Proposition (11.1)**.** If $A \in \mathcal{B}$, $B \in \mathcal{B}$, and $A \subset B$, then

$$\mu(A) \leq \mu(B).$$

Proposition (11.2)**.** If $E_i \in \mathcal{B}$, $\mu(E_1) < \infty$ and $E_i \supset E_{i+1}$, then

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu\left(E_n\right).$$

Proposition (11.3)**.** If $E_i \in \mathcal{B}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{n} \mu\left(E_i\right).$$

Definition. Let $(X, \mathcal{B}, \mu)$ be a measure space. Then

(1) If $\mu(x) < \infty$ and hence $\mu(E) < \infty$ for all $E \in \mathcal{B}$ then μ is called **finite**.

(2) Let $X = \bigcup_{i=1}^{\infty} E_i$ where $E_i \in \mathcal{B}$ and $\mu(E_i) < \infty$ for all $i \in \mathbb{N}$. Then μ is called σ-**finite**.

(3) If for all $E \in \mathcal{B}$ with $\mu(E) = \infty$ and there exists nonempty $F \subset E$ such that $F \in \mathcal{B}$ and $\mu(F) < \infty$, then μ is called **semi-finite**.

Remark. Note if μ is σ-finite, then μ is semi-finite. Additionally, note the following:

(1) Note that

$$somestuff.$$

(2) The triple $(\mathbb{R}, \mathcal{M}, \mathcal{B})$ is σ-finite because

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [n, n+1] \cup \bigcup_{n=-1}^{\infty} [n-1, n] \cup [-1, 1].$$

(3) We will mostly only discuss σ-finite measure.

Definition. A measure space $(X, \mathcal{B}, \mu)$ is said to be **complete** if $\mathcal{B}$ contains all subsets of measure zero i.e., if $B \in \mathcal{B}$, $\mu(B) = 0$, and $A \subset B$, then $A \in \mathcal{B}$.

Proposition (11.4)**.** If $(X, \mathcal{B}, \mu)$ is a measure space, then there exists a complete measure space $(X, \mathcal{B}_0, \mu_0)$ such that

(i) $\mathcal{B} \subset \mathcal{B}_0$.

(ii) If $E \in \mathcal{B}$, then $\mu(E) = \mu_0(E)$.

(iii) $E \in \mathcal{B}_0$ if and only if $E = A \cup B$ where $B \in \mathcal{B}$ and $A \subset C$, $C \in \mathcal{B}$, and $\mu(C) = 0$.

Section 11.2 Measurable Functions

Let $(X, \mathcal{B})$ be a measurable space for any of the following propositions and definitions.

Proposition (11.5)**.** Let $f : X \to \overline{\mathbb{R}}$ be a function, and let $\alpha \in \mathbb{R}$ be fixed. Then the following statements are equivalent:

(i) $\{x : f(x) < \alpha\} \in \mathcal{B}$.

(ii) $\{x : f(x) \leq \alpha\} \in \mathcal{B}$.

(iii) $\{x : f(x) > \alpha\} \in \mathcal{B}$.

(iv) $\{x : f(x) \geq \alpha\} \in \mathcal{B}$.

Definition. The function $f : X \to \overline{\mathbb{R}}$ is a **measurable function** if any of the above statements in Proposition 11.5 hold.

Theorem (11.6)**.** If $c \in \mathbb{R}$ and the functions f and g are measurable, then so are the functions $f + c$, $f + g$, $f \cdot g$, and $f \vee g$. Moreover, if $\{f_n\}$ is a sequence of functions, then $\sup f_n$, $\inf f_n$, $\overline{\lim} f_n$, and $\underline{\lim} f_n$ are all measurable.

Definition. Define a **simple function** by

$$\phi(x) = \sum_{i=1}^{n} a_i X_{E_i}$$

for $a_i \in \mathbb{R}$ and with $E_i \in \mathcal{B}$ where $E_i \cap E_j = \emptyset$ for all $i \neq j$.

Proposition (11.7)**.** Let f be an nonnegative measurable function. Then there is a sequence $\{\phi_n\}$ of simple functions with $\phi_{n+1} \geq \phi_n$ such that $f = \lim_{n\to\infty} \phi_n$ at each point of X, If f is defined on a σ-finite measure space, then we may choose the functions ϕ_n so that each vanishes outside a set of finite measure.

Proposition (11.8)**.** If μ is a complete measure and f is a measurable function, then $f = g$ almost everywhere implies g is measurable.

Section 11.3 Integration

Let $(X, \mathcal{B}, \mu)$ be a measure space.

Definition. Let

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}$$

be a simple function. The integration of ϕ with respect to μ on E is defined as

$$\int_E \phi = \int_E \phi \, \mathrm{d}\mu := \sum_{i=1}^{n} a_i \cdot \mu\left(E_i \cup E\right).$$

Proposition. Let ϕ, ψ be nonnegative simple functions.

(a) If $\alpha, \beta \geq 0$, then

$$\int_E \alpha\phi + \beta \int_E \psi = \alpha \int_E \phi + \beta \int_E \psi.$$

(b) If $0 \geq \phi \leq \psi$, then

$$\int_E \phi \leq \int_E \psi.$$

(c) The map $\eta : \mathcal{B} \to \mathbb{R}^+ \cup \{0\}$ defined by $A \mapsto \displaystyle\int_A \phi \, \mathrm{d}\mu$ is a measure on $\mathcal{B}$.