FCM II Programming Assignment 3

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1 Executive Summary

In this report we explore the computational powers and accuracy of numerical integration or quadrature. The three types of quadrature that we consider through this project are: Composite Trapezoidal rule, Composite Mid-Point rule and Composite Simpsons. To implement these methods we consider the test integral about the function $\int_0^3 e^x dx = e^3 - 1 \approx 19.0855$. We will then discuss the draw backs and advantages to these types of methods with varying error tolerances. We will then compare these results with the theoretical bounds which should be satisfied. Graphs and other visual aids are provided to further emphasize the error analysis as well as draw backs and advantages.

2 Statement of the Problem

The problem we seek to solve in this report is the issue of numerical integration or quadrature.The three types of quadrature that we consider through this project are: Composite Trapezoidal rule, Composite Mid-Point rule and Composite Simpsons. To implement these methods we consider the test integral about the function $\int_0^3 e^x dx = e^3 - 1 \approx 19.0855$. To verify these methods we consider the following theoretical results listed in Section 3.

3 Description of the Algorithms and Implementation

In this section we will discuss the theoretical intuition behind the following quadrature methods. With $H = \frac{b-a}{m}$ with m representing the step size. The Algorithms were given in our assignment sheet.

3.1 Composite Trapezoidal Rule

$$E_{0,m}(f) = \sum_{k=0}^{m-1} f''(\xi_k)(H/2)^3/3 = \sum_{k=0}^{m-1} f''(\xi_k)\frac{H^2}{24}\frac{b-a}{m} = \frac{b-a}{24}H^2 f''(\xi)$$

With the assumption that $f \in C^2([a,b])$ and $\xi \in (a,b)$. With a degree of exactness equal to 1 and 2nd order method.

3.2 Composite Midpoint Rule

Our textbook defines the quadrature error for Composite midpoint rule as:

$$E_{1,m}(f) = \frac{-(b-a)H^2}{12} f''(\xi), H = b - a$$

With the assumption that $f \in C^2([a,b])$ and $\xi \in (a,b)$. With a degree of exactness equal to 1 and 2nd order method.

3.3 Composite Simpsons Rule

Our textbook defines the quadrature error for Composite Simpsons rule as:

$$E_m^{csf}(f) = I(f) - I_{csf} = -(b-a)\frac{1}{2880}H_m^4 f^{(4)} + O\left(H_m^5\right).$$

The convergence of this method can be seen in Table 1.

3.4 Global refinement of Midpoint rule

The global refinement method for the Composite midpoint rule was given as:

$$I_{3m} = \frac{1}{3}\left[I_m + H_m \sum_{i=0}^{m-1}\left(f\left(a + iH_m + \frac{H_m}{6}\right) + f\left(a + iH_m + \frac{5H_m}{6}\right)\right)\right]$$

This method allows us to recursively use the existing points for each of the methods and refines the interval of integration increasing the order of accuracy and speed of convergence. Major discussion is discussed in Section 4.2.

4 Description of the Experimental Design and Results

In this section we move to discuss the Major results and observations of the methods above. Note that for testing we consider the problem of $\int_0^3 e^x dx \approx 19.0855$.

4.1 Task 1

Throughout our implementation we are able to numerically verify and see that the Composite Midpoint ant the Composite Trapezoidal rule are second order methods of convergence via Table 1. But as we move to Simpsons rule we can see that Simpsons obtains a fourth order convergence. The fourth order convergence becomes apparent when we expect a higher order of accuracy. Depiction of accuracy threshold of 10^{-8} is shown for emphasis of this advantage.

4.2 Task 2

For this Task we were required to perform and complete the global refinement algorithms for Composite Midpoint rule.When observing the global refinement algorithm we are able to identify via Table 2 and Figure 1. The computations for r are seen in Table 3. We are able to identify that as k increases from zero to ten then our r value converges to 2. Furthermore, the computed error decreases drastically as this method recursively reuses the previous mesh computations and improves accuracy. Note that this is different than in the previous task when comparing this method to the Composite Trapezoidal Method. As before the Composite Trapezoidal method worked better and was able to achieve the error threshold faster than the standard Composite Midpoint rule. But with this global refinement of the Composite Midpoint rule we are able to achieve better accuracy as well as faster convergence. A graphical representation of these observations can be seen in figures 1-3.

5 Conclusion and Comparison of Methods

Overall, we are able to analyze and compare the computational powers and accuracy of numerical integration or quadrature. The three types of quadrature that we considered through this project were: Composite Trapezoidal rule, Composite Mid-Point rule and Composite Simpsons. To implement these methods we consider the test integral about the function $\int_0^3 e^x dx = e^3 - 1 \approx 19.0855$. Note that we are able to change this function to best fit real world applications and problems. The main takeaways of this project were that Composite Simpsons rule is the best at convergence to a respective threshold as well as error. But when comparing Composite Trapezoidal to the Global refinement of the Midpoint rule we can see that the Global refinement helps improve the accuracy and efficiency as it recursively calls to the previous points on the interpolated interval. Overall, this project was able to verify the in class numerical schemes and results.

6 Theorems, Tables, and Figures

6.1 Theorems, Properties, Citations

1. There were no notable theorems or Properties used in this write-up.

6.2 Tables

1. Table 1: Iterations for |Error = $|I - \tilde{I}| \leq tol = 0.01$ or $|e^3 - 1 - (\tilde{I})| \leq 0.01$ and |Error = $|I - \tilde{I}| \leq tol = 0.01$ or $|e^3 - 1 - (\tilde{I})| \leq 0.0001$. And 10^{-8} for fun ☺.

Error tolerance	0.01	0.0001	10^{-8}
Composite Trapezoidal	38	379	$37,835$
Composite Midpoint	28	269	$26,754$
Composite Simpson's	4	10	87

2. Table 2: Theoretical bounds

Theoretical bounds	0.01	0.0001
Composite Trapezoidal	69	678
Composite Midpoint	49	477
Composite Simpson's	5	13

3. Table 3: Global refinements for Composite midpoint rule, varying k :

k	$m = 3^k$	r	Error
1	3	1.809481808599895	0.7792
2	9	1.809481808591895	0.08812
3	27	1.976705077739497	0.0097
4	81	1.976705077739497	0.0011
5	243	1.997381717263836	0.0001
6	729	1.997381717263836	$1.35 * 10^{-5}$
7	$2,187$	1.999708703907611	$1.51 * 10^{-6}$
8	$6,561$	1.999708703907611	$1.67 * 10^{-7}$
9	$19,683$	1.999967629107464	$1.85 * 10^{-8}$
10	$59,049$	2	$2.06 * 10^{-9}$

6.3 Figures

6.3.1 Figures

Figure 1: Depiction of the convergence to the true solution of $e^3 - 1 \approx 19.0855$

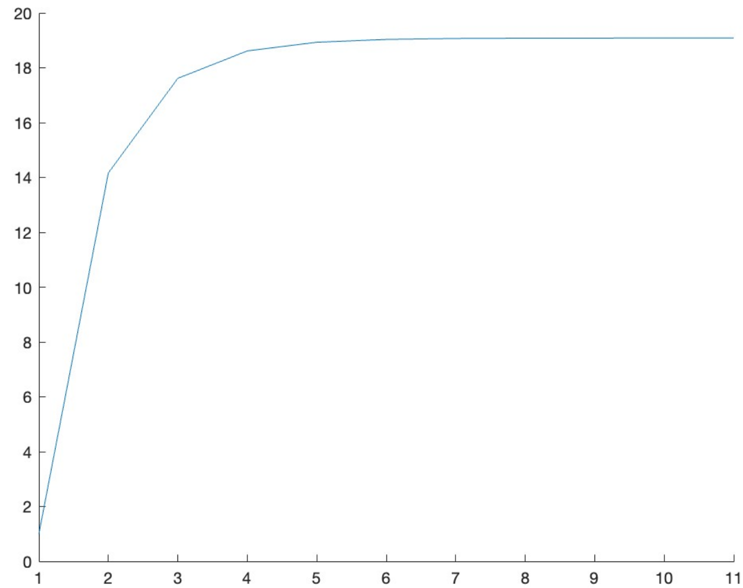

Figure 2: Log Plot of the errors for Global refinement algorithm

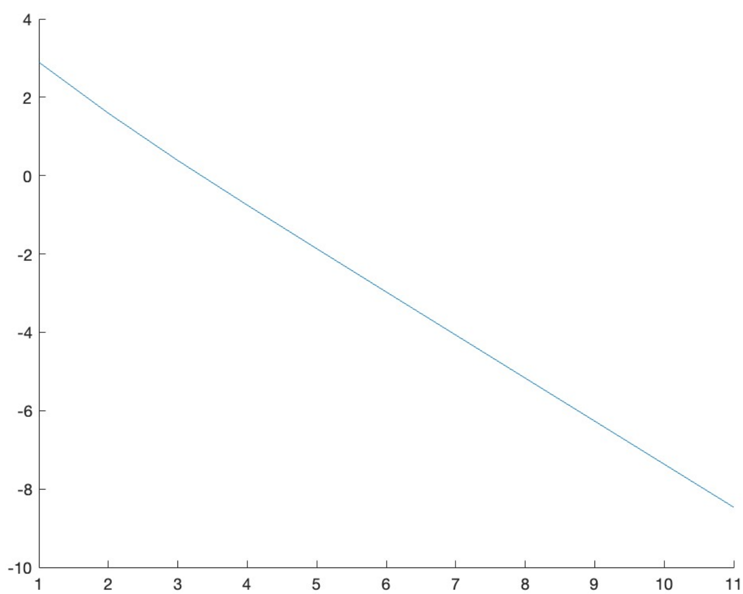

Figure 3: Plot of the errors for Global refinement algorithm