DSCI 551 Review

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Contents

Lecture 1

Basic probability concepts:

In general, the probability of an event A occurring is denoted as $P(A)$ and is defined as

$$P(A) = \frac{\text{Number of times event } A \text{ is observed}}{\text{Total number of events observed}}$$

as the number of events goes to infinity.

- We heavily rely on the "frequency of events" to make estimations of specific parameters of interest in a population or system.
- This is basically the foundation of a frequentist approach: relying on the frequency (or "number"!) of events to estimate your parameters of interest.

Law of total probability: When partitioning the sample space (the set of all possible events), the sum of the probabilities of each event should be one.

$$\sum_{E \in \Omega} P(E) = 1.$$

- In general, for a given event A, the law implies that

$$1 = P(A) + P(A^c).$$

Inclusion-exclusion principle:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C),$$

etc.

Odds: are quite helpful in comparing the probability of two events.

$$o = \frac{p}{1-p},$$

where p is the probability of an event.

- This implies

$$p = \frac{o}{o+1}.$$

Measures of central tendency and uncertainty:

Central tendency: a measure denoting a "typical" value in a random variable.

Uncertainty:a measure of how "spread" a random variable is

- Called **parameters** when it comes to a population
- Are estimated via **sample statistics**

Mode: the outcome having the highest probability (discrete) or highest probability density (continuous)

Entropy: a measure of uncertainty defined by

$$H(X) = \sum_x P(X = x) \ln \left(\frac{1}{P(X = x)} \right)$$

or

$$H(X) = \int_x f_X(x) \ln \left(\frac{1}{f_X(x)} \right) \mathrm{d}x.$$

- Always non-negative in the discrete case
- $H(X) = 0 \iff X$ is constant in the discrete case.

Expectation:

$$\mathbb{E}(X) = \sum_x x \cdot P(X = x).$$

or

$$\mathbb{E}(X) = \int_x x \cdot f_X(x)$$

- Can usually be estimated via the **sample mean**

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Variance:

$$\mathrm{Var}(X) = \mathbb{E}\{[X - \mathbb{E}(X)]^2\}.$$

$$\implies \mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2.$$

- the variance is an expectation (specifically, the squared deviation from the mean)
- can usually be estimated via the **sample variance**

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

- Always non-negative, and $\mathrm{Var}(X) = 0 \iff X$ is constant

Standard deviation: The square root of the variance,

$$\sigma_X = \sqrt{\mathrm{Var}(X)}.$$

Lecture 2

- To maximize entropy, you need equal probabilities for all the outcomes in the sample space. This indicates we have a uniform uncertainty over the whole range of possible outcomes.
- Helpful univariate distribution guide: http://www.math.wm.edu/~leemis/chart/UDR/UDR.html

Binomial distribution:

$$X \sim \text{Binomial}(n, \pi)$$

- X is the number of successes in n trials in which each trial has probability π of success, independent of all other trials.
- PMF:
$$P(X = x \mid n, \pi) = \binom{n}{x} \pi^x (1-\pi)^{n-x} \quad \text{for} \quad x = 0, 1, \ldots, n.$$
- Expected value:
$$\mathbb{E}(X) = n\pi$$
- Variance:
$$\text{Var}(X) = n\pi(1-\pi)$$

Families and Parameters:

- We refer to the entire set of Binomial probability distributions as the **Binomial family of distributions**.
- Specifying a value for both π and n results in a unique Binomial distribution.
- Since π and n fully specify a Binomial distribution, we call them **parameters** of the Binomial family, and we call the Binomial family a **parametric family** of distributions.
- There are other ways we can specify the distribution. For instance, specifying the mean and variance is enough to identify a Binomial distribution.
- Exactly which variables we decide to use to identify a distribution within a family is called the family's parameterization.
- The parameterization you use in practice will depend on the information you can more easily obtain

Geometric distribution:

$$X \sim \text{Geometric}(\pi)$$

X is the number of trials **before** experiencing a success, where each trial has probability π of success, independent of all other trials.
- PMF:
$$P(X = x \mid \pi) = \pi(1-\pi)^x \quad \text{for} \quad x = 0, 1, \ldots$$

- Since there is only one parameter, this means that if you know the mean, you also know the variance!
- Expected value:
$$\mathbb{E}(X) = \frac{1-\pi}{\pi}$$
- Variance:
$$\text{Var}(X) = \frac{1-\pi}{\pi^2}$$

Negative Binomial Distribution:

$$X \sim \text{Negative Binomial}(k, \pi)$$

- X is the number of failed trials before experiencing k successes, where each trial has probability π of success, independent of all other trials. - PMF:

$$P(X = x \mid k, \pi) = \binom{k-1+x}{x} \pi^k (1-\pi)^x \quad \text{for} \quad x = 0, 1, \ldots$$

- The Geometric family results with $k = 1$.
- Expected value:

$$\mathbb{E}(X) = \frac{k(1-\pi)}{\pi}.$$

- Variance:

$$\text{Var}(X) = \frac{k(1-\pi)}{\pi^2}.$$

Poisson Distribution:

$$X \sim \text{Poisson}(\lambda)$$

- X is number of events occurring in a fixed interval of time or space, assuming that these events occur with a known constant mean rate (e.g. 3 events per minute or 5 events per meter) and independently of the time since the last event
- PMF

$$P(X = x \mid \lambda) = \frac{\lambda^x \exp(-\lambda)}{x!} \quad \text{for} \quad x = 0, 1, \ldots$$

- Expected value:

$$\mathbb{E}(X) = \lambda.$$

- Variance:

$$\text{Var}(X) = \lambda.$$

Bernoulli Distribution:

$$X \sim \text{Bernoulli}(\pi)$$

- X is equal to one with probability π and equal to zero with probability $1 - \pi$.
- Basically a weighted coin-flip
- A special case of the Binomial family ($n = 1$)
- PMF:

$$P(X = x \mid \pi) = \pi^x (1-\pi)^{1-x} \quad \text{for} \quad x = 0, 1.$$

- Expected value:

$$\mathbb{E}(X) = \pi.$$

- Variance:

$$\text{Var}(X) = \pi(1-\pi).$$

Lecture 3

Joint distributions and marginal distributions:

- A **joint distribution** is the distribution of n-tuples of random variables, where $n \geq 2$.
- The distribution of an individual variable is called the **marginal distribution** (sometimes just "marginal" or "margin").
- The word "marginal" is not really needed when we are talking about a standalone random variable – there is no difference between the "marginal distribution of X" and the "distribution of X." Therefore, we just use the word "marginal" to emphasize that the distribution is being considered in isolation from other related variables in the same process or system.
- Going from the initial marginal distributions to the joint distribution is not a straightforward procedure.
- It requires us to understand the dependency structure among the random variables
- If we assume that all the RVs are independent, then we can just multiply the probabilities from the marginal distributions to find the joint distribution
- If you have a joint distribution, then the marginal distribution of each individual variable follows as a consequence
- Just sum up (discrete) or integrate (continuous), and apply the law of total probability:

$$P(A) = \sum_n P(A \cap B_n),$$

or

$$P(A) = \int_y P(A \cap Y = y).$$

Independence:

- X and Y are **independent** if

$$P(X = x \cap Y = y) = P(X = x) \cdot P(Y = y) \quad \forall_{x,y}$$

- Equivalently:

$$P(X = x \mid Y = y) = P(X = x) \quad \forall_{x,y}$$

- In other words: X and Y are independent if knowing something about one of them tells us nothing about the other.

Dependence Measures:

Covariance:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)],$$

where $\mu_X = \mathbb{E}(X)$ and $\mu_Y = \mathbb{E}(Y)$.

$$\implies \text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

- Note that

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) \iff \text{Cov}(X, Y) = 0,$$

- Also if X and Y are independent then $\text{Cov}(X, Y) = 0$.
- But the reverse implication does **not** hold in general!
- Zero covariance simply indicates that there is no **linear** trend

Pearson's correlation:

- $\mathrm{Cov}(X, Y)$ is dependent on the scale of X and Y.
- e.g. $\mathrm{Cov}(10X, Y) = 10\,\mathrm{Cov}(X, Y)$.
- Pearson's Correlation standardizes the scale according to the standard deviations of X and Y:

$$\begin{aligned}\mathrm{Corr}(X, Y) &= \mathbb{E}\left[\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right] \\ &= \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}.\end{aligned}$$

- Can show that $-1 \leq \mathrm{Corr}(X, Y) \leq 1$ using the Cauchy–Schwarz inequality
- a value of -1 implies a perfect negative linear relationship
- a value of 0 implies no linear relationship (not necessarily independent tho)
- a value of 1 implies a perfect linear relationship

Kendall's τ_K:

- Another measure of correlation
- Measures **monotonic** dependence instead of linear dependence
- Used on samples of observations, not on entire known distributions
- Measures concordance between each pair of observation (x_i, y_i) and (x_j, y_j) with $i \neq j$.
- Concordant means

$$\begin{aligned} x_i < x_j \quad &\text{and} \quad y_i < y_j, \\ &\text{or} \\ x_i > x_j \quad &\text{and} \quad y_i > y_j;\end{aligned}$$

- Discordant means

$$\begin{aligned} x_i < x_j \quad &\text{and} \quad y_i > y_j, \\ &\text{or} \\ x_i > x_j \quad &\text{and} \quad y_i < y_j;\end{aligned}$$

- The formal definition is then

$$\tau_K = \frac{\text{Number of concordant pairs} - \text{Number of discordant pairs}}{\binom{n}{2}},$$

  where n is the sample size.

Mutual Information:

- Defined as

$$H(X, Y) = \sum_x \sum_y P(X = x \cap Y = y) \log\left[\frac{P(X = x \cap Y = y)}{P(X = x) \cdot P(Y = y)}\right].$$

- Not really used in D551.
- Apparently useful in Machine Learning.

Variance of a linear combination:

$$\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X, Y).$$

Lecture 4

Conditional probabilities and conditional distributions

- In general, for events A and B, the conditional probability of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

- If X and Y are two RVs, then $(X \mid Y = y)$ is a probability distribution

$$\sum_x P(X_{Y=y} = x) = \sum_x P(X = x \mid Y = y) = 1.$$

- And hence $Z = (X \mid Y = y)$ is also a random variable

Law of total expectation:

- A marginal mean can be computed from the conditional means and the probabilities of the conditioning variable.
- The formula, known as the **Law of Total Expectation**, is

$$\begin{aligned}\mathbb{E}_Y(Y) &= \sum_x \mathbb{E}_Y(Y \mid X = x) \cdot P(X = x) \\ &= \mathbb{E}_X[\mathbb{E}_Y(Y \mid X)].\end{aligned}$$

- Also note that the law of total probability implies that

$$\begin{aligned}P(Y = y) &= \sum_x P(Y = y \mid X = x) \cdot P(X = x) \\ &= \mathbb{E}_X[P(Y = y \mid X = x)].\end{aligned}$$

Conditional Independence:

- Note that even if X and Y are independent, they might not be independent when conditioning on some other RV Z.
- We say that X and Y are conditionally independent given Z if and only if

$$P(X = x \cap Y = y \mid Z = z) = P(X = x \mid Z = z) \cdot P(Y = y \mid Z = z) \quad \forall_{x,y,z}$$

- Equivalently:

$$P(X = x \mid Y = y,\ Z = z) = P(X = x \mid Z = z) \quad \forall_{x,y,z}$$

Example of variables that are not independent but are conditionally independent:

- See https://pages.github.ubc.ca/MDS-2022-23/DSCI_551_stat-prob-dsci_students/conditional-probabilities.html#conditional-independence
- Let L be a student's lab grade, Q be a student's quiz grade, and S represent whether the student is a Statistics major
- For simplicity, we will consider only Bernoulli random variables, so L and Q will only take on the values "high" and "low", and S takes on the values of "yes" or "no".
- L and Q appear to have a mild positive correlation: when the lab grade is high, the quiz grade is a bit more likely to also be high.

- So L and Q are not independent.
- But L and Q are conditionally independent given S!!
- Intuition: When L is high, it indicates that the student is more likely to be a statistics major, which in turn indicates that their quiz grade Q is more likely to be high. However, this is the **only** reason why high lab grades are correlated with high quiz grades. If you already know that a person is (or is not) a Statistics major, then their lab and quiz grades are completely independent.
- Note that it is also possible to have the opposite case: two variables that are marginally independent, but not conditionally independent given a third variable.

Lecture 5

Continuous random variables:

- Continuous random variables have an uncountably infinite number of possible outcomes
- In practice, we can never measure anything on a continuous scale since any measuring instrument must always round to some precision.
- But we can use a continuous distribution if the quantity being measured has "enough" precision that the distance between two neighbouring measurements is "not a big deal".
- For example, if dealing with monetary quantities which are in the magnitude of millions of dollars, rounding to the nearest cent is no big deal.

Probability Density:

- Continuous RVs have an associated **probability density function** (PDF)
- It measures the density of probability per unit.
- The density of a RV X is denoted as $f_X(x)$
- In general,
$$P(a \le X \le b) = \int_a^b f_X(x)\mathrm{d}x$$
- By the law of total probability, we have
$$\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1.$$
- Note that
$$P(X = a) = \int_a^a f_X(x)\mathrm{d}x = 0 \quad \forall_a$$
- A PDF must be non-negative everywhere

Distribution properties:

- See lecture 1 notes for definitions of mode, entropy, expectation, variance, and standard deviation.
- Recall that entropy can be negative for continuous RVs.

Median and Quantiles:

- The **median** is the outcome for which there is a 50-50 chance of seeing a greater (or lesser) value.
- By definition,
$$P[X \le \text{Median}(X)] = 0.5 = P[X \ge \text{Median}(X)].$$
- Its empirical definition (i.e. the sample statistic version of median) is the "middle value" after sorting the observations from smallest to largest.
- Better than the mean in some ways because it is not as sensitive to outliers and reduces possibilities to two equally likely outcomes.
- The p**-quantile**, denoted $Q_X(p)$, is the outcome with a probability p of getting a smaller outcome.
- By definition,
$$P[X \le Q_X(p)] = p.$$
- So the 0.5-quantile is just the median.

- An empirically-based definition of the p-quantile is the npth largest (rounded up) observation in a sample of size n.

Special Quantiles:

- The 0.25 , 0.5 , and 0.75 -quantiles are called **quartiles**.
- Often referred to as the 1st, 2nd, and 3rd quartiles, and denoted as Q_1, Q_2, and Q_3, respectively.

- The 0.01 , 0.02 , ..., and 0.99 -quantiles are called **percentiles**.
- The p-quantile will often be called the 100pth percentile; for example, the 40th percentile is the 0.4 -quantile, and the 97th percentile is the 0.97-quantile.

Prediction Intervals

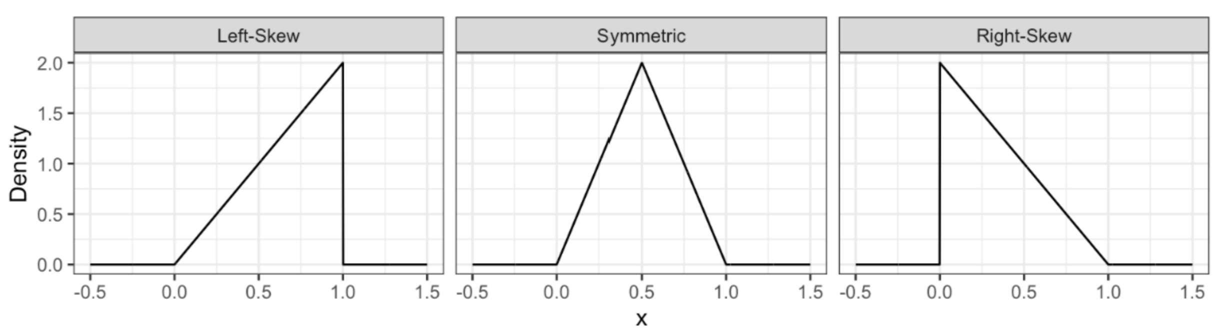
- A $p \times 100\%$ prediction interval $[a, b]$ is such that

$$
\begin{aligned}
P(a \leq X \leq b) &= p, \\
P(X \leq a) &= \frac{1-p}{2}, \\
P(X \geq b) &= \frac{1-p}{2}.
\end{aligned}
$$

- Clearly $a = Q_X(\frac{1-p}{2})$
- Similarly, $b = Q_X(1 - \frac{1-p}{2}) = Q_X(\frac{1+p}{2})$

Skewness

- **Skewness** measures how "lopsided" a distribution is, as well as the direction of the skew.
- If the density is symmetric, then the skewness is 0.
- If the density is more "spread-out" towards the right/positive values, then the distribution is said to be right-skewed (positive skewness).
- If the density is more "spread-out" towards the left/negative values, then the distribution is said to be left-skewed (negative skewness).



- The mathematical definition is given by

$$
\text{Skewness}(X) = \mathbb{E}\left[\left(\frac{X - \mu_X}{\sigma_X}\right)^3\right].
$$

Representing Distributions:

- A PDF (or PMF) is just a representation of a distribution. It is not the distribution itself.
- There are alternative ways to represent a distribution.
- All of these representations capture everything about a distribution, meaning that if one of them is given, the other ones can be derived.

Cumulative Distribution Function

- The **cumulative distribution function** (CDF) is defined by

$$F_X(x) = P(X \leq x)$$

- It can be calculated as

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t.$$

- It is unitless (because probability is unitless)
- A valid CDF $F_X(x)$ must satisfy the following requirements:

1. Must never decrease.
2. It must never evaluate to be < 0 or > 1.
3. $F_X(x) \to 0$ as $x \to -\infty$
4. $F_X(x) \to 1$ as $x \to \infty$.

Survival Function

- The **survival function** $S_X(x)$ is defined by

$$S_X(x) = P(X > x).$$

- It is the CDF "flipped upside down":

$$S_X(x) = 1 - F_X(x)$$

Quantile Function

- The **quantile function** takes a probability p and maps it to the p-quantile.
- Mathematically, this means that

$$Q(p) = F^{-1}(p).$$

- The quantile function is the CDF with the axes swapped, or in other words, the CDF reflected diagonally.

- All these representations (PDF, CDF, survival function, quantile function) fully contain all the information about the distribution; they are different "views" of the same underlying random variable.

Lecture 6

Uniform Distribution:

$$X \sim \text{Uniform}(a, b).$$

- Has uniform density between a and b, where $a < b$. -PDF:

$$f_X(x \mid a, b) = \frac{1}{b-a} \qquad \text{for} \quad a \leq x \leq b.$$

Gaussian (Normal) Distribution:

$$X \sim \mathcal{N}(\mu, \sigma^2).$$

- μ is the mean, σ^2 is the variance
- Follows a "bell-shaped" curve
- PDF:

$$f_X(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \qquad \text{for} \quad -\infty < x < \infty.$$

Log-Normal Distribution:

$$X \sim \text{Log-Normal}(\mu, \sigma^2).$$

- A random variable X is a Log-Normal distribution if the transformation $\ln(X)$ is Normal.
- Can't take the logarithm of a negative number, so must have $X \geq 0$.
- This means that

$$\ln(X) \sim \mathcal{N}(\mu, \sigma^2)$$

- PDF:

$$f_X(x \mid \mu, \sigma^2) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{[\log(x)-\mu]^2}{2\sigma^2}\right\} \qquad \text{for} \quad x \geq 0.$$

Exponential Distribution:

$$X \sim \text{Exponential}(\lambda).$$

- The exponential family is for positive random variables, often interpreted as wait time for some event to happen.
- Characterized by a memoryless property, where after waiting for a certain period of time, the remaining wait time has the same distribution.
- Parameterized by the average rate $\lambda > 0$ at which events happen.
- PDF:

$$f_X(x \mid \lambda) = \lambda e^{-\lambda x} \qquad \text{for} \quad x \geq 0.$$

Beta Distribution:

$$X \sim \text{Beta}(\alpha, \beta).$$

- The Beta family of distributions is defined for random variables taking values between 0 and 1 , so is useful for modelling the distribution of proportions.
- Has the Uniform distribution as a special case.
- PDF:

$$f_X(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad \text{for} \quad 0 \leq x \leq 1.$$

Weibull Distribution:

$$X \sim \text{Weibull}(\lambda, k).$$

- A generalization of the Exponential family, which allows for an event to be more likely the longer you wait.
- characterized by two parameters, a scale parameter $\lambda > 0$ and a shape parameter $k > 0$ (where $k = 1$ results in the Exponential family).
- PDF:

$$f_X(x \mid \lambda, k) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \exp^{-(x/\lambda)^k} \qquad \text{for} \quad x \geq 0.$$

Gamma Distribution:

$$X \sim \text{Gamma}(k, \theta).$$

- A two-parameter distribution family with support on non-negative numbers.
- PDF:

$$f_X(x \mid k, \theta) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} \exp(-x/\theta) \qquad \text{for} \quad x \geq 0.$$

Relevant R Functions:

- Always take the form <x><dist>

- <x> is one of d, p, q, or r:
- d: density function (PDF), i.e. $f_X(x)$
- p: cumulative distribution function (CDF), i.e. $F_X(x)$
- q: quantile function (inverse of CDF), i.e. $Q_X(p)$
- r: random number generator.

- <dist> is an abbreviation of a distribution family
- Some abbreviations for <dist>:
- unif: Uniform (continuous).
- norm: Normal (continuous).
- lnorm: Log-Normal (continuous).
- geom: Geometric (discrete).
- pois: Poisson (discrete).
- binom: Binomial (discrete).

Continuous Conditional Distributions:

- In the continuous world, we replace probabilities with densities.
- For example, if X is continuous then

$$P(X = x \mid X \geq 2500) = \frac{P(X = x)}{P(X \geq 2500)}$$

  is not a useful calculation, since $P(X = x) = 0$.
- Instead, we calculate the conditional density of X:

$$f_{X \mid X \geq 2500}(x) = \begin{cases} \frac{f_X(x)}{P(X \geq 2500)} & \text{for } x \geq 2500, \\ 0 & \text{for } x < 2500 \end{cases}.$$

- In general,

$$f_{X|X\geq a}(x) = \begin{cases} \frac{f_X(x)}{P(X\geq a)} & \text{for } x \geq a, \\ 0 & \text{for } x < a \end{cases}.$$

- Similarly, if Y is continuous then

$$P(X = x \mid Y = 5) = \frac{P(X = x)}{P(Y = 5)}$$

is not a useful calculation, since $P(Y = 5) = 0$.
- Instead, we calculate

$$f_{X|Y=5}(x) = \frac{f_{X,Y}(x, 5)}{f_Y(5)}.$$

- In general,

$$f_{X|Y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)}.$$

Lecture 7

Continuous Independence:

- As per usual, in the continuous case we replace probabilities for densities
- Two continuous RVs X and Y are independent if

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \quad \forall_{x,y} \qquad (1)$$

- Equivalently:

$$f_{Y|X}(y) = f_Y(y) \quad \forall_{x,y} \qquad (2)$$

- Intuitively, (1) means that when slicing the joint density at various points along the x-axis or y-axis, the resulting 1-D function will be the same, up to some multiplication factor.
- e.g $f_{X,Y}(1,y) = f_X(1) \cdot f_Y(y)$ and $f_{X,Y}(2,y) = f_x(2) \cdot f_Y(y)$
- This means that all the slices along an axis will have the same shape
- Note that $f_{X,Y}(1,y)$ is **not** a PDF. The conditional PDF is

$$f_{Y|X=1}(y) = \frac{f_{X,Y}(1,y)}{f_X(1)} = f_Y(y),$$

  which also follows directly from (2).

Direction of Dependence

- Contour plots are useful for determining if two RVs are dependent or not
- They also help determine the direction of dependence
- For two positively correlated variables, there is overall tendency of the contour lines to point up and to the right
- For two negatively correlated variables, there is overall tendency of the contour lines to point down and to the right

Estimating the marginal distribution from the conditional:

- By the law of total probability, we have

$$f_Y(y) = \int_x f_{X,Y}(x,y)\ \mathrm{d}x = \int_x f_{Y|X}(y)\ f_X(x)\ \mathrm{d}x.$$

- This is equivalent to

$$f_Y(y) = \mathbb{E}_X[f_{Y|X}(y)].$$

- If we know the conditional densities and have a sample of X as observed values $x_1, \dots, x_n$, then we can empirically estimate

$$f_Y(y) \approx \frac{1}{n}\sum_{i=1}^{n} f_{Y|X_i}(y).$$

- Similarly, for CDFs we have

$$F_Y(y) = \int_x F_{Y|X}(y)\ f_X(x)\ \mathrm{d}x = \mathbb{E}_X[F_{Y|X}(y)].$$

- And we can estimate the CDF empirically with

$$F_Y(y) \approx \frac{1}{n}\sum_{i=1}^{n} F_{Y|X_i}(y).$$

Estimating the marginal mean from the conditional:

- As in the discrete case, we have

$$\mathbb{E}(Y) = \int_x m(x)\ f_X(x)\ \mathrm{d}x = \mathbb{E}_X[\mathbb{E}_Y(Y \mid X = x)].$$

- Let $m(x) = \mathbb{E}(Y \mid X = x)$. Then the above becomes

$$\mathbb{E}(Y) = \mathbb{E}[m(X)]$$

- In machine learning, a function like $m(x)$ is called a **model function**.
- When you fit a model using Supervised Learning, you usually end up with an estimate of m(x).
- Then, using the empirical mean, we have

$$E(Y) \approx \frac{1}{n}\sum_{i=1}^{n} m(x_i).$$

Estimating marginal quantiles from the conditional:

- Unfortunately, if you have the p-quantile of Y given X = x, then this does not give much info about the p-quantile of Y.
- This is because the average of quantiles is a horrible estimate for quantiles
- e.g. a mean of medians is a horrible estimate for the median.
- So we could calculate the p-quantile of Y as an average using the same ideas as above, but the estimate would be almost worthless.

Mutivariate Gaussian Distribution:

- A multivariate distribution family in which:

1. All RVs must have Gaussian marginals
2. All linear combinations of the marginals are Gaussian (called **Gaussian dependence**)

- In general, it has two parameters:

1. A mean vector, $\vec{\mu}$
2. A covariance matrix, $\mathbf{\Sigma}$

- $\vec{\mu}$ is just a vector of the marginal means.
- e.g. in the bivariate case we have

$$\vec{\mu} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$$

- $\mathbf{\Sigma}$ is a symmetric matrix of marginal covariances, where the diagonal entries are the marginal variances.
- It is symmetric because $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$
- The diagonal entries are the marginal variances because $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$.

- We can also look at a correlation matrix, $\boldsymbol{P}$
- Recall that

$$\rho_{XY} = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{\sigma_{XY}}{\sqrt{\sigma_X^2\sigma_Y^2}}.$$

- This implies that $\rho_{XX} = 1$.
- Thus in the bivariate case the correlation matrix is

$$\begin{aligned}\mathbf{P} &= \begin{pmatrix} \rho_{XX} & \rho_{XY} \\ \rho_{XY} & \rho_{YY} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \rho_{XY} \\ \rho_{XY} & 1 \end{pmatrix}.\end{aligned}$$

- Note that the covariance matrix is always used in the parameterization of a multivariate Gaussian distribution, although the correlation matrix has other uses (e.g PCA)
- The above means that to fully specify a d-dimensional Gaussian distribution, we need:
- the means and variances of all d random variables, and
- the $\binom{d}{2}$ covariance or correlations between each pair of random variables

Properties of the Mutivariate Gaussian Distribution:

- For the multivariate Gaussian distribution, uncorrelated marginals implies independent marginals.

- Marginal distributions are Gaussian, by definition.
- The marginal distribution of a subset of variables can be obtained by just taking the relevant subset of means, and the relevant subset of the covariance matrix.

- Linear combinations of marginals are Gaussian, by definition.
- If we want to find the mean and variance of a linear combination, we apply the rules for expectation and variance of linear combinations:

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$$

  and

$$\operatorname{Var}(aX + bY) = a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y) + 2ab \operatorname{Cov}(X, Y).$$

- In the case of a multivariate Gaussian distribution this yields

$$\mathbb{E}[aX + bY] = a\mu_X + b\mu_Y$$

  and

$$\operatorname{Var}(aX + bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}.$$

- In the case of the variance for linear combinations of $n \geq 3$ marginals, the formula will need to include all pairwise covariances.

- Conditional distributions are Gaussian.
- If (X, Y) have a bivariate Gaussian distribution then the distribution of Y given that $X = x$ is also Gaussian.

- Its mean is
$$\mu_{Y|X=x} = \mu_Y + \frac{\sigma_Y}{\sigma_X}\rho_{XY}(x - \mu_X).$$
- Its variance is
$$\sigma^2_{Y|X=x} = (1 - \rho^2_{XY})\sigma^2_Y.$$
- The conditional mean is linear in x, passes through the mean (μ_X, μ_Y), and has a steeper slope with higher correlation.
- The conditional variance is smaller than the marginal variance, and gets smaller with higher correlation. (But it does not depend on x)