Section 4.3 Integral of Nonnegative Functions

Definition. Let $f \geq 0$ be measurable, and E be a measurable set. The **Lebesgue integral** of f over is defined by

$$\int_E f := \sup_{h \leq f} \int_E h$$

where h is a bounded measurable function and $m\{x : h(x) \neq 0\} < \infty$.

Proposition (4.8). If f and g are nonnegative measurable functions, then:

i. For all $c > 0$,
$$\int_E cf = c \int_E f.$$

ii.
$$\int_E f + g = \int_E f + \int_E g.$$

iii. If $f \leq g$ a.e, then
$$\int_E f \leq \int_E g.$$

Proof. [1] Note that (i) and (iii) come from Proposition 4.5. So for (ii), we first claim that

$$\int_E f + g \leq \int_E f + \int_E g.$$

Let $h \leq f$ be a bounded, measurable function with $m\{x : h \neq 0\} < \infty$, and let $k \leq g$ be a bounded, measurable function with $m\{x : k \neq 0\} < \infty$. Then $h + k \leq f + g$ and

$$\{x : h + k \neq 0\} = \{x : h \neq 0\} \cup \{x : k \neq 0\}$$

and so $m\{x : h + k \neq 0\} < \infty$. By definition of the Lebesgue integral (which is a sup), we have the following:

$$\begin{aligned}
\int_E f + g \geq \int_E h + k &= \int_E h + \int_E k \\
&\geq \int_E h + \int_E g \\
&\geq \int_E h + \int_E k \\
&\geq \int_E f + \int_E g.
\end{aligned}$$

For the other direction, let $l \leq f + g$ be a bounded, measurable function and $m\{x : l(x) \neq 0\} < \infty$. Define $h(x) = \min\{f(x), l(x)\} \leq l(x)$ and so $h(x)$ is bounded as well. Then

$$\int_E f + \int_E g \geq \int_E + \int_E k = \int_E h + k = \int_E l$$

[1] Proof is on page 86 of Royden.

which, by definition of the Lebesgue integral,

$$\int_E f + \int_E g \leq \int_E f + g.$$

□

Theorem (4.9, Fatou's Lemma). If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ pointwise almost everywhere on a set E, then

$$\int_E f \leq \varliminf_{n\to\infty} \int_E f_n.$$

Proof. Without loss of generality, suppose that $f_n(x) \to f(x)$ on E (because the integrals over sets of measure zero are zero.) Suppose that $h \leq f$ is a bounded, measurable function and define $E' = \{x : h(x) \neq 0\}$ and so $m(E') < \infty$. Define $h_n(x) = \min\{h(x), f_n(x)\}$ and so $h_n(x) \to h(x)$ pointwise on E' and $h_n \leq h \leq f_n \leq f$ and so $\{h_n\}$ is a bounded sequence. So, by the Bounded Convergence Theorem,

$$\begin{aligned}
\int_E h &= \int_{E'} h \\
&= \lim_{n\to\infty} \int_E h_n && \text{Bounded Convergence Theorem} \\
&\leq \varliminf_{n\to\infty} \int_E f_n.
\end{aligned}$$

Taking the sup over h,[2]

$$\int_E f \leq \varliminf_{n\to\infty} \int_E f_n.$$

□

Theorem (4.10, Monotone Convergence Theorem). Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions, and let $f = \lim\limits_{n\to\infty} f_n$ almost everywhere. Then

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. By Fatou's Lemma,

$$\int f \leq \varliminf_{n\to\infty} \int f_n.$$

So we just need the other direction for equality. Because $\{f_n\}$ is increasing and converges to f, $f_n \leq f$ for each $n \in \mathbb{N}$ and thus

$$\int f_n \leq \int f.$$

But then the limit inferior is less than than this quantity i.e.,

$$\varliminf_{n\to\infty} \int f_n \leq f$$

[2] Wait, clarify what this means...

and so

$$\lim_{n\to\infty} \int f_n = \int f.$$

□

Corollary (4.11). Let $\{u_n\}$ be a sequence of nonnegative measurable functions, and let $f(x) = \sum_{i=1}^{n} u_n(x)$. Then

$$\int f = \sum_{i=1}^{n} \int u_n.$$

Proposition (4.12). Let f be a nonnegative function and $\{E_i\}$ a disjoint sequence of measurable sets. Let $E = \bigcup_{i=1}^{\infty} E_i$. Then

$$\int_E f = \sum_{i=1}^{\infty} \int_{E_i} f.$$

Definition. Let $f \geq 0$ be a nonnegative measurable function. We say that f is **Lebesgue measurable** over E if

$$\int_E f \leq \infty.$$

Proposition (4.13). Let f and g be two nonnegative measurable functions. If f is integrable over E and $g(x) \leq f(x)$ on E, then g is also integrable on E and,

$$\int_E f - g = \int_E f - \int_E g.$$

Proof. Note that $f - g \geq 0$ on E so we can write this as the sum of two nonnegative functions i.e., $f = (f - g) + g$. Thus, since this is the sum of two nonnegative functions, by Proposition 4.8 part (ii), we have that

$$\int_E f = \int_E (f - g) + \int_E g$$

.

Because the integral of f is finite, the right-hand side must also be finite and so g is measurable.[3] □

Proposition (4.14). Let f be a nonnegative function which is integrable over E. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta$, we have that

$$\int_A f < \varepsilon.$$

[3] This proof does not show the explicit formula, though?

Proof. Let $\varepsilon > 0$ be chosen. If f is bounded, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$. So set $\delta = \dfrac{\varepsilon}{M}$ and estimate $\displaystyle\int_A f$.

If is unbounded then define

$$f_n(x) = \min\{n, f(x)\}.$$

Then $\{f_n\}$ is an increasing sequence and $f_n \to f$ pointwise (i.e, $f_n \uparrow f$ pointwise). By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Thus there exists $N \in \mathbb{N}$ such that if $n \geq N$,

$$\int_E f - \lim_{N\to\infty} \int_E f_N < \frac{\varepsilon}{2}$$

and thus implying that

$$\lim_{N\to\infty} \int_E f_N > \int_E f + \frac{\varepsilon}{2}.$$

Set $\delta = \dfrac{\varepsilon}{2N}$. Choose a set $A \subset E$ such that $m(A) < \delta$. Then

$$\begin{aligned}
\int_A f &= \int_A f - f_N + \int_A f_N \\
&< \frac{\varepsilon}{2} + \int_A f_N \\
&= \frac{\varepsilon}{2} + N \cdot m(A) \\
&< \frac{\varepsilon}{2} + N \cdot \frac{\varepsilon}{2N} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

$\square$