Operations Research - LINMA 2491

Course 5 - Introduction to Stochastic Programming via examples

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A first farmer's problem[1]

- A farmer has 500 acres of land to raise wheat, corn and sugar beets

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- The following table summarizes profits. It is first assumed that scenarios have equal probability of occurence.

Profits ($/acre)	Wheat	Corn	Sugar beets
Future 1	140	210	120
Future 2	250	160	180
Future 3	180	190	260
Average	190	186.6667	186.6666667
Min	140	160	120
Max	250	210	260

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- An optimist would choose to plant 500 acres of sugar beets. Maximax: best of best possible outcomes
- A pessimist would choose to plant 500 acres of corn. Maximin: "best worst case"
- On the long run, better to plant 500 acres of wheat...

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In such a simple setting without second-stage decisions, whatever the probabilities p_1, p_2, p_3 for the three scenarios, to maximize expected gains, choose the crop with best expected profits:

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$$\max_{wheat,corn,beets} p_1(140wheat + 210corn + 120beets)$$

$$+p_2(250wheat+160corn+180beets)+p_3(180wheat+190corn+260beets) \tag{1}$$

s.t.

$$wheat + corn + beets \leq 500 \tag{2}$$

$$wheat, corn, beets \geq 0 \tag{3}$$

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Objective can be rewritten as :

$$(p_1 140 + p_2 250 + p_3 180) wheat$$
$$+ (p_1 210 + p_2 160 + p_3 190) corn + (p_1 120 + p_2 180 + p_3 260) beets \tag{4}$$

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s.t.

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Objective can be rewritten as :

$$(p_1 140 + p_2 250 + p_3 180)wheat$$
$$+ (p_1 210 + p_2 160 + p_3 190)corn + (p_1 120 + p_2 180 + p_3 260)beets \tag{4}$$

- The optimal solution is hence to devote the 500 acres to the crop with highest average profits.

A second farmer's problem

- A farmer has 500 acres of land to raise wheat, corn and sugar beets[2]

[2] The example is taken from the introductory chapter of the book of reference for this part [1].

A second farmer's problem

- A farmer has 500 acres of land to raise wheat, corn and sugar beets[2]
- The farmer must produce at least 200T of wheat and 240T of corn. Yields vary over years, excess production if any can be sold, and crop can be bought to meet requirements.

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A second farmer's problem

- A farmer has 500 acres of land to raise wheat, corn and sugar beets[2]
- The farmer must produce at least 200T of wheat and 240T of corn. Yields vary over years, excess production if any can be sold, and crop can be bought to meet requirements.
- Data of the problem is sumarized in the following table:

	Wheat	Corn	Sugar beets
Planting costs ($/acre)	150	230	260
Selling price ($/T)	170	150	36 below 6000 and 10 above
Purchase price ($/T)	238	210	-
Minimum requirement (T)	200	240	-
Yield (T/acre) Scenario 1	3	3.6	24
Yield (T/acre) Scenario 2	2.5	3	20
Yield (T/acre) Scenario 3	2	2.4	16
Average Yield (T/acre)	2.5	3	20

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Optimizing with average yields

- x_1 land for wheat, x_2 land for corn, x_3 land for sugar beets
- y_1, y_2 Tons bought of wheat and corn respectively
- w_1, w_2 Tons sold of wheat and corn, w_3, w_4 Tons sold of sugar beets, respectively below / above 6000

$$\min 150x_1 + 230x_2 + 260x_3 + 238y_1 + 210y_2 - 170w_1 - 150w_2 - 36w_3 - 10w_4 \quad (5)$$

s.t.

$$x_1 + x_2 + x_3 \leq 500 \quad (6)$$

$$200 \leq 2.5x_1 + y_1 - w_1 \quad (7)$$

$$240 \leq 3x_2 + y_2 - w_2 \quad (8)$$

$$w_3 + w_4 \leq 20x_3 \quad (9)$$

$$w_3 \leq 6000 \quad (10)$$

$$x, y, w \geq 0 \quad (11)$$

Optimizing with above average yields

- x_1 land for wheat, x_2 land for corn, x_3 land for sugar beets
- y_1, y_2 Tons bought of wheat and corn respectively
- w_1, w_2 Tons sold of wheat and corn, w_3, w_4 Tons sold of sugar beets, respectively below / above 6000

$$\min 150x_1 + 230x_2 + 260x_3 \\ + 238y_1 + 210y_2 - 170w_1 - 150w_2 - 36w_3 - 10w_4 \quad (12)$$

s.t.

$$x_1 + x_2 + x_3 \leq 500 \quad (13)$$
$$200 \leq 3x_1 + y_1 - w_1 \quad (14)$$
$$240 \leq 3.6x_2 + y_2 - w_2 \quad (15)$$
$$w_3 + w_4 \leq 24x_3 \quad (16)$$
$$w_3 \leq 6000 \quad (17)$$
$$x, y, w \geq 0 \quad (18)$$

Optimizing with below average yields

- x_1 land for wheat, x_2 land for corn, x_3 land for sugar beets
- y_1, y_2 Tons bought of wheat and corn respectively
- w_1, w_2 Tons sold of wheat and corn, w_3, w_4 Tons sold of sugar beets, respectively below / above 6000

$$\begin{aligned} \min\ & 150x_1 + 230x_2 + 260x_3 \\ & + 238y_1 + 210y_2 - 170w_1 - 150w_2 - 36w_3 - 10w_4 \end{aligned} \tag{19}$$

s.t.

$$x_1 + x_2 + x_3 \leq 500 \tag{20}$$

$$200 \leq 2x_1 + y_1 - w_1 \tag{21}$$

$$240 \leq 2.4x_2 + y_2 - w_2 \tag{22}$$

$$w_3 + w_4 \leq 16x_3 \tag{23}$$

$$w_3 \leq 6000 \tag{24}$$

$$x, y, w \geq 0 \tag{25}$$

Optimal solutions

Average case:

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	120	80	300
Yield (T)	300	240	6000
Sales (T)	100	–	6000
Purchase (T)	–	–	–
Overall profit: $118,600			

Above average case:

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	183.33	66.67	250
Yield (T)	550	240	6000
Sales (T)	350	–	6000
Purchase (T)	–	–	–
Overall profit: $167,667			

Below average case:

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	100	25	375
Yield (T)	200	60	6000
Sales (T)	–	–	6000
Purchase (T)	–	180	–
Overall profit: $59,950			

Average profit under perfect information:

$$\frac{1}{3}118\ 600 + \frac{1}{3}167\ 667 + \frac{1}{3}59\ 950 = 115\ 406\ EUR$$

Expected profit with decisions based on mean values of parameters

In the case of mean values, which also corresponds here to Scenario 2, the decisions are $x_1 = 120, x_2 = 80, x_3 = 300$

The average profits when these decisions are each time taken can be computed via the optimization problem on the next slide. The optimal objective value is 107 240 EUR.

Average profits when using average parameter values

$$\min 150x_1 + 230x_2 + 260x_3 \tag{26}$$
$$+ \frac{1}{3}(238y_{11} + 210y_{21} - 170w_{11} - 150w_{21} - 36w_{31} - 10w_{41}) \tag{27}$$
$$+ \frac{1}{3}(238y_{12} + 210y_{22} - 170w_{12} - 150w_{22} - 36w_{32} - 10w_{42}) \tag{28}$$
$$+ \frac{1}{3}(238y_{13} + 210y_{23} - 170w_{13} - 150w_{23} - 36w_{33} - 10w_{43}) \tag{29}$$

s.t.

$$x_1 = 120 \qquad x_2 = 80 \qquad x_3 = 300$$
$$y, w \geq 0$$

Scenario 1:	Scenario 2:	Scenario 3:
$200 \leq 3x_1 + y_{11} - w_{11}$	$200 \leq 2.5x_1 + y_{12} - w_{12}$	$200 \leq 2x_1 + y_{13} - w_{13}$
$240 \leq 3.6x_2 + y_{21} - w_{21}$	$240 \leq 3x_2 + y_{22} - w_{22},$	$240 \leq 2.4x_2 + y_{23} - w_{23}$
$w_{31} + w_{41} \leq 24x_3$	$w_{32} + w_{42} \leq 20x_3$	$w_{33} + w_{43} \leq 16x_3$
$w_{31} \leq 6000$	$w_{32} \leq 6000$	$w_{33} \leq 6000$

Doing better

- We will now see that we can do better than relying on the average values of the input parameters.

- by allowing other decisions for x_1, x_2, x_3, it is possible to have better profits on average.

Farmer's stochastic program

$$
\begin{aligned}
&\min 150x_1 + 230x_2 + 260x_3 && (30)\\
&+\frac{1}{3}(238y_{11} + 210y_{21} - 170w_{11} - 150w_{21} - 36w_{31} - 10w_{41}) && (31)\\
&+\frac{1}{3}(238y_{12} + 210y_{22} - 170w_{12} - 150w_{22} - 36w_{32} - 10w_{42}) && (32)\\
&+\frac{1}{3}(238y_{13} + 210y_{23} - 170w_{13} - 150w_{23} - 36w_{33} - 10w_{43}) && (33)
\end{aligned}
$$

s.t.

$$
\begin{aligned}
&x_1 + x_2 + x_3 \leq 500\\
&x, y, w \geq 0\\
&Scenario1: && Scenario2: && Scenario3:\\
&200 \leq 3x_1 + y_{11} - w_{11} && 200 \leq 2.5x_1 + y_{12} - w_{12} && 200 \leq 2x_1 + y_{13} - w_{13}\\
&240 \leq 3.6x_2 + y_{21} - w_{21} && 240 \leq 3x_2 + y_{22} - w_{22}, && 240 \leq 2.4x_2 + y_{23} - w_{23}\\
&w_{31} + w_{41} \leq 24x_3 && w_{32} + w_{42} \leq 20x_3 && w_{33} + w_{43} \leq 16x_3\\
&w_{31} \leq 6000 && w_{32} \leq 6000 && w_{33} \leq 6000
\end{aligned}
$$

Optimal solution of the stochastic program

		Wheat	Corn	Sugar Beets
First Stage	Area (acres)	170	80	250
$s = 1$ Above	Yield (T)	510	288	6000
	Sales (T)	310	48	6000 (favor. price)
	Purchase (T)	–	–	–
$s = 2$ Average	Yield (T)	425	240	5000
	Sales (T)	225	–	5000 (favor. price)
	Purchase (T)	–	–	–
$s = 3$ Below	Yield (T)	340	192	4000
	Sales (T)	140	–	4000 (favor. price)
	Purchase (T)	–	48	–
Overall profit: $108,390				

With these decisions for x_1, x_2, x_3, on average, profits will be of 108 390.

EVPI and VSS

With these decisions for x_1, x_2, x_3, on average, profits will be of 108 390.

- On the other hand, with perfect information, we have seen above that profits would on average be 115 406 EUR.
  The difference is called the Expected Value of Perfect Information (EVPI), here 115 406 - 108 390 = 7 016 EUR.

- Also, the decisions based on the average values of the input parameters yielded an average profit of 107 240 EUR (see above).
  The difference is called the Value of the Stochastic Solution (VSS).
  Here, VSS = 108 390 - 107 240 = 1 150 EUR.

General Model Formulation

$$\min cx + \mathbb{E}_s Q(x, s)$$

s.t.

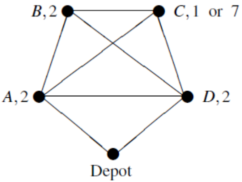
$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

with in the example above
$Q(x, s) = \min\limits_{y,w} 238y_1 - 170w_1 + 210y_2 - 150w_2 - 36w_3 - 10w_4$
s.t.

$$\begin{aligned} 200 \leq t_1(s)x_1 + y_1 - w_1 \\ 240 \leq t_2(s)x_2 + y_2 - w_2 \\ w_3 + w_4 \leq t_3(s)x_3 \\ w_3 \leq 6000 \\ y, w \geq 0 \end{aligned}$$

A routing problem

- A postman has to visit clients A, B, C, D to pick up packages. He can load up to 10 packages in his car.
- Demand is 2 for A, B, D
- For C, demand is 1 or 7 with equal probability



[3]

- Distances between clients are given by:

	0	A	B	C	D
0	–	2	4	4	1
A	2	–	3	4	2
B	4	3	–	1	3
C	4	4	1	–	3
D	1	2	3	3	–

[3] Example and illustration are taken from the reference [1].

A routing problem

It can be checked that the shortest tour to visit all clients and return to the depot is to visit them in the following order $(0, A, B, C, D, 0)$. (N.B. finding such a shortest tour is known as the Traveling Salesman Problem.)

[4] Illustration taken from [1].

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I. Perfect information (wait-and-see solution):

- If demand at C is 7, it's needed to split the trip in two. One can check that the best option is $(0, A, D, 0, B, C, 0)$. Total length is 14.

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- If demand at C is 1, the original shortest tour $(0, A, B, C, D, 0)$ stays feasible and hence the best. Total length is 10.

[4] Illustration taken from [1].

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- On average, tours have a length of $\frac{1}{2}14 + \frac{1}{2}10 = 12$.

[4] Illustration taken from [1].

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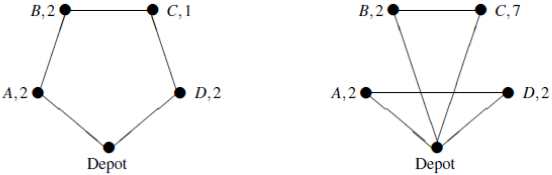
- On average, tours have a length of $\dfrac{1}{2}14 + \dfrac{1}{2}10 = 12$.
- This is the best we can do (taking the best choice relying on perfect information)



[4]

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With a demand of 4, the optimal route is still $(0, A, B, C, D, 0)$. However, the actual demand at C is either 1 or 7. What happens in both cases if we follow this route?

A routing problem

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- If demand at C is 1, ok, the route is still feasible and the total trip length is 10.

A routing problem

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With a demand of 4, the optimal route is still $(0, A, B, C, D, 0)$. However, the actual demand at C is either 1 or 7. What happens in both cases if we follow this route?

- If demand at C is 1, ok, the route is still feasible and the total trip length is 10.
- If load at C is 7, the salesman first visits $(0, A, B)$ and collect 4 units. Arriving at C, he can collect 6 other units and must then return to the depot to unload, then go back to C and resume the trip: $(0, A, B, C, 0, C, D, 0)$. Total length $= 18$. (N.B. $(0, A, B, C, 0, D, C, 0)$ has same length.)

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- On average, trip length $= \frac{1}{2}10 + \frac{1}{2}18 = 14$

A routing problem

III. Recourse solution

We can do better:

We know that the shortest trip is $(0, A, B, C, D, 0)$ of length 10, as for $(0, D, C, B, A, 0)$.

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- Suppose the salesman visits clients in that second order. If demand at C is 1, this is feasible and the length of the trip stays 10.

A routing problem

III. Recourse solution

We can do better:

We know that the shortest trip is $(0, A, B, C, D, 0)$ of length 10, as for $(0, D, C, B, A, 0)$.

- Suppose the salesman visits clients in that second order. If demand at C is 1, this is feasible and the length of the trip stays 10.
- If demand at C turns out to be 7, the load is already of 9 and it won't be feasible to load packages at B and A without returning to the depot, so let's return directly to unload and resume the trip: $(0, D, C, 0, B, A, 0)$ of total length $= 17$.

A routing problem

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We know that the shortest trip is $(0, A, B, C, D, 0)$ of length 10, as for $(0, D, C, B, A, 0)$.

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- With this strategy relying on recourse decisions (after observing what happens at C), average length is $\frac{1}{2}10 + \frac{1}{2}17 = 13.5$

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- If demand at C turns out to be 7, the load is already of 9 and it won't be feasible to load packages at B and A without returning to the depot, so let's return directly to unload and resume the trip: $(0, D, C, 0, B, A, 0)$ of total length $= 17$.
- With this strategy relying on recourse decisions (after observing what happens at C), average length is $\frac{1}{2}10 + \frac{1}{2}17 = 13.5$

It is actually possible to do even better:

- Follow the path $(0, C, B, A, D, 0)$.
- if demand at C is 1, OK, continue, and the total length of the trip is 11.
- if demand at C is 7, do a preventive return at B and then resume the trip: $(0, C, B, 0, A, D, 0)$. Total length is 14.
- With this strategy, average total length is 12.5

A routing problem

The Wait-and-see problem solution (WS) will always be better or equal to the Recourse Problem Solution (RP), which in turn will always be better or equal to the Expectation of the Expected Value Problem solution (EEV) where decisions are based on the average parameter values:

$$WS \leq RP \leq EEV$$

- $RP - WS$ is called the Expected Value of Perfect Information (EVPI)
- $EEV - RP$ is called the Value of the Stochastic Solution (VSS)

The news vendor problem

- A news vendor buys x newspapers at c EUR per copy
- He sells them at q EUR per copy
- Unsold copies can be returned to the publisher, at a price $r < c < q$
- In all cases, the seller won't order more than u newspapers
- Demand is uncertain, randomly distributed

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How many newspapers should the news vendor order?

- x is the number of ordered news papers

The objective is to solve:

$$\min_x cx + D(x) \tag{34}$$

$$0 \leq x \leq u \tag{35}$$

with $D(x) = \mathbb{E}_\xi Q(x, \xi)$

The news vendor problem

$D(x) = \mathbb{E}_\xi Q(x, \xi)$?

- ξ denotes the random event of having a demand $= \xi$
- $y(\xi)$ denotes the copies that can be sold if demand is ξ
- $w(\xi)$ copies returned to the publisher

$Q(x, \xi) = \min_{y,w} -qy(\xi) - rw(\xi)$ s.t.

$$y(\xi) \le \xi \tag{36}$$

$$y(\xi) + w(\xi) \le x \tag{37}$$

$$y, w \ge 0 \tag{38}$$

The news vendor problem

$D(x) = \mathbb{E}_\xi Q(x, \xi)$?

- ξ denotes the random event of having a demand $= \xi$
- $y(\xi)$ denotes the copies that can be sold if demand is ξ
- $w(\xi)$ copies returned to the publisher

$Q(x, \xi) = \min_{y,w} -qy(\xi) - rw(\xi)$ s.t.

$$y(\xi) \leq \xi \tag{36}$$

$$y(\xi) + w(\xi) \leq x \tag{37}$$

$$y, w \geq 0 \tag{38}$$

Optimal solution:

- $y^*(\xi) = min(x, \xi)$
- $w^*(\xi) = x - y^*(\xi) = max(x - \xi, 0)$

$D(x) = \mathbb{E}_\xi Q(x, \xi) = \mathbb{E}_\xi[-q \; min(x, \xi) - r \; max(x - \xi, 0)]$

The news vendor problem

- $f(\xi)$ probability density function
- $F(x)$ cumulative probability distribution (primitive of $f(x)$), i.e. $F(x) = \mathbb{P}[\xi \leq x]$

$$D(x) = \int_{-\infty}^{x} \left[-q\xi - r(x-\xi) \right] f(\xi)d\xi + \int_{x}^{+\infty} -qxf(\xi)d\xi$$

$$= -(q-r)\int_{-\infty}^{x} \xi f(\xi)d\xi - rxF(x) - qx(1-F(x))$$

$$= -qx + (q-r)\int_{-\infty}^{x} F(\xi)d\xi$$

Hence, $D^{'}(x) = -q + (q-r)F(x)$.

The news vendor problem

It can be shown that D is a convex differentiable function when ξ is a continuous random variable (see [1]), and we want to solve $\min_x cx + D(x)$ s.t. $0 \leq x \leq u$.

The optimal solution is:

$$
\begin{cases}
x^* = 0 & \texttt{if} \;\; c + D^{'}(0) > 0 \qquad (39) \\
x^* = u & \texttt{if} \;\; c + D^{'}(u) < 0 \qquad (40) \\
x^* \;\; \texttt{solving} \;\; c + D'(x) = 0 & \textit{otherwise} \qquad (41)
\end{cases}
$$

The news vendor problem

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$$\begin{cases} x^* = 0 & \texttt{if } \; c + D^{'}(0) > 0 & (39) \\ x^* = u & \texttt{if } \; c + D^{'}(u) < 0 & (40) \\ x^* \; \texttt{solving} \; c + D'(x) = 0 & \textit{otherwise} & (41) \end{cases}$$

and using the fact that $D^{'}(x) = -q + (q - r)F(x)$:

$$\begin{cases} x^* = 0 & \texttt{if } \; \frac{q-c}{q-r} < F(0) & (42) \\ x^* = u & \texttt{if } \; \frac{q-c}{q-r} > F(u) & (43) \\ x^* = F^{-1}(\frac{q-c}{q-r}) & \textit{otherwise} & (44) \end{cases}$$

References I

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