

Operations Research - LINMA 2491

Complementary notes on the L-shaped method

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How does a Benders decomposition look like for a 2-stage stochastic program ? The L-shaped method

Consider the 2-stage stochastic program in extensive form:

$$\min_{x,y} c^T x + \sum_{\omega=1}^N p_{\omega} q_{\omega}^T y_{\omega} \quad (1)$$

$$Ax = b \quad (2)$$

$$T_{\omega}x + W_{\omega}y_{\omega} = h_{\omega} \quad \omega = 1, \dots, N \quad (3)$$

$$x \geq 0, y_{\omega} \geq 0 \quad \omega = 1, \dots, N \quad (4)$$

The problem can be reformulated as:

$$\min_{x \in K_1 \cap K_2} \{c^T x + \{\min_y \sum_{\omega=1}^N p_{\omega} q_{\omega}^T y_{\omega} \mid W_{\omega} y_{\omega} = h_{\omega} - T_{\omega} x, y_{\omega} \geq 0, \omega = 1, \dots, N\}\} \quad (5)$$

where $K_1 = \{x \mid Ax = b\}$ and $K_2 = \bigcap_{\omega} K_2(\omega)$ with

$$K_2(\omega) = \{x \mid \exists y : T_{\omega} x + W_{\omega} y = h_{\omega}, y \geq 0\}.$$

- ▶ The blue problem objective value is a function of x , $V : x \rightarrow V(x)$, called the expected value function.
- ▶ This blue problem can be split into N independent sub-problems, by which the value function $V(x)$ can be seen as an expectation of costs corresponding to the best reaction given x and scenario ω :

$$V(x) = \sum_{\omega=1}^N p_{\omega} Q_{\omega}(x) \quad (6)$$

where $Q_{\omega}(x) = \min_{y_{\omega}} q_{\omega}^T y_{\omega} \mid W_{\omega} y_{\omega} = h_{\omega} - T_{\omega} x, y_{\omega} \geq 0$

By LP duality, the original 2-stage stochastic program is equivalent to

$$\min_{x \in K_1 \cap K_2} \{c^T x + \{\max_{\pi} \sum_{\omega=1}^N \pi_{\omega}^T (h_{\omega} - T_{\omega} x) \mid \pi_{\omega}^T W_{\omega} \geq p_{\omega} q_{\omega}^T, \omega = 1, \dots, N\}\} \quad (7)$$

The Benders reformulation of the original 2-stage stochastic program is then:

$$\min_x c^T x + \theta \quad (8)$$

$$Ax = b \quad (9)$$

$$\theta \geq \sum_{\omega=1}^N \pi_{\omega}^T (h_{\omega} - T_{\omega} x) \quad \forall \pi \mid \pi_{\omega}^T W_{\omega} \geq p_{\omega} q_{\omega}^T, \omega = 1, \dots, N \quad (10)$$

where (10) can be replaced by (see exercise 1 in TP 3):

► (Opt. cuts) $\theta \geq \sum_{\omega} (\pi_{\omega}^v)^T (h_{\omega} - T_{\omega} x)$ for each vertex $(\pi_1^v, \dots, \pi_N^v)$ of

$$P := \{\pi \mid \pi_{\omega}^T W_{\omega} \geq p_{\omega} q_{\omega}^T, \omega = 1, \dots, N\}$$

► (Feas. cuts) $0 \geq \sum_{\omega} (\sigma_{\omega}^h)^T (h_{\omega} - T_{\omega} x)$ for each extreme ray (σ^h) of P

Benders decomposition

As with a classic Benders decomposition, at each iteration, for a candidate solution x^*, θ^* solving (8) subject to (9) plus some of the constraints in (10) (the cuts obtained at previous iterations), we can check if (10) fully holds by solving a Benders sub-problem. If (10) holds, we are done and have an optimal solution to the original problem (the values for the y_ω can straightforwardly be recomputed, how ?).

Otherwise, the sub-problem identifies a violated optimality cut or feasibility cut to be added to the master problem. Since there is a finite number of such cuts (because they correspond respectively to vertices or extreme rays of the same polytope P defined in the previous slide), we know that the algorithm will converge to an optimal solution in a finite number of iterations.

The good news is that the Benders sub-problem can be split into N independent sub-problems: one per scenario $\omega = 1, \dots, N$.

► Benders sub-problem. How to test if (10) is satisfied ? Test if

$$\theta^* \geq \max_{\omega} \sum_{\omega} \pi_{\omega}^T (h_{\omega} - T_{\omega} x) \quad s.t. \quad \pi_{\omega}^T W_{\omega} \geq p_{\omega} q_{\omega}^T, \omega = 1, \dots, N \quad (11)$$

For this purpose and to obtain the feasibility/optimality cuts, we will solve for each ω :

$$\max \pi_{\omega}^T (h_{\omega} - T_{\omega} x) \quad (12)$$

$$\pi_{\omega}^T W_{\omega} \geq q_{\omega} \quad (13)$$

Note that p_{ω} is not present in the RHS of (13).

Optimality cuts from split Benders sub-problems

Benders sub-problem split into N
independent sub-problems

Problem (D_ω) , dual of (S_ω) :

$$\max \pi_\omega^T (h_\omega - T_\omega x) \quad (14)$$

$$\pi_\omega^T W_\omega \geq q_\omega \quad (15)$$

Benders sub-problem (D) , dual of
 (S) :

$$\max \sum_{\omega=1}^N \pi_\omega^T (h_\omega - T_\omega x) \quad (16)$$

$$\pi_\omega^T W_\omega \geq p_\omega q_\omega^T, \quad \omega = 1, \dots, N \quad (17)$$

If (14)-(15) admits an optimal solution (π_ω^*) for each ω , $(p_1 \pi_1^*, \dots, p_N \pi_N^*)$ is an optimal solution of (16)-(17). (It is assumed that the $p_{\omega \neq 0}$.)

More precisely, vertices of the set (17) correspond to points of the form $(p_1 \pi_1^*, \dots, p_N \pi_N^*)$ with for each ω , (π_ω^*) a vertex of (15) (show it)

The corresponding optimality cuts (see previous slides) have a nice geometric interpretation: they correspond to supporting hyperplanes of the Expected Value Function $V(x)$, which can be shown to be a convex piece-wise linear function of x (either thanks to sensitivity analysis for linear programs or in view of slide 5: there is 1 opt. cut per vertex of the Benders subproblem feas. set P).

Illustration for optimality cuts

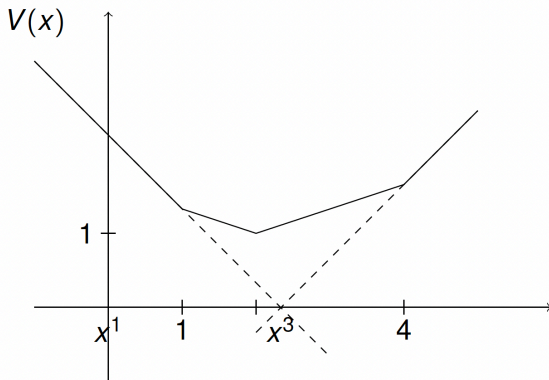


Figure: Illustration for optimality cuts, taken from the slides of Prof. A. Papavasiliou for LINMA2491

N.B. Optimality cuts are needed as long as the candidate θ value underestimates the true value of $V(x) = \text{expected second-stage costs given } x$.

Feasibility cuts from split Benders sub-problems

Benders sub-problem split into N
independent sub-problems

Problem (D_ω) , dual of (S_ω) :

$$\max \pi_\omega^T (h_\omega - T_\omega x) \quad (18)$$

$$\pi_\omega^T W_\omega \geq q_\omega \quad (19)$$

Benders sub-problem (D) , dual of
 (S) :

$$\max \sum_{\omega=1}^N \pi_\omega^T (h_\omega - T_\omega x) \quad (20)$$

$$\pi_\omega^T W_\omega \geq p_\omega q_\omega^T, \quad \omega = 1, \dots, N \quad (21)$$

Extreme rays of the set (21) are of the form $(0, \dots, \sigma_\omega, \dots, 0)$ with for each ω , σ_ω an extreme ray of (19) (show it).

The corresponding feasibility cuts (see previous slides) have a nice geometric interpretation: they correspond to cutting planes (linear constraints cutting off a candidate point outside of the set/domain of interest) describing the domain of the Expected Value Function $V(x)$, that is $K_1 \cap K_2$, i.e. the set of feasible first-stage decisions x for which, for each scenario, there exist second-stage decisions satisfying the second-stage constraints (these second-stage constraints depend on x).

Multi-cut L-shaped method

Instead of the Benders reformulation:

$$\min_x c^T x + \theta \quad (22)$$

$$Ax = b \quad (23)$$

$$\theta \geq \sum_{\omega=1}^N \pi_{\omega}^T (h_{\omega} - T_{\omega} x) \quad \forall \pi = (\pi_1, \dots, \pi_N) \mid \pi_{\omega}^T W_{\omega} \geq p_{\omega} q_{\omega}^T, \omega = 1, \dots, N \quad (24)$$

Consider

$$\min_x c^T x + \sum_{\omega} p_{\omega} \theta_{\omega} \quad (25)$$

$$Ax = b \quad (26)$$

$$\theta_{\omega} \geq \pi_{\omega}^T (h_{\omega} - T_{\omega} x) \quad \forall \pi_{\omega} \mid \pi_{\omega}^T W_{\omega} \geq q_{\omega}^T, \omega = 1, \dots, N \quad (27)$$

and then apply the mechanics.

References I



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Introduction to stochastic programming.

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