Numerik HO5

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$$

a)

$$B_J = D^{-1} \cdot (E + F) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}}}$$

$\rho(B_J)$:

$$\det(B_J - \lambda I) = \det\begin{pmatrix} -\lambda & 0 & -\frac{1}{2} \\ 0 & -\lambda & 0 \\ -\frac{1}{2} & 0 & -\lambda \end{pmatrix} = -\lambda^3 + \frac{1}{4}\lambda$$

$$= -\lambda \cdot \underbrace{\left(\lambda^2 - \frac{1}{4}\right)}_{=0} \overset{!}{=} 0 \qquad \lambda_1 = 0$$

$$\lambda^2 - \frac{1}{4} = 0$$

$$\Leftrightarrow \lambda^2 = \frac{1}{4}$$

$$\lambda_{2,3} = \pm \frac{1}{2}$$

$$\underline{\underline{\rho(B_J) = \frac{1}{2}}}$$

$$B_{GS} = (D - E)^{-1} \cdot F = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}}}$$

$\rho(B_{GS})$:

$$\det(B_{GS} - \lambda I) = \det\begin{pmatrix} -\lambda & 0 & -\frac{1}{2} \\ 0 & -\lambda & 0 \\ 0 & 0 & \frac{1}{4} - \lambda \end{pmatrix} = \lambda^2 \cdot \left(\frac{1}{4} - \lambda\right)$$

$$\lambda_{1,2} = 0 \qquad \lambda_3 = \frac{1}{4}$$

$$\underline{\underline{\rho(B_{GS}) = \frac{1}{4}}}$$

b) Jacobi:

$$\begin{aligned} 2x_1 + \quad\quad x_3 &= 4 \\ x_2 \quad\quad &= 0 \\ x_1 + \quad\quad 2x_3 &= 5 \end{aligned}$$

$$x_1^{(k+1)} = \frac{1}{2}(4 - x_3^{(k)})$$

$$x_2^{(k+1)} = 0$$

$$x_3^{(k+1)} = \frac{1}{2}\cdot(5 - x_1^{(k)})$$

$x^{(0)}$	$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	$x^{(4)}$
0	2	$\frac{3}{4}$	$\frac{5}{4}$	$\frac{15}{16}$
0	0	0	0	0
0	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{17}{8}$	$\frac{15}{8}$

Gauß-Seidel:

$$x_1^{(k+1)} = \frac{1}{2}\cdot(4 - x_3^{(k)})$$

$$x_2^{(k+1)} = 0$$

$$x_3^{(k+1)} = \frac{1}{2}\cdot(5 - x_1^{(k+1)})$$

$x^{(0)}$	$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	$x^{(4)}$
0	2	$\frac{5}{4}$	$\frac{17}{16}$	$\frac{65}{64}$
0	0	0	0	0
0	$\frac{3}{2}$	$\frac{15}{8}$	$\frac{63}{32}$	$\frac{255}{128}$