

## Section 3.5, Measurable Functions

**Proposition (3.18).** Let  $E \subset \mathbb{R}$ , and Let  $f : E \rightarrow [-\infty, \infty]$  be an extended real-valued function whose domain is measurable. Let  $\alpha \in \mathbb{R}$  be any real number. Then the following statements are equivalent:

- (i) The set  $\{x : f(x) > \alpha\}$  is measurable.
- (ii) The set  $\{x : f(x) \geq \alpha\}$  is measurable.
- (iii) The set  $\{x : f(x) < \alpha\}$  is measurable.
- (iv) The set  $\{x : f(x) \leq \alpha\}$  is measurable.

All together, these imply

- (v) The set  $\{x : f(x) = \alpha\}$  is measurable.

*Proof.* <sup>1</sup>

□

**Definition.** An extended real-valued function  $f : E \rightarrow [-\infty, \infty]$  is **(Lebesgue) measurable** if its domain is measurable and satisfies one of the first four statements of Proposition 18.

Note that this means that any continuous function is measurable since then the pre-image of any open set is still an open set.

**Proposition (3.19).** Let  $f$  and  $g$  be two measurable functions defined on the same domain, and let  $c \in \mathbb{R}$ . Then the functions  $f + c$ ,  $cf$ ,  $f + g$ ,  $g - f$ , and  $fg$  are measurable.

*Proof.* Let  $\alpha \in \mathbb{R}$  be any real number. Fix  $c \in \mathbb{R}$ . For  $f(x) + c$ , note that

$$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}$$

and  $\alpha - c$  is a real number, this set is still measurable i.e.,  $f + c$  is measurable. A similar argument shows that  $cf$  is measurable as well.

Take the set

$$\{x : f(x) + g(x) < \alpha\}. \tag{1}$$

Observe that  $f(x) + g(x) < \alpha \Leftrightarrow f(x) < \alpha - g(x)$ . By the density of  $\mathbb{Q}$ , there exists  $r \in \mathbb{Q}$  such that  $f(x) < r < \alpha - g(x)$ . So we can write Equation (1) as

$$\{x : f(x) + g(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} (\{x : f(x) < r\} \cap \{x : g(x) < \alpha - r\}).$$

Because this set is countable, this set is measurable and thus  $f + g$  is measurable.

To show that  $fg$  is measurable, we can show that  $f^2$  is measurable since

$$fg = \frac{1}{2} ((f + g)^2 - f^2 - g^2).$$

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<sup>1</sup>Proof is on page 67.

Take the set

$$\{x : f^2(x) < \alpha\}. \quad (2)$$

For  $\alpha \geq 0$ , note that  $f^2 < \alpha$  is the same as saying  $f(x) > \sqrt{\alpha}$  and  $f(x) < -\sqrt{\alpha}$ . Thus, Equation (2) can be rewritten as

$$\{x : f^2(x) < \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}.$$

This is a measurable set which completes the proof.  $\square$

**Theorem (3.20, Limit of Measurable Functions is Measurable).**<sup>2</sup> Let  $\{f_n\}$  be a sequence of measurable functions with the same domain. Then the functions  $\sup\{f_1(x), \dots, f_n(x)\}$ ,  $\inf\{f_1(x), \dots, f_n(x)\}$ ,  $\sup_n f_n$ ,  $\inf_n f_n$ ,  $\overline{\lim} f_n$ , and  $\underline{\lim} f_n$  are measurable.

*Proof.* Let  $\{f_n\}$  be a sequence of measurable functions. Let  $h(x) = \sup\{f_1(x), \dots, f_n(x)\}$  and we so must show that  $\{x : h(x) < \alpha\}$  for all  $\alpha \in \mathbb{R}$ . To that end, let  $\alpha \in \mathbb{R}$  be chosen. Then

$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^n \{x : f_i(x) > \alpha\}$$

which, because the right-hand side is a union of measurable sets from the  $f_i$ 's being measurable, means that the set  $\{x : h(x) < \alpha\}$  is also measurable.

Let  $g(x) = \sup_n f_n$ . By a similar argument as above,

$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i(x) > \alpha\}$$

is a countable set so  $\{x : g(x) < \alpha\}$  is measurable.  $\square$

**Definition.** A property is said to hold **almost everywhere** (a.e) if the set of points where it fails to hold is a set of measure zero. Thus  $f = g$  a.e if  $f$  and  $g$  have the same domain and  $m\{x : f(x) \neq g(x)\} = 0$ .

**Proposition (3.21).**<sup>3</sup>

If  $f$  is measurable and  $f = g$  a.e, then  $g$  is measurable.

*Proof.* Let  $E = \{x : f(x) \neq g(x)\}$ .

This is equivalent to saying that

$$\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha\}$$

$\square$

The following proposition essentially says that measurable functions are nearly continuous; or, in other words, we can “nicely” approximate measurable functions.

<sup>2</sup>Proof is on bottom of page 68 and top of page 69

<sup>3</sup>Proof is on middle of page 69.

**Proposition (3.22, Littlewood's 2nd Principle).** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a measurable function with  $E \subset \mathbb{R}$  and is equal to  $\pm\infty$  only on sets with measure zero. Then for all  $\varepsilon > 0$ , there exist a step function  $g$  and a continuous function  $h$  such

$$|f - g| < \varepsilon \quad \text{and} \quad |f - h| < \varepsilon$$

except on set of measure less than  $\varepsilon$ ; i.e.,  $m\{x : |f(x) - g(x)| \geq \varepsilon\} < \varepsilon$  and  $m\{x : |f(x) - h(x)| \geq \varepsilon\} < \varepsilon$ . If in addition  $m \leq f \leq M$ , then we may choose the functions  $g$  and  $h$  so that  $m \leq g \leq M$  and  $m \leq h \leq M$ .