2MBS40 Stochastic Simulation and Modeling
Assignment 2: Reflected Brownian Motion with Semipermeable Barriers

Version 1.0

Jaron Sanders

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1 Objective

You are going to write a simulation for a Reflected Brownian Motion that interacts with semipermeable barriers, and then do experiments with it.

Intuitively speaking, the process is a Brownian motion that is reflected hard at the barriers, but with a twist: the process can also tunnel through a barrier whenever it has interacted for long enough with a boundary. How long is long enough, you may ask? Well, using the concept of *local time* we will keep track of the time spent at a boundary, and the Brownian motion will tunnel through the barrier whenever a $\{+,-\}$-valued Markov chain attached to that boundary switches state (which will happen at a rate that depends on which side of the barrier the Brownian motion is, and on how much local time has accumulated at that boundary).

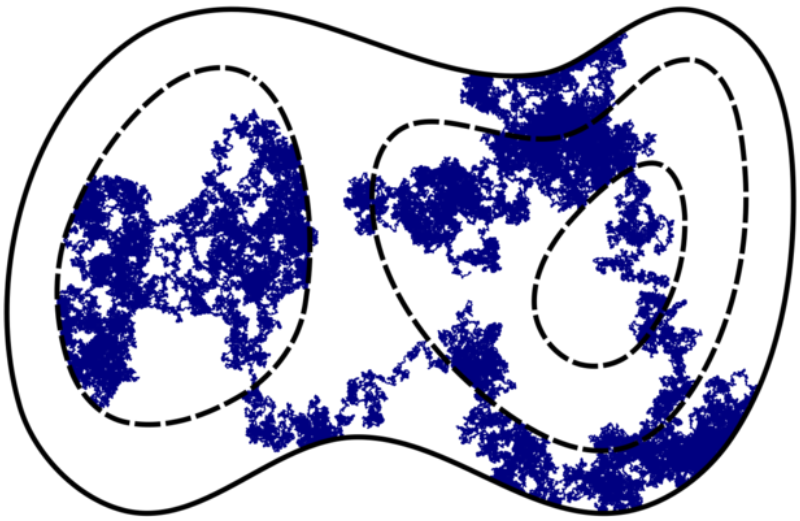

Figure 1: Schematic depiction of a reflected Brownian motion with semipermeable boundaries. Figure courtesy of [2].

2 Formalities

2.1 Reflected Brownian Motion with Semipermeable Barriers

The following quote, taken from [2], describes the necessary mathematical setup for the continuous stochastic process that you will be simulating:

> Fix an open planar set $D_0 \subseteq \mathbb{R}^2$ whose closure D is connected, bounded, and has a smooth boundary ∂D. Additionally, consider $m \geq 0$ smooth curves $B_1, \ldots, B_m \subseteq D_0$ and denote $B_0 := \partial D$. In what follows, the B_i with $i \geq 1$ will be semipermeable barriers and B_0 will be an impermeable barrier, keeping the process in D.

To simplify the presentation and proofs, let us adopt some additional assumptions. First, assume that $B_i \cap B_j = \emptyset$ for every $i \neq j$ so that there are no intersection points which would require separate consideration. Second, assume that the D is simply connected so that B_0 is a single connected curve. Finally, assume that the B_i are simple closed curves so that there are no self-intersections or endpoints. Then, in particular, the Jordan curve theorem yields that $\mathbb{R}^2 \setminus B_i$ has precisely two connected components of which only one is bounded.

For every $i \leq m$ let $\vec{n}_i : B_i \to \mathbb{R}^2$ be the unique vector field which is orthogonal to B_i, points towards the bounded component, and has unit length. We say that $x \in \mathbb{R}^2$ is *on the positive side of* B_i if x lies in the closure of the bounded component of $\mathbb{R}^2 \setminus B_i$. Similarly, the *negative side of* B_i refers to the closure of the unbounded component. Note that x is then both on the positive and negative sides when $x \in B_i$.

The process will be driven by randomness from some auxiliary processes. To drive the movements when away from all barriers, we consider a $\mathbb{R}^2$-valued Wiener process W_t. Further, fix scalars $\lambda_i^+, \lambda_i^- > 0$ for every $i \geq 1$, specifying the permeability of the two sides of each barrier. Then, to regulate the random event where the process switches sides, we consider continuous-time càdlàg Markov chains $s_i(t)$ taking values in $\{+1, -1\}$ with transition rate λ_i^+ (resp. λ_i^-) from -1 to $+1$ (resp. from $+1$ to -1). Finally, let $s_0(t) := +1$ for all $t \geq 0$.

Following [2], we can now give a proper definition of the continuous process:

Definition (Reflected Brownian Motion with semipermeable barriers, [2])**.** *Let X_t and $L_t^{(i)}$ be continuous stochastic processes which take values in D and $\mathbb{R}_{\geq 0}$, respectively. Then, X_t is called* a reflected Brownian motion with semipermeable barriers B_i and local times $L_t^{(i)}$ *if the following properties hold with probability one:*

(i) The following stochastic differential equation is satisfied:

$$\mathrm{d}X_t = \mathrm{d}W_t + \sum_{i=0}^{m} s_i(L_t^{(i)})\vec{n}_i(X_t)\mathbb{1}\{X_t \in B_i\}\mathrm{d}L_t^{(i)}.$$

(ii) For every $i \leq m$, the process $L_t^{(i)}$ is nondecreasing, satisfies $L_0^{(i)} = 0$, and increases at time t if and only if $X_t \in B_i$. That is,

$$L_t^{(i)} = \int_0^t \mathbb{1}\{X_r \in B_i\}\,\mathrm{d}L_r^{(i)}.$$

(iii) For every $i \leq m$, if $s_i(L_t^{(i)}) = +1$ then X_t is on the positive side of B_i. Similarly, if $s_i(L_t^{(i)}) = -1$ then X_t is on the negative side of B_i.

In your simulation, you may use that the local time at the ith barrier admits the following expression [2]:

$$L_t^{(i)} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}\big\{\exists y \in B_i : \|X_r - y\| < \varepsilon\big\}\,\mathrm{d}r,$$

2.2 Environment

For $r \in (0, \infty)$, $x \in \mathbb{R}^2$, let

$$S_{r,x} := \big\{y \in \mathbb{R}^2 : \|y\|_2 = r\big\}$$

refer to a two-dimensional sphere (a circle) at x with radius r. Similarly, for $r \in (0, \infty)$, let

$$S_r := S_{r,0}$$

refer to a two-dimensional sphere centered at the origin with radius r.

Tasks

1. [Easy] You will first experiment with a two-dimensional Wiener process $\{W_t\}_{t \geq 0}$.

(5pts) (a) Program a simulation for the two-dimensional Wiener process that doesn't interact with anything yet. *You are not allowed to use a ready-made software library to simulate stochastic processes. You should actually program the simulation yourselves, without outside help.*

(5pts) (b) Suppose that $W_0 = (0, 0)$. Regarding the hitting time $\tau_1 = \inf\{t : \|W_t\| = 1\}$, it is then known that $\mathbb{E}[\tau_1] = 1/2$. Verify your simulation rigorously against this fact and discuss your findings thoroughly in your report.

(10pts) (c) For $r \in (0, \infty)$ let $\tau_r := \inf\{t \in [0, \infty) : \|W_t\| = r\}$ refer to the hitting time of a circle of radius r at the origin. Let $0 < r_1 < r_2 < \cdots < r_n < \infty$ refer to $n \in \mathbb{N}$ radii satisfying $r_i = 4^{-1}(4 \cdot 10)^{i/n}$ for $i \in [n]$. Now, for these radii, it is known that if $\|W_0\| = r_{i^\star}$ for some $i^\star \in \{2, \dots, n-1\}$, then $\mathbb{P}[\tau_{r_{i^\star+1}} < \tau_{r_{i^\star-1}}] = \frac{1}{2}$; see, for example, [1].

Let $n = 10$. Test your simulation rigorously against the aforementioned fact by generating the following table with confidence intervals, including it in your report, and discussing your findings thoroughly:

$i^\star$	Est. of $\mathbb{E}[\tau_{r_{i^\star-1}} \mid \tau_{r_{i^\star-1}} < \tau_{r_{i^\star+1}}]$	Est. of $\mathbb{P}[\tau_{r_{i^\star+1}} < \tau_{r_{i^\star-1}}]$	Est. of $\mathbb{E}[\tau_{r_{i^\star+1}} \mid \tau_{r_{i^\star-1}} > \tau_{r_{i^\star+1}}]$
2	...	...	...
3	...	...	...
4	...	...	...
...	...	...	...
$n-2$	...	...	...

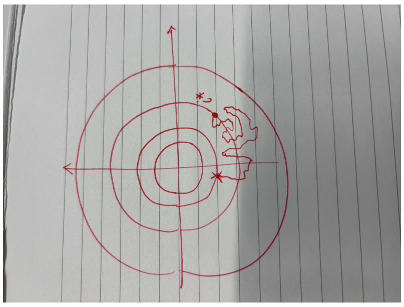

Figure 2: A schematic depiction for Task 1c.

2. [Medium] You will next experiment with a two-dimensional Brownian motion that reflects hard on an inner and outer circle, both centered at the origin. Referring to the setup described in the preliminaries, for $0 < r_1 < r_2 < \infty$, this corresponds to $m = 1$, $D_0 = S_{r_2}$, $B_1 = S_{r_1}$ with $\lambda_1^+ = 0$, $\lambda_1^- = \infty$ and $\lambda_2^+ = \infty$, $\lambda_2^- = 0$.

(5pts) (a) Augment your simulation to handle hard reflections on an inner circle of radius $r_1 \in (0, \infty)$ and outer circle of radius $r_2 \in (r_1, \infty)$, assuming that $\|X_0\|_2 \in (r_1, r_2)$. Describe in detail in your report how you handle hard reflections.

You are not allowed to use a ready-made software library to simulate stochastic processes. You should actually program the simulation yourselves, without outside help.

(10pts) (b) Suppose that $r_2 = 1$. For $i \in \{1, 2\}$, let $\tau_{r_i} = \inf\{t \in [0, \infty) : \|W_t\| = r_i\}$ refer to the hitting times of the inner and outer circle, respectively. It is then known that $\mathbb{E}[\tau_{r_2}] = (1/2)(1 - \|x_0\|^2 + r_1^2 \ln \|x_0\|^2)$. For $r_1 \in \{1/8, 1/4, 1/2\}$, test your simulation rigorously against this fact by plotting $\mathbb{E}[\tau_{r_2}]$ as a function of $\|x_0\| \in (r_1, r_2)$. (So, we expect three plots.) Discuss your findings thoroughly.

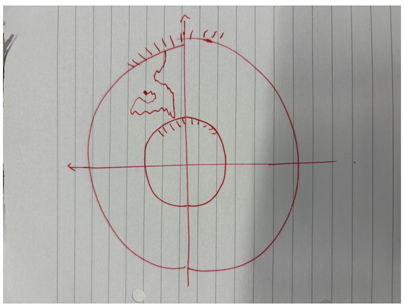

Figure 3: A schematic depiction for Task 2b.

(10pts) (c) Suppose still that $r_2 = 1$. Generate the following table with confidence intervals, include it in your report, and discuss your findings:

3. [Hard] Finally, you will program the semipermeable barriers as in the preliminaries.

	$r_1 = 0.1$	$r_1 = 0.2$	$r_1 = 0.3$	$r_1 = 0.4$	$r_1 = 0.5$
$x_0 = 0.6$	Estimate of $\mathbb{E}[\tau_{r_1}]$	Estimate of $\mathbb{E}[\tau_{r_1}]$	$\cdots$	$\cdots$	$\cdots$
$x_0 = 0.7$	Estimate of $\mathbb{E}[\tau_{r_1}]$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_0 = 0.8$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_0 = 0.9$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Estimate of $\mathbb{E}[\tau_{r_1}]$

(5pts) (a) Augment your simulation to handle circular semipermeable barriers. Describe in detail in your report how you implemented the tunneling.

You are not allowed to use a ready-made software library to simulate stochastic processes. You should actually program the simulation yourselves, without outside help.

(15pts) (b) Suppose that $m = 1$, $D_0 = \text{interior}(S_1)$, $B_1 = S_{1/2}$ with $\lambda_1^+ = 4$, and $\lambda_1^- = 1$. Also, assume that $X_0 = (0, 0)$.

For "$T \to \infty, \varepsilon \downarrow 0$", i.e., for T as large as possible and $\varepsilon \in (0, 1)$ as small as possible, estimate a plot of

$$\frac{1}{2\eta\varepsilon}\mathbb{P}\Big[\big|\|X_T\| - \eta\big| < \frac{\varepsilon}{2}\Big]$$

as a function of $\eta \in (0, 1)$. Be sure to include confidence bands.

(15pts) (c) Suppose that $m = 3$, $D_0 = \text{interior}(S_1)$, and

- $B_1 = S_{1/2,(1/4,0)}$, and $\lambda_1^+ = 2$, $\lambda_1^- = 1/2$,
- $B_2 = S_{1/8,(-1/2,1/2)}$, and $\lambda_2^+ = 3$, $\lambda_2^- = 1/3$,
- $B_3 = S_{1/16,(1/2,1/4)}$, and $\lambda_3^+ = 1/4$, $\lambda_3^- = 4$.

Also, assume that $X_0 = (0, 0)$.

Let $T \in (0, \infty)$ be a simulation horizon, $s \in (0, \infty)$ a sampling period, and $w \in (0, 1]$ a bin width. Let $\{X_t\}_{t \in [0,T]}$ refer to your finely-generated trajectory of the process up to time T. Consider now for $i, j \in \{-\lceil 1/w \rceil, \ldots, -1, 0, 1, \ldots, \lceil 1/w \rceil\}$, the two-dimensional binned counts

$$\hat{N}_{i,j} = \frac{1}{\lceil T/s \rceil} \sum_{t=1}^{\lceil T/s \rceil} \mathbb{1}\big\{X_{ts} \in [iw, (i+1)w) \times [jw, (j+1)w)\big\}.$$

For $s = 1$ and "$T \to \infty, w \downarrow 0$", i.e., for T as large as possible and w as small as possible, plot $\hat{N}_{i,j}$ as a function of (i, j).

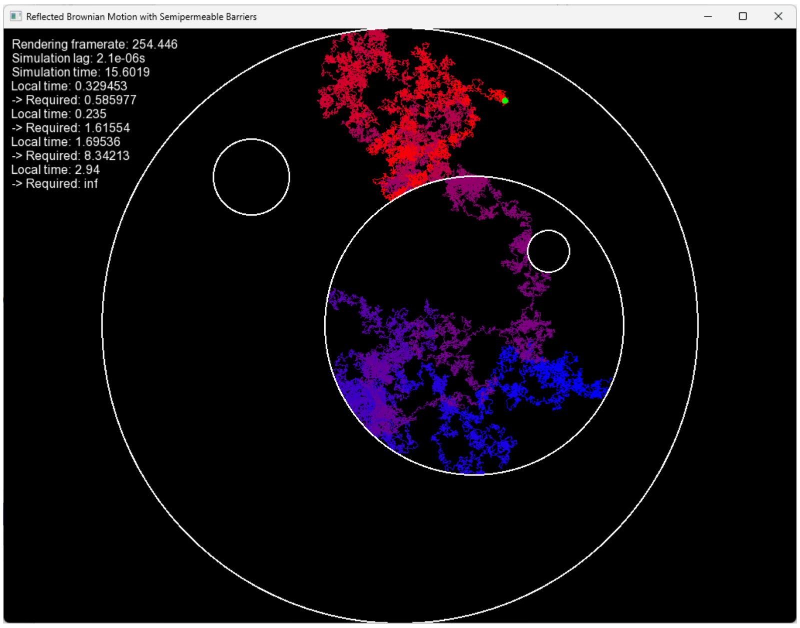

Figure 4: An illustration of Task 3c.

3 Deliverable(s)

Tasks

(20pts) 4. Write an academic report in LaTeX that describes your results and includes an overview of relevant scientific literature, and hand it in as a compiled PDF. Note that this report is subject to knockout criteria that are described below.

These points are awarded to reports (i) that have a well-designed, canonical structure such as "abstract, introduction, methodology, discussion, conclusion", and (ii) in which have all explanations/observations/discussions/conclusions flow naturally from one to the other, have depth, and are well-presented.

Merely handing in a collection of results, e.g. with tasks numbers as section titles, is insufficient.

Knockout criteria

You must do the following to be marked:

(i) Meet the deadline, by:
Submitting a PDF file containing an electronic copy of your report to Canvas, on time.
Submitting a ZIP file containing all of your code to Canvas, on time.

Also, your report must meet the following criteria to be marked:

(i) It includes a title, author names, student numbers, and a bibliography when citing.
(ii) It includes an acknowledgments section, in which any entity that contributed in some manner is mentioned (any entity other than the authors). Examples include other programmers and/or students, but also generative artificial intelligence and public discussion forums from which you may have sourced material. *Remember that as a team you must try to do the entire assignment by yourselves, i.e., without outside help. Still, if you do decide to partake in some discussion or if you opt to use some tool, then you must acknowledge the help that you received.*
(iii) It includes a paragraph in the Appendix, in which every student briefly explains their specific contributions. In particular, tell us which percentage of the (a) analysis, (b) report writing, and (c) programming you did. *Every student is expected to contribute significantly to every component.*
(iv) It finally includes all of your code, verbatim, at the end of the Appendix.
(v) The report is subject to a hard page limit of **seven A4 papers**, single column (and of these, **one A4 paper** must be your literature overview). This page limit excludes the bibliography, assignment contribution statement, acknowledgments section, and code in the Appendix. *Focus on what is important: it is about quality of content, not quantity.*
(vi) Margins should be at least 2 cm, and font size at least 10 pt.
(vii) The overall presentation is clean / neat / orderly.
(viii) Your texts are legible, in academic / formal English, and contain few spelling mistakes.

Grading

Try your best and don't give up if you find this assignment challenging! I will judge you for effort, so understand that you may report difficulties or even failure.

Still, your aim should most certainly be a perfect implementation for that perfect mark.

References

[1] Marius A Schmidt. "A simple proof of the DPRZ theorem for 2d cover times". In: *The Annals of Probability* 48.1 (2020), pp. 445–457.

[2] Alexander Van Werde and Jaron Sanders. "Recovering semipermeable barriers from reflected Brownian motion". In: *arXiv preprint arXiv:2412.14740* (2024).