Foundations of Computational Math 2 Spring 2024

Programming assignment 4

General Task

In this assignment, you have two tasks:

(1) implement the following numerical quadrature methods and compare their observed behavior to theoretical predictions:

- global refinement algorithms for Composite Trapezoidal Rule (CTR)
- composite two-point Gauss-Legendre Method

(2) implement the following methods to work on scalar differential equations:

- Forward Euler:
$$y_n = y_{n-1} + hf_{n-1}$$
- Backward Euler:
$$y_n = y_{n-1} + hf_n$$

Composite Trapezoidal Rule

For the composite Trapezoidal Rule, all of the expressions needed are given here.

Let $f_i = f(b_i)$ for the intervals $[a_i, b_i]$, $1 \leq i \leq m$ and $f_0 = f(a_1)$.

$$I_m^{ctr} = \frac{H_m}{2}\left[f_0 + f_m + 2\sum_{i=1}^{m-1} f_i\right]$$

Assuming $\alpha = 1/2$, there is complete reuse of previous function evaluations to generate the fine grid $(2m)$ intervals quadrature, I_{2m}^{ctr}, from the coarse grid (m intervals) quadrature, I_m^{ctr}:

$$I_{2m}^{ctr} = \frac{1}{2}\left[I_m^{ctr} + H_m \sum_{m \text{ new points}} f_i\right]$$

where the new points are the midpoints of the m coarse grid intervals.

The composite error expression for m intervals is

$$E_m^{ctr} = I(f) - I_m^{ctr}(f) = -(b-a)\frac{H_m^2}{12}f'' + O(H_m^3)$$

and the error estimate from the coarse/fine combination, $\alpha = 1/2$, is easily seen to be the form

$$E_m^{ctr} \approx \frac{2^r}{2^r - 1}(I_{2m}^{ctr} - I_m^{ctr})$$

$$E_{2m}^{ctr} \approx \frac{1}{2^r - 1}(I_{2m}^{ctr} - I_m^{ctr})$$

where $r = 2$ for the Composite Trapezoidal Rule.

Composite Two-point Gauss-Legendre Method

For Gauss-Legendre quadrature on an interval $[a_i, b_i]$, a change of variables to $[-1, 1]$ is needed and this must be taken into account in the local and composite error bound, and the integration of the function $f(x)$ must be written as an integration of a function $F(z) = f\big(x(z)\big)$ using the z_i points defining Gauss-Legendre. We have

$$a_i \le x \le b_i \text{ and } -1 \le z \le 1,$$

$$x = \frac{z(b_i - a_i) + b_i + a_i}{2} \Longrightarrow dx = \frac{b_i - a_i}{2} dz,$$

$$\int_{a_i}^{b_i} f(x)dx = \frac{b_i - a_i}{2} \int_{-1}^{1} f\left(\frac{z(b_i - a_i) + b_i + a_i}{2}\right) dz,$$

$$\begin{aligned} I_{gl}\big(f(x)\big) &= \frac{b_i - a_i}{2} I_{gl}\big(F(z)\big) \\ &= \frac{b_i - a_i}{2}\big(\gamma_0 F(z_0) + \gamma_1 F(z_1)\big) \\ &= \frac{b_i - a_i}{2}\Big(\gamma_0 f\big(x(z_0)\big) + \gamma_1 f\big(x(z_1)\big)\Big) \end{aligned}$$

$$\gamma_0 = \gamma_1 = 1, z_0 = -\frac{1}{\sqrt{3}}, z_1 = \frac{1}{\sqrt{3}}.$$

Task 1: Numerical Quadrature

$$\int_0^3 e^x dx = e^3 - 1$$

1. For the composite Two-Point Gauss-Legendre Method,
   - stop condition: $|\text{error}| = |I - \tilde{I}| < tol$
   - you will test for two tolerances: 0.01 and 0.0001.
   - manually estimate the subinterval size needed to satisfy the above two tolerances using composite methods error expression.
   - numerically calculate the subinterval size needed to satisfy the above tolerances. Compare the true subinterval size with the predicted subinterval size
   - calculate the error using the exact answer of I.
   - Count the function evaluations for each method with two tolerances.
   - Compare and discuss the results.
2. For global refinement algorithms for Composite Trapezoidal Rule (CTR),
   - Starting from $m = 1$, test for $m = 2^k$ with $k = 0, 1, \cdots, 10$. Summarize appropriately and concisely your observations. Discuss and compare the number of function evaluations for each k.
   - Calculate the approximate error for each k (except $k = 0$). Compare this approximation to the error bound using the error expression (involving the derivative of the function) and to the true error.
   - Numerically verify r. For example, since

     $$E_{2m}^{cmr} \approx \frac{1}{2^r - 1}(I_{2m}^{cmr} - I_m^{cmr}),$$

     we will have

     $$r \approx \log_2\left(\frac{I_{2m}^{cmr} - I_m^{cmr}}{I - I_{2m}} + 1\right)$$

   - Compare and discuss the results.

Task 2: Numerical ODE

The Family of Problems

Consider the parameterized family of initial value problems (IVP) given by:

$$
\begin{aligned}
f &= \lambda(y - F(t)) + F'(t) \\
y(0) &= y_0 \\
y(t) &= (y_0 - F(0))e^{\lambda t} + F(t)
\end{aligned}
$$

This family has parameters $\lambda \in \mathbb{R}$, $F : \mathbb{R} \to \mathbb{R}$, and $y_0 \in \mathbb{R}$ to define an initial value problem with solution $y : \mathbb{R} \to \mathbb{R}$ on $T_L \leq t \leq T_U$.

(Note that the integral curve is specificed by the choice of y_0 but all of integral curves contain an exponential and $F(t)$. So even if $y(0) = F(0)$ and $y(t) = F(t)$, integral curves with $y_0 \neq F(0)$ have an exponential component that can be seen by points in the numerical solution y_n since it contains points on many different integral curves of the system.)

The Tasks

0.1 General Comments

In this programming assignment you will explore the behavior of different methods on a finite interval $0 \leq t \leq 10$ with a fixed stepsize h for different choices of λ, y_0 and $F(t)$. You will compare the observed behaviors to those predicted by the theory of local error order, global error (convergence) order, and absolute stability.

For a given IVP, you will solve using a particular method and a series of stepsizes h. For each mesh you should quantify, at least, the error $|y(t_1) - y_1|$ which is the first step's error and therefore a local error, the final global error $|y(t_N) - y_N|$ where $t_N = 10$ and the maximum global error $\max\limits_{0 \leq n \leq N} |y(t_n) - y_n|$. You should use these data to estimate the local error order and the global error (convergence) order of the methods.

You must organize your observations into a compact and clear presentation of evidence to support your conclusions.

0.2 Absolute Stability

Take $F(t) = 0$ and $y(0) = 1$.

- Probe the absolute stability properties of the methods for two $\lambda = \pm 1$
- Identify the intervals on the real axis where the methods are damping and where they are growing. Determine if this is consistent with the theory.

0.3 Accuracy and Stability

Take $F(t) = \sin(\omega t)$ and $y(0) = 0$.

- Test for $\omega = 0.01$ and $\omega = 10$
- Test for $\lambda = -1$ and $\lambda = -0.01$
- Discuss stability and accuracy.

Submission of Results

Expected results comprise:

- A document describing your solutions as prescribed in the notes on writing up a programming solution posted on the Canvas.
- The source code, makefiles, and instructions on how to compile and execute your code including the Math Department machine used, if applicable.
- Code documentation should be included in each routine. (You don't need to paste your code in the writing report).
- All text files that do not contain code or makefiles must be PDF files. **Do not send Microsoft word files of any type.**

These results should be submitted by 11:59 PM on the due date. Submission of results is to be done via Canvas.