Construction of the Real Numbers, $\mathbb{R}$

- We first start from $\mathbb{N} \cup \{0\}$ and add numbers together subsequently (i.e. 1, $\underbrace{1+1}_{2}$, $\underbrace{1+1+1}_{3}$, $\dots$)
- To construct the integers $\mathbb{Z}$, we take the set difference with the natural numbers so that we have
$$\mathbb{Z} = \mathbb{N} \cup \{0\} \setminus \mathbb{N}.$$
- Then the rationals numbers, $\mathbb{Q}$, can be constructed from the integers and are defined by the set
$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \right\}.$$
- To construct the irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$, we can use the dedekind cut to do this. However, this is convoluted and we can go about this in a different way.

Axioms of the Real Numbers

A. The Field Axioms: For all real numbers $x, y \in \mathbb{R}$ we have:

A1. x + y = y + x

A2. (x + y) + z = x + (y + z)

A3. There exists $0 \in \mathbb{R}$ such that $x + 0 =$ for all $x \in \mathbb{R}$.
[**Identity element under addition**]

A4. For each $x \in \mathbb{R}$ there is a $w \in \mathbb{R}$ such that $x + w = 0$.
[**Inverse element under addition**]

A5. xy = yx

A6. (xy)z = x(yz)

A7. There exists $1 \in \mathbb{R}$ such that $1 \neq 0$ and $x \cdot 1 = x$ for all $x \in \mathbb{R}$.

A8. For each $x \in \mathbb{R}$ different from 0 there is $w \in \mathbb{R}$ such that $xw = 1$.

A9. x(y + z) = xy + xz.

We can prove some properties now:

Proposition 1. The additive inverse is unique.

Proof. Let $x \in \mathbb{R}$. Suppose we have two numbers $w_1, w_2 \in \mathbb{R}$ such that $x + w_1 = 0 = x + w_2$. Using the axioms and our assumption, we can show the following:

$$\begin{aligned} w_1 &= w_1 + 0 && \text{Axiom A3} \\ &= w_1 + x + w_2 && \text{Assumption of } 0 = x + w_2 \\ &= w_2 + xw_1 && \text{Axiom A1} \\ &= w_2 \end{aligned}$$

which completes the proof. $\square$

B. The Axioms of Order: The subset P of positive real numbers satisfies the following:

B1. If $x, y \in P$, then $x + y \in P$.

B2. If $x, y \in P$, then $xy \in P$.

B3. If $x \in P$, then $-x \notin P$.

B4. If $x \in \mathbb{R}$, then $x = 0$ or $x \in P$ or $-x \in P$.

Note that any system which satisfies the axioms of groups A and B is called an **ordered field**.

Definition. We can give definitions of the ordered operations $<$, $\leq$, $>$ and $\geq$.

- $x < y$ means that $y - x \in P$.
- $x \leq y$ means that $y - x \in P \cup \{0\}$. Or, this means that $x < y$ or $x = y$.
- $x > y$ means that $x - y \in P$.
- $x \geq y$ means that $x - y \in P \cup \{0\}$. Or, this means that $x > y$ or $x = y$.

From this, we can deduce and prove some which is any set which satisfies the axioms of group A and B.

Definition. Let $x, y \in \mathbb{R}$ and define the absolute value as

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

Proposition 2. Let $a, b, c \in \mathbb{R}$.

1. $a < b$ if and only if $-b < -a$.
2. If $a < b$ and $b < c$, then $a < c$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. For $a, b \in \mathbb{R}$, then only one is true $a = b$, $a > b$ and $a < b$.
5. If $x \neq 0$, then $x^2 = x \cdot x > 0$; in particular, $1 > 0$.
6. If $x, y \in \mathbb{R}$, then $|x + y| \leq |x| + |y|$.

Definition. Let $S \subset \mathbb{R}$. The number $b \in \mathbb{R}$ is an **upper bound** for S if for each $x \in S$, we have $x \leq b$.

Similarly, a number $x \in \mathbb{R}$ is the **least upper bound** for S if it is an upper bound for (S) and if $x \leq b$ for each upper bound b of S. We then call x the **supremum** of S and denote this $x = \sup S$.

Definition. Let $S \subset \mathbb{R}$. The number $l \in \mathbb{R}$ is an **lower bound** for S if for each $x \in S$, we have $l \leq x$.

Similarly, a number $x \in \mathbb{R}$ is the **greatest lower bound** for S if it is a lower bound for S and if $x \leq l$ for each lower bound l of S. We then call x the **infimum** of S and denote this $x = \inf S$.

C. **Completeness Axiom:** Every nonempty set $S \subset \mathbb{R}$ which has an upper bound has a least upper bound.

Proposition 3. Let $L, U \subset \mathbb{R}$ be nonempty subsets with $R = L \cup U$ and such that for each $l \in L$ and each $u \in U$ we have $l < u$. Then either L has a greatest element or L has a least element.

Proposition 4 (Approximation Property.)**.** Let $S \subset \mathbb{R}$ be a nonempty. If $u = \sup S$, then for all $\gamma > 0$, there exists $Sr \in S$ such that $u - r < Sr < u$.