Problem 1 (2.9)**.** Properties of sequences in $\mathbb{R}$.

(a.) Show that $\limsup x_n$ and $\liminf x_n$ are the largest and smallest cluster points of the sequence $\{x_n\}$.

Proof. Let $\{x_n\}$ be any sequence in $\mathbb{R}$. First to show that $l = \limsup x_n$ is indeed a cluster point, let $\varepsilon > 0$ be chosen. Because l is the limit superior, there exists $n_1 \in \mathbb{N}$ such that $x_{k_1} < l + \varepsilon$ for all $k_1 \geq n_1$. Additionally, there are infinitely many value of this value n_1 such that $x_k > l - \varepsilon$ for some $k_1 \geq n_1$, which together with the last sentence implies that $|x_{k_1} - l| < \varepsilon$. To inductively create a subsequence, let $n_1, \ldots, n_j$ and $x_{k_1}, \ldots, x_{k_j}$ be arbitrary. Let n_{j+1} be chosen such that $n_{j+1} > \max\{k_1, \ldots, k_j\}$. Then, because l is the limit superior $x_k < l + \varepsilon$ for any $k \geq n_{j+1}$. Further, for sufficiently large n_{j+1}, there exists $k_{j+1} \geq n_{j+1}$ such that $x_{k_{j+1}} > l - \varepsilon$, which gives us that $|x_{k_{j+1}} - l| < \varepsilon$. Because we can always choose the next point in the subsequence in this manner, this means that the subsequence $\{x_{n_j}\}$ converges to l. By Problem 2.8, this means that l is a cluster point of $\{x_n\}$.

By way of contradiction, suppose that l is not the largest cluster point of the sequence. That is, there exists a cluster point y of $\{x_n\}$ such that $y > l$. Note that by Problem 2.8, this means that there exists a subsequence $\{x_{n_j}\}$ which converges to y. Because l is the limit superior of the sequence, for any $\varepsilon > 0$, we can find $n \in \mathbb{N}$ such that $x_k < l + \varepsilon$ whenever $k \geq n$. Since this is true for any $\varepsilon > 0$, we can choose $\varepsilon > 0$ small enough so that we have

$$l < l + \varepsilon < y - \varepsilon < y.$$

This means that there are a finite number of terms of $\{x_n\}$ contained within the interval $(y - \varepsilon, y + \varepsilon)$. In other words, there does not exist a subsequence $\{x_{n_j}\}$ which converges to y as we would necessarily need an infinite number of terms within ε of y—a contradiction. Therefore, l is the largest cluster point.

By a reverse argument, we can show that $\liminf x_n$ is a cluster point of $\{x_n\}$ as well as the smallest cluster point. □

(b.) Show that every bounded sequence has a convergent subsequence.

Proof. Let $\{x_n\}$ be a bounded sequence. In other words, $\sup x_n$ is a finite real number. By definition of the limit superior, $\limsup x_n \leq \sup x_n$. From part (a), because $lim \sup x_n$ is a cluster point, we can always construct a convergent subsequence (as well as for the limit inferior of the sequence) and so this completes this part. □

Problem 2 (2.43)**.** Let f be defined as

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ p \sin\left(\frac{1}{q}\right) & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

At what points is f continuous? (Please justify your answer.)

Proof. I claim that f is not continuous at the rational numbers. To that end, let $x \in Q$ and choose $\varepsilon = x - f(x)$. Fix $\delta > 0$. Note that we can always find an irrational number $y \in (x, x+\delta)$. Because y is irrational, by definition of the function, $f(y) - f(x) = y - f(x)$. But then $y - f(x) > x - f(x) = \varepsilon$.

For $x = 0$, fix $\varepsilon > 0$. Choose $\delta = \varepsilon$, and a pick a point $y \in \mathbb{R}$ such that $|x - y| = |y - 0| = |y| < \delta$. Now because $\sin(1/q) < 1/q$ for any $q \in \mathbb{N}$, we know that

$$\begin{aligned} |f(y) - f(0)| &\leq |y - 0| \\ &< \delta \\ &= \varepsilon \end{aligned}$$

so f is continuous at 0.

f is also continuous at the irrationals. This is because we can if we pick any point x in the irrationals, we can find sufficiently large q so a rational number $y = \frac{p}{q}$ is close to x (i.e, for a fixed ε, choose δ to be smaller than $f(y) - y$ for this to work). Then this would allow us to bound $|f(y) - f(x)|$ leveraging that we can put the rational numbers in lowest terms

□

Problem 3. Show that $F \subset \mathbb{R}$ is a closed set if and only if F^{c} is open.

Proof. To complete this proof, we will need a forward and backwards implication.

($\Rightarrow$) Suppose $F \subset \mathbb{R}$ is a closed set. Because we desire to show that F^{c} is open, let $x \in F^{\mathsf{c}}$ be a point. This means that $x \notin F$. Since F is a closed set (i.e., $F = \overline{F}$) and $x \notin F$, we know x is not a point of closure of F. So there exists $\delta > 0$ such that for all $y \in F$, we do not have $|x - y| < \delta$. But then if $|x - y| < \delta$, this must mean that $y \in F^{\mathsf{c}}$, and so F is an open set.

($\Leftarrow$) Conversely, suppose that the set F^{c} is open. Let $x \in F^{\mathsf{c}}$. Then there exists $\delta > 0$ such that if $|x - y| < \delta$, then $y \in F^C$. This means that there is no $y \in F$ such that $|x - y| < \delta$ and so x cannot be a point of closure of F. Thus, because x is arbitrary, F necessarily contains all its points of closure; in other words, $F = \overline{F}$ and thus F must be closed, completing this direction.

Having completed both implications, this completes the proof. □

Problem 1 (3.5)**.** Let A be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a **finite** collection of open intervals covering A. Then $\sum_{n=1}^{k} l(I_n) \geq 1$.

Proof. Define the set $A = \mathbb{Q} \cap [0,1]$. Let $\{I_n\}_{n=1}^{k}$ be a **finite** collection of open intervals covering A meaning we have that

$$A \subset \bigcup_{n=1}^{k} I_n$$

We can create the following string of inequalities:

$$\begin{aligned}
1 = l([0,1]) &= m^*([0,1]) \\
&= m^*\left(\overline{A}\right) && \text{Density of } \mathbb{Q} \\
&\leq m^*\left(\overline{\bigcup_{k=1}^{n} I_n}\right) && A \subset B \implies \overline{A} \subset \overline{B} \\
&= m^*\left(\bigcup_{k=1}^{n} \overline{I_n}\right) && \overline{A \cup B} = \overline{A} \cup \overline{B} \\
&\leq \sum_{k=1}^{n} m^*\left(\overline{I_n}\right) && \text{Subadditivity of } m^* \\
&= \sum_{k=1}^{n} l\left(\overline{I_n}\right) \\
&= \sum_{k=1}^{n} l(I_n)
\end{aligned}$$

which shows the desired result, completing the proof. □

Problem 2 (3.10)**.** Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cap E_2) + m(E_1 \cup E_2) = mE_1 + mE_2.$$

Recall that if E is measurable, then $mE := m^*E$.

Proof. Suppose E_1 and E_2 are measurable sets. To show the above equality, the goal will be to rewrite sets as disjoint unions and utilize the subadditivity of m. So note that

$$E_1 \cup E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1) \cup (E_1 \cap E_2)$$

and so by the subadditivity of m,

$$m(E_1 \cup E_2) = m(E_1 \setminus E_2) + m(E_2 \setminus E_1) + m(E_1 \cap E_2).$$

Thus, adding $m(E_1 \cap E_2)$ to the left-hand side

$$\begin{aligned}
m(E_1 \cup E_2) + m(E_1 \cap E_2) &= (m(E_1 \setminus E_2) + m(E_2 \setminus E_1) + m(E_1 \cap E_2)) + m(E_1 \cap E_2) \\
&= (m(E_1 \setminus E_2) + m(E_1 \cap E_2)) + (m(E_2 \setminus E_1) + m(E_1 \cap E_2)).
\end{aligned}$$

Now, let us write E_1 and E_2 as disjoint unions:

$$E_1 = (E_1 \setminus E_2) \cup (E_1 \cap E_2);$$
$$E_2 = (E_2 \setminus E_1) \cup (E_1 \cap E_2)$$

which, again, by the subadditivity of m,

$$m(E_1) = m(E_1 \setminus E_2) + m(E_1 \cap E_2);$$
$$m(E_2) = m(E_2 \setminus E_1) + m(E_1 \cap E_2).$$

Therefore, this gives us that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

showing the desired result. □

Problem 3 (3.13)**.** Prove Proposition 15 by the following steps which I will state below for the record.

Let E be any given set. Then the following are equivalent:

(i) E is measurable.

(ii) For all $\varepsilon > 0$, there is an open set $O \supset E$ with $m^*(O \setminus E) < \varepsilon$.

(iii) For all $\varepsilon > 0$, there is a closed set $F \subset E$ with $m^*(E \setminus F) < \varepsilon$.

(iv) There is a $G \in G_\delta$ with $E \subset G$ such that $m^*(G \setminus E) = 0$.

(v) There is a $F \in F_\sigma$ with $F \subset E$ such that $m^*(E \setminus F) = 0$.

If $m^*(E) < \infty$, the above statements are equivalent:

(vi) For all $\varepsilon > 0$, there is a finite union U of open intervals such that $m^*(U \Delta E) < \varepsilon$.

a. Show that for $m^*E < \infty$, $(i) \Rightarrow (ii) \Leftrightarrow (vi)$.

Proof. To show that the following are equivalent, we will need to create a chain of implications starting with the outline above. For all of these proofs, assume that $m^*(E) < \infty$.

(i) $\Rightarrow$ (ii) Suppose E is a measurable set. Let $\varepsilon > 0$ be chosen. Because E is measurable and thus $m^*(E) = m(E)$, there exists a countable collection of open intervals $\{I_n\}_{n=1}^{\infty}$ so that

$$m(E) + \varepsilon > \sum_{n=1}^{\infty} l(I_n).$$

Since $\{I_n\}_{n=1}^{\infty}$ is open, the set $O = \bigcup_{n=1}^{\infty} I_n$ is an open set as well. By Proposition 3.1, we know that

$$m(O) = m\left(\bigcup_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} l(I_n).$$

Again, because E is measurable, we know that $E \subset O$. Now it is left to show that $m(O \setminus E)$. Because O and E are disjoint, we have that

$$\begin{aligned} m(O \setminus E) &= m(O) - m(E) \\ &= \sum_{n=1}^{\infty} l(I_n) - m(E) \\ &< (m(E) + \varepsilon) - m(E) \\ &= \varepsilon \end{aligned}$$

which completes this direction.

(ii) $\Rightarrow$ (vi) Let $\varepsilon > 0$ be chosen. Then by our hypothesis, there exists an open set O such that $m^*(O \setminus E) < \frac{\varepsilon}{2}$. By the Lindelof Lemma, the set O can be written as countable union of open intervals i.e., there exists a countable collection of intervals $\{I_n\}_{n=1}^{\infty}$ so that $O = \bigcup_{n=1}^{\infty} I_n$. Satisfying the conditions of Proposition 3.5, we can leverage this as suggested and get that

$$\begin{aligned} \sum_{n=1}^{\infty} l(I_n) &= m^* \left(\bigcup_{n=1}^{\infty} I_n \right) \\ &\leq m^*(E) + \frac{\varepsilon}{2}. \end{aligned}$$

This means that there exists $N \in \mathbb{N}$ so that

$$\sum_{n=N+1}^{\infty} l(I_n) = m^* \left(\bigcup_{n=N+1}^{\infty} I_n \right) < \frac{\varepsilon}{2}.$$

Let $\{I_1, \ldots, I_N\}$ be the finite set of collections up to but including $N+1$ and so we let $U = \bigcup_{n=1}^{N} I_n$. We can note that $U \Delta E = (U \setminus E) \cup (E \setminus U)$. Additionally, $U \setminus E \subset O \setminus E$ by construction of U and $E \setminus U \subset O \setminus E$ by hypothesis. Finally, note that

$$O \setminus U = \left(\bigcup_{n=1}^{\infty} I_n \right) \setminus \left(\bigcup_{n=1}^{N} I_n \right) = \bigcup_{n=N+1}^{\infty} I_n.$$

Thus, with of all this, we have that

$$\begin{aligned} m^* (U \Delta E) &= m^* ((U \setminus E) \cup (E \setminus U)) \\ &= m^*(U \setminus E) + m^*(E \setminus U) \\ &\leq m^*(O \setminus E) + m^*(O \setminus U) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

which finishes this direction.

(vi) $\Rightarrow$ (ii) Let $\varepsilon > 0$ be chosen. By assumption, for any set E, there exists a finite union U of open intervals so that

$$m^*(U\Delta E) = m((U \setminus E) \cup (E \setminus U)) < \frac{2\varepsilon}{3}.$$

By Proposition 3.5, there exists an open set $O \supset E \setminus U$ so that

$$m^*(O) \leq m(E \setminus U) + \frac{\varepsilon}{3}$$

which is equivalent to saying that

$$m^*(O \setminus (E \setminus U)) < \frac{\varepsilon}{3}.$$

Note that $E \subset O \cup U$ trivially. Thus, we have that

$$\begin{aligned} m^*(O \setminus E) &\leq m^*((U \cup O) \setminus (E)) \\ &= m^*((U \setminus E) \cup (O \setminus E)) \\ &\leq m((O \setminus (E \setminus U)) \cup (U \setminus E) \cup (E \setminus U)) \\ &= m(O \setminus (E \setminus U)) + m(U \setminus E) + m(E \setminus U) \\ &< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

giving us the desired result.

This completes the first set of our chain of equivalences. $\square$

b. Use part (a) to show that for arbitrary sets, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

Proof. We continue on our journey to a chain of equivalences with this next set! :)

(i) $\Rightarrow$ (ii) Suppose that E is measurable and since we showed that this direction for $m^*(E) < \infty$, suppose $m^*(E) = \infty$. For any $n \in \mathbb{N}$, define the set $E_n = E \cup [-n, n]$. From part(a), there exists an open set $O_n \supset E_n$ for all $n \in \mathbb{N}$ so that

$$m^*(O_n \setminus E_n) < \frac{\varepsilon}{2^n}.$$

Define the set $O = \bigcup_{n=1}^{\infty} O_n$. Then note that $E \subset O$ and $E \subset \bigcup_{n=1}^{\infty} E_n$. Using this, we can show that

$$\begin{aligned} m^*(O \setminus E) &= m^*\left(\bigcup_{n=1}^{\infty} O_n \setminus E\right) \\ &\leq m^*\left(\bigcup_{n=1}^{\infty} O_n \setminus \bigcup_{n=1}^{\infty} E_n\right) \\ &\leq m^*\left(\bigcup_{n=1}^{\infty} O_n \setminus E_n\right) \\ &\leq \sum_{i=1}^{\infty} m^*(O_n \setminus E_n) \\ &< \sum_{i=1}^{\infty} \frac{\varepsilon}{2^k} \\ &= \varepsilon \end{aligned}$$

which completes the proof.

(ii) $\Rightarrow$ (iv) By assumption, we can choose $n \in \mathbb{N}$ so that the open set $O_n \supset E$ implies that $m^*(E \setminus O_n) < \dfrac{1}{n} < \varepsilon$ for any $\varepsilon > 0$, which is possible using the Archimedes principle. Let $G = \bigcap_{n=1}^{\infty} O_n$, which is thus a countable intersection of open sets (i.e., $G \in G_\delta$). Note that $E \subset G \subset O_n$ and so

$$\begin{aligned} m^*(G \setminus E) &\leq m^*(O_n \setminus E) \\ &< \frac{1}{n} \\ &< \varepsilon. \end{aligned}$$

Because we can always find $n \in \mathbb{N}$ for all $\varepsilon < 0$, we have that $m^*(G \setminus E) = 0$. Since we can choose $n \in \mathbb{N}$, certainly $F \subset E$ and $F_n \subset F$ which gives that

$$m^*(E \setminus F) \leq m(E \setminus F_n)$$

(iv) $\Rightarrow$ (i) Assume there exists some $G \in G_\delta$ such that $E \subset G$ and $m^*(G \setminus E) = 0$. Because $G \in G_\delta$ and $m^*(G \setminus E) = 0$, this implies that $G \setminus E$ is a measurable set. But then since $G \setminus E$ is a measurable set, G is a measurable set. Thus since $E = G \setminus (G \setminus E)$, it follows that E is measurable.

This completes this chain of implications. □

c. Use part (b) to show that (i) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (i).

Proof. Finally, can finish the chain on equivalences and finish proving Proposition 3.15.

(i) $\Rightarrow$ (iii) Suppose E is measurable set (i.e., $E \in \mathcal{M}$). Let $\varepsilon > 0$ be chosen. Because $\mathcal{M}$ is a σ-algebra and closed under complement, we know that E^{c} is a measurable set as well. From part (b) (the infinite case of (i) $\Rightarrow$ (ii)), there exists an open set $O \supset E^{\complement}$ so that $m^*\left(O \setminus E^{\complement}\right) < \varepsilon$. Let $F = O^{\complement}$, which is a closed set because its complement is open. Then $F \subset E$ and noting that $O \setminus E^{\complement} = E \cap O = E \setminus F$, we have that

$$m^*(F \setminus E) = m^*(O \setminus E^{\complement}) = m^*(E \cap O) < \varepsilon$$

which completes the proof.

(iii) $\Rightarrow$ (v) Similar to the approach of (ii) $\Rightarrow$ (iv) in part (b), let us choose $n \in \mathbb{N}$ using the Archimedes principle so that a closed $F_n \subset E$ means that

$$m^*(E \setminus F_n) < \frac{1}{n} < \varepsilon \text{ for all } \varepsilon > 0.$$

Let $F = \displaystyle\bigcup_{n=1}^{\infty} F_n$, which is a countable union of closed sets and so $F \in F_\sigma$. Also $F \subset E$ and $F_n \subset F$ for any $n \in \mathbb{N}$ so we know that

$$\begin{aligned} m(E \setminus F) &\leq m(E \setminus F_n) \\ &< \frac{1}{n} \\ &< \varepsilon. \end{aligned}$$

By the same reasoning as the end of the proof of (ii) $\Rightarrow$ (iv) from part (b), we can conclude that $m(E \setminus F) = 0$.

(v) $\Rightarrow$ (i) Again, from part (b), we will use similar logic as (iv) $\Rightarrow$ (i). Because $F \in F_\sigma$ and $m^*(E \setminus F) = 0$, this implies that $E \setminus F$ is a measurable set. But then since $E \setminus F$ is a measurable set, F is a measurable set. Thus since $E = F \cup (E \setminus F)$, it follows that E is measurable.

Finally, having finished the chain of equivalences, we have shown that all of the statements are equivalent, which completes the proof. $\square$