10-601

Machine Learning

http://curtis.ml.cmu.edu/w/courses/index.php/Machine_Learning_10-601_in_Fall_2014

Organizational info

- All up-to-date info is on the course web page (follow links from my page).
- Instructor: for this section
  - Ziv Bar-Joseph
- TAs: See info on website for recitations, office hours etc.
- See web page for contact info, office hours, etc.
- Piazza would be used for questions / comments. Make sure you are subscribed.

- **8/27 - Intro to probability, MLE**
- **9/3 - Classification, KNN**
- **9/8 - Decision trees**
- **9/10 - Naïve Bayes**
- **9/15 - Linear regression**
- **9/17 -**
- **9/22 -**
- **9/24**
- **9/29 - SVM1**
- **10/1 - SVM2**
- **10/6 - Evaluating classifiers**
- **10/8 - PAC learning**
- **10/13 - Bias – Variance decomposition**
- **10/15 - Ensemble learning – Boosting, RF**
- **10/20 - Unsupervised learning – clustering**
- **10/22 - Unsupervised learning – clustering**
- **--**
- **10/27 - review sessions**
- **10/28 - midterm**
- **10/29 - BN**
- **11/3 - BN**
- **11/5 - HMM**
- **11/10 - HMM**
- **11/12 - Matrix factorization / topic models**
- **11/17 - network models**
- **11/19 - Semi-supervised learning**
- **11/24 - scalable learning**
- **12/1 - NLP**
- **12/3 -comp bio**

10/28 (Tuesday): Midterm (4:30-6:30)

Intro and classification (A.K.A. 'supervised learning')

Clustering ('Unsupervised learning')

Probabilistic representation and modeling ('reasoning under uncertainty')

Applications of ML

Grading

- **8 Problem sets - 50%**
- **Project - 20%**
- **Midterm - 30%**

Class assignments

- 8 Problem sets

  - Most containing both theoretical and programming assignments

- Projects

  - Groups of 1-2

  - Engineer a classifier (or collection of classifiers or any other methods learned in this class) to perform supervised learning on large corpora of diverse types of data.

Recitations

  - 4 weekly, 1st half Mon-Wed, 2nd half Tue-Thu.

  - Expand on material learned in class, go over problems from previous classes etc.

What is Machine Learning?

Easy part: Machine

Hard part: Learning

- Short answer: Methods that can help generalize information from the observed data so that it can be used to make better decisions in the future

What is Machine Learning?

Longer answer: The term Machine Learning is used to characterize a number of different approaches for generalizing from observed data:

- Supervised learning
  - Given a set of features and labels learn a model that will predict a label to a new feature set

- Unsupervised learning
  - Discover patterns in data

- Reasoning under uncertainty
  - Determine a model of the world either from samples or as you go along

- Active learning
  - Select not only model but also which examples to use

Paradigms of ML

- Supervised learning
  - Given $D = \{X_i, Y_i\}$ learn a model (or function) $F: X_k \text{ -> } Y_k$

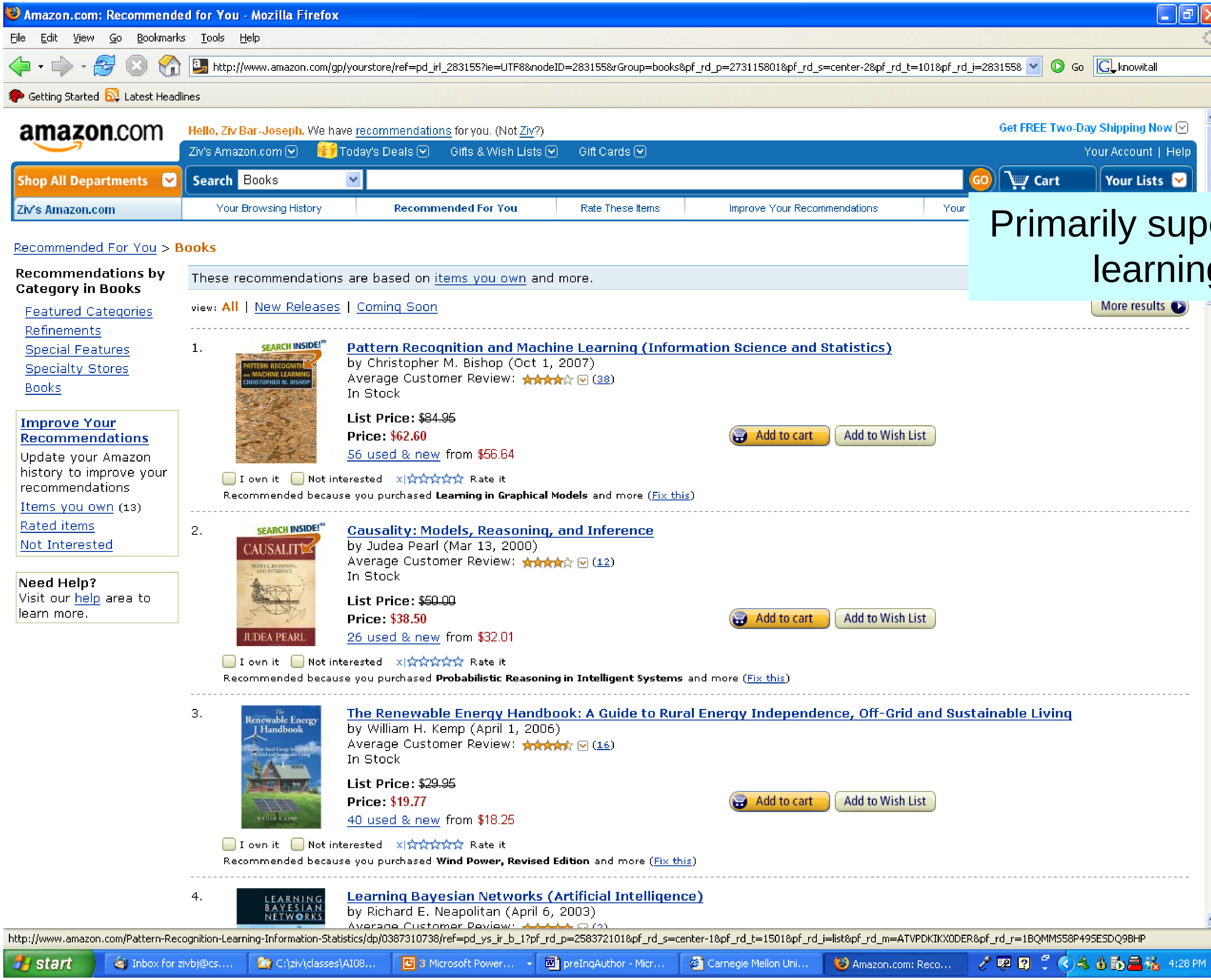
- Unsupervised learning
  Given $D = \{X_i\}$ group the data into Y classes using a model (or function) $F: X_i \text{ -> } Y_j$

- Reinforcement learning (reasoning under uncertainty)
  Given D = {environment, actions, rewards} learn a policy and utility functions:

  policy: $F1: \{e,r\} \text{ - > } a$
  utility: $F2: \{a,e\} \text{ - > } R$

- Active learning
  - Given $D = \{X_i, Y_i\}$, $\{X_j\}$ learn a function $F1 : \{X_j\} \text{ -> } x_k$ *to* maximize the success of the supervised learning function $F2: \{X_i , x_k\} \text{-> } Y$

Recommender systems



Primarily supervised learning

NELL: Never-Ending Language Learning

Can computers learn to read? We think so. "Read the Web" is a research project that attempts to create a computer system that learns over time to read the web. Since January 2010, our computer system called NELL (Never-Ending Language Learner) has been running continuously, attempting to perform two tasks each day:

Browse the Knowledge Base!

- First, it attempts to "read," or extract facts from text found in hundreds of millions of web pages (e.g., `playsInstrument(George_Harrison, guitar)`).
- Second, it attempts to improve its reading competence, so that tomorrow it can extract more facts from the web, more accurately.

Semi supervised learning

At present, NELL has accumulated a knowledge base of 967,123 beliefs that it has read from various web pages. It is not perfect, but NELL is learning. You can track NELL's progress below or @cmunell on Twitter, browse and download its knowledge base, read more about our technical approach, or join the discussion group.

Recently-Learned Facts twitter

Refresh

instance	iteration	date learned	confidence
robert_trent_jones_sr is an Australian person	473	27-dec-2011	100.0
quality_gift is a character trait	475	29-dec-2011	99.5
confectioners_sugar is a food	473	27-dec-2011	95.4
st__petersburg_times is a newspaper	472	26-dec-2011	100.0
scott_olynek is a Canadian person	473	27-dec-2011	94.1
perth is a city that lies on the river swan_river	472	26-dec-2011	99.2
florida_state_university is a sports team also known as state_university	472	26-dec-2011	100.0
press_enterprise is a newspaper in the city riverside	472	26-dec-2011	98.4

Grand and Urban Challenges road race

Supervised and reinforcement learning

Helicopter control

Reinforcement learning

Biology

ACGCTGAGCAATTCGATAGCAATTCGATAACGCTGAGCAACGCTGAGCAATTCGATAGCAATTC
GATAACGCTGAGCAATCGGATAACGCTGAGCAATTCGATAGCAATTCGATAACGCTGAGCAACG
CTGAGCAATTCGATAGCAATTCGATAACGCTGAGCAATCGGATATCGATAGCAATTCGATAAATC
GGATAACGCTGAGCAATTCGATAGCAATTCGATAACGCTGAGCAACGCTGAGCAATTCGATAGC
AATTCGATAACGCTGAGCAATCGGATATCGATAGCAATTCGATAACGCTGAGCAACGCTGAGCA
ATTCGATAGCAATTCGATAACGCTGAGCAATCGGATAACGCTGAGCAATTCGATAGCATTCGAT
AACGCTGAGCAACGCTGAGCAATTCGATAGCAATTCGATAACGCTGAGCAATCGGATAACGCTG
AGCAATTCGATAGCAATTCGATAACGCTGAGCTGAGCAATTCGATAGCAATTCGATAACGCTGA
GCAATCGGATAACGCTGAGCAATTCGATAGCAATTCGATAACGCTGAGCAACGCTGAGCAATTC
GATAGCAATTCGATAGCAATTCGATAGCAATTCGATAACGCTGAGCAACGCTGAGCAATTCGAT
AGCAATTCGATAACGCTGAGCAATCGGATAACGCTGAGCAATTCGATAGCAATTCGATAACGCT
GAGCAACGCTGAGCAATTCGATAGCAATTCGATAACGCTGAGCAATCGGATATCGATAGCAATT
CGATAACGCTGAGCAACGCTGAGCAATTCGATAGCAATTCGATAACGCTGAGCAATCGGATAAC
GC CGCTGAGCTGAGCAATTCGATAGCAATTCGATAACG
CT CGATAGCAATTCGATAACGCTGAGCAACGCTGAGCA
AT AATCGGATATCGATAGCAATTCGATAACGCTGAGCA
ACGCTGAGCAATTCGATAGCAATTCGATAACGCTGAGCAATCGGATAACGCTGAGCAATTCGAT
AGCATTCGATAACGCTGAGCAACGCTGAGCAATTCGATAGCAATTCGATCGGATAACGCTGAGC
AATTCGATAGCAATTCGATAACGCTGAGCTGAGCAATTCGATAGCAATTCGATAACGCTGAGCA
ATCGGATAACGCTGAGCAATTCGATAGCA TGAGCAATTCGAT
AGCAATTCGATAACGCTGAGCAATCGGAT GAGCAACGCTGA
GCAATTCGATAGCAATTCGATAACGCTGA TTCGATAGCATTC
GATAACGCTGAGCAACGCTGAGCAATTCG CAATCGGATAACG
CTGAGCAATTCGATAGCAATTCGATAACG ATTCGATAACGC
TGAGCAATCGGATAACGCTGAGCAATTCGATAGCAATTCGATAACGCTGAGCAACGCTGAGCAA
TTCGATAGCAATTCGATAGCAATTCGATAGCAATTCGATAACGCTGAGCAACGCTGAGCAATTC
GATAGCAATTCGATAACGCTGAGCAATCGGATAACGCTGAGCAATTCGATAGCAATTCGATAAC
GCTGAGCAACGCTGAGCAATTCGATAGCAATTCGATAACGCTGAGCAATCGGATATCGATAGCA
ATTCGATAACGCTGAGCAACGCTGAGCAATTCGATAGCAATTCGATAACGCTGAGCAATCGGAT
AACGCTGAGCAATTCGATAGCAATTCGATAACGCTGAGCTGAGCAATTCGATAGCAATTCGATA
ACGCTGAGCAATCGGA

Which part is the gene?

Supervised and unsupervised learning (can also use active learning)

Common Themes

- Mathematical framework

  - Well defined concepts based on explicit assumptions

- Representation

  - How do we encode text? Images?

- Model selection

  - Which model should we use? How complex should it be?

- Use of prior knowledge

  - How do we encode our beliefs? How much can we assume?

(brief) intro to probability

Basic notations

- Random variable

  - referring to an element / event whose status is unknown:

  A = “it will rain tomorrow”

- Domain (usually denoted by Ω)

  - The set of values a random variable can take:

  - “A = The stock market will go up this year”: Binary

  - “A = Number of Steelers wins in 2012”: Discrete

  - “A = % change in Google stock in 2012”: Continuous

Axioms of probability (Kolmogorov's axioms)

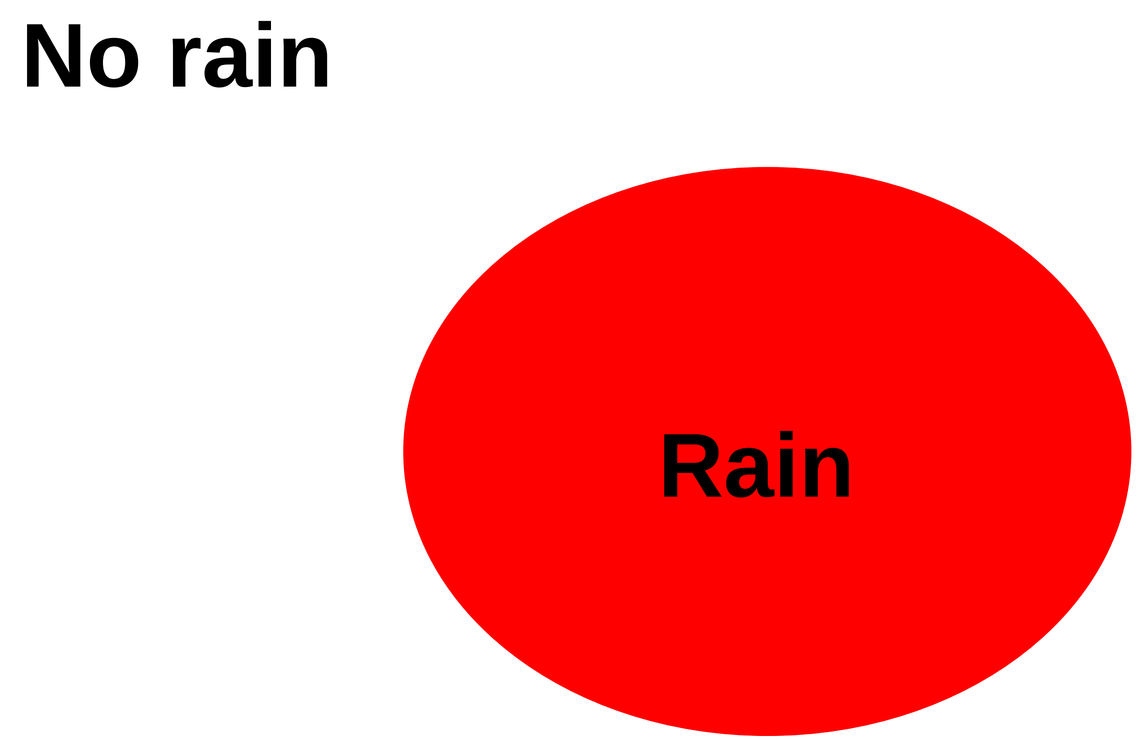
A variety of useful facts can be derived from just three axioms:

1. $0 \leq P(A) \leq 1$
2. $P(\text{true}) = 1$, $P(\text{false}) = 0$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

There have been several other attempts to provide a foundation for probability theory. Kolmogorov's axioms are the most widely used.

Priors

Degree of belief in an event in the absence of any other information



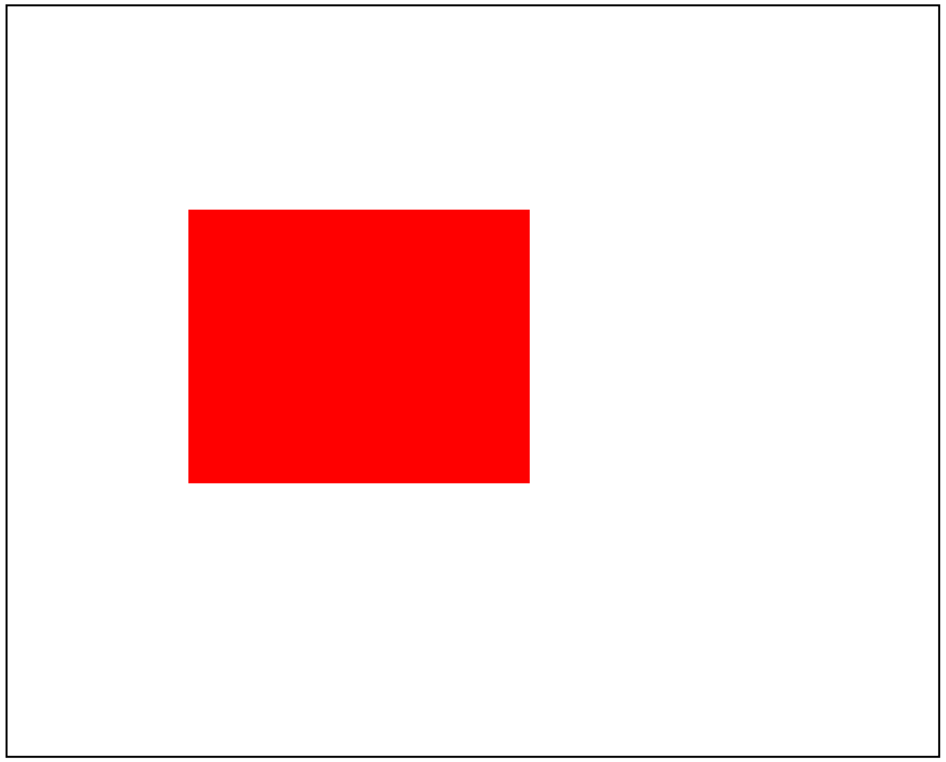
P(rain tomorrow) = 0.2

P(no rain tomorrow) = 0.8

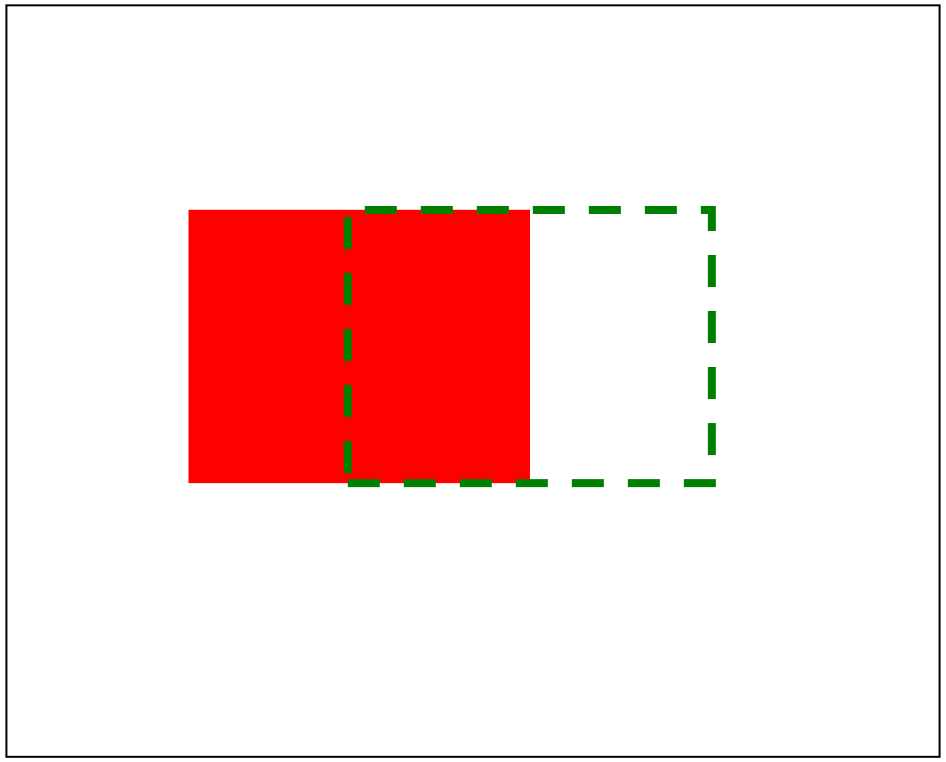
Conditional probability

- P(A = 1 | B = 1): The fraction of cases where A is true if B is true

P(A = 0.2)



P(A|B = 0.5)



Conditional probability

- In some cases, given knowledge of one or more random variables we can improve upon our prior belief of another random variable
- For example:

p(slept in movie) = 0.5

p(slept in movie | liked movie) = 1/4

p(didn't sleep in movie | liked movie) = 3/4

Slept	Liked
1	0
0	1
1	1
1	0
0	0
1	0
0	1
0	1

Joint distributions

- The probability that a *set* of random variables will take a specific value is their joint distribution.
- Notation: $P(A \wedge B)$ or $P(A,B)$
- Example: P(liked movie, slept)

If we assume independence then

$$P(A,B)=P(A)P(B)$$

However, in many cases such an assumption maybe too strong (more later in the class)

Joint distribution (cont)

P(class size > 20) = 0.6

P(summer) = 0.4

P(class size > 20, summer) = ?

Evaluation of classes

Size	Time	Eval
30	R	2
70	R	1
12	S	2
8	S	3
56	R	1
24	S	2
10	S	3
23	R	3
9	R	2
45	R	1

Joint distribution (cont)

P(class size > 20) = 0.6

P(summer) = 0.4

P(class size > 20, summer) = 0.1

Evaluation of classes

Size	Time	Eval
30	R	2
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56	R	1
24	S	2
10	S	3
23	R	3
9	R	2
45	R	1

Joint distribution (cont)

P(class size > 20) = 0.6

P(eval = 1) = 0.3

P(class size > 20, eval = 1) = 0.3

Size	Time	Eval
30	R	2
70	R	1
12	S	2
8	S	3
56	R	1
24	S	2
10	S	3
23	R	3
9	R	2
45	R	1

Joint distribution (cont)

P(class size > 20) = 0.6

P(eval = 1) = 0.3

P(class size > 20, eval = 1) = 0.3

Evaluation of classes

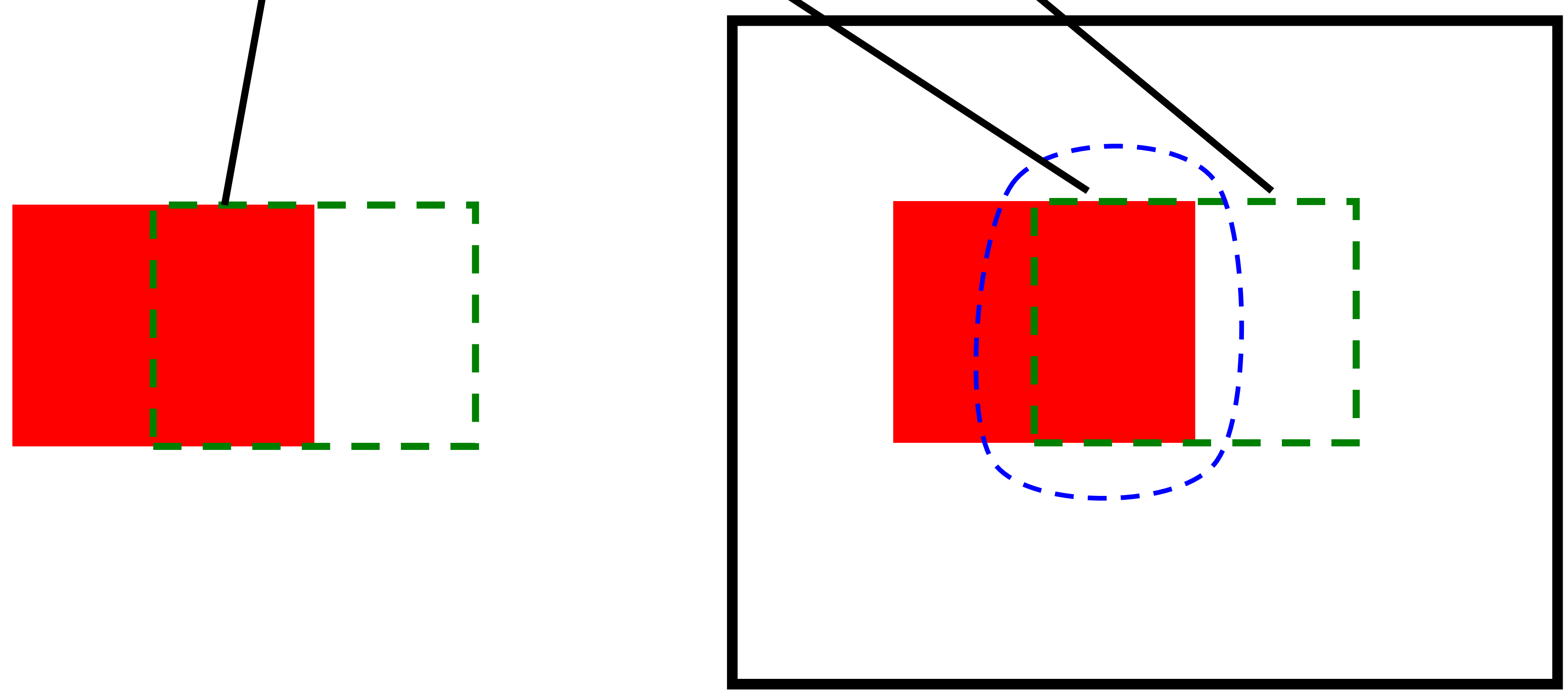
Size	Time	Eval
30	R	2
70	R	1
12	S	2
8	S	3
56	R	1
24	S	2
10	S	3
23	R	3
9	R	2
45	R	1

Chain rule

- The joint distribution can be specified in terms of conditional probability:

  P(A,B) = P(A|B)*P(B)

- Together with Bayes rule (which is actually derived from it) this is one of the most powerful rules in probabilistic reasoning



Bayes rule

- One of the most important rules for this class.
- Derived from the chain rule:

  P(A,B) = P(A | B)P(B) = P(B | A)P(A)
- Thus,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Thomas Bayes was an English clergyman who set out his theory of probability in 1764.

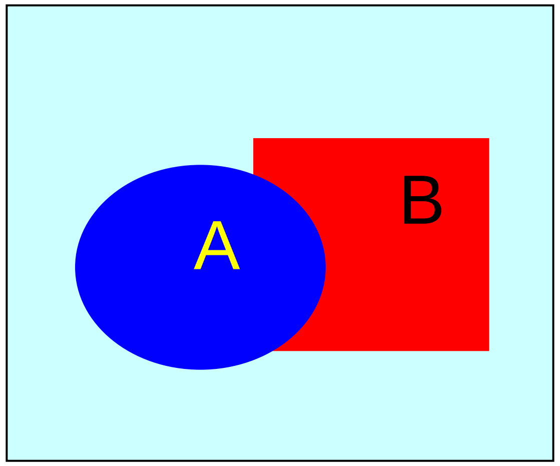
Bayes rule (cont)

Often it would be useful to derive the rule a bit further:

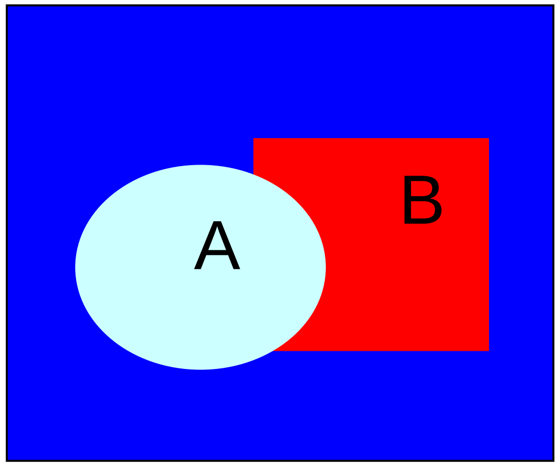
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{\sum_A P(B|A)P(A)}$$

This results from:
$P(B) = \sum_A P(B,A)$

P(B,A=1)



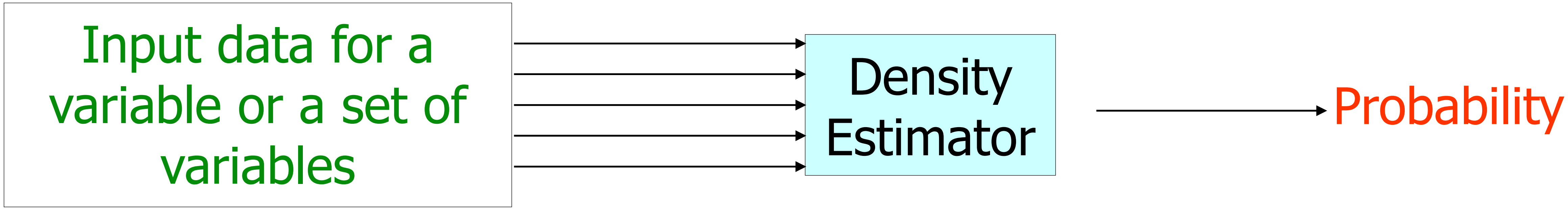
P(B,A=0)



Density estimation

Density Estimation

- A Density Estimator learns a mapping from a set of attributes to a Probability



Density estimation

- Estimate the distribution (or conditional distribution) of a random variable
- Types of variables:

  - Binary

    coin flip, alarm

  - Discrete

    dice, car model year

  - Continuous

    height, weight, temp.,

When do we need to estimate densities?

- Density estimators can do many good things…
  - Can sort the records by probability, and thus spot weird records (anomaly detection)
  - Can do inference: P(E1|E2)

    Medical diagnosis / Robot sensors
  - Ingredient for Bayes networks and other types of ML methods

Density estimation

- Binary and discrete variables:

Easy: Just count!

- Continuous variables:

Harder (but just a bit): Fit a model

Learning a density estimator for discrete variables

$$\hat{P}(x_i = u) = \frac{\#\text{records in which } x_i = u}{\text{total number of records}}$$

A trivial learning algorithm!

But why is this true?

Maximum Likelihood Principle

We can define the likelihood of the data given the model as follows:

$$\hat{P}(\text{dataset} \mid M) = \hat{P}(x_1 \wedge x_2 \cdots \wedge x_n \mid M) = \prod_{k=1}^{n} \hat{P}(x_k \mid M)$$

M is our model (usually a collection of parameters)

For example M is

- The probability of 'head' for a coin flip

- The probabilities of observing 1,2,3,4 and 5 for a dice

- etc.

Maximum Likelihood Principle

$$\hat{P}(\text{dataset} \mid M) = \hat{P}(x_1 \wedge x_2 \cdots \wedge x_n \mid M) = \prod_{k=1}^{n} \hat{P}(x_k \mid M)$$

- Our goal is to determine the values for the parameters in *M*
- We can do this by maximizing the probability of generating the observed samples
- For example, let Θ *be the probabilities for a coin flip*
- Then

$$L(x_1, \ldots, x_n \mid \Theta) = p(x_1 \mid \Theta) \ldots p(x_n \mid \Theta)$$

- The observations (different flips) are assumed to be independent
- For such a coin flip with $P(H)=q$ the best assignment for Θ_h is

$$argmax_q = \#H/\#samples$$

- Why?

Maximum Likelihood Principle: Binary variables

- For a binary random variable A with P(A=1)=q
  $\arg\max_q$ = #1/#samples

- Why?

Data likelihood: $P(D \mid M) = q^{n_1}(1-q)^{n_2}$

We would like to find: $\arg\max_q q^{n_1}(1-q)^{n_2}$

Omitting terms that do not depend on *q*

Maximum Likelihood Principle

Data likelihood: $P(D|M) = q^{n_1}(1-q)^{n_2}$

We would like to find: $\arg\max_q q^{n_1}(1-q)^{n_2}$

$$\frac{\partial}{\partial q} q^{n_1}(1-q)^{n_2} = n_1 q^{n_1-1}(1-q)^{n_2} - q^{n_1} n_2 (1-q)^{n_2-1}$$

$$\frac{\partial}{\partial q} = 0 \Rightarrow$$

$$n_1 q^{n_1-1}(1-q)^{n_2} - q^{n_1} n_2 (1-q)^{n_2-1} = 0 \Rightarrow$$

$$q^{n_1-1}(1-q)^{n_2-1}(n_1(1-q) - qn_2) = 0 \Rightarrow$$

$$n_1(1-q) - qn_2 = 0 \Rightarrow$$

$$n_1 = n_1 q + n_2 q \Rightarrow$$

$$q = \frac{n_1}{n_1 + n_2}$$

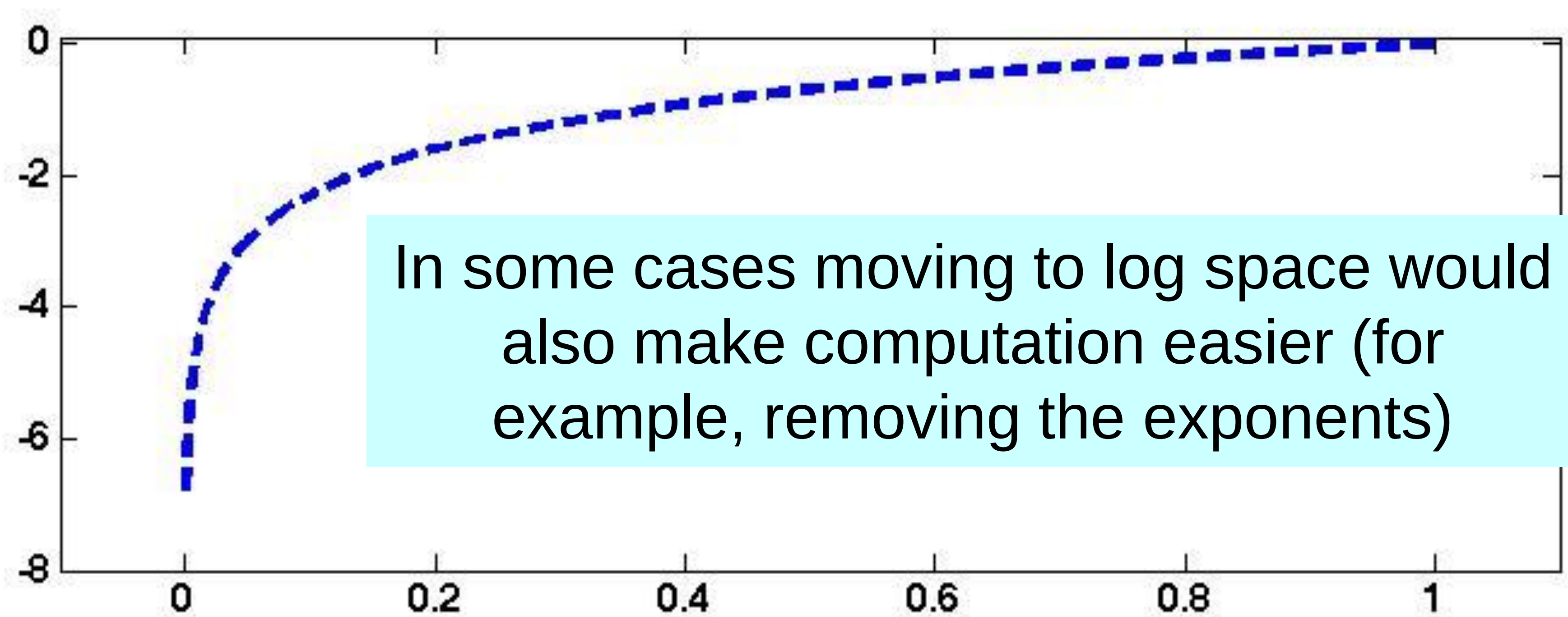
Log Probabilities

When working with products, probabilities of entire datasets often get too small. A possible solution is to use the log of probabilities, often termed ‘log likelihood’

$$\log \hat{P}(\text{dataset} \mid M) = \log \prod_{k=1}^{n} \hat{P}(x_k \mid M) = \sum_{k=1}^{n} \log \hat{P}(x_k \mid M)$$

Maximizing this likelihood function is the same as maximizing P(dataset | M)

Log values between 0 and 1



In some cases moving to log space would also make computation easier (for example, removing the exponents)

Density estimation

- Binary and discrete variables:
- Continuous variables:

Easy: Just count!

Harder (but just a bit): Fit a model

But what if we only have very few samples?

How much do grad students sleep?

- Lets try to estimate the distribution of the time students spend sleeping (outside class).

Possible statistics

- **X**

Sleep time

- **Mean of X:**

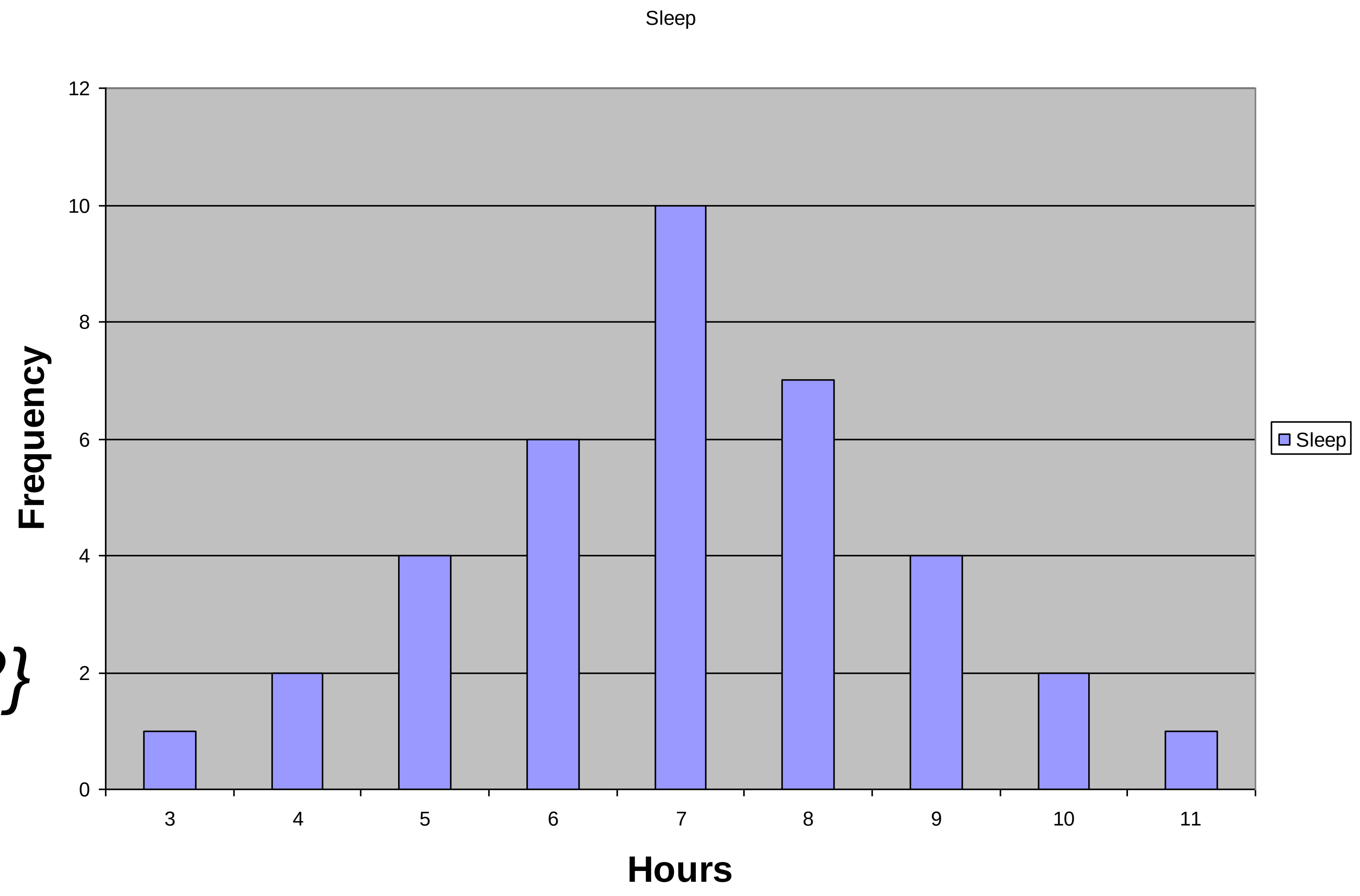
E{X}

7.03

- **Variance of X:**

Var{X} = E{(X-E{X})^2}

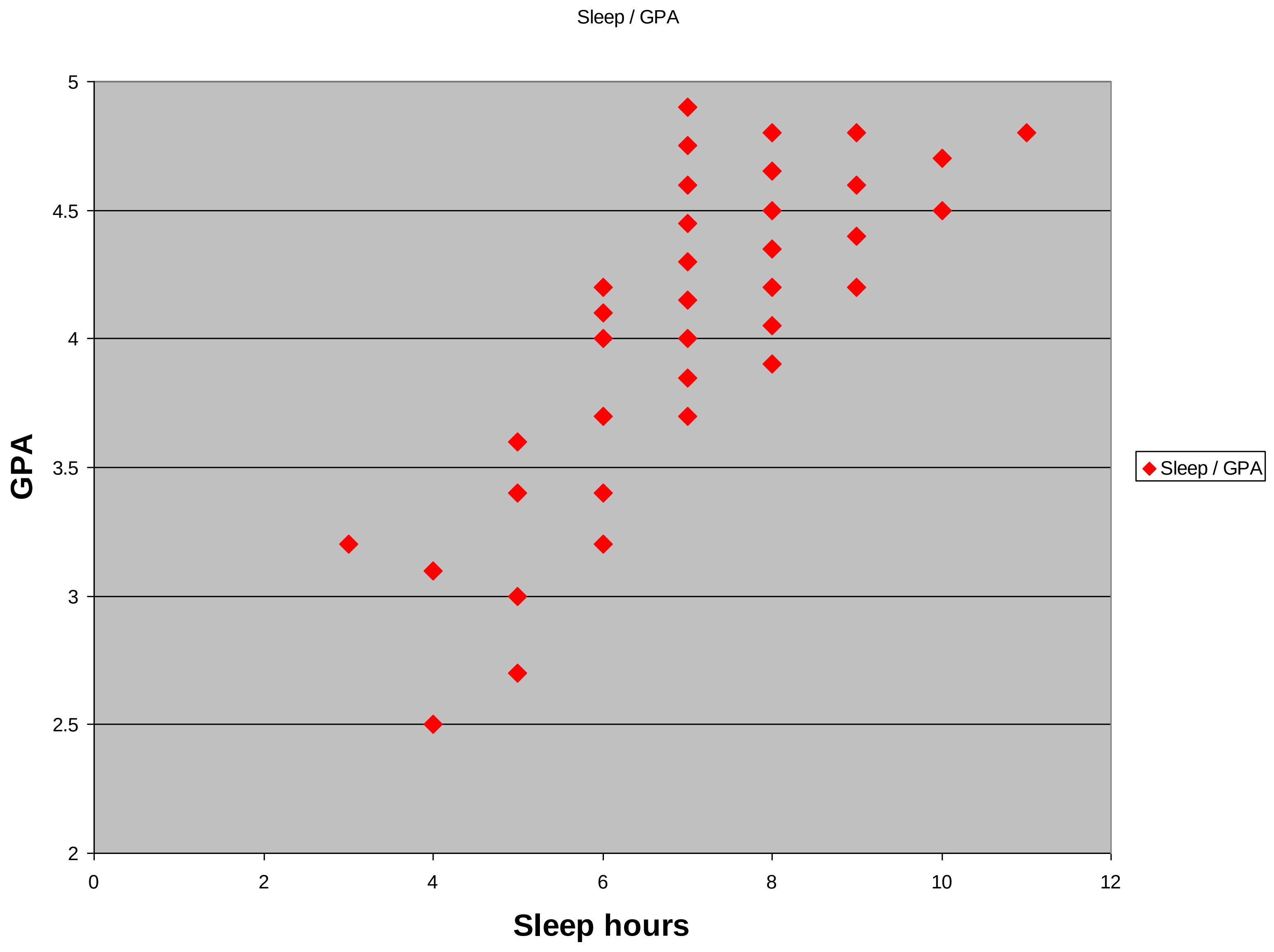
3.05



Covariance: Sleep vs. GPA

- **Co-Variance of X1, X2:**

$$Covariance\{X1,X2\} = E\{(X1-E\{X1\})(X2-E\{X2\})\} = 0.88$$



Statistical Models

• Statistical models attempt to characterize properties of the population of interest

• For example, we might believe that repeated measurements follow a normal (Gaussian) distribution with some mean μ and variance σ^2 , x ~ N(μ,σ^2)

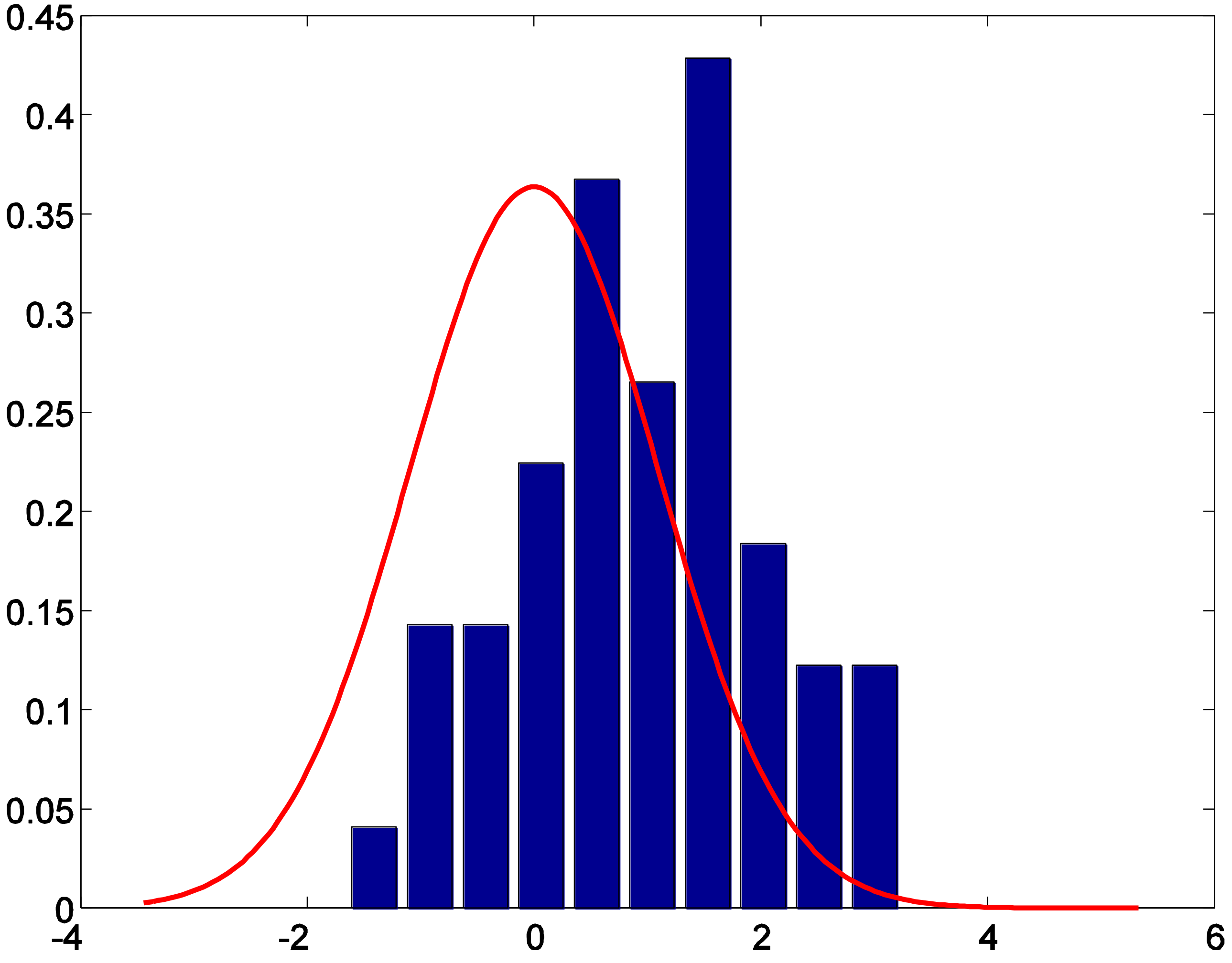
where

$$p(x \mid \Theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

and $\Theta=(\mu,\sigma^2)$ defines the parameters (mean and variance) of the model.

The Parameters of Our Model

- A statistical model is a **collection** of distributions; the **parameters** specify individual distributions x ~ N(μ,σ^2)
- We need to adjust the parameters so that the resulting distribution **fits** the data well



The Parameters of Our Model

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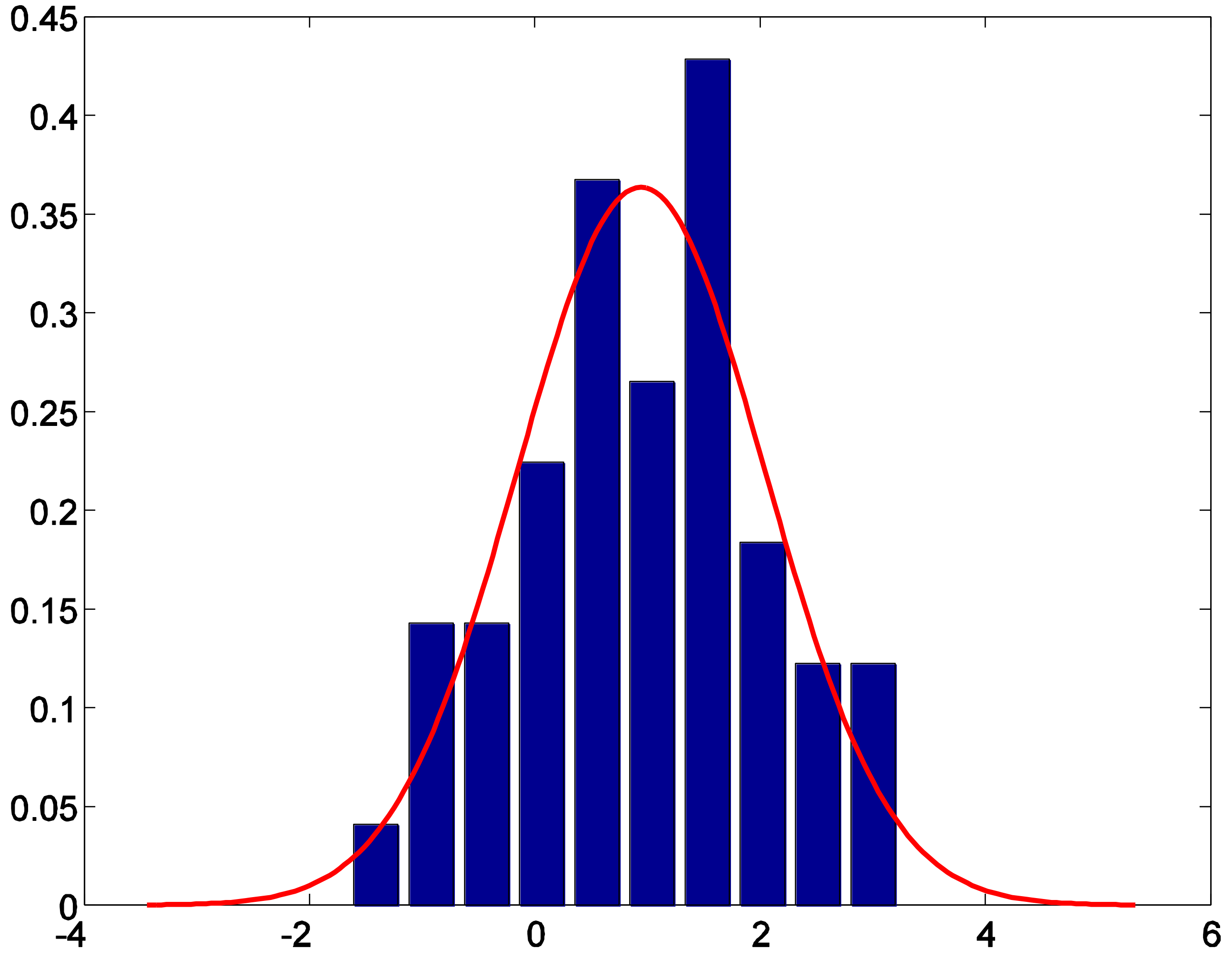
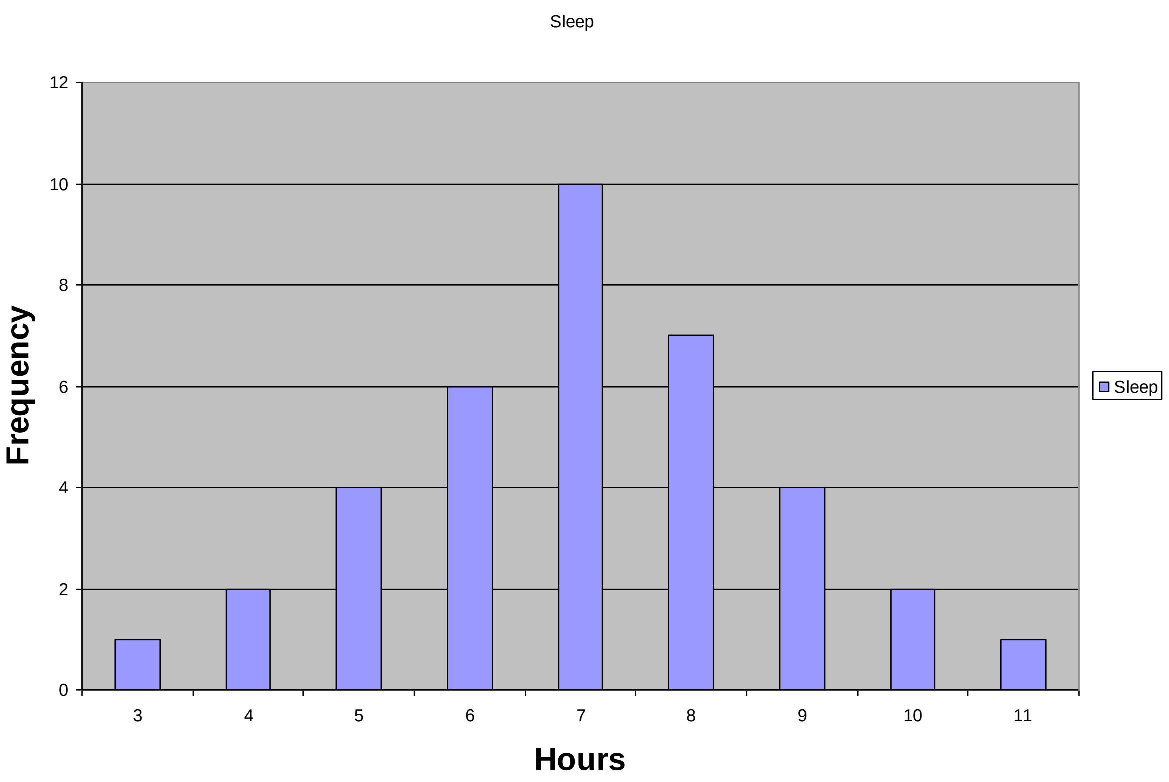


Computing the parameters of our model

- Lets assume a Guassian distribution for our sleep data
- How do we compute the parameters of the model?



Maximum Likelihood Principle

- We can fit statistical models by maximizing the probability of generating the observed samples:
  $L(x_1, \ldots, x_n \mid \Theta) = p(x_1 \mid \Theta) \ldots p(x_n \mid \Theta)$
  (the samples are assumed to be independent)

- In the Gaussian case we simply set the mean and the variance to the sample mean and the sample variance:

$$\overline{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \overline{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{\mu})^2$$

Why?

How much do grad students sleep?

- Lets try to estimate the distribution of the time students spend sleeping (outside class).

Possible statistics

- **X**

  Sleep time

- **Mean of X:**

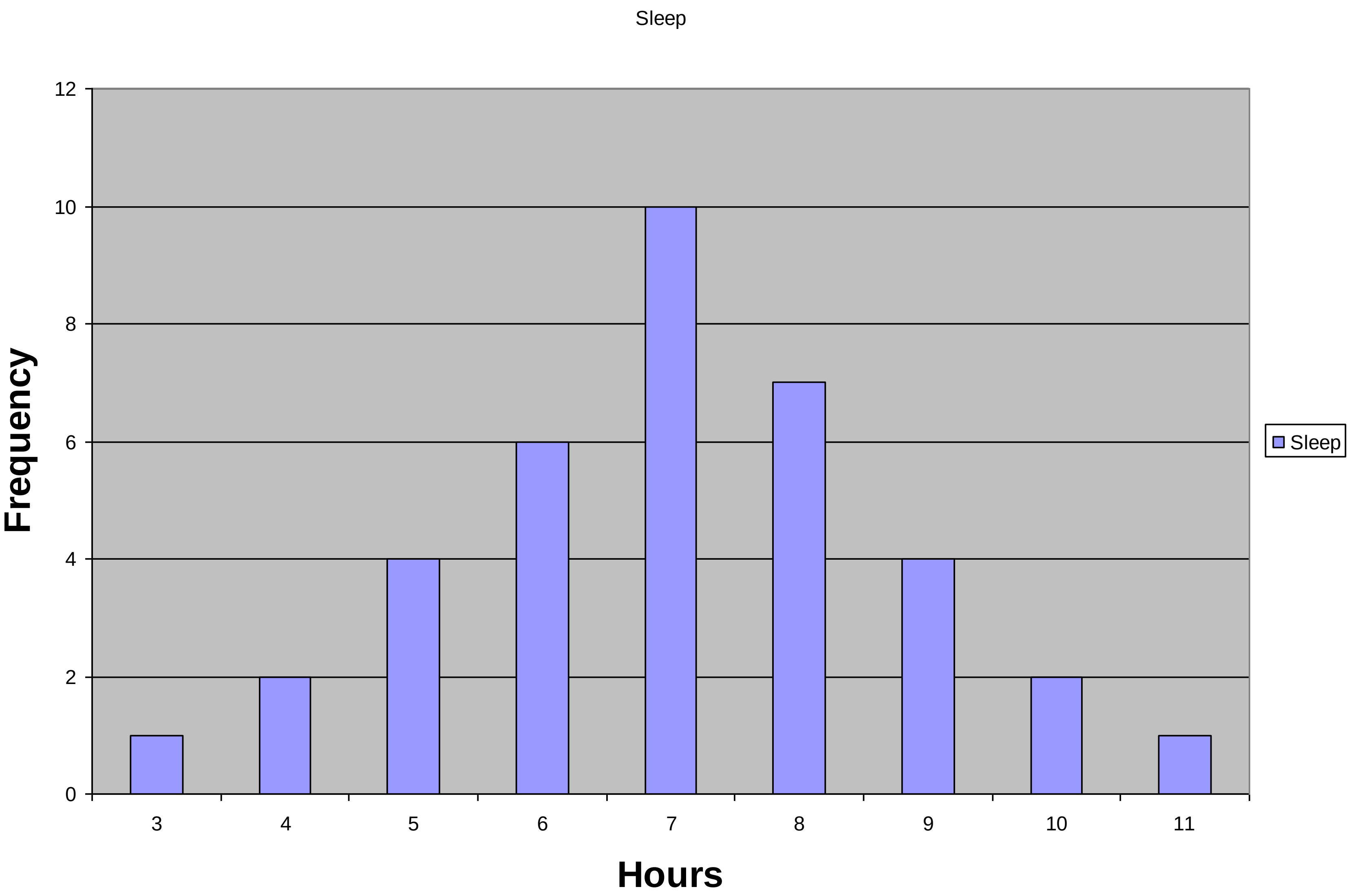
  E{X}

  7.03

- **Variance of X:**

  Var{X} = E{(X-E{X})^2}

  3.05

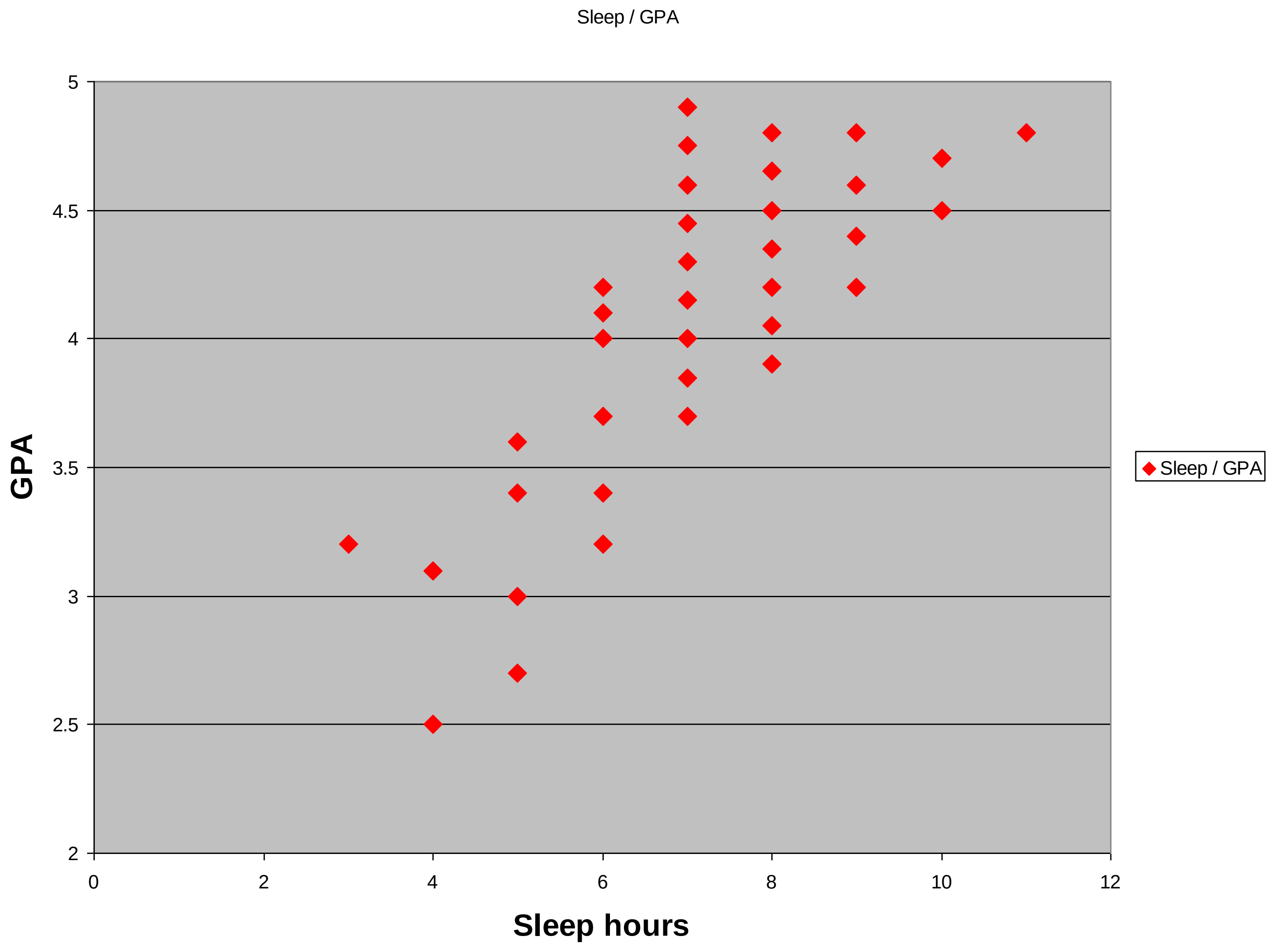


Covariance: Sleep vs. GPA

- **Co-Variance of X1, X2:**

  Covariance{X1,X2} = E{(X1-E{X1})(X2-E{X2})}

  = 0.88



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where

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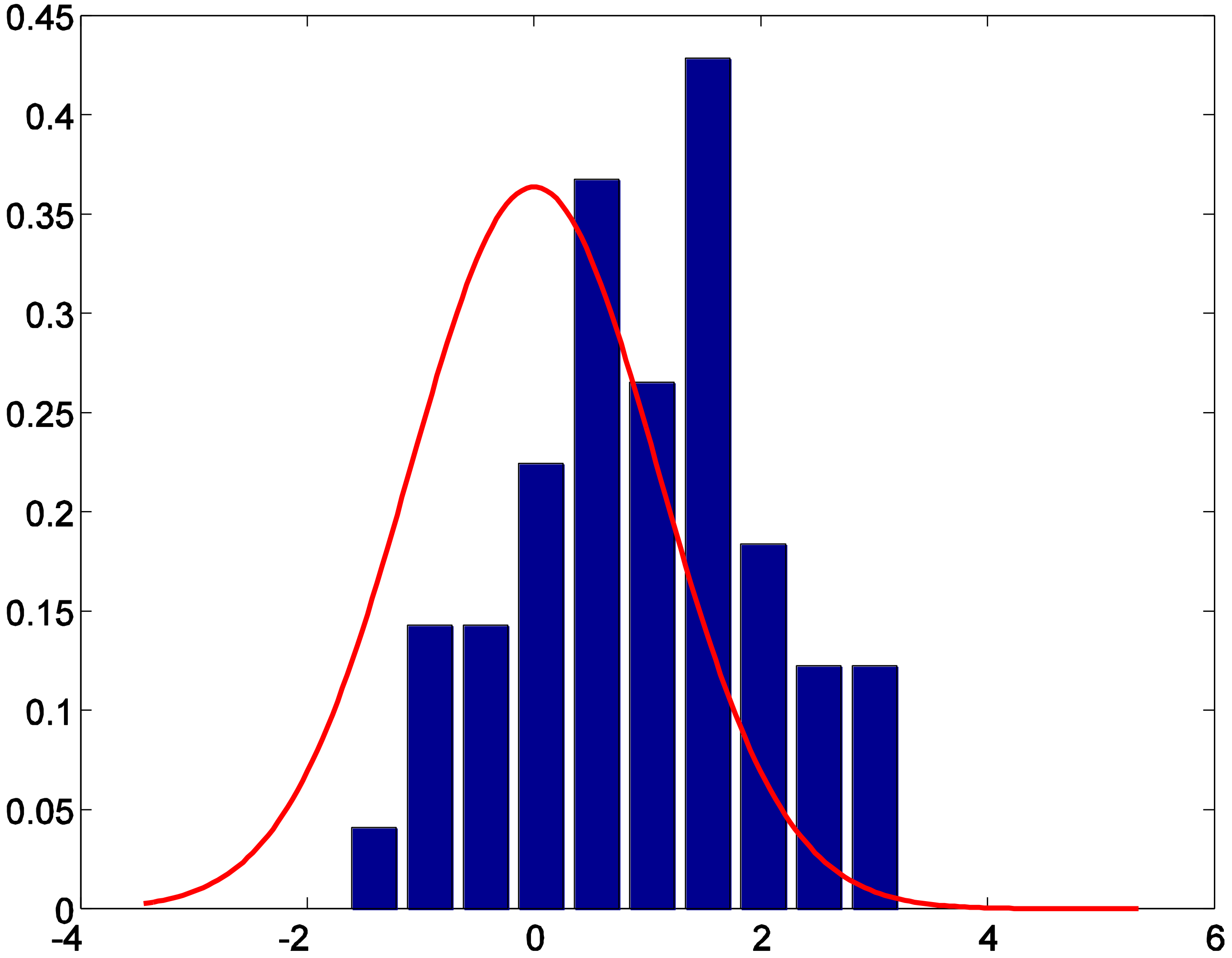
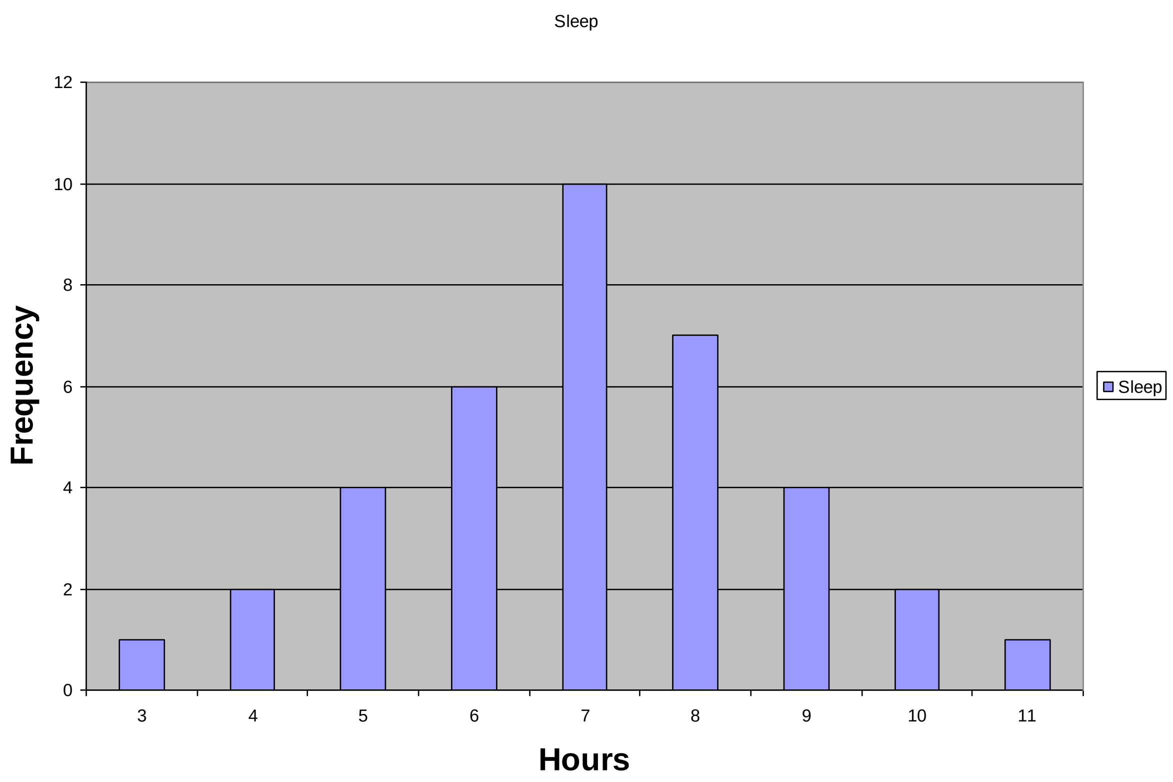


Computing the parameters of our model

- Lets assume a Gaussian distribution for our sleep data
- How do we compute the parameters of the model?



Maximum Likelihood Principle for Gaussian parameter estimation

- We can fit statistical models by maximizing the probability of generating the observed samples:

$L(x_1, \ldots, x_n \mid \Theta) = p(x_1 \mid \Theta) \ldots p(x_n \mid \Theta)$

(the samples are assumed to be independent)

- In the Gaussian case we simply set the mean and the variance to the sample mean and the sample variance:

$$\overline{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \overline{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{\mu})^2$$

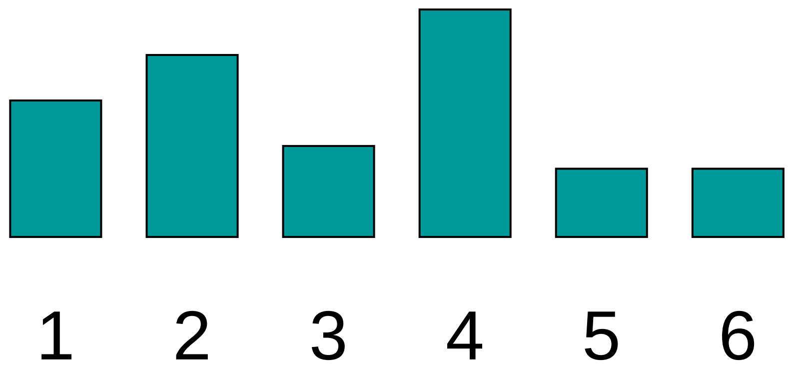
Why?

Important points

- Random variables
- Chain rule
- Bayes rule
- Joint distribution, independence, conditional independence
- MLE

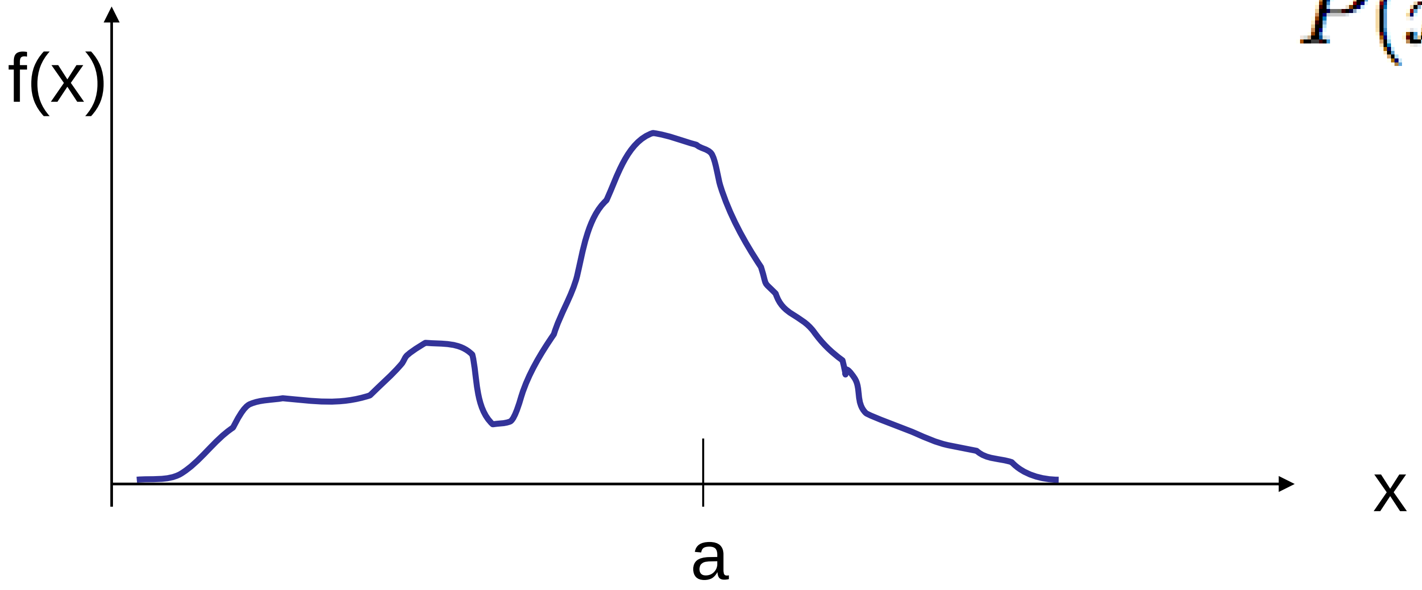
Probability Density Function

- Discrete distributions



$$\sum_i P(X = x_i) = 1$$

- Continuous: Cumulative Density Function (CDF): *F(a)*



$$P(x \leq a) = \int_{-\infty}^{a} f(\tau) d\tau$$

Cumulative Density Functions

- Total probability

$$P(\Omega) = \int_{-\infty}^{\infty} f(x)dx = 1$$

- Probability Density Function (PDF)

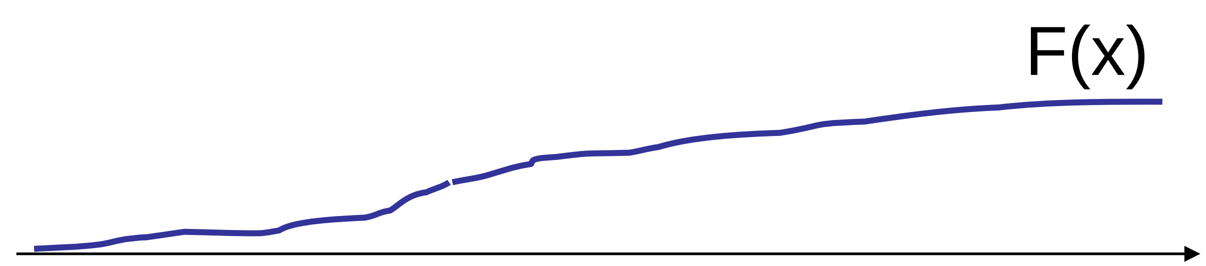
$$\frac{d}{dx}F(x) = f(x)$$

- Properties:

$$P(a \leq x \leq b) = \int_{b}^{a} f(x)dx = F(b) - F(a)$$

$$\lim_{x \to -\infty} F(x) = 0$$

$$\lim_{x \to \infty} F(x) = 1$$



$$F(a) \geq F(b) \ \forall a \geq b$$

Expectations

- Mean/Expected Value:

$$E[x] = \bar{x} = \int x f(x) dx$$

- Variance:

$$Var(x) = E[(x - \bar{x})^2] = E[x^2] - (\bar{x})^2$$

- In general:

$$E[x^2] = \int x^2 f(x) dx$$

$$E[g(x)] = \int g(x) f(x) dx$$

Multivariate

- Joint for (x,y)

$$P\left((x,y)\in A\right)=\int\int_A f(x,y)dxdy$$

- Marginal:

$$f(x)=\int f(x,y)dy$$

- Conditionals:

$$f(x|y)=\frac{f(x,y)}{f(y)}$$

- Chain rule:

$$f(x,y)=f(x|y)f(y)=f(y|x)f(x)$$

Bayes Rule

- Standard form:

$$f(x|y) = \frac{f(y|x)f(x)}{f(y)}$$

- Replacing the bottom:

$$f(x|y) = \frac{f(y|x)f(x)}{\int f(y|x)f(x)dx}$$

Binomial

- Distribution:

$$x \sim Binomial(p, n)$$

$$P(x = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

- Mean/Var:

$$E[x] = np$$

$$Var(x) = np(1 - p)$$

Uniform

- Anything is equally likely in the region [a,b]

- Distribution:

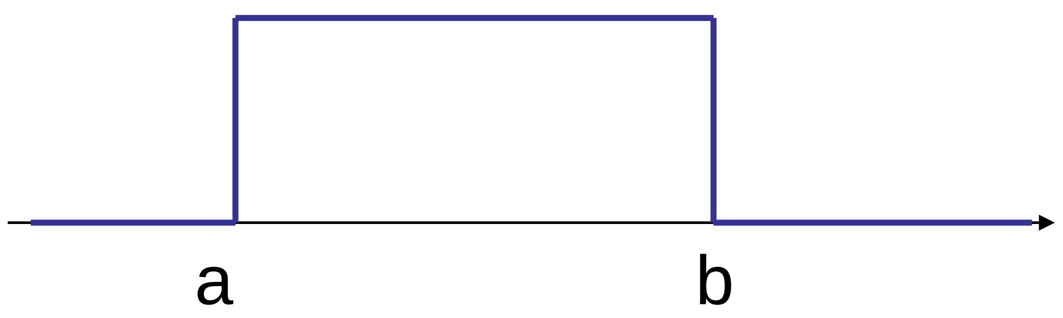
$$x \sim U(a,b)$$

- Mean/Var

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & otherwise \end{cases}$$

$$E[x] = \frac{a+b}{2}$$

$$Var(x) = \frac{a^2+ab+b^2}{3}$$



Gaussian (Normal)

- If I look at the height of women in country xx, it will look approximately Gaussian
- Small random noise errors, look Gaussian/Normal

- Distribution:

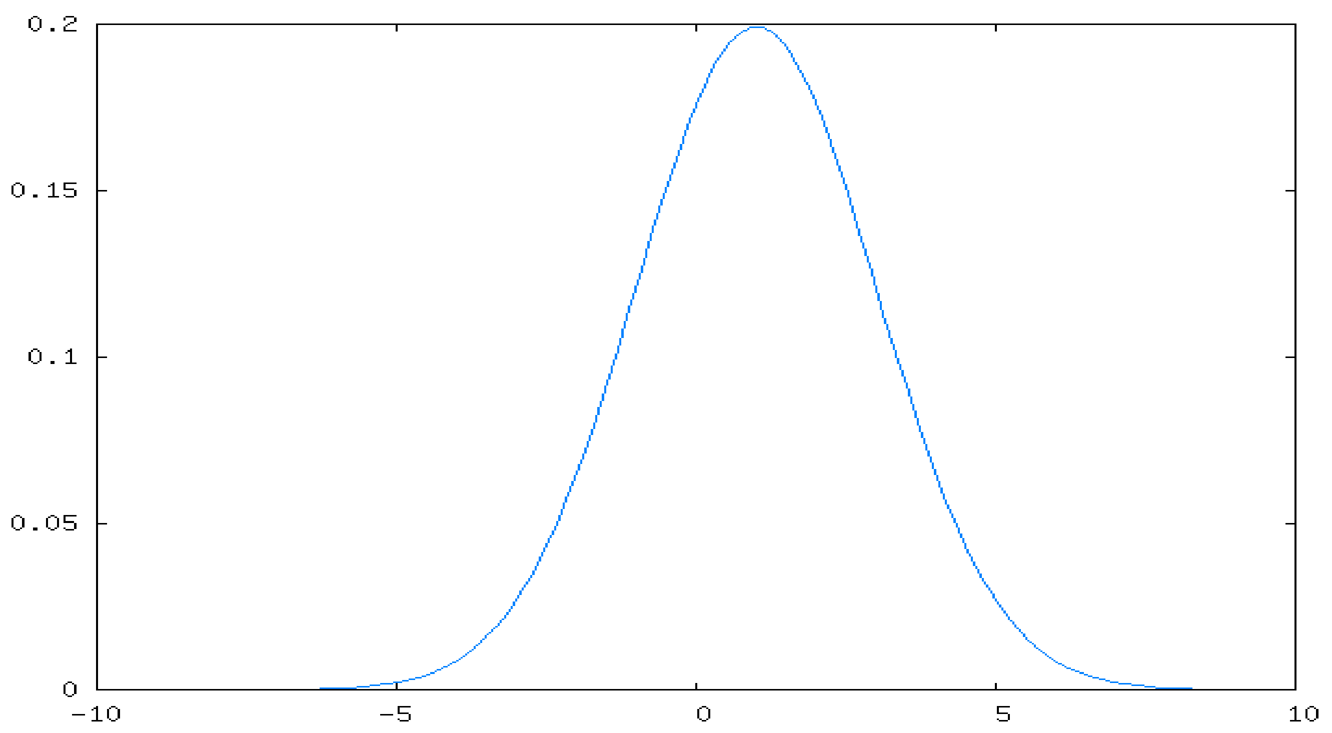
$$x \sim N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

- Mean/var

$$E[x] = \mu$$

$$Var(x) = \sigma^2$$



Why Do People Use Gaussians

- Central Limit Theorem: (loosely)
  - Sum of a large number of IID random variables is approximately Gaussian

Multivariate Gaussians

- Distribution for vector x

$$x = (x_1, \ldots, x_N)^T, \quad x \sim N(\mu, \Sigma)$$

- PDF:

$$f(x) = \frac{1}{(2\pi)^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$$

$$E[x] = \mu = (E[x_1], \ldots, E[x_N])^T$$

$$Var(x) \to \Sigma = \begin{pmatrix} Var(x_1) & Cov(x_1, x_2) & \ldots & Cov(x_1, x_N) \\ Cov(x_2, x_1) & Var(x_2) & \ldots & Cov(x_2, x_N) \\ \vdots & & \ddots & \vdots \\ Cov(x_N, x_1) & Cov(x_N, x_2) & \ldots & Var(x_N) \end{pmatrix}$$

Multivariate Gaussians

$$f(x) = \frac{1}{(2\pi)^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$$

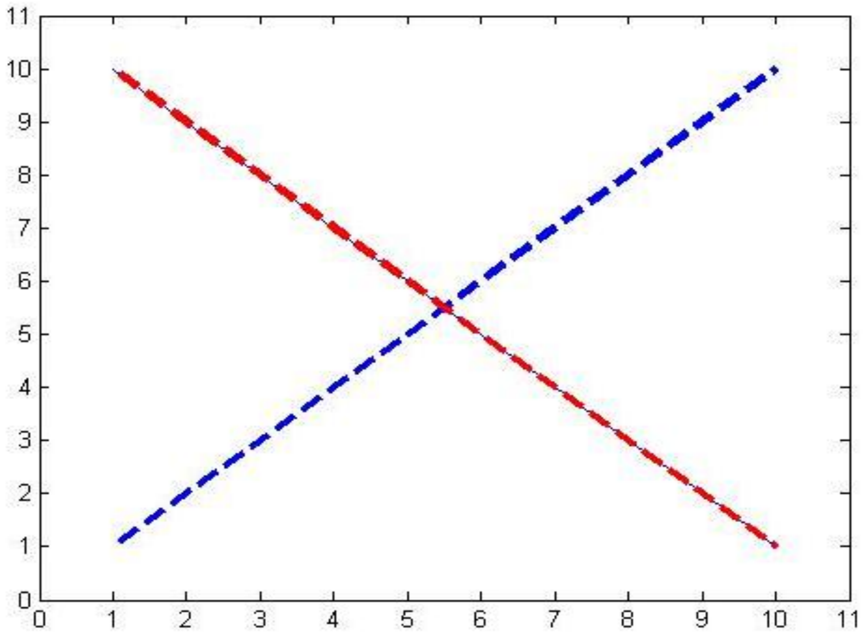
$$E[x] = \mu = (E[x_1], \ldots, E[x_N])^T$$

$$Var(x) \rightarrow \Sigma = \begin{pmatrix} Var(x_1) & Cov(x_1, x_2) & \ldots & Cov(x_1, x_N) \\ Cov(x_2, x_1) & Var(x_2) & \ldots & Cov(x_2, x_N) \\ \vdots & & \ddots & \vdots \\ Cov(x_N, x_1) & Cov(x_N, x_2) & \ldots & Var(x_N) \end{pmatrix}$$

$$\text{cov}(\chi_1, \chi_2) = \frac{1}{n} \sum_{i=1}^{n} (x_{1,i} - \mu_1)(x_{2,i} - \mu_2)$$

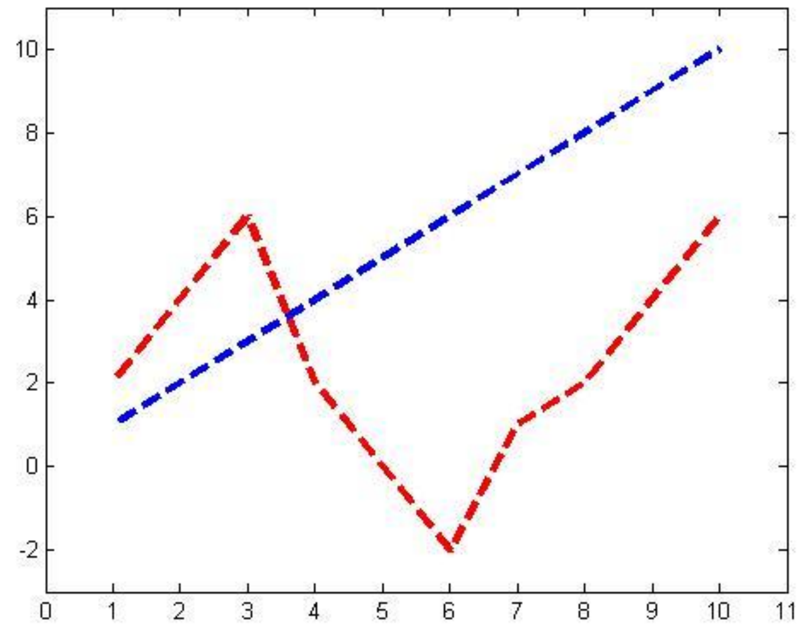
Covariance examples

Anti-correlated



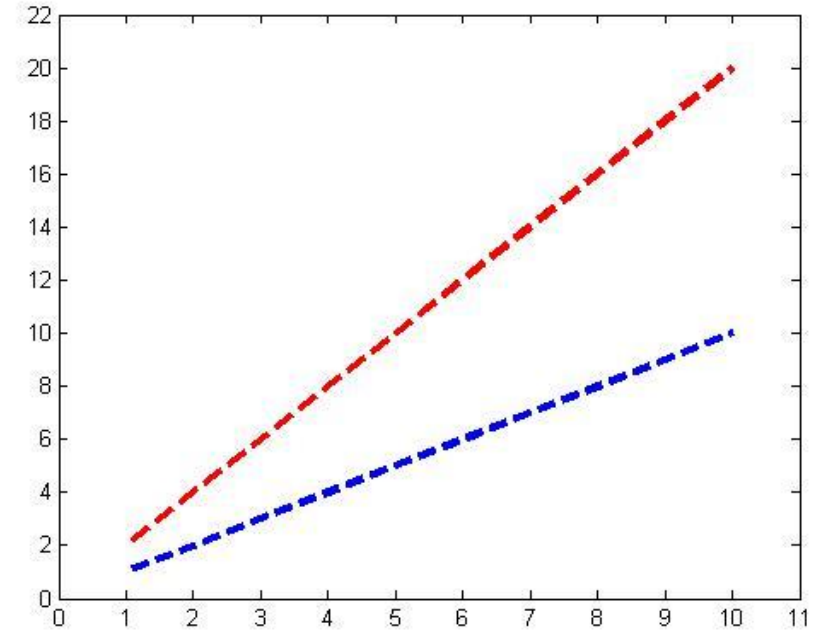
Covariance: -9.2

Independent (almost)



Covariance: 0.6

Correlated



Covariance: 18.33

Sum of Gaussians

- The sum of two Gaussians is a Gaussian:

$$x \sim N(\mu, \sigma^2) \quad y \sim N(\mu_y, \sigma_y^2)$$

$$ax + b \sim N(a\mu + b, (a\sigma)^2)$$

$$x + y \sim N(\mu + \mu_y, \sigma^2 + \sigma_y^2)$$