

# The L-Shaped Method

## Operations Research

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## Extensive Form

Stochastic linear program in **extensive form**:

$$\begin{aligned}(EF) : \quad & \min c^T x + \mathbb{E}_\omega[\min q(\omega)^T y(\omega)] \\ & Ax = b \\ & T(\omega)x + W(\omega)y(\omega) = h(\omega) \\ & x \geq 0, y(\omega) \geq 0\end{aligned}$$

- First-stage decisions:  $x \in \mathbb{R}^{n_1}$
- second-stage decisions:  $y(\omega) \in \mathbb{R}^{n_2}$
- First-stage parameters:  $c \in \mathbb{R}^{n_1}$ ,  $b \in \mathbb{R}^{m_1}$ ,  $A \in \mathbb{R}^{m_1 \times n_1}$
- Second-stage parameters:  $q(\omega) \in \mathbb{R}^{n_2}$ ,  $h(\omega) \in \mathbb{R}^{m_2}$ ,  
 $T(\omega) \in \mathbb{R}^{m_2 \times n_1}$  and  $W(\omega) \in \mathbb{R}^{m_2 \times n_2}$

# Value Function

**Second-stage value function:**

$$\begin{aligned}(S_\omega) : \quad Q_\omega(x) &= \min_y q_\omega^T y \\ W_\omega y &= h_\omega - T_\omega x \\ y &\geq 0.\end{aligned}$$

Interpretation: cost of best possible reaction to  $x$  *and*  $\omega$

**Expected value function:**

$$V(x) = \sum_{\omega=1}^N p_\omega Q_\omega(x).$$

Interpretation: cost of best possible reaction to  $x$  *before* knowing  $\omega$

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## Complete Recourse

Define

- $K_1 = \{x : Ax = b, x \geq 0\}$
- $K_2(\omega) = \{x : \exists y, T_\omega x + W_\omega y = h_\omega, y \geq 0\}$
- $K_2 = \text{dom } V$

Interpretation of  $K_1$ ,  $K_2(\omega)$ ,  $K_2$ ?

**Relative complete recourse:** obeying first-stage constraints ensures feasible second-stage decisions exist:

$$\text{pos } W = \mathbb{R}^{m_2}$$

**Complete recourse:** feasible second-stage decision exists, regardless of first-stage decision and realization of uncertainty:

$$K_2 = \mathbb{R}^{n_1}$$

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# Properties of Value Functions

Dual of  $(S_\omega)$ :

$$(D_\omega) : \max_{\pi} \pi^T (h_\omega - T_\omega x) \\ \pi^T W_\omega \leq q_\omega^T$$

Denote  $\pi_{\omega 0}$  as dual optimal multipliers of  $(S_\omega)$  given  $x_0$ :

- ①  $V(x)$  and  $Q_\omega(x)$  are piecewise linear convex functions of  $x$
- ②  $\pi_{\omega 0}^T (h_\omega - T_\omega x)$  is a supporting hyperplane of  $Q_\omega(x)$  at  $x_0$
- ③  $\sum_{\omega=1}^N p_\omega \pi_{\omega 0}^T (h_\omega - T_\omega x)$  is a supporting hyperplane of  $V(x)$  at  $x_0$

We recall a previous result for the proof

Proof:

- ①  $D_\omega$  has finite number of dual optimal multipliers
- ② Strong duality and  $\pi_{\omega 0} \in \partial Q_\omega(h_\omega - T_\omega x_0)$
- ③ Follows from previous bullet and  $V(x) = \sum_{\omega=1}^N p_\omega Q_\omega(x)$

# The diag operator

Consider a set of matrices  $A_i, i = 1, \dots, n$  (not necessarily square)

The matrix  $\text{diag}(A_1, \dots, A_n)$  is defined as

$$\text{diag}(A_1, A_2, \dots, A_n) = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & A_n \end{pmatrix}$$

Not necessarily a square matrix

$$\begin{aligned} (S) : \quad & \min_y \sum_{\omega=1}^N p_{\omega} q_{\omega}^T y_{\omega} \\ & Wy = h - Tx \\ & y \geq 0 \end{aligned}$$

where

- $y^T = [y_1^T, \dots, y_N^T]$
- $h^T = [h_1^T, \dots, h_N^T]$
- $T = \text{diag} (T_{\omega}, \omega = 1, \dots, N)$
- $W = \text{diag} (W_{\omega}, \omega = 1, \dots, N)$

Is there a relationship between the feasible regions of the duals of  $(S)$  and  $(S_{\omega})$ ?

## Supporting Hyperplanes of $V$

Denote

- $V$ : the set of extreme vertices of  $\{\pi : \pi^T W \leq q^T\}$
- $V_\omega$ : the set of extreme vertices of  $\{\pi : \pi^T W_\omega \leq q_\omega^T\}$

Then

$$V = \{(p_1 \pi_1^T, \dots, p_N \pi_N^T)^T : \pi_1 \in V_1, \dots, \pi_N \in V_N\}$$

## Domain of $V$

Denote

- $R$ : the set of extreme rays of  $\{\pi : \pi^T W \leq q^T\}$
- $R_\omega$ : the set of extreme rays of  $\{\pi : \pi^T W_\omega \leq q_\omega^T\}$

Then

$$R = \{(0, \dots, \sigma_\omega^T, \dots, 0)^T : \sigma_\omega \in R_\omega, \omega = 1, \dots, N\}$$

# Deterministic Equivalent Program

The original problem ( $EF$ ) can be written as a **deterministic equivalent program**:

$$\begin{aligned} \min \quad & c^T x + \theta \\ \text{s.t.} \quad & Ax = b \\ & \sigma^T (h - Tx) \leq 0, \sigma \in R \\ & \theta \geq \pi^T (h - Tx), \pi \in V \\ & x \geq 0 \end{aligned}$$

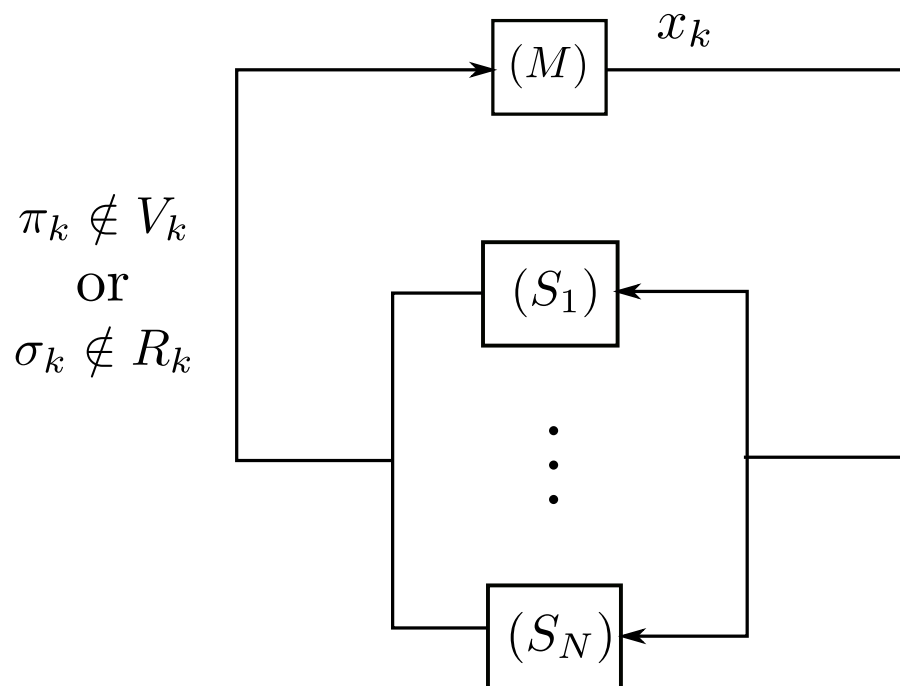
## Master Problem

Define **master problem** as

$$\begin{aligned} (M) : \quad & z_k = \min c^T x + \theta \\ & Ax = b \\ & \sigma^T (h - Tx) \leq 0, \sigma \in R_k \subseteq R & (1) \\ & \theta \geq \pi^T (h - Tx), \pi \in V_k \subseteq V & (2) \\ & x \geq 0 \end{aligned}$$

- Feasibility cuts: equation 1
- Optimality cuts: equation 2

## Overall Scheme



## Bounds

Solution of master provides:

- lower bound  $z_k \leq z^*$
- candidate solution  $x_k$
- under-estimator  $\theta_k \leq V(x_k)$

Solution of *all*  $(S_\omega)$  with input  $x_k$  provides

- upper bound  $c^T x_k + \sum_{\omega=1}^N p_\omega q_\omega^T y_{\omega,k+1} \geq z^*$
- new vertex  $\pi_{k+1} = (p_1 \pi_{1,k+1}^T, \dots, p_N \pi_{N,k+1}^T)^T$  or new extreme ray  $\sigma_{k+1} = (0, \dots, \sigma_\omega^T, \dots, 0)^T$



# The L-Shaped Algorithm

Step 0: Set  $k = 0$ ,  $V_0 = R_0 = \emptyset$

Step 1: Solve  $(M)$

- If  $(M)$  is feasible, store  $x_k$
- If  $(M)$  is infeasible, exit: infeasible

Step 2: For  $\omega = 1, \dots, N$ , solve  $(S_\omega)$  with  $x_k$  as input

- If  $(S_\omega)$  is infeasible, let  $S_{k+1} = S_k \cup \{\sigma_{k+1}\}$ , where  $\sigma_{k+1}$  is an extreme ray of  $(S_\omega)$ , let  $k = k + 1$  and return to step 1
- If  $(S_\omega)$  is feasible, store  $\pi_{\omega,k+1}$

Step 3: Let  $V_{k+1} = V_k \cup \{(p_1\pi_{1,k+1}, \dots, p_N\pi_{N,k+1})\}$

- If  $V_k = V_{k+1}$  then terminate with  $(x_k, y_{k+1})$  as the optimal solution.
- Else, let  $k = k + 1$  and return to step 1

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## Example: Capacity Expansion Planning

$$\begin{aligned} \min_{x, y \geq 0} \quad & \sum_{i=1}^n (I_i \cdot x_i + \mathbb{E}_{\xi} \sum_{j=1}^m C_i \cdot T_j \cdot y_{ij}(\omega)) \\ \text{s.t.} \quad & \sum_{i=1}^n y_{ij}(\omega) = D_j(\omega), j = 1, \dots, m \\ & \sum_{j=1}^m y_{ij}(\omega) \leq x_i, i = 1, \dots, n-1 \end{aligned}$$

- $I_i, C_i$ : fixed/variable cost of technology  $i$
- $D_j(\omega), T_j$ : height/width of load block  $j$
- $y_{ij}(\omega)$ : capacity of  $i$  allocated to  $j$
- $x_i$ : capacity of  $i$

Note:  $D_j$  is not dependent on  $\omega$

## Problem Data

Two possible realizations of load duration curve:

- Reference scenario: 10%
- 10x wind scenario: 90%

	Duration (hours)	Level (MW)	
		Reference scenario	10x wind scenario
Base load	8760	0-7086	0-3919
Medium load	7000	7086-9004	3919-7329
Peak load	1500	9004-11169	7329-10315

## Slave Problem

$$(S_{\omega}) : \min_{y \geq 0} \sum_{i=1}^n \sum_{j=1}^m C_i \cdot T_j \cdot y_{ij}$$

$$(\lambda_j(\omega)) : \sum_{i=1}^n y_{ij} = D_j(\omega), j = 1, \dots, m$$

$$(\rho_i(\omega)) : \sum_{j=1}^m y_{ij} \leq \bar{x}_i, i = 1, \dots, n-1$$

where  $\bar{x}$  has been fixed from the master problem

## Sequence of Investment Decisions

Iteration	Coal (MW)	Gas (MW)	Nuclear (MW)	Oil (MW)
1	0	0	0	0
2	0	0	0	8736
3	0	0	0	15999.6
4	0	14675.5	0	0
5	10673.8	0	0	0
6	10673.8	0	0	13331.8
7	0	163.8	7174.5	3830.8
8	0	3300.6	7868.4	0
9	0	5143.4	7303.9	1679.4
10	3123.9	1948.1	4953.7	1143.3
11	1680	4322.4	6625	0
12	8747.6	1652.8	0	768.6
13	5701.9	464.9	4233.6	768.6
14	4935.9	1405	3994.7	0
15	6552.6	386.3	3173.7	882.9
16	5085	1311	3919	854

# Observations

- Investment candidate in each iteration necessarily different from *all* past iterations
- 'Greedy' behavior: low capital cost in early iterations

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## Example 1

$$z = \min 100x_1 + 150x_2 + \mathbb{E}_\xi(q_1y_1 + q_2y_2)$$

$$\text{s.t. } x_1 + x_2 \leq 120$$

$$6y_1 + 10y_2 \leq 60x_1$$

$$8y_1 + 5y_2 \leq 80x_2$$

$$y_1 \leq d_1, y_2 \leq d_2$$

$$x_1 \geq 40, x_2 \geq 20, y_1, y_2 \geq 0$$

$$\xi = (d_1, d_2, q_1, q_2) = \begin{cases} (500, 100, -24, -28), & p_1 = 0.4 \\ (300, 300, -28, -32), & p_2 = 0.6 \end{cases}$$

## Iteration 1

- *Step 1.*

$$\min\{100x_1 + 150x_2 \mid x_1 + x_2 \leq 120, x_1 \geq 40, x_2 \geq 20\}$$

- $x^1 = (40, 20)^T, \theta^1 = -\infty$

- *Step 3.* For  $\xi = \xi_1$  solve

$$\min\{-24y_1 - 28y_2 \mid 6y_1 + 10y_2 \leq 2400, 8y_1 + 5y_2 \leq 1600$$

$$0 \leq y_1 \leq 500, 0 \leq y_2 \leq 100\}$$

$$w_1 = -6100, y^T = (137.5, 100), \pi_1^T = (0, -3, 0, -13)$$

For  $\xi = \xi_2$  solve

$$\min\{-28y_1 - 32y_2 \mid 6y_1 + 10y_2 \leq 2400, 8y_1 + 5y_2 \leq 1600$$

$$0 \leq y_1 \leq 300, 0 \leq y_2 \leq 300\}$$

$$w_2 = -8384, y^T = (80, 192), \pi_2^T = (-2.32, -1.76, 0, 0)$$

## Iteration 1: Optimality Cut

$$h_1 = (0, 0, 500, 100)^T, h_2 = (0, 0, 300, 300)^T$$

$$T_{.,1} = (-60, 0, 0, 0)^T, T_{.,2} = (0, -80, 0, 0)^T$$

- $e_1 = 0.4 \cdot \pi_1^T \cdot h_1 + 0.6 \cdot \pi_2^T \cdot h_2 = 0.4 \cdot (-1300) + 0.6 \cdot (0) = -520$

- $E_1 = 0.4 \cdot \pi_1^T T + 0.6 \cdot \pi_2^T T =$   
 $0.4(0, 240) + 0.6(139.2, 140.8) = (83.52, 180.48)$

- $w^1 = -520 - (83.52, 180.48) \cdot x^1 = -7470.4$

- $w^1 = -7470.4 > \theta^1 = -\infty$ , therefore add the cut  
 $83.52x_1 + 180.48x_2 + \theta \geq -520$

## Iteration 2

- *Step 1.* Solve master

$$\min\{100x_1 + 150x_2 + \theta \mid x_1 + x_2 \leq 120, x_1 \geq 40, x_2 \geq 20, \\ 83.52x_1 + 180.48x_2 + \theta \geq -520\}$$

$$z = -2299.2, x^2 = (40, 80)^T, \theta^2 = -18299.2$$

- *Step 3.* Add the cut  $211.2x_1 + \theta \geq -1584$

## Iteration 3

- *Step 1.* Solve master.

$$z = -1039.375, x^3 = (66.828, 53.172)^T, \theta^3 = -15697.994$$

- *Step 3.* Add the cut  $115.2x_1 + 96x_2 + \theta \geq -2104$

## Iteration 4

- *Step 1.* Solve master.

$$z = -889.5, x^4 = (40, 33.75)^T, \theta^4 = -9952$$

- *Step 3.* There are multiple solutions for  $\xi = \xi_2$ . Select one, add the cut  $133.44x_1 + 130.56x_2 + \theta \geq 0$

## Iteration 5

- *Step 1.* Solve master

$$\begin{aligned} \min \{ & 100x_1 + 150x_2 + \theta \mid x_1 + x_2 \leq 120, x_1 \geq 40, x_2 \geq 20, \\ & 83.52x_1 + 180.48x_2 + \theta \geq -520, 211.2x_1 + \theta \geq -1584 \\ & 115.2x_1 + 96x_2 + \theta \geq -2104, 133.44x_1 + 130.56x_2 + \theta \geq 0 \} \\ z = & -855.833, x^5 = (46.667, 36.25)^T, \theta^5 = -10960 \end{aligned}$$

- *Step 3.*  $w_5 = -520 - (83.52, 180.48) \cdot x^5 = -10960 = \theta^5$ ,  
stop.  $x = (46.667, 36.25)^T$  is the optimal solution.

## Example 2

$$z = \min \mathbb{E}_\xi(y_1 + y_2)$$

$$\text{s.t. } 0 \leq x \leq 10$$

$$y_1 - y_2 = \xi - x$$

$$y_1, y_2 \geq 0$$

$$\xi = \begin{cases} 1 & p_1 = 1/3 \\ 2 & p_2 = 1/3 \\ 4 & p_3 = 1/3 \end{cases}$$

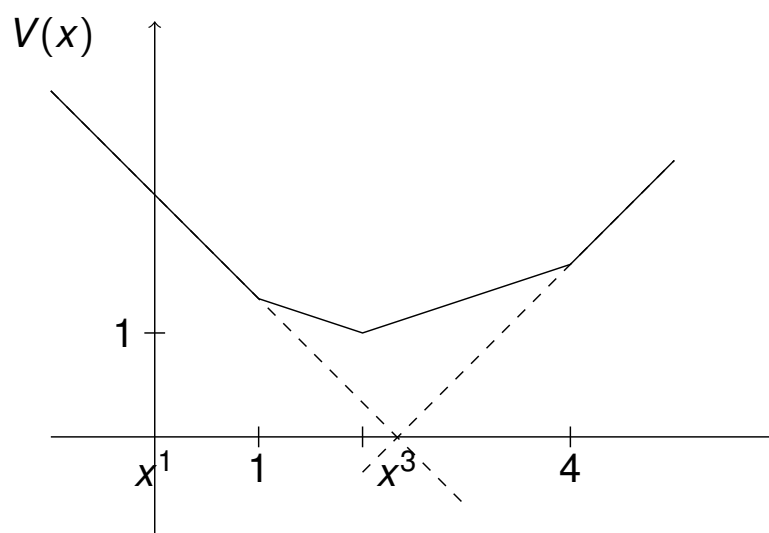
$$K_2 = \mathbb{R}$$



## L-Shaped Method in Example 2

- Iteration 1, Step 1:  $x^1 = 0$
- Iteration 1, Step 3:  $x^1$  not optimal, add cut:  $\theta \geq 7/3 - x$
- Iteration 2, Step 1:  $x^2 = 10$
- Iteration 2, Step 3:  $x^2$  not optimal, add cut:  $\theta \geq x - 7/3$
- Iteration 3, Step 1:  $x^3 = 7/3$
- Iteration 3, Step 3:  $x^3$  not optimal, add cut:  $\theta \geq (x + 1)/3$
- Iteration 4, Step 1:  $x^4 = 1.5$
- Iteration 4, Step 3:  $x^4$  not optimal, add cut:  $\theta \geq (5 - x)/3$
- Iteration 5, Step 1:  $x^5 = 2$
- Iteration 5, Step 3:  $x^5$  is optimal

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- $V(x^1) = 7/3$  and  $V(x) = 7/3 - x$  'around'  $x^1$
- $(7 - x)/3$  is the optimality cut at  $x^1$

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## Feasibility Cuts

Consider the following problem:

$$\begin{aligned} (F) : \min w' &= e^T v^+ + e^T v^- \\ \text{s.t. } Wy + Iv^+ - Iv^- &= h_k - T_k x^v \\ y \geq 0, v^+ \geq 0, v^- &\geq 0 \end{aligned}$$

with dual multipliers  $\sigma^v$ . Define

$$\begin{aligned} D_{r+1} &= (\sigma^v)^T T_k \\ d_{r+1} &= (\sigma^v)^T h_k \end{aligned}$$

Step 2 of L-shaped method: For  $k = 1, \dots, K$  solve (F).

- If  $w' = 0$  for all  $k$ , go to Step 3.
- Else, add  $D_{r+1}x \geq d_{r+1}$ , set  $r = r + 1$  and go to Step 1.

## Example

$$\begin{aligned} \min & 3x_1 + 2x_2 - \mathbb{E}_\xi(15y_1 + 12y_2) \\ \text{s.t. } & 3y_1 + 2y_2 \leq x_1, 2y_1 + 5y_2 \leq x_2 \\ & 0.8\xi_1 \leq y_1 \leq \xi_1, 0.8\xi_2 \leq y_2 \leq \xi_2 \\ & x, y \geq 0 \end{aligned}$$

$$\xi = \begin{cases} (4, 4), p_1 = 0.25 \\ (4, 8), p_2 = 0.25 \\ (6, 4), p_3 = 0.25 \\ (6, 8), p_4 = 0.25 \end{cases}$$

## Generating a Feasibility Cut

For  $x^1 = (0, 0)^T$ ,  $\xi = (6, 8)^T$ , solve

$$\begin{aligned} \min_{v^+, v^-, y} & v_1^+ + v_1^- + v_2^+ + v_2^- + v_3^+ + v_3^- + \\ & v_4^+ + v_4^- + v_5^+ + v_5^- + v_6^+ + v_6^- \\ \text{s.t. } & v_1^+ - v_1^- + 3y_1 + 2y_2 \leq 0, v_2^+ - v_2^- + 2y_1 + 5y_2 \leq 0 \\ & v_3^+ - v_3^- + y_1 \geq 4.8, v_4^+ - v_4^- + y_2 \geq 6.4 \\ & v_5^+ - v_5^- + y_1 \leq 6, v_6^+ - v_6^- + y_2 \leq 8 \end{aligned}$$

We get  $w' = 11.2$ ,  $\sigma^1 = (-3/11, -1/11, 1, 1, 0, 0)$   
 $h = (0, 0, 4.8, 6.4, 6, 8)^T$ ,  $T_{\cdot,1} = (-1, 0, 0, 0, 0, 0)^T$ ,  
 $T_{\cdot,2} = (0, -1, 0, 0, 0, 0)^T$   
 $D_1 = (-3/11, -1/11, 1, 1, 0, 0) \cdot T = (3/11, 1/11)$ ,  
 $d_1 = (-3/11, -1/11, 1, 1, 0, 0) \cdot h = 11.2$   
 $3/11x_1 + 1/11x_2 \geq 11.2$

# Induced Constraints

Going by the book:

- Iteration 2 master problem:  $x^2 = (41.067, 0)^T$
- Iteration 2 feasibility cut:  $x_2 \geq 22.4$
- Iteration 3 master problem:  $x^3 = (33.6, 22.4)^T$
- Iteration 3 feasibility cut:  $x_2 \geq 41.6$
- Iteration 4 master problem:  $x^4 = (27.2, 41.6)^T$  is feasible

## Induced Constraints:

- Observe that for  $\xi = (6, 8)^T$ ,  $y_1 \geq 4.8$ ,  $y_2 \geq 6.4$
- This implies  $x_1 \geq 27.2$ ,  $x_2 \geq 41.6$ , which should be added directly to the master

Personal experience: feasibility cuts are impractical