Geometrically Nonlinear Finite Element Analysis — Stiffness Equations

The Linear Stiffness Matrix Equation is written as:

$$\left[\begin{array}{c|c} K_{\text{ff}} & K_{\text{fs}} \\ \hline K_{\text{sf}} & K_{\text{ss}} \end{array}\right] \cdot \left[\begin{array}{c} u_{\text{f}} \\ \hline u_{\text{s}} \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline 0 \end{array}\right]$$

Solving for the Free DOFs vector $\vec{u_{\text{f}}}$, gives:

$$[K_{\text{ff}}] \cdot \vec{u_{\text{f}}} + [K_{\text{fs}}] \cdot \vec{u_{\text{s}}} = 0$$

Where $\vec{u_{\text{s}}}$ is the known vector of Fixed DOFs (Boundary Conditions). Therefore, the known part of the Equation is moved to the RHS and the Equation is rewritten as:

$$[K_{\text{ff}}] \cdot \vec{u_{\text{f}}} = \vec{b_{\text{f}}}$$

The Non-Linear Stiffness Equation for the i-th timestep is written as:

$$[K_{\text{T}}] \cdot \Delta\vec{u_{\text{f}}} = \vec{R}^i = \vec{f_{\text{ext}}} - \vec{f_{\text{int}}}^{\,i}$$

K_{T} is the Tangent Stiffness Matrix, and is defined as $K_{\text{T}} = K_{\text{L}} + K_{\text{NL}}$. K_{L} is the Linear Stiffness Matrix K_{ff}, as previously defined, whereas K_{NL} is the Non-Linear Stiffness Matrix. Both matrices are calculated on a per-element basis, as a 24-by-24 Local Stiffness Matrix as follows:

$$\begin{aligned} [K_{\text{L}}]_{24\times24} &= \int_V [B_{\text{L}}^T]_{24\times6}[D]_{6\times6}[B_{\text{L}}]_{6\times24}\, dV \\ [K_{\text{NL}}]_{24\times24} &= \int_V [B_{\text{NL}}^T]_{24\times9}[S]^i_{9\times9}[B_{\text{NL}}]_{9\times24}\, dV \end{aligned}$$

Where $[D]$ is the Elasticity Matrix, S^i is the 2nd Piola-Kirchhoff Stress Matrix, defined as:

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} & 0 & 0 & 0 & 0 & 0 & 0 \\ \tau_{xy} & \tau_{yy} & \tau_{yz} & 0 & 0 & 0 & 0 & 0 & 0 \\ \tau_{xz} & \tau_{yz} & \tau_{zz} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_{xx} & \tau_{xy} & \tau_{xz} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_{xy} & \tau_{yy} & \tau_{yz} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_{xz} & \tau_{yz} & \tau_{zz} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau_{xx} & \tau_{xy} & \tau_{xz} \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau_{xy} & \tau_{yy} & \tau_{yz} \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}$$

Where τ_{ij} are the 2nd Piola Stresses. The Linear Deformation Matrix $[B_{\text{L}}]$ and the Non-Linear Deformaiton

Matrix $[B_{\mathrm{NL}}]$ are calculated as:

$$[B_{\mathrm{L}}] = \left[\begin{array}{ccc|c} \frac{\partial N}{\partial x} & 0 & 0 & \cdots \\ 0 & \frac{\partial N}{\partial y} & 0 & \cdots \\ 0 & 0 & \frac{\partial N}{\partial z} & \cdots \\ \frac{\partial N}{\partial y} & \frac{\partial N}{\partial x} & 0 & \cdots \\ 0 & \frac{\partial N}{\partial z} & \frac{\partial N}{\partial y} & \cdots \\ \frac{\partial N}{\partial z} & 0 & \frac{\partial N}{\partial z} & \cdots \end{array}\right], \ [B_{\mathrm{NL}}] = \left[\begin{array}{ccc|c} \frac{\partial N}{\partial x} & 0 & 0 & \cdots \\ \frac{\partial N}{\partial y} & 0 & 0 & \cdots \\ \frac{\partial N}{\partial z} & 0 & 0 & \cdots \\ 0 & \frac{\partial N}{\partial x} & 0 & \cdots \\ 0 & \frac{\partial N}{\partial y} & 0 & \cdots \\ 0 & \frac{\partial N}{\partial z} & 0 & \cdots \\ 0 & 0 & \frac{\partial N}{\partial x} & \cdots \\ 0 & 0 & \frac{\partial N}{\partial y} & \cdots \\ 0 & 0 & \frac{\partial N}{\partial z} & \cdots \end{array}\right]$$

The vector $\vec{f_{\mathrm{ext}}}$ is equal to the vector $\vec{b_f}$ found in the Linear Stiffness Equation. The vector $\vec{f_{\mathrm{int}}}$ is calculated per timestep as:

$$\vec{f_{\mathrm{int}}}_{24\times 6} = \int_V \left[B_{\mathrm{L}}^T\right]_{24\times 6} \vec{S_{\mathrm{v}}}^{i}_{6\times 1} \, dV$$

Where $\vec{S_{\mathrm{v}}}^{i}_{6\times 1}$ is The 2nd Piola Stress Vector (Voigt Notation).
After solving the Nonlinear Equation by iterating, the displacement vector $\vec{u_{\mathrm{f}}}$ is updated as $\vec{u_{\mathrm{f}}}^{i+1} = \Delta\vec{u_{\mathrm{f}}}^{i} + \vec{u_{\mathrm{f}}}^{i}$.