Section 3.2, Outer Measure

Definition. The **outer measure** $m^*(A)$ of a set $A \subset \mathbb{R}$ is given

$$m^*(A) = \inf_{A \subset \bigcup I_n} \sum_n l(I_n)$$

where $\{I_n\}$ is a countable collection of open intervals that cover A.

Note that from this definition, we get that

1. $m^*(\emptyset) = 0$
2. If $A \subset B$, $m^*(A) < m^*(B)$.
3. m^* does not satisfy disjoint additivity.

Proposition (3.1). The outer measure of an interval is its length; that is, $m^*(I) = l(I)$ where $I = [a, b], (a, b), [a, b)$, or $(a, b]$.

Proof. It is sufficient to show that $m^*([a, b]) = l([a, b])$ since every other interval is a subset of $[a, b]$. Let $\varepsilon > 0$. Then $[a, b] \subset \left[a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right]$ which implies, by the definition of the outer measure,

$$m^*([a, b]) \leq l\left(\left[a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right]\right) = b - a + \varepsilon.$$

Because ε was fixed, this means that $m^* \leq b - a$.

Now we must show that $m^* \geq b - a$. Because $[a, b]$ is compact, for any collection $\{I_n\}$ of open intervals covering $[a, b]$, there exists a finite collection of intervals $\{I_1, \ldots, I_k\}$ so that

$$[a, b] \subset \bigcup_{n=1}^{k} I_n.$$

This gives us that

$$\sum_n I_n \geq \sum_{n=1}^{k} I_n \geq b - a$$

and so $b - a$ is a lower bound. But since m^* as the greatest lower bound of all such sums, we have that $m^* \geq b - a$.

Therefore, $m^*([a, b]) = l([a, b]) = b - a$.

□

Proposition (3.2, Subadditivity). Let $\{A_n\}$ be a countable collection of sets on $\mathbb{R}$. Then

$$m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n).$$

Proof. Proof on page 57.

□

Corollary (3.3). If A is a countable set, then $m^*(A) = 0$.

Proof. Proof is on the end of page 57.

$\square$

Section 3.3, Measurable Sets and Lebesgue Measure

Definition. A set $E \subset \mathbb{R}$ is (Lebesgue) **measurable** if for all sets A, we have that

$$m^*(A) = m^*(A \cup E) + m^*(A \cup E^{\mathsf{c}}).$$

Lemma (3.6). If $m^*(E) = 0$, then E is measurable.

Proof. Let A be any chosen set. Because $A \cap E \subset E$ and $m^*(E) = 0$,

$$m^*(A \cap E) \leq m(E) = 0.$$

Note that $A \cap E^{\mathsf{c}} \subset A$ and so $m^*(A) \leq m^*(A \cap E^{\mathsf{c}})$ and so it suffices to show that $m^*(A) \geq m^*(A \cap E^{\mathsf{c}})$. Using this, we can show that

$$m^*(A) \geq m^*(A \cap E^{\mathsf{c}}) + 0 = m^*(A \cap E^{\mathsf{c}}) = m^*(A \cap E)$$

giving us the desired result. $\square$

Definition. Let $\mathcal{M}$ be the set of measurable sets in $\mathbb{R}$

Lemma (3.7). If E_1 and E_2 are measurable sets, then so is $E_1 \cup E_2$.

Proof. Proof on top of page 57. $\square$

Corollary (3.8). The family $\mathcal{M}$ of measurable sets if an algebra of sets. In other words, if $E \in \mathcal{M}$, then $E^{\mathsf{c}} \in \mathcal{M}$. Further, if $E_1, E_2 \in \mathcal{M}$, then $E_1 \cup E_2 \in \mathcal{M}$.