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COMPUTATION OF MODEL-FREE IMPLIED VOLATILITY

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Individuals with an ambition to invest into the stock market are always confronted with the question of how the market will move in the future. One concept to assess the intensity of this movement is model-free implied volatility. The presented thesis explains, why this model-free approach is superior to others and furthermore examines the theoretical concept by proofing that the expected sum of variances of the stock price process can, under certain circumstances, be written as the squared model-free implied volatility. To make this theoretical formula tangible, and to show that the everyday investor can calculate her own volatility measure, it is discretised and applied to options data, in the same manner as the VIX is calculated by the CBOE. By comparing the replicated calculation and the VIX, it is shown that, although the measures do not match precisely, the direction and intensity of the movement is mostly equal.

1. Motivation. Typically asset prices follow highly volatile processes. The measure of this price change is referred to as *volatility*. The volatility is usually measured as the standard deviation of returns and it captures to what extent the price of the asset differs from its mean. This suggests, that volatility is an indicator of the intensity of market movement, however, here it is important to note, that it is not an indicator of the direction of the movement.
Since unpredictability makes the investment in a stock riskier, profound knowledge of market fluctuation at certain times (in the future) is of great importance to investors. Easterling (2020) states, that stock markets perform well (i. e. rise) in lower volatility periods and poorly (decrease) in higher volatility periods. So volatility may also be helpful when trying to determine the likelihood of a stock reaching a specific price by a certain time.
Whereas there are many different types of volatility (historical or realised volatility also being popular measures) we focus on implied volatility.[1]

2. Implied Volatility. Implied Volatility aims to estimate the future markets fluctuation. To do so current option prices are considered. If the market expects minor movements of an asset in the future, this will be reflected in the option prices and therefore the implied volatility will be low. On the contrary, if the market thinks, that price movements will be large, the implied volatility rises with the option prices.
Often, implied volatility is derived from the Black-Scholes-Merton Model, which is used to determine the price of an option or other derivatives. By using as inputs the price of the underlying assets, the strike price, the time to expiration and the

[1]It is important not to confuse implied volatility with historical volatility (aka realised volatility). The latter is a measure of how much the asset has actually moved in the past, whereas implied volatility is the current estimate of future moves. It can however be useful to compare these two.

risk free interest rate and solving for volatility, the Black-Scholes implied volatility is obtained. This, however, can be problematic, as the model assumes that the price of the underlying assets follows a geometric Brownian motion with constant drift and volatility. The latter is problematic in particular, as options with the same underlying asset yield different Black-Scholes Implied Volatilities for different strike prices and different expiration dates.
Hence Britten-Jones and Neuberger (2000) propose a different way of measuring future volatility, called *model-free* implied volatility.

3. Model-Free Implied Volatility. The concept of model-free implied volatility is, unlike the traditional concept, not dependent on any particular model and is theoretically to be derived completely from the definition of European put and call options. There are no assumptions made regarding the underlying stochastic process of the asset, most importantly no assumptions are imposed on the volatility process, and therefore it is referred to as *model free*.
The Chicago Board Options Exchange (CBOE) uses this concept of model-free implied volatility to compute the *Volatility Index*, commonly referred to by its ticker symbol VIX. It typically correlates negatively with the S&P500 index, which it is based on. This justifies its nick-name *fear-gauge*, since volatility tends to rise, when the index collapses, as we have seen during the financial crisis 2008 for example.

The roadmap to the remainder of this thesis is as follows.
A theoretical derivation of the model-free implied volatility from the expected integrated variance is carried out in Section 4. Section 5 shows, what approximations have to be made to the theoretical result of Section 4, to be able to compute the model-free implied volatility in real world instances. Section 6 provides an example on how the VIX is calculated with existing options data.
Finally Section 7 concludes.

4. From Expected Integrated Variance to Model-Free Implied Volatility.

Definition 1. *The model-free implied volatility $MFIV$ is defined as*

$$MFIV := 2e^{rT}\left(\int_0^\infty \frac{M_0\left(K\right)}{K^2}\,\mathrm{d}K\right), \tag{4.1}$$

where r denotes the (risk free) interest rate of the underlying asset. $M_0\left(K\right)$ is defined as $M_0\left(K\right) := \min\left(P_0\left(S_t, K, T\right), C_0\left(S_t, K, T\right)\right)$ representing the minimum of the price of a put option P_0 and the price of a call option C_0 with (asset) price S_t and strike price K at time T

In the following we shall relate the MFIV to the volatility of the underlying.

Definition 2. *To do so we define the integrated variance $\mathbb{IV}(0,T)$ as*

$$\mathbb{IV}(0,T) := \int_0^T \sigma_t^2\,\mathrm{d}t. \tag{4.2}$$

The integrated variance can be seen as the sum of instantaneous variances σ_t^2 realised over time 0 to time T. The following Theorem states under which conditions and how the $MFIV$ relates to the $\mathbb{IV}$.

THEOREM 1. *Assume the dynamics of the stock price process S_t under the risk neutral probability $\mathbb{Q}$ satisfy*

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma_t S_t\,\mathrm{d}W_t, \tag{4.3}$$ [2]

where $(W_t)_{t\geq 0}$ is a Brownian Motion under $\mathbb{Q}$ and $r \geq 0$ is the constant interest rate and $(\sigma_t)_{t\geq 0} > 0$ denotes the volatility process. Assume further, that $(\sigma_t)_{t\geq 0}$ is such that the discounted stock price $Z_t = e^{-rT}S_t$ is a martingale under $\mathbb{Q}$. Then,

$$MFIV = \mathbb{E}_0^{\mathbb{Q}}\left[\mathbb{IV}(0,T)\right]. \tag{4.4}$$

The subsequent proof builds mostly on Andersen, Bondarenko and Gonzalez-Perez (2015), Bondarenko (2007) and Hull (2009).

PROOF. We first show that

$$\mathbb{E}_0^{\mathbb{Q}}\left[\mathbb{IV}\left(0,T\right)\right] = \mathbb{E}_0^{\mathbb{Q}}\left[\int_0^T \sigma_t^2\,\mathrm{d}t\right] = 2rT - 2\,\mathbb{E}_0^{\mathbb{Q}}\left[\ln\left(\frac{S_T}{S_0}\right)\right]. \tag{4.5}$$

From (4.3) and Itô's Lemma follows:

$$\mathrm{d}f\left(S_t\right) = rf'\left(S_t\right)S_t\,\mathrm{d}t + \sigma_t f'\left(S_t\right)S_t\,\mathrm{d}W_t + \frac{1}{2}f''\left(S_t\right)S_t^2\sigma_t^2\,\mathrm{d}t, \tag{4.6}$$

where we assume f to have a finite second derivative which is continuous (almost) everywhere.
By setting $f\left(S_t\right) = -\log\left(S_t\right)$ on the right hand side of (4.5), it reduces to

$$\mathrm{d}f\left(S_t\right) = -r\,\mathrm{d}t - \sigma_t\,\mathrm{d}W_t + \frac{1}{2}\sigma_t^2\,\mathrm{d}t. \tag{4.7}$$

Taking the Itô-Integral on both sides of (4.7), leaves us with

$$f\left(S_T\right) = f\left(S_0\right) - rT - \int_0^T \sigma_t\,\mathrm{d}W_t + \frac{1}{2}\int_0^T \sigma_t^2\,\mathrm{d}t. \tag{4.8}$$

Now, we set $f\left(S_T\right) = -\log\left(S_T\right)$ and $f\left(S_0\right) = -\log\left(S_0\right)$ on both sides of (4.8) and take the expectation under $\mathbb{Q}$. Due to the assumptions imposed on σ_t, the expectation of the stochastic integral $\int_0^T \sigma_t\,\mathrm{d}W_t$ equals 0 and we have

$$\mathbb{E}_0^{\mathbb{Q}}\left[\int_0^T \sigma_t^2\,\mathrm{d}t\right] = 2rT - 2\,\mathbb{E}_0^{\mathbb{Q}}\left[\ln\left(\frac{S_T}{S_0}\right)\right]. \tag{4.9}$$

[2] Note that we can set the drift of the Brownian Motion $\mu = r$, the risk free rate, which follows directly from the definition of risk neutral probability measures $\mathbb{Q}$, respectively the Girsanov Theorem. An empirical investigation of this has been done by Gadidov and Spruill (2011).

Introducing the forward price of the underlying to be paid at time T as $F_0 \coloneqq S_0 e^{rT}$, reduces (4.9) to

$$\mathbb{E}_0^{\mathbb{Q}}\left[\int_0^T \sigma_t^2 \,\mathrm{d}t\right] = -2\,\mathbb{E}_0^{\mathbb{Q}}\left[\ln\left(\frac{S_T}{F_0}\right)\right]. \tag{4.10}$$

By suitable rearrangements we can rewrite $\ln\left(\frac{S_T}{F_0}\right)$ in (4.10) as

$$\ln\left(\frac{S_T}{F_0}\right) = \ln\left(S_T\right) - \ln\left(F_0\right) - S_T\left(\frac{1}{F_0} - \frac{1}{S_T}\right) + \frac{S_T}{F_0} - 1. \tag{4.11}$$

Since $\int_a^b \frac{1}{x}\,\mathrm{d}x = \ln(b) - \ln(a)$ and $\int_a^b \frac{1}{x^2}\,\mathrm{d}x = \frac{1}{a} - \frac{1}{b}$

$$\ln\left(\frac{S_T}{F_0}\right) = \int_{F_0}^{S_T} \frac{1}{K}\,\mathrm{d}K - S_T\int_{F_0}^{S_T} \frac{1}{K^2}\,\mathrm{d}K + \frac{S_T}{F_0} - 1 = -\int_{F_0}^{S_T} \frac{S_T - K}{K^2}\,\mathrm{d}K + \frac{S_T}{F_0} - 1. \tag{4.12}$$

Because $\mathbb{E}_0^{\mathbb{Q}}\left[\frac{S_T}{F_0} - 1\right] = \frac{\mathbb{E}_0^{\mathbb{Q}}[S_T]}{F_0} - 1 = 0$, as $\mathbb{E}_0^{\mathbb{Q}}\left[S_T\right] = F_0$, we only have to consider $\int_{F_0}^{S_T} \frac{S_T-K}{K^2}\,\mathrm{d}K$, which we can rewrite as follows:[3]

$$\begin{aligned}
\int_{F_0}^{S_T} \frac{S_T - K}{K^2}\,\mathrm{d}K &= \mathbb{1}_{S_T > F_0}\int_{F_0}^{S_T} \frac{S_T - K}{K^2}\,\mathrm{d}K + \mathbb{1}_{S_T < F_0}\int_{F_0}^{S_T} \frac{S_T - K}{K^2}\,\mathrm{d}K \\
&= \mathbb{1}_{S_T > F_0}\int_{F_0}^{S_T} \frac{S_T - K}{K^2}\,\mathrm{d}K - \mathbb{1}_{S_T < F_0}\int_{S_T}^{F_0} \frac{S_T - K}{K^2}\,\mathrm{d}K \\
&= \mathbb{1}_{S_T > F_0}\int_{F_0}^{S_T} \frac{S_T - K}{K^2}\,\mathrm{d}K + \mathbb{1}_{S_T < F_0}\int_{S_T}^{F_0} \frac{K - S_T}{K^2}\,\mathrm{d}K \\
&= \int_{F_0}^{\infty} \frac{(S_T - K)^+}{K^2}\,\mathrm{d}K + \int_0^{F_0} \frac{(K - S_T)^+}{K^2}\,\mathrm{d}K,
\end{aligned} \tag{4.13}$$

where $(S_T - K)^+ \coloneqq \max(S_T - K, 0)$ and $(K - S_T)^+ \coloneqq \max(K - S_T, 0)$ respectively, which corresponds to the payoff of a call/put option.
Note that in the last step of (4.13), we assumed that the (unknown) price of the underlying asset S_T can attain extreme low values (such as 0) and extreme high values (∞).

Finally, taking the expectation of (4.13) under $\mathbb{Q}$ and combining it with (4.10), the expected $\mathbb{IV}$ equals

$$\begin{aligned}
\mathbb{E}_0^{\mathbb{Q}}\left[\int_0^T \sigma_t^2\,\mathrm{d}t\right] &= -2\Bigg(-e^{rT}\int_{F_0}^{\infty} \frac{C_0\left(S_t, K, T\right)}{K^2}\,\mathrm{d}K - e^{rT}\int_0^{F_0} \frac{P_0\left(S_t, K, T\right)}{K^2}\,\mathrm{d}K\Bigg) \\
&= 2e^{rT}\Bigg(\int_0^{F_0} \frac{P_0\left(S_t, K, T\right)}{K^2}\,\mathrm{d}K + \int_{F_0}^{\infty} \frac{C_0\left(S_t, K, T\right)}{K^2}\,\mathrm{d}K\Bigg).
\end{aligned} \tag{4.14}$$

[3](Hull, 2009)

Here $\mathbb{E}_0^{\mathbb{Q}}\left[(S_T - K)^+\right] = e^{rT} C_0\left(S_t, K, T\right)$ denotes the price of an European call option and $\mathbb{E}_0^{\mathbb{Q}}\left[(K - S_T)^+\right] = e^{rT} P_0\left(S_t, K, T\right)$ the price of an European put option, with strike K and maturity T.

Define $M_0\left(K\right) := \min\left(P_0\left(S_t, K, T\right), C_0\left(S_t, K, T\right)\right)$ as the price of the option that is currently 'out-of-the-money', that is for a call option: $S_T < K$ and for a put option: $S_T > K$.

Then,

$$\mathbb{E}_0^{\mathbb{Q}}\left[\mathbb{IV}(0,T)\right] = \mathbb{E}_0^{\mathbb{Q}}\left[\int_0^T \sigma_t^2 \,\mathrm{d}t\right] = 2e^{rT}\left(\int_0^\infty \frac{M_0\left(K\right)}{K^2}\,\mathrm{d}K\right) = MFIV. \tag{4.15}$$

□

5. From MFIV to the CBOE's VIX Formula.

5.1. *Discretisation.* Stock prices and therefore stock price processes are discrete and we only have a limited range of options. Therefore, to apply the MFIV to real-world data, we have to make two fundamental approximations to the theoretical formula derived in Section 4. The following exposition builds on Andersen, Bondarenko and Gonzalez-Perez (2015) and Chicago Board Options Exchange (2019).

1. Assume that, instead of a continuum of strike prices, we have K_i, $i = 1, ..., n$ discrete strike prices. This leads us to

$$2e^{rT}\left(\int_{F_0}^\infty \frac{C_0\left(S_t, K, T\right)}{K^2}\,\mathrm{d}K + \int_0^{F_0} \frac{P_0\left(S_t, K, T\right)}{K^2}\,\mathrm{d}K\right) \approx 2e^{rT}\sum_i \frac{\Delta K_i}{K_i^2} Q\left(K_i\right), \tag{5.1}$$

where

$$\Delta K_i = \frac{\left(K_{i+1} - K_{i-1}\right)}{2}, i = 2, ..., n-1 \tag{5.2}$$

and $\Delta K_1 = K_2 - K_1$ and $\Delta K_n = K_n - K_{n-1}$. Here K_i, $i = 1, ..., N$ and $0 < K_1 < ... < K_f < F_0 < K_{f+1} < ... < K_N$, so F_0 presents the forward price of the asset for time T and K_f denotes the first strike price available below the forward price. Furthermore $Q\left(K_i\right) = M_0\left(K\right)$ is defined as

$$Q\left(K_i\right) := \begin{cases} P_0(K_i), & K_i < K_f \\ \frac{P(K_i)+C(K_i)}{2}, & K_i = K_f \\ C_0(K_i), & K_i > K_f \end{cases}. \tag{5.3}$$

Hence, $Q\left(K_i\right)$ equals the put price, when $K_i < K_f$ an the call price, when $K_i > K_f$ and the mean of the two, when $K_i = K_f$.

2. Since there is the possibility that $K_f \neq F_0$, a correction term is introduced. This leads us to the final formula, which is used in practice by the CBOE to compute the VIX:

$$\hat{\sigma}^2_{MF} = \underbrace{\frac{2e^{rT}}{T}\sum_i \frac{\Delta K_i}{K_i^2} Q\left(K_i\right)}_{\text{Discrete Approximation}} - \underbrace{\frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2}_{\text{Correction Term}}. \tag{5.4}$$

The annualisation factor $\frac{1}{T}$ is introduced, since the VIX is defined for a fixed maturity of thirty calendar days.

Except from r (the risk free rate), which is derived from the corresponding U. S. Treasury bill rate, all inputs of the VIX compuation are directly derived from options data. The forward price F_0, which is not directly observable, is imputed by the call-put parity. Using the pair of put and call options with identical strikes, that have the smallest (absolute) price difference, leads to the following equation for F_0:

$$F_0 = \text{Strike Price} + e^{rT} \times (\text{Call Price} - \text{Put Price}). \tag{5.5}$$

As stated before, the VIX is defined for a fixed maturity of $T_M = \frac{30}{365}$ (30 calendar days), but only a few option maturity dates are given at any time. To account for this complication the CBOE takes a weighted average of $\hat{\sigma}^2_1$ (for near-term options) and $\hat{\sigma}^2_2$ (for next-term options) with the two corresponding expiration dates closest to thirty days and tenors T_1 and T_2, but excluding options with less than seven calendar days to expiry.[4] The tenor is composed of

$$T = (M_{\text{current day}} + M_{\text{settlement day}} + M_{\text{other days}})/\text{Minutes in a year}, \tag{5.6}$$

where $M_{\text{current day}}$ denotes the minutes remaining until midnight, $M_{\text{settlement day}}$ are the minutes from midnight until 8:30 a.m and $M_{\text{other days}}$ stands for the number of minutes between current day and expiration day of the options.
The resulting volatility measure is then quoted in annualised percent:

$$VIX = 100 \times \sqrt{\left(T_1\sigma_1^2\left[\frac{N_{T_2} - N_{30}}{N_{T_2} - N_{T_1}}\right] + T_2\sigma_2^2\left[\frac{N_{30} - N_{T_1}}{N_{T_2} - N_{T_1}}\right]\right) \times \frac{N_{365}}{N_{30}}}, \tag{5.7}$$

where N_{T_1} is the number of minutes to settlement of the near-term options, N_{T_2} is the number of minutes to settlement of the next-term options, N_{30} is the number of minutes in 30 days and N_{365} is the number of minutes in a 365-day year.[5]

6. Calculation of VIX. In this section we replicate the CBOE Volatility Index by applying the model-free formula to S&P 500 (SPX) options data derived

[4]Note that tenor and maturity have distinct meanings. Whereas maturity refers to the initial length of the option upon its initiation, tenor refers to the length of time remaining in a contract (time to expiration).

[5]Note that here the square-root time scaling of volatility is implicitly supposed.

from *barchart.com*[6].
Assume, the day of the calculation is February 18, 2020 and the exact time of day is 15:30 Chicago time (UTC -6h). The near-term options expire on March 13, 2020 (28 days) and the next term options expire on March 20, 2020 (35 days).
In the following the VIX is calculated with the statistical software R Version 3.6.0 for one daily observation.

6.1. *The Data.* Table 1 shows a summary of near-term options data. We have

TABLE 1
Summary Statistics of near-term options data

Statistic	N	Mean	St. Dev.	Min	1st Quartile	3rd Quartile	Max
Strike	240	3,049.812	527.311	1,200	2,803.8	3,401.2	4,500
Bid Call	240	394.191	447.525	0	28.85	572.95	2,173.3
Ask Call	240	398.341	449.583	0.100	29.35	579.750	2,180.1
Midpoint Call	240	396.266	448.552	0.05	29.100	576.350	2,176.7
Bid Put	240	65.749	156.292	0	1.0	49.5	1,115
Ask Put	240	67.308	158.163	0.050	1.200	50.200	1,122.0
Midpoint Put	240	66.529	157.225	0.03	1.123	49.850	1,118.55
(Midpoint) Difference	240	329.737	526.441	−1,118.5	−20.75	575.227	2,176.670

240 near-term put and call options. The mean strike is $3,049.812$ and the difference of midpoints of put and call options is on average 329.737 but takes values, as small as $-1,118.5$ and as high as $2,176.67$.

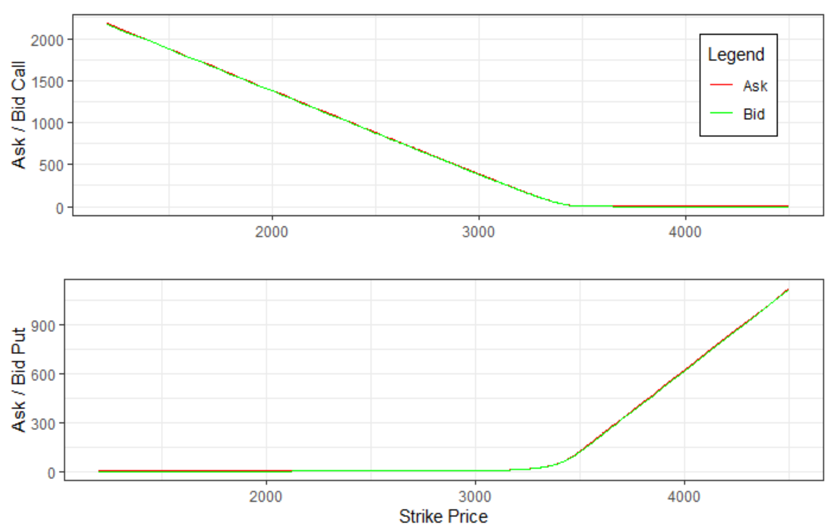

FIG 1. *The upper panel shows ask and bid prices of near-term call options for each strike. The lower panel depicts the same process for ask and bid prices of near-term put options.*

[6]https://www.barchart.com/stocks/quotes/$SPX/options?moneyness=allRows&view=stacked accessed as of February 18. and February 22.

Figure 1 shows near-term call and put options prices split in asks and bids respectively.

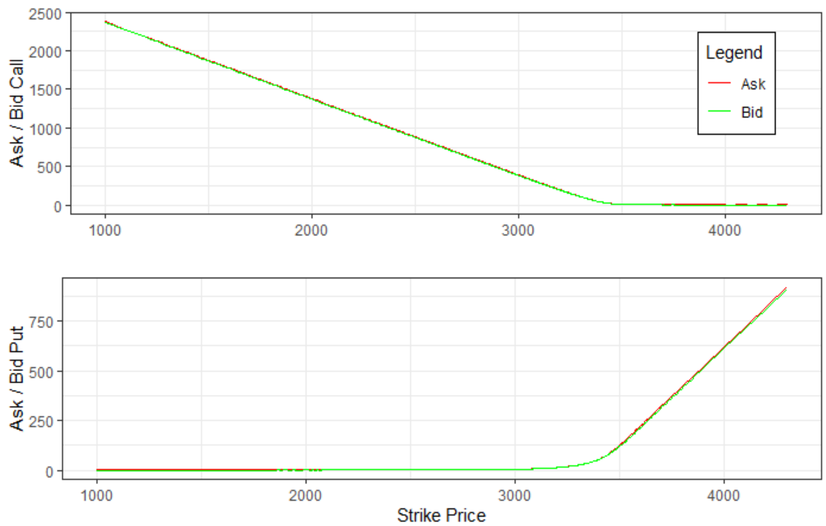

FIG 2. *The upper panel shows ask and bid prices of next-term call options for each strike. The lower panel depicts the same process for ask and bid prices of next-term put options.*

A similar summary is shown in Table 2 for next-term options. Here we have 656 put and call options.

TABLE 2
Summary Statistics of next-term options data

Statistic	N	Mean	St. Dev.	Min	1st Quartile	3rd Quartile	Max
Strike	656	2,836.265	630.928	1,000	2,498.8	3,316.2	4,300
Bid Call	656	590.908	568.637	0	92.475	876.775	2,372.4
Ask Call	656	596.556	571.202	0.05	93.225	885.425	2,381.3
Midpoint Call	656	593.732	569.917	0.03	92.85	881.025	2,376.7
Bid Put	656	49.461	117.155	0	0.4	28.1	914
Ask Put	656	50.902	119.525	0.05	0.55	28.575	922
Midpoint Put	656	50.183	118.337	0.03	0.500	28.35	917.85
(Midpoint) Difference	656	543.550	629.725	−917.82	64.688	880.525	2,376.67

Overall this data is quiet different to near-term options data. The mean strike here is only $2,836.265$, whereas the near term mean strike is $3,049.812$. While the minimum and maximum values of the midpoint difference are quite similar, the average difference differs quite much.

The visualisation of put and call options shows, what can also be observed in the preceding tables: Whereas the overall make up of next-term options is similar to that of near-term options, the scaling is quite different.
The following sections apply this data step-by-step (or rather variable-by-variable) to the discretised formula for model-free implied volatility.

6.2. *Risk-free interest rate r.* With the risk-less bond process B_t proxied by the daily U.S Treasury yield curve rates, the risk-free interest rates with their corresponding maturity t are calculated as

$$r_1 = \frac{\ln B_t}{t} = \frac{\ln 1.61}{28} = 0.0170084,$$

and

$$r_2 = \frac{\ln 1.60}{35} = 0.0134287$$

6.3. *Time to expiration T.* Using the dates and time of day mentioned before and applying them to (5.5), the annualised time to expiration of near-term options equals

$$T_1 = (930 + 570 + 34560)/525600 = 0.068607305936,$$

and for next-term options

$$T_2 = (930 + 570 + 44640)/525600 = 0.087785388121.$$

6.4. *Forward index price F.* To get the forward prices F_1 and F_2, we first have to identify an at-the-money strike price K_f, which the out-of-the-money call and put options are centered around. Therefor, we calculate the midpoint of all asks and bids, and consider the minimum absolute value.

Table 3
Near-term call and put options

	Strike	Bid Call	Ask Call	Bid Put	Ask Put	abs. Difference
...	...	...	...	...	...	...
173	3,365	50.8	51.4	35.4	36.0	15.4
174	3,370	47.4	48.1	37.0	37.6	10.45
175	3,375	44.2	44.8	38.7	39.4	5.45
176	3,380	41.0	41.6	40.6	41.2	0.40
177	3,385	37.9	38.6	42.5	43.1	4.55
178	3,390	35.0	35.6	44.5	45.2	9.55
179	3,395	32.2	32.8	46.6	47.3	14.45
...	...	...	...	...	...	...

Table 3 shows an extract of all near-term put and call options, where the difference of midpoints has been computed. The highlighted row shows the smallest difference. The corresponding at-the-money strike price is $K_{f,1} = 3380$. A similar calculation has been done for next-term options, where the smallest absolute difference is 1.05 and the corresponding at-the-money strike price is also $K_{f,2} = 3380$. Using $K_{f,1}$ and $K_{f,2}$ and the corresponding differences and inserting them into (5.5), yields

$$F_1 = 3380 + e^{0.0170084 \times 0.068607305936} \times 0.4 = 3380.401$$
$$F_2 = 3380 + e^{0.0134287 \times 0.087785388121} \times 1.05 = 3381.053,$$

the forward prices corresponding to near- and next-term options.

6.5. *Selecting out-of-the-money options.* Next, we select all put and call options, that will be included in the calculation. We proceed as follows:

1. For put options select the first strike price $< K_f$ and move step-by-step to lower strike prices. In this process exclude all zero bids. Once two consecutive bids equaling 0 are encountered, no lower strike strike prices are considered. Table 4 shows how this would be done by hand for near-term put options[7]. The highlighted row shows the last considered strike price.

Table 4
Selecting out-of-the-money put options

	Strike	Bid	Ask	Include?
175	3,375	38.700	39.400	Yes
...	...	...	...	...
21	2,350	0.100	0.250	Yes
20	2,300	0.100	0.200	Yes
19	2,250	0.050	0.200	Yes
18	2,200	0.050	0.150	Yes
17	2,150	0.050	0.150	Yes
16	2,100	0	0.150	No
15	2,050	0	0.100	No
14	2,000	0	0.100	Not considered
...	...	...	...	...

2. Then, for call options, select the option with the strike price immediately higher than K_f and move successively upwards. Here too, we leave out all bids equal to zero and stop after encountering two consecutive zero bids. Table 5 shows how this would be done by hand for near-term call options. The highlighted row shows the last considered strike price.

Table 5
Selecting out-of-the-money call options

	Strike	Bid	Ask	Include?
177	3,385	37.900	38.600	Yes
...	...	...	...	...
225	3,625	0.100	0.250	Yes
226	3,630	0.100	0.250	Yes
227	3,640	0.100	0.250	Yes
228	3,650	0.050	0.200	Yes
229	3,675	0.050	0.150	Yes
230	3,700	0	0.150	No
231	3,750	0	0.150	No
233	3,850	0	0.100	Not considered
...	...	...	...	...

A similar process selects next-term out-of-the-money options.

[7]Note that in this particular example no zero-bids occurred and therefore non had to be excluded, except, of course, those that were not considered.

6.6. *Strike price interval* ΔK_i. From (5.2) follows for near-term options, that, for example

$$\Delta K_{2150\text{Put}} = K_{18,\text{Put}} - K_{17,\text{Put}} = 2200 - 2150 = 50,$$

or

$$\Delta K_{3385\text{Call}} = K_{162,\text{Call}} - K_{161,\text{Call}} = 3390 - 3385 = 5,$$

and

$$\Delta K_{3630\text{Call}} = \frac{K_{227,\text{Call}} - K_{225,\text{Call}}}{2} = \frac{(3640 - 3625)}{2} = 7.5.$$

This calculation is similar for next-term options.

6.7. *Midquote prices* $Q(K_i)$. From (5.3) follows for near-term options, that, for example

$$Q(K_{3630,\text{Put}}) \frac{\text{Bid}_{3630,\text{Put}} + \text{Ask}_{3630,\text{Put}}}{2} = \frac{0.10 + 0.25}{2} = 0.175$$

This calculation is similar for next-term options.

6.8. *Contribution by strike.* Using these results, we can calculate the contribution of each strike:

$$\frac{\Delta K_{2150,\text{Put}}}{K^2_{2150,\text{Put}}} e^{r_1 T_1} Q(K_{2150,\text{Put}}) = \frac{50}{2150^2} e^{0.0170084 \times 0.068607305936} \times 0.1 = 0.00000108293.$$

This calculation is done for each strike of a put and call option, for near-term and next-term options equally. Table 6 presents an extract of this procedure:

TABLE 6
Contribution by strike (near-term options)

	Strike	Option Type	Midpoint Price	Contribution
1	2,150	Put	0.1	1.08293e−6
2	2,200	Put	0.1	1.03615e−6
3	2,250	Put	0.125	1.23827e−6
...	...	...	...	...
158	3,370	Put	37.3	1.64710e−5
159	3,375	Put	39.05	1.71927e−5
160	3,380	Put/Call Average	41.1	1.80417e−5
161	3,385	Call	38.25	1.67411e−5
162	3,390	Call	35.3	1.54044e−5
...	...	...	...	...
211	3,640	Call	0.175	1.32475e−7
212	3,650	Call	0.125	1.64688e−7
213	3,675	Call	0.1	1.85663e−7

The contributions by strike summed up and multiplied by $\frac{2e^{rT}}{T}$, yield for near-term options:

$$\frac{2e^{r_1 T_1}}{T_1} \sum_i \frac{\Delta K_i}{K_i^2} Q(K_i) = 0.02053297,$$

and for next-term options:

$$\frac{2e^{r_2 T_2}}{T_2} \sum_i \frac{\Delta K_i}{K_i^2} Q\left(K_i\right) = 0.02075765.$$

6.9. *Correction Term.* For the volatility estimates we now only have to calculate the correction term, which equals for near-term options:

$$\frac{1}{T_1}\left[\frac{F_1}{K_{0,1}} - 1\right]^2 = \frac{1}{0.06909246575}\left[\frac{3380.401}{3380} - 1\right]^2 = 0.001717106,$$

and for next-term options:

$$\frac{1}{T_2}\left[\frac{F_2}{K_{0,2}} - 1\right]^2 = \frac{1}{0.08827054794}\left[\frac{3381.053}{3380} - 1\right]^2 = 0.0001776411.$$

6.10. *Final VIX measure.* With the preceding calculations, (5.4) results in

$$\hat{\sigma}_1^2 = 0.02053297 - 0.0000002037158 = 0.0205327662842,$$

and

$$\hat{\sigma}_2^2 = 0.02075765 - 0.000001099531 = 0.020756550469.$$

Finally, by computing the weighted average of $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$, as in (5.7), we get the VIX value for the corresponding day:[8]

$$\begin{aligned} VIX &= \\ &= 100 \times \sqrt{\left(0.069 \times 0.021\left[\frac{44640 - 43200}{44640 - 34560}\right] + 0.088 \times 0.021\left[\frac{43200 - 34560}{44640 - 34560}\right]\right) \times \frac{525600}{43200}} \\ &= \underline{14.67225 \approx 14.67\%}. \end{aligned}$$

This result is slightly different, from the one calculated by CBOE on the same day, for the same time of day, which is 14.76% (a difference of 0.09%). Reasons for that could be, a different choice of near- and next-term options, or also slightly different data.

6.11. *Further comparison.* In the same manner we can calculate the VIX for the rest of the week, more specifically from February 18 to February 21. The main characteristics of the options used for the calculation are shown in Table 7, where the current day is always February 18.

[8]For styling purposes only rounded values for T_1, T_2, $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ are shown here, the final value, however, is exact.

Table 7
Characteristics of multiple near- and next-term options

	Settlement Day	Min. to expiration	r	F_0	K_f	σ^2	VIX
Near-term	Mar. 16	37440	0.0183167	3340.551	3340	0.02411983	16.03
Next-term	Mar. 23	47520	0.0144313	3340.05	3340	0.02527513	
Near-term	Mar. 18	38880	0.0176383	3340.35	3340	0.02635488	16.58
Next-term	Mar. 25	48960	0.01382364	3340.2	3340	0.02678665	
Near-term	Mar. 20	40320	0.01678584	3340	3335	0.02764996	17.01
Next-term	Mar. 27	50400	0.01342868	3340.551	3340	0.02863699	

The resulting volatility measures are visualised in Figure 3. For comparison, the VIX for the same time of day is also plotted. With the exception of February 19., the replicated calculation increases similar to the VIX.

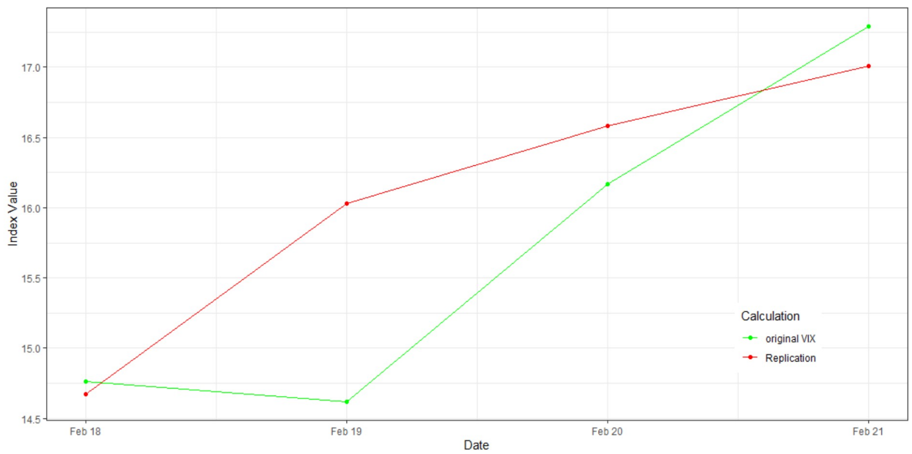

Fig 3. *Visual comparison of the VIX at 15:30 Chicago Time and the replication for the same time of day.*

7. Conclusion. Model-free implied volatility provides a pure market based measure of the future volatility. As such it improves over other implied volatility measures, such as Black-Scholes implied volatility, see Bondarenko (2007). On the other hand, we observed, that there are several approximations and a few assumptions, that have to be made, while deriving the formula and preparing it to be applied to options data. Moreover, the fact that the VIX can only be used for options of the European type, limits it in its application. In addition, it has been shown, that the VIX tends to exaggerate and overestimate actual volatility or does not always predict market fluctuations on time (Kownatzki (2016)). This was, amongst other, the case in 2008 when the market panicked following the Lehman Brothers bankruptcy, as it is shown in Figure 4. In no time it soared to 80 percent, quadrupling its value, only to then immediately fall back to around 40 percent.

As for this thesis, while the theoretical formula may seem obscure at first, it proves

to be quite intuitive. However, when trying to apply this formula to options data, a few things are left open on how to be done. It is, for example, not always clear what exact options with corresponding maturity are to be chosen, or how exactly the U.S Treasury Bill rate is used to approximate the risk free interest rate.
Also, ordinary people trying to replicate this calculation, might be met with complications while extracting options data, since it is not always open to the public completely.

Overall, the forward-looking aspect of the VIX cannot be fully acknowledged, and it has tendencies to over- or under estimate market risk, as well as certain (technical) limitations. Still, the VIX and the underlying model-free implied volatility are, besides other volatility measures, simple and helpful indicators of market movement.

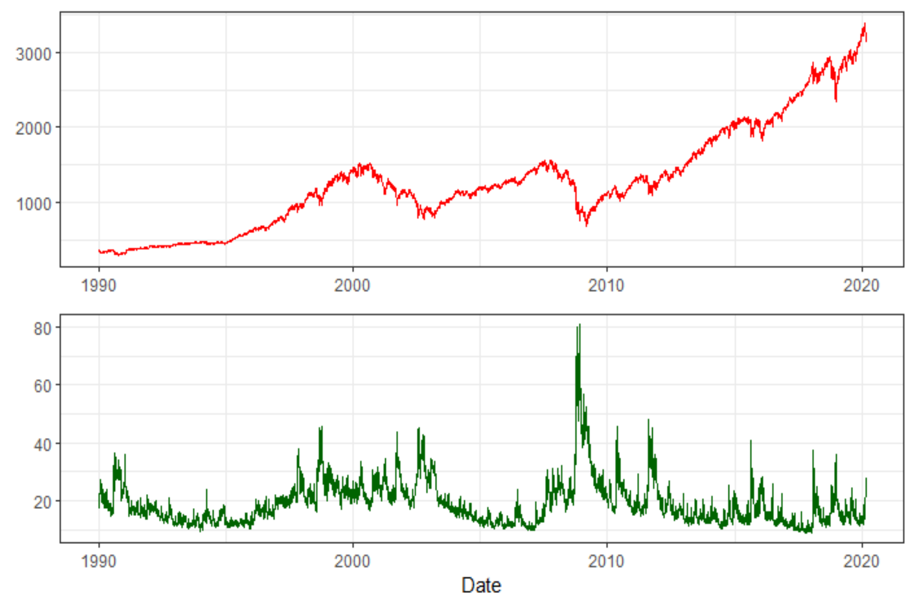

Fig 4. *The upper panel shows the S&P 500 index from Jan. 1990 until Feb. 2020. In the lower panel, we see VIX for the same time span.*

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