Operations Research - LINMA 2491

Course 4 - Benders decompositions

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Benders decomposition

Let us consider

$$\min_{x,y} c^T x + d^T y \tag{1}$$

subject to :

$$Ax + By \geq b \tag{2}$$
$$x \geq 0 \tag{3}$$
$$y \in Y \tag{4}$$

- For fixed y, the remaining part is just an LP that can in nice cases be split into independent smaller LPs.
- One may be tempted to try some values for y, solve for x, "adjust" the values for y and repeat until an optimal solution is found.
- A Benders decomposition approach works along these lines. The condition $y \in Y$ can e.g. be y integral / binary.

Benders decomposition

We will assume that the initial problem is feasible and has an optimal solution. Let us set $Proj_y = \{y \in Y | \texttt{there is x such that } Ax + By \geq b\}$. The problem can be reformulated as :

$$\min_{y \in Proj_y} \{d^T y + \{\min_x c^T x | Ax \geq b - By, x \geq 0\}\} \quad (5)$$

or, using LP duality:

$$\min_{y \in Proj_y} \{d^T y + \{\max_u (b - By)^T u | A^T u \leq c, u \geq 0\}\} \quad (6)$$

Benders decomposition

$$\min_{y \in Proj_y} \{d^T y + \{\max_u (b - By)^T u | A^T u \leq c, u \geq 0\}\}$$

is in turn equivalent (why?) to:

$$\min_y d^T y + x_0 \tag{7}$$

$$y \in Y \tag{8}$$

$$x_0 \geq (b - By)^T u \qquad \texttt{for all } u \geq 0 \textit{ such that } A^T u \leq c \tag{9}$$

- (9) is equivalent to a finite number of inequalities (corresponding to vertices/extreme rays of $A^T u \leq c, u \geq 0$, see exercise session)
  - $x_0 \geq (b - By)^T u^v$ `for all` u^v in the finite set of vertices of P
  - $0 \geq (b - By)^T u^h$ `for all` u^h in the finite set of extreme rays of P
- The idea of a Benders decomposition is to solve (7)-(9), by considering first (7)-(8) and iteratively add cuts to enforce (9)
- Cuts to enforce (9) are obtained from the subproblem in red (or worker problem) which is equivalent to solving the remaining part of the initial problem for fixed y, problem in blue above. See below.

Benders decomposition

Suppose we have a candidate y^*, x_0^* obtained by solving (7)-(8) and some of the constraints in (9) (for example those obtained at previous iterations of the algorithm).

This solution trivially provides a lower bound LB for (7)-(9) since only some of the constraints in (9) are considered, and hence a lower bound for the original problem.

- How to test if (9) is satisfied ? Test if

$$x_0^* \geq \{\max_u (b - By^*)^T u | A^T u \leq c, u \geq 0\} \tag{10}$$

Benders decomposition

- If not, and if the max is finite, add a cut of the form:

$$x_0 \geq (b - By)^T u_k \tag{11}$$

  where u_k solves the right-hand side of (10). In that case, u_k is a vertex of P defined by $A^T u \leq c, u \geq 0$. [Optimality cut]

  Note that this step leads to an upper bound $UB = d^T y^* + \{\max_u (b - By^*)^T u | A^T u \leq c, u \geq 0\}$, since it has been shown above that the original problem is equivalent to:

$$\min_{y \in Proj_y} \{d^T y + \{\max_u (b - By)^T u | A^T u \leq c, u \geq 0\}\}$$

- If the problem is unbounded, the solver can provide an "extreme ray" u_k of the polyhedron P in which direction the objective can be arbitrarily increased, i.e. with $(b - By^*)^T u_k > 0$. In that case, add the cut $(b - By)^T u_k \leq 0$ where u_k is an extreme ray of P [Feasibility cut]

- In all cases, as a polyhedron P has finitely many vertices and extreme rays, there are finitely many cuts to generate.

Benders decomposition

- Note as above with (5) and (7) that the dual of the Benders subproblem is just the remaining part of the initial problem when y has been fixed: $\min_x c^T x \mid Ax + By \geq b, x \geq 0$
- For simplicity (to avoid dealing with unbounded master problems), we will assume that Y is bounded and that we know from the beginning a lower bound M for x_0.

Benders Decomposition Algorithm I

Initialization: $LB := -\infty, UB := +\infty$. While $UB - LB > \epsilon =:$ tolerance

1. Solve the Master Problem:

$$\min_y d^T y + x_0 \tag{7}$$

$$y \in Y, x_0 \geq M \tag{8}$$

Update the lower bound LB, cf. previous slides

M is an initial known lower bound on x_0 (part of the objective corresponding to the x variables)

Benders Decomposition Algorithm II

2. Solve Benders Subproblem $\{\max_u (b - By^*)^T u | A^T u \leq c, u \geq 0\}$
   2.1 If unbounded, add the "feasibility cut":
   $$0 \geq (b - By)^T u_k$$
   where u_k is an extreme ray of P defined by $A^T u \leq c_{ij}, u \geq 0$

   2.2 If bounded and $x_0^* < \{\max_u (b - By^*)^T u | A^T u \leq c, u \geq 0\}$, add the "optimality cut" using u_k an optimal solution to the right-hand side problem:
   $$x_0 \geq (b - By)^T u_k \qquad (11)$$
   Update the upper bound UB, cf. previous slides

   2.3 If (10) $x_0^* \geq \{\max_u (b - By^*)^T u | A^T u \leq c, u \geq 0\}$ is satisfied: stop, optimal solution found

   Repeat: resolve Master Problem with all the added cuts (old and new), make the test with the subproblem, add a cut if needed, etc.

N.B. Instead of solving the Master Problems each time to optimality, often better to include the Benders cuts (coming from the subproblems) in the B&B tree solving the master

Uncapacitated Facility Location Problem

$$\min_{x,y} \sum_{i,j} c_{ij} x_{ij} + \sum_i f_i y_i \tag{12}$$

subject to:

$$\sum_{i=1}^{n} x_{ij} \geq 1 \qquad \forall j = 1, ..., m \tag{13}$$

$$x_{ij} \leq y_i \qquad \forall i = 1, ...n, j = 1, ..., m \tag{14}$$

$$x_{ij} \geq 0 \tag{15}$$

$$y \in \{0, 1\} \tag{16}$$

The costs c_{ij}, f_i are assumed to be non-negative.

Benders Master Problem and Subproblems

Master Problem

$$\min_{x_0, y} x_0 + \sum_i f_i y_i \tag{17}$$

subject to:

$$y \in \{0, 1\} \tag{18}$$

$$x_0 \geq 0 \tag{19}$$

$$\textit{Benders cuts iteratively added} \tag{20}$$

($x_0 \geq 0$ is a known lower bound on x_0, since $c_{ij}x_{ij} \geq 0$)

Benders subproblem "in primal form" (blue problem above) is:

$$\min_x \sum_{i,j} c_{ij}x_{ij} + (\sum_i f_i \overline{y_i}) \tag{21}$$

subject to:

$$\sum_{i=1}^{n} x_{ij} \geq 1 \qquad [v_j] \qquad \forall j = 1, ..., m \tag{22}$$

$$x_{ij} \leq \overline{y_i} \qquad [w_{ij}] \qquad \forall i = 1, ...n, j = 1, ..., m \tag{23}$$

$$x_{ij} \geq 0 \tag{24}$$

Benders subproblems

The "real" Benders subproblem (subproblem in "dual form", "red problem above", dual to the "blue problem") is:

$$\max_{v,w} \sum_j v_j - \sum_{ij} w_{ij}\overline{y_i} \tag{25}$$

subject to:

$$v_j - w_{ij} \leq c_{ij} \qquad \forall i = 1, ...n, j = 1, ..., m \tag{26}$$

$$v_j, w_{ij} \geq 0 \tag{27}$$

Benders subproblems

For a candidate (x_0^*, y^*):

- if this subproblem is unbounded (this will happen at the first iteration where $x_0^* = 0 = y_i^*$, why?), we need to add:

$$0 \geq \sum_j v_j^* - \sum_{ij} w_{ij}^* y_i \tag{28}$$

  where v^*, w^* is an extreme ray of P defined by (26)-(27).

- if this subproblem has an optimal solution (v^*, w^*) with objective greater than x_0, we need to add the cut:

$$x_0 \geq \sum_j v_j^* - \sum_{ij} w_{ij}^* y_i \tag{29}$$

Remarks

It can be shown (see [3], Proposition 10.7) that the following values for v_j, w_{ij} solves the Benders subproblem (25)-(27) when this problem is bounded (always the case for $\overline{y}$ with at least one entry $= 1$):

- $v_j = \min_{i \mid \overline{y_i}=1} c_{ij}$
-
  - $w_{ij} = 0$ for $i \mid \overline{y_i} = 1$
  - $w_{ij} = \max\{(v_j - c_{ij}), 0\}$ for $i \mid \overline{y_i} = 0$

We can hence solve the subproblems by relying on a few comparisons.

An exercise

We take the Example 10.6 from [3]:

		CUSTOMER					
		1	2	3	4	5	FIXED COSTS
PLANT	1	2	3	4	5	7	2
	2	4	3	1	2	6	3
	3	5	4	2	1	3	3

- Solving the initial problem (17)-(19) without any other cut provides the candidate $y_1 = y_2 = y_3 = x_0 = 0$. It can be verified that no x_{ij} exist such as to obtain a feasible solution to the original problem to solve. The Benders subproblem will be unbounded and a Benders cut will be added, equivalent to $y_1 + y_2 + y_3 \geq 1$.

An exercise

- Solve the initial problem (17)-(19), with the additional condition $y_1 + y_2 + y_3 \geq 1$ N.B. The solution can be found "by inspection" without using any solver or advanced computation.
- For the solution $\bar{y}$ obtained, what are the optimal values for the variables v, w in the corresponding Benders subproblem ? (Use the slide "Remarks" above.)
- Does this solution solve the original problem? Make the test by comparing the value obtained for x_0 to the optimal value obtained for the subproblem.
- If the test fails, i.e. the y found doesn't solve the original problem, what is the corresponding Benders cut to add ?

Side question:

- Is this first solution $\bar{y}$ such that we can find values for x_{ij} and obtain a *feasible* solution to the original problem?
- If yes, what are the values for x_{ij} and the corresponding total costs?

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