p.$$

Then

$$\|f_M - \phi\|_p = \int |f_M - \phi|^p$$

$$= \int_{[0,1]\setminus E} \int |f_M - \phi|^p + \int_E |f_M - \phi|^p$$

$$\leq \int_{[0,1]} \left(\frac{\varepsilon}{4}\right)^p + \int_E |f_M - \phi|^p$$

$$= \int_{[0,1]} \left(\frac{\varepsilon}{4}\right)^p = (2M)^p \cdot m(E)$$

$$\leq \int_{[0,1]} \left(\frac{\varepsilon}{4}\right) + (2M)^p \cdot \left(\frac{\varepsilon}{8M}\right)$$

and taking the $p$th root, we know that

$$\|f_M - \phi\|_p \leq \frac{\varepsilon}{2}.$$

Doing the same thing with a continuous functions $f$ than step function, we can show that

$$\|f_M - g\| < \frac{\varepsilon}{2}.$$

$\square$

This means that "step functions" and "continuous functions" are "dense" in $L^p$ (i.e., we can always use step and continuous functions in the limit to approximate functions $f \in L^p$).

# Section 5.1 Differentiation of Monotone Functions

Our goal will be to show if $f$ is monotone increasing over an interval $[a, b]$, then $f$ is differentiable a.e.

**Definition (Vitali Cover).** Let $E \subset \mathbb{R}$ and $\Gamma$ is a collection of intervals. We call $\Gamma$ a **Vitali cover** of $E$ if for all $x \in E$ and all $\varepsilon > 0$, there $I \in \Gamma$ such that $x \in I$ and $0 < |I| < \varepsilon$.

Here is an example. Let $E = [0, 1]$. Then

$$\Gamma_1 = \left\{ \left( x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2} \right) \right\}$$

[there was more to the above... finish this after]

**Lemma (5.1).** Let $E \subset \mathbb{R}$, $m^*(E) < \infty$, and $\Gamma$ is a Vitali cover of $E$. Then for all $\varepsilon > 0$ there a finite disjoint collection $\{I_1, \ldots, I_n\} \subset \Gamma$ such that

$$m^* \left( E \setminus \bigcup_{i=1}^{N} I_i \right) < \varepsilon.$$

**Definition (Dini Derivatives).** We define the **Dini Derivatives** as follows:

$$D^+ f(x) = \varliminf_{h \to 0+} \frac{f(x + h) - f(x)}{h}$$

$$D^- f(x) = \varliminf_{h \to 0+} \frac{f(x) - f(x - h)}{h}$$

$$D_+ f(x) = \varlimsup_{h \to 0+} \frac{f(x + h) - f(x)}{h}$$

$$D_- f(x) = \varlimsup_{h \to 0+} \frac{f(x) - f(x - h)}{h}$$

We can remark by definition the following

1.

$$D^+ f(x) \geq D_+ f(x)$$
$$D^- f(x) \geq D_- f(x).$$

2. Also, $f$ is differentiable if and only if $D_+^+ = D_-^-$.

**Lemma.** Let $f : [a, b] \to \mathbb{R}$ be an increasing function, and let $\gamma < R$. Then

$$E = E_{\gamma, R} = \left\{ x \in (a, b) : D_- f(x) < \gamma < R < D^+ f(x) \right\}.$$

has measure zero.

*Proof.* Write $m^* E = s < \infty$. Let $\varepsilon > 0$ be chosen. By definition of outer measure, there exists an open set $E \subset O$ such that $m(O) < s + \varepsilon$. Let $x \in E$. Then $D_- f(x) < \gamma$ which implies that for any $\delta > 0$, there exists $h \in (0, \delta)$ with

$$\frac{f(x) - f(x - h)}{h} < \gamma$$

coming from the definition of $\liminf$. Then the collection $[x - h, x]$ forms a Vitali cover of $E$. By the definition of Vitalia covering Lemma (**Lemma 5.1**), for our fixed $\varepsilon > 0$, there exists a finite collection $\{I_1 = [x_1 - h, x_1], \ldots, I_N = [x_N - h, x_N]\}$ such that

$$m^* \left( E \setminus \bigcup_{k=1}^{N} I_k \right) < \varepsilon.$$

Let $A = E \cap \bigcup_{k=1}^{N} I_k$. Then $m^*(A) > s - \varepsilon$ (recalling $s = m^*(E)$.) This implies that

$$0 \le \sum_{k=1}^{n} f(x_k) - f(x_k - h_k) < \gamma \sum_{k=1}^{N} h_k$$
$$= m^* \left( \bigcup_{k=1}^{N} I_k \right)$$
$$< \gamma \cdot (s + \varepsilon) \qquad \text{because } \bigcup_{k=1}^{N} I_k \subset O$$

and so we are done with this part.

Let $y \in A$, and by $\limsup$ we know that $D^+ f(x) > R$. Then there exists an arbitrarily small $k > 0$ such that $[y, y + k] \subset I_n$ for some $n \in \mathbb{N}$ such that

$$f(y + k) - f(y) > R \cdot k.$$

Then the collection formed from $[y, y+k]$ forms a Vitali cover on A. Again, by the Vitalia cover lemma (**Lemma 5.1**), there exists a collection

$$J = \{J_1 = [y_1, y_1 + k_1], \ldots, J_M = [y_M, y_M + k_M]\}$$

such that

$$m^* \left( A \setminus \bigcup_{j}^{M} J_j \right) < \varepsilon.$$

This implies that

$$m^* \left( A \cap \setminus \bigcup_{j}^{M} J_j \right) > s - 2\varepsilon.$$

So we can sum across these intervals and get that

$$\sum_{j=1}^{M} f(y_j + k_j) - f(y_j) > R \sum_{j=1}^{M} k_j$$
$$> R(s - 2\varepsilon).$$

Putting this all together and noting that $f$ is an increasing function

$$\sum_{n=1}^{N} f(x_n) - f(x_n - h_n) \ge \sum_{j=1}^{M} f(y_j + k_j) - f(y_j)$$
$$> R(s - 2\varepsilon)$$

and so we get that
$$\gamma(s + \varepsilon) < R(s - 2\varepsilon).$$
Thus $\gamma s \geq Rs$ and implies that $s = 0$ (noting that $R > \gamma$). Now we are done!    □

**Theorem (5.3).** Let $f : [a, b] \to \mathbb{R}$ be monotonic increasing. Then $f$ is differentiable almost everywhere, $f'$ is measurable, and
$$\int_a^b f'(x)\mathrm{d}\, x \leq f(b) - f(a).$$

*Proof.* Let $E = \{x : D_- f(x) < D^+ f(x)\}$. Then
$$E = \bigcup_{r, R \in \mathbb{Q}} E_{r,R}$$
has a measure zero by our lemma above. This tells us that $f$ is differentiable almost everywhere because we showed that the sets where any two derivates are not equal have measure zero. Thus
$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
exists almost everywhere. Define the function
$$f(x) = \begin{cases} f(b) & x \geq b \\ f(x) & x \in [a, b] \end{cases}$$
which extends the function $f$ to the right. Define the sequence $\{G_n\}$ by
$$G_n(x) = \frac{f\left(x + \frac{1}{n}\right) - f(x)}{\frac{1}{n}} \geq 0.$$
This sequence converges a.e. to $f'(x)$ as $n \to \infty$. So by Fatou's lemma,
$$\int_a^b f'(x)\,\mathrm{d}x \leq \varliminf_{n \to \infty} \int_a^b G_n(x)\,\mathrm{d}x$$
$$= \varliminf_{n \to \infty} \int_a^b n\left(f\left(x + \frac{1}{n}\right) - f(x)\right)\,\mathrm{d}x$$
$$= \int_{a + 1/n}^{b + 1/n} f(y)\,\mathrm{d}y$$
$$= \varliminf_{n \to \infty} n \int_{a + 1/n}^{b + 1/n} f(y)\,\mathrm{d}x - \int_a^{a + 1/n} f(x)\,\mathrm{d}x$$
$$= \varliminf_{n \to \infty} n \left(\int_b^{b + 1/n} f(y)\,\mathrm{d}x - \int_a^{a + 1/n} f(x)\,\mathrm{d}x\right)$$
$$= \varliminf_{n \to \infty} n \cdot f(b) \cdot \frac{1}{n} - \varliminf_{n \to \infty} \int_a^{a + 1/n} f(x)\,\mathrm{d}x$$
$$\leq f(b) - \varliminf_{n \to \infty} \int_a^{a + 1/n} f(x)\,\mathrm{d}x$$
$$\leq f(b) - f(a) \qquad\qquad\qquad \text{noting that } f(x) \geq \frac{1}{n}$$

□

The next thing that will be covered is functions of bounded variation in Section 5.2.

**Definition.** Let $f : [a,b] \to \mathbb{R}$. Then $f$ is a function of **bounded variation** if $\|f\|_{TV[a,b]} < \infty$ where $TV[a,b]$ is the total variation of $[a,b]$ and

$$\|f\|_{TV[a,b]} = \sup\left\{\sum_{i=0}^{n} |f(x_{i+1}) - f(x)| : P \text{ is a partition of } [a,b]\right\}.$$