Problem 1 (3.23)**.** Prove Proposition 3.22 by the following lemmas:

a. Given a measurable function f on $[a, b]$ that takes the values $\pm\infty$ only on a set of measure zero, and given $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that $|f| \leq M$ except on a set of measure less than $\frac{\varepsilon}{3}$.

Proof. Suppose f is a measurable function on $[a, b]$ and that $f(x) = \pm\infty$ only on a set of measure zero. Let $\varepsilon > 0$ be chosen. Define the set

$$E_n = \{x \in [a, b] : |f(x)| > n\} \text{ for all } n \in \mathbb{N}.$$

Because the function f is measurable, by definition, this means that each E_i is a measurable set as well. Note that by construction of E_n, we have that $E_i \subset E_{i+1}$ and so $\{E_n\}$ is a decreasing sequence. Since E_1 is a subset of the inverse image of f which is itself a subset of $[a, b]$ i.e., $E_1 \subset [a, b]$, we have that

$$m(E_1) < m([a, b]) = b - a < \infty.$$

Again, by the construction of E_n, we have that

$$\bigcap_{n=1}^{\infty} E_n = \emptyset$$

implying that

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = 0.$$

But having satisfied the conditions of Proposition 3.14, this is the same as saying $E_n \to 0$ as $n \to \infty$ or

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} E_n = 0.$$

Thus, because ε is fixed, we can always find $M \in \mathbb{N}$ such that

$$m(E_M) = m\{x \in [a, b] : |f(x)| > M\} < \frac{\varepsilon}{3}.$$

But this necessarily implies that $|f(x)| \leq M$ for all $x \in [a, b]$ thereby completing the proof. □

b. Let f be a measurable function on $[a, b]$. Given $\varepsilon > 0$ and $M > 0$, there is a simple function ϕ such that $|f(x) - \phi(x)| < \varepsilon$ except where $|f(x)| \geq M$. If $m \leq f \leq M$, then we may take ϕ so that $m \leq \phi \leq M$.

Proof. Suppose f is a measurable function on $[a, b]$. Let $\varepsilon > 0$ and $M > 0$ be chosen. Because ε and M are fixed, by the Archimedes principle, we can choose $N \in \mathbb{N}$ large enough so that $\frac{M}{N} < \varepsilon$. From this, let us define the set

$$E_k = \left\{k\frac{M}{N} \leq f(x) \leq (k+1)\frac{M}{N}\right\}$$

for $k \in [-N, N]$ (integer-valued). Since f is a measurable function, each E_i is a measurable set as well. Let us define the function ϕ by

$$\phi(x) = \sum_{k=-N}^{N} k\left(\frac{M}{N}\right)\chi_{E_k}$$

with $a_i = k\dfrac{M}{N} \in \mathbb{R}$ for each $k \in [-N, N]$. So because ϕ is a linear combination of characteristic functions of E_i and each E_i is a measurable set (in fact, the E_i's are pairwise disjoint), ϕ is a simple function. Suppose that $|f(x)| < M$. Because E_i's are pairwise disjoint, then for all $x \in [a, b]$, $x \in E_k$ for some $k \in [-N, N]$ which implies that

$$k\frac{M}{N} \leq f(x) \leq (k+1)\frac{M}{N}.$$

Thus, $\phi(x) = k\dfrac{M}{N}$ which gives us that

$$\begin{aligned} |f(x) - \phi(x) &= \left|f(x) - k\frac{M}{N}\right| \\ &< \frac{M}{N} \\ &< \varepsilon. \end{aligned}$$

Now suppose that $f(x) \in [m, M]$ for all $x \in [a, b]$ (i.e., f is a bounded function.) Then the same argument holds as before but instead we have that

$$\begin{aligned} |f(x) - \phi(x) &= \left|f(x) - k\frac{M-m}{N}\right| \\ &< \frac{M-m}{N} \\ &< \varepsilon \end{aligned}$$

meaning for all $x \in [a, b]$, we have $\phi(x) = k\dfrac{M-m}{N}$ implying that $\phi(x) \in [m, M]$. □

c. Given a simple function ϕ on $[a, b]$, there is a step function g on $[a, b]$ such that $g(x) = \phi(x)$ except on a set of measure less than $\dfrac{\varepsilon}{3}$.

Proof. Let ϕ by the simple function defined by

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}$$

for measurable, disjoints sets $E_1, \ldots, E_n$ and $a_i \in \mathbb{R}$ for $i = 1, \ldots n$. Let $\varepsilon > 0$ be chosen. Because each E_i is a measurable set, by Proposition 3.15, for each $i = 1, \ldots n$, there exists a finite union U_i of open intervals I_i such that $m(E_i \Delta U_i) < \dfrac{\varepsilon}{3n}$ with

$$U_i = \sum_{k=1}^{N_i} I_{i,k}.[1]$$

[1]This is mostly for myself, but k is the index for the number of intervals N_i associated with each E_i.

Let $A_i = U_i \setminus \left(\bigcup_{j=1}^{i-1} U_j \right)$.[2] For any $x \in [a,b]$, define the function

$$g(x) = \sum_{i=1}^{n} a_i \chi_{A_i}.$$

Because the E_i's are measurable and the difference of measurable sets is measurable, the set $\{A_1, \ldots, A_n\}$ is a set of measurable sets. The A_i's are a subdivision of $[a,b]$ and so g is a step function per the definition on page 76 of Royden. We claim that this function is equal to $\phi(x)$ except on a set of measure less than $\frac{\varepsilon}{3}$. To that end, fix $x \in [a,b]$ so that $\phi(x) \neq g(x)$. Because ϕ and g are linear combinations with the same coefficients, this brings two cases: (i) there is some $i = 1, \ldots n$ so that $g(x) = a_i$ but $\phi(x) \neq a_i$ or (ii) there is some $i = 1, \ldots, n$ so that $g(x) \neq a_i$ but $\phi(x) = a_i$.

For case (i), this means that $x \in A_i \subset U_i \setminus E_i$ for some $i = 1, \ldots, n$. For case (ii), we must have that $x \in E_i \subset E_i \setminus U_i$ for some $i = 1, \ldots n$. So, combining both results,

$$\{x \in [a,b] : \phi(x) \neq g(x)\} \subset \bigcup_{i=1}^{n} U_i \setminus E_i;$$
$$\{x \in [a,b] : \phi(x) \neq g(x)\} \subset \bigcup_{i=1}^{n} E_i \setminus U_i$$

and thereby implies that

$$\{x \in [a,b] : \phi(x) \neq g(x)\} \subset \bigcup_{i=1}^{n} E_i \Delta U_i.$$

Finally, this allows us to show that

$$\begin{aligned} m\left(\{x \in [a,b] : \phi(x) \neq g(x)\}\right) &\leq m\left(\bigcup_{i=1}^{n} E_i \Delta U_i \right) \\ &= \sum_{i=1}^{n} m\left(E_i \setminus U_i\right) \\ &< \sum_{i=1}^{n} \frac{\varepsilon}{3n} \\ &= n \cdot \frac{\varepsilon}{3n} \\ &= \varepsilon \end{aligned}$$

giving us the desired result. $\square$

Problem 2 (3.31)**.** Prove Lusin's Theorem: Let f be a measurable real-valued function on an interval $[a,b]$. Then for all $\delta > 0$, there is a continuous function ϕ on $[a,b]$ such that $m\{x : f(x) \neq \phi(x)\} < \delta$.

[2] Again, mostly for myself, but for each U_i associated with E_i, throw out the preceding $U_i's$.

Proof. Let $\delta > 0$ be chosen. Suppose f is a measurable real-valued function on an interval $[a, b]$. Then by Proposition 3.22, there exists a continuous function h_n for all $n \in \mathbb{N}$ such that

$$|f - h_n| < \frac{\delta}{2^{n+2}}$$

with $m\left\{x \in [a, b] : |f - h_n| \geq \frac{\delta}{2^{n+2}}\right\} < \frac{\delta}{2^{n+2}}$. For convenience, define the sets

$$E_n = \left\{x \in [a, b] : |f - h_n| \geq \frac{\delta}{2^{n+2}}\right\}$$

and

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Note for a fixed $x \in [a, b] \setminus E_n$ and any $n \in \mathbb{N}$, we know by how we defined E_n that

$$|f - h_n| < \frac{\delta}{2^{n+2}}.$$

Thus, since E is the union of the E_n's, we have that

$$\begin{aligned} m(E) &= m\left(\bigcup_{n=1}^{\infty} E_n\right) \\ &\leq \sum_{n=1}^{\infty} m(E_n) \\ &< \sum_{n=1}^{\infty} \frac{\delta}{2^{n+2}} \\ &= \frac{\delta}{4}. \end{aligned}$$

So then on the set $[a, b] \setminus E$, the sequence of continuous, and thereby, measurable functions $\{h_n\}$ converges to f. Having satisfied the conditions of Egoroff's theorem, there exists a set $A \subset [a, b] \setminus E$ with $m(A) < \frac{\delta}{4}$ such that h_n converges uniformly on $([a, b] \setminus E) \setminus A = [a, b] \setminus (E \cup A)$. Since the uniform limit of continuous functions is a continuous function, the function f is continuous on $[a, b] \setminus (E \cup A)$. Because $m(E)$ and $m(A)$ are less than $\frac{\delta}{4}$, $m(E \cup A) < \frac{\delta}{2}$.

Using Proposition 3.15 part (ii), there exists an open set $O \supset (E \cup A)$ with

$$m(O \setminus (E \cup A)) < \frac{\delta}{2}.$$

Because $[a, b] \setminus (E \cup A) \supset [a, b] \setminus O$ and $[a, b] \setminus O = [a, b] \cap O^{\complement}$ (i.e., a closed set), f is continuous on the closed set $[a, b] \setminus O$. Then for any $x \in [a, b] \setminus O$, by Problem 2.40, there exists a continuous function ϕ so that $f(x) = \phi(x)$. But then the set O represents the set of points where ϕ and g are not equal. In particular, we can show that

$$\begin{aligned} m\{x \in [a,b] : f(x) \neq \phi(x)\} &= m(O) \\ &= m((O \setminus (E \cup A)) \cup (E \cup A)) \\ &= m(O \setminus (E \cup A)) + m(E \cup A) \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta \end{aligned}$$

which finally completes the proof. $\square$