Solutions of the book Applied Stochastic Differential Equations of Simmo Särkkä and Arno Solin

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Contents

1 Introduction

2 Some Background on Ordinary Differential Equations

Exercise 2.1. *Consider the initial value problem*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \theta_1 \left(\theta_2 - x(t)\right), \quad x(0) = x_0 \tag{1}$$

where θ_1 and θ_2 are constants.

1. *Derive the solution using the integrating factor method.*
2. *Derive the solution using the Laplace transform.*

Solution

1. The differential equation can be written as

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\theta_1 x + \theta_1\theta_2, \tag{2}$$

and the integral factor is given by $\psi(x) = \exp(\theta_1 x)$. With this integrating factor, and using the equation (2.30) of the book. We have that

$$\begin{aligned} x(t) &= \exp(-\theta_1 t)x_0 + \int_0^t \exp(-\theta_1 \tau)\theta_1\theta_2 d\tau \\ &= \exp(-\theta_1 t)x_0 + [1 - \exp(-\theta_1 t)]\theta_2, \end{aligned} \tag{3}$$

2. The Laplace transform of the ODE is given by

$$\mathfrak{L}\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \theta_1 x - \theta_1\theta_2\right] = 0, \tag{4}$$

and developing the left side we have that

$$\begin{aligned} \mathfrak{L}\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \theta_1 x - \theta_1\theta_2\right] &= \mathfrak{L}\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right] + \theta_1\mathfrak{L}\left[x\right] - \theta_1\theta_2\mathfrak{L}\left[1\right] \\ &= \left[sX(s) - x(0)\right] + \theta_1 X(s) + s^{-1} \end{aligned} \tag{5}$$

Then, combining the previous equations we have that

$$X(s) = (x_0 - \theta_2)(\theta_1 + s)^{-1} + \theta_2 s^{-1}, \tag{6}$$

and applying the inverse Laplace transform we have that

$$\begin{aligned} x(t) &= (x_0 - \theta_2)\mathfrak{L}^{-1}\left[(\theta_1 + s)^{-1}\right] + \theta_2\mathfrak{L}^{-1}\left[s^{-1}\right] \\ &= (x_0 - \theta_2)e^{-\theta_1 t} + \theta_2. \end{aligned} \tag{7}$$

Exercise 2.2. *One way of solving ODEs is to use an ansatz (an educated guess). Show that* $x(t) = c_1 \sin(\omega t) + c_2 \cos(\omega t)$ *is the solution to the second-order ODE*

$$\ddot{x} + \omega^2 x = 0, \quad \omega > 0. \tag{8}$$

Solution

$$\begin{aligned}
\ddot{x} + \omega^2 x &= \frac{d^2}{dt^2}\left(c_1 \sin(\omega t) + c_2 \cos(\omega t)\right) + \omega^2 \left(c_1 \sin(\omega t) + c_2 \cos(\omega t)\right) \\
&= \frac{d}{dt}\left(c_1 \omega \cos(\omega t) - c_2 \omega \sin(\omega t)\right) + \omega^2 \left(c_1 \sin(\omega t) + c_2 \cos(\omega t)\right) \\
&= \left(c_1 \omega \frac{d}{dt}\cos(\omega t) - c_2 \omega \frac{d}{dt}\sin(\omega t)\right) + \omega^2 \left(c_1 \sin(\omega t) + c_2 \cos(\omega t)\right) \\
&= \left(-c_1 \omega^2 \sin(\omega t) - c_2 \omega^2 \cos(\omega t)\right) + \omega^2 \left(c_1 \sin(\omega t) + c_2 \cos(\omega t)\right) \\
&= 0
\end{aligned} \tag{9}$$

Exercise 2.3. *Solve the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y^2 - 1 \tag{10}$$

with initial condition $y(0) = 0$ *using the method of separation of variables.*

Solution The differential equation can be rewriten in integral form as

$$\int_0^t \frac{1}{y^2(\tau) + 1} \frac{\mathrm{d}y(\tau)}{\mathrm{d}\tau} d\tau = -\int_0^t d\tau, \tag{11}$$

and developing the left we have that

$$\begin{aligned}
\int_0^t \frac{1}{y^2(\tau) + 1} \frac{\mathrm{d}y(\tau)}{\mathrm{d}\tau} d\tau &= \int_0^t \frac{d}{d\tau} \arctan(y(\tau)) d\tau \\
&= \arctan(y(t)) - \arctan(y(0)) \\
&= \arctan(y(t)) - \arctan(0) \\
&= \arctan(y(t)).
\end{aligned} \tag{12}$$

Finally the solution of the differential equation is given by the equation

$$\arctan(y(t)) = -t, \tag{13}$$

or equivalentely

$$y(t) = -\tan(t). \tag{14}$$

Exercise 2.4. *In classical ODE literature, ODE systems are typically characterized by the nature of their fixed points (roots of the differential equation). Find out what is meant by this, sketch the behavior of the following ODEs, and classify all their fixed points:*

1. $\dot{x} = 1 + 2\cos x$.

2. $\dot{x} = x - y$ *and* $\dot{y} = 1 - e^x$.

3. $\dot{x} = x - y$ *and* $\dot{y} = x^2 - 4$.

Solution Given the system of autonomous differential equations in $\mathbb{R}^d$

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \tag{15}$$

a fixed point or equilibrium point $\mathbf{x}^*$ is such that $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. It has the property to be invariant independent of the initial condition chosen. When the bidimensional vector field $\mathbf{f}$ is Lipschitz continuous, differentiable, and satisfies some topological properties given by the Hartman-Grobman theorem, then the local behaviour of a 2-dimensional system can be classified depending the trace and determinant of the Jacobian Matrix $\mathbf{Df}(\mathbf{x}^*)$. For it let us define the $\Delta = \mathrm{tr}(\mathbf{Df}(\mathbf{x}^*))^2 - 4\det(\mathbf{Df}(\mathbf{x}^*))$, then the classification is

1. Saddle point: $\det(\mathbf{Df}(\mathbf{x}^*)) < 0$.
2. Stable linear points: $\mathrm{tr}(\mathbf{Df}(\mathbf{x}^*)) < 0$ and $\det(\mathbf{Df}(\mathbf{x}^*)) = 0$.
3. Unstable linear points: $\mathrm{tr}(\mathbf{Df}(\mathbf{x}^*)) > 0$ and $\det(\mathbf{Df}(\mathbf{x}^*)) = 0$.
4. Sink: $\mathrm{tr}(\mathbf{Df}(\mathbf{x}^*)) < 0$, $\det(\mathbf{Df}(\mathbf{x}^*)) \geq 0$ and $\Delta < 0$.
5. Source: $\mathrm{tr}(\mathbf{Df}(\mathbf{x}^*)) > 0$, $\det(\mathbf{Df}(\mathbf{x}^*)) \geq 0$ and $\Delta < 0$.
6. Center: $\mathrm{tr}(\mathbf{Df}(\mathbf{x}^*)) = 0$, $\det(\mathbf{Df}(\mathbf{x}^*)) \geq 0$ and $\Delta > 0$.
7. Degenerate Source: $\mathrm{tr}(\mathbf{Df}(\mathbf{x}^*)) > 0$, $\det(\mathbf{Df}(\mathbf{x}^*)) \geq 0$ and $\Delta = 0$.
8. Spiral Sink: $\mathrm{tr}(\mathbf{Df}(\mathbf{x}^*)) < 0$, $\det(\mathbf{Df}(\mathbf{x}^*)) \geq 0$ and $\Delta > 0$.
9. Spiral Source: $\mathrm{tr}(\mathbf{Df}(\mathbf{x}^*)) > 0$, $\det(\mathbf{Df}(\mathbf{x}^*)) \geq 0$ and $\Delta > 0$.

With this analysis we can classify our systems of ODEs as follow:

1. $\dot{x} = 1 + 2\cos x$.

   We can transform this ODE in a system defining $\dot{y} = 0$, the vector $\mathbf{x} = (x, y)^T$ and the function $\mathbf{f} = (1 + 2\cos(x), 0)^T$. This new system of ODEs has equilibrium point in $\mathbf{x}^* = 2\pi/3$ (modulus 2π), and the Jacobian matrix is given by

$$\mathbf{Df}(\mathbf{x}^*) = \begin{pmatrix} -\sqrt{3} & 0 \\ 0 & 0 \end{pmatrix}. \tag{16}$$

   From here we know that $\mathrm{tr}(\mathbf{Df}(\mathbf{x}^*)) < 0$, $\det(\mathbf{Df}(\mathbf{x}^*)) = 0$ and $\Delta > 0$ and the point is a stable linear point.

2. $\dot{x} = x - y$ and $\dot{y} = 1 - e^x$.

For this case we have a non-linear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = (x, y)^T$ and $\mathbf{f}(\mathbf{x}) = (x - y, 1 - e^x)^T$. The equilibrium point is clearly $\mathbf{x}^* = (0, 0)^T$ and the Jacobian matrix is then

$$\mathbf{Df}(\mathbf{x}^*) = \begin{pmatrix} 1 & -1 \\ 0 & -e^{x^*} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}. \tag{17}$$

It is cleat that $\det(\mathbf{Df}(\mathbf{x}^*)) < 0$ and the equilibrium point is a saddle point.

3. $\dot{x} = x - y$ and $\dot{y} = x^2 - 4$.

Here the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a function given by $\mathbf{f}(\mathbf{x}) = (x - y, x^2 - 4)^T$ with two equilibrium points $\mathbf{x}_1^* = (2, -2)$ and $\mathbf{x}_2^* = (-2, 2)$. The Jacobian matrices in these points are

$$\mathbf{Df}(\mathbf{x}_1^*) = \begin{pmatrix} 1 & -1 \\ 2x_1^* & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}, \tag{18}$$

and

$$\mathbf{Df}(\mathbf{x}_2^*) = \begin{pmatrix} 1 & -1 \\ 2x_2^* & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix}. \tag{19}$$

For the point $\mathbf{x}_1^*$ we have that $\operatorname{tr}(\mathbf{Df}(\mathbf{x}^*)) > 0$, $\det(\mathbf{Df}(\mathbf{x}^*)) > 0$ and $\Delta < 0$, resulting in a source point. While for the point $\mathbf{x}_2^*$ we have that $\det(\mathbf{Df}(\mathbf{x}^*)) < 0$ and the equilibrium point is a saddle point.

Exercise 2.5. *Study the behavior of the following differential equation*

$$\ddot{x} + \dot{x} - \left(\alpha - x^2\right) x = 0, \quad \alpha \geq 0, \tag{20}$$

as follows:

1. *Rewrite the problem in terms of a first-order ODE.*
2. *Find the fixed points of the ODE.*
3. *Characterize the nature of the fixed points.*
4. *Sketch the behavior of trajectories in the $(x, \dot{x})$ plane.*

Solution

1. We define the variables $y = \dot{x}$ and $\mathbf{x} = (x, y)^T$, with them we have the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f}(\mathbf{x}) = (y, (\alpha - x^2)x - y)^T$.
2. The equilibrium points of this system are $\mathbf{x}_1^* = (0, 0)^T$, $\mathbf{x}_2^* = (\alpha, 0)^T$ and $\mathbf{x}_3^* = (-\alpha, 0)^T$.

3. The Jacobian matrix of this system is

$$\mathbf{Df}(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ \alpha - 3x^2 & -1 \end{pmatrix}, \tag{21}$$

where the trace, determinant and discriminant are $\mathrm{tr}(\mathbf{Df}(\mathbf{x})) = -1 < 0$, $\det(\mathbf{Df}(\mathbf{x})) = -\alpha + 3x^2$ and $\Delta(\mathbf{x}) = \mathrm{tr}(\mathbf{Df}(\mathbf{x}))^2 - 4\det(\mathbf{Df}(\mathbf{x})) = 1 + 4\alpha - 12x^2$.

Point $\mathbf{x}_1^*$.

Here $\det(\mathbf{Df}(\mathbf{x}_1^*)) = -\alpha < 0$ and the equilibrium point is a saddle point.

Points $\mathbf{x}_2^*$ and $\mathbf{x}_3^*$.

For this points $\det(\mathbf{Df}(\mathbf{x}^*)) = -\alpha + 3\alpha^2$ and we can split the problem in two cases. If $\alpha < 1/3$, then the determinant is negative and both are saddle points. If $\alpha \in [1/3, 1/2]$ then $\Delta(\mathbf{x}^*) \geq 0$ and the point is an spiral sink, while in the case that $\alpha > 1/2$ we have that $\Delta(\mathbf{x}^*) < 0$ and the equilibrium point is a sink.

Exercise 2.6. *We wish to find the Laplace domain solution to the equation*

$$a\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = u \tag{22}$$

where $y(t)$ is an unknown time-varying function and $u(t)$ is a given function. Solve the equation by following these steps:

1. *Calculate the Laplace transform of the equation.*
2. *Solve the Laplace domain equation for $Y(s)$.*
3. *Take the inverse Laplace transform and provide a solution for $y(t)$.*

Solution

1. Let us define the Laplace transform of the right side as $\mathfrak{L}[u(t)] = U(s)$. Calculating the left side of the ODE we have that

$$\begin{aligned} \mathfrak{L}\left[a\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy\right] &= a\mathfrak{L}\left[\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right] + b\mathfrak{L}\left[\frac{\mathrm{d}y}{\mathrm{d}t}\right] + c\mathfrak{L}[y] \\ &= a\left[s^2Y(s) - sy(0) - y'(0)\right] \\ &\quad + b\left[sY(s) - y(0)\right] + cY(s) \end{aligned} \tag{23}$$

2. The Laplace domain equation is

$$Y(s) = \frac{U(s) + y(0)[as + b] + ay'(0)}{as^2 + bs + c}. \tag{24}$$

3. The solution depends of the signs of the coefficients and the expression for $U(s)$.

Exercise 2.7. *We wish to find the matrix exponential* $\exp(Ft)$*, where* $t \geq 0$ *with*

$$\mathbf{F} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

1. *Solve it using the series expansion definition of the matrix exponential (note that the matrix is nilpotent).*

2. *Solve it using the Laplace transform as in Example 2.5.*

Solution

1. The powers of the matrix $\mathbf{F}$ are

$$\mathbf{F}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{F}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{26}$$

then the exponential matrix is

$$\exp(\mathbf{F}t) = \mathbf{I} + \mathbf{F}t + \mathbf{F}^2 \frac{t^2}{2} = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}. \tag{27}$$

2. Using Laplace transform we have that

$$\begin{aligned} \exp(\mathbf{F}t) &= \mathfrak{L}^{-1}\left[(s\mathbf{I} - \mathbf{F})^{-1}\right] \\ &= \mathfrak{L}^{-1}\left[\begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{pmatrix}\right] \\ &= \mathfrak{L}^{-1}\left[\begin{pmatrix} s^{-1} & s^{-2} & s^{-3} \\ 0 & s^{-1} & s^{-2} \\ 0 & 0 & s^{-1} \end{pmatrix}\right] \\ &= \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{28}$$

Exercise 2.8. *Use a computer algebra system (e.g.,MATLAB) to compute the matrix exponential*

$$\exp\left\{\begin{pmatrix} 0 & 1 \\ -v^2 & -\gamma \end{pmatrix} t\right\}. \tag{29}$$

How can you extract the impulse response (2.47) from the result?

Solution The printing result is

Matrix([[(gamma*exp(t*sqrt(gamma**2 - 4*v**2)) - gamma + sqrt(gamma**2 - 4*v**2)*exp(t*sqrt(gamma**2 - 4*v**2)) + sqrt(gamma**2 - 4*v**2))*exp(-t*(gamma + sqrt(gamma**2 - 4*v**2))/2)/(2*sqrt(gamma**2 - 4*v**2)), (-gamma*exp(t*sqrt(gamma**2 - 4*v**2)) + gamma - sqrt(gamma**2 - 4*v**2)*exp(t*sqrt(gamma**2 - 4*v**2)) + sqrt(gamma**2 - 4*v**2))*exp(-t*(gamma + sqrt(gamma**2 - 4*v**2))/2)/(-gamma**2 - gamma*sqrt(gamma**2 - 4*v**2) + 4*v**2)], [v**2*(1 - exp(t*sqrt(gamma**2 - 4*v**2)))*exp(-t*(gamma + sqrt(gamma**2 - 4*v**2))/2)/sqrt(gamma**2 - 4*v**2), (-gamma**2 - gamma*sqrt(gamma**2 - 4*v**2) + 2*v**2*exp(t*sqrt(gamma**2 - 4*v**2)) + 2*v**2)*exp(-t*(gamma + sqrt(gamma**2 - 4*v**2))/2)/(-gamma**2 - gamma*sqrt(gamma**2 - 4*v**2) + 4*v**2)]])

Where the impulse response is

$$\mathbf{H}(\omega) = ((\mathrm{i}\omega)\mathbf{I} - \mathbf{F})^{-1} = \mathfrak{F}\left[\exp(\mathbf{F}t)u(t)\right]. \tag{30}$$

Exercise 2.9. *Consider the initial value problem $\dot{x} = -x$ where $x(0) = 1$.*

1. *Solve the problem analytically. What is the exact value of $x(1)$?.*
2. *Implement the Euler method for this ODE. Using a step size of 1, es- timate $x(1)$ numerically. Repeat this for step sizes 10^{-n}, where $n = 1, 2, 3, 4$.*

Solution

1. By inspection is clear that solution is $x(t) = \exp(-t)$.
2. The solution is given on the Figure 1. There we can see how the solution is separating from analytical solution for $\Delta t = 0.1$ and large time. This is an indicator of instability.

Exercise 2.10. *Implement the fourth-order Runge-Kutta (RK4) and Heun methods for solving the following second-order ODE:*

$$\ddot{x} + \dot{x} - \left(\alpha - x^2\right)x = 0, \quad \alpha \geq 0 \tag{31}$$

with $\alpha = 1$. How does the choice of Δt affect the results obtained from the methods?

Solution We actually know that the system has the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f}(\mathbf{x}) = (y, (\alpha - x^2)x - y)^T$. Choosing the initial condition $\mathbf{x} = (0.5, 0.5)$ and the time interval $[0, T]$ with $T = 1$ we have the solutions on the Figure 2. There we can see how the solution is approximating an equilibrium point $(1, 0)$ and how the Heun method is more unstable than RK4.

Exercise 2.11. *Show that the Picard iteration converges for th10 linear system*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Fx(t), \tag{32}$$

by showing that it is a contraction mapping. Assume that $|F| < 1$.

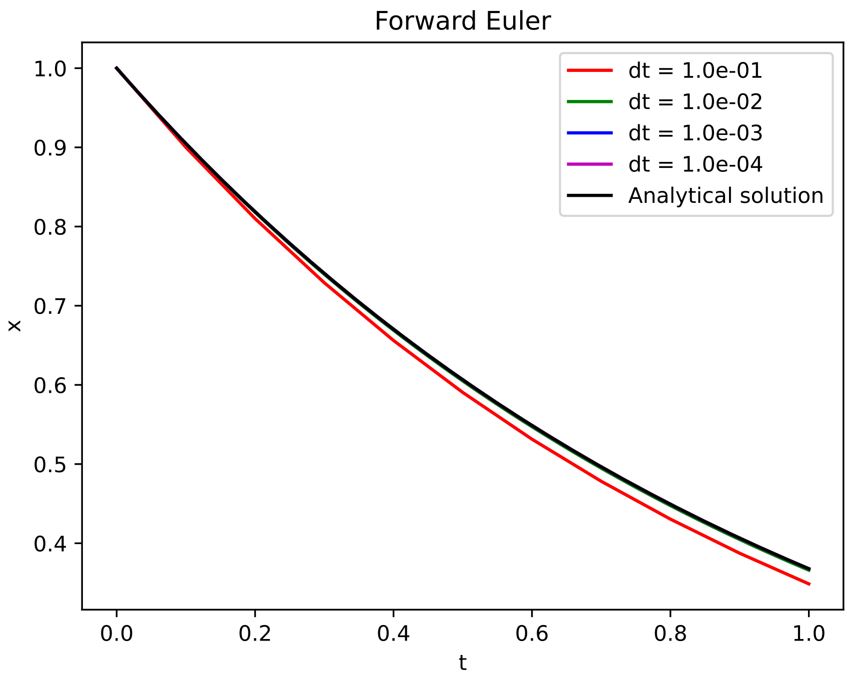


Figure 1: Exercise 2.9 Euler method for diferent step sizes.

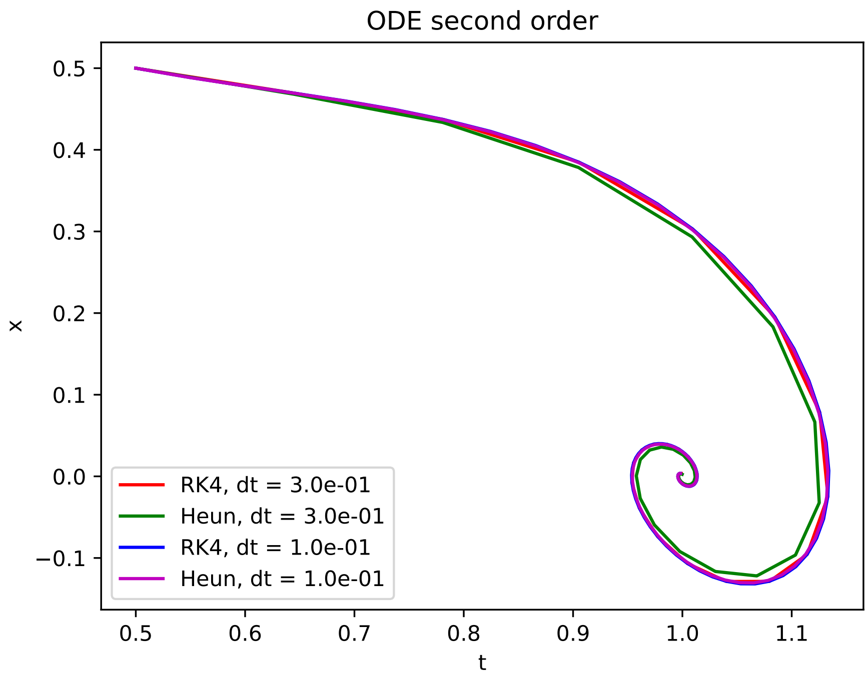


Figure 2: Exercise 2.10. RK4 and Heun method for different step sizes

Solution

Let us define the space of functions $\mathcal{X} := \mathcal{C}([t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b])$, and with $a \leq 1$. It is well known that this space is a Banach Space with the norm $\| \cdot \|_\infty$.

We also define the operator $G : \mathcal{X} \to \mathcal{X}$ such that $G[x(t)] := x_0 + F \int_{t_0}^{t} x(\tau) d\tau$. It is easy to prove that is well defined (proof omitted). And we will prove that it is a contraction, for it let be $x_1(t)$ and $x_2(t)$ two functions with the same initial condition $x_1(0) = x_2(0)$. Then

$$\begin{aligned}
\|G(x_1) - G(x_2)\|_\infty &= \left\| F \int_{t_0}^{t} x_1(\tau) d\tau - F \int_{t_0}^{t} x_2(\tau) d\tau \right\|_\infty \\
&= |F| \left\| \int_{t_0}^{t} (x_1(\tau) - x_2(\tau)) d\tau \right\|_\infty \\
&\leq |F| \int_{t_0}^{t} \|x_1 - x_2\|_\infty \, d\tau \\
&= |F| \, \|x_1 - x_2\|_\infty \int_{t_0}^{t} d\tau \\
&= |F| \, \|x_1 - x_2\|_\infty \, (t - t_0) \\
&\leq |F| a \, \|x_1 - x_2\|_\infty \\
&< \|x_1 - x_2\|_\infty
\end{aligned} \tag{33}$$

Then, the mapping is locally contractive and we have a local solution for the Fixed Point Banach Theorem.

3 Pragmatic Introduction to Stochastic Differential Equations

Exercise 3.1. *Mean and covariance equations of linear SDEs:*

1. *Complete the missing steps in the derivation of the covariance (3.39).*
2. *Derive the mean and covariance differential equations (3.40) by differentiating the equations (3.38) and (3.39).*

Solution

1. Let be $t \in [t_0, T]$, then we have the relation

$$\begin{aligned}[x(t) - m(t)][x(t) - m(t)]^T &= x(t)x^T(t) - m(t)x^T(t) \\ &- x(t)m^T(t) + m(t)m^T(t) \\ &= x(t)x^T(t) - m(t)x^T(t) \\ &- [m(t)x^T(t)]^T + m(t)m^T(t).\end{aligned} \tag{34}$$

We know that $\mathbb{E}[x(t)m^T(t)] = m(t)m^T(t)$ and $\mathbb{E}[x(t)m^T(t)]^T = m(t)m^T(t)$, and it implies that

$$\mathbb{E}[[x - m][x - m]^T] = \mathbb{E}[xx^T] - mm^T. \tag{35}$$

On the other hand

$$\begin{aligned}\mathbb{E}[xx^T] &= \exp(F(t - t_0))\mathbb{E}[x_0 x_0^T] \exp(F(t - t_0))^T \\ &+ \exp(F(t - t_0)) \int_{t_0}^{t} \mathbb{E}[x_0 w(\tau)] L^T \exp(F(t - \tau))^T d\tau \\ &+ \int_{t_0}^{t} \int_{t_0}^{t} \exp(F(t - \tau)) L \mathbb{E}[w(\tau) w(\tau')] L^T \exp(F(t - \tau'))^T d\tau d\tau' \\ &= \exp(F(t - t_0))\mathbb{E}[x_0 x_0^T] \exp(F(t - t_0))^T \\ &+ 0 + 0 + \int_{t_0}^{t} \int_{t_0}^{t} \exp(F(t - \tau)) L \delta(\tau - \tau') Q L^T \exp(F(t - \tau'))^T d\tau d\tau' \\ &= \exp(F(t - t_0))[P_0 + m_0 m_0^T] \exp(F(t - t_0))^T \\ &+ \int_{t_0}^{t} \exp(F(t - t_0)) L Q L^T \exp(F(t - \tau'))^T d\tau.\end{aligned} \tag{36}$$

Finally, we have that

$$\begin{aligned}\mathbb{E}[[x - m][x - m]^T] &= \exp(F(t - t_0)) P_0 \exp(F(t - \tau'))^T \\ &+ mm^T \\ &+ \int_{t_0}^{t} \exp(F(t - t_0)) L Q L^T \exp(F(t - \tau'))^T d\tau \\ &- mm^T.\end{aligned} \tag{37}$$

2. Using the chain rule

$$\frac{dm(t)}{dt} = \frac{d}{dt}\exp(F(t-t_0))m_0 = \exp(F(t-t_0))m_0\frac{d}{dt}F(t-t_0) = Fm, \quad (38)$$

and

$$\begin{aligned}\frac{dP(t)}{dt} &= \frac{d}{dt}\left[\exp(F(t-t_0))P_0\exp(F(t-t_0))^T\right] \\ &+ \frac{d}{dt}\int_{t_0}^{t}\exp(F(t-\tau))LQL^T\exp(F(t-\tau))^T d\tau.\end{aligned} \quad (39)$$

It is straightforward for $\frac{dm(t)}{dt}$ and for $\frac{dP(t)}{dt}$ we use the Leibniz Rule.

Exercise 3.2. *Solution of an Ornstein–Uhlenbeck process:*

1. *Find the complete solution $x(t)$ as well as the mean $m(t)$ and variance $P(t)$ of the following scalar stochastic differential equation:*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\lambda x(t) + w(t), \quad x(0) = x_0, \quad (40)$$

where x_0 and $\lambda > 0$ are given constants and the white noise $w(t)$ has spectral density q.

2. *Compute the limit of the mean and variance when $t \to \infty$ directly via $lim_{t\to\infty} m(t)$, $lim_{t\to\infty} P(t)$, and (ii) by solving the stationary state of the differential equations $dm/dt = 0$ and $dP/dt = 0$.*

Solution

1. Following the notation of the book. The operators are given by $F := \lambda$, $Q := q$ and $L := 1$. Then the mean and covariance ODEs are respectively given by the system

$$\frac{d}{dt}m(t) = -\lambda m(t), \quad (41)$$

and

$$\frac{d}{dt}P(t) = -\lambda P(t) + P(t)(-\lambda)^T + q = -2\lambda P(t) + q. \quad (42)$$

The solution of those ODEs are

$$m(t) = m_0\exp(-\lambda t), \quad (43)$$

and

$$P(t) = q[1 - \exp(-2\lambda(t-t_0))], \quad (44)$$

respectively.

2. i) It is clear that

$$\lim_{t\to\infty} m(t) = 0, \qquad \text{and} \qquad \lim_{t\to\infty} P(t) = q/(2\lambda). \tag{45}$$

ii) On the other hand, the solution to the stationary equations must be constants. It implies that the solutions are

$$m(t) = m_0, \qquad \text{and} \qquad P(t) = P_0. \tag{46}$$

Exercise 3.3. *Simulate 1000 trajectories on the time interval $t \in [0,1]$ from the Ornstein-Uhlenbeck process in the previous exercise using the Euler-Maruyama method with $\lambda = 1/2$, $q = 1$, $\Delta t = 1/100$, $x_0 = 1$, and check that the mean and covariance trajectories approximately agree with the theoretical values*

Solution

The Figure 3 shows trajectories of the SDE with indicated parameters. We can see how the estimated mean value is spliting from the analytical solution.The reason is the instability in Euler-Maruyama with that step size.

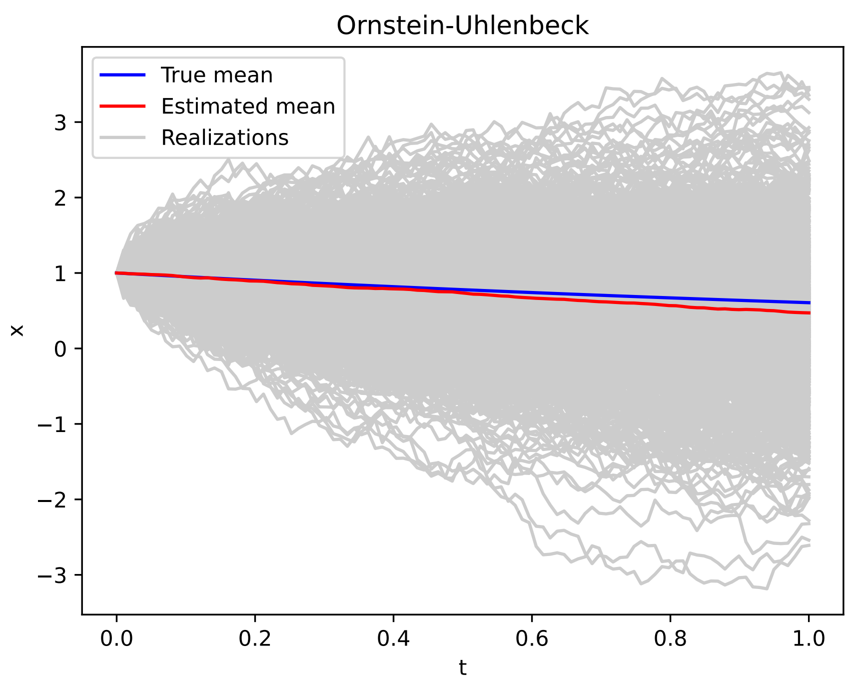


Figure 3: Exercise 3.3. Trajectories of SDE and mean values. Theoretical and estimated.

On the other hand, the Figure 4 shows a the theoretical and estimated variances at any time. It is clear that the instability of Euler-Maruyama is more visible than the mean value.

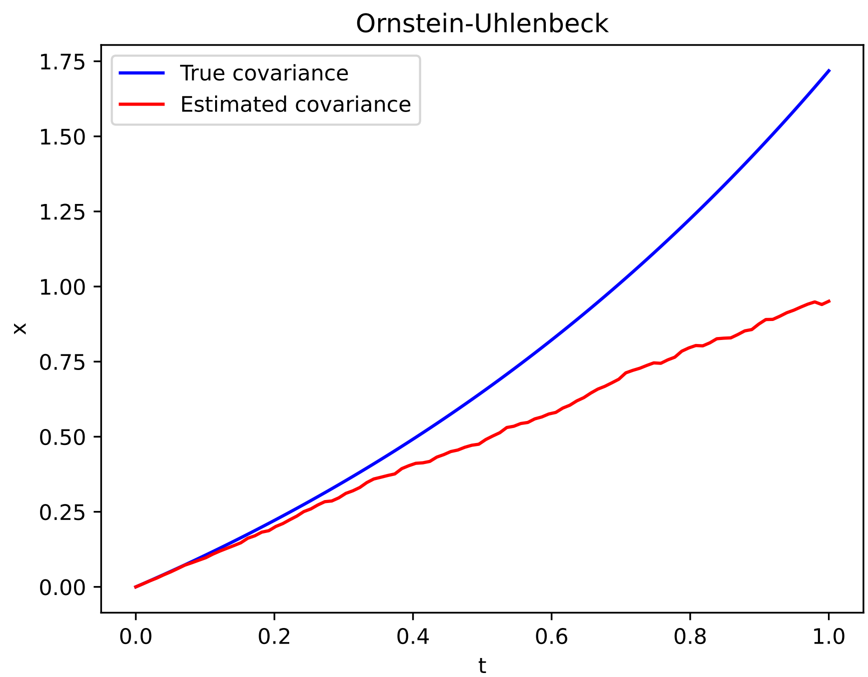
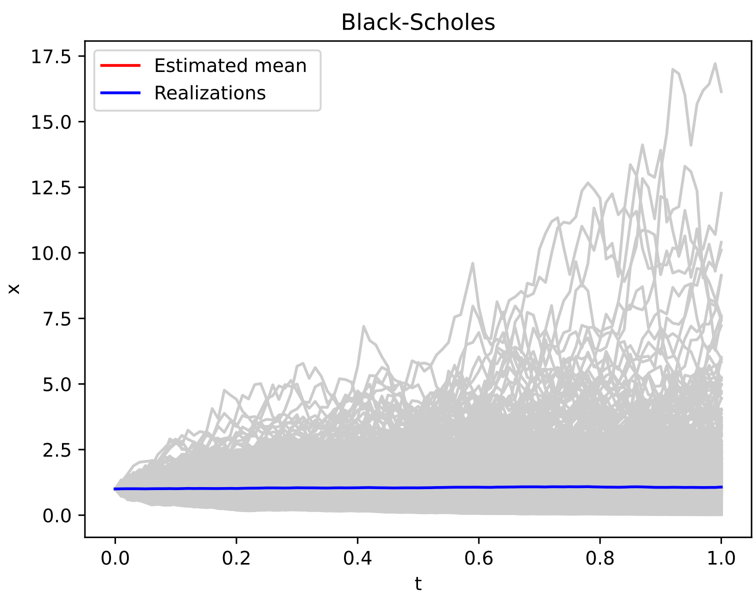


Figure 4: Exercise 3.3. Theoretical and estimated covariances.

Exercise 3.4. *Simulate 1000 trajectories from the Black-Scholes model (see Example 3.8) on the time interval* $[0, 1]$ *using the Euler-Maruyama method with* $\mu = 1/10$ *and* $\sigma = 1$*. By comparing to the exact solution given in Example 4.7 (approximate* $\beta(t)$ *as sum of* $\Delta\beta_k$ *), study the scaling of the error as a function of* Δt*.*

Solution

In Figure 5 we can see the realizations and expected means of the true solution and the solution with Euler-Maruyama. In both we can appreciate how the realizations and means are in the same rank of magnitude. Additionally we have a separated plot with the means in the Figure 6. There is possible to see how for a short time interval the Euler-Maruyama reproduces in an acceptable way the solutions.



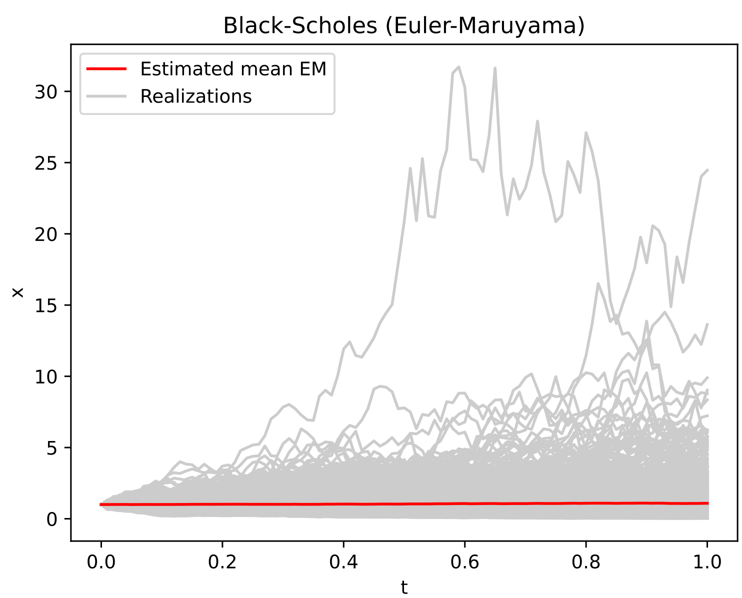


Figure 5: Exercise 3.4. Trajectories of Black-Scholes and estimated means.

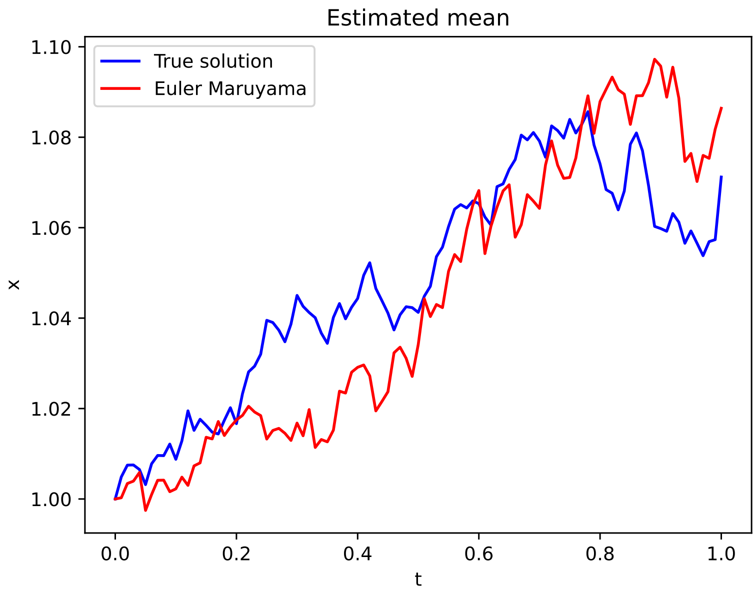


Figure 6: Exercise 3.4. Estimated means in Black-Scholes equation.

Exercise 3.5. *The covariance function of the Ornstein-Uhlenbeck process is*

$$C(t, t') = \frac{q}{2\lambda} \exp\left(-\lambda |t - t'|\right). \tag{47}$$

1. *Show that in the limit of $\lambda \to \infty$, the Ornstein-Uhlenbeck process reverts to white noise.*
2. *Show that the power spectral density of the Ornstein-Uhlenbeck process becomes flat in the limit.*

Solution

1. Let be a test function $\phi \in C_c(\mathbb{R})$ (smooth and with compact support in $\mathbb{R}$). The affirmation is equivalent to prove

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}} \phi(t) C(t, t') dt = \phi(t'). \tag{48}$$

2. It is clear that $C(t, t') \to 0$ when $\lambda \to \infty$.

4 Itô Calculus and Stochastic Differential Equations

Exercise 4.1. *The Itô formula:*

1. *Compute the Itô differential of* $\phi(\beta, t) = t + \exp(\beta)$*, where* $\beta(t)$ *is a Brownian motion with diffusion constant* q*.*
2. *Compute the Ito differential of* $\phi(x) = x^2$*, where* x *solves the scalar SDE*

$$\mathrm{d}x = f(x)\mathrm{d}t + L\ \mathrm{d}\beta,$$

the parameter L *is a constant, and* $\beta(t)$ *is a standard Brownian motion* $(q = 1)$*.*

3. *Compute the Itô differential of* $\phi(\mathbf{x}) = \mathbf{x}^\top \mathbf{x}$*, where*

$$\mathrm{d}\mathbf{x} = \mathbf{F}\mathbf{x}\mathrm{d}t + \mathrm{d}\boldsymbol{\beta},$$

where $\mathbf{F}$ *is a constant matrix and the diffusion matrix of* $\boldsymbol{\beta}$ *is* $\mathbf{Q}$.

Solution

1. Using the Itô formula

$$\begin{aligned} d\phi &= \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial \beta} d\beta + \frac{1}{2}\frac{\partial^2 \phi}{\partial \beta^2} d\beta^2 \\ &= dt + \exp(\beta) d\beta + \frac{1}{2}\exp(\beta) d\beta^2 \\ &= \left(1 + \frac{1}{2}\exp(\beta) q\right) dt + \exp(\beta) d\beta. \end{aligned} \tag{49}$$

2. By Itô formula

$$\begin{aligned} d\phi &= \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial \beta} d\beta + \frac{1}{2}\frac{\partial^2 \phi}{\partial \beta^2} d\beta^2 \\ &= 0 + 2x dx + \frac{1}{2} 2 dx^2 \\ &= 2x dx + dx^2, \end{aligned} \tag{50}$$

then using the expression of the SDE, we have that

$$\begin{aligned} d\phi &= 2x\left[f(x)dt + Ld\beta\right] + \left[f(x)dt + Ld\beta\right]^2 \\ &= 2xf(x)dt + 2xLd\beta + \left[f(x)dt + Ld\beta\right]^2 \\ &= 2xf(x)dt + 2xLd\beta + L^2 q dt \\ &= [2xf(x) + L^2]dt + 2xLd\beta. \end{aligned} \tag{51}$$

3. By definition $\phi(x) = x^T x = \sum_{k=1}^d x_k^2$. Then

$$
\begin{aligned}
d\phi &= \frac{\partial \phi}{\partial t} dt + \sum_{i=1}^d \frac{\partial \phi}{\partial x_i} dx_i + \frac{1}{2} \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial_{x_i} \partial_{x_j}} \phi \right) dx_i dx_j \\
&= \sum_{i=1}^d \frac{\partial}{\partial x_i} \left[\sum_{k=1}^d x_k^2 \right] dx_i + \frac{1}{2} \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial_{x_i} \partial_{x_j}} \left[\sum_{k=1}^d x_k^2 \right] \right) dx_i dx_j \\
&= \sum_{i=1}^d [2x_i]\, dx_i + \frac{1}{2} \sum_{i,j=1}^d \left(\frac{\partial}{\partial_{x_i}} [2x_j] \right) dx_i dx_j \\
&= 2 \sum_{i=1}^d x_i dx_i + \sum_{i,j=1}^d \delta_{i,j} dx_i dx_j \qquad (52) \\
&= 2x^T dx + tr(dx dx^T) \\
&= 2x^T [Fxdt + d\beta] + tr([Fxdt + d\beta][Fxdt + d\beta]^T) \\
&= 2x^T Fxdt + 2x^T d\beta \\
&+ tr(Fxx^T F^T dt^2 + Fxd\beta^T dt + d\beta x^T F^T dt + d\beta d\beta^T) \\
&= 2x^T Fxdt + 2x^T d\beta + tr(d\beta d\beta^T) \\
&= 2x^T Fxdt + 2x^T d\beta + tr(Qdt) \\
&= (2x^T Fx + tr(Q))dt + 2x^T d\beta.
\end{aligned}
$$

Exercise 4.2. *Check that*

$$x(t) = \exp(\beta(t))$$

solves the SDE

$$\mathrm{d}x = \frac{1}{2} x \; \mathrm{d}t + x \; \mathrm{d}\beta,$$

where $\beta(t)$ is a standard Brownian motion ($q = 1$).

Solution

Using the Itô formula

$$
\begin{aligned}
dx &= \frac{\partial x}{\partial t} dt + \frac{\partial x}{\partial t} dt + \frac{\partial x}{\partial \beta} d\beta + \frac{1}{2} \frac{\partial^2 x}{\partial \beta^2} d\beta^2 \\
&= \exp(\beta) d\beta + \frac{1}{2} \exp(\beta) d\beta^2 \\
&= \exp(\beta) d\beta + \frac{1}{2} \exp(\beta) q dt \qquad (53) \\
&= x(t) d\beta + \frac{1}{2} x(t) q dt \\
&= \frac{1}{2} x(t) dt + x(t) d\beta.
\end{aligned}
$$

Exercise 4.3. *Consider the Black-Scholes model*

$$\mathrm{d}x = \mu x \ \mathrm{d}t + \sigma x \ \mathrm{d}\beta,$$

where μ and $\sigma > 0$ are constants, and $\beta(t)$ is a standard Brownian motion.

1. *Solve $x(t)$ by changing the variable to $y = \log x$ and transforming back.*
2. *Notice that the solution is log-normal and compute its mean $m(t)$ and variance $P(t)$.*
3. *Compare (and check) your expressions for the mean and variance against the numerical results obtained by the Euler-Maruyama method (cf. Exercise 3.4).*

Solution

1. Let us define $y = log(x)$. applying the Itô formula we have the expression

$$\begin{aligned}
dy &= \frac{\partial y}{\partial t}dt + \frac{\partial y}{\partial x}dx + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}dx^2 \\
&= \frac{1}{x}dx - \frac{1}{2}\frac{1}{x^2}dx^2 \\
&= \frac{1}{x}[\mu x dt + \sigma x d\beta] - \frac{1}{2}\frac{1}{x^2}[\mu x dt + \sigma x d\beta]^2 \\
&= \mu dt + \sigma d\beta - \frac{1}{2}\frac{1}{x^2}[\mu^2 x^2 dt^2 + 2\mu x^2 \sigma dt d\beta + \sigma^2 x^2 d\beta^2] \qquad (54) \\
&= \mu dt + \sigma d\beta - \frac{1}{2}\sigma^2 d\beta^2 \\
&= \mu dt + \sigma d\beta - \frac{1}{2}\sigma^2 q dt \\
&= \left(\mu - \frac{1}{2}\sigma^2 q\right) dt + \sigma d\beta.
\end{aligned}$$

Integrating respect to the time both sides of the equation we have that

$$\int_0^t dy(t) = \left(\mu - \frac{1}{2}\sigma^2 q\right) \int_0^t dt + \sigma \int_0^t d\beta(t), \qquad (55)$$

or equivalently

$$y(t) = y_0 + \left(\mu - \frac{1}{2}\sigma^2 q\right) t + \sigma\beta(t). \qquad (56)$$

Finally, returning to the original variable the expression for the Black-Scholes equation is

$$x(t) = x_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2 q\right) t + \sigma\beta(t)\right]. \qquad (57)$$

2. We define the variable $\hat{\mu} := (\mu - \frac{1}{2}\sigma^2 q)t$. Then, factorizing the Brownian motion as $\beta(t) = \sqrt{qt}Z(t)$. We can express the solution of the SDE as

$$x = \exp\left(\hat{\mu} + \sigma\beta(t)\right) = \exp\left(\hat{\mu} + \sigma\sqrt{qt}Z(t)\right), \tag{58}$$

which for every t defines a log-normal distribution. The formulas of the expectation and variance are well known, and for this case they are respectively

$$\mathbb{E}[x(t)] = \exp\left(\hat{\mu} + \frac{1}{2}\sigma^2 qt\right) = \exp\left(\mu t\right), \tag{59}$$

and

$$\begin{aligned} Var[x(t)] &= \left[\exp\left(\sigma^2 qt\right) - 1\right]\exp\left(2\hat{\mu} + \sigma^2 qt\right) \\ &= \left[\exp\left(\sigma^2 qt\right) - 1\right]\exp\left(2\mu t\right). \end{aligned} \tag{60}$$

3. Figure 7 shows realizations and estimated means of the analytical solution and Euler-Maruyama with values $\mu = 2$, $\sigma = 0.1$ and $q = 1$. The number of nodes in the temporal grid is 100. In both cases we can see a how the Euler-Maruyama approximates with good accuracy the analytical solution.

   On the other hand, the Figure 8 shows a comparison between the real and estimated covariance with EM. The approximation conserves a good degree of accuracy.

Exercise 4.4. *Find the solution to the mean-reverting Ornstein-Uhlenbeck process given in a parameterization favoured in financial applications (in finance, the model is known as the Vasicek model):*

$$\mathrm{d}x = \theta(\mu - x)\mathrm{d}t + \sigma\mathrm{d}\beta(t), \quad x(0) = x_0,$$

where θ gives the speed of reversion, μ is the long-run equlibrium, σ stands for the volatility, and $\beta(t)$ is a standard Brownian motion with diffusion coefficient $q = 1$.

Solution

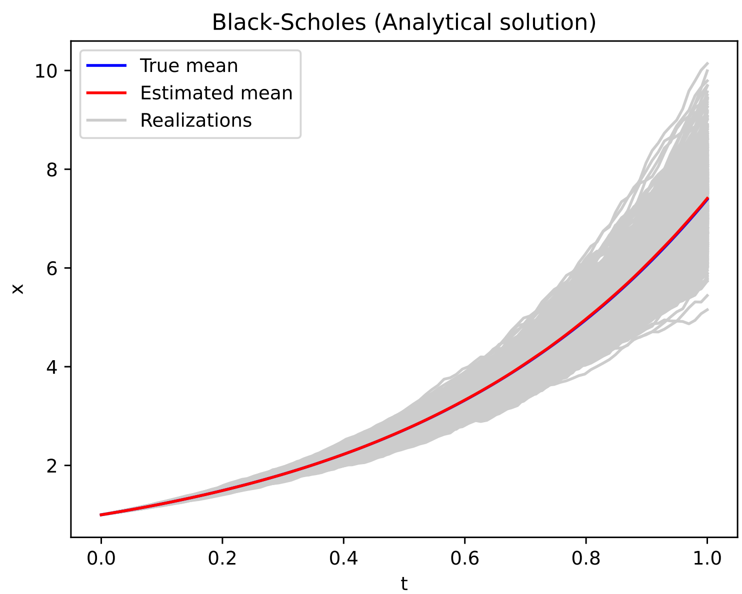
Following the notation of the book, we choose $a = \theta$, $c = \theta\mu$, $b = \sigma$ and $d = 0$. Then

$$\psi(t) = \exp\left(\left(\theta - \frac{1}{2}\sigma^2\right)t + \sigma\beta(t)\right). \tag{61}$$

Using the Kloeden Platen formula, we have that

$$x(t) = \exp\left[\left(\theta - \frac{1}{2}\sigma^2\right)t + \sigma\beta(t)\right]\left(x_0 + \theta\mu\int_0^t \exp\left[-\left(\theta - \frac{1}{2}\sigma^2\right)s - \sigma\beta(s)\right]\right) \tag{62}$$

Exercise 4.5. *Derive the Black-Scholes solution given in Example 4.7 from the solution given for linear time-invariant models with multiplicative noise in Example 4.6.*



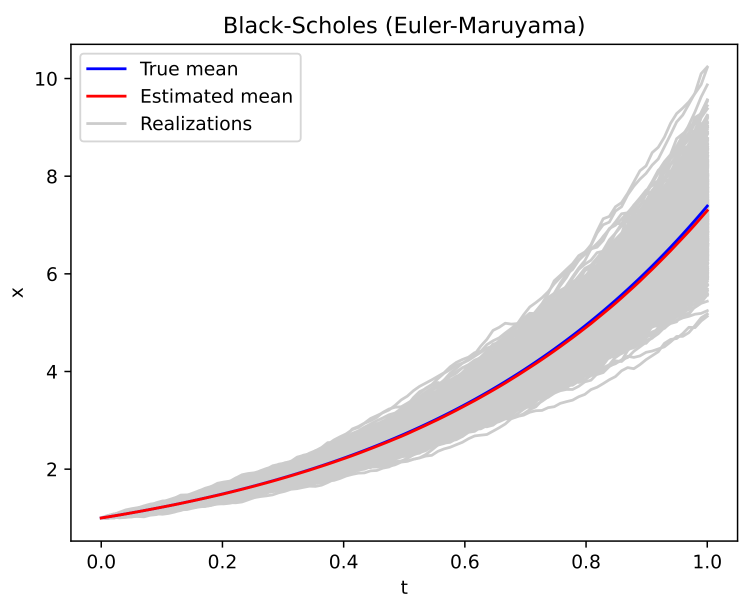


Figure 7: Exercise 4.3. Trajectories of Black-Scholes and estimated means.

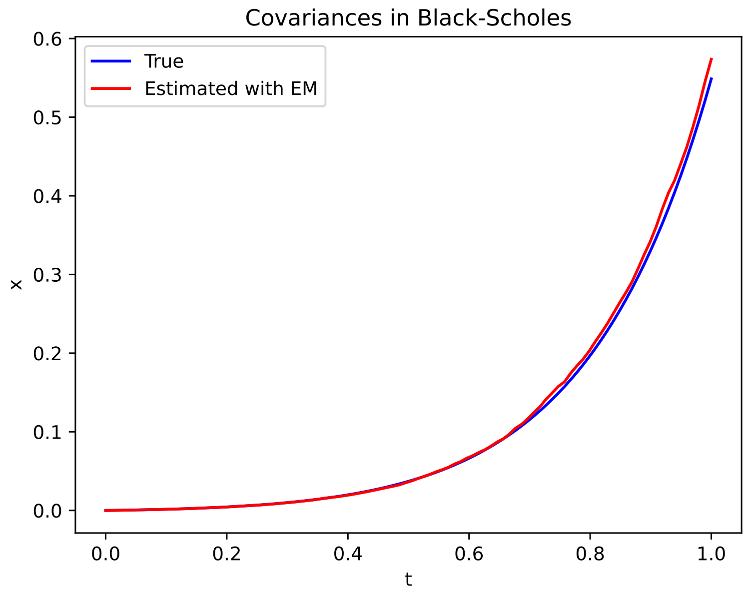


Figure 8: Exercise 4.3. Covariances of Black-Scholes.

Solution
The solution is trivial.

Exercise 4.6. *Consider the nonlinear SDE:*

$$\mathrm{d}x = -\tanh(x)\left(a + \frac{1}{2}\hat{b}^2 \operatorname{sech}^2(x)\right)\mathrm{d}t + \hat{b}\operatorname{sech}(x)\mathrm{d}\beta,$$

where $\beta(t)$ is a standard Brownian motion. Solve it using the methods given for one-dimensional nonlinear SDEs in Section 4.4. Hint: Try $h(x) = \sinh(x)$.

Solution
Let us define $b(x) = \hat{b}\operatorname{sech}(x)$ and $\alpha = a$. Then

$$\frac{d}{dx}b(x) = -\hat{b}\tanh(x)\operatorname{sech}(x), \tag{63}$$

and

$$h(x) = \int_0^x \frac{1}{b(u)}du = \hat{b}^{-1}\sinh(x). \tag{64}$$

It is clear that $b(x)$ and α satisfies the hypothesis of the Kloeden-Platen method.

Then the solution is given by

$$
\begin{aligned}
x(t) &= b \operatorname{arcsinh}\left[\exp(\alpha t)\hat{b}^{-1}(0) + \int_0^t \exp(a(t-s))d\beta(s)\right] \\
&= b \operatorname{arcsinh}\left[\int_0^t \exp(a(t-s))d\beta(s)\right].
\end{aligned} \tag{65}
$$

Remark. The last integral term is an stochastic term such that

$$
\int_0^t \exp(a(t-s))d\beta(s) \sim \mathcal{N}(0, \exp(a(t-s))). \tag{66}
$$

Exercise 4.7. *It is also possible to derive an integration by parts formula for Itô processes. Let $\beta(t)$ be a standard Brownian motion.*

1. *By computing the Itô differential of $\phi(\beta)\zeta(\beta)$ and integrating the result, show that the following holds:*

$$
\begin{aligned}
&\int_0^t \left[\frac{\partial \phi(x)}{\partial x}\zeta(x)\right]_{x=\beta(\tau)} \mathrm{d}\beta(\tau) \\
&= \phi(\beta(t))\zeta(\beta(t)) - \phi(0)\zeta(0) - \int_0^t \left[\phi(x)\frac{\partial \zeta(x)}{\partial x}\right]_{x=\beta(\tau)} \mathrm{d}\beta(\tau) \\
&- \frac{1}{2}\int_0^t \left[\frac{\partial^2 \phi(x)}{\partial x^2}\zeta(x) + 2\frac{\partial \phi(x)}{\partial x}\frac{\partial \zeta(x)}{\partial x} + \phi(x)\frac{\partial^2 \zeta(x)}{\partial x^2}\right]_{x=\beta(\tau)} \mathrm{d}t.
\end{aligned}
$$

2. *Use this result to express the following integral in terms of ordinary integrals (hint: put $\zeta(x) = 1$):*

$$
\int_0^t \tanh(\beta)\mathrm{d}\beta.
$$

Solution

1. Let be $\psi(\beta) = \phi(\beta)\zeta(\beta)$, then

$$
\begin{aligned}
d\psi &= \frac{\partial \psi}{\partial t}dt + \frac{\partial \psi}{\partial \beta}d\beta + \frac{1}{2}\frac{\partial^2 \psi}{\partial \beta^2} \\
&= \frac{q}{2}\left[\zeta\frac{\partial^2 \phi}{\partial \beta^2} + 2\frac{\partial \phi}{\partial \beta}\frac{\partial \zeta}{\partial \beta} + \phi\frac{\partial^2 \zeta}{\partial \beta^2}\right]dt + \left[\zeta\frac{\partial \phi}{\partial \beta} + \phi\frac{\partial \zeta}{\partial \beta}\right]d\beta.
\end{aligned} \tag{67}
$$

Then, integrating the left side we have the term

$$
\begin{aligned}
&\int_0^t d\psi = \psi(t) - \psi(0) \\
&\qquad = \frac{1}{2}\int_0^t \left[\zeta\frac{\partial^2 \phi}{\partial \beta^2} + 2\frac{\partial \phi}{\partial \beta}\frac{\partial \zeta}{\partial \beta} + \phi\frac{\partial^2 \zeta}{\partial \beta^2}\right]dt + \int_0^t \left[\zeta\frac{\partial \phi}{\partial \beta} + \phi\frac{\partial \zeta}{\partial \beta}\right]d\beta,
\end{aligned} \tag{68}
$$

and re-organizing terms we have the result.

2. Using the formula we have that

$$\int_0^t \tanh(\beta)d\beta = \tanh(\beta) - \tanh(0) - \int_0^t \left[\int_0^\tau \tanh(\beta(s))\frac{\partial 1}{\partial x}ds\right]_{x=\beta(\tau)} d\beta(\tau)$$
$$- \frac{1}{2}\int_0^t \left[sech^2(x)\right]_{x=\beta(\tau)} dt$$
$$= \tanh(\beta) - \frac{1}{2}\int_0^t sech^2(\beta(\tau))dt. \tag{69}$$

Exercise 4.8. *Convert the following Stratonovich SDE into the equivalent Itô SDE:*

$$\mathrm{d}x = \tanh(x)\mathrm{d}t + x \circ \mathrm{d}\beta,$$

where $\beta(t)$ is a scalar Brownian motion.

Solution

We define the functions $f(x,t) = \tanh(x)$ and $L(x,t) = x$, then using the Algorithm 4.11 we have

$$\begin{aligned}\tilde{f}(x,t) &= f(x,t) + \frac{1}{2}\frac{\partial L}{\partial x}L \\ &= \tanh(x) + \frac{1}{2}x.\end{aligned} \tag{70}$$

It implies that

$$dx = \left(\tanh(x) + \frac{1}{2}x\right)dt + xd\beta. \tag{71}$$

Exercise 4.9. *Convert the following Stratonovich SDE into the equivalent Itô SDE:*

$$\mathrm{d}x_1 = -x_2 \circ \mathrm{d}\beta,$$
$$\mathrm{d}x_2 = x_1 \circ \mathrm{d}\beta,$$

where $\beta(t)$ is a scalar Brownian motion.

Solution

Let be $x = [x_1, x_2]^T$, $\beta = [\beta_1, \beta_2]^T$, then $dx = L \circ d\beta$ with

$$L = \begin{bmatrix} 0 & -x_2 \\ x_1 & 0 \end{bmatrix}. \tag{72}$$

Using the Algorithm 4.11

$$
\begin{aligned}
dx_i &= \frac{1}{2}\left[\sum_{j,l}^{2} \frac{\partial L_{i,j}}{\partial x_l}(x,t) L_{l,j}(x,t)\right] \\
&= \frac{1}{2}\left[\frac{\partial L_{i,1}}{\partial x_1} L_{1,1} + \frac{\partial L_{i,1}}{\partial x_2} L_{2,1} + \frac{\partial L_{i,2}}{\partial x_1} L_{1,2} + \frac{\partial L_{i,2}}{\partial x_2} L_{2,2}\right] \\
&= \frac{1}{2}\left[\frac{\partial L_{i,1}}{\partial x_2} L_{2,1} + \frac{\partial L_{i,2}}{\partial x_1} L_{1,2}\right],
\end{aligned}
\tag{73}
$$

where finally

$$
dx_1 = \frac{1}{2} x_2 d\beta, \tag{74}
$$

and

$$
dx_2 = \frac{1}{2} x_1 d\beta. \tag{75}
$$

5 Probability Distributions and Statistics of SDEs

Exercise 5.1. *Write the generator $\mathcal{A}$ for the Beneš SDE*

$$\mathrm{d}x = \tanh(x)\mathrm{d}t + \mathrm{d}\beta, \quad x(0) = 0,$$

where $\beta(t)$ is a standard Brownian motion.

Solution

$$\begin{aligned}\mathcal{A} &= f(x)\frac{\partial}{\partial x} + \frac{1}{2}L^2(x)q\frac{\partial^2}{\partial x^2} \\ &= \tanh(x)\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\end{aligned} \tag{76}$$

Exercise 5.2. *FPK equation:*

1. *Write down the FPK for the Beneš Equation (5.66), and check that the following probability density solves it:*

$$p(x,t) = \frac{1}{\sqrt{2\pi t}}\cosh(x)\exp\left(-\frac{1}{2}t\right)\exp\left(-\frac{1}{2t}x^2\right).$$

2. *Plot the evolution of the probability density when $t \in [0, 5]$.*
3. *Simulate 1000 trajectories from the SDE using the Euler-Maruyama method and check visually that the histogram matches the correct density at time $t = 5$.*

Solution

1. The adjoint operator is given by the expression

$$\mathcal{A}^* = -\frac{\partial}{\partial x}\tanh(x)(\cdot) + \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{77}$$

Then, the FPK equation is

$$\frac{d}{dt}P(x,t) = -\frac{\partial}{\partial x}\left[\tanh(x)P(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}P(x,t). \tag{78}$$

Now

$$\begin{aligned}\frac{\partial}{\partial t}P(x,t) &= \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi t}}\cosh(x)\exp\left(-\frac{1}{2}t\right)\exp\left(-\frac{1}{2t}x^2\right)\right] \\ &= -\frac{1}{2}\frac{\cosh(x)}{\sqrt{2\pi t}}\left[\frac{1}{t} + 1 + \frac{x^2}{t^2}\right]\exp\left(-\frac{x^2}{2t} - \frac{1}{2}t\right).\end{aligned} \tag{79}$$

$$\frac{\partial}{\partial x}\tanh(x)P(x,t) = \frac{1}{\sqrt{2\pi t}}\left[-\frac{x}{t}\sinh(x) + \cosh(x)\right]\exp\left(-\frac{x^2}{2t} - \frac{1}{2}t\right) \tag{80}$$

$$\frac{\partial}{\partial x}P(x,t) = \frac{1}{\sqrt{2\pi t}}\left[-\frac{x}{t}\cosh(x) + \sinh(x)\right]\exp\left(-\frac{x^2}{2t} - \frac{1}{2}t\right), \tag{81}$$

and

$$\frac{\partial^2}{\partial x^2}P(x,t) = \frac{1}{\sqrt{2\pi t}}\left[\frac{x^2}{t^2}\cosh(x) - 2\frac{x}{t}\sinh(x) + \cosh(x)\right]\exp\left(-\frac{x^2}{2t} - \frac{1}{2}t\right). \tag{82}$$

2. The Figure 9 shows densities for 5 different profiles and the histograms generated with the sample of the profiles in different times. An expected phenomena is the reduction in magnitude of the densities, this is explained due to the diffusion involved in the PDE.

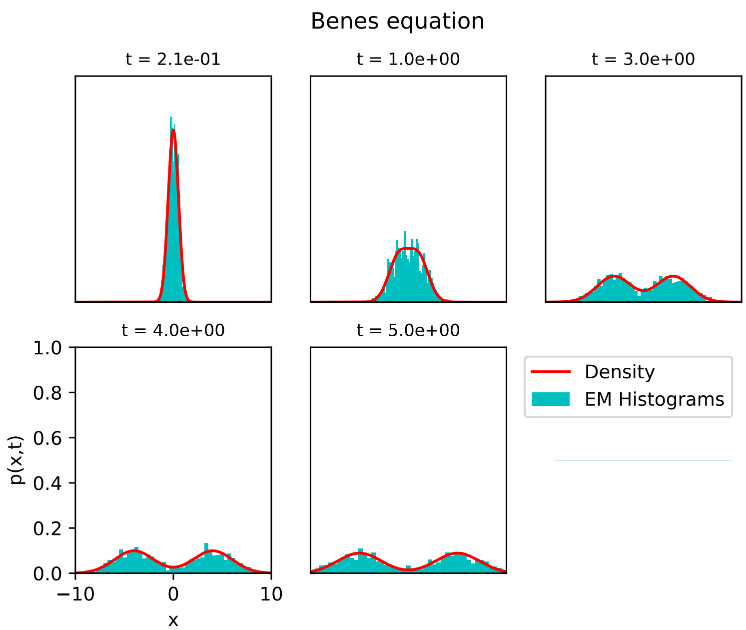


Figure 9: Exercise 5.2. Densities of Benes equation.

3. Figure 10 shows the trajectories of Benes equation and the estimated mean. The approximated value of the mean is cero, this value agrees graphically with the means in Figure 9.

Exercise 5.3. *Numerical solution of the FPK equation: Use a finite-differences method to solve the FPK for the Beneš SDE in Equation (5.66). For simplicity, you can select a finite range* $x \in [-L, L]$ *and use the Dirichlet boundary conditions* $p(-L,t) = p(L,t) = 0$.

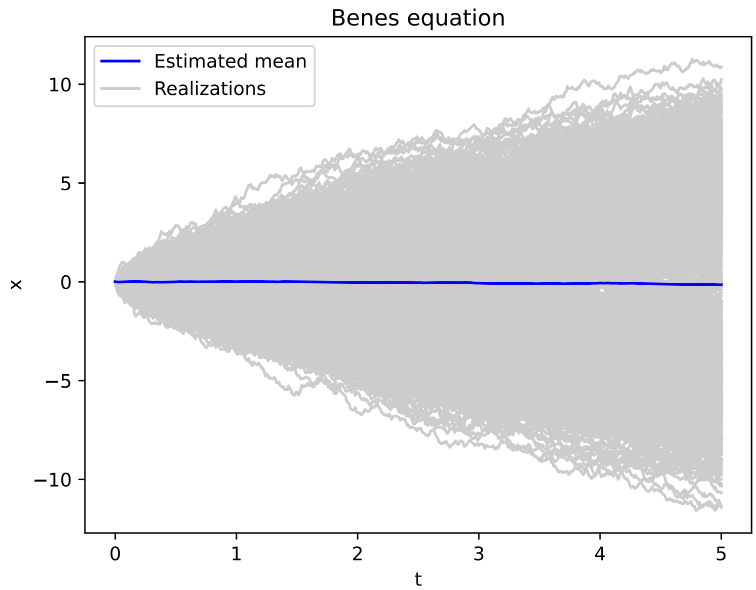


Figure 10: Exercise 5.2. Trajectories of Benes equation.

1. *Divide the range to* n *grid points and let* $h = 2L/(n+1)$. *On the grid, approximate the partial derivatives of* $p(x,t)$ *via*

$$\frac{\partial p(x,t)}{\partial x} \approx \frac{p(x+h,t) - p(x-h,t)}{2h},$$
$$\frac{\partial^2 p(x,t)}{\partial x^2} \approx \frac{p(x+h,t) - 2p(x,t) + p(x-h,t)}{h^2}.$$

2. *Let* $\mathbf{p}(t) = (p(h-L,t), p(2h-L,t), \cdots, p(nh-L,t))^\top$ *and from the preceding, form an equation of the form*

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}\mathbf{p}.$$

3. *Solve the preceding equation using (i) the implicit backward Euler method, (ii) by numerical computation of* $\exp(\mathbf{F}t)$, *and (iii) by the forward Euler method. Check that the results match the solution in the previous exercise.*

Solution

The Figure (11) shows the densities of Benes equation in 5 time profiles. They were calculated with the numerical solution of Fokker-Plank-Kolmogorov equations and wit the exponential matrix. The Numerical solution was calculated with a temporal grid of $N_t = 123$ points and $N_x = 100$ nodes in the axis x.

We can see how those solutions agreed with the analytical solution with all methods for the first two time profiles. Nevertheless, we can see instability in the next time profiles for the method Forward Euler, something that was expected previously.

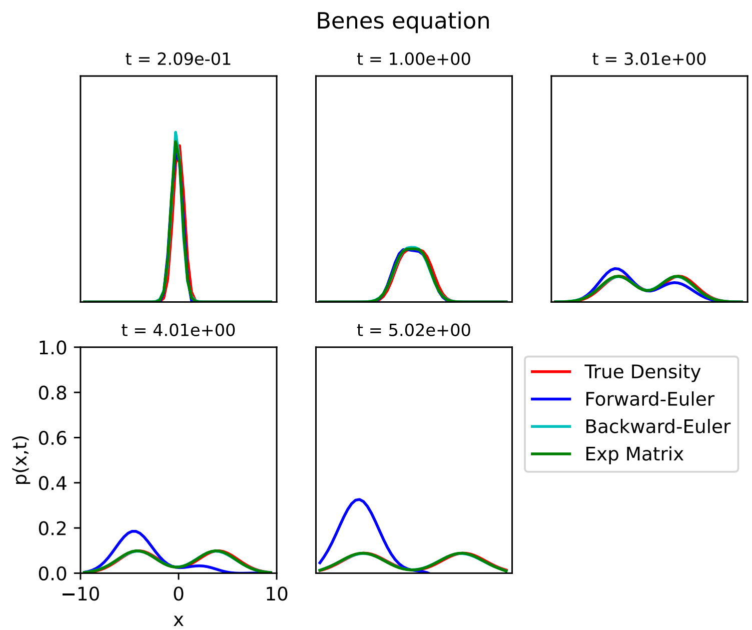


Figure 11: Exercise 5.3. Numerical solution of Benes equation with Fokker-Plank and Exponential Matrix.

Exercise 5.4. *Consider the Langevin model of Brownian motion (with $m = 1$), which we already considered in Example 3.2:*

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -c\frac{\mathrm{d}x}{\mathrm{d}t} + w, \quad x(0) = (\mathrm{d}x/\mathrm{d}t)(0) = 0,$$

where $c = 6\pi\eta r$ and the white noise $w(t)$ has some spectral density q.

1. *Interpret the preceding model as an Itô SDE and write it as a twodimensional state-space form SDE.*

2. *Write down the differential equations for the elements of the mean $\mathbf{m}(t)$ and covariance $\mathbf{P}(t)$. Conclude that the mean is zero and find the closed-form solutions for the elements $P_{11}(t), P_{12}(t), P_{21}(t)$, and $P_{22}(t)$ of the covariance matrix $\mathbf{P}(t)$. Hint: Start by solving $P_{22}(t)$, then use it to find the solutions for $P_{12}(t) = P_{21}(t)$, and finally solve $P_{11}(t)$.*

3. *Find the limiting solution $P_{22}(t)$ when $t \to \infty$, and use the following to determine the diffusion coefficient (spectral density) q :*

$$\mathrm{E}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\right] = \frac{RT}{N}.$$

4. *Plot the solution $P_{11}(t)$ and conclude that it asymptotically approaches a straight line. Compute the asymptotic solution $P_{11}(t)$ when $t \to \infty$, and conclude that it gives Langevin's result.*

Solution

1. Defining the variable $y = \dfrac{dx}{dt}$, we have the SDE

$$\frac{dy}{dt} = -cy + w. \tag{83}$$

We define also the vectors matrices

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \overline{\beta} = \begin{bmatrix} \beta \\ \beta \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \textit{and} \quad \mathbf{L} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{84}$$

Then, the system of SDEs can be written as

$$d\mathbf{x} = \mathbf{F}\mathbf{x}dt + \mathbf{L}d\overline{\beta}, \quad \mathbf{x}(0) = 0. \tag{85}$$

2. The ODE that modelates the mean is given by

$$\frac{dm}{dt} = \mathbb{E}[f(\mathbf{x})] = \mathbf{F}\mathbb{E}[\mathbf{x}] = \mathbf{F}m, \tag{86}$$

then the solution is of the form

$$m = \exp(\mathbf{F}t)m_0 = 0. \tag{87}$$

In the case of the Covariance matrix, the ODE is determined as

$$\begin{aligned}
\frac{dP}{dt} &= \mathbb{E}[f(x,t)(x-m)^T] + \mathbb{E}[(x-m)f^T(x,t)] + \mathbb{E}[LQL] \\
&= FP + PF^T + LQL \\
&= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} + q\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} P_{21} & P_{22} \\ -P_{21} & -P_{22} \end{bmatrix} + \begin{bmatrix} P_{12} & -P_{12} \\ P_{22} & -P_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix} \\
&= \begin{bmatrix} P_{12} + P_{21} & -P_{12} + P_{22} \\ -P_{21} + P_{22} & q - 2P_{22} \end{bmatrix}
\end{aligned} \tag{88}$$

The differential equation of the component (2,2) is given by

$$\frac{dP_{22}}{dt} = q - 2P_{22}, \tag{89}$$

and the solution of this autonomous equation is given by

$$P_{22}(t) = \exp(-2t)[P_{22}(0) - q/2] + q/2. \tag{90}$$

With this solution, we can solve in sequence the ODEs in components (1,2), (2,1) and (1,1). Then, the solutions are

$$P_{12}(t) = \exp(-2t)[P_{22}(0)-q/2]+q/2+[P_{12}(0)+P_{22}(0)-q]\exp(-t), \tag{91}$$

$$P_{21}(t) = P_{12}(t), \tag{92}$$

and

$$\begin{aligned} P_{11}(t) = {} & P_{11}(0) - 2[P_{12}(0) + P_{22}(0) - q][\exp(-t) - 1] \\ & - \frac{1}{2}[q - 2P_{22}(0)][\exp(-2t) - 1] + qt. \end{aligned} \tag{93}$$

3. It is clear that $P_{22}(t) \to q/2$ when $t \to \infty$. Now

$$\mathbb{E}\left[\left(\frac{dx}{dt}\right)^2\right] = \mathbb{E}[y^2] = P_{22}(t) = \frac{RT}{N}, \tag{94}$$

then $q = 2RT/N$.

4. The Figure 12 shows the component P_{11} and how it tends asymptotically to a line.

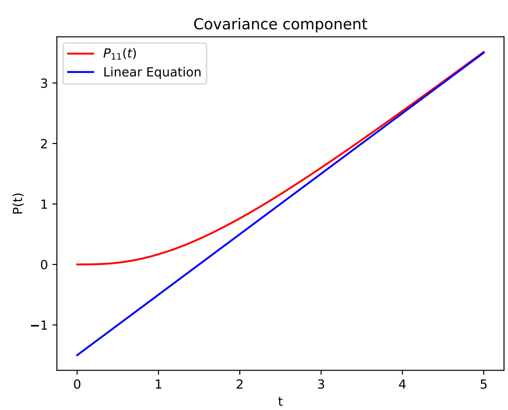


Figure 12: Exercise 5.4. Component $P_{11}(t)$.

Exercise 5.5. *Stationary FPK equation: Show that Equation (5.26) solves the corresponding stationary FPK (Eq. 5.24).*

Solution

The solution is trivial.

Exercise 5.6. *In so-called Metropolis-adjusted Langevin Monte Carlo methods (see, e.g., Girolami and Calderhead, 2011), the idea is to construct an SDE whose stationary solution is a given probability distribution $\pi(x)$.*

1. *Construct an SDE of the form (5.25) such that the stationary solution is the Gamma distribution with the probability density*

$$\pi(x) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx),$$

where $a, b > 0$ are constants and the gamma function is defined as

$$\Gamma(a) = \int_0^\infty x^{a-1} \exp(-x) \mathrm{d}x.$$

2. *Simulate the SDE using the Euler-Maruyama method and check that the solution indeed stabilizes to this distribution.*

Solution

1. If the SDEs is of the form

$$dx = -\frac{1}{2}\nabla v(x) dt + d\beta(t), \tag{95}$$

then the stationary solution is of the form

$$p(x) = \frac{1}{z}\exp(-v(x)/q). \tag{96}$$

By hypothesis $p(x) = \pi(x) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx)$, and we can rewrite $\pi(x)$ as

$$\pi(x) = \frac{b^a}{\Gamma(a)} \exp(-bx + \log(x^{a-1})), \tag{97}$$

then

$$\begin{aligned} v(x) &= -q \left[\log(x^{a-1}e^{-bx})\right] \\ &= -q \left[\log\left(\frac{\Gamma(a)}{b^a}\pi(x)\right)\right] \\ &= -q \left[\log\left(\frac{\Gamma(a)}{b^a}\right) + \log\left(\pi(x)\right)\right] \end{aligned} \tag{98}$$

Finally

$$dx = \frac{1}{2} q \nabla \log\left(\pi(x)\right) dt + d\beta(t). \tag{99}$$

2. Figure 13 shows the trajectories of the Langevin process with Euler Maruyama and the estimated mean. We can see how the mean is becoming asymptotically constant. The Figure 14 shows the stationary density and the histogram at time $t = 20$. There is clear how the histogram is matching with the stationary density.

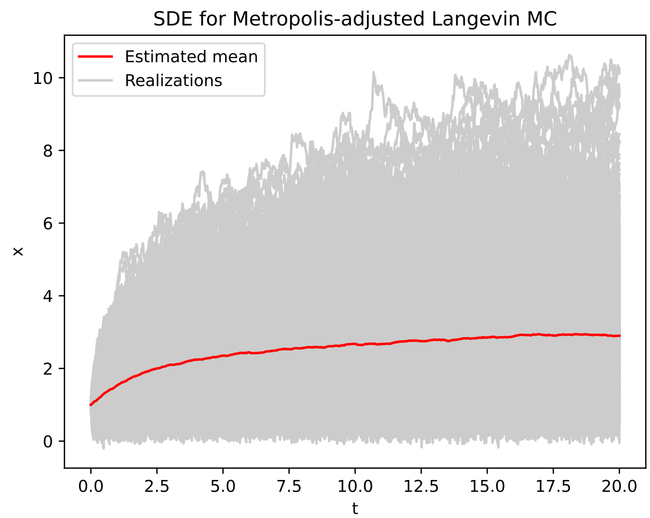


Figure 13: Exercise 5.6. Trajectories.

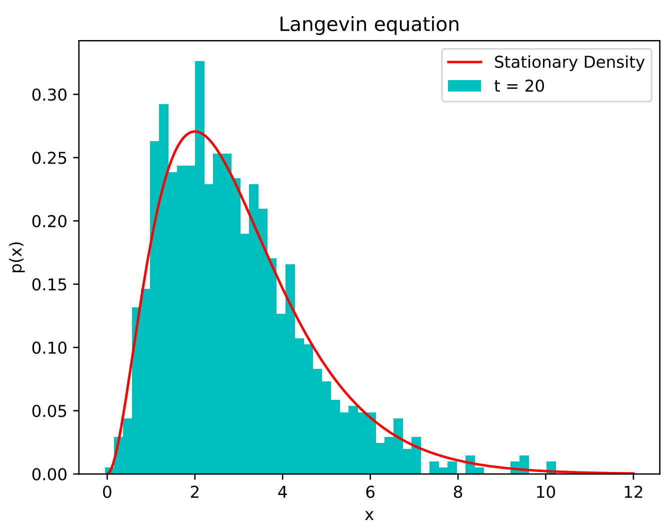


Figure 14: Exercise 5.6. Stationary solution of FPK equations.

Exercise 5.7. *Show that the operator $\mathcal{A}$ in Exercise 5.1 is the adjoint operator of the operator $\mathcal{A}^*$ in the FPK equation in Exercise 5.2.*

Solution

Exercise 5.8. *Recall the mean-reverting Ornstein-Uhlenbeck process is*

$$\mathrm{d}x = \theta(\mu - x)\mathrm{d}t + \sigma \mathrm{d}\beta(t), \quad x(0) = x_0,$$

where θ gives the speed of reversion, μ is the long-run equlibrium, and σ stands for the volatility.

1. *Recall (from Exercise 4.4) or find the complete solution $x(t)$.*
2. *Derive the expressions for the mean $m(t)$ and variance $P(t)$ of the solution.*

Solution

1. The solution given by Kloeden-Platen formula is

$$x(t) = \exp\left[(\theta - \frac{1}{2}\sigma^2) + \sigma\beta(t)\right]\left(x + \theta\mu\int_0^t \exp\left[-\left(\theta - \frac{1}{2}\sigma^2\right)s - \sigma\beta(s)\right]ds\right).$$

2. The differential equation that modelates the mean value is given by

$$\frac{dm}{dt} = \mathbb{E}[f(x,t)] = \mathbb{E}[\theta(\mu - x)] = \theta(\mu - m), \tag{100}$$

and the its solution is

$$m(t) = (m_0 - \mu)e^{-\theta t} + \mu. \tag{101}$$

In the same way the ODE of the covariance matrix is

$$\begin{aligned}\frac{dP}{dt} &= \mathbb{E}[f(x,t)(x-m)^T] + \mathbb{E}[(x-m)f(x,t)^T] + \mathbb{E}[L(x,t)QL^T(x,t)]\\ &= 2\mathbb{E}[f(x,t)(x-m)] + q\mathbb{E}[L^2(x,t)]\\ &= 2\mathbb{E}[\theta(\mu - x)(x-m)] + q\mathbb{E}[\sigma^2]\\ &= 2\theta\mathbb{E}[(\mu-m)(x-m)] - 2\theta\mathbb{E}[(x-m)^2] + q\sigma^2\\ &= -2\theta P + q\sigma^2,\end{aligned} \tag{102}$$

which solution is

$$P(t) = \left(P_0 - \frac{q\sigma^2}{2\theta}\right)\exp(-2\theta t) + \frac{q\sigma^2}{2\theta}. \tag{103}$$

Exercise 5.9. *The Cox-Ingersoll-Ross (CIR) model is used in finance for modeling interest rates. The model is given by*

$$\mathrm{d}x = (\theta_1 - \theta_2 x)\,\mathrm{d}t + \theta_3\sqrt{x}\;\mathrm{d}\beta(t), \quad x(0) = x_0 > 0,$$

where $\theta_1, \theta_2, \theta_3 > 0$. The transition density for the Cox-Ingersoll-Ross model can be written down in closed form: Consider the transformed process $y(t) = 2cx(t)$ with $c = 2\theta_2 / \left(\theta_3^2 \left(1 - e^{-\theta_2 t}\right)\right)$. Its transition density is a noncentral χ^2 probability density

$$\frac{1}{2} \exp\left(-\frac{y+\lambda}{2}\right) \left(\frac{y}{\lambda}\right)^{\nu/4-1/2} \mathrm{I}_{\nu/2-1}(\sqrt{\lambda y})$$

with $v = 4\theta_1/\theta_3^2$ degrees of freedom, and a noncentrality of $\lambda = y_0 \exp\left(-\theta_2 t\right)$. I_α denotes the modified Bessel function of the first kind. Derive the expression for the transition density of the original process $x(t)$.

Solution

Let us remark the formula of change of variable. Given $g : \mathbb{R} \to \mathbb{R}$ and $y = g(x)$. The densities for Y and X follow the relation

$$f_Y(y) = \left|\frac{dx}{dy}\right| f_X(x), \tag{104}$$

then

$$\frac{dy}{dx} = \frac{d}{dx} 2cx = 2c. \tag{105}$$

It implies that $dx/dy = (2c)^{-1}$ and $f_X(x) = 2c f_Y(y)$. Finally

$$f_X(x) = C \exp\left(-\frac{2cx+\lambda}{2}\right) \left(\frac{2cx}{\lambda}\right)^{\nu/4-1/2} \mathbf{1}_{\nu/2-1}(\sqrt{2\lambda cx}), \tag{106}$$

Exercise 5.10. *Show that the modified CIR model*

$$\mathrm{d}x = -\theta_1 x \ \mathrm{d}t + \theta_2 \sqrt{1+x^2} \ \mathrm{d}\beta, \tag{107}$$

where $2\theta_1 > \theta_2^2$, has a stationary distribution with a density proportional to

$$\frac{1}{\left(1+x^2\right)^{1+\theta_1/\theta_2^2}}.$$

Reparametrize this density to show that it is in fact the probability density of a Student's t distribution.

Solution

Let us define $f(x,t) = -\theta_1 x$ and $L(x,t) = \theta_2\sqrt{1+x^2}$. The stationary Fokker-Planck-Kolmogorov equation is

$$-\frac{\partial}{\partial x}[f(x)p(x)] + \frac{q}{2}\frac{\partial^2}{\partial x^2}[L^2(x)p(x)] = 0. \tag{108}$$

Defining

$$h(x) = -\frac{\partial}{\partial x}[f(x)\pi(x)] + \frac{q}{2}\frac{\partial^2}{\partial x^2}[L^2(x)\pi(x)], \tag{109}$$

then

$$h(x) = -\frac{\partial}{\partial x}\left[-\theta_1 x \frac{C}{(1+x^2)^{1+\theta_1\theta_2^{-2}}}\right] + \frac{q}{2}\frac{\partial^2}{\partial x^2}\left[L^2(x)\pi(x)\right]. \tag{110}$$

Developing the partial derivatives we can corroborate that $h(x) = 0$ and as a consequence, it is a stationary solution of the FPK equation.

Exercise 5.11. *Statistics of the Black-Scholes model:*

1. *Solve the differential equation (5.62) for the mean $m(t)$ and variance $P(t)$ with given initial conditions $m(0) = m_0$ and $P(0) = P_0$.*

2. *Verify using the Euler-Maruyama method that your solutions are correct.*

Solution

1. The differential equations governing the mean value m and the covariance matrix P are given by
$$\frac{d}{dt}m(t) = \mu m(t) \tag{111}$$
and
$$\frac{d}{dt}P(t) = (2\mu + \sigma^2)P + \sigma^2 m^2 \tag{112}$$
respectively. The solution of these linear ODEs are given by
$$m(t) = m_0 \exp(\mu t) \tag{113}$$
and
$$P(t) = (m_0^2 + P_0)e^{(2\mu+\sigma^2)t} - m_0^2 e^{2\mu t}. \tag{114}$$

2. The Figures 15 shows the trajectories of the SDEs calculated with Euler-Maruyama with the true and estimated means. While the Figure 16 shows the estimated and true variances. There we can see how the estimated means and variances match in both figures.

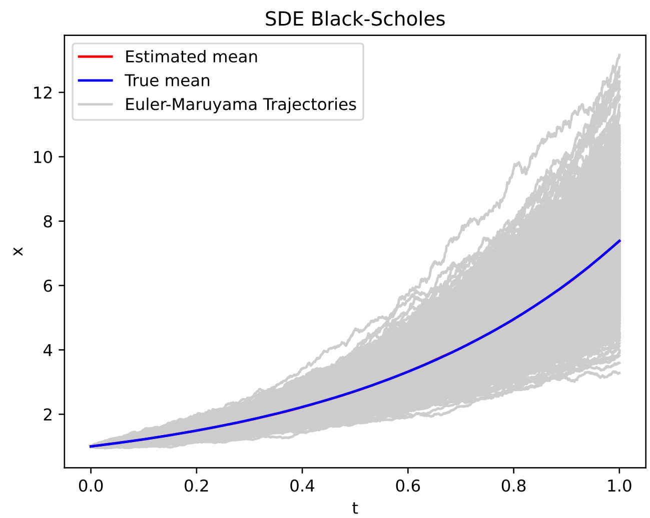


Figure 15: Exercise 5.11. Trajectories of the SDE and means.

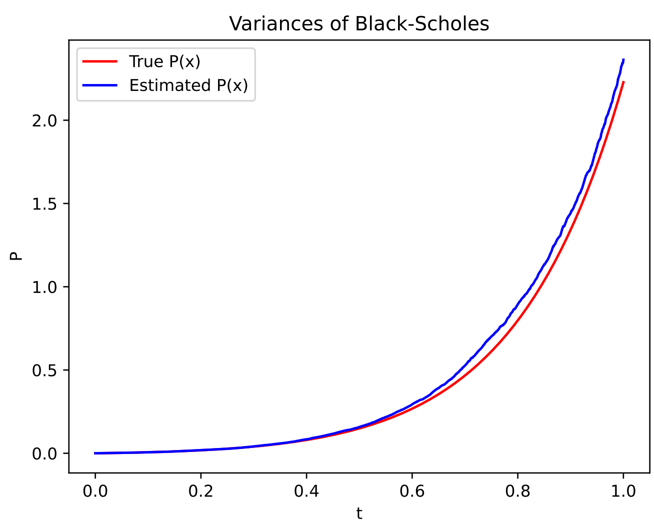


Figure 16: Exercise 5.11. Variances.

6 Statistics of Linear Stochastic Differential Equations

Exercise 6.1. *Consider a "Wiener acceleration model", where the model matrices of the linear time-invariant SDE are*

$$\mathbf{F} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

which corresponds to $\mathrm{d}^3 x(t)/\mathrm{d}t^3 = w(t)$. *Provide the discrete-time solution in terms of* $\mathbf{A}(\Delta t)$ *and* $\mathbf{\Sigma}(\Delta t)$.

Solution
The matrix A_k is given by

$$A_k = \exp(F\Delta t_k) = \sum_{n=0}^{\infty} \frac{(F\Delta t_k)^n}{n!}. \tag{115}$$

Now

$$F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad F^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad F^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{116}$$

Then

$$\exp(F\Delta t_k) = \begin{pmatrix} 1 & \Delta t_k & \Delta t_k^2 \\ 0 & 1 & \Delta t_k \\ 0 & 0 & 1 \end{pmatrix} \tag{117}$$

At the same time, we know that $Q = qI$, then

$$LQL^T = qLL^T = q[0,0,1]^T[0,0,1] = q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{118}$$

Using the last equations, the integral returns the next matrix

$$\begin{aligned} \Sigma(\Delta t) &= \int_0^{\Delta_k} \exp\left(F[\Delta t_k - \tau]\right) LQL^T \exp\left(F[\Delta t_k - \tau]\right)^T d\tau \\ &= q \begin{pmatrix} \frac{\Delta t_k^5}{20} & \frac{\Delta t_k^4}{6} & \frac{\Delta t_k^3}{6} \\ \frac{\Delta t_k^4}{6} & \frac{\Delta t_k^3}{3} & \frac{\Delta t_k^2}{2} \\ \frac{\Delta t_k^2}{6} & \frac{\Delta t_k^2}{2} & \Delta t_k \end{pmatrix}. \end{aligned} \tag{119}$$

Exercise 6.2. *Provide the discrete-time solution in terms of* $\mathbf{A}(\Delta t)$ *and* $\mathbf{\Sigma}(\Delta t)$ *to the linear SDE given for smartphone orientation tracking in Example 3.5:*

$$\mathrm{d}\mathbf{g}_\mathrm{L} = -\omega_\mathrm{L} \times \mathbf{g}_\mathrm{L} \mathrm{d}t + \mathrm{d}\boldsymbol{\beta},$$

where $\mathbf{g}_\mathrm{L}$ *represents the locally seen gravitation,* ω_L *angular velocity (here assumed to be constant), and* $\boldsymbol{\beta}(t)$ *is a Brownian motion with diffusion matrix q I. Recall that the matrix exponential was already solved in Exercise 3.6 using the Rodrigues formula.*

Exercise 6.3. *Calculate the matrix fraction decomposition for the Ornstein-Uhlenbeck process*

$$\mathrm{d}x = -\lambda x \ \mathrm{d}t + \mathrm{d}\beta,$$

where $\beta(t)$ *is a Brownian motion with diffusion constant* q.

Solution

We define $F = -\lambda$, $Q = q$ and $L = 1$. Then

$$\begin{pmatrix} C_\Sigma(\Delta t) \\ D_\Sigma(\Delta t) \end{pmatrix} = \exp\left[\begin{pmatrix} -\lambda & q \\ 0 & \lambda \end{pmatrix} \Delta t\right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{120}$$

The matrix

$$R := \begin{pmatrix} -\lambda & q \\ 0 & \lambda \end{pmatrix} \tag{121}$$

has different eigenvalues and as a consequence it is diagonalizable. Then R can be decomposed as

$$R = T \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix} T^{-1}, \tag{122}$$

with

$$T = \frac{q}{2\lambda} \begin{pmatrix} \frac{2\lambda}{q} & -1 \\ 0 & 1 \end{pmatrix}. \tag{123}$$

Finally, the solution is given by

$$\begin{pmatrix} C_\Sigma(\Delta t) \\ D_\Sigma(\Delta t) \end{pmatrix} = T \begin{pmatrix} \exp(-\lambda \Delta t) & 0 \\ 0 & \exp(\lambda \Delta t) \end{pmatrix} T^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{124}$$

Exercise 6.4. *Derive the expression for* $\mathbf{B}\left(\Delta t_k\right)$ *based on Remark 6.7.*

Solution

The system can be written as

$$\begin{pmatrix} dx \\ du \end{pmatrix} = \begin{pmatrix} F & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} dt + \begin{pmatrix} L \\ 0 \end{pmatrix} d\beta(t), \tag{125}$$

and we define the matrices

$$\mathbf{F} = \begin{pmatrix} F & I \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} L \\ 0 \end{pmatrix}. \tag{126}$$

By induction we have the relation

$$\mathbf{F}^{k+1} = \begin{pmatrix} F^{k+1} & F^k \\ 0 & 0 \end{pmatrix}, \tag{127}$$

and the exponential matrix is

$$
\begin{aligned}
\exp(\mathbf{F}\Delta t) &= \sum_{k=0}^{\infty} \mathbf{F}^k \frac{\Delta t^k}{k!} \\
&= \begin{pmatrix} \sum_{k=1}^{\infty} F^k \frac{\Delta t^k}{k!} & \sum_{k=1}^{\infty} F^{k-1} \frac{\Delta t^k}{k!} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=0}^{\infty} F^k \frac{\Delta t^k}{k!} & \sum_{k=1}^{\infty} F^{k-1} \frac{\Delta t^k}{k!} \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} \exp(F\Delta t) & \sum_{k=1}^{\infty} F^{k-1} \frac{\Delta t^k}{k!} \\ 0 & I \end{pmatrix}
\end{aligned} \tag{128}
$$

From here is clear that $B(\Delta t_k)$ is given by the expression

$$
B(\Delta t_k) = \sum_{k=1}^{\infty} F^{k-1} \frac{\Delta t^k}{k!}. \tag{129}
$$

Exercise 6.5. *Derive the covariance function for the Wiener velocity model*

$$
\mathrm{d}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x}\mathrm{d}t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}\beta \tag{130}
$$

where $\lambda > 0, \beta(t)$ is a Brownian motion with diffusion q and the initial condition is $\mathbf{x}(0) = \mathbf{0}$.

Solution
We know that F is nilpotent and

$$
F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{131}
$$

Then

$$
\exp(Ft) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad LQL^T = q \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{132}
$$

Using the formula of the integral matrix, we have the expression $\Psi(t,\tau) = \exp(F(t-\tau))$ and

$$
\begin{aligned}
P(t) &= \Psi(t,0)P_0\Psi^T(t,0) + \int_0^t \Psi(t,\tau)LQL^T\Psi^T(t,\tau)d\tau \\
&= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} P_0 \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} + q \begin{pmatrix} \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{pmatrix}.
\end{aligned} \tag{133}
$$

Finally

$$
C(t,s) = \begin{cases} P(t)\Psi^T(s,t), & t < s \\ \Psi(t,s)P(s), & \text{otherwise.} \end{cases} \tag{134}
$$

Exercise 6.6. *Derive the covariance function of the nonstationary model*

$$\mathrm{d}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x}\mathrm{d}t,$$

where $\mathbf{x}(0) \sim \mathrm{N}\left(\mathbf{0}, \mathrm{diag}\left(0, \sigma^2\right)\right)$.

Solution

In this case $L = (0,0)^T$ and $P_0 = \mathrm{diag}\left(0, \sigma^2\right)$. The first equation implies that $LQL^T = 0$ and using the results of the previous exercise, we have

$$\begin{aligned} P(t) &= \Psi(t,0) P_0 \Psi^T(t,0) \\ &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \\ &= \sigma^2 \begin{pmatrix} t^2 & t \\ t & 1 \end{pmatrix}. \end{aligned} \tag{135}$$

Finally

$$C(t,s) = \begin{cases} P(t)\Psi^T(s,t), & t < s \\ \Psi(t,s)P(s), & \text{otherwise.} \end{cases} \tag{136}$$

Exercise 6.7. *Theorem 5.6 tells that we can form a stationary solution to the FPK equation in form of Equation (5.26) when the drift can be written as a gradient of a potential function. In case of a linear SDE*

$$\mathrm{d}\mathbf{x} = \mathbf{F}\mathbf{x}\mathrm{d}t + \mathbf{L}\mathrm{d}\boldsymbol{\beta}$$

what kinds of conditions does this imply for the matrix $\mathbf{F}$*? Can you use this result to obtain a class of solutions to Equation (6.69)?*

Solution

By inspection the function such that its gradient is a linear transformation must be a bi-linear form such that

$$v(x) = -\frac{1}{2} x^T A x. \tag{137}$$

Its gradient must be

$$-\frac{1}{2}\nabla v = (A + A^T)x. \tag{138}$$

Then

$$F = A + A^T \tag{139}$$

which implies that F is a symmetric matrix.

Exercise 6.8. *Derive the steady-state mean* $\mathbf{m}_\infty$ *and covariance* $\mathbf{P}_\infty$ *of the Matérn* $(\nu = 3/2)$ *model*

$$\mathrm{d}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -\lambda^2 & -2\lambda \end{pmatrix} \mathbf{x}\mathrm{d}t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}\beta,$$

by solving the stationary states of the mean and covariance differential equations. The parameter $\lambda > 0$ and $\beta(t)$ is a Brownian motion with diffusion q. Also derive the covariance function of the process.

Solution

The matrices of the system are

$$F = \begin{pmatrix} 0 & 1 \\ \lambda^2 & -2\lambda \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{140}$$

Then the equation is

$$FP_\infty + P_\infty F^T + LQL^T = 0, \tag{141}$$

where the last term is reduced as $LQL^T = q\text{diag}(0, 1)$ and

$$\begin{pmatrix} 0 & 1 \\ -\lambda^2 & -2\lambda \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22}\lambda \end{pmatrix} + \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22}\lambda \end{pmatrix} \begin{pmatrix} 0 & -\lambda^2 \\ 1 & -2\lambda \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{142}$$

The solution of this algebraic linear equation is the matrix

$$P_\infty = \frac{q}{4\lambda^2} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}. \tag{143}$$

Now, the Joran canonical form of F is given by

$$\begin{pmatrix} 0 & 1 \\ -\lambda^2 & -2\lambda \end{pmatrix} = \begin{pmatrix} 1 & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix} \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} 0 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix}, \tag{144}$$

and the exponential matrix of F is

$$\exp(F\tau) = \begin{pmatrix} 1 & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix} \exp\left[\begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix}\right] \begin{pmatrix} 0 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix}. \tag{145}$$

The argument in the second matrix of the right side can be decomposed as

$$\begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = -\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{146}$$

and it is easy to prove that the product of the last two matrices commute. As a consequence, the exponential matrix can be decomposed as the product of the exponential matrices as

$$\exp\left[\begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix}\right] = \exp(-\lambda\tau) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} = \exp(-\lambda\tau) \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}. \tag{147}$$

The covariance matrix of the process is finally

$$C(\tau) = \begin{cases} P_\infty \exp(F\tau), & \tau > 0 \\ \exp(-F\tau) P_\infty, & \text{otherwise.} \end{cases} \tag{148}$$

7 Useful Theorems and Formulas for SDEs

Exercise 7.1. *Consider the modified Cox-Ingersoll-Ross model (5.69) being transformed by $y(t) = h(x(t))$ using the Lamperti transform*

$$h(x) = \int_0^x \frac{1}{L(u)} \mathrm{d}u = \int_0^x \frac{1}{\theta_2 \sqrt{1+u^2}} \, \mathrm{d}u.$$

Rewrite the original process in terms of the $y(t)$ process.

Solution
The Cox-Ingersoll-Ross model is given by the SDE

$$dx = -\theta_1 x dt + \theta_2 \sqrt{1+x^2} d\beta. \tag{149}$$

The Lamperti transform can be reduced as

$$h(x) = \int_0^x \frac{1}{\theta_2 \sqrt{1+u^2}} \, \mathrm{d}u = \mathrm{asinh}(x), \tag{150}$$

and as a consequence $x = \sinh(\theta_2 y) = h^{-1}(y)$. The transformed SDE is

$$\begin{aligned} dy &= \left[\frac{\partial}{\partial t} \int_0^x \frac{1}{L(u,t)} du + \frac{f(x,t)}{L(x,t)} - \frac{1}{2} \frac{\partial}{\partial x} L(x,t) \right]_{x=h^{-1}(y)} dt + d\beta(t) \\ &= -\left(\frac{\theta_1}{\theta_2} + \frac{1}{2}\theta_2 \right) \tanh(\theta_2 y) dt + d\beta(t). \end{aligned} \tag{151}$$

Exercise 7.2. *Derive the weak solution to the Ornstein-Uhlenbeck process following the techniques in Example 7.10.*

Solution
The expression of the term z is given by

$$\begin{aligned} z(t) &= \exp\left[-\frac{1}{2} \int_0^t f^T(x_0 + \beta(\tau)) Q^{-1} f(x_0 + \beta(\tau)) d\tau + \int_0^t f^T(x_0 + \beta(\tau)) Q^{-1} d\beta(\tau) \right] \\ &= \exp\left[-\frac{1}{2} \int_0^t \lambda^2 (x_0 + \beta(\tau))^2 d\tau - \int_0^t \lambda (x_0 + \beta(\tau)) d\beta(\tau) \right]. \end{aligned} \tag{152}$$

Let us define now the variable $\mathbf{x} = \lambda(x_0 + \beta(\tau))$ and the function $\phi(\mathbf{x}) = \mathbf{x}^2/2$. Using the Itô formula

$$\begin{aligned} d\Phi(\mathbf{x}) &= \frac{\partial \Phi}{\partial t} dt + \frac{\partial \Phi}{\partial \mathbf{x}} d\mathbf{x} + \frac{1}{2} \frac{\partial^2 \Phi}{\partial \mathbf{x}^2} d\mathbf{x}^2 \\ &= 0 + \mathbf{x} d\mathbf{x} + \frac{1}{2} d\mathbf{x}^2 \\ &= \lambda(x_0 + \beta(t)) d\beta(t) + \frac{1}{2} dt. \end{aligned} \tag{153}$$

Integrating both sides of the previous equation, we have the equation

$$\Phi(t) - \Phi(0) = \int_0^t \lambda(x_0 + \beta(\tau))d\beta(\tau) + \frac{1}{2}t, \tag{154}$$

which is reduced as

$$x_0 + \beta(t) - x_0 = \int_0^t \lambda(x_0 + \beta(\tau))d\beta(\tau) + \frac{1}{2}t, \tag{155}$$

or

$$\frac{1}{2}t - \beta(t) = -\int_0^t \lambda(x_0 + \beta(\tau))d\beta(\tau). \tag{156}$$

Substituting this expression in $z(t)$ we have the equation

$$z(t) = \exp\left[-\frac{1}{2}\int_0^t \lambda^2(x_0 + \beta(\tau))^2 d\tau + \frac{1}{2}t - \beta(t)\right]. \tag{157}$$

Let be the variable $\tilde{\mathbf{x}} = \mathbf{x}_0 + \beta(t)$. It is distributed as $\tilde{\mathbf{x}} \sim \mathcal{N}(\mathbf{x}_0, t)$, then

$$\begin{aligned} p(\tilde{\mathbf{x}}, t) &= z(t)\mathcal{N}(\mathbf{x}_0, t) \\ &= \exp\left[-\frac{1}{2}\int_0^t \lambda^2(x_0 + \beta(\tau))^2 d\tau + \frac{1}{2}t - \beta(t)\right] \\ &\quad (2\pi t)^{-1/2}\exp\left[-\frac{1}{2t}(\tilde{\mathbf{x}} - \mathbf{x}_0)^2\right] \\ &= (2\pi t)^{-1/2}\exp\left[-\frac{1}{2}\int_0^t \lambda^2\tilde{\mathbf{x}}^2 d\tau + \frac{1}{2}t - \beta(t) - \frac{1}{2t}(\tilde{\mathbf{x}} - \mathbf{x}_0)^2\right] \end{aligned} \tag{158}$$

Exercise 7.3. *Derive the weak solution to the SDE model*

$$dx = \sin(x)dt + d\beta,$$

where $\beta(t)$ is a standard Brownian motion, by using the Girsanov theorem.

Solution

Let be $\tilde{\mathbf{x}} = x_0 + \beta$. Using the formula of the function $z(t)$ we have that

$$\begin{aligned} z(t) &= \exp\left[-\frac{1}{2}\int_0^t f^T(x_0 + \beta(\tau))Q^{-1}f(x_0 + \beta(\tau))d\tau + \int_0^t f^T(x_0 + \beta(\tau))Q^{-1}d\beta(\tau)\right] \\ &= \exp\left[-\frac{1}{2}\int_0^t \sin^2(x_0 + \beta(\tau))d\tau - \int_0^t \sin(x_0 + \beta(\tau))d\beta(\tau)\right] \\ &= \exp\left[-\frac{1}{2}\int_0^t \sin^2(\tilde{\mathbf{x}})d\tau - \int_0^t \sin(\tilde{\mathbf{x}})d\beta(\tau)\right] \\ &= \exp\left[-\frac{t}{4} + \frac{1}{4}\int_0^t \cos(2\tilde{\mathbf{x}})d\tau - \int_0^t \sin(\tilde{\mathbf{x}})d\beta(\tau)\right]. \end{aligned} \tag{159}$$

Let us define $\Phi(\tilde{\mathbf{x}}) = -\cos(\tilde{\mathbf{x}})$. Using Itô formula

$$\begin{aligned} d\Phi(\tilde{\mathbf{x}}) &= \frac{\partial \Phi}{\partial t} dt + \frac{\partial \Phi}{\partial \tilde{\mathbf{x}}} d\tilde{\mathbf{x}} + \frac{1}{2} \frac{\partial^2 \Phi}{\partial \tilde{\mathbf{x}}^2} d\tilde{\mathbf{x}}^2 \\ &= \sin(\tilde{\mathbf{x}}) d\beta + \frac{1}{2} \cos(\tilde{\mathbf{x}}) q dt, \end{aligned} \tag{160}$$

and integrating both sides of the SDE results in the equation

$$\int_0^t d\phi = \int_0^t \sin(\tilde{\mathbf{x}}) d\beta + \frac{1}{2} \int_0^t \cos(\tilde{\mathbf{x}}) d\tau. \tag{161}$$

Remarking that the left side of the last equation can be reduced as

$$\int_0^t d\phi = \phi(t) - \phi(0) = \cos(x_0) - \cos(\tilde{\mathbf{x}}), \tag{162}$$

and then the term z can be written as

$$z(t) = \exp\left[-\frac{t}{4} + \frac{1}{4} \int_0^t \cos(2\tilde{\mathbf{x}}) d\tau + \cos(x_0) - \cos(\tilde{\mathbf{x}}) - \frac{1}{2} \int_0^t \cos(\tilde{\mathbf{x}}) d\tau \right]. \tag{163}$$

Finally, the density of the process is given for

$$p(\tilde{\mathbf{x}}(t)) = \frac{z(t)}{\sqrt{2\pi t}} \exp\left(\frac{1}{2t} (x(t) - x_0)^2 \right). \tag{164}$$

Exercise 7.4. *Consider the solution given in Example 7.12:*

1. *Write down the mean and variance expressions for the solution as a function of t.*

2. *Check numerically using the Euler-Maruyama method that the solution matches your mean and variance expressions.*

Solution

1. The conditioned solution of the SDE has the form

$$dx = \left[-\lambda x + q \frac{a(t)}{\sigma^2(t)} (x_T - a(t) x) \right] dt + d\beta, \tag{165}$$

where $a(t) = \exp(-\lambda(T-t))$ and $\sigma^2(t) = \frac{q}{2\lambda}[1 - \exp(-2\lambda(T-t))]$.

The SDE is of the form

$$dx = -\hat{\lambda} x dt + u dt + d\beta, \tag{166}$$

where

$$\hat{\lambda} = \lambda + 2\lambda \frac{\exp\left(-2\lambda(T-t)\right)}{1 - \exp(-2\lambda(T-t))}, \tag{167}$$

and

$$u = \frac{2x_T\lambda \exp\left(-\lambda(T-t)\right)}{1-\exp(-2\lambda(T-t))}. \tag{168}$$

The integrand factor is calculated as $\Psi(s,t) = \exp\left(-\int_t^s \hat{\lambda}(\tau)d\tau\right)$, which is reduced as

$$\Psi(t,s) = \exp\left(-\lambda(t-s)\right)\frac{1-\exp\left(-2\lambda[T-t]\right)}{1-\exp\left(-2\lambda[T-s]\right)}. \tag{169}$$

The mean value is given by

$$\begin{aligned} m(t) &= \Psi(t,0)m(0) + \int_0^t \Psi(t,\tau)u(\tau)d\tau \\ &= 0 + \int_0^t \Psi(t,\tau)u(\tau)d\tau \\ &= \int_0^t \Psi(t,\tau)u(\tau)d\tau \\ &= x_T\frac{\sinh(\lambda t)}{\sinh(\lambda T)}. \end{aligned} \tag{170}$$

2. The Figure 17 shows the solution of the conditioned SDE of the Ornstein-Uhlenbleck. There we can see how the estimated mean and the theoretical value match.

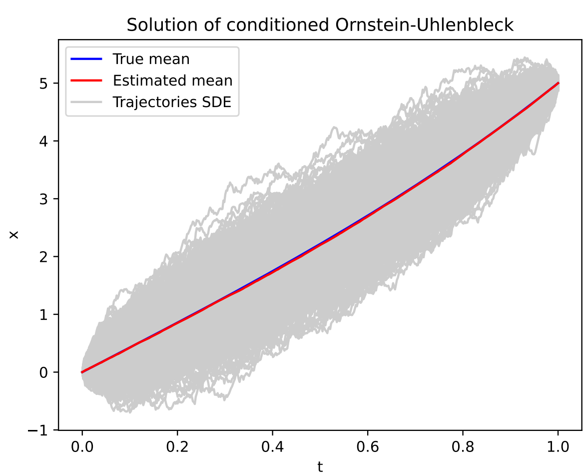


Figure 17: Exercise 7.4. Trajectories of the SDE and means.

Now the Figure 18 shows how the estimated variance grows and decreases until arrive to 0. This phenomena can be explained due to the last point of the trajectory is conditioned to the fixed value (T, x_T). It implies that there is not uncertainty about the position of this last point.

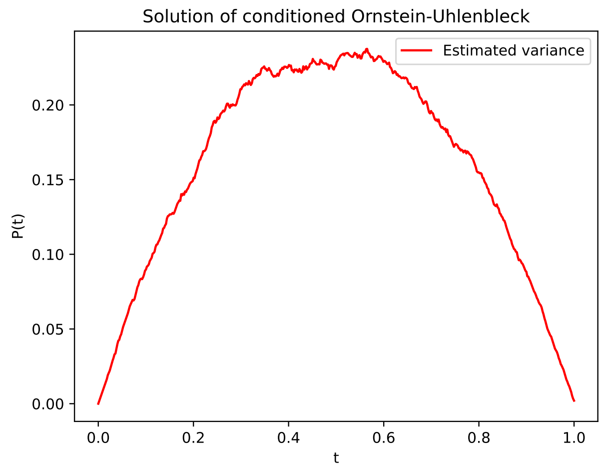


Figure 18: Exercise 7.4. Estimated variance of the SDE.

Exercise 7.5. *Derive the conditioned SDE (similar to Example 7.12) for the Beneš model* $\mathrm{d}x = \tanh(x)\mathrm{d}t + \mathrm{d}\beta$ *conditioned on* $x(T) = x_T$.

Solution

Exercise 7.6. *Solve the following PDE by using finite differences and SDE simulation and check that the results match. The PDE is*

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

with boundary conditions

$$\begin{aligned} u(x,-1) &= \cos(\pi x/2), \quad u(x,1) = -\cos(\pi x/2), \\ u(-1,y) &= \cos(\pi y/2), \quad u(1,y) = -\cos(\pi y/2). \end{aligned}$$

Solution

The Figure 19 shows the solution of the Laplace Equation with Feynman Kack formula and with finite differences method. We can see how the Feynman Kack formula produces an acceptable solution.

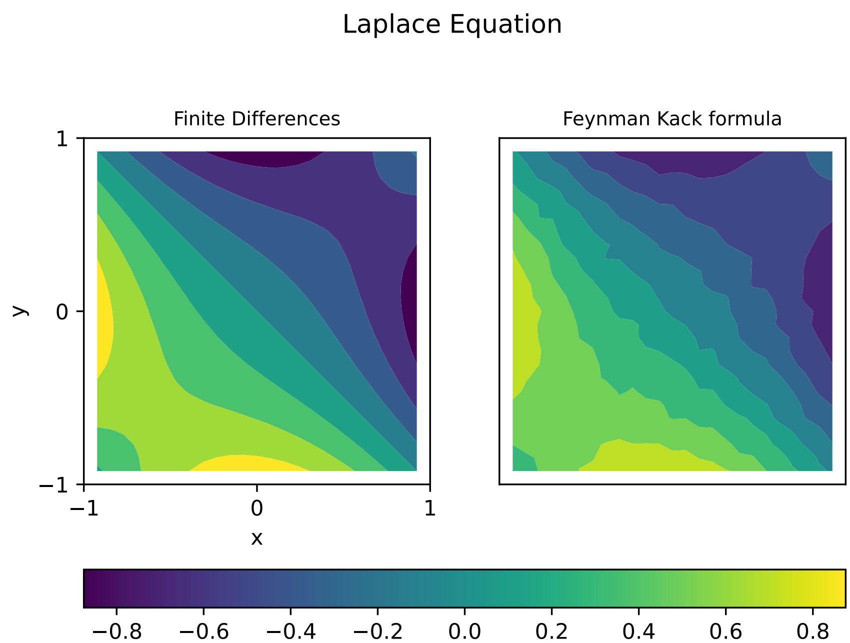


Figure 19: Exercise 7.6. Solution of Laplace equation with Feynmann Kack formula.

8 Numerical Simulation of SDEs

Exercise 8.1. *Consider the following scalar SDE:*

$$\mathrm{d}x = -cx\ \mathrm{d}t + gx\ \mathrm{d}\beta, \quad x(0) = x_0,$$

where $c, g,$ and x_0 are positive constants and $\beta(t)$ is a standard Brownian motion.

1. *Check using the Ito formula that the solution to this equation is*

$$x(t) = x_0 \exp\left[\left(-c - g^2/2\right) t + g\beta(t)\right].$$

2. *Simulate trajectories from the equation using the Milstein method with parameters $x_0 = 1, c = 1/10, g = 1/10$, and check that the histogram at $t = 1$ looks the same as obtained by sampling from the preceding exact solution.*

Solution

1. Applying the Itô formula

$$\begin{aligned}
dx &= \frac{\partial x}{\partial t}dt + \frac{\partial x}{\partial \beta}d\beta + \frac{1}{2}\frac{\partial^2 x}{\partial \beta^2}d\beta^2 \\
&= -(c+g^2/2)x_0 \exp\left[-(c+g^2/2)t + g\beta\right] dt \\
&+ x_0 \exp\left[-(c+g^2/2)t + g\beta\right] gd\beta \\
&+ \frac{1}{2}x_0 \exp\left[-(c+g^2/2)t + g\beta\right] g^2 d\beta^2 \\
&= -(c+g^2/2)xdt + gxd\beta + \frac{1}{2}g^2xdt \\
&= -cxdt + gxd\beta.
\end{aligned} \tag{171}$$

2. The figure 20 shows the comparison between the analytical solution and the Milstein method. It is clear how the method reproduces correctly the SDE and the estimated mean values have the same behaviour.

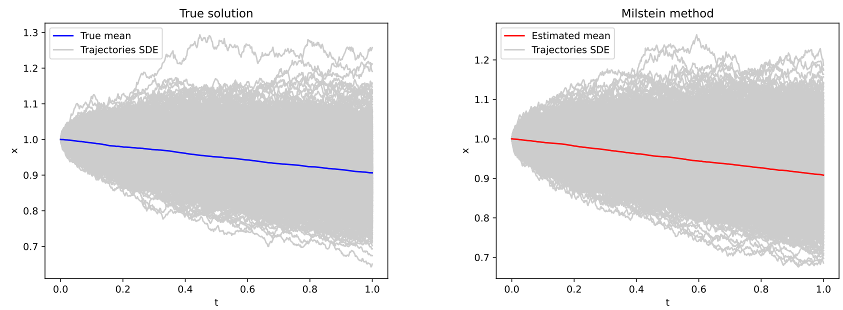


Figure 20: Exercise 8.1. Analytical solution of the SDE and with Milstein method.

Exercise 8.2. *Consider the following Beneš SDE:*

$$\mathrm{d}x = \tanh(x)\mathrm{d}t + \mathrm{d}\beta, \quad x(0) = 0,$$

where $\beta(t)$ is a standard Brownian motion. Recall that it has the probability density

$$p(x,t) = \frac{1}{\sqrt{2\pi t}} \cosh(x) \exp\left(-\frac{1}{2}t\right) \exp\left(-\frac{1}{2t}x^2\right).$$

Use the following numerical methods for simulating from the model:

1. *Simulate 1000 trajectories from the SDE with the strong order 1.5 Itô-Taylor series-based method presented earlier. Compare the histogram to the exact solution at $t = 5$.*

2. *Simulate 1000 trajectories from the SDE with the weak order 2.0 Itô-Taylor series-based method presented earlier using both (i) Gaussian increments and (ii) the three-point distributed random variables. Compare the histograms to the exact distribution at $t = 5$.*

3. *What can you say about the simulated trajectories when $t \in [0, 5]$ for the different methods?*

Solution

The Figure 21 shows the histograms at $t = 5$. There, we can see how the Itô-Taylor methods produce trajectories that adjust correctly the histograms. It is clear that the best adjusting are given by the weak-order 2 methods.

We can see in the Figure 22 how the trajectories have a similar behaviour for the three methods.

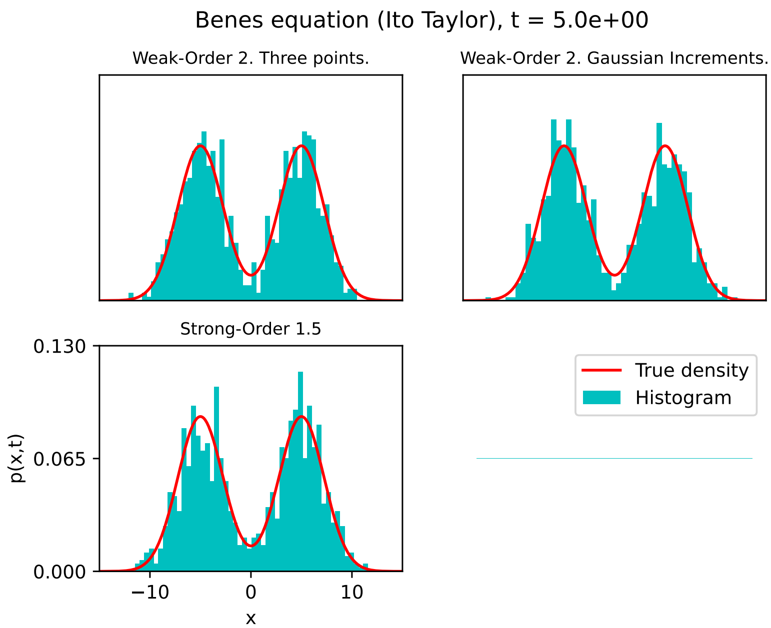


Figure 21: Exercise 8.2. Histograms at $t = 5$ for Itô-Taylor methods.

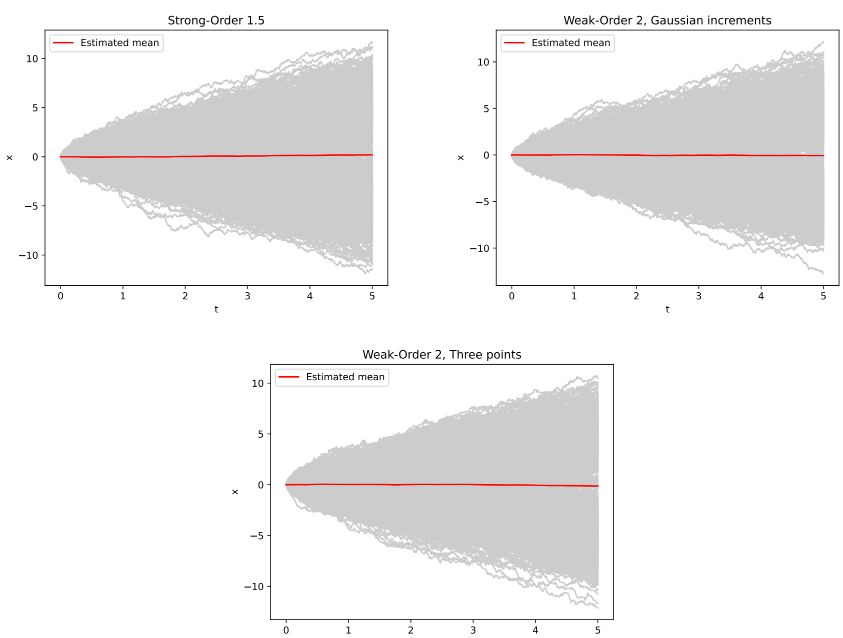


Figure 22: Exercise 8.2. Trajectories with Itô-Taylor methods.

Exercise 8.3. *Consider a simple strong order 1.0 stochastic Runge-Kutta method with the following extended Butcher tableau:*

$$
\begin{array}{c|ccc|ccc|ccc}
0 & & & & & & & & & \\
0 & 0 & & & 0 & & & & & \\
0 & 0 & 0 & & 0 & 0 & & & & \\
\hline
0 & & & & & & & & & \\
0 & 0 & & & 1 & & & & & \\
0 & 0 & 0 & & -1 & 0 & & & & \\
\hline
 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2}
\end{array}
$$

1. *Write down the iteration equations required for evaluating the method corresponding to the table in Equation (8.135).*

2. *Consider the Duffing van der Pol oscillator model:*

$$
\begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1\left(\alpha - x_1^2\right) - x_2 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 \\ x_1 \end{pmatrix} \mathrm{d}\beta,
$$

where $\beta(t)$ is a one-dimensional Brownian motion $\left(q = 0.5^2\right)$ and $\alpha = 1$. Use the method you just constructed for drawing trajectories starting from $x_2(0) = 0$ and $x_1(0) = -4, -3.9, \ldots, -2$. Use a time span $[0, 10]$. Plot the results in the (x_1, x_2) plane.

3. *Experiment with different step sizes $\Delta t = 2^{-k}, k = 0, 2, 4, 6$ and visually compare the trajectories produced by the method implemented in (b) to the Euler-Maruyama scheme.*

Solution

1. The methods can be written as

$$
\hat{x}(t_{k+1}) = \hat{x}(t_k) + f(\hat{x}(t_k), t_k)\Delta t + \sum_{n=1}^{S}\left[\Delta\beta_k^{(n)} L_n(x_1^{(n)}, t_k) + \frac{\sqrt{\Delta t}}{2} L_n(x_2^{(n)}, t_k) - \frac{\sqrt{\Delta t}}{2} L_n(x_3^{(n)}, t_k)\right], \tag{172}
$$

where

$\tilde{x}_i^{(0)} = \hat{x}(t_k)$

$\tilde{x}_1^{(n)} = \hat{x}(t_k)$

$\tilde{x}_2^{(n)} = \hat{x}(t_k) + \sum_{l=1}^{S} L_l(\tilde{x}_1^{(l)}, t_k) \frac{\Delta\chi_k^{l,n}}{\sqrt{\Delta t}}$

$\tilde{x}_3^{(n)} = \hat{x}(t_k) - \sum_{l=1}^{S} L_l(\tilde{x}_1^{(l)}, t_k) \frac{\Delta\chi_k^{l,n}}{\sqrt{\Delta t}}$

For $S=1$

$\Delta\chi_k^{i,j} = \Delta\chi_k = \frac{1}{2}[(\Delta\beta_k)^2 - q\Delta t]$

2. The Figures 23, 24, 25 and 26 show the mean values of solutions for $k = 0, 1, 2, 3$. It is clear how the solutions diverge for $k = 0, 1$, while the solutions improve when k increase after $k \geq 2$. The solutions are concentrated on the equilibrium points as it was expected.

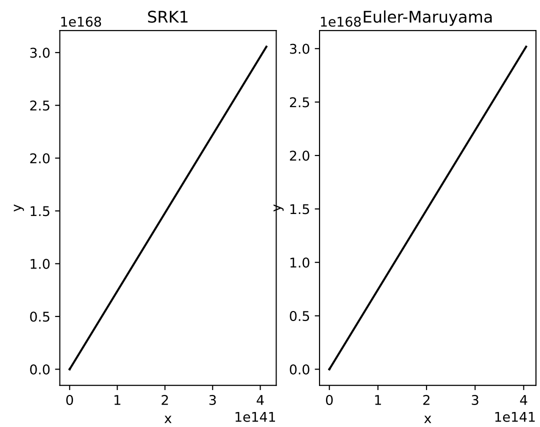


Figure 23: Exercise 8.3. Solutions for $k = 0$.

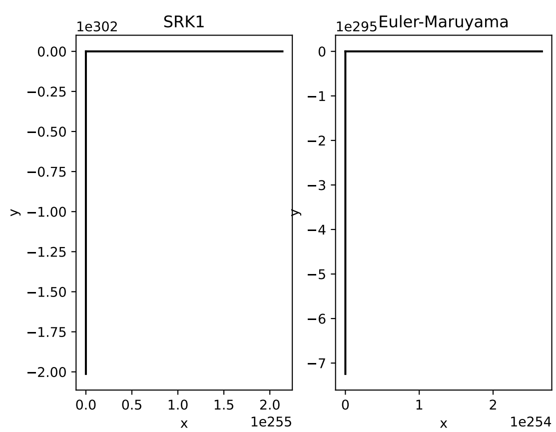


Figure 24: Exercise 8.3. Solutions for $k = 1$.

Exercise 8.4.

Solution

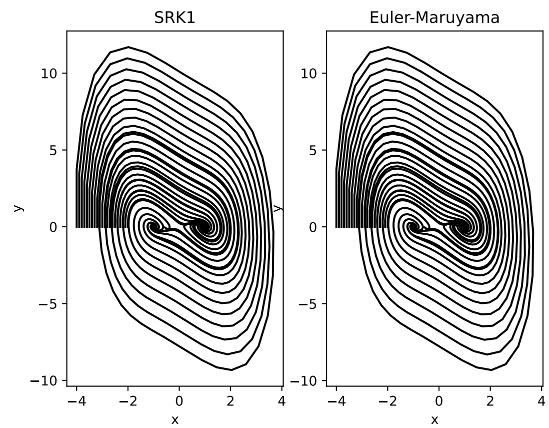


Figure 25: Exercise 8.3. Solutions for $k = 2$.

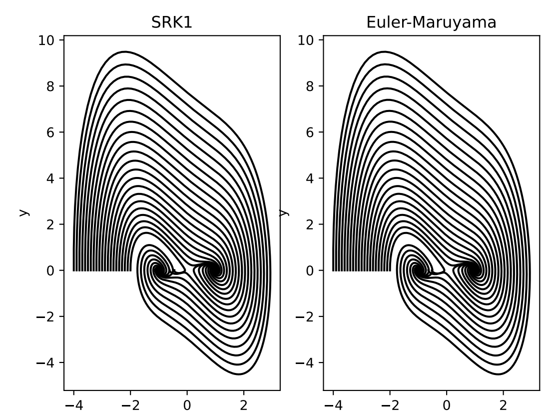


Figure 26: Exercise 8.3. Solutions for $k = 3$.

Exercise 8.5.

Solution

Exercise 8.6. *Consider the double-well potential model*

$$\mathrm{d}x = \left(x - x^3\right) \mathrm{d}t + \frac{1}{2}\sqrt{1 + x^2}\ \mathrm{d}\beta,$$

where $\beta(t)$ is a standard Brownian motion. Simulate trajectories from the stochatic differential equation to characterize the solution at $x(10)$ starting from $x(0) = 1$:

1. *Use the Euler-Maruyama scheme.*
2. *Use the weak order 2.0 stochastic Runge-Kutta scheme in Algorithm 8.18.*
3. *Study the effect of step size Δt.*

Solution

The Figures 27, 28 and 29 show the solutions for $\Delta t = 2^{-2}, 2^{-4}, 2^{-8}$ respectively. It is clear that the solution with Euler-Maruyama tends to be similar to the solution with the Weak SRK-2 method when the step size decrease. The small changes in the behaviour of Weak SRK-2 method respect to the step size indicates that the method converges fast in comparison of Euler-Maruyama.

We can notice that histograms present bimodality near to the equilibrium points $x = -1, 1$ of the deterministic term, with more concentration in $x = 1$. This result is expected due to the initial point is at $x_0 = 1$.

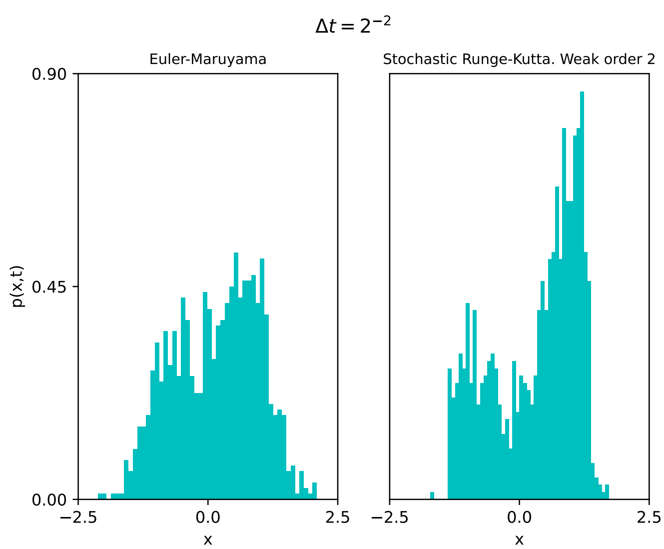


Figure 27: Exercise 8.6. Solutions for $\Delta t = 2^{-2}$.

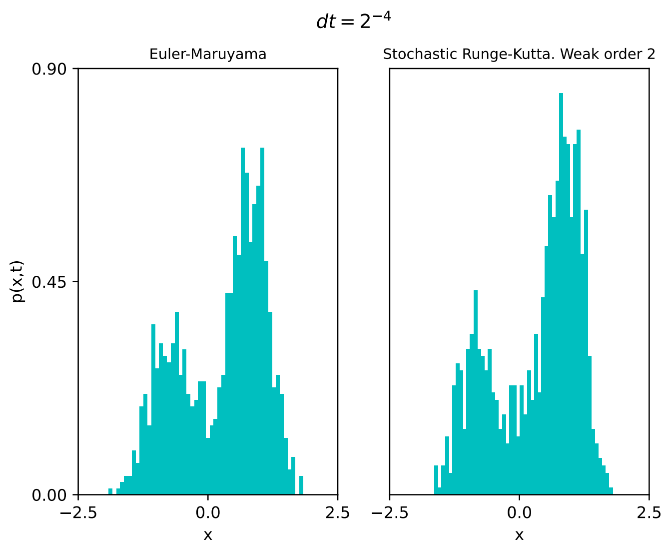


Figure 28: Exercise 8.6. Solutions for $\Delta t = 2^{-4}$.

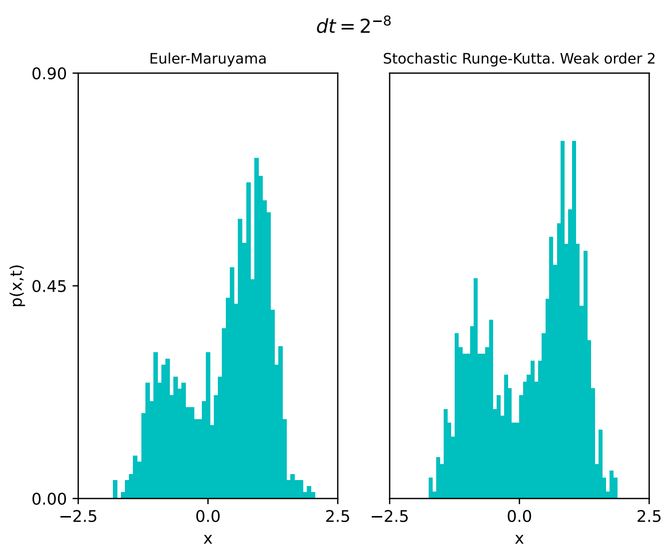


Figure 29: Exercise 8.6. Solutions for $\Delta t = 2^{-8}$.

Exercise 8.7. *Consider the second-order SDE model that fits under the family of models suitable for the Verlet integration scheme with* $f(x) = x - x^3, b(x) = x$, $\eta = 1$, *and* $q = 1$.

1. *Write the Itô SDE corresponding to the given model.*
2. *Simulate 10000 trajectories on the interval* $[0, 10]$ *from the model using (i) the Leapfrog Verlet algorithm and (ii) the Euler-Maruyama method. Use a step size of* $\Delta t = 1/10$ *and the initial value* $\mathbf{x} = (1, 0)$. *Compare the results.*
3. *Study the behavior of the two methods as* Δt *increases.*

Solution

1. The SDE system is
$$d\mathbf{x} = \mathbf{f}(\mathbf{x})dt + \mathbf{L}(\mathbf{x})d\beta, \tag{173}$$
where $\mathbf{x} = [x, v]^T$, $\mathbf{f}(\mathbf{x}) = [v, -\eta b^2(x)v + f(x)]^T$ and $\mathbf{L} = [0, b(x)]^T$.

2. Figures 30, 31 and 32 show the histograms for the step sizes $\Delta t = 0.1, 0.01, 0.001$. Here we can see how the Euler-Maruyama method approximates the Verlet method when the step size decreases. It implies that Verlet method converges faster than Euler-Maruyama. Additionality, there exists a bimodality near to the equilibrium points $x = -1, 1$, $v = 0$ of the deterministic term.

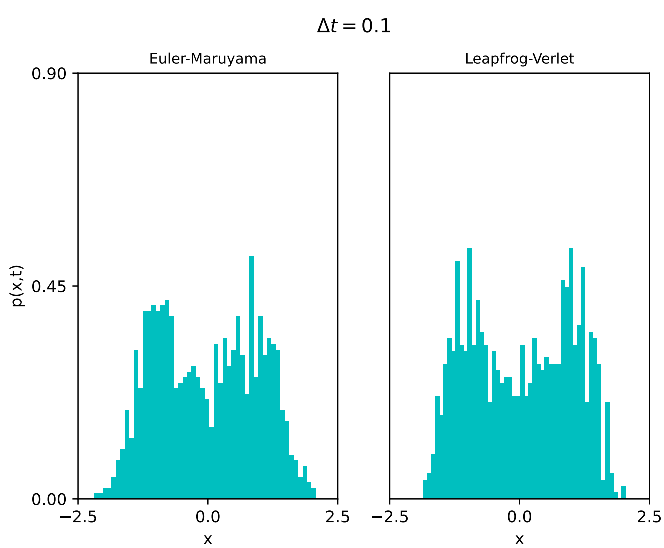


Figure 30: Exercise 8.7. Solutions for $\Delta t = 0.1$.

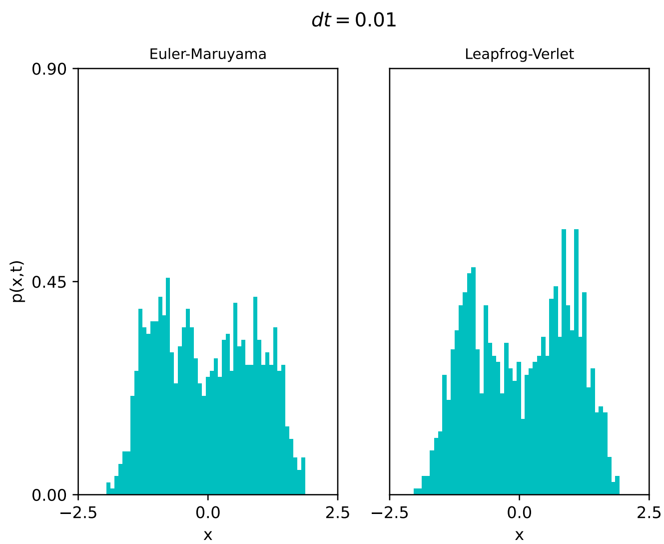


Figure 31: Exercise 8.7. Solutions for $\Delta t = 0.01$.

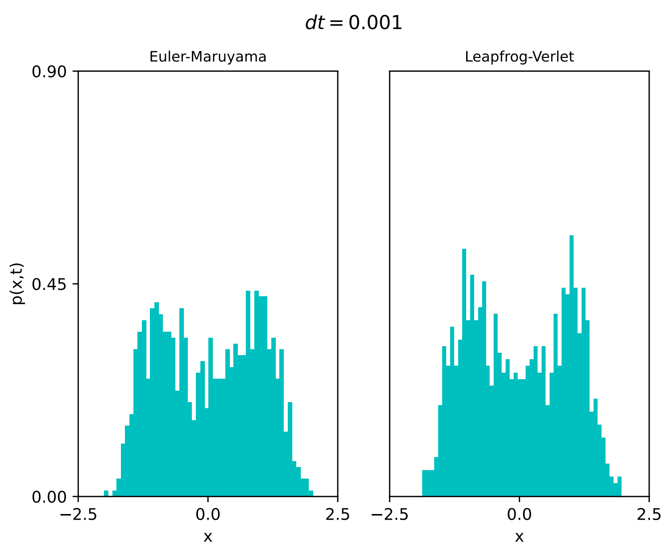


Figure 32: Exercise 8.7. Solutions for $\Delta t = 0.001$.

9 Approximation of Nonlinear SDEs

Exercise 9.1. *Gaussian approximation of SDEs:*

1. *Form a Gaussian assumed density approximation to the SDE in Equation (8.134) in the time interval* $t \in [0, 5]$ *and compare it to the exact solution. Compute the Gaussian integrals numerically on a uniform grid.*

2. *Form a Gaussian assumed density approximation to Equation (8.133) and numerically compare it to the histogram obtained in Exercise 8.1.*

Solution

Exercise 9.2. *Derive a similar series of moments as in Example 9.16 for the sine diffusion model*

$$dx = \sin(x) dt + d\beta,$$

where β *is a standard Brownian motion, and compare it to moments computed from the Euler-Maruyama method.*

Solution

Exercise 9.3. *Derive the local linearization for the model given in Equation (9.124) and numerically compare it to a cubature-based sigma-point approximation.*

Solution

Exercise 9.4. *Use a finite-differences approximation to the FPK equation for the model given in Equation (9.124):*

1. *Select the same basis as in Example 9.20 and write out the evolution equation.*

2. *Solve it numerically and sketch the evolution of the probability density with a suitably chosen initial condition. Compare the result to the Euler-Maruyama method.*

Solution

Exercise 9.5. *Implement a pathwise series approximation to the model (9.124) using the same basis as in Example 9.23.*

Solution

10 Filtering and Smoothing Theory

Exercise 10.1. *Consider the Beneš filtering problem in Example 10.8:*

1. *Compute the normalized posterior distribution by evaluating the normalization constant in Equation (10.36).*
2. *Show that the normalized posterior distribution solves the Kushner-Stratonovich equation.*
3. *Form the corresponding unnormalized distribution $q(x,t)$ and show that it solves the Zakai equation.*

Solution

Exercise 10.2. *Consider the Kushner-Stratonovich equation for the Ornstein-Uhlenbeck model:*

1. *Write down the Kushner-Stratonovich equation for the model*

$$\begin{aligned} \mathrm{d}x &= -\lambda x \; \mathrm{d}t + \mathrm{d}\beta \\ \mathrm{d}z &= x \; \mathrm{d}t + \mathrm{d}\eta \end{aligned}$$

   where β and r are independent standard Brownian motions with diffusion coefficients q and r.

2. *Show that the Kalman-Bucy filter to the Ornstein-Uhlenbeck model in Example 10.10 solves the Kushner-Stratonovich equation.*

Solution

Exercise 10.3. *Write down the equations of the Kalman-Bucy filter for the car tracking model in Example 10.3.*

Solution

Exercise 10.4. *Continuous-time approximate nonlinear filtering: Consider the model*

$$\begin{aligned} \mathrm{d}x &= \tanh(x)\mathrm{d}t + \mathrm{d}\beta \\ \mathrm{d}z &= \sin(x)\mathrm{d}t + \mathrm{d}\eta \end{aligned}$$

where β and η are independent Brownian motions with diffusion coefficients q and $r = 1$, respectively.

1. *Write down the extended Kalman-Bucy filter for this model.*
2. *Simulate data from the model over a time span $[0, 5]$ with $\Delta t = 1/100$, and implement the filtering method numerically. How does it work, and how could you improve the filter?*

Solution

Exercise 10.5. *Show that the Kalman filters in Examples 10.19 and 10.21 are equivalent.*

Solution

Exercise 10.6. *Find the Kalman filter for the car tracking model in the resources of the book Särkkä (2013). Implement the continuous-discrete Kalman filter for the same problem and check that the results match.*

Solution

Exercise 10.7. *Recall the smartphone tracking Example 3.5, where the three-dimensional orientation of the device was modeled with the help of observations of acceleration and angular velocity in the local coordinate frame of the device. Consider that you have (noisy) observations provided by a gyroscope and accelerometer at 100 Hz (you can assume t to be constant).*

1. *Write the corresponding state-space model for the smartphone tracking problem. How would you formulate the model in terms of the two data sources. Also recall that the SDE model was solved in Exercise 6.2.*
2. *Formulate the filtering problem that provides a solution to the tracking problem.*
3. *Implement the filter numerically and check your solution by simulating data (you can assume the device to rotate with a constant speed around one axis in your simulation).*

Solution

Exercise 10.8. *Consider the Black-Scholes model from Example 3.8:*

1. *Formulate the corresponding state-space model, where the state is observed through a log-normal measurement model.*
2. *Formulate the filtering solution to the state-space model.*
3. *Derive the corresponding smoother.*

Solution

Exercise 10.9. *Show that the Rauch-Tung-Striebel smoothers in Examples 10.29 and 10.33 are equivalent.*

Solution

Exercise 10.10. *Show that the smoother for the Ornstein-Uhlenbeck model given in Example 10.33 solves the Leondes equation (Algorithm 10.30).*

Solution

Exercise 10.11. *Write down the Type I and Type III smoothers for Example 10.38.*

Solution

11 Parameter estimation in SDE models

Exercise 11.1. *Consider the following problem with unknown parameter θ:*

$$x_k = \theta + r_k, \quad k = 1, 2, \ldots, T,$$

where $r_k \sim \mathrm{N}\left(0, \sigma^2\right)$ are independent. Then accomplish the following:

1. *Derive the ML estimate of θ given the measurements $x_1, x_2, \ldots, x_T$*
2. *Fix $\theta = 1, \sigma = 1$ and simulate a set of data from the preceding model. Then compute the ML estimate. How close is it to the truth?*
3. *Plot the negative log-likelihood as function of the parameter θ. Is the maximum close to the true value?*

Solution

Exercise 11.2. *Assume that in Equation (11.40) we have a Gaussian prior density $p(\theta) = \mathrm{N}\left(\theta \mid 0, \lambda^2\right)$. Then accomplish the following:*

1. *Derive the MAP estimate of θ.*
2. *How does the estimate behave as function of λ?*
3. *Fix $\theta = 1, \sigma = 1, \lambda = 2$, and simulate a set of data from the preceding model. Then compute the MAP estimate. How close is it to the truth?*
4. *Plot the posterior distribution of the parameter. Is the true parameter value well within the support of the distribution?*

Solution

Exercise 11.3. *Pendiente*

Solution

Exercise 11.4. *Fill in the details in the derivation of the Ornstein-Uhlenbeck ML estimate:*

1. *Derive Equations (11.18) from (11.17).*
2. *Derive Equation (11.19).*

Solution

Exercise 11.5. *Pendiente*

Solution

Exercise 11.6. *Using the data from the preceding exercise, estimate q by numerically finding the ML estimate using the negative log-likelihood expression (11.25).*

Solution

Exercise 11.7. *Pendiente*

Solution

Exercise 11.8. *Derive the derivatives of Equation (11.25) using the matrix fraction decomposition (see, e.g., Mbalawata et al., 2013). Check numerically using the Wiener velocity model that they are correct.*

Solution

Exercise 11.9. *Pendiente*

Solution

Exercise 11.10. *Repeat Exercise 11.9 with the Itô–Taylor method in Algorithm 8.4.*

Solution

Exercise 11.11. *Repeat Exercise 11.9 by replacing the transition density with a linearization approximation given in Algorithm 9.4.*

Solution

Exercise 11.12. *Extend the Kalman filter equations in Example 10.19 so that they can be used to compute the marginal likelihood of parameters λ and q. Using simulated data, estimate the parameters from noisy observations by maximizing the marginal likelihood.*

Solution

12 Stochastic Differential Equations in Machine Learning

Exercise 12.1. *Pendiente*

Solution

Exercise 12.2. *The SDE corresponding to the Matérn covariance function with* $\nu = 3/2$ *was given in Example 12.7. Derive (starting from the spectral density function) the corresponding expression for the Matérn covariance function (12.33) with* $\nu = 5/2$.

Solution

Exercise 12.3. *The squared exponential covariance function:*

1. *Why is it not possible to write an exact finite-dimensional SDE model for the squared exponential covariance function?*
2. *The squared exponential covariance function has the spectral density function*

$$S(\omega) = \sigma^2 \sqrt{2\pi}\ell \exp\left(-\frac{\ell^2\omega^2}{2}\right)$$

   Approximate this function by a rational function by forming a Taylor series expansion in the denominator.

3. *Factor the rational function into an unstable and stable part corresponding to the transfer function (see, e.g., Särkkä et al., 2013).*
4. *Form the so-called companion matrix corresponding to the linear SDE drift function by truncating the series and collecting the coefficients. Truncate your model so that you end up with a six-dimensional SDE model.*
5. *Compare numerically the covariance function of the state-space model to the original squared exponential covariance function.*

Solution

Exercise 12.4. *Pendiente*

Solution

Exercise 12.5. *Pendiente*

Solution

Exercise 12.6. *Consider the polynomial covariance function for* $p = 2$. *Derive the SDE model corresponding to this covariance function.*

Solution

Exercise 12.7. *Gaussian process regression:*

1. *Draw 10 random points from a Gaussian with an exponential (Ornstein-Uhlenbeck) covariance structure. Choose your parameters for the magnitude, length scale, and measurement noise variance as you see fit, and start by first drawing the 10 input locations uniformly from the interval* $[0, 1]$ *Add independent Gaussian measurement noise to your realization trajectory.*

2. *Implement batch Gaussian process regression for an exponential covariance function model, where you use the data you just simulated. Visualize your result by plotting the mean and marginal variance over* $[0, 1]$ *with 100 discretization points.*

3. *The corresponding SDE model was given in Example 12.7. Write the closed-form discrete-time state-space model for this SDE.*

4. *Implement the sequential way of solving the GP regression problem by Kalman filtering and Rauch-Tung-Striebel smoothing (refer to Chapter 10 for implementation details). Check numerically that you get the same solution (up to minor numerical errors) as in the batch solution.*

Solution