ASSIGNMENT 6

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1 Introduction :

- In this assignment, we will look at how to analyse "Linear Time-invariant Systems" with numerical tools in Python
- In this assignment we will use mostly mechanical examples, but will move on to circuits in the next assignment.
- All the problems will be in "continuous time" and will use Laplace Transforms. Python has a Signals toolbox which is very useful and complete.
- In this assignment we will make use of some signal toolbox functions like
  - scipy.signal.lti
  - scipy.signal.impulse
  - scipy.signal.lsim
- A LTI system is described by its impulse response $h(t)$. If the input to the system is $x(t)$ then the output will be $y(t)$

$$y(t) = x(t) * h(t)$$

- Converting the above equation in laplace domain

$$Y(s) = X(s) * H(s)$$

- Y(s) is the Transfer Function of the system

1.1 Importing libraries:

```
import numpy as np
import scipy.signal as sp
from pylab import *
```

2 Question-1 : Time Response of Spring:

- We consider a case of a spring driven by a forcing function f,

$$f(t) = \cos(1.5t)e^{-0.5t}u_0(t)$$

- Laplace transform of f(t) is

$$F(s) = \frac{s+0.5}{(s+0.5)^2+2.25}$$

- The Differential Equation governing the Displacement of the string is

$$\frac{d^2x}{dt^2} + 2.25x = f(t)$$

  Initial coonditions are x(0) = 0 , x'(0) = 0 for t going from zero to 50 seconds.

- Converting equation into laplace domain :

$$s^2X(s) - sx(0^-) - x'(0^-) + 2.25X(s) = F(s)$$

$$s^2X(s) + 2.25X(s) = F(s)$$

$$X(s) = \frac{F(s)}{s^2+2.25}$$

$$X(s) = \frac{s+0.5}{[(s+0.5)^2+2.25][s^2+2.25]}$$

- Inverse laplace Transform can be found by signal.impulse function.

- Code for finding the x(t) :

```
# Question_1

F1 = sp.lti([1,0.5],[1,1,2.5])
X1 = sp.lti([1,0.5],[1,1,4.75,2.25,5.625])
t,x1 = sp.impulse(X1,None,linspace(0,50,501))
figure(0)
plot(t,x1)
xlabel(r'$t$')
ylabel(r'$x(t)$')
title(r' $x(t)$ vs $t$ for decay rate 0.5/sec')
plt.grid(True, color = "grey", linewidth = "1.4", linestyle = "-")
show()
```

- Plot for the x(t) vs time for 0<t<50sec

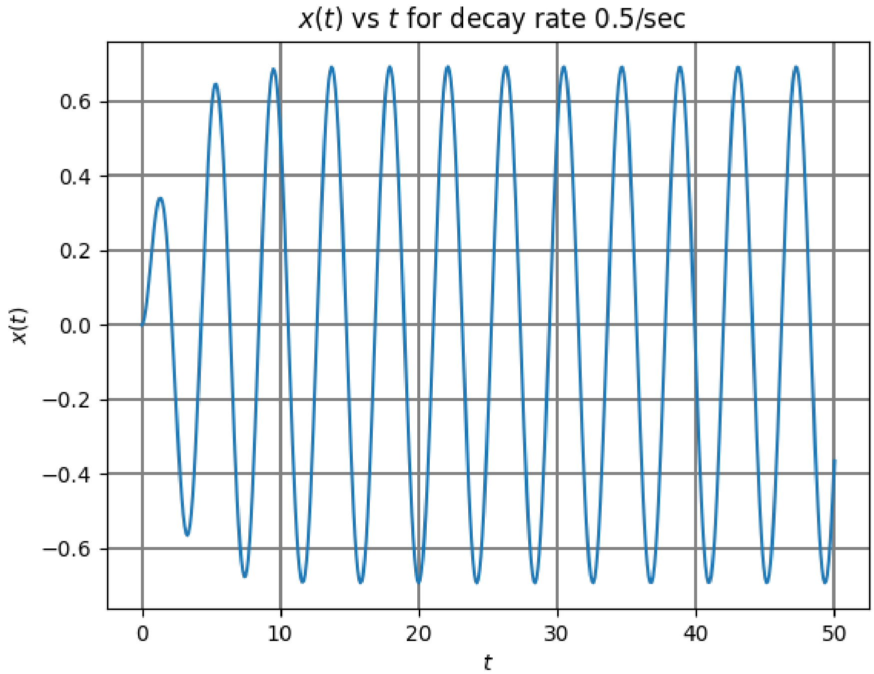

Figure 1: Response for Decay rate = 0.5/sec

3 Question-2 : Time Response for reduced decay constant:

We consider a case of a spring driven by a forcing function f which has decay constant 10 tiimes less than the previous case,

$$f(t) = \cos(1.5t)e^{-0.05t}u_0(t)$$

- Laplace transform of f(t) is

$$F(s) = \frac{s + 0.05}{(s + 0.05)^2 + 2.25}$$

$$X(s) = \frac{F(s)}{s^2 + 2.25}$$

$$X(s) = \frac{s + 0.05}{[(s + 0.05)^2 + 2.25][s^2 + 2.25]}$$

- Code for finding the new time response will be :

```
# Question_2

F2 = sp.lti([1,0.05],[1,0.1,2.2525])
X2 = sp.lti([1,0.05],[1,0.1,4.5025,0.225,5.068125])
t,x2 = sp.impulse(X2,None,linspace(0,50,501))
```

```
figure(1)
plot(t,x2)
xlabel(r'$t$')
ylabel(r'$x(t)$')
title(r' $x(t)$ vs $t$ for decay rate 0.05/sec')
plt.grid(True, color = "grey", linewidth = "1.4", linestyle = "-")
```

- Plot for the new x(t) vs time for 0<t<50sec

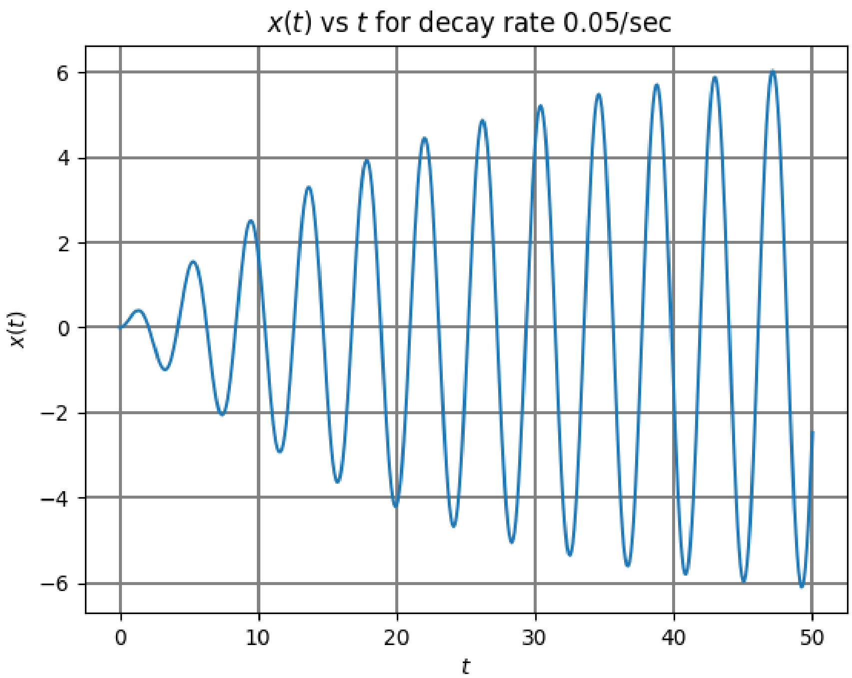

Figure 2: Response for Decay rate = 0.05/sec

4 Question-3 : Response for Various Cosine Frequencies :

- Now we try to change the frequency of the input and observe the output.The frequency of the input is changed from 1.4 to 1.6 in steps of 0.05 and the decay constant is left the same(0.05).
- This can be done using for loop,and varying the frequency of the cosine in $f(t)$ from 1.4 to 1.6 in steps of 0.05
- Code for finding responses :

```
H = sp.lti([1],[1,0,2.25])
dec = linspace(1.4,1.6,5)

figure(2)
for i in dec:
    p = i*i + 0.0025
    F = sp.lti([1,0.05],[1,0.1,p])
```

```
    t,f = sp.impulse(F,None,linspace(0,50,501))
    t,y,svec = sp.lsim(H,f,t)
    plot(t,y,label = 'freq = '+str(i)+' ')


xlabel(r'$t$')
ylabel(r'$y(t)$')
plt.grid(True, color = "grey", linewidth = "1.4", linestyle = "-")
title(r'Responses for different frequency inputs')
legend()
```

4.1 Explination for the plots:

- Initially there is no force on the spring.When the driving force starts acting the spring starts expanding and naturally it opposes this by applying a force of kx in the opposite direction.
- When the frequency of input matches with the natural frequency of the spring which is given to be 1.5 ,both their time periods match and the force is always in the direction of velocity and it always does the positive work.Due to which the amplitude of the spring goes on increasing as it passes through its equilibrium position.
- But after some time, the forcing function dies out ($e^{0.05t}$) and the spring continues to oscillate with its natural frequency.As there is no force thereafter,the spring continues to oscillate with the same amplitude.
- As the decay constant decreases,it takes more time for the forced response to die out,which explains the variation between Q1 and Q2 graphs.
- Plots for various frequencies

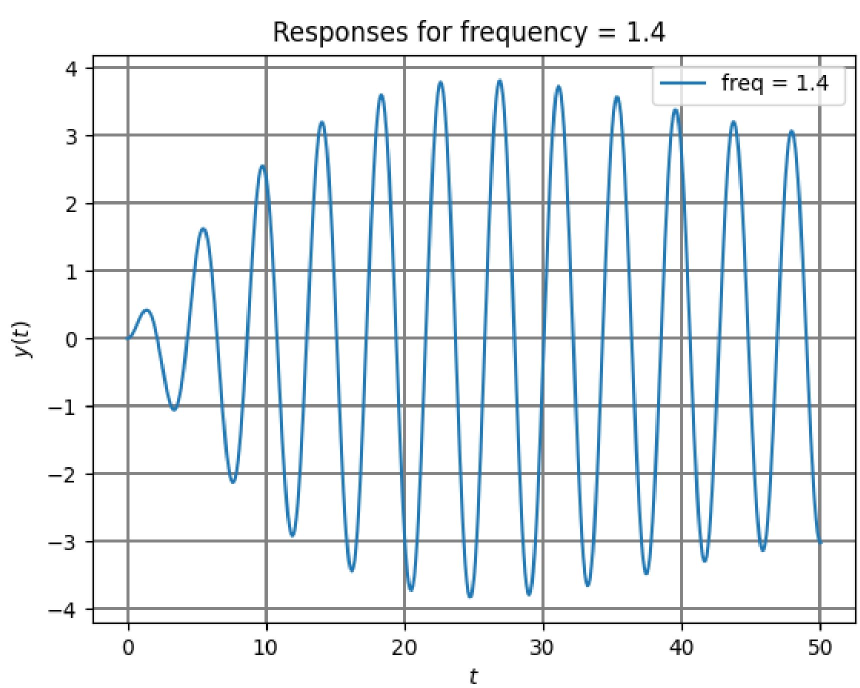

Figure 3: Response for freq = 1.4

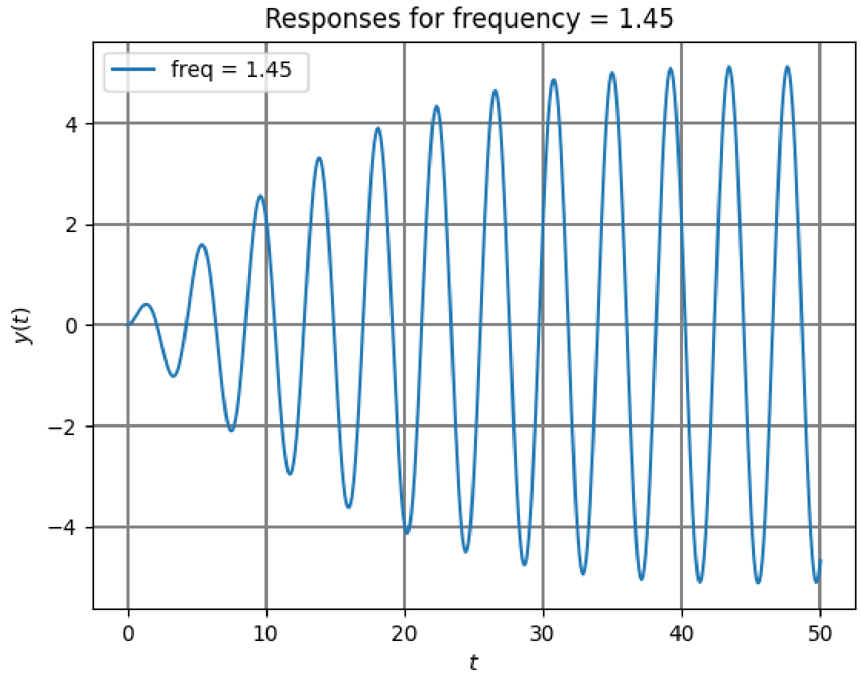

Figure 4: Response for freq = 1.45

- In the above two graphs,the forced response frequency is lesser than the natural frequency of the spring,implying that the forced response time-period is more than the natural time-period.
- First the force will be along velocity direction,after quarter time period, the spring starts compressing back,while the force stays in the same direction.Resulting, force doing negative work for short time delay introduced.
- As positive work dominates initially,the amplitude goes on increasing.But As time proresses,the delay increases which results in force doing more negative work than positive work.So,the amplitude starts falling back.
- The amplitude falls back till the forced response dies out and then it oscillates with its natural frequency.

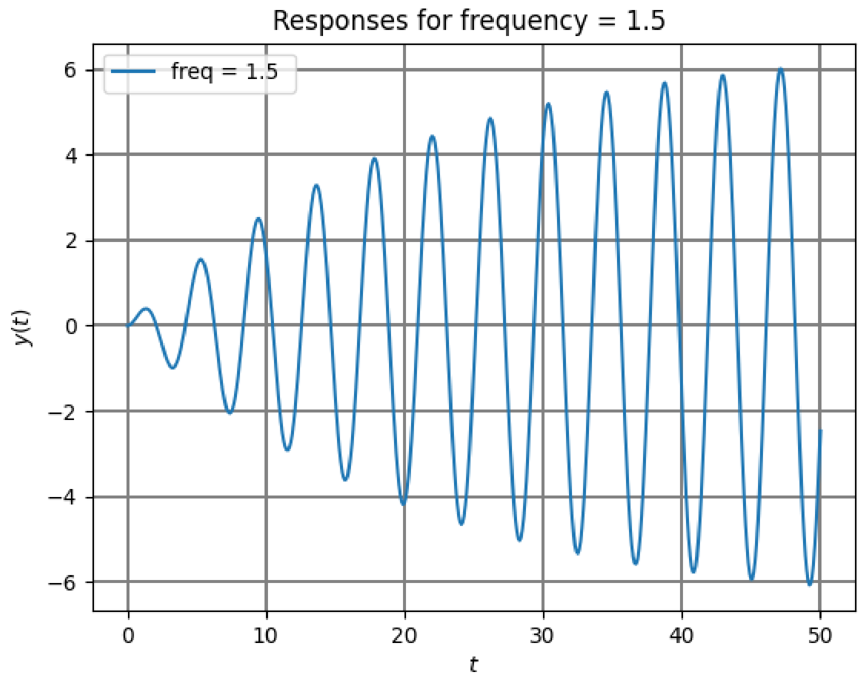

Figure 5: Response for freq = 1.5

- When the input frequency is at the natural frequency, the output amplitude is maximum. In the other cases the output amplitude decreases. This phenomenon is known as resonance.

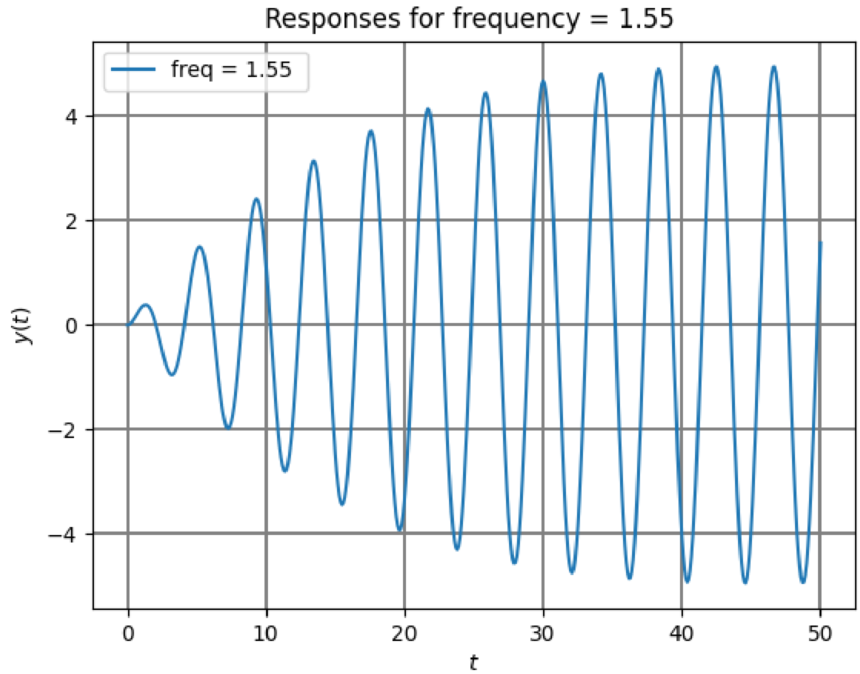

Figure 6: Response for freq = 1.55

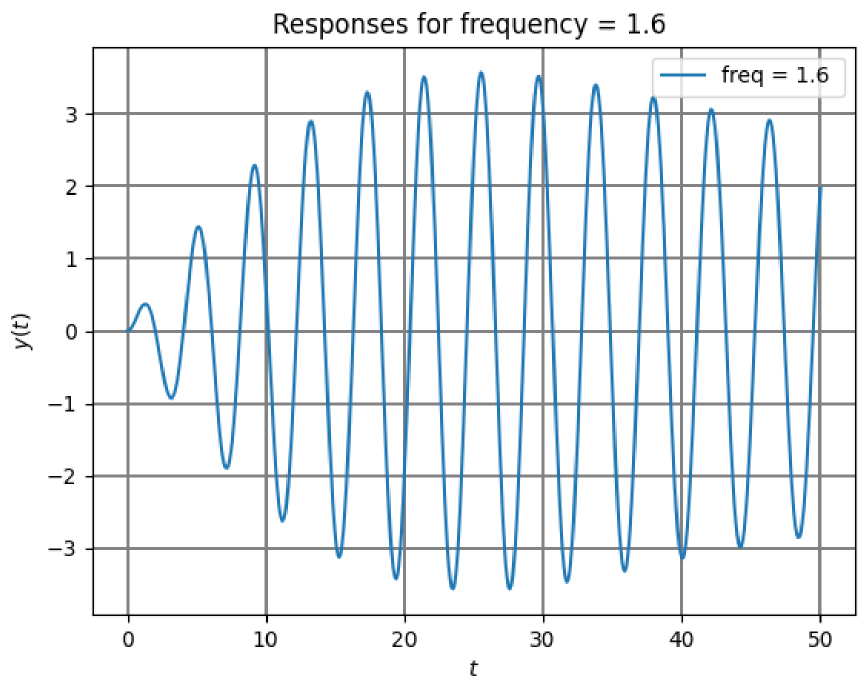

Figure 7: Response for freq = 1.6

- In the above two cases,the forced response frequency is more than the natural frequency.Implying that the time period of forced response is less than the natural time period.

- In the same way ;as when we had smaller frequencies this also works the same way,the forces initially does positive work,then does negative work due to the time-delay.
- So we get the graphs similar to the smaller frequencies.But the amplitudes here are smaller compared to the previous time,as here force stays for smaller period in the direction of velocity,so less positive work is done
- All responses plotted in single plot

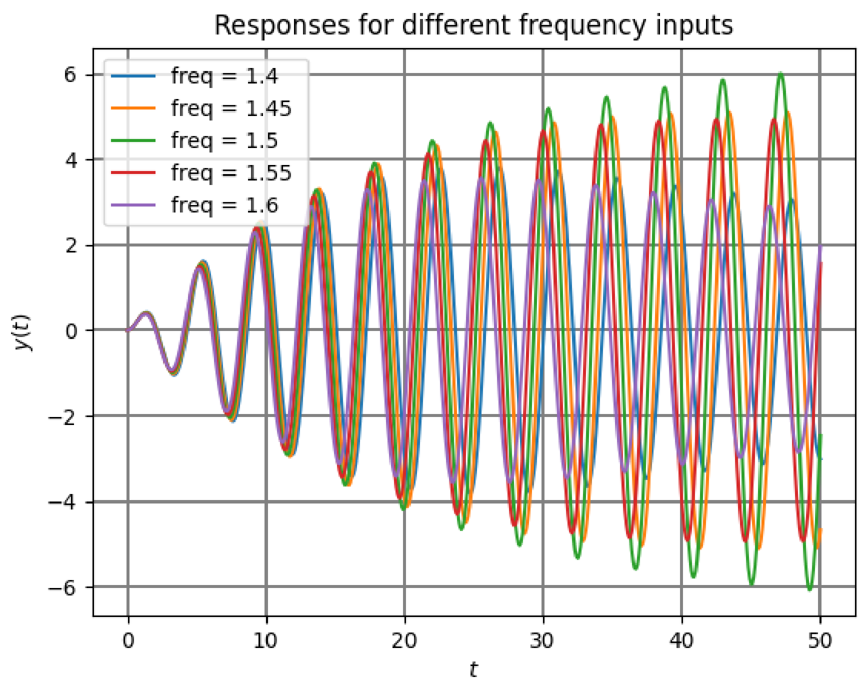

Figure 8: Responses for various Frequencies

5 Question-4 Coupled Spring Problem:

- The displacement in the both (x,y) are governed by the coupled differential equation

$$\frac{d^2x}{dt^2} + x = 0$$

$$\frac{d^2y}{dt^2} + 2y = 0$$

- The initial conditions are given by

$$x(0^-) = 1, x'(0^-) = y(0) = y'(0^-) = 0.$$

- This problem is solved by converting it into laplace domain.
- The equation in the laplace domain are like

$$s^2X(s) - sx(0^-) - x'(0^-) + X(s) - Y(s) = 0$$

$$s^2Y(s) - sy(0^-) - y'(0^-) + 2Y(s) - 2X(s) = 0$$

- By substituting the initial conditions and solving we get :

$$X(s) = \frac{s^2+2}{s^3+3s}$$

$$Y(s) = \frac{2}{s^3+3s}$$

- By doing the inverse laplace and finding the original functions
- Code :

```
# Question_4

X4 = sp.lti([1,0,2],[1,0,3,0])
Y4 = sp.lti([2],[1,0,3,0])

t1,x4 = sp.impulse(X4,None,linspace(0,20,201))
t1,y4 = sp.impulse(Y4,None,linspace(0,20,201))

figure(3)
plot(t1,x4)
xlabel(r'$t$')
ylabel(r'$x(t)$')
plt.grid(True, color = "grey", linewidth = "1.4", linestyle = "-")
title(r'x(t)')

figure(4)
plot(t1,y4)
xlabel(r'$t$')
ylabel(r'$y(t)$')
plt.grid(True, color = "grey", linewidth = "1.4", linestyle = "-")
title(r'y(t)')
```

- Plots for x(t) and y(t)

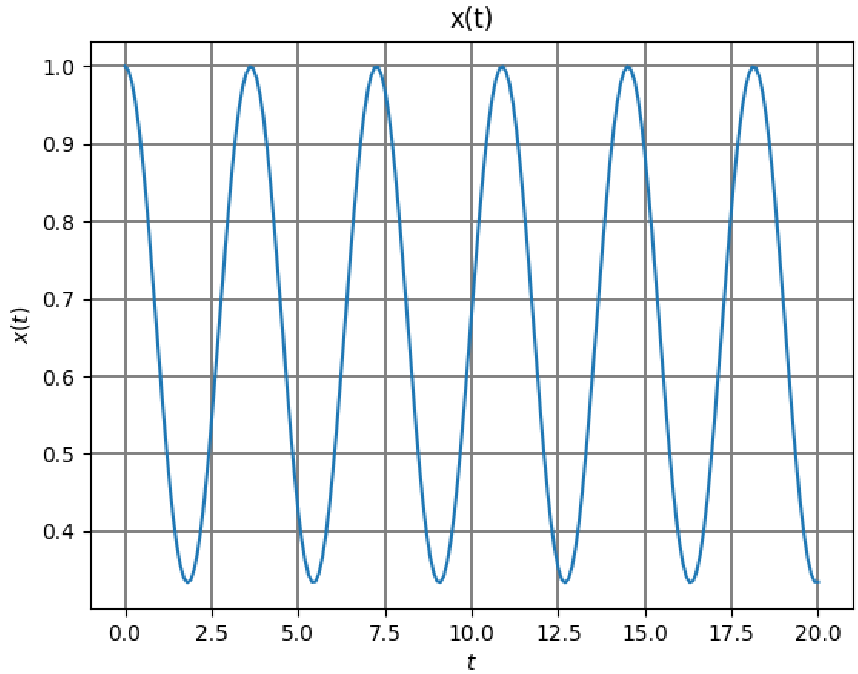

Figure 9: Displacement in the x direction

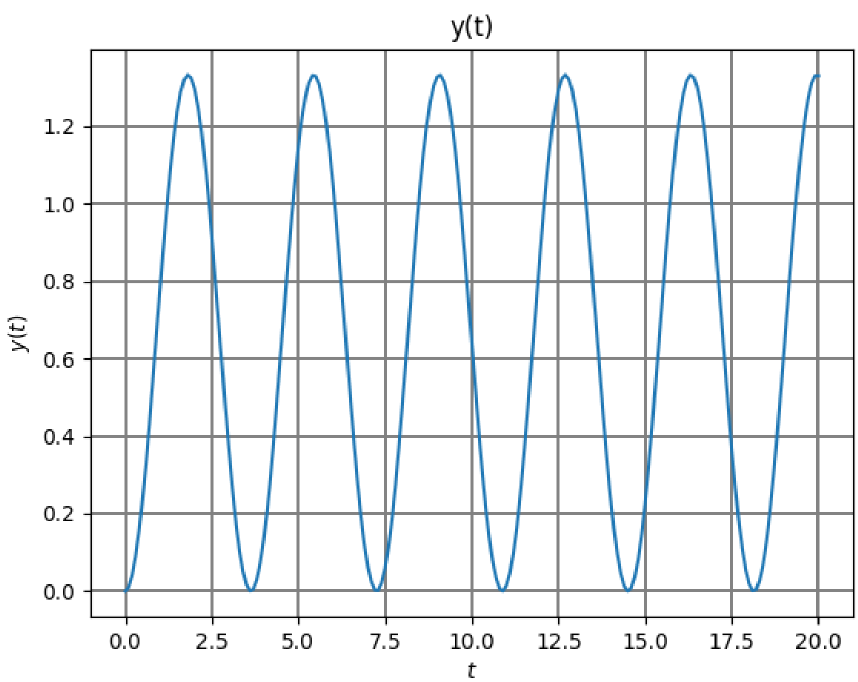

Figure 10: Displacement in the x direction

6 Question-5 Two Port Network :

- consider the RLC circuit shown in the figure
- Input Voltage v_i is applied and the Output Voltage v_o is measured

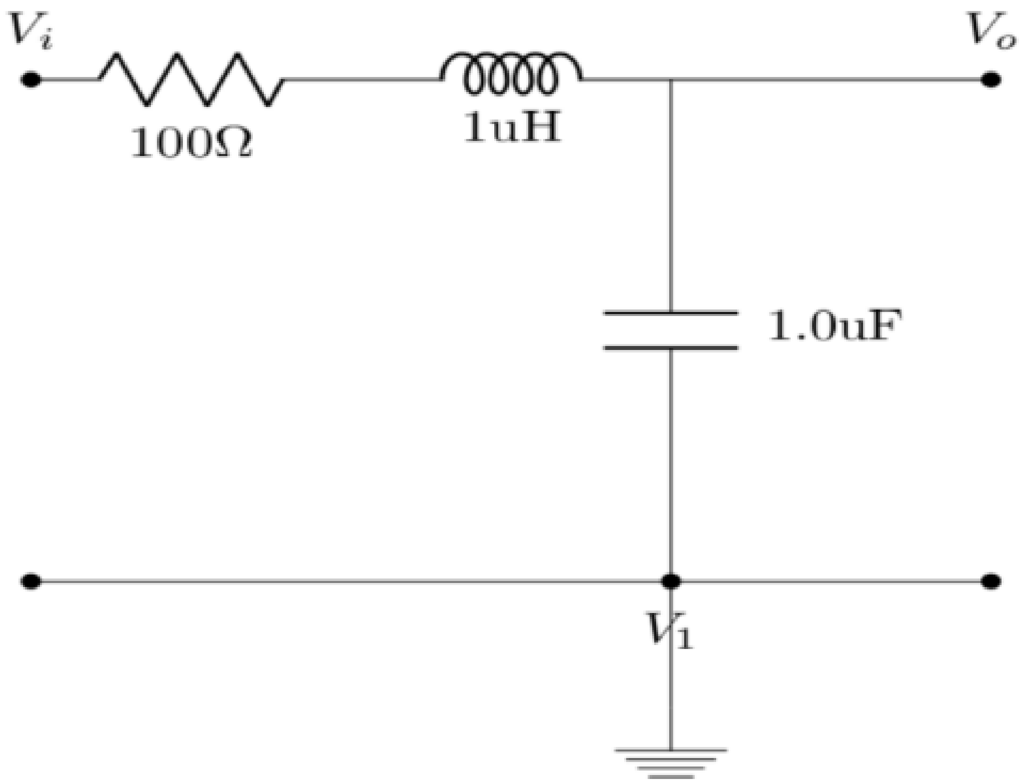

Figure 11: RLC circuit

- In the above circuit,the transfer function $\frac{Vo}{Vi}$ is computed by the nodal equation in S-domain at Vo by taking V1 as ground.

$$\frac{V_o - V_i}{R + sL} + V_o sC = 0$$

$$\frac{V_o}{V_i} = \frac{1}{1 + sRC + s^2 LC}$$

$$\frac{V_o}{V_i} = \frac{1}{1 + 10^{-4}s + 10^{-12}s^2}$$

- Code for finding the Magnitude and Phase response :

```
# Question_5

R = 100
```

```
L = 10**(-6)
C = 10**(-6)

H5 = sp.lti([1],[L*C,R*C,1])

w,S,phi = H5.bode()

figure(5)
plt.semilogx(w,S)
xlabel(r'$w$')
ylabel(r'$|H(s)|$')
plt.grid(True, color = "grey", linewidth = "1.4", linestyle = "-")
title(r'Magnitude_Response')

figure(6)
plt.semilogx(w,phi)
xlabel(r'$w$')
ylabel(r'Phase$(H(s))$')
plt.grid(True, color = "grey", linewidth = "1.4", linestyle = "-")
title(r'Phase_Response')
```

- The Magnitude Response Plot

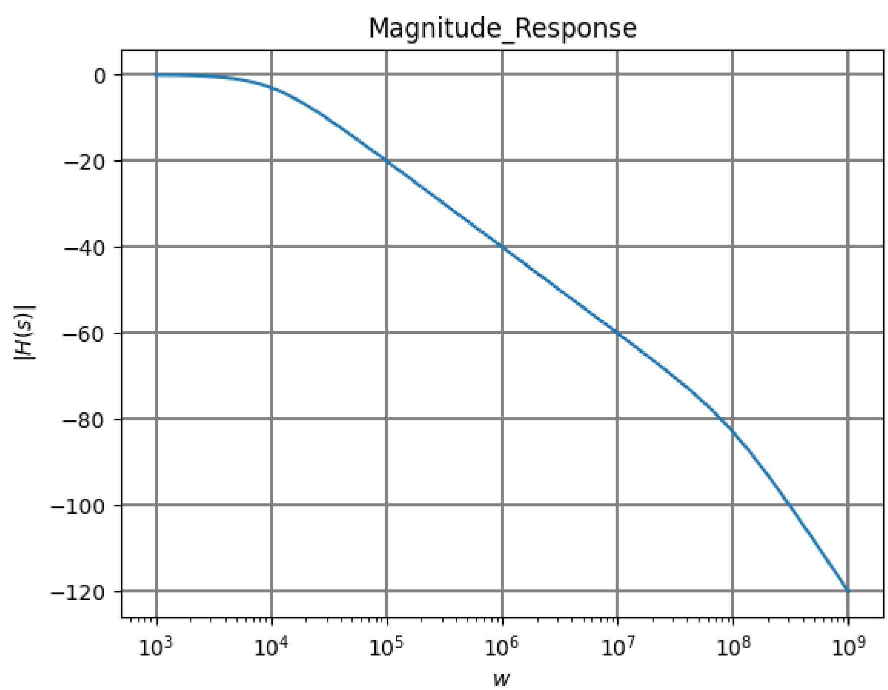

Figure 12: Magnitude Response

- From the graph we can see that there are two distinct poles
- Hence at High frequency the Magnitude Attenuates by -40DB/Decade

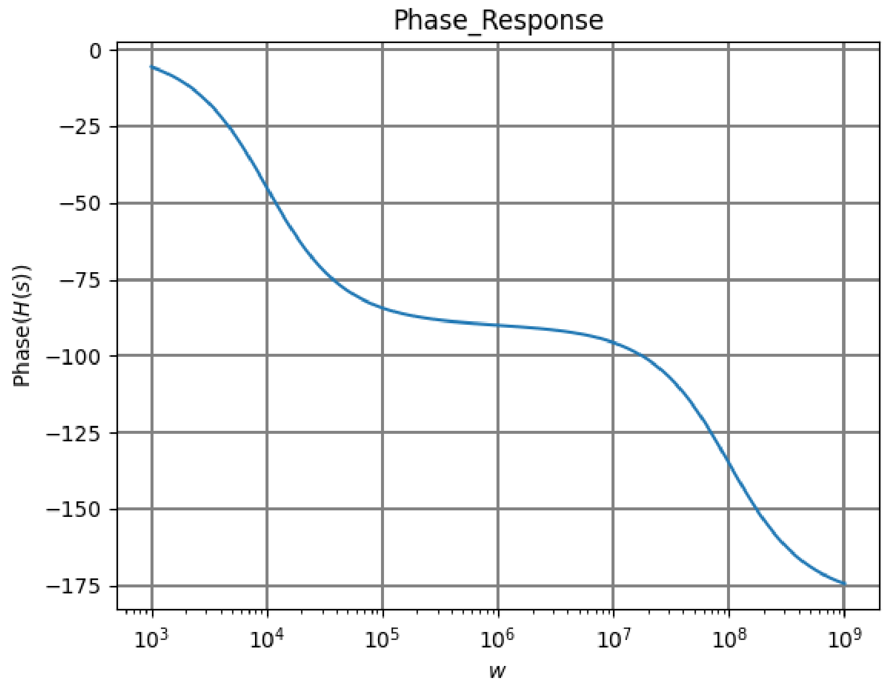

Figure 13: Phase Response

- From the graph as each pole contributes to a phase reduction of 90 degrees
- So at High frequency due to two poles The phase will be reduced by 180 degrees

7 Question-6 Sinusoidal Input to RLC Network :

- The input given to the RLC circuit is

$$V_i = \cos(10^3 t)u(t) - \cos(10^6 t)u(t)$$

- Code for obtaining The output is :

7.1 Steady State Response

```
H5 = sp.lti([1],[L*C,R*C,1])

t2 = linspace(0,0.01,(10**5+1))
vi = np.cos(1000*t2)-np.cos((10**6)*t2)
t2,vo,svec = sp.lsim(H5,vi,t2)

figure(7)
plot(t2,vo)
xlabel(r'$t$')
ylabel(r'V_output')
```

```
plt.grid(True, color = "grey", linewidth = "1.4", linestyle = "-")
title(r'steady state Output of network')
```

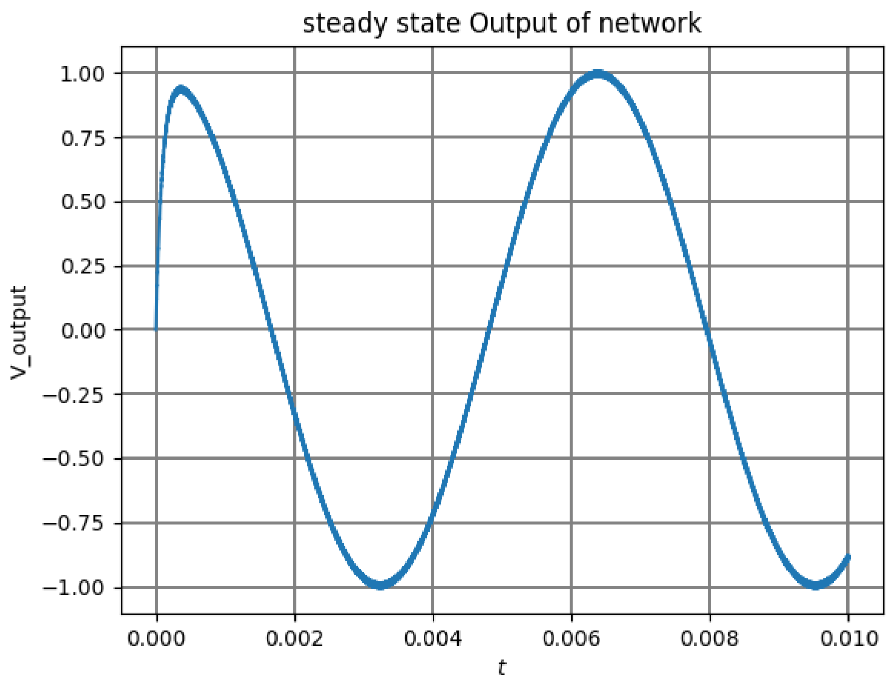

Figure 14: Phase Response

7.2 Transient Response

```
H5 = sp.lti([1],[L*C,R*C,1])

t3 = linspace(0,0.00003,1001)
vi1 = np.cos(1000*t3)-np.cos((10**6)*t3)
t3,vo1,svec = sp.lsim(H5,vi1,t3)

figure(8)
plot(t3,vo1)
xlabel(r'$t$')
ylabel(r'V_output')
plt.grid(True, color = "grey", linewidth = "1.4", linestyle = "-")
title(r'Output of network for T<30usec:Transient response')
show()
```

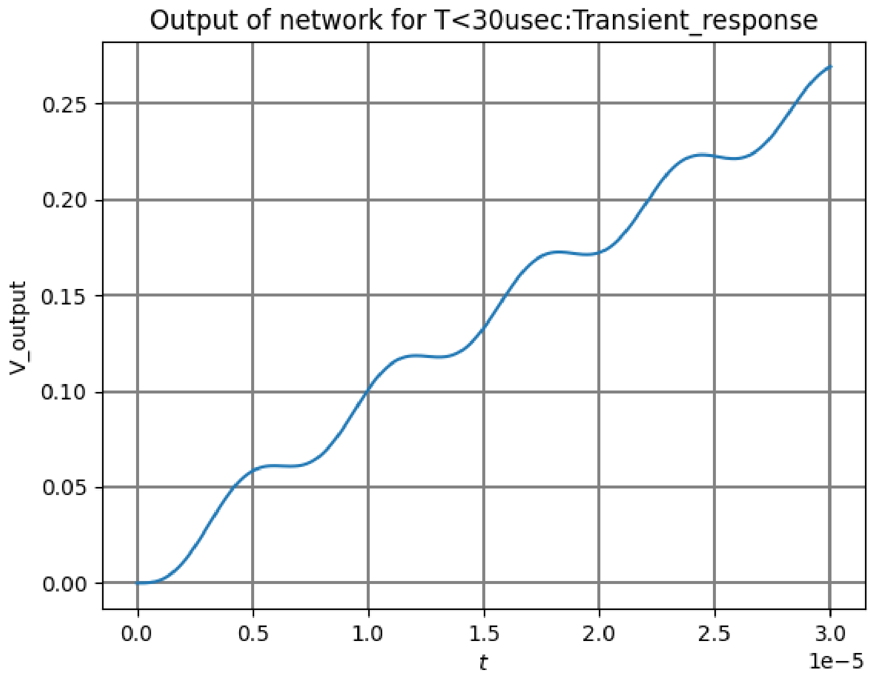

Figure 15: Phase Response

- The poles of the transfer function here are 10^4 and 10^8
- We split the input into two parts : $cos(10^3t)$ and $cos(10^6t)$. So,The frequencies in the input are 10^3 rad/s and 10^6 rad/s.
- From the bode plots,we can infer that there is no effect on the frequencies less than the first pole(10^4rad/s).One of the frequencies of the input,10^3rad/s is thus passed without any change.
- The other frequency 10^6rad/s is between the two poles of the transfer function.So,there will be some effect on it.We can clearly see that the phase at this frequency is -90 .So the cos(106^t) changes to sin(10^6t) in the output.
- Even the magnitude also becomes too small i.e. reduces by around 40dB(100 =0.01) times.

$$\text{Vo(t)} = cos(10^3t) \;\; 0.01sin(10^6t)$$

- It behaves like a low pass filter passing only low frequencies
- From the Bode plot of H(s), we notice that the system provides unity gain for a low frequency of 10^3rad/s . Thus, the low frequency component is more or less preserved in the output.
- However, the system dampens a high frequency of 10^6rad/s, with $|H(s)|_dB \;\; 40$
- This is because of the inherent nature of the given circuit to act as a low pass filter.
- Thus, magnitude of oscillations of these frequency is reduced

8 Conclusions

- We analyzed LTI Systems using Laplace Transform.
- We saw a low pass filter constructed from an RLC circuit.
- We used the scipy signals toolkit to calculate the time domain response and the Bode Plot
- The scipy.signal library provides a useful toolkit of functions for circuit analysis.
- The toolkit was used for the analysis of LTI systems in various domains.
- The forced response of a simple spring body system was obtained over various frequencies of the applied force, and highest amplitude was observed at resonant frequency
- A coupled spring problem was solved using the sp.impulse function to obtain two sinusoids of the same frequency.
- A two-port network, functioning as a low-pass filter was analysed and the output was obtained for a mixed frequency input.