Nr. 1

a) $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$B_J = I - M_J^{-1} A = I - D^{-1} A$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \qquad \det \begin{vmatrix} -\lambda & 0 & -\frac{1}{2} \\ 0 & -\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (-\lambda^2 - \lambda^3) - (-\tfrac{1}{2}\lambda) = 0$$

$$\Leftrightarrow -\lambda(\lambda^2 + \lambda - \tfrac{1}{2}) = 0$$

$$\Rightarrow \lambda_1 = 0$$

$$\lambda_{2,3} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{2}} = \frac{-1 \pm \sqrt{3}}{2}$$

$$\Rightarrow \rho(B_J) = \frac{1+\sqrt{3}}{2}$$

$$D - E = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_{GS} = I - M_{GS}^{-1} A = I - (D-E)^{-1} A = (D-E)^{-1} F$$

$$= \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ -1/4 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix} \qquad \det \begin{vmatrix} -\lambda & 0 & -\frac{1}{2} \\ 0 & -\lambda & 0 \\ 0 & 0 & -\frac{1}{4}-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow -\lambda^3 - \tfrac{1}{4}\lambda^2 = 0$$

$$\Leftrightarrow \lambda^2(\lambda + \tfrac{1}{4}) = 0$$

$$\Rightarrow \lambda_{1,2} = 0$$

$$\lambda_3 = -\frac{1}{4}$$

$$\Rightarrow \rho(B_{GS}) = \frac{1}{4}$$

b)

Jacobi:

$$x_{k+1}^{(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_k^{(2)} - a_{13}x_k^{(3)}\right) = \frac{1}{2}\left(4 - 0x_k^{(2)} - 1x_k^{(3)}\right) = 2 - \frac{1}{2}x_k^{(3)}$$

$$x_{k+1}^{(2)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_k^{(1)} - a_{23}x_k^{(3)}\right) = 1\left(0 - 0x_k^{(1)} - 0x_k^{(3)}\right) = 0$$

$$x_{k+1}^{(3)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_k^{(1)} - a_{32}x_k^{(2)}\right) = \frac{1}{2}\left(5 - 1x_k^{(1)} - 0x_k^{(2)}\right) = \frac{5}{2} - \frac{1}{2}x_k^{(1)}$$

$$\Rightarrow \quad x_0 = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad x_1 = \begin{pmatrix}2\\0\\5/2\end{pmatrix} \quad x_2 = \begin{pmatrix}-1/2\\0\\3/2\end{pmatrix} \quad x_3 = \begin{pmatrix}1/4\\0\\11/4\end{pmatrix} \quad x_4 = \begin{pmatrix}5/8\\0\\19/8\end{pmatrix}$$

Gauß

$$x_{k+1}^{(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_k^{(2)} - a_{13}x_k^{(3)}\right) = \frac{1}{2}\left(4 - 0x_k^{(2)} - 1x_k^{(3)}\right) = 2 - \frac{1}{2}x_k^{(3)}$$

$$x_{k+1}^{(2)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_{k+1}^{(1)} - a_{23}x_k^{(3)}\right) = 1\left(0 - 0x_{k+1}^{(1)} - 0x_k^{(3)}\right) = 0$$

$$x_{k+1}^{(3)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_{k+1}^{(1)} - a_{32}x_{k+1}^{(2)}\right) = \frac{1}{2}\left(5 - 1x_{k+1}^{(1)} - 0x_{k+1}^{(2)}\right) = \frac{5}{2} - \frac{1}{2}x_{k+1}^{(1)}$$

$$\Rightarrow \quad x_0 = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad x_1 = \begin{pmatrix}2\\0\\3/2\end{pmatrix} \quad x_2 = \begin{pmatrix}1/4\\0\\\frac{19}{8}\end{pmatrix} \quad x_3 = \begin{pmatrix}\frac{13}{16}\\0\\\frac{67}{32}\end{pmatrix} \quad x_4 = \begin{pmatrix}\frac{61}{64}\\0\\\frac{259}{128}\end{pmatrix}$$