

**Problem 1** (2.9). Properties of sequences in  $\mathbb{R}$ .

- (a.) Show that  $\limsup x_n$  and  $\liminf x_n$  are the largest and smallest cluster points of the sequence  $\{x_n\}$ .

*Proof.* Let  $\{x_n\}$  be any sequence in  $\mathbb{R}$ . First to show that  $l = \limsup x_n$  is indeed a cluster point, let  $\varepsilon > 0$  be chosen. Because  $l$  is the limit superior, there exists  $n_1 \in \mathbb{N}$  such that  $x_{k_1} < l + \varepsilon$  for all  $k_1 \geq n_1$ . Additionally, there are infinitely many value of this value  $n_1$  such that  $x_k > l - \varepsilon$  for some  $k_1 \geq n_1$ , which together with the last sentence implies that  $|x_{k_1} - l| < \varepsilon$ . To inductively create a subsequence, let  $n_1, \dots, n_j$  and  $x_{k_1}, \dots, x_{k_j}$  be arbitrary. Let  $n_{j+1}$  be chosen such that  $n_{j+1} > \max\{k_1, \dots, k_j\}$ . Then, because  $l$  is the limit superior  $x_k < l + \varepsilon$  for any  $k \geq n_{j+1}$ . Further, for sufficiently large  $n_{j+1}$ , there exists  $k_{j+1} \geq n_{j+1}$  such that  $x_{k_{j+1}} > l - \varepsilon$ , which gives us that  $|x_{k_{j+1}} - l| < \varepsilon$ . Because we can always choose the next point in the subsequence in this manner, this means that the subsequence  $\{x_{n_j}\}$  converges to  $l$ . By Problem 2.8, this means that  $l$  is a cluster point of  $\{x_n\}$ .

By way of contradiction, suppose that  $l$  is not the largest cluster point of the sequence. That is, there exists a cluster point  $y$  of  $\{x_n\}$  such that  $y > l$ . Note that by Problem 2.8, this means that there exists a subsequence  $\{x_{n_j}\}$  which converges to  $y$ . Because  $l$  is the limit superior of the sequence, for any  $\varepsilon > 0$ , we can find  $n \in \mathbb{N}$  such that  $x_k < l + \varepsilon$  whenever  $k \geq n$ . Since this is true for any  $\varepsilon > 0$ , we can choose  $\varepsilon > 0$  small enough so that we have

$$l < l + \varepsilon < y - \varepsilon < y.$$

This means that there are a finite number of terms of  $\{x_n\}$  contained within the interval  $(y - \varepsilon, y + \varepsilon)$ . In other words, there does not exist a subsequence  $\{x_{n_j}\}$  which converges to  $y$  as we would necessarily need an infinite number of terms within  $\varepsilon$  of  $y$ —a contradiction. Therefore,  $l$  is the largest cluster point.

By a reverse argument, we can show that  $\liminf x_n$  is a cluster point of  $\{x_n\}$  as well as the smallest cluster point.  $\square$

- (b.) Show that every bounded sequence has a convergent subsequence.

*Proof.* Let  $\{x_n\}$  be a bounded sequence. In other words,  $\sup x_n$  is a finite real number. By definition of the limit superior,  $\limsup x_n \leq \sup x_n$ . From part (a), because  $\limsup x_n$  is a cluster point, we can always construct a convergent subsequence (as well as for the limit inferior of the sequence) and so this completes this part.  $\square$

**Problem 2** (2.43). Let  $f$  be defined as

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ p \sin\left(\frac{1}{q}\right) & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

At what points is  $f$  continuous? (Please justify your answer.)

*Proof.* I claim that  $f$  is not continuous at the rational numbers. To that end, let  $x \in \mathbb{Q}$  and choose  $\varepsilon = x - f(x)$ . Fix  $\delta > 0$ . Note that we can always find an irrational number  $y \in (x, x + \delta)$ . Because  $y$  is irrational, by definition of the function,  $f(y) - f(x) = y - f(x)$ . But then  $y - f(x) > x - f(x) = \varepsilon$ .

For  $x = 0$ , fix  $\varepsilon > 0$ . Choose  $\delta = \varepsilon$ , and pick a point  $y \in \mathbb{R}$  such that  $|x - y| = |y - 0| = |y| < \delta$ . Now because  $\sin(1/q) < 1/q$  for any  $q \in \mathbb{N}$ , we know that

$$\begin{aligned} |f(y) - f(0)| &\leq |y - 0| \\ &< \delta \\ &= \varepsilon \end{aligned}$$

so  $f$  is continuous at 0.

$f$  is also continuous at the irrationals. This is because we can if we pick any point  $x$  in the irrationals, we can find sufficiently large  $q$  so a rational number  $y = \frac{p}{q}$  is close to  $x$  (i.e., for a fixed  $\varepsilon$ , choose  $\delta$  to be smaller than  $f(y) - y$  for this to work). Then this would allow us to bound  $|f(y) - f(x)|$  leveraging that we can put the rational numbers in lowest terms

□

**Problem 3.** Show that  $F \subset \mathbb{R}$  is a closed set if and only if  $F^c$  is open.

*Proof.* To complete this proof, we will need a forward and backwards implication.

( $\Rightarrow$ ) Suppose  $F \subset \mathbb{R}$  is a closed set. Because we desire to show that  $F^c$  is open, let  $x \in F^c$  be a point. This means that  $x \notin F$ . Since  $F$  is a closed set (i.e.,  $F = \overline{F}$ ) and  $x \notin F$ , we know  $x$  is not a point of closure of  $F$ . So there exists  $\delta > 0$  such that for all  $y \in F$ , we do not have  $|x - y| < \delta$ . But then if  $|x - y| < \delta$ , this must mean that  $y \in F^c$ , and so  $F$  is an open set.

( $\Leftarrow$ ) Conversely, suppose that the set  $F^c$  is open. Let  $x \in F^c$ . Then there exists  $\delta > 0$  such that if  $|x - y| < \delta$ , then  $y \in F^c$ . This means that there is no  $y \in F$  such that  $|x - y| < \delta$  and so  $x$  cannot be a point of closure of  $F$ . Thus, because  $x$  is arbitrary,  $F$  necessarily contains all its points of closure; in other words,  $F = \overline{F}$  and thus  $F$  must be closed, completing this direction.

Having completed both implications, this completes the proof.

□

**Problem 1** (3.5). Let  $A$  be the set of rational numbers between 0 and 1, and let  $\{I_n\}$  be a **finite** collection of open intervals covering  $A$ . Then  $\sum_{n=1}^k l(I_n) \geq 1$ .

*Proof.* Define the set  $A = \mathbb{Q} \cap [0, 1]$ . Let  $\{I_n\}_{n=1}^k$  be a **finite** collection of open intervals covering  $A$  meaning we have that

$$A \subset \bigcup_{n=1}^k I_n$$

We can create the following string of inequalities:

$$\begin{aligned}
 1 = l([0, 1]) &= m^*([0, 1]) \\
 &= m^*(\overline{A}) && \text{Density of } \mathbb{Q} \\
 &\leq m^*\left(\bigcup_{k=1}^n I_n\right) && A \subset B \implies \overline{A} \subset \overline{B} \\
 &= m^*\left(\bigcup_{k=1}^n \overline{I_n}\right) && \overline{A \cup B} = \overline{A} \cup \overline{B} \\
 &\leq \sum_{k=1}^n m^*(\overline{I_n}) && \text{Subadditivity of } m^* \\
 &= \sum_{k=1}^n l(\overline{I_n}) \\
 &= \sum_{k=1}^n l(I_n)
 \end{aligned}$$

which shows the desired result, completing the proof.  $\square$

**Problem 2** (3.10). Show that if  $E_1$  and  $E_2$  are measurable, then

$$m(E_1 \cap E_2) + m(E_1 \cup E_2) = mE_1 + mE_2.$$

Recall that if  $E$  is measurable, then  $mE := m^*E$ .

*Proof.* Suppose  $E_1$  and  $E_2$  are measurable sets. To show the above equality, the goal will be to rewrite sets as disjoint unions and utilize the subadditivity of  $m$ . So note that

$$E_1 \cup E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1) \cup (E_1 \cap E_2)$$

and so by the subadditivity of  $m$ ,

$$m(E_1 \cup E_2) = m(E_1 \setminus E_2) + m(E_2 \setminus E_1) + m(E_1 \cap E_2).$$

Thus, adding  $m(E_1 \cap E_2)$  to the left-hand side

$$\begin{aligned}
 m(E_1 \cup E_2) + m(E_1 \cap E_2) &= (m(E_1 \setminus E_2) + m(E_2 \setminus E_1) + m(E_1 \cap E_2)) + m(E_1 \cap E_2) \\
 &= (m(E_1 \setminus E_2) + m(E_1 \cap E_2)) + (m(E_2 \setminus E_1) + m(E_1 \cap E_2)).
 \end{aligned}$$

Now, let us write  $E_1$  and  $E_2$  as disjoint unions:

$$\begin{aligned} E_1 &= (E_1 \setminus E_2) \cup (E_1 \cap E_2); \\ E_2 &= (E_2 \setminus E_1) \cup (E_1 \cap E_2) \end{aligned}$$

which, again, by the subadditivity of  $m$ ,

$$\begin{aligned} m(E_1) &= m(E_1 \setminus E_2) + m(E_1 \cap E_2); \\ m(E_2) &= m(E_2 \setminus E_1) + m(E_1 \cap E_2). \end{aligned}$$

Therefore, this gives us that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

showing the desired result.  $\square$

**Problem 3 (3.13).** Prove Proposition 15 by the following steps which I will state below for the record.

Let  $E$  be any given set. Then the following are equivalent:

- (i)  $E$  is measurable.
- (ii) For all  $\varepsilon > 0$ , there is an open set  $O \supset E$  with  $m^*(O \setminus E) < \varepsilon$ .
- (iii) For all  $\varepsilon > 0$ , there is a closed set  $F \subset E$  with  $m^*(E \setminus F) < \varepsilon$ .
- (iv) There is a  $G \in G_\delta$  with  $E \subset G$  such that  $m^*(G \setminus E) = 0$ .
- (v) There is a  $F \in F_\sigma$  with  $F \subset E$  such that  $m^*(E \setminus F) = 0$ .

If  $m^*(E) < \infty$ , the above statements are equivalent:

- (vi) For all  $\varepsilon > 0$ , there is a finite union  $U$  of open intervals such that  $m^*(U \Delta E) < \varepsilon$ .
- a. Show that for  $m^*E < \infty$ ,  $(i) \Rightarrow (ii) \Leftrightarrow (vi)$ .

*Proof.* To show that the following are equivalent, we will need to create a chain of implications starting with the outline above. For all of these proofs, assume that  $m^*(E) < \infty$ .

- (i)  $\Rightarrow$  (ii) Suppose  $E$  is a measurable set. Let  $\varepsilon > 0$  be chosen. Because  $E$  is measurable and thus  $m^*(E) = m(E)$ , there exists a countable collection of open intervals  $\{I_n\}_{n=1}^\infty$  so that

$$m(E) + \varepsilon > \sum_{n=1}^{\infty} l(I_n).$$

Since  $\{I_n\}_{n=1}^\infty$  is open, the set  $O = \bigcup_{n=1}^{\infty} I_n$  is an open set as well. By Proposition 3.1, we know that

$$m(O) = m\left(\bigcup_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} l(I_n).$$

Again, because  $E$  is measurable, we know that  $E \subset O$ . Now it is left to show that  $m(O \setminus E)$ . Because  $O$  and  $E$  are disjoint, we have that

$$\begin{aligned} m(O \setminus E) &= m(O) - m(E) \\ &= \sum_{n=1}^{\infty} l(I_n) - m(E) \\ &< (m(E) + \varepsilon) - m(E) \\ &= \varepsilon \end{aligned}$$

which completes this direction.

(ii)  $\Rightarrow$  (vi) Let  $\varepsilon > 0$  be chosen. Then by our hypothesis, there exists an open set  $O$  such that  $m^*(O \setminus E) < \frac{\varepsilon}{2}$ . By the Lindelof Lemma, the set  $O$  can be written as countable union of open intervals i.e., there exists a countable collection of intervals  $\{I_n\}_{n=1}^{\infty}$  so that  $O = \bigcup_{n=1}^{\infty} I_n$ . Satisfying the conditions of Proposition 3.5, we can leverage this as suggested and get that

$$\begin{aligned} \sum_{n=1}^{\infty} l(I_n) &= m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \\ &\leq m^*(E) + \frac{\varepsilon}{2}. \end{aligned}$$

This means that there exists  $N \in \mathbb{N}$  so that

$$\sum_{n=N+1}^{\infty} l(I_n) = m^*\left(\bigcup_{n=N+1}^{\infty} I_n\right) < \frac{\varepsilon}{2}.$$

Let  $\{I_1, \dots, I_N\}$  be the finite set of collections up to but including  $N + 1$  and so we let  $U = \bigcup_{n=1}^N I_n$ . We can note that  $U \Delta E = (U \setminus E) \cup (E \setminus U)$ . Additionally,  $U \setminus E \subset O \setminus E$  by construction of  $U$  and  $E \setminus U \subset O \setminus E$  by hypothesis. Finally, note that

$$O \setminus U = \left(\bigcup_{n=1}^{\infty} I_n\right) \setminus \left(\bigcup_{n=1}^N I_n\right) = \bigcup_{n=N+1}^{\infty} I_n.$$

Thus, with of all this, we have that

$$\begin{aligned} m^*(U \Delta E) &= m^*((U \setminus E) \cup (E \setminus U)) \\ &= m^*(U \setminus E) + m^*(E \setminus U) \\ &\leq m^*(O \setminus E) + m^*(O \setminus U) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

which finishes this direction.

(vi)  $\Rightarrow$  (ii) Let  $\varepsilon > 0$  be chosen. By assumption, for any set  $E$ , there exists a finite union  $U$  of open intervals so that

$$m^*(U \Delta E) = m((U \setminus E) \cup (E \setminus U)) < \frac{2\varepsilon}{3}.$$

By Proposition 3.5, there exists an open set  $O \supset E \setminus U$  so that

$$m^*(O) \leq m(E \setminus U) + \frac{\varepsilon}{3}$$

which is equivalent to saying that

$$m^*(O \setminus (E \setminus U)) < \frac{\varepsilon}{3}.$$

Note that  $E \subset O \cup U$  trivially. Thus, we have that

$$\begin{aligned} m^*(O \setminus E) &\leq m^*((U \cup O) \setminus (E)) \\ &= m^*((U \setminus E) \cup (O \setminus E)) \\ &\leq m((O \setminus (E \setminus U)) \cup (U \setminus E) \cup (E \setminus U)) \\ &= m(O \setminus (E \setminus U)) + m(U \setminus E) + m(E \setminus U) \\ &< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

giving us the desired result.

This completes the first set of our chain of equivalences.  $\square$

b. Use part (a) to show that for arbitrary sets, (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iv)  $\Rightarrow$  (i).

*Proof.* We continue on our journey to a chain of equivalences with this next set! :)

(i)  $\Rightarrow$  (ii) Suppose that  $E$  is measurable and since we showed that this direction for  $m^*(E) < \infty$ , suppose  $m^*(E) = \infty$ . For any  $n \in \mathbb{N}$ , define the set  $E_n = E \cup [-n, n]$ . From part(a), there exists an open set  $O_n \supset E_n$  for all  $n \in \mathbb{N}$  so that

$$m^*(O_n \setminus E_n) < \frac{\varepsilon}{2^n}.$$

Define the set  $O = \bigcup_{n=1}^{\infty} O_n$ . Then note that  $E \subset O$  and  $E \subset \bigcup_{n=1}^{\infty} E_n$ . Using

this, we can show that

$$\begin{aligned}
 m^*(O \setminus E) &= m^*\left(\bigcup_{n=1}^{\infty} O_n \setminus E\right) \\
 &\leq m^*\left(\bigcup_{n=1}^{\infty} O_n \setminus \bigcup_{n=1}^{\infty} E_n\right) \\
 &\leq m^*\left(\bigcup_{n=1}^{\infty} O_n \setminus E_n\right) \\
 &\leq \sum_{i=1}^{\infty} m^*(O_n \setminus E_n) \\
 &< \sum_{i=1}^{\infty} \frac{\varepsilon}{2^k} \\
 &= \varepsilon
 \end{aligned}$$

which completes the proof.

- (ii)  $\Rightarrow$  (iv) By assumption, we can choose  $n \in \mathbb{N}$  so that the open set  $O_n \supset E$  implies that  $m^*(E \setminus O_n) < \frac{1}{n} < \varepsilon$  for any  $\varepsilon > 0$ , which is possible using the Archimedes principle. Let  $G = \bigcap_{n=1}^{\infty} O_n$ , which is thus a countable intersection of open sets (i.e.,  $G \in G_{\delta}$ ). Note that  $E \subset G \subset O_n$  and so

$$\begin{aligned}
 m^*(G \setminus E) &\leq m^*(O_n \setminus E) \\
 &< \frac{1}{n} \\
 &< \varepsilon.
 \end{aligned}$$

Because we can always find  $n \in \mathbb{N}$  for all  $\varepsilon < 0$ , we have that  $m^*(G \setminus E) = 0$ . Since we can choose  $n \in \mathbb{N}$ , certainly  $F \subset E$  and  $F_n \subset F$  which gives that

$$m^*(E \setminus F) \leq m(E \setminus F_n)$$

- (iv)  $\Rightarrow$  (i) Assume there exists some  $G \in G_{\delta}$  such that  $E \subset G$  and  $m^*(G \setminus E) = 0$ . Because  $G \in G_{\delta}$  and  $m^*(G \setminus E) = 0$ , this implies that  $G \setminus E$  is a measurable set. But then since  $G \setminus E$  is a measurable set,  $G$  is a measurable set. Thus since  $E = G \setminus (G \setminus E)$ , it follows that  $E$  is measurable.

This completes this chain of implications. □

- c. Use part (b) to show that (i)  $\Rightarrow$  (iii)  $\Rightarrow$  (v)  $\Rightarrow$  (i).

*Proof.* Finally, can finish the chain on equivalences and finish proving Proposition 3.15.

- (i)  $\Rightarrow$  (iii) Suppose  $E$  is measurable set (i.e.,  $E \in \mathcal{M}$ ). Let  $\varepsilon > 0$  be chosen. Because  $\mathcal{M}$  is a  $\sigma$ -algebra and closed under complement, we know that  $E^c$  is a measurable

set as well. From part (b) (the infinite case of (i)  $\Rightarrow$  (ii)), there exists an open set  $O \supset E^c$  so that  $m^*(O \setminus E^c) < \varepsilon$ . Let  $F = O^c$ , which is a closed set because its complement is open. Then  $F \subset E$  and noting that  $O \setminus E^c = E \cap O = E \setminus F$ , we have that

$$m^*(F \setminus E) = m^*(O \setminus E^c) = m^*(E \cap O) < \varepsilon$$

which completes the proof.

(iii)  $\Rightarrow$  (v) Similar to the approach of (ii)  $\Rightarrow$  (iv) in part (b), let us choose  $n \in \mathbb{N}$  using the Archimedes principle so that a closed  $F_n \subset E$  means that

$$m^*(E \setminus F_n) < \frac{1}{n} < \varepsilon \text{ for all } \varepsilon > 0.$$

Let  $F = \bigcup_{n=1}^{\infty} F_n$ , which is a countable union of closed sets and so  $F \in F_{\sigma}$ . Also  $F \subset E$  and  $F_n \subset F$  for any  $n \in \mathbb{N}$  so we know that

$$\begin{aligned} m(E \setminus F) &\leq m(E \setminus F_n) \\ &< \frac{1}{n} \\ &< \varepsilon. \end{aligned}$$

By the same reasoning as the end of the proof of (ii)  $\Rightarrow$  (iv) from part (b), we can conclude that  $m(E \setminus F) = 0$ .

(v)  $\Rightarrow$  (i) Again, from part (b), we will use similar logic as (iv)  $\Rightarrow$  (i). Because  $F \in F_{\sigma}$  and  $m^*(E \setminus F) = 0$ , this implies that  $E \setminus F$  is a measurable set. But then since  $E \setminus F$  is a measurable set,  $F$  is a measurable set. Thus since  $E = F \cup (E \setminus F)$ , it follows that  $E$  is measurable.

Finally, having finished the chain of equivalences, we have shown that all of the statements are equivalent, which completes the proof.  $\square$



**Problem 1** (3.23). Prove Proposition 3.22 by the following lemmas:

- a. Given a measurable function  $f$  on  $[a, b]$  that takes the values  $\pm\infty$  only on a set of measure zero, and given  $\varepsilon > 0$ , there is an  $M \in \mathbb{R}$  such that  $|f| \leq M$  except on a set of measure less than  $\frac{\varepsilon}{3}$ .

*Proof.* Suppose  $f$  is a measurable function on  $[a, b]$  and that  $f(x) = \pm\infty$  only on a set of measure zero. Let  $\varepsilon > 0$  be chosen. Define the set

$$E_n = \{x \in [a, b] : |f(x)| > n\} \text{ for all } n \in \mathbb{N}.$$

Because the function  $f$  is measurable, by definition, this means that each  $E_i$  is a measurable set as well. Note that by construction of  $E_n$ , we have that  $E_i \subset E_{i+1}$  and so  $\{E_n\}$  is a decreasing sequence. Since  $E_1$  is a subset of the inverse image of  $f$  which is itself a subset of  $[a, b]$  i.e.,  $E_1 \subset [a, b]$ , we have that

$$m(E_1) < m([a, b]) = b - a < \infty.$$

Again, by the construction of  $E_n$ , we have that

$$\bigcap_{n=1}^{\infty} E_n = \emptyset$$

implying that

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = 0.$$

But having satisfied the conditions of Proposition 3.14, this is the same as saying  $E_n \rightarrow 0$  as  $n \rightarrow \infty$  or

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} m(E_n) = 0.$$

Thus, because  $\varepsilon$  is fixed, we can always find  $M \in \mathbb{N}$  such that

$$m(E_M) = m\{x \in [a, b] : |f(x)| > M\} < \frac{\varepsilon}{3}.$$

But this necessarily implies that  $|f(x)| \leq M$  for all  $x \in [a, b]$  thereby completing the proof.  $\square$

- b. Let  $f$  be a measurable function on  $[a, b]$ . Given  $\varepsilon > 0$  and  $M > 0$ , there is a simple function  $\phi$  such that  $|f(x) - \phi(x)| < \varepsilon$  except where  $|f(x)| \geq M$ . If  $m \leq f \leq M$ , then we may take  $\phi$  so that  $m \leq \phi \leq M$ .

*Proof.* Suppose  $f$  is a measurable function on  $[a, b]$ . Let  $\varepsilon > 0$  and  $M > 0$  be chosen. Because  $\varepsilon$  and  $M$  are fixed, by the Archimedes principle, we can choose  $N \in \mathbb{N}$  large enough so that  $\frac{M}{N} < \varepsilon$ . From this, let us define the set

$$E_k = \left\{ k \frac{M}{N} \leq f(x) \leq (k+1) \frac{M}{N} \right\}$$

for  $k \in [-N, N]$  (integer-valued). Since  $f$  is a measurable function, each  $E_i$  is a measurable set as well. Let us define the function  $\phi$  by

$$\phi(x) = \sum_{k=-N}^N k \left( \frac{M}{N} \right) \chi_{E_k}$$

with  $a_i = k \frac{M}{N} \in \mathbb{R}$  for each  $k \in [-N, N]$ . So because  $\phi$  is a linear combination of characteristic functions of  $E_i$  and each  $E_i$  is a measurable set (in fact, the  $E_i$ 's are pairwise disjoint),  $\phi$  is a simple function. Suppose that  $|f(x)| < M$ . Because  $E_i$ 's are pairwise disjoint, then for all  $x \in [a, b]$ ,  $x \in E_k$  for some  $k \in [-N, N]$  which implies that

$$k \frac{M}{N} \leq f(x) \leq (k+1) \frac{M}{N}.$$

Thus,  $\phi(x) = k \frac{M}{N}$  which gives us that

$$\begin{aligned} |f(x) - \phi(x)| &= \left| f(x) - k \frac{M}{N} \right| \\ &< \frac{M}{N} \\ &< \varepsilon. \end{aligned}$$

Now suppose that  $f(x) \in [m, M]$  for all  $x \in [a, b]$  (i.e.,  $f$  is a bounded function.) Then the same argument holds as before but instead we have that

$$\begin{aligned} |f(x) - \phi(x)| &= \left| f(x) - k \frac{M-m}{N} \right| \\ &< \frac{M-m}{N} \\ &< \varepsilon \end{aligned}$$

meaning for all  $x \in [a, b]$ , we have  $\phi(x) = k \frac{M-m}{N}$  implying that  $\phi(x) \in [m, M]$ . □

- c. Given a simple function  $\phi$  on  $[a, b]$ , there is a step function  $g$  on  $[a, b]$  such that  $g(x) = \phi(x)$  except on a set of measure less than  $\frac{\varepsilon}{3}$ .

*Proof.* Let  $\phi$  be the simple function defined by

$$\phi(x) = \sum_{i=1}^n a_i \chi_{E_i}$$

for measurable, disjoint sets  $E_1, \dots, E_n$  and  $a_i \in \mathbb{R}$  for  $i = 1, \dots, n$ . Let  $\varepsilon > 0$  be chosen. Because each  $E_i$  is a measurable set, by Proposition 3.15, for each  $i = 1, \dots, n$ , there exists a finite union  $U_i$  of open intervals  $I_i$  such that  $m(E_i \Delta U_i) < \frac{\varepsilon}{3n}$  with

$$U_i = \sum_{k=1}^{N_i} I_{i,k}.^1$$

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<sup>1</sup>This is mostly for myself, but  $k$  is the index for the number of intervals  $N_i$  associated with each  $E_i$ .

Let  $A_i = U_i \setminus \left( \bigcup_{j=1}^{i-1} U_j \right)$ .<sup>2</sup> For any  $x \in [a, b]$ , define the function

$$g(x) = \sum_{i=1}^n a_i \chi_{A_i}.$$

Because the  $E_i$ 's are measurable and the difference of measurable sets is measurable, the set  $\{A_1, \dots, A_n\}$  is a set of measurable sets. The  $A_i$ 's are a subdivision of  $[a, b]$  and so  $g$  is a step function per the definition on page 76 of Royden. We claim that this function is equal to  $\phi(x)$  except on a set of measure less than  $\frac{\varepsilon}{3}$ . To that end, fix  $x \in [a, b]$  so that  $\phi(x) \neq g(x)$ . Because  $\phi$  and  $g$  are linear combinations with the same coefficients, this brings two cases: (i) there is some  $i = 1, \dots, n$  so that  $g(x) = a_i$  but  $\phi(x) \neq a_i$  or (ii) there is some  $i = 1, \dots, n$  so that  $g(x) \neq a_i$  but  $\phi(x) = a_i$ .

For case (i), this means that  $x \in A_i \subset U_i \setminus E_i$  for some  $i = 1, \dots, n$ . For case (ii), we must have that  $x \in E_i \subset E_i \setminus U_i$  for some  $i = 1, \dots, n$ . So, combining both results,

$$\begin{aligned} \{x \in [a, b] : \phi(x) \neq g(x)\} &\subset \bigcup_{i=1}^n U_i \setminus E_i; \\ \{x \in [a, b] : \phi(x) \neq g(x)\} &\subset \bigcup_{i=1}^n E_i \setminus U_i \end{aligned}$$

and thereby implies that

$$\{x \in [a, b] : \phi(x) \neq g(x)\} \subset \bigcup_{i=1}^n E_i \Delta U_i.$$

Finally, this allows us to show that

$$\begin{aligned} m(\{x \in [a, b] : \phi(x) \neq g(x)\}) &\leq m\left(\bigcup_{i=1}^n E_i \Delta U_i\right) \\ &= \sum_{i=1}^n m(E_i \setminus U_i) \\ &< \sum_{i=1}^n \frac{\varepsilon}{3n} \\ &= n \cdot \frac{\varepsilon}{3n} \\ &= \varepsilon \end{aligned}$$

giving us the desired result.  $\square$

**Problem 2 (3.31).** Prove Lusin's Theorem: Let  $f$  be a measurable real-valued function on an interval  $[a, b]$ . Then for all  $\delta > 0$ , there is a continuous function  $\phi$  on  $[a, b]$  such that  $m\{x : f(x) \neq \phi(x)\} < \delta$ .

<sup>2</sup>Again, mostly for myself, but for each  $U_i$  associated with  $E_i$ , throw out the preceding  $U_i$ 's.

*Proof.* Let  $\delta > 0$  be chosen. Suppose  $f$  is a measurable real-valued function on an interval  $[a, b]$ . Then by Proposition 3.22, there exists a continuous function  $h_n$  for all  $n \in \mathbb{N}$  such that

$$|f - h_n| < \frac{\delta}{2^{n+2}}$$

with  $m \left\{ x \in [a, b] : |f - h_n| \geq \frac{\delta}{2^{n+2}} \right\} < \frac{\delta}{2^{n+2}}$ . For convenience, define the sets

$$E_n = \left\{ x \in [a, b] : |f - h_n| \geq \frac{\delta}{2^{n+2}} \right\}$$

and

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Note for a fixed  $x \in [a, b] \setminus E_n$  and any  $n \in \mathbb{N}$ , we know by how we defined  $E_n$  that

$$|f - h_n| < \frac{\delta}{2^{n+2}}.$$

Thus, since  $E$  is the union of the  $E_n$ 's, we have that

$$\begin{aligned} m(E) &= m \left( \bigcup_{n=1}^{\infty} E_n \right) \\ &\leq \sum_{n=1}^{\infty} m(E_n) \\ &< \sum_{n=1}^{\infty} \frac{\delta}{2^{n+2}} \\ &= \frac{\delta}{4}. \end{aligned}$$

So then on the set  $[a, b] \setminus E$ , the sequence of continuous, and thereby, measurable functions  $\{h_n\}$  converges to  $f$ . Having satisfied the conditions of Egoroff's theorem, there exists a set  $A \subset [a, b] \setminus E$  with  $m(A) < \frac{\delta}{4}$  such that  $h_n$  converges uniformly on  $([a, b] \setminus E) \setminus A = [a, b] \setminus (E \cup A)$ . Since the uniform limit of continuous functions is a continuous function, the function  $f$  is continuous on  $[a, b] \setminus (E \cup A)$ . Because  $m(E)$  and  $m(A)$  are less than  $\frac{\delta}{4}$ ,  $m(E \cup A) < \frac{\delta}{2}$ .

Using Proposition 3.15 part (ii), there exists an open set  $O \supset (E \cup A)$  with

$$m(O \setminus (E \cup A)) < \frac{\delta}{2}.$$

Because  $[a, b] \setminus (E \cup A) \supset [a, b] \setminus O$  and  $[a, b] \setminus O = [a, b] \cap O^c$  (i.e., a closed set),  $f$  is continuous on the closed set  $[a, b] \setminus O$ . Then for any  $x \in [a, b] \setminus O$ , by Problem 2.40, there exists a continuous function  $\phi$  so that  $f(x) = \phi(x)$ . But then the set  $O$  represents the

set of points where  $\phi$  and  $g$  are not equal. In particular, we can show that

$$\begin{aligned} m\{x \in [a, b] : f(x) \neq \phi(x)\} &= m(O) \\ &= m((O \setminus (E \cup A)) \cup (E \cup A)) \\ &= m(O \setminus (E \cup A)) + m(E \cup A) \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta \end{aligned}$$

which finally completes the proof. □

**Problem 1 (4.2).** (a) Let  $f$  be a bounded function on  $[a, b]$ , and let  $h$  be the upper envelope of  $f$  (cf. Problem 2.51). Then  $R \int_a^b f = \int_a^b h$ .

*Proof.* Let  $f$  be a bounded function on  $[a, b]$  with  $h(y) = \inf_{\delta > 0} \sup_{|x-y| < \delta} f(x)$  for all  $x \in [a, b]$  be the upper envelope of  $f$ . Because  $f$  is bounded, by Problem 2.51 part (b),  $h$  is lower semicontinuous. To show equality, we will show that

$$R \int_a^b f \leq \int_a^b h \quad \text{and} \quad R \int_a^b f \geq \int_a^b h.$$

Let  $\phi$  be a step function on  $[a, b]$  such that  $\phi \geq f$ . Then for any  $x \in [a, b]$ ,  $h(x) \leq f(x) \leq \phi(x)$ , except at the defined partition points of  $\phi$ . Thus, by using the definition of Lebesgue integration, we have that

$$\int_a^b h(x) dx \leq \int_a^b f(x) dx = \inf \int_a^b \phi(x) dx \leq R \int_a^b f(x) dx.$$

For the other inequality, we note that because  $h$  is upper semicontinuous, by Problem 2.51 part (g), there exists a monotonically decreasing sequence of step functions  $\{\phi_n\}$  such that  $\phi_n \rightarrow h$  pointwise. Because  $f$  is bounded, we have that for all  $x \in [a, b]$ , there exists some  $M > 0$  such that

$$|\phi_n| \leq |h| \leq |f| \leq M \text{ for all } n \in \mathbb{N}.$$

Thus using the Bounded Convergence Theorem and properties of the upper Riemann integral,

$$\lim_{n \rightarrow \infty} \int_a^b \phi_n(x) dx = \int_a^b h(x) dx \leq R \int_a^b f(x) dx \leq R \int_a^b f(x) dx.$$

Therefore,

$$R \int_a^b f = \int_a^b h$$

which is the desired result. □

(b) Use part (a) to prove Proposition 7 which is stated as follows

**Proposition (4.7).** A bounded function  $f$  on  $[a, b]$  is Riemann integrable if and only if the set of points at which  $f$  is discontinuous has measure zero.

*Proof.* Let  $f$  be a bounded function on  $[a, b]$ . We will need to show a forward and backwards implication to complete this proof. For simplicity, define  $E$  to be the set of discontinuities of  $f$ . Additionally, let  $g(y) = \sup_{\delta > 0} \inf_{|x-y| < \delta} f(x)$  be the lower envelope of  $f$ .

( $\Leftarrow$ ) First, suppose  $m(E) = 0$ . Since  $g$  is the lower envelope of  $f$ , there exists a monotonically increasing sequence of step functions  $\{\phi_n\} \rightarrow g$  pointwise. Thus, by a similar but reverse argument to part (a),

$$\int_a^b g(x) dx = \int_a^b f(x) dx. \tag{1}$$

So because  $f$  is continuous everywhere except on the set  $E$ —namely, continuous on  $[a, b] \setminus E$ —by Problem 2.51,  $g(x) = h(x)$  is continuous on the set  $[a, b] \setminus E$ . But since  $m(E) = 0$ , this means  $g = h$  almost everywhere and thus, using part (a) of this problem and Equation (1),

$$\int_a^b f(x) \, dx = \int_a^b g(x) \, dx = \int_a^b h(x) \, dx = \overline{\int_a^b f(x) \, dx}.$$

Thus,  $f$  is Riemann integrable.

( $\Rightarrow$ ) Now suppose  $f$  is Riemann integrable. Thus, the lower and upper integrals of  $f$  are equal to each other and so using Equation (1)

$$\int_a^b h(x) \, dx = \int_a^b g(x) \, dx.$$

Consider the set  $A_n = \left\{ x : |g(x) - h(x)| > \frac{1}{n} \right\}$  for all  $n \in \mathbb{N}$ . Because the integrals of  $g$  and  $h$  are equal,

$$\int_a^b |h(x) - g(x)| \, dx = 0.$$

So for any fixed  $n \in \mathbb{N}$ ,

$$\int_a^b |h(x) - g(x)| \, dx \geq m(A_n).$$

So  $h(x) = g(x)$  almost everywhere and so by Problem 2.51 part(a), we must that  $f$  is continuous almost everywhere as well. But then this means that the measure of discontinuities is zero—that is,  $m(E) = 0$  which is what we wanted to show.

Having completed both directions, we have shown the desired equivalence.  $\square$

**Problem 2** (4.3). Let  $f$  be a nonnegative measurable function. Show that  $\int f = 0$  implies  $f = 0$  almost everywhere.

*Proof.* Let  $f \geq 0$  be a measurable function, and suppose that  $\int f = 0$ . We want to show that the set  $E = \{x : f(x) > 0\} = \{x : f(x) > 0\}$  has measure 0. Define the set

$$E_n = \left\{ x : f(x) \geq \frac{1}{n} \right\} \text{ for all } n \in \mathbb{N}.$$

Note that  $\bigcup_{n=1}^{\infty} E_n = E$ . Fix  $n \in \mathbb{N}$ . Because the integral of  $f$  is equal to 0,

$$0 = \int_{E_n} f \geq \int_{E_n} \frac{1}{n} = m(E_n) \cdot \frac{1}{n} \geq 0.$$

Thus, because  $n \in \mathbb{N}$  was fixed,  $m\left(\bigcup_{n=1}^{\infty} E_n\right) = 0 = m(E)$ , and therefore  $f = 0$  almost everywhere.  $\square$

**Problem 3** (4.8). Prove the following generalization of Fatou's Lemma: If  $f_n$  is a sequence of nonnegative functions then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

*Proof.* Let  $\{f_n\} \geq 0$  for each  $n \in \mathbb{N}$  on any set  $E$ . Define  $h_n = \inf_{k \geq n} f_k$  for all  $n \in \mathbb{N}$ . Note that as  $n \rightarrow \infty$ ,  $h_n \rightarrow \liminf_{n \rightarrow \infty} f_n$  (i.e.,  $h_n$  converges pointwise on  $E$  to the limit inferior of  $f_n$ ). Thus, by Fatou's lemma, we have that

$$\int_E \liminf_{n \rightarrow \infty} f \leq \int_E \liminf_{n \rightarrow \infty} h_n.$$

But since  $h_n$  is the infimum of the  $f_n$ 's, this implies that  $h_n \leq f_n$  for all  $n \in \mathbb{N}$  and so

$$\int_E h_n \leq \int_E f_n$$

and thus

$$\liminf_{n \rightarrow \infty} \int_E h_n \leq \liminf_{n \rightarrow \infty} \int_E f_n.$$

By combining all three inequalities, we get that

$$\int_E \liminf_{n \rightarrow \infty} f \leq \liminf_{n \rightarrow \infty} \int_E h_n \leq \liminf_{n \rightarrow \infty} \int_E f_n$$

which then completes the proof. □