Operations Research - LINMA 2491

Complementary notes on the L-shaped method

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How does a Benders decomposition look like for a 2-stage stochastic program ? The L-shaped method

Consider the 2-stage stochastic program in extensive form:

$$\min_{x,y} c^T x + \sum_{\omega=1}^{N} p_\omega q_\omega^T y_\omega \tag{1}$$

$$Ax = b \tag{2}$$

$$T_\omega x + W_\omega y_\omega = h_\omega \qquad \omega = 1, ..., N \tag{3}$$

$$x \geq 0, y_\omega \geq 0 \qquad \omega = 1, ..., N \tag{4}$$

The problem can be reformulated as:

$$\min_{x \in K_1 \cap K_2} \{c^T x + \{\min_y \sum_{\omega=1}^{N} p_\omega q_\omega^T y_\omega | W_\omega y_\omega = h_\omega - T_\omega x, y_\omega \geq 0, \ \omega = 1, ..., N\}\} \tag{5}$$

where $K_1 = \{x | Ax = b\}$ and $K_2 = \bigcap_\omega K_2(\omega)$ with $K_2(\omega)\{x | \exists y : T_\omega x + W_\omega y = h_\omega, y \geq 0\}$.

- The blue problem objective value is a function of x, $V : x \rightarrow V(x)$, called the expected value function.
- This blue problem can be split into N independent sub-problems, by which the value function $V(x)$ can be seen as an expectation of costs corresponding to the best reaction given x and scenario ω:

$$V(x) = \sum_{\omega=1}^{N} p_\omega Q_\omega(x) \tag{6}$$

where $Q_\omega(x) = \min_{y_\omega} q_\omega^T y_\omega | W_\omega y_\omega = h_\omega - T_\omega x, y_\omega \geq 0$

By LP duality, the original 2-stage stochastic program is equivalent to

$$\min_{x\in K_1\cap K_2}\{c^Tx+\{\max_{\pi}\sum_{\omega=1}^{N}\pi_\omega^T(h_\omega - T_\omega x) \mid \pi_\omega^T W_\omega \geq p_\omega q_\omega^T,\ \omega=1,...,N\}\} \tag{7}$$

The Benders reformulation of the original 2-stage stochastic program is then:

$$\min_x c^Tx+\theta \tag{8}$$

$$Ax=b \tag{9}$$

$$\theta \geq \sum_{\omega=1}^{N}\pi_\omega^T(h_\omega - T_\omega x) \qquad \forall\ \pi \mid \pi_\omega^T W_\omega \geq p_\omega q_\omega^T,\ \omega=1,...,N \tag{10}$$

where (10) can be replaced by (see exercise 1 in TP 3):

- (Opt. cuts) $\theta \geq \sum_\omega (\pi_\omega^v)^T(h_\omega - T_\omega x)$ `for each` vertex $(\pi_1^v,...,\pi_N^v)$ of $P := \{\pi | \pi_\omega^T W_\omega \geq p_\omega q_\omega^T, \omega=1,...,N\}$
- (Feas. cuts) $0 \geq \sum_\omega (\sigma_\omega^h)^T(h_\omega - T_\omega x)$ `for each` extreme ray (σ^h) of P

Benders decomposition

As with a classic Benders decomposition, at each iteration, for a candidate solution x^*, θ^* solving (8) subject to (9) plus some of the constraints in (10) (the cuts obtained at previous iterations), we can check if (10) fully holds by solving a Benders sub-problem. If (10) holds, we are done and have an optimal solution to the original problem (the values for the y_ω can straightforwardly be recomputed, how ?).

Otherwise, the sub-problem identifies a violated optimality cut or feasibility cut to be added to the master problem. Since there is a finite number of such cuts (because they correspond respectively to vertices or extreme rays of the same polytope P defined in the previous slide), we know that the algorithm will converge to an optimal solution in a finite number of iterations.

The good news is that the Benders sub-problem can be split into N independent sub-problems: one per scenario $\omega = 1, ..., N$.

- Benders sub-problem. How to test if (10) is satisfied ? Test if

$$\theta^* \geq \max \sum_{\omega} \pi_\omega^T (h_\omega - T_\omega x) \quad s.t. \quad \pi_\omega^T W_\omega \geq p_\omega q_\omega^T, \omega = 1, ..., N \quad (11)$$

For this purpose and to obtain the feasibility/optimality cuts, we will solve for each ω:

$$\max \pi_\omega^T (h_\omega - T_\omega x) \quad (12)$$
$$\pi_\omega^T W_\omega \geq q_\omega \quad (13)$$

Note that p_ω is not present in the RHS of (13).

Optimality cuts from split Benders sub-problems

Benders sub-problem split into N independent sub-problems

Problem (D_ω), dual of (S_ω):

$$\max \pi_\omega^T (h_\omega - T_\omega x) \quad (14)$$
$$\pi_\omega^T W_\omega \geq q_\omega \quad (15)$$

Benders sub-problem (D), dual of (S):

$$\max \sum_{\omega=1}^{N} \pi_\omega^T (h_\omega - T_\omega x) \quad (16)$$
$$\pi_\omega^T W_\omega \geq p_\omega q_\omega^T, \ \omega = 1, ..., N \quad (17)$$

If (14)-(15) admits an optimal solution (π_ω^*) for each ω, $(p_1\pi_1^*, ..., p_1\pi_N^*)$ is an optimal solution of (16)-(17). (It is assumed that the $p_{\omega \neq 0}$.)

More precisely, vertices of the set (17) correspond to points of the form $(p_1\pi_1^*, ..., p_1\pi_N^*)$ with for each ω, (π_ω^*) a vertex of (15) (show it)

The corresponding optimality cuts (see previous slides) have a nice geometric interpretation: they correspond to supporting hyperplanes of the Expected Value Function $V(x)$, which can be shown to be a convex piece-wise linear function of x (either thanks to sensitivity analysis for linear programs or in view of slide 5: there is 1 opt. cut per vertex of the Benders subproblem feas. set P).

Illustration for optimality cuts

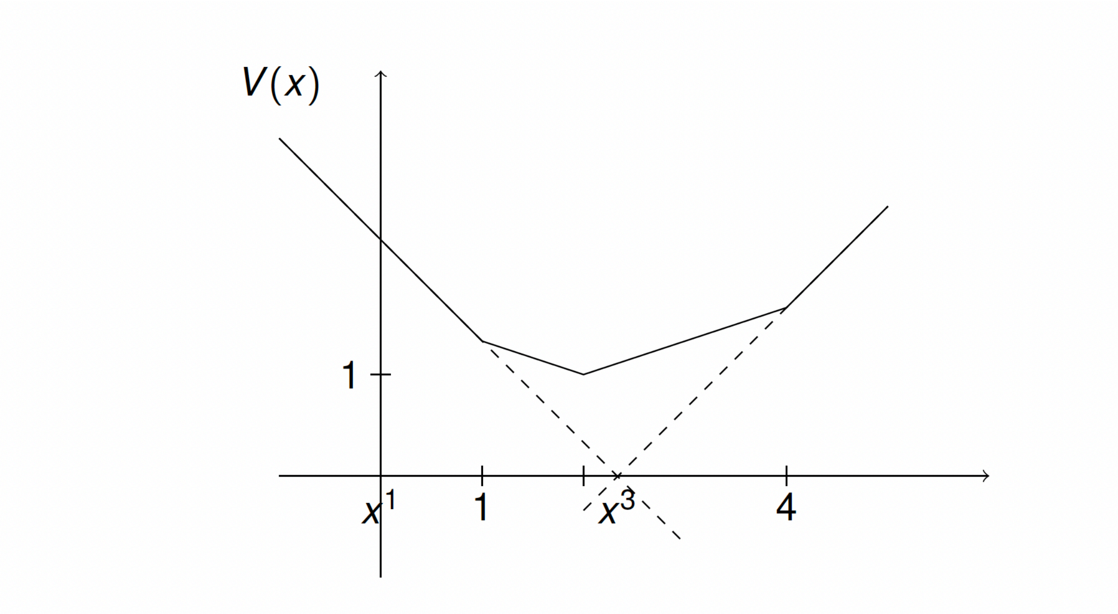

Figure: Illustration for optimality cuts, taken from the slides of Prof. A. Papavasiliou for LINMA2491

N.B. Optimality cuts are needed as long as the candidate θ value underestimates the true value of $V(x)$ = expected second-stage costs given x.

Feasibility cuts from split Benders sub-problems

Benders sub-problem split into N independent sub-problems

Problem (D_ω), dual of (S_ω):

$$\max \pi_\omega^T(h_\omega - T_\omega x) \tag{18}$$

$$\pi_\omega^T W_\omega \geq q_\omega \tag{19}$$

Benders sub-problem (D), dual of (S):

$$\max \sum_{\omega=1}^{N} \pi_\omega^T(h_\omega - T_\omega x) \tag{20}$$

$$\pi_\omega^T W_\omega \geq p_\omega q_\omega^T, \ \omega = 1, ..., N \tag{21}$$

Extreme rays of of the set (21) are of the form $(0, ..., \sigma_\omega, ..., 0)$ with for each ω, σ_ω an extreme ray of (19) (show it).

The corresponding feasibility cuts (see previous slides) have a nice geometric interpretation: they correspond to cutting planes (linear constraints cutting off a candidate point outside of the set/domain of interest) describing the domain of the Expected Value Function $V(x)$, that is $K_1 \cap K_2$, i.e. the set of feasible first-stage decisions x for which, for each scenario, there exist second -stage decisions satisfying the second-stage constraints (these second-stage constraints depend on x).

Multi-cut L-shaped method

Instead of the Benders reformulation:

$$\min_x c^T x + \theta \tag{22}$$

$$Ax = b \tag{23}$$

$$\theta \geq \sum_{\omega=1}^{N} \pi_\omega^T (h_\omega - T_\omega x) \quad \forall\, \pi = (\pi, ..., \pi_N) \mid \pi_\omega^T W_\omega \geq p_\omega q_\omega^T,\ \omega = 1, ..., N \tag{24}$$

Consider

$$\min_x c^T x + \sum_\omega p_\omega \theta_\omega \tag{25}$$

$$Ax = b \tag{26}$$

$$\theta_\omega \geq \pi_\omega^T (h_\omega - T_\omega x) \qquad \forall\, \pi_\omega \mid \pi_\omega^T W_\omega \geq q_\omega^T,\ \omega = 1, ..., N \tag{27}$$

and then apply the mechanics.

References I

J. R. Birge and F. Louveaux.
Introduction to stochastic programming.
Springer Science & Business Media, 2011.