Exercises 1A:

1) Show that $\alpha+\beta=\beta+\alpha$ for all $\alpha,\beta\in\mathbb{C}$

$$\alpha = \alpha_1+\alpha_2 i \qquad \beta=\beta_1+\beta_2 i$$

$$(\alpha_1+\alpha_2 i)+(\beta_1+\beta_2 i) = (\alpha_1+\beta_1)+(\alpha_2 i+\beta_2 i)$$

by the definition of complex addition.

since

$$\alpha_1,\beta_1\in\mathbb{R}$$

$$\therefore\ \alpha_1+\beta_1=\beta_1+\alpha_1$$

$$\alpha_2 i+\beta_2 i = (\alpha_2+\beta_2)i$$

since

$$\alpha_2,\beta_2\in\mathbb{R}\ ,\quad \alpha_2+\beta_2=\beta_2+\alpha_2$$

$$\therefore\ (\alpha_1+\beta_1)+(\alpha_2+\beta_2)i = (\beta_1+\alpha_1)+(\beta_2+\alpha_2)i$$

$$=\beta+\alpha$$

$$\therefore\ \alpha+\beta=\beta+\alpha$$

2) Show that $(\alpha+\beta)+\lambda = \alpha+(\beta+\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$

$$(\alpha+\beta)+\lambda = \left((\alpha_1+\beta_1)+(\alpha_2+\beta_2)i\right) + \lambda_1 + \lambda_2 i$$

Let $(\alpha_1+\beta_1)+(\alpha_2+\beta_2)i = \gamma$, $\gamma \in \mathbb{C}$, $\gamma = \gamma_1 + \gamma_2 i$

$$(\alpha+\beta)+\lambda = \gamma+\lambda = (\gamma_1+\lambda_1)+(\gamma_2+\lambda_2)i$$

$$\gamma_1 = \alpha_1+\beta_1, \quad \gamma_2 = \alpha_2+\beta_2$$

$$\therefore (\alpha+\beta)+\lambda = ((\alpha_1+\beta_1)+\lambda_1) + ((\alpha_2+\beta_2)+\lambda_2)i$$

Since $\alpha_1, \beta_1, \lambda_1, \alpha_2, \beta_2, \lambda_2 \in \mathbb{R}$

$$\therefore \quad = (\alpha_1+\beta_1+\lambda_1) + (\alpha_2+\beta_2+\lambda_2)i$$

using the same logic for $\alpha+(\beta+\lambda)$ we get

$$\alpha+(\beta+\lambda) = (\alpha_1+\beta_1+\lambda_1)+(\alpha_2+\beta_2+\lambda_2)i$$

$$\therefore (\alpha+\beta)+\lambda = \alpha+(\beta+\lambda)$$

3) Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$

$\alpha = \alpha_1 + \alpha_2 i \quad \beta = \beta_1 + \beta_2 i \quad \lambda = \lambda_1 + \lambda_2 i$

$$\alpha\beta = (\alpha_1 + \alpha_2 i)(\beta_1 + \beta_2 i)$$

$$= \alpha_1\beta_1 - \alpha_2\beta_2 + \beta_1\alpha_2 i + \alpha_1\beta_2 i$$

$$= (\alpha_1\beta_1 - \alpha_2\beta_2) + (\beta_1\alpha_2 + \alpha_1\beta_2) i$$

Let $\alpha\beta = \gamma$, $\gamma = \gamma_1 + \gamma_2 i$, $\gamma \in \mathbb{C}$

$$\gamma\lambda = (\gamma_1\lambda_1 - \gamma_2\lambda_2) + (\lambda_1\gamma_2 + \gamma_1\lambda_2) i$$

Since $\alpha_1, \beta_1, \alpha_2, \beta_2, \lambda_1, \lambda_2 \in \mathbb{R}$

$$\gamma\lambda = (\lambda_1\alpha_1\beta_1 - \lambda_1\alpha_2\beta_2 - \beta_1\alpha_2\lambda_2 - \alpha_1\beta_2\lambda_2) + (\lambda_1\beta_1\alpha_2 + \lambda_1\alpha_1\beta_2 + \lambda_2\alpha_1\beta_1 - \lambda_2\alpha_2\beta_2) i$$

Let $\beta\lambda = \omega$

then using the same logic

$$\alpha\omega = (\lambda_1\alpha_1\beta_1 - \lambda_1\alpha_2\beta_2 - \beta_1\alpha_2\lambda_2 - \alpha_1\beta_2\lambda_2) + (\lambda_1\beta_1\alpha_2 + \lambda_1\alpha_1\beta_2 + \lambda_2\alpha_1\beta_1 - \lambda_2\alpha_2\beta_2) i$$

$$\therefore \alpha(\beta\lambda) = (\alpha\beta)\lambda$$

4) Show that $\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta$ for all $\alpha, \beta, \lambda \in \mathbb{C}$

Let $\alpha = \alpha_1 + \alpha_2 i$, $\beta = \beta_1 + \beta_2 i$, $\lambda = \lambda_1 + \lambda_2 i$

$\alpha + \beta = (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) i$

from the definition of addition in $\mathbb{C}$

Let $\alpha + \beta = \gamma$, $\gamma = \gamma_1 + \gamma_2 i$, $\gamma \in \mathbb{C}$

$\therefore \lambda(\alpha+\beta) = \lambda\gamma = (\lambda_1\gamma_1 - \gamma_2\lambda_2) + (\lambda_1\gamma_2 + \lambda_2\gamma_1) i$

since $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2 \in \mathbb{R}$

$\therefore \lambda_1(\alpha_1+\beta_1) - \lambda_2(\alpha_2+\beta_2) = \lambda_1\alpha_1 + \lambda_1\beta_1 - \lambda_2\alpha_2 - \lambda_2\beta_2$

from the commutivity of the real numbers.

by the same:

$\lambda_1(\alpha_2+\beta_2) + \lambda_2(\alpha_1+\beta_1) = \lambda_1\alpha_2 + \lambda_1\beta_2 + \lambda_2\alpha_1 + \lambda_2\beta_1$

$\therefore \lambda(\alpha+\beta) = (\lambda_1\alpha_1 + \lambda_1\beta_1 - \lambda_2\alpha_2 - \lambda_2\beta_2) +$

$(\lambda_1\alpha_2 + \lambda_1\beta_2 + \lambda_2\alpha_1 + \lambda_2\beta_1) i$

$= \left((\lambda_1\alpha_1 - \lambda_2\alpha_2) + (\lambda_1\alpha_2 + \lambda_2\alpha_1) i\right) + \left((\lambda_1\beta_1 - \lambda_2\beta_2) + (\lambda_2\beta_1 + \lambda_1\beta_2) i\right)$

$\lambda\alpha = (\lambda_1 + \lambda_2 i)(\alpha_1 + \alpha_2 i) = (\lambda_1\alpha_1 - \lambda_2\alpha_2) + (\lambda_2\alpha_1 + \lambda_1\alpha_2) i$

$\lambda\beta = (\lambda_1\beta_1 - \lambda_2\beta_2) + (\lambda_2\beta_1 + \lambda_1\beta_2) i$

$\therefore \lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta$

5) Show that $\forall\, \alpha \in \mathbb{C}$, $\exists$ a unique $\beta \in \mathbb{C}$ s.t. $\alpha + \beta = 0$.

Let $\alpha = \alpha_1 + \alpha_2 i$, $\alpha_1, \alpha_2 \in \mathbb{R}$

There exists $-\alpha_1, -\alpha_2 \in \mathbb{R}$ by the inverse addition property of real numbers.

such that $\alpha_1 + (-\alpha_1) = 0$

$\alpha_2 + (-\alpha_2) = 0$

Let $\beta = -\alpha_1 - \alpha_2 i$

$\therefore \quad \alpha + \beta = (\alpha_1 + (-\alpha_1)) + (\alpha_2 + (-\alpha_2))i$

by defn of complex addition,

$\therefore \quad \alpha + \beta = 0 + 0i$

$\therefore \quad = 0 \iff \beta = -\alpha_1 - \alpha_2 i$

check if this is correctly used

6) Show that $\forall \alpha \in \mathbb{C}$ with $\alpha \neq 0$, $\exists$ a unique $\beta \in \mathbb{C}$ s.t. $\alpha\beta = 1$

Let $\alpha = a+bi$ $\quad \beta = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$

$\alpha\beta = \frac{a+bi}{a+bi} = 1 \qquad a, b \in \mathbb{R} \quad \beta, \alpha \in \mathbb{C}$

if $a^2+b^2 \neq 0$ then $\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2} \in \mathbb{R}$
and therefore β exists and is the multiplicative inverse of α

$a^2+b^2 = 0 \iff a$ or $b \in \mathbb{C}$

Since
$a = \frac{0 \pm \sqrt{0-4b^2}}{2}$, $b = \frac{0 \pm \sqrt{0-4a^2}}{2}$

if a or b is complex this contradicts our definition of all complex numbers

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This whole para is very woolly

7) Show that $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1

$$\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$=\left(\frac{1}{4}-\frac{\sqrt{3}}{4}i-\frac{\sqrt{3}}{4}i-\frac{3}{4}\right)\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$=\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$z=\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \quad z^{-1}=\frac{-\frac{1}{2}-\frac{\sqrt{3}}{2}i}{\frac{1}{4}+\frac{3}{4}}=\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$$

$$zz^{-1}=1 \quad \text{So} \quad \left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)^3=1$$

8) find two distinct square roots of i.

Let $\alpha = a+bi$, $\alpha \in \mathbb{C}$, $a, b \in \mathbb{R}$

Such that
$$(a+bi)(a+bi) = i$$

$$b \neq 0$$
$$a \neq 0$$

$$a^2 + 2abi - b^2 = i \qquad a^2 = b^2$$
$$2ab = 1 \qquad b = \frac{1}{2a}$$

$$a = \frac{0 \pm \sqrt{0 - 4b^2}}{2} = \pm b$$

$$b = \frac{1}{2b}, \quad b = \frac{-1}{2b}$$

$$b^2 = \frac{1}{2} \qquad b = \pm\frac{1}{\sqrt{2}} \qquad a = \pm\frac{1}{\sqrt{2}}$$

$$a = -\frac{1}{\sqrt{2}}, \; b = -\frac{1}{\sqrt{2}} \qquad a = \frac{1}{\sqrt{2}}, \; b = \frac{1}{\sqrt{2}}$$

$$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2 = \frac{1}{2} - \frac{1}{2} + \frac{2}{2}i = i$$

$$\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^2 = \frac{1}{2} - \frac{1}{2} + \frac{2}{2}i = i$$

$$\alpha = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \quad \alpha = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

9) find $x \in \mathbb{R}^4$ such that

$$(4,-3,1,7)+2x=(5,9,-6,8)$$

$$2x=(5,9,-6,8)-(4,-3,1,7)$$

$$2x=(1,12,-7,1) \quad x=\left(\tfrac{1}{2},6,-\tfrac{7}{2},\tfrac{1}{2}\right)$$

10) explain why there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2-3i,\ 5+4i,\ -6+7i)=(12-5i,\ 7+22i,\ -32-9i)$$

let $\lambda = a+bi$

$a, b \in \mathbb{R}$

$$\lambda(2-3i)=12-5i$$
$$\lambda(5+4i)=7+22i$$
$$\lambda(-6+7i)=-32-9i$$
$$2a-3ai+2bi+3b=12-5i$$
$$5a+4ai+5bi-4b=7+22i$$
$$-6a+7ai-6bi-7b=-32-9i$$

$\therefore$ $2a+3b=12$

$5a-4b=7$

$-6a-7b=-32$

$a=\frac{12-3b}{2}$

$5\left(\frac{12-3b}{2}\right)-4b=7$

$\frac{60-15b}{2}-4b=7$

$60-15b-8b=14 \Rightarrow -23b=-46$

$\therefore b=2 \quad a=3$

$-6\times(3)-7\times(2)=-18-14=-32$

$\therefore$ $-3a+2b=-5$

$4a+5b=22$

$7a-6b=-9$

✓ $-3\times3+2\times2=-5$

✓ $4\times3+5\times2=22$

✗ $7\times3-6\times2=9$

$9\neq-9$ therefore there exists no complex number λ

this is because the scalar multiple in the λ direction never crosses the point $-32-9i$

11) Show that $(x+y)+z = x+(y+z)$ for all $x, y, z \in F^n$.

let $x = (x_1, \dots, x_n)$ $\quad x_1, x_2, \dots, x_n \in F$

$y = (y_1, \dots, y_n)$ $\quad y_1, y_2, \dots, y_n \in F$

$z = (z_1, \dots, z_n)$ $\quad z_1, z_2, \dots, z_n \in F$

$$x+y = (x_1, \dots, x_n) + (y_1, \dots, y_n)$$

$$= \underline{(x_1+y_1, \dots, x_n+y_n)}$$

from the definition of addition in F^n

let $x+y = w = (w_1, \dots, w_n)$

$w_1, w_2, \dots, w_n \in F$

$w \in F^n$

$$\therefore w+z = (w_1+z_1, \dots, w_n+z_n)$$

$$= (x_1+y_1+z_1, \dots, x_n+y_n+z_n) = x+y+z$$

$x+y+z \in F^n$

$y+z = U \in F^n \Rightarrow U+x \in F^n$

$$= (y_1+z_1+x_1, \dots, y_n+z_n+x_n)$$

$$= (x_1+y_1+z_1, \dots, x_n+y_n+z_n)$$

due to Commutivity in F

thus proof also seems vaolly

$$\therefore (x+y)+z = x+(y+z)$$

12) Show that $(ab)x = a(bx)$ for all $x \in F^n$ and all $a, b \in F$.

Let $x = (x_1, \dots, x_n)$ $x \in F^n$

$ab \in F$ by Multiplication in F $\quad a, b \in F$

$\therefore (ab)x = (abx_1, \dots, abx_n)$ $(ab)x \in F^n$

$bx = (bx_1, \dots, bx_n)$ $bx \in F^n$

from Scalar mult in F^n

$abx = (abx_1, \dots, abx_n)$

13) Show that $1x = x$, $\forall x \in F^n$.

let $x = (x_1, \dots, x_n)$ $\quad x \in F^n$

$1 \in F$

$1x = (1x_1, \dots, 1x_n)$ $\quad 1x \in F^n$

from scalar mult in F^n

$1x_k = x_k$, $\forall k \in [1, n]$

from the definition of scalar multiplication in F

$\therefore$ $1x = x$

needs checking?

14) Show that $\lambda(x+y) = \lambda x + \lambda y$, $\forall \lambda \in F$ and $\forall x, y \in F^n$

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$

$$x + y = (x_1, \dots, x_n) + (y_1, \dots, y_n)$$

$$x + y = (x_1 + y_1, \dots, x_n + y_n) \quad x + y \in F^n$$

from the definition of addition in F^n

$$\lambda(x+y) = (\lambda(x_1 + y_1), \dots, \lambda(x_n + y_n))$$

From the def of Scalar mult in F^n

$$= (\lambda x_1 + \lambda y_1, \dots, \lambda x_n + \lambda y_n)$$

from the distributive property in F

$$\lambda x = (\lambda x_1, \dots, \lambda x_n)$$
$$\lambda y = (\lambda y_1, \dots, \lambda y_n)$$

from the def of Scalar mult in F^n

$\lambda x, \lambda y \in F^n$

$$\lambda x + \lambda y = (\lambda x_1 + \lambda y_1, \dots, \lambda x_n + \lambda y_n)$$

from the def of addition in F^n

$$\therefore \lambda x + \lambda y = \lambda(x+y) \quad \forall x, y \in F^n \quad \forall \lambda \in F$$

15) Show that $(a+b)x = ax + bx$, $\forall\, a,b \in F$ and all $x \in F^n$.

Let $x = (x_1, \dots, x_n)$ $\quad x \in F^n \quad a,b \in F$

$a+b \in F$

from the def addition in F

$$(a+b)x = ((a+b)x_1, \dots, (a+b)x_n)$$
$$= (ax_1 + bx_1, \dots, ax_n + bx_n)$$

from the definition of scalar mult in F^n and the distributive property in F

$$\left.\begin{aligned} ax &= (ax_1, \dots, ax_n) \\ bx &= (bx_1, \dots, bx_n) \end{aligned}\right\}$$ from def of scalar mult in F^n

$ax, bx \in F^n$

$$ax + bx = (ax_1 + bx_1, \dots, ax_n + bx_n)$$

from the def of addition in F^n

$$\therefore (a+b)x = ax + bx, \quad x \in F^n$$
$$a,b \in F$$