

EE2211 Tutorial 4

(Systems of Linear Equations)

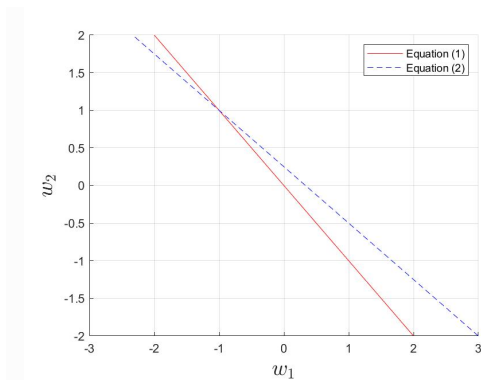
Question 1:

Given $\mathbf{X}\mathbf{w} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- (a) What kind of system is this? (even-, over- or under-determined?)
- (b) Is \mathbf{X} invertible? Why?
- (c) Solve for \mathbf{w} if it is solvable.

Answer:

- (a) This is an even-determined system.
- (b) $\det(\mathbf{X}) = 1 \times 4 - 1 \times 3 = 1 \neq 0$. $\mathbf{X}^{-1} = \frac{1}{1} \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$.
- (c) $\hat{\mathbf{w}} = \mathbf{X}^{-1}\mathbf{y} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.



```
import numpy as np

m_list = [[1, 1], [3, 4]]

X = np.array(m_list)

inv_X = np.linalg.inv(X)

y = np.array([0, 1])

w = inv_X.dot(y)

print(w)
```

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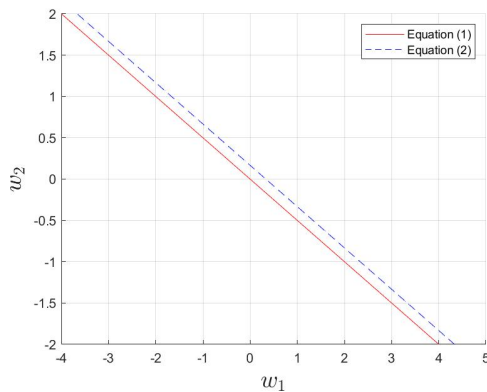
Question 2:

Given $\mathbf{X}\mathbf{w} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- What kind of system is this? (even-, over- or under-determined?)
- Is \mathbf{X} invertible? Why?
- Solve for \mathbf{w} if it is solvable.

Answer:

- This is an even-determined system.
- \mathbf{X} is NOT invertible since the determinant of $\mathbf{X} = 1 \times 6 - 2 \times 3 = 0$.
- There is no solution for \mathbf{w} since the rows/columns of \mathbf{X} are inter-dependent. The two lines shown in the plot are parallel and has no intersection.



(Systems of Linear Equations)

Question 3:

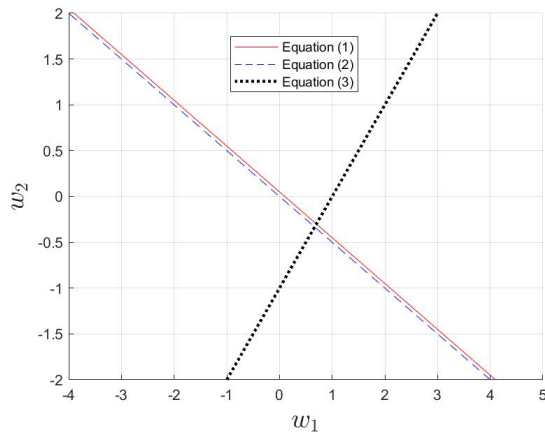
Given $\mathbf{X}\mathbf{w} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix}$.

- What kind of system is this? (even-, over- or under-determined?)
- Is \mathbf{X} invertible? Why?
- Find a solution for \mathbf{w} if it is solvable.

Answer:

- This is an over-determined system.
- \mathbf{X} is NOT invertible but $\mathbf{X}^T\mathbf{X} = \begin{bmatrix} 6 & 9 \\ 9 & 21 \end{bmatrix}$ is. The determinant of $\mathbf{X}^T\mathbf{X} = 6 \times 21 - 9 \times 9 = 45$.
- An approximated solution is given by

$$\hat{\mathbf{w}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} = \begin{bmatrix} 0.4667 & -0.2 \\ -0.2 & 0.1333 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.68 \\ -0.32 \end{bmatrix}.$$



```
# import numpy as np

# m_list = [[1, 2], [2, 4], [1, -1]]

# X = np.array(m_list)

# inv_XTX = np.linalg.inv(X.transpose().dot(X))

# pinv = inv_XTX.dot(X.transpose())

# y = np.array([0, 0.1, 1])

# w = pinv.dot(y)

# print(w)
```

```
import numpy as np

from numpy.linalg import inv

X = np.array([[1, 2], [2, 4], [1, -1]])

y = np.array([0, 0.1, 1])

w = inv(X.T @ X) @ X.T @ y

print(w)
```

(Systems of Linear Equations)

Question 4:

Given $\mathbf{X}\mathbf{w} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

- (a) What kind of system is this? (even-, over- or under-determined?)
- (b) Is \mathbf{X} invertible? Why?
- (c) Solve for \mathbf{w} if it is solvable.

Answer:

- (a) This is an under-determined system.
- (b) \mathbf{X} is NOT invertible but $\mathbf{X}\mathbf{X}^T$ is.

The determinant of $\mathbf{X}\mathbf{X}^T = \det\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}\right) = 2\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = 2 \times 8 - 2 \times 4 + (-4) = 4$.

(c) $\hat{\mathbf{w}} = \mathbf{X}^T(\mathbf{X}\mathbf{X}^T)^{-1}\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0.75 & 0.5 \\ -1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$

(Systems of Linear Equations)

Question 5:

Given $\mathbf{w}^T\mathbf{X} = \mathbf{y}^T$ where $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- (a) What kind of system is this? (even-, over- or under-determined?)
- (b) Is \mathbf{X} invertible? Why?
- (c) Solve for \mathbf{w} if it is solvable.

Answer:

- (a) This is an even-determined system.
- (b) \mathbf{X} is NOT invertible since the determinant of $\mathbf{X} = 1 \times 6 - 2 \times 3 = 0$.
- (c) There is no solution for \mathbf{w} (two parallel lines).

(Systems of Linear Equations)

Question 6:

Given $\mathbf{w}^T\mathbf{X} = \mathbf{y}^T$ where

$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- (a) What kind of system is this? (even-, over- or under-determined?)

- (b) Is \mathbf{X} invertible? Why?
- (c) Solve for \mathbf{w} if it is solvable.

Answer:

- (a) This is an under-determined system (there are 3 unknowns with 2 equations).
- (b) \mathbf{X} is NOT invertible but $\mathbf{X}^T \mathbf{X}$ is. The determinant of $\mathbf{X}^T \mathbf{X} = 6 \times 21 - 9 \times 9 = 45$.
- (c) A constrained solution (exact) is given by

$\hat{\mathbf{w}}^T = (\mathbf{X}\mathbf{a})^T$ (The 3-dimensional vector \mathbf{w} can be constrained by projecting \mathbf{X} onto a 2-dimensional vector \mathbf{a})

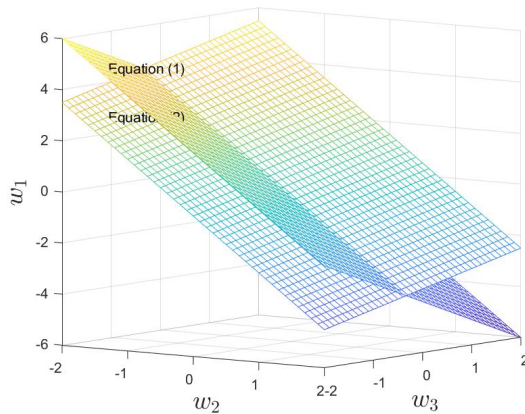
$$= \mathbf{a}^T \mathbf{X}^T$$

$$= \mathbf{y}^T (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T$$

$$= [0 \quad 1] \begin{bmatrix} 0.4667 & -0.2 \\ -0.2 & 0.1333 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}$$

$$= [0.0667 \quad 0.1333 \quad -0.3333]$$

Note: $\dim(\mathbf{X})$ is 3×2 , $\dim(\mathbf{a})$ is 2×1 , estimation is done/constrained on/to the lower dimension of (3×2) and then projected back to the higher dimension 3.



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Question 7:

This question is related to determination of types of system where an appropriate solution can be found subsequently. The following matrix has a left inverse.

$$\mathbf{X} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- a) True
- b) False

Answer: b)

Solution: Left inverse is given by $(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T$ where $\mathbf{X}^T \mathbf{X}$ should be invertible. In this case, $\mathbf{X}^T \mathbf{X}$ is not invertible so the matrix does not have a left inverse.

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Question 8:

MCQ: Which of the following is/are true about matrix **A** below? **There could be more than one answer.**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- a) **A** is invertible
- b) **A** is left invertible
- c) **A** is right invertible
- d) **A** has no determinant
- e) None of the above

Answer: c and d.