# COVER AND HITTING TIMES OF HYPERBOLIC RANDOM GRAPHS

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Abstract. We study random walks on the giant component of Hyperbolic Random Graphs (HRGs), in the regime when the degree distribution obeys a power law with exponent in the range $(2, 3)$. In particular, we focus on the expected times for a random walk to hit a given vertex or visit, i.e. cover, all vertices. We show that up to multiplicative constants: the cover time is $n(\log n)^2$, the maximum hitting time is $n \log n$, and the average hitting time is $n$. The first two results hold in expectation and a.a.s. and the last in expectation (with respect to the HRG).

We prove these results by determining the effective resistance either between an average vertex and the well-connected "center" of HRGs or between an appropriately chosen collection of extremal vertices. We bound the effective resistance by the energy dissipated by carefully designed network flows associated to a tiling of the hyperbolic plane on which we overlay a forest-like structure.

***Keywords***— Random walk, hyperbolic random graph, cover time, hitting time, average hitting time, target time, effective resistance, Kirchhoff index.
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## 1. Introduction

In 2010, Krioukov et al. [49] proposed the Hyperbolic Random Graph (HRG) as a model of "real-world" networks such as the Internet (also referred to as complex networks). Early results via non-rigorous methods indicated that HRGs exhibited several key properties empirically observed in frequently studied networks (such as networks of acquaintances, citation networks, networks of autonomous systems [14], etc.). Many of these properties were later established formally, among these are power-law degree distribution [36], short graph distances [1, 45] (a.k.a. small world phenomena), and strong clustering [18, 31, 36]. Many other fundamental parameters of the HRG model have been studied since its introduction (see the literature review section), however notable exceptions are key quantities concerning the behaviour of random walks. This paper is a first step in redressing this situation. The random walk is the quintessential random process, and studies of random walks have proven relevant for algorithm design and analysis; this coupled with the aforementioned appealing aspects of the HRG model motivates this research.

The (simple) random walk is a stochastic process on the vertices of a graph, which at each time step uniformly samples a neighbour of the current vertex as its next state [3, 53]. A key property of the random walk is that, for any connected graph, the expected time it takes for the walk to visit a given vertex (or to visit all vertices) is polynomial in the number of vertices in the graph. These times are known as the hitting and cover times, respectively. This ability of a random walk to explore an unknown connected graph efficiently using a small amount of memory was, for example, used to solve the undirected $s - t$ connectivity problem in logarithmic space [4]. Other properties such as the ability to sample a vertex independently of the start vertex after a polynomial (often logarithmic) number of steps (mixing time) helped random walks become a fundamental primitive in the design of randomized and approximation algorithms [56]. In particular random walks have been applied in tasks such as load balancing [65], searching [34], resource location [43], property testing [26, 50, 51], graph parameter estimation [9] and biological applications [37].

One of the central aims of this paper is to determine the order of the hitting and cover times of random walks on the HRG. To bound the hitting and cover times of the walk on the giant component we shall appeal to a connection of the former to effective resistances in the graph [53, Section 9]. The effective resistance is a metric and the resistances between all pairs of vertices uniquely determines the graph [39]. The effective resistance has also found applications to graph clustering [5], spectral sparsification [66], graph convolutional networks [2], and flow-based problems in combinatorial optimization [7, 19, 58].

One issue to keep in mind when working with HRGs is that for the most relevant range of parameters of the model (that one for which it exhibits the properties observed in "real-world" networks) the graphs obtained are disconnected with probability that tends to 1 as the order of the graph goes to infinity. Quantities such as average hitting time and commute time are not meaningful for disconnected graphs (i.e., they are trivially equal to infinity). However, again for the range of parameters we are interested in, Bode, Fountoulakis and Müller [12] showed that it is very likely the graph has a component of linear size. This result was then complemented by the first author and Mitsche [45] who showed that all other connected components were of size at most polylogarithmic in the order of the graph. This justifies referring to the linear size component as the *giant component*. With this work being a first study of characteristics of simple random walks in HRGs, it is thus natural and relevant to understand their behavior in the giant component of such graphs. This is the main challenge we undertake in this paper.

1.1. **Main Contributions.** Our main contributions are to determine several quantities related to random walks on the largest connected component $\mathcal{C}_{\alpha,\nu}(n)$ of the (Poissonized) hyperbolic random graph $\mathcal{G}_{\alpha,\nu}(n)$. We refer to this component as the *giant* and note that is known to have $\Theta(n)$ vertices a.a.s. [12]. The primary probability space we will be working in is the one induced by the HRG and we use $\mathbb{P}$ for the associated measure. We also deal with the expected stopping times of random walks, and we use bold type (e.g. $\mathbf{E}$) for the expectation with respect to the random walk on a fixed graph. We say that a sequence of events (w.r.t. the HRG) holds *asymptotically almost surely (a.a.s.)* if it occurs with probability going to 1 as $n \to \infty$. We give brief descriptions of the objects we study here, for full definitions see Section 2.

The *effective resistance* $\mathcal{R}\left(x \leftrightarrow y\right)$ between two vertices $x, y$ of a graph $G$ is the energy dissipated by a unit current flow from $x$ to $y$ when all edges have unit resistances, see Section 2.5 for a formal definition. The sum of all resistances in $G$ is the *Kirchhoff index* $\mathcal{K}(G)$, this has found uses in centrality [54], noisy consensus problems [64], and social recommender systems [70]. Our first result shows the expected effective resistance between two vertices of the giant chosen uniformly at random is bounded, and gives the expected order of the Kirchhoff index.

**Theorem 1.** *For any $\frac{1}{2} < \alpha < 1$ and $\nu > 0$, if $\mathcal{C} := \mathcal{C}_{\alpha,\nu}(n)$, then*

$$\mathbb{E}(\mathcal{K}(\mathcal{C})) = \Theta(n^2), \qquad \textit{and} \qquad \mathbb{E}\Big(\frac{1}{|V(\mathcal{C})|^2} \sum_{u,v \in V(\mathcal{C})} \mathcal{R}\left(u \leftrightarrow v\right)\Big) = \Theta(1).$$

For a graph $G$ we let $\mathbf{E}_x^G[\tau_y]$ denote the expected time for a random walk started from $x \in V(G)$ to first visit $y \in V(G)$. The target time $t_{\odot}(G)$ of $G$ (also known as Kemeny's constant) is the expected time for a random walk to travel between two vertices chosen according to the stationary distribution, that is $t_{\odot}(G) = \sum_{u,v\in V(G)} \mathbf{E}_u^G[\tau_v]\pi(u)\pi(v)$, see Section 2.4. Our next result shows that this "average" hitting time on the giant of the HRG is of order $n$ in expectation.

**Theorem 2.** *For any $\frac{1}{2} < \alpha < 1$ and $\nu > 0$, if $\mathcal{C} := \mathcal{C}_{\alpha,\nu}(n)$, then $\mathbb{E}\big(t_{\odot}(\mathcal{C})\big) = \Theta(n)$.*

Let $t_{\mathsf{hit}}(G) := \max_{x,y\in V(G)} \mathbf{E}_x^G[\tau_y]$ denote the *maximum hitting time*, and the *cover time* $t_{\mathsf{cov}}(G)$ be the expected time for the walk to visit all vertices of $G$ (taken from a worst case start vertex). We show that both of these quantities concentrate on the giant of the HRG.

**Theorem 3.** *For any $\frac{1}{2} < \alpha < 1$ and $\nu > 0$, if $\mathcal{C} := \mathcal{C}_{\alpha,\nu}(n)$, then a.a.s. and in expectation*

$$t_{\mathsf{hit}}(\mathcal{C}) = \Theta(n \ln n) \qquad \textit{and} \qquad t_{\mathsf{cov}}(\mathcal{C}) = \Theta(n \ln^2 n).$$

The result above also shows that the maximum resistance between two vertices of the giant is $\Theta(\log n)$ a.a.s., compared to $\Theta(1)$ for a typical pair by Theorem 1. This discrepancy between the maximum and the average resistances is also seen in graph distances in the giant, as the maximum and average distances are $\Theta(\log n)$ [61] and $\Theta(\log\log n)$ [1] a.a.s., respectively. Interestingly, there are enough pairs of vertices with resistance matching the maximum to ensure that the cover time is a factor $\Theta(\log^2 n)$ larger than the average hitting time, many random graphs (eg., Erdős-Rényi, preferential attachment) are expanders and do not have this feature.

Stating additional contributions of this paper, as well as providing more detail about those already stated, require a bit more terminology and notation which we introduce below after discussing the related literature.

1.2. **Further Related Work and Our Techniques.** Over the last two decades the cover time of many random graph models has been determined. These networks include the binomial random graph [20, 22], random geometric graph [23], preferential attachment model [21], configuration model [25], random digraphs [24] and the binomial random intersection graph [11]. These results were all proven using Cooper and Frieze's *first visit lemma*, see the aforementioned papers or [59]. This result is based on expressing the probability that a vertex has been visited up-to a given time by a function of the return probabilities. One (simplified) condition required to easily apply the first visit lemma is that $t_{\mathsf{rel}} \cdot \max_{v\in V} \pi(v) = o(1)$, where $t_{\mathsf{rel}}$ and $\pi$ are the relaxation time and stationary distribution of the lazy random walk. However inserting the best known bounds on $t_{\mathsf{rel}}$ and $\max_{v\in V} \pi(v)$ for the HRG, by [46] and [36] respectively, gives $t_{\mathsf{rel}} \cdot \max_{v\in V} \pi(v) \leqslant (n^{2\alpha-1}\log n)\cdot n^{\frac{1}{2\alpha}-1+o(1)}$ which is not $o(1)$ for any $1/2 \leqslant \alpha \leqslant 1$. Another obstruction to (easily) applying the first visit lemma (or indeed recovering the leading constant for the cover time more generally) is that the large mixing time and irregular structure of the HRG make it very difficult to calculate the expected number of returns to a vertex before the walk mixes.

Given the perceived difficulty in determining the cover time using the return probabilities as described above, the approach taken in this paper is to determine the hitting and cover times via the effective resistances $\{\mathcal{R}\,(u \leftrightarrow v)\}_{u,v\in V}$, see Section 2.5. There is an intimate connection between reversible Markov chains and electrical networks as certain quantities in each setting are determined by the same harmonic equations. Classically this connection has been exploited to determine whether random walks on infinite graphs are transient or recurrent [57, Chapter 2], and more recently the effective resistance metric has been understood to relate the blanket times of random walks on finite graphs to the Gaussian free field [28]. The main connection we shall use is that the commute time (sum of hitting times in either direction) between two vertices is equal to the number of edges times the effective resistance between the two points [17, 68]. This result has been used to bound hitting and cover times in several random graph models, notably in the binomial random graph [40, 67] and the geometric random graph [8]. Luxburg et al. [69] recently refined a previous bound of Lovász [56] to give

$$\left|\mathcal{R}\,(u \leftrightarrow v) - \frac{1}{d(u)} - \frac{1}{d(v)}\right| \leqslant \frac{t_{\mathsf{rel}}+2}{d_{\mathsf{min}}}\left(\frac{1}{d(u)} + \frac{1}{d(v)}\right), \tag{1}$$

for any non-bipartite graph $G$ with minimum degree $d_{\mathsf{min}}$ and any $u, v \in V(G)$. For the HRG with parameter $1/2 < \alpha < 1$, with high probability, $t_{\mathsf{rel}} \geqslant n^{2\alpha-1}/\log n$ [46] and the average degree is constant - thus (1) does not give a good bound. Instead, we construct a flow explicitly from vertices in peripheries of the hyperbolic disk to the well-connected core around the origin.

Our construction of the flow uses a tiling of the hyperbolic plane, in this regard it bears some similarity to how various authors have obtained bounds on the diameter of the HRG. However, when constructing a desirable flow one often needs multiple paths (as opposed to just one when bounding the diameter) or else the energy dissipated by the flow could be too large to get a tight bound on the effective resistance. Abdullah et al. [1] showed that hyperbolic random graphs of expected size $\Theta(n)$ have typical distances of length $\Theta(\log\log n)$ (within the same component), in contrast we show that typical resistances are $\Theta(1)$. The diameter of the HRG was recently [61] determined to be $\Theta(\log n)$, we show that the maximum effective resistance is also $\Theta(\log n)$.

Since their introduction in 2010 [49], hyperbolic random graphs have been studied by various authors. Apart from the results already mentioned (power-law degree distribution, short graph distances, strong clustering, giant component, spectral gap and diameter), connectivity was investigated by Bode et al. [13]. Further results exist on the number of $k$-cliques and the clique number [32], the existence of perfect matchings and Hamilton cycles [29], the tree-width [10] and sub-tree counts [62]. Two models, commonly considered closely related to the hyperbolic random graphs, are scale-free percolation [27] and geometric inhomogeneous random graphs [15].

Few random processes on HRGs have been rigorously studied. Among the notable exceptions is the work by Linker et al. [55] which studies the contact processes in the HRG model. Bootstrap percolation has been studied by Candellero and Fountoulakis [16] and Marshall et al. [60], and, for geometric inhomogeneous random graphs, by Koch and Lengler [48].

To the best of our knowledge, there is no previous work that explicitly studies simple random walk on HRGs, although they have been analyzed on infinite versions of HRGs. Specifically, Heydenreich et al. study transience and recurrence of random walks in the scale-free percolation model [38] (also known as heterogeneous long-range percolation) which is a "lattice" version of the HRG model. For similar investigations, but for more general graphs on Poisson point processes, see [35]. Additionally the first author, Linker, and Mitsche [44] have studied a dynamic variant of the HRG generated by stationary Brownian motions.

## 2. Preliminaries

In this section we introduce notation, define some objects and terms we will be working with, and collect, for future reference, some known results concerning them. We recall a large deviations bound in Section 2.2, we then describe the HRG model in Section 2.3, random walks in Section 2.4, and finally electrical networks in Section 2.5.

### 2.1. Graphs.

We shall follow standard notation i.e., a graph $G$ has a vertex set $V(G)$ and edge set $E(G)$. We use $d_G(u,v)$ to denote the graph distance between two vertices $u, v \in V(G)$, let $N(v) := \{u \in V \mid d_G(u,v) = 1\}$ denote the neighbourhood of a vertex, and let $d(v) = |N(v)|$.

### 2.2. Poisson Random Variables.

We will be working with a Poissonized model, where the number of points within a given region is Poisson-distributed. Thus, we will need some elementary results for Poisson random variable. The first is a (Chernoff) large deviation bound.

**Lemma 4.** *Let $P$ have a Poisson distribution with mean $\mu$. The following holds*

*(i)* $\mathbb{P}(P \leqslant \frac{1}{2}\mu) \leqslant e^{-\frac{1}{8}\mu}$.

*(ii) If* $\delta \geqslant e^{\frac{3}{2}}$, *then* $\mathbb{P}(P \geqslant \delta\mu) \leqslant e^{-\frac{1}{2}\delta\mu}$.

*Proof.* The first part is a direct application of [6, Theorem A.1.15]. For the second part, by the cited result, note that if $\chi > 0$, then

$$\mathbb{P}(P \geqslant (1+\chi)\mu) \leqslant \big(e^{\chi}(1+\chi)^{-(1+\chi)}\big)^{\mu}.$$

Taking $\chi := \delta - 1$ above (also noticing that by hypothesis on $\delta$, we have $\chi > 0$), we get $e^{\chi}(1+\chi)^{-(1+\chi)} = \frac{1}{e}(e/\delta)^{\delta} \leqslant e^{-\frac{1}{2}\delta}$ and the desired conclusion follows. □

We need the following crude but useful bound on the raw moments of Poisson random variables several times when bounding various expectations.

**Lemma 5.** *Let $X$ be a Poisson random variable with mean $\mu$. Then, for any real $\kappa \geqslant 1$, we have* $\mathbb{E}(X^{\kappa}) \leqslant \mu^{\kappa} \cdot 5\big(40 \cdot \min\big\{\frac{\kappa}{5\mu}, 1\big\}\big)^{\kappa}$.

*Proof.* Observe that for a non-negative random variable $X$ and any $x_0 > 0$ we have

$$\mathbb{E}X^{\kappa} = \sum_{x \in \mathbb{N}} \mathbb{P}(X^{\kappa} > x) \leqslant x_0^{\kappa} + \sum_{p \in \mathbb{N}} 2^{p\kappa} x_0^{\kappa} \cdot \mathbb{P}(X^{\kappa} > 2^{p\kappa} x_0^{\kappa}) = x_0^{\kappa} \cdot (1 + \sum_{p \in \mathbb{N}} 2^{p\kappa} \cdot \mathbb{P}(X \geqslant 2^{p} x_0)).$$

Recall that $\mathbb{P}(P \geqslant \delta\mu) \leqslant e^{-\frac{1}{2}\delta\mu}$ for any $\delta \geqslant e^{\frac{3}{2}}$ by Lemma 4, so choosing $x_0 = 5\mu$ gives $\mathbb{P}(X \geqslant 2^p x_0)) \leqslant e^{-2^{p-1}5\mu} := a_p$ for any $p \in \mathbb{N}$. Observe that $a_{p+1}/a_p = e^{-2^{p-1}5\mu}$ and that for any $p \geqslant \ln_2(\frac{\kappa}{5\mu}) + 1$ we have $2^\kappa \cdot e^{-2^{p-1}5\mu} \leqslant (2/e)^\kappa < 3/4$. Thus, for $\ell = \lceil \min\{\ln_2(\frac{\kappa}{5\mu}),\ 0\} + 1\rceil$,

$$\mathbb{E}X^\kappa \leqslant (5\mu)^\kappa \cdot \Big(1 + \sum_{p=0}^{\ell} 2^{p\kappa} + 2^{(\ell+1)\cdot\kappa} \cdot \sum_{p=0}^{\infty} \big(\tfrac{3}{4}\big)^p\Big) \leqslant (5\mu)^\kappa \cdot 5 \cdot 2^{(\ell+1)\cdot\kappa},$$

the result follows as $2^{(\ell+1)\cdot\kappa} = 2^{(\lceil \min\{\ln_2(\frac{\kappa}{5\mu}),\ 0\}+1\rceil+1)\kappa} \leqslant 8^\kappa \cdot \min\left\{\frac{\kappa}{5\mu},\ 1\right\}^\kappa$. □

2.3. **The HRG model.** We represent the hyperbolic plane, denoted $\mathbb{H}^2$, by points in $\mathbb{R}^2$. Elements of $\mathbb{H}^2$ are referred to by the polar coordinates $(r,\theta)$ of their representation as points in $\mathbb{R}^2$. The point with coordinates $(0,0)$ will be called the *origin* of $\mathbb{H}^2$ and denoted $O$. When alluding to a point $u \in \mathbb{H}^2$ we denote its polar coordinates by $(r_u,\theta_u)$. The hyperbolic distance $d_{\mathbb{H}^2}(u,v)$ between two points $u,v \in \mathbb{H}^2$ is determined via the Hyperbolic Law of Cosines as the unique solution of

$$\cosh d_{\mathbb{H}^2}(u,v) = \cosh r_u \cosh r_v - \sinh r_u \sinh r_v \cos(\theta_u - \theta_v).$$

In particular, the hyperbolic distance between the origin and a point $u \in \mathbb{H}^2$ equals $r_u$. For a point $p \in \mathbb{H}^2$ the ball of radius $\rho > 0$ centered at $p$ will be denoted $B_p(\rho)$, i.e.,

$$B_p(\rho) := \{q \in \mathbb{H}^2 \mid d_{\mathbb{H}^2}(p,q) < \rho\}.$$

We will work in the Poissonized version of the HRG model which we describe next. For a positive integer $n$ and positive constant $\nu$ we consider a Poisson point process on the hyperbolic disk centered at the origin $O$ and of radius $R := 2\ln(n/\nu)$. The intensity function at polar coordinates $(r,\theta)$ for $0 \leqslant r < R$ and $\theta \in \mathbb{R}$ equals

$$\lambda(r,\theta) := \nu e^{\frac{R}{2}} f(r,\theta) = n f(r,\theta) \tag{2}$$

where $f(r,\theta)$ is the joint density function of independent random variables $\theta$ and $r$, where $\theta$ is chosen uniformly at random in $[0,2\pi)$ and $r$ is chosen according to the following density function:

$$f(r) := \frac{\alpha \sinh(\alpha r)}{\cosh(\alpha R) - 1} \cdot \mathbf{1}_{[0,R)}(r) \qquad \text{where } \mathbf{1}_{[0,R)}(\cdot) \text{ is the indicator of } [0,R).$$

We shall need the following useful approximation to this density.

**Lemma 6** ([33, Equation (3)]). $f(r) = \alpha e^{-\alpha(R-r)} \cdot (1 + \Theta(e^{-\alpha R} + e^{-2\alpha r})) \cdot \mathbf{1}_{[0,R)}(r)$.

We denote the point set of the Poisson process by $V$ and we identify elements of $V$ with the vertices of a graph whose edge set $E$ is the collection of vertex pairs $uv$ such that $d_{\mathbb{H}^2}(u,v) < R$. The probability space over graphs $(V,E)$ thus generated is denoted by $\mathcal{G}_{\alpha,\nu}(n)$ and referred to as the HRG. Note in particular that $\mathbb{E}|V| = n$ since

$$\int_{B_O(R)} \lambda(r,\theta)\,\mathrm{d}\theta\mathrm{d}r = \nu e^{\frac{R}{2}} \int_0^\infty f(r)\,\mathrm{d}r = n.$$

The Hyperbolic Law of Cosines turns out to be complicated to work with when computing distances in hyperbolic space. Instead, it is more convenient to consider the maximum angle $\theta_R(r_u,r_v)$ that two points $u,v \in B_O(R)$ can form with the origin $O$ and still be within (hyperbolic) distance at most $R$ provided $u$ and $v$ are at distance $r_u$ and $r_v$ from the origin, respectively.

**Remark 7.** *Replacing in* (7) *the terms $d_{\mathbb{H}^2}(u,v)$ by $R$ and $\theta_u - \theta_v$ by $\theta_R(r_u,r_v)$, taking partial derivatives on both sides with respect to $r_u$ and some basic arithmetic gives that the mapping $r_u \mapsto \theta_R(r_u,r_v)$ is continuous and strictly decreasing in the interval* $[0,R)$. *Since $\theta_R(r_u,r_v) = \theta_R(r_v,r_u)$, the same is true of the mapping $r_v \mapsto \theta_R(r_u,r_v)$. (See* [47, Remark 2.1] *for additional details.)*

The following estimate of $\theta_R(r,r')$, due to Gugelmann, Panagiotou & Peter is very useful and accurate (especially when $R-(r+r')$ goes to infinity with $n$).

**Lemma 8** ([36, Lemma 6]). *If $0 \leqslant r \leqslant R$ and $r+r' \geqslant R$, then $\theta_R(r,r') = 2e^{\frac{1}{2}(R-r-r')}(1+\Theta(e^{R-r-r'}))$.*

We will need estimates for measures of regions of the hyperbolic plane, and specifically for the measure of balls. We denote by $\mu(S)$ the measure of a set $S \subseteq \mathbb{H}^2$, i.e., $\mu(S) := \int_S f(r,\theta)\,\mathrm{d}r\mathrm{d}\theta$. The following approximation of the measures of the ball of radius $\rho$ centered at the origin and the ball of radius $R$ centered at $p \in B_O(R)$, both also due to Gugelmann et al., will be used frequently in our analysis.

**Lemma 9** ([36, Lemma 7]). *For $\alpha > \frac{1}{2}$, $p \in B_O(R)$ and $0 \leqslant r \leqslant R$ we have*

$$\mu(B_O(r)) = e^{-\alpha(R-r)}(1+o(1)),$$
$$\mu(B_p(R) \cap B_O(R)) = \frac{2\alpha e^{-\frac{1}{2}r_p}}{\pi(\alpha-\frac{1}{2})}\big(1+\mathcal{O}(e^{-(\alpha-\frac{1}{2})r_p} + e^{-r_p})\big).$$

Next, we state a result that is implicit in [12] (later refined in [30]) concerning the size and the "geometric location" of the giant component of $\mathcal{G}_{\alpha,\nu}(n)$. First, observe that the set of vertices of $\mathcal{G}_{\alpha,\nu}(n)$ that are inside $B_O(\frac{R}{2})$ are within distance at most $R$ of each other. Hence, they form a clique and in particular belong to the same connected component. The graph induced by the connected component of $\mathcal{G}_{\alpha,\nu}(n)$ to which the vertices in $B_O(\frac{R}{2})$ belong will be referred to as the *center component* of $\mathcal{G}_{\alpha,\nu}(n)$ and denoted $\mathcal{C}_{\alpha,\nu}(n)$.

**Theorem 10** ([12, Theorem 1.4],[47, Theorem 1.1]). *If $\frac{1}{2} < \alpha < 1$, then a.a.s. $\mathcal{C}_{\alpha,\nu}(n)$ has size $\Theta(n)$ and all other connected components of $\mathcal{G}_{\alpha,\nu}(n)$ are of size polylogarithmic in $n$.*

The previous result partly explains why we focus our attention on simple random walks in the center component of $\mathcal{G}_{\alpha,\nu}(n)$.

The condition $\alpha > \frac{1}{2}$ guarantees that $\mathcal{G}_{\alpha,\nu}(n)$ has constant average degree depending on $\alpha$ and $\nu$ only [36, Theorem 2.3]. If on the contrary $\alpha \leqslant \frac{1}{2}$, then the average degree grows with $n$. If $\alpha > 1$, the largest component of $\mathcal{G}_{\alpha,\nu}(n)$ is sublinear in $n$ [12, Theorem 1.4]. For $\alpha = 1$ whether the largest component is of size linear in $n$ depends on $\nu$ [12, Theorem 1.5]. Hence, the parameter range where $\frac{1}{2} < \alpha < 1$ is where HRGs are always sparse, exhibit power law degree distribution with exponent between 2 and 3 and the giant component is, a.a.s., unique and of linear size. All these are characteristics ascribed to so called 'social' or 'complex networks' which HRGs purport to model. Our main motivation is to contribute to understand processes that take place in complex networks, many of which, as already discussed in the introduction, either involve or are related to simple random walks on such networks. Thus, we restrict our study exclusively to the case where $\frac{1}{2} < \alpha < 1$, but in order to avoid excessive repetition, we omit this condition from the statements we establish.

The last known property of HRGs that we recall is that, w.h.p. the center component with a linear in $n$ number of edges.

**Lemma 11** ([46, Lemma 15]). *For $\frac{1}{2} < \alpha < 1$, w.h.p $|E(\mathcal{C}_{\alpha,\nu}(n))|$ is $O(n)$.*

2.4. **Random Walks.** The simple random walk $(X_t)_{t\geqslant 0}$ on a graph $G=(V,E)$ is a discrete time random process on $V(G)$ which at each time moves to a neighbour of the current vertex $u \in V$ uniformly with probability $1/d(u)$. We use $\mathbf{P}(\cdot) := \mathbf{P}^G(\cdot)$ to denote the law of the random walk on a graph $G$ (as opposed to $\mathbb{P}$ for the random graph). For a probability distribution $\mu$ on $V$ we let $\mathbf{P}^G_\mu(\,\cdot\,) := \mathbf{P}^G(\,\cdot \mid X_0 \sim \mu\,)$ be the conditional probability of the walk on $G$ started from a vertex sampled from $\mu$. In the case where the walk starts from a single vertex $u \in V$ we write $u$ instead of $\mu$, for example $\mathbf{E}^G_u(\,\cdot\,) := \mathbf{E}^G(\,\cdot \mid X_0 = u\,)$. We shall drop the superscript $G$

when the graph is clear from the context. We now define the *cover time* $t_{\mathsf{cov}}(G)$ of $G$ by

$$t_{\mathsf{cov}}(G) := \max_{u\in V} \mathbf{E}_u^G(\tau_{\mathsf{cov}}), \qquad \text{where} \qquad \tau_{\mathsf{cov}} := \inf\Big\{t : \bigcup_{i=0}^{t}\{X_i\} = V(G)\Big\}.$$

Similarly for $u, v \in V$ we let $\mathbf{E}_u(\tau_v)$, where $\tau_v := \inf\{t \mid X_t = v\}$, be the *hitting time* of $v$ from $u$. We shall use $\pi$ to denote the *stationary distribution* of a single random walk on a connected graph $G$, given by $\pi(v) := \frac{d(v)}{2|E|}$ for $v \in V$. Let

$$t_{\mathsf{hit}}(G) := \max_{u,v\in V} \mathbf{E}_u^G(\tau_v), \quad \text{and} \quad t_{\odot}(G) := \sum_{u,v\in V(G)} \mathbf{E}_u^G(\tau_v)\,\pi(u)\pi(v),$$

denote the *maximum hitting time* and the *target time* respectively. We define each of these times to be 0 if $G$ is either the empty graph or consists of just a single vertex. The target time $t_{\odot}(G)$, also given by $\mathbf{E}_\pi^G(\tau_\pi)$, is the expected time for a random walk to travel between two vertices sampled independently from the stationary distribution [53, Section 10.2]. We will consider the random walk on the center component $\mathcal{C} := \mathcal{C}_{\alpha,\nu}(n)$ of the HRG and so each of the expected stopping times $t_{\mathsf{cov}}(\mathcal{C}), t_{\mathsf{hit}}(\mathcal{C})$ and $t_{\odot}(\mathcal{C})$ will in fact be random variables.

We shall also refer to $\mathbf{E}_u(\tau_v) + \mathbf{E}_v(\tau_u)$ as the *commute time* between the vertices $u, v$. Let $t_{\mathsf{rel}}(G) = \frac{1}{1-\lambda_2}$ be the relaxation time of $G$, where $\lambda_2$ is the second largest eigenvalue of the transition matrix of the (lazy) random walk on $G$.

2.5. **Electrical Networks & Effective Resistance.** An electrical network, $N := (G, C)$, is a graph $G$ and an assignment of conductances $C : E(G) \to \mathbb{R}^+$ to the edges of $G$. For an undirected graph $G$ we define $\vec{E}(G) := \{\vec{xy} \mid xy \in E(G)\}$, this is the set of all possible oriented edges for which there is an edge in $G$. For some $S, T \subseteq V(G)$, a *flow* from $S$ to $T$ (or $(S,T)$*-flow*, for short) on $N$ is a function $f : \vec{E}(G) \to \mathbb{R}$ satisfying $f(\vec{xy}) = -f(\vec{yx})$ for every $xy \in E(G)$ as well as Kirchoff's node law for every vertex apart from those that belong to $S$ and $T$, i.e.

$$\sum_{u\in N(v)} f(\vec{uv}) = 0 \qquad \text{for each}\, v \in V \setminus (S \cup T).$$

A flow from $S$ to $T$ is called a *unit* flow if, in addition, its strength is 1, that is

$$\sum_{s\in S}\sum_{u\in N(s)} f(\vec{su}) = 1.$$

We say that the $(S,T)$-flow is *balanced* if

$$\sum_{u\in N(s)} f(\vec{su}) = \sum_{u\in N(s')} f(\vec{s'u}) \text{ and } \sum_{u\in N(t)} f(\vec{ut}) = \sum_{u\in N(t')} f(\vec{ut'}) \qquad \text{for all } s,s' \in S \text{ and } t,t' \in T.$$

When $S = \{s\}$ and $T = \{t\}$ we write $(s,t)$-flow instead of $(\{s\},\{t\})$-flow. Note that $(s,t)$-flows are always balanced. Given two flows $f$ and $g$ on $N := (G,C)$, we say that $g$ *can be concatenated to* $f$ if $f+g$ is a flow on $N$ and for every oriented edge $\vec{e} \in \vec{E}$ either $f(\vec{e})$ and $g(\vec{e})$ are both 0, or they have opposite signs, so $(f(\vec{e}) + g(\vec{e}))^2 \leqslant (f(\vec{e}))^2 + (g(\vec{e}))^2$.

For the network $N := (G,C)$ and a flow $f$ on $N$ define the *energy dissipated* by $f$, denoted $\mathcal{E}(f)$, by

$$\mathcal{E}(f) := \sum_{e\in\vec{E}(G)} \frac{f(e)^2}{2C(e)}, \tag{3}$$

For future reference, we state the following easily verified fact:

**Claim 12.** *Let $N := (G,C)$ be an electrical network and $S, M, T \subseteq V(G)$. Let $f$ and $g$ be balanced unit $(S,M)$ and $(M,T)$ flows on $N$, respectively. Then, $g$ can be concatenated to $f$, $f+g$ is a balanced unit $(S,T)$-flow on $N$, and $\mathcal{E}(f+g) \leqslant \mathcal{E}(f) + \mathcal{E}(g)$. Moreover, the same conclusion holds if either $f$ or $g$ (but not both) are everywhere zero flows.*

For $S, T \subseteq V(G)$, the *effective resistance* between $S$ and $T$, denoted $\mathcal{R}_C(S \leftrightarrow T)$, is

$$\mathcal{R}_C(S \leftrightarrow T) := \inf\{\mathcal{E}(f) \mid f \text{ is a unit flow from } S \text{ to } T\}. \tag{4}$$

The set of conductances $C$ define a reversible Markov chain [57, Chapter 2] and in this paper we shall fix $C(e) = 1$ for all $e \in E(G)$ as this corresponds to a simple random walk. In this case we write $\mathcal{R}_G(S \leftrightarrow T)$ (or $\mathcal{R}(S \leftrightarrow T)$ if the graph is clear) instead of $\mathcal{R}_C(S \leftrightarrow T)$. The following is well known.

**Proposition 13** ([53, Corollary 10.8]). *The effective resistances form a metric space on any graph, in particular they satisfy the triangle inequality.*

Choosing a single shortest path $P$ between any two vertices $s, t$ (if one exists) in a network (with $C(e) = 1$ for each $e \in E$) and routing a unit flow down the edges of $P$ we obtain the following basic but useful result straight from the definition (4) of $\mathcal{R}(s \leftrightarrow t)$.

**Lemma 14** ([17, Lemma 3.2]). *For any graph $G = (V, E)$ and $s, t \in V$, we have $\mathcal{R}(s \leftrightarrow t) \leqslant d_G(s, t)$.*

Another very useful tool is Rayleigh's monotonicity law (RML).

**Theorem 15** (Rayleigh's Monotonicity Law [53, Theorem 9.12]). *If $\{C(e)\}_{e\in E}$ and $\{C'(e)\}_{e\in E}$ are sets of conductances on the edges of the same graph $G$ and if $C(e) \geqslant C'(e)$ for all $e \in E$, then*

$$\mathcal{R}_C(S \leftrightarrow T) \leqslant \mathcal{R}_{C'}(S \leftrightarrow T) \qquad \text{for all } S, T \subseteq V(G).$$

The *Kirchhoff index* $\mathcal{K}(G)$ of a graph $G$ is the sum of resistances in the graph, that is

$$\mathcal{K}(G) = \sum_{u,v \in V(G)} \mathcal{R}(u \leftrightarrow v).$$

The Kirchhoff index has applications in mathematical chemistry, see [63] and citing papers.

The following result allows us to relate hitting times to effective resistance.

**Lemma 16** ([53, Proposition 10.6]). *For any graph $G$ and any pair of vertices $u, v \in V$ we have*

$$\mathbf{E}_u(\tau_v) + \mathbf{E}_v(\tau_u) = 2|E(G)| \cdot \mathcal{R}(u \leftrightarrow v). \tag{Commute time identity}$$

**Conventions:** Throughout, we use standard notions and notation concerning the asymptotic behavior of sequences. If $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ are two sequences of real numbers, we write $a_n = \mathcal{O}(b_n)$ to denote that for some constant $C > 0$ and $n_0 \in \mathbb{N}$ such that $|a_n| \leqslant C|b_n|$ for all $n \geqslant n_0$. Also, we write $a_n = \Omega(b_n)$ if $b_n = \mathcal{O}(a_n)$, and $a_n = \Theta(b_n)$ if $a_n = \mathcal{O}(b_n)$ and $a_n = \Omega(b_n)$.

Unless stated otherwise, all asymptotics are as $n \to \infty$ and all other parameters are assumed fixed. Expressions given in terms of other variables that depend on $n$, for example $\mathcal{O}(R)$, are still asymptotics with respect to $n$. As we are interested in asymptotics, we only claim and prove inequalities for $n$ sufficiently large. So, for simplicity, we always assume $n$ sufficiently large. For example, we may write $n^2 > 5n$ without requiring $n > 5$.

An event, more precisely a family of events parameterized by $n \in \mathbb{N}$, is said to hold *asymptotically almost surely (a.a.s.)* if it occurs with probability going to 1 as $n \to \infty$. We say that an event holds *with high probability (w.h.p.)*, if for every $c > 0$ the event holds with probability at least $1 - \mathcal{O}(n^{-c})$. Note that the union of polynomially (in $n$) many events (where the degree of the polynomial is not allowed to depend on $c$) that hold w.h.p. is also an event that holds w.h.p. Furthermore, we say that an event holds *with extremely high probability (w.e.h.p.)*, if the event holds with probability at least $1 - n^{-\omega(1)}$. Finally, for $m \in \mathbb{N}$, we write $[m] := \{1, \ldots, m\}$.

## 3. The Effective Resistance Between Typical Vertices of the Giant

The main goal of this section is to show that in expectation, the effective resistance between two uniformly chosen vertices in the giant component of $\mathcal{G}_{\alpha,\nu}(n)$ is $\mathcal{O}(1)$. By Theorem 10 it suffices to establish the claim for the center component of $\mathcal{G}_{\alpha,\nu}(n)$.

The section is organized in four subsections. In the first one, we partition hyperbolic space into tiles and associate a forest-like structure to the collection of tiles. In Section 3.2, under some conditions on vertices $s,t$ in the center component of $\mathcal{G}_{\alpha,\nu}$ and a tilling, we construct a candidate unit $(s,t)$-flow $f_{s,t}$ whose dissipated energy will give us an upper bound on the effective resistance between vertices $s$ and $t$. Section 3.3 is devoted to establishing conditions under which one can guarantee the existence of the flow $f_{s,t}$ and determining an upper bound on the energy it dissipates. Unfortunately, for most pairs of vertices $s,t$ the existence of flow $f_{s,t}$ is not guaranteed. Significant additional effort is spent in showing that for almost all pairs of center component vertices $s,t$ there are vertices $s',t'$ for which: (i) the flow $f_{s',t}$ exists and its dissipated energy is relatively small, and (ii) the effective resistance between $s$ and $s'$ is relatively small, and similarly for the resistance between $t'$ and $t$. Finally, in Section 3.4, we use the flow constructed in the earlier subsections to determine the expectation of the effective between two uniformly selected vertices, and the expectation of the target time.

**Remark 17.** *For technical reasons we work with the center component rather than the giant, that is the component containing all vertices of $B_O(R/2)$. Thus from now on take $\mathcal{C}_{\alpha,\nu}(n)$ to be the center component rather than the giant. This does not impact the statement of any result since if $\mathcal{S}_n$ is the event that the giant ( largest component) is equal to the center component then $\mathbb{P}(\mathcal{S}_n) = 1 - e^{-\Omega(n)}$ by [12]. It follows immediately that all of our results holding a.a.s. for the center component also hold for the giant component. For statements of the form $\mathbb{E}(X(\mathcal{C}))$, where $X(\mathcal{C})$ is a random variable (function) of interest, such as cover time, on the center component the results also carry over. That is, if $\mathcal{C}'$ is the giant of the HRG then $\mathbb{E}(X(\mathcal{C}')) = (1+o(1))\mathbb{E}(X(\mathcal{C}))$. To see this note that for all the hitting, cover or resistance quantities studied here we have $0 \leqslant X(G) \leqslant |V(G)|^\kappa$ for some fixed $\kappa > 0$. Then, since $\mathbb{E}(|V(G)|^\kappa) = \mathcal{O}(n^\kappa)$ by Lemma 5,*

$$\mathbb{E}(X(\mathcal{C}')) = \mathbb{E}(X(\mathcal{C}'\mathbf{1}_{\mathcal{S}_n})) + \sqrt{\mathbb{E}|V(\mathcal{C}')^{2\kappa}| \cdot \mathbb{P}(\mathcal{S}_n)} \leqslant \mathbb{E}(X(\mathcal{C})) + n^\kappa e^{-\Omega(n)} = (1+o(1))\mathbb{E}(X(\mathcal{C})),$$

*by Cauchy Schwarz. This also holds with the roles of $\mathcal{C}'$ and $\mathcal{C}$ reversed, giving the result.*

3.1. **Tiling.** We start by constructing an infinite tiling of $\mathbb{H}^2$. A region of $\mathbb{H}^2$ between two rays emanating from the origin $O$ will be called a *sector*. A sector centered at the origin with bisector $p$ and (central) angle $\theta$ will be denoted by $\Upsilon_p(\theta)$. Formally,

$$\Upsilon_p(\theta) := \{q \in \mathbb{H}^2 \mid -\tfrac{1}{2}\theta \leqslant \theta_p - \theta_q < \tfrac{1}{2}\theta\}.$$

(The reason for using closed-open angle intervals for defining sectors is due to a minor technical convenience and mostly inconsequential for the ensuing discussion.)

A *tiling* of $\mathbb{H}^2$ is a partition of $\mathbb{H}^2$ into regions topologically equivalent to a disk. The tiling will depend on a parameter $c$ (a positive constant) and will be denoted by $\mathcal{F}(c)$. There will be a distinguished collection of tiles, called *root* tiles, corresponding to the elements of the equipartition of $B_O(\frac{R}{2})$ into $N_0$ sectors, hence each sector with central angle $2\pi/N_0$, for

$$N_0 := \lceil \pi/\theta_R(\tfrac{1}{2}(R+c), \tfrac{1}{2}(R+c)) \rceil \qquad \text{where } c > 0. \tag{5}$$

Also, let

$$N_i := 2^i N_0 \quad \text{and} \quad \theta_i := 2\pi/N_i \qquad \text{for } i \in \mathbb{N}. \tag{6}$$

The rest of the tiling is specified recursively as follows: For $i \in \mathbb{N}\setminus\{0\}$ and $j \in \{0,...,N_i-1\}$ let $\{\Upsilon_{i,2j}, \Upsilon_{i,2j+1}\}$ be an equipartition of $\Upsilon_{i-1,j}$ (thus each sector has central angle $\theta_i$ and there is a total of $N_i = 2N_{i-1}$ sectors for a given $i$). Also, for the same constant $c > 0$ as above, let

$$h_{-1} := 0, \qquad h_0 = \tfrac{1}{2}\cdot R, \qquad h_1 := \tfrac{1}{2}(R+c),$$

and then recursively define

$$h_i := \sup\{h \mid \theta_R(h,h) \geqslant \tfrac{1}{2}\theta_R(h_{i-1},h_{i-1})\} \quad \text{for } i > 1.$$

The fact that $h_i$ is well-defined is a direct consequence of the continuity and monotonicity of the mapping $h \mapsto \theta_R(h,h)$ (see Remark 7). In particular, observe that the sequence $(h_i)_{i\in\mathbb{N}}$ is monotone increasing and unbounded. We can now specify the rest of the tiling. Let the $(i,j)$-*tile* be defined by

$$T_{i,j} := \Upsilon_{i,j} \cap (B_O(h_i) \setminus B_O(h_{i-1})) \qquad \text{for } i \in \mathbb{N}\setminus\{0\}, \text{ and } j \in \{0,...,N_i-1\}.$$

Note that the previous formula remains valid even for $i = 0$. See Figure 1(a) for an illustration of the tiling.

It will also be convenient to introduce various terms that will simplify the exposition. First, a tile $T_{i,j}$ for which $j \in \{0,...,N_i-1\}$ will be referred to as a *level* $i$ tile. We say that $T_{i,j}$ is the *parent* tile of both $T_{i+1,2j}$ and $T_{i+1,2j+1}$ and refer to the latter two tiles as *children* of tile $T_{i,j}$ (root tiles are assumed to be their own parent). We define the mapping $T \mapsto \pi(T)$ that associates to tile $T$ its parent tile and denote the composition of $\pi$ with itself $s$ times by $\pi^s$. A tile that belongs to $\{\pi^s(T) \mid s \in \mathbb{N}\}$ will be called *ancestor* of $T$. A tile $T$ will be said to be a *descendant* of another tile $T'$ if the latter is an ancestor of the former. Note that by definition a tile is always an ancestor and a descendant of itself. From a geometrical perspective, the ancestors of a given tile $T$ are all those tiles that intersect the line segment between the origin and any point in $T$. We say that a tile $T'$ is a *sibling* of a tile $T$ of level $i$, if the former is obtained from the latter by rotation around the origin in either $-\theta_i$ or $+\theta_i$. If the rotation is in $-\theta_i$ (respectively, $+\theta_i$) we say that $T'$ is a clockwise (respectively, *anti-clockwise*) sibling of $T$. One last piece of terminology/notation that we will rely on is the following: Let $\{\Upsilon^0_{i,j}, \Upsilon^1_{i,j}\}$ be the equipartition of $\Upsilon_{i,j}$ (along its bisector) into two isometric sectors. We let $T^b_{i,j} := T_{i,j} \cap \Upsilon^b_{i,j}$ for $b \in \{0,1\}$. Note that $\{T^0_{i,j}, T^1_{i,j}\}$ is the "natural" equipartition of $T_{i,j}$. We refer to the $T^b_{i,j}$'s as *half-tiles*. Given a tile $T$ we call $H$ the *parent half-tile* of $T$ if it is a half-tile of the parent of $T$ and a line segment from the origin to any point in the interior of $T$ intersects $H$. We extend the concept of levels and of (clockwise/anti-clockwise) sibling to half-tiles (for the latter concept, the only difference now is that when dealing with level $i$ half-tiles rotations are in $\pm\frac{1}{2}\theta_i$). Clearly, a half-tile $H$ has two sibling half-tiles, one of which is included in the tile containing $H$, say tile $T$, and the other one which is not contained in $T$. We refer to the former half-tile, i.e., to $T \setminus H$ as the *twin* half-tile of $H$ (or twin of $H$, for short).

**Remark 18.** *There are two key facts about tiles that are worth stressing.*

*(i) Two points in a given tile are within distance at most $R$ of each other (which follows from the fact that a level $i$ tile is contained in $B_O(h_i)$ and that any two such points span an angle at the origin which is at most $\theta_i = \theta_R(h_i,h_i)$).*

*(ii) Any point in a tile is within distance at most $R$ from any point in its parent half-tile (again, this follows from the fact that a level $i$ tile and its parent half-tile are both contained in $B_O(h_i)$ and any two points in their union span an angle at the origin which is at most $\theta_i = \theta_R(h_i,h_i)$).*

3.2. **Definition of the Flow.** We now turn our attention to the construction of a unit $(s,t)$-flow, where $s$ and $t$ are any two distinct vertices in the center component $\mathcal{C}_{\alpha,\nu}(n)$ of $\mathcal{G}_{\alpha,\nu}(n)$. As already discussed, the energy dissipated by such flows yields bounds on the effective resistance between $s$ and $t$.

Throughout, we assume $\frac{1}{2} < \alpha < 1$ and let $\mathcal{G} := \mathcal{G}_{\alpha,\nu}(n)$, $V := V(\mathcal{G})$ and $E := E(\mathcal{G})$. In addition, we assume a constant $c > 0$ is given which determines a tiling $\mathcal{F}(c)$ as described in Section 3.1. All tiles and half-tiles mentioned should be understood with respect to $\mathcal{F}(c)$. Also, given a vertex $w$ of $\mathcal{G}$ we denote by $T(w)$ and $H(w)$ the (unique) tile and half-tile containing $w$, respectively.

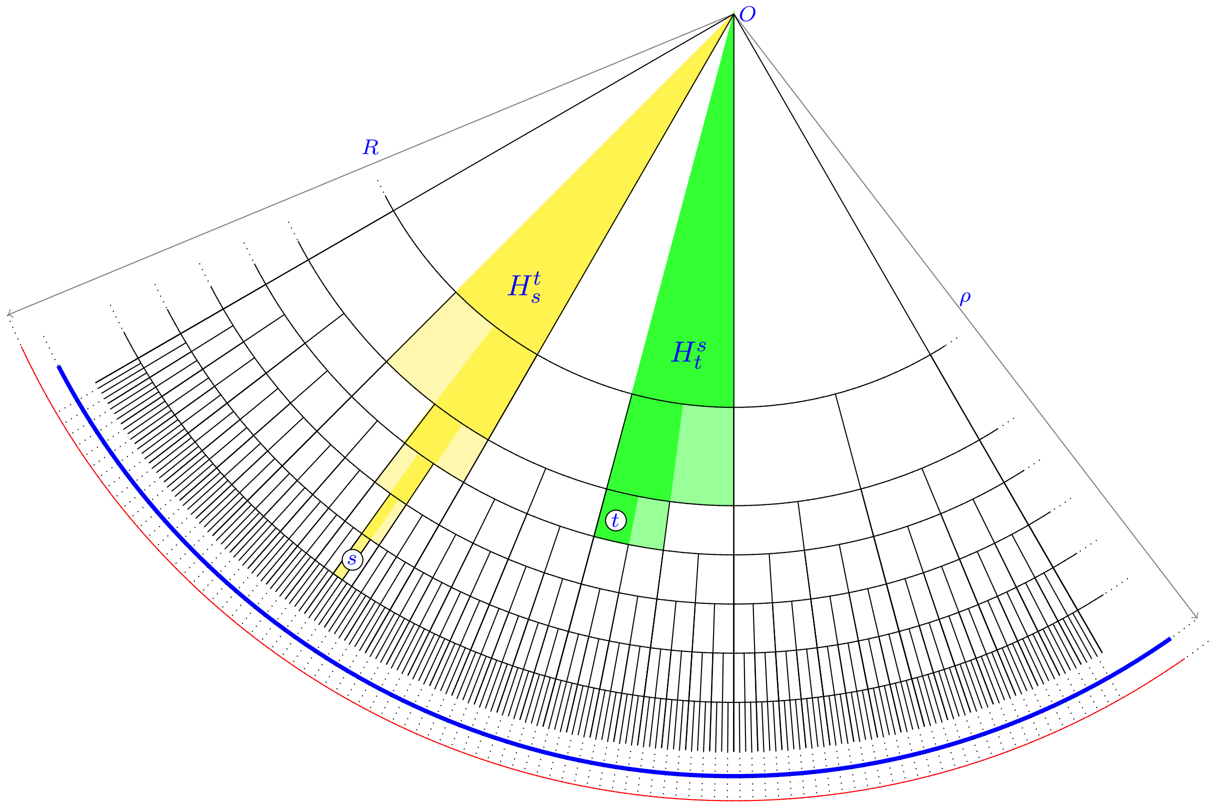


FIGURE 1. (a) Partial illustration of a tiling of $B_O(R)$ (not at scale). (b) Illustration of flow between vertices $s$ and $t$ with no common ancestor tile. Coloured regions contain vertices through which flow from $s$ to $t$ is routed. Flow is pushed radially towards the origin $O$ from a yellow shaded tile to its parent half-tile. Flow is pushed in an angular direction from dark to light yellow shaded half-tiles. Direction of flow is reversed in green shaded region.

In order to make the $(s,t)$-flow construction simpler to grasp and analyze we build it as a combination of several flows which are non-null over disjoint sets of oriented edges.

The $(s,t)$-flow will be obtained as the sum of 5 balanced flows; a source flow $f_s$, a radial flow towards the origin $f_s^t$, a mid-flow $f^{s,t}$, a radial flow away from the origin $f_t^s$, and a sink flow $f_t$, so

$$f_{s,t} := f_s + f_s^t + f^{s,t} + f_t^s + f_t. \tag{7}$$

The term $f_s$ is a $(\{s\}, V \cap H(s))$-flow that equally distributes the unit of flow that is injected in $s$ among all vertices in the half-tile $H(s)$ to which $s$ belongs. Similarly, the term $f_t$ is an $(V \cap H(t), \{t\})$-flow that sends towards $t$ all the flow at vertices in the half-tile $H(t)$ to which $t$ belongs. Formally, we define $f_s : \vec{E}(\mathcal{G}) \to \mathbb{R}$ and $f_t : \vec{E}(\mathcal{G}) \to \mathbb{R}$ as follows:

$$\begin{aligned} f_s(\vec{sv}) &:= \frac{1}{|V \cap H(s)|}, && \text{if } v \in H(s),\, v \neq s, \\ f_t(\vec{ut}) &:= \frac{1}{|V \cap H(t)|}, && \text{if } u \in H(t),\, u \neq t. \end{aligned} \tag{8}$$

The mapping $f_s$ is extended to all $\vec{E}(\mathcal{G})$ so it satisfies the antisymmetric property of flows, i.e., $f_s(\vec{uv}) = -f_s(\vec{vu})$, and so it is null for all remaining oriented edges $\vec{uv} \in \vec{E}(\mathcal{G})$ where the value of $f_s(\vec{uv})$ remains undefined. The mapping $f_t$ is similarly extended.

As we proceed to define the flow $f_{s,t}$ for all pairs of vertices $s, t$, there are several special cases that need to be considered. We introduce the next parts of the flow assuming a specific case holds (*Case (i)* below) since in other cases certain parts of the flow are null. We then discuss how to adapt the flow for the other cases.

*Case (i)*: The vertices $s$ and $t$ belong to tiles that do not share a common ancestor.

Let $H_s^t$ (respectively, $H_t^s$) be the half-tile contained in a root tile which intersects the ray between the origin and $s$ (respectively, $t$). Clearly, $H_s^t$ (respectively, $H_t^s$) belongs to an ancestor tile of $T(s)$ (respectively, $T(t)$). If $s$ or $t$ belong to a root tile, then $H_s^t$ and $H_t^s$ coincide with $H(s)$ or $H(t)$, respectively. The term $f_s^t$ will be defined so it determines a balanced flow from the half-tile to which $s$ belongs (i.e., $H(s)$) to the root half-tile $H_s^t$. The underlying idea in specifying $f_s^t$ consists in pushing the flow from vertices $H := H(s)$ to vertices in its twin half-tile $H'$ so that all vertices in the tile $T := H \cup H'$ end up with the same amount of flow, subsequently pushing the flow through the oriented edges from vertices in $T$ to vertices in the parent half-tile of $T$ so all edges involved carry exactly the same amount of flow, and then repeating the process but taking $H$ as the parent half-tile of $T$, and so on and so forth until $H$ equals $H_s^t$.

Formally, we define $f_s^t : \vec{E}(\mathcal{G}) \to \mathbb{R}$ so that if $H \neq H_s^t$ and $T$ is the tile that contains $H$ (thus, $T \setminus H$ is the twin of $H$), then

$$f_s^t(\vec{uv}) := \frac{1}{|V \cap T|} \cdot \frac{1}{|V \cap H|} \qquad \text{for all } u \in V \cap H \text{ and } v \in V \cap (T \setminus H). \tag{9}$$

If $T$ is an ancestor tile of $T(s)$ that is not a root tile and $H$ is a parent half-tile of $T$, then

$$f_s^t(\vec{uv}) := \frac{1}{|V \cap T|} \cdot \frac{1}{|V \cap H|} \qquad \text{for all } u \in V \cap T \text{ and } v \in V \cap H. \tag{10}$$

The value of $f_s^t$ is extended to all of $\vec{E}(\mathcal{G})$ in the same way as $f_s$ was extended before. Note that when $s$ belongs to a root tile, we get that $f_s^t$ is zero everywhere.

The term $f_t^s$ is defined as $f_s^t$ but replacing $H(s)$ and $H_s^t$ by $H(t)$ and $H_t^s$, respectively, and reversing the direction of the flow in each oriented edge (i.e., multiplying the value of the flow of every oriented edge by $-1$).

It only remains to specify $f^{s,t}$ which corresponds to the balanced $(V \cap H_s^t, V \cap H_t^s)$-flow obtained by equally distributing the flow at vertex $v \in V \cap H_s^t$ among the oriented edges leaving $v$ and incident to vertices in $V \cap H_t^s$. Formally, $f^{s,t} : \vec{E}(G) \to \mathbb{R}$ is defined as follows:

$$f^{s,t}(\vec{uv}) := \frac{1}{|V \cap H_s^t|} \cdot \frac{1}{|V \cap H_t^s|} \qquad \text{for all } u \in V \cap H_s^t \text{ and } v \in V \cap H_t^s. \tag{11}$$

This concludes Case (i) where $s$ and $t$ have no common ancestors and thus we need to route flow though the central clique of the center component. If the tiles containing $s$ and $t$ share a common ancestor, then we distinguish several cases.

*Case (ii)*: The vertices $s$ and $t$ belong to the same tile.

Note that in this case, since the tile a vertex resides in is considered an ancestor tile of itself, both $s$ and $t$ belong to ancestor tiles of each other. There are two subcases, firstly if $s$ and $t$ are in different half-tiles we let $H_s^t = H(s)$ and $H_t^s = H(t)$ and set $f_s^t$, $f_t^s$ to be zero everywhere. The flow $f^{s,t}$ is as described in Case (i). For the second subcase, that is when $s$ and $t$ belong to the same half-tile, we let $H_s^t = H_t^s = H(s)$ (note that $H(s) = H(t)$ in this case) and additionally set $f_s^t$, $f^{s,t}$, $f_t^s$ to be zero everywhere.

*Case (iii)*: The vertices $s$ and $t$ belong to distinct tiles that share a common ancestor.

If $s$ and $t$ belong to half-tiles that lie in a ray, we can assume w.l.o.g. that $t$ is closer to the origin than $s$. We let $H_s^t = H_t^s = H(t)$, set $f^{s,t}$ and $f_t^s$ to be zero everywhere and let $f_s$, $f_s^t$, $f_t$ be again as in Case (i). If they do not then $H_s^t$ and $H_t^s$ are two twin half-tiles whose union is the highest level tile among all common ancestor tiles of $s$ and $t$ and define $f_s$, $f_s^t$, $f^{s,t}$, $f_t^s$, $f_t$ as in Case (i).

In all cases above, each of the 5 terms $f_s, f_s^t, f^{s,t}, f_t^s, f_t$ is well-defined in the sense that no oriented edge is assigned distinct values.

Although our notation does not reflect it, the definition of $f_{s,t}$ depends on the tiling $\mathcal{F}(c)$ and thus implicitly on $c$. If we need to stress the dependency, then we will say $f_{s,t}$ is $\mathcal{F}(c)$-*compatible.* Let $\mathcal{H}_{s,t}$ be the collection of half-tiles $H$ of $\mathcal{F}(c)$ such that $H \in \{H_s^t, H_t^s\}$ or $H$ is a half-tile of an ancestor of either $T(s)$ or $T(t)$ but not of both. We now show that $f_{s,t}$ as defined above is indeed a unit $(s,t)$-flow. Although the proof is straightforward, we include it for completeness.

**Lemma 19.** *Let $c > 0$ be a constant that determines the tilling $\mathcal{F}(c)$. If every half-tile in $\mathcal{H}_{s,t}$ contains a vertex of $\mathcal{G} := \mathcal{G}_{\alpha,\nu}(n)$, then each non-zero $f \in \{f_s, f_s^t, f^{s,t}, f_t^s, f_t\}$ is an $\mathcal{F}(c)$-compatible balanced unit flow in $\mathcal{G}$.*

*Proof.* The hypothesis that $V \cap H \neq \emptyset$ guarantees that none of the denominators that appear in (9)-(11) is null.

The claim is obvious when $f$ equals either $f_s$, $f^{s,t}$, or $f_t$. We argue that the claim holds for $f_s^t$ (the case of $f_t^s$ is analogous). Assuming $f := f_s^t$ is non-zero, vertices $s$ and $t$ can not belong to the same tile.

Let $V := V(\mathcal{G})$. We need to check that Kirchoff's node law is satisfied at every vertex $V \setminus (H(s) \cup H_s^t)$, that the outgoing flow from every vertex $u$ in $H(s)$ is $1/|H(s)|$, and that the incoming flow at every vertex $u$ in $H_s^t$ is $1/|H_s^t|$. Throughout this proof, we let $f(u)$ denote the flow outgoing from $u$, i.e., $f(u) := \sum_{v \in N(u)} f(\vec{uv})$.

First, we verify that $f(u) = 1/|H(s)|$ for all $u \in H(s)$. Let $H'$ be the twin of $H(s)$ and $H$ be the parent half-tile of $T(s) := H(s) \cup H'$. Observe that by (9)-(10), since by construction of the tilling $\mathcal{F}(c)$ we have $N(u) \cap H' = H'$ and $N(u) \cap H = H$, it follows that

$$\begin{aligned}
f(u) &= \sum_{v \in V \cap N(u) \cap H'} f(\vec{uv}) + \sum_{v \in V \cap N(u) \cap H} f(\vec{uv}) \\
&= \sum_{v \in V \cap H'} \frac{1}{|V \cap T(s)|} \cdot \frac{1}{|V \cap H(s)|} + \sum_{v \in V \cap H} \frac{1}{|V \cap T(s)|} \cdot \frac{1}{|V \cap H|} \\
&= \frac{1}{|V \cap T(s)|} \cdot \frac{|V \cap H'|}{|V \cap H(s)|} + \frac{1}{|V \cap T(s)|}.
\end{aligned}$$

Since $|V \cap T(s)| = |V \cap H(s)| + |V \cap H'|$, it follows immediately that $f(u) = 1/|H(s)|$.

Similarly, if $u \in H_s^t$, one can verify that $f(u) = 1/|H_s^t|$.

If $u \in V \setminus (H(s) \cap H_s^t)$, there are two subcases to consider. In the first one, $u$ belongs to a half-tile that intersects the ray between $s$ and the origin $O$. Here, we let $H_a$ denote the parent half-tile of $T(u)$ and let $T_d$ be the tile whose parent half-tile is $H(U)$. Then, again by (9)-(11), since by construction of the tilling $\mathcal{F}(c)$ we have $N(u) \cap (T(u) \setminus H(u)) = T(u) \setminus H(u)$, $N(u) \cap H_a = H_a$ and $N(u) \cap T_d = T_d$, it follows that

$$\begin{aligned}
f(u) &= \sum_{v \in V \cap N(u) \cap (T(u) \setminus H(u))} f(\vec{uv}) + \sum_{v \in V \cap N(u) \cap H_a} f(\vec{uv}) + \sum_{v \in V \cap N(u) \cap T_d} f(\vec{vu}) \\
&= \sum_{v \in V \cap (T(u) \setminus H(u))} \frac{1}{|V \cap T(u)|} \cdot \frac{1}{|V \cap H(u)|} + \sum_{v \in V \cap H_a} \frac{1}{|V \cap T(u)|} \cdot \frac{1}{|V \cap H_a|} \\
&\quad - \sum_{v \in V \cap T_d} \frac{1}{|V \cap T_d|} \cdot \frac{1}{|V \cap H(u)|} \\
&= \frac{1}{|V \cap T(u)|} \cdot \frac{|V \cap (T(u) \setminus H(u))|}{|V \cap H(u)|} + \frac{1}{|V \cap T(u)|} - \frac{1}{|V \cap H(u)|}
\end{aligned}$$

Since $|V \cap (T(u) \setminus H(u))| = |V \cap T(u)| - |V \cap H(u)|$, we get that $f(u) = 0$ as sought.

The other subcase where $u$ does not belong to a half-tile that intersects the ray between $s$ and the origin $O$ is similar but simpler, and left to the reader. $\square$

We finally conclude that provided the minimal necessary conditions for $f_{s,t}$ to exist and be well-defined, it is indeed a unit $(s,t)$-flow and the energy it dissipates can be upper bounded in terms of the one dissipated by $f_s, f_s^t, f^{s,t}, f_t^s, f_t$.

**Lemma 20.** *Under the same assumptions of Lemma 19, we have that $f_{s,t} := f_s+f_s^t+f^{s,t}+f_t^s+f_t$ is an $\mathcal{F}(c)$-compatible unit $(s,t)$-flow in $\mathcal{G}$. Moreover,*

$$\mathcal{E}(f_{s,t}) \leqslant \mathcal{E}(f_s) + \mathcal{E}(f_s^t) + \mathcal{E}(f^{s,t}) + \mathcal{E}(f_t^s) + \mathcal{E}(f_t).$$

*Proof.* Let $\mathcal{G} = (V,E)$. The reader may check that there is no oriented edge in $\vec{E}$ for which two distinct flows among $f_s$, $f_s^t$ , $f^{s,t}$, $f_t^s$, $f_t$ take non-zero values, except when $s$ and $t$ belong to the same half-tile, in which case any orientation $\vec{e}$ of the edge between $s$ and $t$ is such that $f_s(\vec{e}) = -f_t(\vec{e})$. Thus, the flow $f_s^t$ can be concatenated to $f_s$, the flow $f^{s,t}$ to $f_s + f_s^t$, the flow $f_t^s$ to $f_s + f_s^t + f^{s,t}$, and $f_t$ to $f_s + f_s^t + f^{s,t} + f_t^s$. Recalling that, by construction, the flows $f_s$, $f_s^t$ , $f^{s,t}$, $f_t^s$, $f_t$ are balanced and applying Claim 12 repeatedly (4 times) the desired conclusion follows. □

3.3. **Validity of the Flow, and Bounds on the Energy it Dissipates.** The preceding result explains why we need to turn our attention to determining conditions under which $f_{s,t}$ exists (i.e., every half tile of $\mathcal{F}(c)$ contains a vertex), and more importantly is a good flow in the sense that it dissipates a small amount of energy. Clearly, the larger the number of vertices in each element of $\mathcal{H}_{s,t}$, the smaller the energy dissipated by the flow. To make this statement precise, we establish several intermediate results whose aim is to quantify the expected number of vertices per half-tile involved in the definition of $f_{s,t}$ and upper bound the probability that any of them contain far fewer than the number of vertices one expects to find in it.

**Claim 21.** *For any $\varepsilon > 0$, there exists a constant $c := c(\varepsilon) > 0$ such that if $h_1 := \frac{1}{2}(R + c)$,*

$$h_i := \sup\{h \mid \theta_R(h,h) \geqslant \tfrac{1}{2}\theta_R(h_{i-1}, h_{i-1})\} \qquad \textit{for all } i > 1,$$

*then*

$$|h_j - h_i - (j-i)\ln 2| \leqslant \varepsilon \quad \textit{and} \quad \frac{\theta_i}{\sqrt{e^{\varepsilon}}} \leqslant 2e^{\frac{1}{2}(R-2h_i)} \leqslant \theta_i\sqrt{e^{\varepsilon}} \qquad \textit{for all } j \geqslant i \geqslant 1.$$

*Proof.* By Lemma 8, there is a sufficiently large constant $c := c(\varepsilon) > 0$ such that if $a+b \geqslant R+c$, then for all sufficiently large $n$,

$$2e^{\frac{1}{2}(R-a-b+\varepsilon)} \geqslant \theta_R(a,b) \geqslant 2e^{\frac{1}{2}(R-a-b-\varepsilon)}.$$

Hence, for $j \geqslant i \geqslant 1$, by definition of $h_s$, we have $2^{j-i}e^{\frac{1}{2}(R-2h_j+\varepsilon)} \geqslant e^{\frac{1}{2}(R-2h_i-\varepsilon)}$, and moreover $2^{j-i}e^{\frac{1}{2}(R-2h_j-\varepsilon)} \leqslant e^{\frac{1}{2}(R-2h_i+\varepsilon)}$, so $h_j + (j-i)\ln(2) - \varepsilon \leqslant h_i \leqslant h_j + (j-i)\ln(2) + \varepsilon$

Finally, observe that from the above discussion, and given that $2h_i \geqslant R+c$ for all $i \in \mathbb{N}\setminus\{0\}$, it follows that $\theta_i \geqslant 2e^{-\frac{1}{2}\varepsilon}e^{\frac{1}{2}(R-2h_i)}$ and $\theta_i \leqslant 2e^{\frac{1}{2}\varepsilon}e^{\frac{1}{2}(R-2h_i)}$. □

Now, for $C > 0$ a large constant to be determined, let

$$\rho := R - \frac{\ln(C\frac{R}{\nu})}{1-\alpha}. \tag{12}$$

Next, we determine the expected number of vertices in each tile.

**Claim 22.** *Let $0 < \varepsilon < \frac{1}{2\alpha}\ln 2$ and $c := c(\varepsilon)$ be as in Claim 21. Then, for $V := V(\mathcal{G}_{\alpha,\nu}(n))$ and any tile $T$ of level $i \geqslant 2$ in $\mathcal{F}(c)$ such that $h_i < R$, we have*

$$\tfrac{1}{2}\nu e^{(1-\alpha)(R-h_i)} \leqslant \mathbb{E}|V \cap T| \leqslant 2\nu e^{(1-\alpha)(R-h_i)}.$$

*For $i = 0$ and $i = 1$, it holds that $E|V \cap T| = \Omega(e^{(1-\alpha)(R-h_i)})$.*

*Proof.* Assume $T$ is a tile of level $i$, then $\mu(T) = \theta_i \mu(B_O(h_i) \setminus B_O(h_{i-1}))$. By Lemma 9, we get

$$\mathbb{E}|V \cap T| = n\theta_i\mu(B_O(h_i) \setminus B_O(h_{i-1})) = (1 + o(1))n\theta_i e^{-\alpha(R-h_i)}(1 - e^{-\alpha(h_i-h_{i-1})}). \tag{13}$$

Since $n = \nu e^{\frac{R}{2}}$ by definition of $R$, the fact that $\theta_0 = 2\theta_1 = 2\pi/N_0 = \Theta(1)$, $h_0 - h_{-1} = \frac{R}{2}$ and $h_1 - h_0 = \frac{c}{2}$, both when $i = 0$ and $i = 1$ we get $\mathbb{E}|V \cap T| = \Omega(e^{-(1-\alpha)(R-h_i)})$ as claimed.

For $i \geqslant 2$, Claim 21 and the fact that $n = \nu e^{\frac{R}{2}}$ imply $e^{-\alpha(h_i-h_{i-1})} \geqslant (2e^{\varepsilon})^{-\alpha}$ and $n\theta_i \leqslant 2\nu\sqrt{e^{\varepsilon}}e^{R-h_i}$. Thus,

$$\begin{aligned}\mathbb{E}|V \cap T| &\leqslant 2\nu(1 + o(1))\sqrt{e^{\varepsilon}}(1 - (2e^{\varepsilon})^{-\alpha})e^{(1-\alpha)(R-h_i)} \\ &\leqslant \nu(1 + o(1))e^{-(\alpha-\frac{1}{2})\varepsilon}(2e^{\alpha\varepsilon} - 1)e^{(1-\alpha)(R-h_i)}.\end{aligned}$$

Since $(e^{\alpha\varepsilon}-1)^2 > 0$ and $e^{2\alpha\varepsilon} \leqslant 2$ (by hypothesis regarding $\varepsilon$), it follows that $2e^{\alpha\varepsilon}-1 \leqslant e^{2\alpha\varepsilon} \leqslant 2$. To conclude, take $n$ large enough so $1 + o(1) \leqslant e^{(\alpha-\frac{1}{2})\varepsilon}$ above. The proof of the lower bound, for $i \geqslant 2$, is analogous. ☐

Henceforth, we say that a half-tile $H$ is *sparse* if the number of vertices it contains is fewer than half the ones expected, i.e., if $|V \cap H| < \frac{1}{2}\mathbb{E}|V \cap H|$. We say that a tile $T$ is *faulty* if either one of its two associated half-tiles is sparse. Since the number of points in a region $\Omega \subseteq B_O(R)$ is distributed according to a Poisson with mean $n\mu(\Omega) = \mathbb{E}|V \cap \Omega|$, standard large deviation bounds for Poisson distributions yield the following result.

**Claim 23.** *If $c > 0$ and $V := V(\mathcal{G}_{\alpha,\nu}(n))$, then the probability that a half-tile $H$ of $\mathcal{F}(c)$ is sparse is at most $\exp(-\frac{1}{8}\mathbb{E}|V \cap H|)$ and the probability that a tile $T$ of $\mathcal{F}(c)$ is faulty is at most $2\exp(-\frac{1}{16} \cdot \mathbb{E}|V \cap T|)$.*

*Proof.* The first part is a direct application of Part (i) of Lemma 4. By definition of half-tile, the measure of $T$ is twice that of $H$, so we have $\mathbb{E}|T \cap V| = 2\mathbb{E}|H \cap V|$. Hence, the second part follows from the first part and a union bound. ☐

Using standard arguments concerning Poisson point processes we argue that as long as a half-tile $H$ is contained in $B_O(\rho)$ the number of vertices that belong to $H$ is concentrated close to its expected value, thus implying, w.h.p., that none of the tiles $T$ contained in $B_O(\rho)$ are faulty.

**Lemma 24.** *Let $0 < \varepsilon \leqslant \frac{1}{2}\ln 2$ and $c = c(\varepsilon)$ be as in Claim 22. Then, for every $d > 0$ there is a sufficiently large $C > 0$ such that, with probability $1 - o(1/n^d)$, none of the tiles of $\mathcal{F}(c)$ contained in $B_O(\rho)$ is faulty.*

*Proof.* We begin by establishing that if the tile $T$ is contained in $B_O(\rho)$, then $\mathbb{E}|V \cap T| \geqslant \frac{1}{2}CR$. Indeed, if $T$ is at level $i \geqslant 2$, the assertion is an immediate consequence of Claim 22 and our choice of $\rho$ since $T \subseteq B_O(\rho)$ implies that $h_i \leqslant \rho$. If $T$ is a root tile, then

$$\mathbb{E}|V \cap T| = \theta_0\mu(B_O(\tfrac{1}{2}R)) = (1 + o(1))(n/N_0)e^{-\alpha\frac{R}{2}} = (1 + o(1))\nu e^{(1-\alpha)\frac{R}{2}}/N_0,$$

which for large $n$, since $N_0 = \Theta(1)$, is at least $\frac{1}{2}CR$ with room to spare. Similarly, if $T$ is a level $i = 1$ tile, then $\mathbb{E}|V \cap T| = (1 + o(1))(n/N_1)e^{-\alpha\frac{R-c}{2}} = (1 + o(1))\nu e^{\alpha\frac{c}{2}}e^{(1-\alpha)\frac{R}{2}}/N_1$ which, again for $n$ large, is at least $\frac{1}{2}CR$ with plenty of slack. This completes the proof of this paragraph's opening claim.

Let $H \subseteq B_O(\rho)$ be a half-tile of $\mathcal{F}(c)$. By Claim 22 and the assertion of the previous paragraph, taking $C > 0$ large enough, we can guarantee that $\mathbb{E}|V \cap H| = \frac{1}{2}\mathbb{E}|V \cap T| \geqslant 4(d+1)R$, implying that $H$ is sparse with probability at most $\mathcal{O}(e^{-\frac{1}{2}R(d+1)}) = \mathcal{O}(1/n^{d+1})$. By a union bound, it is enough to argue that there are $\mathcal{O}(n)$ tiles contained in $B_O(\rho)$. Indeed, the number of tiles at level $i$ is $N_i$ as defined in (5) and (6). If $\ell$ is the largest integer such that $h_\ell \leqslant \rho$, then the number of tiles up to level $\ell$ is $\sum_{i=0}^{\ell} N_i = N_0 \sum_{i=0}^{\ell} 2^i < 2^{\ell+1} N_0$. Since by hypothesis $\varepsilon \leqslant \frac{1}{2}\ln 2$, Claim 21 applies, so recalling the definition of $\rho$ from (12) we get that

$$R - \ln(C\tfrac{R}{\nu}) \geqslant \rho \geqslant h_\ell \geqslant h_1 + (\ell-1)\ln 2 - \varepsilon \geqslant \tfrac{1}{2}R + (\ell-2)\ln 2.$$

Since by definition $R = 2\ln(n/\nu)$, we conclude that $\ell \leqslant \log_2(4n/(CR))$ and $2^\ell = o(n)$. □

Say that *access to* $T$ *is robust* (or $T$ *is robust* for short) if none of its ancestors (including itself) is faulty. If access to $T$ is not robust we say that *access to* $T$ *is frail* (or $T$ *is frail*). Lemma 24 implies that w.h.p. every tile $T$ contained in $B_O(\rho)$ is robust. The condition $T \subseteq B_O(\rho)$ cannot be relaxed significantly, so we will have to settle for a weaker statement. Indeed, we will show that, with non-negligible probability, if a tile is at least some sufficiently large constant apart from the boundary of $B_O(R)$, then it is robust. Henceforth, for $C' > 0$ let

$$\rho' := R - \frac{\ln(\frac{2C'}{\nu})}{1-\alpha}. \tag{14}$$

For $\rho$ and $\rho'$ given by (12) and (14) we let $\ell$ and $\ell'$ be the largest integers such that

$$h_\ell \leqslant \rho \qquad \text{and} \qquad h_{\ell'} \leqslant \rho'. \tag{15}$$

**Lemma 25.** *Let $\varepsilon$ and $c := c(\varepsilon)$ be as in Lemma 24 and $C' \geqslant 32\ln 2$. Then, for every tile $T$ of level $i$ in the tiling $\mathcal{F}(c)$ where $\ell < i \leqslant \ell'$, the following holds:*

*(i)* $\mathbb{E}|V \cap T| \geqslant C' e^{(1-\alpha)(h_{\ell'} - h_i)}$.

*(ii)* $\mathbb{P}(T \textit{ is frail} \mid \pi^{(\ell-i)}(T) \textit{ is robust}) = \mathcal{O}(\exp(-\frac{1}{23}C' e^{(1-\alpha)(h_{\ell'}-h_i)}))$.

*Proof.* For the first part, by Claim 22, the definition of $\ell'$, and as $h_i \leqslant h_{\ell'}$, we have

$$\mathbb{E}|V \cap T| \geqslant \tfrac{1}{2}\nu e^{(1-\alpha)(R-h_i)} = \tfrac{1}{2}\nu e^{(1-\alpha)(R-h_{\ell'})} \cdot e^{(1-\alpha)(h_{\ell'}-h_i)} \geqslant C' \cdot e^{(1-\alpha)(h_{\ell'}-h_i)}.$$

To establish the second part, observe that by Claim 23, we have

$$\begin{aligned}\mathbb{P}(T \text{ is robust} \mid \pi^{(\ell-i)}(\mathrm{T}) \text{ is robust}) &= \prod_{p=0}^{\ell-i-1} \big(1 - \mathbb{P}(\pi^p(T) \text{ is faulty})\big) \\ &\geqslant \prod_{p=0}^{\ell-i-1} \big(1 - 2\exp\big(-\tfrac{1}{16}\mathbb{E}|V \cap \pi^p(T)|\big)\big).\end{aligned}$$

By Item (i), if we take $C' \geqslant 32\ln 2$, then $\exp(-\frac{1}{16}\mathbb{E}|V \cap \pi^p(T)|) \leqslant \frac{1}{4}$ for all $p \in \{0, ..., \ell-i-1\}$, so recalling that $\ln(1-x) \geqslant -\frac{x}{1-x_0}$ for $0 \leqslant x \leqslant x_0 < 1$,

$$\ln \mathbb{P}(T \text{ is robust} \mid \pi^{\ell-i}(T) \text{ is robust}) \geqslant -4 \sum_{p=0}^{\ell-i-1} \exp(-\tfrac{1}{16}\mathbb{E}|V \cap \pi^p(T)|).$$

Since by hypothesis $0 < \varepsilon \leqslant \frac{1}{2}\ln 2$, Claim 21 holds, so we have that

$$h_{\ell'} - h_{i-p} = (h_{\ell'} - h_i) + (h_i - h_{i-p}) \geqslant (h_{\ell'} - h_i) + p\ln 2 - \varepsilon \geqslant (h_{\ell'} - h_i) + (p-1)\ln 2.$$

Thus, if we let $\delta := (1-\alpha)\ln 2$ and $\gamma := \frac{2^\alpha}{32}C' e^{(1-\alpha)(h_{\ell'}-h_i)} \geqslant \frac{1}{23}C' e^{(1-\alpha)(h_{\ell'}-h_i)}$, then applying Item (i) of the present lemma and noting that $\pi^p(T)$ is a level $i-p$ tile, we get

$$\ln \mathbb{P}(T \text{ is robust} \mid \pi^{\ell-i}(T) \text{ is robust}) \geqslant -4 \sum_{p=0}^{\ell-i-1} \exp\big(-\gamma e^{p\delta}\big). \tag{16}$$

Next, since the mapping $x \mapsto e^{-\gamma e^{x\delta}}$ is non-increasing, applying the substitution $y := \gamma e^{x\delta}$ gives

$$\sum_{p=1}^{\ell-i-1} \exp\big(-\gamma e^{p\delta}\big) \leqslant \int_0^\infty e^{-\gamma e^{x\delta}}\,\mathrm{d}x \leqslant \frac{1}{\delta}\int_\gamma^\infty y^{-1}e^{-y}\,\mathrm{d}y \leqslant \frac{1}{\delta\gamma}e^{-\gamma}$$

where the last inequality holds because $\int_a^\infty y^{-1}e^{-y}\,\mathrm{d}y \leqslant \frac{1}{a}\int_a^\infty e^{-y}\,\mathrm{d}y$ for $a>0$. Thus,

$$\sum_{p=0}^{\ell-i-1} \exp\big(-\gamma e^{p\delta}\big) \leqslant e^{-\gamma} + \frac{1}{\delta\gamma}e^{-\gamma} =: \xi. \tag{17}$$

Combining the above with (16) we have $\ln(\mathbb{P}(T \text{ is frail}) \mid \pi^{\ell-i}(T) \text{ is robust}) \leqslant 1-e^{-4\xi} \leqslant 4\xi$. Since $C' \geqslant 32\ln 2$, the bound in (17) gives $4\xi \leqslant 4(1+\frac{1}{\delta\gamma})e^{-\gamma} = \mathcal{O}(e^{-\gamma})$, as we recall that $\gamma \geqslant \ln 2$ and $\delta := (1-\alpha)\ln 2 > 0$. The claimed result follows. □

Lemma 24 tells us that a.a.s. every tile contained in $B_O(\rho)$ is robust, which combined with Lemma 25 implies that a tile contained in $B_O(\rho')$ has a constant probability of being robust. The significance of this last implication is given by our next result, which shows that between any two vertices $s$ and $t$ belonging to either the same or distinct robust tiles, there is a flow whose dissipated energy is $\mathcal{O}(1)$, and thence by definition of effective resistance (see 3) and the commute time identity (see Lemma 16) we deduce that $\mathbf{E}_u(\tau_{s,t}) = \mathcal{O}(|\mathbb{E}(\mathcal{C})|)$.

**Proposition 26.** *Let $\varepsilon$, $c := c(\varepsilon)$ and $C'$ be as in Lemma 25. If vertices $s$ and $t$ of $\mathcal{G} := \mathcal{G}_{\alpha,\nu}(n)$ belong to robust tiles of $\mathcal{F}(c)$ both of which are contained in $B_O(\rho')$, then $f_{s,t}$ (as defined in Section 3.2) is an $\mathcal{F}(c)$-compatible unit $(s,t)$-flow in $\mathcal{G}$ satisfying $\mathcal{E}(f_{s,t}) = \mathcal{O}(1)$.*

*Proof.* The fact that $f_{s,t}$ is a unit $(s,t)$-flow in $\mathcal{G}$ is a direct consequence of Lemma 20 (the necessary hypothesis are satisfied by definition of robustness for the tiles).

We only treat the case where $s$ and $t$ belong to tiles that do not share a common ancestor, the other cases being simpler are left to the reader.

Let $H_s^t, H_t^s$ and $f_s, f_s^t, f^{s,t}, f_t^s, f_t$ be as defined in Section 3.2. By Lemma 20, it suffices to bound each of $\mathcal{E}(f_s), \mathcal{E}(f_s^t), \mathcal{E}(f^{s,t}), \mathcal{E}(f_t^s), \mathcal{E}(f_t)$. Directly from (8) and (11), we get that

$$\mathcal{E}(f_s) = \frac{1}{|V\cap H(s)|}, \qquad \mathcal{E}(f^{s,t}) = \frac{1}{|V\cap H_s^t|}\cdot\frac{1}{|V\cap H_t^s|}, \qquad \mathcal{E}(f_t) = \frac{1}{|V\cap H(t)|}.$$

By Claim 22, observing that $H_s^t$ and $H_t^s$ are level 0 tiles, it follows that

$$\mathcal{E}(f_s) = \mathcal{O}\big(e^{-(1-\alpha)(R-h_{\ell_s})}\big), \quad \mathcal{E}(f^{s,t}) = \mathcal{O}\big(e^{-2(1-\alpha)\frac{R}{2}}\big), \quad \mathcal{E}(f_t) = \mathcal{O}\big(e^{-(1-\alpha)(R-h_{\ell_t})}\big).$$

Assume $T(s)$ is a level $\ell_s$ tile of $\mathcal{F}(c)$. Note that the set of ancestors of $T(s)$ is $\{\pi^p(T(s)) \mid p\in\{0,...,\ell_s\}\}$. For $0\leqslant p<\ell_s$, let $H_{p+1}(s)\subseteq \pi^{p+1}(T(s))$ be the parent half-tile of $\pi^p(T(s))$. By (9)-(10),

$$\mathcal{E}(f_s^t) = \sum_{p=0}^{\ell_s-1}\frac{1}{|V\cap H_p(s)|}\cdot\frac{1}{|V\cap(\pi^p(T(s))\setminus H_p(s))|} + \sum_{p=0}^{\ell_s-1}\frac{1}{|V\cap\pi^p(T(s))|}\cdot\frac{1}{|V\cap H_{p+1}(s)|}.$$

Since $T$ is by assumption a robust tile, none of its ancestors is faulty, so all terms like $|V\cap\pi^p(T(s))|$, $|V\cap H_p(s)|$, $|V\cap(\pi^p(T(s))\setminus H_p(s))|$, are at least half their expected value, in particular none of them is 0. By Claim 21, we have $h_{\ell_s-p}\leqslant h_{\ell_s}-p\ln 2+\varepsilon$ for every $0\leqslant p<\ell_s$, $h_1 := \frac{R}{2}+c$ and $h_0 := \frac{R}{2}$. Taking all the previous comments into account, using Claim 22 and observing that $\pi^p(T(s))$ is a level $\ell_s-p$ tile, we get

$$\mathcal{E}(f_s^t) = \sum_{p=0}^{\ell_s} O\big(e^{-2(1-\alpha)(R-h_{\ell_s-p})}\big) = \mathcal{O}\Big(e^{-2(1-\alpha)(R-h_{\ell_s})}\sum_{p=0}^{\ell_s}2^{-2(1-\alpha)p}\Big) = \mathcal{O}\big(e^{-2(1-\alpha)(R-h_{\ell_s})}\big).$$

Similarly, if $t$ belongs to a level $\ell_t$ tile, then $\mathcal{E}(f_t) = \mathcal{O}(e^{-2(1-\alpha)(R-h_{\ell_t})})$. Since $h_{\ell_s}, h_{\ell_t}\geqslant h_0 = \frac{R}{2}$, we conclude that

$$\mathcal{E}(f_{s,t}) = \mathcal{O}\big(e^{-(1-\alpha)(R-h_{\ell_s})} + e^{-(1-\alpha)(R-h_{\ell_t})}\big).$$

The desired conclusion follows since by hypothesis, $h_{\ell_s}, h_{\ell_t}\leqslant \rho' = R-\mathcal{O}(1)$. □

We are yet far from done. Indeed, a significant fraction of the vertices of $\mathcal{G} := \mathcal{G}_{\alpha,\nu}(n)$ fall outside the ball $B_O(\rho')$. This holds even if we only take into account the vertices belonging to the center component. To address this issue we show that in expectation (over $\mathcal{C}$ and its vertices) the graph distance between the vertices of $\mathcal{C}$ located outside $B_O(\rho')$ and a vertex belonging to a robust tile is $\mathcal{O}(1)$. To achieve this goal, it will be useful to bound the probability that none of the level $\ell'$ descendants of a given tile $A$ is robust. Unfortunately, the events corresponding to nearby tiles being robust are not independent which complicates our task.

**Lemma 27.** *Let $\varepsilon$, $c := c(\varepsilon)$, and $C'$ be as in Lemma 25 and let $\ell$ and $\ell'$ be given by* (15)*. If $A$ is a level $i := \ell' - p > \ell$ tile of the tiling $\mathcal{F}(c)$, then*

$$\mathbb{P}(\pi^{-p}(\{A\}) \textit{ is frail} \mid A \textit{ is robust}) \leqslant \exp\big(-\tfrac{1}{16}C'e^{(1-\alpha)(h_{\ell'}-h_i)}\big).$$

*Proof.* Let $P_p(A)$ denote the probability in the statement. We claim that for $p \in \{0,...,\ell'-i\}$

$$P_p(A) \leqslant \exp(-\tfrac{1}{16}C'e^{(1-\alpha)(h_{\ell'}-h_{\ell'-p})}).$$

To prove the claim we proceed by induction on $p$. For $p = 0$ the result is obvious since then $\pi^{-p}(\{A\}) = \{A\}$ so $P_p(A) = 0$. Assume now that $p > 0$. Observe that $\pi^{-1}(\{A\}) = \{A', A''\}$ where $A'$ and $A''$ are the two children of tile $A$. Note that $\pi^{-p}(\{A\}) = \pi^{-(p-1)}(\{A'\}) \cup \pi^{-(p-1)}(\{A''\})$. Since a non-faulty tile is robust provided its parent tile is robust, it follows that for $\pi^{-p}(\{A\})$ to be frail given that $A$ is robust it must happen that for every $T \in \{A', A''\}$ either $T$ is faulty or $\pi^{-(p-1)}(\{T\})$ is frail conditioned on $T$ being robust. Formally,

$$P_p(A) \leqslant \prod_{T\in\pi^{-1}(\{A\})} \big(\mathbb{P}(T \text{ is faulty}) + P_{p-1}(T)\big).$$

By the inductive hypothesis, Item (i) of Lemma 25, Claim 23, and since $h_{\ell'-p} \geqslant h_{\ell'-(p-1)} - (\ln 2 + \varepsilon)$,

$$P_p(A) \leqslant \big(3\exp\big(-\tfrac{1}{16}C'e^{(1-\alpha)(h_{\ell'}-h_{\ell'-(p-1)})}\big)\big)^2 \leqslant 9\big(\exp\big(-\tfrac{1}{16}C'e^{(1-\alpha)(h'_{\ell}-h_{\ell'-p})}\big)\big)^{2^\alpha/e^{(1-\alpha)\varepsilon}}.$$

To conclude recall that by hypothesis $\varepsilon \leqslant \frac{1}{2}\ln 2$, thus $2^\alpha/e^{(1-\alpha)\varepsilon} \geqslant 2^{\frac{1}{2}(3\alpha-1)} > 1$, so the induction claim follows provided $C'$ is large enough so $\ln(9) < \frac{1}{16}C'(2^{\frac{1}{2}(3\alpha-1)} - 1)$. $\square$

Consider a vertex $w$ of $\mathcal{G} := \mathcal{G}_{\alpha,\nu}(n)$ which might belong or not to the center component $\mathcal{C}$ of $\mathcal{G}$. In what follows, we abuse terminology and say $w$ is robust if it belongs to a robust tile. Let $w^+$ (respectively, $w^-$) be the robust vertex in $V(\mathcal{G}) \cap (B_O(h_{\ell'}) \setminus B_O(h_{\ell'-1}))$ which is clockwise (respectively, anti-clockwise) from $w$ and closest in angular coordinate to $w$. It is not hard to see that both $w^-$ and $w^+$ exist and are distinct w.e.h.p. Observe that in our definition of $w^-$ and $w^+$ we do not require $w$ being a robust vertex. In fact, $w$ might not even belong to the center component of $\mathcal{G}$. In contrast, vertices $w^-$ and $w^+$ necessarily belong to $\mathcal{C}$ (since by definition, robust vertices always belong to the center component). Henceforth, let $\Upsilon(w)$ be the smallest sector whose closure contains both $w^-$ and $w^+$.

**Lemma 28.** *Let $w$ be a vertex in the center component of $\mathcal{G} := \mathcal{G}_{\alpha,\nu}(n)$ and let $w' \in V(\mathcal{G}) \cap B_O(\rho')$ be a robust vertex which is closest (in graph distance) to $w$. Then,*

$$d(w, w') \leqslant 1 + |V(G) \cap \Upsilon(w)|.$$

*Proof.* Consider a shortest path $P_w$ between $w$ and $w'$. If the path $P_w$ contains no vertex of $\mathcal{G}$ outside $\Upsilon(w)$, then $d(w, w') \leqslant |V(\mathcal{G}) \cap \Upsilon(w)|$ and we are done. Assume then that $P_w$ contains vertices in $V(\mathcal{G}) \setminus \Upsilon(w)$. Let $u$ be the last (starting from $w$) vertex of $P_w$ that belongs to $\Upsilon(w)$. Also, let $v$ be the first vertex of $P_w$ that is not in $\Upsilon(w)$. By definition of $\Upsilon(w)$, we can assume that the angular coordinate of $w^+$ is between those of $u$ and $v$ (the argument is similar if $w^-$ was the vertex "in between" $u$ and $v$ and thus we omit it).

For the ensuing discussion it will be convenient, given two points $q, q' \in \mathbb{H}^2$, to denote by $\theta(q, q') \in [0, \pi]$ the angle spanned by $q$ and $q'$ at the origin.

First, we consider the case where the radial coordinate of $v$ is at least the one of $w^+$, i.e., $r_v \geqslant r_{w^+}$. Since $u$ and $v$ are neighbours in $P_w$, then $\theta(u,v) \leqslant \theta_R(r_u,r_v)$. On the other hand, by Remark 7, we have $\theta_R(r_u,r_v) \leqslant \theta_R(r_u,r_{w^+})$. Since $\theta(u,w^+) \leqslant \theta(u,v)$, it follows that $u$ and $w^+$ are neighbours in $\mathcal{G}$, contradicting the fact that $P_w$ is a shortest path between $u$ and a robust vertex in $B_O(\rho')$ (recall that by definition $w^+ \in B_O(\rho')$ is a robust vertex). Assume then that $r_v < r_{w^+}$ but suppose that $r_u \geqslant r_{w^+}$. Arguing as before, we now deduce that $v$ and $w^+$ are neighbours in $\mathcal{G}$, so the only vertex that $P_w$ has outside $\Upsilon(w)$ is $v$. Hence, $d(w,w^+) \leqslant 1 + |V(P) \cap \Upsilon(w)|$ and we are since $V(P) \subseteq V(\mathcal{G})$ and recalling again that $w^+ \in B_O(\rho')$ is robust so by minimality $d(w,w') \leqslant d(w,w^+)$.

We are left with the case where $r_u, r_v < r_{w^+}$. Let $i$ be the smallest integer such that $r_u, r_v < h_i$ (thus, $\max\{r_u,r_v\} \geqslant h_{i-1}$). By definition of $w^+$, the tile $T(w^+)$ to which it belongs is at level $\ell'$, thus $r_u, r_v < r_{w^+} < h_{\ell'}$ so $i \leqslant \ell'$. Since all tiles intersecting the line segment between the origin $O$ and $w^+$ are ancestors of $T(w^+)$, they must be robust tiles (because $T(w^+)$ is a robust tile). Hence, all ancestor tiles of $T(w^+)$ are non-empty, in particular the one at level $i$, say $T$. Let $H$ be the parent half-tile of $T$ (hence, $H$ belongs to a level $i-1$ tile). Since $T$ is a robust tile, its parent is not faulty and thus $V(\mathcal{G}) \cap H$ is non-empty, so there exists some vertex $\widehat{w} \in V(\mathcal{G}) \cap H$. Since $H$ is at level $i-1$, we have $r_{\widehat{w}} < h_{i-1} \leqslant \max\{r_u,r_v\}$. Note in particular that $\widehat{w}$ is robust (because it belongs to the robust tile $T$) and lies in $B_O(\rho')$ (because $r_{\widehat{w}} < h_{i-1} \leqslant h_{\ell'} \leqslant \rho'$. Observe that $\widehat{w}$ is in between $u$ and $v$, so $\theta(u,\widehat{w}) + \theta(\widehat{w},v) = \theta(u,v) \leqslant \theta_R(r_u,r_v)$. To conclude we consider the following three cases:

(i) Case $r_u \leqslant r_{\widehat{w}} < r_v$: By Remark 7 we get $\theta(u,\widehat{w}) \leqslant \theta_R(r_u,r_v) \leqslant \theta_R(r_u,r_{\widehat{w}})$ implying that $u\widehat{w}$ is an edge of $\mathcal{G}$ and contradicting the minimality of $P_w$.

(ii) Case $r_u > r_{\widehat{w}} \geqslant r_v$: Once more by Remark 7, we get $\theta(\widehat{w},v) \leqslant \theta_R(r_u,r_v) \leqslant \theta_R(r_{\widehat{w}},r_v)$. It follows that $v\widehat{w}$ is an edge of $\mathcal{G}$, so by minimality, $P_w$ contains a single vertex outside $\Upsilon(w)$, and thus $d(w,w') \leqslant d(w,\widehat{w}) \leqslant 1 + |V(\mathcal{G}) \cap \Upsilon(w)|$.

(iii) Case $r_u, r_v > r_{\widehat{w}}$: By Remark 7, we have $\theta(u,\widehat{w}) \leqslant \theta_R(r_u,r_v) \leqslant \theta_R(r_u,r_{\widehat{w}})$ implying that $u\widehat{w}$ is an edge of $\mathcal{G}$ which, as we have already seen in Case (i), contradicts the minimality of $P_w$.

The result follows. □

**Corollary 29.** *Let $s$ and $t$ be vertices of the center component of $\mathcal{G} := \mathcal{G}_{\alpha,\nu}(n)$. If neither $s$ nor $t$ are located in $B_O(\rho)$, then*

$$\mathcal{R}\,(s \leftrightarrow t) \leqslant |V(\mathcal{G}) \cap \Upsilon(s)| + |V(\mathcal{G}) \cap \Upsilon(t)| + \mathcal{O}(1).$$

*If $s$ but not $t$ (respectively, $t$ but not $s$) is located in $B_O(\rho)$, then the first (respectively, second) term in the RHS of the inequality above may be omitted, if both $s$ and $t$ are located in $B_O(\rho)$ then $\mathcal{R}\,(s \leftrightarrow t) = \mathcal{O}(1)$.*

*Proof.* Let $s'$ and $t'$ be robust vertices in $V(\mathcal{G}) \cap B_O(\rho')$ which are closest (in graph distance) to $s$ and $t$, respectively. Such vertices exist because $s$ and $t$ belong to the center component of $\mathcal{G}$. Recalling that $\mathcal{R}\,(s \leftrightarrow t) \leqslant \mathcal{R}\,(s \leftrightarrow s') + \mathcal{R}\,(s' \leftrightarrow t') + \mathcal{R}\,(t' \leftrightarrow t)$ by Proposition 13 and observing that by Proposition 26 we know that $\mathcal{R}\,(s' \leftrightarrow t') = \mathcal{O}(1)$, we get that $\mathcal{R}\,(s \leftrightarrow t) \leqslant \mathcal{R}\,(s \leftrightarrow s') + \mathcal{R}\,(t' \leftrightarrow t) + \mathcal{O}(1)$. The desired conclusion follows applying Lemma 28 twice. □

Corollary 29 shows that we can bound the resistance between two poorly connected vertices in the peripheries of $\mathcal{C}$ by controlling the number of vertices in the smallest sectors between pairs of robust vertices that contains them. The next Lemma helps us achieve this.

**Lemma 30.** *There exist constants $\kappa_1, \kappa_2 > 0$ such that for any $w \in V(\mathcal{G})$ and $p \in \mathbb{N}$,*

$$\mathbb{P}\left(|V(\mathcal{G}) \cap \Upsilon(w)| \geqslant \kappa_1 \cdot 2^p\right) \leqslant \begin{cases} \exp\left(-\kappa_2 \cdot 2^p\right) + \exp\left(-\kappa_2 \cdot 2^{p(1-\alpha)}\right) & \text{if } p \leqslant \ell' \\ \exp\left(-\kappa_2 \cdot 2^p\right) & \text{if } p > \ell' \end{cases}.$$

*Proof.* Let $\Upsilon^+(w)$ and $\Upsilon^-(w)$ be the elements of $\Upsilon(w)$ that are clockwise and anticlockwise from $w$, respectively. Since $\Upsilon(w) = \Upsilon^+(w) \cup \Upsilon^-(w)$, for all $w \in V(\mathcal{G})$, and the distribution of

$|V(\mathcal{G}) \cap \Upsilon^+(w)|$ and $|V(\mathcal{G}) \cap \Upsilon^-(w)|$ are identical by symmetry, it suffices to prove the bound for $\mathbb{P}\left(|V(\mathcal{G}) \cap \Upsilon^+(w)| \geqslant C \cdot 2^p\right)$.

Recall that from (6) we have $2^{\ell'}\theta_{\ell'} = 2\pi/N_0$, where $N_0 := N_0(c)$ is constant that is independent of $n$. Hence, as $2^{\ell'}N_0\theta_{\ell'} = 2\pi > \frac{1}{2}e^{\frac{3}{2}}$, by Lemma 4 we have

$$\mathbb{P}(|V(\mathcal{G}) \cap \Upsilon^+(w)| \geqslant 2^p N_0 n\theta_{\ell'}) \leqslant \mathbb{P}(|V(\mathcal{G})| \geqslant 2^p N_0 n\theta_{\ell'}) \leqslant e^{-2^{p-1}N_0 n\theta_{\ell'}}, \qquad \text{if } p > \ell'. \tag{18}$$

Henceforth, assume $1 \leqslant p \leqslant \ell'$. Let $\Psi_w^+$ be the sector of central angle $2^p N_0\theta_{\ell'}$ whose elements are clockwise from $w$ and contains $w$ in its boundary, that is,

$$\Psi_w^+ := \{q \in \mathbb{H}^2 \mid \theta_w \leqslant \theta_q < \theta_w + 2^p N_0\theta_{\ell'}\}.$$

Clearly, again by (6), for $p \leqslant \ell'$, it holds that $2^p N_0\theta_{\ell'} \leqslant 2\pi$. Furthermore, note that if $\theta(w, w^+) \leqslant 2^p N_0\theta_{\ell'}$, then $\Upsilon^+(w)$ is contained in $\Psi_w^+$, so

$$\mathbb{P}(|V(\mathcal{G}) \cap \Upsilon^+(w)| \geqslant 2^p N_0 n\theta_{\ell'}) \leqslant \mathbb{P}(|V(\mathcal{G}) \cap \Psi_w^+| \geqslant 2^p N_0 n\theta_{\ell'}) + \mathbb{P}(\theta(w, w^+) > 2^p N_0\theta_{\ell'}). \tag{19}$$

By Lemma 4, since $\mathbb{E}|V(\mathcal{G}) \cap \Psi_w^+| = \frac{2^p N_0\theta_{\ell'}}{2\pi}n$ and $2\pi > e^{\frac{3}{2}}$, we have

$$\mathbb{P}(|V(\mathcal{G}) \cap \Psi_w^+| \geqslant 2^p N_0 n\theta_{\ell'}) \leqslant e^{-2^{p-1}N_0 n\theta_{\ell'}}, \qquad \text{for } 0 \leqslant p \leqslant \ell'. \tag{20}$$

To bound the second summand in the RHS of (19) we consider a vertex $w$ of $\mathcal{G}_{\alpha,\nu}(n)$ and let $A_{p-1}^+(w)$ denote the clockwise sibling of the $(p-1)^{\text{th}}$ level ancestor of $T(w)$, i.e., $A_{p-1}^+(w)$ is the clockwise sibling of $A_{p-1}(w) := \pi^{p-1}(T(w))$. Note that every point that belongs to a tile in $\pi^{-(p-1)}(\{A_{p-1}^+(w)\})$ is at a clockwise angular distance at most $2^p\theta_{\ell'}$ from $w$. Note that the factor $2^{p-1}$ is because $\pi^{-(p-1)}(\{A_{p-1}^+(w)\})$ contains that number of tiles, the factor $\theta_{\ell'}$ is because each such tile is contained within a sector of such angle, and the additional factor of 2 accounts for the location of $w$ among the tiles in $\pi^{-(p-1)}(\{A_{p-1}(w)\})$. The relevance of the previous observation is that we can now bound the probability that $\theta(w, w^+)$ exceeds $2^p\theta_{\ell'}$ by the probability that $\pi^{-(p-1)}(\{A_{p-1}^+(w)\})$ is frail. Specifically, by Lemma 27 and Item (ii) of Lemma 25, we conclude that for $p \leqslant \ell'$

$$\begin{aligned}\mathbb{P}(\theta(w, w^+) \geqslant 2^p\theta_{\ell'}) &\leqslant \mathbb{P}(\pi^{-(p-1)}(\{A_{p-1}^+(w)\}) \text{ is frail}) \\ &= \mathcal{O}\left(\exp\left[-\tfrac{1}{8}C'e^{(1-\alpha)\left(h_{\ell'}-h_{\ell'-p+1}\right)}\right]\right) \\ &= e^{-\Omega(2^{p(1-\alpha)})},\end{aligned} \tag{21}$$

where the last equality holds since, for any $\varepsilon > 0$, if $n$ is large enough, we have $h_{\ell'} - h_{\ell'-p+1} \geqslant (p-1)\ln 2 - \varepsilon$ by Claim 21. Thus (21) bounds the second term in (19) since $N_0 \geqslant 1$.

Next, we claim that $\theta_{\ell'} = \Theta(1/n)$. Indeed, for large enough $n$, by Claim 21, $h_{\ell'} + \ln 2 + \varepsilon > h_{\ell'+1} > \rho' \geqslant h_{\ell'}$, so by our choice of $\rho'$ in (14), we have that $\theta_{\ell'} = \theta_R(h_{\ell'}, h_{\ell'}) = \Theta(e^{-\frac{1}{2}R}) = \Theta(1/n)$. It follows that $N_0 n\theta_{\ell'} = \Theta(1)$ and so the results follows. □

The following bound on the raw moments will be useful for bounding the expectation of the target time.

**Lemma 31.** *For any fixed real $1 \leqslant \kappa < \infty$ we have*

$$\mathbb{E}\Big(\sum_{w \in V(\mathcal{G}) \setminus B_O(\rho)} |V(\mathcal{G}) \cap \Upsilon(w)|^\kappa\Big) = \mathcal{O}(n).$$

*Proof.* An application of the Campbell-Mecke formula [52, Theorem 4.4], and a recollection of $\lambda(\cdot,\cdot)$ (the intensity function from (2)), yields the following

$$\mathbb{E}\Big(\sum_{w \in V(\mathcal{G})} |V(\mathcal{G}) \cap \Upsilon^+(w)|^\kappa\Big) = \int_0^{2\pi}\int_0^R \mathbb{E}|V(\mathcal{G}) \cap \Upsilon^+(w)|^\kappa\, \lambda(r_w, \theta_w)\, \mathrm{d}r_w \mathrm{d}\theta_w.$$

It suffices to show that for a fixed $w \in B_O(R)$, $w \neq O$ and any real $\kappa \geqslant 1$, there exists a constant $\eta := \eta_{\alpha,\nu,\kappa} > 0$ such that $\mathbb{E}|V(\mathcal{G}) \cap \Upsilon^+(w)|^\kappa \leqslant \eta$.

Observe that for a non-negative random variable $X$ and any $x_0 > 0$ we have

$$\mathbb{E}X^\kappa = \sum_{x\in\mathbb{N}} \mathbb{P}(X^\kappa > x) \leqslant x_0^\kappa + \sum_{p\in\mathbb{N}} 2^{p\kappa} x_0^\kappa \cdot \mathbb{P}(X^\kappa > 2^{p\kappa} x_0^\kappa) = x_0^\kappa \cdot (1 + \sum_{p\in\mathbb{N}} 2^{p\kappa} \cdot \mathbb{P}(X \geqslant 2^p x_0)),$$

taking $X := \theta(w, w^+)$ and $x_0 := \kappa_1$ to be the constant from Lemma 30, the bound above gives

$$\begin{aligned}\mathbb{E}|V(\mathcal{G}) \cap \Upsilon(w)|^\kappa &\leqslant (\kappa_1)^\kappa \cdot \Big(1 + \sum_{p\in\mathbb{N}} 2^{p\kappa} \cdot \mathbb{P}(|V(\mathcal{G}) \cap \Upsilon(w)| \geqslant \kappa_1 \cdot 2^p)\Big) \\ &\leqslant (\kappa_1)^\kappa \cdot \Big(2 + \sum_{p=1}^{\ell'} 2^{p\kappa} \cdot e^{-\kappa_2 \cdot 2^{p(1-\alpha)}} + \sum_{p\in\mathbb{N}} 2^{p\kappa} \cdot e^{-\kappa_2 \cdot 2^{p-1}}\Big) \leqslant \eta,\end{aligned}$$

for some $\eta := \eta_{\alpha,\nu,\kappa}$ by the bound in Lemma 30. □

3.4. **Average Resistance & Target Time.** We shall now use our flow from to determine the average effective resistance between two vertices and the target time.

**Theorem 1** (restated)**.** *For any $\frac{1}{2} < \alpha < 1$ and $\nu > 0$, if $\mathcal{C} := \mathcal{C}_{\alpha,\nu}(n)$, then*

$$\mathbb{E}(\mathcal{K}(\mathcal{C})) = \Theta(n^2), \qquad \text{and} \qquad \mathbb{E}\Big(\frac{1}{|V(\mathcal{C})|^2} \sum_{u,v\in V(\mathcal{C})} \mathcal{R}(u \leftrightarrow v)\Big) = \Theta(1).$$

*Proof.* For the lower bound observe that in any connected graph $G$ with at most $\kappa|V(G)|$ edges there are at least $|V(G)|/2$ vertices with degree at most $4\kappa$. Thus, there are $\Omega(|V(G)|^2)$ pairs $u, v$ with degree at most $4\kappa$ that do not share an edge. It follows that $\mathcal{R}(u \leftrightarrow v) \geqslant 1/d(u) + 1/d(v) \geqslant 1/(2\kappa)$ for each such pair, and so $\sum_{u,v\in V(G)} \mathcal{R}(u \leftrightarrow v) \geqslant \Omega(|V(G)|^2) \cdot 1/(2\kappa)$. Recall that $|V(\mathcal{C})| = \Theta(n)$ and $|E(\mathcal{C})| = \Theta(n)$ a.a.s. by Lemmas 10 and 11 respectively, giving the lower bound for the expectation of $\mathcal{K}(\mathcal{C})$ and the average resistance.

For the upper bound let $S := V(\mathcal{C}) \setminus B_O(\rho)$ and observe that, by Corollary 29, we have

$$\begin{aligned}\sum_{u,v\in V(\mathcal{C})} \mathcal{R}(u \leftrightarrow v) &\leqslant \sum_{u,v\in V(\mathcal{C})} (|V(\mathcal{G}) \cap \Upsilon(u)|\mathbf{1}_{u\in S} + |V(\mathcal{G}) \cap \Upsilon(v)|\mathbf{1}_{v\in S} + \mathcal{O}(1)) \\ &= |V(\mathcal{C})| \cdot \sum_{w\in S} |V(\mathcal{G}) \cap \Upsilon(w)| + \mathcal{O}(|V(\mathcal{C})|^2).\end{aligned} \tag{22}$$

For some $\kappa > 0$, let $\mathcal{V}_n = \{\kappa n \leqslant |V(\mathcal{C})| \leqslant n/\kappa\}$. Then by (22) and Lemma 31 we have

$$\mathbb{E}\Big(\sum_{u,v\in V(\mathcal{C})} \mathcal{R}(u \leftrightarrow v)\, \mathbf{1}_{\mathcal{V}_n}\Big) \leqslant (n/\kappa) \cdot \mathbb{E}\Big(\sum_{w\in S} |V(\mathcal{G}) \cap \Upsilon(w)|\Big) + \mathcal{O}(\mathbb{E}(|V(\mathcal{C})|^2)) = \mathcal{O}(n^2). \tag{23}$$

By [46, Theorem 16] and Lemma 4 there exists a $\kappa > 0$ such that $\mathbb{P}(\mathcal{E}_n) = e^{-\Omega(n)}$. Thus

$$\mathbb{E}\Big(\sum_{u,v\in V(\mathcal{C})} \mathcal{R}(u \leftrightarrow v)\, \mathbf{1}_{\mathcal{V}_n^c}\Big) \leqslant \mathbb{E}\Big(|V(\mathcal{C})|^4 \cdot \mathbf{1}_{\mathcal{V}_n^c}\Big) \leqslant \sqrt{\mathbb{E}(|V(\mathcal{C})|^8) \cdot \mathbb{P}(\mathcal{V}_n^c)} = o(1), \tag{24}$$

as $\mathcal{R}(u \leftrightarrow v) < |V(\mathcal{C})|$ and $\mathbb{E}(|V(\mathcal{C})|^8) = \mathcal{O}(n^8)$ by Lemma 5, thus $\mathbb{E}(\mathcal{K}(\mathcal{C})) = \mathcal{O}(n^2)$.

For the average resistance, by (22) and Lemma 31 we have

$$\mathbb{E}\Big(\frac{1}{|V(\mathcal{C})|^2} \sum_{u,v\in V(\mathcal{C})} \mathcal{R}(u \leftrightarrow v)\, \mathbf{1}_{\mathcal{V}_n}\Big) \leqslant \frac{1}{\kappa n} \cdot \mathbb{E}\Big(\sum_{w\in S} |V(\mathcal{G}) \cap \Upsilon(w)|\Big) + \mathcal{O}(1) = \mathcal{O}(1).$$

and similarly to before we have

$$\mathbb{E}\Big(\frac{1}{|V(\mathcal{C})|^2} \sum_{u,v\in V(\mathcal{C})} \mathcal{R}(u \leftrightarrow v)\, \mathbf{1}_{\mathcal{V}_n^c}\Big) \leqslant \mathbb{E}\Big(|V(\mathcal{C})|^2 \cdot \mathbf{1}_{\mathcal{V}_n^c}\Big) \leqslant \sqrt{\mathbb{E}(|V(\mathcal{C})|^4) \cdot \mathbb{P}(\mathcal{V}_n^c)} = o(1),$$

thus combining the two bounds above gives us the result for average resistance. □

We will also use our bounds on the effective resistance to determine the target time however first we must control sums of moments of degrees.

**Lemma 32.** *For any fixed $1/2 < \alpha < 1$ and any real $1 \leqslant \kappa < 2\alpha$ we have*

$$\mathbb{E}\Big[\sum_{v\in V(\mathcal{G})} d(v)^\kappa\Big] = \mathcal{O}(n).$$

*Proof.* The number of neighbours $d(v)$ of a vertex $v$, where $v$ is at radial distance $r_v$ from the origin, is Poisson-distributed with mean $\mu_v$ given by

$$\mu_v := n\cdot\mu(B_v(R)\cap B_O(R)) = n\cdot c_\alpha e^{-\frac{1}{2}r_v}\big(1+\mathcal{O}(e^{-(\alpha-\frac{1}{2})r_v}+e^{-r_v})\big),$$

where $c_\alpha := 2\alpha/\pi(\alpha-1/2)$ and the last expression is from Lemma 9. Using Lemma 5, the bound on the moments of a Poisson random variable, for any real $1\leqslant\kappa<2\alpha$ we have

$$\mathbb{E}(d(v)^\kappa) \leqslant \mu_v^\kappa\cdot 5\big(40\cdot\min\big\{\frac{\kappa}{5\mu_v},\ 1\big\}\big)^\kappa \leqslant c' n^\kappa\cdot e^{-\frac{1}{2}r_v\kappa}, \tag{25}$$

for some constant $c' := c'_{\alpha,\kappa}$ as $1/2<\alpha<1$. Recall the function $f(r)$ in our definition of the intensity function $\lambda(r,\theta)$ from (2), and observe that $f(r) = \mathcal{O}(e^{-\alpha(R-r)})$ by Lemma 6. Thus, by the Campbell-Mecke formula [52, Theorem 4.4] and (25) we deduce that

$$\mathbb{E}\big(\sum_{w\in V(\mathcal{G})} d(v)^\kappa\big) = \int_0^{2\pi}\int_0^R \mathbb{E}(d(v)^\kappa)\,\lambda(r_v,\theta_v)\,\mathrm{d}r_v\mathrm{d}\theta_v = \mathcal{O}(n^{\kappa+1}\cdot e^{-\alpha R})\cdot\int_0^R e^{(\alpha-\frac{1}{2}\kappa)r_v}\,\mathrm{d}r_v.$$

Now observe that, since $R = 2\ln(n/\nu)$ and $\alpha-\kappa/2>0$, we have

$$\int_0^R e^{(\alpha-\frac{1}{2}\kappa)r_v}\,\mathrm{d}r_v = \left[\frac{e^{(\alpha-\frac{1}{2}\kappa)r_v}}{\alpha-\frac{1}{2}\kappa}\right]_0^R = \mathcal{O}(e^{(\alpha-\kappa/2)R}) = \mathcal{O}(e^{\alpha R}n^{-\kappa}),$$

the result follows. ☐

We finally have all necessary ingredients to determine the expectation, over the choice of $\mathcal{G} := \mathcal{G}_{\alpha,\nu}(n)$, of the target time of the center component $\mathcal{C}$ of $\mathcal{G}$.

**Theorem 2** (restated)**.** *For any $\frac{1}{2}<\alpha<1$ and $\nu>0$, if $\mathcal{C} := \mathcal{C}_{\alpha,\nu}(n)$, then $\mathbb{E}\big(t_\odot(\mathcal{C})\big) = \Theta(n)$.*

*Proof.* Let $\mathcal{C} := \mathcal{C}_{\alpha,\nu}(n)$ be the center component, and note that we can assume $|V(\mathcal{C})|>1$ holds. By definition of $t_\odot(\cdot)$, and Lemma 16 we have

$$t_\odot(\mathcal{C}) = \frac{1}{2}\sum_{u,v\in V(\mathcal{C})}(\mathbf{E}_u(\,\tau_v\,)+\mathbf{E}_v(\,\tau_u\,))\cdot\pi(u)\pi(v) = \frac{1}{8|E(\mathcal{C})|}\sum_{u,v\in V(\mathcal{C})}\mathcal{R}\,(u\leftrightarrow v)\cdot d(u)d(v). \tag{26}$$

Let $\mathcal{E}_n$ be the event that none of the tiles of $\mathcal{F}(c)$ contained in $B_O(\rho)$ is faulty. Thus, if we condition on the event $\mathcal{E}_n$ and let $S := V(\mathcal{C})\setminus B_O(\rho)$ then, by Corollary 29, we have

$$\begin{aligned}
t_\odot(\mathcal{C}) &\leqslant \frac{1}{8|E(\mathcal{C})|}\sum_{u,v\in V(\mathcal{C})}(|V(\mathcal{G})\cap\Upsilon(u)|\mathbf{1}_{u\in S}+|V(\mathcal{G})\cap\Upsilon(v)|\mathbf{1}_{v\in S}+\mathcal{O}(1))\cdot d(u)d(v)\\
&= \frac{2}{8|E(\mathcal{C})|}\sum_{u,v\in V(\mathcal{C})}|V(\mathcal{G})\cap\Upsilon(u)|\mathbf{1}_{u\in S}\cdot d(u)d(v)+\mathcal{O}(|E(\mathcal{C})|)\\
&= \frac{2}{8|E(\mathcal{C})|}\sum_{x\in V(\mathcal{C})}d(x)\sum_{w\in V(\mathcal{C})\setminus B_O(\rho)}|V(\mathcal{G})\cap\Upsilon(w)|\cdot d(w)+\mathcal{O}(|E(\mathcal{C})|)\\
&= \frac{1}{2}\sum_{w\in V(\mathcal{C})\setminus B_O(\rho)}|V(\mathcal{G})\cap\Upsilon(w)|\cdot d(w)+\mathcal{O}(|E(\mathcal{C})|).
\end{aligned}$$

Recall that $1/2<\alpha<1$ is fixed, so we can set $k = 3/4+\alpha/2$; which satisfies $1<k<2\alpha$, and has Hölder conjugate $1<k/(k-1)<\infty$. Thus, applying Hölder's inequality gives

$$t_\odot(\mathcal{C}) \leqslant \frac{1}{2}\Big(\sum_{w\in V(\mathcal{C})\setminus B_O(\rho)}|V(\mathcal{G})\cap\Upsilon(w)|^{\frac{k}{k-1}}\Big)^{\frac{k-1}{k}}\cdot\Big(\sum_{w\in V(\mathcal{C})\setminus B_O(\rho)}d(u)^k\Big)^{\frac{1}{k}}+\mathcal{O}(|E(\mathcal{C})|).$$

Applying the version of Hölder's inequality for expectations (with the same $k$) yields

$$\mathbb{E}\big[t_{\odot}(\mathcal{C})\cdot \mathbf{1}_{\mathcal{E}_n}\big] \leqslant \frac{1}{2}\mathbb{E}\Big[\sum_{w\in V(\mathcal{C})\setminus B_O(\rho)} |V(\mathcal{G})\cap \Upsilon(w)|^{\frac{k}{k-1}}\Big]^{\frac{k-1}{k}}\cdot \mathbb{E}\Big[\sum_{v\in V} d(v)^k\Big]^{\frac{1}{k}} + \mathcal{O}(|E(\mathcal{C})|)$$
$$= \mathcal{O}(n), \tag{27}$$

where the second line follows by the bounds on moments from Lemmas 31 and 32. We now need to control what happen outside of the "desirable" event $\mathcal{E}_n$. Observe that by (26) we have

$$t_{\odot}(\mathcal{C}) \leqslant |E(\mathcal{C})|\cdot \max_{u,v\in V(\mathcal{C})} \mathcal{R}\,(u\leftrightarrow v) \leqslant |V(\mathcal{C})|\cdot|E(\mathcal{C})| \leqslant |V(\mathcal{G})|\cdot|E(\mathcal{G})|,$$

since $\mathcal{C}\subseteq\mathcal{G}$ is connected. Recall that, by Lemma 24, for every $d>0$ there is a sufficiently large $C>0$ such that, $\mathbb{P}(\mathcal{E}_n^c) = o(1/n^d)$. Thus, again by Hölder's inequality, we have

$$\mathbb{E}\left[t_{\odot}(\mathcal{C})\cdot \mathbf{1}_{\mathcal{E}_n^c}\right] \leqslant \mathbb{E}\left[|V(\mathcal{G})|\cdot|E(\mathcal{G})|\cdot \mathbf{1}_{\mathcal{E}_n^c}\right] \leqslant \mathbb{E}(|V(\mathcal{G})|^4)^{\frac{1}{4}}\cdot \mathbb{E}(|E(\mathcal{G})|^2)^{\frac{1}{2}}\cdot \mathbb{P}(\mathcal{E}_n^c)^{\frac{1}{4}}. \tag{28}$$

We now observe that $\mathbb{E}(|V(\mathcal{C})|^4) = \mathcal{O}(n^4)$ by the fact that $|V(\mathcal{C})|$ is Poisson with mean $n$ and the bound on moments of Poisson random variables from Lemma 5. Also note that $\mathbb{E}(|E(\mathcal{G})|^2) = \mathcal{O}(n^2)$ by [18, Claim 5.2]. Thus we have $\mathbb{E}\left[t_{\odot}(\mathcal{C})\cdot \mathbf{1}_{\mathcal{E}_n^c}\right] = o(1)$ by (28), and so the bound $\mathbb{E}\big(t_{\odot}(\mathcal{C}_{\alpha,\nu}(n))\big) = \mathcal{O}(n)$ follows from (27). □

## 4. Cover Time and Maximum Hitting Time

In this section, we first determine the asymptotic behavior of the cover time of the giant component of HRGs. To do so, we rely on several intermediate structural results concerning HRGs that were established in the previous section. However, the arguments developed here are not flow based. In particular, the ones concerning commute times rely on extensions of Matthew's Bound, a classical useful method for bounding cover times. This result is then complemented, by the determination of the minimum and maximum hitting times also for the giant.

As in the previous sections, we restrict our discussion exclusively to the parameter range where $\frac{1}{2} < \alpha < 1$.

### 4.1. Upper Bounds on Hitting and Cover times.

To determine the cover time of the center component of HRGs we rely on a classical result concerning random walks in graphs know as Matthew's bound (stated below) and one of its extensions (mentioned later). The bound and its extensions relate the cover time of a graph with its minimum and maximum hitting times.

**Theorem 33.** [53, Section 11.2] *For any finite irreducible Markov chain on $n$ states we have*

$$t_{\mathsf{cov}} \leqslant t_{\mathsf{hit}}\cdot\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right).$$

A well known relation between hitting times and diameter of graphs together with established results concerning the diameter of HRGs allows us to easily derive upper bounds on hitting and, via Matthew's bound, also on the cover time of simple random walks on the center component of $\mathcal{G}_{\alpha,\nu}(n)$.

**Proposition 34.** *If $\mathcal{C} := \mathcal{C}_{\alpha,\nu}(n)$, then a.a.s. and in expectation we have*

$$t_{\mathsf{hit}}(\mathcal{C}) = \mathcal{O}(n\ln n) \qquad \text{and} \qquad t_{\mathsf{cov}}(\mathcal{C}) = \mathcal{O}(n\ln^2 n).$$

*Proof.* Recall that $d_G(u,v)$ denotes the length of the shortest path between a pair of vertices $u,v\in V(G)$, of a graph $G$. Then, by Lemmas 16 and 14, we have $\mathbf{E}_u(\,\tau_v\,) \leqslant 2|E(G)|d_G(u,v)$. Let $\mathcal{D}_1 = \{\max_{u,v\in V(\mathcal{C})} d_{\mathcal{C}}(u,v) \leqslant \kappa\ln n\}$ for a suitably large constant $\kappa$, then we have $\mathbb{P}(\mathcal{D}_1^c) = n^{-\kappa'}$ for some fixed $\kappa'>0$ by [61], as $1/2<\alpha<1$ is fixed. We note that Müller and Staps [61] only state that their $\mathcal{O}(\log n)$ diameter bound holds a.a.s., however if one checks the proofs all steps proved in their paper they hold with probability $1-n^{-1000n}$, and they additionally use a coupling [30, Lemmas 27 & 30] that holds with probability $1-n^{-\kappa'}$, where $\kappa'>0$ depends on $\alpha$. Recall also that $|E(\mathcal{C})| = \mathcal{O}(n)$ a.a.s. by Lemma 11. Thus we get that $t_{\mathsf{hit}} = \mathcal{O}(n \ln n)$ holds a.a.s.. We also define the event $\mathcal{L} = \{\sum_{k=1}^{|V(\mathcal{C})|} k^{-1} \leqslant 2\log n\}$ and observe that $\mathbb{P}(\mathcal{L}^c) \leqslant \mathbb{P}(|V(\mathcal{C})| > 100n) \leqslant e^{-\Omega(n)}$ by Lemma 4. The desired a.a.s. upper bound for the cover time then follows from Matthew's bound (see Theorem 33).

We now show that these bounds also hold in expectation. Before we begin we must define an additional event $\mathcal{D}_2 = \{\max_{u,v \in V(\mathcal{C})} d_{\mathcal{C}}(u,v) \leqslant \kappa \ln^{\frac{1}{1-2\alpha}} n\}$ and note that $\mathbb{P}(\mathcal{D}_2^c) = (n^{-100})$ by [33, Theorem 2] (also see the remark on [33, Page 1316] after Theorem 3).

We begin with $\mathbb{E}(t_{\mathsf{hit}})$, conditioning on the event $\mathcal{D}_1 = \{\max_{u,v \in V(\mathcal{C})} d_{\mathcal{C}}(u,v) \leqslant \kappa \ln n\}$ gives

$$\mathbb{E}\left(t_{\mathsf{hit}} \cdot \mathbf{1}_{\mathcal{D}_1}\right) \leqslant (\kappa \log n) \cdot 2\mathbb{E}\left(|E(\mathcal{C})|\right) = \mathcal{O}(n \log n), \tag{29}$$

as $\mathbb{E}\left(|E(G)|\right) = \mathcal{O}(n)$ by Lemma 32. By the Cauchy–Schwarz inequality we have

$$\mathbb{E}\left(t_{\mathsf{hit}} \cdot \mathbf{1}_{\mathcal{D}_1^c \cap \mathcal{D}_2}\right) \leqslant (\kappa \log^{\frac{1}{1-2\alpha}} n) \cdot 2\mathbb{E}\left(|E(\mathcal{C})| \mathbf{1}_{\mathcal{D}_1^c}\right) \leqslant (2\kappa \log^{\frac{1}{1-2\alpha}} n) \cdot \sqrt{\mathbb{E}\left(|E(\mathcal{C})|^2\right) \mathbb{P}(\mathcal{D}_1^c)},$$

then as $\mathbb{E}(|E(\mathcal{G})|^2) = \mathcal{O}(n^2)$ by [18, Claim 5.2] and by the bound on $\mathbb{P}(\mathcal{D}_1^c)$ above we have

$$\mathbb{E}\left(t_{\mathsf{hit}} \cdot \mathbf{1}_{\mathcal{D}_1^c \cap \mathcal{D}_2}\right) \leqslant (2\kappa \log^{\frac{1}{1-2\alpha}} n) \cdot \sqrt{\mathcal{O}(n^2) \cdot n^{-\kappa'}} = o(n). \tag{30}$$

Finally, since $d_{\mathcal{C}}(u,v) \leqslant |V(\mathcal{G})|$ and $|E(\mathcal{C})| \leqslant |V(\mathcal{G})|^2$ we have

$$\mathbb{E}\left(t_{\mathsf{hit}} \cdot \mathbf{1}_{\mathcal{D}_1^c \cap \mathcal{D}_2^c}\right) \leqslant \mathbb{E}\left(|V(\mathcal{C})^3| \cdot \mathbf{1}_{\mathcal{D}_2^c}\right) \leqslant \sqrt{\mathbb{E}\left(|V(\mathcal{C})|^6\right) \mathbb{P}(\mathcal{D}_2^c)} = o(1), \tag{31}$$

since $\mathbb{E}|V(\mathcal{C})|^k = \mathcal{O}(n^k)$ for any fixed $k$ by Lemma 5. The result follows from (29)-(31).

We now bound $\mathbb{E}(t_{\mathsf{cov}})$. Recall the event $\mathcal{L} = \{\sum_{k=1}^{|V(\mathcal{C})|} k^{-1} \leqslant 2\log n\}$ and observe that

$$\mathbb{E}\left(t_{\mathsf{cov}} \cdot \mathbf{1}_{\mathcal{D}_1 \cap \mathcal{L}}\right) \leqslant (\kappa \log n) \cdot 2\mathbb{E}\left(|E(\mathcal{C})|\right) \cdot (2\log n) = \mathcal{O}(n \log^2 n). \tag{32}$$

Now, since $d_{\mathcal{C}}(u,v) < |V(\mathcal{C})|$, $\sum_{k=1}^{|V(\mathcal{C})|} k^{-1} < |V(\mathcal{C})|$, and $|E(\mathcal{C})| \leqslant |V(\mathcal{C})|^2$, we have

$$\mathbb{E}\left(t_{\mathsf{cov}} \cdot \mathbf{1}_{\mathcal{L}^c}\right) \leqslant \mathbb{E}\left(|V(\mathcal{C})|^4 \mathbf{1}_{\mathcal{L}^c}\right) \leqslant \sqrt{\mathbb{E}(|V(\mathcal{C})|^8) \mathbb{P}(\mathcal{L}^c)} = o(1), \tag{33}$$

as $\mathbb{P}(\mathcal{L}^c) = e^{-\Omega(n)}$ and $\mathbb{E}|V(\mathcal{C})|^k = \mathcal{O}(n^k)$ for any fixed $k$ by Lemma 5. Similarly,

$$\mathbb{E}\left(t_{\mathsf{cov}} \cdot \mathbf{1}_{\mathcal{D}_1^c \cap \mathcal{L} \cap \mathcal{D}_2}\right) \leqslant (\kappa \log^{\frac{1}{1-2\alpha}} n) \cdot 2\sqrt{\mathbb{E}\left(|E(\mathcal{C})|^2\right) \mathbb{P}(\mathcal{D}_1^c)} \cdot (2\log n) = o(n). \tag{34}$$

Finally, again similarly to before, we have

$$\mathbb{E}\left(t_{\mathsf{cov}} \cdot \mathbf{1}_{\mathcal{D}_1^c \cap \mathcal{L} \cap \mathcal{D}_2^c}\right) \leqslant \mathbb{E}\left(|V(\mathcal{C})|^4 \cdot \mathbf{1}_{\mathcal{D}_2^c}\right) \leqslant \sqrt{\mathbb{E}\left(|V(\mathcal{C})|^8\right) \mathbb{P}(\mathcal{D}_2^c)} = o(1). \tag{35}$$

The bound on $\mathbb{E}(t_{\mathsf{cov}})$ follows from combining (32)-(35). ☐

4.2. **Lower Bounds on Hitting and Cover times.** The lower bounds in the classical version of Matthew's bound [53, Section 11.2] do not directly (or at least not obviously) imply a lower bound for the cover time of $\mathcal{G}_{\alpha,\nu}(n)$ matching the upper bound obtained in Proposition 34. Fortunately, several extensions of Matthew's bound have been established. Among these, we will take advantage of the following result by Kahn, Kim, Lovász and Vu.

**Theorem 35.** *([41, Theorem 1.3]) If $G = (V, E)$ is a connected graph, then*

$$t_{\mathsf{cov}}(G) \geqslant \frac{1}{2} \cdot \max_{U \subseteq V} \left(\kappa_S \cdot \ln |S|\right) \qquad \text{where } \kappa_U := \min_{u,v \in U} [\mathbf{E}_u(\,\tau_v\,) + \mathbf{E}_v(\,\tau_u\,)].$$

The main difficulty in applying the preceding result, in order to lower bound the cover time of the center component of HRGs, is finding an adequate set $U$ of vertices of the center component of $\mathcal{G}_{\alpha,\nu}(n)$ which is both sufficiently large and such that for every distinct pair of elements $u, v \in U$ the commute time between $u$ and $v$ is large enough, say $|U| = n^{\Omega(1)}$ and $\kappa(u,v) = \Omega(\ln n)$. In order to lower bound the cover time of the center component of HRGs we show that such a set of vertices of $\mathcal{C}_{\alpha,\nu}(n)$ is likely to exist. The existence of such set $U$ and the fact that it satisfies the necessary conditions relies on a structural result about induced paths of length $\Theta(\log n)$ in HRGs. The result implies that a.a.s. there are many (specifically, $n^{\Omega(1)}$) vertices of the center component of $\mathcal{G}_{\alpha,\nu}(n)$ whose removal gives rise to a path component of order $\Omega(\log n)$. The result is implicit in [45, Theorem 4.1] and explicitly known (in an equivalent form) to the first author and Mitsche since their characterization of the size of the second largest component of HRGs [47, Theorem 1.1]. We state and prove it below, since it is essential for our study of the commute time of the giant component of HRGs, might also be useful in other settings, and now seems of interest in its own right. The proof argument is somewhat different (we believe also simpler) from the one implicit in [45, Theorem 4.1] and relies on the tiling structure constructed in the previous section. Before going into its proof, we give an overview of it. To start, $B_O(R)$ is partitioned into several regions. The first region will be a ball centered at the origin with radius as large as possible but still so it will be unlikely that it contains vertices of $\mathcal{G}_{\alpha,\nu}(n)$. This region will be denoted $\mathcal{R}_0$ and correspond to $B_O(\rho_0)$ for a $\rho_0$ appropriately chosen. Next, for a positive integer $m$ to be determined, we consider $m$ sectors of central angle $2\pi/m$ and intersect each one of them with $B_O(R)\backslash\mathcal{R}_0$ thus obtaining a partition $\{\mathcal{R}_1, ..., \mathcal{R}_m\}$ of $B_O(R)\backslash\mathcal{R}_0$. We then show that we can pick $m = \Omega(n^\zeta)$ for some $0 < \zeta < 1$ so that in each region $\mathcal{R}_a$, $a \in [m]$, with a small but non-negligible probability (specifically, with probability $\Omega(n^{-\chi})$ for a small constant $0 < \chi < \zeta$) and independent of what happens in the other regions except for $\mathcal{R}_0$, there is a vertex $v_a$ of the center component of $\mathcal{G}_{\alpha,\nu}(n)$ whose removal disconnects $\mathcal{G}_{\alpha,\nu}(n)$ into two connected components, one of which is a path $\mathcal{P}_a$ of length $\Theta(\log n)$. This suffices for our purpose, since it would imply that the expected number of induced paths of length $\Theta(\log n)$ in $\mathcal{C}_{\alpha,\nu}(n)$ is $\Omega(n^{\zeta-\chi})$, so conditioned on $\mathcal{R}_0$ being empty, the independence of events within different regions together with standard concentration bounds would yield the result we seek to establish. This completes the high-level overview of our proof argument. We discuss its details next.

Before stating the result, we introduce one additional term. We say that a sub-graph $P$ of a graph $G$ is a *dangling path* if $P$ is a path in $G$ and there is a single edge of $G$ whose removal disconnects the graph into two connected components one of which is $P$. Furthermore, we say that $P$ is a *maximal dangling path* if it is not a proper sub-graph of some other dangling path of $G$.

**Lemma 36.** *For all $0 < \zeta < 1 - \frac{1}{2\alpha}$ there is a positive constant $\Lambda := \Lambda(\zeta)$ such that with probability at least $1 - e^{-\Omega(n^\zeta)}$ there are $\Omega(n^\zeta)$ maximal dangling paths of length at least $\Lambda \ln n$ in $\mathcal{C}_{\alpha,\nu}(n)$.*

*Proof.* Henceforth, let $V := V(\mathcal{G}_{\alpha,\nu}(n))$. Set $\rho_0 := (1-\frac{1+\eta}{2\alpha})R$ where $0 < \eta < 2\alpha-1$ is a constant. By Lemma 9, we have $\mu(\mathcal{R}_0) = (1 + o(1))e^{-(1+\eta)\frac{R}{2}}$, as $\mathcal{R}_0 = B_O(\rho_0)$ . Since $R = 2\ln(\frac{n}{\nu})$,

$$\mathbb{P}(V \cap \mathcal{R}_0 = \emptyset) = e^{-n\mu(\mathcal{R}_0)} = e^{-\mathcal{O}(n^{-\eta})} = 1 - \mathcal{O}(n^{-\eta}). \tag{36}$$

Next, we equipartition $B_O(R)$ into $m$ sectors of central angle $2\pi/m$, say $\Upsilon_1, ..., \Upsilon_m$, and focus the ensuing discussion on a specific $\Upsilon_a$, i.e., we fix $a \in [m]$. The intersection of $\Upsilon_a$ with $B_O(R) \setminus \mathcal{R}_0$ gives rise to $\mathcal{R}_a$, i.e., $\mathcal{R}_a = (\Upsilon_a \cap B_O(R)) \setminus \mathcal{R}_0$. For the sake of concreteness, let $\phi_a \in [0, 2\pi)$ be such that

$$\Upsilon_a = \{(r, \theta) \mid \phi_a \leqslant \theta < \phi_a + \tfrac{2\pi}{m}\}.$$

The sector $\Upsilon_a$ will contain three contiguous sub-sectors. Moving clockwise around the origin, the order in which these sub-sectors are encountered are; first a buffer sub-sector $\Upsilon_a^{\mathrm{bf}}$, next a path sub-sector $\Upsilon_a^{\mathrm{pt}}$, and finally a connection sub-sector $\Upsilon_a^{\mathrm{ct}}$. Their intersections with $\mathcal{R}_a$ will be denoted $\mathcal{R}_a^{\mathrm{bf}}$, $\mathcal{R}_a^{\mathrm{pt}}$ and $\mathcal{R}_a^{\mathrm{ct}}$, respectively. Our goal is to show that in a narrow band of $\mathcal{R}_a^{\mathrm{pt}}$ close to the boundary of $B_O(R)$, with non-negligible probability, there is a collection of vertices that induce a path $\mathcal{P}_a$ of length $\Theta(\log n)$. We will establish that with constant probability there is a path all of whose vertices belong to $\mathcal{R}_a^{\mathrm{ct}}$ that guarantees that one of the end vertices of $\mathcal{P}_a$ (and thus also all vertices of $\mathcal{P}_a$) belongs to the center component of $\mathcal{G}_{\alpha,\nu}(n)$. Still, it could be that internal vertices of the path $\mathcal{P}_a$ might have other neighbours not in $\mathcal{P}_a$. So, our choices of path, and buffer regions will be such that we can guarantee that all potential neighbours of vertices of $\mathcal{P}_a$ other than its end vertex $v_a$ would belong to either $\mathcal{R}_0$ or some sub-region of $\mathcal{R}^{\mathrm{pt}}_a \cup \mathcal{R}^{\mathrm{bf}}_a$ which, with non-negligible probability, do not contain vertices of $\mathcal{G}_{\alpha,\nu}(n)$ (we already have shown that something stronger holds for $\mathcal{R}_0$). Thus, we will be able to conclude that $\mathcal{P}_a$ is an induced path of $\mathcal{G}_{\alpha,\nu}(n)$. However, we need to make sure that $\mathcal{P}_a$ is in fact an induced path of $\mathcal{C}_{\alpha,\nu}(n)$. To achieve this we establish that, with constant probability, there is a path from the end vertex $v_a$ of $\mathcal{P}_a$ to a vertex $u_a$ that almost certainly belongs to the center component. Next, we formalize this paragraph's discussion.

We let $\Upsilon^{\mathrm{bf}}_a$ be the sector of $\mathbb{H}^2$ which contains points with angular coordinates at least $\phi_a$ and strictly less than $\phi_a + \theta_R(h_{\ell'}-\delta, \rho_0)$ where (with hindsight) we set $\delta := \ln(9/8)$. Thus, $\Upsilon^{\mathrm{bf}}_a$ and $\Upsilon_a$ share one of their boundaries. Let $\Upsilon^{\mathrm{pt}}_a$ be the sector of $\mathbb{H}^2$ of central angle $3k\theta_{\ell'}$, where $k$ will be fixed later, and contiguous to $\Upsilon^{\mathrm{bf}}_a$ when moving clockwise around the origin, i.e., $\Upsilon^{\mathrm{pt}}_a$ contains all points of $\mathbb{H}^2$ whose angular coordinates are at least $\phi_a + \theta_R(h_{\ell'}-\delta, \rho_0)$ and strictly less than $\phi_a + \theta_R(h_{\ell'}-\delta, \rho_0) + 3k\theta_{\ell'}$. Also, let $\mathcal{R}^{\mathrm{bf}}_a := \Upsilon^{\mathrm{bf}}_a \cap (B_O(R) \setminus \mathcal{R}_0)$ and $\mathcal{R}^{\mathrm{pt}}_a := \Upsilon^{\mathrm{pt}}_a \cap (B_O(R) \setminus \mathcal{R}_0)$. We consider a band $\mathcal{B}_a$ of points of $\Upsilon^{\mathrm{pt}}_a$ with radial distance to the origin between $h_{\ell'} - \delta$ and $h_{\ell'}$, specifically $\mathcal{B}_a := \Upsilon^{\mathrm{pt}}_a \cap (B_O(h_{\ell'}) \setminus B_O(h_{\ell'}-\delta))$. Partitioning $\Upsilon^{\mathrm{pt}}_a$ into sectors of central angle $\frac{1}{4}\theta_{\ell'}$ we end up with $12k$ sectors whose intersections with $\mathcal{B}_a$ partitions the latter into $\mathcal{B}'_1, ..., \mathcal{B}'_{12k}$ (these parts depend on the index $a$, but our notation will not reflect this in order to avoid over cluttering). Without loss of generality the indexing of the $\mathcal{B}'_j$'s reflects the order in which each piece is encountered when moving anti-clockwise around the origin. Thus, $\mathcal{B}'_{12k}$'s boundary intersects the one of $\Upsilon^{\mathrm{bf}}_a$ while $\mathcal{B}'_1$ does not. We claim that our choice of central angle for $\Upsilon^{\mathrm{bf}}_a$ ensures that for every point $p \in \mathcal{B}_a$ the elements of $B_p(R) \setminus \mathcal{R}_0$ with angular coordinate at most $\theta_p$ are completely contained in $\Upsilon^{\mathrm{pt}}_a \cup \Upsilon^{\mathrm{bf}}_a$. Indeed, all such elements span an angle at the origin with $p$ which is at most $\theta_R(r_p, \rho_0)$. By Remark 7, the latter angle is at most $\theta_R(h_{\ell'}, \rho_0)$ (because $p \notin B_0(h_{\ell'})$). Our last claim follows.

Now, suppose the following event occurs: there is exactly one vertex, say $v_j$, that belongs to $\mathcal{B}'_j$ if $j-1$ is a multiple of 3 and there is no vertex in $\mathcal{B}'_j$ for all other $j$'s. We claim that the $v'_j$'s induce a path $\mathcal{P}_a$ on $4k$ vertices. Indeed, note that the angle at the origin between $v'_j$ and $v'_{j+1}$ is at most $\theta_{\ell'}$. Since both $v'_j$ and $v'_{j+1}$ belong to $B_O(h_{\ell'}) \setminus B_O(h_{\ell'} - \delta)$, by definition of $\theta_R(\cdot, \cdot)$, both vertices are within distance at most $R$ of each other, so there is an edge between them in $\mathcal{G}_{\alpha,\nu}(n)$. On the other hand, the angle at the origin spanned by $v'_{j-1}$ and $v'_{j+1}$ is at least $\frac{5}{4}\theta_{\ell'} = \frac{5}{4}(1 + o(1))e^{-\delta}\theta_R(h_{\ell'}-\delta, h_{\ell'}-\delta)$, so by our choice of $\delta$, the said angle is at least $\frac{10}{9}(1 + o(1))\theta_{\ell'}$, hence $v'_{j-1}$ and $v'_{j+1}$ are at distance at least $R$ from each other and no edge between them exists in $\mathcal{G}_{\alpha,\nu}(n)$. This completes the proof of the claim stated at the start of this paragraph. We argue next that the event described at the start of this paragraph happens with probability $\Omega(n^{-\chi})$ provided that $C'$ is chosen a large enough constant. Thus, by Lemma 8 and Lemma 9, the expected number of vertices in any single one of the $\mathcal{B}'_j$'s is

$$\mathbb{E}|V \cap \mathcal{B}'_j| = n\frac{\theta_{\ell'}}{4}\mu(B_O(h_{\ell'}) \setminus B_O(h_{\ell'}-\delta)) = \Theta(n\theta_{\ell'}e^{-\alpha(R-h_{\ell'})}) = \Theta(\nu e^{(1-\alpha)(R-h_{\ell'})}).$$

By our choice of $h_{\ell'}$, we have that $R - h_{\ell'} \geqslant \frac{1}{1-\alpha}\ln(\frac{2C'}{\nu})$ and conclude that $\lambda := \mathbb{E}|V \cap \mathcal{B}'_j| = \Omega(C')$. Thus, the probability that any single one of the $\mathcal{B}'_j$'s does not contain a vertex of $\mathcal{G}_{\alpha,\nu}(n)$ is $e^{-\lambda}$, and the probability that it contains exactly one vertex is $\lambda e^{-\lambda}$. Since the $\mathcal{B}'_j$'s are disjoint, the number of vertices of $\mathcal{G}_{\alpha,\nu}(n)$ contained within each one is independent. Thus, if we choose $C'$ large enough so $(\lambda^2 e^{-3\lambda})^4 \geqslant e^{-1}$, we get that

$$\mathbb{P}(V \cap \mathcal{B}_a \text{ induces a path in } \mathcal{G}_{\alpha,\nu}(n)) \geqslant (\lambda^2 e^{-3\lambda})^{4k} \geqslant e^{-k}. \tag{37}$$

So far we have exposed the band $\mathcal{B}_a$ and shown that with non-negligible probability it contains an induced path of $\mathcal{G}_{\alpha,\nu}(n)$. Any neighbour that a vertex of $\mathcal{P}_a$ could have, except maybe the vertices $\{v'_1, v'_2, v'_3\}$ of $\mathcal{P}_a$, would have to belong to $\mathcal{R}_0$ or $\mathcal{R}^{\mathrm{pt}}_a \setminus \mathcal{B}_a$ or $(B_{v'_4}(R) \cup B_{v'_{4k}}(R) \setminus \mathcal{R}_0) \setminus \mathcal{B}_a$. We have already seen that the first region is unlikely to contain vertices. In contrast, we shall see that for the latter two regions we can only guarantee that with non-negligible probability they do not contain vertices. The measure of the second region is upper bounded by the one of the sector $\Upsilon_a^{\rm pt}$ (because $\mathcal{R}_a^{\rm pt} \subseteq \Upsilon_a^{\rm pt}$), which using Lemma 9 and Lemma 21, equals $3k\theta_{\ell'} = \Theta(k\frac{\nu}{n}(\frac{2C'}{\nu})^{\frac{1}{1-\alpha}})$. Using a union bound and Lemma 9 we get that the measure of the third region is $\mathcal{O}(e^{-\frac{1}{2}h_{\ell'}}) = \mathcal{O}(\frac{\nu}{n}(\frac{2C'}{\nu})^{\frac{1}{1-\alpha}})$. Thus, choosing $C'$ large enough, we conclude that

$$\mathbb{P}(V \cap (\mathcal{R}_a^{\rm pt} \setminus \mathcal{B}_a) \cup ((B_{v_4'}(R) \cup B_{v_{4k}'}(R) \setminus \mathcal{R}_0) \setminus \mathcal{B}_a) = \emptyset) \geqslant e^{-k}. \tag{38}$$

Now, consider the tiling $\mathcal{F}(c)$ of Section 3.1 where $\varepsilon := \frac{1}{2}\ln 2$ and $c := c(\varepsilon)$ is like in Lemma 21. Let $H$ be the half-tile to which $v_1'$ belongs. Since $v_j' \in B_O(h_{\ell'} \setminus B_O(h_{\ell'}-\delta))$, the half-tile $H$ must be a level $\ell'$ tile. Next, we recursively specify a sequence of half-tiles of $\mathcal{F}(c)$. First, we let $H_{\ell'}^{(1)}$ be the half-tile that is contiguous to $H$ obtained by rotating the latter clockwise by the adequate angle (i.e., by $\frac{1}{2}\theta_{\ell'}$). Giving a half-tile $H_s^{(i)}$ at level $s$ of $\mathcal{F}(c)$, $i \in [5]$, let $H_s^{(i+1)}$ be the half-tile, also at level $s$, that is contiguous to $H_s^{(i)}$ and is again obtained by rotating the latter clockwise by the adequate angle (i.e., now by $\theta_s$). The process is repeated up to reaching level $\ell$ where as usual $\ell$ is the largest integer such that $h_\ell \leqslant \rho$ for $\rho$ defined in (12) (i.e., $\rho := R - \frac{1}{1-\alpha}\ln(\frac{CR}{\nu})$). Let $\mathcal{E}_a$ be the event such that for all $s \in \{\ell, ..., \ell'\}$ and $i \in [6]$ the half-tile $H_s^{(i)}$ contains at least one vertex of $\mathcal{G}_{\alpha,\nu}(n)$. By Remark 18, if $\mathcal{E}_a$ occurs, then $v_a := v_1'$ is connected via a path to a vertex, say $u_a \in V \cap H_\ell^{(4)}$ and, since Lemma 24 implies that, a.a.s., all vertices in $B_O(\rho)$ belong to $\mathcal{C}_{\alpha,\nu}(n)$, so does $v_a = v_1'$ and hence also the vertices of $\mathcal{P}_a$. Before calculating the probability that $\mathcal{E}_a$ occurs, we need to make sure that it does not involve regions already exposed. For this, it will suffice to show that the $H_s^{(i)}$'s do not intersect $B_{v_4'}(R)$. By construction the angle spanned at the origin by $v_4'$ and a point in $\bigcup_{i\in[6]} H_{\ell'}^{(i)}$ is at least $\frac{9}{4}\theta_{\ell'} \geqslant \frac{9}{8}\theta_{\ell'-1} \geqslant \theta_R(h_{\ell'-1}, h_{\ell'} - \delta)$ (the last inequality is obtained using Remark 8, since $h_{\ell'} \leqslant h_{\ell'-1} + \ln 2 + \varepsilon$, and because of our choices of $\varepsilon$ and $\delta$). Thus, by definition of $\theta_R(\cdot,\cdot)$, the intersection $B_{v_4'}(R) \cup \bigcup_{i\in[6]} H_{\ell'}^{(i)}$ is empty. For values of $s < \ell'$, we have that the angle spanned by any point in $\bigcup_{i\in[6]} H_s^{(i)}$ and $v_4'$ is at least

$$\frac{9}{4}\theta_{\ell'} + 2\sum_{j=s+1}^{\ell'} \theta_j > 2\theta_{\ell'}\Big(1 + \sum_{j=0}^{\ell'-s-1} 2^j\Big) = \theta_{\ell'}2^{\ell'-s+1} \geqslant \theta_{s-1}.$$

By definition of $\theta_{s-1}$, Remark 7, and since $s < \ell'$, it follows that $\theta_{s-1} = \theta_R(h_{s-1}, h_{s-1}) \geqslant \theta_R(h_{s-1}, h_{\ell'-1})$, so we conclude again that the intersection $B_{v_4'} \cap \bigcup_{i\in[6]} H_s^{(i)}$ is empty as we had claimed. We now determine the probability of the event $\mathcal{E}_a$. Since the $H_s^{(i)}$'s are disjoint, independence and standard arguments imply that

$$\mathbb{P}(\mathcal{E}_a) = \prod_{s=\ell}^{\ell'} \prod_{i\in[6]} \Big(1 - \exp\big(-\mathbb{E}|V \cap H_s^{(i)}|\big)\Big).$$

Since the expected number of vertices in a tile is twice that of a half-tile of the same level, by Claim 22, we have $\mathbb{E}|V \cap H_s^{(i)}| \geqslant \frac{1}{4}\nu e^{(1-\alpha)(R-h_s)}$. Thus, since $h_{\ell'} \leqslant \rho'$, by definition of $\rho'$, we get $\mathbb{E}|V \cap H_s^{(i)}| \geqslant \frac{1}{4}\nu e^{(1-\alpha)(R-h_{\ell'})} \geqslant \frac{1}{2}C'$. So, recalling that $\ln(1-x) \geqslant \frac{x}{1-x_0}$ for $0 \leqslant x \leqslant x_0 < 1$ and provided $C' \geqslant \ln 4$

$$\ln \mathbb{P}(\mathcal{E}_a) \geqslant -2\sum_{s=\ell}^{\ell'} \exp\big(-\tfrac{3}{2}\nu e^{(1-\alpha)(R-h_s)}\big) = -2\sum_{s=\ell}^{\ell'} \exp\big(-\tfrac{3}{4}C' e^{(1-\alpha)(h_{\ell'}-h_s)}\big).$$

By Claim 21, $h_{\ell'} - h_s \geqslant (\ell' - s)(\ln 2 - \varepsilon)$, so taking $\gamma := \frac{3}{4}C'$ and $\delta := (1-\alpha)(\ln 2 - \varepsilon)$,

$$\sum_{s=0}^{\ell-\ell'} \exp\big(-\gamma e^{\delta s}\big) \leqslant e^{-\gamma} + \tfrac{1}{\delta\gamma} e^{-\gamma} =: \xi.$$

Taking $C'$ a large constant we can make $\gamma$ large and $\xi$ as small as needed, say smaller enough so that $\mathbb{P}(\mathcal{E}_a) \geqslant e^{-2\xi} \geqslant \frac{1}{2}$ (any constant instead of $\frac{1}{2}$ will suffice). Summarizing, for $C'$ large enough,

$$\mathbb{P}(\text{vertex } v_a = v'_1 \text{ belongs to } \mathcal{C}_{\alpha,\nu}(n)) \geqslant \frac{1}{2}. \tag{39}$$

On the other hand, we have that all $H_s^{(i)}$'s are contained within a sector contiguous to $\Upsilon_a^{\rm pt}$ of angle

$$3\sum_{j=\ell}^{\ell'} \theta_j = 3\theta_{\ell'} \sum_{j=0}^{\ell'-\ell} 2^j \leqslant 3\theta_{\ell'} 2^{\ell'-\ell+1} = 6\theta_\ell.$$

Let $\Upsilon_a^{\rm ct}$ be the sector clockwise contiguous to $\Upsilon_a^{\rm pt}$ of central angle $6\theta_\ell$ and define $\mathcal{R}_a^{\rm ct} := \Upsilon_a^{\rm ct} \setminus \mathcal{R}_0$. We want to choose $m$ so the sector $\Upsilon_a$ which is of central angle $2\pi/m$ is large enough in order to include $\Upsilon_a^{\rm ct} \cup \Upsilon_a^{\rm pt} \cup \Upsilon_a^{\rm bf}$ which is a sector of central angle $\theta_R(h_{\ell'}, \rho_0) + 3k\theta_{\ell'} + 6\theta_\ell$. Recalling that $\rho_0 = (1 - \frac{1+\eta}{2\alpha})R$, for any $\zeta, \chi > 0$ such that $0 < \zeta + \chi < 1 - \frac{1}{2\alpha}$, we can take $\eta$ small enough so that $\theta_R(h_{\ell'}, \rho_0) = \mathcal{O}(n^{-(\zeta+\chi)})$. Also, fixing $k = \chi \ln n$ we have $k\theta_{\ell'} = \mathcal{O}(n^{-1}\log n)$. Moreover, note that $\theta_\ell = \mathcal{O}(n^{-1}(\log n)^{\frac{1}{1-\alpha}})$. Thus, we need to make sure that the central angle of $\Upsilon_a$ is at least $C'' n^{-(\zeta+\chi)}$ for some appropriately chosen large constant $C''$. Hence, we may choose $m = \lfloor \frac{2\pi}{C''} n^{\zeta+\chi} \rfloor$, partition $B_O(R)$ into $m$ sectors $\Upsilon_1, ..., \Upsilon_m$ so within $\Upsilon_a$ we can accommodate $\Upsilon_a^{\rm ct} \cup \Upsilon_a^{\rm pt} \cup \Upsilon_a^{\rm bf}$. It follows that $\mathcal{R}_a := (\Upsilon_a \cap B_O(R)) \setminus \mathcal{R}_0$ contains $\mathcal{R}_a^{\rm ct} \cup \mathcal{R}_a^{\rm pt} \cup \mathcal{R}_a^{\rm bf}$.

To conclude, let $X_a$ be the indicator of the following event: there is a vertex in the subgraph of $\mathcal{C}_{\alpha,\nu}(n)$ induced by $V \cap \mathcal{R}_a$ whose removal gives rise to a path component on $4k-3 = \Omega(\log n)$ vertices. Let $X_a$ be the indicator of the latter event and let $X = \sum_{a\in[m]} X_a$. From (37)-(39) we conclude that $\mathbb{E}X_a = \Omega(e^{-k}) = \Omega(n^{-\chi})$, we get that $\mathbb{E}X = \Omega(n^{\zeta})$. Thus, by Chernoff bounds (see [6, Theorem A.1.7]), the probability that $X \leqslant \frac{1}{2}\mathbb{E}X$ is $\exp(-\Omega(n^{\zeta}))$. □

We now have all the necessary ingredients to derive the lower bounds matching the bounds in Proposition 34. Together these two propositions prove Theorem 3.

**Proposition 37.** *If $\mathcal{C} := \mathcal{C}_{\alpha,\nu}(n)$, then a.a.s. $t_{\mathsf{hit}}(\mathcal{C}) = \Omega(n \ln n)$ and $t_{\mathsf{cov}}(\mathcal{C}) = \Omega(n \ln^2 n)$.*

*Proof.* The lower bound on $t_{\mathsf{hit}}(\mathcal{G})$ is a direct consequence of the claimed lower bound on $t_{\mathsf{cov}}(\mathcal{G})$ and Matthew's bound, so we concentrate on proving the latter one.

Let $\zeta$ and $\Lambda := \Lambda(\zeta)$ be as in the statement of Lemma 36. Let $U \subseteq V(\mathcal{C})$ be the collection of vertices $u$ of $\mathcal{C}$ which have degree 2 in $\mathcal{C}$ and are end vertices of a maximal dangling path $P_u$ of $\mathcal{C}$ of length at least $\Lambda \ln(n)$. Now, $u, v \in U$, $u \neq v$, and contract into a single vertex all vertices of $\mathcal{C}$ except for those that belong to $P_u$ or $P_v$. One obtains a path of length at least $2\Lambda \ln(n)$. Rayleigh's Monotonicity Law (RML) (Theorem 15) implies that the contraction of edges does not increase the effective resistance between any of the remaining pair of vertices. Since the effective resistance between the endpoints of an $\ell$ edge path is $\ell$ (this follows directly from our working definition of effective resistance – see (4)), by RML we get that $\mathcal{R}(u \leftrightarrow v) \geqslant 2\Lambda \ln(n)$ for all distinct $u, v \in U$. Since $\mathbf{E}_u(\tau_v) + \mathbf{E}_v(\tau_u) = 2|E(\mathcal{C})| \cdot \mathcal{R}(u \leftrightarrow v)$ by Lemma 16,

$$\kappa_U := \min_{u,v\in U} [\mathbf{E}_u(\tau_v) + \mathbf{E}_v(\tau_u)] \geqslant 2\Lambda |E(\mathcal{C})| \cdot \ln(n).$$

By Theorem 10, we know that a.a.s. $\mathcal{C}$ is of order $\Theta(n)$, and since $\mathcal{C}$ is connected, we also have that a.a.s. $|E(\mathcal{C})| = \Omega(n)$. Thus, we conclude that $\kappa_U = \Omega(n \ln n)$ a.a.s., so the sought after lower bound follows directly from (35). □

We also remark that Proposition 37 shows that the maximum effective resistance between any two vertices in the center component is a.a.s. $\Theta(\log n)$.

## 5. Concluding remarks

In this paper we determined the expected order of the maximum hitting time, cover time, target time and effective resistance between two uniform vertices, with the first two results also holding a.a.s. (w.r.t. the HRG). Our main finding to take away is that (in expectation) there are order $\log n$ gaps between each of the quantities. This indicates that most vertices in the giant are well-connected to the center of the graph, but a significant proportion are not.

We restricted our study to the giant component of the graph in the regime $1/2 < \alpha < 1$, although this is arguably the most interesting regime it would still be interesting to determine the aforementioned quantities on the other smaller components or when $\alpha \notin (1/2, 1)$. Another problem we leave open is to discover the leading constants hidden behind our asymptotic notation, if the expression for the clustering coefficient of the HRG [31] is anything to go by these constants may have very rich and complex expressions as functions of $\alpha$ and $\nu$. An interesting problem is to determine the order of meeting time, that is, the expected time it takes two (lazy) random walks to occupy the same vertex when started from the worst case start vertices [42]. Finally, the mixing time of a (lazy) random walk on the giant HRG is known up to polylogarithmic factors by [46]. Closing this gap is of great importance, but it may well be quite challenging.

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