

# Construction of the Real Numbers, $\mathbb{R}$

- We first start from  $\mathbb{N} \cup \{0\}$  and add numbers together subsequently (i.e.  $1, \underbrace{1+1}_2, \underbrace{1+1+1}_3, \dots$ )

- To construct the integers  $\mathbb{Z}$ , we take the set difference with the natural numbers so that we have

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \setminus \mathbb{N}.$$

- Then the rational numbers,  $\mathbb{Q}$ , can be constructed from the integers and are defined by the set

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \right\}.$$

- To construct the irrational numbers,  $\mathbb{R} \setminus \mathbb{Q}$ , we can use the dedekind cut to do this. However, this is convoluted and we can go about this in a different way.

## Axioms of the Real Numbers

**A. The Field Axioms:** For all real numbers  $x, y \in \mathbb{R}$  we have:

A1.  $x + y = y + x$

A2.  $(x + y) + z = x + (y + z)$

A3. There exists  $0 \in \mathbb{R}$  such that  $x + 0 = x$  for all  $x \in \mathbb{R}$ .  
[Identity element under addition]

A4. For each  $x \in \mathbb{R}$  there is a  $w \in \mathbb{R}$  such that  $x + w = 0$ .  
[Inverse element under addition]

A5.  $xy = yx$

A6.  $(xy)z = x(yz)$

A7. There exists  $1 \in \mathbb{R}$  such that  $1 \neq 0$  and  $x \cdot 1 = x$  for all  $x \in \mathbb{R}$ .

A8. For each  $x \in \mathbb{R}$  different from 0 there is  $w \in \mathbb{R}$  such that  $xw = 1$ .

A9.  $x(y + z) = xy + xz$ .

We can prove some properties now:

**Proposition 1.** The additive inverse is unique.

*Proof.* Let  $x \in \mathbb{R}$ . Suppose we have two numbers  $w_1, w_2 \in \mathbb{R}$  such that  $x + w_1 = 0 = x + w_2$ . Using the axioms and our assumption, we can show the following:

$$\begin{aligned} w_1 &= w_1 + 0 && \text{Axiom A3} \\ &= w_1 + x + w_2 && \text{Assumption of } 0 = x + w_2 \\ &= w_2 + xw_1 && \text{Axiom A1} \\ &= w_2 \end{aligned}$$

which completes the proof. □

**B. The Axioms of Order:** The subset  $P$  of positive real numbers satisfies the following:

- B1. If  $x, y \in P$ , then  $x + y \in P$ .
- B2. If  $x, y \in P$ , then  $xy \in P$ .
- B3. If  $x \in P$ , then  $-x \notin P$ .
- B4. If  $x \in \mathbb{R}$ , then  $x = 0$  or  $x \in P$  or  $-x \in P$ .

Note that any system which satisfies the axioms of groups A and B is called an **ordered field**.

**Definition.** We can give definitions of the ordered operations  $<$ ,  $\leq$ ,  $>$  and  $\geq$ .

- $x < y$  means that  $y - x \in P$ .
- $x \leq y$  means that  $y - x \in P \cup \{0\}$ . Or, this means that  $x < y$  or  $x = y$ .
- $x > y$  means that  $x - y \in P$ .
- $x \geq y$  means that  $x - y \in P \cup \{0\}$ . Or, this means that  $x > y$  or  $x = y$ .

From this, we can deduce and prove some which is any set which satisfies the axioms of group A and B.

**Definition.** Let  $x, y \in \mathbb{R}$  and define the absolute value as

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

**Proposition 2.** Let  $a, b, c \in \mathbb{R}$ .

1.  $a < b$  if and only if  $-b < -a$ .
2. If  $a < b$  and  $b < c$ , then  $a < c$ .
3. If  $a < b$  and  $c > 0$ , then  $ac < bc$ .
4. For  $a, b \in \mathbb{R}$ , then only one is true  $a = b$ ,  $a > b$  and  $a < b$ .
5. If  $x \neq 0$ , then  $x^2 = x \cdot x > 0$ ; in particular,  $1 > 0$ .
6. If  $x, y \in \mathbb{R}$ , then  $|x + y| \leq |x| + |y|$ .

**Definition.** Let  $S \subset \mathbb{R}$ . The number  $b \in \mathbb{R}$  is an **upper bound** for  $S$  if for each  $x \in S$ , we have  $x \leq b$ .

Similarly, a number  $x \in \mathbb{R}$  is the **least upper bound** for  $S$  if it is an upper bound for  $S$  and if  $x \leq b$  for each upper bound  $b$  of  $S$ . We then call  $x$  the **supremum** of  $S$  and denote this  $x = \sup S$ .

**Definition.** Let  $S \subset \mathbb{R}$ . The number  $l \in \mathbb{R}$  is an **lower bound** for  $S$  if for each  $x \in S$ , we have  $l \leq x$ .

Similarly, a number  $x \in \mathbb{R}$  is the **greatest lower bound** for  $S$  if it is a lower bound for  $S$  and if  $x \leq l$  for each lower bound  $l$  of  $S$ . We then call  $x$  the **infimum** of  $S$  and denote this  $x = \inf S$ .

**C. Completeness Axiom:** Every nonempty set  $S \subset \mathbb{R}$  which has an upper bound has a least upper bound.

**Proposition 3.** Let  $L, U \subset \mathbb{R}$  be nonempty subsets with  $R = L \cup U$  and such that for each  $l \in L$  and each  $u \in U$  we have  $l < u$ . Then either  $L$  has a greatest element or  $L$  has a least element.

**Proposition 4** (Approximation Property.). Let  $S \subset \mathbb{R}$  be a nonempty. If  $u = \sup S$ , then for all  $\gamma > 0$ , there exists  $Sr \in S$  such that  $u - r < Sr < u$ .

**Theorem (2.3, Axiom of Archimedes).** If  $x \in \mathbb{R}$  is any real number, then there exists  $n \in \mathbb{N}$  such that  $x < n$ .

*Proof.* We can break this into two cases

1. Let  $x < 1$ . If so, then simply choose  $x = 1$ .
2. Let  $x \geq 1$ . Define the set  $S = \{n \in \mathbb{N} : n \leq x\}$ . Then since this set is bounded above, by the Completeness Axiom,  $\sup S = y$  exists. Because  $x$  is an upper bound  $S$ , by definition of the supremum, we have that  $y \leq x$ . Let  $r = \frac{1}{2}$ . Then we can find  $k \in S$  such that  $y - \frac{1}{2} < k \leq y$ . But then we have that  $y < y + \frac{1}{2} < k + 1 \leq y + 1$ . Then this means  $k + 1 \notin S$  and so  $x < k + 1$ , completing this case.

Having exhausted all cases, this completes the proof.  $\square$

**Proposition 1 (Well-Ordering Principle).** Every nonempty subset  $S \subset \mathbb{N}$  has a minimum.

**Proposition 2 (Density of the Rational Numbers).** Let  $x, y \in \mathbb{R}$ . Then if  $x < y$ , there exists  $q \in \mathbb{Q}$  such that  $x < q < y$

## Section 2.4, Sequences in $\mathbb{R}$

**Definition.** We define a **sequence** of real numbers to be a function that maps each natural number  $n$  into the real number  $x$ . That is, a sequence is a function  $s : \mathbb{N} \rightarrow \mathbb{R}$  for  $A \subset \mathbb{R}$ . This is written as  $\{x_n\}$  or  $\{x_n\}_{n=1}^{\infty}$ .

**Definition (Convergence of a Sequence).** A sequence converges to the real number  $l \in \mathbb{R}$  if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$|a_n - l| < \varepsilon.$$

**Definition (Cauchy Sequence).** A sequence  $\{x_n\}$  in  $\mathbb{R}$  is **Cauchy** sequence if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n, m \geq N$ ,

$$|a_n - a_m| < \varepsilon.$$

**Theorem.** Let  $\{x_n\}$  be a sequence in  $\mathbb{R}$ . Then  $\{x_n\}$  is Cauchy if and only if  $\{x_n\}$  is convergent.

**Definition.** The number  $l \in \mathbb{R}$  is called a **cluster point** of  $\{x_n\}$  if there exists a subsequence  $\{x_{n_m}\}$  of  $\{x_n\}$  such that  $x_{n_m} \rightarrow l$ .

We can define this in another way. The number  $l \in \mathbb{R}$  is called **cluster point** of  $\{x_n\}$  if for all  $\varepsilon > 0$  and for all  $N \in \mathbb{N}$ , there exists  $n \geq N$  such that  $|x_n - l| < \varepsilon$ .

**Definition.** We define the **limit superior** of a sequence  $\{x_n\}$  in  $\mathbb{R}$  to be

$$\overline{\lim}_{n \rightarrow \infty} x_n = \inf_n \sup_{k \geq n} x_k.$$

This is also denoted as  $\limsup$ .

**Theorem.** A number  $l \in \mathbb{R}$  is the **limit superior** of the sequence  $\{x_n\}$  if and only if

- (i) For all  $\varepsilon > 0$ , there exists  $n \in \mathbb{N}$  such that for all  $k \geq n$ ,  $x_k < l + \varepsilon$
- (ii) For all  $\varepsilon > 0$  and for all  $n \in \mathbb{N}$ , there exists  $k \geq n$  such that  $x_k > l - \varepsilon$ .

**Definition.** We define the **limit inferior** of a sequence  $\{x_n\}$  in  $\mathbb{R}$  to be

$$\underline{\lim}_{n \rightarrow \infty} x_n = \sup_n \inf_{k \geq n} x_k.$$

This is also denoted as  $\liminf$ .

**Theorem.** A number  $l \in \mathbb{R}$  is the **limit inferior** of the sequence  $\{x_n\}$  if and only if

- (i) For all  $\varepsilon > 0$ , there exists  $n \in \mathbb{N}$  such that for all  $k \geq n$ ,  $x_k > l - \varepsilon$
- (ii) For all  $\varepsilon > 0$  and for all  $n \in \mathbb{N}$ , there exists  $k \geq n$  such that  $x_k < l + \varepsilon$ .

**Proposition 3.** From the last two definitions, we have the following property.

- $\overline{\lim}_{n \rightarrow \infty}$  is the largest cluster point.
- $\underline{\lim}_{n \rightarrow \infty}$  is the smallest cluster point.

## Section 2.5, Open and Closed Sets in $\mathbb{R}$

**Definition.** The set  $O \subset \mathbb{R}$  is called an **open** set if for all  $x \in O$ , there exists  $\delta > 0$  such that  $x - \delta, x + \delta$ .

Equivalently,  $O$  is an **open** set if for all  $x \in O$ , there is a  $\delta > 0$  such that each  $y$  with  $|x - y| < \delta$  belongs to  $O$ .

**Proposition 4.** From this above, we have the following properties:

1. The set  $\bigcup_{\alpha} O_{\alpha}$  is open.
2. The set  $\bigcup_{n=1}^n O_n$  is open.

**Theorem (Lindelof Theorem).** Every open set in  $\mathbb{R}$  is a disjoint union of countable union of open intervals.

*Proof.* This proof is contained on page 42 of Royden. □

**Definition.** A real number  $x \in \mathbb{R}$  is called **point of closure** of a set  $E \subset \mathbb{R}$  if for every  $\delta > 0$  there exists a  $y \in E$  such that  $|x - y| < \delta$ .

The set of points of closure of  $E$  is denoted  $\overline{E}$ .

**Proposition 5.** If  $A \subset B \subset \mathbb{R}$ , then  $\overline{A} \subset \overline{B}$ . Additionally,  $\overline{A \cup B} = \overline{A} \cup \overline{B}$ .

*Proof.* The proof of this is on page 43 of Royden. □

**Definition.** A set  $F \subset \mathbb{R}$  is called a **closed** set if  $\overline{F} = F$ .

Note that because  $F \subset \overline{F}$  always, a set  $F$  is closed if  $\overline{F} \subset F$ —that is,  $F$  contains all of its points of closure.

**Proposition 6.** For any set  $E$ , the set  $\overline{E}$  is closed; that is  $\overline{\overline{E}} = \overline{E}$ .

**Proposition 7.** Let  $E \subset \mathbb{R}$ . Then  $E$  is open if and only if  $E^c$  is closed.

**Definition.** We say that a collection of sets  $\mathcal{C}$  is a **cover** of a set  $F$  if

$$F \subset \bigcup_{O \in \mathcal{C}} O.$$

The collection  $\mathcal{C}$  is a covering of the set  $F$ .

**Theorem (Heine-Borel).** Let  $E \subset \mathbb{R}$  be set. Then  $E$  is compact if and only if  $E$  is closed and bounded.

## Compactness

**Theorem.** Let  $E \subset \mathbb{R}$ . Then  $E$  is compact if and only if  $E$  is sequentially compact. That is, for every  $\{x_n\}$  in  $E$ , there exists a convergent subsequence  $x_{n_m} \rightarrow x_0$  in  $E$ .

**Theorem.** Let  $\{I_n\}$  be a sequence of closed intervals such that  $I_{n+1} \subset I_n$ . Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

If  $[a_n, b_n]$  is an interval and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = a$ , then  $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$ .

## Section 2.6, Continuous Functions

**Definition.** Let  $E \subset \mathbb{R}$ , and let  $f : E \rightarrow \mathbb{R}$  be a real-valued function. Then  $f$  is **continuous** at the point  $x = a \in E$  if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $y \in E$  with  $|x - y| < \delta$  implies that  $|f(x) - f(y)| < \varepsilon$ .

Note that we can have continuity in terms of sequences. I will state it as a theorem here even though it was not in lecture because it is important to be able to use on its own.

**Theorem.** Let  $f : E \rightarrow \mathbb{R}$  be a function with  $E \subset \mathbb{R}$ . Let  $x \in E$  be any point. Then  $f$  is continuous at  $a$  if and only if for every sequence  $\{x_n\}$  in  $E$  converging to  $a$ , the sequence  $\{f(x_n)\}$  in  $f(E)$  (the image of  $E$ ) converges to  $f(a)$ .

**Proposition.** Let  $E \subset \mathbb{R}$  be compact. Let  $f : E \rightarrow \mathbb{R}$  be continuous real-valued function. Then  $f(E)$  is a compact set.

*Proof.* Let  $E \subset \mathbb{R}$  be a compact and suppose the function  $f : E \rightarrow \mathbb{R}$  is continuous. To show that  $f(E)$  is compact, we will use the Heine-Borel theorem and show that it is closed and bounded. To show that  $f(E)$  is closed, suppose we have any sequence  $\{f(x_n)\}$  converging to the point  $f(a) \in \mathbb{R}$ . Additionally, let  $\{x_n\}$  be any sequence in  $E$ . Because  $E$  is compact, there exists a subsequence  $\{x_{n_m}\}$  which converges to a point  $x_0 \in E$ . Since  $f$  is continuous, by the preceding theorem this means that the sequence  $\{f(x_{n_m})\}$  converges to  $f(x_0) \in f(E)$ .  $\square$

**Proposition (2.17, Extreme Value Theorem).** Let  $E \subset \mathbb{R}$  be a compact set, and let  $f : E \rightarrow \mathbb{R}$  be a continuous function. Then there exists  $x_1, x_2 \in E$  such that

$$f(x_1) \leq f(x) \leq f(x_2), \text{ for all } x \in E.$$

**Proposition (2.18).** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. Then  $f$  is **continuous** if and only if  $f^{-1}(O)$  is open for all open sets  $O \subset \mathbb{R}$ .

**Proposition (2.19).** Let  $E \subset \mathbb{R}$ , and let  $f : E \rightarrow \mathbb{R}$  be continuous. Without loss of generality, suppose that  $f(a) \leq f(b)$ . Then for all  $\gamma \in [f(a), f(b)]$ , there exists  $c \in [a, b]$  such that  $f(c) = \gamma$ .

**Definition (Uniform Continuity).** Let  $E \subset \mathbb{R}$ . A function  $f : E \rightarrow \mathbb{R}$  is **uniformly continuous** if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $x, y \in E$  with  $|x - y| < \delta$  implies that  $|f(x) - f(y)| < \varepsilon$ .

**Proposition (2.20).** Let  $E \subset \mathbb{R}$  be a compact set. If  $f : E \rightarrow \mathbb{R}$  is a continuous function on  $E$ , then  $f$  is uniformly continuous on  $E$ .

**Definition.** Let  $f_n : E \rightarrow \mathbb{R}$  be a sequence of functions, and let  $f : E \rightarrow \mathbb{R}$ .

1. The sequence  $\{f_n\}$  **converges pointwise** on  $E$  to  $f$  if for all  $\varepsilon > 0$  and for all  $x \in E$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|f(x) - f_n(x)| < \varepsilon$ .
2. The sequence  $\{f_n\}$  **converges uniformly** if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $x \in E$  and for all  $n \geq N$ ,  $|f(x) - f_n(x)| < \varepsilon$ .

## Section 3.1, Lebesgue Measure

[Perhaps finish these notes another time...]



## Section 3.2, Outer Measure

**Definition.** The **outer measure**  $m^*(A)$  of a set  $A \subset \mathbb{R}$  is given

$$m^*(A) = \inf_{A \subset \bigcup I_n} \sum_n l(I_n)$$

where  $\{I_n\}$  is a countable collection of open intervals that cover  $A$ .

Note that from this definition, we get that

1.  $m^*(\emptyset) = 0$
2. If  $A \subset B$ ,  $m^*(A) \leq m^*(B)$ .
3.  $m^*$  does not satisfy disjoint additivity.

**Proposition (3.1).** The outer measure of an interval is its length; that is,  $m^*(I) = l(I)$  where  $I = [a, b]$ ,  $(a, b)$ ,  $[a, b)$ , or  $(a, b]$ .

*Proof.* It is sufficient to show that  $m^*([a, b]) = l([a, b])$  since every other interval is a subset of  $[a, b]$ . Let  $\varepsilon > 0$ . Then  $[a, b] \subset [a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}]$  which implies, by the definition of the outer measure,

$$m^*([a, b]) \leq l\left([a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}]\right) = b - a + \varepsilon.$$

Because  $\varepsilon$  was fixed, this means that  $m^* \leq b - a$ .

Now we must show that  $m^* \geq b - a$ . Because  $[a, b]$  is compact, for any collection  $\{I_n\}$  of open intervals covering  $[a, b]$ , there exists a finite collection of intervals  $\{I_1, \dots, I_k\}$  so that

$$[a, b] \subset \bigcup_{n=1}^k I_n.$$

This gives us that

$$\sum_n l(I_n) \geq \sum_{n=1}^k l(I_n) \geq b - a$$

and so  $b - a$  is a lower bound. But since  $m^*$  is the greatest lower bound of all such sums, we have that  $m^* \geq b - a$ .

Therefore,  $m^*([a, b]) = l([a, b]) = b - a$ .

□

**Proposition (3.2, Subadditivity).** Let  $\{A_n\}$  be a countable collection of sets on  $\mathbb{R}$ . Then

$$m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n).$$

*Proof.* Proof on page 57.

□

**Corollary (3.3).** If  $A$  is a countable set, then  $m^*(A) = 0$ .

*Proof.* Proof is on the end of page 57. □

## Section 3.3, Measurable Sets and Lebesgue Measure

**Definition.** A set  $E \subset \mathbb{R}$  is (Lebesgue) **measurable** if for all sets  $A$ , we have that

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

**Lemma (3.6).** If  $m^*(E) = 0$ , then  $E$  is measurable.

*Proof.* Let  $A$  be any chosen set. Because  $A \cap E \subset E$  and  $m^*(E) = 0$ ,

$$m^*(A \cap E) \leq m^*(E) = 0.$$

Note that  $A \cap E^c \subset A$  and so  $m^*(A) \leq m^*(A \cap E^c) + m^*(A \cap E)$  and so it suffices to show that  $m^*(A) \geq m^*(A \cap E^c)$ . Using this, we can show that

$$m^*(A) \geq m^*(A \cap E^c) + 0 = m^*(A \cap E^c) = m^*(A \cap E)$$

giving us the desired result. □

**Definition.** Let  $\mathcal{M}$  be the set of measurable sets in  $\mathbb{R}$

**Lemma (3.7).** If  $E_1$  and  $E_2$  are measurable sets, then so is  $E_1 \cup E_2$ .

*Proof.* Proof on top of page 57. □

**Corollary (3.8).** The family  $\mathcal{M}$  of measurable sets is an algebra of sets. In other words, if  $E \in \mathcal{M}$ , then  $E^c \in \mathcal{M}$ . Further, if  $E_1, E_2 \in \mathcal{M}$ , then  $E_1 \cup E_2 \in \mathcal{M}$ .

**Lemma (3.9).** Let  $A$  be any set, and  $E_1, \dots, E_n$  be a finite sequence of sets such that  $E_i \cap E_j = \emptyset$  for all  $i \neq j$ . Then

$$m^* \left( A \cap \left[ \bigcup_{i=1}^n E_i \right] \right) = \sum_{i=1}^n m^*(A \cap E_i).$$

*Proof.* We proceed by induction. For  $n = 1$ , we have the set  $E_1$  and the equality holds. Suppose that we have  $n = k$  sets  $E_1, \dots, E_k$  with  $E_i \cap E_j = \emptyset$  for all  $i \neq j$  so that

$$m^* \left( A \cap \left[ \bigcup_{i=1}^k E_i \right] \right) = \sum_{i=1}^k m^*(A \cap E_i).$$

Consider  $n = k + 1$ . Because each  $E_i$  is disjoint,

$$\begin{aligned} A \cap \left( \bigcup_{i=1}^{k+1} E_i \right) \cap E_{k+1} &= A \cap E_{k+1}; \\ A \cap \left( \bigcup_{i=1}^{k+1} E_i \right) \cap E_{k+1}^c &= A \cap \bigcup_{i=1}^k E_i. \end{aligned}$$

Because the  $E_i$ 's are measurable,

$$\begin{aligned} m^* \left( A \cap \bigcup_{i=1}^{k+1} E_i \right) &= m^*(A \cap E_{k+1}) + m^* \left( A \cap \bigcup_{i=1}^k E_i \right) \\ &= m^*(A \cap E_{k+1}) + \sum_{i=1}^k m^*(A \cap E_i) \quad \text{Induction Hypothesis} \\ &= \sum_{i=1}^{k+1} m^*(A \cap E_i) \end{aligned}$$

which, by induction, completes the proof.  $\square$

**Theorem (3.10).**  $\mathcal{M}$  is a  $\sigma$ -algebra. In other words, in addition to being an algebra of sets, if  $\{E_i\}_{i=1}^\infty \subset \mathcal{M}$ , then  $\bigcup_{i=1}^\infty E_i \in \mathcal{M}$ .

*Proof.* <sup>1</sup>

$\square$

**Lemma (3.11).** The interval  $(a, \infty)$  is measurable for all  $a \in \mathbb{R}$ .

*Proof.* <sup>2</sup>

$\square$

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<sup>1</sup>Proof on bottom of page 59 and top of page 60.

<sup>2</sup>Proof on the bottom of page 60 through the middle of page 61.

**Theorem (3.12).** Every Borel set is measurable. In particular, each open set and each closed set is measurable.

*Proof.* <sup>3</sup>

□

**Definition.** Let  $E \in \mathcal{M}$ . We define  $m(E) := m^*(E)$  to be the **Lebesgue measure** of  $E$ .

**Proposition (3.13, Countable Additivity).** Let  $\{E_i\}_{i=1}^n$  be a sequence of measurable sets. Then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^n m(E_i).$$

If, in addition,  $E_i \cap E_j = \emptyset$  for all  $i \neq j$ , then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^n m(E_i).$$

**Proposition (3.14).** Let  $\{E_i\} \subset \mathcal{M}$  be a decreasing sequence (i.e.,  $E_{i+1} \subset E_i$ ). Let  $m(E_1) < \infty$ . Then

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} m(E_n).$$

**Proposition (3.15).** Let  $E$  be any given set. Then the following are equivalent:

- (i)  $E$  is measurable.
- (ii) For all  $\varepsilon > 0$ , there is an open set  $O \supset E$  with  $m^*(O \setminus E) < \varepsilon$ .
- (iii) For all  $\varepsilon > 0$ , there is a closed set  $F \subset E$  with  $m^*(E \setminus F) < \varepsilon$ .
- (iv) There is a  $G \in G_\delta$  with  $E \subset G$  such that  $m^*(G \setminus E) = 0$ .
- (v) There is a  $F \in F_\sigma$  with  $F \subset E$  such that  $m^*(E \setminus F) = 0$ .

If  $m^*(E) < \infty$ , the above statements are equivalent:

- (vi) For all  $\varepsilon > 0$ , there is a finite union  $U$  of open intervals such that  $m^*(U \Delta E) < \varepsilon$ .

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<sup>3</sup>Proof on the bottom of page 61.

## Section 3.5, Measurable Functions

**Proposition (3.18).** Let  $E \subset \mathbb{R}$ , and Let  $f : E \rightarrow [-\infty, \infty]$  be an extended real-valued function whose domain is measurable. Let  $\alpha \in \mathbb{R}$  be any real number. Then the following statements are equivalent:

- (i) The set  $\{x : f(x) > \alpha\}$  is measurable.
- (ii) The set  $\{x : f(x) \geq \alpha\}$  is measurable.
- (iii) The set  $\{x : f(x) < \alpha\}$  is measurable.
- (iv) The set  $\{x : f(x) \leq \alpha\}$  is measurable.

All together, these imply

- (v) The set  $\{x : f(x) = \alpha\}$  is measurable.

*Proof.*<sup>1</sup>

□

**Definition.** An extended real-valued function  $f : E \rightarrow [-\infty, \infty]$  is **(Lebesgue) measurable** if its domain is measurable and satisfies one of the first four statements of Proposition 18.

Note that this means that any continuous function is measurable since then the pre-image of any open set is still an open set.

**Proposition (3.19).** Let  $f$  and  $g$  be two measurable functions defined on the same domain, and let  $c \in \mathbb{R}$ . Then the functions  $f + c$ ,  $cf$ ,  $f + g$ ,  $g - f$ , and  $fg$  are measurable.

*Proof.* For  $f(x) + c$ , note that

$$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}.$$

□

**Theorem (3.20, Limit of Measurable Functions is Measurable).** Let  $\{f_n\}$  be a sequence of measurable functions with the same domain. Then the functions  $\sup\{f_1(x), \dots, f_n(x)\}$ ,  $\inf\{f_1(x), \dots, f_n(x)\}$ ,  $\sup_n f_n$ ,  $\inf_n f_n$ ,  $\overline{\lim} f_n$ , and  $\underline{\lim} f_n$  are measurable.

*Proof.* Let  $\{f_n\}$  be a sequence of measurable functions. Let  $h(x) = \sup\{f_1(x), \dots, f_n(x)\}$  and we so must show that  $\{x : h(x) < \alpha\}$  for all  $\alpha \in \mathbb{R}$ . To that end, let  $\alpha \in \mathbb{R}$  be chosen. Then

$$\{x : h(x) < \alpha\} = \bigcup_{i=1}^n \{x : f_i(x) > \alpha\}$$

which, because the right-hand side is a union of measurable sets from the  $f_i$ 's being measurable, means that the set  $\{x : h(x) < \alpha\}$  is also measurable. □

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<sup>1</sup>Proof is on page 67.

**Definition.** A property is said to hold **almost everywhere** (a.e) if the set of points where it fails to hold is a set of measure zero. Thus  $f = g$  a.e if  $f$  and  $g$  have the same domain and  $m\{x : f(x) \neq g(x)\} = 0$ .

**Proposition (3.21).** If  $f$  is measurable and  $f = g$  a.e, then  $g$  is measurable.

*Proof.* Let  $\{x : g(x) > \alpha\}$ . This is equivalent to saying that

$$\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha, f(x) \leq \alpha\}$$

□

**Proposition (3.22).** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a measurable function with  $E \subset \mathbb{R}$  and is equal to  $\pm\infty$  only on sets with measure zero. Then for all  $\varepsilon > 0$ , there exist a step function  $g$  and a continuous function  $h$  such

$$|f - g| < \varepsilon \quad \text{and} \quad |f - h| < \varepsilon$$

except on set of measure less than  $\varepsilon$ ; i.e.,  $m\{x : |f(x) - g(x)| \geq \varepsilon\} < \varepsilon$  and  $m\{x : |f(x) - h(x)| \geq \varepsilon\} < \varepsilon$ . If in addition  $m \leq f \leq M$ , then we may choose the functions  $g$  and  $h$  so that  $m \leq g \leq M$  and  $m \leq h \leq M$ .

**Proposition (3.23, (Weak) Egonoff's Theorem).** Let  $E$  be a measurable set of finite measure, and  $\{f_n\}$  be a sequence of measurable functions defined on  $E$ . Let  $f$  be real-valued function such for each  $x \in E$  we have  $f_n(x) \rightarrow f(x)$ . Then for all  $\varepsilon > 0$  and all  $\delta > 0$ , there is measurable set  $A \subset E$  with  $m(A) < \delta$  and  $N \in \mathbb{N}$  such that for all  $x \notin A$  and all  $n \geq N$ ,

$$|f_n(x) - f(x)| < \varepsilon.$$