Section 3.5, Measurable Functions

Proposition (3.18). Let $E \subset \mathbb{R}$, and Let $f : E \to [-\infty, \infty]$ be an extended real-valued function whose domain is measurable. Let $\alpha \in \mathbb{R}$ be any real number. Then the following statements are equivalent:

(i) The set $\{x : f(x) > \alpha\}$ is measurable.

(ii) The set $\{x : f(x) \geq \alpha\}$ is measurable.

(iii) The set $\{x : f(x) < \alpha\}$ is measurable.

(iv) The set $\{x : f(x) \leq \alpha\}$ is measurable.

All together, these imply

(v) The set $\{x : f(x) = \alpha\}$ is measurable.

Proof. [1]

□

Definition. An extended real-valued function $f : E \to [-\infty, \infty]$ is **(Lebesgue) measurable** if its domain is measurable and satisfies one of the first four statements of Proposition 18.

Note that this means that any continuous function is measurable since then the pre-image of any open set is still an open set.

Proposition (3.19). Let f and g be two measurable functions defined on the same domain, and let $c \in \mathbb{R}$. Then the functions $f+c$, cf, $f+g$, $g-f$, and fg are measurable.

Proof. For $f(x)+c$, note that

$$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}.$$

□

Theorem (3.20, Limit of Measurable Functions is Measurable). Let $\{f_n\}$ be a sequence of measurable functions with the same domain. Then the functions $\sup\{f_1(x), \ldots, f_n(x)\}$, $\inf\{f_1(x), \ldots, f_n(x)\}$, $\sup_n f_n$, $\inf_n f_n$, $\overline{\lim} f_n$, and $\underline{\lim} f_n$ are measurable.

Proof. Let $\{f_n\}$ be a sequence of measurable functions. Let $h(x) = \sup\{f_1(x), \ldots, f_n(x)\}$ and we so must show that $\{x : h(x) < \alpha\}$ for all $\alpha \in \mathbb{R}$. To that end, let $\alpha \in \mathbb{R}$ be chosen. Then

$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$$

which, because the right-hand side is a union of measurable sets from the f_i's being measurable, means that the set $\{x : h(x) < \alpha\}$ is also measurable. □

[1] Proof is on page 67.

Definition. A property is said to hold **almost everywhere** (a.e) if the set of points where it fails to hold is a set of measure zero. Thus $f = g$ a.e if f and g have the same domain and $m\{x : f(x) \neq g(x)\}$.

Proposition (3.21). If f is measurable and $f = g$ a.e, then g is measurable.

Proof. Let $\{x : g(x) > \alpha\}$. This is equivalent to saying that

$$\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha\}$$

□

Proposition (3.22). Let $f : [a, b] \to E$ be a measurable function with $E \subset \mathbb{R}$ and is equal to $\pm\infty$ only on sets with measure zero. Then for all $\varepsilon > 0$, there exist a step function g and a continuous function f such

$$|f - g| < \varepsilon \quad \text{and} \quad |f - h| < \varepsilon$$

except on set of measure less than ε; i.e., $m\{x : |f(x) - g(x)| \geq \varepsilon\} < \varepsilon$ and $m\{x : |f(x) - h(x)| \geq \varepsilon\} < \varepsilon$. If in addition $m \leq f \leq M$, then we may choose the functions g and h so that $m \leq g \leq M$ and $m \leq h \leq M$.

Proposition (3.23, (Weak) Egonoff's Theorem). Let E be a measurable set of finite measure, and $\{f_n\}$ be a sequence of measurable functions defined on E. Let f be real-valued function such for each $x \in E$ we have $f_n(x) \to f(x)$. Then for all $\varepsilon > 0$ and all $\delta > 0$, there is measurable set $A \subset E$ with $m(A) < \delta$ and $N \in \mathbb{N}$ such that for all $x \notin A$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$