FCM Project 4

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1 Executive Summary

In this report we look to solve particular types of root finding problems. Root problems have specific importance and interesting results in the field of Numerical Analysis. We move to discuss the implementation and validation of three theoretical root finding tests, the Bisection, Newtons, and Fixed point methods respectively. For our discussion we compare the convergence speed of finding the root with these methods as well as some potential advantages and disadvantages for each of the methods. Figures and tables are shown for visual aid in to the theoretical results discussed in class.

2 Statement of the Problem

For this project the problem we wish to tackle is to implement, analyze and empirically test the efficacy and accuracy of the Bisection, Newtons, and Fixed point methods respectively. For this task we work with two elementary functions which we have been seen to obtain real roots. These two test functions are:

$$f(x) = xe^{-x} - 0.06064 \tag{1}$$

$$g(x) = x^3 - x - 6 \tag{2}$$

To discuss these roots a couple of important pieces of information must be found. First, we must understand what interval we are working on, it would be great if we could just give these methods a wide range of variables to choose from but this is not the case, discussion on how we choose these intervals is discussed bellow. Next, for Newtons method it is crucial to know that there can not be any simple roots. As the Newtons method uses the information of the derivative at each point, there must be no simple roots on the interval we have chosen. For the fixed point discussion we need to obtain and test two particular phi's respective to $f(x)$ and $g(x)$. We then move to compare the difference in convergence speed and plot the results. We now move to a more in-depth discussion on how these Algorithms work.

3 Description of the Algorithms and Implementation

For the two functions listed above we consider three key root finding methods in the Bisection, Newtons, and Fixed point method. We now move to discuss the implementation in detail.

3.1 Bisection Method

The Bisection method is relatively straight forward, for a user prompted function and interval $x \in [a, b]$ where $f(a)f(b) < 0$ then the algorithm takes the midpoint of the interval and evaluates the product of $f(a_{k+1})f(b_{k+1})$ if this product is less than our tolerance i.e relatively small we have found our root and are done with the computations. If this product is greater than our tolerance we take the midpoint once again and evaluate the sign of this product. Note: we choose the negative case when proceeding to match with the theory.

3.2 Newtons Method

The Newtons Method evaluates the following iteration where the algorithm takes in a user initial guess and a predetermined derivative for the desired function.

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}, k > 0$$

This method iterates until the stopping condition where the $|f(x_k)| < 10^{-6}$ where 10^{-6} is our allowed tolerance for all methods. Note that we obtain a stopping condition that prompts the user to try another initial guess if $f'(x_k) < \epsilon$ where epsilon is the computer generated tolerance, in other words $f'(x_k) = 0$.

3.3 Fixed Point Method

For the Fixed Point Method we have the user enter an initial guess of x_0 as well as a pre-determined or calculated $\phi(x)$ where $\phi(\alpha) = \alpha$ yields our root. The algorithm then iterates over the designated interval searching for when $|\phi(x_k) - x| < 10^{-6}$. Where we have obtained our root, the reasoning beihind the selection of this method can be found in **Theorem 6.1**. We now move to discuss the results for many different cases of these methods.

4 Description of the Experimental Design and Results

4.1 Test 1: Double Root Exponential

For the first test of the algorithms we consider our test function to be $f(x) = xe^{-x} - 0.06064$. We plot this graphically to gain intuition on how many roots we expect. As seen in Figure 1 there will be two roots, one at $\alpha_1 = x_1^* = 0.0647, \alpha_2 = x_2^* = 4.5$ where α_i denotes the root.

4.1.1 Bisection method

Assuming that we do not know these roots, we first start off with a significantly large interval with $x, \alpha \in [1, 50]$ which yields a solution of $\alpha = 4.25$ after 45 iterations. Again, assuming that we do not know these roots, we lean on intuition gained from **Property 6.1** in our text. With this information we are able to make an analysis argument to initialize our interval $[a, b]$. Using the fact that $e^{-x} > 0$ we can bound $|f(x)| < |x - 0.06064|$ which shows that for any initial guess of $a < 0.06064$ we obtain that $f(a) < 0$ and for any close enough initial guess of $b > 0.06064$ $f(b) > 0$ which implies that $f(a)f(b) < 0$. However, since we have two roots we must not deviate too far away from this interval, for simplicity we choose $a = -2, b = -2$. In Table 2 we are able to see that the Bisection method converges under an error threshold of 10^{-6} in 22 iterations we are also able to graphically see the actual convergence and the logarithmic convergence. While this particular case shows monotonic reduction, we not that this is not always the case for the bisection method. Also, even though this method is relatively slow we can guarantee convergence for continuous functions.

4.1.2 Newton's Method

For the testing of Newtons method we first take an initial guess of $x_0 = 1$. This initial guess causes our code to yield an error and prompts the user to try an initial guess for x_0. This is due to the fact that our $f'(x) = (1 - x)e^{-x}$ and $f'(1) = (1 - 1)e^{-1} = 0$. Recall, that one of the requirements of the Newtons method is to have a non zero derivative. In addition, the approximation of the condition number is given as $K_{abs}(d) \approx \frac{1}{|f'(x)|}$. if the value of $|f'(x_0)|$ is relatively small then we obtain an ill conditioned problem. Now we see that if we take a new initial guess, call it $\tilde{x_0} = .99$ the Newtons method converges in 92 iterations. We also note that for this initial guess we are driven to the left hand root of $\alpha = 0.0647$ due to the magnitude of the derivative at this point. We can see graphically in Figure 5 that the Newtons method takes a long time to gather itself after our initial guess, this is depicted in the downward jump. After many iterations it finally converges within our tolerance. To see if we can improve this method we once again target the true root of $\alpha = 0.0647$. Our initial guess we have chosen will be $\tilde{x_0} = 0$. The motivation behind this selecting was to make $f'(1) = (1-0)e^0 = 1$ which is much larger relative to the previous initial guess. As the theory suggests this initial point should converge faster than the initial guess of $x_0 = 0.99$. Figure 6 shows that this is the case as for an initial guess of $\tilde{x_0} = 0$ we converge to the root in 4 iterations.

4.1.3 Fixed point method

For this fixed point iteration we consider our$\phi(x)$ to be $\phi(x) = xe^{-x} = 0.06064$. Where $\phi(\alpha) = \alpha e^{-\alpha} = -0.06064 = \alpha$ yields the correct solution of the root. Note that $\phi(x)$ satisfies the conditions found in **Theorem 6.1** (bellow) there are also additional manners for solving for this $\phi(x)$ which can be seen in the correctness test. For our initial guess we work to target the right hand root where $\alpha = 4.25, f(\alpha) = 0$, to do this we consider an initial guess of $x_0 = 2$ (same as correctness test). Figure 7 shows that this method converges to $\alpha = 4.25$ in 231 iterations. Furthermore, we can look at the log plot in Figure 8 which shows the same convergence. This result is expected and further verifies that our method is correct. While this method took more iterations than the Newtons method we recognise that this method is use full since we can compute $\phi(x)$ using in class techniques and does not run into divide by zero errors as the Newtons method does. Note that we can not target the left hand root since it is not in our interval of convergence for this method.

4.2 Test 2: Single Root Polynomial

For the second test of the algorithms we consider our test function to be $g(x) = x^3 - x - 6$. We plot this graphically to gain intuition on how many roots we expect. As seen in Figure 9 there will be one root at $\alpha = 2$.

4.2.1 Bisection method

Assuming that we do not know these roots, we once again lean on intuition gained from **Property 6.1** in our text. With this information we are able to make an analysis argument to initialize our interval $[a, b]$. One can see that $g(1) = 1^3 - 1 - 6 = -6$ which would make a sufficient choice for a. Now looking for our b we perform similar analysis by computing $g(1) = 3^3 - 3 - 6 = 18$. With this we obtain $g(a)g(b) < 0$ and have a sufficient choice for our $[a, b]$. Now we perform the bisection method to obtain Figures 10 and 11, we can see that for this choice we converge to our desired root of $\alpha = 2$ in 20 iterations.

Extra 1: We now wish to demonstrate the convergence order referring to the convergence theorem in our book. We consider the same level of accuracy ($|x_k - x| < 10^{-6}$) on the interval $[-5, 10]$. Our method converges in 23 iterations which is larger than the projected iterations of 21 and can be seen graphically in Figure 12. This shows that our experiment is consistent with the theory from our book.

4.2.2 Newtons method

For the testing of Newtons method we first take an initial guess of $x_0 = \frac{1}{\sqrt{3}}$. This initial guess causes our code to yield an error and prompts the user to try an initial guess for x_0. This is due to the fact that our $g'(x) = 3x^2 - 1$ and $g'(\frac{1}{\sqrt{3}}) = 3\left(\frac{1}{\sqrt{3}}\right)^2 - 1 = 0$. Recall, that one of the requirements of the Newtons method is to have a non zero derivative. We can now test another initial guess $\tilde{x_0} = 0.57735$. While this is still a poor guess as our $g'(0.57735)$ is still very large, we are able to converge in 47 iterations on the interval $[-5, 10]$ this can be seen graphically in Figure 13. We once again note that the spike in error is due to the poor initial guess, after many iterations we are able to see the method converge. With this information we now wish to obtain a better initial guess of x_0, we will choose $x_0 = 3$. Using the fact that $g'(x) = 3(3)^2 - 1 = 26$ which is sufficiently larger than our initial two guesses. With this guess we perform our algorithm and get sufficiently close under the conditions $|x_k - x| < 10^-6$ in 6 iterations. Visually this can be seen in Figure 14.

Extra 2: To test quadratic convergence we first need the definition i.e:

$$\exists C > 0 : |x^{(k+1)} - \alpha| \leq C|x^{(k)} - \alpha|^p, \forall k_0 \leq k$$

So for this case our $p = 2$ and can be seen running the Newtons method for each of these cases. We can observe that we can pick a C which will satisfy this condition for all of the methods. The most explicit case of the quadratic convergence can be seen graphically in Figure 19 and 20.

4.2.3 Fixed point method

For testing the fixed point method where $g(x) = x^3 - x - 6$ we will consider the same initial guess as above of $x_0 = 3$, tolerance of 10^{-6} and derive our $\phi(x)$ using the following in class method:

$$\begin{aligned} 0 &= g(\alpha) = \alpha^3 - \alpha - 6 \\ 0 &= \alpha^3 - \alpha - 6 \\ \alpha &= (\alpha + 6)^{(\frac{1}{3})} \\ &\text{Or} \\ x &= (x + 6)^{(\frac{1}{3})} \end{aligned}$$

This is a sufficient $\phi(x)$ for our problem on the interval and our derivation ensures that $\phi(\alpha) = \alpha$ will give us a correct root. We then insert this $\phi(x)$ into our code and test it with the parameters $x, \alpha \in [-300, 300]$ and initial guess $300 = x_0$ in Table 5. We then proceed with further testing with the same parameters as above in the first two methods. We can see that we obtain the correct root of $\alpha = 2$ in 7 iterations. Graphical representations are depicted in Figure 14 and 15 respectively. This is very fast and efficient.

4.3 Correctness test

For the correctness test we consider the function $f(x) = xe^{-x} - 0.06064$ on the interval $x \in [1, 13]$ with a tolerance of 10^{-6} and initial guess for fixed point and Newtons method to be $x_0 = 2$. For the fixed point method we use $\phi(x) = xe^{-x} + x - 0.06064$.Note that $\phi(x)$ satisfies the conditions found in **Theorem 6.1** (bellow). The iteration steps for all three methods are shown in table 5. We can see that the Bisection method converged in 23 iterations which is larger than when we first tested. This makes sense since our interval was much tighter for testing. For the Newtons method we see convergence under our threshold in 9 iterations. Once again we compare this to our previous testing methods, since our initial guess for testing was closer to the actual root we would expect the faster convergence. Finally, for the fixed point method we obtain convergence in 231 iterations, this is expected since our initial guess was farther away and due to the nature of our selected $\phi(x)$. Graphically these correct results can be seen in Figures 17 through 22.

5 Conclusion and Comparison of Methods

Overall, this project shed light on the pros and cons with respect to root finding of two specific examples in our $f(x) = xe^{-x} - 0.06064$ and $g(x) = x^3 - x - 6$. We were able to apply three key root finding methods in: Bisection, Newtons, and Fixed point. We were able to see that the Bisection is a reliable method but takes many iterations to converge. A pro to the Bisection is that for any continuous function which has $f(a)f(b) < 0$, then there exists an $\alpha \in (a, b)$ such that $f(\alpha) = 0$ ensuring the finding of the root. We also do not need any additional information for Bisection method, such as a derivative or additional function. The Newtons method is a fast and efficient method when considering root finding problems. The pro to the Newtons method is that it is quick and reliable when we know a lot about the function in question. On that same note however, we need to ensure that our $f(x)$ is at least one times differentiable and that the root we are searching for is a simple root i.e $f'(\alpha) \neq 0$. An additional drawback to the Newtons method is that for quicker convergence we wish the derivative around α to be sufficiently large. As seen with $g(x) = x^3 - x - 6$

contains a steep slope around the root where as $f(x) = xe^{-x} - 0.06064$ is relatively flat, leading to a longer convergence. Also the Newtons method is the sufficiently close requirement for our initial guess, this can be difficult and costly with minimal information about a function. Finally, we considered the Fixed Point method in searching for roots of our continuous functions. A pro to this method is the certainty and simplicity of finding $\phi(x)$ and understanding that that when $\phi(x) = x$ implies that we have obtained our root we also have no dependence on our initial guess x_0 (**Theorem 6.1**). Furthermore, drawbacks do exist with choosing a close enough initial guess as well as understanding the interval of convergence. Overall, these methods have their pros and cons and are useful in gaining intuition behind root finding and its importance.

6 Tables, Figures and Theorems

6.1 Theorems and Properties

1. **Property 6.1:** Given a continuous function $f : [a, b] \to \mathbb{R}$ such that $f(a)f(b) < 0$, then there exists an $\alpha \in (a, b)$ such that $f(\alpha) = 0$.

2. **Theorem 6.1: Convergence of fixed-point iterations** Consider the sequence $x^{(k+1)} = \phi(x^{(k)})$ for $k \geq 0$, and a given $x^{(0)}$. Assume that:

   (a) $\phi : [a, b] \to [a, b]$

   (b) $\phi \in C^1([a, b])$

   (c) $\exists K < 1 : |\phi'(x)| \leq K \forall x \in [a, b]$

   Then, ϕ has a unique fixed point $\alpha \in [a, b]$ and the sequence $\{x^{(k)}\}$ converges to α for any choice of $x^{(0)} \in [a, b]$. Moreover,

$$\lim_{k \to \infty} \frac{x^{(k+1)} - \alpha}{x^{(k)} - \alpha} = \phi'(\alpha)$$

3. **Definition of order of Convergence:**

$$\exists C > 0 : |x^{(k+1)} - \alpha| \leq C|x^{(k)} - \alpha|^p, \forall k_0 \leq k$$

6.2 Tables

1. Table 1: Correctness results for $f(x) = -xe^{-x} - 0.06064$ on the interval $[1, 5]$

$f(x) = -xe^{-x} - 0.06064$	Iterations
Bisection method	21
Newtons method	5
Fixed point method	231

2. Table 2: Correctness results for $f(x) = -xe^{-x} - 0.06064$ on the interval $[-3, 3]$

$f(x) = -xe^{-x} - 0.06064$	Iterations
Bisection method	22
Newtons method	5
Fixed point method	231

3. Table 3: Correctness results for $g(x) = x^3 - x - 6$ on the interval $[-3, 3]$ initial guess $x_0 = 0$. This makes sense why newtons takes forever since it has to go through a change in derivative

$f(x) = x^3 - x - 6$	Iterations
Bisection method	21
Newtons method	32
Fixed point method	7

4. Table 4: Correctness results for $g(x) = x^3 - x - 6$ on the interval $[-3, 3]$ initial guess $x_0 = 3$. This makes sense why newtons takes forever since it has to go through a change in derivative

$f(x) = x^3 - x - 6$	Iterations
Bisection method	22
Newtons method	6
Fixed point method	7

5. Table 5: Correctness results for $g(x) = x^3 - x - 6$ on the interval $[-300, 300]$ initial guess $x_0 = 300$. This makes sense why newtons takes forever since it has to go through a change in derivative

$f(x) = x^3 - x - 6$	Iterations
Bisection method	29
Newtons method	17
Fixed point method	9

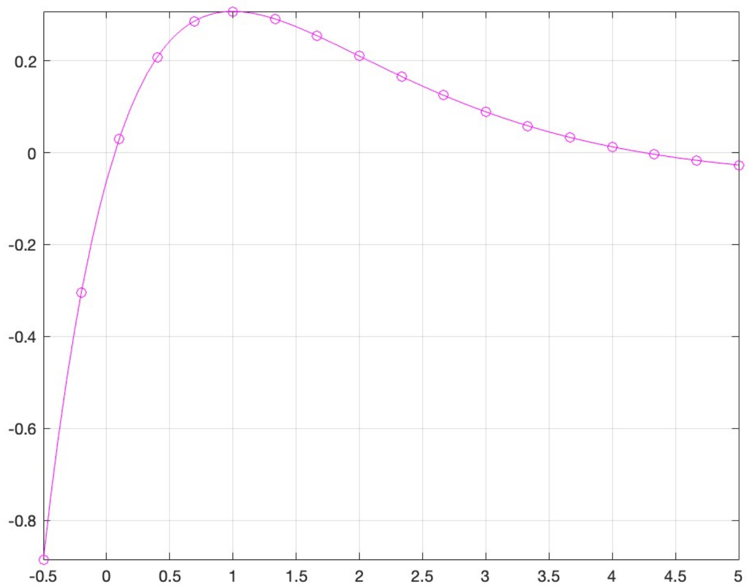
6. Table 6: Correctness test results on the interval $[1, 13]$, For Newtons initial guess $x_0 = 2$, For Fixed point method $\phi(x) = xe^{-x} + x - 0.06064$. Iterations are shown for their respective method

	Iterations
Bisection method	23
Newtons method	9
Fixed point method	231

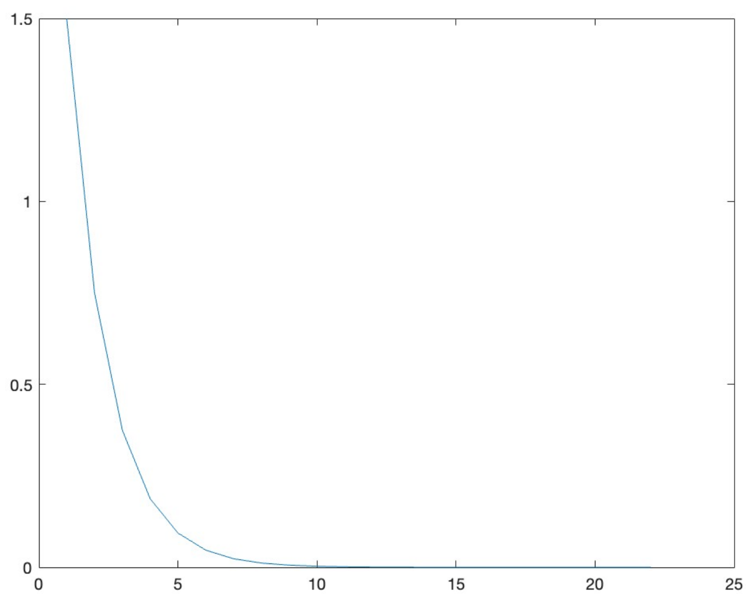
6.3 Figures and Correctness test

6.3.1 Test 1

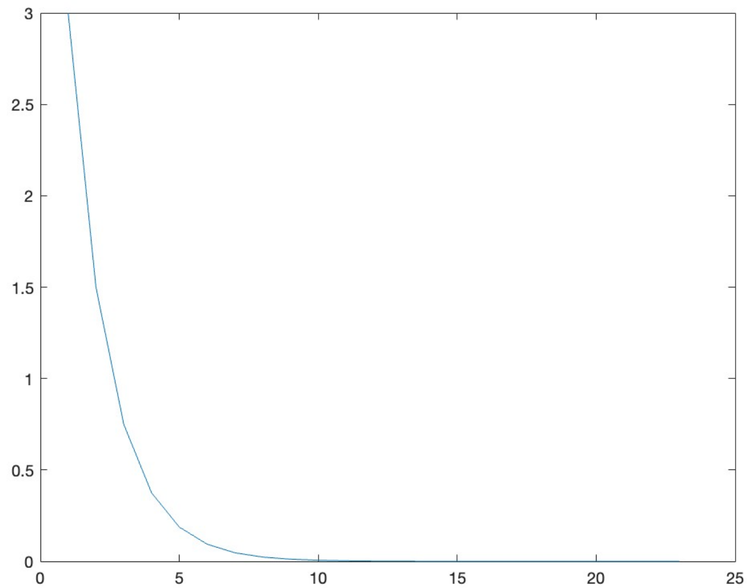
1. **Figure 1:** General graph of $f(x) = xe^{-x} - 0.06064$ on the interval $[-.5, 5]$ for visual clarity.



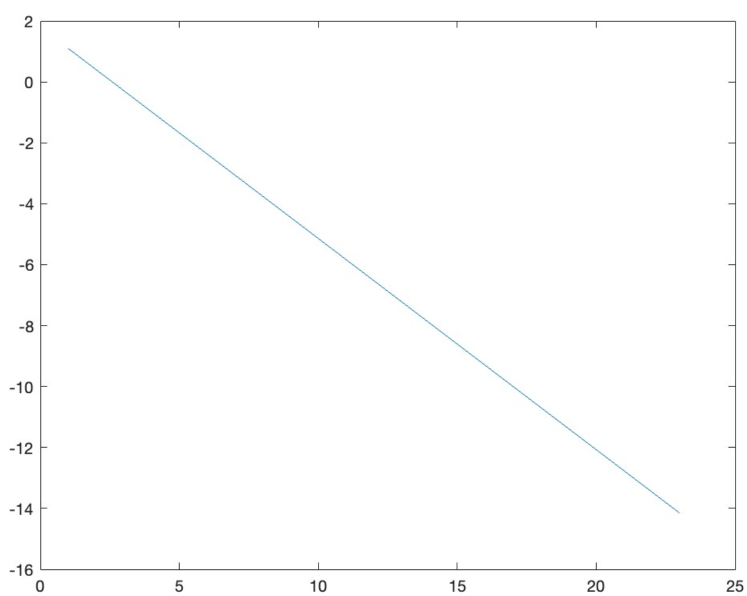
2. **Figure 2:** Bisection convergence plot of$f(x) = xe^{-x} - 0.06064$ $[-3, 3]$.



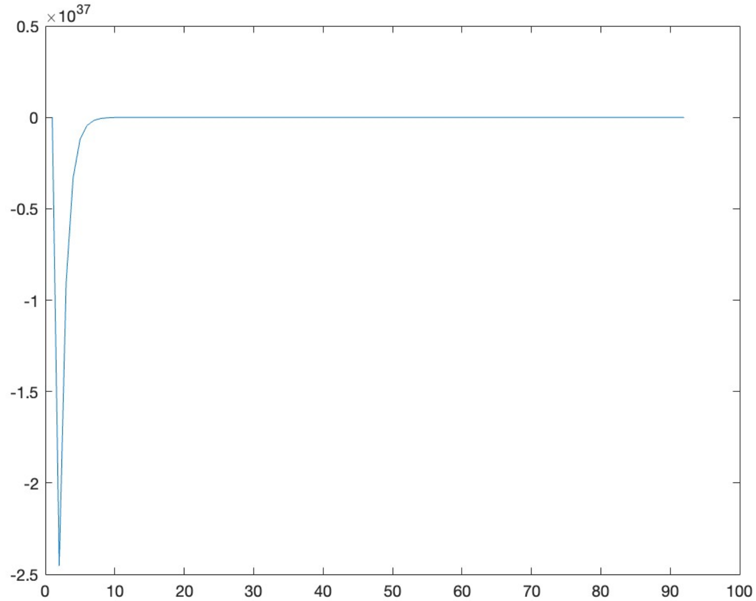
3. **Figure 3:** Figure shows the plot of all iterations for the Bisection method on the interval $[1, 13]$.



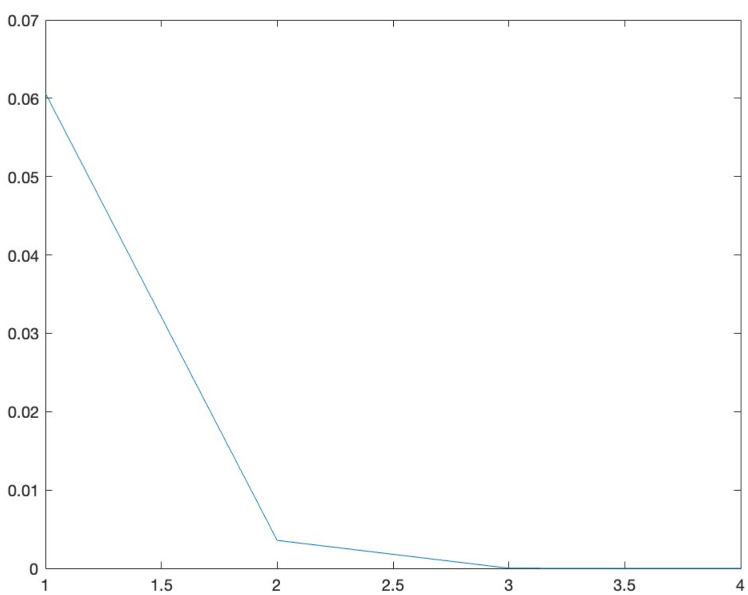
4. **Figure 4:** Figure shows the log plot of all iterations for the Bisection method on the interval $[1, 13]$.



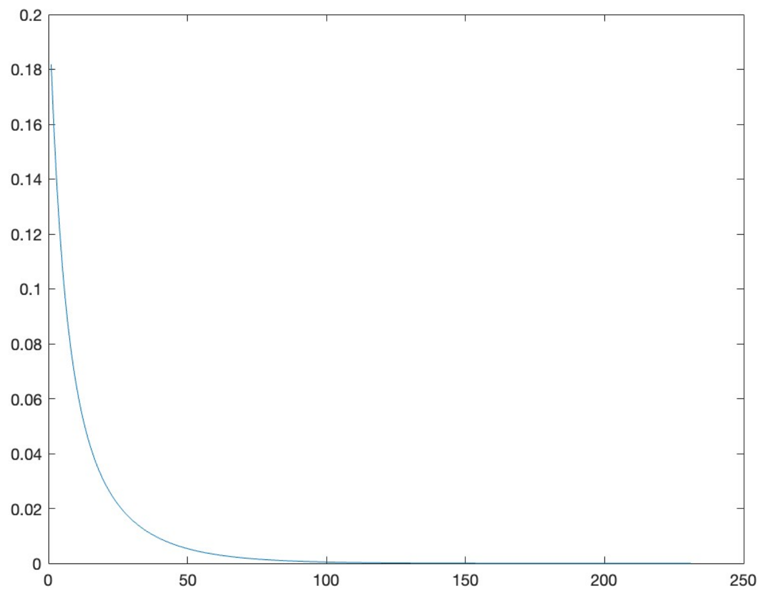
5. **Figure 5:** Figure shows the plot of newtons method with initial guess of $x_0 = 0.99$



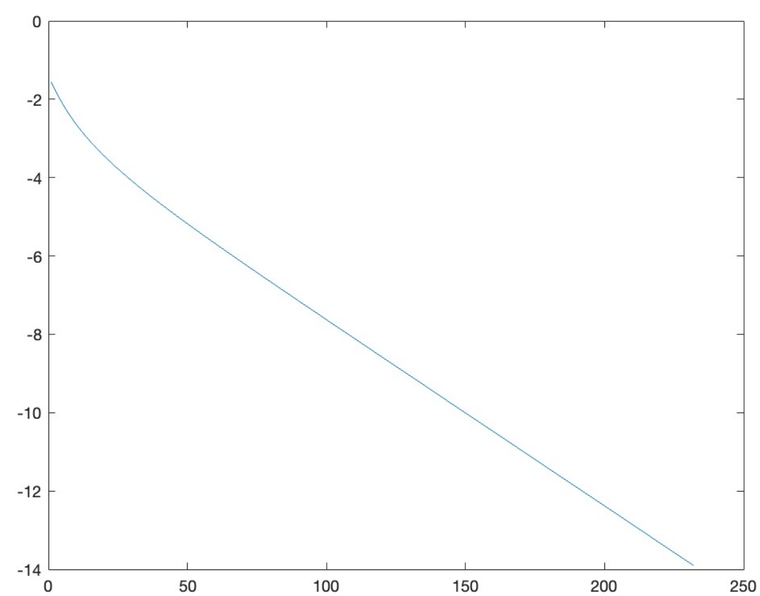
6. **Figure 6:** Figure shows the plot of Newtons method with initial guess of $x_0 = 0$



7. **Figure 7:** Figure shows the plot of all iterations for the fixed point method with initial guess $x_0 = 2$ and where $\phi(x) = xe^{-x} + x - 0.06064$, initial stop condition occurs when $|f(x_k) < 10^-6$. Iterates are on the x-axis and $|f(x_k)|$ is on the y-axis.

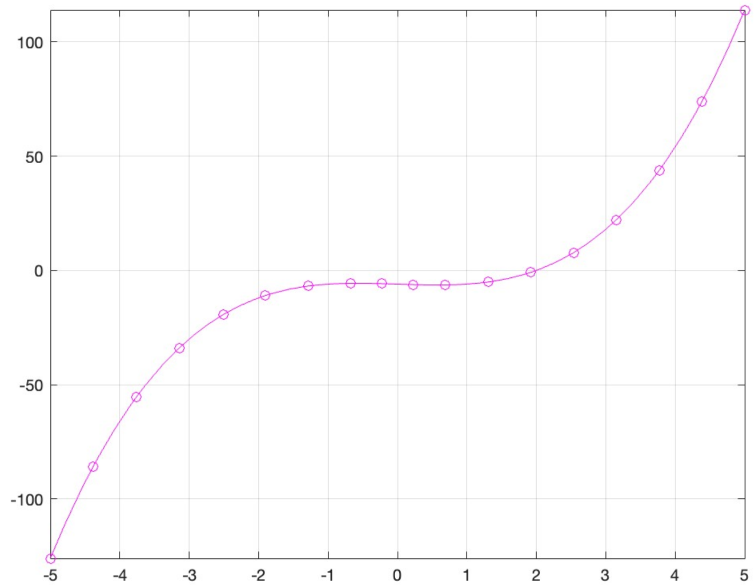


8. **Figure 8:** Figure shows the log plot of all iterations for the fixed point method with initial guess $x_0 = 2$ and where $\phi(x) = xe^{-x} + x - 0.06064$, initial stop condition occurs when $|f(x_k)| < 10^-6$. Iterates are on the x-axis and $ln(f(x_k))$ is on the y-axis

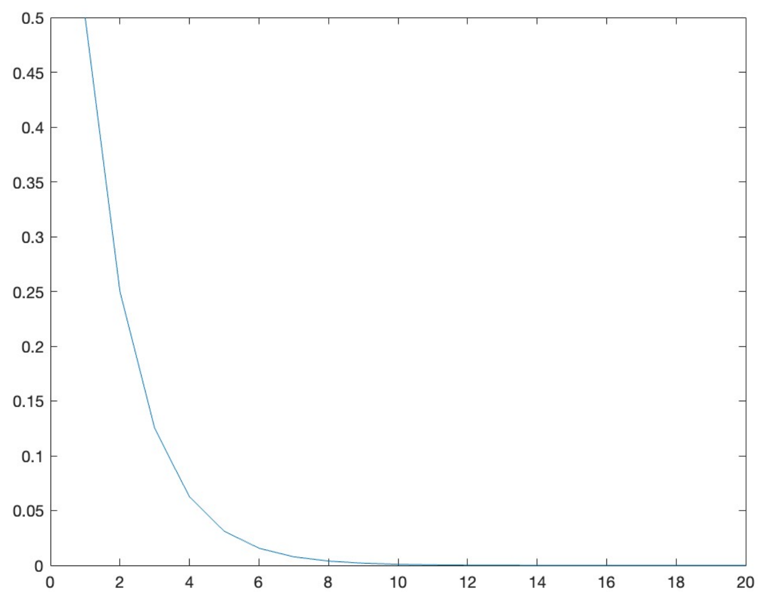


6.3.2 Test 2

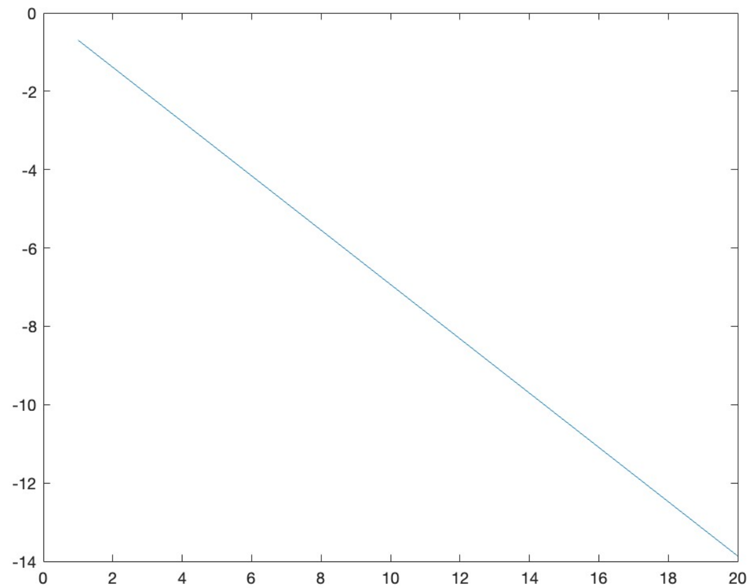
9. **Figure 9:** General graph of $g(x) = x^3 - x - 6$ on the interval $[-5, 5]$ for visual clarity.



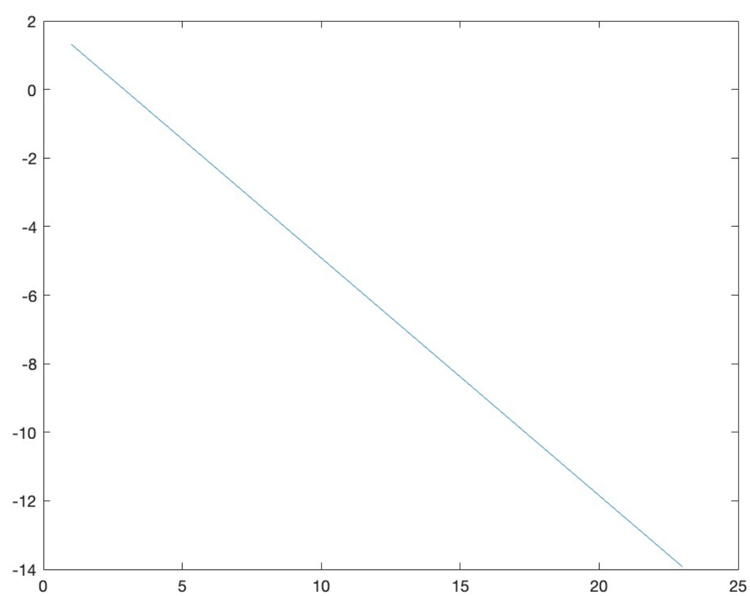
10. **Figure 10:** Bisection convergence plot of$g(x) = x^3 - x - 6$ $[1, 3]$.



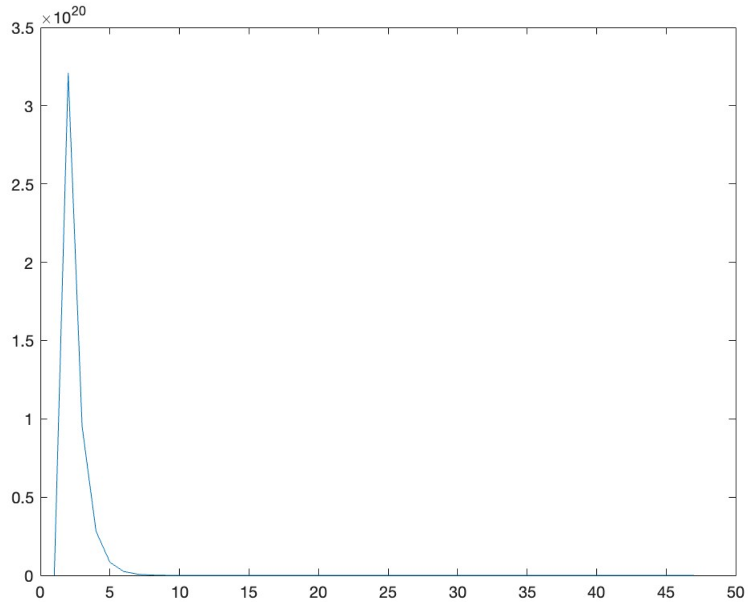
11. **Figure 11:** Bisection convergence log plot of $g(x) = x^3 - x - 6$ $[1, 3]$.



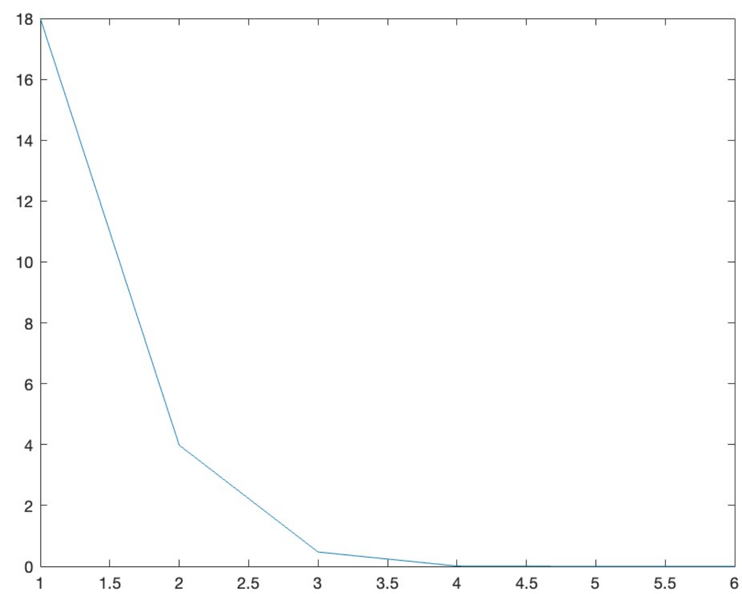
12. **Figure 12: Extra:** Bisection convergence log plot of $g(x) = x^3 - x - 6$ $[-5, 10]$.



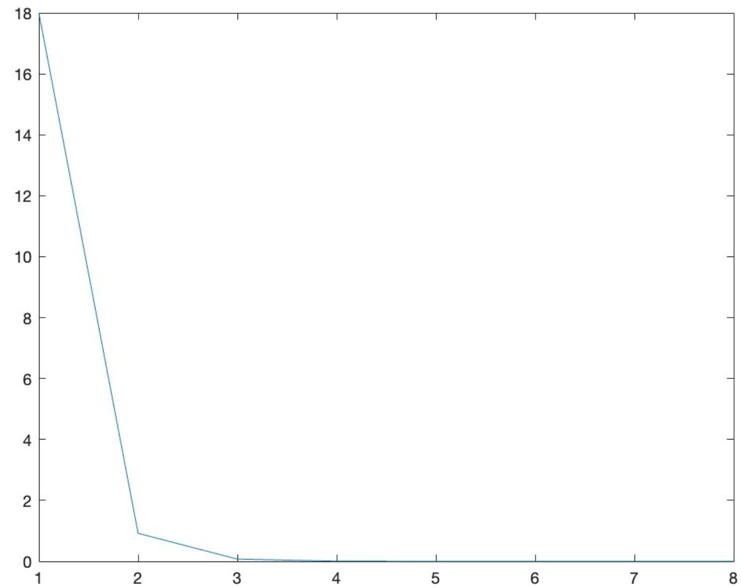
13. **Figure 13:** Newtons method convergence absolute value plot of $g(x) = x^3 - x - 6$ $[-5, 10]$ initial guess of $x_0 = 0.57735$.



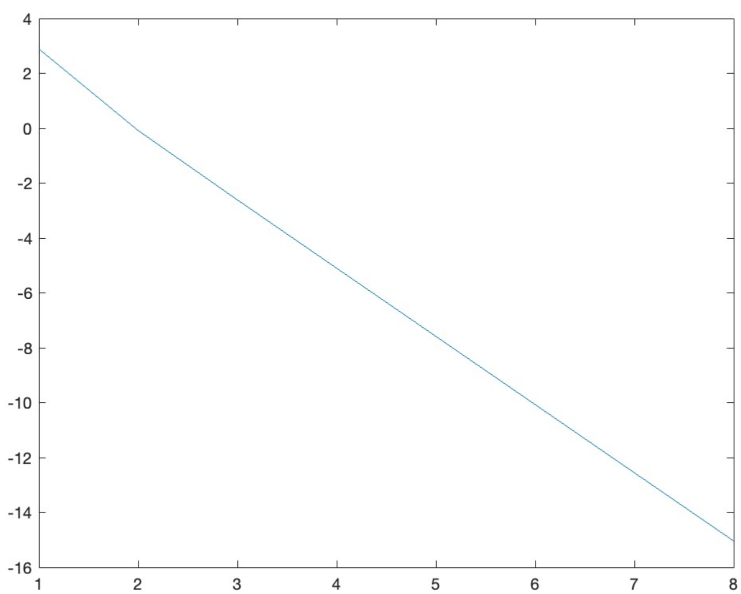
14. **Figure 14:** Newtons method convergence plot of $g(x) = x^3 - x - 6$ $[-5, 10]$ initial guess of $x_0 = 3$.



15. **Figure 15:** Fixed point method convergence plot of $g(x) = x^3 - x - 6$ $[-5, 10]$ initial guess of $x_0 = 3$.



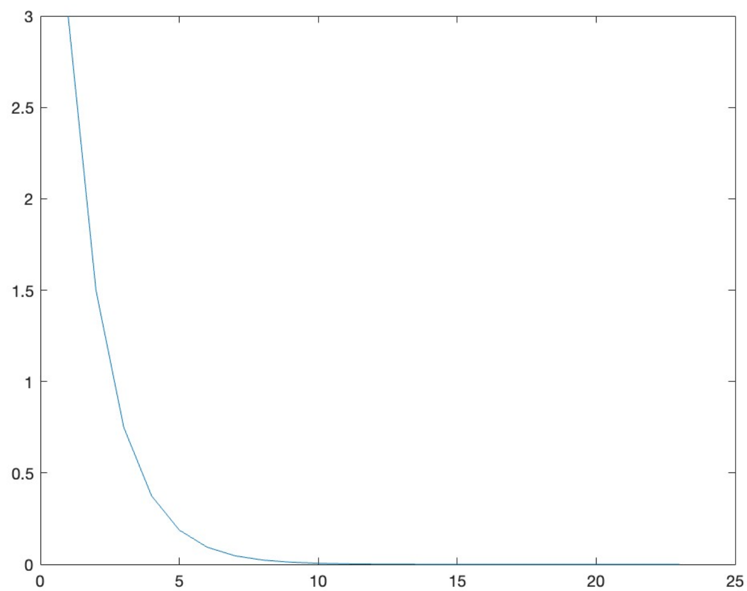
16. **Figure 16:** Fixed point method convergence log plot of $g(x) = x^3 - x - 6$ $[-5, 10]$ initial guess of $x_0 = 3$.



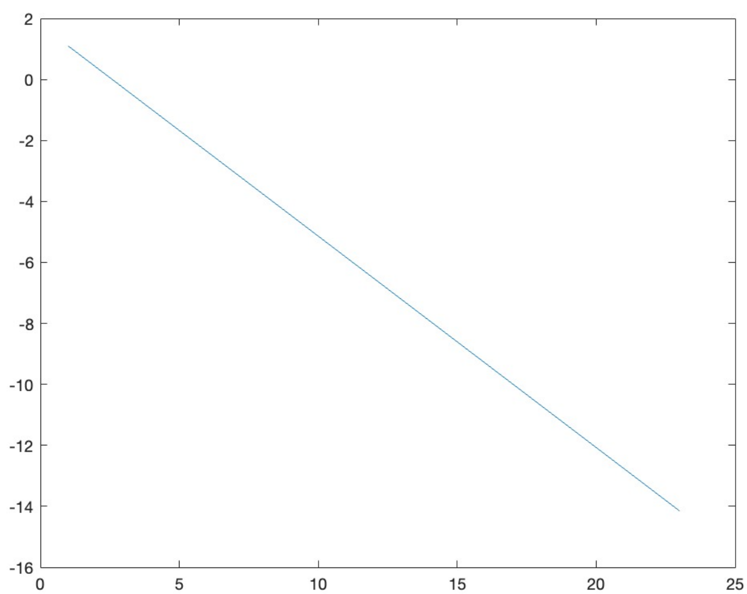
6.3.3 Correctness Test

Bisection Method

17. **Figure 17:** Figure shows the plot of all iterations for the Bisection method on the interval $[1, 13]$.

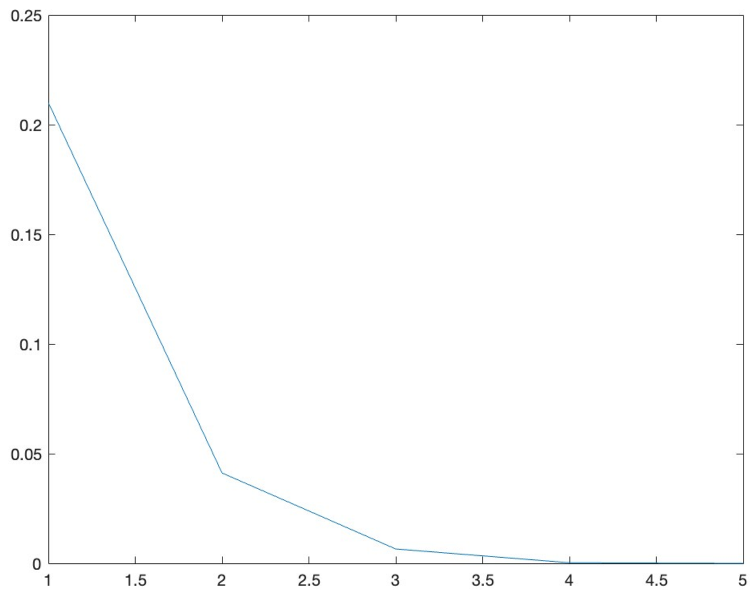


18. **Figure 18:** Figure shows the log plot of all iterations for the Bisection method on the interval $[1, 13]$.

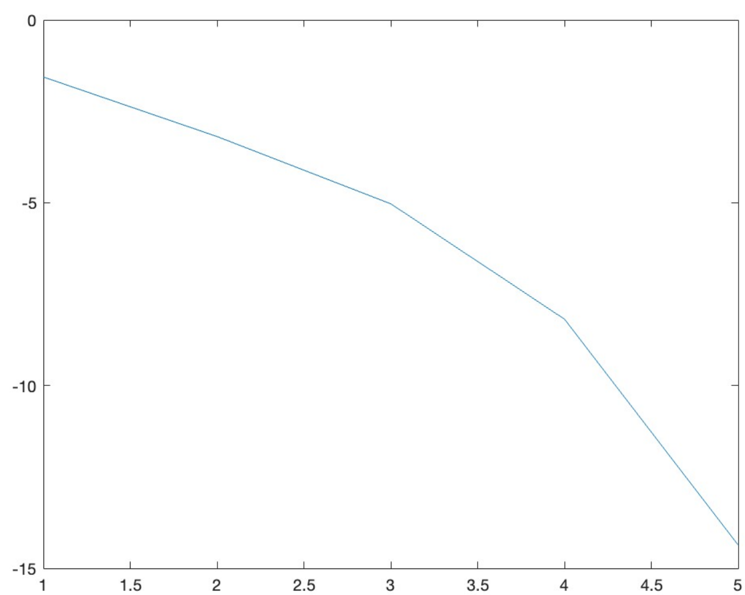


Newtons Method

19. **Figure 19:** Figure shows the plot of all iterations for the Newtons method on the interval $[1, 13]$ with initial guess $x_0 = 2$.

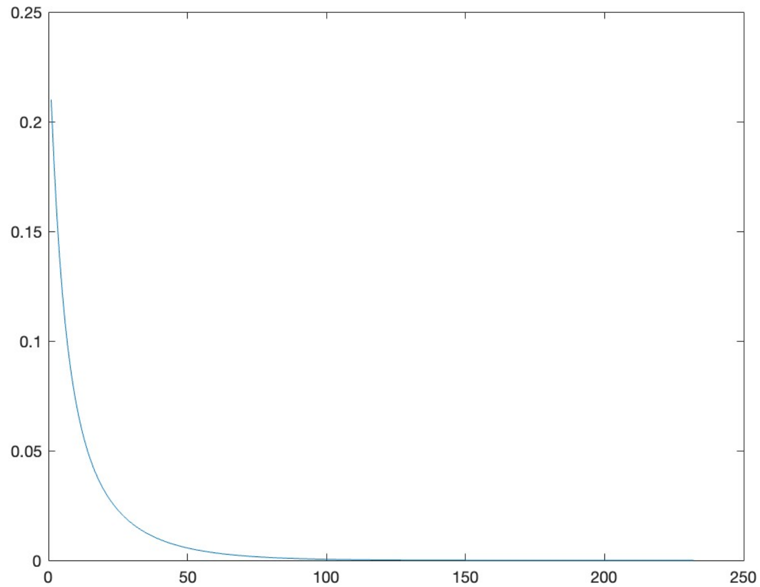


20. **Figure 20:** Figure shows the log plot of all iterations for the Newtons method on the interval $[1, 13]$ with initial guess $x_0 = 2$.



Fixed Point Method

21. **Figure 21:** Figure shows the plot of all iterations for the Fixed Point method on the interval $[1, 13]$ with initial guess $x_0 = 2$.



22. **Figure 22:** Figure shows the log plot of all iterations for the Fixed Point method on the interval $[1, 13]$ with initial guess $x_0 = 2$.