Lagrangian duality and market equilibria

LINMA 2491

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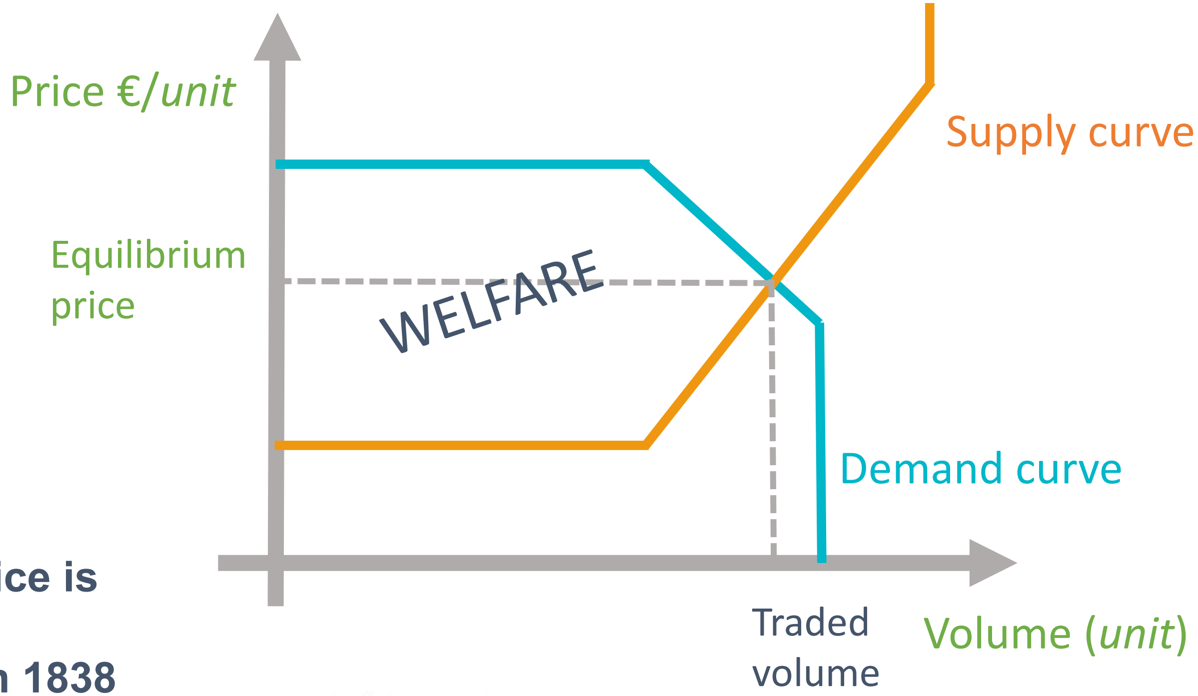
Welfare maximization and competitive equilibrium - Economic principles

Paul Samuelson
(Nobel prize in economics in 1970)

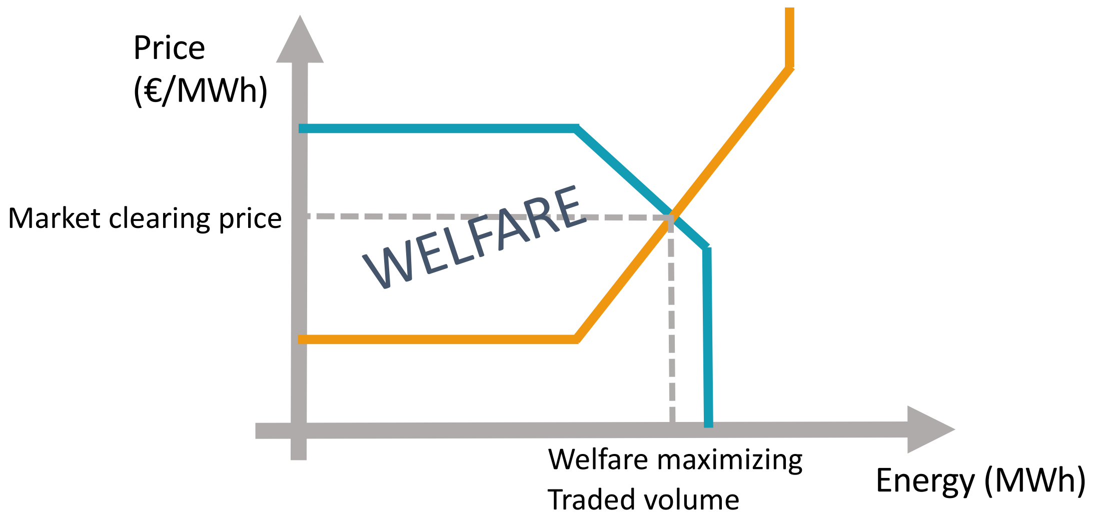
Spatial Price Equilibrium and Linear Programming
Paul A. Samuelson
The American Economic Review
Vol. 42, No. 3 (Jun., 1952), pp. 283-303



"The first explicit statement that competitive market price is determined by the intersection of supply and demand functions seems to have been given by A. A. Cournot in 1838 in connection, curiously enough, with the more complicated problem of price relations between two spatially separated markets – such as Liverpool and New-York. The latter problem, that of "communication of markets", has itself a long history, involving many of the great names of theoretical economics. [...]"

Cournot

Markets with Continuous (convex) Orders: **welfare maximization ⇔ competitive equilibrium**



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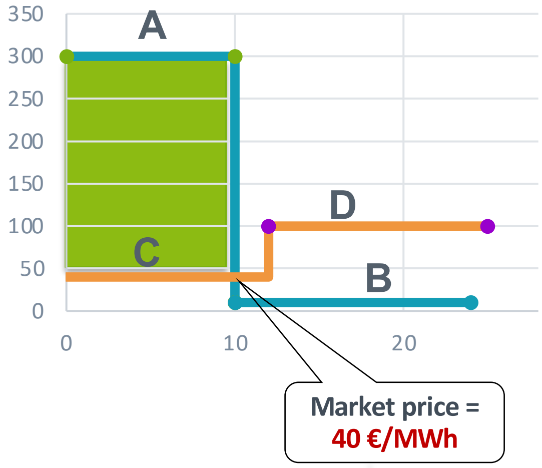
Welfare: total utility of demand – total costs of offers
(taking accepted orders into account)

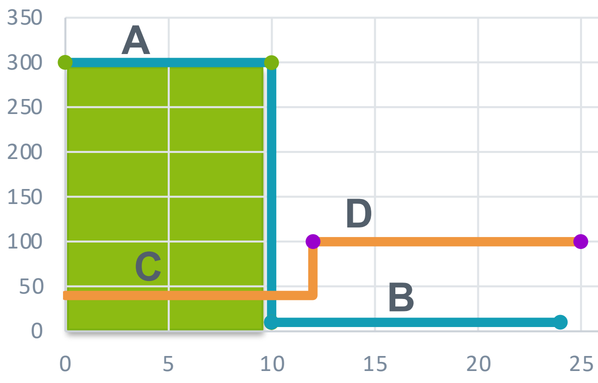
Samuelson coined the term "Cournot-Enke equilibrium" for the concept of spatial price equilibrium

Welfare maximization, competitive equilibrium and duality

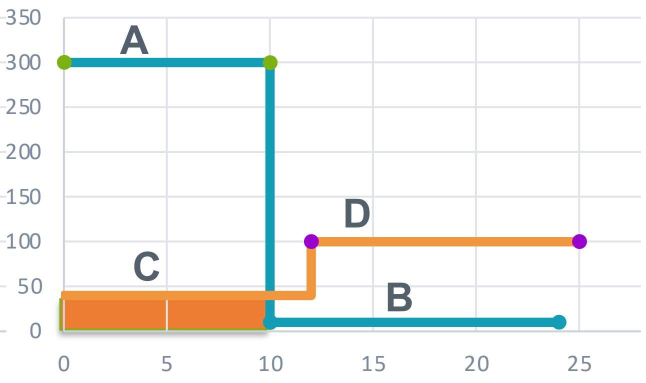
Welfare =



Utility of demand



Costs of production



Bids	Quantity (MWh)	Limit Price (€/MWh)
A buy	10	300
B buy	14	10
C sell	12	40
D sell	13	100

Welfare maximization program

$$\max_{x} \; (10)(300)x_a + (14)(10)x_b - (12)(40)x_c - (13)(100)x_d$$

(1) $10x_a + 14x_b = 12x_c + 13\,x_d$ [optimal dual var. **p* = 40**]

(2) $0 \leq x_a, x_b, x_c, x_d \leq 1$

Optimal (primal) solution

$$x_a = 1, x_b = 0, x_c = \frac{10}{12}, x_d = 0$$

Welfare maximization, competitive equilibrium and duality

Welfare maximization program

$$\max_x \quad (10)(300)x_a + (14)(10)x_b - (12)(40)x_c - (13)(100)x_d$$

$$10x_a + 14x_b = 12x_c + 13\,x_d \qquad [p^* = 40]$$

$$0 \le x_a, x_b, x_c, x_d \le 1$$

Form the LD
By dualizing the
balance constraint

Lagrangian dual by dualization of the balance condition

$$\min_p \left\{ \begin{array}{l} \max_x \quad (10)(300)x_a + (14)(10)x_b \\ \quad -(12)(40)x_c - (13)(100)x_d \\ \quad - \;\; \boldsymbol{p(10\,x_a + 14x_b - 12x_c - 13\,x_d)} \\ \text{Subject to:} \\ 0 \le x_a, x_b, x_c, x_d \le 1 \end{array} \right.$$

$\Longleftrightarrow$ (as the inner problem is separable in x)

Lagrangian dual by dualization of the balance condition

$$\min_p \left[\begin{array}{l} \max_{0 \le x_a \le 1} (10)(300 - \boldsymbol{p})x_a \\ + \max_{0 \le x_b \le 1} (14)(10 - \boldsymbol{p})x_a \\ + \max_{0 \le x_c \le 1} (12)(\boldsymbol{p} - \mathbf{40})x_a \\ + \max_{0 \le x_d \le 1} (13)(\boldsymbol{p} - \mathbf{100})x_a \end{array} \right]$$

Optimal primal solution

$$x_a = 1, x_b = 0, x_c = \frac{10}{12}, x_d = 0$$

Optimal dual var. $p^* = 40$

solves:

Competitive equilibrium

Acceptances x_i^* and a market price p such that:

- **x_i^* solves $\max_{x_i} Q_i(LimitPrice_i - p)x_i$**
- **$\sum_i Q_i\, x_i^* = 0$ (the market clears)**

convention:
$Q_i > 0$ for sell orders, and $Q_i < 0$ for buy orders

Welfare maximization, competitive equilibrium and duality

Welfare maximization program

$$\max_x \sum_i c_i^T x_i$$

$\sum_i A_i x_i = 0 \quad$ [p^*]

$B_i x_i \leq b_i \qquad \forall\ market\ participant\ i$

Lagrangian dual by dualization of the balance conditions

$$\min_p \sum_i \max_{x_i \mid B_i x_i \leq b_i} (c_i^T - {p^*}^T A_i) x_i$$

⇕

Competitive equilibrium

Vectors (of acceptances) x_i^* and market prices p^* such that:

- x_i^* solves $\max_{x_i \mid B_i x_i \leq b_i} (c_i^T - {p^*}^T A_i) x_i$
- $\sum_i A_i x_i^* = 0$ (the market clears)

Welfare maximization and market equilibrium in general

Welfare maximization problem

$$\max_x \sum_k c_k^T x_k, \tag{A.1}$$

$$\text{s.t.} \sum_k A_k x_k = 0, \qquad [\pi] \tag{A.2}$$

$$B_k x_k \le b_k \qquad \forall k \tag{A.3}$$

Definition 1 (*Competitive (or Walrasian) Equilibrium*). A competitive equilibrium consists in decisions $(x_k^*)_{k\in K}$ and prices π^* such that:

- for each k, for the given fixed prices π^*, (x_k^*) solves

  $$\max_{x_k} c_k^T x_k - (\pi^*)^T (A_k x_k), \tag{A.4}$$

  subject to

  $$B_k x_k \le b_k, \tag{A.5}$$

- The market clears: $\sum_k A_k x_k^* = 0$, cf. condition (A.2) above.

Theorem A.1 (*Generalisation of the Main Result in [57] and Special Case without Binary Decisions of Theorem 2 in [55], with Simplified Proof*). *Let us consider an optimal solution $(x_k^*)_{k\in K}$ to the welfare maximisation problem (A.1)–(A.3). Let π^* be obtained as (linear programming) optimal dual variables to the constraints (A.2) in the linear optimisation problem (A.1)–(A.3). The solution $(x_k^*)_{k\in K}$ and the prices π^* form a competitive equilibrium as defined in Definition 1.*

Proof. Balance constraints (A.2) are directly satisfied by $(x_k^*)_{k\in K}$ solving (A.1)–(A.3). It remains to show that $(x_k^*)_{k\in K}$ also solves (A.4)–(A.5).

This follows from the fact, proved below, that π^* and $(x_k^*)_{k\in K}$ solve the following Lagrangian dual problem, or "partial linear programming dual" of (A.1)–(A.3) where the balance conditions (A.2) are dualised

http://refhub.elsevier.com/
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iers π:

$$\min_\pi \left\{ \max_x \sum_k c_k^T x_k - \pi^T (A_k x_k), \ s.t.\ B_k x_k \le b_k \ \forall k \right\}$$

$$= \min_\pi \left\{ \sum_k \{ \max_{x_k} c_k^T x_k - \pi^T (A_k x_k), \ \text{s.t. } B_k x_k \le b_k \} \right\}. \tag{A.6}$$

The equality in (A.6) follows from the fact that the inner maximisation problem in the left-hand side can be separated per market participant k. It can then be seen that in the right-hand side, for the π fixed, the inner problems are exactly (A.4)–(A.5) written for each participant k, that must be solved by $(x_k^*)_{k\in K}$ to obtained the desired result.

Let us verify that $(x_k^*)_{k\in K}$ indeed solve the inner maximisation problems in (A.6). By strong duality for linear programs (see Theorem 1 in [58] and [59], Section 6.1 for more details on strong duality for such a partial dual), π^* solves the left-hand side of (A.6), and:

$$\sum_k c_k^T x_k^* = \min_\pi \left\{ \max_x \sum_k \left(c_k^T x_k - \pi^T (A_k x_k) \right), \ \text{s.t. } B_k x_k \le b_k \ \forall k \right\}. \tag{A.7}$$

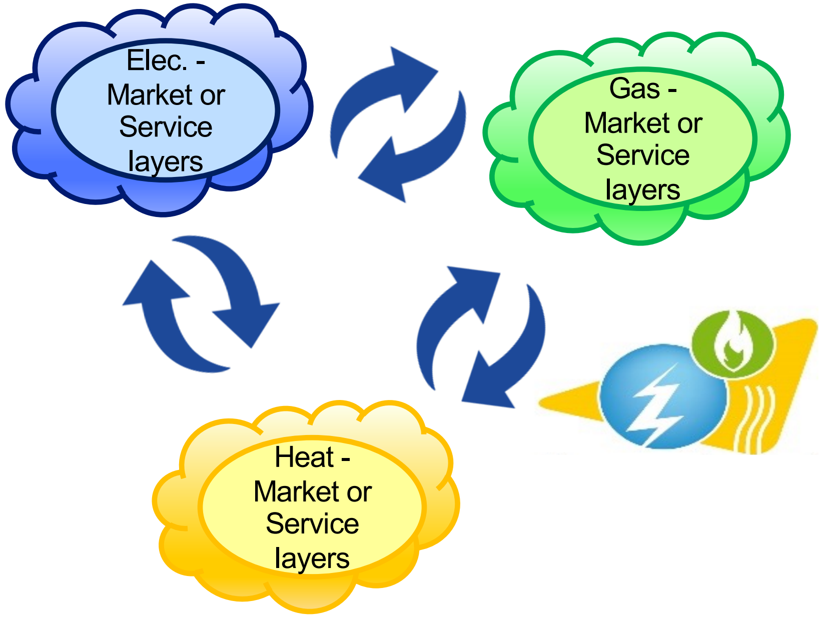
Using (A.2) multiplied by π and the fact that π^* solves (A.6), we have:

$$\sum_k c_k^T x_k^* - (\pi^*)^T (A_k x_k) = \sum_k \left(\max_{x_k} c_k^T x_k - (\pi^*)^T (A_k x_k), \ \text{s.t. } B_k x_k \le b_k \right). \tag{A.8}$$

Since for all k, x_k^* satisfies $B_k x_k \le b_k$, i.e. the x_k^* are feasible for the profit maximisation problems in the right-hand side of (A.8), they must also be optimal solutions for these problems: otherwise, (A.8) would not hold and the right-hand-side would be strictly greater than the left-hand side, contradicting strong duality for linear programs. □

Source: Appendix A in Torbaghan, S.S., Madani, M., Sels, P., Virag, A., Le Cadre, H., Kessels, K. and Mou, Y., 2021. Designing day-ahead multi-carrier markets for flexibility: Models and clearing algorithms. *Applied Energy*, *285*, p.116390.

Magnitude: Multi-energy Markets integration



- Innovative market designs for coupled multi-energy carriers enabling to leverage the value of flexibility services
- Market simulator

- EU 2020 and 2030 targets for the reduction of greenhouse gas emission
- Increasing share of variable renewable energy sources
- Expected increase of electricity demand (new usages such as electric vehicles, heat pumps)

…

➔**Needs for more flexibility**, more active involvement of all the stakeholders and more collaboration… at all levels (from distribution to pan-European)

➔**Need to harness the service provision capabilities** of both **centralized and decentralized resources** in a coordinated way (including both consumers and producers resources).

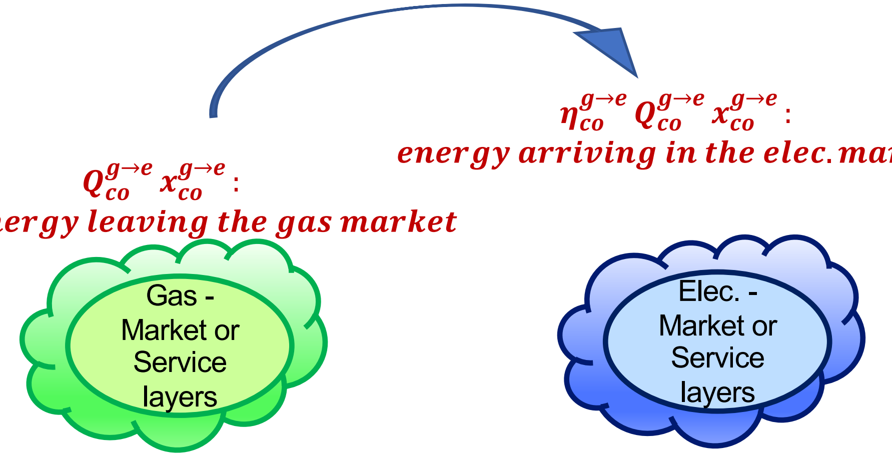
➔ **Enhanced synergies between different energy carriers** appear now as one of the **possible means to provide flexibility to the electricity system** but also to drive efficiency **and business innovation in the energy sector as a whole.**

Horizon 2020
European Union Funding
for Research & Innovation

Conversion orders and market equilibrium

$\boldsymbol{\eta_{co}^{g\to e}\, Q_{co}^{g\to e}\, x_{co}^{g\to e}}$: ***energy arriving in the elec. market***

$\boldsymbol{Q_{co}^{g\to e}\, x_{co}^{g\to e}}$: ***energy leaving the gas market***



$\boldsymbol{P_{co}^{g\to e}}$ ***= marginal cost of conversion co***

Welfare max. program (WMP)

$$\max_{x\in X} \sum_{o\in\mathcal{O}^g} P_o^g Q_o^g x_o + \sum_{o\in\mathcal{O}^e} P_o^e Q_o^e x_o + \sum_{o\in\mathcal{O}^h} P_o^h Q_o^h x_o - \sum_{co\in\mathcal{O}^{g\to e}} P_{co}^{g\to e} Q_{co}^{g\to e} x_{co}^{g\to e}$$

$$\sum_{o\in\mathcal{O}^g} Q_o^g x_o^g = - \sum_{co\in\mathcal{O}^{g\to e}} Q_{co}^{g\to e} x_{co}^{g\to e} \qquad [\pi^g]$$

$$\sum_{o\in\mathcal{O}^e} Q_o^e x_o^e = \sum_{co\in\mathcal{O}^{g\to e}} \eta_{co}^{g\to e} Q_{co}^{g\to e} x_{co}^{g\to e} \qquad [\pi^e]$$

$$\sum_{o\in\mathcal{O}^h} Q_o^h x_o^h = 0 \qquad [\pi^h]$$

$$0 \le x \le 1$$

Balance conditions for gas, electricity and heat

$\boldsymbol{\eta_{co}^{g\to e}}$ ***= efficiency of conversion co***

$\boldsymbol{Q_{co}^{g\to e}}$ ***= total conversion capacity***

Equilibrium for conversion orders

Proposition. Let x^* be an optimal solution to **WMP** and π^g, π^e, π^h the optimal dual variables respectively of the balance constraints for gas, electricity and heat. Then, the $x_{co}^{g\to e*}$ solve:

$$\max_{0\le x_{co}^{g\to e}\le 1} (\pi^e \boldsymbol{\eta_{co}^{g\to e}} - \pi^g - \boldsymbol{P_{co}^{g\to e}}) \boldsymbol{Q_{co}^{g\to e}}\, x_{co}^{g\to e}$$

- $\pi^e \boldsymbol{\eta_{co}^{g\to e}} - \pi^g - \boldsymbol{P_{co}^{g\to e}} > \mathbf{0}$: if the conversion is profitable , the order is fully accepted
- $\pi^e \boldsymbol{\eta_{co}^{g\to e}} - \pi^g - \boldsymbol{P_{co}^{g\to e}} < \mathbf{0}$: if the conversion would incur losses, the bid is fully rejected
- $\pi^e \boldsymbol{\eta_{co}^{g\to e}} - \pi^g - \boldsymbol{P_{co}^{g\to e}} = \mathbf{0}$: case of zero profits/losses, the order can be accepted or rejected

- Conditions are similar to conditions for transmission lines with losses and tariffs.
- Profits of conversion orders: similar to the congestion rent of a transmision system beetween electricity and gas markets.