Exercises 1B:

1) Prove that $-(-v) = v$ for every $v \in V$

$$-1(-v) - 1v = -1(-v+v) = -1(0) = 0$$

$-(-v)$ is therefore the additive inverse of $-v$ which is v

2) Suppose $a \in F$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$

$$av = 0, \quad av + av = 0 \quad \underbrace{(a+a)v = 0}_{\text{distributive property on scalar mult}}$$

$$a + a = a, \quad \underbrace{a + a - a}_{\text{additive inverse in } F} = a - a = 0, \quad a = 0$$

$a \in F$

$$\underbrace{\frac{1}{a}(av)}_{\text{multiplicative inverse in } F} = \underbrace{1v = v}_{\text{multiplicative identity}} = \frac{1}{a}(0) = 0$$

3) Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$

Let $v + 3x = v + 3x' = w \qquad v + 3x - v = v + 3x' - v$

$\underbrace{v - v' + 3x = 0 + 3x}_{\text{commutativity in } V} = 3x = 3x', \quad \underbrace{\tfrac{1}{3}(3x) = \tfrac{1}{3}(3x') = 1x}_{\text{multiplicative inverse in } F} = x = x'$

4) The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space:

1.20 definition: *vector space*

A *vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold.

commutativity
$u + v = v + u$ for all $u, v \in V$.

associativity
$(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and for all $a, b \in \mathbf{F}$.

additive identity
There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$. ?

additive inverse
For every $v \in V$, there exists $w \in V$ such that $v + w = 0$. ?

multiplicative identity
$1v = v$ for all $v \in V$.

distributive properties
$a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbf{F}$ and all $u, v \in V$.

one of these either

$\{\} + 0 = \{\}$

or

$\{\} + 0 = 0$

which one?

The additive identity since the empty set plus 0 is no longer empty but equal to 0

5) Show that in the definition of a vector space the additive inverse condition can be replaced with the condition that: $\underbrace{0v}_{0 \in F} = \underbrace{0}_{0 \in V}$ for all $v \in V$.

Let $v = (v_1, \dots v_n)$ $\quad v \in V$

$0v = (0v_1, \dots, 0v_n)$ by definition of scalar multiplication in V

$\forall v_i \in V, \; 0v_i \in F \quad 0v_i = 0$

$0v = (0, \dots, 0) = 0, \; 0 \in V$

6) Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t<0, \\ 0 & \text{if } t=0, \\ \infty & \text{if } t>0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t<0, \\ 0 & \text{if } t=0, \\ -\infty & \text{if } t>0. \end{cases}$$

and

$$t+\infty = \infty + t = \infty + \infty = \infty,$$
$$t+(-\infty) = (-\infty)+t = (-\infty)+(-\infty) = -\infty,$$
$$\infty + (-\infty) = (-\infty) + \infty = 0.$$

With these operations of addition and scalar multiplication, is $\mathbb{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbb{R}$?

Using these definitions of addition and scalar multiplication, we need to prove that the properties of commutivity, associativity, additive and multiplicative inverses hold

we can see that addition remains commutative $t+\infty = \infty + t = \infty \quad -\infty + t = t - \infty = -\infty$, however addition is not associative because:

given $u \neq 0$
$$(\infty + (-\infty)) + u = 0 + u = u$$
$$\infty + ((-\infty)+u) = \infty + (-\infty) = 0$$

therefore $\mathbb{R} \cup \{\infty, -\infty\}$ is not a vector space over $\mathbb{R}$

7) Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S and show that V^S is a vector space with these definitions.

For $f, g \in V^S$, the sum $f+g \in V^S$ is the function : $(f+g)(x) = f(x) + g(x)$

For $\lambda \in F$ and $f \in V^S$, the product $\lambda f \in V^S$: $(\lambda f)(x) = \lambda f(x)$

$\forall x \in S$

i) Commutativity

$S \neq \emptyset$, for $f, g \in V^S$

$: (f+g)(x) = f(x) + g(x)$

$g(x), f(x) \in V \qquad \therefore = g(x) + f(x)$

By commutivity in V which is a given as V is a vector space.

ii) assosiativity for $f, g, m \in V^S$

$: (f(x) + g(x)) + m(x) = f(x) + (g(x) + m(x))$

$f(x), g(x), m(x) \in V$ given by associativity of V

for $a, b \in F$ $f \in V^S$

$(ab) f(x) = a(bf(x))$

let $bf(x) = g(x)$, $g(x) \in V$

$\therefore a(bf(x)) = ag(x) = abf(x)$

$abf(x) \in V$

let $ab = c$ $c \in F$, $\therefore (ab)f(x) = cf(x)$

$\therefore = abf(x)$ $\therefore (ab)f(x) = a(bf(x))$

iii) additive Identity

$\exists\, 0 \in V^S$ such that $0(x) f(x) = 0$

S is non-empty so this holds.

Let $\underline{0}$ be the additive id for V $0(x) = \underline{0}$ $\underline{0} \in V$

iv) additive Inverse

$\exists\, w, v \in V^S$ such that $w(x) + v(x) = 0$

Let $(w+v)(x) = 0(x) = 0$, $\forall x \in S$

Since $w(x) + v(x) = (w+v)(x)$ S is non empty

V) multiplicative identity

let $f, I \in V^S$ such that $I(x)f(x) = f(x)$ $\forall x \in S$, $I(x), f(x) \in V$

there exists a multiplicative identity in V so let $I(x) = 1$, $1 \in V$

Vi) distributive property

let $f, g \in V^S$, $a, b \in F$, $\forall x \in S$

such that $a((f+g)(x)) = a(f(x) + g(x))$

and $= af(x) + ag(x)$

$$(a+b)f(x) = af(x) + bf(x)$$

since $f(x), g(x) \in V$ the first holds for distributivity in V

and

since $a, b \in F$ the second holds by scalar distributivity in V

8) Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above V_c is a complex vector space.

i) Commutativity

Let $(u_2,v_2),(u_1,v_1) \in V_C$, $u_1,u_2,v_1,v_2 \in V$

$$u_1+v_1 i + u_2+v_2 i = (u_1+u_2)+(v_1+v_2)i$$

Since $u_1,u_2,v_1,v_2 \in V$

$$(u_1+u_2) = (u_2+u_1), \quad (v_1+v_2) = (v_2+v_1)$$

$$u_2+v_2 i + u_1+v_1 i = (u_2+u_1)+(v_1+v_2)i$$

$\therefore$ Commutivity holds in V_c

ii) associativity

Let $(u_3,v_3),(u_2,v_2),(u_1,v_1) \in V_C$, $u_1,u_2,u_3,v_1,v_2,v_3 \in V$

then $(u_1+v_1 i + u_2+v_2 i) + u_3 + v_3 i$

$= (u_1+u_2) + (v_1+v_2)i + u_3 + v_3 i$

Since $(u_1+u_2),(v_1+v_2) \in V$

$\therefore = (u_1+u_2+u_3) + (v_1+v_2+v_3)i$

$u_1+v_1 i + (u_2+v_2 i + u_3+v_3 i)$

by the same logic

$= (u_1+u_2+u_3) + (v_1+v_2+v_3)i$

Let $a,b \in \mathbb{R}$, $u,v \in V$

$((b_1+b_2 i)(a_1+a_2 i))(u+vi)$

$= ((b_1a_1 - b_2a_2) + (b_1a_2 + a_1b_2)i)(u+vi)$

$= (b_1a_1-b_2a_2)u + (b_1a_1-b_2a_2)vi + (b_1a_2+a_1b_2)ui - (b_1a_2+a_1b_2)v$

$= (b_1a_1u - b_2a_2u - b_1a_2v - a_1b_2v)$

$+$

$(b_1a_1v - b_2a_2v + b_1a_2u + a_1b_2u)i$

$(b_1+b_2i)((a_1+a_2i)(u+vi))$

$= (b_1+b_2i)(a_1u + a_2ui + a_1vi - a_2v)$

$= (b_1+b_2i)((a_1u - a_2v) + (a_1v+a_2u)i)$

$= b_1(a_1u - a_2v) + b_1(a_1v+a_2u)i + b_2i(a_1u-a_2v) - b_2(a_1v-a_2u)$

$= (b_1a_1u - b_2a_2u - b_1a_2v - a_1b_2v)$

$+$

 $(b_1a_1v - b_2a_2v + b_1a_2u + a_1b_2u)i$

iii) Additive identity.

$0 \in V_c$ $u_1, u_2 \in V$ such that

$$u_1 + u_2 i + 0 = u_1 + u_2 i$$

$$= (u_1 + 0) + (u_2 + 0) i = u_1 + u_2 i$$

Since $u_1, u_2 \in V$ $u_1 + 0 = u_1$, $0 \in V$
$u_2 + 0 = u_2$

$\therefore \exists \; 0 \in V^c$

iv) Additive inverse

for every $v_1, v_2 \in V$, $v \in V_c$, $\exists \; u_1, u_2 \in V$, $u \in V_c$

such that $v + u = 0$, $0 \in V_c$

$$v_1 + v_2 i + u_1 + u_2 i = 0$$

$$= (v_1 + u_1) + (v_2 + u_2) i = 0$$

Since $v_1, v_2, u_1, u_2 \in V$ using additive inverses in V

$u_1 = -v_1$, $u_2 = -v_2$

such that $(v_1 - v_1) + (v_2 - v_2) i = 0$

therefore additive inverse exists in V_c

V) multiplicative identity

$\exists \; 1 \in V_c$ such that $\forall \; v \in V_c$ $v_1, v_2 \in V$

$$1(v) = 1(v_1 + v_2 i) = v_1 + v_2 i = v \quad \text{let } 1 = a + bi$$

$a, b \in V$

$$a v_1 + b v_1 i + a v_2 i - b v_2 = (a v_1 - b v_2) + (a v_2 + b v_1) i$$

let $b = 0$, $0 \in V$

$= av_1 + av_2 i$, since $v_1, v_2 \in V$

let $a = 1$, $1 \in V$

$= v_1 + v_2 i = v$

So the multiplicative inverse in V_c is the same as for in V

Vi) Distributive properties

let $a, v, u \in V_c$, $a_1, a_2, v_1, v_2, u_1, u_2 \in V$

$$(a_1 + a_2 i)(u_1 + u_2 i + v_1 + v_2 i) = (a_1 + a_2 i)((u_1 + v_1) + (u_2 + v_2)i)$$

Since $u_1, u_2, v_1, v_2 \in V$ let $w_1 = u_1 + v_1$, $w_1 \in V$

let $w_2 = u_2 + v_2$, $w_2 \in V$

$$\therefore = (a_1 + a_2 i)(w_1 + w_2 i)$$

$$= (a_1 w_1 - a_2 w_2) + (a_1 w_2 + a_2 w_1) i$$

by def of mult in V_c

$$= (a_1(u_1 + v_1) - a_2(u_2 + v_2)) + (a_1(u_2 + v_2) + a_2(u_1 + v_1))i$$

since all are in V using distributive property in V we get

$$= (a_1 u_1 + a_1 v_1 - a_2 u_2 - a_2 v_2) + (a_1 u_2 + a_1 v_2 + a_2 u_1 + a_2 v_1)i$$

$$= ((a_1 u_1 - a_2 u_2) + (a_1 u_2 + a_2 u_1)i) + ((a_1 v_1 - a_2 v_2) + (a_2 v_1 + a_1 v_2)i)$$

by associativity in V_c proved to hold earlier.

$= au + av$ therefore distributive properties hold for all $a, v, u \in V_c$

This therefore proves that, since all the properties which define a vector space hold, that V_c is a vector space.