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CSCI 387, §S01/02
Homework 7
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1. In order to formulate this problem as a language, we can phrase it in the following way:

$$\begin{aligned}L = \{\langle G, K\rangle \mid &\text{ For } G = (V, E) \text{ a graph with } K \subseteq V,\\ &\text{there exists a set of vertices } M \subseteq V \backslash K \text{ such that for each } k \in K,\\ &k \text{ is adjacent to exactly } n(k) \text{ vertices in } M \text{ where } n(k) \text{ is the label of } k\}\end{aligned}$$

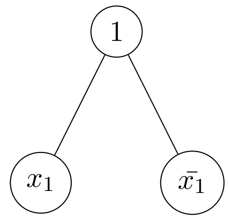
To demonstrate that L is $\mathbb{NP}$-complete, we must first show that L is in $\mathbb{NP}$. To do so, we can construct a polynomial-time verifier D for L. When given an input $\langle G, K, w\rangle$, D can check first that $K, M \subseteq V$ and furthermore that $K \cap M = \varnothing$. Traversing the vertex set of M and K is certainly polynomial time and then comparing them against each other is $O(|V|^2)$ time. Therefore, this step can be done in polynomial time. To verify that an assignment is valid, we can simply iterate through each $k \in K$, check its adjacency list, and then see if there are exactly $n(k)$ vertices adjacent to k that are in M. If there are not, reject. If we have not rejected by the time we have iterated through each k in K, accept. This can be done in $O(|V|^2)$ time because that is the most amount of time it can take to check all of the adjacent vertices of all vertices in G. Therefore, D is polynomial time in the size of $\langle G, K\rangle$.

To check that D verifies L, we see first that any witness that D will accept necessarily has M and K being disjoint. Any valid witness will have this property because if there exists an $e \in M \cap K$, then a mine would have been revealed and the game would be over. Then, we simply check that every revealed node has exactly as many mines adjacent to it as its value says it does. If it does not, then the graph is necessarily inconsistent because there exists a revealed node whose value is more or less than the number of mines it is adjacent to and we reject. If all revealed nodes have precisely the number of mines adjacent to them as their s value indicates, then the graph is consistent by definition. Therefore, we accept. Thus, D decides L and therefore L is in $\mathbb{NP}$.

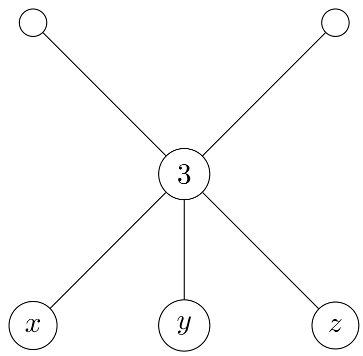
To demonstrate that L is $\mathbb{NP}$-complete, we can perform a reduction from 3SAT. Given a 3-cnf boolean expression X, we are attempting to produce a graph and revealed vertex set G, K such that finding whether or not G, K is in L would prove the satisfiability of X.

We can start by recognizing what our truth value is supposed to be. For my construction, I am going to assign each literal to exactly one unrevealed vertex in G. Then, a variable being marked as True is associated with having a mine marker placed on that variable's associated vertex.

We can start by constructing a variable gadget. We want it to be that for a variable x, either x or $\bar{x}$ is True. Therefore, only x or $\bar{x}$ can have a mine placed on top of it. Therefore, we create a setup whereby x and $\bar{x}$ are adjacent to each other and they are both adjcaent to a revealed vertex whose value is 1. Thus, if there is a satisfying assignment, it cannot be that both x and $\bar{x}$'s vertices have mines. A visual demonstration follows:



Then, to construct a clause gadget, we want it to be that for each clause, at least one of the variable's associated vertices has a mine. To do so, we can construct a gadget whereby if 3 variables are in the same clause, they are all connected to a vertex labelled 3. That vertex labeled 3 is additionally connected to 2 other unrevealed vertices not associated with any variable. For instance, in the clause $x \vee y \vee z$, we would produce the following component:



This guarantees that at least one of x, y, z contains a mine because 3 has already been revealed and therefore cannot contain a mine itself. In order for this graph to be consistent, then because there are only 2 other unrevealed nodes connected to 3, one of x, y, z has to contain a mine.

To show that this construction is actually a reduction, we find that the construction is inherently consistent—either x or $\bar{x}$ contains a mine—and that each clause must contain at least one mine. Therefore, if we let our mines represent True, we have that demonstrating G, K is consistent is equivalent to showing that X has a satisfying assignment.

In this construction, we find that because each variable x has a vertex in K associated with its variable gadget and each clause has 3 vertices associated with its clause gadget, we get $|K| = n \cdot m$ if m is the number of clauses and n is the number of variables. Then, $|V| = 2n + |K|$. These are both polynomial in the length of our boolean expression. Because $|E| \in O(|V|^2)$, we then have that we can also construct all of our edges in polynomial time. Therefore, this reduction is polynomial time.

Therefore, because $L \in \mathbb{NP}$ and there exists a polynomial-time reduction from 3SAT to L and 3SAT is $\mathbb{NP}$-complete, we have that L is $\mathbb{NP}$-complete.