Problem 1 (6.11)**.** Prove that L^∞ is complete.

Proof. To show that L^∞ is complete, we must show every Cauchy sequence converges. To that end, let $\{f_n\}$ be any Cauchy sequence in L^∞ and let $\varepsilon > 0$ be chosen. Since $\{f_n\}$ is Cauchy, there exists $N_1 \in \mathbb{N}$ such that for all $n, m \geq N_1$, we have

$$\|f_n - f_m\|_\infty = \inf\{M : m\{t : |f_n(t) - f_m(t)| > M\} = 0\} < \frac{\varepsilon}{2}.$$

So for any fixed $n, m \geq N_1$, there exists $M < \dfrac{\varepsilon}{2}$ such that $m\{t : |f_n(t) - f_m(t)| > M\} = 0$. implying that $m\left\{t : |f_n(t) - f_m(t)| > \dfrac{\varepsilon}{2}\right\} = 0$. So then on the set $L^\infty \backslash \{t : |f_n(t) - f_m(t)| > \frac{\varepsilon}{2}\}$, the sequence $\{f_n\}$ converges to some function f almost everywhere. We must show that this limit function f is in L^∞

Since $f_n \to f$ almost everywhere, there exists $N_2 \in \mathbb{N}$ such that for any $n > N_2$, $|f_n - f| < \dfrac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then for any fixed $n > N$, we can see that

$$\|f_n - f\|_\infty = \inf\{M : m\{t : |f_n(t) - f(t)| > M\} = 0\} < \varepsilon$$

which, because ε is arbitrary, means that $\|f_n - f\| \to 0$. Thus $f \in L^\infty$ and so L^∞ is complete. □

Problem 2 (6.13)**.** Let $C = C[0,1]$ be the space of all continuous functions on $[0,1]$ and define $\|f\| = \max|f(x)|$. Show that C is a Banach space.

Proof. To show that the space $(C, \|\cdot\|)$ is a Banach space, we must show $\|f\| = \max|f(x)|$ is indeed a norm and that C with this norm is a complete space.

First, we will show that $\|\cdot\|$ defined above on C is a norm. That is, we must show the following the following three properties:

(i) For any $f \in C$, $\|f\| = 0$ if and only if $f = 0$.

(ii) For any $f, g \in C$, $\|f + g\| \leq \|f\| + \|g\|$.

(iii) For any $f \in C$ and for all $\alpha \in \mathbb{R}$, $\|\alpha f\| = |\alpha|\,\|f\|$.

For (i), first suppose that $\|f\| = 0$. Then $\max|f(x) = 0|$ which is true only if $f(x) = 0$ for any $x \in [0,1]$. Conversely, suppose $f = 0$. Then for any $x \in [0,1]$, we have that $\|f\| = \max|f(x)| = \max 0 = 0$.

To prove (ii), fix $f, g \in C$. By the triangle inequality, we know that $|f + g| \leq |f| + |g|$. The max function adheres to the triangle inequality and so

$$\begin{aligned}
\|f + g\| &= \max|f + g| \\
&\leq \max|f| + \max|g| \\
&= \|f\| + \|g\|.
\end{aligned}$$

Finally, let $\alpha \in \mathbb{R}$ and $f \in C$ be chosen. Then

$$\begin{aligned}
\|\alpha f\| = \max|\alpha f| &= \max|\alpha|\,|f| \\
&= |\alpha| \max|f| \\
&= |\alpha|\,\|f\|
\end{aligned}$$

where the last equality follows since α is a scalar and not dependent upon taking the max over $[0, 1]$.

To show C is complete, let $\{f_n\}$ be a Cauchy sequence on C. Let $\varepsilon > 0$ be chosen. Since $\{f_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have that

$$\|f_n - f_m\| < \varepsilon.$$

From how $\|\cdot\|$ is defined, this implies that $|f_n(x) - f_m(x)| < \varepsilon$ for any $x \in [0, 1]$. But this means we can always find $n > N$ large enough so that $|f_n - f| < \varepsilon$ i.e., $\{f_n\}$ converges to a function f pointwise. Because $x \in [0, 1]$, this means that that this convergence is uniform and so $\|f_n - f\| < \varepsilon$ i.e., $\|f_n - f\| \to 0$. Thus, $f \in C$ and so C is a complete space.

Therefore, having shown that $\|f\| = \max |f|$ is a norm and C is a complete space, C is a Banach space. $\square$

Problem 3 (5.1)**.** Let f be the function defined

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Find $D^+f(0)$, $D_+f(0)$, $D^-f(0)$, and $D_-f(0)$.

Proof. First, we will note that

$$D^+(0) = \frac{f(0+h) - f(0)}{h} = \overline{\lim_{h \to 0^+}} \sin\left(\frac{1}{h}\right) = 1.$$

Similarly,

$$D^-(0) = \frac{f(0) - f(0-h)}{h} = \overline{\lim_{h \to 0^+}} \sin\left(\frac{1}{h}\right) = 1.$$

However, this flips once we look at the limit inferior i.e.,

$$D_+(0) = \underline{\lim_{h \to 0^+}} \sin\left(\frac{1}{h}\right) = -1$$

and

$$D_-(0) = \frac{f(0) - f(0-h)}{h} = \overline{\lim_{h \to 0^+}} \sin\left(\frac{1}{h}\right) = -1..$$

$\square$