Module 3: Distributions of random variables

Lecture 7: Continuous Random Variables

Overview

Table 1: Supplementary Reading

Topics	**Relevant Ch**
Continuous Distributions	OIS: 3.5

RbE = R by Example

OIS = OpenIntro Statistics

Continuous Distributions

Introduction

- As mentioned previously in lectures, there are two fundamentally different types of random variables:

  discrete random variables a variable whose value is obtained by counting

  continuous random variables a variable whose value is obtained by measuring

- We have already seen **discrete** probability distributions (Bernoulli, Binomial, Hypergeometric, and the Poisson).
- We can now extend the concepts from last lectures to **continuous** random variables.
- Many of the formulas in the discrete case will replace the summation $\sum$ by integration $\int$ for the continuous case.

Reveiw

- A **discrete** random variable maps the sample space to finite or countably infinite set (a list in which there is a first element, a second element, etc.)
  - eg. the number of times we switch a light-bulb on and off before it dies, X = 0,1,2,3,... (the support of X is countably infinite)

- A **continuous** random variable maps the sample space to an uncountable set
  - An example a continuous RV would be the lifetime of a lightbulb $X = \{x : x \geq 0\}$)

- If the support of a random variable is an entire *interval* of numbers is not discrete.

Continuous RV Examples

Other examples of continuous rv include:

- the depth measurements at randomly chosen locations or a lake. Note that X could assume any value between the minimum and maximum depths of the lake.
- The pH level in a compound. In this case, the set of all possible values is the interval [0,14].
- The lifetime of a lightbulb which has support $[0, \infty)$.

Recall notation:

- $x \in [0, 1] \implies 0 \leq x \leq 1,$
- $x \in [0, 1) \implies 0 \leq x < 1,$
- $x \in (0, 1] \implies 0 < x \leq 1,$
- $x \in (0, 1) \implies 0 < x < 1,$

Continuous RV

- It could be argued that measurements such as height, weight, and temperature are, in practice, are discrete due to the limitations of our measuring instruments.
- In a sense this is true. We are constantly ‘discretizing’ continuous heights to the nearest centimetre, weights to the nearest tenth of pound, temperature to the nearest half Celsius, for example.
- However, continuous models are often easier to deal with and provide a good approximate real-world situations.

Continuous RV

- Take for example our example for the depts of a lake. If we measure to the nearest meter, we can plot the discrete distribution of using a histogram.
- The height of each bar represents the proportion of the lake whose depth is (to the nearest meter) is equal to the integer on the corresponding x-axis.
- Measuring to the nearest centimetre, we would get a similar plot, only now we would have more so-called ‘bins’ (i.e. we would be discretizing our x-axis into more discrete intervals.
- If we extend this to some infinitesimally small measurement, we would eventually get a smooth histogram, akin to a curve. See next slide for a visual. . .

Discrete RV pmf $P(X=x) = f(x)$ Continuous RV

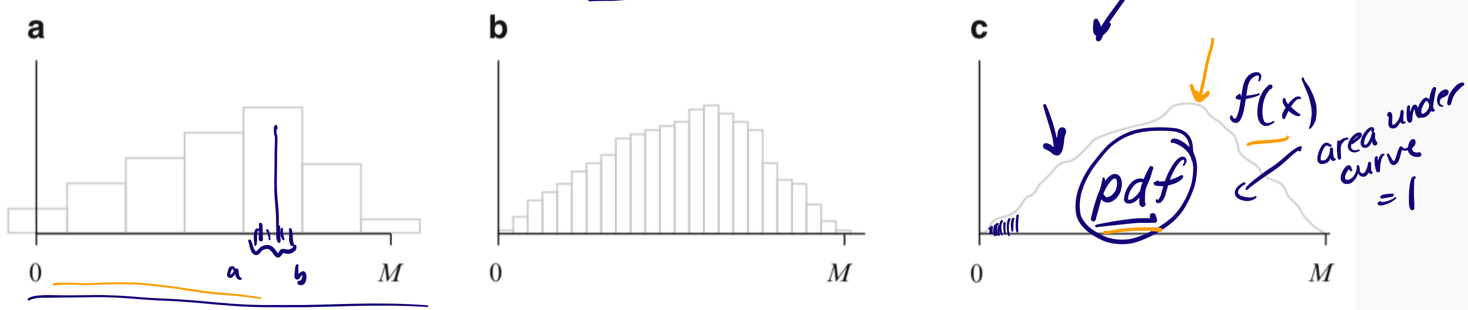


Figure 4.1 (a) Probability histogram of depth measured to the nearest meter; (b) probability histogram of depth measured to the nearest centimeter; (c) a limit of a sequence of discrete histograms

- Note that the heights of the rectangles need to sum to 1.
- Similarly, the *area* under the smooth curve must also be 1.
- The smooth curve pictured in Figure 4.1(c) exemplifies a *continuous probability density function or pdf/PDF*.

Image source: Devore, J. L (2016). *Probability and Statistics for Engineering and the Sciences*

Definitions

Definition 1 (probability density function (pdf))

A function with values $f(x)$, defined over the set of all real numbers, is called a ***probability density function (pdf)*** *of continuous random variable X if and only if*

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

for any real constants a and b with $a \leq b$.

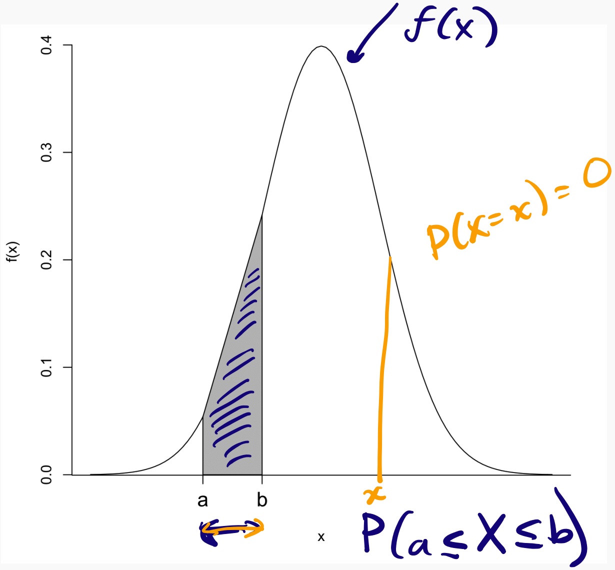
Note that $\boxed{f(x) \neq P(X = x)}$.

It is only once we integrate, that we get probabilities

Graphical presentation of probability densities

A pdf $f(x)$ integrated from a to b (with $a \leq b$), gives the probability that the continuous RV will take on a value on the interval $[a, b]$.

- $P(a \leq X \leq b)$ is represented by the shaded region below $f(x)$ and within $[a, b]$
- $f(x)$ is often referred to as the *density curve*
- N.B. the $\boxed{P(X = x) = 0 \; \forall \; x}$



The last point may seem surprising, however, if we think of probability as the area under a curve, then it begins to make sense.

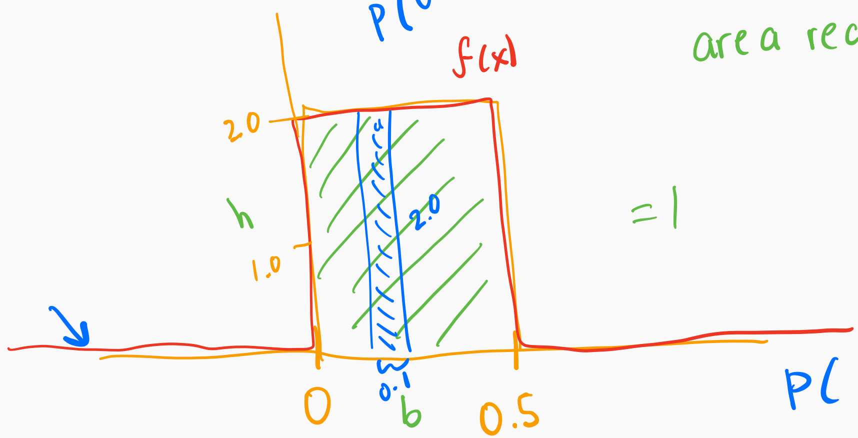
Properties of pdf

A function can serve as a probability density of a continuous random variable X if its values, $f(x)$, satisfy the conditions

(1) $f(x) \geq 0$, for $-\infty < x < \infty$.

(2) $\int_{-\infty}^{\infty} f(x)dx = 1$.

$P(0.2 < x < 0.3) = 0.2$



$= 1$

area a rectangle = b x h

$1 = 0.5 \times h$

$\Rightarrow 2 = h$

$P(X < 0.3) = \int_{-\infty}^{0.3} 2\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{0.3} 2\,dx$

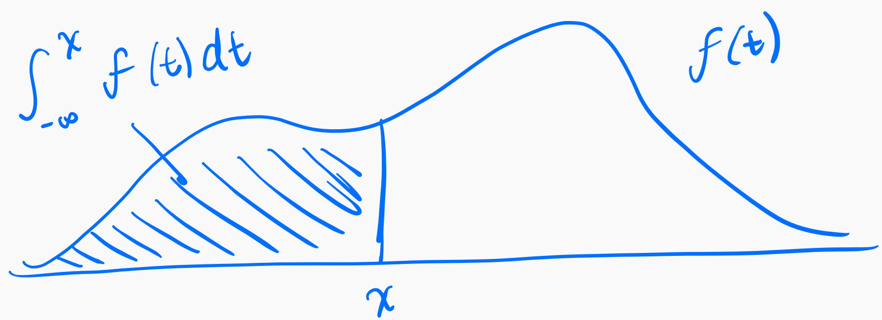
Cumulative distribution function (cdf)

Just as with pmf, we can also define a cumulative distribution function (CDF) that defined $P(X \leq x)$

Definition 2 (Cumulative distribution function (cdf))

For a continuous random variable X that has the probability density (or pdf) at t as $f(t)$, the ***cumulative distribution****, of X is given by*

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt, \qquad \forall -\infty < x < \infty.$$



Visualization of cdf for a continuous RV

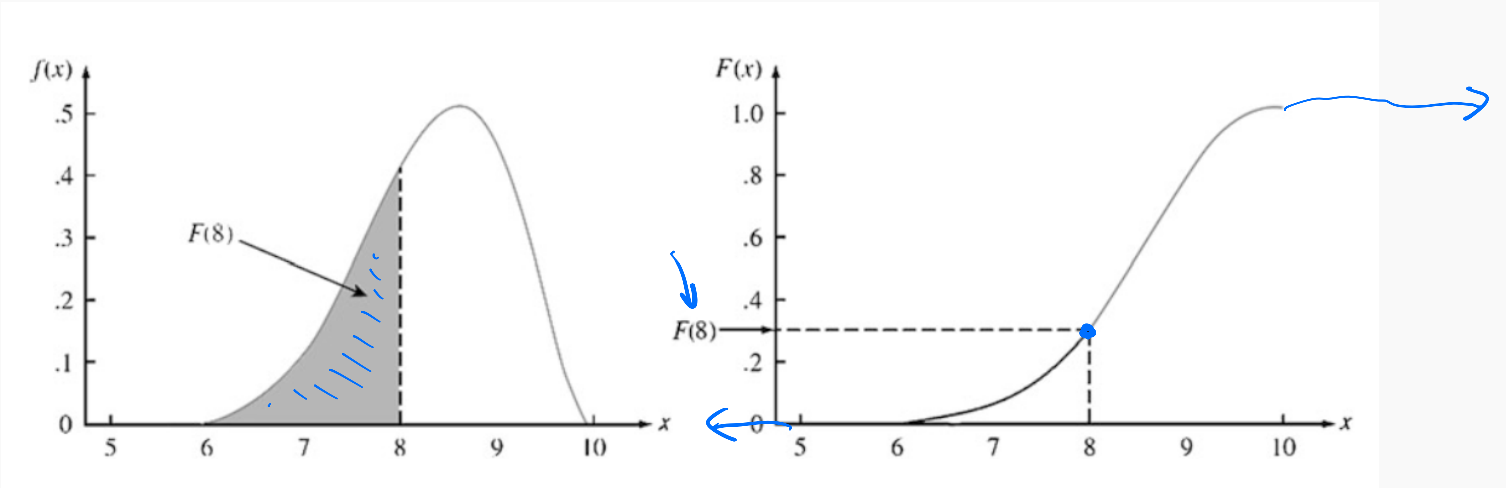


Figure 4.5 A pdf and associated cdf

Figure 1: Taken from Devore, J. L (2016). *Probability and Statistics for Engineering and the Sciences*

For X a continuous random variable, the following hold:

$$\boxed{P(X \leq 8) = P(X < 8)} + P(\cancel{X = 8})$$

0

$$P(X = x) = 0 \quad \text{for all } x,$$

$$P(X \leq x) = P(X < x) = F(x) \quad \text{for all } x,$$

$$\rightarrow \quad P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$$

Proposition 1 Compliment Rule $P(A) = 1 - P(A^c)$

Let X be a continuous rv with pdf $f(x)$ and cdf $F(x)$. Then for any number a

$$= 1 - P(X \leq a)$$

$$P(X > a) = 1 - F(a) \quad \textit{for all } x$$

and for any two numbers a and b with $a < b$,

$$P(a \leq X \leq b) = F(b) - F(a) = \int_a^b f(x)dx,$$

$P(a \leq x \leq b)$ $= F(b)$ $P(x \leq b)$ $- P(x \leq a)$

$f(x)$

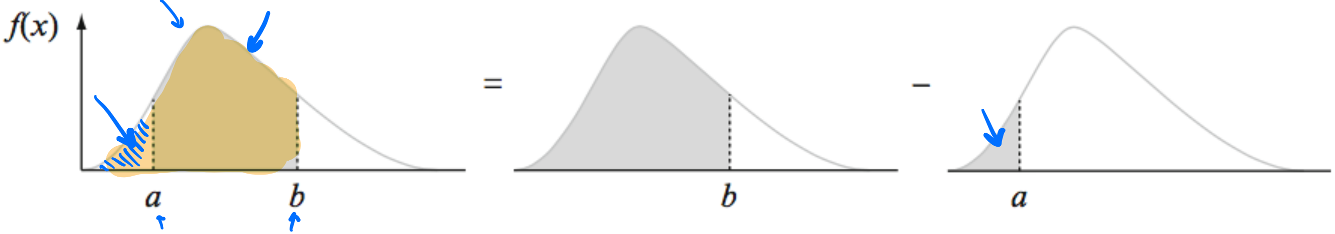


Figure 4.8 Computing $P(a \leq X \leq b)$ from cumulative probabilities

Figure 2: Taken from Devore, J. L (2016). *Probability and Statistics for Engineering and the Sciences*

Example 1

Find the constant c for the following pdf:

$$f(x) = \begin{cases} 0 & \text{elsewhere} \\ cx & 0 \le x \le 1 \end{cases}$$

$f(x) = 2x \quad 0 \le x \le 1$

Aside:

$f(x)$	$\int f(x)\,dx$
x^n	$\frac{x^{n+1}}{n+1} + \text{cons}$
e^x	$e^x + \text{cons}$
$1/x$	$\ln\lvert x\rvert + \text{cons}$

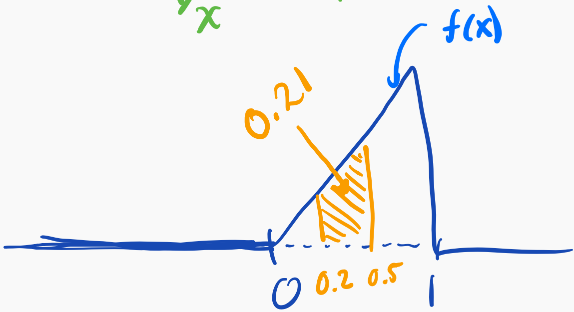
We know $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\cancel{\int_{-\infty}^{0} 0\,dx} + \int_0^1 cx\,dx + \cancel{\int_1^{\infty} 0\,dx} = 1$$

$$\frac{cx^2}{2} + \cancel{\text{cons}} \Big|_0^1 = 1$$

$$\frac{c(1)^2}{2} + \cancel{\text{cons}} - \left[\cancel{\frac{c(0)^2}{2}} + \cancel{\text{cons}}\right] = 1$$

$$\frac{c}{2} = 1 \Rightarrow \boxed{c = 2}$$



Example 1 (cont'd)

Find c and $P(0.2 < X < 0.5)$ using the pdf from Example 1.

$$P(0.2 \leq X \leq 0.5) = \int_{0.2}^{0.5} f(x)\,dx$$

$$= \int_{0.2}^{0.5} 2x\,dx$$

$$= \frac{\cancel{2}x^2}{\cancel{2}}\Bigg|_{0.2}^{0.5}$$

$$= (0.5)^2 - 0.2^2$$

$$= 0.21$$

Example 2

The pdf of a continuous variable X is given below. Find $F(x)$, the cdf of X, and $P(0.2 < X < 0.5)$ using the cdf.

$$f(x) = 2x \quad 0 \le x \le 1$$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} 2t\,dt$$

$$= t^2 \Big|_0^x = x^2$$

$$\boxed{P(0.2 < X < 0.5)} = F(0.5) - F(0.2)$$

$$= 0.5^2 - 0.2^2$$

$$= 0.21$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \boxed{x^2 \quad 0 \le x < 1} & \\ 1 & x \geq 1 \end{cases}$$

Clicker Question

Which of the following calculate the $P(X > 0.4)$

a) $F(0.4)$

b) $1 - F(0.4)$

c) $\int_{0.4}^{\infty} f(x)dx$

d) $\int_{-\infty}^{0.4} f(x)dx$

e) option b) and c)

f) None of the above

Calculating probabilities using the cdf

Notice that one the cdf has been obtained, any probability involving X can easily be calculated without integration!

Clicker Question

Which of the following calculate the $P(X > 0.4)$

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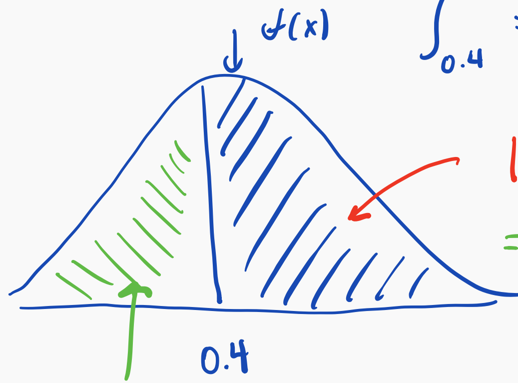


$\int_{0.4}^{\infty} f(x)dx)$

$1 - F(0.4)$

$= 1 - 0.4^2$

$= 0.84$

$P(X \leq 0.4) = F(0.4)$

Calculating probabilities using the cdf

Notice that one the cdf has been obtained, any probability involving X can easily be calculated without integration!

cdf to pdf

- Recall: for X discrete, the pmf is obtained from the cdf by taking the difference between two $F(x)$ values.
- In the continuous case, rather than taking the difference we take a derivative.

Proposition 2

If X is a continuous rv with pdf $f(x)$ and cdf $F(x)$, then at every x at which the derivative $F'(x)$ exists, $\boxed{F'(x) = f(x).}$

CDF to PDF example

Example 3

Let X be a continuous rv with cdf given by (cont'd from Ex. 2):

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ x^2, & \text{for } 0 \le x \le 1 \\ 1, & \text{for } x > 1 \end{cases}$$

Note that $F(x)$ is differentiable except at $x = 0$ and $x = 1$ (where the graph has sharp corners). Since $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$, $F'(x) = 0 = f(x)$ for such x. For $0 < x < 1$:

$F'(x) = f(x) = \frac{d}{dx} x^2 = 2x \qquad 0 \le x \le 1$

Definition 3

Let p be a number between 0 and 1. The (100p)th percentile of the distribution of a continuous rv X, denoted by $\eta(p)$, is defined by

$$p = F[\eta(p)] = \boxed{\int_{-\infty}^{\eta(p)} f(y)dy} \tag{1}$$

Definition 4 (median)

The median of a continuous distribution, denoted by $\tilde{\mu}$, is the 50th percentile, so $\tilde{\mu}$ satisfies $0.5 = F(\tilde{\mu})$. That is, half the area under the density curve is to the left of $\tilde{\mu}$ and half is to the right of $\tilde{\mu}$.

$$0.5 = F[\underbrace{\eta(0.5)}_{\tilde{\mu}}] = P(X \leq \eta(0.5)) = 0.5$$

- According to Eq. (1), $\eta(p)$ is that value on the measurement axis such that $100p\%$ of the area under the graph of $f(x)$ lies to the left of $\eta(p)$ and $100(1-p)\%$ lies to the right.
- Thus $\eta(.75)$, the 75th percentile, is such that the area under the graph of $f(x)$ to the left of $\eta(.75)$ is .75.

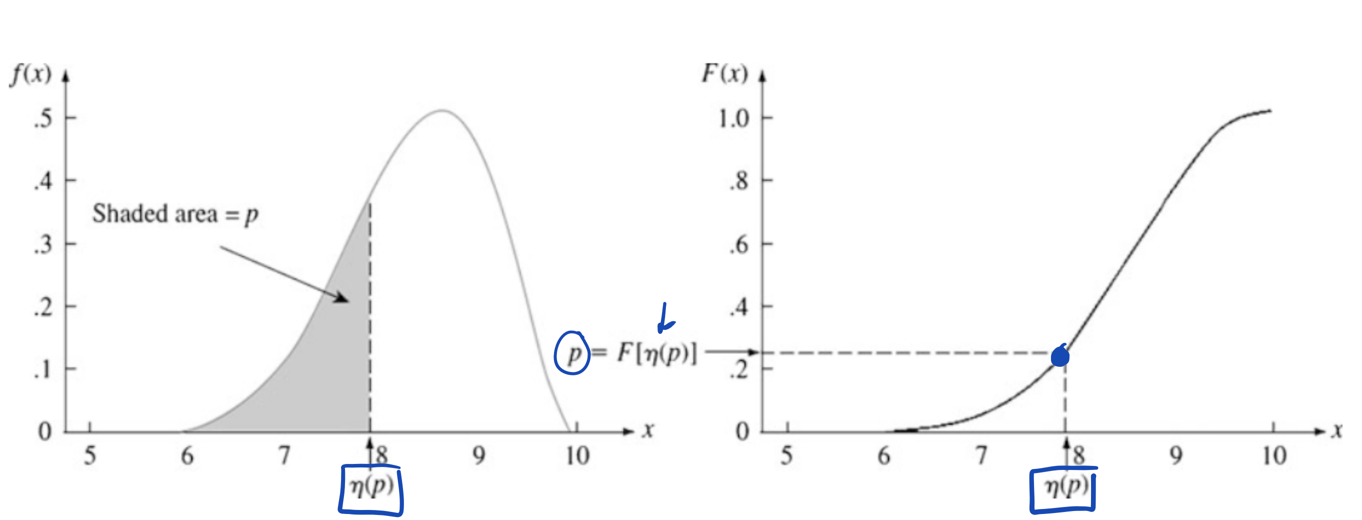


Figure 4.10 The (100p)th percentile of a continuous distribution

Figure 3: From Devore, J. L (2016). *Probability and Statistics for Engineering and the Sciences*

Percentile Example

Example 4

The distribution of the amount of gravel (in tons) sold by a construction supply company in a given week is a continuous rv X with pdf

$$f(x) = \frac{3}{2}(1 - x^2) \quad \textit{if } 0 \leq x \leq 1$$

The cdf of sales for any x between 0 and 1 is

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{3}{2}\left(x - \frac{x^3}{3}\right) & \textit{if } 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Find the 50th percentile. (AKA median)

? $\tilde{\mu}$

$$p = F[\eta(p)]$$

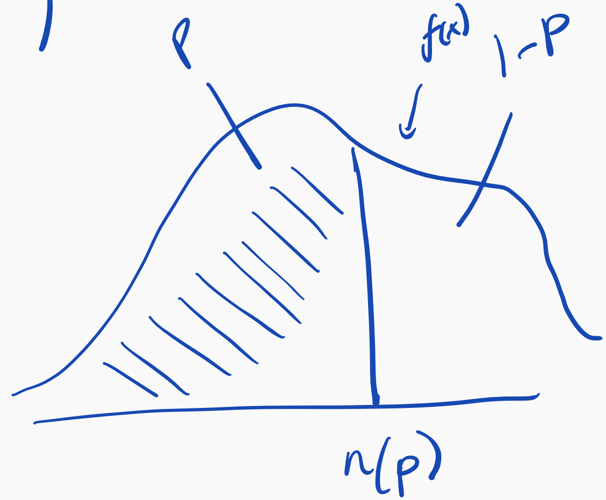
$$0.5 = \frac{3}{2}\left(\tilde{\mu} - \frac{\tilde{\mu}^3}{3}\right)$$

$$1 = 3\tilde{\mu} - \tilde{\mu}^3$$

$$\tilde{\mu}^3 - 3\tilde{\mu} + 1 = 0$$

... math

$$\tilde{\mu} = 0.347$$



$$p = F(n(p))$$

If the distribution remains the same from week to week, then in the long run 50% of all weeks will result in sales of less than .347 tons and 50% in more than .347 tons.

Symmetry

- A continuous distribution whose pdf is symmetric has median $\tilde{\mu}$ equal to the point of symmetry.
- That is, half the area under the curve lies to either side of this point.

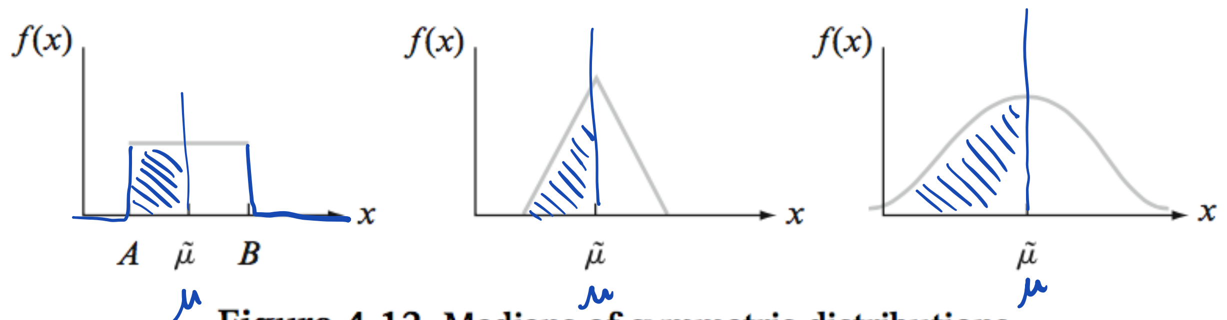


Figure 4.12 Medians of symmetric distributions

Figure 4: From Devore, J. L (2016). *Probability and Statistics for Engineering and the Sciences*

Expected Value & Variance

discrete $E[X] = \sum_x x\, P(X=x) = \boxed{\sum x f(x)}$

Definition 5

The expected value of a continuous RV is defined as follows;

$$\mu = \mathbb{E}[X] = \boxed{\int_{-\infty}^{\infty} x f(x) dx}$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

where $f(x)$ is the pdf of X.

Expected Value & Variance

Definition 5

The expected value of a continuous RV is defined as follows;

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$$
$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

where $f(x)$ is the pdf of X.

When the pdf $f(x)$ specifies a model for the distribution of values in a numerical population, then μ is the population mean, which is the most frequently used measure of population location or center.

Useful shortcuts

Definition 6

The variance of continuous RV X is

$$Var[X] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

As before we have the following shortcuts for linear combinations of rv X and for our calculation of Variance:

- $$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$
- $$Var[aX + b] = a^2 Var[X]$$
- $$\boxed{Var[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

Shortcut

Example 5

Find $\mathbb{E}(X)$ *and* $Var[X]$ *for a random variable* X *with pdf:*

$$f(x) = 2x \quad \textit{if } 0 \leq x \leq 1$$

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x \cdot 2x\,dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$$

$$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \int_0^1 \left(x - \frac{2}{3}\right)^2 2x\,dx$$

$$= \int_0^1 \left(x^2 - \frac{4}{3}x + \left(\frac{2}{3}\right)^2\right) 2x\,dx$$

$$= \int_0^1 \left(2x^3 - \frac{8}{3}x^2 + \frac{8}{9}x\right) dx$$

$$= \left[\frac{1}{2}x^4 - \frac{8}{9}x^3 + \frac{4}{9}x^2\right]_0^1$$

$$= \frac{1}{2} - \frac{8}{9} + \frac{4}{9} = \frac{1}{18} = 0.0\dot{5} = 0.0555555\ldots$$

Verify this w/ shortcut

Example 6

The pdf of weekly gravel sales X is given below. Find $\mathbb{E}[X]$

$$f(x) = \frac{3}{2}(1 - x^2) \quad \textit{if } 0 \le x \le 1$$

$$\mu = E[X] = \int_0^1 x \frac{3}{2}(1-x^2)\,dx$$

$$= \frac{3}{2}\int_0^1 x - x^3\,dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8} = \underline{0.375}$$

Aside

$\tilde{\mu} = 0.347$

If gravel sales are determined week after week according to the given pdf, then the long-run average value of sales per week will be 0.375 tons.

Example 6

Find the $Var(X)$. $= E[X^2] - (E[X])^2$

$$E[X^2] = \int_0^1 x^2 \frac{3}{2}(1-x^2)\,dx$$

$$= \frac{3}{2}\int_0^1 x^2 - x^4\,dx$$

$$= \frac{3}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{2}\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= \frac{3}{15} = \frac{1}{5}$$

$$\sigma_x^2 = Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{1}{5} - \left(\frac{3}{8}\right)^2 = \frac{19}{320} = 0.059$$

$sd(X) = \sqrt{Var(X)}$

$\sigma_x = 0.244$

Exercises I

Exercise 1 The pdf of a continuous variable X is given by

$$f(x) = cx^2 \quad \text{if } 0 \le x \le 2$$

where c is constant.

- Find c.
- Find the cdf $F(x)$.
- What is $P(1 \le X \le 1.5)$?
- Find $\mathbb{E}[X]$.
- Find $Var[X]$.

Exercises II

Exercise 2 The cdf of a continuous variable X is given by

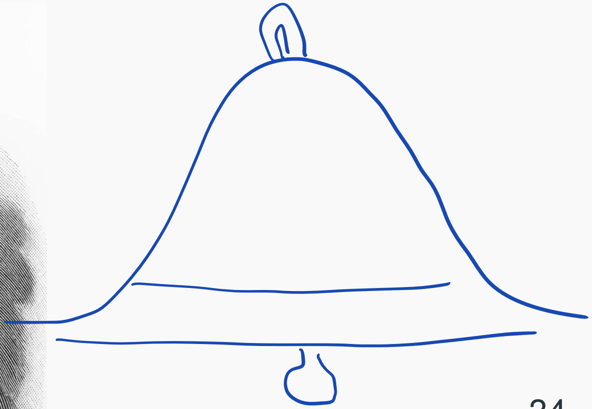
$$F(x) = \begin{cases} 1 - e^{-x} & \text{if } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

- What is $P(X \leq 2.6)$?
- What is $P(1 < X < 4)$?
- Find the pdf $f(x)$.

The Normal Distribution

Background

- The normal distribution is the most frequently used continuous probability distribution.
- Many measurements can be well approximated by a normal distribution.
- The normal distribution is characterized by a bell-shaped curve.
- The normal distribution is also called the Gaussian distribution (after Johann Carl Friedrich Gauss).



PDF of the Normal Distribution

$X \sim N(\mu, \sigma)$

Definition 7 (Gaussian (Normal) Distribution) $X \sim N(\mu, \sigma^2)$

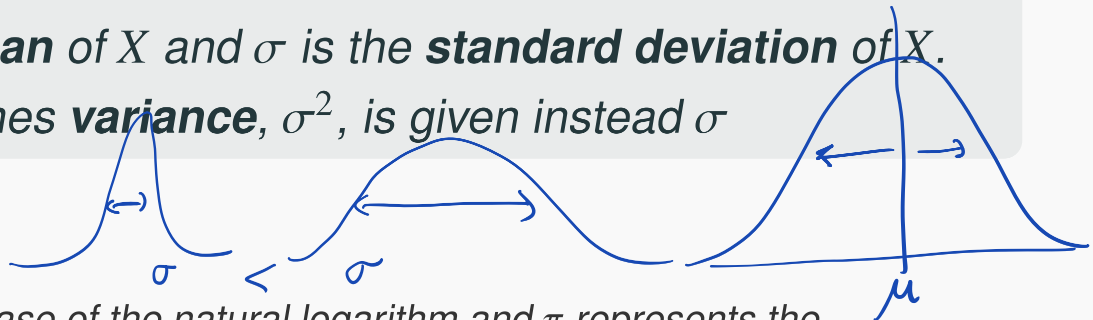
If a rv X follows a normal distribution with parameters μ and σ, then we write $X \sim N(\mu, \sigma)$, and the pdf of X is

$f(x)$

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty, \tag{2}$$

f

*where μ is the **mean** of X and σ is the **standard deviation** of X. Note that sometimes **variance**, σ^2, is given instead σ*

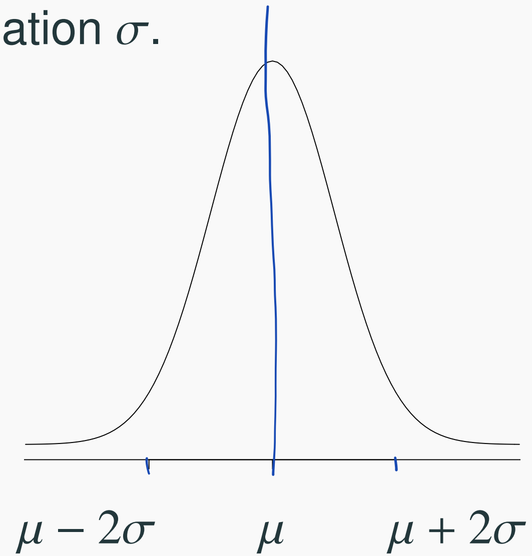


Note: e denotes the base of the natural logarithm and π represents the mathematic constant with approximate value 3.14159. Sometimes I will write e^x as $\exp\{x\}$.

e^x $\exp\{x\}$

Normal Distribution I

- The normal distribution is usually characterized by its mean μ and standard deviation σ.



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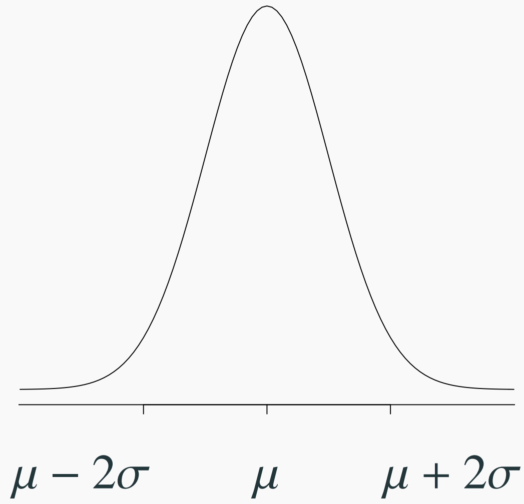
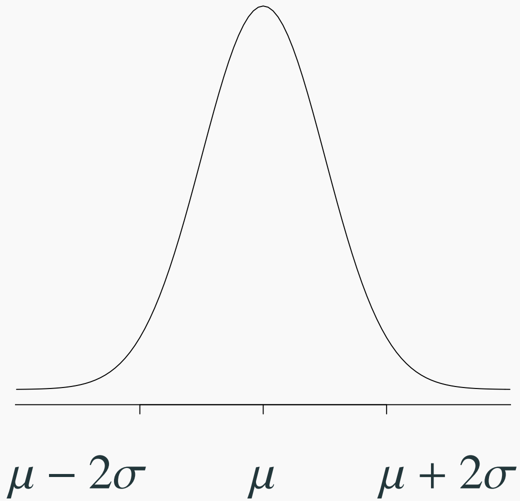


- The normal distribution is symmetric about the mean.

Normal Distribution I

- The normal distribution is usually characterized by its mean μ and standard deviation σ.



- The normal distribution is symmetric about the mean.
- 95% of the density of the distribution lies within $\approx$two (1.96 to be precise) standard deviations of the mean.

- It can be shown that $E(X) = \mu$ and $Var(X) = \sigma^2$, so the parameters are the mean and the standard deviation of X.
- We can see the variants of $f(x; \mu, \sigma)$ for several different values of μ and σ in the figure below.

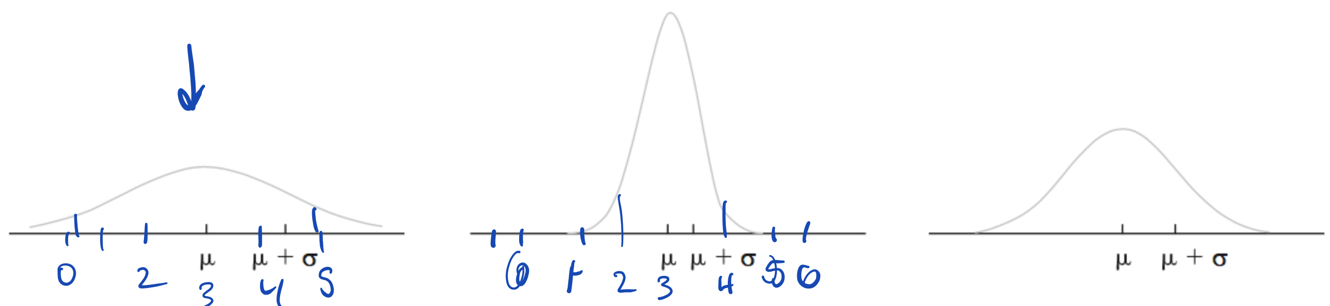


Figure 4.13 Normal density curves

Figure 5: From Devore, J. L (2016). *Probability and Statistics for Engineering and the Sciences*

- Each density curve is symmetric about μ, and bell-shape (the center of the bell is both the mean and median).

- Large values of σ yield density curves that are quite spread out about μ, i.e. a fatter squashed bells like the one seen on the left of Figure 5.

- Small values of σ yield density curves with a high peak above μ and most of the area under the density curve quite close to μ, i.e. a pointer bells like the one seen in the middle of Figure 5.

- Thus a large σ implies that a value of X far from μ may well be observed, whereas such a value is quite unlikely when σ is small.

- The normal or Gaussian distribution in Eq (2) cannot be integrated directly To obtain the $P(X \leq x)$ when $X \sim$ Normal(μ, σ) we need to integrate:

$$\int_{\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

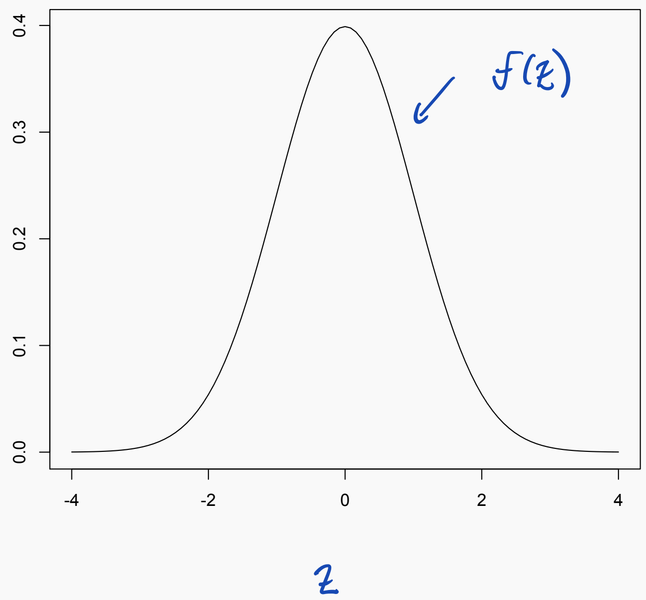
- None of the standard integration techniques can be used to evaluate this expression.

- In order to find probabilities, we need to approximate the area under the curve using numerical approximation.

- While you will see in labs how computer programs like R can be use to find $P(X \leq x)$, on paper, we need to first transform X into a so-called standard normal distribution.

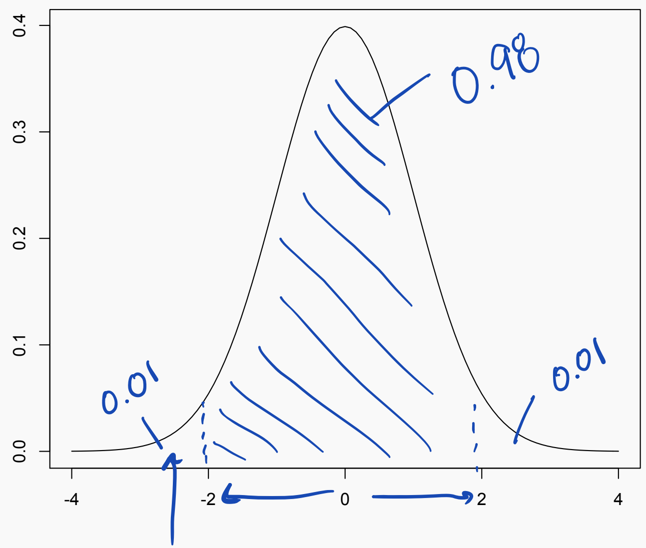
Standard Normal Distribution

- The "**standard normal distribution**" is a special case of the normal distribution when the mean (μ) is 0 and standard deviation (σ) is 1. $Z \sim N(\mu=0, \sigma=1)$



Standard Normal Distribution

- The **standard normal distribution** is a special case of the normal distribution when the mean (μ) is 0 and standard deviation (σ) is 1.



- 95% of the density of the distribution (the area under the curve) lies been ± 1.96.

- It is standard convention to use the rv Z when defining a standard normal; hence, $Z \sim$ Normal$(\mu = 0, \sigma = 1)$ or simply $Z \sim$ Normal$(0, 1)$ or $Z \sim$ Normal$(0, 1^2)$ if we are quoting variance instead of standard deviation.

$P(\cancel{X} \leq 3)$ or paper directly in R

- Integration of the following has to been done numerically and summarized in what we sometimes refer to as *standard normal tables*

$P(Z \leq -1.96)$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \text{ for } -\infty < z < \infty$$

$P(Z \leq z)$

z	P_z
-3	
$\vdots$	
-3	

- Note: we can always transform a normal distribution to a standard normal distribution.

$X \sim N(\mu, \sigma) \longrightarrow Z \sim N(0,1)$

Normal Distribution

Theorem 1

Let $X \sim N(\mu, \sigma^2)$ *and* $Z \sim N(0, 1)$*, then*

$$Z = \boxed{\frac{X - \mu}{\sigma}}.$$

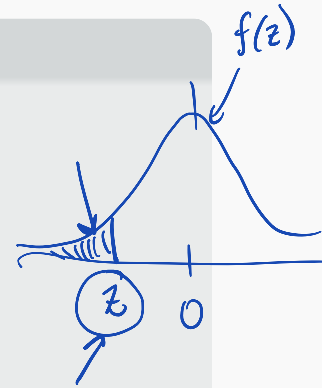
Normal Distribution

Theorem 1

Let $X \sim N(\mu, \sigma^2)$ *and* $Z \sim N(0,1)$, *then*

$$Z = \frac{X - \mu}{\sigma}.$$

This transformation is referred to as ***standardization***.



- Therefore, to compute $P(X \le x)$ we can use the fact that

$$P(X \le x) = P\Big(Z \le \frac{x - \mu}{\sigma}\Big) = P(Z \le z),$$

and then consult the standard normal tables. These tables store the value of $P(Z \le z)$ for many values of z.

Normal Distribution

Example 7

Consider the example of the height of women in Ireland, which we assume is normally distributed with mean 1.62m and standard deviation 0.11m.

- The median and mode of the height of women is 1.62m.
- Theoretically, the proportion of women shorter than 1.52m is the same as proportion of women taller than 1.72m.
- 95% of women are between 1.40m and 1.84m tall

Example 7

$\mu \pm 2\sigma \rightarrow$ ✱

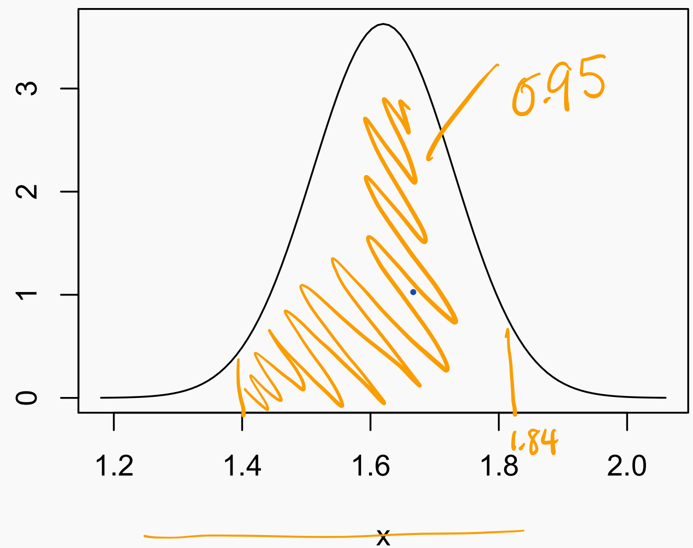
1.96

Consider the example of the height of women in Ireland, which we assume is normally distributed with mean 1.62m and standard deviation 0.11m.

$1.62 + 1.96(0.11) \doteq 1.84$

$1.62 - 1.96(0.11) = 1.40$

- The median and mode of the height of women is 1.62m.
- Theoretically, the proportion of women shorter than 1.52m is the same as proportion of women taller than 1.72m.
- 95% of women are between 1.40m and 1.84m tall



Let see some examples of how we use Z-Tables to find probabilities

$X \sim N(1.62, 0.11^2)$ $\qquad N(1.62, 0.11)$?

Example 7 (Ex 7 cont'd)

Find the probability that a randomly selected woman in Ireland is shorter than 1.52m

$\int_{-\infty}^{1.52} \cancel{2\pi}$

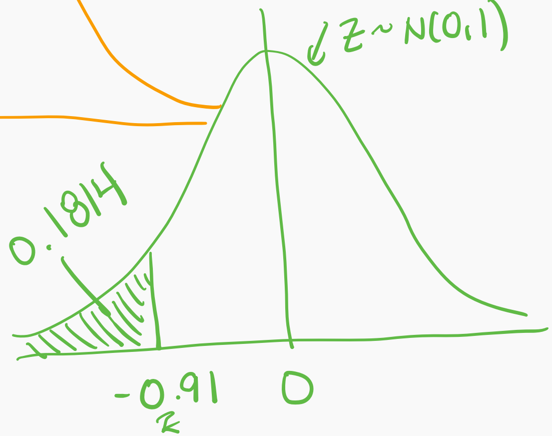
$$P(X \leq 1.52) = P\left(Z \leq \frac{x - \mu_x}{\sigma_x}\right)$$

$Z \sim N(0,1)$

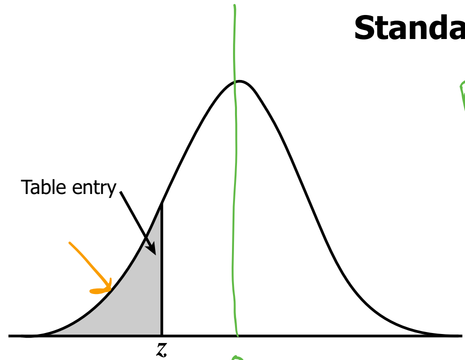
$$= P\left(Z \leq \frac{1.52 - 1.62}{0.11}\right)$$

$$= P(Z \leq -0.91)$$

$$= 0.1814$$



Standard Normal Probabilities

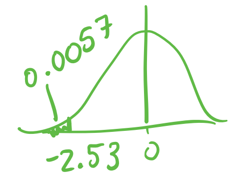


$P(Z \le z) = F(z)$

(−)ve

Table entry for z is the area under the standard normal curve to the left of z.

$P(Z \le -2.53) = 0.0057$



$P(Z \le -0.91) = 0.1814$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Standard Normal Probabilities

$P(Z \le z) = F(z)$ (+)ve

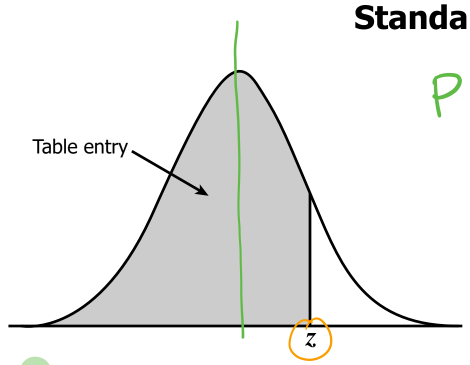
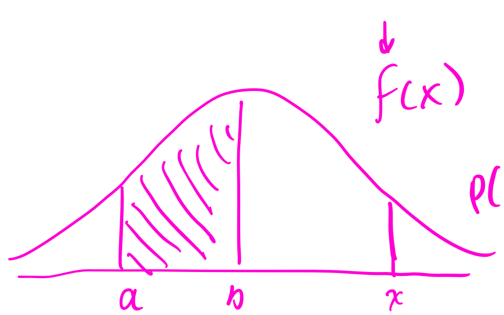


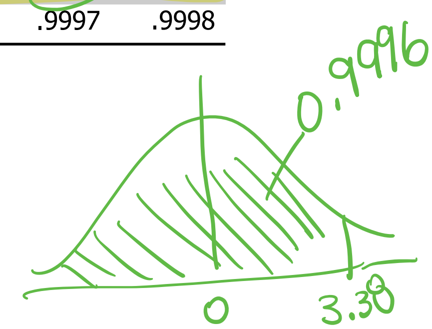
Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

z

$P(Z \le z)$



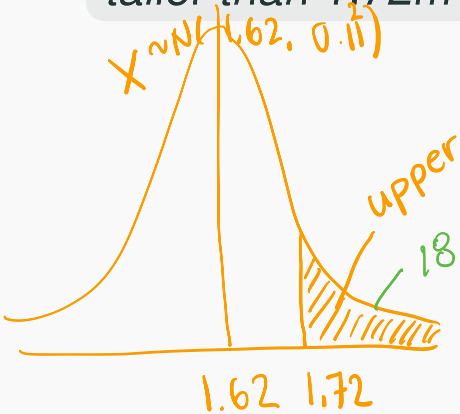


"standardizing"

Example 7 (Ex 7 cont'd)

Find the probability that a randomly selected woman in Ireland is taller than 1.72m

$X \sim N(1.62,\ 0.11^2)$



$$P(X > 1.72) = P\left(Z > \frac{1.72 - 1.62}{0.11}\right)$$

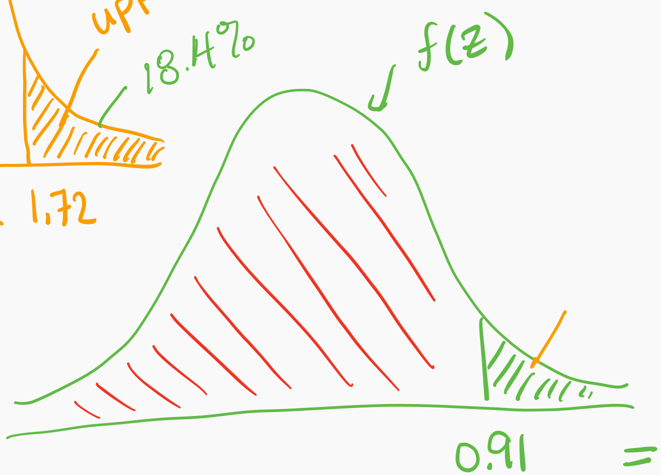
$$= P(Z > 0.91)$$

Z

$$= 1 - \boxed{P(Z \leq 0.91)}$$

$$= 1 - 0.8186$$

$$= 0.1814$$



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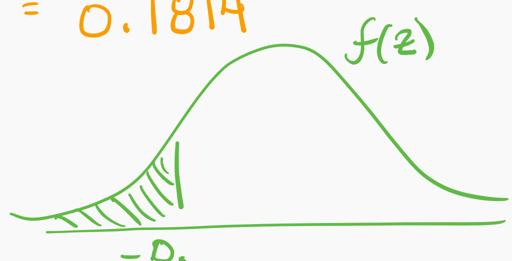
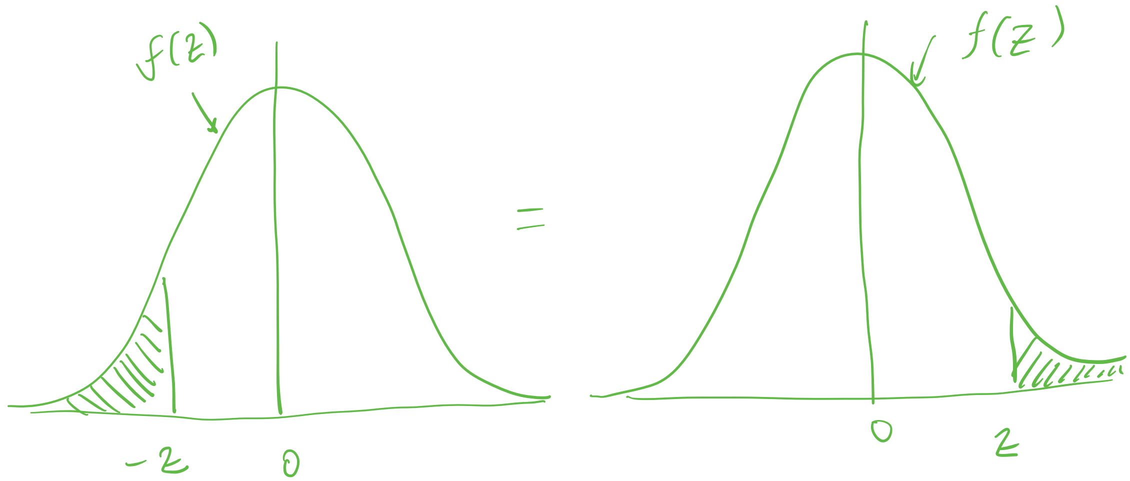
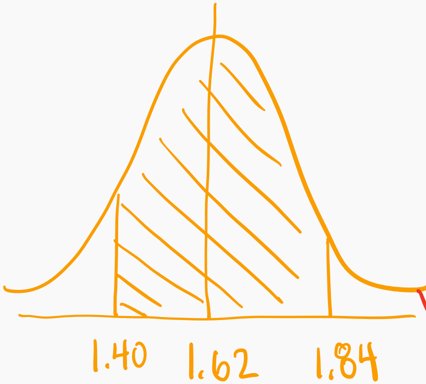




$$P(Z \leq -z) = P(Z \geq z)$$

Example 7 (Ex 7 cont'd)

Find the probability that women are between 1.40 and 1.84 meters tall.



$$P(1.4 \le X \le 1.84) = P\left(\frac{1.4-1.62}{0.11} \le Z \le \frac{1.84-1.62}{0.11}\right)$$

$$= P(-2.00 \le Z \le 2.00)$$

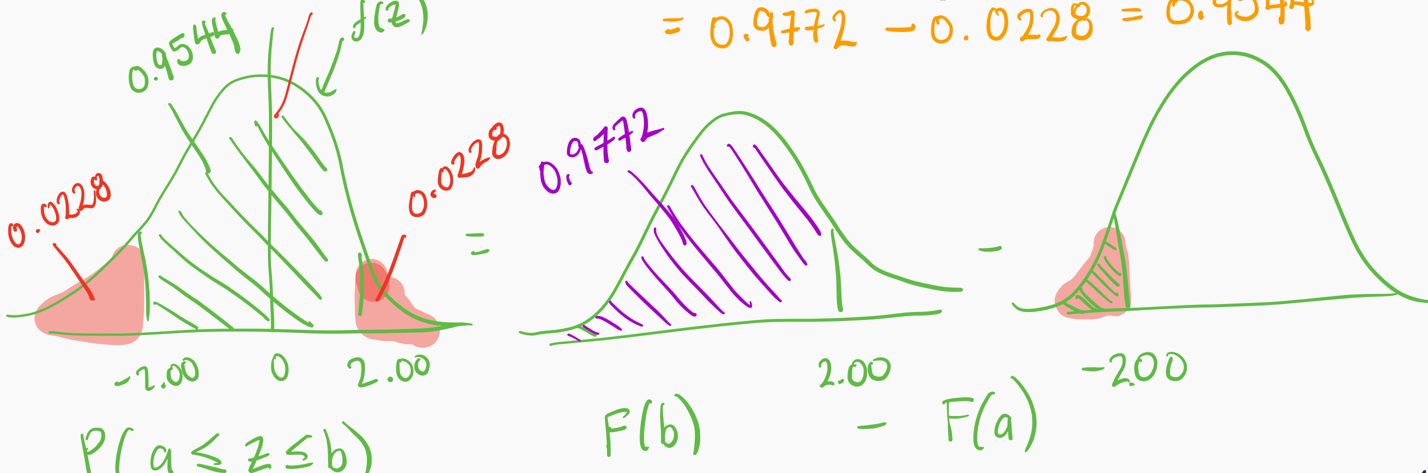
$$= \boxed{P(Z \le 2.00)} - P(Z \le -2)$$

$$= 0.9772 - 0.0228 = 0.9544$$

$1 - 2P(Z \le -2)$

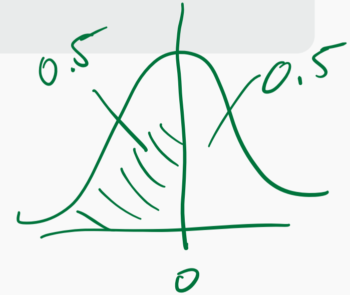
$1 - 2(0.0228)$

$= 0.9544$



Example 7 (Ex 7 cont'd)

Find the median height of women in Ireland.

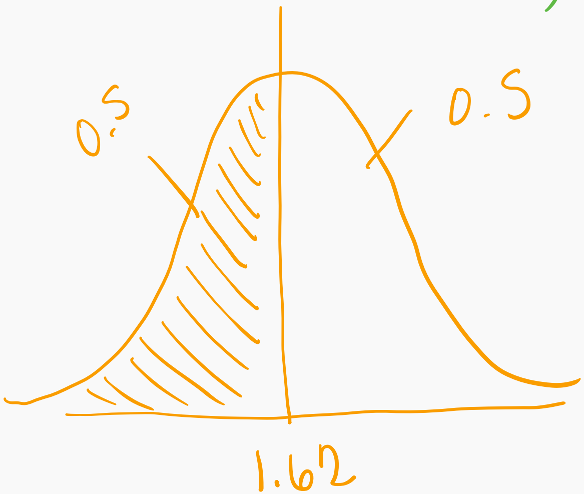


$$F(\underbrace{\eta(0.5)}_{\tilde{\mu}}) = 0.5$$

$$P(X \le \tilde{\mu}) = 0.5$$

$$P\left(Z \le \overbrace{\frac{\tilde{\mu} - 1.62}{0.11}}^{z}\right) = 0.5$$

$$P(Z \le 0) = 0.5$$



$$\frac{\tilde{\mu} - 1.62}{0.11} = 0$$

$$\Rightarrow \boxed{\tilde{\mu} = 1.62}$$

Exercises

Exercise 1 The height of women in a particular region follows a normal distribution with mean 1.55m and standard deviation 0.17m.

- What proportion of women are smaller than 1.61m?
- What proportion of women are taller than 1.8m?
- What proportion of women are between 1.45m and 1.61m tall?

Exercise 2 The wingspans of the males of a certain species of bird of prey form a normal distribution with mean 162.50cm and standard deviation 6.0cm. What is the probability that the wingspan of a randomly selected male will exceed 170cm?

Other Special Continuous Distributions

Other Special Continuous Distributions

- While the normal distribution does well to model many natural phenomenon, for many practical situations, we will require a distribution that is *not* symmetric.
- A family of pdf's that yields a wide variety of skewed distributional shapes is the gamma family.
- Before describing this family, we first need to introduce a function that plays an important role in many branches of mathematics . . . the gamma function.

Definition 8

*The **gamma function** is defined as*

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

gamma(α)

Definition 8

*The **gamma function** is defined as*

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Theorem 2

Using integration by parts, we can show that, for $\alpha > 1$,

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1),$$

Theorem 3

$n! = n \times (n-1) \times \cdots 3 \times 2 \times 1$

When α is a positive integer, it follows that

$$\Gamma(\alpha) = (\alpha - 1)!$$

Proof for Theorem 2

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$$

IBP

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

$$\Gamma(\alpha) = \int_0^\infty \underbrace{x^{\alpha-1}}\,\underbrace{e^{-x}dx}$$

$u = x^{\alpha-1}$ $\qquad$ $dv = e^{-x}dx$

$du = (\alpha-1)x^{\alpha-2}dx$ $\qquad$ $v = -e^{-x}$

$$= -x^{\alpha-1}e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}(\alpha-1)x^{\alpha-2}dx$$

$$= 0 - 0 + (\alpha-1)\boxed{\int_0^\infty e^{-x}x^{\alpha-2}dx}$$

$$= (\alpha-1)\Gamma(\alpha-1) \qquad \blacksquare$$

Proof for Theorem 3

Proof by Induction

Base case: $\Gamma(1) = \int_0^\infty e^{-x} x^{1-1} dx = \int_0^\infty e^{-x} = \left[-e^{-x}\right]_0^\infty = 1$

Induction step

$$\Gamma(n) = (n-1)\underbrace{\Gamma(n-1)}\quad \text{Thrm 2}$$

$$= (n-1)(n-2)\underbrace{\Gamma(n-2)}\quad \text{Thrm 2}$$

$$= (n-1)(n-2)(n-3)\Gamma(n-3)\quad \text{Thrm 2}$$

$$= \quad \ldots$$

$$= (n-1)\cdot(n-2)\cdot(n-3)\cdot\ldots\ldots(1)\cdot\Gamma(1)$$

$$= (n-1)!$$

Definition 9

*If X follows a **gamma distribution**, then*

$$f(x) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{for } x > 0, \alpha > 0, \beta > 0 \\ 0 & \text{otherwise} \end{cases}$$

$X \sim \text{gamma}(\alpha, \beta)$

$f(x; \alpha, \beta)$

$E[X] = \mu$

center

The Gamma Distribution

Definition 9

*If X follows a **gamma distribution**, then*

$$f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \textit{for } x > 0, \alpha > 0, \beta > 0 \\ 0 & \textit{otherwise} \end{cases}$$

- $\mathbb{E}[X] = \alpha\beta$.

The Gamma Distribution

Definition 9

*If X follows a **gamma distribution**, then*

$$f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{for } x > 0, \alpha > 0, \beta > 0 \\ 0 & \text{otherwise} \end{cases}$$

- $\mathbb{E}[X] = \alpha\beta$.
- **Exercise 3** Show $Var[X] = \alpha\beta^2$ [Hint: let $y = x/\beta$].

Show $\mathbb{E}[X] = \alpha\beta$.

$$E[X] = \int_{-\infty}^{\infty} x \underbrace{f(x)} \, dx$$

Thrm 2 $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

$$\boxed{\Gamma(\alpha+1) = \alpha\Gamma(\alpha)}$$

$$= (1)\int_{-\infty}^{\infty} \underbrace{x} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \underbrace{x^{\alpha-1}} e^{-x/\beta} \, dx$$

$$= \int_{-\infty}^{\infty} \frac{\alpha\beta}{\alpha\beta} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha} e^{-x/\beta} \, dx$$

$$= \alpha\beta \int_{-\infty}^{\infty} \frac{1}{\beta^{\alpha+1}\underbrace{\Gamma(\alpha)\alpha}_{\Gamma(\alpha+1)}} x^{(\alpha+1)-1} e^{-x/\beta} \, dx$$

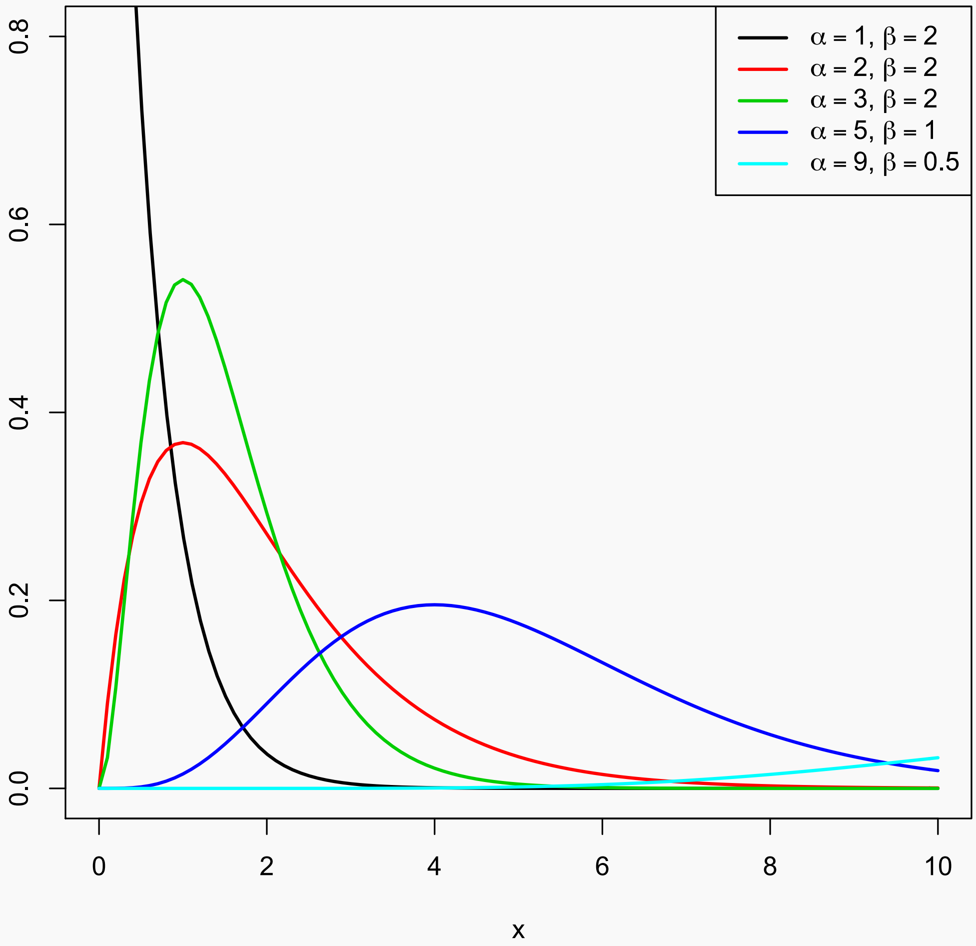
$f(x)$ PDF gamma dist RV with $\alpha+1$, β

$$= \alpha\beta \int_{-\infty}^{\infty} \underbrace{\frac{1}{\beta^{\alpha+1}\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-x/\beta}}_{f(x)} \, dx$$

$$= \alpha\beta \underbrace{\int_{-\infty}^{\infty} f(x) \, dx}_{1}$$

$\square$

Show $\mathbb{E}[X] = \alpha\beta$.



α = 1, β = 2
α = 2, β = 2
α = 3, β = 2
α = 5, β = 1
α = 9, β = 0.5
0.0
0.2
0.4
0.6
0.8
0
2
4
6
8
10
x

The Exponential Distribution

The exponential distribution is a special case of the gamma distribution, with $\alpha = 1$.

It is often used to model the time until next event for processes like those arising from a Poisson distribution.

Definition 10

*If X follows an **exponential distribution**, then*

$$\boxed{f(x) = \frac{1}{\beta} e^{-x/\beta}} \text{ for } x > 0,$$

for $\beta > 0$. From the results from the gamma distribution,

$$\mathbb{E}[X] = \beta \qquad Var[X] = \beta^2.$$

Alternative Parameterization for the Exponential Distribution

$X \sim N(\mu=7, \sigma=3)$
$\rightarrow X \sim N(\mu=7, \sigma^2=9)$

On the previous slide the exponential distribution was parameterized by its mean β.

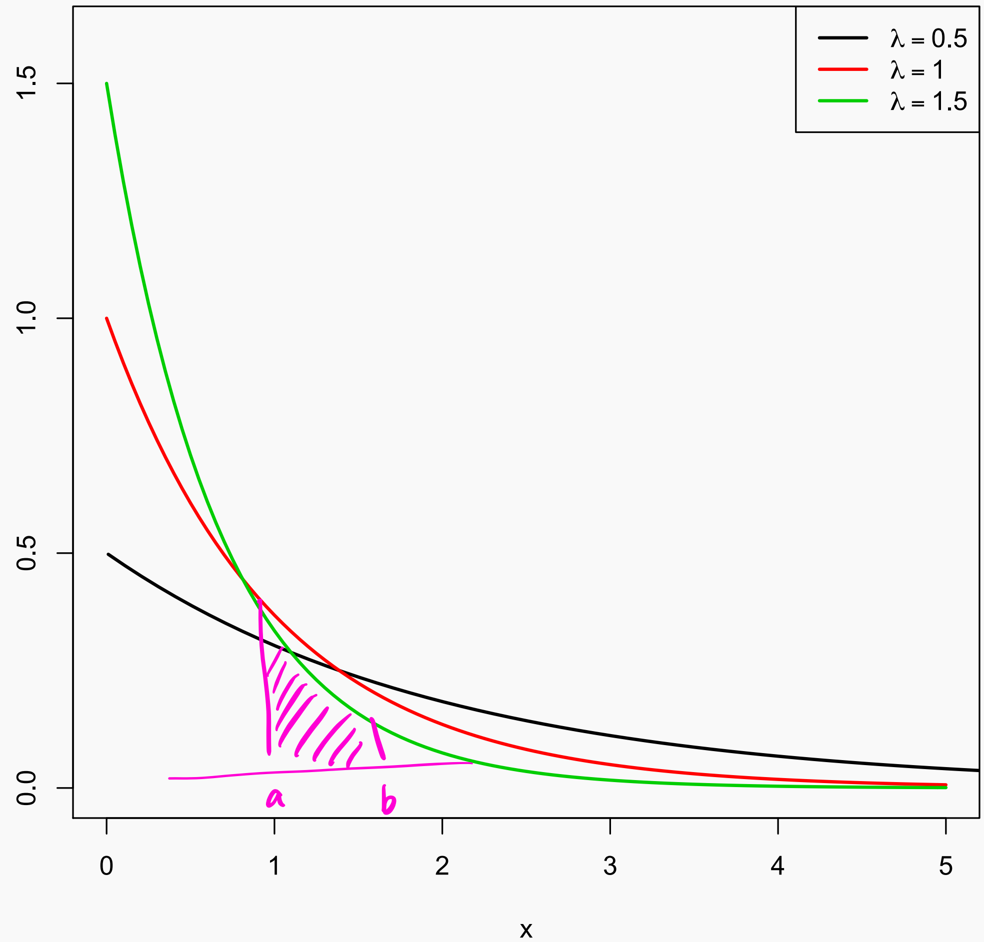
Alternatively, it can be parameterized by its rate parameter λ via the probability density function

$$f(x) = \lambda e^{-\lambda x}, \quad \text{for } x > 0, \text{ with } \lambda > 0.$$

Note that $\boxed{\lambda = 1/\beta}$ so it is often called the inverse scale parameter.

Using this parameterization, our mean and variance would be:

$$\mathbb{E}[X] = \frac{1}{\lambda} \qquad\qquad Var[X] = \frac{1}{\lambda^2}.$$



λ = 0.5
λ = 1
λ = 1.5
0.0
0.5
1.0
1.5
0
1
2
3
4
5
x
a
b

Example 8

Based on data collected from metal shredders across the US, the amount of extractable lead in metal shredder residue has an approximate exponential distribution with $\boxed{\text{mean 2.5mg/liter.}}$ What is the probability that:

$\beta = 2.5$

- *(a) The amount of extractable lead in one liter of metal shredder residue is less than 4.5mg?*
- *(b) The amount of extractable lead in one liter of metal shredder residue is greater than 2mg?*

a) $X \sim Exp(\beta = 2.5)$

$F(x) = P(X \le x)$

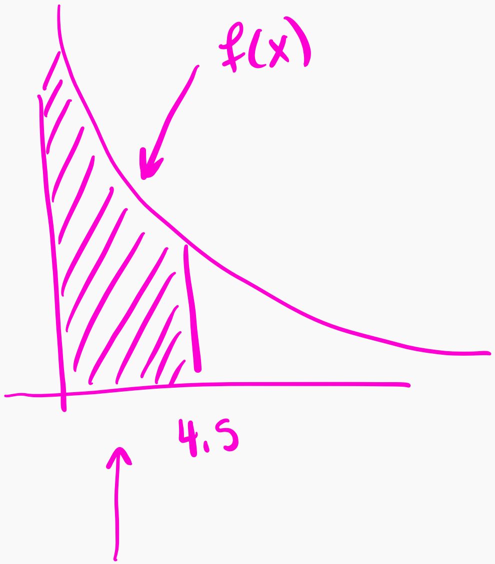
$P(X < 4.5) = P(X \le 4.5)$

$$P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

8 part (a)

$$P(X<4.5) = \int_0^{4.5} \underbrace{\frac{1}{\beta} e^{-x/\beta}}\,dx$$

$$\frac{\cancel{1}}{\cancel{\beta}} \quad \frac{-e^{-x/\beta}}{\cancel{1/\beta}}$$



$$= -e^{-x/\beta}\Big|_0^{4.5}$$

$$= \left(-e^{-4.5/\beta}\right) - \left(-e^{-0/\beta}\right)$$

$$= 1 - e^{-4.5/\beta}$$

$$= 1 - e^{-4.5/2.5}$$

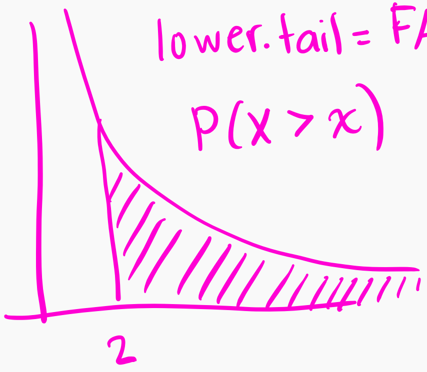
$$= 0.835$$

★ see lab to calculate in R

lower.tail=TRUE

$$F(x) = \boxed{P(X \le x) = 1 - e^{-x/\beta}} \qquad x>0,\ \beta>0$$

8 part (b)



lower.tail = FALSE

$P(X > x)$

$$P(X > 2) = \int_2^\infty \frac{1}{\beta} e^{-x/\beta}\,dx$$

$$= -e^{-x/\beta}\Big|_2^\infty$$

$$= 0 + e^{-2/2.5}$$

$$= 0.449$$

$$P(X > 2) = 1 - P(X \leq 2)$$

$$= 1 - F(2)$$

$$= 1 - \left[1 - e^{-2/2.5}\right] = e^{-2/2.5} = 0.449$$

The Beta Distribution

Definition 11

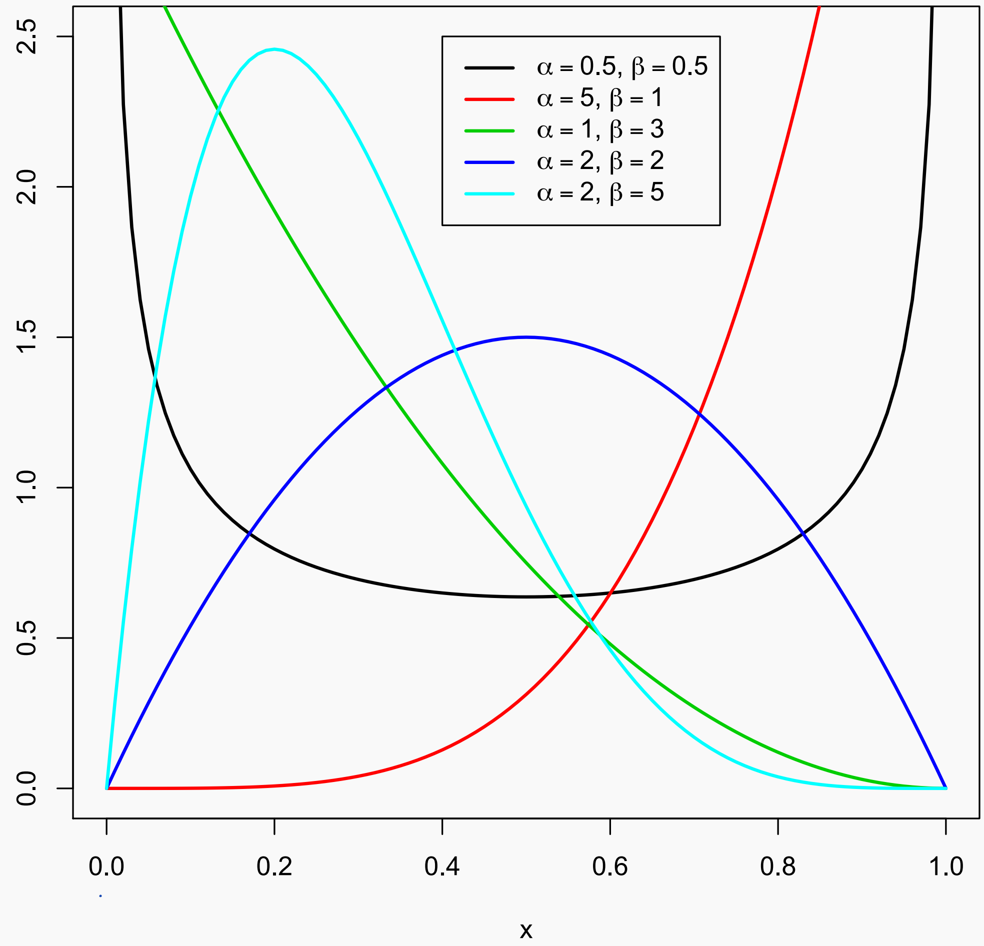
*If X follows a **beta distribution**, then*

$$f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0 < x < 1, \alpha > 0, \beta > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}$$

$$Var[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Since the Beta Distribution is restricted between 0 and 1, it is often used to model proportions.



α = 0.5, β = 0.5
α = 5, β = 1
α = 1, β = 3
α = 2, β = 2
α = 2, β = 5
0.0
0.5
1.0
1.5
2.0
2.5
0.0
0.2
0.4
0.6
0.8
1.0
x

Example 9

Let $X \sim$ *beta*$(\alpha = 3, \beta = 2)$. *What is* $P(X < 0.6)$?

$$P(X < 0.6) = \int_0^{0.6} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

Thrm 1 α = integer

$\Gamma(\alpha) = (\alpha-1)!$

$\Gamma(5) = 4!$

$\Gamma(3) = 2!$

$\Gamma(2) = 1!$

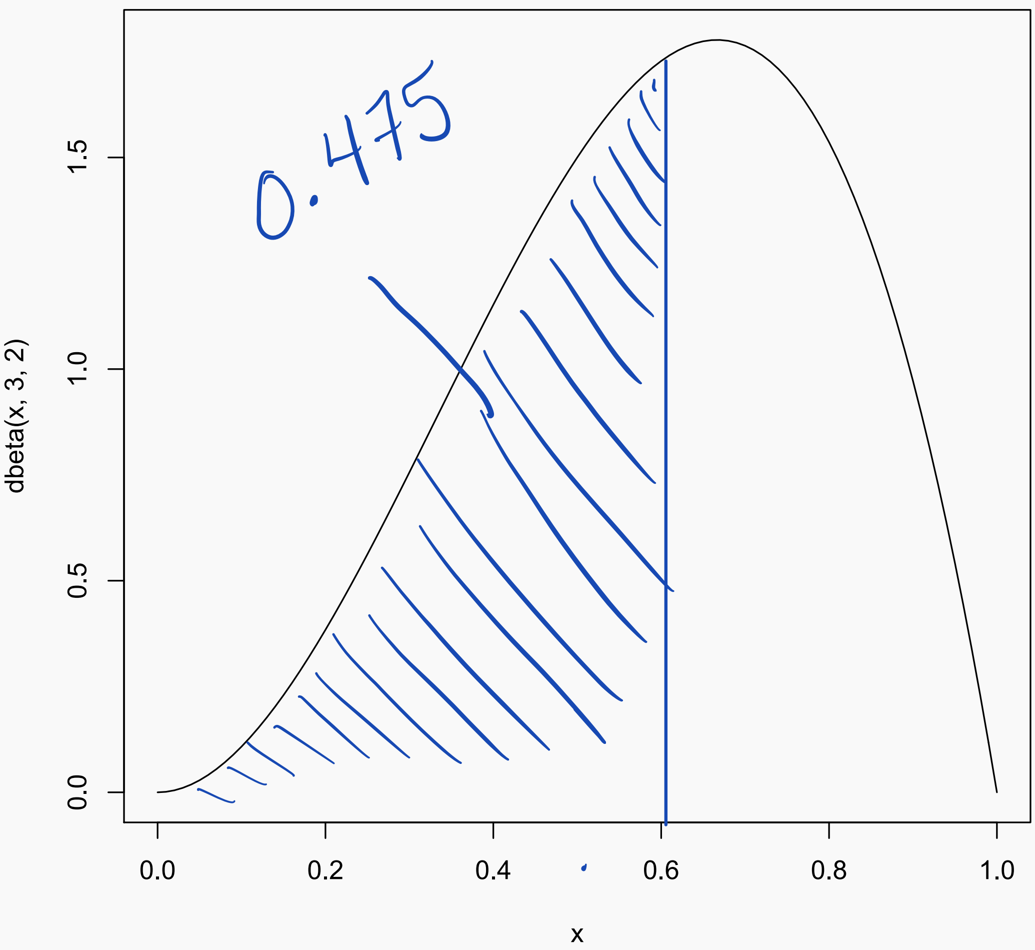
$$= \int_0^{0.6} \frac{\Gamma(5)}{\underbrace{\Gamma(3)\Gamma(2)}} \cdot x^{3-1}(1-x)^{2-1}\,dx$$

$$= \int_0^{0.6} \frac{4\times 3\times \cancel{2}\times 1}{\cancel{2}\times 1\times 1} \underbrace{x^2(1-x)^1}\,dx$$

$$= 12\int_0^{0.6} x^2 - x^3\,dx$$

$$= 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^{0.6}$$

$$= \ldots = 0.475$$



0.475
dbeta(x, 3, 2)
0.0
0.5
1.0
1.5
0.0
0.2
0.4
0.6
0.8
1.0
x

Example 10

In a particular province, the proportion of highway sections requiring repairs in a given year follows a beta distribution with $\alpha = 3$ and $\beta = 2$.

- *(a) What is the probability that, in a given year, less than half of highways sections in the province require repair?*

(a) $P(X < 0.5) = \ldots$ exercise $= 0.313$

(b) $E[X] = \dfrac{\alpha}{\alpha+\beta} = \dfrac{3}{3+2} = 0.6$

$\boxed{\text{pdist}}$ ← $\boxed{P(X \le x)}$

$\boxed{\text{q dist}}$ ←

$\boxed{\text{ddist}}$ ← $\boxed{f(x)}$

rdist ← generate random obs from that

dist

binom
pois
norm
exp
beta

?pbeta(

pdist(....)

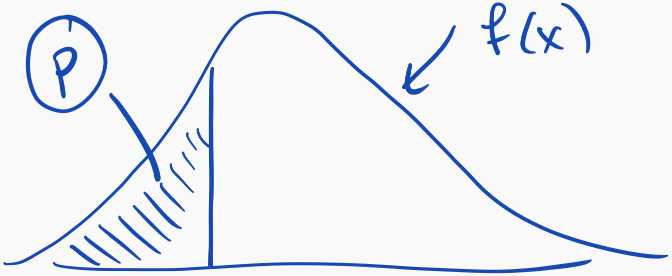
dist.

Example 10 (cont'd)

- *(b) What percentage of highway sections would you expect to require repair in a given year?*

p

f(x)



pdist(q ....) = $P(X \le q)$

q

qdist(p, ...) = $P(X \le q) = p$

dexp

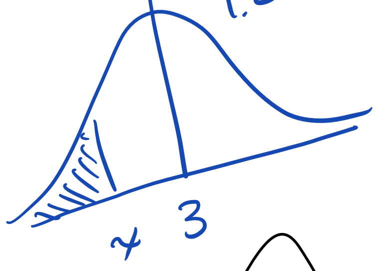
$$f(x) = \frac{1}{\beta} e^{-x/\beta} = \frac{1}{1.1} e^{-2/1.1}$$

Discrete.

$$\boxed{P(X \le x) \ne P(X < x)}$$

dexp(2, beta = 1.1)

~N(0,1)



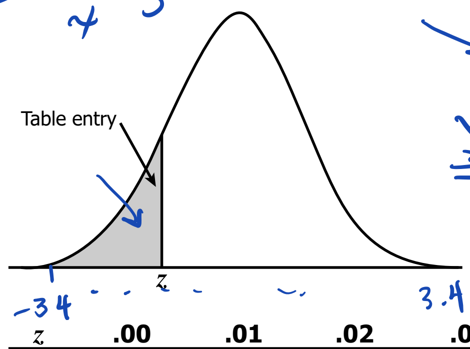
$Z \sim N(0,1)$

$N(\mu, \sigma)$

$N(\mu, \sigma^2)$

$X \sim N(3.0, 1.2^2)$

Standard Normal Probabilities



$P(X \leq x)$ pnorm

$P(Z \leq z)$

Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Standard Normal Probabilities

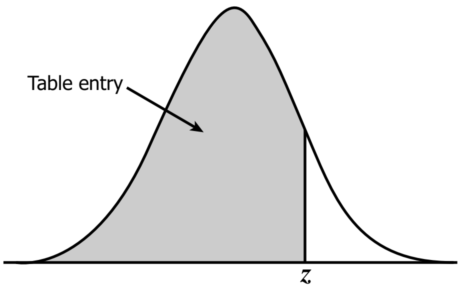


Table entry for z is the area under the standard normal curve to the left of z.

z	**.00**	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998