

Problem 1 (2.9). Properties of sequences in \mathbb{R} .

- (a.) Show that $\limsup x_n$ and $\liminf x_n$ are the largest and smallest cluster points of the sequence $\{x_n\}$.

Proof. Let $\{x_n\}$ be any sequence in \mathbb{R} . First to show that $l = \limsup x_n$ is indeed a cluster point, let $\varepsilon > 0$ be chosen. Because l is the limit superior, there exists $n_1 \in \mathbb{N}$ such that $x_{k_1} < l + \varepsilon$ for all $k_1 \geq n_1$. Additionally, there are infinitely many value of this value n_1 such that $x_k > l - \varepsilon$ for some $k_1 \geq n_1$, which together with the last sentence implies that $|x_{k_1} - l| < \varepsilon$. To inductively create a subsequence, let n_1, \dots, n_j and x_{k_1}, \dots, x_{k_j} be arbitrary. Let n_{j+1} be chosen such that $n_{j+1} > \max\{k_1, \dots, k_j\}$. Then, because l is the limit superior $x_k < l + \varepsilon$ for any $k \geq n_{j+1}$. Further, for sufficiently large n_{j+1} , there exists $k_{j+1} \geq n_{j+1}$ such that $x_{k_{j+1}} > l - \varepsilon$, which gives us that $|x_{k_{j+1}} - l| < \varepsilon$. Because we can always choose the next point in the subsequence in this manner, this means that the subsequence $\{x_{n_j}\}$ converges to l . By Problem 2.8, this means that l is a cluster point of $\{x_n\}$.

By way of contradiction, suppose that l is not the largest cluster point of the sequence. That is, there exists a cluster point y of $\{x_n\}$ such that $y > l$. Note that by Problem 2.8, this means that there exists a subsequence $\{x_{n_j}\}$ which converges to y . Because l is the limit superior of the sequence, for any $\varepsilon > 0$, we can find $n \in \mathbb{N}$ such that $x_k < l + \varepsilon$ whenever $k \geq n$. Since this is true for any $\varepsilon > 0$, we can choose $\varepsilon > 0$ small enough so that we have

$$l < l + \varepsilon < y - \varepsilon < y.$$

This means that there are a finite number of terms of $\{x_n\}$ contained within the interval $(y - \varepsilon, y + \varepsilon)$. In other words, there does not exist a subsequence $\{x_{n_j}\}$ which converges to y as we would necessarily need an infinite number of terms within ε of y —a contradiction. Therefore, l is the largest cluster point.

By a reverse argument, we can show that $\liminf x_n$ is a cluster point of $\{x_n\}$ as well as the smallest cluster point. \square

- (b.) Show that every bounded sequence has a convergent subsequence.

Proof. Let $\{x_n\}$ be a bounded sequence. In other words, $\sup x_n$ is a finite real number. By definition of the limit superior, $\limsup x_n \leq \sup x_n$. From part (a), because $\limsup x_n$ is a cluster point, we can always construct a convergent subsequence (as well as for the limit inferior of the sequence) and so this completes this part. \square

Problem 2 (2.43). Let f be defined as

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ p \sin\left(\frac{1}{q}\right) & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

At what points is f continuous? (Please justify your answer.)

Proof. I claim that f is not continuous at the rational numbers. To that end, let $x \in \mathbb{Q}$ and choose $\varepsilon = x - f(x)$. Fix $\delta > 0$. Note that we can always find an irrational number $y \in (x, x + \delta)$. Because y is irrational, by definition of the function, $f(y) - f(x) = y - f(x)$. But then $y - f(x) > x - f(x) = \varepsilon$.

For $x = 0$, fix $\varepsilon > 0$. Choose $\delta = \varepsilon$, and pick a point $y \in \mathbb{R}$ such that $|x - y| = |y - 0| = |y| < \delta$. Now because $\sin(1/q) < 1/q$ for any $q \in \mathbb{N}$, we know that

$$\begin{aligned} |f(y) - f(0)| &\leq |y - 0| \\ &< \delta \\ &= \varepsilon \end{aligned}$$

so f is continuous at 0.

f is also continuous at the irrationals. This is because we can if we pick any point x in the irrationals, we can find sufficiently large q so a rational number $y = \frac{p}{q}$ is close to x (i.e., for a fixed ε , choose δ to be smaller than $f(y) - y$ for this to work). Then this would allow us to bound $|f(y) - f(x)|$ leveraging that we can put the rational numbers in lowest terms

□

Problem 3. Show that $F \subset \mathbb{R}$ is a closed set if and only if F^c is open.

Proof. To complete this proof, we will need a forward and backwards implication.

(\Rightarrow) Suppose $F \subset \mathbb{R}$ is a closed set. Because we desire to show that F^c is open, let $x \in F^c$ be a point. This means that $x \notin F$. Since F is a closed set (i.e., $F = \overline{F}$) and $x \notin F$, we know x is not a point of closure of F . So there exists $\delta > 0$ such that for all $y \in F$, we do not have $|x - y| < \delta$. But then if $|x - y| < \delta$, this must mean that $y \in F^c$, and so F is an open set.

(\Leftarrow) Conversely, suppose that the set F^c is open. Let $x \in F^c$. Then there exists $\delta > 0$ such that if $|x - y| < \delta$, then $y \in F^c$. This means that there is no $y \in F$ such that $|x - y| < \delta$ and so x cannot be a point of closure of F . Thus, because x is arbitrary, F necessarily contains all its points of closure; in other words, $F = \overline{F}$ and thus F must be closed, completing this direction.

Having completed both implications, this completes the proof.

□

Problem 1 (3.5). Let A be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a **finite** collection of open intervals covering A . Then $\sum_{n=1}^k l(I_n) \geq 1$.

Proof. Define the set $A = \mathbb{Q} \cap [0, 1]$. Let $\{I_n\}_{n=1}^k$ be a **finite** collection of open intervals covering A meaning we have that

$$A \subset \bigcup_{n=1}^k I_n$$

We can create the following string of inequalities:

$$\begin{aligned}
 1 = l([0, 1]) &= m^*([0, 1]) \\
 &= m^*(\overline{A}) && \text{Density of } \mathbb{Q} \\
 &\leq m^*\left(\bigcup_{k=1}^n I_n\right) && A \subset B \implies \overline{A} \subset \overline{B} \\
 &= m^*\left(\bigcup_{k=1}^n \overline{I_n}\right) && \overline{A \cup B} = \overline{A} \cup \overline{B} \\
 &\leq \sum_{k=1}^n m^*(\overline{I_n}) && \text{Subadditivity of } m^* \\
 &= \sum_{k=1}^n l(\overline{I_n}) \\
 &= \sum_{k=1}^n l(I_n)
 \end{aligned}$$

which shows the desired result, completing the proof. \square

Problem 2 (3.10). Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cap E_2) + m(E_1 \cup E_2) = mE_1 + mE_2.$$

Recall that if E is measurable, then $mE := m^*E$.

Proof. Suppose E_1 and E_2 are measurable sets. To show the above equality, the goal will be to rewrite sets as disjoint unions and utilize the subadditivity of m . So note that

$$E_1 \cup E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1) \cup (E_1 \cap E_2)$$

and so by the subadditivity of m ,

$$m(E_1 \cup E_2) = m(E_1 \setminus E_2) + m(E_2 \setminus E_1) + m(E_1 \cap E_2).$$

Thus, adding $m(E_1 \cap E_2)$ to the left-hand side

$$\begin{aligned}
 m(E_1 \cup E_2) + m(E_1 \cap E_2) &= (m(E_1 \setminus E_2) + m(E_2 \setminus E_1) + m(E_1 \cap E_2)) + m(E_1 \cap E_2) \\
 &= (m(E_1 \setminus E_2) + m(E_1 \cap E_2)) + (m(E_2 \setminus E_1) + m(E_1 \cap E_2)).
 \end{aligned}$$

Now, let us write E_1 and E_2 as disjoint unions:

$$\begin{aligned} E_1 &= (E_1 \setminus E_2) \cup (E_1 \cap E_2); \\ E_2 &= (E_2 \setminus E_1) \cup (E_1 \cap E_2) \end{aligned}$$

which, again, by the subadditivity of m ,

$$\begin{aligned} m(E_1) &= m(E_1 \setminus E_2) + m(E_1 \cap E_2); \\ m(E_2) &= m(E_2 \setminus E_1) + m(E_1 \cap E_2). \end{aligned}$$

Therefore, this gives us that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

showing the desired result. \square

Problem 3 (3.13). Prove Proposition 15 by the following steps which I will state below for the record.

Let E be any given set. Then the following are equivalent:

- (i) E is measurable.
- (ii) For all $\varepsilon > 0$, there is an open set $O \supset E$ with $m^*(O \setminus E) < \varepsilon$.
- (iii) For all $\varepsilon > 0$, there is a closed set $F \subset E$ with $m^*(E \setminus F) < \varepsilon$.
- (iv) There is a $G \in G_\delta$ with $E \subset G$ such that $m^*(G \setminus E) = 0$.
- (v) There is a $F \in F_\sigma$ with $F \subset E$ such that $m^*(E \setminus F) = 0$.

If $m^*(E) < \infty$, the above statements are equivalent:

- (vi) For all $\varepsilon > 0$, there is a finite union U of open intervals such that $m^*(U \Delta E) < \varepsilon$.
- a. Show that for $m^*E < \infty$, $(i) \Rightarrow (ii) \Leftrightarrow (vi)$.

Proof. To show that the following are equivalent, we will need to create a chain of implications starting with the outline above. For all of these proofs, assume that $m^*(E) < \infty$.

- (i) \Rightarrow (ii) Suppose E is a measurable set. Let $\varepsilon > 0$ be chosen. Because E is measurable and thus $m^*(E) = m(E)$, there exists a countable collection of open intervals $\{I_n\}_{n=1}^\infty$ so that

$$m(E) + \varepsilon > \sum_{n=1}^{\infty} l(I_n).$$

Since $\{I_n\}_{n=1}^\infty$ is open, the set $O = \bigcup_{n=1}^{\infty} I_n$ is an open set as well. By Proposition 3.1, we know that

$$m(O) = m\left(\bigcup_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} l(I_n).$$

Again, because E is measurable, we know that $E \subset O$. Now it is left to show that $m(O \setminus E)$. Because O and E are disjoint, we have that

$$\begin{aligned} m(O \setminus E) &= m(O) - m(E) \\ &= \sum_{n=1}^{\infty} l(I_n) - m(E) \\ &< (m(E) + \varepsilon) - m(E) \\ &= \varepsilon \end{aligned}$$

which completes this direction.

(ii) \Rightarrow (vi) Let $\varepsilon > 0$ be chosen. Then by our hypothesis, there exists an open set O such that $m^*(O \setminus E) < \frac{\varepsilon}{2}$. By the Lindelof Lemma, the set O can be written as countable union of open intervals i.e., there exists a countable collection of intervals $\{I_n\}_{n=1}^{\infty}$ so that $O = \bigcup_{n=1}^{\infty} I_n$. Satisfying the conditions of Proposition 3.5, we can leverage this as suggested and get that

$$\begin{aligned} \sum_{n=1}^{\infty} l(I_n) &= m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \\ &\leq m^*(E) + \frac{\varepsilon}{2}. \end{aligned}$$

This means that there exists $N \in \mathbb{N}$ so that

$$\sum_{n=N+1}^{\infty} l(I_n) = m^*\left(\bigcup_{n=N+1}^{\infty} I_n\right) < \frac{\varepsilon}{2}.$$

Let $\{I_1, \dots, I_N\}$ be the finite set of collections up to but including $N + 1$ and so we let $U = \bigcup_{n=1}^N I_n$. We can note that $U \Delta E = (U \setminus E) \cup (E \setminus U)$. Additionally, $U \setminus E \subset O \setminus E$ by construction of U and $E \setminus U \subset O \setminus E$ by hypothesis. Finally, note that

$$O \setminus U = \left(\bigcup_{n=1}^{\infty} I_n\right) \setminus \left(\bigcup_{n=1}^N I_n\right) = \bigcup_{n=N+1}^{\infty} I_n.$$

Thus, with of all this, we have that

$$\begin{aligned} m^*(U \Delta E) &= m^*((U \setminus E) \cup (E \setminus U)) \\ &= m^*(U \setminus E) + m^*(E \setminus U) \\ &\leq m^*(O \setminus E) + m^*(O \setminus U) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

which finishes this direction.

(vi) \Rightarrow (ii) Let $\varepsilon > 0$ be chosen. By assumption, for any set E , there exists a finite union U of open intervals so that

$$m^*(U \Delta E) = m((U \setminus E) \cup (E \setminus U)) < \frac{2\varepsilon}{3}.$$

By Proposition 3.5, there exists an open set $O \supset E \setminus U$ so that

$$m^*(O) \leq m(E \setminus U) + \frac{\varepsilon}{3}$$

which is equivalent to saying that

$$m^*(O \setminus (E \setminus U)) < \frac{\varepsilon}{3}.$$

Note that $E \subset O \cup U$ trivially. Thus, we have that

$$\begin{aligned} m^*(O \setminus E) &\leq m^*((U \cup O) \setminus (E)) \\ &= m^*((U \setminus E) \cup (O \setminus E)) \\ &\leq m((O \setminus (E \setminus U)) \cup (U \setminus E) \cup (E \setminus U)) \\ &= m(O \setminus (E \setminus U)) + m(U \setminus E) + m(E \setminus U) \\ &< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

giving us the desired result.

This completes the first set of our chain of equivalences. \square

b. Use part (a) to show that for arbitrary sets, (i) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (i).

Proof. We continue on our journey to a chain of equivalences with this next set! :)

(i) \Rightarrow (ii) Suppose that E is measurable and since we showed that this direction for $m^*(E) < \infty$, suppose $m^*(E) = \infty$. For any $n \in \mathbb{N}$, define the set $E_n = E \cup [-n, n]$. From part(a), there exists an open set $O_n \supset E_n$ for all $n \in \mathbb{N}$ so that

$$m^*(O_n \setminus E_n) < \frac{\varepsilon}{2^n}.$$

Define the set $O = \bigcup_{n=1}^{\infty} O_n$. Then note that $E \subset O$ and $E \subset \bigcup_{n=1}^{\infty} E_n$. Using

this, we can show that

$$\begin{aligned}
 m^*(O \setminus E) &= m^*\left(\bigcup_{n=1}^{\infty} O_n \setminus E\right) \\
 &\leq m^*\left(\bigcup_{n=1}^{\infty} O_n \setminus \bigcup_{n=1}^{\infty} E_n\right) \\
 &\leq m^*\left(\bigcup_{n=1}^{\infty} O_n \setminus E_n\right) \\
 &\leq \sum_{i=1}^{\infty} m^*(O_n \setminus E_n) \\
 &< \sum_{i=1}^{\infty} \frac{\varepsilon}{2^k} \\
 &= \varepsilon
 \end{aligned}$$

which completes the proof.

- (ii) \Rightarrow (iv) By assumption, we can choose $n \in \mathbb{N}$ so that the open set $O_n \supset E$ implies that $m^*(E \setminus O_n) < \frac{1}{n} < \varepsilon$ for any $\varepsilon > 0$, which is possible using the Archimedes principle. Let $G = \bigcap_{n=1}^{\infty} O_n$, which is thus a countable intersection of open sets (i.e., $G \in G_{\delta}$). Note that $E \subset G \subset O_n$ and so

$$\begin{aligned}
 m^*(G \setminus E) &\leq m^*(O_n \setminus E) \\
 &< \frac{1}{n} \\
 &< \varepsilon.
 \end{aligned}$$

Because we can always find $n \in \mathbb{N}$ for all $\varepsilon < 0$, we have that $m^*(G \setminus E) = 0$. Since we can choose $n \in \mathbb{N}$, certainly $F \subset E$ and $F_n \subset F$ which gives that

$$m^*(E \setminus F) \leq m(E \setminus F_n)$$

- (iv) \Rightarrow (i) Assume there exists some $G \in G_{\delta}$ such that $E \subset G$ and $m^*(G \setminus E) = 0$. Because $G \in G_{\delta}$ and $m^*(G \setminus E) = 0$, this implies that $G \setminus E$ is a measurable set. But then since $G \setminus E$ is a measurable set, G is a measurable set. Thus since $E = G \setminus (G \setminus E)$, it follows that E is measurable.

This completes this chain of implications. □

- c. Use part (b) to show that (i) \Rightarrow (iii) \Rightarrow (v) \Rightarrow (i).

Proof. Finally, can finish the chain on equivalences and finish proving Proposition 3.15.

- (i) \Rightarrow (iii) Suppose E is measurable set (i.e., $E \in \mathcal{M}$). Let $\varepsilon > 0$ be chosen. Because \mathcal{M} is a σ -algebra and closed under complement, we know that E^c is a measurable

set as well. From part (b) (the infinite case of (i) \Rightarrow (ii)), there exists an open set $O \supset E^c$ so that $m^*(O \setminus E^c) < \varepsilon$. Let $F = O^c$, which is a closed set because its complement is open. Then $F \subset E$ and noting that $O \setminus E^c = E \cap O = E \setminus F$, we have that

$$m^*(F \setminus E) = m^*(O \setminus E^c) = m^*(E \cap O) < \varepsilon$$

which completes the proof.

(iii) \Rightarrow (v) Similar to the approach of (ii) \Rightarrow (iv) in part (b), let us choose $n \in \mathbb{N}$ using the Archimedes principle so that a closed $F_n \subset E$ means that

$$m^*(E \setminus F_n) < \frac{1}{n} < \varepsilon \text{ for all } \varepsilon > 0.$$

Let $F = \bigcup_{n=1}^{\infty} F_n$, which is a countable union of closed sets and so $F \in F_{\sigma}$. Also $F \subset E$ and $F_n \subset F$ for any $n \in \mathbb{N}$ so we know that

$$\begin{aligned} m(E \setminus F) &\leq m(E \setminus F_n) \\ &< \frac{1}{n} \\ &< \varepsilon. \end{aligned}$$

By the same reasoning as the end of the proof of (ii) \Rightarrow (iv) from part (b), we can conclude that $m(E \setminus F) = 0$.

(v) \Rightarrow (i) Again, from part (b), we will use similar logic as (iv) \Rightarrow (i). Because $F \in F_{\sigma}$ and $m^*(E \setminus F) = 0$, this implies that $E \setminus F$ is a measurable set. But then since $E \setminus F$ is a measurable set, F is a measurable set. Thus since $E = F \cup (E \setminus F)$, it follows that E is measurable.

Finally, having finished the chain of equivalences, we have shown that all of the statements are equivalent, which completes the proof. \square

Problem 1 (3.23). Prove Proposition 3.22 by the following lemmas:

- a. Given a measurable function f on $[a, b]$ that takes the values $\pm\infty$ only on a set of measure zero, and given $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that $|f| \leq M$ except on a set of measure less than $\frac{\varepsilon}{3}$.

Proof. Suppose f is a measurable function on $[a, b]$ and that $f(x) = \pm\infty$ only on a set of measure zero. Let $\varepsilon > 0$ be chosen. Define the set

$$E_n = \{x \in [a, b] : |f(x)| > n\} \text{ for all } n \in \mathbb{N}.$$

Because the function f is measurable, by definition, this means that each E_i is a measurable set as well. Note that by construction of E_n , we have that $E_i \subset E_{i+1}$ and so $\{E_n\}$ is a decreasing sequence. Since E_1 is a subset of the inverse image of f which is itself a subset of $[a, b]$ i.e., $E_1 \subset [a, b]$, we have that

$$m(E_1) < m([a, b]) = b - a < \infty.$$

Again, by the construction of E_n , we have that

$$\bigcap_{n=1}^{\infty} E_n = \emptyset$$

implying that

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = 0.$$

But having satisfied the conditions of Proposition 3.14, this is the same as saying $E_n \rightarrow 0$ as $n \rightarrow \infty$ or

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} m(E_n) = 0.$$

Thus, because ε is fixed, we can always find $M \in \mathbb{N}$ such that

$$m(E_M) = m\{x \in [a, b] : |f(x)| > M\} < \frac{\varepsilon}{3}.$$

But this necessarily implies that $|f(x)| \leq M$ for all $x \in [a, b]$ thereby completing the proof. \square

- b. Let f be a measurable function on $[a, b]$. Given $\varepsilon > 0$ and $M > 0$, there is a simple function ϕ such that $|f(x) - \phi(x)| < \varepsilon$ except where $|f(x)| \geq M$. If $m\{x : |f(x)| \geq M\} \leq \frac{\varepsilon}{M}$, then we may take ϕ so that $m\{x : |f(x) - \phi(x)| \geq \varepsilon\} \leq \frac{\varepsilon}{M}$.

Proof. Suppose f is a measurable function on $[a, b]$. Let $\varepsilon > 0$ and $M > 0$ be chosen. Because ε and M are fixed, by the Archimedes principle, we can choose $N \in \mathbb{N}$ large enough so that $\frac{M}{N} < \varepsilon$. From this, let us define the set

$$E_k = \left\{k \frac{M}{N} \leq f(x) \leq (k+1) \frac{M}{N}\right\}$$

for $k \in [-N, N]$ (integer-valued). Since f is a measurable function, each E_i is a measurable set as well. Let us define the function ϕ by

$$\phi(x) = \sum_{k=-N}^N k \left(\frac{M}{N} \right) \chi_{E_k}$$

with $a_i = k \frac{M}{N} \in \mathbb{R}$ for each $k \in [-N, N]$. So because ϕ is a linear combination of characteristic functions of E_i and each E_i is a measurable set (in fact, the E_i 's are pairwise disjoint), ϕ is a simple function. Suppose that $|f(x)| < M$. Because E_i 's are pairwise disjoint, then for all $x \in [a, b]$, $x \in E_k$ for some $k \in [-N, N]$ which implies that

$$k \frac{M}{N} \leq f(x) \leq (k+1) \frac{M}{N}.$$

Thus, $\phi(x) = k \frac{M}{N}$ which gives us that

$$\begin{aligned} |f(x) - \phi(x)| &= \left| f(x) - k \frac{M}{N} \right| \\ &< \frac{M}{N} \\ &< \varepsilon. \end{aligned}$$

Now suppose that $f(x) \in [m, M]$ for all $x \in [a, b]$ (i.e., f is a bounded function.) Then the same argument holds as before but instead we have that

$$\begin{aligned} |f(x) - \phi(x)| &= \left| f(x) - k \frac{M-m}{N} \right| \\ &< \frac{M-m}{N} \\ &< \varepsilon \end{aligned}$$

meaning for all $x \in [a, b]$, we have $\phi(x) = k \frac{M-m}{N}$ implying that $\phi(x) \in [m, M]$. □

- c. Given a simple function ϕ on $[a, b]$, there is a step function g on $[a, b]$ such that $g(x) = \phi(x)$ except on a set of measure less than $\frac{\varepsilon}{3}$.

Proof. Let ϕ be the simple function defined by

$$\phi(x) = \sum_{i=1}^n a_i \chi_{E_i}$$

for measurable, disjoint sets E_1, \dots, E_n and $a_i \in \mathbb{R}$ for $i = 1, \dots, n$. Let $\varepsilon > 0$ be chosen. Because each E_i is a measurable set, by Proposition 3.15, for each $i = 1, \dots, n$, there exists a finite union U_i of open intervals I_i such that $m(E_i \Delta U_i) < \frac{\varepsilon}{3n}$ with

$$U_i = \sum_{k=1}^{N_i} I_{i,k}.^1$$

¹This is mostly for myself, but k is the index for the number of intervals N_i associated with each E_i .

Let $A_i = U_i \setminus \left(\bigcup_{j=1}^{i-1} U_j \right)$.² For any $x \in [a, b]$, define the function

$$g(x) = \sum_{i=1}^n a_i \chi_{A_i}.$$

Because the E_i 's are measurable and the difference of measurable sets is measurable, the set $\{A_1, \dots, A_n\}$ is a set of measurable sets. The A_i 's are a subdivision of $[a, b]$ and so g is a step function per the definition on page 76 of Royden. We claim that this function is equal to $\phi(x)$ except on a set of measure less than $\frac{\varepsilon}{3}$. To that end, fix $x \in [a, b]$ so that $\phi(x) \neq g(x)$. Because ϕ and g are linear combinations with the same coefficients, this brings two cases: (i) there is some $i = 1, \dots, n$ so that $g(x) = a_i$ but $\phi(x) \neq a_i$ or (ii) there is some $i = 1, \dots, n$ so that $g(x) \neq a_i$ but $\phi(x) = a_i$.

For case (i), this means that $x \in A_i \subset U_i \setminus E_i$ for some $i = 1, \dots, n$. For case (ii), we must have that $x \in E_i \subset E_i \setminus U_i$ for some $i = 1, \dots, n$. So, combining both results,

$$\begin{aligned} \{x \in [a, b] : \phi(x) \neq g(x)\} &\subset \bigcup_{i=1}^n U_i \setminus E_i; \\ \{x \in [a, b] : \phi(x) \neq g(x)\} &\subset \bigcup_{i=1}^n E_i \setminus U_i \end{aligned}$$

and thereby implies that

$$\{x \in [a, b] : \phi(x) \neq g(x)\} \subset \bigcup_{i=1}^n E_i \Delta U_i.$$

Finally, this allows us to show that

$$\begin{aligned} m(\{x \in [a, b] : \phi(x) \neq g(x)\}) &\leq m\left(\bigcup_{i=1}^n E_i \Delta U_i\right) \\ &= \sum_{i=1}^n m(E_i \setminus U_i) \\ &< \sum_{i=1}^n \frac{\varepsilon}{3n} \\ &= n \cdot \frac{\varepsilon}{3n} \\ &= \varepsilon \end{aligned}$$

giving us the desired result. \square

Problem 2 (3.31). Prove Lusin's Theorem: Let f be a measurable real-valued function on an interval $[a, b]$. Then for all $\delta > 0$, there is a continuous function ϕ on $[a, b]$ such that $m\{x : f(x) \neq \phi(x)\} < \delta$.

²Again, mostly for myself, but for each U_i associated with E_i , throw out the preceding U_i 's.

Proof. Let $\delta > 0$ be chosen. Suppose f is a measurable real-valued function on an interval $[a, b]$. Then by Proposition 3.22, there exists a continuous function h_n for all $n \in \mathbb{N}$ such that

$$|f - h_n| < \frac{\delta}{2^{n+2}}$$

with $m \left\{ x \in [a, b] : |f - h_n| \geq \frac{\delta}{2^{n+2}} \right\} < \frac{\delta}{2^{n+2}}$. For convenience, define the sets

$$E_n = \left\{ x \in [a, b] : |f - h_n| \geq \frac{\delta}{2^{n+2}} \right\}$$

and

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Note for a fixed $x \in [a, b] \setminus E_n$ and any $n \in \mathbb{N}$, we know by how we defined E_n that

$$|f - h_n| < \frac{\delta}{2^{n+2}}.$$

Thus, since E is the union of the E_n 's, we have that

$$\begin{aligned} m(E) &= m \left(\bigcup_{n=1}^{\infty} E_n \right) \\ &\leq \sum_{n=1}^{\infty} m(E_n) \\ &< \sum_{n=1}^{\infty} \frac{\delta}{2^{n+2}} \\ &= \frac{\delta}{4}. \end{aligned}$$

So then on the set $[a, b] \setminus E$, the sequence of continuous, and thereby, measurable functions $\{h_n\}$ converges to f . Having satisfied the conditions of Egoroff's theorem, there exists a set $A \subset [a, b] \setminus E$ with $m(A) < \frac{\delta}{4}$ such that h_n converges uniformly on $([a, b] \setminus E) \setminus A = [a, b] \setminus (E \cup A)$. Since the uniform limit of continuous functions is a continuous function, the function f is continuous on $[a, b] \setminus (E \cup A)$. Because $m(E)$ and $m(A)$ are less than $\frac{\delta}{4}$, $m(E \cup A) < \frac{\delta}{2}$.

Using Proposition 3.15 part (ii), there exists an open set $O \supset (E \cup A)$ with

$$m(O \setminus (E \cup A)) < \frac{\delta}{2}.$$

Because $[a, b] \setminus (E \cup A) \supset [a, b] \setminus O$ and $[a, b] \setminus O = [a, b] \cap O^c$ (i.e., a closed set), f is continuous on the closed set $[a, b] \setminus O$. Then for any $x \in [a, b] \setminus O$, by Problem 2.40, there exists a continuous function ϕ so that $f(x) = \phi(x)$. But then the set O represents the

set of points where ϕ and g are not equal. In particular, we can show that

$$\begin{aligned} m\{x \in [a, b] : f(x) \neq \phi(x)\} &= m(O) \\ &= m((O \setminus (E \cup A)) \cup (E \cup A)) \\ &= m(O \setminus (E \cup A)) + m(E \cup A) \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta \end{aligned}$$

which finally completes the proof. □

Problem 1 (4.2). (a) Let f be a bounded function on $[a, b]$, and let h be the upper envelope of f (cf. Problem 2.51). Then $R \int_a^b f = \int_a^b h$.

Proof. Let f be a bounded function on $[a, b]$ with $h(y) = \inf_{\delta > 0} \sup_{|x-y| < \delta} f(x)$ for all $x \in [a, b]$ be the upper envelope of f . Because f is bounded, by Problem 2.51 part (b), h is lower semicontinuous. To show equality, we will show that

$$R \int_a^b f \leq \int_a^b h \quad \text{and} \quad R \int_a^b f \geq \int_a^b h.$$

Let ϕ be a step function on $[a, b]$ such that $\phi \geq f$. Then for any $x \in [a, b]$, $h(x) \leq f(x) \leq \phi(x)$, except at the defined partition points of ϕ . Thus, by using the definition of Lebesgue integration, we have that

$$\int_a^b h(x) dx \leq \int_a^b f(x) dx = \inf \int_a^b \phi(x) dx \leq R \int_a^b f(x) dx.$$

For the other inequality, we note that because h is upper semicontinuous, by Problem 2.51 part (g), there exists a monotonically decreasing sequence of step functions $\{\phi_n\}$ such that $\phi_n \rightarrow h$ pointwise. Because f is bounded, we have that for all $x \in [a, b]$, there exists some $M > 0$ such that

$$|\phi_n| \leq |h| \leq |f| \leq M \text{ for all } n \in \mathbb{N}.$$

Thus using the Bounded Convergence Theorem and properties of the upper Riemann integral,

$$\lim_{n \rightarrow \infty} \int_a^b \phi_n(x) dx = \int_a^b h(x) dx \leq R \int_a^b f(x) dx \leq R \int_a^b f(x) dx.$$

Therefore,

$$R \int_a^b f = \int_a^b h$$

which is the desired result. □

(b) Use part (a) to prove Proposition 7 which is stated as follows

Proposition (4.7). A bounded function f on $[a, b]$ is Riemann integrable if and only if the set of points at which f is discontinuous has measure zero.

Proof. Let f be a bounded function on $[a, b]$. We will need to show a forward and backwards implication to complete this proof. For simplicity, define E to be the set of discontinuities of f . Additionally, let $g(y) = \sup_{\delta > 0} \inf_{|x-y| < \delta} f(x)$ be the lower envelope of f .

(\Leftarrow) First, suppose $m(E) = 0$. Since g is the lower envelope of f , there exists a monotonically increasing sequence of step functions $\{\phi_n\} \rightarrow g$ pointwise. Thus, by a similar but reverse argument to part (a),

$$\int_a^b g(x) dx = \int_a^b f(x) dx. \tag{1}$$

So because f is continuous everywhere except on the set E —namely, continuous on $[a, b] \setminus E$ —by Problem 2.51, $g(x) = h(x)$ is continuous on the set $[a, b] \setminus E$. But since $m(E) = 0$, this means $g = h$ almost everywhere and thus, using part (a) of this problem and Equation (1),

$$\int_a^b f(x) \, dx = \int_a^b g(x) \, dx = \int_a^b h(x) \, dx = \overline{\int_a^b f(x) \, dx}.$$

Thus, f is Riemann integrable.

(\Rightarrow) Now suppose f is Riemann integrable. Thus, the lower and upper integrals of f are equal to each other and so using Equation (1)

$$\int_a^b h(x) \, dx = \int_a^b g(x) \, dx.$$

Consider the set $A_n = \left\{ x : |g(x) - h(x)| > \frac{1}{n} \right\}$ for all $n \in \mathbb{N}$. Because the integrals of g and h are equal,

$$\int_a^b |h(x) - g(x)| \, dx = 0.$$

So for any fixed $n \in \mathbb{N}$,

$$\int_a^b |h(x) - g(x)| \, dx \geq m(A_n).$$

So $h(x) = g(x)$ almost everywhere and so by Problem 2.51 part(a), we must that f is continuous almost everywhere as well. But then this means that the measure of discontinuities is zero—that is, $m(E) = 0$ which is what we wanted to show.

Having completed both directions, we have shown the desired equivalence. \square

Problem 2 (4.3). Let f be a nonnegative measurable function. Show that $\int f = 0$ implies $f = 0$ almost everywhere.

Proof. Let $f \geq 0$ be a measurable function, and suppose that $\int f = 0$. We want to show that the set $E = \{x : f(x) > 0\} = \{x : f(x) > 0\}$ has measure 0. Define the set

$$E_n = \left\{ x : f(x) \geq \frac{1}{n} \right\} \text{ for all } n \in \mathbb{N}.$$

Note that $\bigcup_{n=1}^{\infty} E_n = E$. Fix $n \in \mathbb{N}$. Because the integral of f is equal to 0,

$$0 = \int_{E_n} f \geq \int_{E_n} \frac{1}{n} = m(E_n) \cdot \frac{1}{n} \geq 0.$$

Thus, because $n \in \mathbb{N}$ was fixed, $m\left(\bigcup_{n=1}^{\infty} E_n\right) = 0 = m(E)$, and therefore $f = 0$ almost everywhere. \square

Problem 3 (4.8). Prove the following generalization of Fatou's Lemma: If f_n is a sequence of nonnegative functions then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Proof. Let $\{f_n\} \geq 0$ for each $n \in \mathbb{N}$ on any set E . Define $h_n = \inf_{k \geq n} f_k$ for all $n \in \mathbb{N}$. Note that as $n \rightarrow \infty$, $h_n \rightarrow \liminf_{n \rightarrow \infty} f_n$ (i.e., h_n converges pointwise on E to the limit inferior of f_n). Thus, by Fatou's lemma, we have that

$$\int_E \liminf_{n \rightarrow \infty} f \leq \int_E \liminf_{n \rightarrow \infty} h_n.$$

But since h_n is the infimum of the f_n 's, this implies that $h_n \leq f_n$ for all $n \in \mathbb{N}$ and so

$$\int_E h_n \leq \int_E f_n$$

and thus

$$\liminf_{n \rightarrow \infty} \int_E h_n \leq \liminf_{n \rightarrow \infty} \int_E f_n.$$

By combining all three inequalities, we get that

$$\int_E \liminf_{n \rightarrow \infty} f \leq \liminf_{n \rightarrow \infty} \int_E h_n \leq \liminf_{n \rightarrow \infty} \int_E f_n$$

which then completes the proof. □