EML 6281 – Robot Geometry I — 13 December 2000
Final Exam — 2 hours, open book and notes

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1) (20 pts.) Explain how to solve the following two equations for corresponding solutions for the variables x_1 and x_2.

$$3x_1x_2^2 + 5x_1x_2 + 4x_1^2 + 8x_2 + 5 = 0$$

$$2x_1x_2 + 3x_1^2 + 6 = 0$$

How many solution sets will exist?

The equations can be regrouped as

$$x_1^2\,(4) + x_1\,(3x_2^2 + 5x_2) + (8x_2 + 5) = 0$$

$$x_1^2\,(3) + x_1\,(2x_2) + (6) = 0$$

These equations may be written as

$$L_1\,x_1^2 + M_1\,x_1 + N_1 = 0$$

$$L_2\,x_1^2 + M_2\,x_1 + N_2 = 0$$

where

$$L_1 = 4, \qquad M_1 = 3x_2^2 + 5x_2, \qquad N_1 = 8x_2 + 5$$

$$L_2 = 3, \qquad M_2 = 2x_2, \qquad N_2 = 6$$

The coefficients L_1 through N_2 must satisfy the following condition in order that there exist common roots to the two equations:

$$\begin{vmatrix} L_1 & M_1 \\ L_2 & M_2 \end{vmatrix}\begin{vmatrix} M_1 & N_1 \\ M_2 & N_2 \end{vmatrix} - \begin{vmatrix} L_1 & N_1 \\ L_2 & N_2 \end{vmatrix}^2 = 0$$

$$\begin{vmatrix} 4 & 3x_2^2 + 5x_2 \\ 3 & 2x_2 \end{vmatrix}\begin{vmatrix} 3x_2^2 + 5x_2 & 8x_2 + 5 \\ 2x_2 & 6 \end{vmatrix} - \begin{vmatrix} 4 & 8x_2 + 5 \\ 3 & 6 \end{vmatrix}^2 = 0$$

Expanding this equation gives

$$(8x_2 - 9x_2^2 - 15\,x_2)\,(18x_2^2 + 30x_2 - 16x_2^2 - 10x_2) - (24 - 24x_2 - 15)^2 = 0$$
$$(-9x_2^2 - 7x_2)\,(2x_2^2 + 20x_2) - (9 - 24x_2)^2 = 0$$
$$-18\,x_2^4 - 194\,x_2^3 - 590\,x_2^2 + 432\,x_2 - 81 = 0$$

The above equation must be satisfied in order to guarantee that a common root for x_1 can be obtained. Four values of x_2 can thus be determined.

Corresponding values for x_1 can be determined by evaluating the coefficients L_1 through N_2 for each value of x_2 and then evaluating x_1 from the equations

$$x_1 = \frac{-\begin{vmatrix} M_1 & N_1 \\ M_2 & N_2 \end{vmatrix}}{\begin{vmatrix} L_1 & N_1 \\ L_2 & N_2 \end{vmatrix}} \quad \text{or} \quad x_1 = \frac{-\begin{vmatrix} L_1 & N_1 \\ L_2 & N_2 \end{vmatrix}}{\begin{vmatrix} L_1 & M_1 \\ L_2 & M_2 \end{vmatrix}}$$

as described in Section 8.2.2 of the text.

2. (20 pts.) Body 1 is rotated 60° about the X axis of a fixed coordinate system. A second rotation of angle γ about an axis **m** was then performed. It was observed that the net motion of body 1 could have been accomplished by rotating 90° about an axis parallel to $3\mathbf{i} - 4\mathbf{k}$.

Determine the angle γ and the axis **m** of the second rotation.

The first and second rotations can be modeled by the quaternions q_1 and q_2 where

$$q_1 = \cos 30° + \sin 30° i$$

and q_2 is unknown. The net motion of the body is described by the quaternion $q_2 q_1$. This net rotation must equal the quaternion q_3 where

$$q_3 = \cos 45° + \sin 45°(0.6i - 0.8k)$$
$$q_3 = 0.7071 + 0.7071(0.6i - 0.8k)$$
$$q_3 = 0.7071 + 0.4243i - 0.5657k$$

The quaternion q_2 may be obtained as

$$q_2 = q_3\, q_1^{-1}$$

Since q_1 is a unit quaternion, its inverse will equal its conjugate and thus

$$q_2 = (0.7071 + 0.4243i - 0.5657k)\,(0.866 - 0.5i)$$
$$q_2 = 0.8245 + 0.0139i + 0.2829j - 0.4899k$$

Rearranging the vector part of q_2 gives

$$q_2 = 0.8245 + 0.5659(0.02451i + 0.5j - 0.8658k)$$

The angle γ can now be determined from

$$\cos\gamma = 0.8245$$
$$\sin\gamma = 0.5659$$
$$\gamma = 34.5°$$

The angle of rotation is equal to $2\gamma = 69°$

The axis of rotation is

$$\mathbf{m} = 0.02451\mathbf{i} + 0.5\mathbf{j} - 0.8658\mathbf{k}$$

3. (15 pts.) You have four equations of the form

$$a_i x^2y + b_i xy + c_i y + d_i x^2 + e_i x + f_i = 0 , \quad i = 1..4$$

Determine the condition that the coefficients a_i through f_i must satisfy in order that the four equations have common solutions for x and y.

The four equations can be written in a matrix format as

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 & f_4 \end{bmatrix} \begin{bmatrix} x^2y \\ xy \\ y \\ x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

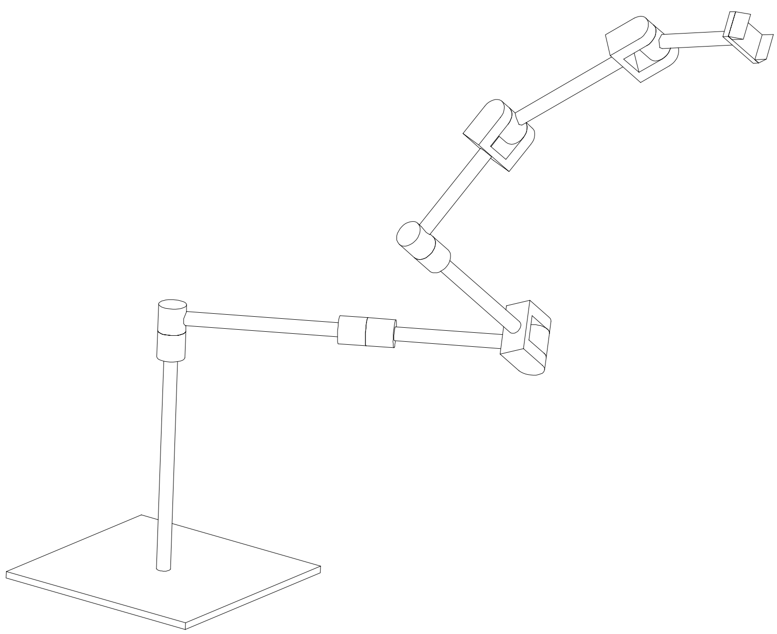
Multiplying the four equations by x will introduce two new terms, i.e. x^3 and x^3y. Writing the equations in matrix format gives

$$\begin{bmatrix} 0 & 0 & a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ 0 & 0 & a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ 0 & 0 & a_4 & b_4 & c_4 & d_4 & e_4 & f_4 \\ a_1 & d_1 & b_1 & c_1 & 0 & e_1 & f_1 & 0 \\ a_2 & d_2 & b_2 & c_2 & 0 & e_2 & f_2 & 0 \\ a_3 & d_3 & b_3 & c_3 & 0 & e_3 & f_3 & 0 \\ a_4 & d_4 & b_4 & c_4 & 0 & e_4 & f_4 & 0 \end{bmatrix} \begin{bmatrix} x^3y \\ x^3 \\ x^2y \\ xy \\ y \\ x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In order for there to be a solution for x and y that will simultaneously satisfy all four of the original equations, it is necessary for the eight equations listed above to be linearly dependent. Thus the coefficients a_i through f_i must satisfy the condition

$$\begin{vmatrix} 0 & 0 & a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ 0 & 0 & a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ 0 & 0 & a_4 & b_4 & c_4 & d_4 & e_4 & f_4 \\ a_1 & d_1 & b_1 & c_1 & 0 & e_1 & f_1 & 0 \\ a_2 & d_2 & b_2 & c_2 & 0 & e_2 & f_2 & 0 \\ a_3 & d_3 & b_3 & c_3 & 0 & e_3 & f_3 & 0 \\ a_4 & d_4 & b_4 & c_4 & 0 & e_4 & f_4 & 0 \end{vmatrix} = 0$$

4. (25 pts.)



A six revolute manipulator is shown in the above drawing. Lines that look perpendicular are perpendicular and lines that look parallel are parallel.

a) Draw joint axis vectors and link length vectors on the figure.

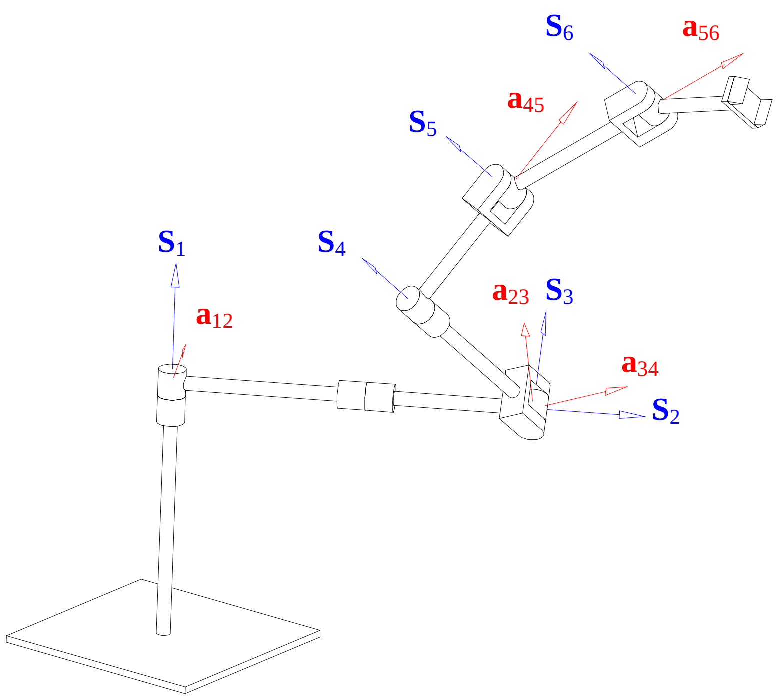
b) Make a table that lists the constant mechanism parameters for this manipulator based on your selection of joint and link vectors. Give numerical values for the parameters where they are known.

c) What free choices must be made in order to identify a ‘standard’ coordinate system on the last link of the robot?

d) Assume you are given the coordinates of a tool point measured in the standard coordinate system attached to the last link as well as the desired position of the tool point measured in the standard fixed coordinate system and the orientation of the vectors $\mathbf{S}_6$ and $\mathbf{a}_{67}$ measured in the fixed coordinate system. What parameters will be known after 'closing the loop'?

e) Perform the first step of the reverse analysis procedure by explaining how to solve for any one of the manipulator joint parameters.

a)



b) Mechanism parameters:

$a_{12} = 0$	$\alpha_{12} = 90°$	
$a_{23} = 0$	$\alpha_{23} = 90°$	S_2
$a_{34} = 0$	$\alpha_{34} = 90°$	$S_3 = 0$
a_{45}	$\alpha_{45} = 0°$	S_4
a_{56}	$\alpha_{56} = 0°$	$S_5 = 0$

Note that the offset distance S_6 is still a free choice.

c) The 6^{th} coordinate system will be defined by selecting the distance S_6 and the direction for the vector $\mathbf{a}_{67}$.

d) The parameters that will be determined upon closing the loop are

$$S_7,\ a_{71},\ S_1,\ \alpha_{71},\ \theta_7,\ \text{and}\ \gamma_1$$

e) The vector loop equation is written as

$$S_1\mathbf{S}_1 + S_2\mathbf{S}_2 + S_4\mathbf{S}_4 + a_{45}\mathbf{a}_{45} + a_{56}\mathbf{a}_{56} + S_6\mathbf{S}_6 + a_{67}\mathbf{a}_{67} + S_7\mathbf{S}_7 + a_{71}\mathbf{a}_{71} = \mathbf{0}$$

Projecting the vector loop equation onto $\mathbf{S}_6$ (which is parallel to $\mathbf{S}_4$ and $\mathbf{S}_5$) gives

$$S_1\ Z_7 + S_2\ Z_{71} + S_4 + S_6 + S_7\ c_{67} + a_{71}\ X_7 = 0$$

This equation contains θ_1 as its only unknown and can be factored in the format

$$A\ c_1 + B\ s_1 + D = 0$$

and can be solved to yield two values for the angle θ_1.

5. (10 pts.) You are given the following function

$$f(s, t) = (s + t^2)\ (s^2 + t)$$

Evaluate the function for the case where $s = 2 + 3\varepsilon$ and $t = 3 - 2\varepsilon$.

The function is expanded as

$$f(s,t) = s^3 + t^3 + s^2t + t^2s$$

The dual numbers can be inserted into the equation using Taylor's series expansion to give

$$f(2+3\varepsilon, 3-2\varepsilon) = f(2,3) + 3\varepsilon \left.\frac{\partial f}{\partial s}\right|_{\substack{s=2\\t=3}} - 2\varepsilon \left.\frac{\partial f}{\partial t}\right|_{\substack{s=2\\t=3}}$$

The partial derivatives of f with respect to s and t are

$$\frac{\partial f}{\partial s} = 3s^2 + 2st + t^2 \qquad \frac{\partial f}{\partial s} = 3t^2 + 2st + s^2$$

$$\left.\frac{\partial f}{\partial s}\right|_{\substack{s=2\\t=3}} = 33 \qquad \frac{\partial f}{\partial s} = 43$$

Thus

$$f(2+3\varepsilon,\ 3-2\varepsilon) = 65 + 13\varepsilon$$

6. (10 pts.) Insert dual angles into the following cosine law for a spherical pentagon

$$Z_{345} = c_{12}$$

The right side of the equation can be written as

$$\hat{c}_{12} = c_{12} - \varepsilon a_{12} s_{12}$$

The left side may be written as

$$\hat{Z}_{345} = Z_{345} + \varepsilon Z_{0345}$$

The term Z_{0345} may be written as

$$Z_{0345} = S_3 \frac{\partial Z_{345}}{\partial \theta_3} + S_4 \frac{\partial Z_{345}}{\partial \theta_4} + S_5 \frac{\partial Z_{345}}{\partial \theta_5} + a_{23} \frac{\partial Z_{345}}{\partial \alpha_{23}} + a_{34} \frac{\partial Z_{345}}{\partial \alpha_{34}} + a_{45} \frac{\partial Z_{345}}{\partial \alpha_{45}} + a_{51} \frac{\partial Z_{345}}{\partial \alpha_{51}}$$

$$Z_{0345} = S_3\, s_{23} X_{543} + S_4\,(-\bar{X}_5 X^*_{34} - \bar{Y}_5 X_{34}) + S_5\, s_{51} X_{345}$$

$$+\, a_{23}\, Y_{543} + a_{34}\,(-s_{23}c_3 Z_{54} + c_{23} Y_{54}) + a_{45}\,(-s_{51}c_5 Z_{34} + c_{51} Y_{34}) + a_{51}\, Y_{345}$$

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Final Exam – 11 December 2001

open book and notes
2 hours

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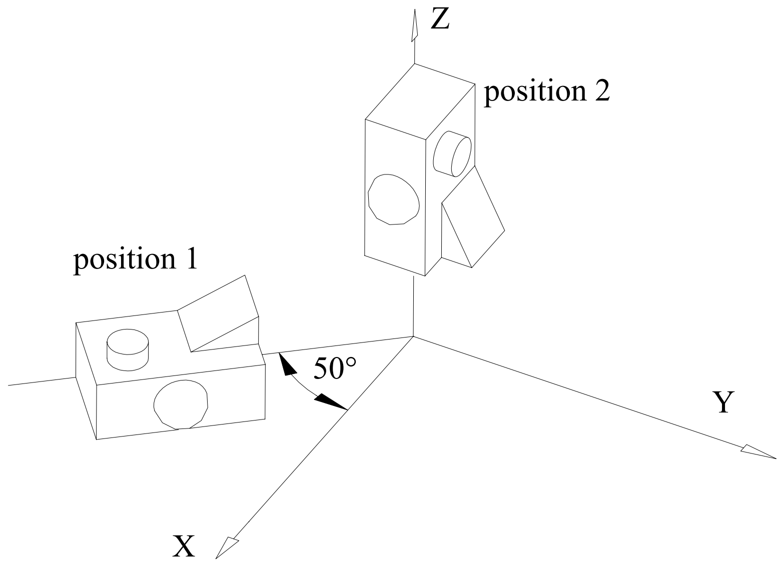
1. (20 pts.) Determine the axis and angle of rotation about which the object shown in the figure can be moved from position 1 to position 2 in one rotation.



The object can be moved from position 1 to position 2 by rotating an angle of 50° about the Z axis, followed by a rotation of -90° about the Y axis, and then followed by a rotation of -90° about the Z axis.

The quaternion associated with the first rotation can be written as

$$q_1 = \cos(25°) + \sin(25°)\, k\ .$$

The quaternion associated with the second rotation can be written as

$$q_2 = \cos(-45°) + \sin(-45°)\, j\ .$$

The quaternion associated with the third rotation can be written as

$$q_3 = \cos(-45°) + \sin(-45°)\, k\ .$$

The quaternion operator that will rotate all points in the object from position 1 to position 2 can be written as

$$q_3\ q_2\ q_1\ (\ )\ q_1^{-1}\ q_2^{-1}\ q_3^{-1}$$

The product $q_3q_2q_1$ is evaluated as

$$q_3\ q_2\ q_1 = (0.664463, -0.664463, -0.241845, -0.241845)$$

This product may be written as

$$q_3\ q_2\ q_1 = 0.664463 + 0.747321\ (-0.889126\ i - 0.323616\ j - 0.323616\ k)$$

This product represents a single rotation about the axis **m** where

$$\mathbf{m} = -0.889126\ \mathbf{i} - 0.323616\ \mathbf{j} - 0.323616\ \mathbf{k}$$

where half the angle of rotation, θ/2, is obtained from

$$\cos\theta/2 = 0.64463$$

$$\sin\theta/2 = 0.747321.$$

The angle of rotation is thus

$$\theta/2 = 48.3589°\ , \qquad \theta = 96.7177°$$

2. *(15 pts.)* *<u>Explain</u> how to solve for solutions for the variables x and y in the following two equations.*

$$23\,y^2 + 3\,x\,y^2 + 3\,x + 8\,y + 4\,y^3\,x + 4 = 0$$

$$4\,y^2 + x\,y^3 + 2 = 0$$

The two equations may be factored as

$$x\,[3\,y^2 + 4\,y^3 + 3] + [23\,y^2 + 8\,y + 4] = 0$$
$$x\,[y^3] + [4\,y^2 + 2] = 0$$

These equations are linear in the variable x and the equations must be linearly dependent if there is to be a common solution for x that solves both equations. The equations will be linearly dependent if

$$(3\,y^2 + 4\,y^3 + 3)\,(4\,y^2 + 2) - (y^3)\,(23\,y^2 + 8\,y + 4) = 0\;.$$

Expanding this equation yields

$$-7\,y^5 + 4\,y^4 + 4\,y^3 + 18\,y^2 + 6 = 0\;.$$

The values of y that satisfy this polynomial are[1]

-.6701874410-.9886366296*I

-.6701874410+.9886366296*I

.6383430811e-1-.5768000698*I

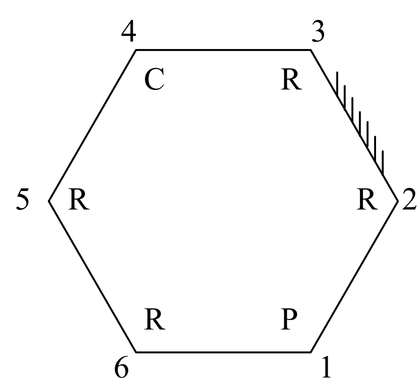
.6383430811e-1+.5768000698*I

1.784134837

The value for x that corresponds to the one real value of y can be obtained from either of the original two equations as

$$x = -2.594$$

3. *(30 pts.) The figure on the right is a planar representation of a spatial hexagon.*



(a) *What is the mobility of this mechanism?*

The sum of the freedoms of the joints is 7 and therefore the mobility is 1.

(b) *What group mechanism is it and why?*

It is a group 2 mechanism since its equivalent spherical mechanism has mobility 2.

(c) *The angle θ_3 is given along with all the constant mechanism parameters. Explain how to obtain values for any one of the other joint angles.*

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[1] (this step is not expected to be accomplished during the exam)

It is necessary to find two equations in two unknown joint angles. The first equation will be generated by inserting dual angles into the subsidiary cosine law

$$Z_{23} = Z_{65} .$$

The dual portion of $\hat{Z}_{23}$ may be written as

$$Z_{023} = S_2 \frac{\partial Z_{23}}{\partial \theta_2} + S_3 \frac{\partial Z_{23}}{\partial \theta_3} + a_{12} \frac{\partial Z_{23}}{\partial \alpha_{12}} + a_{23} \frac{\partial Z_{23}}{\partial \alpha_{23}} + a_{34} \frac{\partial Z_{23}}{\partial \alpha_{34}}$$

The definition of Z_{23} is written as

$$Z_{23} = s_{34}(X_2 s_3 + Y_2 c_3) + c_{34} Z_2$$

Expanding the partial derivatives gives

$$Z_{023} = S_2 s_{12} X_{32} + S_3 s_{34} X_{23} + a_{12} Y_{32} + a_{23}(-s_{34} c_3 Z_2 + c_{34} Y_2) + a_{34} Y_{23}$$

Similarly, the dual portion of $\hat{Z}_{65}$ can be expanded to yield

$$Z_{065} = S_6 s_{61} X_{56} + S_5 s_{45} X_{65} + a_{61} Y_{56} + a_{56}(-s_{45} c_5 Z_6 + c_{45} \overline{Y}_6) + a_{45} Y_{65}$$

The term Z_{023} contains the angle θ_2 as its only unknown. Thus it will be attempted to express all the terms in Z_{065} in terms of the known angles θ_3 and θ_1, the angle θ_2, and one more joint angle.

The following substitutions can be made

$$X_{56} = X_{321}$$

$$Y_{56} = -X^*_{321}$$

$$X_{65} = X_{234}$$

$$Y_{65} = -X^*_{234}$$

$$Z_6 = Z_{234}$$

All the terms in Z_{065} have been expressed in terms of θ_1, θ_2, θ_3, and θ_4 except for $\overline{Y}_6$.

The definition of $\overline{Y}_6$ is expanded as

$$\overline{Y}_6 = -(s_{56} c_{61} + c_{56} s_{61} c_6) .$$

This may be written as

$$\overline{Y}_6 = \frac{-s_{56}^2 c_{61} - s_{56} c_{56} s_{61} c_6}{s_{56}}$$

Adding $(c_{56}^2\, c_{61} - c_{56}^2\, c_{61})$ to the numerator and regrouping gives

$$\overline{Y}_6 = \frac{-c_{61}(s_{56}^2 + c_{56}^2) + c_{56}(c_{56} c_{61} - s_{56} s_{61} c_6)}{s_{56}} = \frac{-c_{61} + c_{56} Z_6}{s_{56}}$$

Substituting $Z_6 = Z_{234}$ yields

$$\overline{Y}_6 = \frac{-c_{61} + c_{56} Z_{234}}{s_{56}}$$

Thus all the terms in Z_{023} and Z_{065} have been expressed in terms of the known angles θ_3 and θ_1, and the unknown joint angles θ_2 and θ_4. This equation can be paired with the fundamental cosine law

$$Z_{4321} = c_{56} \ .$$

Both of the equations can be factored into the form

$$c_4 \, [A_i c_2 + B_i s_2 + D_i] + s_4 \, [E_i c_2 + F_i s_2 + G_i] + [H_i c_2 + I_i s_2 + J_i] = 0, \quad i = 1,2$$

and these two equations can be solved to yield an 8^{th} degree polynomial in the tan-half angle of θ_2 with unique corresponding values of θ_4 then obtained.

4. (20 pts.) The function f(x,y) is given as

$$f(x,y) = 3\,y^3 \sin(x) + 2\,x^2\,y \ .$$

Evaluate this function for the case where $x = 2 + 3\varepsilon$ *and* $y = -3 - 4\varepsilon$.

$$f(2+3\varepsilon, -3-4\varepsilon) = f(2,-3) + \varepsilon \left[3 \left.\frac{\partial f}{\partial x}\right|_{\substack{x=2 \\ y=-3}} - 4 \left.\frac{\partial f}{\partial y}\right|_{\substack{x=2 \\ y=-3}} \right]$$

$$f(2, -3) = 3\,(-3)^3 \sin(2) + 2\,(2)^2\,(-3) = -97.65$$

$$\frac{\partial f}{\partial x} = 3\,y^3 \cos(x) + 4\,x\,y$$

$$\frac{\partial f}{\partial y} = 9\,y^2 \sin(x) + 2\,x^2$$

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=2 \\ y=-3}} = 9.708$$

$$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=2 \\ y=-3}} = 81.653$$

$$f(2+3\varepsilon, -3-4\varepsilon) = -97.65 - 297.49\,\varepsilon$$

5. (15 pts.) You are given the following three functions:

$$J_5\ x^5 + J_4\ x^4 + J_3\ x^3 + J_2\ x^2 + J_1\ x + J_0 = 0$$

$$L_3\ x^3 + L_2\ x^2 + L_1\ x + L_0 = 0$$

$$M_4\ x^4 + M_3\ x^3 + M_2\ x^2 + M_1\ x + M_0 = 0$$

where the terms J_i and L_i are quadratic in the variable y and the terms M_i are linear in the variable y. <u>Explain</u> how to obtain a polynomial in the variable y that must be satisfied in order for common solutions to exist for the three equations. What degree is this polynomial?

Multiplying the 2nd equation by x and x^2 and the third equation by x will give six equations that can be written in matrix form as

$$\begin{bmatrix} J_5 & J_4 & J_3 & J_2 & J_1 & J_0 \\ 0 & 0 & L_3 & L_2 & L_1 & L_0 \\ 0 & M_4 & M_3 & M_2 & M_1 & M_0 \\ 0 & L_3 & L_2 & L_1 & L_0 & 0 \\ L_3 & L_2 & L_1 & L_0 & 0 & 0 \\ M_4 & M_3 & M_2 & M_1 & M_0 & 0 \end{bmatrix} \begin{bmatrix} x^5 \\ x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

A solution will exist only if the equations are linearly dependent. This occurs if

$$\begin{vmatrix} J_5 & J_4 & J_3 & J_2 & J_1 & J_0 \\ 0 & 0 & L_3 & L_2 & L_1 & L_0 \\ 0 & M_4 & M_3 & M_2 & M_1 & M_0 \\ 0 & L_3 & L_2 & L_1 & L_0 & 0 \\ L_3 & L_2 & L_1 & L_0 & 0 & 0 \\ M_4 & M_3 & M_2 & M_1 & M_0 & 0 \end{vmatrix} = 0 .$$

Expanding this determinant will yield a 10th degree polynomial in y.

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1. A rigid body is rotated an angle of 60° about an axis parallel to 3**i** – 2**j** that passes through the origin. A second rotation is then performed. A third rotation of 220° about an axis parallel to 2**j** – 3**k** that passes through the origin returns the body to its original position and orientation. Determine the angle and axis of the second rotation.

The quaternion operator for the first rotation can be written as $q_1()q_1^{-1}$ where

$$q_1 = \cos\left(\frac{60°}{2}\right) + \sin\left(\frac{60°}{2}\right)\frac{3i - 2j}{\sqrt{13}} \ ,$$
$$q_1 = 0.866 + 0.416\ I – 0.277\ j \ .$$

The quaternion operator for the third rotation can be written as $q_3()q_3^{-1}$ where

$$q_3 = \cos\left(\frac{220°}{2}\right) + \sin\left(\frac{220°}{2}\right)\frac{2j - 3k}{\sqrt{13}} \ ,$$
$$q_3 = -0.342 + 0.521\ j – 0.782\ k \ .$$

The quaternion operator that represents the three may be written as

$$q_3\ q_2\ q_1\ ()\ q_1^{-1}\ q_2^{-1}\ q_3^{-1} \ .$$

Since the three rotations result in the rigid body returning to its original position and orientation, it must be the case that

$$q_3\ q_2\ q_1 = 1 \ .$$

Post multiplying both sides by q_1^{-1} gives

$$q_3\ q_2 = q_1^{-1} \ .$$

Pre multiplying both sides by q_3^{-1} gives

$$q_2 = q_3^{-1}\ q_1^{-1} \ .$$

Since q_1 and q_3 are unit quaternions,

$$q_1^{-1} = K_{q1} \ ,$$
$$q_3^{-1} = K_{q3} \ .$$

Solving for q_2 gives

$$q_2 = -0.1516 – 0.0746\ i – 0.8716\ j + 0.4603\ k$$

Interpreting this unit quaternion as a rotation about an axis, it can be determined that

$$\cos\frac{\theta}{2} = -0.1516$$
$$\sin\frac{\theta}{2} = \pm\sqrt{0.0746^2 + 0.8716^2 + 0.4603^2} = 0.9884$$

Thus the angle and axis of rotation of the second rotation are

$$\theta = 197.44°$$
$$\mathbf{s} = -0.075\ \mathbf{i} - 0.8818\ \mathbf{j} + 0.4657\ \mathbf{k}$$

2. You have two equations in the variables L_1 and L_2 that are written as follows

$$A\, L_2^4 + B\, L_2^2 + C = 0\ ,$$

$$D\, L_2^5 + E\, L_2^4 + F\, L_2^3 + G\, L_2^2 + H\, L_2 + J = 0$$

where the terms A, B, and C are quadratic in L_1 and the terms D, E, F, G, H, and J are linear in L_1, i.e.

$$A = A_2\, L_1^2 + A_1\, L_1 + A_0\ ,$$
$$B = B_2\, L_1^2 + B_1\, L_1 + B_0\ ,$$
$$C = C_2\, L_1^2 + C_1\, L_1 + C_0\ ,$$

$$D = D_1\, L_1 + D_0\ ,$$
$$E = E_1\, L_1 + E_0\ ,$$
$$F = F_1\, L_1 + F_0\ ,$$
$$G = G_1\, L_1 + G_0\ ,$$
$$H = H_1\, L_1 + H_0$$

where the coefficients A_2 through H_0 are given constants.

a) Determine the condition that the coefficients A through J must satisfy in order for there to be common solutions for L_2 for the two original equations.

The two given equations can be written in matrix form as

$$\begin{bmatrix} 0 & A & 0 & B & 0 & C \\ D & E & F & G & H & J \end{bmatrix} \begin{bmatrix} L_2^5 \\ L_2^4 \\ L_2^3 \\ L_2^2 \\ L_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

Multiplying the first equation by L_2, L_2^2, L_2^3, and L_2^4 and the second equation by L_2, L_2^2, L_2^3 to yield a total of nine equations that can be written in matrix form as

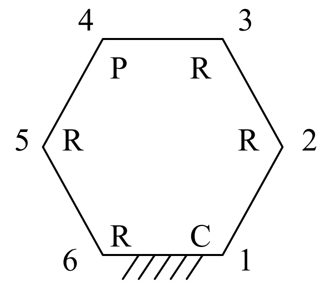
$$\begin{bmatrix} 0 & 0 & 0 & D & E & F & G & H & J \\ 0 & 0 & 0 & 0 & A & 0 & B & 0 & C \\ 0 & 0 & 0 & A & 0 & B & 0 & C & 0 \\ 0 & 0 & D & E & F & G & H & J & 0 \\ 0 & 0 & A & 0 & B & 0 & C & 0 & 0 \\ 0 & D & E & F & G & H & J & 0 & 0 \\ 0 & A & 0 & B & 0 & C & 0 & 0 & 0 \\ D & E & F & G & H & J & 0 & 0 & 0 \\ A & 0 & B & 0 & C & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} L_2^8 \\ L_2^7 \\ L_2^6 \\ L_2^5 \\ L_2^4 \\ L_2^3 \\ L_2^2 \\ L_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{22}$$

A solution to this set of equations can only occur if

$$\begin{vmatrix} 0 & 0 & 0 & D & E & F & G & H & J \\ 0 & 0 & 0 & 0 & A & 0 & B & 0 & C \\ 0 & 0 & 0 & A & 0 & B & 0 & C & 0 \\ 0 & 0 & D & E & F & G & H & J & 0 \\ 0 & 0 & A & 0 & B & 0 & C & 0 & 0 \\ 0 & D & E & F & G & H & J & 0 & 0 \\ 0 & A & 0 & B & 0 & C & 0 & 0 & 0 \\ D & E & F & G & H & J & 0 & 0 & 0 \\ A & 0 & B & 0 & C & 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

b) What is the degree of the polynomial in the variable L_1 that will result from the condition written in part (a)?

Five rows of the determinant contain A, B, and C. Four rows contain D through J. The degree of the resulting polynomial in L_1 will be equal to $2(5) + 4 = 14$.



3. The figure shows a planar representation of an RRPRRC spatial mechanism where link 61 is fixed to ground.

(a) What group mechanism is this and why?

The mobility of the spatial mechanism is 1 and the mobility of the equivalent spherical mechanism is 2. Therefore it is a group 2 mechanism.

(b) List the constant mechanism parameters.

a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{61}
α_{12}, α_{23}, α_{34}, α_{45}, α_{56}, α_{61}
S_2, S_3, S_5, S_6
θ_4

(c) The angle θ_6 is given. You wish to determine an equation that contains the constant link lengths and joint offsets together with the unknown joint angles. Substitute dual angles into an appropriate cosine law such that the resulting equation will not contain the unknown joint offsets S_1 or S_4. Expand your partial derivatives.

The desired equation can be obtained by substituting dual angles into the subsidiary cosine law $Z_{56} = Z_{23}$ to yield

$$Z_{056} = Z_{023}\ .$$

The term Z_{056} can be written as

$$Z_{056} = S_5 \frac{\partial Z_{56}}{\partial \theta_5} + S_6 \frac{\partial Z_{56}}{\partial \theta_5} + a_{45} \frac{\partial Z_{56}}{\partial \alpha_{45}} + a_{56} \frac{\partial Z_{56}}{\partial \alpha_{56}} + a_{61} \frac{\partial Z_{56}}{\partial \alpha_{61}}\ .$$

The definition of Z_{56} is written as

$$Z_{56} = s_{61}(X_5 s_6 + Y_5 c_6) + c_{61} Z_5$$

where

$$X_5 = s_{45}\, s_5,$$
$$Y_5 = -(s_{56}\, c_{45} + c_{56}\, s_{45}\, c_5)\ ,$$
$$Z_5 = c_{56}\, c_{45} - s_{56}\, s_{45}\, c_5\ .$$

Expanding the partial derivatives gives

$$Z_{056} = S_5\, s_{45}\, X_{65} + S_6\, s_{61}\, X_{56} + a_{45}\, Y_{65} + a_{56}\, (-s_{61}\, Z_5\, c_6 + c_{61}\, Y_5) + a_{61}\, Y_{56}\ .$$

The definition of Z_{23} is written as

$$Z_{23} = s_{34}(X_2s_3 + Y_2c_3) + c_{34}Z_2$$

where

$$X_2 = s_{12}\, s_2,$$
$$Y_2 = -(s_{23}\, c_{12} + c_{23}\, s_{12}\, c_2)\ ,$$
$$Z_5 = c_{23}\, c_{12} - s_{23}\, s_{12}\, c_2\ .$$

Expanding the partial derivatives for Z_{023} gives

$$Z_{023} = S_2\, s_{12}\, X_{32} + S_3\, s_{34}\, X_{23} + a_{12}\, Y_{32} + a_{23}\, (-s_{34}\, Z_2\, c_3 + c_{34}\, Y_2) + a_{34}\, Y_{23}\ .$$

The expansion of Z_{056} and Z_{023} completes the exam problem. However, to complete the solution of the mechanism would require that the equation $Z_{056} = Z_{023}$ contain only two unknown joint angles. The term Z_{056} contains θ_5 and the term Z_{023} contains θ_2 and θ_3. Thus there are currently three extra joint angles. Efforts will be made to eliminate θ_2. The following subsidiary equations can be substituted into Z_{023}:

$$X_{23} = X_{654}\ ,$$
$$Y_{23} = -\ X^*_{654}\ ,$$
$$Z_2 = Z_{654}\ ,$$
$$Y_2 = -X^*_{6543}\ .$$

There are two terms that must yet be converted, i.e. X_{32} and Y_{32}. Y_{32} can be converted to Z_{32} and this then replaced by Z_{56}. X_{32} is more difficult and will not be expanded further here.

4. Express the determinant of matrix **M** as the sum of products of only 2×2 determinants:

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & H_1 & 0 & I_1 & J_1 \\ 0 & 0 & H_2 & 0 & I_2 & J_2 \\ 0 & E_1 & F_1 & G_1 & 0 & 0 \\ 0 & E_2 & F_2 & G_2 & 0 & 0 \\ A_1 & B_1 & 0 & 0 & C_1 & 0 \\ A_2 & B_2 & 0 & 0 & C_2 & 0 \end{bmatrix}.$$

The expansion of the determinant will first be done using Laplace's theorem as follows:

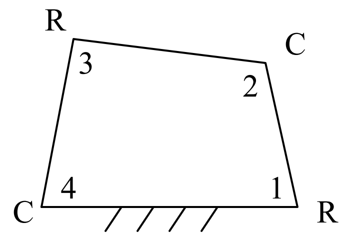
$$|\mathbf{M}| = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \begin{vmatrix} H_1 & 0 & I_1 & J_1 \\ H_2 & 0 & I_2 & J_2 \\ F_1 & G_1 & 0 & 0 \\ F_2 & G_2 & 0 & 0 \end{vmatrix} - \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} \begin{vmatrix} 0 & H_1 & 0 & J_1 \\ 0 & H_2 & 0 & J_2 \\ E_1 & F_1 & G_1 & 0 \\ E_2 & F_2 & G_2 & 0 \end{vmatrix}.$$

The two 4×4 matrices can then be expanded as products of 2×2 matrices to yield

$$|\mathbf{M}| = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \left(\begin{vmatrix} F_1 & G_1 \\ F_2 & G_2 \end{vmatrix} \begin{vmatrix} I_1 & J_1 \\ I_2 & J_2 \end{vmatrix} \right) - \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} \left(- \begin{vmatrix} E_1 & G_1 \\ E_2 & G_2 \end{vmatrix} \begin{vmatrix} H_1 & J_1 \\ H_2 & J_2 \end{vmatrix} \right) ,$$

$$|\mathbf{M}| = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \begin{vmatrix} F_1 & G_1 \\ F_2 & G_2 \end{vmatrix} \begin{vmatrix} I_1 & J_1 \\ I_2 & J_2 \end{vmatrix} + \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} \begin{vmatrix} E_1 & G_1 \\ E_2 & G_2 \end{vmatrix} \begin{vmatrix} H_1 & J_1 \\ H_2 & J_2 \end{vmatrix} .$$

1. A planar representation of a spatial CRCR closed-loop chain is shown in the figure.



(a) Determine the mobility of the device and of the equivalent spherical device.

The mobility of the closed-loop device is equal to the sum of the freedoms of the joints minus six. Thus

$$M = 6 - 6 = 0$$

The mobility of the equivalent spherical mechanism is equal to the sum of the freedom of the joints minus 3. Thus

$$M_{equiv} = 4 - 3 = 1$$

(b) List the constant mechanism parameters.

a_{12}, $a_{23,}$ a_{34}, a_{41}

α_{12}, α_{23}, α_{34}, α_{41}

S_1, S_3

(c) <u>Explain</u> how the remaining variable parameters can be determined.

The remaining parameters to be determined are listed as follows:

θ_1, θ_2, θ_3, θ_4, S_2, S_4

Any cosine law that is written for the equivalent spherical mechanism will contain two unknown joint angles. Therefore, the approach to solve this problem will be to obtain another equation from the spatial structure that contains two unknown joint angles. This will be done by inserting dual angles into the following cosine law:

$$Z_1 = Z_3 \ .$$

Equation the dual parts of the result will yield

$$Z_{01} = Z_{03}$$

where

$$Z_{01} = S_1 \frac{\partial Z_1}{\partial \theta_1} + a_{41} \frac{\partial Z_1}{\partial \alpha_{41}} + a_{12} \frac{\partial Z_1}{\partial \alpha_{12}} \ ,$$

$$Z_{03} = S_3 \frac{\partial Z_1}{\partial \theta_3} + a_{34} \frac{\partial Z_1}{\partial \alpha_{34}} + a_{23} \frac{\partial Z_1}{\partial \alpha_{23}} \ .$$

Expanding the partial derivatives yields

$$Z_{01} = S_1 \, s_{12} \, X_1 + a_{41} \, \overline{Y}_1 + a_{12} \, Y_1 \ ,$$

$$Z_{03} = S_3\, s_{34}\, X_3 + a_{23}\, \overline{Y}_3 + a_{34}\, Y_3\,.$$

Writing the equation $Z_{01} - Z_{03} = 0$ yields

$$S_1\, s_{12}\, X_1 + a_{41}\, \overline{Y}_1 + a_{12}\, Y_1 - S_3\, s_{34}\, X_3 - a_{23}\, \overline{Y}_3 - a_{34}\, Y_3 = 0$$

This equation contains θ_1 and θ_3 as its only unknowns. It can be combined with the following spherical cosine law which also contains these two joint angles:

$$Z_3 - Z_1 = 0$$

In general, the solution of these two equations would yield 8 possible solution sets for (θ_1, θ_3) that would satisfy both equations. However, for this problem, only four solutions will exist (this is beyond the scope of the exam).

Once sets of values for (θ_1,θ_3) are determined, corresponding values for θ_2 can be obtained from the fundamental sine and sine-cosine laws:

$$X_{32} = s_{41}\, s_1\,,$$

$$Y_{32} = s_{41}\, c_1\,.$$

Expanding these equations will yield two scalar equations in the two unknowns, s_2 and c_2 from which a unique corresponding value for θ_2 can be obtained.

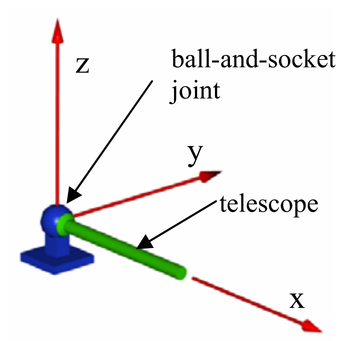
Next, the following fundamental sine and sine-cosine laws can be used to obtain corresponding values for θ_4:

$$X_{23} = s_{41}\, s_4\,,$$

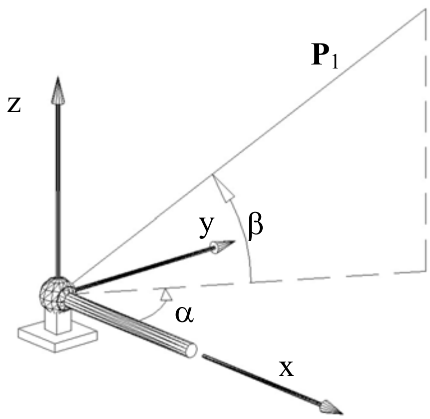
$$Y_{23} = s_{41}\, c_4\,.$$

Lastly, corresponding values for the joint offsets S_4 and S_2 can be obtained by writing the vector loop equation and projecting it onto two directions. This will yield two scalar equations in the two unknown joint offsets.

2. *A telescope is initially pointed along the X axis as shown in the accompanying figure. You are given the coordinates of a point,* $\boldsymbol{P}_1=[x_1,y_1,z_1]^T$. *Determine the coordinates of a quaternion that will rotate the telescope so it is pointing at the point* $\boldsymbol{P}_1$.



The solution to this problem is not unique since the telescope only has to point at $\mathbf{P}_1$. While pointed at $\mathbf{P}_1$, the telescope could have any rotation angle about its own axis. One solution approach will be presented.



The telescope can be pointed at $\mathbf{P}_1$ if it is rotated by an angle α about the Z axis followed by a rotation of $-\beta$ about the <u>modified</u> Y axis.

The first step of the solution will be to determine values for the angles α and β in terms of the given coordinates of $\mathbf{P}_1$. The angle α can be obtained from

$$\cos\alpha = \frac{x_1}{\sqrt{x_1^2 + y_1^2}}$$

$$\sin\alpha = \frac{y_1}{\sqrt{x_1^2 + y_1^2}}$$

The angle β can be obtained from

$$\cos\beta = \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_1^2 + y_1^2 + z_1^2}} \qquad \sin\beta = \frac{z_1^2}{\sqrt{x_1^2 + y_1^2 + z_1^2}}$$

The quaternion q_1 that models the first rotation can be written as

$$q_1 = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}k\ .$$

The quaternion q_2 that models the second rotation can be written as

$$q_2 = \cos\frac{-\beta}{2} + \sin\frac{-\beta}{2}(-\sin\alpha\, i + \cos\alpha\, j)\ .$$

The net quaternion that will rotate the telescope to point at $\mathbf{P}_1$ can now be written as

$$q = q_2\, q_1$$

$$\begin{aligned} q = {} & \cos\frac{\beta}{2}\cos\frac{\alpha}{2} \\ & + i\left(\sin\frac{\beta}{2}\cos\frac{\alpha}{2}\sin\alpha - \sin\frac{\beta}{2}\sin\frac{\alpha}{2}\cos\alpha\right) \\ & + j\left(-\sin\frac{\beta}{2}\cos\frac{\alpha}{2}\cos\alpha - \sin\frac{\beta}{2}\sin\frac{\alpha}{2}\sin\alpha\right) \\ & + k\left(\cos\frac{\beta}{2}\sin\frac{\alpha}{2}\right) \end{aligned}$$

3. *You are given the following two equations in the variables x and y. Explain how to determine sets of values for x and y that simultaneously satisfy both equations assuming that the coefficients a through g are known.*

$$a\,y^3 + b\,y^2\,x^2 + c\,x = 0$$

$$d\,x^3\,y^2 + e\,y^2\,x + f\,x\,y + g = 0$$

Writing the two equations in matrix format yields

$$\begin{bmatrix} a & b x^2 & 0 & c x \\ 0 & d x^3 + e x & f x & g \end{bmatrix} \begin{bmatrix} y^3 \\ y^2 \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Multiplying the first equation by y and the second by y and y^2 gives three additional equations. The total of five equations can be written in matrix form as

$$\begin{bmatrix} 0 & a & b x^2 & 0 & c x \\ a & b x^2 & 0 & c x & 0 \\ 0 & 0 & d x^3 + e x & f x & g \\ 0 & d x^3 + e x & f x & g & 0 \\ d x^3 + e x & f x & g & 0 & 0 \end{bmatrix} \begin{bmatrix} y^4 \\ y^3 \\ y^2 \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The condition that these five homogeneous equations have solutions for y is that they be linearly dependent. Thus it is necessary that

$$\begin{vmatrix} 0 & a & b x^2 & 0 & c x \\ a & b x^2 & 0 & c x & 0 \\ 0 & 0 & d x^3 + e x & f x & g \\ 0 & d x^3 + e x & f x & g & 0 \\ d x^3 + e x & f x & g & 0 & 0 \end{vmatrix} = 0$$

Expansion of this determinant will yield a polynomial in the variable x. The solutions of this polynomial will yield the possible values of x.

Corresponding values of y can be determined by substituting each solution for x into the original two equations to yield two polynomials in the variable y. The corresponding value for y can be found as the value that is a solution to both of the new polynomials.

4. *Evaluate the following function when $x = 4 + 6\varepsilon$ and $y = -2$*

$$f(x, y) = 2\,x^4\,y^3 + 3\,x^2\,y^2 + 20\,x + 10\,y + 4$$

The function can be evaluated by inserting the dual numbers into the Taylor series expansion of the function. The Taylor series expansion for a function with two variables can be written as

$$f(a+\varepsilon a_0, b+\varepsilon b_0) = f(a,b) + \varepsilon a_0 \left.\frac{\partial f}{\partial x}\right|_{\substack{x=a\\y=b}} + \varepsilon b_0 \left.\frac{\partial f}{\partial y}\right|_{\substack{x=a\\y=b}} .$$

Thus

$$f(4+6\varepsilon,-2) = f(4,-2) + 6\varepsilon \left.\frac{\partial f}{\partial x}\right|_{\substack{x=4\\y=-2}}$$

$$f(4, -2) = -3840$$

$$\frac{\partial f}{\partial x} = 8\,x^3\,y^3 + 6\,x\,y^2 + 20$$

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=4\\y=-2}} = -3980 .$$

Thus

$$f(4+6\varepsilon, -2) = -3840 + 6\varepsilon\,(-3980) = -3840 - 23{,}880\,\varepsilon$$

5. *True or False*

(a) *When performing the reverse analysis for a serial robot manipulator, it is always possible to determine the hypothetical closure link even if the desired position of the tool point is outside the reachable workspace of the manipulator.*

True. Determination of the close-the-loop parameters does not depend on the manipulator's mechanism parameters or the desired position and orientation of the end effector.

(b) *A quaternion is equal to its inverse times its norm.*

False.

(c) *It is not possible to design a five-link group three closed-loop mechanism that is comprised only of some combination of prismatic, revolute, and cylindric joints.*

True. A group 3 mechanism has an equivalent spherical mechanism that has three degrees of freedom. A minimum of six joints are required for a spherical mechanism to have three degrees of freedom.

(d) *It is not possible to design a spatial closed-loop mechanism that is comprised only of some combination of prismatic, revolute, and cylindric joints whose mobility is greater than that of its equivalent spherical mechanism.*

False.

(e) *Industrial robots are built with limitations on the range of joint motion so that only one configuration can occur that will position and orient the end effector as desired.*

False.

(f) *A body has been rotated about an axis that passes through the origin of the reference coordinate system. The quaternion that models this rotation is unique.*

True. Although the rotation could be interpreted as either a rotation of θ about some axis **s** or by a rotation of $-\theta$ about $-$**s**, the values of the quaternion, $q = d + ai + bj + ck$, will be the same.

1. (25 pts.) A coordinate system is attached to the end effector of a serial robot manipulator which will be referred to as the 6th coordinate system. The 6th coordinate system is initially coincident with the fixed coordinate system. It is translated to the point $[5, 6, 9]^T$ and then rotated by the quaternion [0.8090, -0.3919, -0.1959, 0.3919] where the axis of rotation passes through the point $[5, 6, 9]^T$. Determine the transformation matrix ${}^F_6\mathbf{T}$.

The problem will be solved by first identifying the angle and axis of rotation that is represented by the unit quaternion. Any unit quaternion can be interpreted as

$$[d, a, b, c] = [\cos\frac{\theta}{2}, s_x \sin\frac{\theta}{2}, s_y \sin\frac{\theta}{2}, s_z \sin\frac{\theta}{2}]$$

where θ is the angle of rotation and $[s_x, s_y, s_z]^T$ is a unit vector along the axis of rotation. Thus

$$\cos\frac{\theta}{2} = 0.8090$$

$$s_x \sin\frac{\theta}{2} = -0.3919$$

$$s_y \sin\frac{\theta}{2} = -0.1959$$

$$s_z \sin\frac{\theta}{2} = 0.3919$$

Solving for θ, s_x, s_y, and s_z (and choosing the value of θ between 0 and π) gives

$\theta = 72.0^o$

$s_x = -0.6667$ $s_y = -0.3333$ $s_z = 0.6667$

Equation 2.61 of the text can be used to determine the rotation matrix component of ${}^F_6\mathbf{T}$ as

$${}^F_6\mathbf{R} = \begin{bmatrix} s_x{}^2(1-\cos\theta)+\cos\theta & s_x s_y(1-\cos\theta)-s_z\sin\theta & s_x s_z(1-\cos\theta)+s_y\sin\theta \\ s_x s_y(1-\cos\theta)+s_z\sin\theta & s_y{}^2(1-\cos\theta)+\cos\theta & s_y s_z(1-\cos\theta)-s_x\sin\theta \\ s_x s_z(1-\cos\theta)-s_y\sin\theta & s_y s_z(1-\cos\theta)+s_x\sin\theta & s_z{}^2(1-\cos\theta)+\cos\theta \end{bmatrix},$$

$${}^F_6\mathbf{R} = \begin{bmatrix} 0.6161 & -0.4805 & -0.6241 \\ 0.7876 & 0.3858 & 0.4805 \\ 0.010 & -0.7876 & 0.6161 \end{bmatrix}.$$

Since the coordinates of the origin of the 6th coordinate system are known in the fixed system, the 4×4 transformation matrix ${}^F_6\mathbf{T}$ can be written as

$$ {}^{F}_{6}\mathbf{T} = \begin{bmatrix} 0.6161 & -0.4805 & -0.6241 & 5 \\ 0.7876 & 0.3858 & 0.4805 & 6 \\ 0.010 & -0.7876 & 0.6161 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

2. (20 pts) You are given the following two equations in the unknown variables x_1 and x_2.

$$ A\, x_1 + B\, x_1^2\, x_2 + C\, x_1\, x_2^2 + D\, x_1\, x_2 = 0 $$

$$ E\, x_1^4\, x_2^3 + F\, x_1\, x_2^2 + G\, x_1\, x_2 + H\, x_2 + J = 0 $$

Explain how to determine all solution sets (x_1, x_2) that satisfy both equations.

The original equations can be regrouped as

$$ (C\, x_1)\, x_2^2 + (B\, x_1^2 + D\, x_1)\, x_2 + (A\, x_1) = 0 $$

$$ (E\, x_1^3)\, x_2^3 + (F\, x_1)\, x_2^2 + (G\, x_1 + H)\, x_2 + J = 0\ . $$

These equations can be written in matrix form as

$$ \begin{bmatrix} 0 & C\,x_1 & B\,x_1^2 + D\,x_1 & A\,x_1 \\ E\,x_1^3 & F\,x_1 & G\,x_1 + H & J \end{bmatrix} \begin{bmatrix} x_2^3 \\ x_2^2 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. $$

Multiplying the first equation by x_2 and x_2^2 and the second equation by x_2 will give a total of five equations that can be written in the following matrix format:

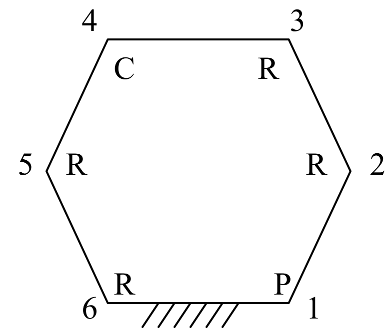
$$ \begin{bmatrix} 0 & 0 & C\,x_1 & B\,x_1^2 + D\,x_1 & A\,x_1 \\ 0 & C\,x_1 & B\,x_1^2 + D\,x_1 & A\,x_1 & 0 \\ C\,x_1 & B\,x_1^2 + D\,x_1 & A\,x_1 & 0 & 0 \\ 0 & E\,x_1^3 & F\,x_1 & G\,x_1 + H & J \\ E\,x_1^3 & F\,x_1 & G\,x_1 + H & J & 0 \end{bmatrix} \begin{bmatrix} x_2^4 \\ x_2^3 \\ x_2^2 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. $$

A solution to these equations will only exist if they are linearly dependent. This condition will occur if the determinant of the coefficient matrix is equal to zero as

$$\begin{vmatrix} 0 & 0 & C x_1 & B x_1^2 + D x_1 & A x_1 \\ 0 & C x_1 & B x_1^2 + D x_1 & A x_1 & 0 \\ C x_1 & B x_1^2 + D x_1 & A x_1 & 0 & 0 \\ 0 & E x_1^3 & F x_1 & G x_1 + H & J \\ E x_1^3 & F x_1 & G x_1 + H & J & 0 \end{vmatrix} = 0 \ .$$

Expansion of this determinant will yield a 12th degree polynomial in the variable x_1. Corresponding values for x_2 can be found for each value of x_1 by solving the first equation for two values of x_2 and then substituting each of these into the second equation to see which one will satisfy the second equation.

3. (30 pts) A planar representation of a spatial group 2 mechanism is shown in the accompanying figure.



Assuming that all the constant mechanism parameters are known together with the angle θ_6 you wish to determine all the unknown variable parameters.

(a) Explain how to obtain two equations that can be factored into the form

$$(A_i\, c_j + B_i\, s_j + D_i)\, c_k + (E_i\, c_j + F_i\, s_j + G_i)\, s_k + (H_i\, c_j + I_i\, s_j + J_i) = 0, \quad i = 1,2$$

where s_j, c_j, s_k, c_k represent the sines and cosines of θ_j and θ_k which are two of the variable joint parameters and the coefficients A_i through J_i are expressed in terms of given quantities.

You do not have to factor the equations into the format. Also you may have one term that is difficult to express in terms of θ_j and θ_k. Just identify this term.

The first equation will be obtained by substituting dual angles into the following spherical cosine law:

$$Z_{56} = Z_{23} \ .$$

This equation will be expanded to determine which two unknown angles that it will contain. Substituting dual angles into Z_{56} will give

$$Z_{056} = S_5 \frac{\partial Z_{56}}{\partial \theta_5} + S_6 \frac{\partial Z_{56}}{\partial \theta_6} + a_{45} \frac{\partial Z_{56}}{\partial \alpha_{45}} + a_{56} \frac{\partial Z_{56}}{\partial \alpha_{56}} + a_{61} \frac{\partial Z_{56}}{\partial \alpha_{61}} \ .$$

Evaluating the partial derivatives gives

$$Z_{056} = S_5\, s_{45} X_{65} + S_6\, s_{61} X_{56} + a_{45}\, Y_{65} + a_{56}\, (c_{61} Y_5 - s_{61} c_6 Z_5) + a_{61}\, Y_{56} \ .$$

Substituting dual angles into Z_{23} will give

$$Z_{023} = S_2 \frac{\partial Z_{23}}{\partial \theta_2} + S_3 \frac{\partial Z_{23}}{\partial \theta_3} + a_{12} \frac{\partial Z_{23}}{\partial \alpha_{12}} + a_{23} \frac{\partial Z_{23}}{\partial \alpha_{23}} + a_{34} \frac{\partial Z_{23}}{\partial \alpha_{34}}$$

and evaluating the partial derivatives gives

$$Z_{023} = S_2\, s_{12}X_{32} + S_3\, s_{34}X_{23} + a_{12}\, Y_{32} + a_{23}\, (c_{34}Y_2 - s_{34}c_3Z_2) + a_{34}\, Y_{23}\ .$$

Equating Z_{056} and Z_{023} yields

$$\begin{aligned}&S_5\, s_{45}X_{65} + S_6\, s_{61}X_{56} + a_{45}\, Y_{65} + a_{56}\, (c_{61}Y_5 - s_{61}c_6Z_5) + a_{61}\, Y_{56}\\ &= S_2\, s_{12}X_{32} + S_3\, s_{34}X_{23} + a_{12}\, Y_{32} + a_{23}\, (c_{34}Y_2 - s_{34}c_3Z_2) + a_{34}\, Y_{23}\ .\end{aligned}$$

This equation contains the unknowns θ_5, θ_2, and θ_3. Subsidiary spherical laws will be used to substitute for the terms containing θ_2 and θ_3 to give

$$\begin{aligned}&S_5\, s_{45}X_{65} + S_6\, s_{61}X_{56} + a_{45}\, Y_{65} + a_{56}\, (c_{61}Y_5 - s_{61}c_6Z_5) + a_{61}\, Y_{56}\\ &= S_2\, s_{12}X_{561} + S_3\, s_{34}X_{654} - a_{12}\, X^*_{561} + a_{23}\, (c_{34}Y_2 - s_{34}c_3Z_2) - a_{34}\, X^*_{654}\ .\end{aligned}$$

All the terms, except for the a_{23} term, have been expressed in terms of θ_4 and θ_5. After considerable manipulation, this term can also be expressed in terms of the same unknown joint angles (see equation (8.222) through equation (8.324) in text).

The second equation that contains the two unknown joint parameters θ_4 and θ_5 is the following fundamental cosine law:

$$Z_{4561} = c_{23}\ .$$

(b) Assuming that you used Bezout's method to solve the two equations for corresponding sets of angles (θ_j, θ_k), explain how you would obtain corresponding values for the remaining unknown variable parameters.

Corresponding values for θ_2 can be obtained from the following fundamental sine and sine-cosine laws:

$$X_{4561} = s_{23}\, s_2\ ,$$
$$Y_{4561} = s_{23}\, c_2\ .$$

Corresponding values for θ_3 can be obtained from the following fundamental sine and sine-cosine laws:

$$X_{1654} = s_{23}\, s_3\ ,$$
$$Y_{1654} = s_{23}\, c_3\ .$$

Corresponding values for the joint offsets S_1 and S_4 can be obtained by projecting the vector loop equation twice to obtain two scalar equations in the two unknowns.

4. (15 pts.) Evaluate the function

$$f(x,y) = e^x [\cos(3xy) + 5y]$$

when $x = 3+4\varepsilon$ and $y = 2 - 5\varepsilon$.

The function can be evaluated as follows:

$$f(3+4\varepsilon, 2-5\varepsilon) = f(3,2) + 4\varepsilon \left.\frac{\partial f}{\partial x}\right|_{\substack{x=3\\y=2}} - 5\varepsilon \left.\frac{\partial f}{\partial y}\right|_{\substack{x=3\\y=2}}$$

The function evaluated at x=3, y=2 is calculated as

$$f(3,2) = 214.12 .$$

The partial derivatives are evaluated as

$$\frac{\partial f}{\partial x} = e^x[\cos(3xy) + 5y] - 3\,y\,e^x \sin(3xy) ,$$

$$\frac{\partial f}{\partial y} = e^x [-3\,x\sin(3xy) + 5] .$$

Evaluating the partial derivatives at x=3, y=2 gives

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=3\\y=2}} = 304.62 ,$$

$$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=3\\y=2}} = 236.18 .$$

The function can be now be evaluated as

$$f(3+4\varepsilon, 2-5\varepsilon) = 214.12 + \varepsilon[4(304.62) - 5(236.18)] ,$$

$$f(3+4\varepsilon, 2-5\varepsilon) = 214.12 + 37.57\,\varepsilon$$

5. (10 pts) True or False

(a) Typical industrial manipulators can be classified as group 1 mechanisms due to their simplified geometry.

False. Typical industrial manipulators have six revolute joints so they would be classified as group 4.

(b) The reverse analysis of the PUMA robot will always result in either eight possible configurations or zero possible configurations for a given desired position and orientation of the end effector.

False. It is possible to specify a desired position and orientation of the PUMA manipulator that will yield only four configurations.

(c) All unit quaternions can be interpreted as representing a rigid body rotation of some angle about some axis that passes through the origin of the reference coordinate system.

True.

(d) For a six revolute axis manipulator, the reverse analysis position problem solution will fail whenever a solution would require that the manipulator be in a singular configuration.

False. Typically in a singular configuration there will be a free choice for one of the variable joint parameters.

(e) If the unit quaternion q represents the rotation of a rigid body, the conjugate of q, after being normalized, will represent a rotation that will return the body to its original orientation.

True. The conjugate of a unit quaternion is equal to its inverse.

1. (20 pts.) A rigid body was rotated 70° about an axis through the origin that is parallel to the vector $\mathbf{m}_1 = 3\mathbf{i}+4\mathbf{j}$. Second, it was rotated an angle θ about an axis through the origin that is parallel to the vector $\mathbf{m}_2$. Third the rigid body was rotated 130° about an axis through the origin that is parallel to the vector $\mathbf{m}_3 = \mathbf{i}-\mathbf{k}$. After the three rotations the body is at the same position and orientation as when it started. Use a quaternion approach to determine θ and $\mathbf{m}_2$, the angle and axis of the second rotation.

The first rotation can be modeled by the quaternion

$$q_1 = \cos 35° + \sin 35° \left(\frac{3i + 4j}{5}\right)$$
$$= 0.8192 + 0.3441i + 0.4589j\,.$$

The third rotation can be modeled by the quaternion

$$q_3 = \cos 65° + \sin 65° \left(\frac{i - k}{\sqrt{2}}\right)$$
$$= 0.4226 + 0.6409i - 0.6409k\,.$$

The second rotation is modeled by the quaternion q_2 which is to be determined.

The coordinates of some arbitrary point r can be determined from the three rotations as

$$r' = q_3\, q_2\, q_1\, (r)\, q_1^{-1}\, q_2^{-1}\, q_3^{-1}$$

Since r' = r,

$$q_3\, q_2\, q_1 = 1\,.$$

Solving for q_2 gives

$$q_2 = q_3^{-1}\, q_1^{-1}$$
$$q_2 = (0.4226 - 0.6409i + 0.6409k)(0.8192 - 0.3441i - 0.4589j)$$
$$= 0.1256 - 0.3763i - 0.4145j + 0.8190k\,.$$

The angle and axis of the second rotation can be determined from q_2 as

$$\cos\frac{\theta}{2} = 0.1256, \quad m_x \sin\frac{\theta}{2} = -0.3763, \quad m_y \sin\frac{\theta}{2} = -0.4145, \quad m_z \sin\frac{\theta}{2} = 0.8190\,.$$

Solving for θ_2 and $\mathbf{m}_2$ gives

$$\theta_2 = 165.6°$$
$$m_x = -0.3793$$
$$m_y = -0.4178$$
$$m_z = 0.8256\,.$$

2. (20 pts.) You are given three equations of the form

$$A_0 + A_1 u + A_2 v + A_3 u v + A_4 u^2 v^2 = 0$$
$$B_0 + B_1 u^2 + B_2 v^2 + B_3 u v + B_4 u v^2 = 0$$
$$C_0 + C_1 u^2 + C_2 v^2 + C_3 u v + C_4 u v^3 = 0$$

where the coefficients A_0 through C_4 are known constants. Explain how to determine solution sets (u,v) that simultaneously satisfy the three equations.

The three equations may be written as

$$\begin{aligned} (A_4v^2)\ u^2 + (A_1+A_3v)\ u + (A_0+A_2v) &= 0 \\ (B_1)\ u^2 + (B_3v+B_4v^2)\ u + (B_0+B_2v^2) &= 0 \\ (C_1)\ u^2 + (C_3v+C_4v^3)\ u + (C_0+C_2v^2) &= 0\ . \end{aligned}$$

These three equations can be written in matrix form as

$$\begin{bmatrix} A_4v^2 & A_1+A_3v & A_0+A_2v \\ B_1 & B_3v+B_4v^2 & B_0+B_2v^2 \\ C_1 & C_3v+C_4v^3 & C_0+C_2v^2 \end{bmatrix} \begin{bmatrix} u^2 \\ u \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\ .$$

The three equations must be linearly dependent and thus the determinant of the coefficient matrix must equal zero. This is written as

$$\begin{vmatrix} A_4v^2 & A_1+A_3v & A_0+A_2v \\ B_1 & B_3v+B_4v^2 & B_0+B_2v^2 \\ C_1 & C_3v+C_4v^3 & C_0+C_2v^2 \end{vmatrix} = 0\ .$$

This will yield a polynomial in the unknown term *v*. Corresponding values of the term *u* can be obtained by substituting each value of *v* into the first two equations and then solving the two equations that are quadratic in *u* and selecting the common value.

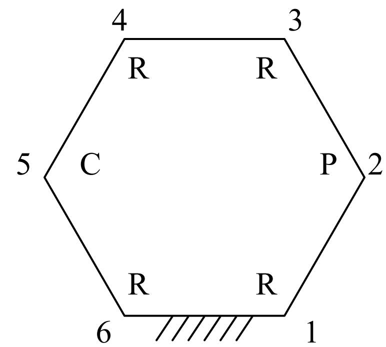
Alternatively, the corresponding value of *u* can be found by first writing the first two equations as

$$\begin{bmatrix} A_4v^2 & A_1+A_3v \\ B_1 & B_3v+B_4v^2 \end{bmatrix} \begin{bmatrix} u^2 \\ u \end{bmatrix} = \begin{bmatrix} -(A_0+A_2v) \\ -(B_0+B_2v^2) \end{bmatrix}\ .$$

The values for *u* and u^2 can then be determined as

$$\begin{bmatrix} u^2 \\ u \end{bmatrix} = \begin{bmatrix} A_4v^2 & A_1+A_3v \\ B_1 & B_3v+B_4v^2 \end{bmatrix}^{-1} \begin{bmatrix} -(A_0+A_2v) \\ -(B_0+B_2v^2) \end{bmatrix}\ .$$

3. (25 pts.) You are analyzing a six-link RCRRPR spatial mechanism where the link a_{61} is fixed to ground. The constant mechanism parameters are known together with the angle θ_6. Obtain two equations in two unknown joint parameters.



The first equation will be obtained by inserting dual angles into the subsidiary cosine law

$$Z_{61} = Z_{34} .$$

The secondary cosine law can be written as

$$Z_{061} = Z_{034}$$

where

$$Z_{061} = S_6 \frac{\partial Z_{61}}{\partial \theta_6} + S_1 \frac{\partial Z_{61}}{\partial \theta_1} + a_{56} \frac{\partial Z_{61}}{\partial \alpha_{56}} + a_{61} \frac{\partial Z_{61}}{\partial \alpha_{61}} + a_{12} \frac{\partial Z_{61}}{\partial \alpha_{12}} ,$$

$$Z_{034} = S_3 \frac{\partial Z_{34}}{\partial \theta_3} + S_4 \frac{\partial Z_{34}}{\partial \theta_4} + a_{23} \frac{\partial Z_{34}}{\partial \alpha_{23}} + a_{34} \frac{\partial Z_{34}}{\partial \alpha_{34}} + a_{45} \frac{\partial Z_{34}}{\partial \alpha_{45}} .$$

Evaluating the partial derivatives gives

$$Z_{061} = S_6\, s_{56}X_{16} + S_1\, s_{12}X_{61} + a_{56}\, Y_{16} + a_{61}\, (c_{12}Y_6 - s_{12}c_1Z_6) + a_{12}\, Y_{61} ,$$
$$Z_{034} = S_3\, s_{23}X_{43} + S_4\, s_{45}X_{34} + a_{23}\, Y_{43} + a_{34}\, (c_{45}Y_3 - s_{45}c_4Z_3) + a_{45}\, Y_{34} .$$

The following subsidiary laws are used to substitute into Z_{034} :

$$X_{43} = X_{612}, \quad X_{34} = X_{165}, \quad Y_{43} = -X^*_{612}, \quad Y_{34} = -X^*_{165}, \quad Z_3 = Z_{561} .$$

All terms are now expressed in the unknowns θ_1 and θ_5 except for Y_3. The definition of Y_3 is written as

$$Y_3 = -(s_{34}c_{23}+c_{34}s_{23}c_3) .$$

Multiplying this by s_{34}/s_{34} and regrouping gives

$$Y_3 = \frac{-s_{34}^{\,2}c_{23} - c_{34}s_{23}(s_{34}c_3)}{s_{34}} .$$

The fundamental sine-cosine law $Y_{5612} = s_{34}c_3$ can be used to express Y_3 in terms of the unknowns θ_1 and θ_5.

The secondary cosine law can now be used together with the spherical cosine law

$$Z_{5612} = c_{34}$$

to result in two equations in the two unknowns θ_1 and θ_5.

4. (15 pts.) The sine of the difference of two angles can be expressed as

$$\sin(\sigma - \beta) = \sin(\sigma)\cos(\beta) - \cos(\sigma)\sin(\beta)\ .$$

Functions 1 and 2 are now defined as the right and left side of the above expression as

$$f_1(x,y) = \sin(x)\cos(y) - \cos(x)\sin(y),$$
$$f_2(z) = \sin(z)\ .$$

Evaluate f_1 when $x = \theta + \varepsilon S$ and $y = \alpha + \varepsilon a$. Evaluate f_2 when $z = x\text{-}y = (\theta\text{-}\alpha) + \varepsilon(S\text{-}a)$. Show that the results are equal which will prove that the result of inserting dual angles into the given trig identity yields a valid answer.

The evaluation of the first function can be written as

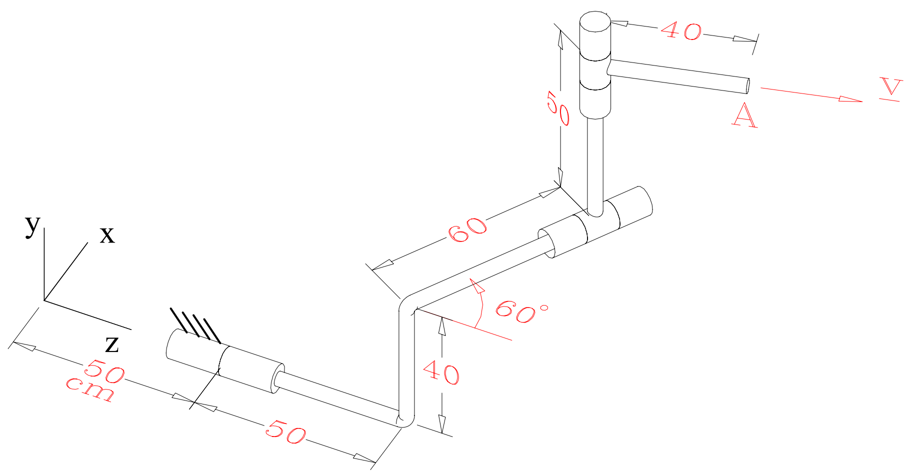
$$\begin{aligned}
f_1(\theta+\varepsilon S, \alpha+\varepsilon a) &= f_1(\theta,\alpha) + \varepsilon S \left.\frac{\partial f_1(x,y)}{\partial x}\right|_{\substack{x=\theta \\ y=\alpha}} + \varepsilon a \left.\frac{\partial f_1(x,y)}{\partial y}\right|_{\substack{x=\theta \\ y=\alpha}} \\
&= \sin\theta\cos\alpha - \cos\theta\sin\alpha + \varepsilon S(\cos\theta\cos\alpha + \sin\theta\sin\alpha) \\
&\quad + \varepsilon a(-\sin\theta\sin\alpha - \cos\theta\cos\alpha) \\
&= \sin\theta\cos\alpha - \cos\theta\sin\alpha + \varepsilon(S-a)(\cos\theta\cos\alpha + \sin\theta\sin\alpha) \\
&= \sin(\theta-\alpha) + \varepsilon(S-a)\cos(\theta-\alpha)
\end{aligned}$$

The second function can be evaluated as

$$\begin{aligned}
f_2((\theta-\alpha)+\varepsilon(S-a)) &= f_1(\theta-\alpha) + \varepsilon(S-a)\left.\frac{df_2(x)}{dx}\right|_{x=\theta+\alpha} \\
&= \sin(\theta-\alpha) + \varepsilon(S-a)\cos(\theta-\alpha)
\end{aligned}$$

Since the function evaluations are equal, it has been proven that the trig identity associated with the sine of the difference of two angles is valid when dual angles are inserted.

5. (20 pts.) A three axis manipulator and a fixed coordinate system are shown in the figure below.



Suppose you are given the desired orientation of the last link as defined by specifying the direction of vector **v** and the direction of the line along the third joint axis, both measured with respect to the fixed coordinate system. The position of point A is not specified.

Explain how you would determine the three joint angles that would orient the last link as desired. (It is not necessary to perform any numerical calculations.)

How many solution sets for the angles φ_1, θ_2, and θ_3 exist that will orient the last link as desired.

The directions of the three axes of the standard third coordinate system are known with respect to the fixed coordinate system and can be written as

$$ {}^F\mathbf{x}_3 = {}^F\mathbf{v}, \quad {}^F\mathbf{z}_3 = {}^F\mathbf{S}_3, \quad {}^F\mathbf{y}_3 = {}^F\mathbf{z}_3 \times {}^F\mathbf{x}_3 \,. $$

Note that ${}^F\mathbf{S}_3 = {}^F\mathbf{z}_3$ and that ${}^F\mathbf{a}_{34} = {}^F\mathbf{x}_3$. The vectors $\mathbf{S}_3$ and $\mathbf{a}_{34}$ can be expressed in terms of the standard fixed coordinate system by using set 1 of the direction cosine tables for the quadrilateral, pentagon, etc as

$$ {}^1\mathbf{S}_3 = \begin{bmatrix} \overline{X}_2 \\ \overline{Y}_2 \\ \overline{Z}_2 \end{bmatrix}, \quad {}^1\mathbf{a}_{34} = \begin{bmatrix} W_{32} \\ -U^*_{321} \\ U_{321} \end{bmatrix}. $$

These vectors can be expressed in terms of the fixed coordinate system as

$$ {}^F\mathbf{S}_3 = {}^F_1\mathbf{R}\,{}^1\mathbf{S}_3, \quad {}^F\mathbf{a}_{34} = {}^F_1\mathbf{R}\,{}^1\mathbf{a}_{34} $$

where

$$
{}^{F}_{1}\mathbf{R} = \begin{bmatrix} \cos\phi_1 & -\sin\phi_1 & 0 \\ \sin\phi_1 & \cos\phi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

Thus

$$
{}^{F}\mathbf{S}_3 = \begin{bmatrix} \cos\phi_1\,\overline{X}_2 - \sin\phi_1\,\overline{Y}_2 \\ \sin\phi_1\,\overline{X}_2 + \cos\phi_1\,\overline{Y}_2 \\ \overline{Z}_2 \end{bmatrix}, \quad {}^{F}\mathbf{a}_{34} = \begin{bmatrix} \cos\phi_1\,W_{32} + \sin\phi_1\,U^*_{321} \\ \sin\phi_1\,W_{32} - \cos\phi_1\,U^*_{321} \\ U_{321} \end{bmatrix}.
$$

Two possible values for θ_2 can be obtained by equating the term $\overline{Z}_2$ with the known third component of the vector ${}^{F}\mathbf{S}_3$. Corresponding values for φ_1 can be obtained from the two equations corresponding to the first and second components of the vector ${}^{F}\mathbf{S}_3$. Corresponding values for θ_3 can be obtained by solving any two of the three equations associated with the components of ${}^{F}\mathbf{a}_{34}$ for the values of s_3 and c_3. Thus two solution sets for the angles φ_1, θ_2, and θ_3 exist that will orient the last link as desired.

1. (30 pts.) A six axis manipulator is mounted on a ground vehicle. The *vehicle coordinate system* is attached to the vehicle frame. The *manipulator coordinate system* is also attached to the vehicle frame with its origin at the intersection point of the line along the first joint axis and the line along the first link length. The Z axis of the manipulator coordinate system is along the first joint axis.

   The coordinates of the origin point of the manipulator coordinate system measured in the vehicle coordinate system are given as $[180, 60, 90]^T$ cm. The orientation of the manipulator coordinate system is defined as initially being aligned with the vehicle coordinate system. It is then rotated by the quaternion operator $q()q^{-1}$ where q= 0.79335 + 0.27225 j + 0.54449 k.

   It is desired to position the origin point of the standard 6th coordinate system at the coordinates $[220, 60, 200]^T$ cm with respect to the vehicle coordinate system. The desired orientation of the 6th coordinate system is defined as initially being aligned with the vehicle coordinate system. It is then rotated by the quaternion operator $q()q^{-1}$ where q= -0.81915 + 0.51302 i + 0.25651 k.

   Determine the transformation matrix ${}^{manipulator}_{6}\mathbf{T}$ that defines the relationship between the 6th coordinate system and the manipulator coordinate system.

The problem will be solved by determining the relation between the manipulator coordinate system and the vehicle coordinate system as defined by ${}^{vehicle}_{manipulator}\mathbf{T}$ and the relation between the 6th coordinate system and the vehicle coordinate system as defined by ${}^{vehicle}_{6}\mathbf{T}$. The solution can then be obtained as

$$ {}^{manipulator}_{6}\mathbf{T} = \left({}^{vehicle}_{manipulator}\mathbf{T} \right)^{-1} \; {}^{vehicle}_{6}\mathbf{T} \, . $$

The rotation matrix ${}^{vehicle}_{manipulator}\mathbf{R}$ is obtained by determining the angle of rotation and axis of rotation associated write the quaternion q= 0.79335 + 0.27225 j + 0.54449 k. The angle of rotation θ is determined from

$$ \cos\frac{\theta}{2} = 0.79335, \quad \sin\frac{\theta}{2} = \sqrt{0.27225^2 + 0.54449^2} = 0.60876 \quad . $$

Solving for θ gives θ = 75°. The unitized axis of rotation is gives by the vector part of the quaternion as

$$ \mathbf{m} = 0.44722\,\mathbf{j} + 0.89442\,\mathbf{k} \, . $$

Equation 2.61 of the text shows how to determine the rotation matrix when coordinate system B has been rotated an angle θ about a unit axis **m** relative to coordinate system A (note that v=1-c).

$$
{}^{A}_{B}\mathbf{R} = \begin{bmatrix} m_x m_x v + c & m_x m_y v - m_z s & m_x m_z v + m_y s \\ m_x m_y v + m_z s & m_y m_y v + c & m_y m_z v - m_x s \\ m_x m_z v - m_y s & m_y m_z v + m_x s & m_z m_z v + c \end{bmatrix} \qquad \textbf{(2.61)}
$$

From this equation, and given the coordinates of the origin point of the manipulator coordinate system in terms of the vehicle coordinate system, the transformation matrix ${}^{vehicle}_{manipulator}\mathbf{T}$ can be written as

$$
{}^{vehicle}_{manipulator}\mathbf{T} = \begin{bmatrix} 0.25882 & -0.86395 & 0.43198 & 180 \\ 0.86395 & 0.40706 & 0.29648 & 60 \\ -0.43198 & 0.29648 & 0.85176 & 90 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

The orientation of the 6th coordinate system relative to the vehicle coordinate system is defined by the quaternion q= -0.81915 + 0.51302 i + 0.25651 k. The angle θ' and the axis m' defined by this quaternion are

θ' = 290°,

m' = 0.89443 **i** + 0.44721 **k**.

The transformation matrix ${}^{vehicle}_{6}\mathbf{T}$ can be written as

$$
{}^{vehicle}_{6}\mathbf{T} = \begin{bmatrix} 0.86840 & 0.42024 & 0.26319 & 220 \\ -0.42024 & 0.34202 & 0.84049 & 60 \\ 0.26319 & -0.84049 & 0.47362 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

The relation between the 6th coordinate system and the vehicle coordinate system is then determined as

$$
{}^{manipulator}_{6}\mathbf{T} = \left({}^{vehicle}_{manipulator}\mathbf{T}\right)^{-1} {}^{vehicle}_{6}\mathbf{T}
= \begin{bmatrix} -0.25200 & 0.76733 & 0.58966 & -37.1652 \\ -0.84329 & -0.47303 & 0.25516 & -1.9456 \\ 0.47472 & -0.43295 & 0.76628 & 110.9728 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

2. (30 pts.) You are to build a spatial triangle from three links that are defined by

α_{12} = 60°, a_{12} = 6 cm,

α_{23} = 110°, a_{23} = 8 cm,

α_{31} = 80°, a_{31} = 10 cm.

Determine one solution for the joint angles θ_1, θ_2, θ_3 and the joint offsets S_1, S_2, S_3 for your assembled spatial triangle.

The joint angles θ_1, θ_2, θ_3 can be solved for first by analyzing the equivalent spherical triangle. A cosine law is written as

$$Z_3 = c_{12} .$$

Expanding the definition for Z_3 and solving for c_3 gives

$$c_3 = -\frac{c_{12} - c_{31}c_{23}}{s_{31}s_{23}} = -0.60447 .$$

Two values of θ_3 satisfy this equation:

$$\theta_{3a} = 2.2199 \text{ radians} = 127.19°,$$

$$\theta_{3b} = -2.2199 \text{ radians} = -127.19° .$$

Corresponding values for θ_2 may be obtained from the sine and sine-cosine laws

$$\bar{X}_3 = s_{12}s_2 ,$$
$$\bar{Y}_3 = s_{12}c_2 .$$

The corresponding values for θ_2 are

$$\theta_{2a} = 2.0081 \text{ radians} = 115.06°,$$

$$\theta_{2b} = -2.0081 \text{ radians} = -115.06° .$$

Corresponding values for θ_1 may be obtained from the sine and sine-cosine laws

$$X_3 = s_{12}s_1 ,$$

$$Y_3 = s_{12}c_1 .$$

The corresponding values for θ_1 are

$$\theta_{1a} = 1.0439 \text{ radians} = 59.81°,$$

$$\theta_{1b} = -1.0439 \text{ radians} = -59.81° .$$

The joint offsets S_1, S_2, and S_3 may be obtained by writing the vector loop equation as

$$S_1\,\mathbf{S}_1 + a_{12}\,\mathbf{a}_{12} + S_2\,\mathbf{S}_2 + a_{23}\,\mathbf{a}_{23} + S_3\,\mathbf{S}_3 + a_{31}\,\mathbf{a}_{31} = \mathbf{0} .$$

Expressing all the vectors in terms of the 1st standard coordinate system gives

$$S_1\begin{bmatrix}0\\0\\1\end{bmatrix}+a_{12}\begin{bmatrix}1\\0\\0\end{bmatrix}+S_2\begin{bmatrix}0\\-s_{12}\\c_{12}\end{bmatrix}+a_{23}\begin{bmatrix}c_2\\s_2c_{12}\\U_{21}\end{bmatrix}+S_3\begin{bmatrix}\bar{X}_2\\\bar{Y}_2\\\bar{Z}_2\end{bmatrix}+a_{31}\begin{bmatrix}c_1\\-s_1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Evaluating these equations for the A solution case gives

$$S_{1a}\begin{bmatrix}0\\0\\1\end{bmatrix}+6\begin{bmatrix}1\\0\\0\end{bmatrix}+S_{2a}\begin{bmatrix}0\\-0.8660\\0.5\end{bmatrix}+8\begin{bmatrix}-0.4235\\0.4529\\0.7845\end{bmatrix}+S_{3a}\begin{bmatrix}0.8513\\0.4952\\0.1736\end{bmatrix}+10\begin{bmatrix}0.5028\\-0.8644\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

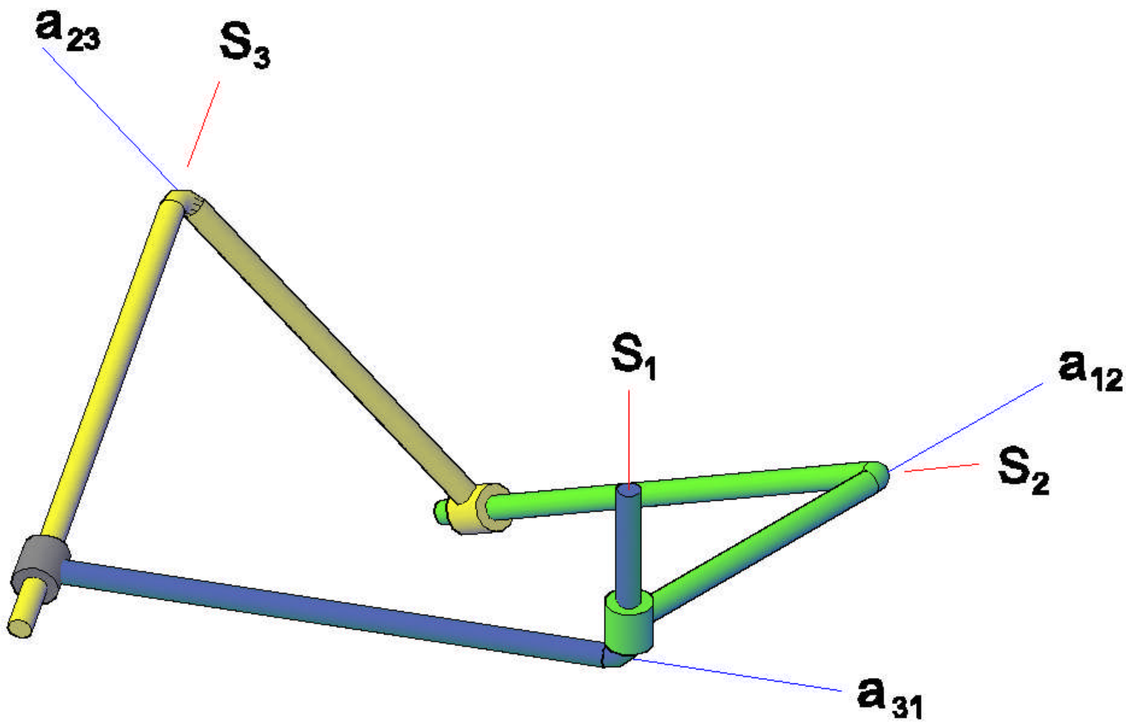
Solving for the joint offsets yields

S_{1a} = 0.7468 cm,

S_{2a} = -10.9289 cm,

S_{3a} = -8.9751 cm .

The A solution is depicted in the following figure.



For the B solution case, the joint offset values were determined from the vector loop equation as

S_{1a} = -0.7468 cm,

S_{2a} = 10.9289 cm,

S_{3a} = 8.9751 cm .

3. (20 pts.) Explain how to obtain values for *s* and *t* that simultaneously satisfy the following equations:

$$A s^3 t^2 + B t^5 + C s^2 t + D = 0$$

$$E s^2 t^4 + F s t + G t^3 = 0$$

The equations can be factored as

$$(A t^2) s^3 + (C t) s^2 + (B t^5 + D) = 0$$

$$(E t^4) s^2 + (F t) s + (G t^3) = 0 .$$

Multiplying the first equation by *s* and the second equation by *s* and s^2 and writing the set of equations in matrix format gives

$$\begin{bmatrix} 0 & At^2 & Ct & 0 & Bt^5+D \\ 0 & 0 & Et^4 & Ft & Gt^3 \\ At^2 & Ct & 0 & Bt^5+D & 0 \\ 0 & Et^4 & Ft & Gt^3 & 0 \\ Et^4 & Ft & Gt^3 & 0 & 0 \end{bmatrix} \begin{bmatrix} s^4 \\ s^3 \\ s^2 \\ s \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

For a common solution, it is necessary that the equations be linearly dependent. Thus

$$\begin{vmatrix} 0 & At^2 & Ct & 0 & Bt^5+D \\ 0 & 0 & Et^4 & Ft & Gt^3 \\ At^2 & Ct & 0 & Bt^5+D & 0 \\ 0 & Et^4 & Ft & Gt^3 & 0 \\ Et^4 & Ft & Gt^3 & 0 & 0 \end{vmatrix} = 0$$

which will yield a polynomial in the variable *t*.

Corresponding values for the parameter s can be found from solving any four of the equations as, for example, the first four, as

$$\begin{bmatrix} 0 & At^2 & Ct & 0 \\ 0 & 0 & Et^4 & Ft \\ At^2 & Ct & 0 & Bt^5+D \\ 0 & Et^4 & Ft & Gt^3 \end{bmatrix} \begin{bmatrix} s^4 \\ s^3 \\ s^2 \\ s \end{bmatrix} = \begin{bmatrix} -(Bt^5+D) \\ -Gt^3 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} s^4 \\ s^3 \\ s^2 \\ s \end{bmatrix} = \begin{bmatrix} 0 & At^2 & Ct & 0 \\ 0 & 0 & Et^4 & Ft \\ At^2 & Ct & 0 & Bt^5+D \\ 0 & Et^4 & Ft & Gt^3 \end{bmatrix}^{-1} \begin{bmatrix} -(Bt^5+D) \\ -Gt^3 \\ 0 \\ 0 \end{bmatrix} .$$

4. (20 pts.) Evaluate the function

$$f(x,y) = (x+y)^2 + y\sin(x) + x\sin(y)$$

when x = 2+3ε and y =-3 + 5ε .

The function can be expended using Taylor series expansion as

$$f(2+3\varepsilon,-3+5\varepsilon) = f(2,-3) + 3\varepsilon \left.\frac{\partial f}{\partial x}\right|_{\substack{x=2\\y=-3}} + 5\varepsilon \left.\frac{\partial f}{\partial y}\right|_{\substack{x=2\\y=-3}} .$$

The partial derivatives are written as

$$\frac{\partial f}{\partial x} = 2x + 2y + y\cos(x) + \sin(y) ,$$

$$\frac{\partial f}{\partial y} = 2x + 2y + \sin(x) + x\cos(y) .$$

Evaluating the function gives

$$f(2+3\varepsilon,-3+5\varepsilon) = \text{-}2.0101 - 18.0315\varepsilon .$$

EML 6281 – Robot Geometry 1 Open book and notes.
Final Exam, Fall 2010 2 hour exam.

1. Explain how to determine all the possible solution sets for the variables *x* and *y* for the following two equations

$$(a_1 x^2 + a_2 x + a_3) y + (a_4 x^2 + a_5 x + a_6) = 0$$

$$(b_1 x^2 + b_2 x + b_3) y^2 + (b_4 x^2 + b_5 x + b_6) y + (b_7 x^2 + b_8 x + b_9) = 0 .$$

The coefficients a_i, i=1..6, and b_j, j=1..9, can be assumed to be known values.

The equations may be written as

$$A_1 y + A_2 = 0 \tag{1}$$

$$B_1 y^2 + B_2 y + B_3 = 0 \tag{2}$$

where

$$A_1 = (a_1 x^2 + a_2 x + a_3)$$

$$A_2 = (a_4 x^2 + a_5 x + a_6)$$

$$B_1 = (b_1 x^2 + b_2 x + b_3)$$

$$B_2 = (b_4 x^2 + b_5 x + b_6)$$

$$B_3 = (b_7 x^2 + b_8 x + b_9) .$$

Multiplying equation (1) by y gives

$$A_1 y^2 + A_2 y = 0 . \tag{3}$$

Equations (1), (2), and (3) may be written in matrix form as

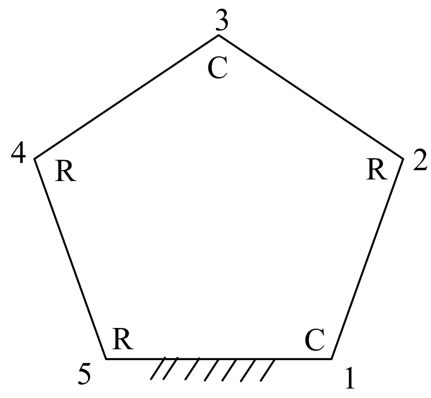
$$\begin{bmatrix} 0 & A_1 & A_2 \\ A_1 & A_2 & 0 \\ B_1 & B_2 & B_3 \end{bmatrix} \begin{bmatrix} y^2 \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} .$$

In order for solutions to exist the set of equations must be linearly dependent and thus

$$\begin{vmatrix} 0 & A_1 & A_2 \\ A_1 & A_2 & 0 \\ B_1 & B_2 & B_3 \end{vmatrix} = 0 .$$

Expansion of the determinant will yield a sixth order polynomial in *x*. Corresponding values for *y* can be obtained directly from equation (1).

2. A planar representation of a group 2 mechanism is shown in the figure to the right.



a) List the constant mechanism parameters.

a_{12}, a_{23}, a_{34}, a_{45}, a_{51},

α_{12}, α_{23}, α_{34}, α_{45}, α_{51},

S_2, S_4, S_5

b) Assuming that the constant mechanism parameters are known, together with the value for the angle θ_4, obtain two equations that contain the sines and cosines of the angles θ_2 and θ_5 as the only unknowns.

Factor these equations into the format

$$(A_i c_5+B_i s_5+D_i)c_2 + (E_i c_5+F_i s_5+G_i)s_2 + (H_i c_5+I_i s_5+J_i) = 0, \quad i=1..2.$$

In other words, obtain expressions for the coefficients A_i through J_i.

This problem is similar to the spatial mechanism analyzed in Section 8.4.1 of the text. A cyclic exchange of subscript can be performed on the text solution, i.e. *1,2,3,4,5* can be cycled to *5,1,2,3,4*. Thus two equations will be obtained in the format

$$(A_i c_5+B_i s_5+D_i)c_2 + (E_i c_5+F_i s_5+G_i)s_2 + (H_i c_5+I_i s_5+J_i) = 0, \quad i=1..2$$

where by cyclic exchange

$A_1 = 0$, $B_1 = 0$, $D_1 = s_{12}s_{23}$,

$E_1 = 0$, $F_1 = 0$, $G_1 = 0$,

$H_1 = s_{51}Y_4$, $I_1 = s_{51}X_4$, $J_1 = c_{51}Z_4 - c_{12}c_{23}$,

$A_2 = 0$, $B_2 = 0$, $D_2 = a_{12}c_{12}s_{23} + a_{23}c_{23}s_{12}$,

$E_2 = 0$, $F_2 = 0$, $G_2 = -S_2 s_{23}s_{12}$,

$H_2 = a_{51}c_{51}Y_4 + S_5 s_{51}X_4 + s_{51}Y_{04}$,

$I_2 = a_{51}c_{51}X_4 - S_5 s_{51}Y_4 + s_{51}X_{04}$,

$J_2 = -a_{51}s_{51}Z_4 + c_{51}Z_{04} + a_{12}s_{12}c_{23} + a_{23}s_{23}c_{12}$

where

$$X_{04} = S_4 s_{34}c_4 + a_{34}c_{34}s_4 \ ,$$

$$Y_{04} = S_4 c_{45}s_{34}s_4 + a_{34}(s_{45}s_{34}\text{-}c_{45}c_{34}c_4) - a_{45}Z_4 \ ,$$

$$Z_{04} = S_4 s_{45}s_{34}s_4 + a_{34}\bar{Y}_4 + a_{45}Y_4 \ .$$

3. Evaluate the function

$$f(x,y) = 3x^2 \cos(y) - 2x \sin(y) + 5$$

when x = 3-5ε and y = -4+2ε.

The function can be evaluated as

$$
\begin{aligned}
f(3-5\varepsilon, -4+2\varepsilon) &= f(3,-4) - 5\varepsilon \left.\frac{\partial f}{\partial x}\right|_{\substack{x=3\\y=-4}} + 2\varepsilon \left.\frac{\partial f}{\partial y}\right|_{\substack{x=3\\y=-4}} \\
&= 27\cos(-4) - 6\sin(-4) + 5 \\
&\quad -5\varepsilon\left[6(3)\cos(-4) - 2\sin(-4)\right] \\
&\quad +2\varepsilon\left[-3(3)^2\sin(-4) - 2(3)\cos(-4)\right] \\
&= -17.19 - 5\varepsilon[-13.28] + 2\varepsilon[-16.51] \\
&= -17.19 + 33.37\varepsilon
\end{aligned}
$$

4. The origin of coordinate system B is coincident with the origin of coordinate system A. The orientation of coordinate system B is given by the matrix ${}^A_B\mathbf{R}$ where

$$ {}^A_B\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. $$

a) Since ${}^A_B\mathbf{R}$ is a rotation matrix, the terms r_{ij} must satisfy six constraint equations. What are these equations?

The three columns must be unit vectors, i.e

$$r_{1i}^2 + r_{2i}^2 + r_{3i}^2 = 1, \ \ i=1..3$$

and the three columns must be orthogonal to one another, i.e.

$$r_{1(i)}r_{1(i+1)} + r_{2(i)}r_{2(i+1)} + r_{2(i)}r_{2(i+1)} = 0, \ \ i=1..3 \ .$$

b) The rotation matrix ${}^A_B\mathbf{R}$ transforms the coordinates of a point from the B coordinate system to the A coordinate system according to the equation

$$ {}^A\mathbf{P}_1 = {}^A_B\mathbf{R} \; {}^B\mathbf{P}_1 \ . $$

A quaternion q will also transform the coordinates of a point from the B coordinate system to the A coordinate system according to the equation

$$ {}^A p_1 = q\,({}^B p_1)\,q^{-1} $$

where ${}^{A}p_1$ and ${}^{B}p_1$ are quaternions whose scalar component equals zero and whose vector part represents the coordinates of point 1 in the respective coordinate system.

Explain how to obtain the quaternion q when given the rotation matrix ${}_{B}^{A}\mathbf{R}$.

The matrix ${}_{B}^{A}\mathbf{R}$ may be written as

$$ {}_{B}^{A}\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. $$

The cosine of the angle of rotation, θ, may be obtained from text equation (2.65) as

$$ \cos\theta = \frac{r_{11} + r_{22} + r_{33} - 1}{2} $$

and the value of θ may be chosen in the range of 0 to π.

The axis of rotation will be referred to as **m** where

$$ \mathbf{m} = m_x\mathbf{i} + m_y\mathbf{j} + m_z\mathbf{k}. $$

The values of m_x, m_y, and m_z may be obtained from text equations (2.66) through (2.68) as

$$ m_x = \frac{r_{32} - r_{23}}{2\sin\theta}, \quad m_y = \frac{r_{13} - r_{31}}{2\sin\theta}, \quad m_z = \frac{r_{21} - r_{12}}{2\sin\theta}. $$

The quaternion q which will transform points from the B coordinate system to the A coordinate system can be formed by a rotation of θ about **m** and can thus be written as

$$ q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left(m_x i + m_y j + m_z k\right). $$

5. A rigid body has been rotated by an angle θ about an axis parallel to the unit vector $\mathbf{m} = m_x\mathbf{i} + m_y\mathbf{j} + m_z\mathbf{k}$ that passes through the origin of the fixed reference coordinate system.

   The same change in orientation could be accomplished by rotating the body by the angle γ_1 about the x axis of the reference coordinate system followed by a rotation of γ_2 about the y axis of the fixed reference coordinate system followed by a rotation of γ_3 about the z axis of the fixed reference coordinate system.

   Explain how to determine γ_1, γ_2, and γ_3 in terms of θ, m_x, m_y, and m_z.

   The original rotation can be modeled by the quaternion operator $q()q^{-1}$ where

$$ q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left(m_x i + m_y j + m_z k\right). $$

   The three successive rotations can be modeled by $q_3\, q_2\, q_1\, ()\, q_1^{-1}\, q_2^{-1}\, q_3^{-1}$ where

$$q_1 = \cos\frac{\gamma_1}{2} + \sin\frac{\gamma_1}{2} i,$$

$$q_2 = \cos\frac{\gamma_2}{2} + \sin\frac{\gamma_2}{2} j,$$

$$q_3 = \cos\frac{\gamma_3}{2} + \sin\frac{\gamma_3}{2} k.$$

It must be the case that

$$q = q_3\, q_2\, q_1 \,.$$

The product $q_3\, q_2\, q_1$ is evaluated as

$$\begin{aligned} q_3 q_2 q_1 = & \left[\cos\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2}\cos\frac{\gamma_3}{2} + \sin\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}\sin\frac{\gamma_3}{2} \right] \\ & + i\left[\sin\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2}\cos\frac{\gamma_3}{2} - \cos\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}\sin\frac{\gamma_3}{2} \right] \\ & + j\left[\cos\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}\cos\frac{\gamma_3}{2} + \sin\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2}\sin\frac{\gamma_3}{2} \right] \\ & + k\left[\cos\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2}\sin\frac{\gamma_3}{2} - \sin\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}\cos\frac{\gamma_3}{2} \right]. \end{aligned}$$

Equating the scalar and vector parts of q and the product $q_3q_2q_1$ gives the four equations

$$\begin{aligned} \cos\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2}\cos\frac{\gamma_3}{2} + \sin\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}\sin\frac{\gamma_3}{2} &= \cos\frac{\theta}{2} \\ \sin\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2}\cos\frac{\gamma_3}{2} - \cos\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}\sin\frac{\gamma_3}{2} &= \sin\frac{\theta}{2} m_x \\ \cos\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}\cos\frac{\gamma_3}{2} + \sin\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2}\sin\frac{\gamma_3}{2} &= \sin\frac{\theta}{2} m_y \\ \cos\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2}\sin\frac{\gamma_3}{2} - \sin\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}\cos\frac{\gamma_3}{2} &= \sin\frac{\theta}{2} m_z. \end{aligned}$$

The equations are rewritten as

$$c_1c_2c_3 + s_1s_2s_3 = K_1$$

$$s_1c_2c_3 - c_1s_2s_3 = K_2$$

$$c_1s_2c_3 + s_1c_2s_3 = K_3$$

$$c_1c_2s_3 - s_1s_2c_3 = K_4$$

where

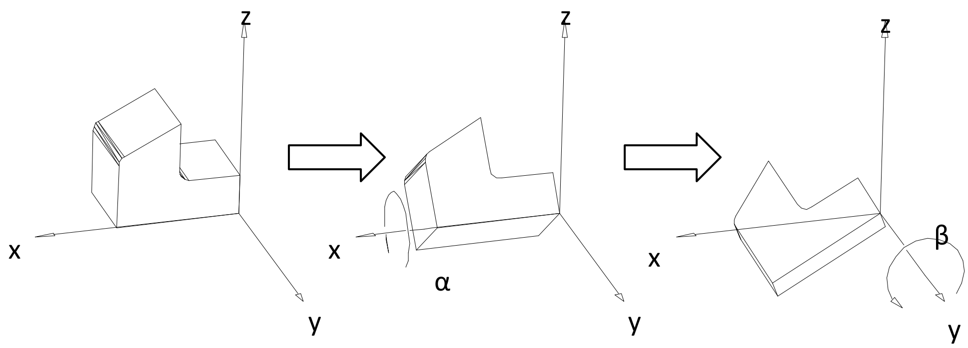
$$s_i = \sin\frac{\gamma_i}{2}, \quad c_i = \cos\frac{\gamma_i}{2}, i = 1..3$$

$$K_1 = \cos\frac{\theta}{2}, \quad K_2 = m_x \sin\frac{\theta}{2}, \quad K_3 = m_y \sin\frac{\theta}{2}, \quad K_4 = m_z \sin\frac{\theta}{2}.$$

It remains to solve these four equations for the angles γ_1, γ_2, and γ_3 which is beyond the current scope of the exam question.

6. A rigid body is rotated an angle α about an axis along the *X* axis followed by a rotation of an angle β about an axis along the *Y* axis (see figure).

   Can the body be moved from the starting pose to the same final pose by first rotating an angle γ about the *Y* axis followed by a rotation of angle θ about the *X* axis? If so, explain how to determine γ and θ.



No. In the figure it is apparent that the edge along the *X* axis will remain in the *XZ* plane after the two rotations. Rotating about the *Y* axis first and then the *X* axis would move the edge out of the *XZ* plane.

Another argument is that the two original rotations can be modeled by the quaternion operator $q\,()\,q^{-1}$ where

$$q=\left[\cos\frac{\beta}{2}+\sin\frac{\beta}{2}j\right]\left[\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}i\right]$$
$$=\cos\frac{\beta}{2}\cos\frac{\alpha}{2}+i\left(\cos\frac{\beta}{2}\sin\frac{\alpha}{2}\right)+j\left(\sin\frac{\beta}{2}\cos\frac{\alpha}{2}\right)+k\left(-\sin\frac{\beta}{2}\sin\frac{\alpha}{2}\right)$$

The rotations of γ about the *Y* axis followed by θ about the *X* axis can be modeled by the quaternion operator $q'\,()\,q'^{-1}$ where

$$q'=\left[\cos\frac{\theta}{2}+\sin\frac{\theta}{2}i\right]\left[\cos\frac{\gamma}{2}+\sin\frac{\gamma}{2}j\right]$$
$$=\cos\frac{\theta}{2}\cos\frac{\gamma}{2}+i\left(\sin\frac{\theta}{2}\cos\frac{\gamma}{2}\right)+j\left(\cos\frac{\theta}{2}\sin\frac{\gamma}{2}\right)+k\left(\sin\frac{\theta}{2}\sin\frac{\gamma}{2}\right)$$

For the rotations to be equivalent it is necessary that q = q' and thus

$$\cos\frac{\beta}{2}\cos\frac{\alpha}{2} = \cos\frac{\theta}{2}\cos\frac{\gamma}{2}$$
$$\cos\frac{\beta}{2}\sin\frac{\alpha}{2} = \sin\frac{\theta}{2}\cos\frac{\gamma}{2}$$
$$\sin\frac{\beta}{2}\cos\frac{\alpha}{2} = \cos\frac{\theta}{2}\sin\frac{\gamma}{2}$$
$$-\sin\frac{\beta}{2}\sin\frac{\alpha}{2} = \sin\frac{\theta}{2}\sin\frac{\gamma}{2}$$

Dividing the second equation by the first equation and the fourth equation by the third equation gives

$$\tan\frac{\alpha}{2} = \tan\frac{\theta}{2}$$
$$\tan\frac{\alpha}{2} = -\tan\frac{\theta}{2}$$

The only solution to these two equations is the trivial solution of α and θ equal to 0. Thus it is not possible to find general values for θ and γ that will satisfy the four original equations.

It is also possible to show that the four equations do not have a solution by considering the rotation of a point that is on the x axis, i.e. the point $[L,0,0]^T$.

The product q (L) q^{-1} can be evaluated as

$$q(L)q^{-1} = i\,L\left[2\left(\cos\frac{\beta}{2}\right)^2 - 1\right] - 2k\,L\left[\cos\frac{\beta}{2}\sin\frac{\beta}{2}\right].$$

The product q' (L) q'^{-1} can be evaluated as

$$q'(L)q'^{-1} = i\,L\left[2\left(\cos\frac{\gamma}{2}\right)^2 - 1\right]$$
$$+ j4L\cos\frac{\gamma}{2}\sin\frac{\gamma}{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$
$$-k\,2\,L\cos\frac{\gamma}{2}\sin\frac{\gamma}{2}\left[2\left(\cos\frac{\theta}{2}\right)^2 - 1\right].$$

Note that the product $q(L)q^{-1}$ has not j component (since the point has remained in the xz plane) while the product $q'(L)q'^{-1}$ does have a j component.

EMl 6281 – Robot Geometry I
Exam 3 – Fall 2011

open book and notes
2 hours

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1. (20 pts.) A rigid body is rotated from its starting position (position 0) to its final position (position 1) by a rotation of α about the fixed reference frame X axis followed by a rotation of β about the fixed Z axis.

   Similarly, a second position (position 2) is defined by having the body initialized at its starting position (position 0) and then rotating it by an angle θ about the fixed X axis followed by a rotation by an angle γ about the fixed Y axis.

   Explain how to determine the axis and angle of rotation that would move the body directly from position 1 to position 2.

The transformation of points from position 0 to position 1 can be defined by the quaternion operator $q_2\, q_1\, ()\, q_1^{-1}\, q_2^{-1}$ where

$$q_1 = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} i$$

$$q_2 = \cos\frac{\beta}{2} + \sin\frac{\beta}{2} k.$$

The transformation of points from position 0 to position 2 can be defined by the quaternion operator $q_4\, q_3\, ()\, q_3^{-1}\, q_4^{-1}$ where

$$q_3 = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} i$$

$$q_4 = \cos\frac{\gamma}{2} + \sin\frac{\gamma}{2} j.$$

The quaternion that would rotate the body from position 1 to position 0 is the conjugate of the product q_2q_1 which equals the conjugates of the individual quaternions written in opposite order, i.e. $K_{q1}K_{q2}$. Thus the quaternion q, where the operator $q()q^{-1}$ rotates the body from position 1 to position 2, can be written as the product of the quaternion that rotates the body from position 1 to position 0 times the quaternion that rotates the body from position 0 to position 2 as

$$q = q_4\, q_3\, K_{q1}\, K_{q2}\ .$$

The scalar component of q will equal the cosine of half the angle of rotation and the vector part of q will be parallel to the axis of rotation as the body rotates directly from position 1 to position 2.

2. (20 pts.) You are given the following two equations where the coefficients A_i through B_i, i=1..6, are known and the terms x and y are unknown.

$$(A_6\, x^2 + A_5\, x + A_4)\, y + (A_3\, x^2 + A_2\, x + A_1) = 0$$

$$(B_6\, x + B_5)\, y^2 + (B_4\, x + B_3)\, y + (B_2\, x + B_1) = 0$$

Explain how to determine the sets of solutions (x,y) that simultaneously satisfy both equations.

The two equations may be written as

$$L_1\, y + L_2 = 0$$

$$M_1\, y^2 + M_2\, y + M_3 = 0$$

where

$$L_1 = A_6\, x^2 + A_5\, x + A_4$$

$$L_2 = A_3\, x^2 + A_2\, x + A_1$$

$$M_1 = B_6\, x + B_5$$

$$M_2 = B_4\, x + B_3$$

$$M_3 = B_2\, x + B_1\ .$$

Multiplying the first equation by y will yield a total of three equations that can be written in matrix form as

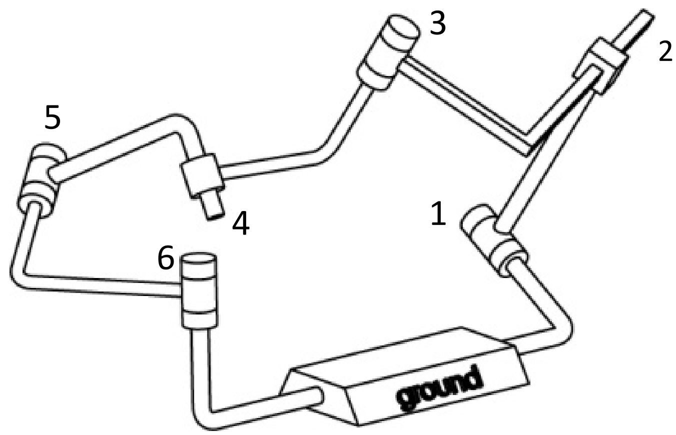
$$\begin{bmatrix} 0 & L_1 & L_2 \\ M_1 & M_2 & M_3 \\ L_1 & L_2 & 0 \end{bmatrix} \begin{bmatrix} y^2 \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These three equations must be linearly dependent and thus the determinant of the coefficients must equal zero. This may be written as

$$\begin{vmatrix} 0 & L_1 & L_2 \\ M_1 & M_2 & M_3 \\ L_1 & L_2 & 0 \end{vmatrix} = 0$$

Expansion of the determinant will give a 5^{th} order polynomial in x. Corresponding values for y can be found by substituting each value for x into the first equation and solving for y.

3. (40 pts.) A RRCRPR spatial mechanism is shown in the following figure.



(a) What group mechanism is it and why?

This is a spatial mechanism that has one degree of freedom. Its equivalent spherical mechanism has 2 degrees of freedom. Therefore it is a group 2 spatial mechanism.

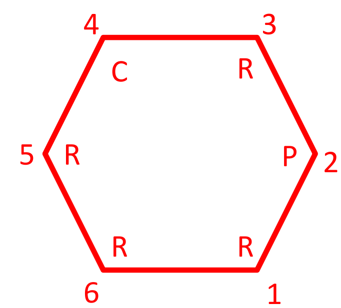
(b) List the constant mechanism parameters.

α_{12}, α_{23}, α_{34}, α_{45}, α_{56}, α_{61}

a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{61}

S_1, S_3, S_5, S_6

θ_2



(c) Assume the angle θ_6 is given. Obtain an equation that does not contain the unknown joint offsets. Express all the terms in this equation in terms of the two unknowns θ_1 and θ_5 (with the exception of the term multiplied by the link length a_{34}).

Dual angles are substituted into the cosine law

$$Z_{561} = Z_3$$

The dual parts of the result are written as

$$Z_{0561} = Z_{03}$$

where

$$\begin{aligned} Z_{03} &= S_3 \frac{\partial Z_3}{\partial \theta_3} + a_{23} \frac{\partial Z_3}{\partial \alpha_{23}} + a_{34} \frac{\partial Z_3}{\partial \alpha_{34}} \\ &= S_3 s_{23} \underbrace{\bar{X}_3}_{X_{5612}} + a_{23} \underbrace{\bar{X}_3}_{-X^*_{5612}} + a_{34} Y_3 \end{aligned}$$

$$Z_{0561} = S_5 \frac{\partial Z_{561}}{\partial \theta_5} + S_6 \frac{\partial Z_{561}}{\partial \theta_6} + S_1 \frac{\partial Z_{561}}{\partial \theta_1} + a_{45} \frac{\partial Z_{561}}{\partial \alpha_{45}} + a_{56} \frac{\partial Z_{561}}{\partial \alpha_{56}} + a_{61} \frac{\partial Z_{561}}{\partial \alpha_{61}} + a_{12} \frac{\partial Z_{561}}{\partial \alpha_{12}}$$

$$= S_5 s_{45} X_{165} + S_6 \left[-\bar{X}_1 X^*_{56} - \bar{Y}_1 X_{56} \right] + S_1 s_{12} X_{561} + a_{45} Y_{165} + a_{56} \left[c_{45} Y_{16} - s_{45} c_5 Z_{16} \right] + a_{61} \left[c_{12} Y_{56} - s_{12} c_1 Z_{56} \right] + a_{12} Y_{561}$$

(d) Assuming that the term multiplied by the link length a_{34} in the previous equation could be expressed in terms of only the unknowns θ_1 and θ_5, write a second equation that also contains θ_1 and θ_5 as the only unknowns.

$$Z_{5612} = c_{34}$$

(e) How many sets of values for θ_1 and θ_5 do you expect to obtain which will simultaneously satisfy the equations specified in (c) and (d) above?

In general, 8 solutions will exist.

(f) Assuming that solutions have been obtained for the angles θ_5 and θ_1, explain how to obtain corresponding values for the angle θ_3.

$$X_{5612} = s_{34}\, s_3$$

$$Y_{5612} = s_{34}\, c_3$$

(g) Assuming that solutions have been obtained for the angles θ_5 and θ_1, explain how to obtain corresponding values for the angle θ_4.

$$X_{2165} = s_{34}\, s_4$$

$$Y_{2165} = s_{34}\, c_4$$

(h) Explain how to obtain corresponding values for the variable joint offsets.

Write the vector loop equation. Project it onto $\mathbf{a}_{23}$ to obtain an equation in the unknown S_4. Next, project the vector loop equation onto $\mathbf{a}_{34}$ to obtain an equation in the unknown S_2.

4. (20 pts.) The reverse kinematic analysis of a serial-linked manipulator can be performed using closed-form or numerical solutions.

a) The numerical solution approach has at least two important "disadvantages" compared to the closed-form approach. Name one of these disadvantages.

- Only one solution is obtained.
- The obtained solution is dependent on the starting values chosen for the joint variables.
- The function to be minimized may have terms with different units.

b) The numerical solution approach has at least two "advantages" compared to the closed-form approach. Name one of these advantages.

- A common computer program can be used to solve any manipulator geometry.
- The numerical approach can be used on redundant robots.

EMl 6281 – Robot Geometry I
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open book and notes
2 hours

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1. (25 pts.) A body is rotated from its initial pose by a rotation of 30° about an axis that passes through the origin and which is parallel to **i**+**j** followed by a rotation of θ about an axis parallel to the vector **m**. It is then returned to its original orientation by rotating it 45° about the fixed Y axis. Determine θ and **m**.

   The first rotation can be modeled by the quaternion q_1 as

$$q_1 = \cos\frac{30}{2} + \frac{1}{\sqrt{2}}\sin\frac{30}{2}(i+j)$$
$$= 0.9659 + 0.1830(i+j)\,.$$

   The second rotation will be represented by the quaternion q_2 and the third rotation can be modeled by the quaternion q_3 as

$$q_3 = \cos\frac{45}{2} + \sin\frac{45}{2}(j)$$
$$= 0.9239 + 0.3827(j)\,.$$

   The three rotations return the body to its original position. Thus,

$$q_3\, q_2\, q_1 = 1\,.$$

   The quaternion q_2 can be solved for as

$$q_2 = q_3^{-1}\, q_1^{-1}$$
$$= 0.8224 - 0.1691(i) - 0.5387(j) - 0.0700(k)\,.$$

   It is now necessary to determine the angle of rotation, θ, and the axis of rotation, **m**, which is represented by q_2. The scalar part of q_2 is equal to cos(θ/2) and the vector part of q_2 is equal to sin(θ/2)**m**. The rotation angle and axis are determined as

$$\theta = 69.26°$$

$$\mathbf{m} = -0.2972\,\mathbf{i} - 0.9469\,\mathbf{j} - 0.1231\,\mathbf{k}\,.$$

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2. (20 pts.) You are given the following three equations.

$$(A_2)\, y^2 + (A_1)\, y + (A_0) = 0$$
$$(B_3)\, y^3 + (B_2)\, y^2 + (B_1)\, y + (B_0) = 0$$
$$(C_3)\, y^3 + (C_2)\, y^2 + (C_1)\, y + (C_0) = 0$$

What condition must the coefficients A_2 through C_0 satisfy in order that values of *y* can be determined which satisfy the three equations simultaneously?

Multiplying the first equation by *y* will yield four equations that can be written in matrix form as

$$\begin{bmatrix} 0 & A_2 & A_1 & A_0 \\ A_2 & A_1 & A_0 & 0 \\ B_3 & B_2 & B_1 & B_0 \\ C_3 & C_2 & C_1 & C_0 \end{bmatrix} \begin{bmatrix} y^3 \\ y^2 \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

These equations will have a solution for *y* only if the equations are linearly dependent. Thus the necessary condition that the coefficients A_2 through C_0 must satisfy is expressed by the determinant

$$\begin{vmatrix} 0 & A_2 & A_1 & A_0 \\ A_2 & A_1 & A_0 & 0 \\ B_3 & B_2 & B_1 & B_0 \\ C_3 & C_2 & C_1 & C_0 \end{vmatrix} = 0 .$$

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3. (25 pts.) You are given the function

$$f(x) = 2\,x^2 + x - 6\ .$$

For what values of *x* will the function evaluate to the dual number $4 + 4\varepsilon$?

The dual number which represents the solution to the problem may be written as

$$x = a + \varepsilon\, b\ .$$

Substituting this into the function gives

$$f(a+\varepsilon b) = f(a) + \varepsilon b \left.\frac{df}{dx}\right|_{x=a}$$
$$= (2a^2 + a - 6) + \varepsilon b(4a+1)$$

Equating the real and dual parts gives

$$2a^2 + a - 6 = 4$$
$$b\,(4a + 1) = 4\ .$$

The first equation gives two values for *a* as

$$a = 2 \quad \text{or} \quad a = \text{-}2.5\ .$$

The second equation gives corresponding values for *b* as

$$b = \frac{4}{9} = 0.4444 \quad , \quad b = -\frac{4}{9} = -0.4444$$

Thus, the values of x that will evaluate the function $f(x) = 4 + 4\varepsilon$ are

$$x = 2 + 0.4444\varepsilon$$

$$x = -2.5 - 0.4444\varepsilon \ .$$

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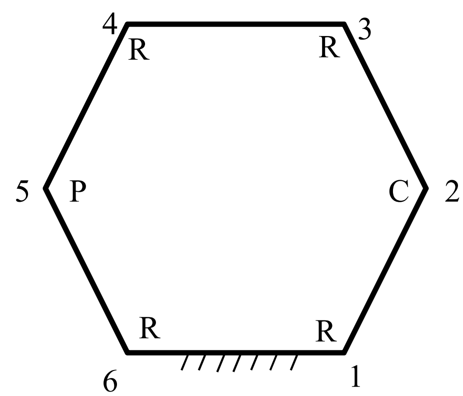
4. (30 pts.) A planar representation of a group 2 spatial mechanism is shown in the figure.



a) List the constant mechanism parameters.

a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{61}

α_{12}, α_{23}, α_{34}, α_{45}, α_{56}, a_{61}

S_1, S_3, S_4, S_6

θ_5

b) Assuming that the angle θ_6 is given, obtain an equation in terms of the unknowns θ_1 and θ_2 (with the exception of the term multiplied by a_{34}).

Dual angles will be substituted into the subsidiary spherical cosine law

$$Z_{61} = Z_{43} \ .$$

The dual part of Z_{61} can be written as

$$\begin{aligned} Z_{061} &= S_6 \frac{\partial Z_{61}}{\partial \theta_6} + S_1 \frac{\partial Z_{61}}{\partial \theta_1} + a_{56} \frac{\partial Z_{61}}{\partial \alpha_{56}} + a_{61} \frac{\partial Z_{61}}{\partial \alpha_{61}} + a_{12} \frac{\partial Z_{61}}{\partial \alpha_{12}} \\ &= S_6 \left[s_{56} X_{16} \right] + S_1 \left[s_{12} X_{61} \right] + a_{56} \left[Y_{16} \right] + a_{61} \left[c_{12} Y_6 - s_{12} Z_6 c_1 \right] + a_{12} \left[Y_{61} \right]. \end{aligned}$$

The term Z_{061} contains θ_1 as its only unknown.

The dual part of Z_{43} can be written as

$$\begin{aligned} Z_{043} &= S_4 \frac{\partial Z_{43}}{\partial \theta_4} + S_3 \frac{\partial Z_{43}}{\partial \theta_3} + a_{45} \frac{\partial Z_{43}}{\partial \alpha_{45}} + a_{34} \frac{\partial Z_{43}}{\partial \alpha_{34}} + a_{23} \frac{\partial Z_{43}}{\partial \alpha_{23}} \\ &= S_4 \left[s_{12} X_{34} \right] + S_3 \left[s_{23} X_{43} \right] + a_{45} \left[Y_{34} \right] + a_{34} \left[c_{23} \bar{Y}_4 - s_{23} \bar{Z}_4 c_3 \right] + a_{23} \left[Y_{43} \right]. \end{aligned}$$

Substituting for expressions in Z_{043} gives

$$Z_{043} = S_4 \left[s_{12} X_{165} \right] + S_3 \left[s_{23} X_{612} \right] + a_{45} \left[-X^*_{165} \right] + a_{34} \left[c_{23} \bar{Y}_4 - s_{23} \bar{Z}_4 c_3 \right] + a_{23} \left[-X^*_{612} \right].$$

The term multiplied by a_{34} could be manipulated further to express it in terms of θ_1 and θ_2, but it is not required for this problem.

c) Write a spherical equation which also contains the angles θ_1 and θ_2 as the only unknowns.

The spherical cosine law that contains θ_1 and θ_2 as the only unknowns is

$$Z_{5612} = c_{34}.$$

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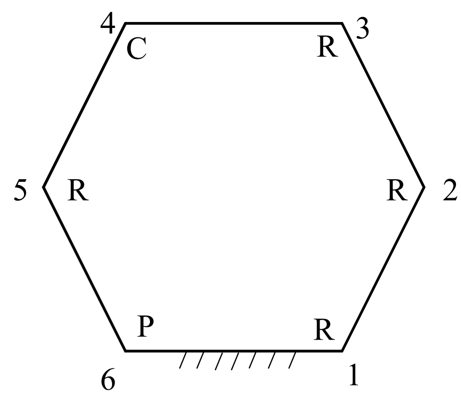
Exam 3 – Fall 2013

open book and notes
2 hours

1. (30 pts.) A planar representation of a group 2 spatial mechanism is shown in the figure.



a) List the constant mechanism parameters.

a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{61}

α_{12}, α_{23}, α_{34}, α_{45}, α_{56}, a_{61}

S_1, S_2, S_3, S_5

θ_6

b) Assuming that the angle θ_1 is given, obtain an equation for the spatial mechanism in terms of the unknowns θ_2 and θ_3.

Dual angles will be substituted into the subsidiary spherical cosine law

$$Z_{321} = Z_5 \ .$$

The dual part of Z_5 can be written as

$$\begin{aligned} Z_{05} &= S_5 \frac{\partial Z_5}{\partial \theta_5} + a_{45} \frac{\partial Z_5}{\partial \alpha_{45}} + a_{56} \frac{\partial Z_6}{\partial \alpha_{56}} \\ &= S_5 \left[s_{56} X_5 \right] + a_{45} \left[\bar{Y}_5 \right] + a_{56} \left[Y_5 \right]. \end{aligned}$$

The dual part of Z_{321} can be written as

$$\begin{aligned} Z_{0321} &= S_3 \frac{\partial Z_{321}}{\partial \theta_3} + S_2 \frac{\partial Z_{321}}{\partial \theta_2} + S_1 \frac{\partial Z_{321}}{\partial \theta_1} + a_{34} \frac{\partial Z_{321}}{\partial \alpha_{34}} + a_{23} \frac{\partial Z_{321}}{\partial \alpha_{23}} + a_{12} \frac{\partial Z_{321}}{\partial \alpha_{12}} + a_{61} \frac{\partial Z_{321}}{\partial \alpha_{61}} \\ &= S_3 \left[s_{34} X_{123} \right] + S_2 \left[-\bar{X}_1 X^*_{32} - \bar{Y}_1 X_{32} \right] + S_1 \left[s_{61} X_{321} \right] \\ &\quad + a_{34} \left[Y_{123} \right] + a_{23} \left[-s_{34} c_3 Z_{12} + c_{34} Y_{12} \right] + a_{12} \left[-s_{61} c_1 Z_{32} + c_{61} Y_{32} \right] + a_{61} \left[Y_{321} \right]. \end{aligned}$$

All the terms in Z_{0321} expressed in terms of the given parameters and the angles θ_2 and θ_3. Subsidiary spherical laws can be used to substitute for the S_6 and a_{56} terms to give

$$Z_{05} = S_6 \left[s_{56} X_{3216} \right] + a_{45} \left[\bar{Y}_5 \right] + a_{56} \left[-X^*_{3216} \right].$$

The term $\bar{Y}_5$ may be written as

$$\bar{Y}_5 = -\frac{s_{45}{}^2 c_{56} + s_{45} c_{45} s_{56} c_5}{s_{45}}.$$

Adding and subtracting $c_{45}^2 c_{56}$ to the numerator gives

$$\bar{Y}_5 = -\frac{s_{45}{}^2 c_{56} + c_{45}{}^2 c_{56} - c_{45}\left(c_{45}c_{56} - s_{45}s_{56}c_5\right)}{s_{45}}$$

$$= -\frac{c_{56} - c_{45}Z_5}{s_{45}}.$$

Substituting $Z_5 = Z_{321}$ gives the final result

$$Z_{05} = S_5\left[s_{56}X_{3216}\right] + a_{45}\left[-\frac{c_{56} - c_{45}Z_{321}}{s_{45}}\right] + a_{56}\left[-X^*_{3216}\right].$$

All terms in Z_{05} and Z_{0321} have been expressed in terms of the unknown angles θ_2 and θ_3.

c) Obtain a second equation from the equivalent spherical mechanism that has the angles θ_2 and θ_3 as the only unknowns.

$$Z_{3216} = c_{45}\ .$$

2. (20 pts.) You are given the following two equations.

$$A_1\, x^2\, y + A_2\, x^2 + A_3\, x\, y + A_4\, x + A_5 = 0$$
$$B_1\, x^2\, y^2 + B_2\, x\, y + B_3\, y + B_4 = 0$$

where the coefficients A_1 through B_4 are known constants. Explain how to obtain solutions for x and y.

The two equations can be written as

$$(A_1\, x^2 + A_3\, x)\, y + (A_2\, x^2 + A_4\, x + A_5) = 0$$
$$(B_1\, x^2)\, y^2 + (B_2\, x + B_3)\, y + (B_4) = 0\ .$$

These two equations can be written as

$$L_1\, y + L_2 = 0$$
$$M_1\, y^2 + M_2\, y + M_3 = 0$$

where

$$L_1 = A_1\, x^2 + A_3\, x$$
$$L_2 = A_2\, x^2 + A_4\, x + A_5$$
$$M_1 = B_1\, x^2$$
$$M_2 = B_2\, x + B_3$$
$$M_3 = B_4\ .$$

Multiplying the first equation by y and writing the three equations in matrix form gives

$$\begin{bmatrix} 0 & L_1 & L_2 \\ L_1 & L_2 & 0 \\ M_1 & M_2 & M_3 \end{bmatrix} \begin{bmatrix} y^2 \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} .$$

For a solution to exist, the three equations must be linearly dependent. Thus it must be the case that

$$\begin{vmatrix} 0 & L_1 & L_2 \\ L_1 & L_2 & 0 \\ M_1 & M_2 & M_3 \end{vmatrix} = 0 .$$

Expanding the determinant will yield a 6th order polynomial in x.

Corresponding values for y can be obtained from the first equation as

$$y = \frac{-\left(\mathrm{A}_2 x^2 + \mathrm{A}_4 x + \mathrm{A}_5\right)}{\mathrm{A}_1 x^2 + \mathrm{A}_3 x} .$$

3. (25 pts.) A body is rotated 42° about an axis parallel to 3**i**+3**j**-2**k** which passes through the origin. It is then rotated 30° about the Y axis. Finally, it is returned to its original orientation by rotating 20° about the X axis followed by a rotation of angle θ about an axis **m** which passes through the origin. Determine θ and **m**.

The first rotation can be represented by the quaternion q_1 which is written as

$$q_1 = \cos\left(\frac{42°}{2}\right) + \sin\left(\frac{42°}{2}\right) \frac{[3i + 3j - 2k]}{\sqrt{22}}$$
$$= 0.9336 + 0.2292i + 0.2292j - 0.1528k .$$

The second rotation can be represented by the quaternion q_2 which is written as

$$q_2 = \cos\left(\frac{30°}{2}\right) + \sin\left(\frac{30°}{2}\right) j$$
$$= 0.9659 + 0.2588j .$$

The third rotation can be represented by the quaternion q_3 which is written as

$$q_3 = \cos\left(\frac{20°}{2}\right) + \sin\left(\frac{20°}{2}\right) i$$
$$= 0.9848 + 0.1736i .$$

The fourth rotation will be represented by quaternion q_4 where

$$q_4 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\left[m_x i + m_y j + m_z k\right].$$

The body returns to its original orientation after the four rotations. Thus it must be the case that

$$q_4\, q_3\, q_2\, q_1 = 1 \,.$$

Solving for q_4 gives

$$\begin{aligned} q_4 &= q_1^{-1}\, q_2^{-1}\, q_3^{-1} \\ &= 0.7981 - 0.3254i - 0.4919\,j + 0.1234k \,. \end{aligned}$$

The angle θ and the axis **m** can be obtained as

$$\cos\left(\frac{\theta}{2}\right) = 0.7981$$

$$\sin\left(\frac{\theta}{2}\right) m_x = -0.3254$$

$$\sin\left(\frac{\theta}{2}\right) m_y = -0.4919$$

$$\sin\left(\frac{\theta}{2}\right) m_z = 0.1234 \,.$$

Thus

$$\theta = 1.2934 \quad \text{rad} = 74.1°$$

$$\boldsymbol{m} = \begin{bmatrix} -0.5400 \\ -0.8164 \\ 0.2048 \end{bmatrix}.$$

4. (25 pts.) Evaluate the function $f(x,y) = 20\, x^3\, y^2 + 11\, x^2\, y - 20\, x + 5\, y + 7$ for the case where $x = 3+2\varepsilon$ and $y = -4 - 7\varepsilon$.

The function can be evaluated using the Taylor series expansion as

$$\mathrm{f}(\mathrm{a}+\mathrm{a}_0\,\varepsilon, \mathrm{b}+\mathrm{b}_0\,\varepsilon) = \mathrm{f}(\mathrm{a},\mathrm{b}) + \mathrm{a}_0\,\varepsilon \left.\frac{\partial f}{\partial x}\right|_{\substack{x=a\\y=b}} + \mathrm{b}_0\,\varepsilon \left.\frac{\partial f}{\partial y}\right|_{\substack{x=a\\y=b}} .$$

Thus

$$\begin{aligned} \mathrm{f}(3+2\varepsilon, -4-7\varepsilon) &= \mathrm{f}(3, -4) + 2\varepsilon \left.\frac{\partial f}{\partial x}\right|_{\substack{x=3\\y=-4}} - 7\varepsilon \left.\frac{\partial f}{\partial y}\right|_{\substack{x=3\\y=-4}} \\ &= 8171 + 2\varepsilon\,(8356) - 7\varepsilon\,(-4216) \\ &= 8171 + 46224\,\varepsilon \end{aligned}$$

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1. (15 pts.) Evaluate the function

$$f(x,y) = 2x\cos(y) + 3x^2 y^4$$

when $x = 4 - 3\varepsilon$ and $y = 2 + 5\varepsilon$.

The function can be evaluated using the Taylor series expansion as

$$f(a + a_0\,\varepsilon, b + b_0\,\varepsilon) = f(a,b) + a_0\,\varepsilon \left.\frac{\partial f}{\partial x}\right|_{\substack{x=a\\y=b}} + b_0\,\varepsilon \left.\frac{\partial f}{\partial y}\right|_{\substack{x=a\\y=b}} .$$

Thus

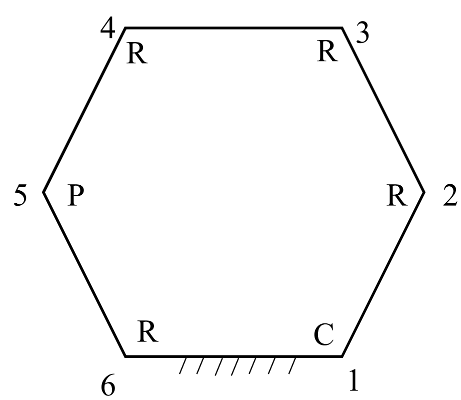
$$f(4\text{-}3\varepsilon,\ 2\text{+}5\varepsilon) = f(4,\ 2) - 3\varepsilon \left.\frac{\partial f}{\partial x}\right|_{\substack{x=4\\y=2}} + 5\varepsilon \left.\frac{\partial f}{\partial y}\right|_{\substack{x=4\\y=2}}$$

$$= [8\cos(2) + 768] - 3\varepsilon\,[2\cos(2) + 384] + 5\varepsilon\,[-8\sin(2) + 1536]$$

$$= [764.67] - 3\varepsilon\,[383.17] + 5\varepsilon\,[1528.73]$$

$$= 764.67 + 6494.12\,\varepsilon$$

2. (30 pts.) A planar representation of a group 2 spatial mechanism is shown in the figure.



a) List the constant mechanism parameters.

a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{61}

α_{12}, α_{23}, α_{34}, α_{45}, α_{56}, a_{61}

S_2, S_3, S_4, S_6

θ_5

b) Assuming that the angle θ_2 is given, obtain an equation for the spatial mechanism in terms of the unknowns θ_3 and θ_4.

Dual angles will be substituted into the subsidiary spherical cosine law

$$Z_{234} = Z_6 .$$

The dual part of Z_6 can be written as

$$Z_{06} = S_6 \frac{\partial Z_6}{\partial \theta_6} + a_{56} \frac{\partial Z_6}{\partial \alpha_{56}} + a_{61} \frac{\partial Z_6}{\partial \alpha_{61}}$$
$$= S_6 \left[s_{56} \bar{X}_6 \right] + a_{56} \left[\bar{Y}_6 \right] + a_{61} \left[Y_6 \right].$$

The dual part of Z_{234} can be written as

$$Z_{0234} = S_2 \frac{\partial Z_{234}}{\partial \theta_2} + S_3 \frac{\partial Z_{234}}{\partial \theta_3} + S_4 \frac{\partial Z_{234}}{\partial \theta_4} + a_{12} \frac{\partial Z_{234}}{\partial \alpha_{12}} + a_{23} \frac{\partial Z_{234}}{\partial \alpha_{23}} + a_{34} \frac{\partial Z_{234}}{\partial \alpha_{34}} + a_{45} \frac{\partial Z_{234}}{\partial \alpha_{45}}$$
$$= S_2 \left[s_{12} X_{432} \right] + S_3 \left[-\bar{X}_4 X_{23}^* - \bar{Y}_4 X_{23} \right] + S_4 \left[s_{45} X_{234} \right]$$
$$+ a_{12} \left[Y_{432} \right] + a_{23} \left[-s_{12} c_2 Z_{43} + c_{12} Y_{43} \right] + a_{34} \left[-s_{45} c_4 Z_{23} + c_{45} Y_{23} \right] + a_{45} \left[Y_{234} \right].$$

All the terms in Z_{0234} expressed in terms of the given parameters and the angles θ_3 and θ_4. Subsidiary spherical laws can be used to substitute for the S_6 and a_{56} terms to give

$$Z_{06} = S_6 \left[s_{56} X_{2345} \right] + a_{56} \left[-X_{2345}^* \right] + a_{61} \left[Y_6 \right].$$

The term Y_6 may be written as

$$Y_6 = -\frac{s_{61}{}^2 c_{56} + s_{61} c_{61} s_{56} c_6}{s_{61}}.$$

Adding and subtracting $c_{61}{}^2 c_{56}$ to the numerator gives

$$Y_6 = -\frac{s_{61}{}^2 c_{56} + c_{61}{}^2 c_{56} - c_{61} \left(c_{61} c_{56} - s_{61} s_{56} c_6 \right)}{s_{61}}$$
$$= -\frac{c_{56} - c_{61} Z_6}{s_{61}}.$$

Substituting $Z_6 = Z_{432}$ gives the final result

$$Z_{06} = S_6 \left[s_{56} X_{2345} \right] + a_{56} \left[-\frac{c_{56} - c_{61} Z_{432}}{s_{61}} \right] + a_{56} \left[-X_{2345}^* \right].$$

All terms in Z_{06} and Z_{0234} have been expressed in terms of the unknown angles θ_3 and θ_4.

c) Obtain a second equation from the equivalent spherical mechanism that has the angles θ_3 and θ_4 as the only unknowns.

$$Z_{2345} = c_{61}.$$

3. (15 pts.) Use LaPlace's theorem to evaluate the following determinant as the sum of products of 2×2 determinants times 3×3 determinants:

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 & d_1 \\ a_2 & b_2 & c_2 & 0 & d_2 \\ e_1 & f_1 & 0 & g_1 & h_1 \\ e_2 & f_2 & 0 & g_2 & h_2 \\ e_3 & f_3 & 0 & g_3 & h_3 \end{vmatrix}.$$

The determinant can be evaluated as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 & d_1 \\ a_2 & b_2 & c_2 & 0 & d_2 \\ e_1 & f_1 & 0 & g_1 & h_1 \\ e_2 & f_2 & 0 & g_2 & h_2 \\ e_3 & f_3 & 0 & g_3 & h_3 \end{vmatrix} = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\begin{vmatrix} e_1 & g_1 & h_1 \\ e_2 & g_2 & h_2 \\ e_3 & g_3 & h_3 \end{vmatrix} - \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}\begin{vmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \\ e_3 & f_3 & g_3 \end{vmatrix}.$$

4. (20 pts.) A body is rotated 28° about the Z axis of a fixed coordinate system. It is then rotated 76° about its modified X axis.

   You wish to return it to its original position by rotating it 50° about the original fixed coordinate system Y axis and then by θ about an axis $\mathbf{s} = s_x\mathbf{i} + s_y\,\mathbf{j} + s_z\,\mathbf{k}$.

   Determine θ and **s**.

   The rotation matrix that will transform points from the body coordinate system to the fixed coordinate system after the first two rotations can be written as

$$ {}^F_2\mathbf{R} = \begin{bmatrix} \cos(28°) & -\sin(28°) & 0 \\ \sin(28°) & \cos(28°) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(76°) & -\sin(76°) \\ 0 & \sin(76°) & \cos(76°) \end{bmatrix} = \begin{bmatrix} 0.8829 & -0.1136 & 0.4555 \\ 0.4695 & 0.2136 & -0.8567 \\ 0 & 0.9703 & 0.2419 \end{bmatrix}.$$

   This rotation matrix is equivalent to having the body perform a single rotation of angle α about a unit vector **m**. The cosine of the angle α is calculated as

$$\cos\alpha = \frac{r_{11} + r_{22} + r_{33} - 1}{2} = 0.1692$$

   and

$$\alpha = 80.26° .$$

   The coordinates of the vector **m** are calculated from text equations (2.66) through (2.68) as

$$m_x = 0.9269, \quad m_y = 0.2311, \quad m_z = 0.2958 .$$

   The result of the first two rotations can thus be represented by the quaternion q_1 where

$$q_1 = \cos\left(\frac{80.26°}{2}\right) + \sin\left(\frac{80.26°}{2}\right)(0.9269i + 0.2311j + 0.2958k)$$
$$= 0.7646 + 0.5974i + 0.1489j + 0.1906k \quad .$$

The third rotation can be represented by the quaternion q_2 where

$$q_2 = \cos\left(\frac{50°}{2}\right) + \sin\left(\frac{50°}{2}\right)j \quad .$$
$$= 0.9063 + 0.4226j$$

The final rotation will be represented by the quaternion q_3. Since the body is returned to its original orientation, it must be the case that

$$q_3\ q_2\ q_1 = 1$$

and thus

$$q_3 = q_1^{-1}\ \ q_2^{-1}\ .$$

Solving for q_3 gives

$$q_3 = 0.6300 - 0.6220\ i - 0.4581\ j + 0.0797\ k\ .$$

This quaternion represents a rotation of 101.90° about the axis $\mathbf{s} = -0.8009\mathbf{i} - .5899\mathbf{j} + 0.1026\mathbf{k}$.

Note: Alternatively, the second rotation can be modeled by a quaternion where the angle of rotation is 76° about the axis $\cos(28°)\mathbf{i} + \sin(28°)\mathbf{j}$ as measured in the fixed coordinate system.

5. (20 pts.) You are given the following two equations.

$$A_1\, x y^3 + A_2\, x^2 + A_3\, x + A_4\, y + A_5 = 0$$
$$B_1\, x^4 y^2 + B_2\, x\, y + B_3\, y + B_4 = 0$$

where the coefficients A_1 through B_4 are known constants. Explain how to obtain solutions for x and y.

The two equations can be written as

$$(A_1\, x)\, y^3 + (A_4)\, y + (A_2\, x^2 + A_4\, x + A_5) = 0$$
$$(B_1\, x^4)\, y^2 + (B_2\, x + B_3)\, y + (B_4) = 0\ .$$

These two equations can be written as

$$L_1\, y^3 + L_2\, y + L_3 = 0$$
$$M_1\, y^2 + M_2\, y + M_3 = 0$$

where

$L_1 = A_1 x$
$L_2 = A_4$
$L_3 = A_2 x^2 + A_4 x + A_5$
$M_1 = B_1 x^4$
$M_2 = B_2 x + B_3$
$M_3 = B_4$.

Multiplying the first equation by y and the second equation by y and y^2 and writing the total of five equations in matrix form gives

$$\begin{bmatrix} 0 & L_1 & 0 & L_2 & L_3 \\ L_1 & 0 & L_2 & L_3 & 0 \\ 0 & 0 & M_1 & M_2 & M_3 \\ 0 & M_1 & M_2 & M_3 & 0 \\ M_1 & M_2 & M_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} y^4 \\ y^3 \\ y^2 \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

For a solution to exist, the five equations must be linearly dependent. Thus it must be the case that

$$\begin{vmatrix} 0 & L_1 & 0 & L_2 & L_3 \\ L_1 & 0 & L_2 & L_3 & 0 \\ 0 & 0 & M_1 & M_2 & M_3 \\ 0 & M_1 & M_2 & M_3 & 0 \\ M_1 & M_2 & M_3 & 0 & 0 \end{vmatrix} = 0 .$$

Expanding the determinant will yield a polynomial in x.

Corresponding values for y can be obtained by writing the first four equations as

$$\begin{bmatrix} 0 & L_1 & 0 & L_2 \\ L_1 & 0 & L_2 & L_3 \\ 0 & 0 & M_1 & M_2 \\ 0 & M_1 & M_2 & M_3 \end{bmatrix} \begin{bmatrix} y^4 \\ y^3 \\ y^2 \\ y \end{bmatrix} = \begin{bmatrix} -L_3 \\ 0 \\ -M_3 \\ 0 \end{bmatrix} .$$

The 4×4 matrix is evaluated for each value of x and the corresponding value for y can be found as the fourth element of the following vector:

$$\begin{bmatrix} y^4 \\ y^3 \\ y^2 \\ y \end{bmatrix} = \begin{bmatrix} 0 & L_1 & 0 & L_2 \\ L_1 & 0 & L_2 & L_3 \\ 0 & 0 & M_1 & M_2 \\ 0 & M_1 & M_2 & M_3 \end{bmatrix}^{-1} \begin{bmatrix} -L_3 \\ 0 \\ -M_3 \\ 0 \end{bmatrix} .$$

EMl 6281 – Robot Geometry I open book and notes
Exam 3 – Fall 2016 2 hours

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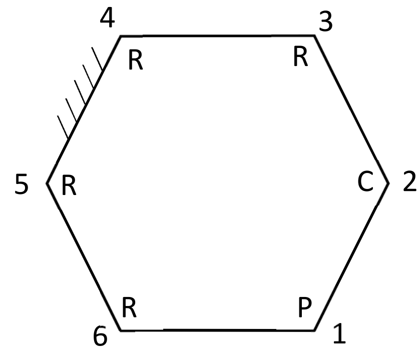
1. (25 pts.) A planar representation of a spatial mechanism is show in the figure. Link 45 is fixed to ground.



(a) What group is this spatial mechanism? Explain.

It is a group 2 spatial mechanism. It is a one degree of freedom spatial mechanism whose equivalent spherical mechanism has two degrees of freedom.

(b) List the constant mechanism parameters.

$a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{61}$

$\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{45}, \alpha_{56}, \alpha_{61}$

S_3, S_4, S_5, S_6

θ_1

(c) The angle θ_5 is given. Obtain an equation from the spatial mechanism that contains the angles θ_4 and θ_6 as the only unknowns.

Since the prismatic and cylindric joint are adjacent, the vector loop equation will be projected onto the vector $\mathbf{a}_{12}$ in order to obtain an equation that does not have the unknown joint offsets S_1 and S_2.

$$a_{12}(\mathbf{a}_{12} \cdot \mathbf{a}_{12}) + a_{23}(\mathbf{a}_{23} \cdot \mathbf{a}_{12}) + S_3(\mathbf{S}_3 \cdot \mathbf{a}_{12}) + a_{34}(\mathbf{a}_{34} \cdot \mathbf{a}_{12}) + S_4(\mathbf{S}_4 \cdot \mathbf{a}_{12}) + a_{45}(\mathbf{a}_{45} \cdot \mathbf{a}_{12})$$

$$+ S_5(\mathbf{S}_5 \cdot \mathbf{a}_{12}) + a_{56}(\mathbf{a}_{56} \cdot \mathbf{a}_{12}) + S_6(\mathbf{S}_6 \cdot \mathbf{a}_{12}) + a_{61}(\mathbf{a}_{61} \cdot \mathbf{a}_{12}) = 0$$

The direction cosines from set 1 and set 12 are listed here for reference.

Set 1

$\underline{S}_1$	0,	0,	1	$\underline{a}_{12}$	1,	0,	0
$\underline{S}_2$	0,	$-s_{12}$,	c_{12}	$\underline{a}_{23}$	c_2,	s_2c_{12},	U_{21}
$\underline{S}_3$	$\bar{X}_2$,	$\bar{Y}_2$,	$\bar{Z}_2$	$\underline{a}_{34}$	W_{32},	$-U^*_{321}$,	U_{321}
$\underline{S}_4$	X_{32},	Y_{32},	Z_{32}	$\underline{a}_{45}$	W_{432},	$-U^*_{4321}$,	U_{4321}
$\underline{S}_5$	X_{432},	Y_{432},	Z_{432}	$\underline{a}_{56}$	W_{5432},	$-U^*_{54321}$,	U_{54321}
$\underline{S}_6$	X_{5432},	Y_{5432},	Z_{5432}	$\underline{a}_{61}$	c_1,	$-s_1$,	0

Set 12								
$\underline{S}_2$	(0,	0,	1)		$\underline{a}_{12}$	(1,	0,	0)
$\underline{S}_1$	(0,	s_{12},	c_{12})		$\underline{a}_{61}$	(c_1,	$-s_1c_{12}$,	U_{12})
$\underline{S}_6$	(X_1,	$-Y_1$,	Z_1)		$\underline{a}_{56}$	(W_{61},	U^*_{612},	U_{612})
$\underline{S}_5$	(X_{61},	$-Y_{61}$,	Z_{61})		$\underline{a}_{45}$	(W_{561},	U^*_{5612},	U_{5612})
$\underline{S}_4$	(X_{561},	$-Y_{561}$,	Z_{561})		$\underline{a}_{34}$	(W_{4561},	U^*_{45612},	U_{45612})
$\underline{S}_3$	(X_{4561},	$-Y_{4561}$,	Z_{4561})		$\underline{a}_{23}$	(c_2,	s_2,	0)

Evaluating the scalar products gives

$$a_{12} + a_{23}(c_2) + S_3(X_{4561}) + a_{34}(W_{4561}) + S_4(X_{561}) + a_{45}(W_{561})$$

$$+ S_5(X_{61}) + a_{56}(W_{61}) + S_6(X_1) + a_{61}(c_1) = 0$$

All terms with the exception of the term multiplied by a_{23} are expressed in terms of the known angles (θ_5 and θ_1) and the two unknown angles (θ_4 and θ_6).

One fundamental sine-cosine law for a spherical hexagon is

$$Y_{4561} = s_{23}\, c_2.$$

Solving for c_2 and substituting into the previous equation gives the result

$$a_{12} + a_{23}\left(\frac{Y_{4561}}{s_{23}}\right) + S_3(X_{4561}) + a_{34}(W_{4561}) + S_4(X_{561}) + a_{45}(W_{561})$$

$$+ S_5(X_{61}) + a_{56}(W_{61}) + S_6(X_1) + a_{61}(c_1) = 0$$

(d) Write an equation from the equivalent spherical mechanism that also has the angles θ_4 and θ_6 as the only unknowns.

$$Z_{4561} = c_{23}$$

2. (20 pts.) You are given the following two equations in the unknowns *x* and *y*. Explain how to obtain all the values of *x* and *y* that will simultaneously solve both equations.

$$A\, x^3 y^3 + B\, x^2 y^3 + C\, x\, y^2 + D\, x + E = 0$$

$$F\, x^4 y^2 + G\, y + H = 0$$

The two equations are regrouped as

$$(A\, x^3 + B\, x^2)\, y^3 + (C\, x)\, y^2 + (D\, x + E) = 0$$

$$(F\, x^4)\, y^2 + (G)\, y + (H) = 0$$

Multiplying the first equation by *y* and the second by *y* and y^2 will give a total of five equations that can be written in matrix form as

$$\begin{bmatrix} 0 & Ax^3+Bx^2 & Cx & 0 & Dx+E \\ 0 & 0 & Fx^4 & G & H \\ Ax^3+Bx^2 & Cx & 0 & Dx+E & 0 \\ 0 & Fx^4 & G & H & 0 \\ Fx^4 & G & H & 0 & 0 \end{bmatrix} \begin{bmatrix} y^4 \\ y^3 \\ y^2 \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For solutions for y to exist, it is necessary that the five equations be linearly dependent. This will occur if the determinant of the coefficient matrix equals zero. Thus it is necessary that

$$\begin{vmatrix} 0 & Ax^3+Bx^2 & Cx & 0 & Dx+E \\ 0 & 0 & Fx^4 & G & H \\ Ax^3+Bx^2 & Cx & 0 & Dx+E & 0 \\ 0 & Fx^4 & G & H & 0 \\ Fx^4 & G & H & 0 & 0 \end{vmatrix} = 0 .$$

This will yield a polynomial in x. The solution of this polynomial will yield all the solutions for x.

Corresponding values for y for each value of x can be obtained by writing the first four equations as

$$\begin{bmatrix} 0 & Ax^3+Bx^2 & Cx & 0 \\ 0 & 0 & Fx^4 & G \\ Ax^3+Bx^2 & Cx & 0 & Dx+E \\ 0 & Fx^4 & G & H \end{bmatrix} \begin{bmatrix} y^4 \\ y^3 \\ y^2 \\ y \end{bmatrix} = \begin{bmatrix} -(Dx+E) \\ -H \\ 0 \\ 0 \end{bmatrix}$$

and corresponding values for y are obtained from the fourth component of the vector obtained as

$$\begin{bmatrix} y^4 \\ y^3 \\ y^2 \\ y \end{bmatrix} = \left(\begin{bmatrix} 0 & Ax^3+Bx^2 & Cx & 0 \\ 0 & 0 & Fx^4 & G \\ Ax^3+Bx^2 & Cx & 0 & Dx+E \\ 0 & Fx^4 & G & H \end{bmatrix} \right)^{-1} \begin{bmatrix} -(Dx+E) \\ -H \\ 0 \\ 0 \end{bmatrix}.$$

3. (20 pts.) You have a spherical pentagon. Write the equation you would obtain if you substituted dual angles into the subsidiary cosine law $Z_{21} = Z_4$.

The dual part of Z_4 can be written as

$$Z_{04} = S_4 \frac{\partial Z_4}{\partial \theta_4} + a_{34} \frac{\partial Z_4}{\partial \alpha_{34}} + a_{45} \frac{\partial Z_4}{\partial \alpha_{45}}$$
$$= S_4 \left[s_{34} \bar{X}_4 \right] + a_{34} \left[\bar{Y}_4 \right] + a_{45} \left[Y_4 \right].$$

The dual part of Z_{21} can be written as

$$Z_{021} = S_2 \frac{\partial Z_{21}}{\partial \theta_2} + S_1 \frac{\partial Z_{21}}{\partial \theta_1} + a_{23} \frac{\partial Z_{21}}{\partial \alpha_{23}} + a_{12} \frac{\partial Z_{21}}{\partial \alpha_{12}} + a_{51} \frac{\partial Z_{21}}{\partial \alpha_{51}}$$
$$= S_2 \left[s_{23} X_{12} \right] + S_1 \left[s_{51} X_{21} \right] + a_{23} \left[Y_{12} \right] + a_{12} \left[-s_{51} c_1 \bar{Z}_2 + c_{51} \bar{Y}_2 \right] + a_{51} \left[Y_{21} \right].$$

The resulting expression can be written as

$$S_2 \left[s_{23} X_{12} \right] + S_1 \left[s_{51} X_{21} \right] + a_{23} \left[Y_{12} \right] + a_{12} \left[-s_{51} c_1 \bar{Z}_2 + c_{51} \bar{Y}_2 \right] + a_{51} \left[Y_{21} \right]$$
$$= S_4 \left[s_{34} \bar{X}_4 \right] + a_{34} \left[\bar{Y}_4 \right] + a_{45} \left[Y_4 \right].$$

4. (20 pts.) A rigid body has been rotated 90° about its *X* axis, then 40° about its *Y* axis, and then finally by an angle θ about an axis parallel to the unit vector $\mathbf{s} = s_x\mathbf{i} + s_y\mathbf{j} + s_k\mathbf{j}$. The body could have moved directly from the initial orientation to the final orientation by rotating 90° about the *Z* axis. Determine θ and **s**.

The first three rotations can be modeled by the three quaternions q_1, q_2, and q_3 where

$$q_1 = \cos 45° + \sin 45°\, i = 0.7071 + 0.7071 i$$
$$q_2 = \cos 20° + \sin 20°\, j = 0.9397 + 0.3420\, j$$
$$q_3 = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left(s_x i + s_y j + s_z k \right).$$

The direct rotation will be modeled by the quaternion q_4 where

$$q_4 = \cos 45° + \sin 45°\, k = 0.7071 + 0.7071 k\ .$$

Since the first three rotations are equivalent to the direct rotation, it must be the case that

$q_3\ q_2\ q_1 = q_4$.

Solving this equation for q_3 gives

$q_3 = q_4\ (q_1)^{-1}\ (q_2)^{-1}$.

$q_3 = 0.2988 – 0.2988\ i\ -0.6409\ j + 0.6409\ k$.

The quaternion q_3 represents the rotation of angle θ about the vector **s**. The cosine of θ/2 is written as

cos θ/2 = 0.2988

and thus

$$\theta = 145.22° .$$

The unit vector along the direction of the rotation axis is

$$\mathbf{s} = -0.3132\ \mathbf{i} - 0.6715\ \mathbf{j} + 0.6715\ \mathbf{k} .$$

5. (15 pts.) Evaluate the function

$$f(x,y,z) = 2\,x\,y\cos(z) + 3\,x\,y^2\,z$$

when $x = 2 - 3\varepsilon$, $y = 3 - 1\varepsilon$, and $z = 6\varepsilon$.

The function can be evaluated using the Taylor series expansion as

$$\mathrm{f}(\mathrm{a}+\mathrm{a}_0\,\varepsilon, \mathrm{b}+\mathrm{b}_0\,\varepsilon, \mathrm{c}+\mathrm{c}_0\varepsilon) = \mathrm{f}(\mathrm{a},\mathrm{b},\mathrm{c}) + \mathrm{a}_0\,\varepsilon \left.\frac{\partial f}{\partial x}\right|_{\substack{x=a\\y=b\\z=c}} + \mathrm{b}_0\,\varepsilon \left.\frac{\partial f}{\partial y}\right|_{\substack{x=a\\y=b\\z=c}} + \mathrm{c}_0\varepsilon \left.\frac{\partial f}{\partial z}\right|_{\substack{x=a\\y=b\\z=c}} .$$

Thus

$$\mathrm{f}(2\text{-}3\varepsilon, 3\text{-}1\varepsilon, 0\text{+}6\varepsilon) = \mathrm{f}(2, 3, 0) - 3\varepsilon \left.\frac{\partial f}{\partial x}\right|_{\substack{x=2\\y=3\\z=0}} - 1\varepsilon \left.\frac{\partial f}{\partial y}\right|_{\substack{x=2\\y=3\\z=0}} + 6\varepsilon \left.\frac{\partial f}{\partial z}\right|_{\substack{x=2\\y=3\\z=0}}$$

$$= [2\,(2)(3)\cos(0) + 3\,(2)(3)^2(0)] - 3\varepsilon\,[2\,(3)\cos(0)] - 1\varepsilon\,[2\,(2)\cos(0)]$$

$$+ 6\varepsilon\,[-2\,(2)(3)\sin(0) + 3(2)(3)^2]$$

$$= [12] - 3\varepsilon\,[6] - 1\varepsilon\,[4] + 6\varepsilon\,[54]$$

$$= 12 + 302\,\varepsilon$$

EML 6281 – Robot Geometry I — 15 December 1998

Final Exam — open book and notes

1) *(15 pts) Evaluate the function $f(x,y) = 2x^3y + 4e^xy^2$ when $x=-3-2\epsilon$ and $y=1+3\epsilon$.*

$$f(-3-2\varepsilon, 1+3\varepsilon) = f(-3,1) - 2\varepsilon \left.\frac{\partial f}{\partial x}\right|_{\substack{x=-3\\y=1}} + 3\varepsilon \left.\frac{\partial f}{\partial y}\right|_{\substack{x=-3\\y=1}}$$

$$f(-3,1) = -53.80$$

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=-3\\y=1}} = \left.(6x^2y + 4e^xy^2)\right|_{\substack{x=-3\\y=1}} = 54.20$$

$$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=-3\\y=1}} = \left.(2x^3y + 8e^xy)\right|_{\substack{x=-3\\y=1}} = -53.60$$

Therefore

$$f(-3-2\epsilon, 1+3\epsilon) = -53.80 - 269.2\epsilon$$

2) *(15 pts) Determine all values of x and y that satisfy the following two equations:*

$$x^2y+2y-4x^2-8 = 0$$
$$-x^2y+5y+3x^2+5 = 0$$

Show your work.

Regroup the equations as

$$(x^2+2)\,y + (-4x^2-8) = 0$$
$$(-x^2+5)\,y + (3x^2+5) = 0$$

These equations are linear in the variable y. In order for there to be a common root for y, these two equations must be linearly dependent. Thus

$$(x^2+2)(3x^2+5) - (-x^2+5)(-4x^2-8) = 0$$

Expanding this equation gives

$$3x^4 + 11x^2 + 10 - (4x^4-12x^2-40) = 0$$
$$-x^4 + 23x^2+50 = 0$$

This equation is quadratic in the variable (x^2). Thus

$$x^2 = \frac{-23 \pm \sqrt{23^2+200}}{-2}$$
$$x^2 = \frac{-23 \pm 27}{-2}$$

Thus

$$x^2 = 25 \text{ or } x^2 = -2$$

When $x^2=25$, $y = 4$. When $x^2=-2$, $y=\frac{1}{7}$. Thus all the solutions for x and y are

(5,4)

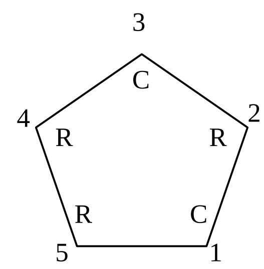
(-5,4)

$(\sqrt{2}\,i, \frac{1}{7})$

$(-\sqrt{2}\,i, \frac{1}{7})$

3) *(30 pts.) Assume that all the constant mechanism parameters are known for a spatial RRCRC mechanism (note that the cylindric joints are located along axes 1 and 3). Also assume that the angle θ_4 is given.*

Explain how to solve for all the unknown mechanism parameters.



The unknown parameters are θ_1, θ_2, θ_3, θ_5, S_1, and S_3.

One equation that will not contain the offsets S_1 and S_3 will be the secondary cosine law $Z_{045}=Z_{02}$. Expanding this equation gives

$$Z_{02} = S_2\frac{\partial Z_2}{\partial \theta_2} + a_{12}\frac{\partial Z_2}{\partial \alpha_{12}} + a_{23}\frac{\partial Z_2}{\partial \alpha_{23}}$$
$$Z_{02} = S_2 s_{12} s_{23} s_2 + a_{12}\overline{Y}_2 + a_{23} Y_2$$

$$Z_{045} = S_4\frac{\partial Z_{45}}{\partial \theta_4} + S_5\frac{\partial Z_{45}}{\partial \theta_5} + a_{34}\frac{\partial Z_{45}}{\partial \alpha_{34}} + a_{45}\frac{\partial Z_{45}}{\partial \alpha_{45}} + a_{51}\frac{\partial Z_{45}}{\partial \alpha_{51}}$$

$$Z_{045} = S_4 s_{34} X_{54} + S_5 s_{51} X_{45} + a_{34} Y_{54} + a_{45}(-s_{51} c_5 Z_4 + c_{51} Y_4) + a_{51} Y_{45}$$

Thus, the secondary cosine law contains the sines and cosines of θ_5 and θ_2 as the only unknowns. This equation can be paired with the subsidiary cosine law $Z_{45}=Z_2$ to yield two equations in the sines and cosines of θ_5 and θ_2. Upon substituting the tan-half angle identities for these variables, the two equations can be factored into the format

$$(a_i x_2^2+b_i x_2+d_i)x_5^2 + (e_i x_2^2+f_i x_2+g_i)x_5 + (h_i x_2^2+i_i x_2+j_i) = 0, \quad i = 1,2$$

where $x_5=\tan\frac{\theta_5}{2}$ and $x_2=\tan\frac{\theta_2}{2}$. Bezout's method or Sylvester's method can be used to obtain an eighth degree polynomial in x_2. This polynomial represents the condition that must exist for the two original equations, which are quadratic in x_5, to have a common root.

Unique corresponding values for θ_2 for each value of θ_5 can be determined as the common root of the original two equations. Bezout's method provides a simple expression for this common root.

Corresponding values for the angle θ_3 may be determined from the pair of equations

$$X_{54} = X_{23}$$
$$Y_{54} = -X^*_{23}$$

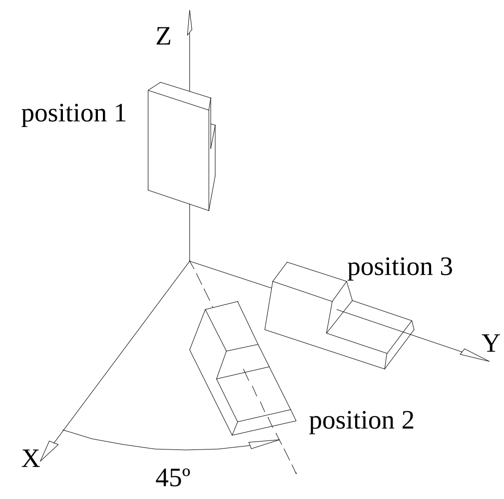
These equations represent two equations in the two unknowns s_3 and c_3.

Corresponding values for the angle θ_1 may be determined from the two equations

$$X_{345} = s_{12} s_1$$
$$Y_{345} = s_{12} c_1$$

The joint offsets S_1 and S_3 can be obtained by projecting the vector loop equation onto any two directions. This will yield two equations in the two unknown offset values.

4) (25 pts.)



A rigid body is to be moved from position 1 to position 2 and then to position 3.

(a) *Determine the rotation axis and angle that will move the body directly from position 1 to position 2.*

The body can be moved from position 1 to position 2 by rotating 90º about the Y axis followed by 45º about the Z axis. Let q_1 be the quaternion that performs the first rotation and q_2 be the quaternion that performs the second rotation. Thus

$$q_1 = \cos 45^\circ + \sin 45^\circ\, j = 0.707 + 0.707\, j$$
$$q_2 = \cos 22.5^\circ + \sin 22.5^\circ\, k = 0.924 + 0.383\, k$$

The product $q_2 q_1$ is equal to

$$q_2 q_1 = 0.653 - 0.271\, i + 0.653\, j + 0.271\, k$$

The net angle of rotation is equal to 98.46º about an axis parallel to $-0.271\,\mathbf{i} + 0.653\,\mathbf{j} + 0.271\,\mathbf{k}$.

(b) *Determine the rotation axis and angle that will move the body directly from position 2 to position 3.*

It is obvious that the body is rotated 45º about the Z axis.

(c) *Determine the rotation axis and angle that will move the body directly from position 3 back to position 1.*

The body can be moved from position 3 to position 1 by rotating 90º about the X axis followed by -90º about the Z axis. Let q_3 be the quaternion that performs the first rotation and q_4 be the quaternion that performs the second rotation. Thus

$$q_3 = \cos 45º + \sin 45º\ j = 0.707 + 0.707\ i$$
$$q_4 = \cos 45º - \sin 45º\ k = 0.707 - 0.707\ k$$

The product $q_4 q_3$ is equal to

$$q_4 q_3 = 0.5 + 0.5\ i - 0.5\ j - 0.5\ k$$

Thus the net angle of rotation is 120º about an axis parallel to 0.5 **i** - 0.5 **j** - 0.5 **k**.

5) *(15 pts) You are given four equations of the format*

$$(a_i y^2 + b_i y + d_i)\, x + (e_i\, y^2 + f_i\, y + g_i) = 0\ ,\ \ i = 1..4$$

What condition must the coefficients a_i through g_i satisfy in order to guarantee that the four equations will have common roots?

Multiplying the four equations by y will give a total of eight equations in eight unknowns. These equations can be written in matrix form as

$$\begin{bmatrix} 0 & 0 & a_1 & b_1 & d_1 & e_1 & f_1 & g_1 \\ 0 & 0 & a_2 & b_2 & d_2 & e_2 & f_2 & g_2 \\ 0 & 0 & a_3 & b_3 & d_3 & e_3 & f_3 & g_3 \\ 0 & 0 & a_4 & b_4 & d_4 & e_4 & f_4 & g_4 \\ a_1 & e_1 & b_1 & d_1 & 0 & f_1 & g_1 & 0 \\ a_2 & e_2 & b_2 & d_2 & 0 & f_2 & g_2 & 0 \\ a_3 & e_3 & b_3 & d_3 & 0 & f_3 & g_3 & 0 \\ a_4 & e_4 & b_4 & d_4 & 0 & f_4 & g_4 & 0 \end{bmatrix} \begin{bmatrix} xy^3 \\ y^3 \\ xy^2 \\ xy \\ x \\ y^2 \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

In order for a solution to exist, other than the trivial solution, it is necessary that the eight equations be linearly dependent. Thus, the condition that the coefficients must satisfy to ensure that there are common roots for x and y is

$$\begin{vmatrix} 0 & 0 & a_1 & b_1 & d_1 & e_1 & f_1 & g_1 \\ 0 & 0 & a_2 & b_2 & d_2 & e_2 & f_2 & g_2 \\ 0 & 0 & a_3 & b_3 & d_3 & e_3 & f_3 & g_3 \\ 0 & 0 & a_4 & b_4 & d_4 & e_4 & f_4 & g_4 \\ a_1 & e_1 & b_1 & d_1 & 0 & f_1 & g_1 & 0 \\ a_2 & e_2 & b_2 & d_2 & 0 & f_2 & g_2 & 0 \\ a_3 & e_3 & b_3 & d_3 & 0 & f_3 & g_3 & 0 \\ a_4 & e_4 & b_4 & d_4 & 0 & f_4 & g_4 & 0 \end{vmatrix} = 0$$

EML 6281 – Robot Geometry 1 Exam 3 – 2 hours

Open book and notes ; Computer allowed. 15 Dec 2017

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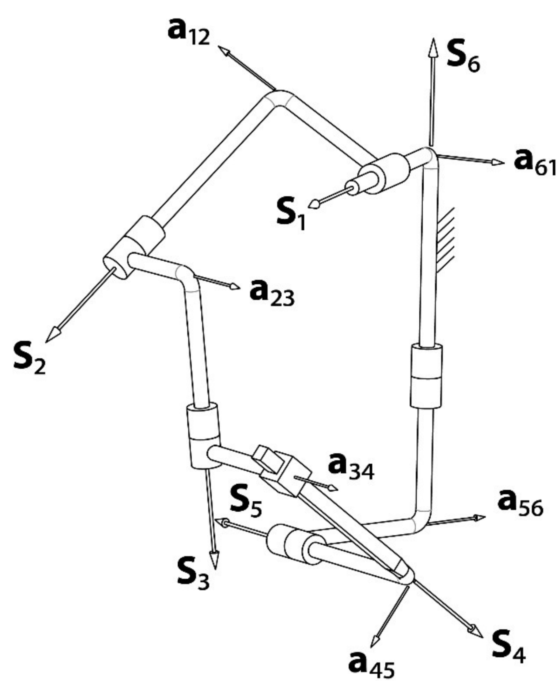
1. (35 pts.) A spatial mechanism is shown in the figure. The link a_{61} is fixed to ground.



(a) What is the mobility of this mechanism?

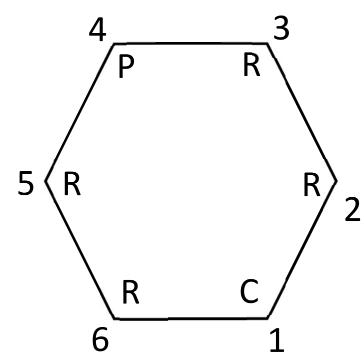
$$M = \sum_{i=1}^{6} f_i - 6 = 1$$

(b) What group mechanism is this? Explain.

This is a group 2 mechanism because it is a one degree of freedom spatial mechanism whose equivalent spherical mechanism has two degrees of freedom.

(c) You are given the value of the angle θ_3. Explain how to obtain an equation that contains the constant link lengths and joint angles as well as the angles θ_2 and θ_1. Note: There will be one term that requires much manipulation to express it in terms of θ_2 and θ_1. You must identify this term, but you do not have to do all the manipulation to convert this one term.

A planar representation of the spatial mechanism is shown. A secondary cosine law will be written which will not have the unknown joint offset values S_1 and S_4. This is written as



$$Z_{056} = Z_{023}$$

$$Z_{056} = S_5 \frac{\partial Z_{56}}{\partial \theta_5} + S_6 \frac{\partial Z_{56}}{\partial \theta_6} + a_{45} \frac{\partial Z_{56}}{\partial \alpha_{45}} + a_{56} \frac{\partial Z_{56}}{\partial \alpha_{56}} + a_{67} \frac{\partial Z_{56}}{\partial \alpha_{67}}$$
$$= S_5(s_{45}X_{65}) + S_6(s_{61}X_{56}) + a_{45}(Y_{65}) + a_{56}(-s_{61}c_6 Z_5 + c_{61}Y_5) + a_{61}(Y_{56})$$

$$Z_{023} = S_2 \frac{\partial Z_{23}}{\partial \theta_2} + S_3 \frac{\partial Z_{23}}{\partial \theta_3} + a_{12} \frac{\partial Z_{23}}{\partial \alpha_{12}} + a_{23} \frac{\partial Z_{23}}{\partial \alpha_{23}} + a_{34} \frac{\partial Z_{23}}{\partial \alpha_{34}}$$
$$= S_2(s_{12}X_{32}) + S_3(s_{34}X_{23}) + a_{12}(Y_{32}) + a_{23}(-s_{34}c_3 Z_2 + c_{34}Y_2) + a_{34}(Y_{23})$$

Substituting

$$X_{65} = X_{234}$$
$$X_{56} = X_{321}$$
$$Y_{65} = -X^*_{234}$$
$$Y_{56} = -X^*_{321}$$

gives an expression in terms of the unknown angles θ_2 and θ_1 with the exception of the term multiplied by a_{56}. It is not necessary to convert this term for this problem.

(d) Obtain a spherical cosine law that contains the angles θ_2 and θ_1 as its only unknowns.

$$Z_{4321} = c_{56}$$

(e) Explain how to obtain the values for θ_1 and θ_2 that satisfy the equations you obtained in (c) and (d).

The equations obtained in parts (c) and (d) are linear in the sines and cosines of θ_2 and θ_1. Tan-half angle substitutions are used to create two bi-quadratics in the variables x_1 and x_2. These equations can be written as

$$L_1 x_2^{\,2} + M_1 x_2 + N_1 = 0$$
$$L_2 x_2^{\,2} + M_2 x_2 + N_2 = 0$$

where the terms L_1 through N_2 are quadratic in x_1. The condition that these two equations have common roots for x_2 is

$$\begin{vmatrix} L_1 & M_1 \\ L_2 & M_2 \end{vmatrix}\begin{vmatrix} M_1 & N_1 \\ M_2 & N_2 \end{vmatrix} - \begin{vmatrix} L_1 & N_1 \\ L_2 & N_2 \end{vmatrix}^2 = 0 .$$

This yields an eighth degree polynomial in x_1. Corresponding values of x_2 are obtained from

$$x_2 = \frac{-\begin{vmatrix} M_1 & N_1 \\ M_2 & N_2 \end{vmatrix}}{\begin{vmatrix} L_1 & N_1 \\ L_2 & N_2 \end{vmatrix}} .$$

2. (25 pts.) Obtain the values of x and y that simultaneously satisfy the following two equations

$$4xy + y + 3x + 2 = 0$$
$$2xy^2 - 5y + 6 = 0 .$$

These equations may be written as

$$(4x+1)y+(3x+2)=0$$
$$(2x)y^2-(5)y+(6)=0\,.$$

Multiplying the first equation by y and writing the three equations in matrix format gives

$$\begin{bmatrix} 0 & 4x+1 & 3x+2 \\ 2x & -5 & 6 \\ 4x+1 & 3x+2 & 0 \end{bmatrix} \begin{bmatrix} y^2 \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The condition that there is a common solution for these three equations is that they be linearly dependent. Thus it is necessary that

$$\begin{vmatrix} 0 & 4x+1 & 3x+2 \\ 2x & -5 & 6 \\ 4x+1 & 3x+2 & 0 \end{vmatrix} = 0\,.$$

Expanding this determinant gives

$$18x^3+180x^2+111x+16=0\,.$$

Solving for x gives

$$x_A=-0.2230\,,$$
$$x_B=-0.4264\,,$$
$$x_C=-9.3507\,.$$

Corresponding values for y can be obtained by substituting each value of x into the first equation. This gives

$$y_A=-12.3062\,,$$
$$y_B=1.0219\,,$$
$$y_C=-0.7157\,.$$

3. (20 pts.) You are given the function

$$f(x)=4x^2-3x+6\ .$$

For what dual number values will the function evaluate to $12 + 8\varepsilon$?

The dual number value will be written as $a + \varepsilon b$. Substituting this into the function gives

$$f(a+\varepsilon b) = f(a) + (\varepsilon b)\left.\frac{df}{dx}\right|_{x=a}$$
$$= \left[4a^2 - 3a + 6\right] + \varepsilon\left[b(8a-3)\right]$$

Thus it is necessary that

$$4a^2 - 3a + 6 = 12$$
$$b(8a-3) = 8 \quad .$$

From the first equation, two possible values for *a* exist, i.e.

$$a_A = 1.6559$$
$$a_B = -0.9059\,.$$

Corresponding values for *b* may be obtained from the second equation as

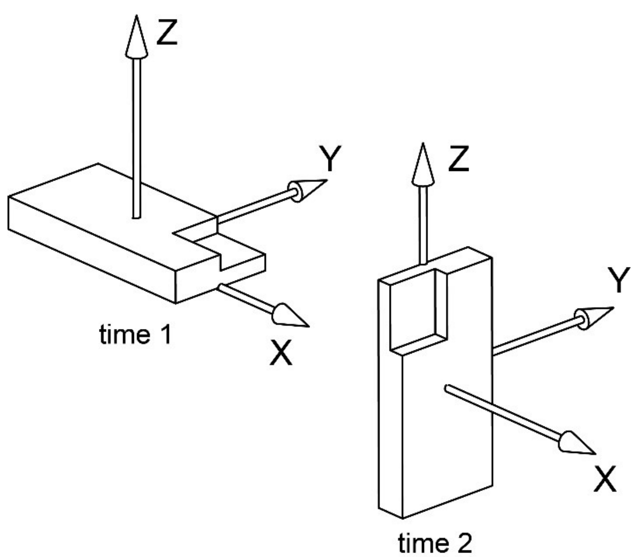
$$b_A = 0.7807$$
$$b_B = -0.7087\,.$$

Thus the two dual number values are

$$1.6559 + 0.7807\,\varepsilon$$

$$-0.9059 - 0.7087\,\varepsilon\,.$$

4. (20 pts.) A rigid body is shown at time 1 and in another orientation at time 2. Determine the axis and angle about which to rotate the body from its original orientation to its final orientation in one step.



The body can be moved by rotating it -90° about the Y axis followed by 180° about the Z axis. The first and second rotations can be modeled by the quaternions q_1 and q_2 where

$$q_1 = \cos(-45°) + \sin(-45°)\, j = 0.7071 - 0.7071\, j$$
$$q_2 = \cos(90°) + \sin(90°) k = k \quad .$$

The quaternion that represents the orientation change via a single rotation, q_3, can be written as

$$q_3 = q_2 q_1 = k\left(0.7071 - 0.7071\, j\right) = 0 + 0.7071 i + 0.7071 k\, .$$

Quaternion q_3 is interpreted as $q_3 = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left(s_x i + s_y j + s_z k\right)$. Thus the change in orientation can be accomplished by a single rotation of θ = 180° about the axis **s** = +0.7071 **i** + 0.7071 **k** .

EML 6281 – Robot Geometry I
Two hours – open book and notes

Final Exam
14 December 1999

1. (15 pts.) Write the determinant of the following 6×6 matrix as the sum of products of 3×3 determinants.

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & 0 & d_3 & e_3 & f_3 \\ a_4 & b_4 & 0 & 0 & e_4 & f_4 \\ a_5 & b_5 & 0 & 0 & e_5 & 0 \\ a_6 & b_6 & 0 & 0 & e_6 & 0 \end{bmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & 0 & d_3 & e_3 & f_3 \\ a_4 & b_4 & 0 & 0 & e_4 & f_4 \\ a_5 & b_5 & 0 & 0 & e_5 & 0 \\ a_6 & b_6 & 0 & 0 & e_6 & 0 \end{vmatrix} = -\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}\begin{vmatrix} b_4 & e_4 & f_4 \\ b_5 & e_5 & 0 \\ b_6 & e_6 & 0 \end{vmatrix}$$

$$-\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}\begin{vmatrix} a_4 & e_4 & f_4 \\ a_5 & e_5 & 0 \\ a_6 & e_6 & 0 \end{vmatrix}$$

$$+\begin{vmatrix} c_1 & d_1 & e_1 \\ c_2 & d_2 & e_2 \\ c_3 & d_3 & e_3 \end{vmatrix}\begin{vmatrix} a_4 & b_4 & f_4 \\ a_5 & b_5 & 0 \\ a_6 & b_6 & 0 \end{vmatrix}$$

$$-\begin{vmatrix} c_1 & d_1 & f_1 \\ c_2 & d_2 & f_2 \\ c_3 & d_3 & f_3 \end{vmatrix}\begin{vmatrix} a_4 & b_4 & e_4 \\ a_5 & b_5 & e_5 \\ a_6 & b_6 & e_6 \end{vmatrix}$$

2. (20 pts.) A rigid body is to be rotated three times. After the third rotation, the body is at its original position. The first rotation was an angle of 30° about the Z axis of the fixed coordinate system. The third rotation was an angle of 100° about an axis along the vector ($\underline{i}+\underline{j}$) of the fixed coordinate system. What was the second rotation? Determine the axis (as measured in the fixed coordinate system) and the angle of rotation.

The first rotation can be modeled by the quaternion q_1 where

$$q_1 = \cos 15° + \sin 15°\ k.$$

The third rotation can be modeled by the quaternion q_3 where

$$q_3 = \cos 50° + (\sin 50°/\sqrt{2})\ (i + j)$$

The coordinates of any arbitrary point, r, after the three rotations can be written as

$$r' = q_3\ q_2\ q_1\ r\ q_1^{-1}\ q_2^{-1}\ q_3^{-1}$$

Since the body is back at its starting position after the three rotations, the net rotation, $q_3\ q_2\ q_1$, must equal the quaternion 1.

Thus

$$q_3\ q_2\ q_1 = 1\ .$$

Solving for q_2 gives

$$q_2\ q_1 = q_3^{-1}$$
$$q_2 = q_3^{-1}\ q_1^{-1}$$

$$q_2 = [\cos 50° - (\sin 50°/\sqrt{2})\ (i + j)]\ [\cos 15° - \sin 15°\ k]$$
$$q_2 = [0.6428 - 0.5417\ (i+j)]\ [0.9659 - 0.2588\ k]$$
$$q_2 = 0.6209 - 0.1664\ k - 0.5232\ (i+j) + 0.1402\ i - 0.1402\ j$$
$$q_2 = 0.6209 - 0.3830\ i - 0.6634\ j - 0.1664\ k$$
$$q_2 = 0.6209 + 0.7839\ (-0.4886\ i - 0.8463\ j - 0.2123\ k)$$

The angle for the second rotation can be calculated from

$$\cos \gamma/2 = 0.6209, \quad \sin \gamma/2 = .7839$$

The second rotation was therefore a rotation of 103.2° about an axis parallel to ($-0.4886\ \underline{i} - 0.8463\ \underline{j} - 0.2123\ \underline{k}$).

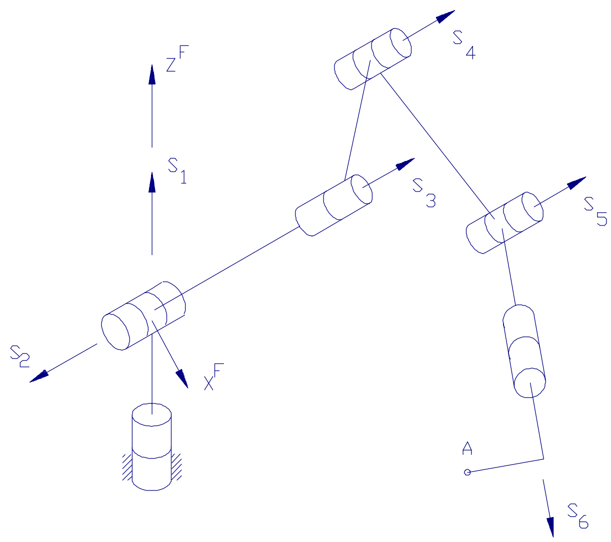
3. (25 pts.) A 6-R manipulator is shown in the figure. The following facts are known:

(i) The 1st and 2nd axes intersect and are perpendicular.

(ii) The 2nd and 3rd axes intersect and are perpendicular.

(iii) The 3rd, 4th, and 5th axes are parallel. The 4th and 5th offset values are zero.

(iv) The 5th and 6th axes intersect and are perpendicular.



(a) Tabulate the constant mechanism parameters. Indicate which distances are equal to zero and which angles are equal to 0, 90, or 270°.

$a_{12}=0$	$a_{23}=0$	a_{34}	a_{45}	$a_{56}=0$
$\alpha_{12}=90°$	$\alpha_{23}=90°$	$\alpha_{34}=0$	$\alpha_{45}=0$	$\alpha_{56}=90°$
	$S_2=0$	S_3	$S_4=0$	$S_5=0$

(b) What parameters must you specify in order to define the sixth standard coordinate system?

S_6 and direction of $\mathbf{a}_{67}$

(c) Assume that the coordinates of point A are given together with the direction cosines of $\underline{S}_6$ and $\underline{a}_{67}$ (all in terms of the fixed coordinate system). List the names of the parameters that you are free to choose and the variables that become known when you 'close-the-loop'.

free choices: α_{67} (usually 90°), a_{67} (usually 0)
close-the-loop parameters: a_{71}, S_7, S_1, α_{71}, θ_7, γ_1

(d) Obtain an equation that contains only the variables θ_7 and θ_6. Expand the equation as far as necessary in order to show that the only unknowns in the equation are θ_7 and θ_6. How many values of θ_6 will satisfy this equation?

Project the vector loop equation onto the direction of the three parallel axes, i.e. $\mathbf{S}_3$.

Vector loop equation:

$$S_1\mathbf{S}_1 + S_3\mathbf{S}_3 + a_{34}\mathbf{a}_{34} + a_{45}\mathbf{a}_{45} + S_6\mathbf{S}_6 + S_7\mathbf{S}_7 + a_{71}\mathbf{a}_{71} = \mathbf{0}$$

Project vector loop equation onto $\mathbf{S}_3$

$$S_1 Z_{7654} + S_3 + S_7 Z_{654} + a_{71} U_{76543} = 0$$

Expanding Z_{7654} gives

$$Z_{7654} = s_{34}(X_{765}s_4+Y_{765}c_4) + c_{34} Z_{765}$$

Substituting for α_{34} gives

$$Z_{7654} = Z_{765}$$

Expanding Z_{765} gives

$$Z_{7654} = s_{45}(X_{76}s_5+Y_{76}c_5) + c_{45} Z_{76}$$

Substituting for α_{45} gives

$$Z_{7654} = Z_{76}$$

Expanding the term Z_{654} gives

$$Z_{654} = s_{34}(X_{65}s_4+Y_{65}c_4) + c_{34}Z_{65}$$

Substituting for α_{34} gives

$$Z_{654} = Z_{65}$$

Expanding Z_{65} gives

$$Z_{654} = s_{45}(\overline{X}_6 s_5+\overline{Y}_6 c_5) + c_{45}Z_6$$

Substituting for α_{45} gives

$$Z_{654} = Z_6$$

Expanding U_{76543} gives

$$U_{76543} = U_{7654}c_{34} - V_{7654}s_{34}$$

Substituting for α_{34} gives

$$U_{76543} = U_{7654}$$

Expanding U_{7654} gives

$$U_{76543} = U_{765}c_{45} - V_{765}s_{45}$$

Substituting for α_{45} gives

$$U_{76543} = U_{765}$$

The projection of the vector loop equation can thus be written as

$$S_1 Z_{76} + S_3 + S_7 Z_6 + a_{71} U_{765} = 0$$

This equation contains only θ_6 as the only unknown. The equation can be factored into the format A c_6 + B s_6 + D = 0. Two values of θ_6 will satisfy this equation.

4. (20 pts.) You are given the two cubic equations:

$$J_1x^3 + K_1x^2 + L_1x + M_1 = 0$$
$$J_2x^3 + K_2x^2 + L_2x + M_2 = 0$$

where the J,K,L,M coefficients are quadratic in the variable y.

(a) What condition must the coefficients satisfy in order to guarantee that a common value of x will satisfy both equations? What degree polynomial in y does this condition represent?

Multiplying the set of equations by x and then by x^2 gives (in matrix form)

$$\begin{bmatrix} 0 & 0 & J_1 & K_1 & L_1 & M_1 \\ 0 & 0 & J_2 & K_2 & L_2 & M_2 \\ 0 & J_1 & K_1 & L_1 & M_1 & 0 \\ 0 & J_2 & K_2 & L_2 & M_2 & 0 \\ J_1 & K_1 & L_1 & M_1 & 0 & 0 \\ J_2 & K_2 & L_2 & M_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^5 \\ x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In order for common roots to exist for the two equations, the above set of six equations must be linearly dependent. Thus

$$\begin{vmatrix} 0 & 0 & J_1 & K_1 & L_1 & M_1 \\ 0 & 0 & J_2 & K_2 & L_2 & M_2 \\ 0 & J_1 & K_1 & L_1 & M_1 & 0 \\ 0 & J_2 & K_2 & L_2 & M_2 & 0 \\ J_1 & K_1 & L_1 & M_1 & 0 & 0 \\ J_2 & K_2 & L_2 & M_2 & 0 & 0 \end{vmatrix} = 0$$

This condition will result in a 12th degree polynomial in y.

(b) Assuming that the coefficients satisfy the condition such that a common value of x will satisfy both equations, explain how to calculate this common root.

Method 1:
For each value of y, calculate numerical values for the coefficients J_i, K_i, L_i, M_i. Then root each cubic polynomial for three values of x. Find the common value of x that solves both equations.

Method 2:

Write the two original equations in the format

$$(J_1x + K_1)\, x^2 + (L_1x + M_1) = 0$$
$$(J_2x + K_2)\, x^2 + (L_2x + M_2) = 0$$

The condition that these two equations have a common root of x^2 is

$$(J_1x + K_1)(L_2x + M_2) - (J_2x + K_2)(L_1x + M_1) = 0$$

$$(J_1L_2 - J_2L_1)\, x^2 + (J_1M_2 + K_1L_2 - J_2L_1 - K_2L_1)\, x + (K_1M_2 - K_2M_1) = 0$$

$$|JL|\, x^2 + (|JM| + |KL|)\, x + |KM| = 0$$

where $|JL| = J_1L_2\text{-}J_2L_1$ and the other 2×2 determinants are similarly defined.

Now write the original equations in the format

$$(J_1)\, x^3 + (K_1x^2 + L_1x + M_1) = 0$$
$$(J_2)\, x^3 + (K_2x^2 + L_2x + M_2) = 0$$

The condition that these two equations have a common root of x^3 is

$$|JK|\, x^2 + |JL|\, x + |JM| = 0$$

The two quadratic equations in x may now be written as

$$|JL|\, x^2 + [(|JM| + |KL|)\, x + |KM|] = 0$$
$$|JK|\, x^2 + [|JL|\, x + |JM|] = 0$$

The condition that these two equations have a common root of x^2 is

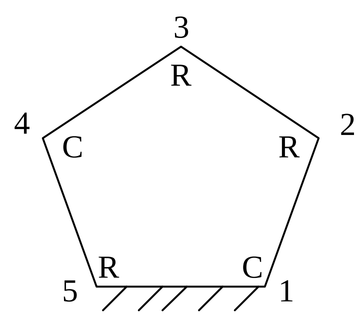
$$|JL|\, (|JL|\, x + |JM|) - |JK|\, [(|JM| + |KL|)\, x + |KM|] = 0$$

$$[|JL|^2 - |JK|(|JM| + |KL|)]\, x + [|JL|\, |JM| - |JK|\, |KM|] = 0$$

Solving for x gives

$$x = \frac{-|JL||JM| + |JK||KM|}{|JL|^2 - |JK|(|JM| + |KL|)}$$

5. (20 pts.) A planar representation of a spatial mechanism is shown in the accompanying figure. You may assume that all the constant mechanism parameters are known as well as the angle θ_5. To solve this mechanism for the variable parameters it is necessary to obtain two equations that contain the same two unknown variables. Write these equations and expand them as necessary to verify that they do contain only the same two unknowns.



Since this is a group 2 mechanism, it will be necessary to obtain two equations in two unknown variables. The first equation can be obtained by substituting dual angles into the subsidiary spherical cosine law $Z_{23} = Z_5$ to obtain

$$Z_{023} = Z_{05}$$

Expanding Z_{05} gives

$$Z_{05} = S_5 \frac{\partial Z_5}{\partial \theta_5} + a_{45} \frac{\partial Z_5}{\partial \alpha_{45}} + a_{51} \frac{\partial Z_5}{\partial \alpha_{51}}$$
$$Z_{05} = S_5 s_{51} X_5 + a_{45} \overline{Y}_5 + a_{51} Y_5$$

Expanding Z_{023} gives

$$Z_{023} = S_2 \frac{\partial Z_{23}}{\partial \theta_2} + S_3 \frac{\partial Z_{23}}{\partial \theta_3} + a_{12} \frac{\partial Z_{23}}{\partial \alpha_{12}} + a_{23} \frac{\partial Z_{23}}{\partial \alpha_{23}} + a_{34} \frac{\partial Z_{23}}{\partial \alpha_{34}}$$

The definition of Z_{23} is written as

$$Z_{23} = s_{34}(X_2 s_3 + Y_2 c_3) + c_{34} Z_2$$

Expanding the partial derivatives gives

$$Z_{023} = S_2 s_{12} X_{32} + S_3 s_{34} X_{23} + a_{12} Y_{32} + a_{23}(-s_{34} c_3 Z_2 + c_{34} Y_2) + a_{34} Y_{23}$$

The resulting equation may now be written as

$$S_2 s_{12} X_{32} + S_3 s_{34} X_{23} + a_{12} Y_{32} + a_{23}(-s_{34} c_3 Z_2 + c_{34} Y_2) + a_{34} Y_{23}$$
$$- S_5 s_{51} X_5 - a_{45} \overline{Y}_5 - a_{51} Y_5 = 0$$

This equation contains only θ_2 and θ_3 as the only unknowns.

A second equation that contains these two unknowns is the subsidiary spherical cosine law $Z_{23} = Z_5$.

These two equations can be factored into the format

$$(A_i c_3 + B_i s_3 + D_i)\, c_2 + (E_i c_3 + F_i s_3 + G_i)\, s_2 + (H_i c_3 + I_i s_3 + J_i) = 0, \quad i = 1,2$$

Upon substituting tan-half angle expressions, two equations can be obtained in the variables x_2 and x_3. In the general case, the solution of these two equations would yield an eighth degree polynomial in one of the variables. In this case, however, the solution will simplify to a fourth degree polynomial.