Problem 1 (4.14)**.** Some sequence and integral convergence problems.

(a) Show that under the hypotheses of Theorem 4.17 we have

$$\int |f_n - f| \to 0.$$

(b) Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ almost everywhere with f integrable. Then $\displaystyle\int |f - f_n| \to 0$ if and only if $\displaystyle\int |f_n| \to \int |f|$.

Problem 2 (4.16)**.** Establish the *Riemann-Lebesgue Theorem*: If f is an integrable function on $-\infty, \infty$, then $\displaystyle\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\cos(nx)\,\mathrm{d}x = 0$. [Hint: The theorem is easy if f is a step function. Use Problem 15.]

Problem 3 (4.25)**.** A sequence $\{f_n\}$ of measurable functions is said to be a Cauchy sequence in measure if given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have

$$m\left\{x : |f_n(x) - f_m(x)| \geq \varepsilon\right\} < \varepsilon.$$

Show that if $\{f_n\}$ is a Cauchy sequence in measure, then there is a function f to which the sequence $\{f_n\}$ converges in measure.

Problem 4. Compute $\displaystyle\lim_{n\to\infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-2}\,\mathrm{d}x$. Justify your answer.