Operations Research - LINMA 2491

Introduction & Course 1

LP/QP duality review, LP Lagrangian duality, and market equilibria

Mehdi Madani

UCLouvain & N-SIDE

February 8, 2023

Table of contents

Overview

- Objective: cover a range of useful classic results and techniques in OR.
- The application selected here for the motivation: application to electricity markets and power system economics.
- Running examples in electricity markets: econonmic dispatches, optimal power flows and unit commitment problems (variants of).

N.B. References for further reading will be provided.

Outline I

Part I: Deterministic models (4 courses)

Course 1 **Review of LP fundamentals, duality and applications** Review of linear programming duality, Dorn's dual for convex quadratic programs, dualizing a subset of constraints (Lagrangian duality), applications to market equilibria and Financial Transmission Rights.

Outline I

Part I: Deterministic models (4 courses)

Course 1 **Review of LP fundamentals, duality and applications** Review of linear programming duality, Dorn's dual for convex quadratic programs, dualizing a subset of constraints (Lagrangian duality), applications to market equilibria and Financial Transmission Rights.

Course 2 **Lagrangian duality for MIP and solution methods** Lagrangian duality + applications: Convex Hull Pricing in electricity markets. Primal approach to solving Lagrangian duals, properties of the Lagangian dual and brief oveview of some solution methods (Dantzig-Wolfe, subgradient methods and recovery of primal solutions, other non-smooth convex optimization techniques).

Outline I

Part I: Deterministic models (4 courses)

Course 1 **Review of LP fundamentals, duality and applications** Review of linear programming duality, Dorn's dual for convex quadratic programs, dualizing a subset of constraints (Lagrangian duality), applications to market equilibria and Financial Transmission Rights.

Course 2 **Lagrangian duality for MIP and solution methods** Lagrangian duality + applications: Convex Hull Pricing in electricity markets. Primal approach to solving Lagrangian duals, properties of the Lagangian dual and brief oveview of some solution methods (Dantzig-Wolfe, subgradient methods and recovery of primal solutions, other non-smooth convex optimization techniques).

Course 3 **Extended Formulations.** Extended Formulations and examples (Balas EF for disjunctions and EF for lot sizing problems), generalizing EF for disjunctions and application to unit commitment problems, dedicated solution methods.

Outline I

Part I: Deterministic models (4 courses)

Course 1 **Review of LP fundamentals, duality and applications** Review of linear programming duality, Dorn's dual for convex quadratic programs, dualizing a subset of constraints (Lagrangian duality), applications to market equilibria and Financial Transmission Rights.

Course 2 **Lagrangian duality for MIP and solution methods** Lagrangian duality + applications: Convex Hull Pricing in electricity markets. Primal approach to solving Lagrangian duals, properties of the Lagangian dual and brief oveview of some solution methods (Dantzig-Wolfe, subgradient methods and recovery of primal solutions, other non-smooth convex optimization techniques).

Course 3 **Extended Formulations.** Extended Formulations and examples (Balas EF for disjunctions and EF for lot sizing problems), generalizing EF for disjunctions and application to unit commitment problems, dedicated solution methods.

Course 4 **Benders decompositions** Projections and Benders reformulations, optimality and feasibility cuts, choice of Benders cuts, examples.

Outline II

Outline II

Part II: Stochastic programming (6 courses)

- Stochastic Linear Programming
- Performance of Stochastic Programming solutions
- The L-shaped method (and relation to Benders decompositions)
- Multi-cut L-shaped method
- Nested decompositions
- Stochastic Dual Dynamic Programming

Outline II

Part II: Stochastic programming (6 courses)

- Stochastic Linear Programming
- Performance of Stochastic Programming solutions
- The L-shaped method (and relation to Benders decompositions)
- Multi-cut L-shaped method
- Nested decompositions
- Stochastic Dual Dynamic Programming

Part III: Insights on other topics (3 courses)

- Robust optimization

Table of contents

Weak Duality

Consider the linear program

$$\max\{c^T x \mid Ax \leq b, x \geq 0\} \tag{1}$$

Its dual is

$$\min\{b^T y \mid A^T y \geq c, y \geq 0\} \tag{2}$$

Weak Duality

Consider the linear program

$$\max\{c^T x \mid Ax \leq b, x \geq 0\} \tag{1}$$

Its dual is

$$\min\{b^T y \mid A^T y \geq c, y \geq 0\} \tag{2}$$

Theorem (Weak duality theorem for linear programs)

For x primal feasible, i.e. $x|Ax \leq b$ and y dual feasible, i.e. $y|A^T y \geq c$:

$$c^T x \leq b^T y \tag{3}$$

Weak Duality

Consider the linear program

$$\max\{c^T x \mid Ax \leq b, x \geq 0\} \tag{1}$$

Its dual is

$$\min\{b^T y \mid A^T y \geq c, y \geq 0\} \tag{2}$$

Theorem (Weak duality theorem for linear programs)

For x primal feasible, i.e. $x|Ax \leq b$ and y dual feasible, i.e. $y|A^T y \geq c$:

$$c^T x \leq b^T y \tag{3}$$

Proof.

- $c^T x \leq (y^T A)x$ by dual feasibility
- $y^T(Ax) \leq y^T b$ by primal feasibility
- hence $c^T x \leq b^T y$

$\square$

Weak Duality

Consider the linear program

$$\max\{c^T x \mid Ax \leq b, x \geq 0\} \tag{1}$$

Its dual is

$$\min\{b^T y \mid A^T y \geq c, y \geq 0\} \tag{2}$$

Theorem (Weak duality theorem for linear programs)

For x primal feasible, i.e. $x|Ax \leq b$ and y dual feasible, i.e. $y|A^T y \geq c$:

$$c^T x \leq b^T y \tag{3}$$

Proof.

- $c^T x \leq (y^T A)x$ by dual feasibility
- $y^T(Ax) \leq y^T b$ by primal feasibility
- hence $c^T x \leq b^T y$

$\square$

From the weak duality theorem, if either the primal or the dual is unbounded, the other problem is infeasible (why?). Note that both can be infeasible (give an example).

Strong duality for linear programs

Consider the linear program

$$\max\{c^T x \mid Ax \leq b, x \geq 0\} \tag{4}$$

and its dual

$$\min\{b^T y \mid A^T y \geq c, y \geq 0\} \tag{5}$$

Strong duality for linear programs

Consider the linear program

$$\max\{c^T x \mid Ax \leq b, x \geq 0\} \tag{4}$$

and its dual

$$\min\{b^T y \mid A^T y \geq c, y \geq 0\} \tag{5}$$

Theorem (Strong duality theorem for linear programs)

If both $\{x | Ax \leq b, x \geq 0\}$ *and* $\{y | A^T y \geq c, y \geq 0\}$ *are non-empty, the primal* (9) *admits an optimal solution* x^**, the dual* (13) *admits an optimal solution* y^**, and*

$$c^T x^* = b^T y^* \tag{6}$$

Strong duality for linear programs

Consider the linear program

$$\max\{c^T x \mid Ax \leq b, x \geq 0\} \tag{4}$$

and its dual

$$\min\{b^T y \mid A^T y \geq c, y \geq 0\} \tag{5}$$

Theorem (Strong duality theorem for linear programs)

If both $\{x|Ax \leq b, x \geq 0\}$ *and* $\{y|A^T y \geq c, y \geq 0\}$ *are non-empty, the primal* (9) *admits an optimal solution* x^**, the dual* (13) *admits an optimal solution* y^**, and*

$$c^T x^* = b^T y^* \tag{6}$$

Proof.

This fundamental result can be proved from the simplex algorithm or from the Farkas lemma, see e.g. [8]. □

Optimality conditions for linear programs

Consider again the primal (9)

$$\max\{c^T x \mid Ax \leq b, x \geq 0\}$$

and its dual (13)

$$\min\{b^T y \mid A^T y \geq c, y \geq 0\} \tag{7}$$

Optimality conditions for linear programs

Consider again the primal (9)

$$\max\{c^T x \mid Ax \leq b, x \geq 0\}$$

and its dual (13)

$$\min\{b^T y \mid A^T y \geq c, y \geq 0\} \tag{7}$$

It follows easily from the strong duality theorem (see next slide) that:

Theorem (Optimality conditions for linear programs: equality of objective values and Complementarity Slackness)

The following conditions are equivalent:

1. x^* *is optimal for the primal and* y^* *is optimal for the dual*

Optimality conditions for linear programs

Consider again the primal (9)

$$\max\{c^T x \mid Ax \leq b, x \geq 0\}$$

and its dual (13)

$$\min\{b^T y \mid A^T y \geq c, y \geq 0\} \tag{7}$$

It follows easily from the strong duality theorem (see next slide) that:

Theorem (Optimality conditions for linear programs: equality of objective values and Complementarity Slackness)

The following conditions are equivalent:

1. *x^* is optimal for the primal and y^* is optimal for the dual*
2. *x^*, y^* are respectively primal and dual feasible, and:*

$$c^T x \geq b^T y \tag{8}$$

(equality of primal and dual objective functions, why equality?)

Optimality conditions for linear programs

Consider again the primal (9)

$$\max\{c^T x \mid Ax \leq b, x \geq 0\}$$

and its dual (13)

$$\min\{b^T y \mid A^T y \geq c, y \geq 0\} \quad (7)$$

It follows easily from the strong duality theorem (see next slide) that:

Theorem (Optimality conditions for linear programs: equality of objective values and Complementarity Slackness)

The following conditions are equivalent:

1. x^* *is optimal for the primal and* y^* *is optimal for the dual*
2. x^*, y^* *are respectively primal and dual feasible, and:*

   $$c^T x \geq b^T y \quad (8)$$

   (equality of primal and dual objective functions, why equality?)
3. x^*, y^* *are respectively primal and dual feasible, and (complementarity slackness), for each row* i *and each column* j *of* A*,*
   (a) $(b_i - A_i x) y_i = 0$
   (b) $(A_j^T y - c_j) x_j = 0$

Optimality conditions for LPs

Proof.

1. Equivalence between condition 1 and condition 2 follows directly from the weak and strong duality theorems (why?)

Optimality conditions for LPs

Proof.

1. Equivalence between condition 1 and condition 2 follows directly from the weak and strong duality theorems (why?)
2. The equivalence between condition 1 and condition 2 is also direct: One has:

$$\sum_i (b_i - A_i x) y_i + \sum_j (A_j^T y - c_j) x_j \tag{9}$$

$$= y^T b - y^T A x + x^T A^T y - c^T x \tag{10}$$

$$= y^T b - c^T x = 0 \tag{11}$$

For x^* primal feasible and y^* dual feasible, each term in (9) is non-negative, hence this sum is null if and only if each term is null (conditions 3(a) and 3(b)), if and only if $b^T y - c^T x = 0$.

□

Optimality conditions for LPs

Proof.

1. Equivalence between condition 1 and condition 2 follows directly from the weak and strong duality theorems (why?)
2. The equivalence between condition 1 and condition 2 is also direct: One has:

$$\sum_i (b_i - A_i x) y_i + \sum_j (A_j^T y - c_j) x_j \tag{9}$$

$$= y^T b - y^T A x + x^T A^T y - c^T x \tag{10}$$

$$= y^T b - c^T x = 0 \tag{11}$$

For x^* primal feasible and y^* dual feasible, each term in (9) is non-negative, hence this sum is null if and only if each term is null (conditions 3(a) and 3(b)), if and only if $b^T y - c^T x = 0$.

□

N.B. Primal, dual constraints, and the equality of objective functions (here see (11)) altogether give *linear* optimality conditions: solving an LP is equivalent to solving a system of linear inequalities.

Dualization recipe

	Primal linear program	Dual linear program
Variables	$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$
Matrix	A	A^T
Right-hand side	$\mathbf{b}$	$\mathbf{c}$
Objective function	$\max \mathbf{c}^T \mathbf{x}$	$\min \mathbf{b}^T \mathbf{y}$
Constraints	ith constraint has $\leq$	$y_i \geq 0$
	$\geq$	$y_i \leq 0$
	$=$	$y_i \in \mathbb{R}$
	$x_j \geq 0$	jth constraint has $\geq$
	$x_j \leq 0$	$\leq$
	$x_j \in \mathbb{R}$	$=$

Figure: Dualization recipe reproduced from [8]

Table of contents

Dorn's dual for quadratic programs

First non-linear programming dual, proposed in [4, 5]. The related duality results can be obtained via Lagrangian duality, see [6]

Consider a primal convex quadratic program

$$\max\{\frac{1}{2}x^TQx + c^Tx \mid Ax \leq b, x \geq 0\}, \tag{12}$$

with Q a semi-definite negative matrix.

Dorn's dual for quadratic programs

First non-linear programming dual, proposed in [4, 5]. The related duality results can be obtained via Lagrangian duality, see [6]

Consider a primal convex quadratic program

$$\max\{\frac{1}{2}x^TQx + c^Tx \mid Ax \leq b, x \geq 0\}, \tag{12}$$

with Q a semi-definite negative matrix.

Its (Dorn's) dual is (13)

$$\min\{b^Tu - \frac{1}{2}v^TQv \mid A^Tu - Qv \geq c, u \geq 0\} \tag{13}$$

Weak and strong duality for convex QPs

Theorem (Weak duality theorem for linear programs)

For x primal feasible, and (u, v) dual feasible:

$$\frac{1}{2}x^T Qx + c^T x \leq b^T u - \frac{1}{2}v^T Qv \tag{14}$$

Proof.

See [5] or notes on Moodle. $\square$

Weak and strong duality for convex QPs

Theorem (Weak duality theorem for linear programs)

For x primal feasible, and (u, v) dual feasible:

$$\frac{1}{2}x^T Qx + c^T x \leq b^T u - \frac{1}{2}v^T Qv \tag{14}$$

Proof.

See [5] or notes on Moodle. □

Theorem (Strong duality theorem for linear programs)

If both $\{x | Ax \leq b, x \geq 0\}$ and $\{(u, v) | A^T u - Qv \geq c, u \geq 0\}$ are non-empty, the primal (9) admits an optimal solution x^*, the dual (13) admits an optimal solution (u^*, v^*), and

$$\frac{1}{2}(x^*)^T Qx^* + c^T x^* = b^T u^* - \frac{1}{2}(v^*)^T Qv^* \tag{15}$$

Proof.

See [5] or notes on Moodle. □

Optimality conditions for convex QPs

Optimality conditions similar to optimality conditions for LPs:

Theorem

The following conditions are equivalent:

1. x^* *is optimal for the primal and* (u^*, v^*) *is optimal for the dual*
2. x^*, (u^*, v^*) *are respectively primal and dual feasible, and satisfy*

$$x^T Qx + c^T x \geq b^T u - \frac{1}{2} v^T Qv \tag{16}$$

(equality of primal and dual objective functions, why equality?)

3. x^*, (u^*, v^*) *are respectively primal and dual feasible, and*
   3.1 $Qx^* = Qv^*$
   3.2 *(complementarity slackness), for each row* i *and each column* j *of* A,
      (a) $(b_i - A_i x)u_i = 0$
      (b) $(A_j^T u - Qv_j - c_j)x_j = 0$

Proof.

Exercise. (See also later, notes on Moodle.)

Table of contents

Lagrangian duality

Let us consider

$$\max c^T x \tag{17}$$

subject to :

$$Ax \leq b \qquad [\pi \geq 0] \tag{18}$$

$$Cx \leq d \qquad [\mu \geq 0] \tag{19}$$

and consider the Lagrangian function:

$$L(\pi) = \max_{x | Cx \leq d} c^T x + \pi^T (b - Ax) \tag{20}$$

The Lagrangian dual (or here partial linear programming dual) is:

$$\min_{\pi \geq 0} L(\pi) = \min_{\pi \geq 0} \max_{x | Cx \leq d} c^T x + \pi^T (b - Ax) \tag{21}$$

Lagrangian duality: the linear programming case

For any $\pi \geq 0$ (feasible for the Lagrangian dual), and x feasible for (18)-(19),

$$L(\pi) = \max_{x \mid Cx \leq d} c^T x + \pi^T (b - Ax) \geq c^T x \tag{22}$$

Proof.

Exercise. $\square$

Lagrangian duality: the linear programming case

For any $\pi \geq 0$ (feasible for the Lagrangian dual), and x feasible for (18)-(19),

$$L(\pi) = \max_{x|Cx \leq d} c^T x + \pi^T (b - Ax) \geq c^T x \tag{22}$$

Proof.

Exercise. □

Theorem

Assume that the LP (17)-(19) *has an optimal solution* x^* *and let* (π^*, μ^*) *be an optimal solution to the classic linear programming dual of* (17)-(19)*. Then* π^* *solves the Lagrangian dual* (21)*:*

$$\min_{\pi \geq 0} L(\pi) = \min_{\pi \geq 0} \max_{x|Cx \leq d} c^T x + \pi^T (b - Ax)$$

and (strong Lagrangian duality)

$$c^T x^* = L(\pi^*) \tag{23}$$

Lagrangian duality: the linear programming case

Proof.

By assumption, (π^*, μ^*) solves the classic linear programming dual of (17)-(19), $\max\{c^T x \mid Ax \leq b, Cx \leq d\}$, i.e. solves:

$$\min\{b^T\pi + d^T\mu \mid A^T\pi + C^T\mu = c, \pi \geq 0, \mu \geq 0\} \tag{24}$$

Lagrangian duality: the linear programming case

Proof.

By assumption, (π^*, μ^*) solves the classic linear programming dual of (17)-(19), $\max\{c^T x \mid Ax \leq b, Cx \leq d\}$, i.e. solves:

$$\min\{b^T\pi + d^T\mu \mid A^T\pi + C^T\mu = c, \pi \geq 0, \mu \geq 0\} \tag{24}$$

$$
\begin{aligned}
&= \min_{\pi \geq 0}\{b^T\pi + \min\{d^T\mu \mid C^T\mu = c - A^T\pi, \mu \geq 0\}\} && (25)\\
&= \min_{\pi \geq 0}\{b^T\pi + \max_{x \mid Cx \leq d} c^T x - \pi^T A x\} && (26)\\
&= \min_{\pi \geq 0} \max_{x \mid Cx \leq d} c^T x + \pi^T(b - Ax)\} && (27)\\
&= \min_{\pi \geq 0} L(\pi) && (28)
\end{aligned}
$$

$\square$

Table of contents

References I

[1] M. S. Bazaraa, J. J. Jarvis, and H. D. Sherali.
Linear Programming and Network Flows.
Wiley, second edition, 1990.

[2] D. Bertsimas and J. N. Tsitsiklis.
Introduction to Linear Optimization.
Athena Scientific and Dynamic Ideas, 1997.

[3] M. Conforti, G. Cornuéjols, G. Zambelli, et al.
Integer programming, volume 271.
Springer, 2014.

[4] W. S. Dorn.
Duality in quadratic programming.
Courant Institute of Mathematical Sciences, New York University, 1958.

[5] W. S. Dorn.
Duality in quadratic programming.
Quarterly of applied mathematics, 18(2):155–162, 1960.

References II

[6] A. M. Geoffrion.
Duality in nonlinear programming: a simplified applications-oriented development.
SIAM review, 13(1):1–37, 1971.

[7] R. K. Martin.
Large Scale Linear and Integer Optimization: a unified approach.
Springer, 1999.

[8] J. Matousek and B. Gärtner.
Understanding and using linear programming.
Springer Science & Business Media, 2007.

[9] Y. Pochet and L. A. Wolsey.
Production Planning by Mixed Integer Programming.
Springer, 2006.