Compactness

Theorem. Let $E \subset \mathbb{R}$. Then E is compact if and only E is sequentially compact. That is, for every $\{x_n\}$ in E, there exists a convergent subsequence $x_{n_m} \to x_0$ in E.

Theorem. Let $\{I_n\}$ be a sequence of closed intervals such that $I_{n+1} \subset I_n$. Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

If $[a_n, b_n]$ is an interval and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} = a$, then $\bigcap_{n=1}^{\infty}$.

Section 2.6, Continuous Functions

Definition. Let $E \subset \mathbb{R}$, and let $f : E \to \mathbb{R}$ be a real-valued function. Then f is **continuous** at the point $x = a \in E$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in E$ with $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

Note that we can have continuity in terms of sequences. I will state it as a theorem here even though it was not in lecture because it is important to be able to use on its own.

Theorem. Let $f : E \to \mathbb{R}$ be a function with $E \subset \mathbb{R}$. Let $x \in E$ be any point. Then f is continuous at a if and only for every sequence $\{x_n\}$ in E converging to a, the sequence $\{f(x_n)\}$ in $f(E)$ (the image of E) converges to $f(a)$.

Proposition. Let $E \subset \mathbb{R}$ be compact. Let $f : E \to \mathbb{R}$ be continuous real-valued function. Then $f(E)$ is a compact set.

Proof. Let $\subset \mathbb{R}$ be a compact and suppose the function $f : E \to \mathbb{R}$ is continuous. To show that $f(E)$ is compact, we will use the Heine-Borel theorem and show that it is closed and bounded. To show that $f(E)$ is closed, suppose we have any sequence $\{f(x_n)\}$ converging to the point $f(a) \in \mathbb{R}$. Additionally, let $\{x_n\}$ be any sequence in E. Because E is compact, there exists a subsequence $\{x_{n_m}\}$ which converges to a point $x_0 \in E$. Since f is continuous, by the preceding theorem this means that the sequence $\{f(x_{n_m})\}$ converges to $f(x_0) \in f(E)$. $\square$

Proposition (2.17, Extreme Value Theorem). Let $E \subset \mathbb{R}$ be a compact set, and let $f : E \to \mathbb{R}$ be a continuous function. Then there exists $x_1, x_2 \in E$ such that

$$f(x_1) \leq f(x) \leq (x_2), \text{ for all x} \in E.$$

Proposition (2.18). Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is **continuous** if and only if $f^{-1}(O)$ is open for all open sets $O \subset \mathbb{R}$.

Proposition (2.19). Let $E \subset \mathbb{R}$, and let $f : E \to \mathbb{R}$ be continuous. Without loss of generality, suppose that $f(a) \leq f(b)$. Then for all $\gamma \in [f(a), f(b)]$, there exists $c \in [a, b]$ such that $f(c) = \gamma$.

Definition (Uniform Continuity). Let $E \subset \mathbb{R}$. A function $f : E \to \mathbb{R}$ is **uniformly continuous** if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

Proposition (2.20). Let $E \subset \mathbb{R}$ be a compact set. If $f : E \to \mathbb{R}$ is a continuous function on E, then f is uniformly continuous on E.

Definition. Let $f_n : E \to \mathbb{R}$ be a sequence of functions, and let $f : E \to \mathbb{R}$.

1. The sequence $\{f_n\}$ **converges pointwise** on E to f if for all $\varepsilon > 0$ and for all $x \in E$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f(x) - f_n(x)| < \varepsilon$.

2. The sequence $\{f_n\}$ **converges uniformly** if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in E$ and for all $n \geq N$, $|f(x) - f_n(z)| < \varepsilon$.

Section 3.1, Lebesgue Measure

[Perhaps finish these notes another time...]