Operations Research - LINMA 2491

Course 3 - Insights on Extended Formulations

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Polyhedra, polytopes and cones

Definition (Polyhedron)

A polyhedron $P \subseteq \mathbb{R}^n$ is a set that can be described by a finite set of linear inequalities:

$$P = \{x \mid A_i x \leq b_i, i = 1, ..., m\} \tag{1}$$

N.B. Alternative definitions can be given, see the Minkowski-Weyl theorem below.

Definition

A polytope is a bounded polyhedron

Definition

The finitely generated cone $C \subseteq \mathbb{R}^n$ by a finite set of vectors $r_1, ..., r_k$ is the cone $C = cone(\{r_1, ..., r_k\}) = \{x \in \mathbb{R}^n \mid x = \mu_1 r_1 + ... + \mu_k r_k, \mu \geq 0\}$

The Minkowski-Weyl theorem

Theorem (The Minkowski-Weyl theorem)

A set P is a polyhedron if and only if P can be described as

$$P = Q + C \tag{2}$$

with $Q = conv(\{v_1, ..., v_K\})$ and $C = cone(\{r_1, ..., r_H\})$

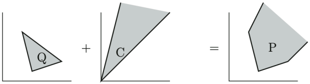

Figure: Illustration of the Minkowski-Weyl theorem taken from [1]

A polyhedron is a polytope (i.e. is bounded) if and only if it is the convex hull of a finite set of points

$\mathcal{H}$-representations and $\mathcal{V}$-representations

In view of the Minkowski-Weyl theorem, a polyhedron can be described in two different ways:

1. $\mathcal{H}$-representations: as the intersection of a finite number of half-spaces

2. $\mathcal{V}$-representations: as the convex hull of a finite set of points (only vertices = extreme points are necessary) + the conic hull of a finite set of vectors (only the "extreme rays" are necessary)

Moving from a $\mathcal{H}$-representation to a $\mathcal{V}$-representation is called a vertex enumeration problem.

Moving from a $\mathcal{V}$-representation to a $\mathcal{H}$-representation is called a convex hull problem (standard problem in computational geometry).

Perfect formulations of Mixer Integer Linear Sets

A perfect formulation for a set $X \subseteq \mathbb{R}^n$ is a description $Ax \leq b$ such that $conv(X) = \{x \mid Ax \leq b\}$

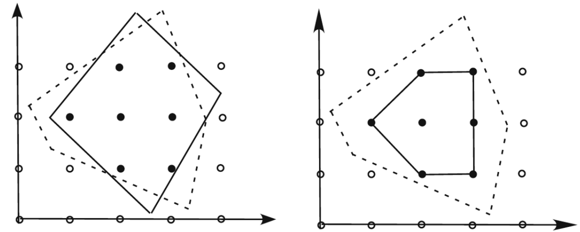

Figure 6.1. Two formulations for X. **Figure 6.2.** The convex hull of X.

Figure: A formulation and a perfect formulation for X, taken from [3].

A perfect formulation might require a large number of constraints and fully characterizing such constraints might be difficult (there are problems for which a tight explicit formulation of X in the "original space" of the variables x is not known), and it may be easier to describe $conv(X)$ as the projection of a higher-dimensional, but easier to describe, polyhedron.

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Projections

Consider a polyhedron $P = \{(x, y) \mid Ax + Gy \leq b\}$.

Its orthogonal projection on the x space (i.e. on the subspace $\{(x, y) \mid y = 0\}$) is $Proj_x(P) = \{x | \exists y, Ax + Gy \leq b\}$

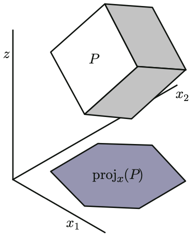


Figure: Illustration of an orthogonal projection, taken from [1]

Observation

$$\max\{c^T x \mid x \in Proj_x(P)\} = \max\{c^T x + 0y \mid (x, y) \in P\} \tag{3}$$

Extended Formulations

Definition (Extended formulation of a polyhedron)

An extended formulation $P = \{(x, y) \mid Ax + Gy \leq b\}$ of a polyhedron Q is a (higher-dimensional) polyhedron such that $Proj_x(P) = Q$.

Definition (Extended formulation for mixed integer linear sets)

A polyhedron $P = \{(x, y) \mid Ax + Gy \leq b\}$ is an extended formulation of a mixed integer linear set $X = \{x \mid Cx \leq d, x_1, ..., x_p \in \mathbb{Z}\}$ if $Proj_x(P) \cap \mathbb{Z}^p = X$, i.e. $Proj_x(P)$ is a *formulation* of X (in the sense that $Proj_x(P)$ combined with the integer requirements describe X)

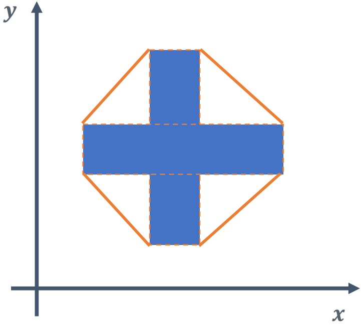
Definition (Tight Extended formulations)

A polyhedron $P = \{(x, y) \mid Ax + Gy \leq b\}$ is a *tight* extended formulation of a mixed integer linear set $X = \{x \mid Cx \leq d, x_1, ..., x_p \in \mathbb{Z}\}$ if $Proj_x(P) = conv(X)$.

Extended Formulations are interesting when they are given by a "small" number of constraints and variables, compared to their projection of interest which would require much more constraints/variables.

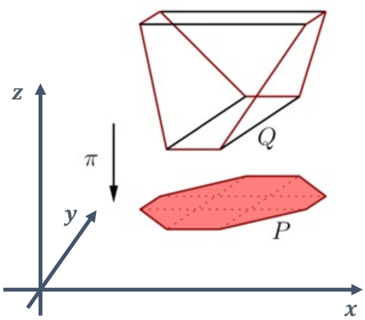
Illustration

Convex hull of the union of 2 polytopes



Convex hull described by 8 constraints and 2 variables (x,y)

Same convex hull described as the projection of an Extended Formulation



Same convex hull can be described by an Extended Formulation with 6 constraints and 3 vars (x,y,z)

Illustration on the right is from: https://arxiv.org/pdf/1107.0371.pdf

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Production Planning example

A manufacturer produces bicycles[1]. Forecasts of the demand are given, corresponding to a non-constant demand over the coming months. The objective is to set a production plan given the following information:

- 1 batch per month at most: due to special equipment and tools to install at the beginning of each batch, at most one batch of bicycles is produced per month
- Setup cost of 5000 €: installation of equipment and tools
- Marginal cost of 100 €: in a batch, each bicycle costs 100€
- Forecasted demand:

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug
400	400	800	800	1200	1200	1200	1200

- Inventory costs: 5€ per bicycle (capital and storage costs)
- Initial stock contains 200 bicycles
- Assumption: inventory evolves linearly from one period to the next, so that inventory costs depend on the average $\dfrac{INV_{t-1} + INV_t}{2}$

[1] Small example taken from [3]

A MIP for our tiny example

MC marginal cost, *FC* the setup cost, D_t is the demand given above for each period and *h* the inventory cost.

- Decision variables?
  - x_t number of bicycles produced at period t
  - y_t binary variable *indicating* if a batch is produced at period t (indicator variable)
  - s_t, inventory level at period t
- Objective? Minimize total costs :=
  - $$\sum_{t=1}^{N}(MC\, x_t + FC\, y_t) + \sum_{t=1}^{N} hs_t$$
- Constraints
  - $s_{t-1} + x_t = D_t + s_t$, for all $t = 1, ..., N$ (demand satisfaction)
  - $x_t \leq \ y_t(\sum_{k=t}^{N} D_t)$, for all $t = 1, ..., N$ (variable upper bound)
  - $s_0 = 200, s_N = 0$ (zero initial and final inventories)
  - $x_t, s_t \geq 0, y_t \in \{0, 1\}$

Uncapacitated Lot Sizing problem (LS-U)

$$\min_{x,y,s} \sum_{t=1}^{N} (p_t \; x_t + q_t \; y_t + h_t s_t) \tag{4}$$

subject to:

$$s_{t-1} + x_t = D_t + s_t \qquad \forall t \in PERIODS \tag{5}$$

$$x_t \leq (\sum_{k=t}^{N} D_t) y_t \qquad \forall t \in PERIODS \tag{6}$$

$$s_0 = s_{ini}, s_N = 0 \tag{7}$$

$$x_t, s_t \geq 0 \qquad \forall t \in PERIODS \tag{8}$$

$$y_t \in \{0, 1\} \qquad \forall t \in PERIODS \tag{9}$$

In the small example: $N = 8, s_{init} = 200, p = 100, q = 5000, h = 5,$
$D = [400, 400, 800, 800, 1200, 1200, 1200, 1200]$

"Facility Location" extended formulation for LS-U

It can be shown that the following extended formulation for LS-U is tight:

$$s_{u-1} + x_u = D_u + s_u \qquad \forall u \in PERIODS \qquad (10)$$

$$\sum_{u=1}^{t} w_{ut} = D_t \qquad \forall u \in PERIODS \qquad (11)$$

$$w_{ut} \leq D_t y_u \qquad \forall 1 \leq u \leq t \leq NT \qquad (12)$$

$$x_u = \sum_{t=u}^{NT} w_{ut} \qquad \forall u \in PERIODS \qquad (13)$$

$$s_0 = s_{ini}, s_N = 0 \qquad (14)$$

$$w_{ut}, s_t \geq 0 \qquad \forall t \in PERIODS \qquad (15)$$

$$y_t \in \{0, 1\} \qquad \forall t \in PERIODS \qquad (16)$$

- For any objective function, we can optimize the continuous relaxation and obtain an optimal solution with y_t binary, solving the original problem.
- Let us again verify this on the small example, using a computer

Retrieving an integer feasible optimal solution

Consider a mixed integer linear set $Q = \{x \mid Ax \leq b, x_1, ..., x_p \in \mathbb{Z}\}$ and a tight extended formulation P for Q. One has:

$$\max\{c^T x \mid x \in Q\} = \max\{c^T x \mid x \in conv(Q)\} = \max\{c^T x \mid x \in Proj_x(P)\}\} \tag{17}$$

With $conv(Q)$, we know that vertices are "integral" (the variable values $x_1, ..., x_p$ are integer values). However, even though $Proj_x(P) = conv(Q)$, the projection of a vertex of $Proj_x(P)$ may not be a vertex of $conv(Q)$.

Hence, we may need some extra work to be able to retrieve an optimal solution which is a vertex $conv(Q)$, see Figure 5.

Vertices of an EF versus vertices of the projection

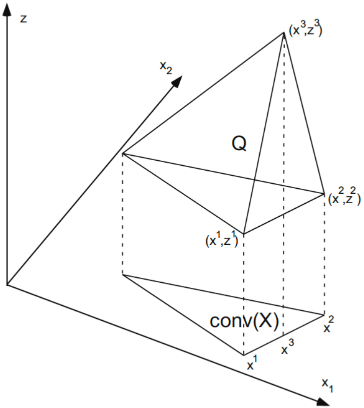

Figure: Illustration of vertices of an extended formulation not necessarily vertices of the projection, taken from [3].

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Union of polyhedra

The notes focus on polytopes (i.e. bounded polyhedra), but also point how to consider more generally polyhedra.
Consider N *non-empty* polytopes $P_i = \{x \mid A^i x^i \leq b^i\}, i = 1, ..., N$. One can easily express that a point x belongs to the union of these polytopes as follows:

$$\cup_i P_i = Proj_x(X), where\ X\ is\ given\ by: \tag{18}$$

$$x = \sum_i x^i \tag{19}$$

$$A^i x^i \leq b^i\ y_i \tag{20}$$

$$\sum_i y_i = 1 \tag{21}$$

$$y \in \{0, 1\}^N \tag{22}$$

N.B. Since we consider non-empty polytopes, if $y_i = 0$, (20) will imply $x_i = 0$ (why?).

N.B. A union of polytopes/polyhedra correspond to a logical disjunction ($\wedge$ means 'OR') $p_1 \wedge ... \wedge p_N$ where p_i is the statement $x_i \in P_i$

Balas formulation for the convex hull of an union of polyhedra

It turns out that the continuous relaxation below (dropping the binary requirements and letting $y_i \in [0,1]$) describes $\overline{conv(\cup_i P_i)}$:

$$\overline{conv(\cup_i P_i)} = Proj_x(X), where\ X\ is\ given\ by: \tag{23}$$

$$x = \sum_i x^i \tag{24}$$

$$A^i x^i \leq b^i\ y_i \qquad \forall i \tag{25}$$

$$\sum_i y_i = 1 \tag{26}$$

$$y_i \leq 1 \qquad \forall i \tag{27}$$

$$y \geq 0 \tag{28}$$

N.B. The above result holds true for polyhedra as well, in which case considering $\overline{conv(\cup_i P_i)}$ instead of $conv(\cup_i P_i)$ is important (see next slides). For polytopes however, $\overline{conv(\cup_i P_i)} = conv(\cup_i P_i)$.

Proof: Exercise. (Hint. Start for example with 2 polytopes and express that a point is a convex combination of points in both polytopes.) □

Importance of considering $\overline{conv(\cup_i P_i)}$ instead of $conv(\cup_i P_i)$ for general polyhedra in Balas formulation

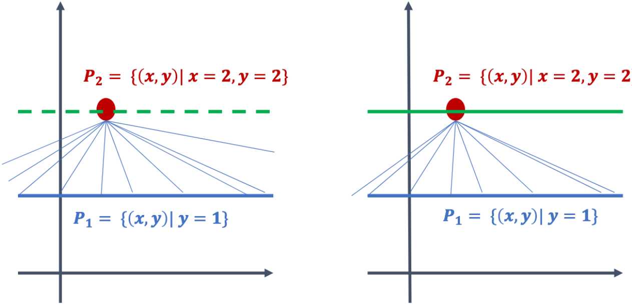

Figure: Illustration of a difference between $\overline{conv(\cup_i P_i)}$ and $conv(\cup_i P_i)$

$$conv(P_1 \cup P_2) = \{(x, y) \mid y \geq 1, y < 2\} \cup \{(2, 2)\}$$

$$\overline{conv(P_1 \cup P_2)} = \{(x, y) \mid 1 \leq y \leq 2\}$$

Application to Financial (electricity) Transmission Rights (FTRs)

Separate deck of slides.

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Generalization of Balas formulation for the union of polyhedra

The Balas formulation above for the convex hull of the union of polytopes (restated below) essentially describes $y_1 P_1 + \ldots + y_N P_N$ for

$$y \in \{y_i \mid y_i \in [0,1], \sum_i y_i = 1\} = conv(\{y_i \mid y_i \in \{0,1\}, \sum_i y_i = 1\}) \qquad (29)$$

.

Balas formulation:

$$x = \sum_i x^i \qquad (30)$$

$$A^i x^i \leq b^i \; y_i \qquad \forall i \qquad (31)$$

$$\sum_i y_i = 1 \qquad (32)$$

$$y_i \leq 1 \qquad \forall i \qquad (33)$$

$$y \geq 0 \qquad (34)$$

N.B. $y_1 P_1 + \ldots + y_N P_N$ denotes $\{y_1 x_1 + \ldots + y_N x_N \mid x_i \in P_i, \; i = 1, \ldots, N\}$
What can we say if we replace (29) by $y \in Y$ with Y some other integral polyhedron Y (i.e. whose vertices are integral) ?

Generalization of Balas formulation for the union of polyhedra

Consider non-empty polytopes $P_i = \{x \mid A^i x \leq b^i\}$, $i = 1, ..., m$ and $Y \subseteq \mathbb{R}^m_+$ a non-empty polytope (in the non-negative orthant) with integer vertices and which contains at least one $y > 0$ (i.e., $y_i > 0 \forall i = 1, ..., m$).

Consider Q defined by

$$A^i x^i \leq b^i y_i \qquad i = 1, ..., m \tag{35}$$

$$x = \sum_{i=1}^{m} x^i \tag{36}$$

$$(y_1, ..., y_m) \in Y \tag{37}$$

Then

1. $Proj(Q) = P := \bigcup_{y \in Y} (\sum_{i=1}^{m} y_i P_i)$
2. all the vertices of Q have integer $(y_1, ..., y_m)$

See [2] and references therein.

References I

[1] M. Conforti, G. Cornuéjols, G. Zambelli, et al.
Integer programming, volume 271.
Springer, 2014.

[2] B. Knueven, J. Ostrowski, and J. Wang.
The ramping polytope and cut generation for the unit commitment problem.
INFORMS Journal on Computing, 30(4):739–749, 2018.

[3] Y. Pochet and L. A. Wolsey.
Production Planning by Mixed Integer Programming.
Springer, 2006.